PLANE

TRIGONOMETRY.

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PLANE

TRIGONOMETRY

BY

S. L. LONEY, M.A.

LATE FELLOW OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE,
PROFESSOR AT THE ROYAL HOLLOWAY COLLEGE.

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PREFACE.

THE following work will, I hope, be found to be a fairly complete elementary text-book on Plane Trigonometry, suitable for Schools and the Pass and Junior Honour classes of Universities. In the higher portion of the book I have endeavoured to present to the student, as simply as possible, the modern treatment of complex quantities, and I hope it will be found that he will have little to unlearn when he commences to read treatises of a more difficult character.

As Trigonometry consists largely of formulæ and the applications thereof, I have prefixed (on pages x to xvi) a list of the principal formulæ which the student should commit to memory. These more important formulæ are distinguished in the text by the use of thick type. Other formulæ are subsidiary and of less importance.

The number of examples is very large. A selection only should be solved by the student on a first reading.

On a first reading also the articles marked with an asterisk should be omitted.

Considerable attention has been paid to the printing of the book and I am under great obligation to the Syndics of the Press for their liberality in this matter, and to the officers and workmen of the Press for the trouble they have taken.

I am indebted to Mr W. J. Dobbs, B.A., late Scholar of St John's College, for his kindness in reading and correcting the proof-sheets and for many valuable suggestions.

For any corrections and suggestions for improvement I shall be thankful.

S. L. LONEY.

ROYAL HOLLOWAY COLLEGE,
EGHAM, SURREY.
September 12, 1893.

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$\pi = 3{\cdot}14159\ldots \left[\text{Approximations are } \frac{22}{7} \text{ and } \frac{355}{113}\right]$. (Art. 13.)

A Radian $= 57°\ 17'\ 44{\cdot}8''$ nearly. (Art. 16.)

Two right angles $= 180° = 200^g = \pi$ radians. (Art. 19.)

Angle $= \frac{\text{arc}}{\text{radius}} \times$ Radian. (Art. 21.)

II.

$$\sin^2\theta + \cos^2\theta = 1\ ;$$
$$\sec^2\theta = 1 + \tan^2\theta\ ;$$
$$\operatorname{cosec}^2\theta = 1 + \cot^2\theta. \qquad \text{(Art. 27.)}$$

III.

$$\sin 0° = 0\ ;\ \cos 0° = 1. \qquad \text{(Art. 36.)}$$

$$\sin 30° = \frac{1}{2}\ ;\ \cos 30° = \frac{\sqrt{3}}{2}. \qquad \text{(Art. 34.)}$$

$$\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}. \qquad \text{(Art. 33.)}$$

$$\sin 60° = \frac{\sqrt{3}}{2}\ ;\ \cos 60° = \frac{1}{2}. \qquad \text{(Art. 35.)}$$

$$\sin 90° = 1\ ;\ \cos 90° = 0. \qquad \text{(Art. 37.)}$$

$$\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}\ ;\ \cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}. \qquad \text{(Art. 106.)}$$

$$\sin 18° = \frac{\sqrt{5}-1}{4}\ ;\ \cos 36° = \frac{\sqrt{5}+1}{4}. \qquad \text{(Arts. 120, 121.)}$$

IV. $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$. (Art. 68.)

$\sin(90° - \theta) = \cos\theta$; $\cos(90° - \theta) = \sin\theta$. (Art. 69.)

$\sin(90° + \theta) = \cos\theta$; $\cos(90° + \theta) = -\sin\theta$. (Art. 70.)

$\sin(180° - \theta) = \sin\theta$; $\cos(180° - \theta) = -\cos\theta$. (Art. 72.)

$\sin(180° + \theta) = -\sin\theta$; $\cos(180° + \theta) = -\cos\theta$. (Art. 73.)

V. If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$. (Art. 82.)

If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$. (Art. 83.)

If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$. (Art. 84.)

VI. $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

$\cos(A + B) = \cos A \cos B - \sin A \sin B$. (Art. 88.)

$\sin(A - B) = \sin A \cos B - \cos A \sin B$.

$\cos(A - B) = \cos A \cos B + \sin A \sin B$. (Art. 90.)

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}.$$

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}.$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}.$$

$$\cos D - \cos C = 2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}.$$ (Art. 94.)

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$.

$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$.

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$.

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$. (Art. 97.)

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}. \qquad \text{(Art. 98.)}$$

$$\sin 2A = 2 \sin A \cos A.$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1.$$

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A}; \quad \cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}. \qquad \text{(Art. 109.)}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}. \qquad \text{(Art. 105.)}$$

$$\sin 3A = 3\sin A - 4\sin^3 A.$$

$$\cos 3A = 4\cos^3 A - 3\cos A.$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}. \qquad \text{(Art. 107.)}$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}; \quad \cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}. \qquad \text{(Art. 110.)}$$

$$2\sin\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A}.$$

$$2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \mp \sqrt{1-\sin A}. \qquad \text{(Art. 113.)}$$

$$\tan(A_1 + A_2 + \ldots + A_n) = \frac{s_1 - s_3 + s_5 - \ldots}{1 - s_2 + s_4 - \ldots}. \qquad \text{(Art. 125.)}$$

VII.

$$\log_a mn = \log_a m + \log_a n.$$

$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

$$\log_a m^n = n \log_a m. \qquad \text{(Art. 136.)}$$

$$\log_a m = \log_b m \times \log_a b. \qquad \text{(Art. 147.)}$$

VIII. $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$ (Art. 163.)

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \ldots$$ (Art. 164.)

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \ldots\ldots$$ (Art. 165.)

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}, \ldots\ldots$$ (Art. 166.)

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \ldots\ldots$$ (Art. 167.)

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}, \ldots$$ (Art. 169.)

$$a = b \cos C + c \cos B, \ldots\ldots$$ (Art. 170.)

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}, \ldots\ldots$$ (Art. 171.)

$$S = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C.$$

(Art. 198.)

IX. $$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4S}.$$ (Arts. 200, 201.)

$$r = \frac{S}{s} = (s-a) \tan \frac{A}{2} = \ldots = \ldots$$ (Arts. 202, 203.)

$$r_1 = \frac{S}{s-a} = s \tan \frac{A}{2}.$$ (Arts. 205, 206.)

Area of a quadrilateral inscribable in a circle

$$= \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$ (Art. 219.)

$$\frac{\sin \theta}{\theta} = 1, \text{ when } \theta \text{ is very small.}$$ (Art. 228.)

Area of a circle $= \pi r^2$. (Art. 233.)

X. $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots$ to n terms

$$= \frac{\sin\left\{\alpha + \frac{n-1}{2}\beta\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}. \qquad \text{(Art. 241.)}$$

$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \dots$ to n terms

$$= \frac{\cos\left\{\alpha + \frac{n-1}{2}\beta\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}. \qquad \text{(Art. 242.)}$$

XI. $$\underset{n=\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n = e = 2{\cdot}71828\dots\dots \qquad \text{(Arts. 250, 251.)}$$

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \dots\dots\text{ad inf.}$$

$$a^x = 1 + x\log_e a + \frac{x^2}{\underline{|2}}(\log_e a)^2 + \dots\text{ad inf.} \qquad \text{(Art. 253.)}$$

$$\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots\dots\text{ad inf.}$$

when $x > -1$ and $\not> 1$. (Art. 257.)

$$\underset{n=\infty}{\text{Lt}}\left(\cos\frac{a}{n}\right)^n = \underset{n=\infty}{\text{Lt}}\left(\frac{\sin\frac{a}{n}}{\frac{a}{n}}\right)^n = 1. \qquad \text{(Arts. 262, 263.)}$$

XII. $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin\theta. \qquad \text{(Art. 268.)}$$

XIII. $$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}\theta\sin^3\theta + \ldots\ldots$$

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{1.2}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cos^{n-4}\theta\sin^4\theta - \ldots\ldots \quad \text{(Art. 274.)}$$

$$\tan n\theta = \frac{s_1 - s_3 + s_5 - s_7\ldots\ldots}{1 - s_2 + s_4 - s_6\ldots\ldots} \quad \text{(Art. 277.)}$$

XIV. $$\sin\alpha = \alpha - \frac{\alpha^3}{\underline{|3}} + \frac{\alpha^5}{\underline{|5}} - \ldots\ldots \text{ ad inf.} \quad \text{(Art. 280.)}$$

$$\cos\alpha = 1 - \frac{\alpha^2}{\underline{|2}} + \frac{\alpha^4}{\underline{|4}} - \ldots\ldots \text{ ad inf.} \quad \text{(Art. 279.)}$$

$$\sin x = \frac{e^{xi} - e^{-xi}}{2i}; \ \cos x = \frac{e^{xi} + e^{-xi}}{2}. \quad \text{(Art. 308.)}$$

XV. $$\text{Log}(\alpha + \beta i) = \log_e \sqrt{\alpha^2 + \beta^2} + i(2n\pi + \theta),$$

where $\cos\theta = \dfrac{\alpha}{+\sqrt{\alpha^2+\beta^2}}$ and $\sin\theta = \dfrac{\beta}{+\sqrt{\alpha^2+\beta^2}}$.

(Art. 329.)

XVI. $$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots\ldots \text{ ad inf.},$$

where x is numerically not greater than unity. (Art. 344.)

$$\theta - p\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots\ldots \text{ ad inf.}$$

where θ lies between $p\pi - \dfrac{\pi}{4}$ and $p\pi + \dfrac{\pi}{4}$. (Art. 343.)

XVII.

$$x^{2n} - 2a^n x^n \cos n\theta + a^{2n}$$

$$= \prod_{r=0}^{r=n-1} \left\{x^2 - 2ax\cos\left(\theta + \frac{2r\pi}{n}\right) + a^2\right\} \quad \text{(Art. 362.)}$$

$$x^n - 1 = (x^2 - 1) \prod_{r=1}^{r=\frac{n}{2}-1} \left(x^2 - 2x\cos\frac{2r\pi}{n} + 1\right), \ (n \text{ even})$$

and
$$= (x - 1) \prod_{r=1}^{r=\frac{n-1}{2}} \left(x^2 - 2x\cos\frac{2r\pi}{n} + 1\right), \ (n \text{ odd}).$$

(Art. 366.)

$$x^n + 1 = \prod_{r=0}^{r=\frac{n}{2}-1} \left(x^2 - 2x\cos\frac{2r+1}{n}\pi + 1\right), \ (n \text{ even})$$

and
$$= (x + 1) \prod_{r=0}^{r=\frac{n-3}{2}} \left(x^2 - 2x\cos\frac{2r+1}{n}\pi + 1\right), \ (n \text{ odd}).$$

(Art. 367.)

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right) \ldots\ldots \text{ad inf.} \quad \text{(Art. 369.)}$$

$$\cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right) \ldots\ldots \text{ad inf.} \quad \text{(Art. 370.)}$$

PART I.

GEOMETRICAL TRIGONOMETRY.

CHAPTER I.

MEASUREMENT OF ANGLES, SEXAGESIMAL, CENTESIMAL, AND CIRCULAR MEASURE.

1. IN geometry angles are measured in terms of a right angle. This, however, is an inconvenient unit of measurement on account of its size.

2. In the **Sexagesimal** system of measurement a right angle is divided into 90 equal parts called **Degrees.** Each degree is divided into 60 equal parts called **Minutes**, and each minute into 60 equal parts called **Seconds.**

The symbols $1^\circ, 1', $ and $1''$ are used to denote a degree, a minute, and a second respectively.

Thus 60 Seconds $(60'')$ make One Minute $(1')$,
60 Minutes $(60')$ „ „ Degree (1°),
and 90 Degrees (90°) „ „ Right Angle.

This system is well established and is always used in the practical applications of Trigonometry. It is not however very convenient on account of the multipliers 60 and 90.

3. On this account another system of measurement called the **Centesimal**, or French, system has been proposed. In this system the right angle is divided into 100 equal parts, called **Grades**; each grade is subdivided into 100 **Minutes**, and each minute into 100 **Seconds.**

The symbols 1^g, $1^{\backprime}$, and $1^{\backprime\backprime}$ are used to denote a Grade, a Minute, and a Second respectively.

Thus 100 Seconds ($100^{\backprime\backprime}$) make One Minute ($1^{\backprime}$),
100 Minutes ($100^{\backprime}$) „ „ Grade, (1^g),
100 Grades (100^g) „ „ Right angle.

4. This system would be much more convenient to use than the ordinary Sexagesimal System.

As a preliminary, however, to its practical adoption, a large number of tables would have to be recalculated. For this reason the system has in practice never been used.

5. *To convert Sexagesimal into Centesimal Measure, and vice versa.*

Since a right angle is equal to 90° and also to 100^g, we have

$$90^\circ = 100^g.$$

$$\therefore\ 1^\circ = \frac{10^g}{9}, \text{ and } 1^g = \frac{9^\circ}{10}.$$

Hence, to change degrees into grades, add on one-ninth; to change grades into degrees, subtract one-tenth.

Ex. $$36^\circ = \left(36 + \frac{1}{9} \times 36\right)^g = 40^g,$$

and $$64^g = \left(64 - \frac{1}{10} \times 64\right)^\circ = (64 - 6{\cdot}4)^\circ = 57{\cdot}6^\circ.$$

If the angle do not contain an integral number of degrees, we may reduce it to a fraction of a degree and then change to grades.

In practice it is generally found more convenient to reduce any angle to a fraction of a right angle. The method will be seen in the following examples;

Ex. 1. *Reduce* $63°\,14'\,51''$ *to Centesimal Measure.*

We have $$51'' = \frac{17'}{20} = \cdot 85',$$

and $$14'\,51'' = 14\cdot 85' = \frac{14\cdot 85°}{60} = \cdot 2475°,$$

$$\therefore\ 63°\,14'\,51'' = 63\cdot 2475° = \frac{63\cdot 2475}{90} \text{ rt. angle}$$

$$= \cdot 70275 \text{ rt. angle}$$

$$= 70\cdot 275^g = 70^g\,27\cdot 5^{\backprime} = 70^g\,27^{\backprime}\,50^{\backprime\backprime}.$$

Ex. 2. *Reduce* $94^g\,23^{\backprime}\,87^{\backprime\backprime}$ *to Sexagesimal Measure.*

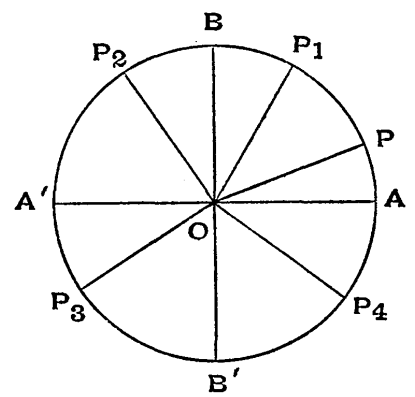
$$94^g\,23^{\backprime}\,87^{\backprime\backprime} = \cdot 942387 \text{ right angle}$$

$$\begin{array}{r} 90 \\ \hline 84\cdot 81483 \text{ degrees} \\ 60 \\ \hline 48\cdot 8898 \text{ minutes} \\ 60 \\ \hline 53\cdot 3880 \text{ seconds} \end{array}$$

$$\therefore\ 94^g\,23^{\backprime}\,87^{\backprime\backprime} = 84°\,48'\,53\cdot 388''.$$

6. Angles of any size.

Suppose AOA' and BOB' to be two fixed lines meeting at right angles in O, and suppose a revolving line OP (turning about a fixed point at O) to start from OA and revolve in a direction opposite to that of the hands of a watch.



For any position of the revolving line between OA and OB, such as OP_1, it will have turned through an angle AOP_1, which is less than a right angle.

For any position between OB and OA', such as OP_2, the angle AOP_2 through which it has turned is greater than a right angle.

For any position OP_3, between OA' and OB', the angle traced out is AOP_3, *i.e.* $AOB+BOA'+A'OP_3$, *i.e.* 2 right angles $+A'OP_3$, so that the angle described is greater than two right angles.

For any position OP_4, between OB' and OA, the angle turned through is similarly greater than three right angles.

When the revolving line has made a complete revolution, so that it coincides once more with OA, the angle through which it has turned is 4 right angles.

If the line OP still continue to revolve, the angle through which it has turned, when it is for the second time in the position OP_1, is not AOP_1 but 4 right angles $+AOP_1$.

Similarly when the revolving line, having made two complete revolutions, is once more in the position OP_2, the angle it has traced out is 8 right angles $+AOP_2$.

7. If the revolving line OP be between OA and OB it is said to be in the first quadrant; if it be between OB and OA' it is in the second quadrant; if between OA' and OB' it is in the third quadrant; if it is between OB' and OA it is in the fourth quadrant.

8. Ex. *What is the position of the revolving line when it has turned through* (1) 225°, (2) 480°, *and* (3) 1050°?

(1) Since $225°=180°+45°$, the revolving line has turned through 45° more than two right angles and is therefore halfway between OA' and OB'.

(2) Since $480°=360°+120°$, the revolving line has turned through 120° more than one complete revolution, and is therefore between OB and OA', and makes an angle of 30° with OB.

(3) Since $1050° = 11 \times 90° + 60°$, the revolving line has turned through 60° more than eleven right angles and is therefore between OB' and OA and makes 60° with OB'.

EXAMPLES. I.

Express in terms of a right angle the angles

1. 60°. 2. 75° 15′. 3. 63° 17′ 25″.
4. 130° 30′. 5. 210° 30′ 30″. 6. 370° 20′ 48″.

Express in grades, minutes, and seconds the angles

7. 30°. 8. 81°. 9. 138° 30′. 10. 35° 47′ 15″.
11. 235° 12′ 36″. 12. 475° 13′ 48″.

Express in terms of right angles and also in degrees, minutes, and seconds the angles

13. 120^g. 14. $45^g\,35^{\backprime}\,24^{\backprime\backprime}$. 15. $39^g\,45^{\backprime}\,36^{\backprime\backprime}$.
16. $255^g\,48^{\backprime}\,81^{\backprime\backprime}$. 17. $759^g\,45^{\backprime}\,60^{\backprime\backprime}$.

Mark the position of the revolving line when it has traced out the following angles:

18. $\frac{4}{3}$ right angle. 19. $3\frac{1}{2}$ right angles. 20. $13\frac{1}{3}$ right angles.
21. 120°. 22. 315°. 23. 745°. 24. 1185°. 25. 150^g.
26. 420^g. 27. 875^g.

28. How many degrees, minutes and seconds are respectively passed over in $11\frac{1}{9}$ minutes by the hour and minute hands of a watch?

29. The number of degrees in one acute angle of a right-angled triangle is equal to the number of grades in the other; express both the angles in degrees.

30. Prove that the number of Sexagesimal minutes in any angle is to the number of Centesimal minutes in the same angle as 27 : 50.

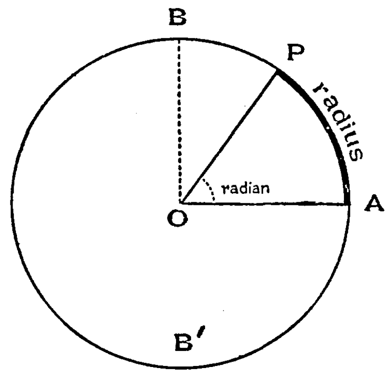
31. Divide 44° 8′ into two parts such that the number of Sexagesimal seconds in one part may be equal to the number of Centesimal seconds in the other part.

Circular Measure.

9. A third system of measurement of angles has been devised, and it is this system which is used in all the higher branches of Mathematics.

The unit used is obtained thus;

Take *any* circle $APBB'$, whose centre is O, and from any point A measure off an arc AP whose length is equal to the radius of the circle. Join OA and OP.



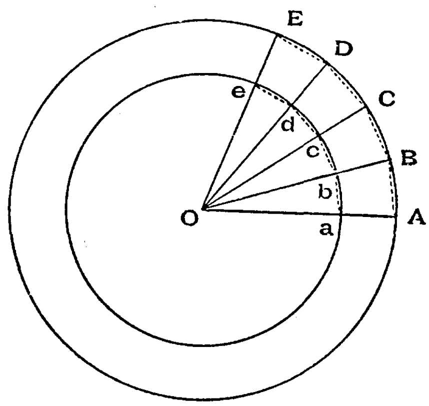
The angle AOP is the angle which is taken as the unit of circular measurement, *i.e.* it is the angle in terms of which in this system we measure all others.

This angle is called **A Radian** and is often denoted by 1^c.

10. It is clearly essential to the proper choice of a unit that it should be a *constant* quantity; hence we must shew that the Radian is a constant angle. This we shall do in the following articles.

11. Theorem. *The length of the circumference of a circle always bears a constant ratio to its diameter.*

Take any two circles whose common centre is O. In the large circle inscribe a regular polygon of n sides, $ABCD....$



Let OA, OB, OC,... meet the smaller circle in the points a, b, c, d... and join ab, bc, cd,....

Then, by Euc. VI. 2, $abcd$... is a regular polygon of n sides inscribed in the smaller circle.

Since $Oa = Ob$, and $OA = OB$,

the lines ab and AB must be parallel, and hence

$$\frac{AB}{ab} = \frac{OA}{Oa}. \qquad \text{(Euc. VI. 4).}$$

Also the polygon $ABCD$... being regular, its perimeter, *i.e.* the sum of its sides, is equal to $n.AB$. Similarly for the inner polygon.

Hence we have

$$\frac{\text{Perimeter of the outer polygon}}{\text{Perimeter of the inner polygon}} = \frac{n.AB}{n.ab} = \frac{AB}{ab} = \frac{OA}{Oa} \quad \text{...........(1).}$$

This relation exists whatever be the number of sides in the polygons.

Let then the number of sides be indefinitely increased (*i.e.* let n become inconceivably great) so that finally the perimeter of the outer polygon will be the same as the circumference of the outer circle, and the perimeter of the inner polygon the same as the circumference of the inner circle.

The relation (1) will then become

$$\frac{\text{Circumference of outer circle}}{\text{Circumference of inner circle}} = \frac{OA}{Oa}$$

$$= \frac{\text{Radius of outer circle}}{\text{Radius of inner circle}}.$$

Hence $\dfrac{\text{Circumference of outer circle}}{\text{Radius of outer circle}}$

$$= \frac{\text{Circumference of inner circle}}{\text{Radius of inner circle}}.$$

Since there was no restriction whatever as to the sizes of the two circles, it follows that the quantity

$$\frac{\text{Circumference of a circle}}{\text{Radius of the circle}}$$

is **the same for all circles.**

Hence the ratio of the circumference of a circle to its radius, and therefore also to its diameter, is a constant quantity.

12. In the previous article we have shewn that the ratio $\frac{\text{Circumference}}{\text{Diameter}}$ is the same for all circles. The value of this constant ratio is always denoted by the Greek letter π (pronounced Pi), so that π is a number.

Hence $\frac{\text{Circumference}}{\text{Diameter}}$ = the constant number π.

We have therefore the following theorem; **The circumference of a circle is always equal to π times its diameter or 2π times its radius.**

13. Unfortunately the value of π is not a whole number, nor can it be expressed in the form of a vulgar fraction, and hence not in the form of a decimal fraction, terminating or recurring.

The number π is an incommensurable magnitude, *i.e.* a magnitude whose value cannot be exactly expressed as the ratio of two whole numbers.

Its value, correct to 8 places of decimals, is

$$3{\cdot}14159265\ldots.$$

The fraction $\frac{22}{7}$ gives the value of π correctly for the first two decimal places; for $\frac{22}{7} = 3{\cdot}14285\ldots.$

The fraction $\frac{355}{113}$ is a more accurate value of π being correct to 6 places of decimals; for $\frac{355}{113} = 3{\cdot}14159203\ldots.$

[N.B. The fraction $\frac{355}{113}$ may be remembered thus; write down the first three odd numbers repeating each twice, thus 113355; divide the number thus obtained into portions and let the first part be divided into the second, thus 113) 355(.

The quotient is the value of π to 6 places of decimals.]

To sum up. **An approximate value of π, correct to 2 places of decimals, is the fraction $\frac{22}{7}$; a more accurate value is 3·14159....**

By division we can shew that

$$\frac{1}{\pi} = \cdot 3183098862....$$

14. Ex. 1. *The diameter of a tricycle wheel is 28 inches; through what distance does its centre move during one revolution of the wheel?*

The radius r is here 14 inches.

The circumference therefore $= 2 . \pi . 14 = 28\pi$ inches.

Taking $\pi = \frac{22}{7}$, the circumference $= 28 \times \frac{22}{7}$ inches $= 7$ ft. 4 inches approximately.

Giving π the more accurate value 3·14159265... the circumference

$$= 28 \times 3\cdot 14159265... \text{ inches} = 7 \text{ ft. } 3\cdot 96459... \text{ inches.}$$

Ex. 2. *What must be the radius of a circular running path, round which an athlete must run 5 times in order to describe one mile?*

The circumference must be $\frac{1}{5} \times 1760$, i.e. 352, yards.

Hence, if r be the radius of the path in yards, we have $2\pi r = 352$,

i.e. $$r = \frac{176}{\pi} \text{ yards.}$$

Taking $\pi = \frac{22}{7}$, we have $r = \frac{176 \times 7}{22} = 56$ yards nearly.

Taking the more accurate value $\frac{1}{\pi} = \cdot 31831$, we have

$$r = 176 \times \cdot 31831 = 56\cdot 02256 \text{ yards.}$$

EXAMPLES. II.

1. If the radius of the earth be 4000 miles, what is the length of its circumference?

2. The wheel of a railway carriage is 3 feet in diameter and makes 3 revolutions in a second; how fast is the train going?

3. A mill sail whose length is 18 feet makes 10 revolutions per minute. What distance does its end travel in an hour?

4. The diameter of a halfpenny is an inch; what is the length of a piece of string which would just surround its curved edge?

5. Assuming that the earth describes in one year a circle, of 92500000 miles radius, whose centre is the sun, how many miles does the earth travel in a year?

6. The radius of a carriage wheel is 1 ft. 9 ins., and it turns through 80° in $\frac{1}{9}$th of a second; how many miles does the wheel travel in one hour?

15. Theorem. *The radian is a constant angle.*

Take the figure of Art. 9. Let the arc AB be a quadrant of the circle, *i.e.* one quarter of the circumference.

By Art. 12, the length of AB is therefore $\frac{\pi r}{2}$, where r is the radius of the circle.

By Euc. VI. 33, we know that angles at the centre of any circle are to one another as the arcs on which they stand.

Hence $$\frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } AB} = \frac{r}{\frac{\pi}{2} r} = \frac{2}{\pi},$$

i.e. $$\angle AOP = \frac{2}{\pi} \cdot \angle AOB.$$

But we defined the angle AOP to be a Radian.

Hence a Radian $= \frac{2}{\pi} . \angle AOB$

$$= \frac{2}{\pi} \times \text{a right angle.}$$

Since a right angle is a constant angle and since we have shewn (Art. 12) that π is a constant quantity, it follows that a Radian is a constant angle, and is therefore the same whatever be the circle from which it is derived.

16. Magnitude of a Radian.

By the previous article a radian

$$= \frac{2}{\pi} \times \text{a right angle} = \frac{180^\circ}{\pi}$$

$$= \frac{180^\circ}{3 \cdot 14159265 \ldots} = 57 \cdot 2957795^\circ$$

$$= 57^\circ \, 17' \, 44 \cdot 8'' \text{ nearly.}$$

17. Since a Radian $= \frac{2}{\pi} \times$ a right angle,

therefore a right angle $= \frac{\pi}{2}$. radians,

so that $180^\circ = 2$ right angles $= \pi$ radians,

and $\quad 360^\circ = 4$ right angles $= 2\pi$ radians.

Hence when the revolving line (Art. 6) has made a complete revolution it has described an angle equal to 2π radians; when it has made three complete revolutions it has described an angle of 6π radians; when it has made n revolutions it has described an angle of $2n\pi$ radians.

18. In practice the symbol "c" is generally omitted and instead of "an angle π^c" we find written "an angle π."

The student must notice this point carefully. If the unit, in terms of which the angle is measured, be not mentioned, he must mentally supply the word "radians." Otherwise he will easily fall into the mistake of supposing that π stands for 180°. It is true that π radians (π^c) is the same as 180°, but π itself is a number, and a number only.

19. *To convert circular measure into sexagesimal measure or centesimal measure and vice versa.*

The student should remember the relations,

Two right angles $= 180^\circ = 200^g = \pi$ radians.

The conversion is then merely Arithmetic.

Ex. (1) $\cdot 45\pi^c = \cdot 45 \times 180^\circ = 81^\circ = 90^g$.

(2) $$3^c = \frac{3}{\pi} \times \pi^c = \frac{3}{\pi} \times 180^\circ = \frac{3}{\pi} \times 200^g.$$

(3) $$40^\circ\,15'\,36'' = 40^\circ\,15\tfrac{3}{5}' = 40{\cdot}26^\circ$$
$$= 40{\cdot}26 \times \frac{\pi^c}{180} = \cdot 223\dot{6}\pi \text{ radians.}$$

(4) $$40^g\,15^`\,36^{``} = 40{\cdot}1536^g = 40{\cdot}1536 \times \frac{\pi}{200} \text{ radians}$$
$$= \cdot 200768\pi \text{ radians.}$$

20. Ex. 1. *The angles of a triangle are in* A. P. *and the number of grades in the least is to the number of radians in the greatest as* $40 : \pi$; *find the angles in degrees.*

Let the angles be $(x-y)^\circ$, x°, and $(x+y)^\circ$.

Since the sum of the three angles of a triangle is 180°, we have
$$180 = x - y + x + x + y = 3x,$$
so that
$$x = 60.$$

The required angles are therefore
$$(60-y)^\circ,\ 60^\circ,\ \text{and}\ (60+y)^\circ.$$

Now
$$(60-y)^\circ = \frac{10}{9} \times (60-y)^g,$$
and
$$(60+y)^\circ = \frac{\pi}{180} \times (60+y) \text{ radians.}$$

Hence $$\frac{10}{9}(60-y) : \frac{\pi}{180}(60+y) :: 40 : \pi,$$

$$\therefore \quad \frac{200}{\pi}\frac{60-y}{60+y} = \frac{40}{\pi},$$

i.e. $$5(60-y) = 60+y,$$

i.e. $$y = 40.$$

The angles are therefore 20°, 60°, and 100°.

Ex. 2. *Express in the* 3 *systems of angular measurement the magnitude of the angle of a regular decagon.*

The corollary to Euc. I. 32 states that all the interior angles of any rectilinear figure together with four right angles are equal to twice as many right angles as the figure has sides.

Let the angle of a decagon contain x right angles, so that all the angles are together equal to $10x$ right angles.

The corollary therefore states that

$$10x + 4 = 20,$$

so that $$x = \frac{8}{5} \text{ right angles.}$$

But one right angle

$$= 90^\circ = 100^g = \frac{\pi}{2} \text{ radians.}$$

Hence the required angle

$$= 144^\circ = 160^g = \frac{4\pi}{5} \text{ radians.}$$

EXAMPLES. III.

Express in degrees, minutes, and seconds the angles,

1. $\frac{\pi^c}{3}$. 2. $\frac{4\pi^c}{3}$. 3. $10\pi^c$. 4. 1^c. 5. 8^c.

Express in grades, minutes, and seconds the angles,

6. $\frac{4\pi^c}{5}$. 7. $\frac{7\pi^c}{6}$. 8. $10\pi^c$.

Express in radians the following angles:

9. 60°. 10. $110^\circ\,30'$. 11. $175^\circ\,45'$. 12. $47^\circ\,25'\,36''$.
13. 395°. 14. 60^g. 15. $110^g\,30^{\backprime}$. 16. $345^g\,25^{\backprime}\,36^{\backprime\backprime}$.

17. The difference between the two acute angles of a right-angled triangle is $\frac{2}{5}\pi$ radians; express the angles in degrees.

18. One angle of a triangle is $\frac{2}{3}x$ grades and another is $\frac{3}{2}x$ degrees, whilst the third is $\frac{\pi x}{75}$ radians; express them all in degrees.

19. The circular measure of two angles of a triangle are respectively $\frac{1}{2}$ and $\frac{1}{3}$; what is the number of degrees in the third angle?

20. The angles of a triangle are in A. P. and the number of degrees in the least is to the number of radians in the greatest as 60 to π; find the angles in degrees.

21. The angles of a triangle are in A. P. and the number of radians in the least angle is to the number of degrees in the mean angle as 1 : 120. Find the angles in radians.

22. Find the magnitude, in radians and degrees, of the interior angle of (1) a regular pentagon, (2) a regular heptagon, (3) a regular octagon, (4) a regular duodecagon, and (5) a regular polygon of 17 sides.

23. The angle in one regular polygon is to that in another as 3 : 2; also the number of sides in the first is twice that in the second; how many sides have the polygons?

24. The number of sides in two regular polygons are as 5 : 4, and the difference between their angles is 9°; find the number of sides in the polygons.

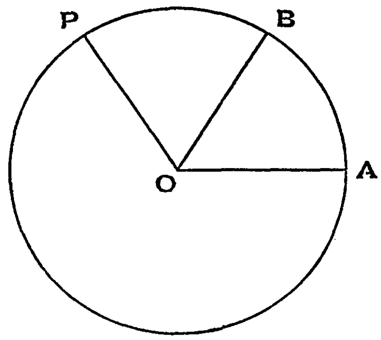
25. Find two regular polygons such that the number of their sides may be as 3 to 4 and the number of degrees in an angle of the first to the number of grades in an angle of the second as 4 to 5.

26. The angles of a quadrilateral are in A. P. and the greatest s double the least; express the least angle in radians.

27. Find in radians, degrees, and grades the angle between the hour-hand and the minute-hand of a clock at (1) half-past three, (2) twenty minutes to six, (3) a quarter past eleven.

21. Theorem. *The number of radians in any angle whatever is equal to a fraction, whose numerator is the arc which the angle subtends at the centre of any circle, and whose denominator is the radius of that circle.*

Let AOP be the angle which has been described by a line starting from OA and revolving into the position OP.



With centre O and any radius describe a circle cutting OA and OP in the points A and P.

Let the angle AOB be a radian, so that the arc AB is equal to the radius OA.

By Euc. VI. 33, we have

$$\frac{\angle AOP}{A \text{ Radian}} = \frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } AB} = \frac{\text{arc } AP}{\text{Radius}},$$

so that
$$\angle AOP = \frac{\text{arc } AP}{\text{Radius}} \times A \text{ Radian}.$$

Hence the theorem is proved.

22. **Ex. 1.** *Find the angle subtended at the centre of a circle of radius* 3 *feet by an arc of length* 1 *foot.*

The number of radians in the angle $= \dfrac{\text{arc}}{\text{radius}} = \dfrac{1}{3}$.

Hence the angle

$$= \frac{1}{3} \text{ radian} = \frac{1}{3} \cdot \frac{2}{\pi} \text{ right angle} = \frac{2}{3\pi} \times 90^\circ = \frac{60^\circ}{\pi} = 19\tfrac{1}{11}^\circ,$$

taking π equal to $\dfrac{22}{7}$.

Ex. 2. *In a circle of* 5 *feet radius what is the length of the arc which subtends an angle of* 33° 15′ *at the centre?*

If x feet be the required length, we have

$$\frac{x}{5} = \text{number of radians in } 33^\circ 15'$$

$$= \frac{33\frac{1}{4}}{180}\pi \quad \text{(Art. 19).}$$

$$= \frac{133}{720}\pi.$$

$$\therefore \; x = \frac{133}{144}\pi \text{ feet} = \frac{133}{144} \times \frac{22}{7} \text{ feet nearly}$$

$$= 2\tfrac{65}{72} \text{ feet nearly.}$$

Ex. 3. *Assuming the average distance of the earth from the sun to be* 92500000 *miles, and the angle subtended by the sun at the eye of a person on the earth to be* 32′, *find the sun's diameter.*

Let D be the diameter of the sun in miles.

The angle subtended by the sun being very small, its diameter is very approximately equal to a small arc of a circle whose centre is the eye of the observer. Also the sun subtends an angle of 32′ at the centre of this circle.

Hence, by Art. 21, we have

$$\frac{D}{92500000} = \text{the number of radians in } 32'$$

$$= \text{the number of radians in } \frac{8^\circ}{15}$$

$$= \frac{8}{15} \times \frac{\pi}{180} = \frac{2\pi}{675}.$$

$$\therefore \; D = \frac{185000000}{675} \pi \text{ miles}$$

$$= \frac{185000000}{675} \times \frac{22}{7} \text{ miles approximately}$$

$$= \text{about } 862000 \text{ miles.}$$

Ex. 4. *Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of* 5′ *at his eye, find what is the height of the letters that he can read at a distance* (1) *of* 12 *feet, and* (2) *of a quarter of a mile.*

Let x be the required height in feet.

In the first case, x is very nearly equal to the arc of a circle, of radius 12 feet, which subtends an angle of 5′ at its centre.

Hence
$$\frac{x}{12} = \text{number of radians in } 5'$$

$$= \frac{1}{12} \times \frac{\pi}{180}.$$

$$\therefore \; x = \frac{\pi}{180} \text{ feet} = \frac{1}{180} \times \frac{22}{7} \text{ feet nearly}$$

$$= \frac{1}{15} \times \frac{22}{7} \text{ inches} = \text{about } \frac{1}{5} \text{ inch.}$$

In the second case the height y is given by

$$\frac{y}{440 \times 3} = \text{number of radians in } 5'$$

$$= \frac{1}{12} \times \frac{\pi}{180},$$

so that

$$y = \frac{11}{18}\pi = \frac{11}{18} \times \frac{22}{7} \text{ feet nearly}$$

$$= \text{about } 23 \text{ inches.}$$

EXAMPLES. IV.

1. Find the number of degrees subtended at the centre of a circle by an arc whose length is ·357 times the radius, taking $\pi = 3{\cdot}1416$.

2. Express in radians and degrees the angle subtended at the centre of a circle by an arc whose length is 15 feet, the radius of the circle being 25 feet.

3. The value of the divisions on the outer rim of a graduated circle is $5'$ and the distance between successive graduations is ·1 inch. Find the radius of the circle.

4. The diameter of a graduated circle is 6 feet and the graduations on its rim are $5'$ apart; find the distance from one graduation to another.

5. Find the radius of a globe which is such that the distance between two places on the same meridian whose latitude differs by $1° \, 10'$ may be half-an-inch.

6. Taking the radius of the earth as 4000 miles find the difference in latitude of two places, one of which is 100 miles north of the other.

7. Assuming the earth to be a sphere and the distance between two parallels of latitude, which subtends an angle of 1° at the earth's centre, to be $69\frac{1}{9}$ miles, find the radius of the earth.

8. The radius of a certain circle is 3 feet; find approximately the length of an arc of this circle, if the length of the chord of the arc be 3 feet also.

9. What is the ratio of the radii of two circles at the centre of which two arcs of the same length subtend angles of 60° and 75°?

10. If an arc, of length 10 feet, on a circle of 8 feet diameter subtend at the centre an angle of $143° \, 14' \, 22''$; find the value of π to 4 places of decimals.

11. If the circumference of a circle be divided into 5 parts which are in A. P., and if the greatest part be 6 times the least, find in radians the magnitudes of the angles that the parts subtend at the centre of the circle.

12. The perimeter of a certain sector of a circle is equal to the length of the arc of a semicircle having the same radius; express the angle of the sector in degrees, minutes, and seconds.

13. At what distance does a man, whose height is 6 feet, subtend an angle of $10'$?

14. Find the length which at a distance of one mile will subtend an angle of $1'$ at the eye.

15. Find approximately the distance at which a globe, $5\frac{1}{2}$ inches in diameter, will subtend an angle of $6'$.

16. Find approximately the distance of a tower whose height is 51 feet and which subtends at the eye an angle of $5\frac{5}{11}'$.

17. A church spire, whose height is known to be 45 feet, subtends an angle of $9'$ at the eye; find approximately its distance.

18. Find approximately in minutes the inclination to the horizon of an incline which rises $3\frac{1}{2}$ feet in 210 yards.

19. The radius of the earth being taken to be 3960 miles, and the distance of the moon from the earth being 60 times the radius of the earth, find approximately the radius of the moon which subtends at the earth an angle of $16'$.

20. When the moon is setting at any given place the angle that is subtended at its centre by the radius of the earth passing through the given place is $57'$. If the earth's radius be 3960 miles, find approximately the distance of the moon.

21. Prove that the distance of the sun is about 81 million geographical miles, assuming that the angle which the earth's radius subtends at the distance of the sun is $8{\cdot}76''$, and that a geographical mile subtends $1'$ at the earth's centre. Find also the circumference and diameter of the earth in geographical miles.

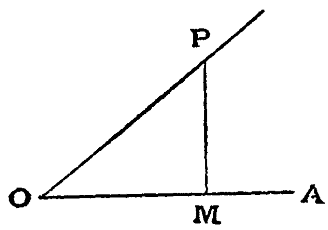
22. The radius of the earth's orbit, which is about 92700000 miles, subtends at the star Sirius an angle of about $\cdot 4''$; find roughly the distance of Sirius.

CHAPTER II.

TRIGONOMETRICAL RATIOS FOR ANGLES LESS THAN A RIGHT ANGLE.

23. In the present chapter we shall only consider angles which are less than a right angle.

Let a revolving line OP start from OA and revolve into the position OP, thus tracing out the angle AOP.



In the revolving line take any point P and draw PM perpendicular to the initial line OA.

In the triangle MOP, OP is the hypothenuse, PM is the perpendicular, and OM is the base.

The trigonometrical ratios, or functions, of the angle AOP are defined as follows;

$\frac{MP}{OP}$, *i.e.* $\frac{\text{Perp.}}{\text{Hyp.}}$, is called the **Sine** of the angle AOP;

$\frac{OM}{OP}$, *i.e.* $\frac{\text{Base}}{\text{Hyp.}}$, " " **Cosine** " "

$\frac{MP}{OM}$, *i.e.* $\frac{\text{Perp.}}{\text{Base}}$, " " **Tangent** " "

$\frac{OM}{MP}$, *i.e.* $\frac{\text{Base}}{\text{Perp.}}$, " " **Cotangent** " "

$\frac{OP}{MP}$, *i.e.* $\frac{\text{Hyp.}}{\text{Perp.}}$, " " **Cosecant** " "

$\frac{OP}{OM}$, *i.e.* $\frac{\text{Hyp.}}{\text{Base}}$, " " **Secant** " "

The quantity by which the cosine falls short of unity, *i.e.* $1 - \cos AOP$, is called the **Versed Sine** of AOP; also the quantity $1 - \sin AOP$, by which the sine falls short of unity, is called the **Coversed Sine** of AOP.

24. It will be noted that the trigonometrical ratios are all **numbers.**

The names of these eight ratios are written, for brevity,

sin AOP, cos AOP, tan AOP, cot AOP, cosec AOP, sec AOP, vers AOP, and covers AOP respectively.

The two latter ratios are seldom used.

25. It will be noticed, from the definitions, that the cosecant is the reciprocal of the sine, so that

$$\text{cosec}\, AOP = \frac{1}{\sin AOP}.$$

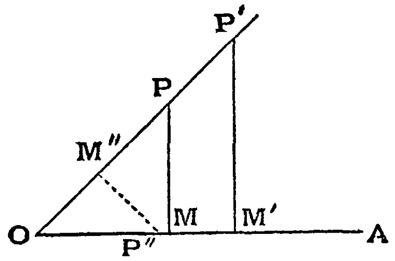
So the secant is the reciprocal of the cosine, *i.e.*

$$\sec AOP = \frac{1}{\cos AOP},$$

and the cotangent is the reciprocal of the tangent, *i.e.*

$$\cot AOP = \frac{1}{\tan AOP}.$$

26. *To shew that the trigonometrical ratios are always the same for the same angle.*



We have to shew that if in the revolving line OP any other point P' be taken and $P'M'$ be drawn perpendicular to OA, the ratios derived from the triangle

$OP'M'$ are the same as those derived from the triangle OPM.

In the two triangles the angle at O is common, and the angles at M and M' are both right angles and therefore equal.

Hence the two triangles are equiangular and therefore, by Euc. VI. 4, we have $\frac{MP}{OP} = \frac{M'P'}{OP'}$, *i.e.* the sine of the angle AOP is the same whatever point we take on the revolving line.

Since, by the same proposition, we have

$$\frac{OM}{OP} = \frac{OM'}{OP'} \text{ and } \frac{MP}{OM} = \frac{M'P'}{OM'},$$

it follows that the cosine and tangent are the same whatever point be taken on the revolving line. Similarly for the other ratios.

If OA be considered as the revolving line and in it be taken any point P'' and $P''M''$ be drawn perpendicular to OP, the functions as derived from the triangle $OP''M''$ will have the same values as before.

For, since in the two triangles OPM and $OP''M''$, the two angles $P''OM''$ and $OM''P''$ are respectively equal to POM and OMP, these two triangles are equiangular and therefore similar, and we have

$$\frac{M''P''}{OP''} = \frac{MP}{OP}, \text{ and } \frac{OM''}{OP''} = \frac{OM}{OP}.$$

27. *Fundamental relations between the trigonometrical ratios of an angle.*

We shall find that if one of the trigonometrical ratios of an angle be known, the numerical magnitude of each of the others is known also.

Let the angle AOP (Fig., Art. 23) be denoted by θ.

In the triangle AOP we have, by Euc. I. 47,

$$MP^2 + OM^2 = OP^2 \text{..................(1).}$$

Hence, dividing by OP^2, we have

$$\left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = 1,$$

i.e. $$(\sin\theta)^2 + (\cos\theta)^2 = 1.$$

The quantity $(\sin\theta)^2$ is always written $\sin^2\theta$ and so for the other ratios.

Hence this relation is

$$\mathbf{\sin^2\theta + \cos^2\theta = 1} \text{.................(2).}$$

Again, dividing both sides of equation (1) by OM^2, we have

$$\left(\frac{MP}{OM}\right)^2 + 1 = \left(\frac{OP}{OM}\right)^2,$$

i.e. $$(\tan\theta)^2 + 1 = (\sec\theta)^2,$$

so that $$\mathbf{\sec^2\theta = 1 + \tan^2\theta} \text{ .... (3).}$$

Again, dividing equations (1) by MP^2 we have

$$1 + \left(\frac{OM}{MP}\right)^2 = \left(\frac{OP}{MP}\right)^2,$$

i.e. $$1 + (\cot\theta)^2 = (\operatorname{cosec}\theta)^2,$$

so that $$\mathbf{\operatorname{cosec}^2\theta = 1 + \cot^2\theta} \text{............(4).}$$

Also, since $\sin\theta = \frac{MP}{OP}$ and $\cos\theta = \frac{OM}{OP}$,

we have $$\frac{\sin\theta}{\cos\theta} = \frac{MP}{OP} \div \frac{OM}{OP} = \frac{MP}{OM} = \tan\theta.$$

Hence $$\mathbf{\tan\theta = \frac{\sin\theta}{\cos\theta}} \text{ .... (5).}$$

Similarly $$\mathbf{\cot\theta = \frac{\cos\theta}{\sin\theta}} \text{ .... (6).}$$

28. Ex. 1. *Prove that* $\sqrt{\dfrac{1-\cos A}{1+\cos A}} = \operatorname{cosec} A - \cot A$.

We have
$$\sqrt{\frac{1-\cos A}{1+\cos A}} = \sqrt{\frac{(1-\cos A)^2}{1-\cos^2 A}},$$
$$= \frac{1-\cos A}{\sqrt{1-\cos^2 A}} = \frac{1-\cos A}{\sin A},$$

by relation (1) of the last article,
$$= \frac{1}{\sin A} - \frac{\cos A}{\sin A} = \operatorname{cosec} A - \cot A.$$

Ex. 2. *Prove that*
$$\sqrt{\sec^2 A + \operatorname{cosec}^2 A} = \tan A + \cot A.$$

We have seen that $\sec^2 A = 1 + \tan^2 A$,

and $\operatorname{cosec}^2 A = 1 + \cot^2 A$.

$$\therefore \sec^2 A + \operatorname{cosec}^2 A = \tan^2 A + 2 + \cot^2 A$$
$$= \tan^2 A + 2 \tan A \cot A + \cot^2 A$$
$$= (\tan A + \cot A)^2,$$

so that
$$\sqrt{\sec^2 A + \operatorname{cosec}^2 A} = \tan A + \cot A.$$

Ex. 3. *Prove that*
$$(\operatorname{cosec} A - \sin A)(\sec A - \cos A)(\tan A + \cot A) = 1.$$

The given expression
$$= \left(\frac{1}{\sin A} - \sin A\right)\left(\frac{1}{\cos A} - \cos A\right)\left(\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\right)$$
$$= \frac{1-\sin^2 A}{\sin A} \cdot \frac{1-\cos^2 A}{\cos A} \cdot \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}$$
$$= \frac{\cos^2 A}{\sin A} \cdot \frac{\sin^2 A}{\cos A} \cdot \frac{1}{\sin A \cos A}$$
$$= 1.$$

EXAMPLES. V.

Prove the following statements.

1. $\cos^4 A - \sin^4 A + 1 = 2\cos^2 A$.

2. $(\sin A + \cos A)(1 - \sin A\cos A) = \sin^3 A + \cos^3 A$.

3. $\dfrac{\sin A}{1+\cos A} + \dfrac{1+\cos A}{\sin A} = 2\,\mathrm{cosec}\,A$.

4. $\cos^6 A + \sin^6 A = 1 - 3\sin^2 A\cos^2 A$.

5. $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = \sec A - \tan A$.

6. $\dfrac{\mathrm{cosec}\,A}{\mathrm{cosec}\,A - 1} + \dfrac{\mathrm{cosec}\,A}{\mathrm{cosec}\,A + 1} = 2\sec^2 A$.

7. $\dfrac{\mathrm{cosec}\,A}{\cot A + \tan A} = \cos A$.

8. $(\sec A + \cos A)(\sec A - \cos A) = \tan^2 A + \sin^2 A$.

9. $\dfrac{1}{\cot A + \tan A} = \sin A\cos A$.

10. $\dfrac{1}{\sec A - \tan A} = \sec A + \tan A$.

11. $\dfrac{1-\tan A}{1+\tan A} = \dfrac{\cot A - 1}{\cot A + 1}$.

12. $\dfrac{1+\tan^2 A}{1+\cot^2 A} = \dfrac{\sin^2 A}{\cos^2 A}$.

13. $\dfrac{\sec A - \tan A}{\sec A + \tan A} = 1 - 2\sec A\tan A + 2\tan^2 A$.

14. $\dfrac{\tan A}{1-\cot A} + \dfrac{\cot A}{1-\tan A} = \sec A\,\mathrm{cosec}\,A + 1$.

15. $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} = \sin A + \cos A$.

16. $(\sin A + \cos A)(\cot A + \tan A) = \sec A + \mathrm{cosec}\,A$.

17. $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A$.

18. $\cot^4 A + \cot^2 A = \operatorname{cosec}^4 A - \operatorname{cosec}^2 A.$

19. $\sqrt{\operatorname{cosec}^2 A - 1} = \cos A \operatorname{cosec} A.$

20. $\sec^2 A \operatorname{cosec}^2 A = \tan^2 A + \cot^2 A + 2.$

21. $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A.$

22. $(1 + \cot A - \operatorname{cosec} A)(1 + \tan A + \sec A) = 2.$

23. $\dfrac{1}{\operatorname{cosec} A - \cot A} - \dfrac{1}{\sin A} = \dfrac{1}{\sin A} - \dfrac{1}{\operatorname{cosec} A + \cot A}.$

24. $\dfrac{\cot A \cos A}{\cot A + \cos A} = \dfrac{\cot A - \cos A}{\cot A \cos A}.$

25. $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \cot A \tan B.$

26. $\left(\dfrac{1}{\sec^2 \alpha - \cos^2 \alpha} + \dfrac{1}{\operatorname{cosec}^2 \alpha - \sin^2 \alpha}\right) \cos^2 \alpha \sin^2 \alpha = \dfrac{1 - \cos^2 \alpha \sin^2 \alpha}{2 + \cos^2 \alpha \sin^2 \alpha}.$

27. $\sin^8 A - \cos^8 A = (\sin^2 A - \cos^2 A)(1 - 2 \sin^2 A \cos^2 A).$

28. $\dfrac{\cos A \operatorname{cosec} A - \sin A \sec A}{\cos A + \sin A} = \operatorname{cosec} A - \sec A.$

29. $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1 + \sin A}{\cos A}.$

30. $(\tan \alpha + \operatorname{cosec} \beta)^2 - (\cot \beta - \sec \alpha)^2 = 2 \tan \alpha \cot \beta (\operatorname{cosec} \alpha + \sec \beta).$

31. $2 \sec^2 \alpha - \sec^4 \alpha - 2 \operatorname{cosec}^2 \alpha + \operatorname{cosec}^4 \alpha = \cot^4 \alpha - \tan^4 \alpha.$

32. $\dfrac{1 - \sin A}{1 + \sin A} = 1 + 2 \tan A (\tan A - \sec A).$

33. $(\operatorname{cosec} A + \cot A) \operatorname{covers} A - (\sec A + \tan A) \operatorname{vers} A$
$= (\operatorname{cosec} A - \sec A)(2 - \operatorname{vers} A \operatorname{covers} A).$

34. $(1 + \cot A + \tan A)(\sin A - \cos A) = \dfrac{\sec A}{\operatorname{cosec}^2 A} - \dfrac{\operatorname{cosec} A}{\sec^2 A}.$

35. $2 \operatorname{versin} A + \cos^2 A = 1 + \operatorname{versin}^2 A.$

29. *Limits to the values of the trigonometrical ratios.*

From equation (2) of Art. 27 we have

$$\sin^2\theta + \cos^2\theta = 1.$$

Now $\sin^2\theta$ and $\cos^2\theta$, being both squares, are both necessarily positive. Hence, since their sum is unity, neither of them can be greater than unity.

[For if one of them, say $\sin^2\theta$, were greater than unity, the other, $\cos^2\theta$, would have to be negative, which is impossible.]

Hence neither the sine nor the cosine can be numerically greater than unity.

Since $\sin\theta$ cannot be greater than unity therefore $\operatorname{cosec}\theta$, which equals $\frac{1}{\sin\theta}$, cannot be numerically less than unity.

So $\sec\theta$, which equals $\frac{1}{\cos\theta}$, cannot be numerically less than unity.

30. The foregoing results follow easily from the figure of Art. 23.

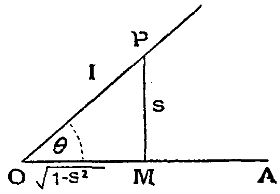
For, whatever be the value of the angle AOP, neither the side OM nor the side MP is ever greater than OP.

Since MP is never greater than OP the ratio $\frac{MP}{OP}$ is never greater than unity, so that the sine of an angle is never greater than unity.

Also since OM is never greater than OP, the ratio $\frac{OM}{OP}$ is never greater than unity, *i.e.* the cosine is never greater than unity.

31. We can express the trigonometrical ratios of an angle in terms of any one of them.

The simplest method of procedure is best shewn by examples.



Ex. 1. *To express all the trigonometrical ratios in terms of the sine.*

Let AOP be any angle θ.

Let the length OP be unity and let the corresponding length of MP be s.

By Euc. I. 47, $OM = \sqrt{OP^2 - MP^2} = \sqrt{1 - s^2}$.

Hence $$\sin\theta = \frac{MP}{OP} = \frac{s}{1} = s,$$

$$\cos\theta = \frac{OM}{OP} = \sqrt{1 - s^2} = \sqrt{1 - \sin^2\theta},$$

$$\tan\theta = \frac{MP}{OM} = \frac{s}{\sqrt{1 - s^2}} = \frac{\sin\theta}{\sqrt{1 - \sin^2\theta}},$$

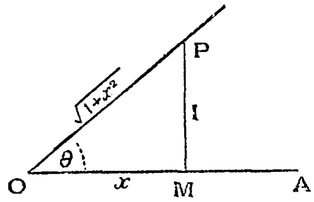
$$\cot\theta = \frac{OM}{MP} = \frac{\sqrt{1 - s^2}}{s} = \frac{\sqrt{1 - \sin^2\theta}}{\sin\theta},$$

$$\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{1}{s} = \frac{1}{\sin\theta},$$

and $$\sec\theta = \frac{OP}{OM} = \frac{1}{\sqrt{1 - s^2}} = \frac{1}{\sqrt{1 - \sin^2\theta}}.$$

The last five equations give what is required.

Ex. 2. *To express all the trigonometrical relations in terms of the cotangent.*



Taking the usual figure let the length MP be unity, and let the corresponding value of OM be x.

By Euc. I. 47,

$$OP = \sqrt{OM^2 + MP^2} = \sqrt{1 + x^2}.$$

Hence $$\cot\theta = \frac{OM}{MP} = \frac{x}{1} = x,$$

$$\sin\theta = \frac{MP}{OP} = \frac{1}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+\cot^2\theta}},$$

$$\cos\theta = \frac{OM}{OP} = \frac{x}{\sqrt{1+x^2}} = \frac{\cot\theta}{\sqrt{1+\cot^2\theta}},$$

$$\tan\theta = \frac{MP}{OM} = \frac{1}{x} = \frac{1}{\cot\theta},$$

$$\sec\theta = \frac{OP}{OM} = \frac{\sqrt{1+x^2}}{x} = \frac{\sqrt{1+\cot^2\theta}}{\cot\theta},$$

and $$\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+\cot^2\theta}.$$

The last five equations give what is required.

It will be noticed that, in each case, the denominator of the fraction which defines the trigonometrical ratio was taken equal to unity. For example, the sine is $\frac{MP}{OP}$, and hence in Ex. 1 the denominator OP is taken equal to unity.

The cotangent is $\frac{OM}{MP}$, and hence in Ex. 2 the side MP is taken equal to unity.

Similarly suppose we had to express the other ratios in terms of the cosine, we should, since the cosine is equal to $\frac{OM}{OP}$, put OP equal to unity and OM equal to c. The working would then be similar to that of Exs. 1 and 2.

In the following examples the sides have numerical values.

Ex. 3. *If* $\cos\theta$ *equal* $\frac{3}{5}$, *find the values of the other ratios.*

Along the initial line OA take OM equal to 3, and erect a perpendicular MP.

Let a line OP, of length 5, revolve round O until its other end meets this perpendicular in the point P. Then AOP is the angle θ.

By Euc. I. 47, $MP=\sqrt{OP^2-OM^2}=\sqrt{5^2-3^2}=4$.

Hence clearly

$$\sin\theta=\frac{4}{5},\quad \tan\theta=\frac{4}{3},\quad \cot\theta=\frac{3}{4},\quad \operatorname{cosec}\theta=\frac{5}{4}\quad\text{and}\quad \sec\theta=\frac{5}{3}.$$

Ex. 4. *Supposing* θ *to be an angle whose sine is* $\frac{1}{3}$, *to find the numerical magnitude of the other trigonometrical ratios.*

Here $\sin\theta=\frac{1}{3}$, so that the relation (2) of Art. 27 gives

$$\left(\frac{1}{3}\right)^2+\cos^2\theta=1,$$

i.e.

$$\cos^2\theta=1-\frac{1}{9}=\frac{8}{9},$$

i.e.

$$\cos\theta=\frac{2\sqrt{2}}{3}.$$

Hence

$$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4},$$

$$\cot\theta=\frac{1}{\tan\theta}=2\sqrt{2},$$

$$\operatorname{cosec}\theta=\frac{1}{\sin\theta}=3,$$

$$\sec\theta=\frac{1}{\cos\theta}=\frac{3}{2\sqrt{2}}=\frac{3\sqrt{2}}{4},$$

$$\operatorname{vers}\theta=1-\cos\theta=1-\frac{2\sqrt{2}}{3},$$

and

$$\operatorname{covers}\theta=1-\sin\theta=1-\frac{1}{3}=\frac{2}{3}.$$

32. In the following table is given the result of expressing each trigonometrical ratio in terms of each of the others.

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\dfrac{1}{\sqrt{1+\cot^2\theta}}$	$\dfrac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\dfrac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\dfrac{1}{\sqrt{\tan^2\theta+1}}$	$\dfrac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\dfrac{1}{\sec\theta}$	$\dfrac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\dfrac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\dfrac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\cot\theta$	$\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\dfrac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\dfrac{1}{\tan\theta}$	$\cot\theta$	$\dfrac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2\theta-1}$
$\sec\theta$	$\dfrac{1}{\sqrt{1-\sin^2\theta}}$	$\dfrac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\dfrac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\operatorname{cosec}\theta$	$\dfrac{1}{\sin\theta}$	$\dfrac{1}{\sqrt{1-\cos^2\theta}}$	$\dfrac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\dfrac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\operatorname{cosec}\theta$

EXAMPLES. VI.

1. Express all the other trigonometrical ratios in terms of the cosine.

2. Express all the ratios in terms of the tangent.

3. Express all the ratios in terms of the cosecant.

4. Express all the ratios in terms of the secant.

5. The sine of a certain angle is $\frac{1}{4}$; find the numerical values of the other trigonometrical ratios of this angle.

6. If $\sin\theta=\frac{12}{13}$, find $\tan\theta$ and $\text{versin}\,\theta$.

7. If $\sin A=\frac{11}{61}$, find $\tan A$, $\cos A$, and $\sec A$.

8. If $\cos\theta=\frac{4}{5}$, find $\sin\theta$ and $\cot\theta$.

9. If $\cos A=\frac{9}{41}$, find $\tan A$ and $\text{cosec}\,A$.

10. If $\tan\theta=\frac{3}{4}$, find the sine, cosine, versine and cosecant of θ.

11. If $\tan\theta=\frac{1}{\sqrt{7}}$, find the value of $\frac{\text{cosec}^2\theta-\sec^2\theta}{\text{cosec}^2\theta+\sec^2\theta}$.

12. If $\cot\theta=\frac{15}{8}$, find $\cos\theta$ and $\text{cosec}\,\theta$.

13. If $\sec A=\frac{3}{2}$, find $\tan A$ and $\text{cosec}\,A$.

14. If $2\sin\theta=2-\cos\theta$, find $\sin\theta$.

15. If $8\sin\theta=4+\cos\theta$, find $\sin\theta$.

16. If $\tan\theta+\sec\theta=1\cdot5$, find $\sin\theta$.

17. If $\cot\theta+\text{cosec}\,\theta=5$, find $\cos\theta$.

18. If $3\sec^4\theta+8=10\sec^2\theta$, find the values of $\tan\theta$.

19. If $\tan^2\theta+\sec\theta=5$, find $\cos\theta$.

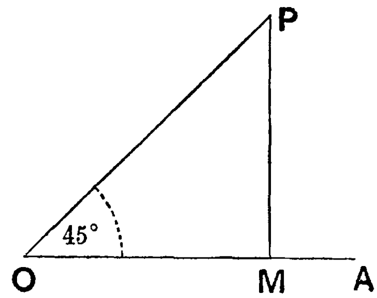
20. If $\tan\theta+\cot\theta=2$, find $\sin\theta$.

21. If $\sec^2\theta=2+2\tan\theta$, find $\tan\theta$.

22. If $\tan\theta=\frac{2x(x+1)}{2x+1}$, find $\sin\theta$ and $\cos\theta$.

Values of the trigonometrical ratios in some useful cases.

33. *Angle of* 45°.



Let the angle AOP traced out be 45°.

Then, since the three angles of a triangle are together equal to two right angles,

$$\angle OPM = 180° - \angle POM - \angle PMO$$

$$= 180° - 45° - 90° = 45° = \angle POM.$$

$$\therefore\ OM = MP = a \text{ (say)},$$

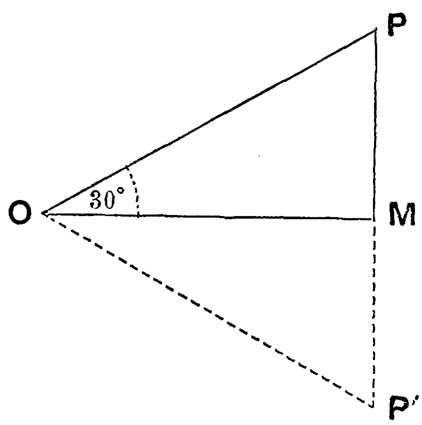
and $$OP = \sqrt{OM^2 + MP^2} = \sqrt{2}\,.\,a.$$

$$\therefore\ \sin 45° = \frac{MP}{OP} = \frac{a}{\sqrt{2}\,.\,a} = \frac{1}{\sqrt{2}},$$

$$\cos 45° = \frac{OM}{OP} = \frac{a}{\sqrt{2}\,.\,a} = \frac{1}{\sqrt{2}},$$

and $$\tan 45° = 1.$$

34. *Angle of* 30°.



Let the angle AOP traced out be 30°.

Produce PM to P' making MP' equal to PM.

The two triangles OMP and OMP' have their sides OM and MP' equal to OM and MP and also the contained angles equal.

Therefore $OP' = OP$, and $\angle OP'P = \angle OPP' = 60°$, so that the triangle $P'OP$ is equilateral.

Hence $$OP^2 = PP'^2 = 4PM^2 = 4OP^2 - 4a^2,$$

where OM equals a.

$$\therefore\ 3OP^2 = 4a^2,$$

so that $$OP = \frac{2a}{\sqrt{3}},\ \text{and}\ MP = \tfrac{1}{2}OP = \frac{a}{\sqrt{3}}.$$

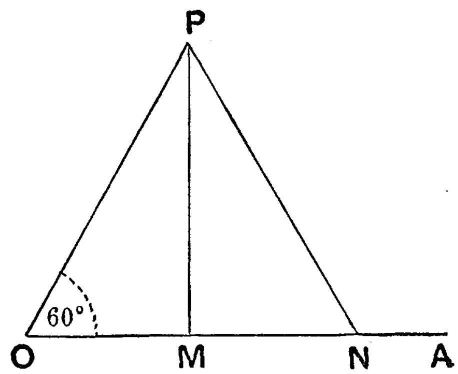
$$\therefore\ \sin 30^\circ = \frac{MP}{OP} = \frac{1}{2},$$

$$\cos 30^\circ = \frac{OM}{OP} = a \div \frac{2a}{\sqrt{3}} = \frac{\sqrt{3}}{2},$$

and $$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{1}{\sqrt{3}}.$$

35. *Angle of* 60°.

Let the angle AOP traced out be 60°.



Take a point N on OA, so that

$$MN = OM = a\ \text{(say)}.$$

The two triangles OMP and NMP have now the sides OM and MP equal to NM and MP respectively, and the included angles equal, so that the triangles are equal.

$$\therefore\ PN = OP,\ \text{and}\ \angle PNM = \angle POM = 60^\circ.$$

The triangle OPN is therefore equilateral, and hence

$$OP = ON = 2OM = 2a.$$

$$\therefore\ MP = \sqrt{OP^2 - OM^2} = \sqrt{4a^2 - a^2} = \sqrt{3}\,.\,a.$$

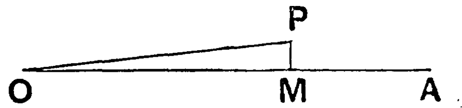
Hence $$\sin 60^\circ = \frac{MP}{OP} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2},$$

$$\cos 60^\circ = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2},$$

and $$\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \sqrt{3}.$$

36. *Angle of* 0°.

Let the revolving line OP have turned through a very small angle, so that the angle MOP is very small.



The magnitude of MP is then very small and initially, before OP had turned through an angle big enough to be perceived, the quantity MP was smaller than any quantity we could assign, *i.e.* was what we denote by 0.

Also, in this case, the two points M and P very nearly coincide, and the smaller the angle AOP the more nearly do they coincide.

Hence, when the angle AOP is actually zero, the two lengths OM and OP are equal and MP is zero.

Hence $$\sin 0^\circ = \frac{MP}{OP} = \frac{0}{OP} = 0,$$

$$\cos 0^\circ = \frac{OM}{OP} = \frac{OP}{OP} = 1,$$

and $$\tan 0^\circ = \tfrac{0}{1} = 0.$$

Also $\cot 0^\circ =$ the value of $\frac{OM}{MP}$ when M and P coincide

= the ratio of a finite quantity to something infinitely small

= a quantity which is infinitely great.

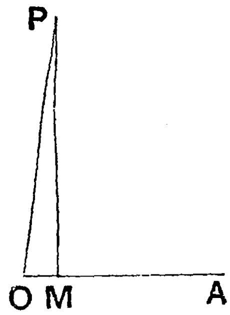
Such a quantity is usually denoted by the symbol ∞.

Hence $\cot 0^\circ = \infty$.

Similarly $\operatorname{cosec} 0^\circ = \frac{OP}{MP} = \infty$ also.

And $\sec 0^\circ = \frac{OP}{OM} = 1$.

37. *Angle of* 90°.



Let the angle AOP be very nearly, but not quite, a right angle.

When OP has actually described a right angle the point M coincides with O, so that then OM is zero and OP and MP are equal.

Hence $\sin 90^\circ = \frac{MP}{OP} = \frac{OP}{OP} = 1$,

$$\cos 90^\circ = \frac{OM}{OP} = \frac{0}{OP} = 0,$$

$$\tan 90^\circ = \frac{MP}{OM} = \frac{\text{a finite quantity}}{\text{an infinitely small quantity}}.$$

$$= \text{a number infinitely large} = \infty.$$

$$\cot 90^\circ = \frac{OM}{MP} = \frac{0}{MP} = 0,$$

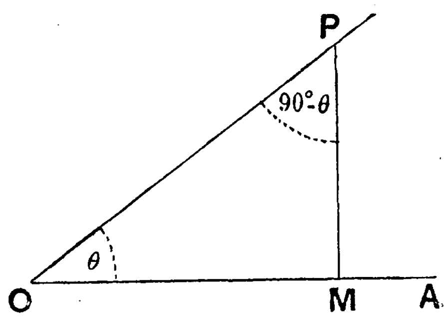
$\sec 90^\circ = \frac{OP}{OM} = \infty$, as in the case of the tangent,

and $\operatorname{cosec} 90^\circ = \frac{OP}{MP} = \frac{OP}{OP} = 1$.

38. Complementary Angles. Def. Two angles are said to be complementary when their sum is equal to a right angle. Thus any angle θ and the angle $90^\circ - \theta$ are complementary.

39. *To find the relations between the trigonometrical ratios of two complementary angles.*

Let the revolving line, starting from OA, trace out any acute angle AOP, equal to θ. From any point P on it draw PM perpendicular to OA.



Since the three angles of a triangle are together equal to two right angles, and since OMP is a right angle, the sum of the two angles MOP and OPM is a right angle.

They are therefore complementary and

$$\angle OPM = 90^\circ - \theta.$$

[When the angle OPM is considered, the line PM is the "base" and MO is the "perpendicular."]

We then have

$$\mathbf{sin}\,(\mathbf{90}^\circ - \boldsymbol{\theta}) = \sin MPO = \frac{MO}{PO} = \cos AOP = \mathbf{cos}\,\boldsymbol{\theta},$$

$$\mathbf{cos}\,(\mathbf{90}^\circ - \boldsymbol{\theta}) = \cos MPO = \frac{PM}{PO} = \sin AOP = \mathbf{sin}\,\boldsymbol{\theta},$$

$$\tan(90^\circ - \theta) = \tan MPO = \frac{MO}{PM} = \cot AOP = \cot\theta,$$

$$\cot(90^\circ - \theta) = \cot MPO = \frac{PM}{MO} = \tan AOP = \tan\theta,$$

$$\text{cosec}\,(90^\circ - \theta) = \text{cosec}\, MPO = \frac{PO}{MO} = \sec AOP = \sec\theta,$$

and $\sec(90^\circ - \theta) = \sec MPO = \dfrac{PO}{PM} = \text{cosec}\, AOP = \text{cosec}\,\theta.$

Hence we observe that
the Sine of any angle = the Cosine of its complement,
the Tangent of any angle = the Cotangent of its complement,
and the Secant of an angle = the Cosecant of its complement.

From this is apparent what is the derivation of the names **Co**sine, **Co**tangent, and **Co**secant.

40. The student is advised before proceeding any further to make himself quite familiar with the following table. [For an extension of this table, see Art. 76.]

Angle	0^0	30^0	45^0	60^0	90^0
Sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
Cotangent	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
Cosecant	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
Secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

If the student commits accurately to memory the portion of the above table included between the thick lines, he should be able to easily reproduce the rest.

For

(1) the sines of 60° and 90° are respectively the cosines of 30° and 0°. (Art. 39.)

(2) the cosines of 60° and 90° are respectively the sines of 30° and 0°. (Art. 39.)

Hence the second and third lines are known.

(3) The tangent of any angle is the result of dividing the sine by the cosine.

Hence any quantity in the fourth line is obtained by dividing the corresponding quantity in the second line by the corresponding quantity in the third line.

(4) The cotangent of any angle is the reciprocal of the tangent, so that the quantities in the fifth row are the reciprocals of the quantities in the fourth row.

(5) Since $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, the sixth row is obtained by inverting the corresponding quantities in the second row.

(6) Since $\sec\theta = \frac{1}{\cos\theta}$, the seventh row is similarly obtained from the third row.

EXAMPLES. VII.

1. If $A = 30^\circ$, verify that

(1) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$,

(2) $\sin 2A = 2\sin A \cos A$,

(3) $\cos 3A = 4\cos^3 A - 3\cos A$,

(4) $\sin 3A = 3\sin A - 4\sin^3 A$,

and (5) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$.

2. If $A=45^\circ$, verify that

(1) $\sin 2A = 2 \sin A \cos A$,

(2) $\cos 2A = 1 - 2\sin^2 A$,

and (3) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$.

Verify that

3. $\sin^2 30^\circ + \sin^2 45^\circ + \sin^2 60^\circ = \dfrac{3}{2}$.

4. $\tan^2 30^\circ + \tan^2 45^\circ + \tan^2 60^\circ = 4\frac{1}{3}$.

5. $\sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ = 1$.

6. $\cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ = -\dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

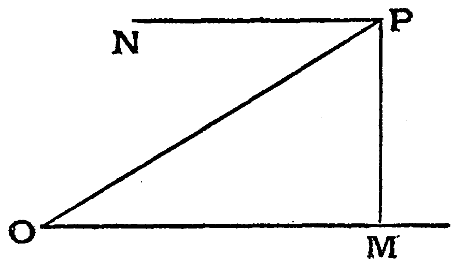
CHAPTER III.

SIMPLE PROBLEMS IN HEIGHTS AND DISTANCES.

41. One of the objects of Trigonometry is to find the distances between points, or the heights of objects, without actually measuring these distances or these heights.

42. Suppose O and P to be two points, P being at a higher level than O.

Let OM be a horizontal line drawn through O to meet in M the vertical line drawn through P.



The angle MOP is called the **Angle of Elevation** of the point P as seen from O.

Draw PN parallel to MO, so that PN is the horizontal line passing through P. The angle NPO is the **Angle of Depression** of the point O as seen from P.

43. Two of the instruments used in practical work are the Theodolite and the Sextant.

The Theodolite is used to measure angles in a vertical plane.

The Theodolite, in its simple form, consists of a telescope attached to a flat piece of wood. This piece of wood is supported by three legs and can be arranged so as to be accurately horizontal.

This table being at O and horizontal and the telescope being initially pointing in the direction OM, the latter can be made to rotate in a vertical plane until it points accurately towards P. A graduated scale shews the angle through which it has been turned from the horizontal, *i.e.* gives us the angle of elevation MOP.

Similarly, if the instrument were at P, the angle NPO through which the telescope would have to be turned, downward from the horizontal, would give us the angle NPO.

The instrument can also be used to measure angles in a horizontal plane.

