TRIGONOMETRY

SPARK CHARTS™

BACKGROUND AND DEFINITIONS

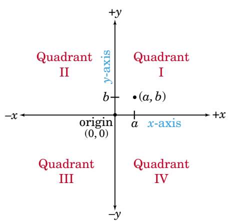
TRIGONOMETRY DEFINED

Trigonometry is the study of relationships between angles and lengths, especially in triangles.

THE COORDINATE PLANE

The **Cartesian** (or **coordinate**) **plane** is an infinite plane with two special perpendicular lines (called **axes**). A **point** on the plane is identified by an **ordered pair** of **coordinates**—the distances from the two axes.

- ***x*-axis:** Usually, the **horizontal** axis of the coordinate plane.
- ***y*-axis:** Usually, the **vertical** axis of the coordinate plane.
- **Origin:** $(0,0)$; the point of intersection of the x-axis and the y-axis.
- **Quadrants:** The four regions of the coordinate plane created by the intersection of the two axes. By convention, they are numbered I, II, III, and IV counterclockwise starting with the upper right quadrant.



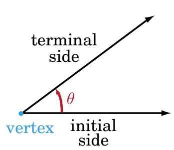
FUNCTIONS

For more about functions, see the SparkChart on Pre-calculus.

- **Function:** A rule for generating values: for every value you plug into the function, there's a unique value that comes out. Often denoted as $f(x)$: for every value $x = a$ that you plug in, $f(a)$ is the result.
- **Domain:** The set of possible incoming values, x, for a function $f(x)$.
- **Range:** The set of possible outcomes of $f(x)$.
- **Graphing a function:** The process of plotting all the points $(x, f(x))$ in the coordinate plane.
- **Vertical line test:** Checks if something is a function by looking at its graph: any vertical line in the coordinate plane must intersect the graph no more than once. For every x-value, there is at most one y-value.

ANGLES

- **Ray:** Part of a line with one fixed endpoint extending without bound in one direction.
- Two rays that share a common endpoint create an **angle.** The common endpoint is called the **vertex** of the angle.
- Often we think of an angle as being formed by rotating a ray **clockwise** or **counterclockwise.** We then distinguish between the **initial side** (starting position of the ray) and the **terminal side** (end position of the ray) of the angle.



MEASURING ANGLES

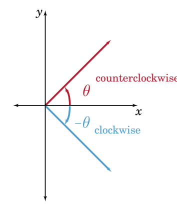
We measure angles to see how wide or narrow they are. There are two standard ways of measuring angles:

- **Degree (°):** A unit of angular measure in which a complete revolution is 360°; each degree is subdivided into 60 **minutes**, and each minute is subdivided into 60 **seconds**.
- **Radians (rad):** A unit of angular measure in which a complete revolution measures 2π radians. The radian measure of an angle is the length of an arc of the unit circle cut off by that angle.
- Converting between degrees and radians:

$$1 \text{ revolution} = 360° = 2\pi \text{ rad}$$

$$1 \text{ rad} = \frac{180°}{\pi} = \frac{1}{2\pi} \text{ revolution}$$

$$1° = \frac{\pi}{180} \text{ rad} = \frac{1}{360} \text{ revolution}$$



θ and $-\theta$ are both in standard position.

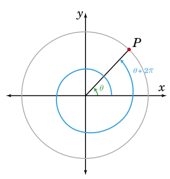
- **Labeling angles:** When viewed as geometric shapes, angles are usually named with capital letters—often after the vertex (**Ex:** $\angle A$)—and often measured in degrees. When viewed as rotations, angle measures are often given Greek letters (**Ex:** θ, ϕ) and are usually specified in radians. In trigonometry—unlike in geometry—the distinction between angles and their measure is often not made (**Ex:** $A = 45°$).

ANGLES IN THE COORDINATE PLANE

- **Standard position:** An angle is in standard position when its vertex lies at the origin and its initial side lies along the positive x-axis.
- **Unit circle:** The circle of radius 1 centered at the origin.