44. The Sextant is used to find the angle subtended by any two points D and E at a third point F. It is an instrument much used on board ships.

Its construction and application are too complicated to be here considered.

45. We shall now solve a few simple examples in heights and distances.

Ex. 1. *A vertical flagstaff stands on a horizontal plane; from a point distant* 150 *feet from its foot the angle of elevation of its top is found to be* 30°; *find the height of the flagstaff.*

Let MP (Fig. Art. 42) represent the flagstaff and O the point from which the angle of elevation is taken.

Then $OM = 150$ feet, and $\angle MOP = 30°$.

Since PMO is a right angle, we have

$$\frac{MP}{OM} = \tan MOP = \tan 30° = \frac{1}{\sqrt{3}} \text{ (Art. 33)}.$$

$$\therefore\ MP = \frac{OM}{\sqrt{3}} = \frac{150}{\sqrt{3}} = \frac{150\sqrt{3}}{3} = 50\sqrt{3}.$$

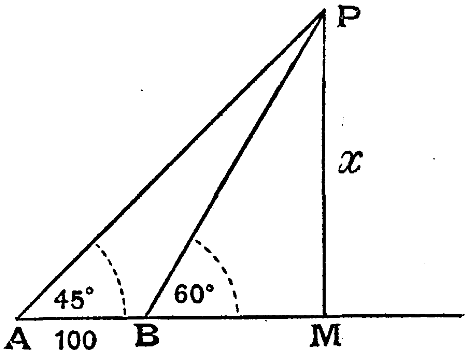
Now, by extraction of the square root, we have

$$\sqrt{3} = 1{\cdot}73205....$$

Hence $\qquad MP = 50 \times 1{\cdot}73205...\text{ feet} = 86{\cdot}6025...\text{ feet}.$

Ex. 2. *A man wishes to find the height of a church spire which stands on a horizontal plane; at a point on this plane he finds the angle of elevation of the top of the spire to be* 45°; *on walking* 100 *feet toward the tower he finds the corresponding angle of elevation to be* 60°; *deduce the height of the tower and also his original distance from the foot of the spire.*

Let P be the top of the spire and A and B the two points at which the angles of elevation are taken. Draw PM perpendicular to AB produced and let MP be x.



We are given $AB=100$ feet,

$$\angle MAP=45°,$$

and $$\angle MBP=60°.$$

We then have

$$\frac{AM}{x}=\cot 45°,$$

and $$\frac{BM}{x}=\cot 60°=\frac{1}{\sqrt{3}}.$$

Hence $$AM=x, \text{ and } BM=\frac{x}{\sqrt{3}}.$$

$$\therefore\ 100=AM-BM=x-\frac{x}{\sqrt{3}}=x\frac{\sqrt{3}-1}{\sqrt{3}}.$$

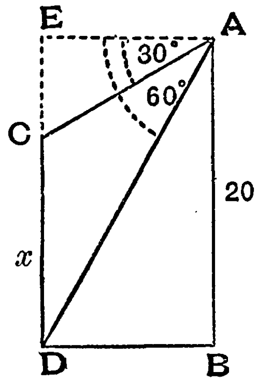
$$\therefore\ x=\frac{100\sqrt{3}}{\sqrt{3}-1}=\frac{100\sqrt{3}(\sqrt{3}+1)}{3-1}=50(3+\sqrt{3})$$

$$=50[3+1\cdot73205\ldots]=236\cdot6\ldots \text{ feet.}$$

Also $AM=x$, so that both of the required distances are equal to 236·6... feet.

Ex. 3. *From the top of a cliff,* 200 *feet high, the angles of depression of the top and bottom of a tower are observed to be* 30° *and* 60°; *find the height of the tower.*

Let A be the point of observation and BA the height of the cliff and let CD be the tower.



Draw AE horizontally, so that $\angle EAC=30°$ and $\angle EAD=60°$.

Let x feet be the height of the tower and produce DC to meet AE in E, so that $CE=AB-x=200-x$.

Since $\angle ADB=\angle DAE=60°$ (Euc. I. 29),

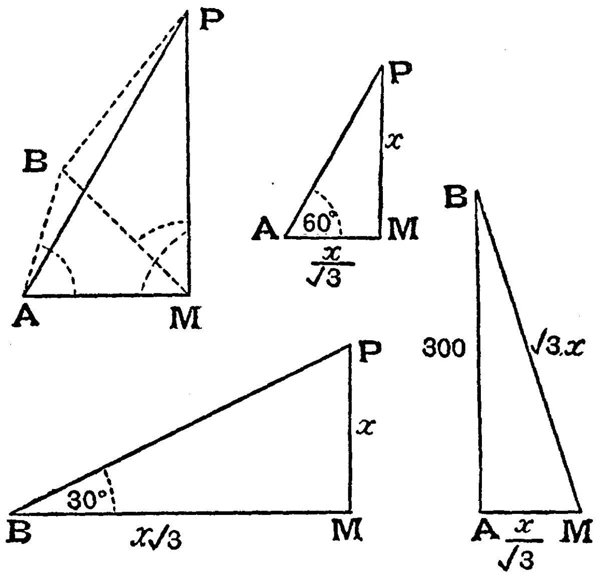
$$\therefore\ DB=AB\cot ADB=200\cot 60°=\frac{200}{\sqrt{3}}.$$

Also $$\frac{200-x}{DB}=\frac{CE}{EA}=\tan 30°=\frac{1}{\sqrt{3}}.$$

$$\therefore\ 200 - x = \frac{DB}{\sqrt{3}} = \frac{200}{3},$$

so that
$$x = 200 - \frac{200}{3} = 133\tfrac{1}{3} \text{ feet.}$$

Ex. 4. *A man observes that at a point due south of a certain tower its angle of elevation is* 60° ; *he then walks* 300 *feet due west on a horizontal plane and finds that the angle of elevation is then* 30° ; *find the height of the tower and his original distance from it.*



Let P be the top, and PM the height, of the tower, A the point due south of the tower and B the point due west of A.

The angles PMA, PMB, and MAB are therefore all right angles.

For simplicity, since the triangles PAM, PBM, and ABM are in different planes, they are reproduced in the second, third, and fourth figures and drawn to scale.

We are given $AB = 300$ feet, $\angle PAM = 60°$, and $\angle PBM = 30°$.

Let the height of the tower be x feet.

From the second figure
$$\frac{AM}{x} = \cot 60° = \frac{1}{\sqrt{3}},$$

so that
$$AM = \frac{x}{\sqrt{3}}.$$

From the third figure
$$\frac{BM}{x} = \cot 30° = \sqrt{3},$$

so that
$$BM = \sqrt{3} \,.\, x,$$

From the last figure we have

$$BM^2 = AM^2 + AB^2,$$

i.e.

$$3x^2 = \frac{1}{3}x^2 + 300^2.$$

$$\therefore\ 8x^2 = 3 \times 300^2.$$

$$\therefore\ x = \frac{300\sqrt{3}}{2\sqrt{2}} = 150 \,.\, \frac{\sqrt{6}}{2} = 75 \times \sqrt{6}$$

$$= 75 \times 2\cdot44949\ldots = 183\cdot71\ldots \text{ feet.}$$

Also his original distance from the tower

$$= x \cot 60^\circ = \frac{x}{\sqrt{3}} = 75 \times \sqrt{2}$$

$$= 75 \times (1\cdot4142\ldots) = 106\cdot065\ldots \text{ feet.}$$

EXAMPLES. VIII.

1. A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60°; when he retires 40 feet from the bank he finds the angle to be 30°; find the height of the tree and the breadth of the river.

2. At a certain point the angle of elevation of a tower is found to be such that its cotangent is $\frac{3}{5}$; on walking 32 feet directly toward the tower its angle of elevation is an angle whose cotangent is $\frac{2}{5}$. Find the height of the tower.

3. At a point A the angle of elevation of a tower is found to be such that its tangent is $\frac{5}{12}$; on walking 240 feet nearer the tower the tangent of the angle of elevation is found to be $\frac{3}{4}$; what is the height of the tower?

4. Find the height of a chimney when it is found that on walking towards it 100 feet in a horizontal line through its base, the angular elevation of its top changes from 30° to 45°.

5. An observer on the top of a cliff, 200 feet above the sea-level, observes the angles of depression of two ships at anchor to be 45° and 30° respectively; find the distances between the ships if the line joining them points to the base of the cliff.

6. From the top of a cliff an observer finds that the angles of depression of two buoys in the sea are 39° and 26° respectively; the buoys are 300 yards apart and the line joining them points straight at the foot of the cliff; find the height of the cliff and the distance of the nearest buoy from the foot of the cliff, given that $\cot 26° = 2{\cdot}0503$, and $\cot 39° = 1{\cdot}2349$.

7. The upper part of a tree broken over by the wind makes an angle of 30° with the ground, and the distance from the root to the point where the top of the tree touches the ground is 50 feet; what was the height of the tree?

8. The horizontal distance between two towers is 60 feet and the angular depression of the top of the first as seen from the top of the second which is 150 feet high is 30°; find the height of the first.

9. The angle of elevation of the top of an unfinished tower from a point distant 120 feet from its base is 45°; how much higher must the tower be raised so that its angle of elevation at the same point may be 60°?

10. Two pillars of equal height stand on either side of a roadway which is 100 feet wide; at a point in the roadway between the pillars the elevations of the tops of the pillars are 60° and 30°; find their height and the position of the point.

11. The angle of elevation of the top of a tower is observed to be 60°; at a point 40 feet above the first point of observation the elevation is found to be 45°; find the height of the tower and its horizontal distance from the points of observation.

12. At the foot of a mountain the elevation of its summit is found to be 45°; after ascending one mile up a slope of 30° inclination the elevation is found to be 60°. Find the height of the mountain.

13. What is the angle of elevation of the sun when the length of its shadow is $\sqrt{3}$ times its height?

14. The shadow of a tower standing on a level plane is found to be 60 feet longer when the sun's altitude is 30° than when it is 45°. Prove that the height of the tower is $30(1+\sqrt{3})$ feet.

15. On a straight coast there are three objects A, B, and C such that $AB = BC = 2$ miles. A vessel approaches B in a line perpendicular to the coast and at a certain point AC is found to subtend an angle of 60°; after sailing in the same direction for ten minutes AC is found to subtend 120°; find the rate at which the ship is going.

16. Two flagstaffs stand on a horizontal plane. A and B are two points on the line joining the bases of the flagstaffs and between them. The angles of elevation of the tops of the flagstaffs as seen from A are 30° and 60° and, as seen from B, they are 60° and 45°. If the length AB be 30 feet, find the heights of the flagstaffs and the distance between them.

17. P is the top and Q the foot of a tower standing on a horizontal plane. A and B are two points on this plane such that AB is 32 feet and QAB is a right angle. It is found that $\cot PAQ = \frac{2}{5}$ and

$$\cot PBQ = \frac{3}{5};$$

find the height of the tower.

18. A square tower stands upon a horizontal plane. From a point in this plane from which three of its upper corners are visible their angular elevations are respectively 45°, 60°, and 45°. Shew that the height of the tower is to the breadth of one of its sides as $\sqrt{6}(\sqrt{5}+1)$ to 4.

19. A lighthouse, facing north, sends out a fan-shaped beam of light extending from north-east to north-west. A steamer sailing due west first sees the lighthouse when it is 5 miles away from the lighthouse and continues to see it for $30\sqrt{2}$ minutes. What is the speed of the steamer?

20. A man stands at a point X on the bank XY of a river with straight and parallel banks and observes that the line joining X to a point Z on the opposite bank makes an angle of 30° with XY. He then goes along the bank a distance of 200 yards to Y and finds that the angle ZYX is 60°. Find the breadth of the river.

21. A man, walking due north, observes that the elevation of a balloon, which is due east of him and is sailing toward the north-west, is then 60°; after he has walked 400 yards the balloon is vertically over his head; find its height supposing it to have always remained the same.

CHAPTER IV.

APPLICATION OF ALGEBRAIC SIGNS TO TRIGONOMETRY.

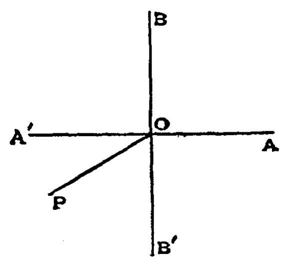
46. Positive and Negative Angles. In Art. 6 in treating of angles of any size we spoke of the revolving line as if it always revolved in a direction opposite to that in which the hands of a watch revolve, when the watch is held with its face uppermost.

This direction is called counter-clockwise.

When the revolving line turns in this manner it is said to revolve in the positive direction and to trace out a positive angle.

When the line OP revolves in the opposite direction, *i.e.* in the same direction as the hands of the watch, it is said to revolve in the negative direction and to trace out a negative angle. This negative direction is clockwise.

47. Let the revolving line start from OA and revolve until it reaches a position OP which lies between OA' and OB' and which bisects the angle $A'OB'$.



If it has revolved in the positive direction it has traced out the positive angle whose measure is $+ 225°$.

If it has revolved in the negative direction it has traced out the negative angle $-135°$.

Again, suppose we only know that the revolving line is in the above position. It may have made one, two, three ... complete revolutions and then have described the positive angle $+225°$. Or again it may have made one, two, three... complete revolutions in the negative direction and then have described the negative angle $-135°$.

In the first case the angle it has described is either $225°$, or $360° + 225°$, or $2 \times 360° + 225°$, or $3 \times 360° + 225°$ ......*i.e.* $225°$, or $585°$, or $945°$, or $1305°$....

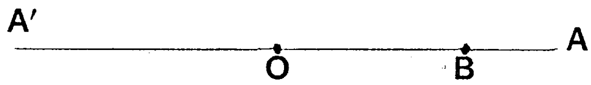
In the second case the angle it has described is $-135°$, or $-360° - 135°$, or $-2 \times 360° - 135°$, or $-3 \times 360° - 135°$ ...... *i.e.* $-135°$, or $-495°$, or $-855°$, or $-1215°$....

48. Positive and Negative Lines. Suppose that a man is told to start from a given milestone on a straight road and to walk 1000 yards along the road and then to stop. Unless we are told the *direction* in which he started we do not know his position when he stops. All we know is that he is either at a distance 1000 yards on one side of the milestone or at the same distance on the other side.

In measuring distances along a straight line it is therefore convenient to have a standard direction; this direction is called the positive direction and all distances measured along it are said to be positive. The opposite direction is the negative direction and all distances measured along it are said to be negative.

The standard or positive directions for horizontal lines is towards the right.

The length OA is in the positive direction. The length OA' is in the negative direction. If the magnitude of the distance OA or OA' be a, the point A is at a distance $+a$ from O and the point A' is at a distance $-a$ from O.



All lines measured to the right have then the positive sign prefixed; all lines to the left have the negative sign prefixed.

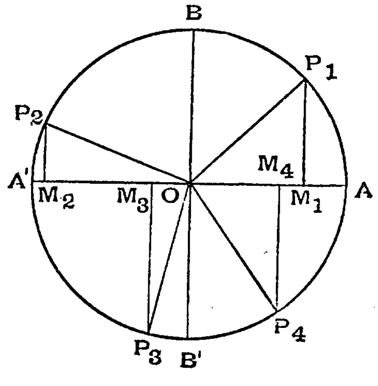
If a point start from O and describe a positive distance OA and then a distance AB back again toward O, equal numerically to b, the total distance it has described measured in the positive direction is $OA + AB$

$$i.e.\ \ +a+(-b),\ \ i.e.\ a-b.$$

49. For lines at right angles to AA' the positive direction is from O towards the top of the page, *i.e.* the direction of OB (Fig. Art. 47). All lines measured from O towards the foot of the page, *i.e.* in the direction OB', are negative.

50. *Trigonometrical ratios for an angle of any magnitude.*

Let OA be the initial line (drawn in the positive direction) and let OA' be drawn in the opposite direction to OA.



Let BOB' be a line at right angles to OA, its positive direction being OB.

Let a revolving line OP start from OA and revolving in either direction, positive or negative, trace out an angle of any magnitude whatever. From a point P in the revolving line draw PM perpendicular to AOA'.

[Four positions of the revolving line are given in the figure, one in each of the four quadrants, and the suffixes 1, 2, 3 and 4 are attached to P for the purpose of distinction.]

We then have the following definitions, which are the same as those given in Art. 23 for the simple case of an acute angle:

$\frac{MP}{OP}$ is called the **Sine** of the angle AOP,

$\frac{OM}{OP}$ " " **Cosine** " "

$\frac{MP}{OM}$ " " **Tangent** " "

$\frac{OM}{MP}$ " " **Cotangent** " "

$\frac{OP}{OM}$ " " **Secant** " "

$\frac{OP}{MP}$ " " **Cosecant** " "

The quantities $1 - \cos AOP$, and $1 - \sin AOP$ are respectively called the **Versed Sine** and the **Coversed Sine** of AOP.

51. In exactly the same manner as in Art. 27 it may be shewn that, for all values of the angle AOP $(= \theta)$, we have

$$\sin^2\theta + \cos^2\theta = 1,$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta,$$

$$\sec^2\theta = 1 + \tan^2\theta,$$

and

$$\operatorname{cosec}^2\theta = 1 + \cot^2\theta.$$

52. Signs of the trigonometrical ratios.

First quadrant. Let the revolving line be in the first quadrant, as OP_1. This revolving line is always positive.

Here OM_1 and M_1P_1 are both positive, so that all the trigonometrical ratios are then positive.

Second quadrant. Let the revolving line be in the second quadrant, as OP_2. Here M_2P_2 is positive and OM_2 is negative.

The sine, being equal to the ratio of a positive quantity to a positive quantity, is therefore positive.

The cosine, being equal to the ratio of a negative quantity to a positive quantity, is therefore negative.

The tangent, being equal to the ratio of a positive quantity to a negative quantity, is therefore negative.

The cotangent is negative.

The cosecant is positive.

The secant is negative.

Third quadrant. If the revolving line be, as OP_3, in the third quadrant, we have both M_3P_3 and OM_3 negative.

The sine is therefore negative.

The cosine is negative.

The tangent is positive.

The cotangent is positive.

The cosecant is negative.

The secant is negative.

Fourth quadrant. Let the revolving line be in the fourth quadrant, as OP_4. Here M_4P_4 is negative and OM_4 is positive.

The sine is therefore negative.

The cosine is positive.

The tangent is negative.

The cotangent is negative.

The cosecant is negative.

The secant is positive.

The annexed table shews the signs of the trigonometrical ratios according to the quadrant in which lies the revolving line, which bounds the angle considered.

B

sin	+	sin	+
cos	−	cos	+
tan	−	tan	+
cot	−	cot	+
cosec	+	cosec	+
sec	−	sec	+

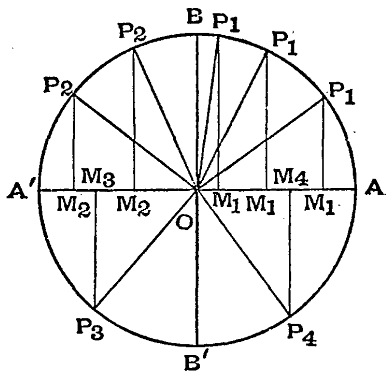
A′ O A

sin	−	sin	−
cos	−	cos	+
tan	+	tan	−
cot	+	cot	−
cosec	−	cosec	−
sec	−	sec	+

B′

53. *Tracing of the changes in the sign and magnitude of the trigonometrical ratios of an angle, as the angle increases from* 0° *to* 360°.

Let the revolving line OP be of constant length a.



When it coincides with OA the length OM_1 is equal to a and, when it coincides with OB, the point M_1 coincides with O and OM_1 vanishes. Also, as the revolving line turns from OA to OB, the distance OM_1 decreases from a to zero.

Whilst the revolving line is in the second quadrant and is revolving from OB to OA', the distance OM_2 is negative and increases numerically from 0 to a [*i.e.* it *decreases algebraically* from 0 to $-a$].

In the third quadrant the distance OM_3 *increases* algebraically from $-a$ to 0, and in the fourth quadrant the distance OM_4 increases from 0 to a.

In the first quadrant the length M_1P_1 increases from 0 to a; in the second quadrant M_2P_2 decreases from a to 0; in the third quadrant M_3P_3 decreases algebraically from 0 to $-a$; whilst in the fourth quadrant M_4P_4 increases algebraically from $-a$ to 0.

54. Sine. In the first quadrant as the angle increases from 0 to 90°, the sine, *i.e.* $\frac{M_1P_1}{a}$, increases from $\frac{0}{a}$ to $\frac{a}{a}$, *i.e.* from 0 to 1.

In the second quadrant as the angle increases from 90° to 180°, the sine decreases from $\frac{a}{a}$ to $\frac{0}{a}$, *i.e.* from 1 to 0.

In the third quadrant as the angle increases from 180° to 270°, the sine *decreases* from $\frac{0}{a}$ to $\frac{-a}{a}$, *i.e.* from 0 to -1.

In the fourth quadrant as the angle increases from 270° to 360°, the sine *increases* from $\frac{-a}{a}$ to $\frac{0}{a}$, *i.e.* from -1 to 0.

55. Cosine. In the first quadrant the cosine, which is equal to $\frac{OM}{a}$, decreases from $\frac{a}{a}$ to $\frac{0}{a}$, *i.e.* from 1 to 0.

In the second quadrant it decreases from $\frac{0}{a}$ to $\frac{-a}{a}$, *i.e.* from 0 to -1.

In the third quadrant it increases from $\frac{-a}{a}$ to $\frac{0}{a}$, *i.e.* from -1 to 0.

In the fourth quadrant it increases from $\frac{0}{a}$ to $\frac{a}{a}$, *i.e.* from 0 to 1.

56. Tangent. In the first quadrant M_1P_1 increases from 0 to a and OM_1 decreases from a to 0, so that $\frac{M_1P_1}{OM_1}$ continually increases (for its numerator continually increases and its numerator continually decreases).

When OP_1 coincides with OA, the tangent is 0; when the revolving line has turned through an angle which is slightly less than a right angle, so that OP_1 nearly coincides with OB, then M_1P_1 is very nearly equal to a and OM_1 is very small. The ratio $\frac{M_1P_1}{OM_1}$ is therefore very large, and the nearer OP_1 gets to OB the larger does the ratio become, so that, by taking the revolving line near enough to OB, we can make the tangent as large as we please. This is expressed by saying that when the angle is equal to 90° its tangent is infinite.

The symbol ∞ is used to denote an infinitely great quantity.

Hence in the first quadrant the tangent increases from 0 to ∞.

In the second quadrant when the revolving line has described an angle AOP_2 slightly greater than a right angle, M_2P_2 is very nearly equal to a and OM_2 is very small and negative, so that the corresponding tangent is very large and negative.

Also, as the revolving line turns from OB to OA', M_1P_1 decreases from a to 0 and OM_2 is negative and decreases from 0 to $-a$, so that when the revolving line coincides with OA' the tangent is zero.

Hence in the second quadrant the tangent increases from $-\infty$ to 0.

In the third quadrant both M_3P_3 and OM_3 are negative, and hence their ratio is positive. Also, when the revolving line coincides with OB', the tangent is infinite.

Hence in the third quadrant the tangent increases from 0 to ∞.

In the fourth quadrant M_4P_4 is negative and OM_4 is positive, so that their ratio is negative. Also, as the revolving line passes through OB' the tangent changes from $+\infty$ to $-\infty$ [just as in passing through OB].

Hence in the fourth quadrant the tangent increases from $-\infty$ to 0.

57. Cotangent. When the revolving line coincides with OA, M_1P_1 is very small and OM_1 is very nearly equal to a, so that the cotangent, *i.e.* the ratio $\frac{OM_1}{M_1P_1}$, is infinite to start with. Also, as the revolving line rotates from OA to OB, the quantity M_1P_1 increases from 0 to a and OM_1 decreases from a to 0.

Hence in the first quadrant the cotangent decreases from ∞ to 0.

In the second quadrant M_2P_2 is positive and OM_2 negative, so that the cotangent decreases from 0 to $\frac{-a}{0}$, *i.e.* from 0 to $-\infty$.

In the third quadrant it is positive and decreases from ∞ to 0 [for as the revolving line crosses OB' the cotangent changes from $-\infty$ to ∞].

In the fourth quadrant it is negative and decreases from 0 to $-\infty$.

58. Secant. When the revolving line coincides with OA the value of OM_1 is a, so that the value of the secant is then unity.

As the revolving line turns from OA to OB, OM_1 decreases from a to 0, and when the revolving line coincides with OB the value of the secant is $\frac{a}{0}$, *i.e.* ∞.

Hence in the first quadrant the secant increases from 1 to ∞.

In the second quadrant OM_2 is negative and decreases from 0 to $-a$. Hence in this quadrant the secant increases from $-\infty$ to -1 [for as the revolving line crosses OB the quantity OM_1 changes sign and therefore the secant changes from $+\infty$ to $-\infty$].

In the third quadrant OM_3 is always negative and increases from $-a$ to 0; therefore the secant decreases from -1 to $-\infty$. In the fourth quadrant OM_1 is always positive and increases from 0 to a. Hence in this quadrant the secant decreases from ∞ to $+1$.

59. Cosecant. The change in the cosecant may be traced in a similar manner to that in the secant.

In the first quadrant it decreases from ∞ to $+1$.
In the second quadrant it increases from $+1$ to $+\infty$.
In the third quadrant it increases from $-\infty$ to -1.
In the fourth quadrant it decreases from -1 to $-\infty$.

60. The foregoing results are collected in the annexed table.

B

In the second quadrant the			In the first quadrant the		
sine	decreases from	1 to 0	sine	increases from	0 to 1
cosine	decreases from	0 to -1	cosine	decreases from	1 to 0
tangent	increases from	$-\infty$ to 0	tangent	increases from	0 to ∞
cotangent	decreases from	0 to $-\infty$	cotangent	decreases from	∞ to 0
secant	increases from	$-\infty$ to -1	secant	increases from	1 to ∞
cosecant	increases from	1 to ∞	cosecant	decreases from	∞ to 1

A′ O A

In the third quadrant the			In the fourth quadrant the		
sine	decreases from	0 to -1	sine	increases from	-1 to 0
cosine	increases from	-1 to 0	cosine	increases from	0 to 1
tangent	increases from	0 to ∞	tangent	increases from	$-\infty$ to 0
cotangent	decreases from	∞ to 0	cotangent	decreases from	0 to $-\infty$
secant	decreases from	-1 to $-\infty$	secant	decreases from	∞ to 1
cosecant	increases from	$-\infty$ to -1	cosecant	decreases from	-1 to $-\infty$

B′

61. Periods of the trigonometrical functions.

As an angle increases from 0 to 2π radians *i.e.*, whilst the revolving line makes a complete revolution its sine first increases from 0 to 1, then decreases from 1 to -1, and finally increases from -1 to 0, and thus the sine goes through all its changes returning to its original value.

Similarly as the angle increases from 2π radians to 4π radians, the sine goes through the same series of changes.

Also the sines of any two angles which differ by four right angles, *i.e.* 2π radians, are the same.

This is expressed by saying that the **period of the sine is 2π.**

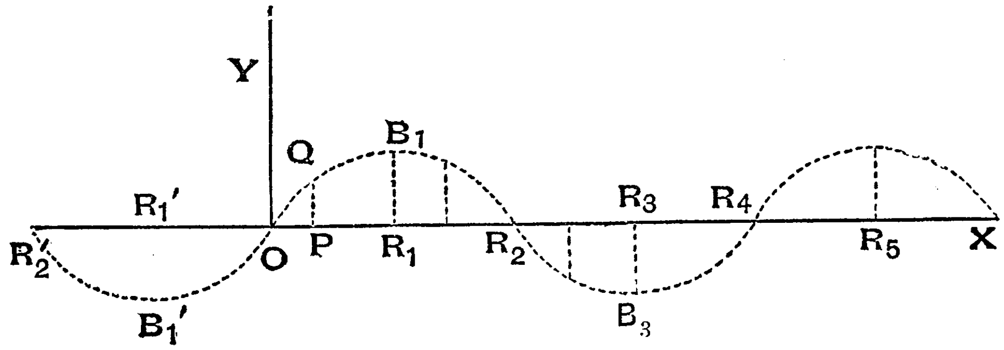
Similarly the cosine, secant, and cosecant go through all their changes as the angle increases by 2π.

The tangent, however, goes through all its changes as the angle increases from 0 to π radians, *i.e.* whilst the revolving line turns through two right angles. Similarly for the cotangent.

The period of the sine, cosine, secant and cosecant is therefore 2π radians; the period of the tangent and cotangent is π radians.

Since the values of the trigonometrical functions repeat over and over again as the angle increases, they are called **periodic functions.**

***62.** The variations in the values of the trigonometrical ratios may be graphically represented to the eye by means of curves constructed in the following manner.



Sine-Curve.

Let OX and OY be two straight lines at right angles and let the magnitudes of **angles** be represented by **lengths** measured along OX.

Let R_1, R_2, R_3,... be points such that the distances OR_1, R_1R_2, R_2R_3,... are equal. If then the distance OR_1 represent a right angle, the distances OR_2, OR_3, OR_4,... must represent two, three, four,... right angles.

Also if P be *any* point on the line OX, then OP represents an angle which bears the same ratio to a right angle that OP bears to OR_1.

[For example, if OP be equal to $\frac{1}{3}OR_1$ then OP would represent one-third of a right angle; if P bisected R_3R_4 then OP would represent $3\frac{1}{2}$ right angles.]

Let also OR_1 be so chosen that one unit of length represents one radian; since OR_2 represents two right angles, *i.e.* π radians, the length OR_2 must be π units of length, *i.e.* about $3\frac{1}{7}$ units of length.

In a similar manner negative angles are represented by distances OR_1', OR_2',... measured from O in a negative direction.

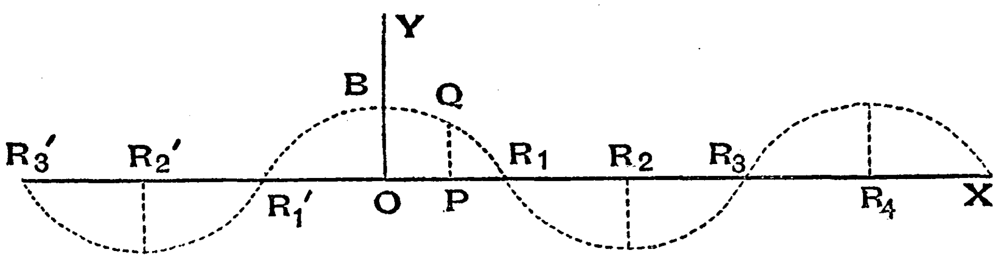
At each point P erect a perpendicular PQ to represent the sine of the angle which is represented by OP; if the sine be positive the perpendicular is to be drawn parallel to OY in the positive direction; if the sine be negative the line is to be drawn in the negative direction.

[For example, since OR_1 represents a right angle, the sine of which is 1, we erect a perpendicular R_1B_1 equal to one unit of length; since OR_2 represents an angle equal to two right angles, the sine of which is zero, we erect a perpendicular of length zero; since OR_3 represents three right angles, the sine of which is -1, we erect a perpendicular equal to -1, *i.e.* we draw R_3B_3 downward and equal to a unit of length; if OP were equal to one-third of OR_1 it would represent $\frac{1}{3}$ of a right angle, *i.e.* 30°, the sine of which is $\frac{1}{2}$, and so we should erect a perpendicular PQ equal to one-half the unit of length.]

The ends of all these lines, thus drawn, would be found to lie on a curve similar to the one drawn above.

It would be found that the curve consisted of portions, similar to $OB_1R_2B_3R_4$, placed side by side. This corresponds to the fact that each time the angle increases by 2π, the sine repeats the same value.

*63. Cosine-Curve.



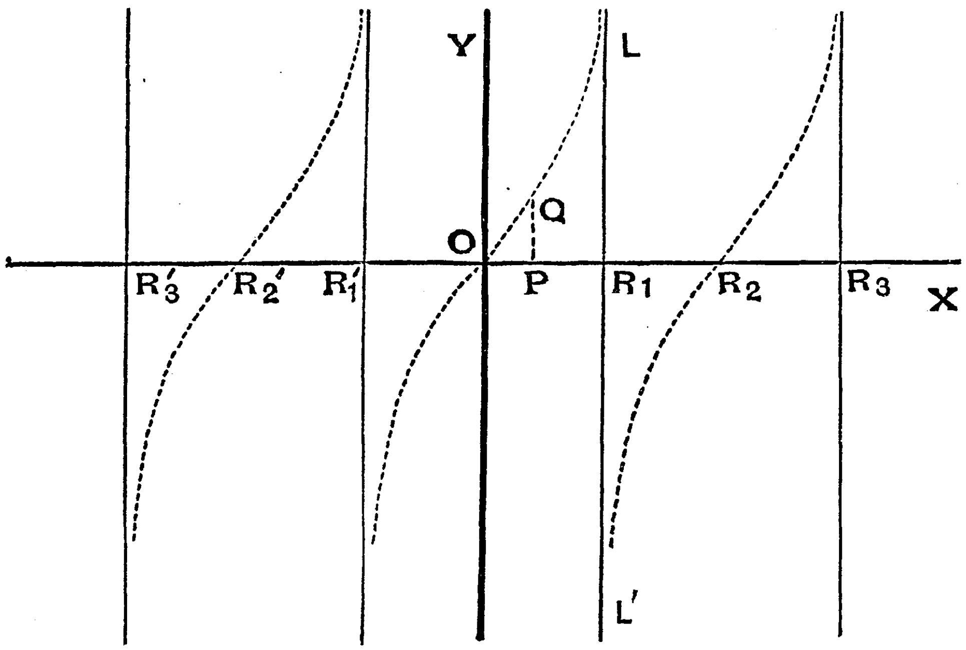
The Cosine-Curve is obtained in the same manner as the Sine-Curve, except that in this case the perpendicular PQ represents the cosine of the angle represented by OP.

The curve obtained is the same as that of Art. 62 if in that curve we move O to R_1 and let OY be drawn along R_1B_1.

*64. Tangent-Curve.

In this case, since the tangent of a right angle is infinite and since OR_1 represents a right angle, the perpendicular drawn at R_1 must be of infinite length and the dotted curve will only meet the line R_1L at an infinite distance.

Since the tangent of an angle slightly greater than a right angle is negative and almost infinitely great, the



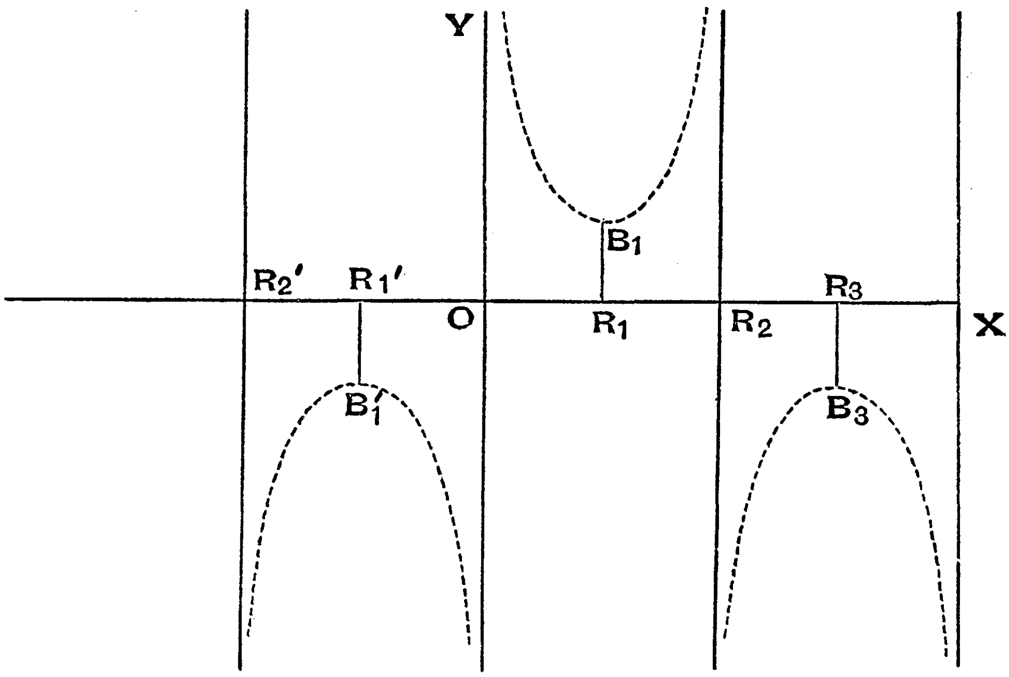
dotted curve immediately beyond LR_1L' commences at an infinite distance on the negative side, *i.e.* below, OX.

The Tangent-Curve will clearly consist of an infinite number of similar but disconnected portions, all ranged parallel to one another. Such a curve is called a Discontinuous Curve. Both the Sine-Curve and the Cosine-Curve are, on the other hand, Continuous Curves.

*65. **Cotangent-Curve.** If the curve to represent the cotangent be drawn in a similar manner, it will be found to meet OY at an infinite distance above O; it will pass through the point R_1 and touch the vertical line through R_2 at an infinite distance on the negative side of OX. Just beyond R_2 it will start at an infinite distance above R_2, and proceed as before.

The curve is therefore discontinuous and will consist of an infinite number of portions all ranged side by side.

*66. Cosecant-Curve.



When the angle is zero the sine is zero, and the cosecant is therefore infinite.

Hence the curve meets OY at infinity.

When the angle is a right angle the cosecant is unity, and hence R_1B_1 is equal to the unit of length.

When the angle is equal to two right angles its cosecant is infinity, so that the curve meets the perpendicular through R_2 at an infinite distance.

Again, as the angle increases from slightly less to slightly greater than two right angles, the cosecant changes from $+\infty$ to $-\infty$.

Hence just beyond R_2 the curve commences at an infinite distance on the negative side of, *i.e.* below, OX.

***67. Secant-Curve.** If, similarly, the Secant-Curve be traced it will be found to be the same as the Cosecant-Curve would be if we moved OY to R_1B_1.

MISCELLANEOUS EXAMPLES. IX.

1. In a triangle one angle contains as many grades as another contains degrees, and the third contains as many centesimal seconds as there are sexagesimal seconds in the sum of the other two; find the number of radians in each angle.

2. Find the number of degrees in the angle at the centre of a circle whose radius is 5 feet which is subtended by an arc of length 6 feet.

3. To turn radians into seconds prove that we must multiply by 206265 nearly, and to turn seconds into radians the multiplier must be ·0000048.

4. If $\sin\theta$ equal $\frac{x^2-y^2}{x^2+y^2}$, find the values of $\cos\theta$ and $\cot\theta$.

5. If
$$\sin\theta=\frac{m^2+2mn}{m^2+2mn+2n^2},$$
prove that
$$\tan\theta=\frac{m^2+2mn}{2mn+2n^2}.$$

6. If
$$\cos\theta-\sin\theta=\sqrt{2}\sin\theta,$$
prove that
$$\cos\theta+\sin\theta=\sqrt{2}\cos\theta.$$

7. Prove that
$$\operatorname{cosec}^6\alpha-\cot^6\alpha=3\operatorname{cosec}^2\alpha\cot^2\alpha+1.$$

8. Express $2\sec^2 A-\sec^4 A-2\operatorname{cosec}^2 A+\operatorname{cosec}^4 A$ in terms of $\tan A$.

9. Solve the equation $3\operatorname{cosec}^2\theta=2\sec\theta$.

10. A man on a cliff observes a boat at an angle of depression of 30°, which is making for the shore immediately beneath him. Three minutes later the angle of depression of the boat is 60°. How soon will it reach the shore?

11. Prove that the equation $\sin\theta=x+\frac{1}{x}$ is impossible if x be real.

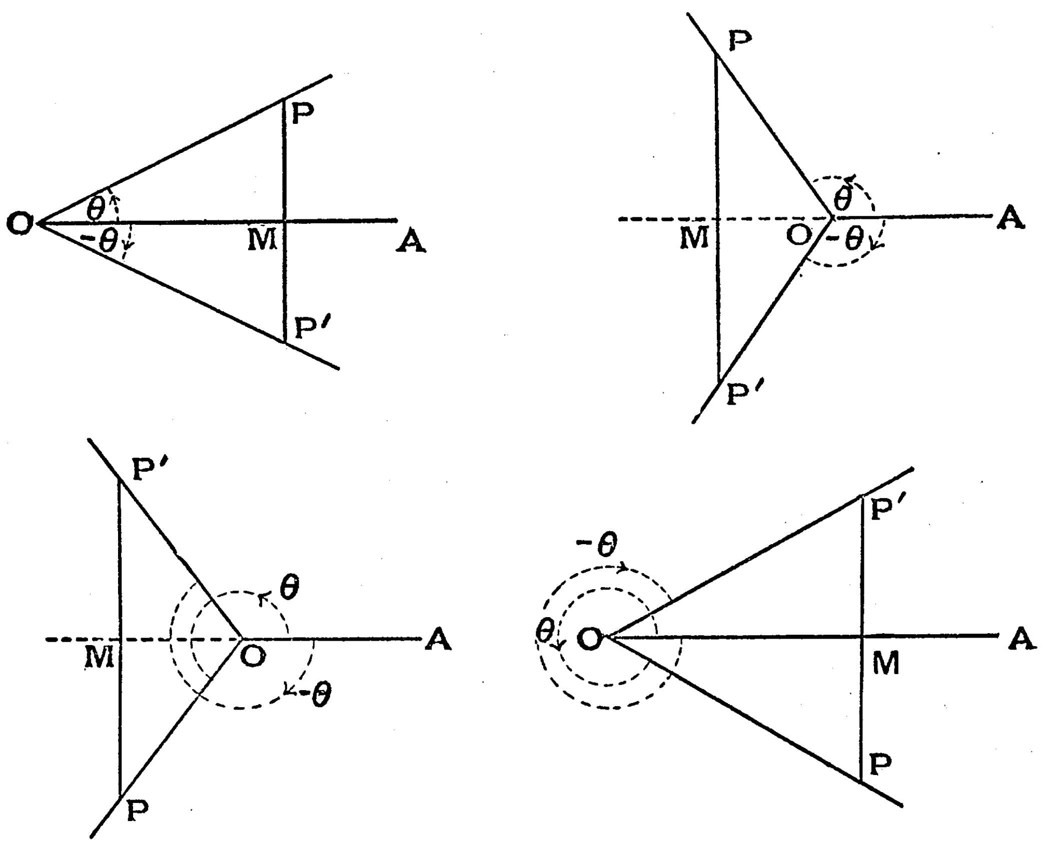
12. Shew that the equation $\sec^2\theta=\frac{4xy}{(x+y)^2}$ is only possible when $x=y$.

CHAPTER V.

TRIGONOMETRICAL FUNCTIONS OF ANGLES OF ANY SIZE AND SIGN.

[On a first reading of the subject, the student is recommended to confine his attention to the first of the four figures given in Arts. 68, 69 and 72.]

68. *To find the trigonometrical ratios of an angle* $(-\theta)$ *in terms of those of* θ, *for all values of* θ.



Let the revolving line, starting from OA, revolve through any angle θ and stop in the position OP.

Draw PM perpendicular to OA (or OA produced) and produce it to P', so that the lengths of PM and MP' are equal.

In the geometrical triangles MOP and MOP' we have the two sides OM and MP equal to the two OM and MP', and the included angles OMP and OMP' are right angles.

Hence (Euc. I. 4), the magnitudes of the angles MOP and MOP' are the same and OP is equal to OP'.

In each of the four figures, the magnitudes of the angle AOP (measured counter-clockwise) and of the angle AOP' (measured clockwise) are the same.

Hence the angle AOP' (measured clockwise) is denoted by $-\theta$.

Also MP and MP' are equal in magnitude but are opposite in sign. (Art. 49.) We have therefore

$$\mathbf{sin}\,(-\boldsymbol{\theta}) = \frac{MP'}{OP'} = \frac{-MP}{OP} = -\mathbf{sin}\,\boldsymbol{\theta},$$

$$\mathbf{cos}\,(-\boldsymbol{\theta}) = \frac{OM}{OP'} = \frac{OM}{OP} = \mathbf{cos}\,\boldsymbol{\theta},$$

$$\tan(-\theta) = \frac{MP'}{OM} = \frac{-MP}{OM} = -\tan\theta,$$

$$\cot(-\theta) = \frac{OM}{MP'} = \frac{OM}{-MP} = -\cot\theta,$$

$$\operatorname{cosec}(-\theta) = \frac{OP'}{MP'} = \frac{OM}{-MP} = -\operatorname{cosec}\theta,$$

and $$\sec(-\theta) = \frac{OP'}{OM} = \frac{OP}{OM} = \sec\theta.$$

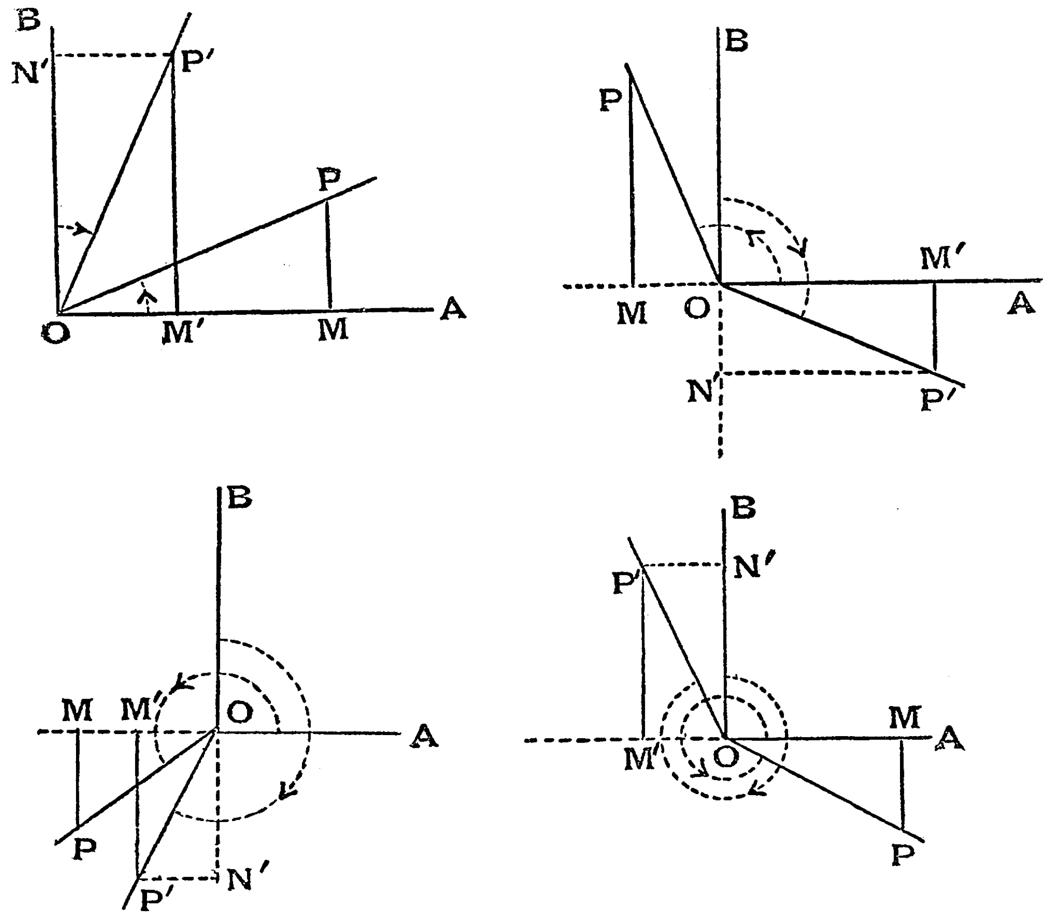
Exs. $\sin(-30°) = -\sin 30° = -\frac{1}{2}$,

$\tan(-60°) = -\tan 60° = -\sqrt{3}$,

and $\cos(-45°) = \cos 45° = \frac{1}{\sqrt{2}}$.

69. *To find the trigonometrical ratios of the angle $(90° - \theta)$ in terms of those of θ, for all values of θ.*

The relations have already been discussed in Art. 39, for values of θ less than a right angle.



Let the revolving line, starting from OA, trace out any angle AOP denoted by θ.

To obtain the angle $90° - \theta$, let the revolving line rotate to B and then rotate from B in the opposite direction through the angle θ, and let the position of the revolving line be then OP'.

The angle AOP' is then $90° - \theta$.

Take OP' equal to OP and draw $P'M'$ and PM perpendicular to OA, produced if necessary. Also draw $P'N'$ perpendicular to OB, produced if necessary.

In each figure the angles AOP and BOP' are numerically equal, by construction.

Hence in each figure

$$\angle MOP = \angle N'OP' = \angle OP'M',$$

since ON' and $M'P'$ are parallel.

Hence the triangles MOP and $M'P'O$ are equal in all respects, and therefore $OM = M'P'$ numerically,

and $$OM' = MP \text{ numerically.}$$

Also in each figure OM and $M'P'$ are of the same sign, and so also are MP and OM',

$$\text{i.e. } OM = +M'P', \text{ and } OM' = +MP.$$

Hence

$$\mathbf{sin\,(90° - \theta)} = \sin AOP' = \frac{M'P'}{OP'} = \frac{OM}{OP} = \mathbf{cos\,\theta},$$

$$\mathbf{cos\,(90° - \theta)} = \cos AOP' = \frac{OM'}{OP'} = \frac{MP}{OP} = \mathbf{sin\,\theta},$$

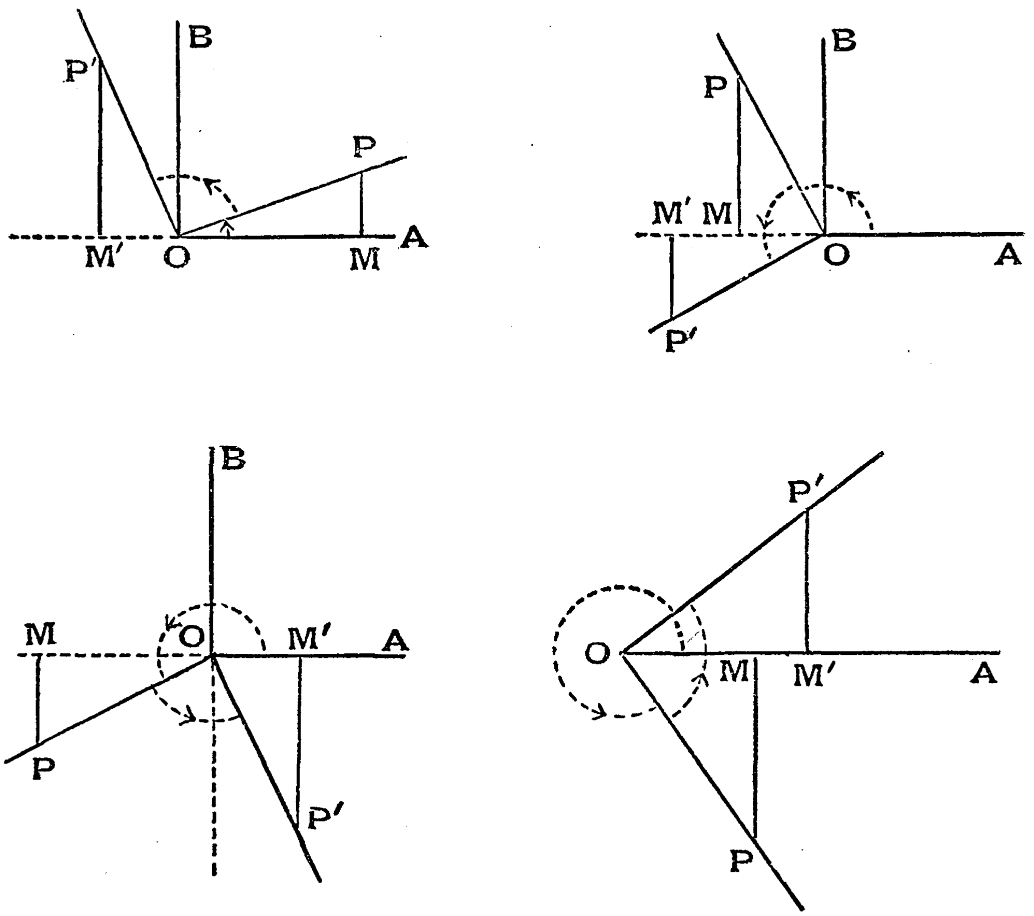
$$\tan(90° - \theta) = \tan AOP' = \frac{M'P'}{OM'} = \frac{OM}{MP} = \cot\theta,$$

$$\cot(90° - \theta) = \cot AOP' = \frac{OM'}{M'P'} = \frac{MP}{OM} = \tan\theta,$$

$$\sec(90° - \theta) = \sec AOP' = \frac{OP'}{OM'} = \frac{OP}{MP} = \operatorname{cosec}\theta,$$

and $$\operatorname{cosec}(90° - \theta) = \operatorname{cosec} AOP' = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta.$$

70. *To find the trigonometrical ratios of the angle* $(90^\circ + \theta)$ *in terms of those of* θ, *for all values of* θ.



Let the revolving line, starting from OA, trace out any angle θ and let OP be the position of the revolving line then, so that the angle AOP is θ.

Let the revolving line turn through a right angle from OP in the positive direction to the position OP', so that the angle AOP' is $(90^\circ + \theta)$.

Take OP' equal to OP and draw PM and $P'M'$ perpendicular to AOA'. In each figure, since POP' is a right angle, the sum of the angles MOP and $P'OM'$ is always a right angle.

Hence $\quad \angle MOP = 90^\circ - \angle P'OM' = \angle OP'M'$.

The two triangles MOP and $M'P'O$ are therefore equal in all respects.

Hence OM and $M'P'$ are numerically equal, as also MP and OM' are numerically equal.

In each figure OM and $M'P'$ have the same sign, whilst MP and OM' have the opposite, so that

$$M'P' = +OM, \text{ and } OM' = -MP.$$

We therefore have

$$\mathbf{sin}\,(\mathbf{90}^\circ + \boldsymbol{\theta}) = \sin AOP' = \frac{M'P'}{OP'} = \frac{OM}{OP} = \mathbf{cos}\,\boldsymbol{\theta},$$

$$\mathbf{cos}\,(\mathbf{90}^\circ + \boldsymbol{\theta}) = \cos AOP' = \frac{OM'}{OP'} = \frac{-MP}{OP} = -\,\mathbf{sin}\,\boldsymbol{\theta},$$

$$\tan(90^\circ + \theta) = \tan AOP' = \frac{M'P'}{OM'} = \frac{OM}{-MP} = -\cot\theta,$$

$$\cot(90^\circ + \theta) = \cot AOP' = \frac{OM'}{M'P'} = \frac{-MP}{OM} = -\tan\theta,$$

$$\sec(90^\circ + \theta) = \sec AOP' = \frac{OP'}{OM'} = \frac{OP}{-MP} = -\operatorname{cosec}\theta,$$

and $$\operatorname{cosec}(90^\circ + \theta) = \operatorname{cosec} AOP' = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta.$$

Exs. $\sin 150^\circ = \sin(90^\circ + 60^\circ) = \cos 60^\circ = \frac{1}{2}$,

$\cos 135^\circ = \cos(90^\circ + 45^\circ) = -\sin 45^\circ = -\frac{1}{\sqrt{2}}$,

and $\tan 120^\circ = \tan(90^\circ + 30^\circ) = -\cot 30^\circ = -\sqrt{3}$.

71. Supplementary Angles.

Two angles are said to be supplementary when their sum is equal to two right angles, *i.e.* the supplement of any angle θ is $180^\circ - \theta$.

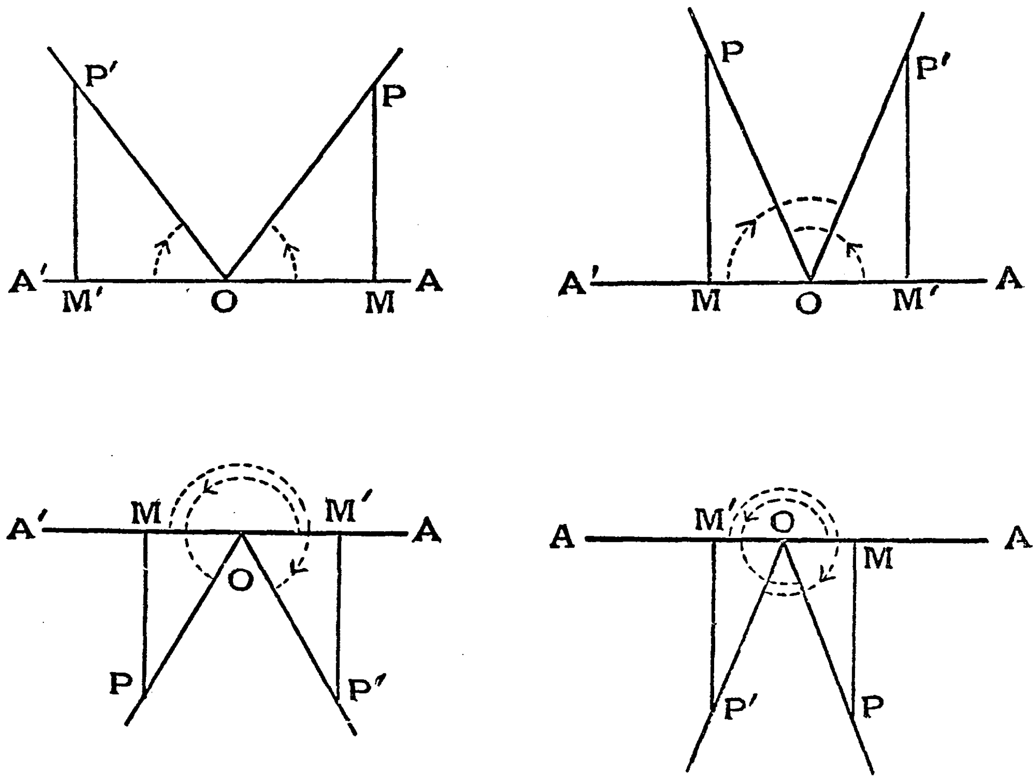
Exs. The supplement of $30^\circ = 180^\circ - 30^\circ = 150^\circ$.

The supplement of $120^\circ = 180^\circ - 120^\circ = 60^\circ$.

The supplement of $275^\circ = 180^\circ - 275^\circ = -95^\circ$.

The supplement of $-126^\circ = 180^\circ - (-126^\circ) = 306^\circ$.

72. *To find the values of the trigonometrical ratios of the angle* $(180^\circ - \theta)$ *in terms of those of the angle* θ, *for all values of* θ.



Let the revolving line start from OA and describe any angle AOP $(=\theta)$.

To obtain the angle $180^\circ - \theta$, let the revolving line start from OA and, after revolving through two right angles (*i.e.* into the position OA'), then revolve back through an angle θ into the position OP', so that the angle $A'OP'$ is equal in magnitude but opposite in sign to the angle AOP.

The angle AOP' is then $180^\circ - \theta$.

Take OP' equal to OP and draw $P'M'$ and PM perpendicular to AOA'.

The angles MOP and $M'OP'$ are equal and hence the triangles MOP and $M'OP'$ are equal in all respects.

Hence OM and OM' are equal in magnitude and so also are MP and $M'P'$.

In each figure OM and OM' are drawn in opposite directions, whilst MP and $M'P'$ are drawn in the same direction, so that

$$OM' = -OM, \text{ and } M'P' = +MP.$$

Hence we have

$$\mathbf{sin\,(180^\circ - \theta)} = \sin AOP' = \frac{M'P'}{OP'} = \frac{MP}{OP} = \mathbf{sin\,\theta},$$

$$\mathbf{cos\,(180^\circ - \theta)} = \cos AOP' = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\mathbf{cos\,\theta},$$

$$\tan(180^\circ - \theta) = \tan AOP' = \frac{M'P'}{OM'} = \frac{MP}{-OM} = -\tan\theta,$$

$$\cot(180^\circ - \theta) = \cot AOP' = \frac{OM'}{M'P'} = \frac{-OM}{MP} = -\cot\theta,$$

$$\sec(180^\circ - \theta) = \sec AOP' = \frac{OP'}{OM'} = \frac{OP}{-OM} = -\sec\theta,$$

and $$\operatorname{cosec}(180^\circ - \theta) = \operatorname{cosec} AOP' = \frac{OP'}{M'P'} = \frac{OP}{MP} = \operatorname{cosec}\theta.$$

Exs. $\sin 120^\circ = \sin(180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$,

$\cos 135^\circ = \cos(180^\circ - 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$,

and $\tan 150^\circ = \tan(180^\circ - 30^\circ) = -\tan 30^\circ = -\frac{1}{\sqrt{3}}$.

73. *To find the trigonometrical ratios of* $(180^\circ + \theta)$ *in terms of those of* θ, *for all values of* θ.

The required relations may be obtained geometrically, as in the previous articles. The figures for this proposition are easily obtained and are left as an example for the student.

They may also be deduced from the results of Art. 70, which have been proved true for all angles. For putting $90^\circ + \theta = B$, we have

$$\mathbf{sin\,(180^\circ + \theta)} = \sin(90^\circ + B) = \cos B \qquad \text{(Art. 70)}$$

$$= \cos(90^\circ + \theta) = -\mathbf{sin\,\theta}, \qquad \text{(Art. 70)}$$

and $$\mathbf{cos\,(180^\circ + \theta)} = \cos(90^\circ + B) = -\sin B \qquad \text{(Art. 70)}$$

$$= -\sin(90^\circ + \theta) = -\mathbf{cos\,\theta}. \qquad \text{(Art. 70)}.$$

So $$\tan(180^\circ + \theta) = \tan(90^\circ + B) = -\cot B$$

$$= -\cot(90^\circ + \theta) = \tan\theta,$$

and similarly $$\cot(180^\circ + \theta) = \cot\theta,$$

$$\sec(180^\circ + \theta) = -\sec\theta,$$

and $$\operatorname{cosec}(180^\circ + \theta) = -\operatorname{cosec}\theta.$$

74. *To find the trigonometrical ratios of an angle* $(360^\circ + \theta)$ *in terms of those of* θ, *for all values of* θ.

In whatever position the revolving line may be when it has described any angle θ, it will be in exactly the same position when it has made one more complete revolution in the positive direction, *i.e.* when it has described an angle $360^\circ + \theta$.

Hence the trigonometrical ratios for an angle $360° + \theta$ are the same as those for θ.

It follows that the addition or subtraction of 360°, or any multiple of 360°, to or from any angle does not alter its trigonometrical ratios.

75. From the theorems of this chapter it follows that the trigonometrical ratios of any angle whatever can be reduced to the determination of the trigonometrical ratios of an angle which lies between 0° and 45°.

For example,

$$\sin 1765° = \sin [4 \times 360° + 325°] = \sin 325° \quad \text{(Art. 74)}$$

$$= \sin (180° + 145°) = -\sin 145° \quad \text{(Art. 73)}$$

$$= -\sin (180° - 35°) = -\sin 35° \quad \text{(Art. 72)};$$

$$\tan 1190° = \tan (3 \times 360° + 110°) = \tan 110° \quad \text{(Art. 74)}$$

$$= \tan (90° + 20°) = -\cot 20° \quad \text{(Art. 70)};$$

and
$$\text{cosec} (-1465°) = -\text{cosec}\, 1465° \quad \text{(Art. 68)}$$

$$= -\text{cosec} (4 \times 360° + 25°) = -\text{cosec}\, 25° \quad \text{(Art. 74)}.$$

Similarly any other such large angles may be treated. First, multiples of 360° should be subtracted until the angle lies between 0° and 360°; if it be then greater than 180° it should be reduced by 180°; if then greater than 90° the formulae of Art. 70 should be used, and finally, if necessary, the formulae of Art. 69 applied.

76. The table of Art. 40 may now be extended to some important angles greater than a right angle.

Angle	0°	30°	45°	60°	90°	120°	135°	150°	180°
Sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
Tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
Cotangent	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	∞
Cosecant	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
Secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1

EXAMPLES. X.

Prove that

1. $\sin 420^\circ \cos 390^\circ + \cos(-300^\circ) \sin(-330^\circ) = 1.$

2. $\cos 570^\circ \sin 510^\circ - \sin 330^\circ \cos 390^\circ = 0.$

and **3.** $\tan 225^\circ \cot 405^\circ + \tan 765^\circ \cot 675^\circ = 0.$

What are the values of $\cos A - \sin A$ and $\tan A + \cot A$ when A has the values

4. $\frac{\pi}{3}$, **5.** $\frac{2\pi}{3}$, **6.** $\frac{5\pi}{4}$, **7.** $\frac{7\pi}{4}$ and **8.** $\frac{11\pi}{3}$?

What values between 0° and 360° may A have when

9. $\sin A = \frac{1}{\sqrt{2}}$, **10.** $\cos A = -\frac{1}{2}$, **11.** $\tan A = -1$,

12. $\cot A = -\sqrt{3}$, **13.** $\sec A = -\frac{2}{\sqrt{3}}$ and **14.** $\operatorname{cosec} A = -2$?

Express in terms of the ratios of a positive angle, which is less than 45°, the quantities

15. $\sin(-65^\circ)$. **16.** $\cos(-84^\circ)$. **17.** $\tan 137^\circ$.

18. $\sin 168^\circ$. **19.** $\cos 287^\circ$. **20.** $\tan(-246^\circ)$.

21. $\sin 843^\circ$. **22.** $\cos(-928^\circ)$. **23.** $\tan 1145^\circ$.

24. $\cos 1410^\circ$. **25.** $\cot(-1054^\circ)$. **26.** $\sec 1327^\circ$ and

27. $\operatorname{cosec}(-756^\circ)$.

What sign has $\sin A + \cos A$ for the following values of A?

28. 140°. **29.** 278°. **30.** -356° and **31.** -1125°.

What sign has $\sin A - \cos A$ for the following values of A?

32. 215°. **33.** 825°. **34.** -634° and **35.** -457°.

36. Find the sines and cosines of all angles in the first four quadrants whose tangents are equal to $\cos 135^\circ$.

Prove that

37. $\sin(270^\circ + A) = -\cos A$, and $\tan(270^\circ + A) = -\cot A$.

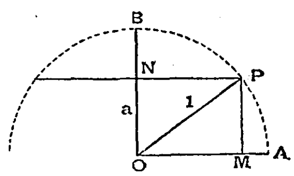
38. $\cos(270^\circ - A) = -\sin A$, and $\cot(270^\circ - A) = \tan A$.

CHAPTER VI.

GENERAL EXPRESSIONS FOR ALL ANGLES HAVING A GIVEN TRIGONOMETRICAL RATIO.

77. *To construct the least positive angle whose sine is equal to a, where a is a proper fraction.*

Let OA be the initial line and let OB be drawn in the positive direction perpendicular to OA.



Measure off along OB a distance ON which is equal to a units of length. [If a be negative the point N will lie in BO produced.]

Through N draw NP parallel to OA. With centre O and radius equal to the unit of length describe a circle and let it meet NP in P.

Then AOP will be the required angle.

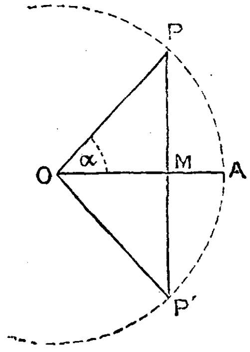
Draw PM perpendicular to OA, so that

$$\sin AOP = \frac{MP}{OP} = \frac{ON}{OP} = \frac{a}{1} = a.$$

The sine AOP is therefore equal to the given quantity and AOP is therefore the angle required.

78. *To construct the least positive angle whose cosine is equal to b where b is a proper fraction.*

Along the initial line measure off a distance OM equal to b and draw MP perpendicular to OA. [If b be negative M will lie on the other side of O in the line AO produced.]

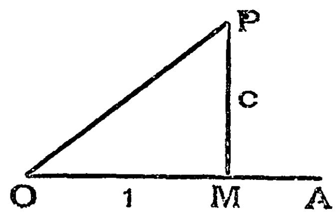


With centre O and radius equal to unity, describe a circle and let it meet MP in P.

Then AOP is the angle required. For

$$\cos AOP = \frac{OM}{OP} = \frac{b}{1} = b.$$

79. *To construct the least positive angle whose tangent is equal to c.*



Along the initial line measure off OM equal to unity and erect a perpendicular MP. Measure off MP equal to c.

Then

$$\tan AOP = \frac{MP}{OM} = c,$$

so that AOP is the required angle.

80. It is clear from the definition given in Art. 50, that, when an angle is given, so also is its sine. The converse statement is not correct; there is more than one angle having a given sine; for example, the angles 30°, 150°, 390°, − 210°,... all have their sine equal to $\frac{1}{2}$.

Hence, when the sine of an angle is given, we do not definitely know the angle; all we know is that the angle is one out of a large number of angles.

Similar statements are true if the cosine, tangent, or any other trigonometrical function of the angle be given.

Hence, simply to give *one* of the trigonometrical functions of an angle does not determine it without ambiguity.

81. Suppose we know that the revolving line OP coincides with the initial line OA. All we know is that the revolving line has made 0, or 1, or 2, or 3,... complete revolutions, either positive or negative.

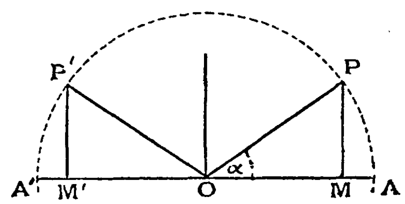
But when the revolving line has made one complete revolution the angle it has described is (Art. 17) equal to 2π radians.

Hence when the revolving line OP coincides with the initial line OA, the angle that it has described is 0, or 1, or 2, or 3... times 2π radians, in either the positive or negative directions, *i.e.* either 0, or $\pm 2\pi$, or $\pm 4\pi$, or $\pm 6\pi$... radians.

This is expressed by saying that when the revolving line coincides with the initial line the angle it has described is $\mathbf{2n\pi}$, where n is some positive or negative whole number.

82. Theorem. *To find a general expression to include all angles which have the same sine.*

Let AOP be the smallest positive angle having the given sine and let it be denoted by α.



Draw PM perpendicular to OA and produce MO to M' making MO equal to OM' and draw $M'P'$ parallel and equal to MP.

As in Art. 72 the angle AOP' is equal to $\pi - \alpha$.

When the revolving line is in either of the positions OP or OP', and in no other position, the sine of the angle traced out is equal to the given sine.

When the revolving line is in the position OP it has made a whole number of complete revolutions and then described an angle α, *i.e.* by the last article it has described an angle equal to

$$2r\pi + \alpha \quad \text{..........................(1)}$$

where r is zero or some positive or negative integer.

When the revolving line is in the position OP' it has, similarly, described an angle $2r\pi + AOP'$, *i.e.* an angle $2r\pi + \pi - \alpha$,

$$i.e. \ (2r+1)\pi - \alpha \quad \text{..................(2)}$$

where r is zero or some positive or negative integer.

All these angles will be found to be included in the expression

$$\mathbf{n\pi + (-1)^n \alpha} \quad \text{..................(3),}$$

where n is zero or a positive or negative integer.

For, when $n = 2r$, since $(-1)^{2r} = +1$, the expression (3) gives $2r\pi + \alpha$, which is the same as the expression (1).

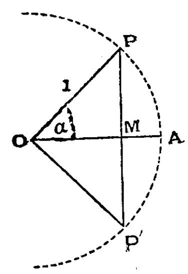
Also, when $n = 2r+1$, since $(-1)^{2r+1} = -1$, the expression (3) gives $(2r+1)\pi - \alpha$, which is the same as the expression (2).

Cor. Since all angles which have the same sine have also the same cosecant, the expression (3) includes all angles which have the same cosecant as α.

83. Theorem. *To find a general expression to include all angles which have the same cosine.*

Let AOP be the smallest angle having the given cosine and let it be denoted by α.

Draw PM perpendicular to OA and produce it to P', making PM equal to MP'.



When the revolving line is in the position OP or OP', and in no other position, then, as in Art. 78, the cosine of the angle traced out is equal to the given cosine.

When the revolving line is in the position OP it has made a whole number of complete revolutions and then described an angle α, *i.e.* it has described an angle $2n\pi + \alpha$, where n is zero or some positive or negative integer.

When the revolving line is in the position OP' it has made a whole number of complete revolutions and then described an angle $-\alpha$, *i.e.* it has described an angle $2n\pi - \alpha$.

All these angles are included in the expression

$$\mathbf{2n\pi \pm \alpha} \text{.......................} (1)$$

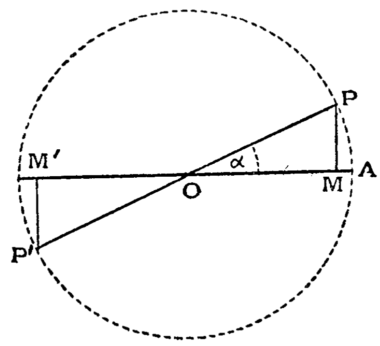
where n is some positive or negative number.

Cor. The expression (1) includes all angles having the same secant as α.

84. Theorem. *To find a general expression for all angles which have the same tangent.*

Let AOP be the smallest angle having the given tangent, and let it be denoted by α.

Produce PO to P' making OP' equal to OP and draw $P'M'$ perpendicular to OM'.



As in Art. 73 the angles AOP and AOP' have the same tangent; also the angle $AOP' = \pi + \alpha$.

When the revolving line is in

the position OP, it has described a whole number of complete revolutions and then turned through an angle α, *i.e.* it has described an angle

$$2r\pi + \alpha \dots\dots\dots\dots\dots\dots\dots\dots (1),$$

where r is zero or some positive or negative integer.

When the revolving line is in the position OP', it has similarly described an angle $2r\pi + (\pi + \alpha)$,

i.e. $$(2r + 1)\pi + \alpha \dots\dots\dots\dots\dots\dots (2).$$

All these angles are included in the expression

$$\mathbf{n\pi + \alpha} \dots\dots\dots\dots\dots\dots\dots\dots (3).$$

For, when n is even, $(= 2r$ say), the expression (3) gives the same angles as the expression (1).

Also, when n is odd, $(= 2r + 1$ say), it gives the same angles as the expression (2).

Cor. The expression (3) includes all angles which have the same cotangent as α.

85. Ex. 1. *Write down the general expression for all angles,*

(1) *whose sine is equal to* $\frac{\sqrt{3}}{2}$,

(2) *whose cosine is equal to* $-\frac{1}{2}$,

and (3) *whose tangent is equal to* $\frac{1}{\sqrt{3}}$.

(1) The smallest angle, whose sine is $\frac{\sqrt{3}}{2}$, is 60°, *i.e.* $\frac{\pi}{3}$.

Hence, by Art. 82, the general expression for all the angles which have this sine is

$$n\pi + (-1)^n \frac{\pi}{3}.$$

(2) The smallest positive angle, whose cosine is

$$-\frac{1}{2}, \text{ is } 120°, \textit{ i.e. } \frac{2\pi}{3}.$$

Hence, by Art. 83, the general expression for all the angles which have this cosine is

$$2n\pi \pm \frac{2\pi}{3}.$$

(3) The smallest positive angle, whose tangent is

$$\frac{1}{\sqrt{3}}, \text{ is } 30°, \textit{ i.e. } \frac{\pi}{6}.$$

Hence, by Art. 84, the general expression for all the angles which have this tangent is

$$n\pi + \frac{\pi}{6}.$$

Ex. 2. *What is the most general value of θ satisfying the equation* $\sin^2\theta = \frac{1}{4}$?

Here we have $\sin\theta = \pm\frac{1}{2}$.