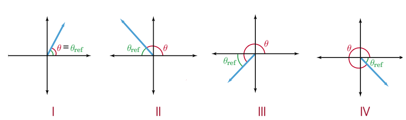
Any angle gives a point on the unit circle.

- **Negative and positive angles:** By convention, positive angles are measured counterclockwise; negative angles are measured clockwise.
- **Multiple rotations:** The angles $\theta,\ \theta + 2\pi,\ \theta - 2\pi,\ \theta \pm 4\pi, \ldots$ etc., all define the same point on the unit circle.



θ and $\theta + 2\pi$ both give point P on the unit circle.

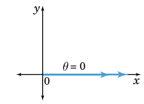
- **Reference angle:** For any angle in standard position, the reference angle is the positive acute angle formed by its terminal side and the x-axis.



θ and its reference angle θ_{ref}

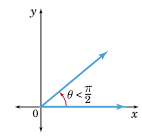
TYPES OF ANGLES

- **Zero** angle: $0° = 0$ rad. The initial side and the terminal side coincide. In standard position, the terminal side is on the positive x-axis.



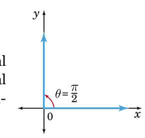
Zero angle

- **Acute** angle: less than $90° = \frac{\pi}{2}$ rad. Between a zero angle and a right angle. In standard position the terminal side is in Quadrant I.



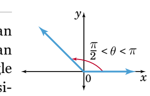
Acute angle

- **Right** angle: $90° = \frac{\pi}{2}$ rad. The initial side is perpendicular to the terminal side. In standard position, the terminal side is on the positive y-axis.



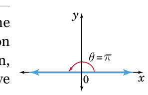
Right angle

- **Obtuse** angle: greater than $90° = \frac{\pi}{2}$ rad and less than $180° = \pi$ rad. Between a right angle and a straight angle. In standard position, the terminal side is in Quadrant II.



Obtuse angle

- **Straight** angle: $180° = \pi$ rad. The initial side and the terminal side lie on the same line. In standard position, the terminal side is on the negative x-axis.



Straight angle

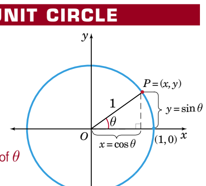
- **Oblique** angle: either acute or obtuse (not zero, right, or straight).
- **Complementary** angles: Two angles that sum to a right angle.
- **Supplementary** angles: Two angles that sum to a straight angle.
- **Coterminal** angles: Two angles in standard position whose terminal sides coincide. Angles θ and ϕ are coterminal if $\theta = \phi + 2k\pi$ for some integer k (positive or negative).

TRIGONOMETRIC FUNCTIONS

The two ways of thinking about angles—as rotations from the standard position, or as static shapes in a geometric figure—give two ways of thinking about trigonometric functions.

...BASED ON THE UNIT CIRCLE

Any angle θ defines a point $P = (x, y)$ on the unit circle. Manipulating the coordinates of this point give the trigonometric functions of θ.



Sine and cosine of θ

Sine: $\sin\theta = y$, the y-coordinate of P. For all θ, $-1 \le \sin\theta \le 1$.

Cosine: $\cos\theta = x$, the x-coordinate of P. For all θ, $-1 \le \cos\theta \le 1$.

Tangent: $\tan\theta = \frac{y}{x}$, the slope of the line $\overline{OP}$.

Secant: $\sec\theta = \frac{1}{x} = \frac{1}{\cos\theta}$

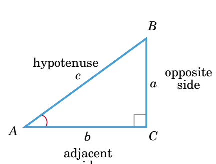
Cosecant: $\csc\theta = \frac{1}{y} = \frac{1}{\sin\theta}$

Cotangent: $\cot\theta = \frac{x}{y} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

Because θ and $\theta + 2k\pi$ define the same point on the unit circle, all trigonometric functions are **periodic** with a **period** of 2π (sin, cos, sec, csc), or π (tan, cot).

...BASED ON A RIGHT TRIANGLE

For an acute angle A, we can define the trigonometric functions by looking at the ratios of the side lengths of a right triangle ABC with a right angle at C. We will use "A" to refer to the point A, the angle $\angle CAB$, and the measure of angle $\angle CAB$.