Taking the upper sign,

$$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$$

$$\therefore\ \theta = n\pi + (-1)^n\frac{\pi}{6}.$$

Taking the lower sign,

$$\sin\theta = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right)$$

$$\therefore\ \theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right).$$

Putting both solutions together we have

$$\theta = n\pi \pm (-1)^n\frac{\pi}{6}$$

or, what is the same expression,

$$\theta = n\pi \pm \frac{\pi}{6}.$$

Ex. 3. *What is the most general value of θ which satisfies both of the equations* $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$?

Considering only angles between 0° and 360° the only angles, whose sine is $-\frac{1}{2}$, are 210° and 330°. Similarly the only angles, whose tangent is $\frac{1}{\sqrt{3}}$, are 30° and 210°.

The only angle, between 0° and 360°, satisfying both conditions is therefore 210°, *i.e.* $\frac{7\pi}{6}$.

The most general value is hence obtained by adding any multiple of four right angles to this angle, and hence is $2n\pi+\frac{7\pi}{6}$ where n is any positive or negative integer.

EXAMPLES. XI.

What are the most general values of θ which satisfy the equations,

1. $\sin\theta=\frac{1}{2}$.
2. $\sin\theta=-\frac{\sqrt{3}}{2}$.
3. $\sin\theta=\frac{1}{\sqrt{2}}$.
4. $\cos\theta=-\frac{1}{2}$.
5. $\cos\theta=\frac{\sqrt{3}}{2}$.
6. $\cos\theta=-\frac{1}{\sqrt{2}}$.
7. $\tan\theta=\sqrt{3}$.
8. $\tan\theta=-1$.
9. $\cot\theta=1$.
10. $\sec\theta=2$.
11. $\operatorname{cosec}\theta=\frac{2}{\sqrt{3}}$.
12. $\sin^2\theta=1$.
13. $\cos^2\theta=\frac{1}{4}$.
14. $\tan^2\theta=\frac{1}{3}$.
15. $4\sin^2\theta=3$.
16. $2\cot^2\theta=\operatorname{cosec}^2\theta$.
17. $\sec^2\theta=\frac{4}{3}$?

18. What is the most general value of θ that satisfies both of the equations

$$\cos\theta=-\frac{1}{\sqrt{2}} \text{ and } \tan\theta=1?$$

19. What is the most general value of θ that satisfies both of the equations

$$\cot\theta=-\sqrt{3} \text{ and } \operatorname{cosec}\theta=-2?$$

20. If $\cos(A-B)=\frac{1}{2}$, and $\sin(A+B)=\frac{1}{2}$, find the smallest positive values of A and B and also their most general values.

21. If $\tan(A-B)=1$, and $\sec(A+B)=\frac{2}{\sqrt{3}}$, find the smallest positive values of A and B and also their most general values.

22. Find the angles between 0° and 360° which have respectively (1) their sines equal to $\frac{\sqrt{3}}{2}$, (2) their cosines equal to $-\frac{1}{2}$, and (3) their tangents equal to $\frac{1}{\sqrt{3}}$.

23. Taking into consideration only angles less than 180°, how many values of x are there if (1) $\sin x = \frac{5}{7}$, (2) $\cos x = \frac{1}{5}$, (3) $\cos x = -\frac{4}{5}$, (4) $\tan x = \frac{2}{3}$, and (5) $\cot x = -7$?

24. Given the angle x construct the angle y if (1) $\sin y = 2\sin x$, (2) $\tan y = 3\tan x$, (3) $\cos y = \frac{1}{2}\cos x$, and (4) $\sec y = \operatorname{cosec} x$.

25. Shew that the same angles are indicated by the two following formulae: (1) $(2n-1)\frac{\pi}{2} + (-1)^n \frac{\pi}{3}$, and (2) $2n\pi \pm \frac{\pi}{6}$, n being any integer.

26. Prove that the two formulae

$$(1)\ \left(2n+\frac{1}{2}\right)\pi \pm \alpha \text{ and } (2)\ n\pi + (-1)^n\left(\frac{\pi}{2} - \alpha\right)$$

denote the same angles, n being any integer.

Illustrate by a figure.

27. If $\theta - \alpha = n\pi + (-1)^n\beta$ prove that $\theta = 2m\pi + \alpha + \beta$ or else that $\theta = (2m+1)\pi + \alpha - \beta$ where m and n are any integers.

28. If $\cos p\theta + \cos q\theta = 0$, prove that the different values of θ form two arithmetical progressions in which the common differences are $\frac{2\pi}{p+q}$ and $\frac{2\pi}{p \sim q}$ respectively.

29. Construct the angle whose sine is $\frac{3}{2+\sqrt{5}}$.

86. An equation involving the trigonometrical ratios of an unknown angle is called a trigonometrical equation.

The equation is not completely solved unless we obtain an expression for all the angles which satisfy it.

Some elementary types of equations are solved in the following article.

87. Ex. 1. *Solve the equation* $2\sin^2 x+\sqrt{3}\cos x+1=0$.

The equation may be written

$$2-2\cos^2 x+\sqrt{3}\cos x+1=0,$$

i.e. $$2\cos^2 x-\sqrt{3}\cos x-3=0,$$

i.e. $$(\cos x-\sqrt{3})(2\cos x+\sqrt{3})=0.$$

The equation is therefore satisfied by $\cos x=\sqrt{3}$, or $\cos x=-\frac{\sqrt{3}}{2}$.

There is no angle whose cosine is $\sqrt{3}$, so that the first factor gives no solution.

The smallest positive angle, whose cosine is $-\frac{\sqrt{3}}{2}$, is 150°, *i.e.* $\frac{5\pi}{6}$.

Hence the most general value of the angle, whose cosine is $-\frac{\sqrt{3}}{2}$, is $2n\pi\pm\frac{5\pi}{6}$. (Art. 83.)

This is the general solution of the given equation.

Ex. 2. *Solve the equation* $\tan 5\theta=\cot 2\theta$.

The equation may be written

$$\tan 5\theta=\tan\left(\frac{\pi}{2}-2\theta\right).$$

Now the most general value of the angle, that has the same tangent as

$$\frac{\pi}{2}-2\theta, \text{ is, by Art. 84,} \quad n\pi+\frac{\pi}{2}-2\theta,$$

where n is any positive or negative integer.

The most general solution of the equation is therefore

$$5\theta=n\pi+\frac{\pi}{2}-2\theta$$

$$\therefore\ \theta=\frac{1}{7}\left(n\pi+\frac{\pi}{2}\right),$$

where n is any integer.

EXAMPLES. XII.

Solve the equations

1. $\cos^2\theta-\sin\theta-\frac{1}{4}=0$.
2. $2\sin^2\theta+3\cos\theta=0$.
3. $2\sqrt{3}\cos^2\theta=\sin\theta$.
4. $\cos\theta+\cos^2\theta=1$.

5. $4\cos\theta - 3\sec\theta = 2\tan\theta$.

6. $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$.

7. $\tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0$.

8. $\cot^2\theta + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\cot\theta + 1 = 0$.

9. $\cot\theta - ab\tan\theta = a - b$.

10. $\tan^2\theta + \cot^2\theta = 2$.

11. $\sec\theta - 1 = (\sqrt{2} - 1)\tan\theta$.

12. $\sin 5\theta = \frac{1}{\sqrt{2}}$.

13. $\sin 9\theta = \sin\theta$.

14. $\sin 3\theta = \sin 2\theta$.

15. $\cos m\theta = \cos n\theta$.

16. $\sin 2\theta = \cos 3\theta$.

17. $\cos 5\theta = \cos 4\theta$.

18. $\cos m\theta = \sin n\theta$.

19. $\cot\theta = \tan 8\theta$.

20. $\cot\theta = \tan n\theta$.

21. $\tan 2\theta = \tan\frac{2}{\theta}$.

22. $\tan 2\theta \tan\theta = 1$.

23. $\tan^2 3\theta = \cot^2\alpha$.

24. $\tan 3\theta = \cot\theta$.

25. $\tan^2 3\theta = \tan^2\alpha$.

26. $3\tan^2\theta = 1$.

27. $\tan mx + \cot nx = 0$.

28. $\tan(\pi\cot\theta) = \cot(\pi\tan\theta)$.

29. $\sin(\theta - \phi) = \frac{1}{2}$, and $\cos(\theta + \phi) = \frac{1}{2}$.

30. $\cos(2x + 3y) = \frac{1}{2}$, $\cos(3x + 2y) = \frac{\sqrt{3}}{2}$.

31. Find all the angles between 0° and 90° which satisfy the equation

$$\sec^2\theta \operatorname{cosec}^2\theta + 2\operatorname{cosec}^2\theta = 8.$$

32. If $\tan^2\theta = \frac{5}{4}$, find versin θ and explain the double result.

33. If the coversin of an angle be $\frac{1}{3}$, find its cosine and cotangent.

CHAPTER VII.

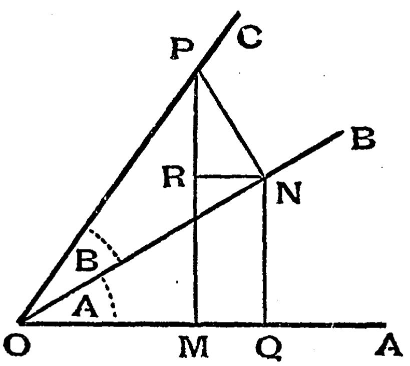
TRIGONOMETRICAL RATIOS OF THE SUM AND DIFFERENCE OF TWO ANGLES.

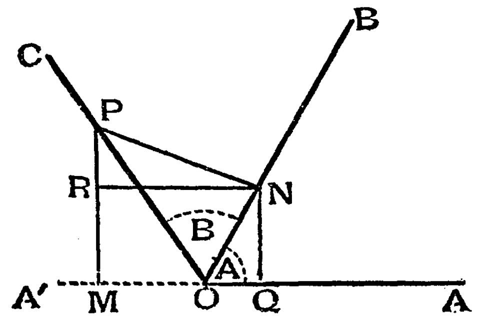
88. Theorem. *To prove that*

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$

and

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$





Let the revolving line start from OA and trace out the angle AOB $(=A)$, and then trace out the further angle BOC $(=B)$.

In the final position of the revolving line take any point P, and draw PM and PN perpendicular to OA and OB respectively; through N draw NR parallel to AO to meet MP in R and draw NQ perpendicular to OA.

The angle

$$RPN = 90^\circ - \angle PNR = \angle RNO = \angle NOQ = A.$$

Hence $\sin(A+B) = \sin AOP = \dfrac{MP}{OP} = \dfrac{MR+RP}{OP}$

$$= \frac{QN}{OP} + \frac{RP}{OP} = \frac{QN}{ON}\frac{ON}{OP} + \frac{RP}{NP}\frac{NP}{OP}$$

$$= \sin A \cos B + \cos RPN \sin B.$$

$$\therefore\ \mathbf{\sin(A+B) = \sin A \cos B + \cos A \sin B.}$$

Again $\cos(A+B) = \cos AOP = \dfrac{OM}{OP} = \dfrac{OQ-MQ}{OP}$

$$= \frac{OQ}{OP} - \frac{RN}{OP} = \frac{OQ}{ON}\frac{ON}{OP} - \frac{RN}{NP}\frac{NP}{OP}$$

$$= \cos A \cos B - \sin RPN \sin B.$$

$$\therefore\ \mathbf{\cos(A+B) = \cos A \cos B - \sin A \sin B.}$$

89. The figures in the last article have been drawn only for the case in which A and B are acute angles.

The same proof will be found to apply to angles of any size, due attention being paid to the signs of the quantities involved.

The results may however be shewn to be true of all angles, without drawing any more figures, as follows.

Let A and B be acute angles, so that, by Art. 88, we know that the theorem is true for A and B.

Let $A_1 = 90^\circ + A$, so that, by Art. 70, we have

$$\sin A_1 = \cos A, \text{ and } \cos A_1 = -\sin A.$$

Then $\sin(A_1+B) = \sin\{90^\circ + (A+B)\} = \cos(A+B)$, by Art. 70,

$$= \cos A \cos B - \sin A \sin B = \sin A_1 \cos B + \cos A_1 \sin B.$$

Also $\cos(A_1+B) = \cos[90^\circ + (A+B)] = -\sin(A+B)$

$$= -\sin A \cos B - \cos A \sin B = \cos A_1 \cos B - \sin A_1 \sin B.$$

Similarly, we may proceed if B be increased by 90°.

Hence the formulae of Art. 88 are true if either A or B be increased by 90°, *i.e.* they are true if the component angles lie between 0° and 180°.

Similarly, by putting $A_2 = 90° + A_1$, we can prove the truth of the theorems when either or both of the component angles have values between 0° and 270°.

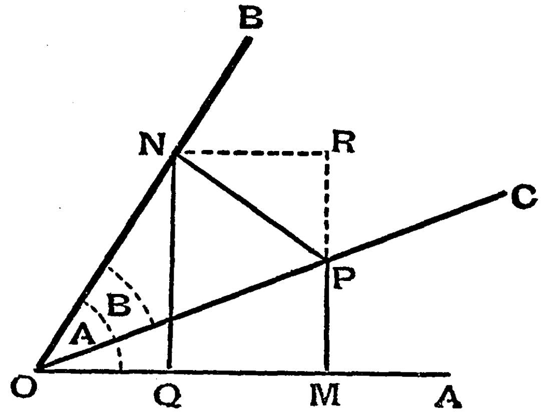
By proceeding in this way we see that the theorems are true universally.

90. Theorem. *To prove that*

$$\sin(A - B) = \sin A \cos B - \cos A \sin B,$$

and
$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

Let the revolving line starting from the initial line OA trace out the angle $AOB \,(= A)$ and then revolving in the opposite direction, trace out the angle BOC, whose magnitude is B. The angle AOC is therefore $A - B$.



Take a point P in the final position of the revolving line, and draw PM and PN perpendicular to OA and OB respectively; from N draw NQ and NR perpendicular to OA and MP respectively.

The angle $RPN = 90° - \angle PNR = \angle RNB = \angle QON = A$.

Hence

$$\sin(A - B) = \sin AOC = \frac{MP}{OP} = \frac{MR - PR}{OP} = \frac{QN}{OP} - \frac{PR}{OP}$$

$$= \frac{QN}{ON}\frac{ON}{OP} - \frac{PR}{PN}\frac{PN}{OP}$$

$$= \sin A \cos B - \cos RPN \sin B,$$

so that $\quad$ **sin (A − B) = sin A cos B − cos A sin B.**

Also $\cos(A-B)=\frac{OM}{OP}=\frac{OQ+QM}{OP}=\frac{OQ}{OP}+\frac{NR}{OP}$

$$=\frac{OQ}{ON}\,\frac{ON}{OP}+\frac{NR}{NP}\,\frac{NP}{OP}=\cos A\cos B+\sin NPR\sin B,$$

so that $\mathbf{\cos(A-B)=\cos A\cos B+\sin A\sin B.}$

91. The proofs of the previous article will be found to apply to angles of any size, provided that due attention be paid to the signs of the quantities involved.

Assuming the truth of the formulae for acute angles, we can shew them to be true universally without drawing any more figures.

For, putting $A_1=90°+A$, we have,

$$(\text{since } \sin A_1=\cos A, \text{ and } \cos A_1=-\sin A),$$

$$\sin(A_1-B)=\sin[90°+(A-B)]=\cos(A-B) \qquad \text{(Art. 70)}$$

$$=\cos A\cos B+\sin A\sin B$$

$$=\sin A_1\cos B-\cos A_1\sin B.$$

Also $\cos(A_1-B)=\cos[90°+(A-B)]=-\sin(A-B)$ (Art. 70)

$$=-\sin A\cos B+\cos A\sin B$$

$$=\cos A_1\cos B+\sin A_1\sin B.$$

Similarly we may proceed if B be increased by 90°.

Hence the theorem is true for all angles which are not greater than two right angles.

So, by putting $A_2=90°+A_1$, we may shew the theorems to be true for all angles less than three right angles, and so on.

Hence, by proceeding in this manner, we may shew that the theorems are true for all angles whatever.

92. The theorems of Arts. 88 and 90 which give respectively the trigonometrical functions of the sum and differences of two angles in terms of the functions of the angles themselves are often called the Addition and Subtraction Theorems.

93. **Ex. 1.** *Find the values of* sin 75° *and* cos 75°.

$$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}},$$

and $$\cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Ex. 2. *Assuming the formulae for* $\sin(x+y)$ *and* $\cos(x+y)$, *deduce the formulae for* $\sin(x-y)$ *and* $\cos(x-y)$.

We have

$$\sin x = \sin\{(x-y)+y\} = \sin(x-y)\cos y + \cos(x-y)\sin y \text{......(1)},$$

and $$\cos x = \cos\{(x-y)+y\} = \cos(x-y)\cos y - \sin(x-y)\sin y \text{......(2)}.$$

Multiplying (1) by $\cos y$ and (2) by $\sin y$ and subtracting, we have

$$\sin x \cos y - \cos x \sin y = \sin(x-y)\{\cos^2 y + \sin^2 y\} = \sin(x-y).$$

Multiplying (1) by $\sin y$ and (2) by $\cos y$ and adding, we have

$$\sin x \sin y + \cos x \cos y = \cos(x-y)\{\cos^2 y + \sin^2 y\} = \cos(x-y).$$

Hence the two formulae required are proved.

These two formulae are true for all values of the angles since the formulae from which they are derived are true for all values.

EXAMPLES. XIII.

1. If $\sin\alpha = \frac{3}{5}$ and $\cos\beta = \frac{9}{41}$, find the value of $\sin(\alpha-\beta)$ and $\cos(\alpha+\beta)$.

2. If $\sin\alpha = \frac{45}{53}$ and $\sin\beta = \frac{33}{65}$, find the values of $\sin(\alpha-\beta)$ and $\sin(\alpha+\beta)$.

3. If $\sin\alpha = \frac{15}{17}$ and $\cos\beta = \frac{12}{13}$, find the values of $\sin(\alpha+\beta)$, $\cos(\alpha-\beta)$, and $\tan(\alpha+\beta)$.

Prove that

4. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$.

5. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$.

6. $\cos(45^\circ - A)\cos(45^\circ - B) - \sin(45^\circ - A)\sin(45^\circ - B) = \sin(A+B)$.

7. $\sin(45^\circ + A)\cos(45^\circ - B) + \cos(45^\circ + A)\sin(45^\circ - B) = \cos(A - B)$.

8. $\dfrac{\sin(A-B)}{\cos A \cos B} + \dfrac{\sin(B-C)}{\cos B \cos C} + \dfrac{\sin(C-A)}{\cos C \cos A} = 0$.

9. $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ$.

10. $\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$.

11. $\cos\alpha \cos(\gamma - \alpha) - \sin\alpha \sin(\gamma - \alpha) = \cos\alpha$.

12. $\cos(\alpha + \beta)\cos\gamma - \cos(\beta + \gamma)\cos\alpha = \sin\beta \sin(\gamma - \alpha)$.

13. $\sin(n+1)A \sin(n-1)A + \cos(n+1)A \cos(n-1)A = \cos 2A$.

14. $\sin(n+1)A \sin(n+2)A + \cos(n+1)A \cos(n+2)A = \cos A$.

94. From Arts. 88 and 90, we have, for all values of A and B,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$

and $$\sin(A-B) = \sin A \cos B - \cos A \sin B.$$

Hence, by addition and subtraction, we have

$$\sin(A+B) + \sin(A-B) = 2\sin A \cos B \ldots\ldots(1),$$

and $$\sin(A+B) - \sin(A-B) = 2\cos A \sin B \ldots\ldots(2).$$

From the same articles we have, for all values of A and B,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

and $$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Hence, by addition and subtraction, we have

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B \ldots\ldots(3),$$

and $$\cos(A-B) - \cos(A+B) = 2\sin A \sin B \ldots\ldots(4).$$

Put $A + B = C$, and $A - B = D$, so that

$$A = \frac{C+D}{2}, \text{ and } B = \frac{C-D}{2}.$$

On making these substitutions the relations (1) to (4) become, for all values of C and D,

$$\sin \mathbf{C} + \sin \mathbf{D} = 2 \sin \frac{\mathbf{C}+\mathbf{D}}{2} \cos \frac{\mathbf{C}-\mathbf{D}}{2} \quad \ldots\ldots\mathbf{I},$$

$$\sin \mathbf{C} - \sin \mathbf{D} = 2 \cos \frac{\mathbf{C}+\mathbf{D}}{2} \sin \frac{\mathbf{C}-\mathbf{D}}{2} \quad \ldots\ldots\mathbf{II},$$

$$\cos \mathbf{C} + \cos \mathbf{D} = 2 \cos \frac{\mathbf{C}+\mathbf{D}}{2} \cos \frac{\mathbf{C}-\mathbf{D}}{2} \quad \ldots\ \mathbf{III},$$

and $$\cos \mathbf{D} - \cos \mathbf{C} = 2 \sin \frac{\mathbf{C}+\mathbf{D}}{2} \sin \frac{\mathbf{C}-\mathbf{D}}{2} \quad \ldots\ \mathbf{IV}.$$

[The student should carefully notice that the left-hand member of IV is $\cos D - \cos C$ and not $\cos C - \cos D$.]

95. These relations I to IV are extremely important and should be very carefully committed to memory.

On account of their great importance we give a geometrical proof for the case when C and D are acute angles.

Let AOC be the angle C and AOD the angle D. Bisect the angle COD by the straight line OE. On OE take a point P and draw QPR perpendicular to OP to meet OC and OD in Q and R respectively.

Draw PL, QM and RN perpendicular to OA, and through R draw RST perpendicular to PL or QM to meet them in S and T respectively.

Since the angle DOC is $C - D$, each of the angles DOE and EOC is $\frac{C-D}{2}$, and also

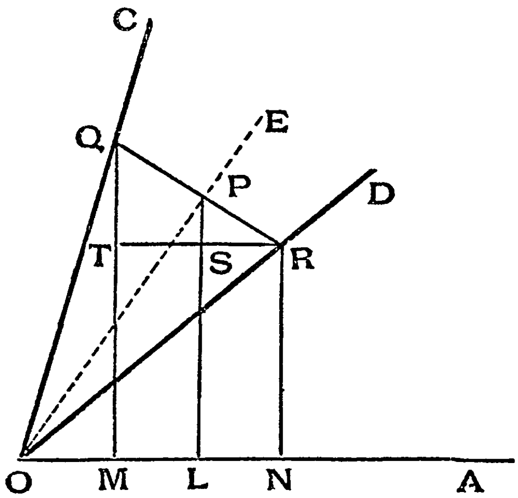
$$\angle AOE = \angle AOD + \angle DOE = D + \frac{C-D}{2} = \frac{C+D}{2}.$$

Since the two triangles POR and POQ are equal in all respects we have $OQ = OR$, and $PR = PQ$, so that

$$RQ = 2RP.$$

Hence $QT = 2PS$, and $RT = 2RS$, *i.e.* $MN = 2ML$.
Therefore $MQ + NR = TQ + 2LS = 2SP + 2LS = 2LP$.
Also $OM + ON = 2OM + MN = 2OM + 2ML = 2OL$.

Hence $\sin C + \sin D = \dfrac{MQ}{OQ} + \dfrac{NR}{OR} = \dfrac{MQ + NR}{OR}$



$$= \frac{2LP}{OR} = 2\frac{LP}{OP}\frac{OP}{OR} = 2 \sin LOP \cos POR$$

$$= 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}.$$

Again $\sin C - \sin D = \dfrac{MQ}{OQ} - \dfrac{NR}{OR} = \dfrac{MQ - NR}{OR}$

$$= 2\frac{SP}{OR} = 2\frac{SP}{RP} \cdot \frac{RP}{OR} = 2 \cos SPR \sin ROP$$

$$= 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2},$$

$$\left[\text{for } \angle SPR = 90^\circ - \angle SPO = \angle LOP = \frac{C + D}{2}\right].$$

Also $\cos C + \cos D = \dfrac{OM}{OQ} + \dfrac{ON}{OR} = \dfrac{OM + ON}{OR}$

$$= 2\frac{OL}{OR} = 2\frac{OL}{OP}\frac{OP}{OR}$$

$$= 2 \cos LOP \cos POR = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2}.$$

Finally $$\cos D - \cos C = \frac{ON}{OR} - \frac{OM}{OQ} = \frac{ON - OM}{OR}$$

$$= \frac{MN}{OR} = 2\frac{SR}{OR} = \frac{2SR}{PR}\frac{PR}{OR}$$

$$= 2 \sin SPR . \sin POR$$

$$= 2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}.$$

96. The student is strongly urged to make himself perfectly familiar with the formulae of the last article and to carefully practise himself in their application; perfect familiarity with these formulae will considerably facilitate his further progress.

The formulae are very useful because they change sums and differences of certain quantities into products of certain other quantities, and products of quantities are, as the student probably knows from Algebra, easily dealt with by the help of logarithms.

We subjoin a few examples of their use.

Ex. 1. $\sin 6\theta + \sin 4\theta = 2 \sin \frac{6\theta + 4\theta}{2} \cos \frac{6\theta - 4\theta}{2} = 2 \sin 5\theta \cos \theta.$

Ex. 2. $\cos 3\theta - \cos 7\theta = 2 \sin \frac{3\theta + 7\theta}{2} \sin \frac{7\theta - 3\theta}{2} = 2 \sin 5\theta \sin 2\theta.$

Ex. 3. $$\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \frac{2 \cos \frac{75^\circ + 15^\circ}{2} \sin \frac{75^\circ - 15^\circ}{2}}{2 \cos \frac{75^\circ + 15^\circ}{2} \cos \frac{75^\circ - 15^\circ}{2}}$$

$$= \frac{2 \cos 45^\circ \sin 30^\circ}{2 \cos 45^\circ \cos 30^\circ} = \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \cdot 57735 \ldots\ldots$$

[This is an example of the simplification given by these formulae; it would be a very long and tiresome process to look out from the tables the values of $\sin 75^\circ$, $\sin 15^\circ$, $\cos 75^\circ$, and $\cos 15^\circ$, and then to perform the division of one long decimal fraction by another.]

Ex. 4. *Simplify the expression*

$$\frac{(\cos\theta-\cos 3\theta)(\sin 8\theta+\sin 2\theta)}{(\sin 5\theta-\sin\theta)(\cos 4\theta-\cos 6\theta)}.$$

On applying the formulae of Art. 94 this expression

$$=\frac{2\sin\frac{\theta+3\theta}{2}\sin\frac{3\theta-\theta}{2}\times 2\sin\frac{8\theta+2\theta}{2}\cos\frac{8\theta-2\theta}{2}}{2\cos\frac{5\theta+\theta}{2}\sin\frac{5\theta-\theta}{2}\times 2\sin\frac{4\theta+6\theta}{2}\sin\frac{6\theta-4\theta}{2}}$$

$$=\frac{4\,.\,\sin 2\theta\sin\theta\,.\,\sin 5\theta\cos 3\theta}{4\,.\,\cos 3\theta\sin 2\theta\,.\,\sin 5\theta\sin\theta}=1.$$

EXAMPLES. XIV.

Prove that

1. $\dfrac{\sin 7\theta-\sin 5\theta}{\cos 7\theta+\cos 5\theta}=\tan\theta.$

2. $\dfrac{\cos 6\theta-\cos 4\theta}{\sin 6\theta+\sin 4\theta}=-\tan\theta.$

3. $\dfrac{\sin A+\sin 3A}{\cos A+\cos 3A}=\tan 2A.$

4. $\dfrac{\sin 7A-\sin A}{\sin 8A-\sin 2A}=\cos 4A\sec 5A.$

5. $\dfrac{\cos 2B+\cos 2A}{\cos 2B-\cos 2A}=\cot(A+B)\cot(A-B).$

6. $\dfrac{\sin 2A+\sin 2B}{\sin 2A-\sin 2B}=\dfrac{\tan(A+B)}{\tan(A-B)}.$

7. $\dfrac{\sin A+\sin 2A}{\cos A-\cos 2A}=\cot\dfrac{A}{2}.$

8. $\dfrac{\sin 5A-\sin 3A}{\cos 3A+\cos 5A}=\tan A.$

9. $\dfrac{\cos 2B-\cos 2A}{\sin 2B+\sin 2A}=\tan(A-B).$

10. $\cos(A+B)+\sin(A-B)=2\sin(45^\circ+A)\cos(45^\circ+B).$

11. $\dfrac{\cos 3A-\cos A}{\sin 3A-\sin A}+\dfrac{\cos 2A-\cos 4A}{\sin 4A-\sin 2A}=\dfrac{\sin A}{\cos 2A\cos 3A}.$

12. $\dfrac{\sin(4A-2B)+\sin(4B-2A)}{\cos(4A-2B)+\cos(4B-2A)}=\tan(A+B).$

13. $\dfrac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta}=4\cos 2\theta\cos 4\theta.$

14. $\dfrac{\cos 3\theta + 2\cos 5\theta + \cos 7\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta} = \cos 2\theta - \sin 2\theta \tan 3\theta.$

15. $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A.$

16. $\dfrac{\sin(\theta+\phi) - 2\sin\theta + \sin(\theta-\phi)}{\cos(\theta+\phi) - 2\cos\theta + \cos(\theta-\phi)} = \tan\theta.$

17. $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}.$

18. $\dfrac{\sin(A-C) + 2\sin A + \sin(A+C)}{\sin(B-C) + 2\sin B + \sin(B+C)} = \dfrac{\sin A}{\sin B}.$

19. $\dfrac{\sin A - \sin 5A + \sin 9A - \sin 13A}{\cos A - \cos 5A - \cos 9A + \cos 13A} = \cot 4A.$

20. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \tan\dfrac{A+B}{2}\cot\dfrac{A-B}{2}.$

21. $\dfrac{\cos A + \cos B}{\cos B - \cos A} = \cot\dfrac{A+B}{2}\cot\dfrac{A-B}{2}.$

22. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\dfrac{A+B}{2}.$

23. $\dfrac{\sin A - \sin B}{\cos B - \cos A} = \cot\dfrac{A+B}{2}.$

24. $\dfrac{\cos(A+B+C) + \cos(-A+B+C) + \cos(A-B+C) + \cos(A+B-C)}{\sin(A+B+C) + \sin(-A+B+C) - \sin(A-B+C) + \sin(A+B-C)}$
$= \cot B.$

25. $\cos 3A + \cos 5A + \cos 7A + \cos 15A = 4\cos 4A\cos 5A\cos 6A.$

26. $\cos(-A+B+C) + \cos(A-B+C) + \cos(A+B-C) + \cos(A+B+C)$
$= 4\cos A\cos B\cos C.$

27. $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0.$

28. $\sin 10^\circ + \sin 20^\circ + \sin 40^\circ + \sin 50^\circ = \sin 70^\circ + \sin 80^\circ.$

29. $\sin\alpha + \sin 2\alpha + \sin 4\alpha + \sin 5\alpha = 4\cos\dfrac{\alpha}{2}\cos\dfrac{3\alpha}{2}\sin 3\alpha.$

Simplify

30. $\cos\left\{\theta + \left(n - \dfrac{3}{2}\right)\phi\right\} - \cos\left\{\theta + \left(n + \dfrac{3}{2}\right)\phi\right\}.$

31. $\sin\left\{\theta + \left(n - \dfrac{1}{2}\right)\phi\right\} + \sin\left\{\theta + \left(n + \dfrac{1}{2}\right)\phi\right\}.$

97. The formulae (1), (2), (3), and (4) of Art. 94 are also very important. They should be remembered in the form

$$\mathbf{2 \sin A \cos B = \sin (A + B) + \sin (A - B)}\ldots(\mathbf{1}),$$
$$\mathbf{2 \cos A \sin B = \sin (A + B) - \sin (A - B)}\ldots(\mathbf{2}),$$
$$\mathbf{2 \cos A \cos B = \cos (A + B) + \cos (A - B)}\ldots(\mathbf{3}),$$
$$\mathbf{2 \sin A \sin B = \cos (A - B) - \cos (A + B)}\ldots(\mathbf{4}).$$

They may be looked upon as the converse of the formulae I—IV. of Art. 94.

Ex. 1. $2 \sin 3\theta \cos \theta = \sin 4\theta + \sin 2\theta$.

Ex. 2. $2 \sin 5\theta \sin 3\theta = \cos 2\theta - \cos 8\theta$.

Ex. 3. $2 \cos 11\theta \cos 2\theta = \cos 13\theta + \cos 9\theta$.

Ex. 4. Simplify

$$\frac{\sin 8\theta \cos \theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos \theta - \sin 3\theta \sin 4\theta}.$$

By the above formulae the expression

$$= \frac{\frac{1}{2}[\sin 9\theta + \sin 7\theta] - \frac{1}{2}[\sin 9\theta + \sin 3\theta]}{\frac{1}{2}[\cos 3\theta + \cos \theta] - \frac{1}{2}[\cos \theta - \cos 7\theta]}$$

$$= \frac{\sin 7\theta - \sin 3\theta}{\cos 3\theta + \cos 7\theta}$$

$$= \frac{2 \cos 5\theta \sin 2\theta}{2 \cos 5\theta \cos 2\theta}, \text{ by the formulae of Art. 94,}$$

$$= \tan 2\theta.$$

[The student should carefully notice the artifice of first employing the formulae of this article and then, to obtain a further simplification, employing the *converse* formulae of Art. 94. This artifice is often successful in simplifications.]

EXAMPLES. XV.

Express as a sum or difference the following

1. $2 \sin 5\theta \sin 7\theta$.
2. $2 \cos 7\theta \sin 5\theta$.
3. $2 \cos 11\theta \cos 3\theta$.
4. $2 \sin 54^\circ \sin 66^\circ$.

Prove that

5. $\sin\frac{\theta}{2}\sin\frac{7\theta}{2}+\sin\frac{3\theta}{2}\sin\frac{11\theta}{2}=\sin 2\theta\sin 5\theta.$

6. $\cos 2\theta\cos\frac{\theta}{2}-\cos 3\theta\cos\frac{9\theta}{2}=\sin 5\theta\sin\frac{5\theta}{2}.$

7. $\sin A\sin(A+2B)-\sin B\sin(B+2A)=\sin(A-B)\sin(A+B).$

8. $(\sin 3A+\sin A)\sin A+(\cos 3A-\cos A)\cos A=0.$

9. $\dfrac{2\sin(A-C)\cos C-\sin(A-2C)}{2\sin(B-C)\cos C-\sin(B-2C)}=\dfrac{\sin A}{\sin B}.$

10. $\dfrac{\sin A\sin 2A+\sin 3A\sin 6A+\sin 4A\sin 13A}{\sin A\cos 2A+\sin 3A\cos 6A+\sin 4A\cos 13A}=\tan 9A.$

11. $\dfrac{\cos 2A\cos 3A-\cos 2A\cos 7A+\cos A\cos 10A}{\sin 4A\sin 3A-\sin 2A\sin 5A+\sin 4A\sin 7A}=\cot 6A\cot 5A.$

12. $\cos(36^\circ-A)\cos(36^\circ+A)+\cos(54^\circ+A)\cos(54^\circ-A)=\cos 2A.$

13. $\cos A\sin(B-C)+\cos B\sin(C-A)+\cos C\sin(A-B)=0.$

14. $\sin(45^\circ+A)\sin(45^\circ-A)=\frac{1}{2}\cos 2A.$

15. $\text{versin}(A+B)\,\text{versin}(A-B)=(\cos A-\cos B)^2.$

16. $\sin(\beta-\gamma)\cos(\alpha-\delta)+\sin(\gamma-\alpha)\cos(\beta-\delta)+\sin(\alpha-\beta)\cos(\gamma-\delta)=0.$

17. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0.$

98. *To prove that* $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$, *and that* $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$.

By Art. 88, we have, for all values of A and B,

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$=\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}},\quad\text{by dividing both}$$

numerator and denominator by $\cos A\cos B$.

$$\therefore\ \mathbf{\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}}$$

Again, by Art. 90,

$$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}$$

$$=\frac{\dfrac{\sin A}{\cos A}-\dfrac{\sin B}{\cos B}}{1+\dfrac{\sin A\sin B}{\cos A\cos B}},\text{ by dividing as before.}$$

$$\therefore\ \mathbf{\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}}.$$

99. The formulae of the preceding article may be obtained geometrically from the figures of Arts. 88 and 90.

(1) Taking the figure of Art. 88 we have

$$\tan(A+B)=\frac{MP}{OM}=\frac{QN+RP}{OQ-RN}$$

$$=\frac{\dfrac{QN}{OQ}+\dfrac{RP}{OQ}}{1-\dfrac{RN}{OQ}}=\frac{\tan A+\dfrac{RP}{OQ}}{1-\dfrac{RN}{RP}\dfrac{RP}{OQ}}.$$

But, since the angles RPN and QON are equal, the triangles RPN and QON are similar, so that

$$\frac{RP}{PN}=\frac{OQ}{ON},$$

and therefore $$\frac{RP}{OQ}=\frac{PN}{ON}=\tan B.$$

Hence $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan RPN\tan B}=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

(2) Taking the figure of Art. 90, we have

$$\tan(A-B)=\frac{MP}{OM}=\frac{QN-PR}{OQ+NR}$$

$$=\frac{\dfrac{QN}{OQ}-\dfrac{PR}{OQ}}{1+\dfrac{NR}{OQ}}=\frac{\tan A-\dfrac{PR}{OQ}}{1+\dfrac{NR}{PR}\dfrac{PR}{OQ}}.$$

But, since the angles RPN and NOQ are equal, we have $\frac{RP}{PN}=\frac{OQ}{ON}$

and therefore $$\frac{PR}{OQ}=\frac{PN}{ON}=\tan B.$$

Hence $$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan RPN\tan B}=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

100. As particular cases of the preceding formulae, we have, by putting B equal to 45°,

$$\tan(A+45°)=\frac{\tan A+1}{1-\tan A}=\frac{1+\tan A}{1-\tan A},$$

and $$\tan(A-45°)=\frac{\tan A-1}{1+\tan A}.$$

Similarly as in Art. 98 we may prove that

$$\cot(A+B)=\frac{\cot A\cot B-1}{\cot A+\cot B}$$

and $$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}.$$

101. **Ex. 1.** $\tan 75°=\tan(45°+30°)=\frac{\tan 45°+\tan 30°}{1-\tan 45°\tan 30°}$

$$=\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}+1}{\sqrt{3}-1}=\frac{(\sqrt{3}+1)^2}{3-1}=\frac{4+2\sqrt{3}}{2}=2+\sqrt{3}$$

$=2+1\cdot73205...=3\cdot73205....$

Ex. 2. $\tan 15°=\tan(45°-30°)=\frac{\tan 45°-\tan 30°}{1+\tan 45°\tan 30°}$

$$=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}-1)^2}{3-1}=\frac{4-2\sqrt{3}}{2}=2-\sqrt{3}$$

$=2-1\cdot73205...=\cdot26795....$

EXAMPLES. XVI.

1. If $\tan A=\frac{1}{2}$ and $\tan B=\frac{1}{3}$, find the values of $\tan(2A+B)$ and $\tan(2A-B)$.

2. If $\tan A=\frac{\sqrt{3}}{4-\sqrt{3}}$ and $\tan B=\frac{\sqrt{3}}{4+\sqrt{3}}$, prove that

$$\tan(A-B)=\cdot 375.$$

3. If $\tan A=\frac{n}{n+1}$ and $\tan B=\frac{1}{2n+1}$, find $\tan(A+B)$.

4. If $\tan\alpha=\frac{5}{6}$ and $\tan\beta=\frac{1}{11}$, prove that $\alpha+\beta=\frac{\pi}{4}$.

Prove that

5. $\tan\left(\frac{\pi}{4}+\theta\right)\times\tan\left(\frac{3\pi}{4}+\theta\right)=-1.$

6. $\cot\left(\frac{\pi}{4}+\theta\right)\cot\left(\frac{\pi}{4}-\theta\right)=1.$

7. $1+\tan A\tan\frac{A}{2}=\tan A\cot\frac{A}{2}-1=\sec A.$

102. As further examples of the use of the formulae of the present chapter we shall find the general value of the angle which has a given sine, cosine or tangent. This has been already found in Arts. 82—84.

Find the general value of all angles having a given sine.

Let α be any angle having the given sine and θ any other angle having the same sine.

We have then to find the most general value of θ which satisfies the equation

$$\sin\theta=\sin\alpha,$$

$$i.e.\ \sin\theta-\sin\alpha=0.$$

This may be written

$$2\cos\frac{\theta+\alpha}{2}.\sin\frac{\theta-\alpha}{2}=0,$$

and it is therefore satisfied by

$$\cos\frac{\theta+\alpha}{2}=0, \text{ and by } \sin\frac{\theta-\alpha}{2}=0,$$

i.e. by $\dfrac{\theta+\alpha}{2}=$ any odd multiple of $\dfrac{\pi}{2}$

and by $\dfrac{\theta-\alpha}{2}=$ any multiple of π,

i.e. by $\theta=-\alpha+$ any odd multiple of π......(1),

and $\theta=\alpha+$ any even multiple of π ......(2),

i.e. θ must $=(-1)^n\alpha+n\pi$, where n is any positive or negative integer.

For when n is odd this expression agrees with (1), and when n is even it agrees with (2).

103. *Find the general value of all angles having the same cosine.*

The equation we have now to solve is

$$\cos\theta=\cos\alpha,$$

i.e. $$\cos\alpha-\cos\theta=0,$$

i.e. $$2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}=0,$$

and it is therefore satisfied by

$$\sin\frac{\theta+\alpha}{2}=0, \text{ and by } \sin\frac{\theta-\alpha}{2}=0,$$

i.e. by $\dfrac{\theta+\alpha}{2}=$ any multiple of π,

and by $\dfrac{\theta-\alpha}{2}=$ any multiple of π, *i.e.* by $\theta = -\alpha +$ any multiple of 2π,

and by $\theta = \alpha +$ any multiple of 2π.

Both these sets of values are included in the solution $\theta = 2n\pi \pm \alpha$, where n is any positive or negative integer.

104. *Find the general value of all angles having the same tangent.*

The equation we have now to solve is

$$\tan\theta - \tan\alpha = 0,$$

i.e. $$\sin\theta\cos\alpha - \cos\theta\sin\alpha = 0,$$

i.e. $$\sin(\theta - \alpha) = 0.$$

$$\therefore\ \theta - \alpha = \text{any multiple of } \pi$$

$= n\pi$, where n is any positive or negative integer,

so that the most general solution is $\theta = n\pi + \alpha$.

CHAPTER VIII.

THE TRIGONOMETRICAL RATIOS OF MULTIPLE AND SUBMULTIPLE ANGLES.

105. *To find the trigonometrical ratios of an angle 2A in terms of those of the angle A.*

If in the formulae of Art. 88 we put $B = A$, we have

$$\mathbf{sin\ 2A} = \sin A \cos A + \cos A \sin A = \mathbf{2\ sin\ A\ cos\ A},$$

$$\mathbf{cos\ 2A} = \cos A \cos A - \sin A \sin A = \mathbf{cos^2 A - sin^2 A}$$

$$= (1 - \sin^2 A) - \sin^2 A = \mathbf{1 - 2\ sin^2 A},$$

and also

$$= \cos^2 A - (1 - \cos^2 A) = \mathbf{2\ cos^2 A - 1}\ ;$$

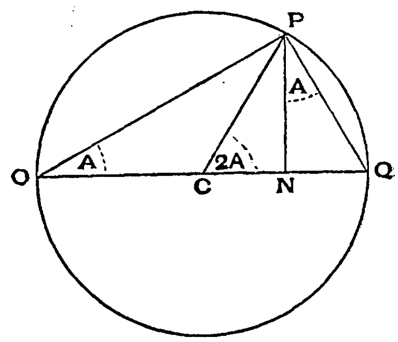
and

$$\mathbf{tan\ 2A} = \frac{\tan A + \tan A}{1 - \tan A \cdot \tan A} = \frac{\mathbf{2\ tan\ A}}{\mathbf{1 - tan^2 A}}.$$

Now the formulae of Art. 88 are true for all values of A and B; hence any formulae derived from them are true for all values of the angles.

In particular the above formulae are true for all values of A.

106. An independent geometrical proof of the formulae of the preceding article may be given, for values of A which are less than a right angle.



Let QCP be the angle $2A$.

With centre C and radius CP describe a circle and let QC meet it again in O.

Join OP and PQ, and draw PN perpendicular to OQ.

By Euc. III. 20, the angle

$$QOP = \tfrac{1}{2} \angle QCP = A,$$

and the angle $\quad NPQ = \angle QOP = A.$

Hence

$$\sin 2A = \frac{NP}{CP} = \frac{2NP}{2CQ} = 2\frac{NP}{OQ} = 2\frac{NP}{OP} \cdot \frac{OP}{OQ}$$

$= 2 \sin NOP \cos POQ$, since OPQ is a right angle,

$= 2 \sin A \cos A$;

also

$$\cos 2A = \frac{CN}{CP} = \frac{2CN}{OQ} = \frac{(OC + CN) - (OC - CN)}{OQ}$$

$$= \frac{ON - NQ}{OQ} = \frac{ON}{OP}\frac{OP}{OQ} - \frac{NQ}{PQ}\frac{PQ}{OQ}$$

$$= \cos^2 A - \sin^2 A;$$

and $\quad \tan 2A = \dfrac{NP}{CN} = \dfrac{2NP}{ON - NQ} = \dfrac{2\dfrac{NP}{ON}}{1 - \dfrac{NQ}{PN}\dfrac{PN}{ON}}$

$$= \frac{2 \tan A}{1 - \tan^2 A}.$$

Ex. *To find the values of sin* 15° *and cos* 15°.

Let the angle $2A$ be 30°, so that A is 15°.
Let the radius CP be $2a$, so that we have

$$CN = 2a \cos 30° = a\sqrt{3},$$

and $$NP = 2a \sin 30° = a.$$

Hence $$ON = OC + CN = a(2 + \sqrt{3}),$$

and $$NQ = CQ - CN = a(2 - \sqrt{3}).$$

$$\therefore \quad OP^2 = ON \cdot OQ = a(2+\sqrt{3}) \times 4a \qquad \text{(Euc. VI. 8)},$$

so that $$OP = a\sqrt{2}(\sqrt{3}+1),$$

and $$PQ^2 = QN \cdot QO = a(2-\sqrt{3}) \times 4a,$$

so that $$PQ = a\sqrt{2}(\sqrt{3}-1).$$

Hence $$\sin 15° = \frac{PQ}{OQ} = \frac{\sqrt{2}(\sqrt{3}-1)}{4} = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

and $$\cos 15° = \frac{OP}{OQ} = \frac{\sqrt{2}(\sqrt{3}+1)}{4} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

107. *To find the trigonometrical functions of* $3A$ *in terms of those of* A.

By Art. 88, putting B equal to $2A$, we have

$$\sin 3A = \sin(A + 2A) = \sin A \cos 2A + \cos A \sin 2A$$
$$= \sin A(1 - 2\sin^2 A) + \cos A \cdot 2\sin A \cos A$$
$$= \sin A(1 - 2\sin^2 A) + 2\sin A(1 - \sin^2 A).$$

Hence $$\mathbf{\sin 3A = 3\sin A - 4\sin^3 A} \quad \text{.........(1).}$$

So

$$\cos 3A = \cos(A + 2A) = \cos A \cos 2A - \sin A \sin 2A$$
$$= \cos A(2\cos^2 A - 1) - \sin A \cdot 2\sin A \cos A$$
$$= \cos A(2\cos^2 A - 1) - 2\cos A(1 - \cos^2 A).$$

Hence $$\mathbf{\cos 3A = 4\cos^3 A - 3\cos A} \quad \text{.........(2).}$$

Also $$\tan 3A = \tan(A+2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$= \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A \, . \dfrac{2\tan A}{1-\tan^2 A}} = \frac{\tan A(1-\tan^2 A) + 2\tan A}{(1-\tan^2 A) - 2\tan^2 A}.$$

Hence $$\mathbf{\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}}.$$

108. By a process similar to that of the last article, the trigonometrical ratios of any higher multiples of θ may be expressed in terms of those of θ. The method is however long and tedious. In a later chapter better methods will be pointed out.

As an example let us express $\cos 5\theta$ in terms of $\cos\theta$. We have

$$\cos 5\theta = \cos(3\theta + 2\theta)$$
$$= \cos 3\theta \cos 2\theta - \sin 3\theta \sin 2\theta$$
$$= (4\cos^3\theta - 3\cos\theta)(2\cos^2\theta - 1)$$
$$- (3\sin\theta - 4\sin^3\theta) \,.\, 2\sin\theta\cos\theta$$
$$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta)$$
$$- 2\cos\theta \,.\, \sin^2\theta(3 - 4\sin^2\theta)$$
$$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta)$$
$$- 2\cos\theta(1-\cos^2\theta)(4\cos^2\theta - 1)$$
$$= (8\cos^5\theta - 10\cos^3\theta + 3\cos\theta)$$
$$- 2\cos\theta(5\cos^2\theta - 4\cos^4\theta - 1)$$
$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

EXAMPLES. XVII.

1. Find the value of $\sin 2\alpha$ when

(1) $\cos\alpha=\frac{3}{5}$, (2) $\sin\alpha=\frac{12}{13}$ and (3) $\tan\alpha=\frac{16}{63}$.

2. Find the value of $\cos 2\alpha$, when

(1) $\cos\alpha=\frac{15}{17}$, (2) $\sin\alpha=\frac{4}{5}$, and (3) $\tan\alpha=\frac{5}{12}$.

3. If $\tan\theta=\frac{b}{a}$, find the value of $a\cos 2\theta+b\sin 2\theta$.

Prove that

4. $\frac{\sin 2A}{1+\cos 2A}=\tan A$.

5. $\frac{\sin 2A}{1-\cos 2A}=\cot A$.

6. $\frac{1-\cos 2A}{1+\cos 2A}=\tan^2 A$.

7. $\tan A+\cot A=2\operatorname{cosec} 2A$.

8. $\tan A-\cot A=-2\cot 2A$.

9. $\operatorname{cosec} 2A+\cot 2A=\cot A$.

10. $\frac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)}=\tan\frac{A}{2}\cot\frac{B}{2}$.

11. $\frac{\cos A}{1\mp\sin A}=\tan\left(45^\circ\pm\frac{A}{2}\right)$.

12. $\frac{\sec 8A-1}{\sec 4A-1}=\frac{\tan 8A}{\tan 2A}$.

13. $\frac{1+\tan^2(45^\circ-A)}{1-\tan^2(45^\circ-A)}=\operatorname{cosec} 2A$.

14. $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$.

15. $\frac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}=\tan(A+B)$.

16. $\tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)=2\tan 2\theta$.

17. $\frac{\cos A+\sin A}{\cos A-\sin A}-\frac{\cos A-\sin A}{\cos A+\sin A}=2\tan 2A$.

18. $\cot(A+15^\circ)-\tan(A-15^\circ)=\frac{4\cos 2A}{1+2\sin 2A}$.

19. $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}=\tan\theta.$ 20. $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=\tan\dfrac{\theta}{2}.$

21. $\dfrac{\sin(n+1)A-\sin(n-1)A}{\cos(n+1)A+2\cos nA+\cos(n-1)A}=\tan\dfrac{A}{2}.$

22. $\dfrac{\sin(n+1)A+2\sin nA+\sin(n-1)A}{\cos(n-1)A-\cos(n+1)A}=\cot\dfrac{A}{2}.$

23. $\sin(2n+1)A\sin A=\sin^2(n+1)A-\sin^2 nA.$

24. $\dfrac{\sin(A+3B)+\sin(3A+B)}{\sin 2A+\sin 2B}=2\cos(A+B).$

25. $\sin 3A+\sin 2A-\sin A=4\sin A\cos\dfrac{A}{2}\cos\dfrac{3A}{2}.$

26. $\tan 2A=(\sec 2A+1)\sqrt{\sec^2 A-1}.$

27. $\cos^3 2\theta+3\cos 2\theta=4(\cos^6\theta-\sin^6\theta).$

28. $1+\cos^2 2\theta=2(\cos^4\theta+\sin^4\theta).$

29. $\sec^2 A(1+\sec 2A)=2\sec 2A.$

30. $\operatorname{cosec} A-2\cot 2A\cos A=2\sin A.$

31. $\cot A=\dfrac{1}{2}\left(\cot\dfrac{A}{2}-\tan\dfrac{A}{2}\right).$

32. $\sin\alpha\sin(60^\circ-\alpha)\sin(60^\circ+\alpha)=\dfrac{1}{4}\sin 3\alpha.$

33. $\cos\alpha\cos(60-\alpha)\cos(60^\circ+\alpha)=\dfrac{1}{4}\cos 3\alpha.$

34. $\cot\alpha+\cot(60+\alpha)-\cot(60^\circ-\alpha)=3\cot 3\alpha.$

35. $\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ=\dfrac{1}{16}.$

36. $\sin 20^\circ\sin 40^\circ\sin 60^\circ\sin 80^\circ=\dfrac{3}{16}.$

37. $\cos 4\alpha=1-8\cos^2\alpha+8\cos^4\alpha.$

38. $\sin 4A=4\sin A\cos^3 A-4\cos A\sin^3 A.$

39. $\cos 6\alpha=32\cos^6\alpha-48\cos^4\alpha+18\cos^2\alpha-1.$

40. $\tan 3A\tan 2A\tan A=\tan 3A-\tan 2A-\tan A.$

41. $\dfrac{2\cos 2^n\theta+1}{2\cos\theta+1}=(2\cos\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)$

$\ldots\ldots(2\cos 2^{n-1}\theta-1).$

Submultiple angles.

109. Since the relations of Art. 105 are true for *all* values of the angle A, they will be true if instead of A we substitute $\frac{A}{2}$, and therefore if instead of $2A$ we put $2 . \frac{A}{2}$, *i.e.* A.

Hence we have the relations

$$\mathbf{\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\mathbf{\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}}$$

$$\mathbf{= 2 \cos^2 \frac{A}{2} - 1 = 1 - 2 \sin^2 \frac{A}{2}} \ldots\ldots\ldots (2),$$

and

$$\mathbf{\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1) we also have

$$\sin A = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2}}$$

$$= \frac{2 \tan \frac{A}{2}}{1 + \tan^2 \frac{A}{2}},$$ by dividing numerator and denominator by $\cos^2 \frac{A}{2}$.

So
$$\cos A = \frac{\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2}}$$

$$= \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}.$$

110. *To express the trigonometrical ratios of the angle* $\frac{A}{2}$ *in terms of* $\cos A$.

From equation (2) of the last article we have

$$\cos A = 1 - 2 \sin^2 \frac{A}{2},$$

so that
$$2 \sin^2 \frac{A}{2} = 1 - \cos A,$$

and therefore
$$\mathbf{\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}} \quad \ldots\ldots\ldots\ldots (1).$$

Again,
$$\cos A = 2 \cos^2 \frac{A}{2} - 1,$$

so that
$$2 \cos^2 \frac{A}{2} = 1 + \cos A,$$

and therefore
$$\mathbf{\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}} \quad \ldots\ldots\ldots\ldots (2).$$

Hence,
$$\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}} \quad \ldots\ldots\ldots (3).$$

111. In each of the preceding formulae it will be noted that there is an ambiguous sign. In any particular case the proper sign can be determined as the following examples will shew.

Ex. 1. *Given* $\cos 45° = \frac{1}{\sqrt{2}}$, *find the values of* $\sin 22\frac{1}{2}°$ *and* $\cos 22\frac{1}{2}°$.

The equation (1) of the last article gives, by putting A equal to 45°,

$$\sin 22\tfrac{1}{2}° = \pm\sqrt{\frac{1-\cos 45°}{2}} = \pm\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \pm\sqrt{\frac{2-\sqrt{2}}{4}}$$

$$= \pm\frac{1}{2}\sqrt{2-\sqrt{2}}.$$

Now $\sin 22\frac{1}{2}°$ is necessarily positive, so that the upper sign must be taken.

Hence $$\sin 22\tfrac{1}{2}° = \frac{1}{2}\sqrt{2-\sqrt{2}}.$$

So $$\cos 22\tfrac{1}{2}° = \pm\sqrt{\frac{1+\cos 45°}{2}} = \pm\sqrt{\frac{2+\sqrt{2}}{4}} = \pm\frac{1}{2}\sqrt{2+\sqrt{2}};$$

also $\cos 22\frac{1}{2}°$ is positive;

$$\therefore \cos 22\tfrac{1}{2}° = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

Ex. 2. *Given* $\cos 330° = \frac{\sqrt{3}}{2}$, *find the values of* $\sin 165°$ *and* $\cos 165°$.

The equation (1) gives

$$\sin 165° = \pm\sqrt{\frac{1-\cos 330°}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \pm\sqrt{\frac{4-2\sqrt{3}}{8}}$$

$$= \pm\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Also

$$\cos 165° = \pm\sqrt{\frac{1+\cos 330°}{2}} = \pm\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \pm\sqrt{\frac{4+2\sqrt{3}}{8}}$$

$$= \pm\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Now 165° lies between 90° and 180°, so that, by Art. 52, its sine is positive and its cosine is negative.

Hence $$\sin 165^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

and $$\cos 165^\circ = -\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

From the above examples it will be seen that, when the angle A and its cosine are given, the ratios for the angle $\frac{A}{2}$ may be determined without any ambiguity of sign.

When however only $\cos A$ is given, there is an ambiguity in finding $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$. The explanation of this ambiguity is given in the next article.

112. *To explain why there is ambiguity when $\cos\frac{A}{2}$ and $\sin\frac{A}{2}$ are found from the value of $\cos A$.*

We know that, if n be any integer,

$$\cos A = \cos(2n\pi \pm A) = k \text{ (say)}.$$

Hence any formula which gives us $\cos\frac{A}{2}$ in terms of k, should give us also the cosine of $\frac{2n\pi \pm A}{2}$.

Now $$\cos\frac{2n\pi \pm A}{2} = \cos\left(n\pi \pm \frac{A}{2}\right)$$

$$= \cos n\pi \cos\frac{A}{2} \mp \sin n\pi \sin\frac{A}{2} = \cos n\pi \cos\frac{A}{2}$$

$$= \pm\cos\frac{A}{2},$$

according as n is even or odd.

Similarly any formula giving us $\sin\frac{A}{2}$ in terms of k, should give us also the sine of $\frac{2n\pi \pm A}{2}$.

Also $$\sin\frac{2n\pi \pm A}{2} = \sin\left(n\pi \pm \frac{A}{2}\right)$$

$$= \sin n\pi \cos\frac{A}{2} \pm \cos n\pi \sin\frac{A}{2} = \pm \cos n\pi \sin\frac{A}{2}$$

$$= \pm \sin\frac{A}{2}.$$

Hence in each case we should expect to obtain two values for $\cos\frac{A}{2}$ and $\sin\frac{A}{2}$, and this is the number which the formulae of Art. 110 give.

113. *To express the trigonometrical ratios of the angle* $\frac{A}{2}$ *in terms of* $\sin A$.

From equation (1) of Art. 109 we have

$$2\sin\frac{A}{2}\cos\frac{A}{2} = \sin A \text{ .... } (1).$$

Also $$\sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1, \text{ always .... } (2).$$

First adding these equations, and then subtracting them, we have

$$\sin^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2} + \cos^2\frac{A}{2} = 1 + \sin A,$$

and $$\sin^2\frac{A}{2} - 2\sin\frac{A}{2}\cos\frac{A}{2} + \cos^2\frac{A}{2} = 1 - \sin A;$$

i.e. $$\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)^2=1+\sin A,$$

and $$\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)^2=1-\sin A;$$

so that $$\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{1+\sin A}\ \ldots\ldots\ldots\ldots(3),$$

and $$\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A}\ \ldots\ldots\ldots\ldots(4).$$

By adding, and then subtracting, we have

$$\mathbf{2\sin\frac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}}\ \ldots\ldots(5),$$

and $$\mathbf{2\cos\frac{A}{2}=\pm\sqrt{1+\sin A}\mp\sqrt{1-\sin A}}\ \ldots\ldots(6).$$

The other ratios of $\frac{A}{2}$ are then easily obtained.

114. In each of the formulae (5) and (6) there are two ambiguous signs. In the following examples it is shewn how to determine the ambiguity in any particular case.

Ex. 1. *Given that sin 30° is $\frac{1}{2}$, find the values of sin 15° and cos 15°.*

Putting $A=30°$, we have from relations (3) and (4),

$$\sin 15°+\cos 15°=\pm\sqrt{1+\sin 30°}=\pm\frac{\sqrt{3}}{\sqrt{2}},$$

$$\sin 15°-\cos 15°=\pm\sqrt{1-\sin 30°}=\pm\frac{1}{\sqrt{2}}.$$

Now $\sin 15°$ and $\cos 15°$ are both positive and $\cos 15°$ is greater than $\sin 15°$. Hence the expressions $\sin 15°+\cos 15°$ and $\sin 15°-\cos 15°$ are respectively positive and negative.

Hence the above two relations should be

$$\sin 15^\circ + \cos 15^\circ = +\frac{\sqrt{3}}{\sqrt{2}},$$

and $$\sin 15^\circ - \cos 15^\circ = -\frac{1}{\sqrt{2}}.$$

Hence $$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}, \text{ and } \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

Ex. 2. *Given that sin* 570° *is equal to* $-\frac{1}{2}$, *find the values of sin* 285° *and cos* 285°.

Putting A equal to 570°, we have

$$\sin 285^\circ + \cos 285^\circ = \pm\sqrt{1+\sin 570^\circ} = \pm\frac{1}{\sqrt{2}},$$

and $$\sin 285^\circ - \cos 285^\circ = \pm\sqrt{1-\sin 570^\circ} = \pm\sqrt{\frac{3}{2}}.$$

Now $\sin 285^\circ$ is negative, $\cos 285^\circ$ is positive, and the former is numerically greater than the latter, as may be seen by a figure.

Hence $\sin 285^\circ + \cos 285^\circ$ is negative and $\sin 285^\circ - \cos 285^\circ$ is also negative.

$$\therefore \sin 285^\circ + \cos 285^\circ = -\frac{1}{\sqrt{2}},$$

and $$\sin 285^\circ - \cos 285^\circ = -\frac{\sqrt{3}}{\sqrt{2}}.$$

Hence $$\sin 285^\circ = -\frac{\sqrt{3}+1}{2\sqrt{2}},$$

and $$\cos 285^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

115. *To explain why there is ambiguity when sin* $\frac{A}{2}$ *and cos* $\frac{A}{2}$ *are found from the value of sin* A.

We know that, if n be any integer,

$$\sin\{n\pi + (-1)^n A\} = \sin A = k \text{ (say)}. \quad \text{(Art. 82.)}$$

Hence any formula which gives us $\sin\frac{A}{2}$ in terms of k, should give us also the sine of $\frac{n\pi+(-1)^n A}{2}$.

First, let n be even and equal to $2m$. Then

$$\sin\frac{n\pi+(-1)^n A}{2}=\sin\left(m\pi+\frac{A}{2}\right)$$

$$=\sin m\pi\cos\frac{A}{2}+\cos m\pi\sin\frac{A}{2}=\cos m\pi\sin\frac{A}{2}$$

$$=\pm\sin\frac{A}{2},$$

according as m is even or odd.

Secondly, let n be odd and equal to $2p+1$.
Then

$$\sin\frac{n\pi+(-1)^n A}{2}=\sin\frac{2p\pi+\pi-A}{2}=\sin\left[p\pi+\frac{\pi-A}{2}\right]$$

$$=\sin p\pi\cos\frac{\pi-A}{2}+\cos p\pi\sin\frac{\pi-A}{2}=\cos p\pi\cos\frac{A}{2}$$

$$=\pm\cos\frac{A}{2},$$

according as p is even or odd.

Hence any formula which gives us $\sin\frac{A}{2}$ in terms of $\sin A$ should be expected to give us, in addition, the values of

$$-\sin\frac{A}{2},\quad \cos\frac{A}{2}\quad\text{and}\quad -\cos\frac{A}{2},$$

i.e. 4 values in all. This is the number of values which we get from the formulae of Art. 113, by giving all possible values to the ambiguities.

In a similar manner it may be shewn that when $\cos\frac{A}{2}$ is found from $\sin A$, we should expect 4 values.

116. In any general case we can shew how the ambiguities in relations (3) and (4) of Art. 113 may be found.

We have

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\frac{A}{2}+\frac{1}{\sqrt{2}}\cos\frac{A}{2}\right)$$

$$=\sqrt{2}\left[\sin\frac{A}{2}\cos\frac{\pi}{4}+\cos\frac{A}{2}\sin\frac{\pi}{4}\right]=\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{A}{2}\right).$$

The right-hand member of this equation is positive if

$$\frac{\pi}{4}+\frac{A}{2} \text{ lie between } 2n\pi \text{ and } 2n\pi+\pi,$$

i.e. if $\quad \frac{A}{2}$ lie between $2n\pi-\frac{\pi}{4}$ and $2n\pi+\frac{3\pi}{4}$.

Hence $\sin\frac{A}{2}+\cos\frac{A}{2}$ is positive if $\frac{A}{2}$ lie between

$$2n\pi-\frac{\pi}{4} \text{ and } 2n\pi+\frac{3\pi}{4};$$

it is negative otherwise.

Similarly we can prove that

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\sqrt{2}\sin\left(\frac{A}{2}-\frac{\pi}{4}\right).$$

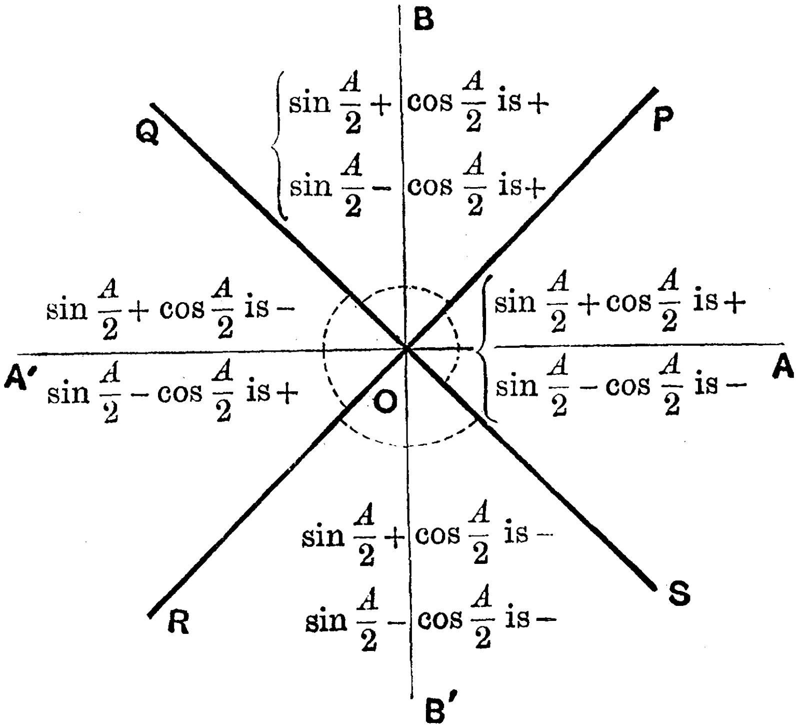
Therefore $\sin\frac{A}{2}-\cos\frac{A}{2}$ is positive if

$$\left(\frac{A}{2}-\frac{\pi}{4}\right) \text{ lie between } 2n\pi \text{ and } 2n\pi+\pi,$$

i.e. if $\frac{A}{2}$ lie between $2n\pi+\frac{\pi}{4}$ and $2n\pi+\frac{5\pi}{4}$.

It is negative otherwise.

The results of this article are shewn graphically in the following figure.



OA is the initial line and *OP*, *OQ*, *OR* and *OS* bisect the angles in the first, second, third and fourth quadrants respectively.

Numerical Example. *Within what limits must* $\frac{A}{2}$ *lie if*

$$2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}.$$

In this case the formulae of Art. 113 must clearly be

$$\sin\frac{A}{2}+\cos\frac{A}{2} = -\sqrt{1+\sin A} \quad \ldots\ldots\ldots\ldots(1),$$

and

$$\sin\frac{A}{2}-\cos\frac{A}{2} = -\sqrt{1-\sin A} \quad \ldots\ldots\ldots\ldots(2).$$

For the addition of these two formulae gives the given formula.

From (1) it follows that the revolving line which bounds the angle $\frac{A}{2}$ must be between *OQ* and *OR* or else between *OR* and *OS*.

From (2) it follows that the revolving line must lie between OR and OS or else between OS and OP.

Both these conditions are satisfied only when the revolving line lies between OR and OS, and therefore the angle $\frac{A}{2}$ lies between

$$2n\pi - \frac{3\pi}{4} \text{ and } 2n\pi - \frac{\pi}{4}.$$

117. *To express the trigonometrical ratios of $\frac{A}{2}$ in terms of tan A.*

From equation (3) of Art. 109 we have

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}.$$

$$\therefore\ 1-\tan^2\frac{A}{2} = \frac{2}{\tan A}\tan\frac{A}{2}.$$

Hence
$$\tan^2\frac{A}{2} + \frac{2}{\tan A}\tan\frac{A}{2} + \frac{1}{\tan^2 A} = 1 + \frac{1}{\tan^2 A}$$
$$= \frac{1+\tan^2 A}{\tan^2 A}.$$

$$\therefore\ \tan\frac{A}{2} + \frac{1}{\tan A} = \pm\frac{\sqrt{1+\tan^2 A}}{\tan A}.$$

$$\therefore\ \tan\frac{A}{2} = \frac{\pm\sqrt{1+\tan^2 A}-1}{\tan A} \quad \text{...........(1).}$$

118. The ambiguous sign in equation (1) can only be determined when we know something of the magnitude of A.

Ex. *Given* $\tan 15° = 2-\sqrt{3}$, *find* $\tan 7\frac{1}{2}°$.

Putting $A = 15°$ we have, from equation (1), of the last article,

$$\tan 7\tfrac{1}{2}° = \frac{\pm\sqrt{1+(2-\sqrt{3})^2}-1}{2-\sqrt{3}} = \frac{\pm\sqrt{8-4\sqrt{3}}-1}{2-\sqrt{3}} \quad \text{...........(1).}$$

Now $\tan 7\frac{1}{2}^\circ$ is positive so that we must take the upper sign.

Hence $$\tan 7\tfrac{1}{2}^\circ = \frac{+(\sqrt{6}-\sqrt{2})-1}{2-\sqrt{3}},$$

$$=(\sqrt{6}-\sqrt{2}-1)(2+\sqrt{3})=\sqrt{6}-\sqrt{3}+\sqrt{2}-2=(\sqrt{3}-\sqrt{2})(\sqrt{2}-1).$$

Since $\tan 15^\circ = \tan 195^\circ$, the equation which gives us $\tan\frac{15^\circ}{2}$ in terms of $\tan 15^\circ$ may be expected to give us $\tan\frac{195^\circ}{2}$ in terms of $\tan 195^\circ$. In fact the value obtained from (1) by taking the negative sign before the radical is $\tan\frac{195^\circ}{2}$.

Hence $$\tan\frac{195^\circ}{2} = \frac{-\sqrt{8-4\sqrt{3}}-1}{2-\sqrt{3}} = \frac{-(\sqrt{6}-\sqrt{2})-1}{2-\sqrt{3}}$$

$$=(-\sqrt{6}+\sqrt{2}-1)(2+\sqrt{3}) = -(\sqrt{3}+\sqrt{2})(\sqrt{2}+1),$$

so that $$-\cot 7\tfrac{1}{2}^\circ = \tan 97\tfrac{1}{2}^\circ = -(\sqrt{3}+\sqrt{2})(\sqrt{2}+1).$$

****119.** *To explain why there is ambiguity when $\tan\frac{A}{2}$ is found from the value of $\tan A$.*

We know, by Art. 84, that, if n be any integer,

$$\tan(n\pi + A) = \tan A = k \text{ (say)}.$$

Hence any equation which gives us $\tan\frac{A}{2}$ in terms of k may be expected to give us $\tan\frac{n\pi + A}{2}$ also.

First, let n be even and equal to $2m$.

Then

$$\tan\frac{n\pi + A}{2} = \tan\frac{2m\pi + A}{2} = \tan\left(m\pi + \frac{A}{2}\right)$$

$$= \tan\frac{A}{2}, \text{ as in Art. 84.}$$

Secondly, let n be odd and equal to $2p+1$.

Then $$\tan\frac{n\pi + A}{2} = \tan\frac{(2p+1)\pi + A}{2}$$

$$= \tan\left(p\pi + \frac{\pi + A}{2}\right) = \tan\frac{\pi + A}{2} \text{ (Art. 84)}$$

$$= -\cot\frac{A}{2}. \quad \text{(Art. 70.)}$$

Hence the formula which gives us the value of $\tan\frac{A}{2}$ should be expected to give us also the value of $-\cot\frac{A}{2}$.

An illustration of this is seen in the example of the last article.

EXAMPLES. XVIII.

1. If $\sin\theta = \frac{1}{2}$ and $\sin\phi = \frac{1}{3}$, find the values of $\sin(\theta + \phi)$ and $\sin(2\theta + 2\phi)$.

2. The tangent of an angle is 2·4. Find its cosecant, the cosecant of half the angle and the cosecant of the supplement of double the angle.

3. If $\cos\alpha = \frac{11}{61}$ and $\sin\beta = \frac{4}{5}$, find the values of $\sin^2\frac{\alpha - \beta}{2}$ and $\cos^2\frac{\alpha + \beta}{2}$, the angles α and β being positive acute angles.

4. If $\cos\alpha = \frac{3}{5}$ and $\cos\beta = \frac{4}{5}$, find the value of $\cos\frac{\alpha - \beta}{2}$, the angles α and β being positive acute angles.

5. Given $\sec\theta = 1\frac{1}{4}$, find $\tan\frac{\theta}{2}$ and $\tan\theta$.

6. If $\cos A = {\cdot}28$, find the value of $\tan\frac{A}{2}$, and explain the resulting ambiguity.

7. Find the values of (1) $\sin 7\frac{1}{2}^\circ$, (2) $\cos 7\frac{1}{2}^\circ$, (3) $\tan 22\frac{1}{2}^\circ$, and (4) $\tan 11\frac{1}{4}^\circ$.

8. If $\sin\theta + \sin\phi = a$ and $\cos\theta + \cos\phi = b$, find the value of $\tan\frac{\theta-\phi}{2}$.

Prove that

9. $(\cos\alpha + \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\cos^2\frac{\alpha+\beta}{2}$.

10. $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\frac{\alpha-\beta}{2}$.

11. $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\frac{\alpha-\beta}{2}$.

12. $\sin A = \dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$.

13. $\cos A = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$.

14. $\sec\left(\frac{\pi}{4}+\theta\right)\sec\left(\frac{\pi}{4}-\theta\right) = 2\sec 2\theta$.

15. $\tan\left(45^\circ + \frac{A}{2}\right) = \sqrt{\dfrac{1+\sin A}{1-\sin A}} = \sec A + \tan A$.

16. $\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right) = \frac{1}{\sqrt{2}}\sin A$.

17. $\cos^2\alpha + \cos^2(\alpha+120^\circ) + \cos^2(\alpha-120^\circ) = \frac{3}{2}$.

18. $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \frac{3}{2}$.

19. $\sin^4\frac{\pi}{8} + \sin^4\frac{3\pi}{8} + \sin^4\frac{5\pi}{8} + \sin^4\frac{7\pi}{8} = \frac{3}{2}$.

20. $\cos 2\theta\cos 2\phi + \sin^2(\theta-\phi) - \sin^2(\theta+\phi) = \cos(2\theta+2\phi)$.