Sine: $\sin A = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}$

Here, $\sin A < 1$ *for all acute angles* A.

Cosine: $\cos A = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}}$

Again, $\cos A < 1$ *for all acute angles* A.

Tangent: $\tan A = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}$

MNEMONIC: **SOHCAHTOA: S**ine is **O**pposite over **H**ypotenuse; **C**osine is **A**djacent over **H**ypotenuse; **T**angent is **O**pposite over **A**djacent.

Cosecant: $\csc A = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{1}{\sin A}$

Secant: $\sec A = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{1}{\cos A}$

Cotangent: $\cot A = \frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{1}{\tan A}$

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“A FIGURE WITH CURVES ALWAYS OFFERS A LOT OF INTERESTING ANGLES.”

WESLEY RUGGLES

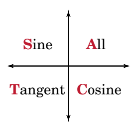
TRIGONOMETRIC FUNCTIONS (CONTINUED)

COMPARING THE TWO DEFINITIONS

The right-triangle definitions give the same trig values as the unit-circle definitions for acute angles. For angles greater than $90°$, apply the right-triangle definition to a reference angle and attach the appropriate $\pm$ sign.

Func.	Unit circle	Right triangle	Domain	Range	Period	Sign in quadrant I	II	III	IV
$\sin\theta$	y	$\frac{\text{opp}}{\text{hyp}}$	all real numbers	$[-1,1]$	2π	+	+	−	−
$\cos\theta$	x	$\frac{\text{adj}}{\text{hyp}}$	all real numbers	$[-1,1]$	2π	+	−	−	+
$\tan\theta$	$\frac{y}{x}$	$\frac{\text{opp}}{\text{adj}}$	all reals except $k\pi+\frac{\pi}{2}$	all real numbers	π	+	−	+	−
$\csc\theta$	$\frac{1}{y}$	$\frac{\text{hyp}}{\text{opp}}$	all reals except $k\pi$	$(-\infty,-1]\cup[1,+\infty)$	2π	+	+	−	−
$\sec\theta$	$\frac{1}{x}$	$\frac{\text{hyp}}{\text{adj}}$	all reals except $k\pi+\frac{\pi}{2}$	$(-\infty,-1]\cup[1,+\infty)$	2π	+	−	−	+
$\cot\theta$	$\frac{x}{y}$	$\frac{\text{adj}}{\text{opp}}$	all reals except $k\pi$	all real numbers	π	+	−	+	−

MNEMONIC: **A**ll **S**tudents **T**ake **C**alculus tells you which of the three primary trig functions (sine, cosine, and tangent) are positive in which Quadrant: I: **A**ll; II: **S**ine only; III: **T**angent only; IV: **C**osine only.

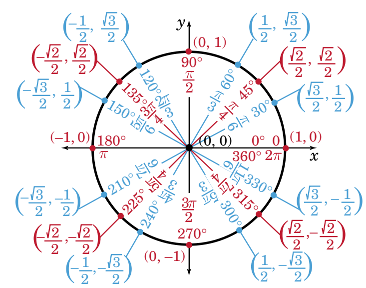


SPECIAL TRIGONOMETRIC VALUES

θ (°)	θ (rad)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0°	0	$0=\frac{\sqrt{0}}{2}$	1	0	undefined	1	undefined
30°	$\frac{\pi}{6}$	$\frac{1}{2}=\frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	$1=\frac{\sqrt{4}}{2}$	0	undefined	1	undefined	0
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2	$-\frac{\sqrt{3}}{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
180°	π	0	-1	0	undefined	-1	undefined
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	-2	$-\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	-2	$\frac{\sqrt{3}}{3}$
270°	$\frac{3\pi}{2}$	-1	0	undefined	-1	undefined	0
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2	$-\frac{\sqrt{3}}{3}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	-2	$\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
360° = 0°	$2\pi=0$	0	1	0	undefined	1	undefined

The angle multiples of $30°$ and $45°$ have easy-to-write trig functions and come up often. The trig functions of most other angles are difficult to write exactly; they are most often given as decimal approximations.