21. $(\tan 4A + \tan 2A)(1 - \tan^2 3A\tan^2 A) = 2\tan 3A\sec^2 A$.

22. $\left(1+\tan\frac{\alpha}{2}-\sec\frac{\alpha}{2}\right)\left(1+\tan\frac{\alpha}{2}+\sec\frac{\alpha}{2}\right) = \sin\alpha\sec^2\frac{\alpha}{2}$.

Find the proper signs to be applied to the radicals in the 3 following formulae.

23. $2\cos\frac{A}{2} = \pm\sqrt{1-\sin A} \pm \sqrt{1+\sin A}$, when $\frac{A}{2} = 278^\circ$.

24. $2\sin\frac{A}{2} = \pm\sqrt{1-\sin A} \pm \sqrt{1+\sin A}$, when $\frac{A}{2} = \frac{19\pi}{11}$.

25. $2\cos\frac{A}{2} = \pm\sqrt{1-\sin A} \pm \sqrt{1+\sin A}$, when $\frac{A}{2} = -140°$.

26. If $A = 340°$, prove that

$$2\sin\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A},$$

and

$$2\cos\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}.$$

27. If $A = 460°$, prove that

$$2\cos\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}.$$

28. If $A = 580°$, prove that

$$2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}.$$

29. Within what respective limits must $\frac{A}{2}$ lie when

(1) $2\sin\frac{A}{2} = \sqrt{1+\sin A} + \sqrt{1-\sin A}$.

(2) $2\sin\frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}$,

(3) $2\sin\frac{A}{2} = +\sqrt{1+\sin A} - \sqrt{1-\sin A}$,

and (4) $2\cos\frac{A}{2} = \sqrt{1+\sin A} - \sqrt{1-\sin A}$.

30. In the formula

$$2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A},$$

find within what limits $\frac{A}{2}$ must lie when

(1) the two positive signs are taken,

(2) the two negative " " "

and (3) the first sign is negative and the second positive.

31. Prove that the sine is algebraically less than the cosine for any angle between $2n\pi - \frac{3\pi}{4}$ and $2n\pi + \frac{\pi}{4}$ where n is any integer.

32. If $\sin\frac{A}{3}$ be determined from the equation

$$\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{2},$$

prove that we should expect to obtain also the values of

$$\sin\frac{\pi - A}{3} \text{ and } -\sin\frac{\pi + A}{3}.$$

33. If $\cos\frac{A}{3}$ be found from the equation

$$\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}$$

prove that we should expect to obtain also the values of

$$\cos\frac{2\pi - A}{3} \text{ and } \cos\frac{2\pi + A}{3}.$$

120. By the use of the formulae of the present chapter we can now find the trigonometrical ratios of some important angles.

To find the trigonometrical functions of an angle of 18°.

Let θ stand for 18°, so that 2θ is 36° and 3θ is 54°.

Hence $$2\theta = 90° - 3\theta,$$

and therefore

$$\sin 2\theta = \sin(90° - 3\theta) = \cos 3\theta.$$

$$\therefore\ 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta \text{ (Arts. 105 and 107).}$$

Hence, either $\cos\theta = 0$, which gives $\theta = 90°$, or

$$2\sin\theta = 4\cos^2\theta - 3 = 1 - 4\sin^2\theta.$$

$$\therefore\ 4\sin^2\theta + 2\sin\theta = 1.$$

By solving this quadratic equation, we have

$$\sin\theta = \frac{\pm\sqrt{5} - 1}{4}.$$

In our case $\sin\theta$ is necessarily a positive quantity. Hence we take the upper sign, and have

$$\mathbf{\sin 18^\circ = \frac{\sqrt{5}-1}{4}}.$$

Hence

$$\cos 18^\circ = \sqrt{1-\sin^2 18^\circ} = \sqrt{1-\frac{6-2\sqrt{5}}{16}} = \sqrt{\frac{10+2\sqrt{5}}{16}}$$

$$= \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

The remaining trigonometrical ratios of 18° may be now found.

Since 72° is the complement of 18°, the values of the ratios for 72° may be obtained by the use of Art. 69.

121. *To find the trigonometrical functions of an angle of* 36°.

Since $\cos 2\theta = 1 - 2\sin^2\theta$, (Art. 105),

$$\therefore\ \cos 36^\circ = 1 - 2\sin^2 18^\circ = 1 - 2\left(\frac{6-2\sqrt{5}}{16}\right)$$

$$= 1 - \frac{3-\sqrt{5}}{4},$$

so that

$$\mathbf{\cos 36^\circ = \frac{\sqrt{5}+1}{4}}.$$

Hence

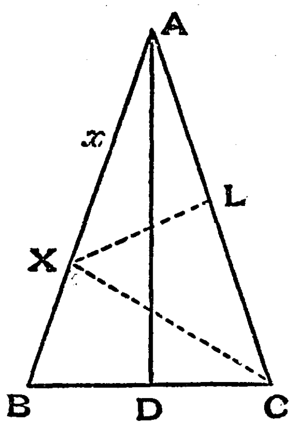
$$\sin 36^\circ = \sqrt{1-\cos^2 36^\circ} = \sqrt{1-\frac{6+2\sqrt{5}}{16}} = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

The remaining trigonometrical functions of 36° may now be found.

Also, since 54° is the complement of 36°, the values of the functions for 54° may be found by the help of Art. 69.

122. The value of sin 18° and cos 36° may also be found geometrically as follows.

Let ABC be a triangle constructed, as in Euc. IV. 10, so that each of the angles B and C is double of the angle A. Then



$$180° = A + B + C = A + 2A + 2A,$$

so that $$A = 36°.$$

Hence, if AD be drawn perpendicular to BC, we have

$$\angle BAD = 18°.$$

By Euclid's construction we know that BC is equal to AX where X is a point on AB, such that

$$AB \,.\, BX = AX^2.$$

Let $AB = a$, and $AX = x$.

This relation then gives

$$a(a - x) = x^2,$$

i.e. $$x^2 + ax = a^2,$$

i.e. $$x = a\frac{\sqrt{5} - 1}{2}.$$

Hence $$\sin 18° = \sin BAD = \frac{BD}{BA} = \frac{1}{2}\frac{BC}{BA}$$

$$= \frac{1}{2}\frac{x}{a} = \frac{\sqrt{5} - 1}{4}.$$

Again (by Euc. IV. 10) we know that AX and XC are equal; hence if XL be perpendicular to AC, then L bisects AC.

Hence

$$\cos 36° = \frac{AL}{AX} = \frac{a}{2} \div x = \frac{1}{\sqrt{5} - 1}$$

$$= \frac{\sqrt{5} + 1}{(\sqrt{5} - 1)(\sqrt{5} + 1)} = \frac{\sqrt{5} + 1}{4}.$$

123. *To find the trigonometrical functions for an angle of* 9°.

Since sin 9° and cos 9° are both positive the relation (3) of Art. 113 gives

$$\sin 9^\circ + \cos 9^\circ = \sqrt{1+\sin 18^\circ} = \sqrt{1+\frac{\sqrt{5}-1}{4}} = \frac{\sqrt{3+\sqrt{5}}}{2} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

Also, since cos 9° is greater than sin 9° (Art. 53), the quantity sin 9° − cos 9° is negative. Hence the relation (4) of Art. 113 gives

$$\sin 9^\circ - \cos 9^\circ = -\sqrt{1-\sin 18^\circ} = -\sqrt{1-\frac{\sqrt{5}-1}{4}}$$

$$= -\frac{\sqrt{5-\sqrt{5}}}{2} \quad \ldots\ldots\ldots\ldots\ldots(2).$$

By adding (1) and (2), we have

$$\sin 9^\circ = \frac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{4},$$

and, by subtracting (2) from (1), we have

$$\cos 9^\circ = \frac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4}.$$

The remaining functions for 9° may now be found.

Also, since 81° is the complement of 9°, the values of the functions for 81° may be obtained by the use of Art. 69.

EXAMPLES. XIX.

Prove that

1. $\sin^2 72^\circ - \sin^2 60^\circ = \dfrac{\sqrt{5}-1}{8}$.

2. $\cos^2 48^\circ - \sin^2 12^\circ = \dfrac{\sqrt{5}+1}{8}$.

3. $\cos 12° + \cos 60° + \cos 84° = \cos 24° + \cos 48°$.

4. $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$.

5. $\sin\frac{\pi}{10} + \sin\frac{13\pi}{10} = -\frac{1}{2}$.

6. $\sin\frac{\pi}{10}\sin\frac{13\pi}{10} = -\frac{1}{4}$.

7. $\tan 6° \tan 42° \tan 66° \tan 78° = 1$.

8. $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{2^7}$.

9. $16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15} = 1$.

10. Two parallel chords of a circle, which are on the same side of the centre, subtend angles of 72° and 144° respectively at the centre. Prove that the perpendicular distance between the chords is half the radius of the circle.

11. In any circle prove that the chord which subtends 108° at the centre is equal to the sum of the two chords which subtend angles of 36° and 60°.

12. Construct the angle whose cosine is equal to its tangent.

13. Solve the equation

$$4\cos\theta - 3\sec\theta = 2\tan\theta.$$

CHAPTER IX.

IDENTITIES AND TRIGONOMETRICAL EQUATIONS.

124. THE formulae of Arts. 88 and 90 can be used to obtain the trigonometrical ratios of the sum of more than two angles.

For example

$$\sin(A+B+C) = \sin(A+B)\cos C + \cos(A+B)\sin C$$

$$= [\sin A\cos B + \cos A\sin B]\cos C$$

$$+ [\cos A\cos B - \sin A\sin B] \times \sin C$$

$$= \sin A\cos B\cos C + \cos A\sin B\cos C$$

$$+ \cos A\cos B\sin C - \sin A\sin B\sin C.$$

So

$$\cos(A+B+C) = \cos(A+B)\cos C - \sin(A+B)\sin C$$

$$= (\cos A\cos B - \sin A\sin B)\cos C$$

$$- (\sin A\cos B + \cos A\sin B)\sin C$$

$$= \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C$$

$$- \sin A\sin B\cos C.$$

Also $$\tan(A+B+C)=\frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$$

$$=\frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\dfrac{\tan A+\tan B}{1-\tan A\tan B}\tan C}$$

$$=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B}.$$

125. The last formula of the previous article is a particular case of a very general theorem which gives the tangent of the sum of any number of angles in terms of the tangents of the angles themselves. The theorem is

$$\mathbf{tan}\,(\mathbf{A}_1+\mathbf{A}_2+\mathbf{A}_3+\ldots+\mathbf{A}_n)$$
$$=\frac{\mathbf{s}_1-\mathbf{s}_3+\mathbf{s}_5-\mathbf{s}_7+\ldots}{\mathbf{1}-\mathbf{s}_2+\mathbf{s}_4-\mathbf{s}_6+\ldots}\ldots\ldots\ldots\ldots\ldots(1),$$

where

$s_1=\tan A_1+\tan A_2+\ldots+\tan A_n$

= the sum of the tangents of the separate angles,

$s_2=\tan A_1\tan A_2+\tan A_1\tan A_3+\ldots$

= the sum of the tangents taken two at a time,

$s_3=\tan A_1\tan A_2\tan A_3+\tan A_2\tan A_3\tan A_4+\ldots$

= the sum of the tangents taken three at a time, and so on.

Assume the relation (1) to hold for n angles and add on another angle A_{n+1}.

Then $$\tan(A_1+A_2+\ldots+A_{n+1})$$
$$=\tan[(A_1+A_2+\ldots+A_n)+A_{n+1}]$$
$$=\frac{\tan(A_1+A_2+\ldots+A_n)+\tan A_{n+1}}{1-\tan(A_1+A_2+\ldots+A_n)\,.\,\tan A_{n+1}}$$

$$=\frac{\dfrac{s_1-s_3+s_5-s_7+\dots}{1-s_3+s_4\dots}+\tan A_{n+1}}{1-\dfrac{s_1-s_3+s_5\dots}{1-s_2+s_4\dots}\tan A_{n+1}}.$$

Let $\tan A_1$, $\tan A_2$, ... $\tan A_{n+1}$ be respectively called $t_1, t_2, \dots t_{n+1}$.

Then $\tan(A_1+A_2+\dots+A_{n+1})$

$$=\frac{(s_1-s_3+s_5\dots)+t_{n+1}(1-s_2+s_4\dots)}{(1-s_2+s_4\dots)-(s_1-s_3+s_5\dots)t_{n+1}}$$

$$=\frac{(s_1+t_{n+1})-(s_3+s_2t_{n+1})+(s_5+s_4t_{n+1})\dots}{1-(s_2+s_1t_{n+1})+(s_4+s_3t_{n+1})-(s_6+s_5t_{n+1})\dots}.$$

But $s_1+t_{n+1}=(t_1+t_2+\dots t_n)+t_{n+1}$

= the sum of the $(n+1)$ tangents,

$s_2+s_1t_{n+1}=(t_1t_2+t_2t_3+\dots)+(t_1+t_2+\dots+t_n)t_{n+1}$

= the sum, two at a time, of the $(n+1)$ tangents.

$s_3+s_2t_{n+1}=(t_1t_2t_3+t_2t_3t_4+\dots)+(t_1t_2+t_2t_3+\dots)t_{n+1}$

= the sum three at a time of the $(n+1)$ tangents and so on.

Hence we see that the same rule holds for $(n+1)$ angles as for n angles.

Hence, if the theorem be true for n angles, it is true for $(n+1)$ angles.

But, by Arts. 98 and 124, it *is* true for 2 and 3 angles.

Hence the theorem is true for 4 angles; hence for 5 angles .... Hence it is true universally.

Cor. If the angles be all equal and there be n of them and each equal to θ, then

$$s_1=n\,.\tan\theta;\ s_2={}^nC_2\tan^2\theta;\ s_3={}^nC_3\tan^3\theta\dots\dots$$

Ex. *Write down the value of tan 4θ.*

Here $\tan 4\theta = \dfrac{s_1 - s_3}{1 - s_2 + s_4} = \dfrac{4\tan\theta - {}^4C_3 \tan^3\theta}{1 - {}^4C_2 \tan^2\theta + {}^4C_4 \tan^4\theta}$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

Ex. Prove that $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$.

126. By a method similar to that of the last article it may be shewn that $\sin(A_1 + A_2 + \ldots + A_n)$

$$= \cos A_1 \cos A_2 \ldots \cos A_n (s_1 - s_3 + s_5 - \ldots),$$

and that $\cos(A_1 + A_2 + \ldots + A_n)$

$$= \cos A_1 \cos A_2 \ldots \cos A_n (1 - s_2 + s_4 - \ldots),$$

where $s_1, s_2, s_3, \ldots$ have the same values as in that article.

127. Identities holding between the trigonometrical ratios of the angles of a triangle.

When three angles A, B and C, are such that their sum is 180°, many identical relations are found to hold between their trigonometrical ratios.

The method of proof is best seen from the following examples.

Ex. 1. *If $A + B + C = 180°$, to prove that*

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

$$\sin 2A + \sin 2B + \sin 2C$$

$$= 2\sin(A + B)\cos(A - B) + 2\sin C \cos C.$$

Since $A + B + C = 180°$,

we have $A + B = 180° - C$,

and therefore $\sin(A + B) = \sin C$,

and $\cos(A + B) = -\cos C$. (Art. 72)

Hence the expression

$$=2\sin C\cos(A-B)+2\sin C\cos C$$

$$=2\sin C[\cos(A-B)+\cos C]$$

$$=2\sin C[\cos(A-B)-\cos(A+B)]$$

$$=2\sin C\,.\,2\sin A\sin B$$

$$=4\sin A\sin B\sin C.$$

Ex. 2. *If* $A+B+C=180°,$

prove that $\cos A+\cos B-\cos C=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$

The expression $=\cos A+(\cos B-\cos C)$

$$=2\cos^2\frac{A}{2}-1+2\sin\frac{B+C}{2}\sin\frac{C-B}{2}.$$

Now $B+C=180°-A,$

so that $\frac{B+C}{2}=90°-\frac{A}{2},$

and therefore $\sin\frac{B+C}{2}=\cos\frac{A}{2},$

and $\cos\frac{B+C}{2}=\sin\frac{A}{2}.$

Hence the expression

$$=2\cos^2\frac{A}{2}-1+2\cos\frac{A}{2}\sin\frac{C-B}{2}$$

$$=2\cos\frac{A}{2}\left[\cos\frac{A}{2}+\sin\frac{C-B}{2}\right]-1$$

$$=2\cos\frac{A}{2}\left[\sin\frac{B+C}{2}+\sin\frac{C-B}{2}\right]-1$$

$$=2\cos\frac{A}{2}\,.\,2\sin\frac{C}{2}\cos\frac{B}{2}-1$$

$$=-1+4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$$

Ex. 3. *If* $A+B+C=180°,$

prove that $\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C.$

Let $$S = \sin^2 A + \sin^2 B + \sin^2 C,$$

so that $$2S = 2\sin^2 A + 1 - \cos 2B + 1 - \cos 2C$$
$$= 2\sin^2 A + 2 - 2\cos(B+C)\cos(B-C)$$
$$= 2 - 2\cos^2 A + 2 - 2\cos(B+C)\cos(B-C).$$
$$\therefore\ S = 2 + \cos A\,[\cos(B-C) + \cos(B+C)],$$

since $$\cos A = \cos\{180^\circ - (B+C)\} = -\cos(B+C).$$
$$\therefore\ S = 2 + \cos A \,.\, 2\cos B\cos C.$$
$$= 2 + 2\cos A\cos B\cos C.$$

Ex. 4. *If* $$A + B + C = 180^\circ,$$

prove that $$\tan A + \tan B + \tan C = \tan A\tan B\tan C.$$

By the third formula of Art. 124, we have

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - (\tan B\tan C + \tan C\tan A + \tan A\tan B)}.$$

But $$\tan(A+B+C) = \tan 180^\circ = 0.$$

Hence $$0 = \tan A + \tan B + \tan C - \tan A\tan B\tan C,$$

i.e. $$\tan A + \tan B + \tan C = \tan A\tan B\tan C.$$

This may also be proved independently. For

$$\tan(A+B) = \tan(180^\circ - C) = -\tan C.$$
$$\therefore\ \frac{\tan A + \tan B}{1 - \tan A\tan B} = -\tan C.$$
$$\therefore\ \tan A + \tan B = -\tan C + \tan A\tan B\tan C,$$

i.e. $$\tan A + \tan B + \tan C = \tan A\tan B\tan C.$$

Ex. 5. *If* $x + y + z = xyz$, prove that

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}.$$

Put $x = \tan A$, $y = \tan B$, and $z = \tan C$, so that we have

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C.$$
$$\therefore\ \frac{\tan A + \tan B}{1 - \tan A\tan B} = -\tan C,$$

so that $$\tan(A+B) = \tan(\pi - C).$$ [Art. 72.]

Hence $$A+B+C=n\pi+\pi,$$

$$\therefore\ \frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2\tan A}{1-\tan^2 A}+\frac{2\tan B}{1-\tan^2 B}+\frac{2\tan C}{1-\tan^2 C}$$

$$=\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C,$$

(by a proof similar to that of the last example)

$$=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}.$$

EXAMPLES. XX.

If $A+B+C=180°$, prove that

1. $\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C.$

2. $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C.$

3. $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C.$

4. $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

5. $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$

6. $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

7. $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C.$

8. $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C.$

9. $\cos^2 A+\cos^2 B-\cos^2 C=1-2\sin A\sin B\cos C.$

10. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

11. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}.$

12. $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$

13. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$

14. $\cot B\cot C+\cot C\cot A+\cot A\cot B=1.$

15. $\sin(B+2C)+\sin(C+2A)+\sin(A+2B)$

$$=4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}.$$

16. $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1=4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}.$

17. $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

18. $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)$

$$=4\sin A\sin B\sin C.$$

If $A+B+C=2S$ prove that

19. $\sin(S-A)\sin(S-B)+\sin S\sin(S-C)=\sin A\sin B.$

20. $\sin S\sin(S-A)\sin(S-B)\sin(S-C)$

$$=1-\cos^2 A-\cos^2 B-\cos^2 C+2\cos A\cos B\cos C.$$

21. $\sin(S-A)+\sin(S-B)+\sin(S-C)-\sin S$

$$=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

22. $\cos^2 S+\cos^2(S-A)+\cos^2(S-B)+\cos^2(S-C)$

$$=2+2\cos A\cos B\cos C.$$

23. $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C$

$$=1+4\cos S\cos(S-A)\cos(S-B)\cos(S-C).$$

24. If $\alpha+\beta+\gamma+\delta=2\pi$, prove that

$$\cos\alpha+\cos\beta+\cos\gamma+\cos\delta+4\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2}=0,$$

and $$\sin\alpha-\sin\beta+\sin\gamma-\sin\delta+4\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2}=0.$$

25. If the sum of four angles be 180° prove that the sum of their cosines taken two and two together is equal to the sum of their sines taken similarly.

Prove

26. $1-\cos^2\theta-\cos^2\phi-\cos^2\psi+2\cos\theta\cos\phi\cos\psi$

$$=4\sin\frac{\theta+\phi+\psi}{2}\sin\frac{\theta+\phi-\psi}{2}\sin\frac{\theta-\phi+\psi}{2}\sin\frac{-\theta+\phi+\psi}{2}.$$

27. $\sin 2\alpha + \sin 2\beta + \sin 2\gamma$

$$= 2(\sin\alpha + \sin\beta + \sin\gamma)(1 + \cos\alpha + \cos\beta + \cos\gamma)$$

if $$\alpha + \beta + \gamma = 0.$$

28. Verify that

$$\sin^3 a \sin(b-c) + \sin^3 b \sin(c-a) + \sin^3 c \sin(a-b)$$
$$+ \sin(a+b+c)\sin(b-c)\sin(c-a)\sin(a-b) = 0.$$

If A, B, C, and D be any angles prove that

29. $\sin A \sin B \sin(A-B) + \sin B \sin C \sin(B-C)$
$$+ \sin C \sin A \sin(C-A) + \sin(A-B)\sin(B-C)\sin(C-A) = 0.$$

30. $\sin(A-B)\cos(A+B) + \sin(B-C)\cos(B+C)$
$$+ \sin(C-D)\cos(C+D) + \sin(D-A)\cos(D+A) = 0.$$

31. $\sin(A+B-2C)\cos B - \sin(A+C-2B)\cos C$
$$= \sin(B-C)\{\cos(B+C-A) + \cos(C+A-B) + \cos(A+B-C)\}.$$

32. $\sin(A+B+C+D) + \sin(A+B-C-D) + \sin(A+B-C+D)$
$$+ \sin(A+B+C-D) = 4\sin(A+B)\cos C \cos D.$$

33. If any theorem be true for values of A, B, and C such that

$$A + B + C = 180^\circ,$$

prove that the theorem is still true if we substitute for A, B, and C respectively the quantities

$$(1)\quad 90^\circ - \frac{A}{2},\ 90^\circ - \frac{B}{2},\ \text{and } 90^\circ - \frac{C}{2},$$

or $$(2)\quad 180^\circ - 2A,\ 180^\circ - 2B,\ \text{and } 180^\circ - 2C.$$

If $x + y + z = xyz$ prove that

34. $$\frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} = \frac{3x - x^3}{1 - 3x^2} \cdot \frac{3y - y^3}{1 - 3y^2} \cdot \frac{3z - z^3}{1 - 3z^2}$$

and 35. $x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) = 4xyz.$

128. The Addition and Subtraction Theorems may be used to solve some kinds of trigonometrical equations.

Ex. *Solve the equation*

$$\sin x + \sin 5x = \sin 3x.$$

By the formulae of Art. 94 the equation is

$$2 \sin 3x \cos 2x = \sin 3x.$$

$$\therefore \sin 3x = 0, \text{ or } 2 \cos 2x = 1.$$

If $\sin 3x = 0$, then $3x = n\pi$.

If $\cos 2x = \frac{1}{2}$, then $2x = 2n\pi \pm \frac{\pi}{3}$.

Hence $x = \frac{n\pi}{3}$, or $n\pi \pm \frac{\pi}{6}$.

129. *To solve an equation of the form*

$$a \cos \theta + b \sin \theta = c.$$

Divide both sides of the equation by $\sqrt{a^2 + b^2}$, so that it may be written

$$\frac{a}{\sqrt{a^2 + b^2}} \cos \theta + \frac{b}{\sqrt{a^2 + b^2}} \sin \theta = \frac{c}{\sqrt{a^2 + b^2}}.$$

Find from the table of tangents the angle whose tangent is $\frac{b}{a}$ and call it α.

Then $\tan \alpha = \frac{b}{a}$, so that

$$\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}, \text{ and } \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}.$$

The equation can then be written

$$\cos \alpha \cos \theta + \sin \alpha \sin \theta = \frac{c}{\sqrt{a^2 + b^2}},$$

i.e. $$\cos (\theta - \alpha) = \frac{c}{\sqrt{a^2 + b^2}}.$$

Next find from the tables, or otherwise, the angle β

whose cosine is $$\frac{c}{\sqrt{a^2+b^2}},$$

so that $$\cos\beta = \frac{c}{\sqrt{a^2+b^2}},$$

[N.B. This can only be done when c is $< \sqrt{a^2+b^2}$.]

The equation is then $\cos(\theta - \alpha) = \cos\beta$.

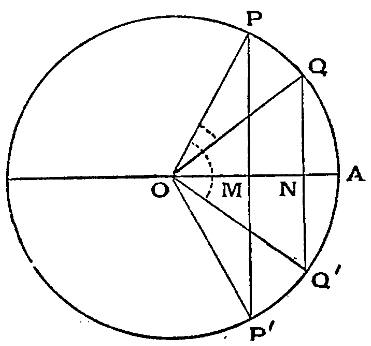
The solution of this is $\theta - \alpha = 2n\pi \pm \beta$, so that

$$\theta = 2n\pi + \alpha \pm \beta,$$

where n is any integer.

Angles, such as α and β, which are introduced into trigonometrical work to facilitate computation are called **Subsidiary Angles.**

130. The above solution may be illustrated graphically as follows;



Measure OM along the initial line equal to a, and MP perpendicular to it, and equal to b. The angle MOP is then the angle whose tangent is $\frac{b}{a}$, i.e. α.

With centre O and radius OP, i.e. $\sqrt{a^2+b^2}$, describe a circle and measure ON along the initial line equal to c.

Draw QNQ' perpendicular to ON to meet the circle in Q and Q'; the angles NOQ and $Q'ON$ are therefore each equal to β.

The angle QOP is therefore $\alpha - \beta$ and $Q'OP$ is $\alpha + \beta$.

Hence the solutions of the equation are respectively

$$2n\pi + QOP \text{ and } 2n\pi + Q'OP.$$

The construction clearly fails if c be $> \sqrt{a^2+b^2}$, for then the point N would fall outside the circle.

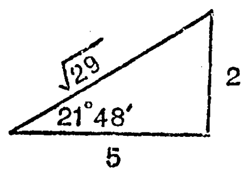
131. As a numerical example let us solve the equation

$$5\cos\theta - 2\sin\theta = 2.$$

given that $\tan 21^\circ 48' = \frac{2}{5}$.

Dividing both sides of the equation by

$$\sqrt{5^2+2^2} \text{ i.e. } \sqrt{29},$$



we have

$$\frac{5}{\sqrt{29}}\cos\theta - \frac{2}{\sqrt{29}}\sin\theta = \frac{2}{\sqrt{29}}.$$

Hence $\cos\theta\cos 21^\circ 48' - \sin\theta \sin 21^\circ 48'$

$= \sin 21^\circ 48' = \sin(90 - 68^\circ 12')$

$= \cos 68^\circ 12'$.

$\therefore \cos(\theta + 21^\circ 48') = \cos 68^\circ 12'$.

Hence $\theta + 21^\circ 48' = 2n\pi \pm 68^\circ 12'$. (Art. 83)

$\therefore \theta = 2n\pi - 21^\circ 48' \pm 68^\circ 12'$

$= 2n\pi - \frac{\pi}{2}$ or $2n\pi + 46^\circ 24'$,

where n is any integer.

EXAMPLES. XXI.

Solve the equations

1. $\sin\theta + \sin 7\theta = \sin 4\theta$.
2. $\cos\theta + \cos 7\theta = \cos 4\theta$.
3. $\cos\theta + \cos 3\theta = 2\cos 2\theta$.
4. $\sin 4\theta - \sin 2\theta = \cos 3\theta$.
5. $\cos\theta - \sin 3\theta = \cos 2\theta$.
6. $\sin 7\theta = \sin\theta + \sin 3\theta$.
7. $\cos\theta + \cos 2\theta + \cos 3\theta = 0$.
8. $\sin\theta + \sin 3\theta + \sin 5\theta = 0$.
9. $\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0$.
10. $\sin(3\theta + \alpha) + \sin(3\theta - \alpha) + \sin(\alpha - \theta) - \sin(\alpha + \theta) = \cos\alpha$.

11. $\cos(3\theta+\alpha)\cos(3\theta-\alpha)+\cos(5\theta+\alpha)\cos(5\theta-\alpha)=\cos 2\alpha$.

12. $\cos n\theta=\cos(n-2)\theta+\sin\theta$.

13. $\sin\frac{n+1}{2}\theta=\sin\frac{n-1}{2}\theta+\sin\theta$.

14. $\sin m\theta+\sin n\theta=0$.

15. $\cos m\theta+\cos n\theta=0$.

16. $\sin^2 n\theta-\sin^2(n-1)\theta=\sin^2\theta$.

17. $\sin 3\theta+\cos 2\theta=0$.

18. $\sqrt{3}\cos\theta+\sin\theta=\sqrt{2}$.

19. $\sin\theta+\cos\theta=\sqrt{2}$.

20. $\sqrt{3}\sin\theta-\cos\theta=\sqrt{2}$.

21. $\sin x+\cos x=\sqrt{2}\cos A$.

22. $5\sin\theta+2\cos\theta=5$ (given $\tan 21^\circ 48'=\cdot 4$).

23. $6\cos x+8\sin x=9$ (given $\tan 53^\circ 8'=1\frac{1}{3}$).

24. $1+\sin^2\theta=3\sin\theta\cos\theta$ (given $\tan 71^\circ 34'=3$).

25. $\operatorname{cosec}\theta=\cot\theta+\sqrt{3}$.

26. $\operatorname{cosec} x=1+\cot x$.

27. $(2+\sqrt{3})\cos\theta=1-\sin\theta$.

28. $\tan\theta+\sec\theta=\sqrt{3}$.

29. $\cos 2\theta=\cos^2\theta$.

30. $4\cos\theta-3\sec\theta=\tan\theta$.

31. $\cos 2\theta+3\cos\theta=0$.

32. $\cos 3\theta+2\cos\theta=0$.

33. $\cos 2\theta=(\sqrt{2}+1)\left(\cos\theta-\frac{1}{\sqrt{2}}\right)$.

34. $\cot\theta-\tan\theta=2$.

35. $4\cot 2\theta=\cot^2\theta-\tan^2\theta$.

36. $3\tan(\theta-15^\circ)=\tan(\theta+15^\circ)$.

37. $\tan\theta+\tan 2\theta+\tan 3\theta=0$.

38. $\tan\theta+\tan 2\theta+\sqrt{3}\tan\theta\tan 2\theta=\sqrt{3}$.

39. $\sin 3\alpha=4\sin\alpha\sin(x+\alpha)\sin(x-\alpha)$.

40. Prove that the equation $x^3-2x+1=0$ is satisfied by putting for x either of the values

$$\sqrt{2}\sin 45^\circ,\ 2\sin 18^\circ,\ \text{and}\ 2\sin 234^\circ.$$

132. Ex. *To trace the changes in the sign and magnitude of the expression $\sin\theta+\cos\theta$ as θ increases from 0 to 360°.*

We have $\sin\theta+\cos\theta=\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta\right]$

$=\sqrt{2}\left[\sin\theta\cos 45^\circ+\cos\theta\sin 45^\circ\right]=\sqrt{2}\sin(\theta+45^\circ)$.

As θ increases from 0 to 45°, $\sin(\theta + 45°)$ increases from sin 45° to sin 90°, and hence the expression increases from 1 to $\sqrt{2}$.

As θ increases from 45° to 135°, $\theta + 45°$ increases from 90° to 180°, and hence the expression is positive and decreases from $\sqrt{2}$ to 0.

As θ increases from 135° to 225°, the expression changes from $\sqrt{2}\sin 180°$ to $\sqrt{2}\sin 270°$, *i.e.* it is negative and decreases from 0 to $-\sqrt{2}$.

As θ increases from 225° to 315°, the expression changes from $\sqrt{2}\sin 270°$ to $\sqrt{2}\sin 360°$, *i.e.* it is negative and increases from $-\sqrt{2}$ to 0.

As θ increases from 315° to 360°, the expression changes from $\sqrt{2}\sin 360°$ to $\sqrt{2}\sin 405°$, *i.e.* it is positive and increases from 0 to 1.

133. Ex. *To trace the changes in the sign and magnitude of $a\cos\theta + b\sin\theta$, and to find the greatest value of the expression.*

We have

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right].$$

Let α be the smallest positive angle such that

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \text{ and } \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

The expression therefore

$$= \sqrt{a^2+b^2}\,[\cos\theta\cos\alpha + \sin\theta\sin\alpha] = \sqrt{a^2+b^2}\cos(\theta-\alpha).$$

As θ changes from α to $360° + \alpha$, the angle $\theta - \alpha$ changes from 0 to 360°, and hence the changes in the sign and magnitude of the expression are easily obtained.

Since the greatest value of the quantity $\cos(\theta - \alpha)$ is unity, *i.e.* when θ equals α, the greatest value of the expression is $\sqrt{a^2 + b^2}$.

Also the value of θ which gives this greatest value is such that its cosine is $\frac{a}{\sqrt{a^2 + b^2}}$.

EXAMPLES. XXII.

As θ increases from 0 to 360°, trace the changes in the sign and magnitude of

1. $\sin\theta - \cos\theta$,

2. $\sin\theta + \sqrt{3}\cos\theta$,

[N. B. $\sin\theta + \sqrt{3}\cos\theta = 2\left[\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta\right] = 2\sin(\theta + 60°)$.]

3. $\sin\theta - \sqrt{3}\cos\theta$.

4. $\cos^2\theta - \sin^2\theta$.

5. $\sin\theta\cos\theta$.

6. $\frac{\sin\theta + \sin 2\theta}{\cos\theta + \cos 2\theta}$.

7. $\sin(\pi\sin\theta)$.

8. $\cos(\pi\sin\theta)$.

9. $\frac{\sin(\pi\cos\theta)}{\cos(\pi\sin\theta)}$.

10. Trace the changes in the sign and magnitude of $\frac{\sin 3\theta}{\cos 2\theta}$ as the angle increases from 0 to 90°.

CHAPTER X.

LOGARITHMS.

134. SUPPOSING that we know that

$$10^{2\cdot4031205} = 253, \quad 10^{2\cdot6095944} = 407,$$

and
$$10^{5\cdot0127149} = 102971,$$

we can shew that $253 \times 407 = 102971$ without performing the operation of multiplication. For

$$253 \times 407 = 10^{2\cdot4031205} \times 10^{2\cdot6095944}$$
$$= 10^{2\cdot4031205 + 2\cdot6095944}$$
$$= 10^{5\cdot0127149} = 102971.$$

Here it will be noticed that the process of multiplication has been replaced by the simpler process of addition.

Again supposing that we know that

$$10^{4\cdot9004055} = 79507,$$

and that
$$10^{1\cdot6334685} = 43,$$

we can easily shew that the cube root of 79507 is 43.

For
$$\sqrt[3]{79507} = [79507]^{\frac{1}{3}} = (10^{4\cdot9004055})^{\frac{1}{3}}$$
$$= 10^{\frac{1}{3} \times 4\cdot9004055} = 10^{1\cdot6334685} = 43.$$

Here it will be noticed that the difficult process of extracting the cube root has been replaced by the simpler process of division.

135. Logarithm. Def. *If a be any number and x and N two other numbers such that $a^x = N$, then x is called the logarithm of N to the base a and is written $\log_a N$.*

Exs. Since $10^2 = 100$, therefore $2 = \log_{10} 100$.

Since $10^5 = 100000$, therefore $5 = \log_{10} 100000$.

Since $2^4 = 16$, therefore $4 = \log_2 16$.

Since $8^{\frac{2}{3}} = [8^{\frac{1}{3}}]^2 = 2^2 = 4$, therefore $\frac{2}{3} = \log_8 4$.

Since $9^{-\frac{3}{2}} = \frac{1}{9^{\frac{3}{2}}} = \frac{1}{3^3} = \frac{1}{27}$, therefore

$$-\frac{3}{2} = \log_9 \left(\frac{1}{27}\right).$$

N.B. Since $a^0 = 1$ always, the logarithm of unity to any base is always zero.

136. In Algebra, if m and n be any real quantities whatever, the following laws, known as the laws of indices, are found to be true:

(i) $a^m \times a^n = a^{m+n}$,

(ii) $a^m \div a^n = a^{m-n}$,

and (iii) $(a^m)^n = a^{mn}$.

Corresponding to these we have three fundamental laws of logarithms, viz.

(i) $\mathbf{\log_a (mn) = \log_a m + \log_a n}$,

(ii) $\mathbf{\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n}$,

and (iii) $\mathbf{\log_a m^n = n \log_a m}$.

The proofs of these laws are given in the following articles.

137. *The logarithm of the product of two quantities is equal to the sum of the logarithms of the quantities to the same base, i.e.*

$$\log_a (mn) = \log_a m + \log_a n.$$

Let $x = \log_a m$, so that $a^x = m$,

and $y = \log_a n$, so that $a^y = n$.

Then $mn = a^x \times a^y = a^{x+y}$.

$$\therefore \log_a mn = x + y \quad \text{(Art. 135, Def.)}$$
$$= \log_a m + \log_a n.$$

138. *The logarithm of the quotient of two quantities is equal to the difference of their logarithms, i.e.*

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n.$$

Let $x = \log_a m$, so that $a^x = m$, (Art. 135, Def.)

and $y = \log_a n$, so that $a^y = n$.

Then $\frac{m}{n} = a^x \div a^y = a^{x-y}$.

$$\therefore \log_a \left(\frac{m}{n}\right) = x - y \quad \text{(Art. 135, Def.)}$$
$$= \log_a m - \log_a n.$$

139. *The logarithm of a quantity raised to any power is equal to the logarithm of the quantity multiplied by the index of the power, i.e.*

$$\log_a (m^n) = n \log_a m.$$

Let $x = \log_a m$, so that $a^x = m$. Then

$$m^n = (a^x)^n = a^{nx}.$$

$$\therefore \log_a (m^n) = nx \quad \text{(Art. 135, Def.)}$$
$$= n \log_a m.$$

140. Common system of logarithms. In the system of logarithms which we practically use the base is always 10, so that, if no base be expressed, the base 10 is always understood. The advantage of using 10 as the base is seen in the three following articles.

141. Characteristic and Mantissa. Def. If the logarithm of any number be partly integral and partly fractional, the integral portion of the logarithm is called its characteristic and the decimal portion is called its mantissa.

Thus, supposing that $\log 795 = 2{\cdot}9003671$, the number 2 is the characteristic and $\cdot 9003671$ is the mantissa.

Negative characteristics. Suppose we know that

$$\log 2 = \cdot 30103.$$

Then

$$\log \tfrac{1}{2} = \log 1 - \log 2 = 0 - \log 2 = -\cdot 30103,$$

so that $\log \frac{1}{2}$ is negative.

Now it is found convenient, as will be seen in Art. 143, that the mantissæ of all logarithms should be kept positive. We therefore instead of $-\cdot 30103$ write $-[1 - \cdot 69897]$, so that

$$\log \tfrac{1}{2} = -(1 - \cdot 69897) = -1 + \cdot 69897.$$

For shortness this latter expression is written $\bar{1}{\cdot}69897$.

The horizontal line over the 1 denotes that the integral part is negative; the decimal part however is positive.

As another example $\bar{3}{\cdot}4771213$ stands for

$$-3 + \cdot 4771213.$$

142. *The characteristic of the logarithm of any number can always be determined by inspection.*

(i) Let the number be greater than unity.

Since $10^0 = 1$, therefore $\log 1 = 0$;

since $10^1 = 10$, therefore $\log 10 = 1$;

since $10^2 = 100$, therefore $\log 100 = 2$,

and so on.

Hence the logarithm of any number lying between 1 and 10 must lie between 0 and 1, that is, it will be a decimal fraction and therefore have 0 as its characteristic.

So the logarithm of any number between 10 and 100 must lie between 1 and 2, *i.e.* it will have a characteristic equal to 1.

Similarly the logarithm of any number between 100 and 1000 must lie between 2 and 3, *i.e.* it will have a characteristic equal to 2.

So, if the number lie between 1000 and 10000, the characteristic will be 3.

Generally, *the characteristic of the logarithm of any number will be one less than the number of digits in its integral part.*

Exs. The number 296·3457 has 3 figures in its integral part and therefore the characteristic of its logarithm is 2.

The characteristic of the logarithm of 29634·57 will be $5-1$, *i.e.* 4.

(ii) Let the number be less than unity.

Since $10^0 = 1$, therefore $\log 1 = 0$;

since $10^{-1} = \frac{1}{10} = \cdot 1$, therefore $\log \cdot 1 = -1$;

since $10^{-2} = \frac{1}{10^2} = \cdot 01$, therefore $\log \cdot 01 = -2$;

since $10^{-3} = \frac{1}{10^3} = \cdot 001$, therefore $\log \cdot 001 = -3$;

and so on.

The logarithm of any number between 1 and ·1 therefore lies between 0 and -1, and so is equal to $-1+$ some decimal, *i.e.* its characteristic is $\bar{1}$.

So the logarithm of any number between ·1 and ·01 lies between -1 and -2, and hence it is equal to $-2+$ some decimal, *i.e.* its characteristic is $\bar{2}$.

Similarly the logarithm of any number between ·01 and ·001 lies between -2 and -3, *i.e.* its characteristic is $\bar{3}$.

Generally, *the characteristic of the logarithm of any decimal fraction will be negative and* **numerically** *will be greater by unity than the number of cyphers following the decimal point.*

For any fraction between 1 and ·1 (*e.g.* ·5) has no cypher following the decimal point and we have seen that its characteristic is $\bar{1}$.

Any fraction between ·1 and ·01 (*e.g.* ·07) has 1 cypher following the decimal point and we have seen that its characteristic is $\bar{2}$.

Any fraction between ·01 and ·001 (*e.g.* ·003) has two cyphers following the decimal point and we have seen that its characteristic is $\bar{3}$.

Similarly for any fraction.

Exs. The characteristic of the logarithm of the number ·00835 is $\bar{3}$.

The characteristic of the logarithm of the number ·0000053 is $\bar{6}$.

The characteristic of the logarithm of the number ·34567 is $\bar{1}$.

143. *The mantissæ of the logarithm of all numbers, consisting of the same digits, are the same.*

This will be made clear by an example.

Suppose we are given that

$$\log 66818 = 4{\cdot}8248935.$$

Then

$$\log 668{\cdot}18 = \log \frac{66818}{100} = \log 66818 - \log 100 \text{ (Art. 138)}$$
$$= 4{\cdot}8248935 - 2 = 2{\cdot}8248935\,;$$

$$\log {\cdot}66818 = \log \frac{66818}{100000} = \log 66818 - \log 100000$$
(Art. 135)
$$= 4{\cdot}8248935 - 5 = \bar{1}{\cdot}8248935.$$

So $$\log {\cdot}00066818 = \log \frac{66818}{10^8} = \log 66818 - \log 10^8$$
$$= 4{\cdot}8248935 - 8 = \bar{4}{\cdot}8248935.$$

Now the numbers 66818, 668·18, ·66818, and ·00066818 consist of the same significant figures and only differ in the position of the decimal point. We observe that their logarithms have the same decimal portion, *i.e.* the same mantissa, and they only differ in the characteristic.

The value of this characteristic is in each case determined by the rule of the previous article.

It will be noted that the mantissa of a logarithm is always positive.

144. Tables of logarithms. The logarithms of all numbers from 1 to 108000 are given in Chambers' Tables of Logarithms. Their values are there given correct to seven places of decimals.

The student should have access to a copy of the above table of logarithms or to some other suitable table. It will be required for many examples in the course of the next few chapters.

On the opposite page is a specimen page selected from Chambers' Tables. It gives the mantissæ of the logarithms of all whole numbers from 52500 to 53000.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
5250	720 1593	1676	1758	1841	1924	2007	2089	2172	2255	2337	
51	2420	2503	2586	2668	2751	2834	2916	2999	3082	3164	
52	3247	3330	3413	3495	3578	3661	3743	3826	3909	3991	
53	4074	4157	4239	4322	4405	4487	4570	4653	4735	4818	
54	4901	4983	5066	5149	5231	5314	5397	5479	5562	5645	
55	5727	5810	5892	5975	6058	6140	6223	6306	6388	6471	
56	6554	6636	6719	6801	6884	6967	7049	7132	7215	7297	
57	7380	7462	7545	7628	7710	7793	7875	7958	8041	8123	
58	8206	8288	8371	8454	8536	8619	8701	8784	8867	8949	
59	9032	9114	9197	9279	9362	9445	9527	9610	9692	9775	
60	9857	9940	0023	0105	0188	0270	0353	0435	0518	0600	
5261	721 0683	0766	0848	0931	1013	1096	1178	1261	1343	1426	
62	1508	1591	1674	1756	1839	1921	2004	2086	2169	2251	
63	2334	2416	2499	2581	2664	2746	2829	2911	2994	3076	
64	3159	3241	3324	3406	3489	3571	3654	3736	3819	3901	
65	3984	4066	4149	4231	4314	4396	4479	4561	4644	4726	
66	4809	4891	4973	5056	5138	5221	5303	5386	5468	5551	
67	5633	5716	5798	5881	5963	6045	6128	6210	6293	6375	
68	6458	6540	6623	6705	6787	6870	6952	7035	7117	7200	
69	7282	7364	7447	7529	7612	7694	7777	7859	7941	8024	
70	8106	8189	8271	8353	8436	8518	8601	8683	8765	8848	
5271	8930	9013	9095	9177	9260	9342	9424	9507	9589	9672	82
72	9754	9836	9919	0001	0084	0166	0248	0331	0413	0495	1 8
73	722 0578	0660	0742	0825	0907	0990	1072	1154	1237	1319	2 16
74	1401	1484	1566	1648	1731	1813	1895	1978	2060	2142	3 25
75	2225	2307	2389	2472	2554	2636	2719	2801	2883	2966	4 33
76	3048	3130	3212	3295	3377	3459	3542	3624	3706	3789	5 41
77	3871	3953	4036	4118	4200	4282	4365	4447	4529	4612	6 49
78	4694	4776	4858	4941	5023	5105	5188	5270	5352	5434	7 57
79	5517	5599	5681	5763	5846	5928	6010	6092	6175	6257	8 66
80	6339	6421	6504	6586	6668	6750	6833	6915	6997	7079	9 74
5281	7162	7244	7326	7408	7491	7573	7655	7737	7820	7902	
82	7984	8066	8148	8231	8313	8395	8477	8559	8642	8724	
83	8806	8888	8971	9053	9135	9217	9299	9382	9464	9546	
84	9628	9710	9792	9875	9957	0039	0121	0203	0286	0368	
85	723 0450	0532	0614	0696	0779	0861	0943	1025	1107	1189	
86	1272	1354	1436	1518	1600	1682	1765	1847	1929	2011	
87	2093	2175	2257	2340	2422	2504	2586	2668	2750	2832	
88	2914	2997	3079	3161	3243	3325	3407	3489	3571	3654	
89	3736	3818	3900	3982	4064	4146	4228	4310	4393	4475	
90	4557	4639	4721	4803	4885	4967	5049	5131	5213	5296	
5291	5378	5460	5542	5624	5706	5788	5870	5952	6034	6116	
92	6198	6280	6362	6445	6527	6609	6691	6773	6855	6937	
93	7019	7101	7183	7265	7347	7429	7511	7593	7675	7757	
94	7839	7921	8003	8085	8167	8250	8332	8414	8496	8578	
95	8660	8742	8824	8906	8988	9070	9152	9234	9316	9398	
96	9480	9562	9644	9726	9808	9890	9972	0054	0136	0218	
97	724 0300	0382	0464	0546	0628	0710	0792	0874	0956	1038	
98	1120	1202	1283	1365	1447	1529	1611	1693	1775	1857	
99	1939	2021	2103	2185	2267	2349	2431	2513	2595	2677	
5300	2759	2841	2923	3005	3086	3168	3250	3332	3414	3496	

145. To obtain the logarithm of any such number, such as 52687, we proceed as follows. Run the eye down the extreme left-hand column until it arrives at the number 5268. Then look horizontally until the eye sees the figures 7035 which are vertically beneath the number 7 at the top of the page. The number corresponding to 52687 is therefore 7217035. But this last number consists only of the digits of the mantissa, so that the mantissa required is ·7217035. But the characteristic for 52687 is 4.

Hence $\log 52687 = 4{\cdot}7217035.$

So $\log {\cdot}52687 = \bar{1}{\cdot}7217035,$

and $\log {\cdot}00052687 = \bar{4}{\cdot}7217035.$

If again the logarithm of 52725 be required the student will find (on running his eye vertically down the extreme left-hand column as far as 5272 and then horizontally along the row until he comes to the column under the digit 5) the number $\overline{0166}$. The bar which is placed over these digits denotes that to them must be prefixed not 721 but 722. Hence the mantissa corresponding to the number 52725 is ·7220166.

Also the characteristic of the logarithm of the number 52725 is 4.

Hence $\log 52725 = 4{\cdot}7220166.$

So $\log {\cdot}052725 = \bar{2}{\cdot}7220166.$

We shall now work a few numerical examples to shew the efficiency of the application of logarithms for purposes of calculation.

146. Ex. 1. *Find the value of* $\sqrt[5]{23{\cdot}4}$.

Let $x = \sqrt[5]{23{\cdot}4} = (23{\cdot}4)^{\frac{1}{5}}$,

so that $\log x = \frac{1}{5} \log (23{\cdot}4),$ by Art. 139.

In the table of logarithms we find, opposite the number 234, the logarithm 3692159.

Hence $$\log 23{\cdot}4 = 1{\cdot}3692159.$$

Therefore $$\log x = \frac{1}{5}[1{\cdot}3692159] = {\cdot}2738432.$$

Again in the table of logarithms we find, corresponding to the logarithm 2738432, the number 187864, so that

$$\log 1{\cdot}87864 = {\cdot}2738432.$$

$$\therefore\ x = 1{\cdot}87864.$$

Ex. 2. *Find the value of*

$$\frac{(6{\cdot}45)^3 \times \sqrt[3]{{\cdot}00034}}{(9{\cdot}37)^2 \times \sqrt[4]{8{\cdot}93}}.$$

Let x be the required value so that, by Arts. 138 and 139,

$$\log x = \log (6{\cdot}45)^3 + \log ({\cdot}00034)^{\frac{1}{3}} - \log (9{\cdot}37)^2 - \log \sqrt[4]{8{\cdot}93}$$

$$= 3 \log (6{\cdot}45) + \frac{1}{3}\log ({\cdot}00034) - 2 \log (9{\cdot}37) - \frac{1}{4}\log 8{\cdot}93.$$

Now in the table of logarithms we find

opposite	the	number	645	the	logarithm	8095597,
,,	,,	,,	34	,,	,,	5314789,
,,	,,	,,	937	,,	,,	9717396,
,,	,,	,,	893	,,	,,	9508515.

Hence

$$\log x = 3 \times {\cdot}8095597 + \frac{1}{3}(\bar{4}{\cdot}5314789)$$

$$- 2 \times {\cdot}9717396 - \frac{1}{4} \times {\cdot}9508515.$$

But $$\frac{1}{3}(\bar{4}{\cdot}5314789) = \frac{1}{3}[\bar{6} + 2{\cdot}5314789]$$

$$= \bar{2} + {\cdot}8438263.$$

$$\therefore\ \log x = 2{\cdot}4286791 + [\bar{2} + {\cdot}8438263] - 1{\cdot}9434792 - {\cdot}2377129$$

$$= 3{\cdot}2725054 - 4{\cdot}1811921$$

$$= \bar{1} + 4{\cdot}2725054 - 4{\cdot}1811921$$

$$= \bar{1}{\cdot}0913133.$$

In the table of logarithms we find, opposite the number 12340 the logarithm 0913152, so that

$$\log \cdot 12340 = \bar{1}\cdot 0913152.$$

Hence $\log x = \log \cdot 12340$ nearly,

and therefore $x = \cdot 12340$ nearly.

When the logarithm of any number does not quite agree with any logarithm in the tables but lies between two consecutive logarithms, it will be shewn in the next chapter how the number may be accurately found.

Ex. 3. *Having given $\log 2 = \cdot 30103$, find the number of digits in 2^{67} and the position of the first significant figure in 2^{-37}.*

We have
$$\log 2^{67} = 67 \times \log 2 = 67 \times \cdot 30103$$
$$= 20\cdot 16901.$$

Since the characteristic of the logarithm of 2^{67} is 20 it follows, by Art. 142, that in 2^{67} there are 21 digits.

Again
$$\log 2^{-37} = -37 \log 2 = -37 \times \cdot 30103$$
$$= -11\cdot 13811 = \overline{12}\cdot 86189.$$

Hence by Art. 142, in 2^{-37} there are 11 cyphers following the decimal point, *i.e.* the first significant figure is in the twelfth place of decimals.

Ex. 4. *Given $\log 3 = \cdot 4771213$, $\log 7 = -8450980$ and $\log 11 = 1\cdot 0413927$, solve the equation*

$$3^x \times 7^{2x+1} = 11^{x+5}.$$

Taking logarithms of both sides we have

$$\log 3^x + \log 7^{2x+1} = \log 11^{x+5}.$$

$$\therefore\ x \log 3 + (2x+1) \log 7 = (x+5) \log 11.$$

$$\therefore\ x [\log 3 + 2 \log 7 - \log 11] = 5 \log 11 - \log 7.$$

$$\therefore\ x = \frac{5 \log 11 - \log 7}{\log 3 + 2 \log 7 - \log 11}$$

$$= \frac{5\cdot 2069635 - \cdot 8450980}{\cdot 4771213 + 1\cdot 6901960 - 1\cdot 0413927}$$

$$= \frac{4\cdot 3618655}{1\cdot 1259246} = 3\cdot 87\ldots.$$

147. *To prove that*

$$\log_a m = \log_b m \times \log_a b.$$

Let $\log_a m = x$, so that $a^x = m$.

Also let $\log_b m = y$, so that $b^y = m$.

$$\therefore\ a^x = b^y.$$

Hence $\log_a (a^x) = \log_a (b^y)$.

$$\therefore\ x = y \log_a b \text{ (Art. 139).}$$

Hence $\log_a m = \log_b m \times \log_a b$.

By the theorem of the foregoing article we can from the logarithm of any number to a base b find its logarithm to any other base a. It is found convenient, as will appear in a subsequent chapter, not to calculate the logarithms to base 10 directly, but to calculate them first to another base and then to transform them by this theorem.

EXAMPLES. XXIII.

1. Given $\log 4 = \cdot 60206$ and $\log 3 = \cdot 4771213$, find the logarithms of $\cdot 8$, $\cdot 003$, $\cdot 0108$, and $(\cdot 00018)^{\frac{1}{7}}$.

2. Given $\log 11 = 1\cdot 0413927$ and $\log 13 = 1\cdot 1139434$, find the values of (1) $\log 1\cdot 43$, (2) $\log 133\cdot 1$, (3) $\log \sqrt[4]{143}$ and (4) $\log \sqrt[3]{\cdot 00169}$.

3. What are the characteristics of the logarithms of $243\cdot 7$, $\cdot 0153$, $2\cdot 8713$, $\cdot 00057$, $\cdot 023$, $\sqrt[5]{24615}$, and $(24589)^{\frac{3}{4}}$?

4. Find the 5th root of $\cdot 003$, having given $\log 3 = \cdot 4771213$ and

$$\log 312936 = 5\cdot 4954243.$$

5. Find the value of (1) $7^{\frac{1}{7}}$, (2) $(84)^{\frac{2}{5}}$ and (3) $(\cdot 021)^{\frac{1}{5}}$, having given

$$\log 2 = \cdot 30103,\ \log 3 = \cdot 4771213,$$

$$\log 7 = \cdot 8450980,\ \log 132057 = 5\cdot 1207283,$$

$$\log 588453 = 5\cdot 7697117 \text{ and } \log 461791 = 5\cdot 6644438.$$

6. Having given $\log 3 = \cdot 4771213,$

find the number of digits in

$$(1)\ 3^{43},\quad (2)\ 3^{27},\ \text{and}\ (3)\ 3^{62},$$

and the position of the first significant figure in

$$(4)\ 3^{-13},\quad (5)\ 3^{-43},\ \text{and}\ (6)\ 3^{-65}.$$

7. Given $\log 2 = \cdot 30103$, $\log 3 = \cdot 4771213$ and $\log 7 = \cdot 8450980$, solve the equations

$$2^x \,.\, 3^{x+4} = 7^x,$$

$$2^{2x+1} \,.\, 3^{3x+2} = 7^{4x},$$

and

$$7^{2x} \div 2^{x-4} = 3^{3x-7}.$$

8. From the tables find the seventh root of $\cdot 000026751$.

Making use of the tables find the approximate values of

9. $\sqrt[3]{645 \cdot 3}$. **10.** $\sqrt[5]{82357}$. **11.** $\dfrac{\sqrt{5} \times \sqrt[3]{7}}{\sqrt[4]{8} \times \sqrt[5]{9}}$.

12. $\sqrt[3]{\dfrac{7 \cdot 2 \times 8 \cdot 3}{9 \cdot 4 \div 16 \cdot 5}}$. **13.** $\sqrt{\dfrac{8^{\frac{1}{5}} \times 11^{\frac{1}{3}}}{\sqrt{74} \times \sqrt[5]{62}}}$.

CHAPTER XI.

TABLES OF LOGARITHMS AND TRIGONOMETRICAL RATIOS. PRINCIPLE OF PROPORTIONAL PARTS.

148. WE have pointed out that the logarithms of all numbers from 1 to 108000 may be found in Chambers' Mathematical Tables, so that, for example, the logarithms of 74583 and 74584 may be obtained directly therefrom.

Suppose however we wanted the logarithm of a number lying between these two, *e.g.* the number 74583·3.

To obtain the logarithm of this number we use the Principle of Proportional Parts which states that the increase in the logarithm of a number is proportional to the increase in the number itself.

Thus from the tables we find

$$\log 74583 = 4{\cdot}8726398 \ldots\ldots\ldots\ldots\ldots(1),$$

and

$$\log 74584 = 4{\cdot}8726457 \ldots\ldots\ldots\ldots\ldots(2).$$

The quantity log 74583·3 will clearly lie between log 74583 and log 74584.

Let then $\log 74583{\cdot}3 = \log 74583 + x$

$$= 4{\cdot}8726398 + x \ldots\ldots\ldots\ldots(3).$$

From (1) and (2) we see that for an increase 1 in the number the increase in the logarithm is $\cdot 0000059$.

The Theory of Proportional Parts then states that for an increase of $\cdot 3$ in the number the increase in the logarithm is

$$\cdot 3 \times \cdot 0000059, \text{ i.e., } \cdot 00000177.$$

$$\text{Hence} \quad \log 74583\cdot 3 = 4\cdot 87226398 + \cdot 00000177$$
$$= 4\cdot 87264157.$$

149. As another example we shall find the value of $\log \cdot 0382757$ and shall exhibit the working in a more concise form.

From the tables we obtain

$$\log \cdot 038275 = \bar{2}\cdot 5829152$$
$$\log \cdot 038276 = \bar{2}\cdot 5829265.$$

Hence difference for

$$\cdot 000001 = \cdot 0000113.$$

Therefore the difference for

$$\cdot 0000007 = \cdot 7 \times \cdot 0000113$$
$$= \cdot 00000791,$$

$$\therefore \quad \log \cdot 0382757 = \bar{2}\cdot 5829152$$
$$+ \cdot 00000791$$
$$= \bar{2}\cdot 58292311.$$

Since we only require logarithms to seven places of decimals we omit the last digit and the answer is $\bar{2}\,5829231$.

150. The converse question is often met with, viz., to find the number whose logarithm is given. If the logarithm be one of those tabulated the required number is easily found. The method to be followed when this is not the case is shewn in the following examples.

Find the number whose logarithm is 2·6283924.

On reference to the tables we find that the logarithm 6283924 is not tabulated, but that the nearest logarithms are 6283889 and 6283991, between which our logarithm lies.

We have then $\log 425{\cdot}00 = 2{\cdot}6283889$ ....(1),

and $\log 425{\cdot}01 = 2{\cdot}6283991$ ....(2).

Let $\log(425{\cdot}00 + x) = 2{\cdot}6283924$ ....(3).

From (1) and (2) we see that corresponding to a difference ·01 in the number there is a difference ·0000102 in the logarithm.

From (1) and (3) we see that corresponding to a difference x in the number there is a difference ·0000035 in the logarithm.

Hence we have $x : {\cdot}01 :: {\cdot}0000035 : {\cdot}0000102$.

$$\therefore\ x = \frac{35}{102} \times {\cdot}01 = \frac{{\cdot}35}{102} = {\cdot}00343 \text{ nearly.}$$

Hence the required number $= 425{\cdot}00 + {\cdot}00343 = 425{\cdot}00343$.

151. Where logarithms are taken out of the tables the labour of subtracting successive logarithms may be avoided. On reference to page 153 there is found at the extreme right a column headed *Diff.* The number 82 at the head of the figures in this column gives the difference corresponding to a difference unity in the numbers on that page.

This number 82 means ·0000082.

The rows below the 82 give the differences corresponding to ·1, ·2,.... Thus the fifth of these rows means that the difference for ·5 is ·0000041.

As an example let us find the logarithm of 52746·74.

From page 153 we have

$$\begin{aligned} \log 52746 \quad &= 4{\cdot}7221895 \\ \text{diff. for } \quad {\cdot}7 &= {\cdot}0000057 \\ \text{diff. for } \quad {\cdot}04 & \\ \left(= \frac{1}{10} \times \text{diff. for } 4\right) &= {\cdot}0000003 \\ \hline \therefore \quad \log 52746{\cdot}74 &= 4{\cdot}7221955. \end{aligned}$$

152. The proof of the Principle of Proportional Parts will not be given at this stage. It is not strictly true without certain limitations.

The numbers to which the principle is applied must contain not less than five significant figures, and then we may rely on the result as correct to seven places of decimals.

For example, we must *not* apply the principle to obtain the value of log 2·5 from the values of log 2 and log 3.

For, if we did, since these logarithms are ·30103 and ·4771213, the logarithm of 2·5 would be ·389075.

But from the tables the value of log 2·5 is found to be ·3979400.

Hence the result which we should obtain would be manifestly quite incorrect.

Tables of trigonometrical ratios.

153. In Chambers' Tables will be found tables giving the values of the trigonometrical ratios of angles between 0° and 45°, the angles increasing by differences of $1'$.

It is unnecessary to separately tabulate the ratios for angles between 45° and 90°, since the ratios of angles between 45° and 90° can be reduced to those of angles between 0° and 45°. (Art. 75.)

For example,

$$[\sin 76^\circ\, 11' = \sin(90^\circ - 13^\circ\, 49') = \cos 13^\circ\, 49',$$

and is therefore known].

Such a table is called a table of natural sines, cosines, etc. to distinguish it from the table of logarithmic sines, cosines, etc.

If we want to find the sine of an angle which contains an integral number of degrees and minutes we can obtain it from the tables. If, however, the angle contain seconds we must use the principle of proportional parts.

Ex. 1. *Given* $\sin 29^\circ\, 14' = \cdot 4883674$,

and $\sin 29^\circ\, 15' = \cdot 4886212$,

find the value of $\sin 29^\circ\, 14'\, 32''$.

By subtraction we have

$$\text{difference in the sine for } 1' = \cdot 0002538.$$

$$\therefore \text{ difference in the sine for } 32'' = \frac{32}{60} \times \cdot 0002538 = \cdot 00013536,$$

$$\begin{aligned} \therefore \sin 29^\circ\, 14'\, 32'' &= \cdot 4883674 \\ &\quad + \underline{\cdot 00013536} \\ &= \cdot 488502\underline{76}. \end{aligned}$$

Since we want our answer only to seven places of decimals we omit the last 6, and, since 76 is nearer to 80 than 70, we write

$$\sin 29^\circ\, 14'\, 32'' = \cdot 488502\underline{8}.$$

N.B. When we omit a figure in the eighth place of decimals we add 1 to the figure in the seventh place, if the omitted figure be 5 or a number greater than 5.

Ex. 2. *Given* $\cos 16^\circ\, 27' = \cdot 9590672$,

and $\cos 16^\circ\, 28' = \cdot 9589848$,

find $\cos 16^\circ\, 27'\, 47''$.

We note that, as was shewn in Art. 55, the cosine decreases as the angle increases.

Hence for an **increase** of $1'$, *i.e.* $60''$, in the angle, there is a **decrease** of $\cdot 0000824$ in the cosine.

Hence for an **increase** of $47''$ in the angle there is a **decrease** of $\frac{47}{60} \times \cdot 0000824$ in the cosine.

$$\begin{aligned} \therefore \cos 16^\circ\, 27'\, 47'' &= \cdot 9590672 - \frac{47}{60} \times \cdot 0000824 \\ &= \cdot 9590672 - \cdot 0000645 \\ &= \cdot 9590672 \\ &\quad - \underline{\cdot 0000645} \\ &= \cdot 9590027. \end{aligned}$$

154. The inverse question, to find the angle, when one of its trigonometrical ratios is given, will now be easy.

Ex. *Find the angle whose cotangent is* 1·4109325, *having given cot* $35° 19' = 1·4114799$, *and cot* $35° 20' = 1·4106098$.

Let the required angle be $35° 19' + x$,

so that $$\cot(35° 19' + x) = 1·4109325.$$

From these three equations we have

For an increase of $60''$ in the angle a decrease of ·0008701 in the cotangent,
,, ,, x'' ,, ,, ,, ,, ·0005474 ,, ,,

$$\therefore x : 60 :: 5474 : 8701, \text{ so that } x = 37·7.$$

Hence the required angle $= 35° 19' 37·7''$.

155. In working all questions involving the application of the Principle of Proportional Parts the student must be very careful to note whether the trigonometrical ratios increase or decrease as the angle increases. As a help to his memory he may observe that in the first quadrant the 3 trigonometrical ratios whose names begin with co-, *i.e.* the cosine, the cotangent, and the cosecant, all decrease as the angle increases.

Tables of logarithmic sines, cosines, etc.

156. In many kinds of trigonometric calculation, as in the solution of triangles, we often require the logarithms of trigonometrical ratios. To avoid the inconvenience of first finding the sine of any angle from the tables and then obtaining the logarithm of this sine by a second application of the tables, it has been found desirable to have separate tables giving the logarithms of the various trigonometrical functions of angles. As before it is only necessary to construct the tables for angles between 0° and 45°.

Since the sine of an angle is always less than unity, the logarithm of its sine is always negative (Art. 142).

Again, since the tangent of an angle between 0° and 45° is less than unity its logarithm is negative, whilst the logarithm of the tangent of an angle between 45° and 90° is the logarithm of a number greater than unity and is therefore positive.

157. To avoid the trouble and inconvenience of printing the proper sign to the logarithms of the trigonometric functions, the logarithms as tabulated are not the true logarithms, but the true logarithms *increased by* 10.

For example, sine $30° = \frac{1}{2}$.

Hence $$\log \sin 30° = \log \tfrac{1}{2} = -\log 2$$
$$= -\cdot 30103 = \bar{1}\cdot 69897.$$

The logarithm tabulated is therefore

$$10 + \log \sin 30°, \text{ i.e. } 9\cdot 69897.$$

Again, $$\tan 60° = \sqrt{3}.$$

Hence $$\log \tan 60° = \tfrac{1}{2} \log 3 = \tfrac{1}{2}(\cdot 4771213)$$
$$= \cdot 2385606.$$

The logarithm tabulated is therefore

$$10 + \cdot 2385606, \text{ i.e. } 10\cdot 2385606.$$

The symbol L is used to denote these "tabular logarithms," *i.e.* the logarithms as found in the English books of tables.

Thus $$L \sin 15° 25' = 10 + \log \sin 15° 25',$$
and $$L \sec 48° 23' = 10 + \log \sec 48° 23'.$$

158. If we want to find the tabular logarithm of any function of an angle, which contains an integral number of degrees and minutes, we can obtain it directly from the tables. If, however, the angle contain seconds we must use the principle of proportional parts. The method of procedure is similar to that of Art. 152. We give an example and also one of the inverse question.

Ex. 1. *Given* $L \operatorname{cosec} 32° 21' = 10{\cdot}2715733$,

and $L \operatorname{cosec} 32° 22' = 10{\cdot}2713740$,

find $L \operatorname{cosec} 32° 21' 51''$.

For an increase of $60''$ in the angle there is a *decrease* of ·0001993 in the logarithm.

Hence for an increase of $51''$ in the angle the corresponding *decrease* is $\frac{51}{60} \times {\cdot}0001993$, *i.e.* ·0001694.

Hence
$$L \operatorname{cosec} 32° 21' 51'' = 10{\cdot}2715733 - {\cdot}0001694 = 10{\cdot}2714039.$$

Ex. 2. *Find the angle such that the tabular logarithm of its tangent is* 9·4417250.

From the tables we have

$$L \tan 15° 27' = 9{\cdot}4415145,$$

and
$$L \tan 15° 28' = 9{\cdot}4420062.$$

Let
$$L \tan (15° 27' + x'') = 9{\cdot}4417250.$$

We then have
$$\frac{x''}{60''} = \frac{9{\cdot}4417250 - 9{\cdot}4415145}{9{\cdot}4420062 - 9{\cdot}4415145} = \frac{{\cdot}0002105}{{\cdot}0004917},$$

so that
$$x = 60 \times \frac{2105}{4917} = \text{nearly } 26.$$

Hence the required angle is $15° 27' 26''$.

Ex. 3. *Given* $L\sin 14^\circ 6' = 9\cdot3867040$
find $L\ cosec\ 14^\circ 6'$.

Here $\log\sin 14^\circ 6' = L\sin 14^\circ 6' - 10$

$$= -1 + \cdot3867040.$$

Now $\log\operatorname{cosec} 14^\circ 6' = \log\dfrac{1}{\sin 14^\circ 6'}$

$$= -\log\sin 14^\circ 6'$$

$$= 1 - \cdot3867040 = \cdot6132960.$$

Hence $L\operatorname{cosec} 14^\circ 6' = 10\cdot6132960.$

The error to be avoided is this; the student sometimes assumes that because

$$\log\operatorname{cosec} 14^\circ 6' = -\log\sin 14^\circ 6',$$

he may therefore assume that

$$L\operatorname{cosec} 14^\circ 6' = -L\sin 14^\circ 6'.$$

This is obviously untrue.

EXAMPLES. XXIV.

1. Given $\log 35705 = 4\cdot5527290$
and $\log 35706 = 4\cdot5527142$,
find the values of $\log 35705\cdot7$ and $\log 35\cdot70585$.

2. Given $\log 5\cdot8743 = \cdot7689487$
and $\log 587\cdot44 = 2\cdot7689561$,
find the values of $\log 58743\cdot57$ and $\log \cdot00587432$.

3. Given $\log 47847 = 4\cdot6798547$
and $\log 47848 = 4\cdot6798638$,
find the numbers whose logarithms are respectively

$$2\cdot6798593 \text{ and } \bar{3}\cdot6798617.$$

4. Given $\log 258{\cdot}36 = 2{\cdot}4122253$

and $\log 2{\cdot}5837 = {\cdot}4122421$

find the numbers whose logarithms are

$${\cdot}4122378 \text{ and } \bar{2}{\cdot}4122287.$$

5. From the table on page 153 find the logarithms of

(1) 52538·97, (2) 527·286, (3) ·000529673,

and the numbers whose logarithms are

(4) 3·7221098, (5) $\bar{2}{\cdot}7240075$ and (6) ·7210386.

6. Given $\sin 43^\circ 23' = {\cdot}6868761$

and $\sin 43^\circ 24' = {\cdot}6870875,$

find the value of $\sin 43^\circ 23' 47''$

7. Find also the angle whose sine is ·6870349.

8. Given $\cos 32^\circ 16' = {\cdot}8455726$

and $\cos 32^\circ 17' = {\cdot}8454172,$

find the values of $\cos 32^\circ 16' 24''$ and of $\cos 32^\circ 16' 47''$.

9. Find also the angles whose cosines are

$${\cdot}8454832 \text{ and } {\cdot}8455176.$$

10. Given $\tan 76^\circ 21' = 4{\cdot}1177784$

and $\tan 76^\circ 22' = 4{\cdot}1230079,$

find the values of $\tan 76^\circ 21' 29''$ and $\tan 76^\circ 21' 47''$.

11. Given $\text{cosec}\, 13^\circ 8' = 4{\cdot}4010616$

and $\text{cosec}\, 13^\circ 9' = 4{\cdot}3955817,$

find the values of $\text{cosec}\, 13^\circ 8' 19''$ and $\text{cosec}\, 13^\circ 8' 37''$.

12. Find also the angle whose cosecant is 4·396789.

13. Given $L \cos 34^\circ 44' = 9{\cdot}9147729$

and $L \cos 34^\circ 45' = 9{\cdot}9146852,$

find the value of $L \cos 34^\circ 44' 27''$.

14. Find also the angle θ, where

$$L \cos \theta = 9{\cdot}9147328.$$

15. Given $L \cot 71^\circ 27' = 9{\cdot}5257779$

and $L \cot 71^\circ 28' = 9{\cdot}5253589$,

find the value of $L \cot 71^\circ 27' 47$

and solve the equation $L \cot \theta = 9{\cdot}5254782$.

16. Given $L \sec 18^\circ 27' = 10{\cdot}0229168$

and $L \sec 18^\circ 28' = 10{\cdot}0229590$,

find the value of $L \sec 18^\circ 27' 35''$.

17. Find also the angle whose L sec is $10{\cdot}0229285$.

18. Find in degrees, minutes, and seconds the angle whose sine is ·6, given that

$$\log 6 = 7781513,\ L \sin 36^\circ 52' = 9{\cdot}7781186$$

and $L \sin 36^\circ 53' = 9{\cdot}7782870$.

159. On the next page is printed a specimen page taken from Chambers' tables. It gives the tabular logarithms of the ratios of angles between 32° and 33° and also between 57° and 58°.

The first column gives the L sine for each minute between 32° and 33°.

In the second column under the word Diff. is found the number 2021. This means that ·0002021 is the difference between $L \sin 32^\circ 0'$ and $L \sin 32^\circ 1'$; this may be verified by subtracting 9·7242097 from 9·7244118. It will also be noted that the figures 2021 are printed halfway between the numbers 9·7242097 and 9·7244118, thus clearly shewing between what numbers it is the difference.

This same column of Differences also applies to the column on its right-hand side which is headed Cosec.

Similarly the fifth column, which is also headed Diff., may be used with the two columns on the right and left of it.