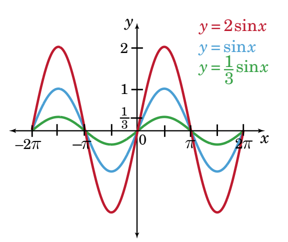
Common angles and the points they define on the unit circle



GRAPHING SINUSOIDAL FUNCTIONS

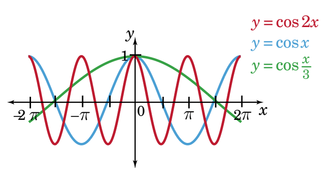
A **sinusoidal** function is any function that looks like a sine or cosine curve.

Amplitude: The amplitude of a sinusoidal function is half the vertical distance from a crest (highest point) to a trough (lowest point).



The amplitudes of these functions are 2, 1, and $\frac{1}{3}$.
The period is 2π for all three functions.

Period: The period of any repeating function is the length of the smallest repeating unit. The period p is the smallest number such that $f(x)=f(x+p)$ for all x.



The periods of these functions are π, 2π, and 6π.
The amplitude is 1 for all three functions.

GRAPHING $y = A\sin B(x-h)+k$ AND $y = A\cos B(x-h)+k$

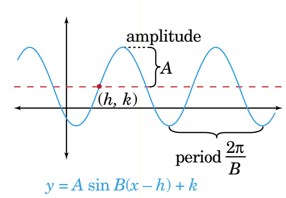
$|A|$ is the **amplitude.**

k is the is the **average value:** halfway between the maximum and the minimum value of the function.

$\frac{2\pi}{B}$ is the **period.** There are B cycles in every interval of length 2π; so $\frac{B}{2\pi}$ is the **frequency**.

h is the **phase shift,** or how far the beginning of the cycle is from the y-axis.

The basic shape of the function will stay the same. The sine curve will start at (h,k) as though it were the origin and go up if A is positive (down if A is negative). A cosine curve will start at (h,k) at the crest if A is positive (trough if A is negative).



CONVERTING EQUATIONS

Cosine and sine functions differ only by a phase shift.

$$\cos\theta=\sin\left(\theta+\frac{\pi}{2}\right)$$

$$\sin\theta=\cos\left(\theta-\frac{\pi}{2}\right)$$

INVERSE FUNCTIONS

INVERSE TRIGONOMETRIC FUNCTIONS

An **inverse function** f^{-1} undoes what the original functions did: if $y=f(x)$, then $x=f^{-1}(y)$. The domain of $f^{-1}(x)$ is the range of $f(x)$ and vice versa. **Ex:** The inverse function of $f(x)=2x+3$ is $f^{-1}(x)=\frac{x-3}{2}$.

- If the original function does not pass the “horizontal line test”—i.e., if it takes on the same value more than once—we restrict the domain of the original function before we take the inverse. **Ex:** $f(x)=x^2$ on the whole real line has no inverse, but the function $f(x)=x^2$ on only the positive reals has the inverse $f^{-1}(x)=\sqrt{x}$.
- Graphically, the inverse function $y=f^{-1}(x)$ has the same shape as the original function, but is reflected over the slanted line $y=x$.

All the trig functions take on the same value many times. To construct inverse functions, we restrict the domains as follows:

Function	Domain	Range
$\sin^{-1}x=\arcsin x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1}x=\arccos x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}x=\arctan x$	all real numbers	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$\csc^{-1}x=\text{arccsc}x$	$(-\infty,-1]\cup[1,+\infty)$	$\left(0,\frac{\pi}{2}\right]\cup\left(\pi,\frac{3\pi}{2}\right]^*$
$\sec^{-1}x=\text{arcsec}x$	$(-\infty,-1]\cup[1,+\infty)$	$\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)^*$
$\cot^{-1}\theta=\text{arccot}x$	all real numbers	$(0,\pi)$

*There is no uniform agreement about which branch of cosecant and secant the inverse functions should follow for $x<0$. Those given here work well with slope formulas from calculus.

CONTINUED ON OTHER SIDE

TRIGONOMETRIC IDENTITIES

These identities are true for all angles.

PYTHAGOREAN IDENTITIES

$\sin^2 A + \cos^2 A = 1$

$1 + \tan^2 A = \sec^2 A$

$\cot^2 A + 1 = \csc^2 A$

ANGLE SUM IDENTITIES

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

ANGLE DIFFERENCE IDENTITIES

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

DOUBLE-ANGLE IDENTITIES

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$
$= 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

TRIPLE-ANGLE IDENTITIES

$\sin 3A = 3 \sin A - 4 \sin^3 A$

$\cos 3A = 4 \cos^3 A - 3 \cos A$

$\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3\tan^2 A}$

RECIPROCAL AND QUOTIENT IDENTITIES

$\sin\theta = \dfrac{1}{\csc\theta} = \dfrac{\tan\theta}{\sec\theta}$

$\cos\theta = \dfrac{1}{\sec\theta} = \dfrac{\cot\theta}{\csc\theta}$

$\tan\theta = \dfrac{1}{\cot\theta} = \dfrac{\sin\theta}{\cos\theta}$

$\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{\cot\theta}{\cos\theta}$

$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\tan\theta}{\sin\theta}$

$\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

SYMMETRIES

Periodicity

$\sin\theta = \sin(\theta + 2k\pi)$

$\cos\theta = \cos(\theta + 2k\pi)$

$\tan\theta = \tan(\theta + k\pi)$

Even functions

Unchanged if flipped over the x-axis.

$\cos(-\theta) = \cos\theta$

$\sec(-\theta) = \sec\theta$

Odd functions

Unchanged if rotated $180°$. Equivalently, flipping over x-axis is the same as flipping over y-axis.

$\sin(-\theta) = -\sin\theta$

$\tan(-\theta) = -\tan\theta$

$\csc(-\theta) = -\csc\theta$

$\cot(-\theta) = -\cot\theta$

COFUNCTION IDENTITIES

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$

$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta \qquad \csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

PRODUCT-TO-SUM IDENTITIES

$\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$

$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$

Products of like terms use cosines; unlike terms use sines.

SUM-TO-PRODUCT IDENTITIES

$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

HALF-ANGLE IDENTITIES

$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$

$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$

$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$
$= \dfrac{\sin A}{1+\cos A} = \dfrac{1-\cos A}{\sin A}$

Choice of $\pm$ sign depends on the quadrant in which $\frac{A}{2}$ lies.

SQUARE-TO-LINEAR IDENTITIES

$\sin^2 A = \dfrac{1-\cos 2A}{2}$

$\cos^2 A = \dfrac{1+\cos 2A}{2}$

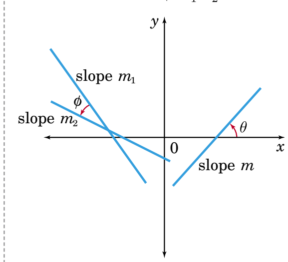
$\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$

ANGLE BETWEEN TWO LINES

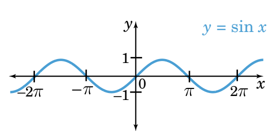
A line with slope m makes an angle of $\arctan m$ with the positive x-axis.

The (counterclockwise) angle ϕ from a line of slope m_1 to a line of slope m_2 is defined by

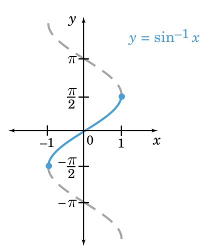
$$\tan\phi = \frac{m_2 - m_1}{1 + m_1 m_2}.$$



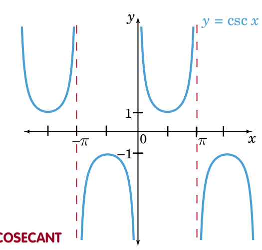
GRAPHING TRIGONOMETRIC FUNCTIONS



SINE



ARCSINE



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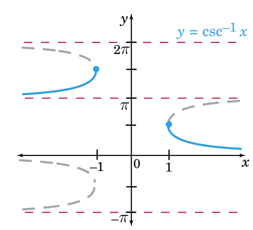