32 Deg.

′	Sine	Diff.	Cosec.	Tang.	Diff.	Cotang.	Secant	Diff.	Cosine	′
0	9·7242097		10·2757903	9·7957892		10·2042108	10·0715795		9·9284205	60
1	9·7244118	2021	10·2755882	9·7960703	2811	10·2039297	10·0716585	790	9·9283415	59
2	9·7246138	2020	10·2753862	9·7963513	2810	10·2036487	10·0717375	790	9·9282625	58
3	9·7248156	2018	10·2751844	9·7966322	2809	10·2033678	10·0718166	791	9·9281834	57
4	9·7250174	2018	10·2749826	9·7969130	2808	10·2030870	10·0718957	791	9·9281043	56
5	9·7252189	2015	10·2747811	9·7971938	2808	10·2028062	10·0719749	792	9·9280251	55
6	9·7254204	2015	10·2745796	9·7974745	2807	10·2025255	10·0720541	792	9·9279459	54
7	9·7256217	2013	10·2743783	9·7977551	2806	10·2022449	10·0721334	793	9·9278666	53
8	9·7258229	2012	10·2741771	9·7980356	2805	10·2019644	10·0722127	793	9·9277873	52
9	9·7260240	2011	10·2739760	9·7983160	2804	10·2016840	10·0722921	794	9·9277079	51
10	9·7262249	2009	10·2737751	9·7985964	2804	10·2014036	10·0723715	794	9·9276285	50
11	9·7264257	2008	10·2735743	9·7988767	2803	10·2011233	10·0724510	795	9·9275490	49
12	9·7266264	2007	10·2733736	9·7991569	2802	10·2008431	10·0725305	795	9·9274695	48
13	9·7268269	2005	10·2731731	9·7994370	2801	10·2005630	10·0726101	796	9·9273899	47
14	9·7270273	2004	10·2729727	9·7997170	2800	10·2002830	10·0726897	796	9·9273103	46
15	9·7272276	2003	10·2727724	9·7999970	2800	10·2000030	10·0727694	797	9·9272306	45
16	9·7274278	2002	10·2725722	9·8002769	2799	10·1997231	10·0728491	797	9·9271509	44
17	9·7276278	2000	10·2723722	9·8005567	2798	10·1994433	10·0729289	798	9·9270711	43
18	9·7278277	1999	10·2721723	9·8008365	2798	10·1991635	10·0730087	798	9·9269913	42
19	9·7280275	1998	10·2719725	9·8011161	2796	10·1988839	10·0730886	799	9·9269114	41
20	9·7282271	1996	10·2717729	9·8013957	2796	10·1986043	10·0731686	800	9·9268314	40
21	9·7284267	1996	10·2715733	9·8016752	2795	10·1983248	10·0732486	800	9·9267514	39
22	9·7286260	1993	10·2713740	9·8019546	2794	10·1980454	10·0733286	800	9·9266714	38
23	9·7288253	1993	10·2711747	9·8022340	2794	10·1977660	10·0734087	801	9·9265913	37
24	9·7290244	1991	10·2709756	9·8025133	2793	10·1974867	10·0734888	801	9·9265112	36
25	9·7292234	1990	10·2707766	9·8027925	2792	10·1972075	10·0735690	802	9·9264310	35
26	9·7294223	1989	10·2705777	9·8030716	2791	10·1969284	10·0736493	803	9·9263507	34
27	9·7296211	1988	10·2703789	9.8033506	2790	10·1966494	10·0737296	803	9·9262704	33
28	9·7298197	1986	10·2701803	9·8036296	2790	10·1963704	10·0738099	803	9·9261901	32
29	9·7300182	1985	10·2699818	9·8039085	2789	10·1960915	10·0738904	805	9·9261096	31
30	9·7302165	1983	10·2697835	9·8041873	2788	10·1958127	10·0739708	804	9·9260292	30
31	9·7304148	1983	10·2695852	9·8044661	2788	10·1955339	10·0740513	805	9·9259487	29
32	9·7306129	1981	10·2693871	9·8047447	2786	10·1952553	10·0741319	806	9·9258681	28
33	9·7308109	1980	10·2691891	9·8050233	2786	10·1949767	10·0742125	806	9·9257075	27
34	9·7310087	1978	10·2689913	9·8053019	2786	10·1946981	10·0742931	806	9·9257069	26
35	9·7313064	1977	10·2687936	9·8055803	2784	10·1944197	10·0743739	808	9·9256261	25
36	9·7314040	1976	10·2685960	9·8058587	2784	10·1941413	10·0744546	807	9·9255454	24
37	9·7316015	1975	10·2683985	9·8061370	2783	10·1938630	10·0745354	808	9·9254646	23
38	9·7317989	1974	10·2682011	9·8064152	2782	10·1935848	10·0746166	809	9·9253837	22
39	9·7319961	1972	10·2680039	9·8066933	2781	10·1933067	10·0746972	809	9·9253028	21
40	9·7321932	1971	10·2678068	9·8069714	2781	10·1930286	10·0747782	810	9·9252218	20
41	9·7323902	1970	10·2676098	9·8072494	2780	10·1927506	10·0748592	810	9·9251408	19
42	9·7325870	1968	10·2674130	9·8075273	2779	10·1924727	10·0749403	811	9·9250597	18
43	9·7327837	1967	10·2672163	9·8078052	2779	10·1921948	10·0750214	811	9·9249786	17
44	9·7329803	1966	10·2670197	9·8080829	2777	10·1919171	10·0751026	812	9·9248974	16
45	9·7331768	1965	10·2668232	9·8083606	2777	10·1916394	10·0751839	813	9·9248161	15
46	9·7333731	1963	10·2666269	9·8086383	2777	10·1913617	10·0752651	812	9·9247349	14
47	9·7335693	1962	10·2664307	9·8089158	2775	10·1910842	10·0753465	814	9·9246535	13
48	9·7337654	1961	10·2662346	9·8091933	2775	10·1908067	10·0754279	814	9·9245721	12
49	9·7339614	1960	10·2660386	9·8094707	2774	10·1905293	10·0755093	814	9·9244907	11
50	9·7341572	1958	10·2658428	9·8097480	2773	10·1902520	10·0755908	815	9·9244092	10
51	9·7343529	1957	10·2656471	9·8100253	2773	10·1899747	10·0756723	815	9·9243277	9
52	9·7345485	1956	10·2654515	9·8103025	2772	10·1896975	10·0757539	816	9·9242461	8
53	9·7347440	1955	10·2652560	9·8105796	2771	10·1894204	10·0758356	817	9·9241644	7
54	9·7349393	1953	10·2650607	9·8108566	2770	10·1891434	10·0759173	817	9·9240827	6
55	9·7351345	1952	10·2648655	9·8111336	2770	10·1888664	10·0759990	817	9·9240010	5
56	9·7353296	1951	10·2646704	9·8114105	2769	10·1885895	10·0760809	819	9·9239191	4
57	9·7355246	1950	10·2644754	9·8116873	2768	10·1883127	10·0761627	818	9·9238373	3
58	9·7357195	1949	10·2642805	9·8119641	2768	10·1880359	10·0762446	819	9·9237554	2
59	9·7359142	1947	10·2640858	9·8122408	2767	10·1877592	10·0763266	820	9·9236734	1
60	9·7361088	1946	10·2638912	9·8125174	2766	10·1874826	10·0764086	820	9·9235914	0
′	Cosine	Diff.	Secant	Cotang.	Diff.	Tang.	Cosec.	Diff.	Sine	′

57 Deg.

160. There is one point to be noticed in using the columns headed Diff. It has been pointed out that 2021 (at the top of the second column) means ·0002021. Now the 790 (at the top of the eighth column) means *not* ·000790, but ·0000790. The rule is this; the right-hand figure of the Diff. must be placed in the seventh place of decimals and the requisite number of cyphers prefixed. Thus

Diff. =	9	means that the difference is	·0000009,
Diff. =	74	" " "	·0000074,
Diff. =	735	" " "	·0000735,
Diff. =	2021	" " "	·0002021,
whilst Diff. =	12348	" " "	·0012348.

161. Page 170 also gives the tabular logs. of ratios between 57° and 58°. Suppose we wanted $L\tan 57^\circ 20'$. We now start with the line at the *bottom* of the page and run our eye *up* the column which has Tang. at its foot. We go up this column until we arrive at the number which is on the same level as the number 20 in the extreme *right-hand* column. This number we find to be 10·1930286, which is therefore the value of

$$L\tan 57^\circ 20'.$$

EXAMPLES. XXV.

1. Find θ given that $\cos\theta = \cdot 9725382$,

$$\cos 13^\circ 27' = \cdot 9725733, \text{ diff. for } 1' = 677.$$

2. Find the angle whose sine is $\frac{3}{8}$, given

$$\sin 22^\circ 1' = \cdot 3748763, \text{ diff. for } 1' = 2696.$$

3. Given $\text{cosec}\, 65^\circ 24' = 1{\cdot}0998243$,

diff. for $1' = 1464$,

find the value of $\text{cosec}\, 65^\circ 24' 37''$

and the angle whose cosec is $1{\cdot}0997938$.

4. Given $L \tan 22^\circ 37' = 9{\cdot}6197205$,

diff. for $1' = 3557$,

find the value of $L \tan 22^\circ 37' 22''$

and the angle whose L tan is $9{\cdot}6195283$.

5. Find the angle whose L cos is $9{\cdot}993$, given

$$L \cos 10^\circ 15' = 9{\cdot}9930131, \text{ diff. for } 1' = 229.$$

6. Find the angle whose L sec is $10{\cdot}15$, given

$$L \sec 44^\circ 55' = 10{\cdot}1498843, \text{ diff. for } 1' = 1260.$$

7. From the table on page 170 find the values of

(1) $L \sin 32^\circ 18' 23''$, (2) $L \cos 32^\circ 16' 49''$,

(3) $L \cot 32^\circ 29' 43''$, (4) $L \sec 32^\circ 52' 27''$,

(5) $L \tan 57^\circ 45' 28''$, (6) $L\, \text{cosec}\, 57^\circ 48' 21''$,

and (7) $L \cos 57^\circ 58' 29''$.

8. With the help of the same page solve the equations

(1) $L \tan \theta = 10{\cdot}1959261$, (2) $L\, \text{cosec}\, \theta = 10{\cdot}0738125$,

(3) $L \cos \theta = 9{\cdot}9259283$, and (4) $L \sin \theta = 9{\cdot}9241352$.

9. Take out of the tables $L \tan 16^\circ 6' 23''$ and calculate the value of the square root of the tangent.

10. Change into a form more convenient for logarithmic computation (*i.e.* express in the form of products of quantities) the quantities

(1) $1 + \tan x \tan y$, (2) $1 - \tan x \tan y$,

(3) $\cot x + \tan y$, (4) $\cot x - \tan y$,

(5) $\dfrac{1 - \cos 2x}{1 + \cos 2x}$, and (6) $\dfrac{\tan x + \tan y}{\cot x + \cot y}$.

CHAPTER XII.

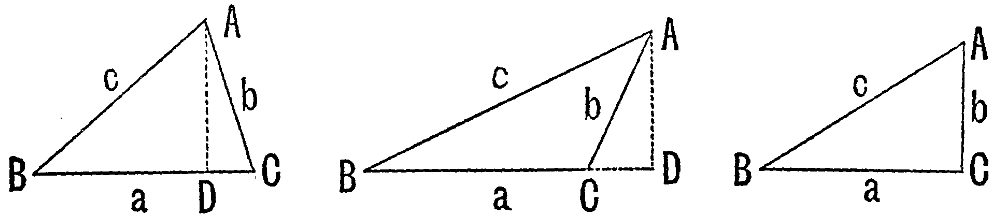
RELATIONS BETWEEN THE SIDES AND THE TRIGONOMETRICAL RATIOS OF THE ANGLES OF ANY TRIANGLE.

162. In any triangle ABC, the side BC, opposite to the angle A, is denoted by a; the sides CA and AB, opposite to the angles B and C respectively, are denoted by b and c.

163. Theorem. *In any triangle ABC,*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

i.e. the sines of the angles are proportional to the opposite sides.



Draw AD perpendicular to the opposite side meeting it, produced if necessary, in the point D.

In the triangle ABD, we have

$$\frac{AD}{AB} = \sin B, \text{ so that } AD = c \sin B.$$

In the triangle ACD, we have

$$\frac{AD}{AC} = \sin C, \text{ so that } AD = b \sin C.$$

[If the angle C be obtuse, as in the second figure, we have

$$\frac{AD}{b} = \sin ACD = \sin (180° - C) = \sin C \qquad \text{(Art. 72)},$$

so that $$AD = b \sin C.]$$

Equating these two values of AD, we have

$$c \sin B = b \sin C,$$

i.e. $$\frac{\sin B}{b} = \frac{\sin C}{c}.$$

In a similar manner by drawing a perpendicular from B upon CA we have

$$\frac{\sin C}{c} = \frac{\sin A}{a}.$$

If one of the angles, C, be a right angle as in the third figure we have $\sin C = 1$,

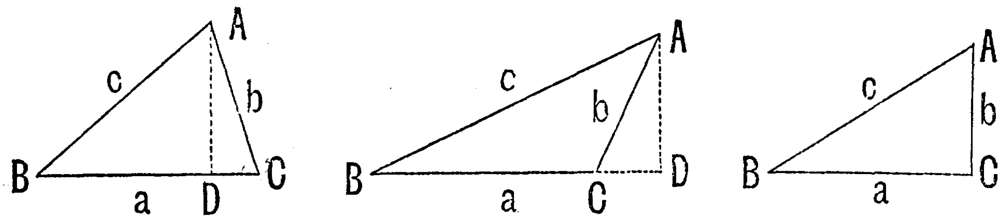
$$\sin A = \frac{a}{c}, \text{ and } \sin B = \frac{b}{c}.$$

Hence $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{1}{c} = \frac{\sin C}{c}.$$

We therefore have, in all cases,

$$\frac{\mathbf{sin\ A}}{\mathbf{a}} = \frac{\mathbf{sin\ B}}{\mathbf{b}} = \frac{\mathbf{sin\ C}}{\mathbf{c}}.$$

164. *In any triangle to find the cosine of an angle in terms of the sides.*



Let ABC be the triangle and let the perpendicular from A on BC meet it, produced if necessary, in the point D.

First, let the angle C be **acute**, as in the left-hand figure.

By Euc. II. 13, we have

$$AB^2 = BC^2 + CA^2 - 2BC \,.\, CD \text{..........(i).}$$

But $$\frac{CD}{CA} = \cos C, \text{ so that } CD = b \cos c.$$

Hence (i) becomes

$$c^2 = a^2 + b^2 - 2a \,.\, b \cos C,$$

i.e. $$2ab \cos C = a^2 + b^2 - c^2,$$

i.e. $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Secondly, let the angle C be **obtuse**, as in the right-hand figure.

By Euc. II. 12, we have

$$AB^2 = BC^2 + CA^2 + 2BC \,.\, CD \text{ ....(ii).}$$

But $$\frac{CD}{CA} = \cos ACD = \cos (180^\circ - C) = -\cos C,$$

(Art. 72)

so that $$CD = -b \cos C.$$

Hence (ii) becomes

$$c^2 = a^2 + b^2 + 2a(-b\cos C) = a^2 + b^2 - 2ab\cos C,$$

so that, as in the first case, we have

$$\mathbf{cos\ C} = \frac{\mathbf{a^2 + b^2 - c^2}}{\mathbf{2ab}}.$$

In a similar manner it may be shewn that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

and $$\cos B = \frac{c^2 + a^2 - b^2}{2ca}.$$

If one of the angles, C, be a right angle, the above formula would give $c^2 = a^2 + b^2$, so that $\cos C = 0$. This is correct, since C is a right angle.

The above formula is therefore true for all values of C.

Ex. If $a = 15$, $b = 36$, and $c = 39$,

then $$\cos A = \frac{36^2 + 39^2 - 15^2}{2 \times 36 \times 39} = \frac{3^2(12^2 + 13^2 - 5^2)}{2 \times 3^2 \times 12 \times 13} = \frac{288}{24 \times 13} = \frac{12}{13}.$$

165. *To find the sines of half the angles in terms of the sides.*

In any triangle we have, by Art. 164,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

By Art. 109, we have

$$\cos A = 1 - 2\sin^2\frac{A}{2}.$$

Hence $$2\sin^2\frac{A}{2} = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b^2 + c^2 - 2bc)}{2bc} = \frac{a^2 - (b-c)^2}{2bc}$$

$$= \frac{[a + (b-c)][a - (b-c)]}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc} \quad \ldots(1).$$

Let $2s$ stand for $a+b+c$, so that s is equal to half the sum of the sides of the triangle, *i.e.* s is equal to the semi-perimeter of the triangle.

We then have

$$a+b-c=a+b+c-2c=2s-2c=2(s-c),$$

and $$a-b+c=a+b+c-2b=2s-2b=2(s-b).$$

The relation (1) therefore becomes

$$2\sin^2\frac{A}{2}=\frac{2(s-c)\times 2(s-b)}{2bc}=2\frac{(s-b)(s-c)}{bc}.$$

$$\therefore \quad \mathbf{\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}} \quad \text{.........(2)}.$$

Similarly,

$$\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}, \text{ and } \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}.$$

166. *To find the cosines of half the angles in terms of the sides.*

By Art. 109, we have

$$\cos A=2\cos^2\frac{A}{2}-1.$$

Hence $$2\cos^2\frac{A}{2}=1+\cos A=1+\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{2bc+b^2+c^2-a^2}{2bc}=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{[(b+c)+a][(b+c)-a]}{2bc}=\frac{(a+b+c)(b+c-a)}{2bc} \quad \text{...(1)}.$$

Now $$b+c-a=a+b+c-2a=2s-2a=2(s-a),$$

so that (1) becomes

$$2\cos^2\frac{A}{2}=\frac{2s\times 2(s-a)}{2bc}=2\frac{s(s-a)}{bc}.$$

$$\therefore\ \mathbf{\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}}\ \ldots\ldots\ldots\ldots\ldots(2).$$

Similarly,

$$\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}},\ \text{and}\ \cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}.$$

167. *To find the tangents of half the angles in terms of the sides.*

Since $$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}},$$

we have, by (2) of Arts. 165 and 166,

$$\mathbf{\tan\frac{A}{2}}=\sqrt{\frac{(s-b)(s-c)}{bc}}\div\sqrt{\frac{s(s-a)}{bc}}=\mathbf{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}$$

Similarly,

$$\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}},\ \text{and}\ \tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Since A is always $< 180°$, $\frac{A}{2}$ is always $< 90°$.

The sine, cosine, and tangent of $\frac{A}{2}$ are therefore always positive (Art. 52).

The positive sign must therefore always be prefixed to the radical sign in the formulae of this and the last two articles.

168. **Ex.** If $a=13$, $b=14$ and $c=15$

then $$s=\frac{13+14+15}{2}=21,\ s-a=8,\ s-b=7,$$

and $$s-c=6.$$

Hence $$\sin\frac{A}{2}=\sqrt{\frac{7\times 6}{14\times 15}}=\frac{1}{\sqrt{5}}=\frac{1}{5}\sqrt{5},$$

$$\sin\frac{B}{2}=\sqrt{\frac{6\times 8}{15\times 13}}=\frac{4}{\sqrt{65}}=\frac{4}{65}\sqrt{65},$$

$$\cos\frac{C}{2}=\sqrt{\frac{21\times 6}{13\times 14}}=\frac{3}{\sqrt{13}}=\frac{3}{13}\sqrt{13},$$

and $$\tan\frac{B}{2}=\sqrt{\frac{6\times 8}{21\times 7}}=\frac{4}{7}.$$

169. *To express the sine of any angle of a triangle in terms of the sides.*

We have, by Art. 109,

$$\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}.$$

But, by the previous articles,

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},\text{ and }\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}.$$

Hence

$$\sin A=2\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-a)}{bc}}.$$

$$\therefore\ \mathbf{\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}}.$$

EXAMPLES. XXVI.

In a triangle

1. Given $a=25$, $b=52$ and $c=63$,

find $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, and $\tan\frac{C}{2}$.

2. Given $a=125$, $b=123$ and $c=62$,
find the sines of half the angles and the sines of the angles.

3. Given $a=18$, $b=24$ and $c=30$,
find $\sin A$, $\sin B$, and $\sin C$.

4. Given $a=35$, $b=84$ and $c=91$,
find $\tan A$, $\tan B$, and $\tan C$.

5. Given $a=13$, $b=14$ and $c=15$,
find the sines of the angles.

6. Given $a=287$, $b=816$ and $c=865$,
find the values of $\tan\frac{A}{2}$ and $\tan A$.

7. Given $a=\sqrt{3}$, $b=\sqrt{2}$ and $c=\frac{\sqrt{6}+\sqrt{2}}{2}$,
find the angles.

170. *In any triangle to prove that*

$$a = b\cos C + c\cos B.$$

Take the figures of Art. 164.

In the first case, we have

$$\frac{BD}{BA} = \cos B, \text{ so that } BD = c\cos B,$$

and
$$\frac{CD}{CA} = \cos C, \text{ so that } CD = b\cos C.$$

Hence $a = BC = BD + DC = c\cos B + b\cos C$.

In the second case, we have

$$\frac{BD}{BA} = \cos B, \text{ so that } BD = c\cos B,$$

and
$$\frac{CD}{CA} = \cos ACD = \cos(180^\circ - C)$$

$$= -\cos C \text{ (Art. 72)},$$

so that
$$CD = -b\cos C.$$

Hence, in this case,

$$a = BC = BD - CD = c\cos B - (-b\cos C),$$

so that in each case

$$\mathbf{a = b\cos C + c\cos B.}$$

Similarly, $\qquad b = c\cos A + a\cos C,$

and $\qquad c = a\cos B + b\cos A.$

171. *In any triangle to prove that*

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}.$$

In any triangle, we have

$$\frac{b}{c} = \frac{\sin B}{\sin C}.$$

$$\therefore \quad \frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$= \frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}} = \frac{\tan\frac{B-C}{2}}{\tan\left(90^\circ - \frac{A}{2}\right)}$$

$$= \frac{\tan\frac{B-C}{2}}{\cot\frac{A}{2}} \text{ (Art. 69).}$$

Hence $\qquad \mathbf{\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}.}$

172. Ex. *From the formulae of Art.* 164 *deduce those of Art.* 170 *and vice versâ.*

The first and third formulae of Art. 164 give

$$b\cos C+c\cos B=\frac{a^2+b^2-c^2}{2a}+\frac{c^2+a^2-b^2}{2a}$$

$$=\frac{2a^2}{2a}=a,$$

so that $$a=b\cos C+c\cos B.$$

Similarly, the other formulae of Art. 170 may be obtained.

Again, the three formulae of Art. 170 give

$$a=b\cos C+c\cos B,$$

$$b=c\cos A+a\cos C,$$

and $$c=a\cos B+b\cos A.$$

Multiplying these in succession by a, b, and $-c$ we have, by addition,

$$a^2+b^2-c^2=a\,(b\cos C+c\cos B)+b\,(c\cos A+a\cos C)-c\,(a\cos B+b\cos A)$$

$$=2ab\cos C.$$

$$\therefore\ \cos C=\frac{a^2+b^2-c^2}{2ab}.$$

Similarly, the other formulae of Art. 162 may be found.

173. The student will often meet with identities, which he is required to prove, which involve both the sides and the angles of a triangle.

It is, in general, desirable in the identity to substitute for the sides in terms of the angles, or to substitute for the ratios of the angles in terms of the sides.

Ex. 1. *Prove that* $a\cos\frac{B-C}{2}=(b+c)\sin\frac{A}{2}$.

By Art. 163 we have

$$\frac{b+c}{a}=\frac{\sin B+\sin C}{\sin A}=\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$=\frac{\cos\frac{A}{2}\cos\frac{B-C}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}.$$

$$\therefore\ (b+c)\sin\frac{A}{2}=a\cos\frac{B-C}{2}.$$

Ex. 2. *In a triangle prove that*

$$(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0.$$

By Art. 163 we have

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k \text{ (say)}.$$

Hence the given expression

$$=(b^2-c^2)\frac{\cos A}{ak}+(c^2-a^2)\frac{\cos B}{bk}+(a^2-b^2)\frac{\cos C}{ck}$$

$$=\frac{1}{k}\left[(b^2-c^2)\frac{b^2+c^2-a^2}{2abc}+(c^2-a^2)\frac{c^2+a^2-b^2}{2abc}+(a^2-b^2)\frac{a^2+b^2-c^2}{2abc}\right]$$

$$=\frac{1}{2abck}[b^4-c^4-a^2(b^2-c^2)+c^4-a^4-b^2(c^2-a^2)+a^4-b^4-c^2(a^2-b^2)]$$

$$=0.$$

Ex. 3. *In a triangle prove that*

$$(a+b+c)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=2c\cot\frac{C}{2}.$$

The left-hand member

$$=2s\left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}+\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}\right], \quad \text{by Art. 167,}$$

$$=2s\sqrt{\frac{s-c}{s}}\left[\sqrt{\frac{s-b}{s-a}}+\sqrt{\frac{s-a}{s-b}}\right]=2\sqrt{s(s-c)}\left[\frac{s-b+s-a}{\sqrt{(s-a)(s-b)}}\right]$$

$$=\frac{2\sqrt{s(s-c)}\,.\,c}{\sqrt{(s-a)(s-b)}}, \quad \text{since } 2s=a+b+c,$$

$$=2c\cot\frac{C}{2}.$$

This identity may also be proved by substituting for the sides. We have, by Art. 163,

$$\frac{a+b+c}{c}=\frac{\sin A+\sin B+\sin C}{\sin C}$$

$$=\frac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}, \text{ as in Art. 127, } =\frac{2\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}}.$$

Also $$\frac{2\cot\frac{C}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}=\frac{2\cos\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}\left[\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}\right]}$$

$$=\frac{2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{C}{2}\sin\frac{A+B}{2}}=\frac{2\cos\frac{A}{2}\cos\frac{B}{2}}{\sin\frac{C}{2}}. \qquad \text{(Art. 69.)}$$

We have therefore

$$\frac{a+b+c}{c}=\frac{2\cot\frac{C}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}},$$

so that $$(a+b+c)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=2c\cot\frac{C}{2}.$$

Ex. 4. *If the sides of a triangle be in Arithmetical Progression, prove that so also are the cotangents of half the angles.*

We have given that $$a+c=2b\,\ldots\ldots\ldots\ldots(1),$$

and we have to prove that

$$\cot\frac{A}{2}+\cot\frac{C}{2}=2\cot\frac{B}{2}\,\ldots\ldots\ldots\ldots(2).$$

Now (2) is true if

$$\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=2\sqrt{\frac{s(s-b)}{(s-c)(s-a)}},$$

or, by multiplying both sides by

$$\sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

if $$(s-a)+(s-c)=2(s-b),$$

i.e. if $$2s-(a+c)=2s-2b,$$

i.e. if $a+c=2b$, which is relation (1).

Hence if relation (1) be true, so also is relation (2).

EXAMPLES. XXVII.

In any triangle ABC, prove that

1. $\sin\frac{B-C}{2}=\frac{b-c}{a}\cos\frac{A}{2}$.

2. $a(\cos B+\cos C)=2(b+c)\sin^2\frac{A}{2}$.

3. $a(\cos C-\cos B)=2(b-c)\cos^2\frac{A}{2}$.

4. $\frac{a+b}{a-b}=\tan\frac{A+B}{2}\cot\frac{A-B}{2}$.

5. $(b+c-a)\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)=2a\cot\frac{A}{2}$.

6. $a^2+b^2+c^2=2(bc\cos A+ca\cos B+ab\cos C)$.

7. $(a^2-b^2+c^2)\tan B=(a^2+b^2-c^2)\tan C$.

8. $c^2=(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2}$.

9. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$.

10. $\frac{a\sin(B-C)}{b^2-c^2}=\frac{b\sin(C-A)}{c^2-a^2}=\frac{c\sin(A-B)}{a^2-b^2}$.

11. $a\sin\frac{A}{2}\sin\frac{B-C}{2}+b\sin\frac{B}{2}\sin\frac{C-A}{2}+c\sin\frac{C}{2}\sin\frac{A-B}{2}=0$.

12. $a^2(\cos^2 B-\cos^2 C)+b^2(\cos^2 C-\cos^2 A)+c^2(\cos^2 A-\cos^2 B)=0$.

13. $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$.

14. $\frac{(a+b+c)^2}{a^2+b^2+c^2}=\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}$.

15. $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc$.

16. In a triangle whose sides are 3, 4, and $\sqrt{38}$ feet respectively, prove that the largest angle is greater than 120°.

17. The sides of a right-angled triangle are 21 and 28 feet; find the length of the perpendicular drawn to the hypothenuse from the right angle.

18. If in any triangle the angles be to one another as 1 : 2 : 3, prove that the corresponding sides are as 1 : $\sqrt{3}$: 2.

19. In any triangle if

$$\tan\frac{A}{2}=\frac{5}{6},\ \tan\frac{B}{2}=\frac{20}{37},$$

find $\tan\frac{C}{2}$, and prove that in this triangle $a+c=2b$.

20. In an isosceles right-angled triangle a straight line is drawn from the middle point of one of the equal sides to the opposite angle. Shew that it divides the angle into parts whose cotangents are 2 and 3.

21. The perpendicular AD to the base of a triangle ABC divides it into segments such that BD, CD and AD are in the ratio of 2, 3 and 6; prove that the vertical angle of the triangle is 45°.

22. A ring, ten inches in diameter, is suspended from a point one foot above its centre by 6 equal strings attached to its circumference at equal intervals. Find the cosine of the angle between consecutive strings.

23. If a^2, b^2 and c^2 be in A.P., prove that $\cot A$, $\cot B$ and $\cot C$ are in A.P. also.

24. If a, b and c be in A.P., prove that $\cos A\cot\frac{A}{2}$, $\cos B\cot\frac{B}{2}$ and $\cos C\cot\frac{C}{2}$ are in A.P.

25. If a, b and c are in H.P. prove that $\sin^2\frac{A}{2}$, $\sin^2\frac{B}{2}$ and $\sin^2\frac{C}{2}$ are also in H.P.

26. The sides of a triangle are in A.P. and the greatest and least angles are θ and ϕ; prove that

$$4(1-\cos\theta)(1-\cos\phi)=\cos\theta+\cos\phi.$$

27. The sides of a triangle are in A.P. and the greatest angle exceeds the least by 90°; prove that the sides are proportional to $\sqrt{7}+1$, $\sqrt{7}$ and $\sqrt{7}-1$.

28. If $C=60°$, then prove that

$$\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}.$$

29. In any triangle ABC if D be any point of the base BC, such that $BD : DC :: m : n$, prove that

$$(m+n)\cot ADC=n\cot B-m\cot C,$$

and $$(m+n)^2 AD^2=(m+n)(mb^2+nc^2)-mna^2.$$

30. If in a triangle the bisector of the side c be perpendicular to the side b, prove that

$$2\tan A+\tan C=0.$$

31. In any triangle prove that, if θ be any angle, then

$$b\cos\theta=c\cos(A-\theta)+a\cos(C+\theta).$$

32. If p and q be the perpendiculars from the angular points A and B on any line passing through the vertex C of the triangle ABC, then prove that

$$a^2p^2+b^2q^2-2abpq\cos C=a^2b^2\sin^2 C.$$

33. In the triangle ABC, lines OA, OB, and OC are drawn so that the angles OAB, OBC, and OCA are each equal to ω; prove that

$$\cot\omega=\cot A+\cot B+\cot C,$$

and $$\text{cosec}^2\,\omega=\text{cosec}^2 A+\text{cosec}^2 B+\text{cosec}^2 C.$$

CHAPTER XIII.

SOLUTION OF TRIANGLES.

174. In any triangle the 3 sides and the 3 angles are often called the elements of the triangle. When any 3 elements of the triangle are given, provided they be not the 3 angles, the triangle is in general completely known, *i.e.* its other angles and sides can be calculated. When the 3 angles are given, only the ratios of the lengths of the sides can be found, so that the triangle is given in *shape* only and not in *size*. When 3 elements of a triangle are given the process of calculating its other 3 elements is called the **Solution of the Triangle.**

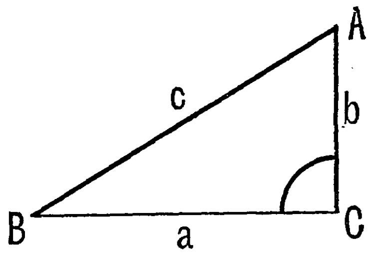
We shall first discuss the solution of right-angled triangles, *i.e.* triangles which have one angle given equal to a right angle.

The next four articles refer to such triangles, and C denotes the right angle.

175. *Case I. Given the hypothenuse and one side, to solve the triangle.*

Let b be the given side and c the given hypothenuse.

The angle B is given by the relation



$$\sin B = \frac{b}{c}.$$

$$\therefore\ L \sin B = 10 + \log b - \log c.$$

Since b and c are known we thus have $L \sin B$ and therefore B.

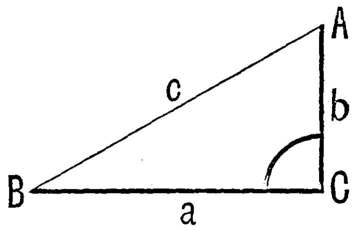
The angle A $(= 90° - B)$ is then known.

The side a is obtained from either of the relations

$$\cos B = \frac{a}{c},\quad \tan B = \frac{b}{a},\ \text{or}\ a = \sqrt{(c-b)(c+b)}.$$

176. *Case II. Given the two sides a and b, to solve the triangle.*

Here B is given by



$$\tan B = \frac{b}{a},$$

so that

$$L \tan B = 10 + \log b - \log a.$$

Hence $L \tan B$, and therefore B, is known.

The angle A $(= 90° - B)$ is then known.

The hypothenuse c is given by the relation $c = \sqrt{a^2 + b^2}$.

This relation is not however very suitable for logarithmic calculation, and c is best given by

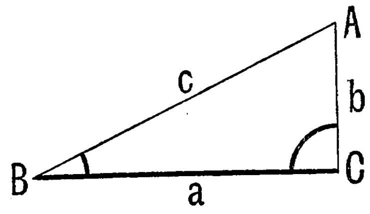
$$\sin B = \frac{b}{c},\ \textit{i.e.}\ c = \frac{b}{\sin B}.$$

$$\therefore\ \log c = \log b - \log \sin B$$

$$= 10 + \log b - L \sin B.$$

Hence c is obtained.

177. *Case III. Given an angle B and one of the sides a, to solve the triangle.*



Here $A\ (= 90^\circ - B)$ is known.

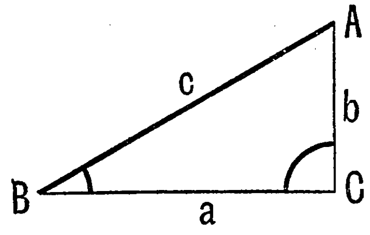
The side b is found from the relation

$$\frac{b}{a} = \tan B,$$

and c from the relation

$$\frac{a}{c} = \cos B.$$

178. *Case IV. Given an angle B and the hypothenuse c, to solve the triangle.*



Here A is known and a and b are obtained from the relations

$$\frac{a}{c} = \cos B, \text{ and } \frac{b}{c} = \sin B.$$

EXAMPLES. XXVIII.

1. In a right-angled triangle ABC, where C is the right angle, if $a=50$ and $B=75^\circ$, find the sides. $(\tan 75^\circ = 2+\sqrt{3}.)$

2. Solve the triangle of which two sides are equal to 10 and 50 feet and of which the included angle is 90°; given that $\log 20 = 1{\cdot}30103$, and

$$L \tan 26^\circ\, 33' = 9{\cdot}6986847, \text{ diff. for } 1' = 3160.$$

3. The length of the perpendicular from one angle of a triangle upon the base is 3 inches and the lengths of the sides containing this angle are 4 and 5 inches. Find the angles, having given

$$\log 2 = {\cdot}30103,\ \log 3 = {\cdot}4771213,$$

$$L \sin 36^\circ\, 52' = 9{\cdot}7781186, \text{ diff. for } 1' = 1684,$$

$$L \sin 48^\circ\, 35' = 9{\cdot}8750142, \text{ diff. for } 1' = 1115.$$

4. Find the acute angles of a right-angled triangle whose hypothenuse is four times as long as the perpendicular drawn to it from the opposite angle.

179. We now proceed to the case of the triangle which is not given to be right angled.

The different cases to be considered are;

Case I. The three sides given;

Case II. Two sides and the included angle given;

Case III. Two sides and the angle opposite one of them given;

Case IV. One side and two angles given;

Case V. The three angles given.

180. *Case I. The three sides a, b, and c given.*

Since the sides are known, the semi-perimeter s is known and hence also the quantities $s-a$, $s-b$, and $s-c$.

The half-angles $\frac{A}{2}$, $\frac{B}{2}$, and $\frac{C}{2}$ are then found from the formulae

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\quad \tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}},$$

and
$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Only two of the angles need be found, the third being known since the sum of the three angles is always 180°.

The angles may also be found by using the formulae for the sine or cosine of the semi-angles.

(Arts. 165 and 166.)

The above formulae are all suited for logarithmic computation.

The angle A may also be obtained from the formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}. \quad \text{(Art. 164.)}$$

This formula is not, in general, suitable for logarithmic calculation. It may be conveniently used however when the sides a, b, and c are small numbers.

Ex. *The sides of a triangle are* 32, 40, *and* 66 *feet; find the angle opposite the greater side, having given that*

$$\log 207 = 2{\cdot}3159703, \quad \log 1073 = 3{\cdot}0305997,$$

$$L \cot 66^\circ\, 18' = 9{\cdot}6424341, \text{ tabulated difference for } 1' = 3431.$$

Here $a = 32, \; b = 40 \text{ and } c = 66,$

so that $s = \dfrac{32 + 40 + 66}{2} = 69, \; s - a = 37, \; s - b = 29 \text{ and } s - c = 3.$

Hence $\cot \dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{(s-a)(s-b)}} = \sqrt{\dfrac{69 \times 3}{37 \times 29}} = \sqrt{\dfrac{207}{1073}}.$

$$L \cot \frac{C}{2} = 10 + \frac{1}{2}[\log 207 - \log 1073]$$

$$= 10 + 1{\cdot}15798515 - 1{\cdot}51529985$$

$$= 9{\cdot}6426853.$$

$L \cot \dfrac{C}{2}$ is therefore *greater* than $L \cot 66^\circ\, 18'$,

so that $\dfrac{C}{2}$ is *less* than $66^\circ\, 18'$.

Let then $\dfrac{C}{2} = 66^\circ\, 18' - x''$.

The difference in the logarithm corresponding to difference of x'' in the angle therefore

$$= \begin{array}{r} 9{\cdot}6426853 \\ -\,9{\cdot}6424341 \\ \hline {\cdot}0002512 \end{array}$$

Also the difference for $60'' = {\cdot}0003431$.

Hence $\dfrac{x}{60} = \dfrac{{\cdot}0002512}{{\cdot}0003431},$

so that $x = \dfrac{2512}{3431} \times 60 = \text{nearly } 44.$

$\therefore \; \dfrac{C}{2} = 66^\circ\, 18' - 44'' = 66^\circ\, 17'\, 16''$, and hence $C = 132^\circ\, 34'\, 32''$.

EXAMPLES. XXIX.

1. If the sides of a triangle be 56, 65, and 33 feet, find the greatest angle.

2. The sides of a triangle are 7, $4\sqrt{3}$, and $\sqrt{13}$ yards respectively. Find the number of degrees in its smallest angle.

3. The sides of a triangle are x^2+x+1, $2x+1$ and x^2-1; prove that the greatest angle is 120°.

4. The sides of a triangle are a, b, and $\sqrt{a^2+ab+b^2}$ feet; find the greatest angle.

5. If $a=2$, $b=\sqrt{6}$ and $c=\sqrt{3}-1$, solve the triangle.

6. If $a=2$, $b=\sqrt{6}$ and $c=\sqrt{3}+1$, solve the triangle.

7. If $a=9$, $b=10$ and $c=11$, find B, given

$$\log 2 = \cdot 30103,\quad L\tan 29^\circ\, 29' = 9\cdot 7523472,$$

and
$$L\tan 29^\circ\, 30' = 9\cdot 7526420.$$

8. The sides of a triangle are 130, 123 and 77 feet. Find the greatest angle, having given

$$\log 2 = \cdot 30103,\quad L\tan 38^\circ\, 39' = 9\cdot 9029376,$$

and
$$L\tan 38^\circ\, 40' = 9\cdot 9031966.$$

9. Find the greatest angle of a triangle whose sides are 242, 188, and 270 feet, having given

$$\log 2 = \cdot 30103,\quad \log 3 = \cdot 4771213,\quad \log 7 = \cdot 8450780,$$

$$L\tan 38^\circ\, 20' = 9\cdot 8980104, \text{ and } L\tan 38^\circ\, 19' = 9\cdot 8977507.$$

10. The sides of a triangle are 2, 3, and 4; find the greatest angle, having given

$$\log 2 = \cdot 30103,\ \log 3 = \cdot 4771213,$$

$$L\tan 52^\circ\, 14' = 10\cdot 1108395,$$

and
$$L\tan 52^\circ\, 15' = 10\cdot 1111004.$$

Making use of the tables, find all the angles when

11. $a=25$, $b=26$ and $c=27$.

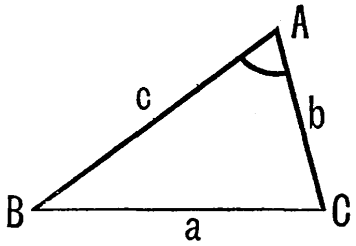
12. $a=17$, $b=20$ and $c=27$.

13. $a=2000$, $b=1050$ and $c=1150$.

181. *Case II. Given two sides b and c and the included angle A.*

Taking b to be the greater of the two given sides, we have

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}\text{ (Art. 171)...(1),}$$



and $$\frac{B+C}{2}=90^\circ-\frac{A}{2}\text{ ......(2).}$$

These two relations give us

$$\frac{B-C}{2}\text{ and }\frac{B+C}{2},$$

and therefore, by addition and subtraction, B and C.

The third side a is then known from the relation

$$\frac{a}{\sin A}=\frac{b}{\sin B},$$

which gives $$a=b\frac{\sin A}{\sin B},$$

and thus determines a.

The side a may also be found from the formula

$$a^2=b^2+c^2-2bc\cos A.$$

This is not adapted to logarithmic calculation but is sometimes useful, especially when the sides a and b are small numbers.

182. **Ex. 1.** *If $b=\sqrt{3}$, $c=1$, and $A=30^\circ$, solve the triangle.*

We have

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{\sqrt{3}-1}{\sqrt{3}+1}\cot 15^\circ.$$

Now $$\tan 15^\circ=\frac{\sqrt{3}-1}{\sqrt{3}+1}\text{ (Art. 101),}$$

so that $$\cot 15^\circ=\frac{\sqrt{3}+1}{\sqrt{3}-1}.$$

Hence $$\tan\frac{B-C}{2}=1.$$

$$\therefore \frac{B-C}{2}=45^\circ \;\ldots\ldots\ldots\ldots\ldots\ldots \;(1).$$

Also $$\frac{B+C}{2}=90^\circ-\frac{A}{2}=90^\circ-15^\circ=75^\circ \;\ldots\ldots\ldots\ldots \;(2).$$

By addition $B=120^\circ$.
By subtraction $C=30^\circ$.
Since $A=C$, we have $a=c=1$.

Otherwise. We have

$$a^2=b^2+c^2-2bc\cos A=3+1-2\sqrt{3}\,.\,\frac{\sqrt{3}}{2}=1,$$

so that $$a=1=c.$$

$$\therefore\; C=A=30^\circ,$$

and $$B=180^\circ-A-C=120^\circ.$$

Ex. 2. *If* $b=215$, $c=105$, *and* $A=74^\circ\,27'$, *find the remaining angles, having given*

$$log\,2=\cdot30103,\;\; log\,11=1\cdot041393,$$

$$log\,105=2\cdot0211893,\;\; log\,212\cdot486=2\cdot3273103,$$

$$L\,cot\,37^\circ\,13'\,30''=10\cdot119341,\quad L\,tan\,24^\circ\,20'\,40''=9\cdot655572,$$

$$L\,tan\,24^\circ\,20'\,50''=9\cdot655626,\quad L\,sin\,74^\circ\,27'=9\cdot9838052,$$

and $$L\,sin\,28^\circ\,25'\,48''=9\cdot6776842.$$

Here $$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{11}{32}\cot 37^\circ\,13'\,30''.$$

$$\therefore\; L\tan\frac{B-C}{2}=\log 11-5\log 2+L\cot 37^\circ\,13'\,30''$$

$$\begin{array}{rrr} & 1\cdot041393 & \cdot30103 \\ & +10\cdot119341 & 5 \\ \hline = & 11\cdot160734 - & 1\cdot50515 \\ = & 9\cdot655584, & \end{array}$$

so that $\frac{B-C}{2}$ lies between $24^\circ\,20'\,40''$ and $24^\circ\,20'\,50''$.

Let then $\frac{B-C}{2} = 24° \, 20' \, 40'' + x''$.

The difference for $\left.\begin{array}{rr} x'' = & 9{\cdot}655584 \\ & -9{\cdot}655572 \end{array}\right\} = {\cdot}000012.$

The difference for $\left.\begin{array}{rr} 10'' = & 9{\cdot}655626 \\ & -9{\cdot}655572 \end{array}\right\} = {\cdot}000054.$

Hence $\frac{x}{10} = \frac{{\cdot}000012}{{\cdot}000054} = \frac{2}{9}$, so that $x = 2\frac{2}{9}''$.

$$\therefore \quad \frac{B-C}{2} = 24° \, 20' \, 42\tfrac{2}{9}'' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But $\frac{B+C}{2} = 90° - \frac{A}{2} = 90° - 37° \, 13' \, 30'' = 52° \, 46' \, 30''$ .... (2).

By adding (1) and (2), we have $B = 77° \, 7' \, 12''$.
By subtracting (1) from (2), we have $C = 28° \, 25' \, 48''$.
To get a we have

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

$$\begin{aligned} \therefore \quad \log a &= \log c + L \sin A - L \sin C \\ &= \log 105 + L \sin 74° \, 27' - L \sin 28° \, 25' \, 48'' \\ & \quad\; 2{\cdot}0211893 \\ & \quad + 9{\cdot}9838052 \\ &= 12{\cdot}0049945 \\ & \quad - 9{\cdot}6776842 \\ &= \; 2{\cdot}3273103 \\ &= \log 212{\cdot}486. \end{aligned}$$

$$\therefore \quad a = 212{\cdot}486.$$

The triangle is therefore completely determined.

***183.** There are ways of finding the third side a of the triangle in the previous case without first finding the angles B and C.

Two methods are as follows :

(1) Since $a^2 = b^2 + c^2 - 2bc \cos A.$

$$= b^2 + c^2 - 2bc\left(2\cos^2\frac{A}{2} - 1\right)$$

$$= (b+c)^2 - 4bc\cos^2\frac{A}{2},$$

$$\therefore\ a^2=(b+c)^2\left[1-\frac{4bc}{(b+c)^2}\cos^2\frac{A}{2}\right].$$

Hence, if $$\sin^2\theta=\frac{4bc}{(b+c)^2}\cos^2\frac{A}{2},$$

we have $$a^2=(b+c)^2[1-\sin^2\theta]=(b+c)^2\cos^2\theta,$$

so that $$a=(b+c)\cos\theta.$$

If then $\sin\theta$ be calculated from the relation

$$\sin\theta=\frac{2\sqrt{bc}}{b+c}\cos\frac{A}{2},$$

we have $$a=(b+c)\cos\theta.$$

(2) We have

$$a^2=b^2-2bc+c^2-2bc(\cos A-1)$$

$$=(b-c)^2+4bc\sin^2\frac{A}{2}$$

$$=(b-c)^2\left[1+\frac{4bc}{(b-c)^2}\sin^2\frac{A}{2}\right].$$

Let $$\frac{4bc}{(b-c)^2}\sin^2\frac{A}{2}=\tan^2\phi,$$

so that $$\tan\phi=\frac{2\sqrt{bc}}{b-c}\sin\frac{A}{2},$$

and hence ϕ is known.

Then $$a^2=(b-c)^2[1+\tan^2\phi]=\frac{(b-c)^2}{\cos^2\phi},$$

so that $$a=(b-c)\sec\phi,$$

and is therefore easily found.

An angle, such as θ or ϕ above, introduced for the purpose of facilitating calculation is called a subsidiary angle (Art. 129).

EXAMPLES. XXX.

1. If $b=90$, $c=70$, and $A=72^\circ\,48'\,30''$, find B and C, given

$$\log 2=\cdot 30103,\quad L\cot 36^\circ\,24'\,15''=10\cdot 1323111,$$

$$L\tan 9^\circ\,37'=9\cdot 2290071$$

and $$L\tan 9^\circ\,38'=9\cdot 2297735.$$

2. If $a=21$, $b=11$, and $C=34° 42' 30''$, find A and B, given

$$\log 2 = \cdot 30103$$

and
$$L \tan 72° 38' 45'' = 10\cdot 50515.$$

3. If the angles of a triangle be in A. P. and the lengths of the greatest and least sides be 24 and 16 feet respectively, find the lengths of the third side and the other angles, given

$$\log 2 = \cdot 30103, \ \log 3 = \cdot 4771213,$$

$$L \tan 19° 6' = 9\cdot 5394287, \text{ diff. for } 1' = 4084.$$

4. If $a=13$, $b=7$, and $C=60°$, find A and B, given that

$$\log 3 = \cdot 4771213,$$

$$L \tan 27° 27' = 9\cdot 7155508, \text{ tabulated diff. for } 1' = 3087.$$

5. If $a=2b$, and $C=120°$, find the values of A, B, and the ratio of c to a, given that

$$\log 3 = \cdot 4771213,$$

$$L \tan 10° 53' = 9\cdot 283907, \text{ diff. for } 1' = 6808.$$

6. If $b=14$, $c=11$, and $A=60°$, find B and C, given that

$$\log 2 = \cdot 30103, \ \log 3 = \cdot 4771213,$$

$$L \tan 11° 44' = 9\cdot 3174299,$$

and
$$L \tan 11° 45' = 9\cdot 3180640.$$

7. The two sides of a triangle are 540 and 420 yards long respectively and include an angle of $52° 6'$. Find the remaining angles, given that

$$\log 2 = \cdot 30103, \ L \tan 26° 3' = 9\cdot 6891430,$$

$$L \tan 14° 20' = 9\cdot 4074189, \ L \tan 14° 21' = 9\cdot 4079453.$$

8. If $b=2\frac{1}{2}$ ft., $c=2$ ft., and $A=22° 20'$, find the other angles, and shew that the third side is nearly one foot, given

$$\log 2 = \cdot 30103, \ \log 3 = \cdot 4771213,$$

$$L \cot 11° 10' = 10\cdot 70465, \ L \sin 22° 20' = 9\cdot 57977,$$

$$L \tan 29° 22' 20'' = 9\cdot 75038, \ L \tan 29° 22' 30'' = 9\cdot 75043,$$

and
$$L \sin 49° 27' 34'' = 9\cdot 88079.$$

9. If $a=2$, $b=1+\sqrt{3}$, and $C=60°$, solve the triangle.

10. Two sides of a triangle are $\sqrt{3}+1$ and $\sqrt{3}-1$, and the included angle is 60°; find the other side and angles.

11. If $b=1$, $c=\sqrt{3}-1$, and $A=60°$, find the length of the side a.

12. If $b=91$, $c=125$ and $\tan\frac{A}{2}=\frac{17}{6}$, prove that $a=204$.

13. If $a=5$, $b=4$, and $\cos(A-B)=\frac{31}{32}$, prove that the third side c will be 6.

14. One angle of a triangle is 30° and the lengths of the sides adjacent to it are 40 and $40\sqrt{3}$ yards. Find the length of the third side and the number of degrees in the other angles.

15. The sides of a triangle are 9 and 3, and the difference of the angles opposite to them is 90°. Find the base and the angles, having given

$$\log 2 = \cdot 30103,\quad \log 3 = \cdot 4771213,$$

$$\log 75894 = 4\cdot 8802074,\quad \log 75895 = 4\cdot 8802132,$$

$$L \tan 26° \, 33' = 9\cdot 6986847$$

and

$$L \tan 26° \, 34' = 9\cdot 6990006.$$

16. Two sides of a triangle are 237 and 158 feet and the contained angle is 66° 40′; find the base and the other angles, having given

$$\log 2 = \cdot 30103,\quad \log 79 = 1\cdot 89763,$$

$$\log 22687 = 4\cdot 35578,\quad L \cot 33° \, 20' = 10\cdot 18197$$

$$L \sin 33° \, 20' = 9\cdot 73998,\quad L \tan 16° \, 54' = 9\cdot 48262,$$

$$L \tan 16° \, 55' = 9\cdot 48308,\quad L \sec 16° \, 54' = 10\cdot 01917,$$

and

$$L \sec 16° \, 55' = 10\cdot 01921.$$

[*Use the formula* $\cos\frac{B-C}{2}=\frac{b+c}{a}\sin\frac{A}{2}$.]

17. If

$$\tan\phi=\frac{a-b}{a+b}\cot\frac{C}{2},$$

prove that

$$c=(a+b)\frac{\sin\frac{C}{2}}{\cos\phi}.$$

If $a=3$, $b=1$, and $C=53°\,7'\,48''$, find c without getting A and B, given

$$\log 2 = \cdot 30103,\ \log 25298 = 4\cdot 4030862,$$

$$\log 25299 = 4\cdot 4031034,\ L\cos 26°\,33'\,54'' = 9\cdot 9515452,$$

and
$$L\tan 26°\,33'\,54'' = 9\cdot 6989700.$$

In the following 4 examples, the required logarithms must be taken from the tables.

18. If $a=242\cdot 5$, $b=164\cdot 3$, and $C=54°\,36'$, solve the triangle.

19. If $b=130$, $c=63$, and $A=42°\,15'\,30''$, solve the triangle.

20. Two sides of a triangle being 2265·4 and 1779 feet and the included angle 58° 17′; find the remaining angles.

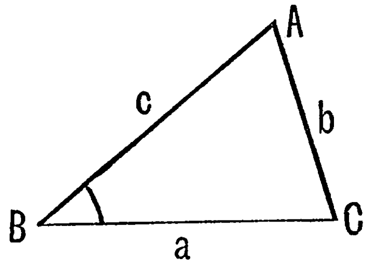
21. Two sides of a triangle being 237·09 and 130·96 feet and the included angle 57° 59′, find the remaining angles.

184. *Case III. Given two sides b and c and the angle B opposite to one of them.*

The angle C is given by the relation

$$\frac{\sin C}{c} = \frac{\sin B}{b},$$

i.e.
$$\sin C = \frac{c}{b}\sin B \;\ldots\ldots(1).$$



Taking logarithms we determine C and then $A\ (=180° - B - C)$ is found.

The remaining side a is then found from the relation

$$\frac{a}{\sin A} = \frac{b}{\sin B},$$

i.e.
$$a = b\,\frac{\sin A}{\sin B} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

185. The equation (1) of the previous article gives in some cases no value, in some cases one, and sometimes two values, for C.

If $c \sin B > b$, the right-hand member of (1) is greater than unity, and hence there is no corresponding value for C.

If $c \sin B = b$, the right-hand member of (1) is equal to unity and the corresponding value of C is 90°.

If $c \sin B < b$, there are two values of C having $\frac{c \sin B}{b}$ as its sine, one value lying between 0° and 90° and the other between 90° and 180°.

Both of these values are not however always admissible.

For if $b > c$, then $B > C$. The obtuse-angled value of C is now not admissible; for, in this case, C cannot be obtuse unless B be obtuse also, and it is manifestly impossible to have *two* obtuse angles in a triangle.

If $b < c$ and B be an acute angle, both values of C are admissible. Hence there are two values found for A and hence the relation (2) gives two values for a. In this case there are therefore two triangles satisfying the given conditions.

Since, for some values of b, c and B, there is a doubt or ambiguity in the determination of the triangle, this case is called the **Ambiguous Case** of the solution of triangles.

186. The Ambiguous Case may also be discussed in a geometrical manner.

Suppose we were given the elements b, c and B and that we proceeded to construct, or attempted to construct, the triangle.

We first measure an angle ABD equal to the given angle B.

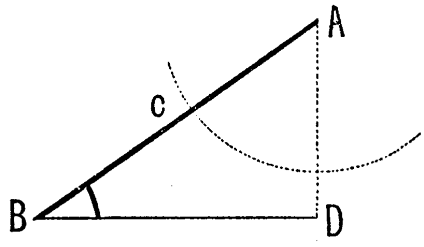


Fig. 1

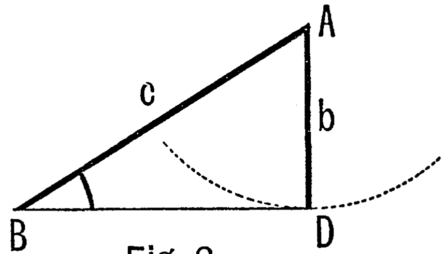


Fig. 2

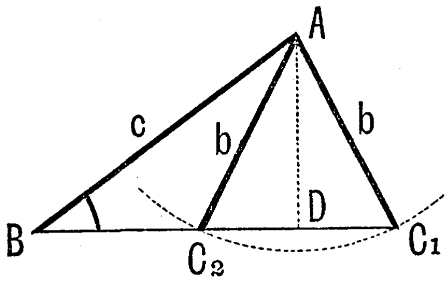


Fig. 3

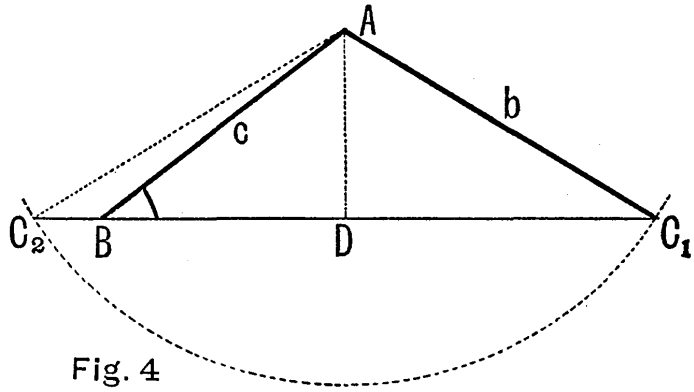


Fig. 4

We then measure along BA a distance BA equal to the given distance c, and thus determine the angular point A.

We have now to find a third point C, which must lie on BD and must also be such that its distance from A shall be equal to b.

To obtain it, we describe with centre A a circle whose radius is b.

The point or points, if any, in which this circle meets BD will determine the position of C.

Draw AD perpendicular to BD, so that

$$AD = AB \sin B = c \sin B.$$

One of the following events will happen.

The circle may never reach BD (Fig. 1) or it may touch BD (Fig. 2), or it may meet BD in two points C_1 and C_2 (Figs. 3 and 4).

In the case of Fig. 1, it is clear that there is no triangle satisfying the given condition.

Here $b < AD$, *i.e.* $< c \sin B$.

In the case of Fig. 2, there is one triangle ABD which is right-angled at D. Here

$$b = AD = c \sin B.$$

In the case of Fig. 3, there are two triangles ABC_1 and ABC_2. Here b lies in magnitude between AD and c, *i.e.* b is $> c \sin B$ and $< c$.

In the case of Fig. 4, there is only one triangle ABC_1 satisfying the given conditions [the triangle ABC_2 is inadmissible; for its angle at B is not equal to B but is equal to $180° - B$]. Here b is greater than both $c \sin B$ and c.

To sum up:

Given the elements b, c, and B of a triangle,

(α) If b be $< c \sin B$, there is no triangle.

(β) If $b = c \sin B$, there is one triangle right-angled.

(γ) If b be $> c \sin B$ and $< c$ and B be acute, there are two triangles satisfying the given conditions.

(δ) If b be $> c$, there is only one triangle.

Clearly if $b = c$, the points B and C_2 in Fig. 3 coincide and there is only one triangle.

If B be obtuse, there is no triangle except when $b > c$.

187. The ambiguous case may also be considered algebraically as follows.

From the figure of Art. 184 we have

$$b^2 = c^2 + a^2 - 2ca \cos B.$$

$$\therefore \quad a^2 - 2ac \cos B + c^2 \cos^2 B = b^2 - c^2 + c^2 \cos^2 B$$
$$= b^2 - c^2 \sin^2 B.$$

$$\therefore \quad a - c \cos B = \pm \sqrt{b^2 - c^2 \sin^2 B},$$

i.e. $$a = c \cos B \pm \sqrt{b^2 - c^2 \sin^2 B} \text{ ....(1).}$$

Now (1) is an equation to determine the value of a when b, c and B are given.

(α) If $b < c \sin B$, the quantity $\sqrt{b^2 - c^2 \sin^2 B}$ is imaginary and (1) gives no real value for a.

(β) If $b = c \sin B$, there is only one value, $c \cos B$, for a; there is thus only one triangle which is right-angled.

(γ) If $b > c \sin B$, there are two values for a. But, since a must be positive, the value obtained by taking the lower sign affixed to the radical is inadmissible unless

$$c \cos B - \sqrt{b^2 - c^2 \sin^2 B} \text{ is positive,}$$

i.e. unless $$\sqrt{b^2 - c^2 \sin^2 B} < c \cos B,$$

i.e. unless $$b^2 - c^2 \sin^2 B < c^2 \cos^2 B,$$

i.e. unless $$b^2 < c^2.$$

There are therefore two triangles only when b is $> c \sin B$ and at the same time $< c$.

188. **Ex.** *Given $b = 16$, $c = 25$, and $B = 33° 15'$, prove that the triangle is ambiguous and find the other angles, having given*

$$\log 2 = \cdot 30103, \quad L \sin 33° 15' = 9 \cdot 7390129,$$

$$L \sin 58° 56' = 9 \cdot 9327616,$$

and $$L \sin 58° 57' = 9 \cdot 9328376.$$

We have

$$\sin C = \frac{c}{b}\sin B = \frac{25}{16}\sin B = \frac{100}{64}\sin B = \frac{10^2}{2^6}\sin 33^\circ 15'.$$

Hence $$L \sin C = 2 + L \sin 33^\circ 15' - 6 \log 2$$
$$= 9{\cdot}9328329.$$

C therefore lies between $58^\circ 56'$ and $58^\circ 57'$, so that

$$C = 58^\circ 56' + x''.$$

For a difference of x'' in the angle the difference in the log

$$= 9{\cdot}9328329 - 9{\cdot}9327616 = {\cdot}0000713.$$

For a difference of $60''$ in the angle the difference

$$= 9{\cdot}9328376 - 9{\cdot}9327616 = {\cdot}0000760.$$

Hence $$\frac{x}{60} = \frac{{\cdot}0000713}{{\cdot}0000760} = \frac{713}{760}.$$

$$\therefore\ x = \frac{6 \times 713}{76} = 56 \text{ nearly,}$$

so that $$L \sin C = L \sin 58^\circ 56' 56''.$$

$$\therefore\ C = 58^\circ 56' 56'' \text{ or } 180^\circ - 58^\circ 56' 56''.$$

Hence (Fig. 3, Art. 186) we have

$$C_1 = 58^\circ 56' 56'', \text{ and } C_2 = 121^\circ 3' 4''.$$

$$\therefore\ \angle BAC_1 = 180^\circ - 33^\circ 15' - 58^\circ 56' 56'' = 87^\circ 48' 4'',$$

and $$\angle BAC_2 = 180^\circ - 33^\circ 15' - 121^\circ 3' 4'' = 25^\circ 41' 56''.$$

EXAMPLES. XXXI.

1. If $a = 5$, $b = 7$, and $\sin A = \frac{3}{4}$, is there any ambiguity?

2. If $a = 2$, $c = \sqrt{3} + 1$, and $A = 45^\circ$, solve the triangle.

3. If $a = 100$, $c = 100\sqrt{3}$ and $A = 30^\circ$, solve the triangle.

4. If $2b = 3a$, and $\tan^2 A = \frac{3}{5}$, prove that there are two values to the third side, one of which is double the other.

5. If $A = 30^\circ$, $b = 8$, and $a = 6$, find c.

6. Given $B=30°$, $c=150$, and $b=150\sqrt{3}$, prove that of the two triangles which satisfy the data one will be isosceles and the other right-angled. Find the greater value of the third side.

Would the solution have been ambiguous had

$$B=30°,\ c=150,\ \text{and}\ b=75?$$

7. In the ambiguous case given a, b, and A, prove that the difference between the two values of c is $2\sqrt{a^2-b^2\sin^2 A}$.

8. If $a=5$, $b=4$, and $A=45°$, find the other angles, having given

$$\log 2=\cdot 30103,\ L\sin 33°\,29'=9\cdot 7520507,$$

and $$L\sin 33°\,30'=9\cdot 7530993.$$

9. If $a=9$, $b=12$, and $A=30°$, find c, having given

$$\log 2=\cdot 30103, \qquad \log 3=\cdot 47712,$$

$$\log 171=2\cdot 23301, \qquad \log 368=2\cdot 56635,$$

$$L\sin 11°\,48'\,39''=9\cdot 31108,\ L\sin 41°\,48'\,39''=9\cdot 82391,$$

and $$L\sin 108°\,11'\,21''=9\cdot 977774.$$

10. Point out whether or no the solutions of the following triangles are ambiguous.

Find the smaller value of the third side in the ambiguous case and the other angles in both cases.

(1) $A=30°$, $c=250$ feet, and $a=125$ feet;

(2) $A=30°$, $c=250$ feet, and $a=200$ feet.

Given $$\log 2=\cdot 30103,\ \log 6\cdot 03893=\cdot 7809601,$$

$$L\sin 38°\,41'=9\cdot 7958800,$$

and $$L\sin 8°\,41'=9\cdot 1789001.$$

11. Given $a=250$, $b=240$, and $A=72°\,4'\,48''$, find the angles B and C, and state whether they can have more than one value, given

$$\log 2\cdot 5=\cdot 3979400, \qquad \log 2\cdot 4=\cdot 3802112,$$

$$L\sin 72°\,4'=9\cdot 9783702,\ L\sin 72°\,5'=9\cdot 9784111,$$

and $$L\sin 65°\,54'=9\cdot 9606739.$$

12. Two straight roads intersect at an angle of 30°; from the point of junction two pedestrians A and B start at the same time, A walking

along one road at the rate of 5 miles per hour and B walking uniformly along the other road. At the end of 3 hours they are 9 miles apart. Shew that there are two rates at which B may walk to fulfil this condition and find them.

For the following 3 examples, a book of tables will be required.

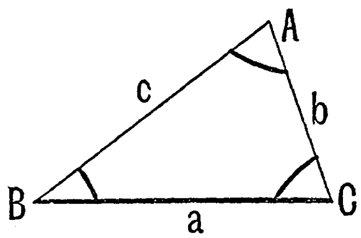
13. Two sides of a triangle are 1015 feet and 732 feet and the angle opposite the latter side is 40°; find the angle opposite the former and prove that more than one value is admissible.

14. Two sides of a triangle being 5374·5 and 1586·6 feet, and the angle opposite the latter being 15° 11′, calculate the other angles of the triangle or triangles.

15. Given $A=10°$, $a=2308·7$, and $b=7903·2$, find the smaller value of c.

189. *Case IV. Given one side and two angles, viz. a, B, and C.*

Since the three angles of a triangle are together equal to two right angles, the third angle is given also.



The sides b and c are now obtained from the relations

$$\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{a}{\sin A},$$

giving
$$b=a\frac{\sin B}{\sin A}, \text{ and } c=a\frac{\sin C}{\sin A}.$$

190. *Case V. The three angles A, B* and *C given.*

Here the ratios only of the sides can be determined by the formulae

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}.$$

Their absolute magnitudes cannot be found.

EXAMPLES. XXXII.

1. If $\cos A = \frac{17}{22}$ and $\cos C = \frac{1}{14}$, find the ratio of $a : b : c$.

2. The angles of a triangle are as $1 : 2 : 7$; prove that the ratio of the greatest side to the least side is $\sqrt{5}+1 : \sqrt{5}-1$.

3. If $A = 45°$, $B = 75°$, and $C = 60°$, prove that $a + c\sqrt{2} = 2b$.

4. Two angles of a triangle are $41° 13' 22''$ and $71° 19' 5''$ and the side opposite the first angle is 55; find the side opposite the latter angle, given

$$\log 55 = 1{\cdot}7403627, \quad \log 79063 = 4{\cdot}8979775,$$
$$L \sin 41° 13' 22'' = 9{\cdot}8188779,$$
and
$$L \sin 71° 19' 5'' = 9{\cdot}9764927.$$

5. From each of two ships, one mile apart, the angle is observed which is subtended by another ship and a beacon on shore; these angles are found to be $52° 25' 15''$ and $75° 9' 30''$ respectively. Given

$$L \sin 75° 9' 30'' = 9{\cdot}9852635,$$
$$L \sin 52° 25' 15'' = 9{\cdot}8990055, \quad \log 1{\cdot}2197 = {\cdot}0862530$$
and
$$\log 1{\cdot}2198 = {\cdot}0862886,$$
find the distance of the beacon from each of the ships.

6. The base angles of a triangle are $22\frac{1}{2}°$ and $112\frac{1}{2}°$; prove that the base is equal to twice the height.

For the following 5 questions a book of tables is required.

7. The base of a triangle being seven feet and the base angles $129° 23'$ and $38° 36'$, find the length of its shorter side.

8. If the angles of a triangle be as $5 : 10 : 21$, and the side opposite the smaller angle be 3 feet, find the other sides.

9. The angles of a triangle being $150°$, $18° 20'$, and $11° 40'$, and the longest side being 1000 feet, find the length of the shortest side.

10. To get the distance of a point A from a point B, a line BC and the angles ABC and BCA are measured, and are found to be 287 yards and $55° 32' 10''$ and $51° 8' 20''$ respectively. Find the distance AB.

11. To find the distance from A to P a distance, AB, of 1000 yards is measured in a convenient direction. At A the angle PAB is found to be $41° 18'$ and at B the angle PBA is found to be $114° 38'$. What is the required distance to the nearest yard?

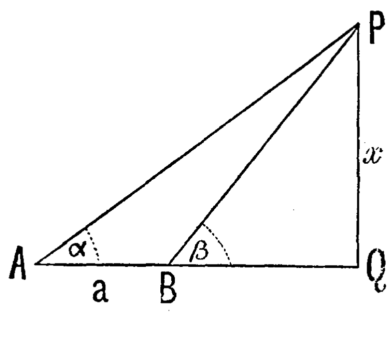
CHAPTER XIV.

HEIGHTS AND DISTANCES.

191. IN the present chapter we shall consider some questions of the kind which occur in land-surveying. Simple questions of this kind have already been considered in Chapter III.

192. *To find the height of an inaccessible tower by means of observations made at distant points.*

Suppose PQ to be the tower and that the ground passing through the foot Q of the tower is horizontal. At a point A on this ground measure the angle of elevation α of the top of the tower.



Measure off a distance $AB(=a)$ from A directly toward the foot of the tower, and at B measure the angle of elevation β.

To find the unknown height x of the tower, we have to connect it with the measured length a. This is best done as follows:

From the triangle PBQ we have

$$\frac{x}{BP} = \sin\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and from the triangle PAB we have

$$\frac{PB}{a} = \frac{\sin PAB}{\sin BPA} = \frac{\sin\alpha}{\sin(\beta-\alpha)} \quad \ldots\ldots\ldots\ldots(2),$$

since $\angle BPA = \angle QBP - \angle QAP = \beta - \alpha$.

From (1) and (2), by multiplication, we have

$$\frac{x}{a} = \frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)},$$

i.e.

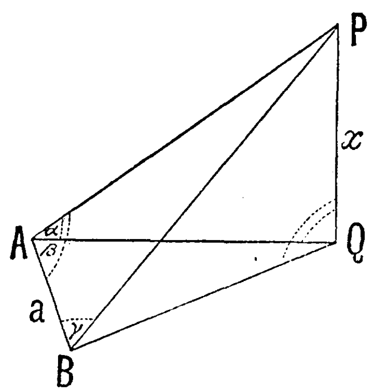
$$x = a\frac{\sin\alpha\sin\beta}{\sin(\beta-\alpha)}.$$

The height x is therefore given in a form suitable for logarithmic calculation.

Numerical Example. If $a=100$ feet, $\alpha=30°$, and $\beta=60°$, then

$$x=100\frac{\sin 30°\sin 60°}{\sin 30°}=100\times\frac{\sqrt{3}}{2}=86{\cdot}6 \text{ feet.}$$

193. It is often not convenient to measure AB directly towards Q.



Measure therefore AB in any other suitable direction on the horizontal ground and at A measure the angle of elevation α of P, and also the angle $PAB\,(=\beta)$.

At B measure the angle PBA $(=\gamma)$.

In the triangle PAB we have then

$$\angle APB = 180° - \angle PAB - \angle PBA = 180° - (\beta+\gamma).$$

Hence $$\frac{AP}{a} = \frac{\sin PBA}{\sin BPA} = \frac{\sin \gamma}{\sin(\beta + \gamma)}.$$

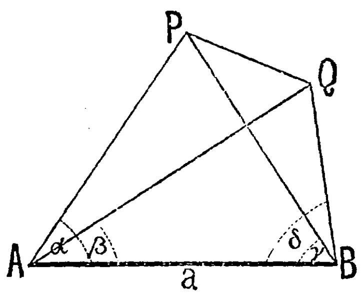
From the triangle PAQ, we have

$$x = AP \sin \alpha = a \frac{\sin \alpha \sin \gamma}{\sin(\beta + \gamma)}.$$

Hence x is found by an expression suitable for logarithmic calculation.

194. *To find the distance between two inaccessible points by means of observations made at two points the distance between which is known, all four points being supposed to be in one plane.*

Let P and Q be two points whose distance apart, PQ, is required.



Let A and B be the two known points whose distance apart, AB, is given to be equal to a.

At A measure the angles PAB and QAB, and let them be α and β respectively.

At B measure the angle PBA and QBA, and let them be γ and δ respectively.

Then in the triangle PAB we have one side a and the two adjacent angles α and γ given, so that, as in Art. 163, we have AP given by the relation

$$\frac{AP}{a} = \frac{\sin \gamma}{\sin APB} = \frac{\sin \gamma}{\sin(\alpha + \gamma)} \quad \text{..........(1).}$$

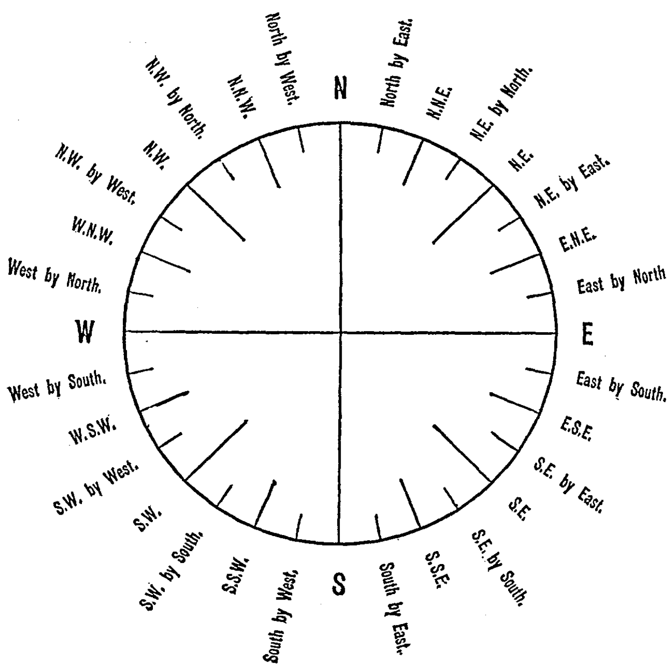
In the triangle QAB we have, similarly,

$$\frac{AQ}{a} = \frac{\sin \delta}{\sin(\beta + \delta)} \quad \text{..................(2).}$$

In the triangle APQ we have now determined the sides AP and AQ; also the included angle $PAQ (= \alpha - \beta)$ is known. We can therefore find the side PQ by the method of Art. 181.