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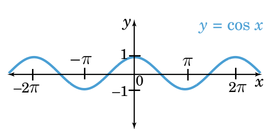
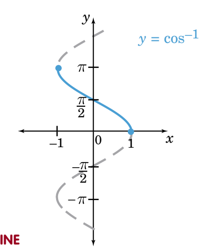


COSINE



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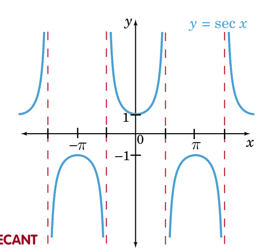
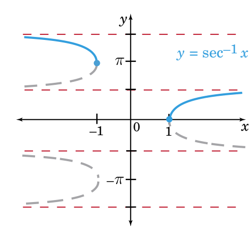


SECANT



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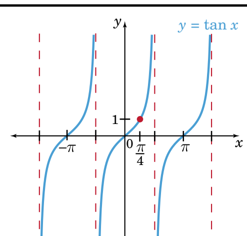
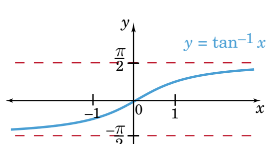
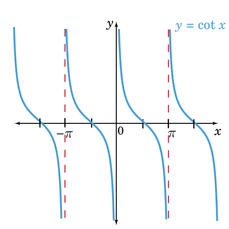


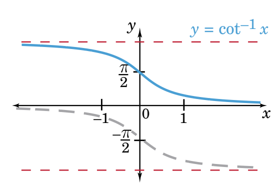
TANGENT



ARCTANGENT



COTANGENT

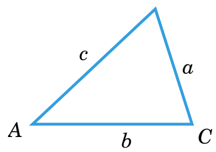


ARCCOTANGENT

TRIANGLE FORMULAS

For more about triangles, see the SparkChart on Geometry.

Triangle with sides of length a, b, c with opposite angles of measure A, B, C, respectively.



SUM OF ANGLES

In any triangle, the sum of the angles is the same:

$$A + B + C = 180°$$

LAW OF SINES

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The side of a triangle is proportional to the sine of the opposite angle.

Beware of ambiguity when using the Law of Sines to calculate angles, since $\sin A = \sin(180° - A)$.

The largest angle is always opposite the longest side.

LAW OF COSINES

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C \qquad \text{Also, } \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

PYTHAGOREAN THEOREM

The Law of Cosines reduces to the Pythagorean theorem when the angle cosined is a right angle. If $C = 90°$, then

$$a^2 + b^2 = c^2.$$

AREA FORMULAS

Area $= \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B.$

Heron's Formula:

$$A = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the **semiperimeter**: $s = \frac{a+b+c}{2}$.

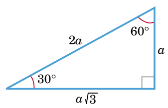
SOLVING RIGHT TRIANGLES

"Solving" a triangle means knowing all six measurements—the lengths of the three sides and the measures of the three angles.

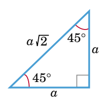
BASICS

- **Adjacent side:** Side adjacent to a given acute angle.
- **Opposite side:** Side opposite a given acute angle.
- **Hypotenuse:** Side opposite the right angle.

SPECIAL RIGHT TRIANGLES



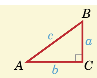
$30°$–$60°$–$90°$ right triangle



$45°$–$45°$–$90°$ right triangle

TECHNIQUES FOR SOLVING RIGHT TRIANGLES

Let's say $C = 90°$. There are five unknown quantities: a, b, c, A, B.

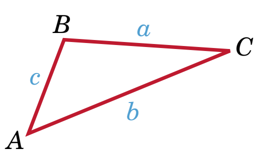
If you know…	…you can use…
Acute angle and opposite side (say, A and a)	$c = \frac{a}{\sin A}$ $b = \frac{a}{\tan A}$ $B = 90° - A$ 
Acute angle and adjacent side (say, A and b)	$c = \frac{b}{\cos A}$ $a = b\tan A$ $B = 90° - A$
Acute angle only (say, A)	$B = 90° - A$ Trig functions give ratios of any two sides.
Both side lengths a and b	Use inverse trig function to find one angle: $\tan A = \frac{a}{b}$. Use $A + B = 90°$ to find the other angle. Use Pythagorean theorem to find hypotenuse $c = \sqrt{a^2 + b^2}$.