If the four points A, B, P, and Q be not in the same plane, we must, in addition, measure the angle PAQ; for in this case PAQ is not equal to $\alpha - \beta$. In other respects the solution will be the same as above.

195. Bearings and Points of the Compass. The Bearing of a given point B as seen from a given point O is the direction in which B is seen from O. Thus if



the direction of OB bisect the angle between East and North, the bearing of B is said to be North-East.

If a line is said to bear 20° West of North we mean that it is inclined to the North direction at an angle of 20°, this angle being measured from the North towards the West.

To facilitate the statement of the bearing of a point the circumference of the mariner's compass-card is divided into 32 equal portions, as in the above figure, and the subdivisions marked as indicated. Consider only the quadrant between East and North. The middle point of the arc between N. and E. is marked North-East (N.E.). The bisectors of the arcs between N.E. and N. and E. are respectively called North-North-East and East-North-East (N.N.E. and E.N.E.). The other four subdivisions, reckoning from N., are called North by East, N.E. by North, N.E. by East, and East by North. Similarly the other three quadrants are subdivided.

It is clear that the arc between two subdivisions of the card subtends an angle of $\frac{360^\circ}{32}$, *i.e.* $11\frac{1}{4}^\circ$, at the centre O.

EXAMPLES. XXXIII.

1. A flagstaff stands on the middle of a square tower. A man on the ground opposite the middle of one face and distant from it 100 feet just sees the flag; receding another 100 feet the tangents of elevation of the top of the tower and the top of the flagstaff are found to be $\frac{1}{2}$ and $\frac{5}{9}$. Find the dimensions of the tower and the height of the flagstaff, the ground being horizontal.

2. A man, walking on a level plane towards a tower, observes that at a certain point the angular height of the tower is 10° and after going 50 yards nearer the tower the elevation is found to be 15°. Having given

$$L\sin 15^\circ = 9\cdot4129962, \quad L\cos 5^\circ = 9\cdot9983442,$$

$$\log 25\cdot783 = 1\cdot4113334 \text{ and } \log 25\cdot784 = 1\cdot4113503,$$

find, to 4 places of decimals, the height of the tower in yards.

3. DE is a tower standing on a horizontal plane and $ABCD$ is a straight line in the plane. The height of the tower subtends an angle θ at A, 2θ at B, and 3θ at C. If AB and BC be respectively 50 and 20 feet, find the height of the tower and the distance CD.

4. A tower, 50 feet high, stands on the top of a mound; from a point on the ground the angles of elevation of the top and bottom of the tower are found to be 75° and 45° respectively; find the height of the mound.

5. A vertical pole (more than 100 feet high) consists of two parts, the lower being $\frac{1}{3}$rd of the whole. From a point in a horizontal plane through the foot of the pole and 40 feet from it, the upper part subtends an angle whose tangent is $\frac{1}{2}$. Find the height of the pole.

6. A tower subtends an angle α at a point on the same level as the foot of the tower and at a second point, h feet above the first, the depression of the foot of the tower is β. Find the height of the tower.

7. A person in a balloon, which has ascended vertically from flat land at the sea level, observes the angle of depression of a ship at anchor to be 30°; after descending vertically for 600 feet he finds the angle of depression to be 15°; find the horizontal distance of the ship from the point of ascent.

8. PQ is a tower standing on a horizontal plane, Q being its foot; A and B are two points on the plane such that the $\angle QAB$ is 90°, and AB is 40 feet. It is found that

$$\cot PAQ = \frac{3}{10} \text{ and } \cot PBQ = \frac{1}{2}.$$

Find the height of the tower.

9. A column is E.S.E. of an observer and at noon the end of the shadow is North-East of him. The shadow is 80 feet long and the elevation of the column at the observer's station is 45°. Find the height of the column.

10. A tower is observed from two stations A and B. It is found to be due north of A and north-west of B. B is due east of A and distant from it 100 feet. The elevation of the tower as seen from A is the complement of the elevation as seen from B. Find the height of the tower.

11. The elevation of a steeple at a place due south of it is 45° and at another place due west of it the elevation is 15°. If the distance between the two places be a, prove that the height of the steeple is

$$\frac{a(\sqrt{3}-1)}{2\sqrt[4]{3}}.$$

12. A person stands in the diagonal produced of the square base of a church tower, at a distance $2a$ from it, and observes the angles of elevation of each of the two outer corners of the top of the tower to be 30°, whilst that of the nearest corner is 45°. Prove that the breadth of the tower is $a(\sqrt{10}-\sqrt{2})$.

13. A person standing at a point A due south of a tower built on a horizontal plane observes the altitude of the tower to be 60°. He then walks to B due west of A and observes the altitude to be 45°, and again at C in AB produced he observes it to be 30°. Prove that B is midway between A and C.

14. At each end of a horizontal base of length $2a$ it is found that the angular height of a certain peak is θ and that at the middle point it is ϕ. Prove that the vertical height of the peak is

$$\frac{a\sin\theta\sin\phi}{\sqrt{\sin(\phi+\theta)\sin(\phi-\theta)}}.$$

15. A and B are two stations 1000 feet apart; P and Q are two stations in the same plane as AB and on the same side of it; the angles PAB, PBA, QAB, and QBA are respectively 75°, 30°, 45°, and 90°; find how far P is from Q and how far each is from A and B.

For the following 4 examples a book of tables will be wanted.

16. At a point on a horizontal plane the elevation of the summit of a mountain is found to be 22° 15′ and at another point on the plane a mile further away in a direct line its elevation is 10° 12′; find the height of the mountain.

17. From the top of a hill the angles of depression of two successive milestones, on level ground and in the same vertical plane with the observer, are found to be 5° and 10° respectively. Find the height of the hill and the horizontal distance to the nearest milestone.

18. A castle and a monument stand on the same horizontal plane. The height of the castle is 140 feet and the angles of depression of the top and bottom of the monument as seen from the top of the castle are 40° and 80° respectively. Find the height of the monument.

19. A flagstaff PN stands on level ground. A base AB is measured at right angles to AN, the points A, B and N being in the same horizontal plane, and the angles PAN and PBN are found to be α and β respectively. Prove that the height of the flagstaff is

$$AB \frac{\sin\alpha\sin\beta}{\sqrt{\sin(\alpha-\beta)\sin(\alpha+\beta)}}.$$

If $AB=100$ feet, $\alpha=70°$, and $\beta=50°$, calculate the height.

20. A man standing due south of a tower on a horizontal plane through its foot finds the elevation of the top of the tower to be $54° 16'$; he goes east 100 yards and finds the elevation to be then $50° 8'$. Find the height of the tower.

21. A man in a balloon observes that the angle of depression of an object on the ground bearing due north is $33°$; the balloon drifts 3 miles due west and the angle of depression is now found to be $21°$. Find the height of the balloon.

22. From the extremities of a horizontal base-line AB, whose length is 1000 feet, the bearings of the foot C of a tower are observed and it is found that $\angle CAB=56° 23'$, $\angle CBA=47° 15'$, and that the elevation of the tower from A is $9° 25'$; find the height of the tower.

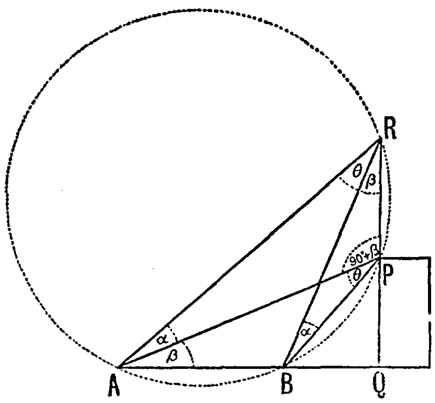
196. Ex. *A flagstaff is on the top of a tower which stands on a horizontal plane. A person observes the angles, α and β, subtended at a point on the horizontal plane by the flagstaff and the tower; he then walks a known distance a toward the tower and finds that the flagstaff subtends the same angle as before; prove that the height of the tower and the length of the flagstaff are respectively*

$$\frac{a\sin\beta\cos(\alpha+\beta)}{\cos(\alpha+2\beta)} \quad \textit{and} \quad \frac{a\sin\alpha}{\cos(\alpha+2\beta)}.$$

Let P and Q be the top and foot of the tower, and let PR be the flagstaff. Let A and B be the points at which the measurements are taken, so that $\angle PAQ=\beta$ and $\angle PAR=\angle PBR=\alpha$. Since the two latter angles are equal, a circle will go through the four points A, B, P, and R.

To get the height of the flagstaff we have to connect the unknown length PR with the known length AB.

This may be done by connecting each with the length AR.



To do this, we must first determine the angles of the triangles ARP and ARB.

Since A, B, P, and R lie on a circle, we have

$$\angle BRP = \angle BAP = \beta,$$

and $\angle APB = \angle ARB = \theta$ (say).

Also $\angle APR = 90^\circ + \angle PAQ = 90^\circ + \beta$.

Hence, since the angles of the triangle APR are together equal to two right angles, we have

$$180^\circ = \alpha + (90^\circ + \beta) + (\theta + \beta),$$

so that $$\theta = 90^\circ - (\alpha + 2\beta) \text{..................(1)}.$$

From the triangles APR and ABR we then have

$$\frac{PR}{\sin\alpha} = \frac{AR}{\sin RPA} = \frac{AR}{\sin RBA} = \frac{a}{\sin\theta} \quad \text{(Art. 163).}$$

[It will be found in Chap. XV. that each of these quantities is equal to the radius of the circle.]

Hence the height of the flagstaff

$$= PR = \frac{a\sin\alpha}{\sin\theta} = \frac{a\sin\alpha}{\cos(\alpha+2\beta)}, \text{ by (1).}$$

Again $$\frac{PQ}{PB} = \cos BPQ = \cos(\alpha+\beta) \text{...........(2),}$$

and $$\frac{PB}{a} = \frac{\sin PAB}{\sin APB} = \frac{\sin\beta}{\sin\theta} \text{...............(3).}$$

Hence, from (2) and (3), by multiplication,

$$\frac{PQ}{a} = \frac{\sin\beta\cos(\alpha+\beta)}{\sin\theta} = \frac{\sin\beta\cos(\alpha+\beta)}{\cos(\alpha+2\beta)}, \text{ by (1).}$$

Also $BQ = PQ \tan BPQ = PQ \tan(\alpha + \beta)$

$$= a\frac{\sin\beta\sin(\alpha+\beta)}{\cos(\alpha+2\beta)},$$

and $$AQ = a + BQ = a\frac{\cos(\alpha+2\beta)+\sin\beta\sin(\alpha+\beta)}{\cos(\alpha+2\beta)}$$

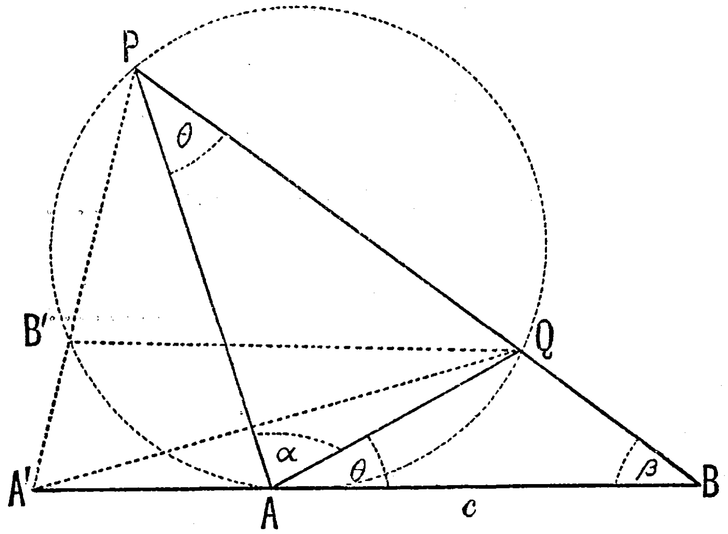
$$= a\frac{\cos\beta\cos(\alpha+\beta)}{\cos(\alpha+2\beta)}.$$

If a, α, and β be given numerically these results are all in a form suitable for logarithmic computation.

197. Ex. *A man walks along a straight road and observes that the greatest angle subtended by two objects is* α; *from the point where this greatest angle is subtended he walks a distance c along the road and finds that the two objects are now in a straight line which makes an angle* β *with the road; prove that the distance between the objects is*

$$c\sin\alpha\sin\beta\sec\frac{\alpha+\beta}{2}\sec\frac{\alpha-\beta}{2}.$$

Let P and Q be the two points and let PQ meet the road in B.



If A be the point at which the greatest angle is subtended then A must be the point where a circle drawn through P and Q touches the road.

[For, take *any* other point A' on AB and join it to P cutting the circle in B' and join $A'Q$ and $B'Q$.

Then $\qquad \angle PA'Q < \angle PB'Q$ (Euc. I. 16),

and therefore $\qquad < \angle PAQ$ (Euc. III. 21).]

Let the angle QAB be called θ. Then (Euc. III. 32) the angle APQ is θ also.

Hence $180° =$ sum of the angles of the triangle PAB

$$= \theta + (\alpha + \theta) + \beta,$$

so that $$\theta = 90° - \frac{\alpha + \beta}{2}.$$

From the triangles PAQ and QAB we have

$$\frac{PQ}{AQ} = \frac{\sin \alpha}{\sin \theta}, \text{ and } \frac{AQ}{c} = \frac{\sin \beta}{\sin AQB} = \frac{\sin \beta}{\sin (\theta + \alpha)}.$$

Hence, by multiplication, we have

$$\frac{PQ}{c} = \frac{\sin \alpha \sin \beta}{\sin \theta \sin (\theta + \alpha)}.$$

$$= \frac{\sin \alpha \sin \beta}{\cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}}.$$

$$\therefore \; PQ = c \sin \alpha \sin \beta \sec \frac{\alpha + \beta}{2} \sec \frac{\alpha - \beta}{2}.$$

EXAMPLES. XXXIV.

1. A bridge has 5 equal spans, each of 100 feet measured from the centre of the piers, and a boat is moored in a line with one of the middle piers. The whole length of the bridge subtends a right angle as seen from the boat. Prove that the distance of the boat from the bridge is $100\sqrt{6}$ feet.

2. A ladder placed at an angle of $75°$ just reaches the sill of a window at a height of 27 feet above the ground on one side of a street. On turning the ladder over without moving its foot, it is found that when it rests against a wall on the other side of the street it is at an angle of $15°$ with the ground. Prove that the breadth of the street and the length of the ladder are respectively

$$27(3+\sqrt{3}) \text{ and } 27(\sqrt{6}-\sqrt{2}) \text{ feet.}$$

3. From a house on one side of a street observations are made of the angle subtended by the height of the opposite house; from the level of the street the angle subtended is the angle whose tangent is 3; from two windows one above the other the angle subtended is found to be the angle whose tangent is -3; the height of the opposite house being 60 feet, find the height above the street of each of the two windows.

4. A rod of given length can turn in a vertical plane passing through the sun, one end being fixed on the ground; find the longest shadow it can cast on the ground.

Calculate the altitude of the sun when the longest shadow it can cast is $3\frac{1}{2}$ times the length of the rod.

5. A ship A observes another ship B leaving a harbour, whose bearing is then N.W. After 10 minutes A, having sailed one mile N.E., sees B due west and the harbour then bears $60°$ West of North. After another 10 minutes B is observed to bear S.W. Find the distances between A and B at the first observation and also the direction and rate of B.

6. A ship sailing north sees two lighthouses, which are 6 miles apart, in a line due west; after an hour's sailing one of them bears S.W. and the other S.S.W. Find the ship's rate.

7. A ship sees a lighthouse N.W. of itself. After sailing for 12 miles in a direction 15° south of W. the lighthouse is seen due N. Find the distance of the lighthouse from the ship in each position.

8. A man, travelling west along a straight road, observes that when he is due south of a certain windmill the straight line drawn to a distant church makes an angle of 30° with the road. A mile further on the bearings of the windmill and tower are respectively N.E. and N.W. Find the distances of the tower from the windmill and from the nearest point of the road.

9. An observer on a headland sees a ship due north of him; after a quarter of an hour he sees it due east and after another half-hour he sees it due south-east; find the direction that the ship's course makes with the meridian and the time after the ship is first seen until it is nearest the observer, supposing that it sails uniformly in a straight line.

10. A man walking along a straight road which runs in a direction 30° east of north notes when he is due south of a certain house; when he has walked a mile further he observes that the house lies due west and that a windmill on the opposite side of the road is N.E. of him; three miles further on he finds that he is due north of the windmill; prove that the line joining the house and the windmill makes with the road the angle whose tangent is

$$\frac{48 - 25\sqrt{3}}{11}.$$

11. A, B, and C are three consecutive milestones on a straight road from each of which a distant spire is visible. The spire is observed to bear north-east at A, east at B, and 60° east of south at C. Prove that the shortest distance of the spire from the road is $\frac{7+5\sqrt{3}}{13}$ miles.

12. Two stations due south of a tower, which leans towards the north, are at distances a and b from its foot; if α and β be the elevations of the top of the tower from these stations, prove that its inclination to the vertical is

$$\cot^{-1}\frac{b\cot\alpha - a\cot\beta}{b-a}.$$

13. From a point A on a level plane the angle of elevation of a balloon is α, the balloon being south of A; from a point B which is at a distance C south of A the balloon is seen northwards at an elevation of β; find the distance of the balloon from A and its height above the ground.

14. A statue on the top of a pillar subtends the same angle α at distances of 9 and 11 yards from the pillar; if $\tan \alpha = \frac{1}{10}$, find the height of the pillar and of the statue.

15. A tower and a spire on the top of the tower subtend equal angles at a point whose distance from the foot of the tower is a; if h be the height of the tower, prove that the height of the spire is

$$h\frac{a^2+h^2}{a^2-h^2}.$$

16. A flagstaff on the top of a tower is observed to subtend the same angle at two points on a horizontal plane, which lie on a line passing through the centre of the base of the tower and whose distance from one another is $2a$, and an angle β at a point halfway between them. Prove that the height of the flagstaff is

$$a \sin \alpha \sqrt{\frac{2 \sin \beta}{\cos \alpha \sin (\alpha - \beta)}}.$$

17. An observer in the first place stations himself at a distance a feet from a column standing upon a mound. He finds that the column subtends an angle, whose tangent is $\frac{1}{2}$, at his eye which may be supposed to be on the horizontal plane through the base of the mound. On moving $\frac{2}{3}a$ feet nearer the column he finds that the angle subtended is unchanged. Find the height of the mound and of the column.

18. A church tower stands on the bank of a river which is 150 feet wide and on the top of the tower is a spire 30 feet high. To an observer on the opposite bank of the river the spire subtends the same angle that a pole six feet high subtends when placed upright on the ground at the foot of the tower. Prove that the height of the tower is nearly 285 feet.

19. A person, wishing to ascertain the height of a tower, stations himself on a horizontal plane through its foot at a point at which the elevation of the top is 30°. On walking a distance a in a certain direction he finds that the elevation of the top is the same as before, and on then walking a distance $\frac{5}{3}a$ at right angles to his former direction he finds the elevation of the top to be 60°. Prove that the height of the tower is either $\sqrt{\frac{5}{6}}a$ or $\sqrt{\frac{85}{48}}a$.

20. The angles of elevation of the top of a tower, standing on a horizontal plane, from two points distant a and b from the base and in the same straight line with it are complementary. Prove that the height of the tower is $\sqrt{ab}$ feet, and, if θ be the angle subtended at the top of the tower by the line joining the two points, then $\sin\theta = \frac{a-b}{a+b}$.

21. A tower 150 feet high stands on the top of a cliff 80 feet high. At what point on the plane passing through the foot of the cliff must an observer place himself so that the tower and the cliff may subtend equal angles, the height of his eye being 5 feet?

22. A statue on the top of a pillar, standing on level ground, is found to subtend the greatest angle α at the eye of an observer when his distance from the pillar is c feet; prove that the height of the statue is $2c \tan\alpha$ feet, and find the height of the pillar.

23. A tower stood at the foot of an inclined plane whose inclination to the horizon was 9°. A line 100 feet in length was measured straight up the incline from the foot of the tower, and at the end of this line the tower subtended an angle of 54°. Find the height of the tower, having given

$$\log 2 = \cdot 30103, \quad \log 114\cdot 122 = 2\cdot 0584726,$$

and

$$L \sin 54° = 9\cdot 9079576.$$

24. A vertical tower stands on a declivity which is inclined at 15° to the horizon. From the foot of the tower a man ascends the declivity for 80 feet, and then finds that the tower subtends an angle of 30°. Prove that the height of the tower is $40(\sqrt{6}-\sqrt{2})$ feet.

25. The altitude of a certain rock is 47° and after walking towards it 1000 feet up a slope inclined at 30° to the horizon an observer finds its altitude to be 77°. Find the vertical height of the rock above the first point of observation, given that $\sin 47° = \cdot 73135$.

26. A man observes that when he has walked c feet up an inclined plane the angular depression of an object in a horizontal plane through the foot of the slope is α, and that, when he has walked a further distance of c feet the depression is β. Prove that the inclination of the slope to the horizon is the angle whose cotangent is

$$(2 \cot\beta - \cot\alpha).$$

27. A regular pyramid on a square base has an edge 150 feet long and the length of the side of its base is 200 feet. Find the inclination of its face to the base.

28. A pyramid has for base a square of side a; its vertex lies on a line through the middle point of the base and perpendicular to it, and at a distance h from it; prove that the angle α between the two lateral faces is given by the equation

$$\sin \alpha = \frac{2h\sqrt{2a^2+4h^2}}{a^2+4h^2}.$$

29. A flagstaff, 100 feet high, stands in the centre of an equilateral triangle which is horizontal. From the top of the flagstaff each side subtends an angle of 60°; prove that the length of the side of the triangle is $50\sqrt{6}$ feet.

30. The extremity of the shadow of a flagstaff, which is 6 feet high and stands on the top of a pyramid on a square base, just reaches the side of the base and is distant 56 and 8 feet respectively from the extremities of that side. Find the sun's altitude if the height of the pyramid be 34 feet.

31. The extremity of the shadow of a flagstaff, which is 6 feet high and stands on the top of a pyramid on a square base, just reaches the side of the base and is distant x feet and y feet respectively from the ends of that side; prove that the height of the pyramid is

$$\sqrt{\frac{x^2+y^2}{2}}\tan \alpha - 6,$$

where α is the elevation of the sun.

32. The angle of elevation of a cloud from a point h feet above a lake is α and the angle of depression of its reflexion in the lake is β; prove that its height is $h\dfrac{\sin(\beta+\alpha)}{\sin(\beta-\alpha)}$.

33. The shadow of a tower is observed to be half the known height of the tower and sometime afterwards it is equal to the known height; how much will the sun have gone down in the interval, given

$$\log 2 = \cdot 30103, \quad L \tan 63° \, 24' = 10 \cdot 3009994,$$

and diff. for $1' = 3159$?

34. An isosceles triangle of wood is placed in a vertical plane, vertex upwards, and faces the sun. If $2a$ be the base of the triangle, h its height, and 30° the altitude of the sun, prove that the tangent of the angle at the apex of the shadow is $\dfrac{2ah\sqrt{3}}{3h^2 - a^2}$.

35. A rectangular target faces due south, being vertical and standing on a horizontal plane. Compare the area of the target with that of its shadow on the ground when the sun is $\beta°$ from the south at an altitude of $\alpha°$.

36. A spherical ball, of diameter δ, subtends an angle α at a man's eye when the elevation of its centre is β; prove that the height of the centre of the ball is $\frac{1}{2}\delta \sin\beta \operatorname{cosec}\frac{\alpha}{2}$.

37. A man standing a plane observes a row of equal and equidistant pillars, the 10th and 17th of which subtend the same angle that they would do if they were in the position of the first and were respectively $\frac{1}{2}$ and $\frac{1}{3}$ of their height. Prove that, neglecting the height of the man's eye, the line of pillars is inclined to the line drawn to the first at an angle whose secant is nearly 2·6.

For the following 4 examples a book of tables will be wanted.

38. A and B are two points on the opposite bank of a river 1000 feet wide and between them is the mast of a ship PN; the vertical elevation of P at A is 14° 20′ and at B it is 8° 10′. What is the height of P above AB?

39. AB is a line 1000 yards long; B is due north of A and from B a distant point P bears 70° east of north; at A it bears 41° 22′ east of north; find the distance from A to P.

40. A is a station exactly 10 miles west of B. The bearing of a particular rock from A is 74° 19′ east of north and its bearing from B is 26° 51′ west of north. How far is it north of the line AB?

41. The summit of a spire is vertically over the middle point of a horizontal square enclosure whose side is of length a feet; the height of the spire is h feet above the level of the square. If the shadow of the spire just reach a corner of the square when the sun has an altitude θ, prove that

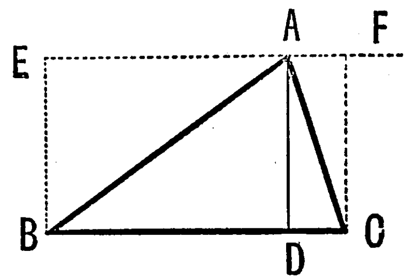
$$h\sqrt{2} = a \tan\theta.$$

Calculate h, having given $a = 1000$ feet and $\theta = 25° 15'$.

CHAPTER XV.

PROPERTIES OF A TRIANGLE.

198. Area of a given triangle. Let ABC be any triangle and AD the perpendicular drawn from A upon the opposite side.



Through A draw EAF parallel to BC and draw BE and CF perpendicular to it. By Euc. I. 41, the area of the triangle ABC

$$= \tfrac{1}{2}\text{ rectangle } BF = \tfrac{1}{2}BC \cdot CF = \tfrac{1}{2}a \cdot AD.$$

But $AD = AB \sin B = c \sin B$.

The area of the triangle ABC therefore $= \frac{1}{2}ca \sin B$. This area is denoted by Δ.

Hence $\boldsymbol{\Delta = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C = \tfrac{1}{2}bc \sin A}$ ...(1).

By Art. 169, we have $\sin A = \dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$,

so that $\quad \Delta = \frac{1}{2}bc \sin A = \boldsymbol{\sqrt{s(s-a)(s-b)(s-c)}}$...(2).

This latter quantity is often called S.

EXAMPLES. XXXV.

Find the area of the triangle ABC when

1. $a=13$, $b=14$, and $c=15$.
2. $a=18$, $b=24$, and $c=30$.
3. $a=25$, $b=52$, and $c=63$.
4. $a=125$, $b=123$, and $c=62$.
5. $a=15$, $b=36$, and $c=39$.
6. $a=287$, $b=816$, and $c=865$.
7. $a=35$, $b=84$, and $c=91$.
8. $a=\sqrt{3}$, $b=\sqrt{2}$, and $c=\dfrac{\sqrt{6}+\sqrt{2}}{2}$.

9. If $B=45^\circ$, $C=60^\circ$, and $a=2(\sqrt{3}+1)$ inches, prove that the area of the triangle is $6+2\sqrt{3}$ sq. inches.

10. The sides of a triangle are 119, 111, and 92 yards; prove that its area is 10 sq. yards less than an acre.

11. The sides of a triangular field re 242, 1212 and 1450 yards; prove that the area of the field is 6 acres.

12. A workman is told to make a triangular enclosure of sides 51, 41, and 21 yards respectively; having made the first side one yard too long, what length must he make the other two sides in order to enclose the prescribed area with the prescribed length of fencing?

13. Find, correct to ·0001 of an inch, the length of one of the equal sides of an isosceles triangle on a base of 14 inches having the same area as a triangle whose sides are 13·6, 15, and 15·4 inches.

14. Prove that the area of a triangle is $\frac{1}{2}a^2\dfrac{\sin B\sin C}{\sin A}$.

If one angle of a triangle be 60°, the area $10\sqrt{3}$ square feet, and the perimeter 20 feet, find the lengths of the sides.

15. The sides of a triangle are in A.P. and its area is $\dfrac{3}{5}$ths of an equal triangle of the same perimeter; prove that its sides are in the ratio 3 : 5 : 7, and find the greatest angle of the triangle.

16. In a triangle the least angle is 45° and the tangents of the angles are in A.P. If its area be 3 square yards, prove that the lengths of the sides are $3\sqrt{5}$, $6\sqrt{2}$, and 9 feet, and that the tangents of the other angles are respectively 2 and 3.

17. The lengths of two sides of a triangle are one foot and $\sqrt{2}$ feet respectively and the angle opposite the shorter side is 30°; prove that there are two triangles satisfying these conditions, find their angles, and shew that their areas are in the ratio

$$\sqrt{3}+1 : \sqrt{3}-1.$$

18. Find by the aid of the tables the area of the larger of the two triangles given by the data

$$A=31^\circ 15', \ a=5 \text{ ins. and } b=7 \text{ ins.}$$

199. On the circles connected with a given triangle.

The circle which passes through the angular points of a triangle ABC is called its circumscribing circle or, more briefly, its **circumcircle.** The centre of this circle is found by the construction of Euc. IV. 5. Its radius is always called R.

The circle which can be inscribed within the triangle so as to touch each of the sides is called its inscribed circle or, more briefly, its **incircle.** The centre of this circle is found by the construction of Euc. IV. 4. Its radius will be denoted by r.

The circle which touches the side BC and the two sides AB and AC produced is called the escribed circle opposite the angle A. Its radius will be denoted by r_1.

Similarly r_2 denotes the radius of the circle which touches the side CA and the two sides BC and BA produced. Also r_3 denotes the radius of the circle touching AB and the two sides CA and CB produced.

200. *To find the magnitude of R, the radius of the circumcircle of any triangle ABC.*

Bisect the two sides BC and CA in D and E respectively, and draw DO and EO perpendicular to BC and CA.

By Euc. IV. 5, O is the centre of the circumcircle. Join OB and OC.

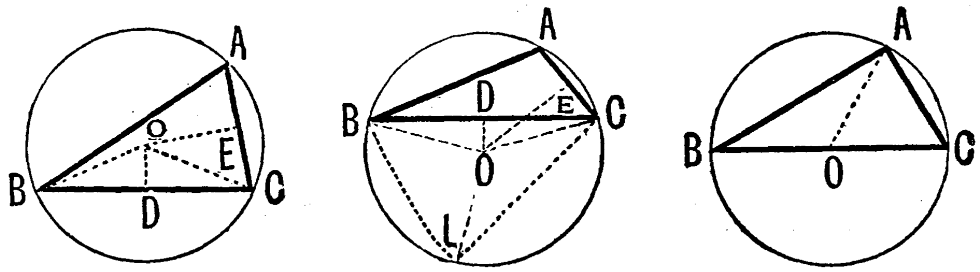


Fig. 1. Fig. 2. Fig. 3.

The point O may either lie within the triangle as in Fig. I., or without it as in Fig. II., or upon one of the sides as in Fig. III.

Taking the first figure, the two triangles BOD and COD are equal in all respects, so that

$$\angle BOD = \angle COD,$$

$$\therefore \angle BOD = \tfrac{1}{2}\angle BOC = \angle BAC \quad \text{(Euc. III. 20)},$$
$$= A.$$

Also $$BD = BO \sin BOD.$$

$$\therefore \frac{a}{2} = R \sin A.$$

If A be obtuse, as in Fig. II., we have

$$\angle BOD = \tfrac{1}{2}\angle BOC = \angle BLC = 180^\circ - A \quad \text{(Euc. III. 22)},$$

so that, as before, $\sin BOD = \sin A$,

and $$R = \frac{a}{2 \sin A}.$$

If A be a right angle, as in Fig. III., we have

$$R = OA = OC = \frac{a}{2}$$

$$= \frac{a}{2 \sin A}, \text{ since in this case } \sin A = 1.$$

The relation found above is therefore true for all triangles.

Hence, in all three cases, we have

$$\mathbf{R} = \frac{\mathbf{a}}{\mathbf{2\sin A}} = \frac{\mathbf{b}}{\mathbf{2\sin B}} = \frac{\mathbf{c}}{\mathbf{2\sin C}} \quad \text{(Art. 163).}$$

201. In Art. 169 we have shewn that

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2S}{bc},$$

where S is the area of the triangle.

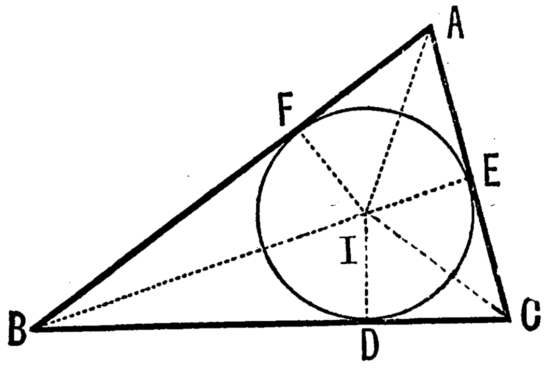
Substituting this value of $\sin A$ in (1), we have

$$\mathbf{R} = \frac{\mathbf{abc}}{\mathbf{4S}},$$

giving the radius of the circumcircle in terms of the sides.

202. *To find the value of r, the radius of the incircle of the triangle ABC.*

Bisect the two angles B and C by the two lines BI and CI meeting in I.



By Euc. III. 4, I is the centre of the incircle. Join IA, and draw ID, IE and IF perpendicular to the three sides.

Then $ID = IE = IF = r$.

We have

$$\text{area of } \triangle\, IBC = \tfrac{1}{2} ID \,.\, BC = \tfrac{1}{2} r \,.\, a,$$

$$\text{area of } \triangle\, ICA = \tfrac{1}{2} IE \,.\, CA = \tfrac{1}{2} r \,.\, b,$$

and

$$\text{area of } \triangle\, IAB = \tfrac{1}{2} IF \,.\, AB = \tfrac{1}{2} r \,.\, c.$$

Hence, by addition, we have

$\frac{1}{2}r \cdot a + \frac{1}{2}r \cdot b + \frac{1}{2}r \cdot c =$ sum of the areas of the triangles IBC, ICA, and IAB

$= \text{area of the } \triangle ABC.$

i.e. $$r\frac{a+b+c}{2} = S,$$

so that $$r \cdot s = S.$$

$$\therefore\ \mathbf{r} = \frac{\mathbf{S}}{\mathbf{s}}.$$

203. Since the angles IBD and IDB are respectively equal to the angles IBF and IFB, the two triangles IDB and IFB are equal in all respects.

Hence $BD = BF$, so that $2BD = BD + BF$.

So also $AE = AF$, so that $2AE = AE + AF$,

and $CE = CD$, so that $2CE = CE + CD$.

Hence, by addition, we have

$$2BD + 2AE + 2CE = (BD + CD) + (CE + AE) + (AF + FB),$$

i.e. $$2BD + 2AC = BC + CA + AB.$$

$$\therefore\ 2BD + 2b = a + b + c = 2s.$$

Hence $$BD = s - b = BF;$$

so $$CE = s - c = CD,$$

and $$AF = s - a = AE.$$

Now $$\frac{ID}{BD} = \tan IBD = \tan\frac{B}{2}.$$

$$\therefore\ r = ID = BD\tan\frac{B}{2} = (s-b)\tan\frac{B}{2}.$$

So $\quad r = IE = CE \tan ICE = (s-c)\tan\frac{C}{2}$,

and also $r = IF = FA \tan IAF = (s-a)\tan\frac{A}{2}$.

Hence $\quad \mathbf{r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}}$.

204. A third value for r may be found as follows: we have $\quad a = BD + DC = ID\cot IBD + ID\cot ICD$

$$= r\cot\frac{B}{2} + r\cot\frac{C}{2}$$

$$= r\left[\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}} + \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\right],$$

$$\therefore\ a\sin\frac{B}{2}\sin\frac{C}{2} = r\left[\sin\frac{C}{2}\cos\frac{B}{2} + \cos\frac{C}{2}\sin\frac{B}{2}\right]$$

$$= r\sin\left(\frac{B}{2}+\frac{C}{2}\right) = r\sin\left[90^\circ - \frac{A}{2}\right] = r\cos\frac{A}{2}.$$

$$\therefore\ r = a\,\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}.$$

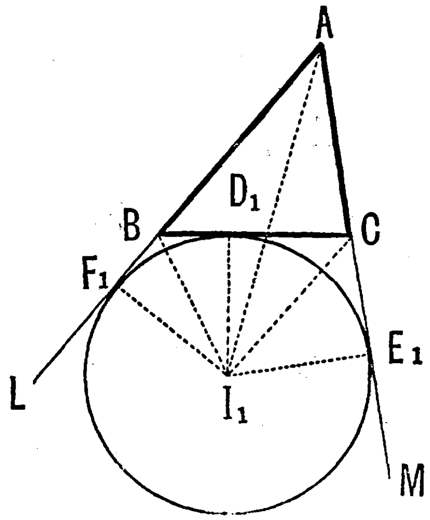
Cor. Since $a = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2}$,

we have $$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

205. *To find the value of r_1, the radius of the escribed circle opposite the angle A of the triangle ABC.*

Produce AB and AC to L and M.

Bisect the angles CBL and BCM by the lines BI_1 and CI_1 and let these lines meet in I_1.



Draw I_1D_1, I_1E_1, and I_1F_1 perpendicular to the three sides respectively.

The two triangles I_1D_1B and I_1F_1B are equal in all respects,

so that $I_1F_1 = I_1D_1$.

Similarly $I_1E_1 = I_1D_1$.

The three perpendiculars I_1D_1, I_1E_1 and I_1F_1 being equal, the point I_1 is the centre of the required circle.

Now the area ABI_1C is equal to the sum of the triangles ABC and I_1BC; it is also equal to the sum of the triangles I_1BA and I_1CA.

Hence

$$\triangle ABC + \triangle I_1BC = \triangle I_1CA + \triangle I_1AB.$$

$$\therefore\ S + \tfrac{1}{2}I_1D_1 \,.\, BC = \tfrac{1}{2}I_1E_1 \,.\, CA + \tfrac{1}{2}I_1F_1 \,.\, AB,$$

i.e. $$S + \tfrac{1}{2}r_1 \,.\, a = \tfrac{1}{2}r_1 \,.\, b + \tfrac{1}{2}r_1 \,.\, c.$$

$$\therefore\ S = r_1\left[\frac{b+c-a}{2}\right] = r_1\left[\frac{b+c+a}{2} - a\right] = r_1(s-a).$$

$$\therefore\ \mathbf{r}_1 = \frac{\mathbf{S}}{\mathbf{s}-\mathbf{a}}.$$

Similarly it can be shewn that

$$\mathbf{r}_2 = \frac{\mathbf{S}}{\mathbf{s}-\mathbf{b}}, \text{ and } \mathbf{r}_3 = \frac{\mathbf{S}}{\mathbf{s}-\mathbf{c}}.$$

206. Since AE_1 and AF_1 are tangents, we have, as in Art. 203, $AE_1 = AF_1$.

Similarly $BF_1 = BD_1$, and $CE_1 = CD_1$.

$$\therefore\ 2AE_1 = AE_1 + AF_1 = AB + BF_1 + AC + CE_1$$

$$= AB + BD_1 + AC + CD_1 = AB + BC + CA = 2s.$$

$$\therefore\ AE_1 = s = AF_1.$$

Also $\quad BD_1 = BF_1 = AF_1 - AB = s - c,$

and $\quad CD_1 = CE_1 = AE_1 - AC = s - b.$

$$\therefore\ I_1E_1 = AE_1 \tan I_1AE_1,$$

i.e. $\quad \mathbf{r_1 = s \tan \frac{A}{2}}.$

207. A third value may be obtained for r_1 in terms of a and the angles B and C.

For, since I_1C bisects the angle BCE_1, we have

$$\angle I_1CD_1 = \tfrac{1}{2}(180^\circ - C) = 90^\circ - \frac{C}{2}.$$

So $\quad \angle I_1BD_1 = 90^\circ - \frac{B}{2}.$

$$\therefore\ a = BC = BD_1 + D_1C$$

$$= I_1D_1 \cot I_1BD_1 + I_1D_1 \cot I_1CD_1$$

$$= r_1 \left(\tan \frac{B}{2} + \tan \frac{C}{2}\right)$$

$$= r_1 \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}\right),$$

$$\therefore\ a\cos\frac{B}{2}\cos\frac{C}{2}=r_1\left(\sin\frac{B}{2}\cos\frac{C}{2}+\cos\frac{B}{2}\sin\frac{C}{2}\right)$$

$$=r_1\sin\left(\frac{B}{2}+\frac{C}{2}\right)=r_1\sin\left(90^\circ-\frac{A}{2}\right)=r_1\cos\frac{A}{2}.$$

$$\therefore\ r_1=a\,\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}.$$

Cor. Since $a=2R\sin A=4R\sin\frac{A}{2}\cos\frac{A}{2}$,

we have $$r_1=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

EXAMPLES. XXXVI.

1. In a triangle whose sides are 18, 24, and 30 inches respectively, prove that the circumradius, the inradius, and the radii of the three escribed circles are respectively 15, 6, 12, 18, and 36 inches.

2. The sides of a triangle are 13, 14, and 15 feet; prove that

(1) $R=8\frac{1}{8}$ ft., (2) $r=4$ ft., (3) $r_1=10\frac{1}{2}$ ft.,

(4) $r_2=12$ ft., and (5) $r_3=14$ ft.

3. In a triangle ABC if $a=13$, $b=4$, and $\cos C=-\frac{5}{13}$, find

R, r, r_1, r_2, and r_3.

4. In the ambiguous case of the solution of triangles prove that the circumcircles of the two triangles are equal.

Prove that

5. $r_1+r_2+r_3-r=4R$. **6.** $r_1r_2+r_2r_3+r_3r_1=s^2$.

7. $r_1r_2r_3=r^3\cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}$. **8.** $rr_1r_2r_3=S^2$.

9. $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r}=0$. **10.** $S=2R^2\sin A\sin B\sin C$.

11. $4R \sin A \sin B \sin C = a \cos A + b \cos B + c \cos C$.

12. $S = 4Rr \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$.

13. $\frac{rr_1}{r_2 r_3} = \tan^2\frac{A}{2}$.

14. $r_1 (s-a) = r_2 (s-b) = r_3 (s-c) = rs = S$.

15. $a (rr_1 + r_2 r_3) = b (rr_2 + r_3 r_1) = c (rr_3 + r_1 r_2)$.

16. $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2+b^2+c^2}{S^2}$.

17. $rr_1 \cot\frac{A}{2} = S$.

18. $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$.

19. $(r_1 + r_2) \tan\frac{C}{2} = (r_3 - r) \cot\frac{C}{2} = c$.

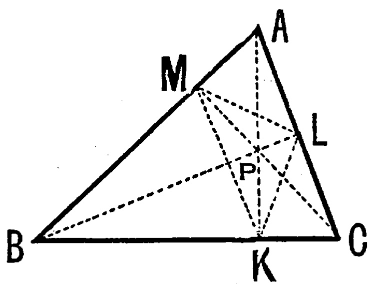
20. $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}$.

21. $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{r} - \frac{1}{2R}$.

22. $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 - b^2 - c^2$.

208. Orthocentre and pedal triangle of any triangle.

Let ABC be any triangle and let AK, BL, and CM be the perpendiculars from A, B, and C upon the opposite sides of the triangle. It can be easily shewn, as in most editions of Euclid, that these three perpendiculars meet in a common point P. This point P is called the **orthocentre** of the triangle. The triangle KLM, which is formed by joining the feet of these perpendiculars, is called the **pedal triangle** of ABC.



209. *Distances of the orthocentre from the angular points of the triangle.*

We have $PK = KB \tan PBK = KB \tan (90° - C)$

$$= AB \cos B \cot C = \frac{c}{\sin C} \cos B \cos C$$

$$= 2R \cos B \cos C \qquad \text{(Art. 200).}$$

Again $\quad AP = AK - PK = c \sin B - PK$

$$= 2R \sin C \sin B - 2R \cos B \cos C$$

$$= - 2R \cos (B + C)$$

$$= 2R \cos A \qquad \text{(Art. 72).}$$

So $\quad BP = 2R \cos B$, and $CP = 2R \cos C$.

The distances of the orthocentre from the angular points are therefore $2R \cos A$, $2R \cos B$ and $2R \cos C$; its distances from the sides are $2R \cos B \cos C$, $2R \cos C \cos A$, and $2R \cos A \cos B$.

210. *To find the sides and angles of the pedal triangle.*

Since the angles PKC and PLC are right angles, the points P, L, C, and K lie on a circle.

$$\therefore \angle PKL = \angle PCL \qquad \text{(Euc. III. 21)}$$

$$= 90° - A.$$

Similarly P, K, B, and M lie on a circle, and therefore

$$\angle PKM = \angle PBM$$

$$= 90° - A.$$

Hence $\quad \angle MKL = 180° - 2A$

$$= \text{the supplement of } 2A.$$

So $\quad \angle KLM = 180° - 2B,$

and $\quad \angle LMK = 180° - 2C.$

Again, from the triangle ALM, we have

$$\frac{LM}{\sin A} = \frac{AL}{\sin AML} = \frac{AB\cos A}{\cos PML}$$

$$= \frac{c\cos A}{\cos PAL} = \frac{c\cos A}{\sin C}.$$

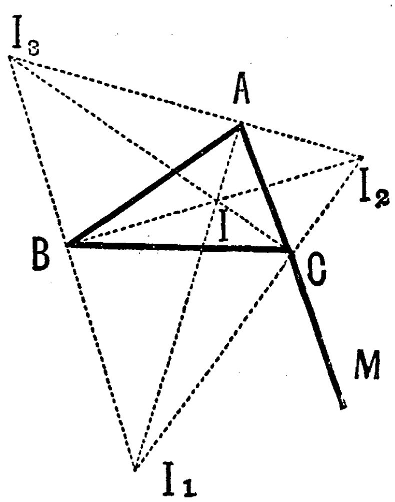
$$\therefore\ LM = \frac{c}{\sin C}\sin A\cos A$$

$$= a\cos A \qquad \text{(Art. 163).}$$

So $\quad MK = b\cos B$, and $KL = c\cos B$.

The sides of the pedal triangle are therefore $a\cos A$, $b\cos B$, and $c\cos C$; also its angles are the supplements of twice the angles of the triangle.

211. Let I be the centre of the incircle and I_1, I_2 and I_3 the centres of the escribed circles which are opposite to A, B and C respectively. As in Arts. 202 and 205 IC bisects the angle ACB, and I_1C bisects the angle BCM.



$$\therefore\ \angle ICI_1 = \angle ICB + \angle I_1CB$$

$$= \tfrac{1}{2}\angle ACB + \tfrac{1}{2}\angle MCB$$

$$= \tfrac{1}{2}[\angle ACB + \angle MCB]$$

$$= \tfrac{1}{2}\,.\,180^\circ = \text{a right angle.}$$

Similarly $\angle ICI_2$ is a right angle.

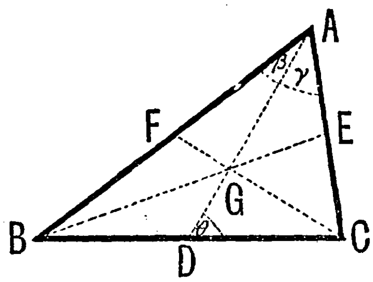
Hence I_1CI_2 is a straight line to which IC is perpendicular.

So I_2AI_3 is a straight line to which IA is perpendicular, and I_3BI_1 is a straight line to which IB is perpendicular.

Also, since IA and I_1A both bisect the angle BAC, the three points A, I, and I_1 are in a straight line. Similarly BII_2 and CII_3 are straight lines. Hence $I_1I_2I_3$ is a triangle which is such that A, B, and C are the feet of the perpendiculars drawn from its vertices upon the opposite sides and such that I is the intersection of these perpendiculars, *i.e.* ABC is its pedal triangle and I is its orthocentre.

212. Centroid and Medians of any Triangle.

If ABC be any triangle, and D, E, and F respectively the middle points of BC, CA, and AB, the lines AD, BE, and CF are called the **medians** of the triangle.



It is shewn in any edition of Euclid that the medians meet in a common point G, such that

$$AG = \tfrac{2}{3}AD, \quad BG = \tfrac{2}{3}BE,$$

and $$CG = \tfrac{2}{3}CF.$$

This point G is called the **centroid** of the triangle.

213. *Length of the medians.* We have, if $AD = x$,

$$b^2 = AC^2 = AD^2 + DC^2 - 2AD \, . \, DC \cos ADC$$

$$= x^2 + \frac{a^2}{4} - ax \cos ADC,$$

and $$c^2 = AB^2 = x^2 + \frac{a^2}{4} - ax \cos ADB$$

$$= x^2 + \frac{a^2}{4} + ax \cos ADC.$$

Hence, by addition, we have

$$b^2 + c^2 = 2x^2 + \frac{a^2}{2}.$$

$$\therefore\ AD = x = \tfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}.$$

Hence also $AD = \frac{1}{2}\sqrt{b^2 + c^2 + 2bc\cos A}$ (Art. 164).

So also

$$BE = \tfrac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}, \text{ and } CF = \tfrac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}.$$

214. *Angles that the median AD makes with the sides.*

If the $\angle BAD = \beta$, and $\angle CAD = \gamma$, we have

$$\frac{\sin\gamma}{\sin C} = \frac{DC}{AD} = \frac{a}{2x}.$$

$$\therefore\ \sin\gamma = \frac{a\sin C}{2x} = \frac{a\sin C}{\sqrt{2b^2 + 2c^2 - a^2}}.$$

Similarly $\sin\beta = \dfrac{a\sin B}{\sqrt{2b^2 + 2c^2 - a^2}}.$

Again, if the $\angle ADC$ be θ, we have

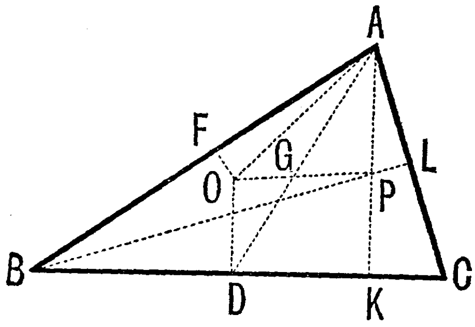
$$\frac{\sin\theta}{\sin C} = \frac{AC}{AD} = \frac{b}{x}.$$

$$\therefore\ \sin\theta = \frac{b\sin C}{x} = \frac{2b\sin C}{\sqrt{2b^2 + 2c^2 - a^2}}.$$

The angles that AD makes with the sides are therefore found.

215. *The centroid lies on the line joining the circumcentre to the orthocentre.*

Let O and P be the circumcentre and orthocentre respectively. Draw OD and PK perpendicular to BC.



Let AD and OP meet in G.

The triangles OGD and PGA are clearly equiangular.

Also, by Art. 200,

$$OD = R\cos A$$

and, by Art. 209,

$$AP = 2R\cos A.$$

Hence, by Euc. VI. 4,

$$\frac{AG}{GD} = \frac{AP}{OD} = 2.$$

The point G is therefore the centroid of the triangle.

Also, by the same proposition,

$$\frac{OG}{GP} = \frac{OD}{AP} = \frac{1}{2}.$$

The centroid therefore lies on the line joining the circumcentre to the orthocentre and divides it in the ratio 1 : 2.

It may be shewn by geometry that the centre of the nine-point circle (which passes through the feet of the perpendiculars, the middle points of the sides, and the middle points of the lines joining the angular points to the orthocentre) lies on OP and bisects it.

The circumcentre, the centroid, the centre of the nine-point circle, and the orthocentre therefore all lie on a straight line.

216. *Distance between the circumcentre and the orthocentre.*

If OF be perpendicular to AB, we have

$$\angle OAF = 90^\circ - \angle AOF = 90^\circ - C.$$

Also $\quad \angle PAL = 90^\circ - C.$

$$\therefore \angle OAP = A - \angle OAF - \angle PAL$$
$$= A - 2(90^\circ - C) = A + 2C - 180^\circ$$
$$= A + 2C - (A + B + C) = C - B.$$

Also $OA = R$, and, by Art. 209,

$$PA = 2R \cos A.$$

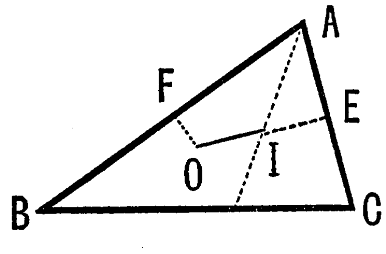
$$\therefore OP^2 = OA^2 + PA^2 - 2OA \,.\, PA \cos OAP$$
$$= R^2 + 4R^2 \cos^2 A - 4R^2 \cos A \cos (C - B)$$
$$= R^2 + 4R^2 \cos A [\cos A - \cos (C - B)]$$
$$= R^2 - 4R^2 \cos A [\cos (B + C) + \cos (C - B)]$$

(Art. 72),

$$= R^2 - 8R^2 \cos A \cos B \cos C.$$

$$\therefore OP = R\sqrt{1 - 8 \cos A \cos B \cos C}.$$

***217.** *To find the distance between the circumcentre and the incentre.*



Let O be the circumcentre and OF perpendicular to AB.

Let I be the incentre and IE perpendicular to AC.

Then, as in the last article,

$$\angle OAF = 90^\circ - C.$$

$$\therefore \angle OAI = \angle IAF - \angle OAF$$

$$= \frac{A}{2} - (90^\circ - C) = \frac{A}{2} + C - \frac{A + B + C}{2} = \frac{C - B}{2}.$$

Also $AI = \dfrac{IE}{\sin\frac{A}{2}} = \dfrac{r}{\sin\frac{A}{2}} = 4R\sin\frac{B}{2}\sin\frac{C}{2}$ (Art. 204. Cor.).

$$\therefore\ OI^2 = OA^2 + AI^2 - 2OA\,.\,AI\cos OAI$$

$$= R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{C-B}{2}.$$

$$\therefore\quad \frac{OI^2}{R^2} = 1 + 16\sin^2\frac{B}{2}\sin^2\frac{C}{2}$$

$$- 8\sin\frac{B}{2}\sin\frac{C}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right]$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B+C}{2}$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2} \quad \text{(Art. 69)}.........(1).$$

$$\therefore\ OI = R\sqrt{1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}}.$$

Also (1) may be written

$$OI^2 = R^2 - 2R \times 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= R^2 - 2Rr. \qquad \text{(Art. 204. Cor.)}$$

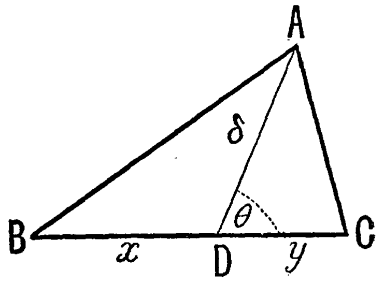
In a similar manner it may be shewn that, if I_1 be the centre of the escribed circle opposite the angle A, we shall have

$$OI_1 = R\sqrt{1 + 8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}},$$

and hence $\qquad OI_1^2 = R^2 + 2Rr_1. \qquad$ (Art. 207. Cor.)

218. Bisectors of the angles.

If AD bisect the angle A and divide the base into portions x and y, we have, by Euc. VI. 3,



$$\frac{x}{y} = \frac{AB}{AC} = \frac{c}{b}.$$

$$\therefore \frac{x}{c} = \frac{y}{b} = \frac{x+y}{b+c} = \frac{a}{b+c} \quad \ldots\ldots (1),$$

giving x and y.

Also, if δ be the length of AD and θ the angle it makes with BC, we have

$$\triangle ABD + \triangle ACD = \triangle ABC.$$

$$\therefore \quad \frac{1}{2} c\delta \sin \frac{A}{2} + \frac{1}{2} b\delta \sin \frac{A}{2} = \frac{1}{2} bc \sin A,$$

i.e.
$$\delta = \frac{bc}{b+c} \frac{\sin A}{\sin \frac{A}{2}} = \frac{2bc}{b+c} \cos \frac{A}{2} \quad \ldots\ldots\ldots (2).$$

Also
$$\theta = 180^\circ - C - \frac{A}{2} = A + B - \frac{A}{2} = \frac{A+2B}{2} \ \ldots \ (3).$$

We thus have the length of the bisector and its inclination to BC.

EXAMPLES. XXXVII.

If I, I_1, I_2, and I_3 be respectively the centres of the incircle and the three escribed circles of a triangle ABC, prove that

1. $AI = r \operatorname{cosec} \frac{A}{2}$.

2. $IA \,.\, IB \,.\, IC = abc \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$.

3. $AI_1 = r_1 \operatorname{cosec} \frac{A}{2}$.

4. $I I_1 = a \sec \frac{A}{2}$.

5. $I_2I_3 = a \operatorname{cosec} \frac{A}{2}$.

6. $II_1 . II_2 . II_3 = 16R^2r$.

7. $I_2I_3^2 = 4R(r_2 + r_3)$.

8. $\angle I_3I_1I_2 = \frac{B+C}{2}$.

9. $II_1^2 + I_2^2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2$.

10. Area of $\Delta I_1I_2I_3 = 8R^2 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2} = \frac{abc}{2r}$.

11. $\frac{II_1 . I_2I_3}{\sin A} = \frac{II_2 . I_3I_1}{\sin B} = \frac{II_3 . I_1I_2}{\sin C}$.

If I, O, and P be respectively the incentre, circumcentre, and orthocentre, and G the centroid of the triangle ABC, prove that

12. $IO^2 = R^2 (3 - 2\cos A - 2\cos B - 2\cos C)$.

13. $IP^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$.

14. $OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$.

15. Area of $\Delta IOP = 2R^2 \sin\frac{B-C}{2} \sin\frac{C-A}{2} \sin\frac{A-B}{2}$.

16. Area of $\Delta IPG = \frac{2}{3} R^2 \sin\frac{B-C}{2} \sin\frac{C-A}{2} \sin\frac{A-B}{2}$.

17. Prove that the distance of the centre of the nine-point circle from the angle A is $\frac{R}{2}\sqrt{1 + 8\cos A \sin B \sin C}$.

18. DEF is the pedal triangle of ABC; prove that

(1) its area is $2S \cos A \cos B \cos C$,

(2) the radius of its circumcircle is $\frac{R}{2}$,

and (3) the radius of its incircle is $2R \cos A \cos B \cos C$.

19. $O_1O_2O_3$ is the triangle formed by the centres of the escribed circles of the triangle ABC; prove that

(1) its sides are $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$, and $4R\cos\frac{C}{2}$,

(2) its angles are $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$, and $\frac{\pi}{2} - \frac{C}{2}$,

and (3) its area is $2Rs$.

20. *DEF* is the triangle formed by joining the points of contact of the incircle with the sides of the triangle *ABC*; prove that

(1) its sides are $2r\cos\frac{A}{2}$, $2r\cos\frac{B}{2}$, and $2r\cos\frac{C}{2}$,

(2) its angles are $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$, and $\frac{\pi}{2}-\frac{C}{2}$,

and (3) its area is $\frac{2S^3}{abcs}$, i.e. $\frac{1}{2}\frac{r}{R}S$.

21. *D*, *E*, and *F* are the middle points of the sides of the triangle *ABC*; prove that the centroid of the triangle *DEF* is the same as that of *ABC* and that its orthocentre is the circumcentre of *ABC*.

In any triangle *ABC*, prove that

22. The perpendicular from *A* divides *BC* into portions which are proportional to the cotangent of the adjacent angles, and that it divides the angle *A* into portions whose cosines are inversely proportional to the adjacent sides.

23. The median through *A* divides it into angles whose cotangents are $2\cot A+\cot C$ and $2\cot A+\cot B$, and makes with the base an angle whose cotangent is $\frac{1}{2}(\cot C-\cot B)$.

24. The distance between the middle point of *BC* and the foot of the perpendicular from *A* is $\frac{b^2-c^2}{2a}$.

25. *O* is the orthocentre of a triangle *ABC*; prove that the radii of the circles circumscribing the triangles *BOC*, *COA*, *AOB* and *ABC* are all equal.

26. *AD*, *BE* and *CF* are the perpendiculars from the angular points of a triangle *ABC* upon the opposite sides; prove that the diameters of the circumcircles of the triangles *AEF*, *BDF* and *CDE* are respectively $a\cot A$, $b\cot B$, and $c\cot C$, and that the perimeters of the triangles *DEF* and *ABC* are in the ratio $r:R$.

27. Prove that the product of the distances of the incentre from the angular points of a triangle is $4Rr^2$.

28. The triangle *DEF* circumscribes the three escribed circles of the triangle *ABC*; prove that

$$\frac{EF}{a\cos A}=\frac{FD}{b\cos B}=\frac{DE}{c\cos C}.$$

29. If a circle be drawn touching the inscribed and circumscribed circles of a triangle and the side BC externally, prove that its radius is

$$\frac{\Delta}{a}\tan^2\frac{A}{2}.$$

30. If a, b, c be the radii of three circles which touch one another externally and r_1 and r_2 be the radii of the two circles that can be drawn to touch these three, prove that

$$\frac{1}{r_1}+\frac{1}{r_2}=\frac{2}{a}+\frac{2}{b}+\frac{2}{c}.$$

31. If Δ_0 be the area of the triangle formed by joining the points of contact of the inscribed circle with the sides of the given triangle, whose area is Δ, and Δ_1, Δ_2, and Δ_3 the corresponding areas for the escribed circles, prove that

$$\Delta_1+\Delta_2+\Delta_3-\Delta_0=2\Delta.$$

32. If the bisectors of the angles of a triangle ABC meet the opposite sides in A', B', and C', prove that the ratio of the areas of the triangles $A'B'C'$ and ABC is

$$2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} : \cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}.$$

33. Through the angular points of a triangle are drawn straight lines which make the same angle α with the opposite sides of the triangle; prove that the area of the triangle formed by them is to the area of the original triangle as $4\cos^2\alpha : 1$.

34. Two circles, of radii a and b, cut each other at an angle θ. Prove that the length of the common chord is

$$\frac{2ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}.$$

35. Three equal circles touch one another; find the radius of the circle which touches all three.

36. Three circles whose radii are a, b and c touch one another and the tangents at their points of contact meet in a point; prove that the distance of this point from either of their points of contact is

$$\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}.$$

37. In the sides BC, CA, AB are taken three points A', B', C' such that

$$BA' : A'C = CB' : B'A = AC' : C'B = m : n;$$

prove that if AA', BB', and CC' be joined they will form by their intersections a triangle whose area is to that of the triangle ABC as

$$(m-n)^2 : m^2+mn+n^2.$$

38. The circle inscribed in the triangle ABC touches the sides BC, CA, and AB in the points A_1, B_1, and C_1 respectively; similarly the circle inscribed in the triangle $A_1B_1C_1$ touches the sides in A_2, B_2, C_2 respectively and so on; if $A_nB_nC_n$ be the nth triangle so formed, prove that its angles are

$$\frac{\pi}{3}+(-2)^{-n}\left(A-\frac{\pi}{3}\right),\ \frac{\pi}{3}+(-2)^{-n}\left(B-\frac{\pi}{3}\right),$$

and

$$\frac{\pi}{3}+(-2)^{-n}\left(C-\frac{\pi}{3}\right).$$

Hence prove that the triangle so formed is ultimately equilateral.

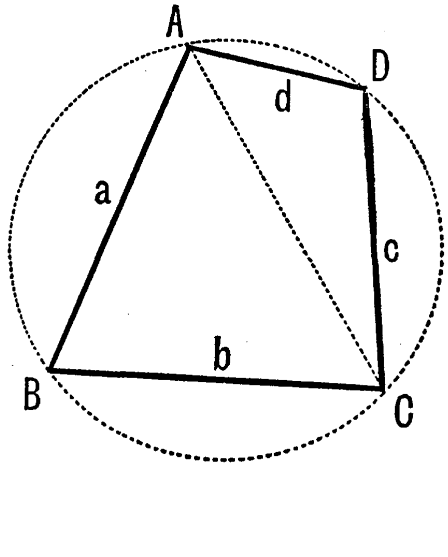
39. $A_1B_1C_1$ is the triangle formed by joining the feet of the perpendiculars drawn from ABC upon the opposite sides; in like manner $A_2B_2C_2$ is the triangle obtained by joining the feet of the perpendiculars from A_1, B_1, and C_1 on the opposite sides and so on. Find the values of the angles A_n, B_n, and C_n in the nth of these triangles.

CHAPTER XVI.

ON QUADRILATERALS AND REGULAR POLYGONS.

219. *To find the area of a quadrilateral which is inscribable in a circle.*

Let $ABCD$ be the quadrilateral, the sides being a, b, c and d as marked in the figure.



The area of the quadrilateral

$=$area of $\triangle ABC +$ area of $\triangle ADC$

$= \frac{1}{2}ab \sin B + \frac{1}{2}cd \sin D$ (Art. 198.)

$= \frac{1}{2}(ab + cd) \sin B$,

since, by Euc. III. 22,

$$\angle B = 180° - \angle D,$$

and therefore

$$\sin B = \sin D.$$

We have to express $\sin B$ in terms of the sides.

We have

$$a^2 + b^2 - 2ab \cos B = AC^2 = c^2 + d^2 - 2cd \cos D.$$

But $\qquad \cos D = \cos(180° - B) = -\cos B.$

Hence

$$a^2 + b^2 - 2ab \cos B = c^2 + d^2 + 2cd \cos B,$$

so that $$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}.$$

Hence

$$\sin^2 B = 1 - \cos^2 B = 1 - \frac{(a^2 + b^2 - c^2 - d^2)^2}{\{2(ab + cd)\}^2}$$

$$= \frac{\{2(ab + cd)\}^2 - \{a^2 + b^2 - c^2 - d^2\}^2}{4(ab + cd)^2}$$

$$= \frac{\{2(ab + cd) + (a^2 + b^2 - c^2 - d^2)\}\{2(ab + cd) - (a^2 + b^2 - c^2 - d^2)\}}{4(ab + cd)^2}$$

$$= \frac{\{(a^2 + 2ab + b^2) - (c^2 - 2cd + d^2)\}\{(c^2 + 2cd + d^2) - (a^2 + b^2 - 2ab)\}}{4(ab + cd)^2}$$

$$= \frac{\{(a + b)^2 - (c - d)^2\}\{(c + d)^2 - (a - b)^2\}}{4(ab + cd)^2}$$

$$= \frac{\{(a + b + c - d)(a + b - c + d)\}\{(c + d + a - b)(c + d - a + b)\}}{4(ab + cd)^2}.$$

Let

$$a + b + c + d = 2s,$$

so that

$$a + b + c - d = (a + b + c + d) - 2d = 2(s - d),$$
$$a + b - c + d = 2(s - c),$$
$$a - b + c + d = 2(s - b),$$

and $$-a + b + c + d = 2(s - a).$$

Hence

$$\sin^2 B = \frac{2(s - d) \times 2(s - c) \times 2(s - b) \times 2(s - a)}{4(ab + cd)^2},$$

so that

$$(ab + cd) \sin B = 2\sqrt{(s - a)(s - b)(s - c)(s - d)}.$$

Hence the area of the quadrilateral

$$= \tfrac{1}{2}(ab + cd) \sin B = \sqrt{(\mathbf{s} - \mathbf{a})(\mathbf{s} - \mathbf{b})(\mathbf{s} - \mathbf{c})(\mathbf{s} - \mathbf{d})}.$$

220. Since $\cos B = \dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}$,

we have
$$AC^2 = a^2+b^2-2ab\cos B$$
$$= a^2+b^2-ab\,\frac{a^2+b^2-c^2-d^2}{ab+cd}$$
$$= \frac{(a^2+b^2)\,cd+ab\,(c^2+d^2)}{ab+cd}$$
$$= \frac{(ac+bd)(ad+bc)}{ab+cd}.$$

Similarly it could be proved that
$$BD^2 = \frac{(ab+cd)(ac+bd)}{ad+bc}.$$

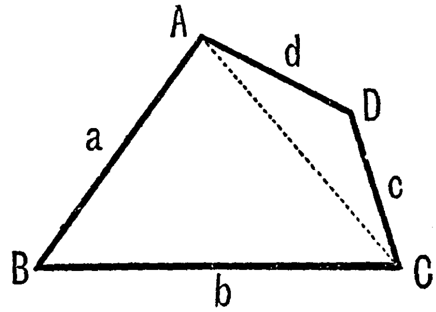
It follows by multiplication that
$$AC^2\,.\,BD^2 = (ac+bd)^2,$$

i.e. $$AC\,.\,BD = AB\,.\,CD + BC\,.\,AD.$$

This is Euc. VI. Prop. D.

221. If we have any quadrilateral, not necessarily inscribable in a circle, we can express its area in terms of its sides and the sum of any two opposite angles.

For let the sum of the two angles B and D be denoted by 2α, and denote the area of the quadrilateral by Δ.



Then

Δ = area of ABC + area of ACD

$= \frac{1}{2}ab\sin B + \frac{1}{2}cd\sin D$,

so that
$$4\Delta = 2ab\sin B + 2cd\sin D \ldots (1).$$

Also $a^2 + b^2 - 2ab\cos B = c^2 + d^2 - 2cd\cos D$,
so that

$$a^2 + b^2 - c^2 - d^2 = 2ab\cos B - 2cd\cos D \;\ldots\ldots(2).$$

Squaring (1) and (2) and adding, we have

$$16\Delta^2 + (a^2 + b^2 - c^2 - d^2)^2 = 4a^2b^2 + 4c^2d^2 - 8abcd(\cos B\cos D - \sin B\sin D)$$
$$= 4a^2b^2 + 4c^2d^2 - 8abcd\cos(B + D)$$
$$= 4a^2b^2 + 4c^2d^2 - 8abcd\cos 2\alpha$$
$$= 4a^2b^2 + 4c^2d^2 - 8abcd(2\cos^2\alpha - 1)$$
$$= 4(ab + cd)^2 - 16abcd\cos^2\alpha,$$

so that

$$16\Delta^2 = 4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2 - 16abcd\cos^2\alpha \;\ldots\ldots\ldots\ldots(3).$$

But, as in Art. 219, we have

$$4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$$
$$= 2(s - a) \,.\, 2(s - b) \,.\, 2(s - c) \,.\, 2(s - d)$$
$$= 16(s - a)(s - b)(s - c)(s - d).$$

Hence (3) becomes

$$\Delta^2 = (s - a)(s - b)(s - c)(s - d) - abcd\cos^2\alpha,$$

giving the required area.

Cor. 1. If d be zero the quadrilateral becomes a triangle, and the formula above becomes that of Art. 198.

Cor. 2. If the sides of the quadrilateral be given in length, we know a, b, c, d and therefore s. The area Δ is hence greatest when $abcd\cos^2\alpha$ is least, that is when $\cos^2\alpha$ is zero, and then $\alpha = 90°$. In this case the sum of two opposite angles of the quadrilateral is 180° and the figure inscribable in a circle. (Euc. III. 22.)

The quadrilateral, whose sides are given, has therefore the greatest area when it can be inscribed in a circle.

222. Ex. *Find the area of a quadrilateral which can have a circle inscribed in it.*

If the quadrilateral $ABCD$ can have a circle inscribed in it so as to touch the sides AB, BC, CD, and DA in the points P, Q, R, and S, we should have

$$AP=AS,\ BP=BQ,\ CQ=CR, \text{ and } DR=DS.$$

$$\therefore\ AP+BP+CR+DR=AS+BQ+CQ+DS,$$

i.e. $$AB+CD=BC+DA,$$

i.e. $$a+c=b+d.$$

Hence $$s=\frac{a+b+c+d}{2}=a+c=b+d.$$

$$\therefore\ s-a=c,\ s-b=d,\ s-c=a, \text{ and } s-d=b.$$

The formula of the last article therefore gives in this case

$$\Delta^2=abcd-abcd\cos^2\alpha=abcd\sin^2\alpha,$$

i.e. $$\text{the area required}=\sqrt{abcd}\sin\alpha.$$

If in addition the quadrilateral be also inscribable in a circle, we have

$$2\alpha=180^\circ, \text{ so that } \sin\alpha=\sin 90^\circ=1.$$

Hence the area of a quadrilateral which can be both inscribed in a circle and circumscribed about another circle is $\sqrt{abcd}$.

EXAMPLES. XXXVIII.

1. Find the area of a quadrilateral, which can be inscribed in a circle, whose sides are

(1) 3, 5, 7, and 9 feet;

and (2) 7, 10, 5, and 2 feet.

2. The sides of a quadrilateral are respectively 3, 4, 5, and 6 feet, and the sum of a pair of opposite angles is 120°; prove that the area of the quadrilateral is $3\sqrt{30}$ square feet.

3. The sides of a quadrilateral which can be inscribed in a circle are 3, 3, 4, and 4 feet; find the radii of the incircle and circumcircle.

4. Prove that the area of any quadrilateral is one-half the product of the two diagonals and the sine of the angle between them.

5. If a quadrilateral can be inscribed in one circle and circumscribed about another circle, prove that its area is $\sqrt{abcd}$ and that the radius of the latter circle is

$$\frac{2\sqrt{abcd}}{a+b+c+d}.$$

6. A quadrilateral $ABCD$ is described about a circle; prove that

$$AB \sin\frac{A}{2} \sin\frac{B}{2} = CD \sin\frac{C}{2} \sin\frac{D}{2}.$$

7. a, b, c, and d are the sides of a quadrilateral taken in order, and α is the angle between the diagonals opposite to b or d; prove that the area of the quadrilateral is

$$\frac{1}{4}(a^2 - b^2 + c^2 - d^2) \tan\alpha.$$

8. If a, b, c, d be the sides and x and y the diagonals of a quadrilateral, prove that its area is

$$\frac{1}{4}[4x^2y^2 - (b^2 + d^2 - a^2 - c^2)^2]^{\frac{1}{2}}.$$

9. If a quadrilateral can be inscribed in a circle, prove that the angle between its diagonals is

$$\sin^{-1}[2\sqrt{(s-a)(s-b)(s-c)(s-d)} \div (ac+bd)].$$

If the same quadrilateral can also be circumscribed about a circle, prove that this angle is then

$$\cos^{-1}\frac{ac-bd}{ac+bd}.$$

10. The sides of a quadrilateral are divided in order in the ratio $m : n$, and a new quadrilateral is formed by joining the points of division; prove that its area is to the area of the original figure as $m^2 + n^2$ to $(m+n)^2$.

11. If a quadrilateral can be inscribed in a circle, prove that the radius of the circle is

$$\frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}.$$

12. If a, b, c, d be the sides of a quadrilateral, taken in order, prove that

$$d^2 = a^2 + b^2 + c^2 - 2ab\cos\alpha - 2bc\cos\beta - 2ca\cos\gamma,$$

where α, β and γ denote the angles between the sides a and b, b and c, and c and a respectively.

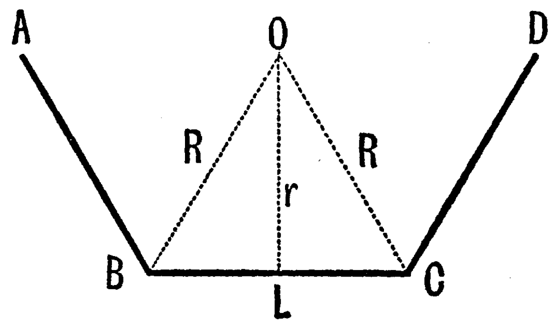
223. Regular Polygons. A regular polygon is a polygon which has all its sides equal and all its angles equal.

If the polygon have n angles we have, by Euc. I. 32, Cor., n times its angle + 4 right angles = twice as many right angles as the figure has sides = $2n$ right angles.

Hence each angle $= \frac{2n-4}{n}$ right angles $= \frac{2n-4}{n} \times \frac{\pi}{2}$ radians.

224. *Radii of the inscribed and circumscribing circles of a regular polygon.*

Let AB, BC, and CD be three successive sides of the polygon, and let n be the number of its sides.



Bisect the angles ABC and BCD by the lines BO and CO which meet in O, and draw OL perpendicular to BC.

It is easily seen that O is the centre of both the incircle and the circumcircle of the polygon and that BL equals LC.

Hence we have $OB = OC = R$, the radius of the circum-circle and $OL = r$, the radius of the incircle.

The angle BOC is $\frac{1}{n}$th of the sum of all the angles subtended at O by the sides,

i.e. $$\angle BOC = \frac{4 \text{ right angles}}{n} = \frac{2\pi}{n} \text{ radians.}$$

Hence $$\angle BOL = \tfrac{1}{2} \angle BOC = \frac{\pi}{n}.$$

If a be a side of the polygon, we have

$$a = BC = 2BL = 2R \sin BOL = 2R \sin \frac{\pi}{n}.$$

$$\therefore \quad R = \frac{a}{2 \sin \frac{\pi}{n}} = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n} \quad \ldots\ldots\ldots\ldots(1).$$

Again,

$$a = 2BL = 2OL \tan BOL = 2r \tan \frac{\pi}{n}.$$

$$\therefore \quad r = \frac{a}{2 \tan \frac{\pi}{n}} = \frac{a}{2} \cot \frac{\pi}{n} \quad \ldots\ldots\ldots\ldots(2).$$

225. Area of a Regular Polygon.

The area of the polygon is n times the area of the triangle BOC.

Hence the area of the polygon

$$= n \times \tfrac{1}{2} OL \,.\, BC = n \,.\, OL \,.\, BL = n \,.\, BL \cot LOB \,.\, BL$$

$$= n \,.\, \frac{a^2}{4} \cot \frac{\pi}{n} \quad \ldots\ldots (1),$$

giving the area in terms of the side.

Also the area

$$= n \,.\, OL \,.\, BL = n \,.\, OL \,.\, OL \tan BOL = nr^2 \tan \frac{\pi}{n} \ldots(2).$$

Again, the area

$$= n \,.\, OL \,.\, BL = n \,.\, OB \cos LOB \,.\, OB \sin LOB$$

$$= nR^2 \cos\frac{\pi}{n} \sin\frac{\pi}{n} = \frac{n}{2} R^2 \sin\frac{2\pi}{n} \qquad \text{.................(3).}$$

226. Ex. *The length of each side of a regular dodecagon is* 20 *feet; find* (1) *the radius of its inscribed circle,* (2) *the radius of its circumscribing circle, and* (3) *its area.*

The angle subtended by a side at the centre of the polygon

$$= \frac{360^\circ}{12} = 30^\circ.$$

Hence we have $\qquad 10 = r \tan 15^\circ = R \sin 15^\circ.$

$$\therefore \; r = 10 \cot 15^\circ$$

$$= \frac{10}{2-\sqrt{3}} \quad \text{(Art. 101)}$$

$$= 10\,(2+\sqrt{3}) = 37\cdot32\ldots \text{ feet.}$$

Also $\qquad R = \dfrac{10}{\sin 15^\circ} = 10 \times \dfrac{2\sqrt{2}}{\sqrt{3}-1} \quad \text{(Art. 106)}$

$$= 10 \,.\, \sqrt{2}\,(\sqrt{3}+1) = 10\,(\sqrt{6}+\sqrt{2})$$

$$= 10\,(2\cdot4495\ldots + 1\cdot4142\ldots) = 38\cdot637\ldots \text{ feet.}$$

Again, the area $\qquad = 12 \times r \times 10$ square feet

$$= 1200\,(2+\sqrt{3}) = 4478\cdot46\ldots \text{ square feet.}$$

EXAMPLES. XXXIX.

1. Find, correct to ·01 of an inch, the length of the perimeter of a regular decagon which surrounds a circle of radius one foot.

2. Find to 3 places of decimals the length of the side of a regular polygon of 12 sides which is circumscribed to a circle of unit radius.

3. Find the area of (1) a pentagon, (2) a hexagon, (3) an octagon, (4) a decagon and (5) a dodecagon, each being a regular figure of side 1 foot.

4. Find the difference between the areas of a regular octagon and a regular hexagon if the perimeter of each be 24 feet.

5. A square, whose side is 2 feet, has its corners cut away so as to form a regular octagon; find its area.

6. Compare the areas and perimeters of octagons which are respectively inscribed in and circumscribed to a given circle, and shew that the areas of the inscribed hexagon and octagon are nearly as $\sqrt{27}$ to $\sqrt{32}$.

7. Prove that the radius of the circle described about a regular pentagon is nearly $\frac{17}{20}$ths of the side of the pentagon.

8. If an equilateral triangle and a regular hexagon have the same perimeter, prove that their areas are as 2 : 3.

9. If a regular pentagon and a regular decagon have the same perimeter, prove that their areas are as $2 : \sqrt{5}$.

10. Prove that the sum of the radii of the circles, which are respectively inscribed in and circumscribed about a regular polygon of n sides, is

$$\frac{a}{2} \cot \frac{\pi}{2n},$$

where a is a side of the polygon.

11. Of two regular polygons of n sides, one circumscribes and the other is inscribed in a given circle. Prove that the three perimeters are in the ratio

$$\sec \frac{\pi}{n} : \frac{\pi}{n} \operatorname{cosec} \frac{\pi}{n} : 1,$$

and that the areas of the polygons are in the ratio $\cos^2 \frac{\pi}{n} : 1$.

12. Given that the area of a polygon of n sides circumscribed about a circle is to the area of the circumscribed polygon of $2n$ sides as 3 : 2, find n.

13. Prove that the area of a regular polygon of $2n$ sides inscribed in a circle is a mean proportional between the areas of the regular inscribed and circumscribed polygons of n sides.

14. The area of a regular polygon of n sides inscribed in a circle is to that of the same number of sides circumscribing the same circle as 3 is to 4. Find the value of n.

15. The interior angles of a polygon are in A. P.; the least angle is 120° and the common difference is 5°; find the number of sides.

16. There are two regular polygons the number of sides in one being double the number in the other, and an angle of one polygon is to an angle of the other as 9 to 8; find the number of sides of each polygon.

17. Show that there are eleven pairs of regular polygons such that the number of degrees in the angle of one is to the number in the angle of the other as 10 : 9. Find the number of sides in each.

18. The side of a base of a square pyramid is a feet and its vertex is at a height of h feet above the centre of the base; if θ and ϕ be respectively the inclinations of any face to the base, and of any two faces to one another, prove that

$$\tan\theta = \frac{2h}{a} \text{ and } \tan\frac{\phi}{2} = \sqrt{1 + \frac{a^2}{2h^2}}.$$

19. A pyramid stands on a regular hexagon as base. The perpendicular from the vertex of the pyramid on the base passes through the centre of the hexagon and its length is equal to that of a side of the base. Find the tangent of the angle between the base and any face of the pyramid and also of half the angle between any two side faces.

20. A regular pyramid has for its base a polygon of n sides, each of length a, and the length of each slant side is l; prove that the cosine of the angle between two adjacent lateral faces is

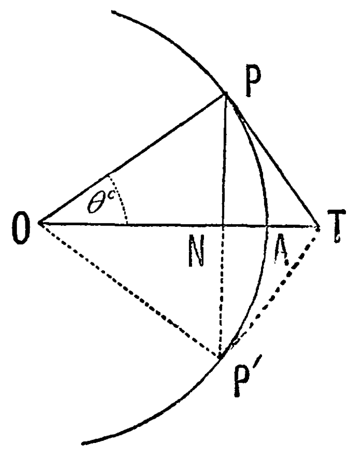
$$\frac{4l^2\cos\frac{2\pi}{n} + a^2}{4l^2 - a^2}.$$

CHAPTER XVII.

TRIGONOMETRICAL RATIOS OF SMALL ANGLES. AREA OF A CIRCLE. DIP OF THE HORIZON.

227. *If θ be the number of radians in any angle, which is less than a right angle, then sin θ, θ and tan θ are in ascending order of magnitude.*

Let TOP be any angle which is less than a right angle.



With centre O and any radius OP describe an arc PAP' meeting OT in A.

Draw PN perpendicular to OA and produce it to meet the arc of the circle in P'.

Draw the tangent PT at P to meet OA in T and join TP'.

The triangles PON and $P'ON$ are equal in all respects, so that $PN = NP'$ and

$$\text{arc } PA = \text{arc } AP'.$$

Also the triangles TOP and TOP' are equal in all respects, so that

$$TP = TP'.$$

The straight line PP' is less than the arc PAP', so that NP is $<$ arc PA.

We shall *assume* that the arc PAP' is less than the sum of PT and TP', so that arc $PA < PT$.

Hence NP, the arc AP, and PT are in ascending order of magnitude.

Therefore $\frac{NP}{OP}$, $\frac{\text{arc } AP}{OP}$ and $\frac{PT}{OP}$ are in ascending order of magnitude.

But $$\frac{NP}{OP} = \sin AOP = \sin\theta,$$

$$\frac{\text{arc } AP}{OP} = \text{number of radians in } \angle AOP = \theta \text{ (Art. 21)},$$

and $$\frac{PT}{OP} = \tan POT = \tan AOP = \tan\theta.$$

Hence $\sin\theta$, θ, and $\tan\theta$ are in ascending order of magnitude, provided that

$$\theta < \frac{\pi}{2}.$$

228. Since $\sin\theta < \theta < \tan\theta$, we have, by dividing each by the positive quantity $\sin\theta$,

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Hence $\frac{\theta}{\sin\theta}$ always lies between 1 and $\frac{1}{\cos\theta}$.

This holds however small θ may be.

Now when θ is very small $\cos\theta$ is very nearly unity, and the smaller θ becomes, the more nearly does $\cos\theta$ become unity, and hence the more nearly does $\frac{1}{\cos\theta}$ become unity.

Hence when θ is very small the quantity $\frac{\theta}{\sin\theta}$ lies between 1 and a quantity which differs from unity by an indefinitely small quantity.

In other words, when θ is made indefinitely small the quantity $\frac{\theta}{\sin\theta}$, and therefore $\frac{\sin\theta}{\theta}$, is ultimately equal to unity, *i.e.* the smaller an angle becomes the more nearly is its sine equal to the number of radians in it.

This is often shortly expressed thus;

$$\sin\theta = \theta, \text{ when } \theta \text{ is very small.}$$

So also $\tan\theta = \theta$, when θ is very small.

Cor. Putting $\theta = \frac{\alpha}{n}$, it follows that, when θ is indefinitely small, n is indefinitely great.

Hence $\dfrac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}}$ is unity, when n is indefinitely great.

So $n\sin\frac{\alpha}{n} = \alpha$, when n is indefinitely great.

229. In the preceding article it must be particularly noticed that θ is the number of radians in the angle considered.

The value of $\sin\alpha^\circ$, when α is small, may be found. For, since $\pi^c = 180^\circ$, we have

$$\alpha^\circ = \left(\pi\frac{\alpha}{180}\right)^c.$$

$$\therefore\ \sin\alpha^\circ = \sin\left(\frac{\pi\alpha}{180}\right)^c = \frac{\pi\alpha}{180},$$

by the result of the last article.

230. From the tables it will be seen that the sine of an angle and its circular measure agree to 7 places of decimals so long as the angle is not greater than 18′. They agree to the 5th place of decimals so long as the angle is less than about 2°.

231. *If θ be the number of radians in an angle, which is less than a right angle, then sin θ* is $>\theta - \frac{\theta^3}{4}$ *and* $\cos\theta$ *is* $>1 - \frac{\theta^2}{2}$.

By Art. 227 we have

$$\tan\frac{\theta}{2} > \frac{\theta}{2}.$$

$$\therefore\ \sin\frac{\theta}{2} > \frac{\theta}{2}\cos\frac{\theta}{2}.$$

Hence, since $$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

we have $$\sin\theta > \theta\cos^2\frac{\theta}{2},\ i.e. > \theta\left(1 - \sin^2\frac{\theta}{2}\right).$$

But since, by Art. 227,

$$\sin\frac{\theta}{2} < \frac{\theta}{2},$$

therefore $$1 - \sin^2\frac{\theta}{2} > 1 - \left(\frac{\theta}{2}\right)^2,\ i.e. > 1 - \frac{\theta^2}{4}.$$

$$\therefore\ \sin\theta > \theta\left(1 - \frac{\theta^2}{4}\right),\ i.e.\ > \theta - \frac{\theta^3}{4}.$$

Again, $$\cos\theta = 1 - 2\sin^2\frac{\theta}{2};$$

therefore, since $$\sin^2\frac{\theta}{2} < \left(\frac{\theta}{2}\right)^2,$$

we have $$1 - 2\sin^2\frac{\theta}{2} > 1 - 2\left(\frac{\theta}{2}\right)^2,\ i.e.\ > 1 - \frac{\theta^2}{2}.$$

It will be found in a later chapter that

$$\sin\theta > \theta - \frac{\theta^3}{6},\ \text{and}\ \cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

232. **Ex. 1.** *Find the values of* sin 10′ *and* cos 10′.

Since $$10' = \frac{1^c}{6} = \frac{\pi^c}{180 \times 6},$$

we have $$\sin 10' = \sin\left(\frac{\pi}{180\times 6}\right)^c = \frac{\pi}{180\times 6}$$

$$=\frac{3\cdot 14159265\ldots}{180\times 6} = \cdot 0029089 \text{ nearly.}$$

Also $$\cos 10' = \sqrt{1-\sin^2 10'}$$

$$=[1-\cdot 000008468\ldots]^{\frac{1}{2}}$$

$$=1-\frac{1}{2}[\cdot 000008468\ldots],$$

approximately by the Binomial Theorem,

$$=1-\cdot 000004234\ldots$$

$$=\cdot 9999958\ldots.$$

Ex. 2. *Solve approximately the equation*

$$\sin\theta = \cdot 52.$$

Since $\sin\theta$ is very nearly equal to $\frac{1}{2}$, θ must be nearly equal to $\frac{\pi}{6}$.

Let then $\theta = \frac{\pi}{6} + x$, where x is small.

$$\therefore\ \cdot 52 = \sin\left(\frac{\pi}{6}+x\right) = \sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x$$

$$=\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x.$$

Since x is very small, we have

$$\cos x = 1 \text{ and } \sin x = x \text{ nearly.}$$

$$\therefore\ \cdot 52 = \frac{1}{2} + \frac{\sqrt{3}}{2}x.$$

$$\therefore\ x = \cdot 02 \times \frac{2}{\sqrt{3}} \text{ radians} = \frac{\sqrt{3}^c}{75} = 1\cdot 32^\circ \text{ nearly.}$$

Hence $$\theta = 31^\circ\ 19' \text{ nearly.}$$

EXAMPLES. XL.

Taking π equal to $3\cdot 14159265$, find to 5 places of decimals the value of

1. $\sin 7'$.
2. $\sin 15''$.
3. $\sin 1'$.
4. $\cos 15'$.
5. $\operatorname{cosec} 8''$.
6. $\sec 5'$.

Solve approximately the equations

7. $\sin\theta = \cdot 01$. **8.** $\sin\theta = \cdot 48$.

9. $\cos\left(\frac{\pi}{3}+\theta\right) = \cdot 49$. **10.** $\cos\theta = \cdot 999$.

11. Find approximately the distance at which a halfpenny, which is an inch in diameter, must be placed so as to just hide the moon, the angular diameter of the moon, that is the angle its diameter subtends at the observer's eye, being taken to be 30′.

12. A person walks in a straight line toward a very distant object and observes that at three points A, B, and C the angles of elevation of the top of the object are α, 2α, and 3α respectively; prove that

$$AB = 3BC \text{ nearly.}$$

13. If θ be the number of radians in an angle which is less than a right angle, prove that

$$\cos\theta \text{ is } < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{16}.$$

14. Prove the theorem of Euler, viz. that

$$\sin\theta = \theta \,.\, \cos\frac{\theta}{2} \,.\, \cos\frac{\theta}{2^2} \,.\, \cos\frac{\theta}{2^3} \ldots\ldots \text{ad. inf.}$$

[We have $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2^2\sin\frac{\theta}{2^2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2}$

$$= 2^3\sin\frac{\theta}{2^3}\cos\frac{\theta}{2^3}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2} = \ldots\ldots$$

$$= 2^n\sin\frac{\theta}{2^n} \times \cos\frac{\theta}{2} \,.\, \cos\frac{\theta}{2^2} \,.\, \cos\frac{\theta}{2^3} \ldots\ldots \cos\frac{\theta}{2^n}.$$

Make n indefinitely great so that, by Art. 228 Cor.,

$$2^n\sin\frac{\theta}{2^n} = \theta.$$

Hence $\sin\theta = \theta \,.\, \cos\frac{\theta}{2} \,.\, \cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3} \ldots\ldots \text{ad inf.}$]

15. Prove that

$$\left(1-\tan^2\frac{\theta}{2}\right)\left(1-\tan^2\frac{\theta}{2^2}\right)\left(1-\tan^2\frac{\theta}{2^3}\right)\ldots\ldots \text{ad inf.}$$

$$= \theta \,.\, \cot\theta.$$

233. Area of a circle.

By Art. 225 the area of a regular polygon of n sides, which is inscribed in a circle of radius R, is

$$\frac{n}{2} R^2 \sin \frac{2\pi}{n}.$$

Let now the number of sides of this polygon be indefinitely increased, the polygon always remaining regular.

It is clear that the perimeter of the polygon must more and more approximate to the circumference of the circle.

Hence, when the number of sides of the polygon is infinitely great, the area of the circle must be the same as that of the polygon.

$$\text{Now } \frac{n}{2} R^2 \sin \frac{2\pi}{n} = \frac{n}{2} R^2 . \frac{2\pi}{n} . \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = \pi R^2 . \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}}$$

$$= \pi R^2 . \frac{\sin \theta}{\theta}, \text{ where } \theta = \frac{2\pi}{n}.$$

When n is made infinitely great the value of θ becomes infinitely small and then, by Art. 228, $\frac{\sin \theta}{\theta}$ is unity.

The area of the circle therefore $= \boldsymbol{\pi} \mathbf{R}^2 = \pi$ times the square of its radius.

234. *Area of the sector of a circle.*

Let O be the centre of a circle, AB the bounding arc of the sector, and let $\angle AOB = \alpha$ radians.

By Euc. VI. 33, since sectors are to one another as the arcs on which they stand, we have

$$\frac{\text{area of sector } AOB}{\text{area of whole circle}} = \frac{\text{arc } AB}{\text{circumference}}$$

$$= \frac{R\alpha}{2\pi R} = \frac{\alpha}{2\pi}.$$

$$\therefore \text{ area of sector } AOB = \frac{\alpha}{2\pi} \times \text{area of whole circle}$$

$$= \frac{\alpha}{2\pi} \times \pi R^2 = \frac{1}{2} R^2 \,.\, \alpha.$$

EXAMPLES. XLI.

1. Find the area of a circle whose circumference is 74 feet.

2. The diameter of a circle is 10 feet; find the area of a sector whose arc is $22\frac{1}{2}°$.

3. The area of a certain sector of a circle is 10 square feet; if the radius of the circle be 3 feet, find the angle of the sector.

4. The perimeter of a certain sector of a circle is 10 feet; if the radius of the circle be 3 feet, find the area of the sector.

5. A strip of paper two miles long and ·003 of an inch thick is rolled up into a solid cylinder; find approximately the radius of the circular ends of the cylinder.

6. A strip of paper, one mile long, is rolled tightly up into a solid cylinder, the diameter of whose circular ends is 6 inches; find the thickness of the paper.

7. Given two concentric circles of radii r and $2r$; two parallel tangents to the inner circle cut off an arc from the outer circle; find its length.

8. The circumference of a semicircle is divided into two arcs such that the chord of one is double that of the other. Prove that the sum of the areas of the two segments cut off by these chords is to the area of the semicircle as 27 is to 55.

$$\left[\pi = \frac{22}{7}.\right]$$

9. If each of 3 circles, of radius a, touch the other two, prove that the area included between them is nearly equal to $\frac{4}{25}a^2$.

10. Six equal circles, each of radius a, are placed so that each touches two others, their centres being all on the circumference of another circle; prove that the area which they enclose is

$$2a^2(3\sqrt{3}-\pi).$$

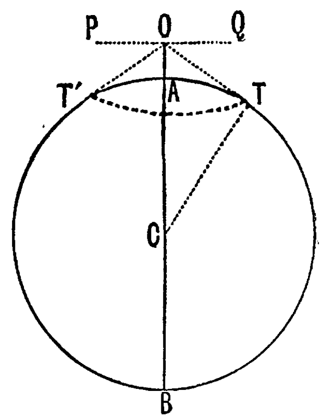
11. From the vertex A of a triangle a straight line AD is drawn making an angle θ with the base and meeting it at D. Prove that the area common to the circumscribing circles of the triangles ABD and ACD is

$$\frac{1}{4}(b^2\gamma+c^2\beta-2bc\sin A)\operatorname{cosec}^2\theta,$$

where β and γ are the number of radians in the angles B and C respectively.

235. Dip of the Horizon.

Let O be a point at a distance h above the earth's surface. Draw tangents, such as OT and OT', to the surface of the earth. The ends of all these tangents all clearly lie on a circle. This circle is called the **Offing or Visible Horizon.** The angle that each of these tangents OT makes with a horizontal plane POQ is called the **Dip** of the Horizon.



Let r be the radius of the earth and let B be the other end of the diameter through A.

We then have, by Euc. III. 36,

$$OT^2 = OA \cdot OB = h(2r+h),$$

so that $$OT = \sqrt{h(2r+h)}.$$

This gives an accurate value for OT.

In all practical cases, however, h is very small compared with r.

[$r = 4000$ miles nearly and h is never greater, and generally is very considerably less, than 5 miles.]

Hence h^2 is very small compared with hr.

As a close approximation we have then

$$OT = \sqrt{2hr}.$$

The dip $= \angle TOQ$

$= 90^\circ - \angle COT = \angle OCT.$

Also $\tan OCT = \frac{OT}{CT} = \frac{\sqrt{2hr}}{r} = \sqrt{\frac{2h}{r}}$,

so that, very approximately, we have the angle

$$OCT = \sqrt{\frac{2h}{r}} \text{ radians}$$

$$= \left(\sqrt{\frac{2h}{r}}\,\frac{180}{\pi}\right)^\circ = \left[\frac{180 \times 60 \times 60}{\pi}\sqrt{\frac{2h}{r}}\right]''.$$

236. **Ex.** *Taking the radius of the earth as* 4000 *miles, find the dip at the top of a lighthouse which is* 264 *feet above the sea and the distance of the offing.*

Here $r = 4000$ miles, and $h = 264$ feet $= \frac{1}{20}$ mile.

Hence h is very small compared with r, so that

$$OT = \sqrt{\frac{1}{10} \times 4000} = \sqrt{400} = 20 \text{ miles}.$$

Also the dip $= \sqrt{\frac{2h}{r}}$ radians $= \frac{1}{200}$ radian

$$= \left(\frac{1}{200} \times \frac{180 \times 60}{\pi}\right)' = \left(\frac{54}{\pi}\right)' = 17'\,11'' \text{ nearly}.$$

EXAMPLES. XLII.

1. Find in degrees, minutes, and seconds the dip of the horizon from the top of a mountain 4400 feet high, the earth's radius being 21×10^6 feet.

2. The lamp of a lighthouse is 196 feet high; how far off can it be seen?

3. If the radius of the earth be 4000 miles, find the height of a balloon when the dip is 1°.

Find also the dip when the balloon is 2 miles high.

4. Prove that, if the height of the place of observation be n feet, the distance that the observer can see is $\sqrt{\frac{3n}{2}}$ miles nearly.

5. There are 10 million metres in a quadrant of the earth's circumference. Find approximately the distance at which the top of the Eiffel tower should be visible, its height being 300 metres.

6. Three vertical posts are placed at intervals of a mile along a straight canal each rising to the same height above the surface of the water. The visual line joining the tops of the two extreme posts cuts the middle post at a point 8 inches below its top. Find the radius of the earth to the nearest mile.

CHAPTER XVIII.

INVERSE CIRCULAR FUNCTIONS.

237. If $\sin\theta = a$, where a is a known quantity, we know from Art. 82, that θ is not definitely known. We only know that θ is some one of a definite series of angles.

The symbol "$\sin^{-1} a$" is used to denote the *smallest* angle, whether positive or negative, that has a for its sine.

The symbol "$\sin^{-1} a$" is read in words as "sine minus one a," and must be carefully distinguished from $\frac{1}{\sin a}$ which would be written, if so desired, in the form $(\sin a)^{-1}$.

It will therefore be carefully noted that "$\sin^{-1} a$" is an **angle** and denotes the **smallest numerical** angle whose sine is a.

So "$\cos^{-1} a$" means the smallest numerical angle whose cosine is a. Similarly "$\tan^{-1} a$," "$\cot^{-1} a$," "$\text{cosec}^{-1} a$," "$\sec^{-1} a$," "$\text{vers}^{-1} a$," and "$\text{covers}^{-1} a$," are defined.

Hence $\sin^{-1} a$ and $\tan^{-1} a$ (and therefore $\text{cosec}^{-1} a$ and $\cot^{-1} a$) always lie between $-90°$ and $+90°$.

But $\cos^{-1} a$ (and therefore $\sec^{-1} a$) always lies between $0°$ and $180°$.

238. The quantities $\sin^{-1} a$, $\cos^{-1} a$, $\tan^{-1} a, \ldots$ are called Inverse Circular Functions.

The symbol $\sin^{-1} a$ is often, especially in foreign mathematical books, written as "arc sin a"; similarly $\cos^{-1} a$ is written "arc cos a," and so for the other inverse ratios.

239. When a is positive, $\sin^{-1} a$ clearly lies between $0°$ and $90°$; when a is negative it lies between $-90°$ and $0°$.

Ex. $\sin^{-1}\frac{1}{2}=30°$; $\sin^{-1}\frac{-\sqrt{3}}{2}=-60°$.

When a is positive, there are two angles, one lying between $0°$ and $90°$ and the other lying between $-90°$ and $0°$, each of which has its cosine equal to a. [For example both $30°$ and $-30°$ have their cosine equal to $\frac{\sqrt{3}}{2}$.] In this case we take the smallest *positive* angle. Hence $\cos^{-1} a$, when a is positive, lies between $0°$ and $90°$.

So $\cos^{-1} a$, when a is negative, lies between $90°$ and $180°$.

Ex. $\cos^{-1}\frac{1}{\sqrt{2}}=45°$; $\cos^{-1}\left(-\frac{1}{2}\right)=120°$.

When a is positive, the angle $\tan^{-1} a$ lies between $0°$ and $90°$; when a is negative, it lies between $-90°$ and $0°$.

Ex. $\tan^{-1}\sqrt{3}=60°$; $\tan^{-1}(-1)=-45°$.

240. Ex. 1. *Prove that* $\sin^{-1}\frac{3}{5}-\cos^{-1}\frac{12}{13}=\sin^{-1}\frac{16}{65}$.

Let $\sin^{-1}\frac{3}{5}=\alpha$, so that $\sin\alpha=\frac{3}{5}$,

and therefore $\cos\alpha=\sqrt{1-\frac{9}{25}}=\frac{4}{5}$.

Let $\cos^{-1}\frac{12}{13}=\beta$, so that $\cos\beta=\frac{12}{13}$,

and therefore $\sin\beta=\sqrt{1-\frac{144}{169}}=\frac{5}{13}$.

Let $\sin^{-1}\frac{16}{65}=\gamma$, so that $\sin\gamma=\frac{16}{65}$.

We have then to prove that

$$\alpha-\beta=\gamma,$$

i.e. to shew that $\sin(\alpha-\beta)=\sin\gamma$.

Now $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$$=\frac{3}{5}\cdot\frac{12}{13}-\frac{4}{5}\cdot\frac{5}{13}=\frac{36-20}{65}=\frac{16}{65}=\sin\gamma.$$

Hence the relation is proved.

Ex. 2. *Prove that* $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$.

Let $\tan^{-1}\frac{1}{3}=\alpha$, so that $\tan\alpha=\frac{1}{3}$,

and let $\tan^{-1}\frac{1}{7}=\beta$, so that $\tan\beta=\frac{1}{7}$.

We have then to shew that

$$2\alpha+\beta=\frac{\pi}{4}.$$

Now $$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$=\frac{\frac{2}{3}}{1-\frac{1}{9}}=\frac{6}{8}=\frac{3}{4}.$$

Also $$\tan(2\alpha+\beta)=\frac{\tan 2\alpha+\tan\beta}{1-\tan 2\alpha\tan\beta}$$

$$=\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}=\frac{21+4}{28-3}=\frac{25}{25}=1=\tan\frac{\pi}{4}.$$

$$\therefore\ 2\alpha+\beta=\frac{\pi}{4}.$$

Ex. 3. *Prove that*

$$\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}.$$

Let $\cos^{-1}\frac{63}{65}=\alpha$, so that $\cos\alpha=\frac{63}{65}$,

and therefore $\sin\alpha=\sqrt{1-\frac{63^2}{65^2}}=\frac{\sqrt{65^2-63^2}}{65}=\frac{16}{65}$.

Let $\tan^{-1}\frac{1}{5}=\beta$, so that $\tan\beta=\frac{1}{5}$,

and therefore (as in Art. 32),

$$\sin\beta=\frac{1}{\sqrt{26}} \text{ and } \cos\beta=\frac{5}{\sqrt{26}}.$$

Also let $\sin^{-1}\frac{3}{5}=\gamma$, so that $\sin\gamma=\frac{3}{5}$.

We have then to prove that

$$\alpha+2\beta=\gamma.$$

Now $\sin(\alpha+2\beta)=\sin\alpha\cos2\beta+\cos\alpha\sin2\beta$

$$=\frac{16}{65}\times(\cos^2\beta-\sin^2\beta)+\frac{63}{65}\times2\sin\beta\cos\beta$$

$$=\frac{16}{65}\times\left(\frac{25-1}{26}\right)+\frac{63}{65}\times\frac{2\times1\times5}{26}$$

$$=\frac{16\times24+63\times10}{65\times26}=\frac{1014}{65\times26}=\frac{3\times338}{65\times26}=\frac{3\times13}{65}=\frac{3}{5}=\sin\gamma.$$

Hence $\alpha+2\beta=\gamma$, so that the relation is proved.

Ex. 4. *Prove that*

$$4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\frac{\pi}{4}.$$

Let $\tan^{-1}\frac{1}{5}=\alpha$, so that $\tan\alpha=\frac{1}{5}$.

Then $$\tan2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}=\frac{\frac{2}{5}}{1-\frac{1}{25}}=\frac{5}{12},$$

and $$\tan4\alpha=\frac{\frac{10}{12}}{1-\frac{25}{144}}=\frac{120}{119},$$

so that $\tan4\alpha$ is nearly unity and 4α therefore nearly $\frac{\pi}{4}$.

Let $$4\alpha=\frac{\pi}{4}+\tan^{-1}x.$$

$$\therefore\ \frac{120}{119}=\tan\left(\frac{\pi}{4}+\tan^{-1}x\right)=\frac{1+x}{1-x}\quad \text{(Art. 100).}$$

$$\therefore\ x=\frac{1}{239}.$$

Hence $$4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\frac{\pi}{4}.$$

Ex. 5. *Prove that*

$$\tan^{-1}a+\tan^{-1}b=\tan^{-1}\frac{a+b}{1-ab}.$$

Let $\tan^{-1}a=\alpha$, so that $\tan\alpha=a$.

Let $\tan^{-1}b=\beta$, so that $\tan\beta=b$.

Also let $\tan^{-1}\left(\frac{a+b}{1-ab}\right)=\gamma$, so that $\tan\gamma=\frac{a+b}{1-ab}$.

We have then to prove that

$$\alpha+\beta=\gamma.$$

Now $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{a+b}{1-ab}=\tan\gamma,$$

so that the relation is proved.

The above relation is merely the formula

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y},$$

expressed in inverse notation.

For put $\tan x=a$, so that $x=\tan^{-1}a$,

and $\tan y=b$, so that $y=\tan^{-1}b$.

Then $$\tan(x+y)=\frac{a+b}{1-ab}.$$

$$\therefore\ x+y=\tan^{-1}\frac{a+b}{1-ab},$$

i.e. $$\tan^{-1}a+\tan^{-1}b=\tan^{-1}\frac{a+b}{1-ab}.$$

In the above we have tacitly assumed that $ab<1$, so that $\frac{a+b}{1-ab}$ is positive, and therefore $\tan^{-1}\frac{a+b}{1-ab}$ lies between 0° and 90°.

If however ab be >1, then $\frac{a+b}{1-ab}$ and therefore according to our definition $\tan^{-1}\frac{a+b}{1-ab}$ is a negative angle. Here γ is therefore a negative angle and, since $\tan(\pi+\gamma)=\tan\gamma$, the formula should be

$$\tan^{-1}a+\tan^{-1}b=\pi+\tan^{-1}\frac{a+b}{1-ab}.$$

Ex. 6. *Solve the equation*

$$\tan^{-1}\frac{x+1}{x-1}+\tan^{-1}\frac{x-1}{x}=\tan^{-1}(-7).$$

Here we have

$$\alpha+\beta=\gamma,$$

where $\alpha=\tan^{-1}\frac{x+1}{x-1}$ and hence $\tan\alpha=\frac{x+1}{x-1}$,

$\beta=\tan^{-1}\frac{x-1}{x}$ and hence $\tan\beta=\frac{x-1}{x}$,

and $\gamma=\tan^{-1}(-7)$ and hence $\tan\gamma=-7$.

Since $\tan(\alpha+\beta)=\tan\gamma$,

$$\therefore\ \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=-7,$$

i.e.

$$\frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\frac{x-1}{x}}=-7,$$

i.e.

$$\frac{2x^2-x+1}{1-x}=-7,$$

so that $x=2$.

This value makes the left-hand side of the given equation positive, so that there is no value of x strictly satisfying the given equation.

The value $x=2$ is a solution of the equation

$$\tan^{-1}\frac{x+1}{x-1}+\tan^{-1}\frac{x-1}{x}=\pi+\tan^{-1}(-7).$$

EXAMPLES. XLIII.

Prove that

1. $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$.

2. $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{7}{25}=\cos^{-1}\left(-\frac{253}{325}\right)$.

3. $\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{3}{5}=\tan^{-1}\frac{27}{11}$.

4. $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$.

5. $\cos^{-1}x=2\sin^{-1}\sqrt{\frac{1-x}{2}}=2\cos^{-1}\sqrt{\frac{1+x}{2}}$.

6. $2\cos^{-1}\frac{3}{\sqrt{13}}+\cot^{-1}\frac{16}{63}+\frac{1}{2}\cos^{-1}\frac{7}{25}=\pi$.

7. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3=45^\circ$.

8. $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$.

9. $\tan^{-1}\frac{2}{3}=\frac{1}{2}\tan^{-1}\frac{12}{5}$.

10. $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}$.

11. $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

12. $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\frac{\pi}{4}$.

13. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

14. $3\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{20}=\frac{\pi}{4}-\tan^{-1}\frac{1}{1985}$.

15. $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\frac{\pi}{4}$.

16. $\tan^{-1}\frac{120}{119}=2\sin^{-1}\frac{5}{13}$.

17. $\tan^{-1}\frac{m}{n}-\tan^{-1}\frac{m-n}{m+n}=\frac{\pi}{4}$.

18. $\tan^{-1}t+\tan^{-1}\frac{2t}{1-t^2}=\tan^{-1}\frac{3t-t^3}{1-3t^2}$,

if $t<\frac{1}{\sqrt{3}}$ and $=\pi+\tan^{-1}\frac{3t-t^3}{1-3t^2}$ if $t>\frac{1}{\sqrt{3}}$.

19. $\tan^{-1}\sqrt{\dfrac{a(a+b+c)}{bc}}+\tan^{-1}\sqrt{\dfrac{b(a+b+c)}{ca}}$

$$+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}=\pi.$$

20. $\cot^{-1}\dfrac{ab+1}{a-b}+\cot^{-1}\dfrac{bc+1}{b-c}+\cot^{-1}\dfrac{ca+1}{c-a}=0.$

21. $\tan^{-1}n+\cot^{-1}(n+1)=\tan^{-1}(n^2+n+1).$

22. $\cos\left(2\tan^{-1}\dfrac{1}{7}\right)=\sin\left(4\tan^{-1}\dfrac{1}{3}\right).$

23. $2\tan^{-1}\left[\tan(45^\circ-\alpha)\tan\dfrac{\beta}{2}\right]=\cos^{-1}\left[\dfrac{\tan 2\alpha+\cos\beta}{1+\tan 2\alpha\cos\beta}\right].$

24. $\tan^{-1}x=2\tan^{-1}[\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x].$

25. $2\tan^{-1}\left[\tan\dfrac{\alpha}{2}\tan\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)\right]=\tan^{-1}\dfrac{\sin\alpha\cos\beta}{\sin\beta+\cos\alpha}.$

26. Shew that

$$\cos^{-1}\sqrt{\frac{a-x}{a-b}}=\sin^{-1}\sqrt{\frac{x-b}{a-b}}=\cot^{-1}\sqrt{\frac{a-x}{x-b}}$$

$$=\frac{1}{2}\sin^{-1}\frac{2\sqrt{(a-x)(x-b)}}{a-b}.$$

27. If $\cos^{-1}\dfrac{x}{a}+\cos^{-1}\dfrac{y}{b}=\alpha$, prove that

$$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$$

Solve

28. $\tan^{-1}\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}=\beta.$

29. $\tan^{-1}2x+\tan^{-1}3x=\dfrac{\pi}{4}.$

30. $\tan^{-1}\dfrac{x-1}{x-2}+\tan^{-1}\dfrac{x+1}{x+2}=\dfrac{\pi}{4}.$

31. $\tan^{-1}(x+1)+\cot^{-1}(x-1)=\sin^{-1}\dfrac{4}{5}+\cos^{-1}\dfrac{3}{5}.$

32. $\tan^{-1}x+\tan^{-1}(x-1)=\tan^{-1}\dfrac{8}{31}.$

33. $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec}x).$

34. $\tan^{-1} x + 2\cot^{-1} x = \frac{2}{3}\pi.$ 35. $\tan\cos^{-1} x = \sin\cot^{-1}\frac{1}{2}.$

36. $\cot^{-1} x - \cot^{-1}(x+2) = 15°.$

37. $\cos^{-1}\frac{x^2-1}{x^2+1} + \tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3}.$

38. $\cot^{-1} x + \cot^{-1}(n^2 - x + 1) = \cot^{-1}(n-1).$

39. $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}.$ 40. $\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}.$

41. $\tan^{-1}\frac{a}{x} + \tan^{-1}\frac{b}{x} + \tan^{-1}\frac{c}{x} + \tan^{-1}\frac{d}{x} = \frac{\pi}{2}.$

42. $\sec^{-1}\frac{x}{a} - \sec^{-1}\frac{x}{b} = \sec^{-1} b - \sec^{-1} a.$

43. $\operatorname{cosec}^{-1} x = \operatorname{cosec}^{-1} a + \operatorname{cosec}^{-1} b.$

44. $2\tan^{-1} x = \cos^{-1}\frac{1-a^2}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2}.$

CHAPTER XIX.

ON SOME SIMPLE TRIGONOMETRICAL SERIES.

241. *To find the sum of the sines of a series of angles, the angles being in arithmetical progression.*

Let the angles be

$$\alpha,\ \alpha+\beta,\ \alpha+2\beta, \ldots\ldots \{\alpha+(n-1)\beta\}.$$

Let

$$S \equiv \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) \ldots + \sin\{\alpha+(n-1)\beta\}.$$

By Art. 97 we have

$$2\sin\alpha\sin\frac{\beta}{2} = \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{\beta}{2}\right),$$

$$2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{3\beta}{2}\right),$$

$$2\sin(\alpha+2\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{3\beta}{2}\right) - \cos\left(\alpha+\frac{5\beta}{2}\right),$$

....

$$2\sin\{\alpha+(n-2)\beta\}\sin\frac{\beta}{2} = \cos\{\alpha+(n-\tfrac{5}{2})\beta\} - \cos\{\alpha+(n-\tfrac{3}{2})\beta\},$$

and

$$2\sin\{\alpha+(n-1)\beta\}\sin\frac{\beta}{2} = \cos\{\alpha+(n-\tfrac{3}{2})\beta\} - \cos\{\alpha+(n-\tfrac{1}{2})\beta\}.$$

By adding together these n lines, we have

$$2\sin\frac{\beta}{2}\,.\,S = \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\{\alpha+(n-\tfrac{1}{2})\beta\},$$

the other terms on the right-hand sides cancelling one another.

Hence, by Art. 94, we have

$$2\sin\frac{\beta}{2}\,.\,S = 2\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\frac{n\beta}{2},$$

i.e.

$$\mathbf{S} = \frac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

Ex. By putting $\beta = 2\alpha$, we have

$$\sin\alpha + \sin 3\alpha + \sin 5\alpha + \ldots + \sin(2n-1)\alpha$$

$$= \frac{\sin\{\alpha + (n-1)\alpha\}\sin n\alpha}{\sin\alpha} = \frac{\sin^2 n\alpha}{\sin\alpha}.$$

242. *To find the sum of the cosines of a series of angles, the angles being in arithmetical progression.*

Let the angles be

$$\alpha,\ \alpha+\beta,\ \alpha+2\beta,\ \ldots\ \alpha+(n-1)\beta.$$

Let

$$S \equiv \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\{\alpha+(n-1)\beta\}.$$

By Art. 97, we have

$$2\cos\alpha\sin\frac{\beta}{2} = \sin\left(\alpha+\frac{\beta}{2}\right) - \sin\left(\alpha-\frac{\beta}{2}\right),$$

$$2\cos(\alpha+\beta)\sin\frac{\beta}{2} = \sin\left(\alpha+\frac{3\beta}{2}\right) - \sin\left(\alpha+\frac{\beta}{2}\right),$$

$$2\cos(\alpha+2\beta)\sin\frac{\beta}{2} = \sin\left(\alpha+\frac{5\beta}{2}\right) - \sin\left(\alpha+\frac{3\beta}{2}\right),$$

....

$$2\cos\{\alpha+(n-2)\beta\}\sin\frac{\beta}{2} = \sin\{\alpha+(n-\tfrac{3}{2})\beta\} - \sin\{\alpha+(n-\tfrac{5}{2})\beta\},$$

and

$$2\cos\{\alpha+(n-1)\beta\}\sin\frac{\beta}{2}=\sin\{\alpha+(n-\tfrac{1}{2})\beta\}-\sin\{\alpha+(n-\tfrac{3}{2})\beta\}.$$

By adding together these n lines, we have

$$2S\times\sin\frac{\beta}{2}=\sin\{\alpha+(n-\tfrac{1}{2})\beta\}-\sin\left\{\alpha-\frac{\beta}{2}\right\},$$

the other terms on the right-hand sides cancelling one another.

Hence, by Art. 94, we have

$$2S\times\sin\frac{\beta}{2}=2\cos\left\{\alpha+\frac{n-1}{2}\beta\right\}\sin\frac{n\beta}{2},$$

i.e.

$$\mathbf{S}=\frac{\cos\left\{\alpha+\frac{n-1}{2}\beta\right\}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

243. Both the expressions for S in Arts. 241 and 242 vanish when $\sin\frac{n\beta}{2}$ is zero, *i.e.* when $\frac{n\beta}{2}$ is equal to any multiple of π,

i.e. when $$\frac{n\beta}{2}=p\pi,$$

where p is any integer,

i.e. when $$\beta=p\,.\frac{2\pi}{n}.$$

Hence the sum of the sines (or cosines) of n angles, which are in arithmetical progression, vanishes when the common difference of the angles is any multiple of $\frac{2\pi}{n}$.

Exs. $$\cos\alpha+\cos\left(\alpha+\frac{2\pi}{n}\right)+\cos\left(\alpha+\frac{4\pi}{n}\right)+\dots\text{ to } n \text{ terms}=0,$$

and $$\sin\alpha+\sin\left(\alpha+\frac{4\pi}{n}\right)+\sin\left(\alpha+\frac{8\pi}{n}\right)+\ldots \text{ to } n \text{ terms}=0.$$

244. **Ex. 1.** *Find the sum of*

$$\sin\alpha-\sin(\alpha+\beta)+\sin(\alpha+2\beta)-\ldots \text{ to } n \text{ terms}.$$

We have, by Art. 73,

$$\sin(\alpha+\beta+\pi)=-\sin(\alpha+\beta),$$
$$\sin(\alpha+2\beta+2\pi)=\sin(\alpha+2\beta),$$
$$\sin(\alpha+3\beta+3\pi)=-\sin(\alpha+3\beta),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence the series

$$=\sin\alpha+\sin(\alpha+\beta+\pi)+\sin\{\alpha+2(\beta+\pi)\}+\sin\{\alpha+3(\beta+\pi)\}+\ldots$$

$$=\frac{\sin\left\{\alpha+\frac{n-1}{2}(\beta+\pi)\right\}\sin\frac{n(\beta+\pi)}{2}}{\sin\frac{\beta+\pi}{2}}, \text{ by Art. 241},$$

$$=\frac{\sin\left\{\alpha+\frac{n-1}{2}(\beta+\pi)\right\}\sin\frac{n(\beta+\pi)}{2}}{\cos\frac{\beta}{2}}.$$

Ex. 2. *Find the sum of the series*

$$\cos^3\alpha+\cos^3 2\alpha+\cos^3 3\alpha+\ldots\ldots \text{ to } n \text{ terms}.$$

By Art. 107, we have

$$\cos 3\alpha=4\cos^3\alpha-3\cos\alpha,$$

so that $$4\cos^3\alpha=3\cos\alpha+\cos 3\alpha.$$

So $$4\cos^3 2\alpha=3\cos 2\alpha+\cos 6\alpha,$$
$$4\cos^3 3\alpha=3\cos 3\alpha+\cos 9\alpha,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence, if S be the given series, we have

$$4S=(3\cos\alpha+\cos 3\alpha)+(3\cos 2\alpha+\cos 6\alpha)+(3\cos 3\alpha+\cos 9\alpha)+\ldots$$
$$=3(\cos\alpha+\cos 2\alpha+\cos 3\alpha+\ldots)+(\cos 3\alpha+\cos 6\alpha+\cos 9\alpha+\ldots)$$

$$=3\frac{\cos\left\{\alpha+\frac{n-1}{2}\cdot\alpha\right\}\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}+\frac{\cos\left\{3\alpha+\frac{n-1}{2}\cdot 3\alpha\right\}\sin\frac{n\cdot 3\alpha}{2}}{\sin\frac{3\alpha}{2}}$$

$$=3\frac{\cos\frac{n+1}{2}\alpha\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}+\frac{\cos\frac{3(n+1)}{2}\alpha\sin\frac{3n\alpha}{2}}{\sin\frac{3\alpha}{2}}.$$

In a similar manner we can obtain the sum of the cubes of the sines of a series of angles in A.P.

Cor. Since

$$2\sin^2\alpha = 1 - \cos 2\alpha, \text{ and } 2\cos^2\alpha = 1 + \cos 2\alpha,$$

we can obtain the sum of the squares.

Since again $\quad 8\sin^4\alpha = 2\,[1 - \cos 2\alpha]^2$

$$= 2 - 4\cos 2\alpha + 2\cos^2 2\alpha = 3 - 4\cos 2\alpha + \cos 4\alpha,$$

we can obtain the sum of the 4th powers of the sines. Similarly for the cosines.

Ex. 3. *Sum to n terms the series*

$$\cos\alpha\sin\beta + \cos 3\alpha\sin 2\beta + \cos 5\alpha\sin 3\beta + \ldots \text{ to } n \text{ terms.}$$

Let S denote the series.

Then

$$2S = \{\sin(\alpha+\beta) - \sin(\alpha-\beta)\} + \{\sin(3\alpha+2\beta) - \sin(3\alpha-2\beta)\} + \{\sin(5\alpha+3\beta) - \sin(5\alpha-3\beta)\} + \ldots$$

$$= \{\sin(\alpha+\beta) + \sin(3\alpha+2\beta) + \sin(5\alpha+3\beta) + \ldots\}$$

$$- \{\sin(\alpha-\beta) + \sin(3\alpha-2\beta) + \sin(5\alpha-3\beta) + \ldots\}$$

$$= \frac{\sin\left\{(\alpha+\beta) + \frac{n-1}{2}(2\alpha+\beta)\right\}\sin n\frac{2\alpha+\beta}{2}}{\sin\frac{2\alpha+\beta}{2}}$$

$$- \frac{\sin\left\{(\alpha-\beta) + \frac{n-1}{2}(2\alpha-\beta)\right\}\sin n\frac{2\alpha-\beta}{2}}{\sin\frac{2\alpha-\beta}{2}}, \text{ by Art. 241,}$$

$$= \frac{\sin\left\{n\alpha + \frac{n+1}{2}\beta\right\}\sin\frac{n(2\alpha+\beta)}{2}}{\sin\frac{2\alpha+\beta}{2}}$$

$$- \frac{\sin\left\{n\alpha - \frac{n+1}{2}\beta\right\}\sin\frac{n(2\alpha-\beta)}{2}}{\sin\frac{2\alpha-\beta}{2}}.$$

Ex. 4. *$A_1A_2 \ldots A_n$ is a regular polygon of n sides inscribed in a circle, whose centre is O, and P is any point on the arc A_nA_1 such that the angle POA_1 is θ; find the sum of the lengths of the lines joining P to the angular points of the polygon.*

Each of the angles A_1OA_2, $A_2OA_3, \ldots A_nOA_1$ is $\frac{2\pi}{n}$, so that the angles POA_1, $POA_2, \ldots$ are respectively

$$\theta, \quad \theta+\frac{2\pi}{n}, \quad \theta+\frac{4\pi}{n}, \ldots$$

Hence, if r be the radius of the circle, we have

$$PA_1=2r\sin\frac{POA_1}{2}=2r\sin\frac{\theta}{2},$$

$$PA_2=2r\sin\frac{POA_2}{2}=2r\sin\left(\frac{\theta}{2}+\frac{\pi}{n}\right)$$

$$PA_3=2r\sin\frac{POA_3}{2}=2r\sin\left(\frac{\theta}{2}+\frac{2\pi}{n}\right).$$

....

Hence the required sum

$$=2r\left[\sin\frac{\theta}{2}+\sin\left(\frac{\theta}{2}+\frac{\pi}{n}\right)+\sin\left(\frac{\theta}{2}+\frac{2\pi}{n}\right)+\ldots\ldots \text{to } n \text{ terms}\right]$$

$$=2r\frac{\sin\left[\frac{\theta}{2}+\frac{n-1}{2}\frac{\pi}{n}\right]\sin\frac{n}{2}\cdot\frac{\pi}{n}}{\sin\frac{\pi}{2n}} \qquad \text{(Art. 241)}$$

$$=2r\operatorname{cosec}\frac{\pi}{2n}\cdot\sin\left[\frac{\pi}{2}+\frac{\theta}{2}-\frac{\pi}{2n}\right]$$

$$=2r\operatorname{cosec}\frac{\pi}{2n}\cos\left(\frac{\theta}{2}-\frac{\pi}{2n}\right).$$

EXAMPLES. XLIV.

Sum the series:

1. $\cos\theta+\cos 3\theta+\cos 5\theta+\ldots$ to n terms.

2. $\cos\frac{A}{2}+\cos 2A+\cos\frac{7A}{2}+\ldots$ to n terms.

3. $\dfrac{\sin\alpha+\sin 2\alpha+\sin 3\alpha+\ldots+\sin n\alpha}{\cos\alpha+\cos 2\alpha+\ldots+\cos n\alpha}=\tan\dfrac{n+1}{2}\alpha.$

4. $\dfrac{\sin\alpha+\sin 3\alpha+\sin 5\alpha+\ldots+\sin(2n-1)\alpha}{\cos\alpha+\cos 3\alpha+\cos 5\alpha+\ldots+\cos(2n-1)\alpha}=\tan n\alpha.$

5. $\cos\dfrac{\pi}{2n+1}+\cos\dfrac{3\pi}{2n+1}+\cos\dfrac{5\pi}{2n+1}+\ldots$ to n terms.

6. $\cos\alpha-\cos(\alpha+\beta)+\cos(\alpha+2\beta)-\ldots$ to $2n$ terms.

7. $\dfrac{\sin\alpha-\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots \text{ to } n \text{ terms}}{\cos\alpha-\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots \text{ to } n \text{ terms}}$

$$=\tan\left\{\alpha+\frac{n-1}{2}(\pi+\beta)\right\}.$$

8. $\sin\theta+\sin\dfrac{n-4}{n-2}\theta+\sin\dfrac{n-6}{n-2}\theta+\ldots$ to n terms.

9. $\cos x+\sin 3x+\cos 5x+\sin 7x+\ldots+\sin(4n-1)x.$

10. $\sin\alpha\sin 2\alpha+\sin 2\alpha\sin 3\alpha+\sin 3\alpha\sin 4\alpha+\ldots$ to n terms.

11. $\cos\alpha\sin 2\alpha+\sin 2\alpha\cos 3\alpha+\cos 3\alpha\sin 4\alpha$
$+\sin 4\alpha\cos 5\alpha+\ldots$ to $2n$ terms.

12. $\sin\alpha\sin 3\alpha+\sin 2\alpha\sin 4\alpha+\sin 3\alpha\sin 5\alpha+\ldots$ to n terms.

13. $\cos\alpha\cos\beta+\cos 3\alpha\cos 2\beta+\cos 5\alpha\cos 3\beta\ldots$ to n terms.

14. $\sin^2\alpha+\sin^2 2\alpha+\sin^2 3\alpha+\ldots$ to n terms.

15. $\sin^2\theta+\sin^2(\theta+\alpha)+\sin^2(\theta+2\alpha)+\ldots$ to n terms.

16. $\sin^3\alpha+\sin^3 2\alpha+\sin^3 3\alpha+\ldots$ to n terms.

17. $\sin^4\alpha+\sin^4 2\alpha+\sin^4 3\alpha+\ldots$ to n terms.

18. $\cos^4\alpha+\cos^4 2\alpha+\cos^4 3\alpha+\ldots$ to n terms.

19. $\cos\theta\cos 2\theta\cos 3\theta+\cos 2\theta\cos 3\theta\cos 4\theta+\ldots$ to n terms.

20. $\sin\alpha\sin(\alpha+\beta)-\sin(\alpha+\beta)\sin(\alpha+2\beta)+\ldots$ to $2n$ terms.

21. From the sum of the series

$$\sin\alpha+\sin 2\alpha+\sin 3\alpha+\ldots \text{ to } n \text{ terms},$$

deduce (by making α very small) the sum of the series

$$1+2+3+\ldots+n.$$

22. From the result of the example of Art. 241 deduce the sum of

$$1+3+5\ldots \text{ to } n \text{ terms}.$$

23. If
$$\alpha=\frac{2\pi}{17},$$
prove that $\quad 2(\cos\alpha+\cos 2\alpha+\cos 4\alpha+\cos 5\alpha)$

and $\quad 2(\cos 3\alpha+\cos 5\alpha+\cos 6\alpha+\cos 7\alpha)$

are the roots of the equation

$$x^2+x-4=0.$$

24. $ABCD...$ is a regular polygon of n sides which is inscribed in a circle, whose centre is O and whose radius is r, and P is any point on the arc AB such that POA is θ. Prove that

$$PA \cdot PB + PA \cdot PC + PA \cdot PD + PB \cdot PC + \ldots$$
$$= r^2 \left[2\cos^2\left(\frac{\theta}{2} - \frac{\pi}{2n}\right) \operatorname{cosec}^2 \frac{\pi}{2n} - n \right].$$

25. Two regular polygons, each of n sides, are circumscribed to and inscribed in a given circle. If an angular point of one of them be joined to each of the angular points of the other then the sum of the squares of the straight lines so drawn is to the sum of the areas of the polygons as

$$2 : \sin \frac{2\pi}{n}.$$

26. A_1, $A_2 \ldots A_{2n+1}$ are the angular points of a regular polygon inscribed in a circle and O is any point on the circumference between A_1 and A_{2n+1}; prove that

$$OA_1 + OA_3 + \ldots + OA_{2n+1} = OA_2 + OA_4 + \ldots + OA_{2n}.$$

27. If perpendiculars be drawn on the sides of a regular polygon of n sides from any point on the inscribed circle whose radius is a, prove that

$$\frac{2}{n} \Sigma \left(\frac{p}{a}\right)^2 = 3, \text{ and } \frac{2}{n} \Sigma \left(\frac{p}{a}\right)^3 = 5.$$

CHAPTER XX.

ELIMINATION.

245. It sometimes happens that we have two equations each containing one unknown quantity. In this case there must clearly be a relation between the constants of the equations in order that the same value of the unknown quantity may satisfy both. For example, suppose we knew that an unknown quantity x satisfied both of the equations

$$ax + b = 0 \text{ and } cx^2 + dx + e = 0.$$

From the first equation we have

$$x = -\frac{b}{a},$$

and this satisfies the second if

$$c\left(-\frac{b}{a}\right)^2 + d\left(-\frac{b}{a}\right) + e = 0,$$

i.e. if $$b^2c - abd + a^2e = 0.$$

This latter equation is the result of eliminating x between the above two equations, and is often called their eliminant.

246. Again, suppose we knew that an angle θ satisfied both of the equations

$$\sin^3 \theta = b, \text{ and } \cos^3 \theta = c,$$

so that $$\sin \theta = b^{\frac{1}{3}}, \text{ and } \cos \theta = c^{\frac{1}{3}}.$$

Now we always have, for all values of θ,

$$\sin^2 \theta + \cos^2 \theta = 1,$$

so that in this case $$b^{\frac{2}{3}} + c^{\frac{2}{3}} = 1.$$

This is the result of eliminating θ.

247. Between any two equations involving one unknown quantity we can, in theory, always eliminate that quantity. In practice a considerable amount of artifice and ingenuity is often required in seemingly simple cases.

So between any three equations involving two unknown quantities we can theoretically eliminate both of the unknown quantities.

248. Some examples of elimination are appended.

Ex. 1. *Eliminate θ from the equations*

$$a \cos \theta + b \sin \theta = c,$$

and $$b \cos \theta + c \sin \theta = a.$$

Solving for $\cos\theta$ and $\sin\theta$ by cross multiplication, or otherwise, we have

$$\frac{\cos \theta}{c^2 - ab} = \frac{\sin \theta}{a^2 - bc} = \frac{1}{ac - b^2}.$$

$$\therefore \ 1 = \cos^2 \theta + \sin^2 \theta = \frac{(c^2 - ab)^2 + (a^2 - bc)^2}{(ac - b^2)^2},$$

i.e. $$(a^2 - bc)^2 + (c^2 - ab)^2 = (b^2 - ac)^2.$$

Ex. 2. *Eliminate θ between*

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \quad\text{.........................}(1),$$

and

$$\frac{ax\sin\theta}{\cos^2\theta} + \frac{by\cos\theta}{\sin^2\theta} = 0 \quad\text{.........................}(2).$$

From (2) we have $ax\sin^3\theta = -by\cos^3\theta$.

$$\therefore \quad \frac{\sin\theta}{-(by)^{\frac{1}{3}}} = \frac{\cos\theta}{(ax)^{\frac{1}{3}}} = \frac{\sqrt{\sin^2\theta + \cos^2\theta}}{\sqrt{(by)^{\frac{2}{3}} + (ax)^{\frac{2}{3}}}}$$

(Hall and Knight's *Higher Algebra*, Art. 12)

$$= \frac{1}{\sqrt{(by)^{\frac{2}{3}} + (ax)^{\frac{2}{3}}}}.$$

Hence

$$\frac{1}{\sin\theta} = -\frac{\sqrt{(by)^{\frac{2}{3}} + (ax)^{\frac{2}{3}}}}{(by)^{\frac{1}{3}}},$$

and

$$\frac{1}{\cos\theta} = \frac{\sqrt{(by)^{\frac{2}{3}} + (ax)^{\frac{2}{3}}}}{(ax)^{\frac{1}{3}}},$$

so that (1) becomes

$$a^2 - b^2 = \sqrt{(by)^{\frac{2}{3}} + (ax)^{\frac{2}{3}}}\left[ax \cdot \frac{1}{(ax)^{\frac{1}{3}}} - by\left\{-\frac{1}{(by)^{\frac{1}{3}}}\right\}\right]$$

$$= \sqrt{(by)^{\frac{2}{3}} + (ax)^{\frac{2}{3}}}\{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}\}$$

$$= \{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}\}^{\frac{3}{2}},$$

i.e.

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

The student who shall afterwards become acquainted with Analytical Geometry will find that the above is the solution of an important problem concerning normals to an ellipse.

Ex. 3. *Eliminate θ from the equations*

$$\frac{x}{a}\cos\theta - \frac{y}{b}\sin\theta = \cos 2\theta \quad\text{.......................}(1),$$

and

$$\frac{x}{a}\sin\theta + \frac{y}{b}\cos\theta = 2\sin 2\theta \quad\text{.......................}(2).$$

Multiplying (1) by $\cos\theta$, (2) by $\sin\theta$, and adding, we have

$$\frac{x}{a}=\cos\theta\cos 2\theta+2\sin\theta\sin 2\theta$$

$$=\cos\theta+\sin\theta\sin 2\theta=\cos\theta+2\sin^2\theta\cos\theta \quad\text{..............(3).}$$

Multiplying (2) by $\cos\theta$, (1) by $\sin\theta$, and subtracting, we have

$$\frac{y}{b}=2\sin 2\theta\cos\theta-\cos 2\theta\sin\theta$$

$$=\sin 2\theta\cos\theta+\sin\theta=\sin\theta+2\sin\theta\cos^2\theta \quad\text{..............(4).}$$

Adding (3) and (4), we have

$$\frac{x}{a}+\frac{y}{b}=(\sin\theta+\cos\theta)\,[1+2\sin\theta\cos\theta]$$

$$=(\sin\theta+\cos\theta)\,[\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta]$$

$$=(\sin\theta+\cos\theta)^3,$$

so that $$\sin\theta+\cos\theta=\left(\frac{x}{a}+\frac{y}{b}\right)^{\frac{1}{3}} \quad\text{........................ (5).}$$

Subtracting (4) from (3), we have

$$\frac{x}{a}-\frac{y}{b}=(\cos\theta-\sin\theta)\,(1-2\sin\theta\cos\theta)$$

$$=(\cos\theta-\sin\theta)^3,$$

so that $$\cos\theta-\sin\theta=\left(\frac{x}{a}-\frac{y}{b}\right)^{\frac{1}{3}} \quad\text{........................ (6).}$$

Squaring and adding (5) and (6), we have

$$2=\left(\frac{x}{a}+\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{x}{a}-\frac{y}{b}\right)^{\frac{2}{3}}.$$

EXAMPLES. XLV.

Eliminate θ from the equations

1. $a\cos\theta+b\sin\theta=c$, and $b\cos\theta-a\sin\theta=d$.
2. $x=a\cos(\theta-\alpha)$, and $y=b\cos(\theta-\beta)$.
3. $a\cos 2\theta=b\sin\theta$, and $c\sin 2\theta=d\cos\theta$.
4. $a\sin\alpha-b\cos\alpha=2b\sin\theta$, and $a\sin 2\alpha-b\cos 2\theta=a$.
5. $x\sin\theta-y\cos\theta=\sqrt{x^2+y^2}$, and $\dfrac{\sin^2\theta}{a^2}+\dfrac{\cos^2\theta}{b^2}=\dfrac{1}{x^2+y^2}$.

6. $$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1,$$

and $$x\sin\theta-y\cos\theta=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}.$$

7. $\sin\theta-\cos\theta=p$, and $\operatorname{cosec}\theta-\sin\theta=q$.

8. If $$m=\operatorname{cosec}\theta-\sin\theta,\ \text{and}\ n=\sec\theta-\cos\theta,$$

prove that $$m^{\frac{2}{3}}+n^{\frac{2}{3}}=(mn)^{-\frac{2}{3}}.$$

9. Prove that the result of eliminating θ from the equations

$$x\cos(\theta+\alpha)+y\sin(\theta+\alpha)=a\sin 2\theta,$$

and $$y\cos(\theta+\alpha)-x\sin(\theta+\alpha)=2a\cos 2\theta,$$

is $$(x\cos\alpha+y\sin\alpha)^{\frac{2}{3}}+(x\sin\alpha-y\cos\alpha)^{\frac{2}{3}}=(2a)^{\frac{2}{3}}.$$

Eliminate θ and ϕ from the equations

10. $$a\cos^2\theta+b\sin^2\theta=c,\ \ b\cos^2\phi+a\sin^2\phi=d,$$

and $$a\tan\theta=b\tan\phi.$$

11. $\cos\theta+\cos\phi=a$, $\cot\theta+\cot\phi=b$, and $\operatorname{cosec}\theta+\operatorname{cosec}\phi=c$.

12. $a\sin\theta=b\sin\phi$, $a\cos\theta+b\cos\phi=c$, and $x=y\tan(\theta+\phi)$.

13. $$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1,\ \ \frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1,$$

and $$a^2\sin\frac{\theta}{2}\sin\frac{\phi}{2}+b^2\cos\frac{\theta}{2}\cos\frac{\phi}{2}=c^2$$

PART II.

ANALYTICAL TRIGONOMETRY.

CHAPTER XXI.

EXPONENTIAL AND LOGARITHMIC SERIES.

249. In the following chapter we are about to obtain an expansion in powers of x for the expression a^x, where both a and x are real, and also to obtain an expansion for $\log_e(1+x)$, where x is real and less than unity, and e stands for a quantity to be defined.

250. *To find the value of the quantity* $\left(1+\frac{1}{n}\right)^n$, *when* n *becomes infinitely great and is real.*

Since $\frac{1}{n} < 1$, we have, by the Binomial Theorem,

$$\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{1.2}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{1.2.3}\frac{1}{n^3} + \ldots$$

$$= 1 + 1 + \frac{1-\frac{1}{n}}{1.2} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{\lfloor 4} + \ldots\ldots$$

This series is true for all values of n, however great. Make then n infinite and the right-hand side

$$= 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \ldots \text{ ad inf.}$$

Hence the limiting value, when n is infinite, of $\left(1+\frac{1}{n}\right)^n$ is the sum of the series

$$1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\ldots \text{ ad inf.}$$

The sum of this series is always denoted by the quantity e.

Hence we have

$$\underset{n=\infty}{\mathbf{Lt}}\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{n}}\right)^{\mathbf{n}}=\mathbf{e},$$

where $\underset{n=\infty}{\text{Lt}}$ stands for "the limit when $n=\infty$."

Cor. By putting $n=\frac{1}{m}$, it follows (since m is zero when n is infinity) that

$$\underset{m=0}{\text{Lt}}(1+m)^{\frac{1}{m}}=\underset{n=\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n=e.$$

251. This quantity e is finite.

For since
$$\frac{1}{\underline{|3}}<\frac{1}{2\,.\,2}<\frac{1}{2^2},$$
$$\frac{1}{\underline{|4}}<\frac{1}{2\,.\,2\,.\,2}<\frac{1}{2^3},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

we have

$$e<1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}\ldots\ldots \text{ ad inf.}$$
$$<1+\frac{1}{1-\frac{1}{2}}$$
$$<1+2, \text{ i.e. } <3.$$

Also clearly $e>2$.

Hence it lies between 2 and 3.

By taking a sufficient number of terms in the series, it can be shewn that

$$e = 2{\cdot}7182818285\ldots$$

252. *The quantity e is incommensurable.*

For, if possible, suppose it to be equal to a fraction $\frac{p}{q}$, where p and q are whole numbers.

We have then

$$\frac{p}{q} = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots + \frac{1}{\lfloor q} + \frac{1}{\lfloor q+1} + \frac{1}{\lfloor q+2} + \ldots \quad \ldots\ldots(1).$$

Multiply this equation by $\lfloor q$, so that all the terms of the series (1) become integers except those commencing with $\frac{\lfloor q}{\lfloor q+1}$. Hence we have

$$p\,\lfloor q-1 = \text{whole number} + \frac{\lfloor q}{\lfloor q+1} + \frac{\lfloor q}{\lfloor q+2} + \frac{\lfloor q}{\lfloor q+3} + \ldots$$

i.e. an integer $= \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \ldots \quad \ldots(2).$

But the right-hand side of this equation is $> \frac{1}{q+1}$, and

$$< \frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \ldots$$

i.e. is $$< \frac{1}{q+1} \div \left(1 - \frac{1}{q+1}\right),$$

i.e. is $$< \frac{1}{q}.$$

Hence the right-hand side of (2) lies between $\frac{1}{q+1}$ and $\frac{1}{q}$, and is therefore a fraction and so cannot be equal to the left-hand side.

Hence our supposition that e was commensurable is incorrect and it therefore must be incommensurable.

253. Exponential Series. *When x is real, to prove that*

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots \textit{ ad inf.}$$

and that

$$a^x = 1 + x \log_e a + \frac{x^2}{\underline{|2}}(\log_e a)^2 + \dots \text{ ad inf.}$$

When n is greater than unity, we have

$$\left\{\left(1+\frac{1}{n}\right)^n\right\}^x = \left(1+\frac{1}{n}\right)^{nx}$$

$$= 1 + nx\frac{1}{n} + \frac{nx(nx-1)}{1.2}\frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{1.2.3}\frac{1}{n^3} + \dots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{1.2} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1.2.3} + \dots$$

In this expression make n infinitely great. The left-hand becomes, as in Art. 250, e^x.

The right-hand becomes

$$1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \dots$$

Hence we have

$$\mathbf{e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \dots \text{ ad inf.}} \quad \dots\dots (1).$$

Let $\qquad a = e^c$, so that $c = \log_e a$.

$$\therefore\ a^x = e^{cx} = 1 + cx + \frac{c^2x^2}{\underline{|2}} + \frac{c^3x^3}{\underline{|3}} + \dots,$$

by substituting cx for x in the series (1).

$$\therefore\ \mathbf{a^x = 1 + x\log_e a + \frac{x^2}{\underline{|2}}(\log_e a)^2 + \frac{x^3}{\underline{|3}}(\log_e a)^3 + \dots \text{ ad inf.}}$$

$$\dots\dots\dots\dots\dots(2).$$

254. It can be shewn (as in C. Smith's *Algebra*, Art. 274) that the series (1), and therefore (2), of the last article is convergent for all real values of x.

255. Ex. 1. *Prove that* $\frac{1}{2}\left(e-\frac{1}{e}\right)=1+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\dots$ *ad inf.*

By equation (1) of Art. 253 we have, by putting x in succession equal to 1 and -1,

$$e=1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\dots \textit{ ad inf.}$$

and $$e^{-1}=1-\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}-\dots \textit{ ad inf.}$$

Hence, by subtraction,

$$e-e^{-1}=2\left(1+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\dots\right),$$

i.e. $$\frac{1}{2}\left(e-\frac{1}{e}\right)=1+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\dots$$

Ex. 2. *Find the sum of the series*

$$1+\frac{1+2}{\lfloor 2}+\frac{1+2+3}{\lfloor 3}+\frac{1+2+3+4}{\lfloor 4}+\dots \textit{ ad inf.}$$

The nth term $$=\frac{1+2+3+\dots+n}{\lfloor n}=\frac{\frac{1}{2}n(n+1)}{\lfloor n}$$

$$=\frac{1}{2}\frac{n+1}{\lfloor n-1}=\frac{1}{2}\left[\frac{(n-1)+2}{\lfloor n-1}\right]=\frac{1}{2}\left[\frac{1}{\lfloor n-2}+\frac{2}{\lfloor n-1}\right],$$

provided that $n>2$.

Similarly

$$\text{the } (n-1)\text{th term}=\frac{1}{2}\left[\frac{1}{\lfloor n-3}+\frac{2}{\lfloor n-2}\right],$$

....

$$\text{the 4th term}=\frac{1}{2}\left[\frac{1}{\lfloor 2}+\frac{2}{\lfloor 3}\right],$$

$$\text{the 3rd term}=\frac{1}{2}\left[\frac{1}{\lfloor 1}+\frac{2}{\lfloor 2}\right].$$

Also $$\text{the 2nd term}=\frac{1}{2}\left[1+\frac{2}{\lfloor 1}\right],$$

$$\text{and the 1st term}=\frac{1}{2}\left[\quad\frac{2}{1}\right].$$

Hence, by addition, the whole series

$$=\frac{1}{2}\left[1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\dots\text{ ad inf.}\right]$$

$$+\frac{1}{2}\cdot 2\left[1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\dots\text{ ad inf.}\right]$$

$$=\frac{1}{2}\cdot e+e=\frac{3e}{2}.$$

256. Logarithmic Series. *To prove that, when y is real and numerically* <1, *then*

$$\log_e(1+y)=y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+\dots\text{ ad inf.}$$

In the equation (2) of Art. 253, put

$$a=1+y,$$

and we have

$$(1+y)^x=1+x\log_e(1+y)+\frac{x^2}{\underline{|2}}\{\log_e(1+y)\}^2+\dots(1).$$

But, since y is real and numerically $<$ unity, we have

$$(1+y)^x=1+x\,.\,y+\frac{x(x-1)}{1\,.\,2}y^2+\frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}y^3+\dots$$
$$\dots\dots\dots\dots(2).$$

The series on the right-hand side of (1) and (2) are equal to one another and both convergent, when y is numerically <1. Hence we may equate like powers of x.

Thus we have

$$\log_e(1+y)=y-\frac{y^2}{1\,.\,2}+\frac{(-1)(-2)}{1\,.\,2\,.\,3}y^3+\frac{(-1)(-2)(-3)}{1\,.\,2\,.\,3\,.\,4}y^4$$
$$+\dots\text{ ad inf.},$$

i.e. $$\mathbf{\log_e(1+y)=y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+\dots\dots(3).}$$

257. If $y=1$, the series (3) of the previous article is equal to

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$$

which is known to be convergent.

If $y=-1$, it equals $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}\dots$ which is known to be divergent.

In addition therefore to being true for all values of y between -1 and $+1$, it is true for the value $y=1$; it is not however true for the value $y=-1$.

258. Calculation of logarithms to base e.

In the logarithmic series, if we put $y=1$, we have

$$\log_e 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots\dots\dots\dots \quad (1).$$

If we put $$y=\frac{1}{2},$$

we have

$$\log_e 3-\log_e 2=\log_e\frac{3}{2}=\log_e\left(1+\frac{1}{2}\right)$$

$$=\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{3}\cdot\frac{1}{2^3}-\frac{1}{4}\cdot\frac{1}{2^4}+\dots \quad (2).$$

If we put $$y=\frac{1}{3},$$

we have

$$\log_e 4-\log_e 3=\log_e\left(1+\frac{1}{3}\right)=\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1}{3}\cdot\frac{1}{3^3}-\frac{1}{4}\cdot\frac{1}{3^4}+ \dots\dots\dots\dots\dots\dots (3).$$

From these equations we could, by taking a sufficient number of terms, calculate $\log_e 2$, $\log_e 3$, and $\log_e 4$.