SOLVING OBLIQUE TRIANGLES

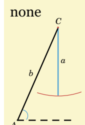
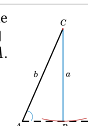
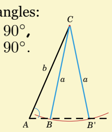
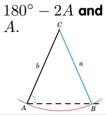
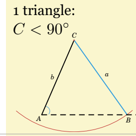
TECHNIQUES FOR SOLVING OBLIQUE TRIANGLES

Suppose you know…	Type	No solution if…	To solve the triangle…
One side and any two angles (say, A and B)	ASA SAA	$A + B \geq 180°$	1. Use $A + B + C = 180°$ to find the third angle. 2. Use Law of Sines to find the other two sides.
Two sides and the angle between them (say, a, b, and C)	SAS		1. Use Law of Cosines to find the third side: $c = \sqrt{a^2 + b^2 - 2ab\cos C}$. 2. *Less work:* Use Law of Sines to calculate one unknown angle. Choose angle so that "largest angle opposite longest side." *Less thinking:* Alternatively, use Law of Cosines a second time to find that angle. 3. Use $A + B + C = 180°$ to find the third angle.
Two sides and an angle not between them (say, a, b, and A)	ASS	A is acute and $a < b\sin A$ A is obtuse and $a < b$	1. Use Law of Sines to find B: $\sin B = \frac{b\sin A}{a}$. Potential ambiguity (see ASS: Ambiguous case, below). 2. Use $A + B + C = 180°$ to find C. 3. Use Law of Sines to find c.
All three sides	SSS	$a + b \leq c$ $a + c \leq b$ $b + c \leq a$	1. Use Law of Cosines to find one angle: $A = \arccos\left(\frac{b^2+c^2-a^2}{2bc}\right)$. 2. *Less work:* Use the Law of Sines to find a second angle: $B = \arcsin\left(\frac{b\sin A}{a}\right)$. Choose angle so "largest angle is opposite longest side." *Less thinking:* Alternatively, use Law of Cosines one more time. 3. Use $A + B + C = 18°$ to compute the third angle.

- An **oblique triangle** is a triangle with no right angles.
- An oblique triangle either has three acute angles, or one obtuse angle and two acute angles.



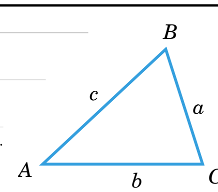
ASS: Ambiguous case: When two sides and an angle opposite one of them are known (say, a, b, A) and that angle is acute, $A < 90°$, the triangle is not always uniquely determined; there may be no solutions or there may be two solutions.

Cases:	$a < b\sin A$	$a = b\sin A$	$b > a > b\sin A$	$b = a$	$b < a$
$A < 90°$	none 	1 right triangle $B = 90°$ **and** $C = 90° - A$. 	2 triangles: $C < 90°$, $C > 90°$. 	1 triangle: $C = 180° - 2A$ **and** $B = A$. 	1 triangle: $C < 90°$ 
$A \geq 90°$	none	none	none	none	1 triangle: $\cos C = \frac{b}{a}$.

AREA OF A TRIANGLE

It is possible to calculate the area of a triangle knowing three of the six measurements (three sides, three angles), provided that one of them is a side.

Suppose you know…	Type	To calculate the area…
One side (say, a) and any two angles	ASA SAA	Use $A + B + C = 180°$ to find the third angle. Area $= \frac{a^2 \sin B \sin C}{2\sin A}$
Two sides and the angle between them (say, a, b, and C)	SAS	Area $= \frac{1}{2}ab\sin C$
Two sides and an angle not between them (say, a, b, and A)	ASS	Use $\sin B = \frac{b\sin A}{a}$ to find B. Keep in mind potential ambiguity since $\sin B = \sin(180° - B)$. Use $A + B + C = 180°$ to find C. Area $= \frac{1}{2}ab\sin C$
All three sides	SSS	Area $= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}$



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