It would be found that a large number of terms would have to be taken to give the values of these logarithms to the required degree of accuracy. We shall therefore obtain more convenient series.

259. By Art. 256 we have

$$\log_e(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \ldots\ldots(1),$$

and, by changing the sign of y,

$$\log_e(1-y) = -y - \frac{1}{2}y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4 + \ldots(2).$$

In both these series y must be numerically less than unity.

By subtraction, we have

$$\log_e(1+y) - \log_e(1-y) = \log_e\frac{1+y}{1-y} = 2\left[y + \frac{1}{3}y^3 + \frac{1}{5}y^5 + \ldots\right]$$
$$\ldots\ldots\ldots\ldots\ldots(3).$$

Let $$y = \frac{m-n}{m+n},$$

where m and n are positive integers and $m > n$, so that

$$\frac{1+y}{1-y} = \frac{m}{n}.$$

The equation (3) becomes

$$\log_e\frac{m}{n} = 2\left[\left(\frac{m-n}{m+n}\right) + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \ldots\right]\ldots(4).$$

Put $m = 2, n = 1$ in (4) and we get $\log_e 2$.

Put $m = 3$, $n = 2$ and we get $\log_e 3 - \log_e 2$, and therefore $\log_e 3$.

By proceeding in this way we get the value of the logarithm of any number to base e.

260. Logarithms to base 10. The logarithms of the previous article, to base e, are called Napierian or natural logarithms.

We can convert these logarithms into logarithms to base 10.

For, by Art. 147, we have, if N be any number,

$$\log_e N = \log_{10} N \times \log_e 10.$$

$$\therefore \log_{10} N = \log_e N \times \frac{1}{\log_e 10}.$$

Now $\log_e 10$ can be found as in the last article and then $\dfrac{1}{\log_e 10}$ is found to be ·4342944819...

Hence $\quad \log_{10} N = \log_e N \times {\cdot}43429448\ldots,$

so that the logarithm of any number to base 10 is found by multiplying its logarithm to base e by the quantity ·43429448.... This quantity is called the Modulus.

EXAMPLES. XLVI.

Prove that

1. $\frac{1}{2}(e+e^{-1}) = 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 6} + \ldots$

2. $\left(1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\ldots\right)\left(1-\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\ldots\right)=1.$

3. $\left(1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 6}+\ldots\right)^2 = 1+\left(1+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\ldots\right)^2.$

4. $1+\frac{2}{\lfloor 3}+\frac{3}{\lfloor 5}+\frac{4}{\lfloor 7}+\ldots=\frac{e}{2}.$ 5. $\frac{2}{\lfloor 3}+\frac{4}{\lfloor 5}+\frac{6}{\lfloor 7}+\ldots=e^{-1}.$

6. $\dfrac{\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 6}+\ldots}{1+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\ldots}=\dfrac{e-1}{e+1}.$

7. $1+\frac{2^3}{\lfloor 2}+\frac{3^3}{\lfloor 3}+\frac{4^3}{\lfloor 4}+\ldots=5e.$

Find the sum of the series

8. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots.$

9. $\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{3}\cdot\frac{1}{2^3}-\frac{1}{4}\cdot\frac{1}{2^4}+\ldots.$

Prove that

10. $\frac{a-b}{a}+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\ldots=\log_e a-\log_e b.$

11. $\log_e\frac{1+x}{1-x}=2\left(x+\frac{1}{3}x^3+\frac{1}{5}x^5+\ldots\right).$

12. $\log_e\frac{x+1}{x-1}=2\left(\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\ldots\right)$, if $x>1$.

13. $$\log_e(1+3x+2x^2)=3x-\frac{5x^2}{2}+\frac{9x^3}{3}-\frac{17x^4}{4}+\ldots$$
$$+(-1)^{n-1}\frac{2^n+1}{n}x^n+\ldots,$$

provided that $2x$ be not >1.

14. $2\log_e x-\log_e(x+1)-\log_e(x-1)=\frac{1}{x^2}+\frac{1}{2x^4}+\frac{1}{3x^6}+\ldots$, if $x>1$.

15. $\log_e 2=\frac{1}{1\,.\,2}+\frac{1}{3\,.\,4}+\frac{1}{5\,.\,6}+\ldots.$

16. $\log_e 2-\frac{1}{2}=\frac{1}{1\,.\,2\,.\,3}+\frac{1}{3\,.\,4\,.\,5}+\frac{1}{5\,.\,6\,.\,7}+\ldots.$

17. $\tan\theta+\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta+\ldots=\frac{1}{2}\log\frac{\cos\left(\theta-\frac{\pi}{4}\right)}{\cos\left(\theta+\frac{\pi}{4}\right)}$, if $\theta<\frac{\pi}{4}$.

18. If θ be $>\frac{\pi}{2}$ and $<\pi$, prove that

$$(1)\quad \sin\theta+\frac{1}{3}\sin^3\theta+\frac{1}{5}\sin^5\theta+\ldots\text{ ad inf.}$$
$$=2\left[\cot\frac{\theta}{2}+\frac{1}{3}\cot^3\frac{\theta}{2}+\frac{1}{5}\cot^5\frac{\theta}{2}+\ldots\text{ ad inf.}\right],$$

and
$$(2)\quad \frac{1}{2}\sin^2\theta+\frac{1}{4}\sin^4\theta+\frac{1}{6}\sin^6\theta+\ldots\text{ ad inf.}$$
$$=2\left[\tan^2\frac{\theta}{2}+\frac{1}{3}\tan^6\frac{\theta}{2}+\frac{1}{5}\tan^{10}\frac{\theta}{2}+\ldots\text{ ad inf.}\right]$$

19. If $\tan^2\theta < 1$, prove that

$$\tan^2\theta - \frac{1}{2}\tan^4\theta + \frac{1}{3}\tan^6\theta - \dots \text{ ad inf.}$$

$$= \sin^2\theta + \frac{1}{2}\sin^4\theta + \frac{1}{3}\sin^6\theta + \dots \text{ ad inf.}$$

20. Prove that, if 2θ be not a multiple of π,

$$\log\cot\theta = \cos 2\theta + \frac{1}{3}\cos^3 2\theta + \frac{1}{5}\cos^5 2\theta + \dots \text{ ad inf.}$$

21. Prove that the coefficient of x^n in the expansion of

$$\{\log_e(1+x)\}^2$$

is

$$\frac{2(-1)^n}{n}\left[1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right].$$

22. Use the methods of Arts. 259 and 260 to prove that

$$\log_{10}2 = \cdot 30103\dots$$

and

$$\log_{10}3 = \cdot 47712\dots.$$

23. Draw the curve $y = \log_e x$.

[If x be negative, y is imaginary; when x is zero, y equals $-\infty$; when x is unity, y is nothing; when x is positive and >1, y is always positive; when x is infinity, y is infinity also.]

24. Draw the curve $y = \log_{10} x$ and state the geometrical relation between it and the curve of the last example.

[Use Art. 147.]

25. Draw the curve $y = a^x$.

261. The two following limits will be required in the next chapter but one.

262. *To prove that the value of* $\left(\cos\frac{\alpha}{n}\right)^n$, *when n is infinite, is unity.*

We have $$\cos\frac{\alpha}{n} = \left(1 - \sin^2\frac{\alpha}{n}\right)^{\frac{1}{2}}.$$

$$\therefore \left(\cos\frac{\alpha}{n}\right)^n = \left(1 - \sin^2\frac{\alpha}{n}\right)^{\frac{n}{2}} = \left[\left(1 - \sin^2\frac{\alpha}{n}\right)^{-\frac{1}{\sin^2\frac{\alpha}{n}}}\right]^{-\frac{n}{2}\sin^2\frac{\alpha}{n}}$$

Now, by putting

$$-\sin^2\frac{\alpha}{n} = m,$$

we have

$$\underset{n=\infty}{\text{Lt}}\left\{1-\sin^2\frac{\alpha}{n}\right\}^{-\frac{1}{\sin^2\frac{\alpha}{n}}} = \underset{m=0}{\text{Lt}}\{1+m\}^{\frac{1}{m}} = e. \qquad \text{(Art. 250, Cor.)}$$

Also, by Art. 228,

$$\frac{n}{2}\sin^2\frac{\alpha}{n}$$

$$=\left(\frac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}}\right)^2 \times \frac{\alpha^2}{2n} = 1\times 0 = 0,$$

when n is infinite.

Hence, when n is infinite,

$$\left[\cos\frac{\alpha}{n}\right]^n = e^0 = 1.$$

Aliter. This limit may also be found by using the logarithmic series.

For, putting $\left(\cos\frac{\alpha}{n}\right)^n = u$, we have

$$\log_e u = n\log_e\cos\frac{\alpha}{n} = \frac{n}{2}\log_e\cos^2\frac{\alpha}{n}$$

$$=\frac{n}{2}\log_e\left(1-\sin^2\frac{\alpha}{n}\right)$$

$$=-\frac{n}{2}\left(\sin^2\frac{\alpha}{n}+\frac{1}{2}\sin^4\frac{\alpha}{n}+\frac{1}{3}\sin^6\frac{\alpha}{n}+\dots\right).$$

(Art. 256.)

The series inside the bracket lies between $\sin^2 \frac{\alpha}{n}$ and the series

$$\sin^2 \frac{\alpha}{n} + \sin^4 \frac{\alpha}{n} + \sin^6 \frac{\alpha}{n} + \ldots \text{ ad inf.},$$

i.e. lies between

$$\sin^2 \frac{\alpha}{n} \text{ and } \frac{\sin^2 \frac{\alpha}{n}}{1 - \sin^2 \frac{\alpha}{n}},$$

i.e. lies between $\quad \sin^2 \frac{\alpha}{n}$ and $\tan^2 \frac{\alpha}{n}$.

Hence $-\log u$ lies between

$$\frac{n}{2} \sin^2 \frac{\alpha}{n} \text{ and } \frac{n}{2} \tan^2 \frac{\alpha}{n} \ldots\ldots\ldots\ldots\ldots(1).$$

But

$$\operatorname*{Lt}_{n=\infty} \frac{n}{2} \sin^2 \frac{\alpha}{n} = \operatorname*{Lt}_{n=\infty} \left(\frac{\sin \frac{\alpha}{n}}{\frac{\alpha}{n}} \right)^2 \times \frac{\alpha^2}{2n} = 1 \times 0 = 0. \text{ (Art. 228.)}$$

And

$$\operatorname*{Lt}_{n=\infty} \frac{n}{2} \tan^2 \frac{\alpha}{n} = \operatorname*{Lt}_{n=\infty} \left\{ \left(\frac{\sin \frac{\alpha}{n}}{\frac{\alpha}{n}} \right)^2 \times \frac{1}{\cos^2 \frac{\alpha}{n}} \times \frac{\alpha^2}{2n} \right\} = 1 \times 1 \times 0 = 0.$$

(Art. 228.)

Hence in the limit both quantities (1) become 0, so that $\log u$ becomes zero also, and therefore, in the limit,

$$u = 1.$$

263. *To prove that the limiting value of* $\left(\dfrac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}}\right)^n$, *when n is infinite, is unity.*

We have shewn, in Art. 227, that $\sin\theta$, θ and $\tan\theta$ are in ascending order of magnitude.

Hence $$\sin\frac{\alpha}{n},\ \frac{\alpha}{n},\ \text{and}\ \tan\frac{\alpha}{n}$$

are in ascending order.

Hence $$1,\ \frac{\frac{\alpha}{n}}{\sin\frac{\alpha}{n}},\ \text{and}\ \frac{1}{\cos\frac{\alpha}{n}}$$

are in ascending order.

Therefore $\left(\dfrac{\frac{\alpha}{n}}{\sin\frac{\alpha}{n}}\right)^n$ lies between 1 and $\left(\dfrac{1}{\cos\frac{\alpha}{n}}\right)^n$, so that $\left(\dfrac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}}\right)^n$ lies between 1 and $\left(\cos\frac{\alpha}{n}\right)^n$.

But, by the last article, the value of $\left(\cos\frac{\alpha}{n}\right)^n$ is unity, when n is infinite.

Hence, when n is infinite, the value of $\left(\dfrac{\sin\frac{\alpha}{n}}{\frac{\alpha}{n}}\right)^n$ is unity.

CHAPTER XXII.

COMPLEX QUANTITIES. DE MOIVRE'S THEOREM.

264. Complex quantities. The quantity $x + y\sqrt{-1}$, where x and y are both real, is called a complex quantity. A complex quantity consists therefore of the sum of two quantities, one of which is wholly real and the other of which is wholly imaginary.

265. A complex quantity can always be put into the form $r(\cos\theta + \sqrt{-1}\sin\theta)$, where r and θ are both real. For assume that

$$x + y\sqrt{-1} = r(\cos\theta + \sqrt{-1}\sin\theta)$$

$$= r\cos\theta + \sqrt{-1} \,.\, r\sin\theta.$$

Equating the real and imaginary parts on the two sides of this equation, we have

$$r\cos\theta = x \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and $$r\sin\theta = y \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Hence, by squaring and adding, we have $r^2 = x^2 + y^2$,

so that $$r = \sqrt{x^2 + y^2}.$$

It is customary to take the positive square root of x^2+y^2 and hence r is known.

From (1) and (2) we then have

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \sin\theta = \frac{y}{\sqrt{x^2+y^2}}.$$

Whatever be the values of x and y, there is one value of θ, and only one value, lying between $-\pi$ radians and $+\pi$ radians which satisfies these two equations.

The quantity $x+y\sqrt{-1}$ can therefore always be expressed in the form $r(\cos\theta+\sqrt{-1}\sin\theta)$.

Def. The quantity $+\sqrt{x^2+y^2}$ is called the **Modulus** of the complex quantity, and that value of θ (lying between $-\pi$ and $+\pi$) which satisfies the relations

$$\cos\theta = \frac{x}{+\sqrt{x^2+y^2}} \text{ and } \sin\theta = \frac{y}{+\sqrt{x^2+y^2}}$$

is called the principal value of the **Amplitude** of

$$x+y\sqrt{-1}.$$

266. Ex. 1. *Express in the above form the quantity* $1+\sqrt{-1}$.

Here $\quad 1+\sqrt{-1}=r(\cos\theta+\sqrt{-1}\sin\theta)$,

so that $\quad r\cos\theta=1$,

and $\quad r\sin\theta=1$.

We therefore have $\quad r=+\sqrt{1+1}=+\sqrt{2}$,

and then $\quad \cos\theta=\frac{1}{\sqrt{2}}$ and $\sin\theta=\frac{1}{\sqrt{2}}$,

so that $\quad \theta=\frac{\pi}{4}$.

Hence $\quad 1+\sqrt{-1}=\sqrt{2}\left[\cos\frac{\pi}{4}+\sqrt{-1}\sin\frac{\pi}{4}\right]$,

so that $\sqrt{2}$ is the modulus and $\frac{\pi}{4}$ is the principal value of the amplitude of the given expression.

Ex. 2. *Quantity* $-1+\sqrt{-3}$.

Here $$-1+\sqrt{-1}\sqrt{3}=r(\cos\theta+\sqrt{-1}\sin\theta),$$

so that $$r\cos\theta=-1,\text{ and } r\sin\theta=\sqrt{3}.$$

$$\therefore\quad r=+\sqrt{1+3}=+2,$$

and then $$\cos\theta=-\frac{1}{2}\text{ and }\sin\theta=\frac{\sqrt{3}}{2},$$

so that $$\theta=\frac{2\pi}{3}.$$

$$\therefore\quad -1+\sqrt{-3}=2\left[\cos\frac{2\pi}{3}+\sqrt{-1}\sin\frac{2\pi}{3}\right].$$

Ex. 3. *Quantity* $-1-\sqrt{-3}$.

Here $$r\cos\theta=-1,\text{ and } r\sin\theta=-\sqrt{3},$$

so that $$r=+\sqrt{1+3}=+2,\ \cos\theta=-\frac{1}{2}\text{ and }\sin\theta=-\frac{\sqrt{3}}{2}.$$

Hence (since we choose for θ that value which lies between $-\pi$ and $+\pi$) we have $$\theta=-\frac{2\pi}{3}.$$

$$\therefore\quad -1-\sqrt{-3}=2\left[\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right].$$

267. In Art. 265 the equations

$$\cos\theta=\frac{x}{+\sqrt{x^2+y^2}}\text{ and }\sin\theta=\frac{y}{+\sqrt{x^2+y^2}}$$

are satisfied by more than one value of θ. For the cosine and sine of an angle repeat the same values when the angle is increased by any multiple of 2π radians, so that, if θ denote the value between $-\pi$ and $+\pi$ satisfying the above relations, the general solution is

$$2n\pi+\theta,$$

where n is any integer.

This is expressed by saying that the amplitude of a complex quantity is **many-valued.** The principal value is that particular value of the amplitude that lies between $-\pi$ and $+\pi$.

If to the principal value of θ we add any multiple of 2π we obtain one of its many values.

To sum up; If θ be that value, lying between $-\pi$ and $+\pi$, which satisfies the equations

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \sin\theta = \frac{y}{\sqrt{x^2+y^2}} \quad \ldots\ldots \text{ (1)},$$

then

$$x + y\sqrt{-1} = \sqrt{x^2+y^2}\,[\cos(2n\pi+\theta) + \sqrt{-1}\sin(2n\pi+\theta)].$$

The quantity $2n\pi + \theta$ is called the amplitude and θ is called its principal value.

For brevity we often write equations (1) in the form

$$\tan\theta = \frac{y}{x}, \; i.e. \; \theta = \tan^{-1}\frac{y}{x},$$

but it must be understood that here the angle denoted is the one that satisfies the conditions (1).

268. De Moivre's Theorem. *Whatever may be the value of n, positive or negative, integral or fractional, the value, or* **one** *of the values, of*

$$(\cos\theta + \sqrt{-1}\sin\theta)^n \text{ is } \cos n\theta + \sqrt{-1}\sin n\theta.$$

Case I. Let n be a positive integer.

By simple multiplication we have

$$[\cos\alpha + \sqrt{-1}\sin\alpha]\,[\cos\beta + \sqrt{-1}\sin\beta]$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + \sqrt{-1}\,[\sin\alpha\cos\beta + \cos\alpha\sin\beta]$$

$$= \cos(\alpha+\beta) + \sqrt{-1}\sin(\alpha+\beta).$$

So

$$[\cos\alpha+\sqrt{-1}\sin\alpha][\cos\beta+\sqrt{-1}\sin\beta][\cos\gamma+\sqrt{-1}\sin\gamma]$$
$$=[\cos(\alpha+\beta)+\sqrt{-1}\sin(\alpha+\beta)][\cos\gamma+\sqrt{-1}\sin\gamma]$$
$$=[\cos(\alpha+\beta)\cos\gamma-\sin(\alpha+\beta)\sin\gamma]$$
$$+\sqrt{-1}[\sin(\alpha+\beta)\cos\gamma+\cos(\alpha+\beta)\sin\gamma]$$
$$=\cos(\alpha+\beta+\gamma)+\sqrt{-1}\sin(\alpha+\beta+\gamma).$$

This process may evidently be continued indefinitely, so that

$$[\cos\alpha+\sqrt{-1}\sin\alpha][\cos\beta+\sqrt{-1}\sin\beta][\cos\gamma+\sqrt{-1}\sin\gamma] \ldots\ldots \text{ to } n \text{ factors}$$
$$=\cos(\alpha+\beta+\gamma+\ldots \text{ to } n \text{ terms})+\sqrt{-1}\sin[\alpha+\beta+\gamma+\ldots \text{ to } n \text{ terms}].$$

In this expression put

$$\alpha=\beta=\gamma=\ldots\ldots=\theta,$$

so that we have

$$[\cos\theta+\sqrt{-1}\sin\theta]^n=\cos n\theta+\sqrt{-1}\sin n\theta.$$

Case II. Let n be a negative integer and equal to $-m$.

We have, by the ordinary law of indices,

$$(\cos\theta+\sqrt{-1}\sin\theta)^n=(\cos\theta+\sqrt{-1}\sin\theta)^{-m}$$
$$=\frac{1}{(\cos\theta+\sqrt{-1}\sin\theta)^m}=\frac{1}{\cos m\theta+\sqrt{-1}\sin m\theta},$$

by Case I,

$$=\frac{\cos m\theta-\sqrt{-1}\sin m\theta}{(\cos m\theta+\sqrt{-1}\sin m\theta)(\cos m\theta-\sqrt{-1}\sin m\theta)}$$
$$=\frac{\cos m\theta-\sqrt{-1}\sin m\theta}{\cos^2 m\theta+\sin^2 m\theta}=\cos m\theta-\sqrt{-1}\sin m\theta$$
$$=\cos(-m)\theta+\sqrt{-1}\sin(-m)\theta$$
$$=\cos n\theta+\sqrt{-1}\sin n\theta.$$

Case III. Let n be fractional and equal to $\frac{p}{q}$, where q is a positive integer and p is an integer, positive or negative.

By the previous cases, we have

$$\left[\cos\frac{\theta}{q} + \sqrt{-1}\,\sin\frac{\theta}{q}\right]^q = \cos\left(q\,.\,\frac{\theta}{q}\right) + \sqrt{-1}\sin\left(q\,.\,\frac{\theta}{q}\right)$$
$$= \cos\theta + \sqrt{-1}\sin\theta.$$

Therefore $\cos\frac{\theta}{q} + \sqrt{-1}\,\sin\frac{\theta}{q}$ is such that when multiplied by itself q times it gives $\cos\theta + \sqrt{-1}\sin\theta$.

Hence $\cos\frac{\theta}{q} + \sqrt{-1}\,\sin\frac{\theta}{q}$ is one of the qth roots of

$$\cos\theta + \sqrt{-1}\,\sin\theta,$$

i.e.
$$\cos\frac{\theta}{q} + \sqrt{-1}\,\sin\frac{\theta}{q}$$
is **one** of the values of

$$(\cos\theta + \sqrt{-1}\,\sin\theta)^{\frac{1}{q}}.$$

Raise each of these quantities to the pth power.

We then have that one of the values of

$$[\cos\theta + \sqrt{-1}\,\sin\theta]^{\frac{p}{q}} \text{ is } \left(\cos\frac{\theta}{q} + \sqrt{-1}\,\sin\frac{\theta}{q}\right)^p,$$

i.e. is
$$\cos\frac{p\theta}{q} + \sqrt{-1}\,\sin\frac{p\theta}{q}.$$

269. The quantity i is always used to denote $\sqrt{-1}$ and will be often so used hereafter. The expression $\cos\theta + i\sin\theta$ therefore means $\cos\theta + \sqrt{-1}\sin\theta$.

Ex. 1. *Simplify*

$$\frac{(\cos 3\theta + i \sin 3\theta)^5 (\cos \theta - i \sin \theta)^3}{(\cos 5\theta + i \sin 5\theta)^7 (\cos 2\theta - i \sin 2\theta)^5}.$$

We have $\cos 3\theta + i \sin 3\theta = (\cos \theta + i \sin \theta)^3$,

$$\cos \theta - i \sin \theta = \cos(-\theta) + i \sin(-\theta) = (\cos \theta + i \sin \theta)^{-1},$$

$$\cos 5\theta + i \sin 5\theta = (\cos \theta + i \sin \theta)^5,$$

and $\cos 2\theta - i \sin 2\theta = \cos(-2\theta) + i \sin(-2\theta) = (\cos \theta + i \sin \theta)^{-2}$.

The given expression therefore

$$= \frac{(\cos \theta + i \sin \theta)^{15} (\cos \theta + i \sin \theta)^{-3}}{(\cos \theta + i \sin \theta)^{35} (\cos \theta + i \sin \theta)^{-10}}$$

$$= (\cos \theta + i \sin \theta)^{-13} = \cos 13\theta - i \sin 13\theta.$$

Ex. 2. *If* $2 \cos \theta = x + \frac{1}{x}$ *and* $2 \cos \phi = y + \frac{1}{y}$,

prove that $2 \cos (m\theta + n\phi) = x^m y^n + \frac{1}{x^m y^n}$.

We have $x^2 - 2x \cos \theta = -1$.

$$\therefore \quad (x - \cos \theta)^2 = -1 + \cos^2 \theta = -\sin^2 \theta.$$

$$\therefore \quad x = \cos \theta + i \sin \theta,$$

so that $x^m = \cos m\theta + i \sin m\theta$,

and $\frac{1}{x^m} = \cos m\theta - i \sin m\theta$.

Similarly $y = \cos \phi + i \sin \phi$,

so that $y^n = \cos n\phi + i \sin n\phi$,

and $\frac{1}{y^n} = \cos n\phi - i \sin n\phi$.

$$\therefore \quad x^m y^n + \frac{1}{x^m y^n}$$

$$= (\cos m\theta + i \sin m\theta)(\cos n\phi + i \sin n\phi)$$

$$+ (\cos m\theta - i \sin m\theta)(\cos n\phi - i \sin n\phi)$$

$$= \cos(m\theta + n\phi) + i \sin(m\theta + n\phi)$$

$$+ \cos(m\theta + n\phi) - i \sin(m\theta + n\phi)$$

$$= 2 \cos(m\theta + n\phi).$$

Similarly it could be shewn that

$$\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2 \cos(m\theta - n\phi).$$

Ex. 3. *If* $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0,$

prove that $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma),$

and $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma).$

This is an example of the many trigonometrical identities which are derived from algebraical identities.

For we know that if $a+b+c=0,$

then $a^3+b^3+c^3=3abc.$

Let $a=\cos\alpha+i\sin\alpha$, $b=\cos\beta+i\sin\beta$, and $c=\cos\gamma+i\sin\gamma$,

so that we have $a+b+c=0.$

$$\therefore\ (\cos\alpha+i\sin\alpha)^3+(\cos\beta+i\sin\beta)^3+(\cos\gamma+i\sin\gamma)^3$$
$$=3(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma),$$

so that, by De Moivre's Theorem,

$$(\cos 3\alpha+\cos 3\beta+\cos 3\gamma)+i(\sin 3\alpha+\sin 3\beta+\sin 3\gamma)$$
$$=3\cos(\alpha+\beta+\gamma)+3i\sin(\alpha+\beta+\gamma).$$

Hence, by equating real and imaginary parts, we have the required results.

EXAMPLES. XLVII.

Put into the form $r(\cos\theta+i\sin\theta)$ the quantities

1. $1+i.$ 2. $-1-i.$ 3. $-\sqrt{3}+i.$

4. $3+4i.$ 5. $1+\sqrt{2}+i.$ 6. $2-\sqrt{3}+i.$

Simplify

7. $\dfrac{(\cos\theta-i\sin\theta)^{10}}{(\cos\alpha+i\sin\alpha)^{12}}.$ 8. $\dfrac{(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)}{(\cos\gamma+i\sin\gamma)(\cos\delta+i\sin\delta)}.$

9. $\dfrac{(\cos 2\theta-i\sin 2\theta)^7(\cos 3\theta+i\sin 3\theta)^{-5}}{(\cos 4\theta+i\sin 4\theta)^{12}(\cos 5\theta-i\sin 5\theta)^{-6}}.$

10. $\dfrac{\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^{1\frac{1}{2}}}{\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{\frac{1}{2}}}.$ 11. $\dfrac{(\cos\alpha+i\sin\alpha)^4}{(\sin\beta+i\cos\beta)^5}.$

12. $\{(\cos\theta-\cos\phi)+i(\sin\theta-\sin\phi)\}^n+\{\cos\theta-\cos\phi-i(\sin\theta-\sin\phi)\}^n$

13. Prove that

$$(\sin x + i\cos x)^n = \cos n\left(\frac{\pi}{2} - x\right) + i\sin n\left(\frac{\pi}{2} - x\right),$$

and that $\left(\dfrac{1+\sin\phi + i\cos\phi}{1+\sin\phi - i\cos\phi}\right)^n = \cos\left(\dfrac{n\pi}{2} - n\phi\right) + i\sin\left(\dfrac{n\pi}{2} - n\phi\right)$.

If x, y, z and u stand respectively for

$\cos\alpha + i\sin\alpha$, $\cos\beta + i\sin\beta$, $\cos\gamma + i\sin\gamma$, and $\cos\delta + i\sin\delta$,

prove that

14. $$(x+y)(z+u) = 4\cos\frac{\alpha-\beta}{2}\cos\frac{\gamma-\delta}{2}\left[\cos\frac{\alpha+\beta+\gamma+\delta}{2} + i\sin\frac{\alpha+\beta+\gamma+\delta}{2}\right].$$

15. $$\frac{1}{(x-y)(z-u)} = -\frac{1}{4}\operatorname{cosec}\frac{\alpha-\beta}{2}\operatorname{cosec}\frac{\gamma-\delta}{2}\left[\cos\frac{\alpha+\beta+\gamma+\delta}{2} - i\sin\frac{\alpha+\beta+\gamma+\delta}{2}\right].$$

16. $$xy + zu = 2\cos\frac{\alpha+\beta-\gamma-\delta}{2}\left[\cos\frac{\alpha+\beta+\gamma+\delta}{2} + i\sin\frac{\alpha+\beta+\gamma+\delta}{2}\right].$$

17. From the identity

$$(a^2-b^2)(c^2-d^2) = (c^2-b^2)(a^2-d^2) + (a^2-c^2)(b^2-d^2)$$

prove, by putting $a = \cos\alpha + i\sin\alpha$ and similar expressions for the other letters, the identity

$$\sin(\alpha-\beta)\sin(\gamma-\delta) = \sin(\alpha-\delta)\sin(\gamma-\beta) + \sin(\alpha-\gamma)\sin(\beta-\delta).$$

18. From the identity

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)} = 1$$

deduce, by assuming $x = \cos 2\theta + i\sin 2\theta$ and corresponding quantities for a, b, and c, that

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\sin 2(\theta-\alpha) + \text{two similar expressions} = 0.$$

Similarly, deduce identities from the identity

$$\frac{1}{(x-a)(x-b)} = \frac{1}{(a-b)(x-a)} - \frac{1}{(a-b)(x-b)}.$$

19. Prove that

$$(a+bi)^{\frac{m}{n}}+(a-bi)^{\frac{m}{n}}=2\,(a^2+b^2)^{\frac{m}{2n}}\cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right).$$

20. If $$2\cos\theta=x+\frac{1}{x},$$

prove that $$2\cos r\theta=x^r+\frac{1}{x^r}.$$

21. If $$2\cos\theta=x+\frac{1}{x},\ 2\cos\phi=y+\frac{1}{y},\ldots\ldots$$

prove that $$2\cos(\theta+\phi+\ldots)=xyz\ldots+\frac{1}{xyz\ldots}.$$

22. If $$x_r=\cos\frac{\pi}{2^r}+\sqrt{-1}\sin\frac{\pi}{2^r};$$

prove that $$x_1\,.\,x_2\,.\,x_3\,.\ldots\text{ ad inf.}=\cos\pi.$$

23. Using De Moivre's Theorem solve the equation

$$x^4-x^3+x^2-x+1=0.$$

270. In Art. 269 we have only shewn that

$$\cos\frac{\theta}{q}+\sqrt{-1}\sin\frac{\theta}{q}$$

is one of the values of

$$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{1}{q}}.$$

The other values may be easily obtained. For

$$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{1}{q}}=[\cos(2n\pi+\theta)+\sqrt{-1}\sin(2n\pi+\theta)]^{\frac{1}{q}},$$

where n is any integer, and one of the values of the latter quantity is

$$\cos\frac{2n\pi+\theta}{q}+\sqrt{-1}\sin\frac{2n\pi+\theta}{q}.$$

By giving n the successive values 0, 1, 2, 3, ... $(q-1)$, we see that each of the quantities

$$\cos\frac{\theta}{q}+\sqrt{-1}\sin\frac{\theta}{q},$$

$$\cos\frac{2\pi+\theta}{q}+\sqrt{-1}\sin\frac{2\pi+\theta}{q},$$

$$\cos\frac{4\pi+\theta}{q}+\sqrt{-1}\sin\frac{4\pi+\theta}{q},$$

$$\cos\frac{6\pi+\theta}{q}+\sqrt{-1}\sin\frac{6\pi+\theta}{q}\ \ldots\ldots\ldots\ldots(1),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

is equal to one of the values of

$$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{1}{q}}.$$

The highest value that we need assign to n is $q-1$; for the values q, $q+1$, $q+2$,... will be found to give the same result as the values 0, 1, 2,....

Also no two of the quantities (1) will be the same. For all the angles involved therein differ from one another by less than 2π and no two angles, differing by less than 2π, have their cosines the same and also their sines the same.

To sum up; By giving to n the successive values $0, 1, 2, \ldots q-1$ in the expression

$$\cos\frac{2n\pi+\theta}{q}+\sqrt{-1}\sin\frac{2n\pi+\theta}{q}$$

we obtain q, and only q, different values for

$$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{1}{q}}.$$

271. By the use of the last article we can now obtain trigonometrical expressions for any root of a quantity of the form $x+yi$.

For we proved in Art. 267 that

$$x + yi = \rho\ [\cos (2n\pi + \theta) + \sqrt{-1} \sin (2n\pi + \theta)],$$

where $$\rho = + \sqrt{x^2 + y^2},$$

and θ is such that

$$\cos \theta = \frac{x}{\rho} \text{ and } \sin \theta = \frac{y}{\rho}.$$

Hence

$$(x + yi)^{\frac{1}{q}} = \rho^{\frac{1}{q}} \left[\cos \frac{2n\pi + \theta}{q} + \sqrt{-1} \sin \frac{2n\pi + \theta}{q} \right].$$

By giving n in succession the values $0, 1, 2, \ldots q - 1$, we obtain the q required roots.

272. **Ex. 1.** *Find the values of*

$$\left(\cos \frac{\pi}{3} + \sqrt{-1} \sin \frac{\pi}{3} \right)^{\frac{1}{4}}.$$

We have

$$\left(\cos \frac{\pi}{3} + \sqrt{-1} \sin \frac{\pi}{3} \right)^{\frac{1}{4}} = \left[\cos \left(2n\pi + \frac{\pi}{3} \right) + \sqrt{-1} \sin \left(2n\pi + \frac{\pi}{3} \right) \right]^{\frac{1}{4}},$$

where n is any integer,

$$= \cos \left(\frac{2n\pi}{4} + \frac{\pi}{12} \right) + \sqrt{-1} \sin \left(\frac{2n\pi}{4} + \frac{\pi}{12} \right).$$

Giving n in succession the values 0, 1, 2, and 3 we have as our answers the quantities

$$\cos \frac{\pi}{12} + \sqrt{-1} \sin \frac{\pi}{12},\ \cos \frac{7\pi}{12} + \sqrt{-1} \sin \frac{7\pi}{12},$$

$$\cos \frac{13\pi}{12} + \sqrt{-1} \sin \frac{13\pi}{12}, \text{ and } \cos \frac{19\pi}{12} + \sqrt{-1} \sin \frac{19\pi}{12}.$$

The student will note that the value $n = 4$ will not give us an additional value. For it gives

$$\cos \left(2\pi + \frac{\pi}{12} \right) + \sqrt{-1} \sin \left(2\pi + \frac{\pi}{12} \right),$$

which is the same as $\cos\frac{\pi}{12}+\sqrt{-1}\sin\frac{\pi}{12}$,

and this is the first of the quantities already found. Similarly the values $n=5$, $n=6$, $n=7$ would only give respectively the remaining three quantities, and so on.

Ex. 2. *Find all the values of* $(-1)^{\frac{1}{3}}$.

Since $\cos\pi=-1$, and $\sin\pi=0$,

we have
$$(-1)^{\frac{1}{3}}=(\cos\pi+\sqrt{-1}\sin\pi)^{\frac{1}{3}}$$
$$=[\cos(2n\pi+\pi)+\sqrt{-1}\sin(2n\pi+\pi)]^{\frac{1}{3}}$$
$$=\cos\frac{2n\pi+\pi}{3}+\sqrt{-1}\sin\frac{2n\pi+\pi}{3}.$$

Giving n the values 0, 1, and 2, the required values are

$$\cos\frac{\pi}{3}+\sqrt{-1}\sin\frac{\pi}{3},\ \cos\pi+\sqrt{-1}\sin\pi,\ \text{and}\ \cos\frac{5\pi}{3}+\sqrt{-1}\sin\frac{5\pi}{3},$$

i.e.
$$\frac{1+\sqrt{-3}}{2},\ -1,\ \text{and}\ \frac{1-\sqrt{-3}}{2}.$$

EXAMPLES. XLVIII.

Find all the values of

1. $1^{\frac{1}{3}}$.
2. $(-1)^{\frac{1}{6}}$.
3. $(-i)^{\frac{1}{6}}$.
4. $(-1)^{\frac{1}{10}}$.
5. $(1+\sqrt{-1})^{\frac{1}{6}}$.
6. $(1+\sqrt{-3})^{\frac{11}{3}}$.
7. $(1-\sqrt{-3})^{\frac{1}{4}}$.
8. $(\sqrt{3}+\sqrt{-1})^{\frac{1}{3}}$.
9. $(\sqrt{3}-\sqrt{-1})^{\frac{2}{5}}$.
10. $16^{\frac{1}{4}}$.
11. $32^{\frac{1}{5}}$.
12. $(1+\sqrt{-3})^{10}+(1-\sqrt{-3})^{10}$.

13. Simplify $\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^{\frac{1}{4}}$

and express the results in a form free from trigonometrical expressions.

14. Find the continued product of the four values of

$$\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}.$$

15. Prove that the roots of the equation $x^{10}+11x^5-1=0$ are

$$\frac{\sqrt{5}-1}{2}\left[\cos\frac{2r\pi}{5}\pm i\sin\frac{2r\pi}{5}\right].$$

16. Solve the equation $x^{12}-1=0$ and find which of its roots satisfy the equation $x^4+x^2+1=0$.

17. Prove that $$\sqrt[n]{a+bi}+\sqrt[n]{a-bi}$$
has n real values and find those of
$$\sqrt[3]{1+\sqrt{-3}}+\sqrt[3]{1-\sqrt{-3}}.$$

18. Prove that the n nth roots of unity form a series in G.P.

19. Find the seven 7th roots of unity and prove that the sum of their nth powers always vanishes unless n be a multiple of 7, n being an integer, and that then the sum is 7.

273. Binomial Theorem for Complex Quantities. It is known that for any real values of n and z, provided that z be less than unity, we have

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{1.2} z^2 + \frac{n(n-1)(n-2)}{1.2.3} z^3 + \dots \qquad \dots\dots\dots(1).$$

When z is complex $(= x + y\sqrt{-1})$ and n is a positive integer, the ordinary proof applies and the theorem (1) is still true.

When z is complex, and n is a fraction or negative, it can be shewn that

$$1 + nz + \frac{n(n-1)}{1.2} z^2 + \dots\dots\dots\dots\dots\dots (2)$$

is **one** of the values of $(1+z)^n$, provided that the modulus of z, *i.e.* $\sqrt{x^2+y^2}$, is less than unity. When this modulus is equal to unity, the theorem is only true (1) when n is positive, and (2) when n is a negative fraction and z is not equal to -1.

The proof is difficult and beyond the range of the present book. We shall therefore assume the result. The student may hereafter refer to Hobson's *Trigonometry*, Arts. 211 and 212.

CHAPTER XXIII.

EXPANSIONS OF $\sin n\theta$ AND $\cos n\theta$. SERIES FOR $\sin\theta$ AND $\cos\theta$ IN POWERS OF θ.

274. BY the use of De Moivre's Theorem we can obtain the expansion of $\cos n\theta$ and $\sin n\theta$ in terms of the trigonometrical functions of θ.

For we have

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n.$$

Since n is a positive integer, the Binomial Theorem holds for $(\cos\theta + i\sin\theta)^n$.

Hence, by expanding, we have

$$\cos n\theta + i\sin n\theta = \cos^n\theta + n\cos^{n-1}\theta \,.\, i\sin\theta$$

$$+\frac{n(n-1)}{1\,.\,2}\cos^{n-2}\theta\,.\,i^2\sin^2\theta + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\cos^{n-3}\theta\,.\,i^3\sin^3\theta \ldots$$

Hence, since

$$i^2 = -1,\quad i^3 = -i,\quad i^4 = 1,\quad i^5 = i, \ldots$$

we have

$$\cos n\theta + i\sin n\theta = \cos^n\theta - \frac{n(n-1)}{1\,.\,2}\cos^{n-2}\theta\sin^2\theta$$

$$+\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}\cos^{n-4}\theta\sin^4\theta + \ldots$$

$$+ i\left[n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\cos^{n-3}\theta\sin^3\theta + \ldots\right].$$

By equating real and imaginary parts, we have

$$\cos n\theta = \cos^n \theta - \frac{n(n-1)}{1.2} \cos^{n-2}\theta \sin^2\theta + \ldots\ldots (1),$$

and

$$\sin n\theta = n \cos^{n-1}\theta \sin\theta - \frac{n(n-1)(n-2)}{1.2.3} \cos^{n-3}\theta \sin^3\theta$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5} \cos^{n-5}\theta \sin^5\theta - \ldots\ldots (2).$$

The terms in each of these series are alternately positive and negative. Also each series continues till one of the factors in the numerator is zero and then ceases.

275. From equations (1) and (2) of the last article we have, by division,

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$$

$$= \frac{n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}\theta\sin^3\theta + \ldots\ldots}{\cos^n\theta - \frac{n(n-1)}{1.2}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cos^{n-4}\theta\sin^4\theta\ldots\ldots}$$

Divide the numerator and denominator of the right-hand member of this equation by $\cos^n\theta$, and we have

$$\tan n\theta = \frac{n\tan\theta - \frac{n(n-1)(n-2)}{1.2.3}\tan^3\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}\tan^5\theta \ldots\ldots}{1 - \frac{n(n-1)}{1.2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\tan^4\theta\ldots\ldots}$$

276. The values for $\cos n\theta$ and $\sin n\theta$ in Art. 274 may also be obtained, by Induction, without the use of imaginary quantities.

For assume (1) and (2) to be true for any value of n. Then, since

$$\cos(n+1)\theta = \cos n\theta\cos\theta - \sin n\theta\sin\theta,$$

we obtain the value of $\cos(n+1)\theta$, which, after rearrangement, is found to be obtained from (1) by changing n into $(n+1)$.

Similarly for $\sin(n+1)\theta$.

Hence, if the formulæ (1) and (2) are true for one value of n, they are true for the next greater value.

But it is easy to shew that they are true for the values $n=2$ and $n=3$. Hence, by Induction, they can be proved to be true for all values of n.

277. From De Moivre's Theorem may be deduced expressions for the sine, cosine and tangent of the sum of any number of unequal angles in terms of the tangents of these angles.

For we have

$$\cos(\alpha+\beta+\gamma+\ldots)+i\sin(\alpha+\beta+\gamma+\ldots)$$
$$=(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)\ldots(1).$$

Now $\qquad \cos\alpha+i\sin\alpha=\cos\alpha\,[1+i\tan\alpha],$

$$\cos\beta+i\sin\beta=\cos\beta\,(1+i\tan\beta),$$

....

Hence (1) may be written

$$\cos(\alpha+\beta+\gamma+\ldots)+i\sin(\alpha+\beta+\gamma+\ldots)$$
$$=\cos\alpha\cos\beta\cos\gamma\ldots(1+i\tan\alpha)(1+i\tan\beta)(1+i\tan\gamma)\ldots$$
$$=\cos\alpha\cos\beta\cos\gamma\ldots[1+i(\tan\alpha+\tan\beta+\tan\gamma+\ldots)$$
$$+i^2(\tan\alpha\tan\beta+\tan\beta\tan\gamma+\ldots)$$
$$+i^3(\tan\alpha\tan\beta\tan\gamma+\tan\beta\tan\gamma\tan\delta\ldots)$$
$$+\ldots\ldots]\ldots\ldots\ldots\ldots(2).$$

Using the notation of Art. 125, this equation may be written

$$\cos(\alpha+\beta+\gamma+\ldots)+i\sin(\alpha+\beta+\gamma+\ldots)$$
$$=\cos\alpha\cos\beta\cos\gamma\ldots[1+is_1-s_2-is_3+s_4+is_5-s_6\ldots]$$

Hence equating real and imaginary parts, we have
$\sin(\alpha+\beta+\gamma\ldots)=\cos\alpha\cos\beta\cos\gamma\ldots[s_1-s_3+s_5-s_7\ldots]\ldots(3)$,
and
$\cos(\alpha+\beta+\gamma\ldots)=\cos\alpha\cos\beta\cos\gamma\ldots(1-s_2+s_4-s_6\ldots)\ldots(4)$.

Hence, by division,

$$\mathbf{tan}\ (\alpha+\beta+\gamma+\ldots)=\frac{\mathbf{s}_1-\mathbf{s}_3+\mathbf{s}_5-\mathbf{s}_7\ldots}{\mathbf{1}-\mathbf{s}_2+\mathbf{s}_4-\mathbf{s}_6\ldots}\ \ldots(5).$$

The signs in the expressions on the right hand of (3) and (4) are alternately positive and negative.

The relation (5) was shewn, by Induction, to be true in Art. 125.

278. **Ex.** *Prove that the equation*

$$a^2\cos^2\theta+b^2\sin^2\theta+2ga\cos\theta+2fb\sin\theta+c=0$$

has 4 *roots, and that the sum of the values of* θ *which satisfy it is an even multiple of* π *radians.*

Let $t\equiv\tan\frac{\theta}{2}$.

Then since (Art. 109), $\sin\theta=\dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$ and $\cos\theta=\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$,

the equation above becomes

$$a^2\left(\frac{1-t^2}{1+t^2}\right)^2+b^2\left(\frac{2t}{1+t^2}\right)^2+2ga\frac{1-t^2}{1+t^2}+2fb\frac{2t}{1+t^2}+c=0,$$

or, on reduction and simplification,

$$t^4(a^2-2ga+c)+4fbt^3+t^2(4b^2-2a^2+2c)+4fbt+a^2+2ga+c=0\ldots\ldots(1).$$

This is an equation having 4 roots.

Also $\qquad s_1=\text{sum of the roots}=-\dfrac{4fb}{a^2-2ga+c}$,

$s_2=\text{sum taken two at a time}=\dfrac{4b^2-2a^2+2c}{a^2-2ga+c}$,

$s_3=\text{sum taken three at a time}=-\dfrac{4fb}{a^2-2ga+c}$,

and $\qquad s_4=\text{sum taken four at a time}=\dfrac{a^2+2ga+c}{a^2-2ga+c}$.

Since $s_1 = s_3$, it follows, by the last article, that

$$\tan\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right) = \frac{s_1 - s_3}{1 - s_2 + s_4} = 0 = \tan n\pi.$$

[The denominator $1 - s_2 + s_4$ does not vanish unless $a^2 = b^2$.]

$$\therefore\ \theta_1+\theta_2+\theta_3+\theta_4 = 2 \,.\, n\pi \text{ radians}$$

$$= \text{an even multiple of } \pi \text{ radians.}$$

EXAMPLES. XLIX.

Prove that

1. $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$

2. $\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta.$

3. $\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta.$

4. $\cos 9\theta = \cos^9\theta - 36\cos^7\theta\sin^2\theta + 126\cos^5\theta\sin^4\theta$
$- 84\cos^3\theta\sin^6\theta + 9\cos\theta\sin^8\theta.$

5. $\cos 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta$
$- 28\cos^2\theta\sin^6\theta + \sin^8\theta.$

Write down, in terms of $\tan\theta$, the values of

6. $\tan 5\theta$. 7. $\tan 7\theta$. 8. $\tan 9\theta$.

9. Prove that the last terms in the expressions for $\cos 11\theta$ and $\sin 11\theta$ are

$$-11\cos\theta\sin^{10}\theta \text{ and } -\sin^{11}\theta.$$

10. Prove that the last terms in the expressions for $\sin 8\theta$ and $\sin 9\theta$ are $-8\cos\theta\sin^7\theta$ and $\sin^9\theta$ respectively.

11. When n is odd, prove that the last terms in the expansions of $\sin n\theta$ and $\cos n\theta$ are respectively

$$(-1)^{\frac{n-1}{2}}\sin^n\theta \text{ and } n(-1)^{\frac{n-1}{2}}\cos\theta\sin^{n-1}\theta.$$

12. When n is even, prove that the last terms in the expansion of $\sin n\theta$ and $\cos n\theta$ are respectively

$$n(-1)^{\frac{n-2}{2}}\cos\theta\sin^{n-1}\theta \text{ and } (-1)^{\frac{n}{2}}\sin^n\theta.$$

13. If α, β, and γ be the roots of the equation

$$x^3 + px^2 + qx + p = 0,$$

prove that $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ radians,

except in one particular case.

14. Prove that the equation

$$\sin 3\theta = a \sin\theta + b\cos\theta + c$$

has six roots and that the sum of the six values of θ, which satisfy it, is equal to an even multiple of $\frac{\pi}{2}$ radians.

15. Prove that the equation

$$ah \sec\theta - bk \operatorname{cosec}\theta = a^2 - b^2$$

has four roots, and that the sum of the four values of θ, which satisfy it, is equal to an odd multiple of π radians.

16. If $\alpha, \beta, \gamma, \ldots$ be the roots of the equation

$$\sin mx - nx \cos mx = 0,$$

prove that $$\tan^{-1}\frac{x}{\alpha} + \tan^{-1}\frac{x}{\beta} + \ldots + \tan^{-1}\frac{x}{\nu} = 0.$$

EXPANSIONS OF THE SINE AND COSINE OF AN ANGLE IN SERIES OF ASCENDING POWERS OF THE ANGLE.

279. As in Art. 274 we have

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{1.2}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cos^{n-4}\theta\sin^4\theta - \ldots.$$

Put $n\theta = \alpha$, and we have

$$\cos\alpha = \cos^n\theta - \frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)}{1.2}\cos^{n-2}\theta\sin^2\theta + \frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)\left(\frac{\alpha}{\theta}-3\right)}{1.2.3.4}\cos^{n-4}\theta\sin^4\theta - \ldots$$

$$= \cos^n\theta - \frac{\alpha(\alpha-\theta)}{1.2}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2 + \frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)}{1.2.3.4}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4 - \ldots\ldots(1).$$

In equation (1) make θ indefinitely small, α remaining constant and therefore n becoming indefinitely great.

Then $\frac{\sin\theta}{\theta}$ is, in the limit, equal to unity and so is every power of $\left(\frac{\sin\theta}{\theta}\right)$. (Art. 263.)

Also $\cos\theta$ is, in the limit, equal to unity and so also is every power of $\cos\theta$. (Art. 262.)

Hence (1) becomes

$$\cos\alpha = 1 - \frac{\alpha^2}{\underline{|2}} + \frac{\alpha^4}{\underline{|4}} - \frac{\alpha^6}{\underline{|6}} \dots \text{ ad inf.}$$

280. *To expand $\sin\alpha$ in terms of α.*

As in Art. 274, we have

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}\theta\sin^3\theta + \dots$$

As before put $n\theta = \alpha$, and we have

$$\sin\alpha = \frac{\alpha}{\theta}\cos^{n-1}\theta\sin\theta - \frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)}{1.2.3}\cos^{n-3}\theta\sin^3\theta$$

$$+\frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)\left(\frac{\alpha}{\theta}-3\right)\left(\frac{\alpha}{\theta}-4\right)}{1.2.3.4.5}\cos^{n-5}\theta\sin^5\theta \dots$$

$$=\alpha\cos^{n-1}\theta.\left(\frac{\sin\theta}{\theta}\right) - \frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{1.2.3}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3 + \dots.$$

As in the last article make θ indefinitely small, keeping α finite, and we have

$$\sin\alpha = \alpha - \frac{\alpha^3}{\underline{|3}} + \frac{\alpha^5}{\underline{|5}} - \frac{\alpha^7}{\underline{|7}} + \dots \text{ ad inf.}$$

281. There is no series, proceeding according to a simple law, for the expansion of $\tan\theta$ in terms of θ, similar to those of Arts. 279 and 280.

We shall find the series for $\tan\theta$ as far as the term involving θ^5.

For $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \dots}{1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}}}$$

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \dots\right)\left[1 - \left(\frac{\theta^2}{2} - \frac{\theta^4}{24} + \dots\right)\right]^{-1}$$

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \dots\right)\left[1 + \left(\frac{\theta^2}{2} - \frac{\theta^4}{24} + \dots\right) + \left(\frac{\theta^2}{2} - \frac{\theta^4}{24}\dots\right)^2 \dots\right],$$

by the Binomial Theorem,

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \dots\right)\left[1 + \frac{\theta^2}{2} - \frac{\theta^4}{24}\dots + \frac{\theta^4}{4}\dots\right],$$

neglecting θ^6 and higher powers of θ,

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \dots\right)\left(1 + \frac{\theta^2}{2} + \frac{5}{24}\theta^4 \dots\right)$$

$$= \theta + \frac{\theta^3}{3} + \frac{2}{15}\theta^5,$$

on reduction and neglecting powers of θ above θ^5.

A similar method would give the series for $\tan\theta$ to as many terms as we please. The method however soon becomes very cumbrous and troublesome.

282. In Arts. 279 and 280 we tacitly assumed that α was equal to the number of radians in the angle considered. For, unless this be the case, the limit of $\frac{\sin\theta}{\theta}$ is not unity when θ is made indefinitely small.

When the angle is expressed in degrees we proceed as follows.

Let $\alpha^\circ = x$ radians, so that

$$\frac{\alpha}{180} = \frac{x}{\pi},$$

and hence

$$x = \frac{\pi}{180}\alpha.$$

Then

$$\cos\alpha^\circ = \cos x^c$$

$$= 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \dots$$

$$= 1 - \frac{1}{\lfloor 2}\frac{\pi^2\alpha^2}{180^2} + \frac{1}{\lfloor 4}\frac{\pi^4\alpha^4}{180^4} - \frac{1}{\lfloor 6}\frac{\pi^6\alpha^6}{180^6} + \dots.$$

So also

$$\sin\alpha^\circ = \sin x^c = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \dots$$

$$= \frac{\pi\alpha}{180} - \frac{1}{\lfloor 3}\left(\frac{\pi\alpha}{180}\right)^3 + \frac{1}{\lfloor 5}\left(\frac{\pi\alpha}{180}\right)^5 \dots.$$

283. Sines and cosines of small angles. The series of Arts. 279 and 280 may be used to find the sines and cosines of small angles.

For example, let us find the values of sin 10″ and cos 10″.

Since

$$10'' = \left(\frac{1}{6\times 60}\times\frac{\pi}{180}\right) \text{ radians}$$

$$= \left(\frac{\pi}{64800}\right)^c,$$

we have

$$\sin 10'' = \frac{\pi}{64800} - \frac{1}{\underline{|3}}\left(\frac{\pi}{64800}\right)^3 + \frac{1}{\underline{|5}}\left(\frac{\pi}{64800}\right)^5 - \ldots$$

and $$\cos 10'' = 1 - \frac{1}{\underline{|2}}\left(\frac{\pi}{64800}\right)^2 + \frac{1}{\underline{|4}}\left(\frac{\pi}{64800}\right)^4 - \ldots.$$

Now $$\frac{\pi}{64800} = \cdot000048481368\ldots,$$

$$\left(\frac{\pi}{64800}\right)^2 = \cdot0000000023504\ldots,$$

and $$\left(\frac{\pi}{64800}\right)^3 = \cdot000000000000113928\ldots.$$

Hence, to twelve places of decimals, we have

$$\sin 10'' = \cdot000048481368,$$

and $$\cos 10'' = 1 - \frac{\cdot0000000023504}{2}$$

$$= 1 - \cdot000000001175$$

$$= \cdot999999998825.$$

284. Approximate value of the root of an equation. The series of Art. 280 may also be used to find an approximate value of the root of an equation. The method will be best shewn by examples.

Ex. 1. *If* $\frac{\sin\theta}{\theta} = \frac{1349}{1350}$, *prove that the angle* θ *is very nearly equal to* $\frac{1}{15}$*th radian.*

We know that, the smaller θ is, the more nearly is $\frac{\sin\theta}{\theta}$ equal to unity. Conversely in our case we see that θ is small.

In the series for $\sin\theta$ (Art. 280) let us omit the powers of θ above the third, and we have

$$\frac{\theta - \dfrac{\theta^3}{\underline{|3}}}{\theta} = \frac{1349}{1350} = 1 - \frac{1}{1350}.$$

$$\therefore \quad \theta^2 = \frac{6}{1350} = \frac{1}{225}.$$

Hence $\theta = \frac{1}{15}$, so that the angle is $\frac{1}{15}$ of a radian nearly.

If we desire a nearer approximation, we take the series for $\sin\theta$ and omit powers above the 5th. We then have

$$\frac{\theta - \dfrac{\theta^3}{\underline{|3}} + \dfrac{\theta^5}{\underline{|5}}}{\theta} = 1 - \frac{1}{1350}.$$

This gives $\qquad \theta^4 - 20\theta^2 = -\frac{120}{1350} = -\frac{20}{225}.$

Hence, by solving,

$$\theta^2 = 10 \pm \frac{\sqrt{22480}}{15} = \frac{150 - 149{\cdot}933312\ldots}{15} = \frac{{\cdot}066688}{15}$$

$$= \frac{1{\cdot}00032}{15^2}.$$

$$\therefore \quad \theta = \frac{1{\cdot}00016}{15} \text{ radian.}$$

This differs from the first approximation by about $\frac{1}{6000}$th part.

Ex. 2. *Solve approximately the equation*

$$\cos\left(\frac{\pi}{3} + \theta\right) = {\cdot}49.$$

Since ·49 is very nearly equal to $\frac{1}{2}$, which is the value of $\cos\frac{\pi}{3}$, it follows that θ must be small.

The equation may be written

$$\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta = {\cdot}49 = \frac{1}{2} - \frac{1}{100} \quad\text{.....................(1).}$$

For a first approximation omit squares and higher powers of θ. By Art. 280 this equation then becomes

$$\frac{1}{2} \cdot 1 - \frac{\sqrt{3}}{2} \cdot \theta = \frac{1}{2} - \frac{1}{100},$$

so that

$$\theta = \frac{2}{\sqrt{3}} \cdot \frac{1}{100} = \frac{2\sqrt{3}}{300} = \frac{3{\cdot}4641...}{300} = {\cdot}011547... \text{ radian.}$$

For a still nearer approximation, omit cubes and higher powers of θ. The equation (1) then becomes

$$\frac{1}{2}\left(1 - \frac{\theta^2}{2}\right) - \frac{\sqrt{3}}{2}\theta = \frac{1}{2} - \frac{1}{100},$$

i.e.

$$\theta^2 + 2\sqrt{3}\theta = \frac{4}{100}.$$

$$\therefore \quad \theta = -\sqrt{3} + \frac{\sqrt{304}}{10} = {\cdot}0115086... \text{ radian.}$$

The first approximation is therefore correct to 4 places of decimals.

The angle θ is therefore very nearly equal to ·0115 radian, i.e. to about 40′.

The accurate answer is found, from the tables, to be ·0115075... radian.

285. Evaluation of quantities apparently indeterminate. We often have to obtain the value of quantities which are apparently indeterminate.

Suppose we required the value of the expression

$$\frac{3 \sin \theta - \sin 3\theta}{\theta (\cos \theta - \cos 3\theta)},$$

when θ is zero.

If we substitute the value 0 for θ, we have

$$\frac{0 - 0}{0 \times 0},$$

which is apparently indeterminate.

The expression however, for all values of θ,

$$= \frac{3 \sin \theta - (3 \sin \theta - 4 \sin^3 \theta)}{\theta \{\cos \theta - (4 \cos^3 \theta - 3 \cos \theta)\}} = \frac{4 \sin^3 \theta}{\theta \{4 \cos \theta - 4 \cos^3 \theta\}}$$

$$= \frac{\sin^3 \theta}{\theta \cos \theta \sin^2 \theta} = \frac{\sin \theta}{\theta \cos \theta} = \frac{1}{\cos \theta} \times \frac{\sin \theta}{\theta}.$$

Now, the smaller θ is, the more nearly do both

$$\frac{1}{\cos\theta} \text{ and } \frac{\sin\theta}{\theta}$$

approach to unity. Hence, when θ is actually zero, the given expression $= 1 \times 1 = 1$.

Such an expression as the one we have discussed is said to be indeterminate. We should more properly say that the expression is "at first sight" indeterminate.

286. In many cases the real value is very easily found by using the series for $\sin\theta$ and $\cos\theta$. The method is shewn in the following examples, of the first of which the example in the preceding article is a particular case.

Ex. 1. *Find the value of*

$$\frac{n\sin\theta - \sin n\theta}{\theta(\cos\theta - \cos n\theta)}.$$

The expression

$$= \frac{n\left(\theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \dots\right) - \left(n\theta - \frac{n^3\theta^3}{\lfloor 3} + \frac{n^5\theta^5}{\lfloor 5} - \dots\right)}{\theta\left[\left(1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4}\dots\right) - \left(1 - \frac{n^2\theta^2}{\lfloor 2} + \frac{n^4\theta^4}{\lfloor 4}\dots\right)\right]}$$

$$= \frac{\frac{n^3-n}{\lfloor 3}\theta^3 - \frac{n^5-n}{\lfloor 5}\theta^5 + \text{higher powers of } \theta}{\theta\left[\frac{n^2-1}{\lfloor 2}\theta^2 - \frac{n^4-1}{\lfloor 4}\theta^4 + \text{higher powers of } \theta\right]}$$

$$= \frac{\frac{n^3-n}{\lfloor 3} - \frac{n^5-n}{\lfloor 5}\theta^2 + \text{higher powers}}{\frac{n^2-1}{\lfloor 2} - \frac{n^4-1}{\lfloor 4}\theta^2 + \text{higher powers}}.$$

When θ is zero, this expression

$$= \frac{n^3-n}{\lfloor 3} \div \frac{n^2-1}{\lfloor 2} = \frac{n}{3}.$$

Ex. 2. *Find the value, when x is zero, of the expression*

$$\frac{\cos x - \log_e(1+x) + \sin x - 1}{e^x - (1+x)}.$$

Since $$\log_e(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \ldots,$$

and $$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} \ldots \text{ (Arts. 253 and 256)},$$

this expression

$$= \frac{\left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} \ldots\right) - \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \ldots\right) + \left(x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \ldots\right) - 1}{\left(1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\right) - (1+x)}$$

$$= \frac{-\frac{x^3}{\lfloor 2} + \text{higher powers of } x}{\frac{x^2}{\lfloor 2} + \text{higher powers of } x} = \frac{-\frac{x}{\lfloor 2} + \text{powers of } x}{\frac{1}{\lfloor 2} + \text{powers of } x}.$$

When x is zero, this latter expression

$$= \frac{0}{1} = 0.$$

Ex. 3. *Find the value, when x is zero, of*

$$\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}.$$

When x is zero, this expression is of the form $\left(\frac{0}{0}\right)^{\infty}$.

But it also $$= \left(\frac{x + \frac{x^3}{3} + \ldots}{x}\right)^{\frac{1}{x}} \text{ (Art. 281).}$$

Now, by Art. 250, the value of

$$\left(1 + \frac{x^2}{3}\right)^{\frac{3}{x^2}}$$

is e, when x is zero.

Hence the expression $$= e^{\frac{x}{3}} = e^0 = 1.$$

The value of the expression may be also found by finding the value of its logarithm.

EXAMPLES. L.

1. If $$\frac{\sin\theta}{\theta}=\frac{1013}{1014},$$
prove that θ is the number of radians in 4° 24′ nearly.

2. If $$\frac{\sin\theta}{\theta}=\frac{863}{864},$$
prove that θ is equal to 4° 47′ nearly.

3. If $$\frac{\sin\theta}{\theta}=\frac{5045}{5046},$$
prove that the angle θ is 1° 58′ nearly.

4. If $$\frac{\sin\theta}{\theta}=\frac{2165}{2166},$$
prove that θ is equal to 3° 1′ nearly.

5. If $$\frac{\sin\theta}{\theta}=\frac{19493}{19494},$$
prove that θ is equal to 1° nearly.

6. If $$\tan\theta=\frac{1}{15},$$
find an approximate value for θ.

Find the value, when x is zero, of the expressions

7. $\dfrac{x-\sin x}{x^3}$.
8. $\dfrac{x^2}{1-\cos mx}$.
9. $\dfrac{\sin ax}{\sin bx}$.
10. $\dfrac{\tan x-\sin x}{\sin^3 x}$.
11. $\dfrac{\tan 2x-2\sin x}{x^3}$.
12. $\dfrac{\text{versin } ax}{\text{versin } bx}$.
13. $\dfrac{m\sin x-\sin mx}{m(\cos x-\cos mx)}$.
14. $\dfrac{a^2\sin ax-b^2\sin bx}{b^2\tan ax-a^2\tan bx}$.
15. $\dfrac{b^2\sin^2 ax-a^2\sin^2 bx}{b^2\tan^2 ax-a^2\tan^2 bx}$.
16. $\dfrac{x\log_e(1+x)}{1-\cos x}$.
17. $\dfrac{e^x-1+\log_e(1-x)}{\sin^3 x}$.
18. $\dfrac{x+2\sin x-\sin 3x}{x+\tan x-\tan 2x}$.
19. $\dfrac{\sin x+\sin 6x-7x}{x^5}$.
20. $\dfrac{\sin^2 nx-\sin^2 px}{1-\cos px}$.

21. $\frac{1}{\theta^4}\left[\frac{\sin\theta}{\theta}+\frac{e^{\theta}-e^{-\theta}}{2\theta}-2\right]$.

22. $\frac{\sin^2\sqrt{mn}\,\theta-\sin m\theta\sin n\theta}{(1-\cos m\theta)(1-\cos n\theta)}$.

23. $\frac{3\sin x-\sin 3x}{x-\sin x}$.

24. $\frac{\left(\sin x-2\sin\frac{x}{2}\right)^2+(1-\cos x)^3}{\sin x\sin 2x-8\cos x\sin^2\frac{x}{2}-\frac{4}{3}\sin^4 x}$.

25. $\frac{a^x-b^x}{x}$.

26. $\left(\frac{\tan x}{x}\right)^{\frac{3}{x^2}}$.

27. $\left(\cos\frac{x}{m}+\sin\frac{3x}{m}\right)^{\frac{m}{x}}$.

Find the value, when x equals $\frac{\pi}{2}$, of

28. $\frac{(\cos x+\sin 2x+\cos 3x)^2}{(\sin x+2\cos 2x-\sin 3x)^3}$.

29. $(\sin\theta)^{\tan\theta}$.

30. $\sec x-\tan x$.

Find the value, when n is infinite, of

31. $\left(\cos\frac{x}{n}\right)^n$.

32. $\left(\cos\frac{x}{n}\right)^{n^2}$.

33. $\left(\cos\frac{x}{n}\right)^{n^3}$.

34. If n be >1 and $\theta=\frac{\pi}{2}$ nearly, prove that $(\sin\theta)^{\frac{1}{n}}$ is very nearly equal to

$$\frac{(n-1)+(n+1)\sin\theta}{(n+1)+(n-1)\sin\theta}.$$

35. In the limit, when $\beta=\alpha$, prove that

$$\frac{\alpha\sin\beta-\beta\sin\alpha}{\alpha\cos\beta-\beta\cos\alpha}=\tan(\alpha-\tan^{-1}\alpha).$$

36. Prove that

$$4\tan^{-1}\frac{1}{5}-\frac{\pi}{4}=\tan^{-1}\frac{1}{239}$$

and deduce that in a triangle ABC, in which C is a right angle and CA is five times CB, the angle A exceeds the eighth part of a right angle by $3'\,36''$, correct to the nearest second.

37. Find a and b so that the expression $a\sin x + b\sin 2x$ may be as close an approximation as possible to the number of radians in the angle x, when x is small.

38. If $y = x - e\sin x$, where e is very small, prove that

$$\tan\frac{y}{2} = \tan\frac{x}{2}\left(1 - e + e^2\sin^2\frac{x}{2}\right),$$

and that

$$\tan\frac{x}{2} = \tan\frac{y}{2}\left(1 + e + e^2\cos^2\frac{y}{2}\right),$$

where powers of e above the second are neglected.

39. If in the equation $\sin(\omega - \theta) = \sin\omega\cos\alpha$, θ be very small, prove that its approximate value is

$$2\tan\omega\sin\frac{\alpha}{2}\left(1 - \tan^2\frac{\omega}{2}\sin^2\frac{\alpha}{2}\right).$$

40. If ϕ be known by means of $\sin\phi$ to be an angle not $> 15'$, prove that its value differs from the fraction

$$\frac{28\sin 2\phi + \sin 4\phi}{12(3 + 2\cos 2\phi)}$$

by less than the number of radians in $1'$.

CHAPTER XXIV.

EXPANSIONS OF SINES AND COSINES OF MULTIPLE ANGLES, AND OF POWERS OF SINES AND COSINES.

[On a first reading of the subject the student is recommended to omit from the beginning of Art. 293 to the end of the chapter.]

287. In this chapter we shall shew how to expand powers of cosines and sines of an angle in terms of cosines and sines of multiples of that angle, and also how to express cosines and sines of multiple angles in terms of powers of cosines and sines.

288. Let $x \equiv \cos\theta + i\sin\theta$, so that

$$\frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} = \cos\theta - i\sin\theta.$$

Hence
$$x + \frac{1}{x} = 2\cos\theta,$$

and
$$x - \frac{1}{x} = 2i\sin\theta.$$

Also, by De Moivre's Theorem, we have

$$x^n = \cos n\theta + i\sin n\theta,$$

and $$\frac{1}{x^n} = \cos n\theta - i \sin n\theta,$$

so that $$x^n + \frac{1}{x^n} = 2 \cos n\theta,$$

and $$x^n - \frac{1}{x^n} = 2i \sin n\theta.$$

289. *To expand $\cos^n \theta$ in a series of cosines of multiples of θ, n being a positive integer.*

From the previous article we have

$$(2 \cos \theta)^n = \left(x + \frac{1}{x}\right)^n$$

$$= x^n + nx^{n-1} \cdot \frac{1}{x} + \frac{n(n-1)}{1 \cdot 2} x^{n-2} \cdot \frac{1}{x^2} + \dots$$

$$+ \frac{n(n-1)}{1 \cdot 2} x^2 \cdot \frac{1}{x^{n-2}} + nx \cdot \frac{1}{x^{n-1}} + \frac{1}{x^n}$$

$$= x^n + nx^{n-2} + \frac{n(n-1)}{1 \cdot 2} x^{n-4} + \dots$$

$$+ \frac{n(n-1)}{1 \cdot 2} \frac{1}{x^{n-4}} + n \cdot \frac{1}{x^{n-2}} + \frac{1}{x^n} \dots\dots\dots \quad (1).$$

Taking together the first and last of these terms, the second and next to last, and so on, we have

$$(2 \cos \theta)^n = \left(x^n + \frac{1}{x^n}\right) + n\left(x^{n-2} + \frac{1}{x^{n-2}}\right)$$

$$+ \frac{n(n-1)}{1 \cdot 2} \left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \dots.$$

But by the last article we have

$$x^n + \frac{1}{x^n} = 2 \cos n\theta, \quad x^{n-2} + \frac{1}{x^{n-2}} = 2 \cos (n-2)\, \theta, \dots.$$

Hence

$$2^n \cos^n \theta = 2 \cos n\theta + n \,.\, 2 \cos (n-2)\,\theta + \frac{n(n-1)}{1\,.\,2} \,.\, 2 \cos (n-4)\,\theta + \ldots\ldots,$$

i.e.

$$2^{n-1} \cos^n \theta = \cos n\theta + n \cos (n-2)\,\theta + \frac{n(n-1)}{1\,.\,2} \cos (n-4)\,\theta + \ldots \quad \ldots (2).$$

If n be odd, there are an even number of terms on the right-hand side of (1), so that the terms take together in pairs and the last term contains $\cos \theta$.

If n be even, there are an odd number of terms on the right-hand side of (1), so that after all the possible pairs have been taken there is a term left not containing x. This term will, when divided by 2, form the last term on the right-hand of (2).

290. Ex. 1. *Expand $\cos^8 \theta$ in a series of cosines of multiples of θ.*

We have
$$(2 \cos \theta)^8 = \left(x + \frac{1}{x}\right)^8$$
$$= x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + 56 \,.\, \frac{1}{x^2} + 28 \,.\, \frac{1}{x^4} + 8 \,.\, \frac{1}{x^6} + \frac{1}{x^8}$$
$$= \left(x^8 + \frac{1}{x^8}\right) + 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) + 56\left(x^2 + \frac{1}{x^2}\right) + 70$$
$$= 2 \,.\, \cos 8\theta + 8 \,.\, 2 \cos 6\theta + 28 \,.\, 2 \cos 4\theta + 56 \,.\, 2 \cos 2\theta + 70,$$
$$\therefore \quad 2^7 \cos^8 \theta = \cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 35.$$

Ex. 2. *Expand $\cos^7 \theta$ in a series of cosines of multiples of θ.*

We have
$$(2 \cos \theta)^7 = \left(x + \frac{1}{x}\right)^7$$
$$= x^7 + 7 \,.\, x^5 + 21x^3 + 35x + 35 \,.\, \frac{1}{x} + 21 \,.\, \frac{1}{x^3} + 7 \,.\, \frac{1}{x^5} + \frac{1}{x^7}$$
$$= \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + \frac{1}{x}\right)$$
$$= 2 \,.\, \cos 7\theta + 7 \,.\, 2 \cos 5\theta + 21 \,.\, 2 \cos 3\theta + 35 \,.\, 2 \cos \theta,$$
$$\therefore \quad 2^6 \cos^7 \theta = \cos 7\theta + 7 \cos 5\theta + 21 \cos 3\theta + 35 \cos \theta.$$

291. *To express $\sin^n\theta$ in a series of cosines or sines of multiples of θ according as n is an even or odd integer.*

By Art. 288 we have

$$2i\sin\theta = x - \frac{1}{x},$$

so that

$$2^n i^n \sin^n\theta = \left(x - \frac{1}{x}\right)^n \text{ .... (1).}$$

Case I. Let n be **even,** so that the last term in the expansion is

$$+\frac{1}{x^n}, \quad \text{and} \quad i^n = (-1)^{\frac{n}{2}}.$$

The equation (1) is therefore

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta = x^n - nx^{n-1}\cdot\frac{1}{x} + \frac{n(n-1)}{1\,.\,2}x^{n-2}\cdot\frac{1}{x^2} - \dots\dots$$

$$+\frac{n(n-1)}{1\,.\,2}x^2\cdot\frac{1}{x^{n-2}} - nx\cdot\frac{1}{x^{n-1}} + \frac{1}{x^n} \text{ ...... (2)}$$

$$= \left(x^n + \frac{1}{x^n}\right) - n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \frac{n(n-1)}{1\,.\,2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) - \dots\dots$$

$$= 2\,.\cos n\theta - n\,.\,2\cos(n-2)\theta + \frac{n(n-1)}{1\,.\,2}\,.\,2\cos(n-4)\theta - \dots\dots,$$

as in Art. 289.

$$\therefore\; 2^{n-1}(-1)^{\frac{n}{2}}\sin^n\theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{1\,.\,2}\cos(n-4)\theta - \dots \quad \dots(3).$$

Since n is even, there are an odd number of terms in (2), so that there will be a middle term which does not contain x. This term, on being divided by 2, will be the last term in equation (3).

Case II. Let n be **odd,** so that the last term in the expansion (1) will be

$$-\frac{1}{x^n}, \text{ and } i^n = i \,.\, i^{n-1} = i\,(-1)^{\frac{n-1}{2}}.$$

The equation (1) then becomes

$$2^n \,.\, i \,.\, (-1)^{\frac{n-1}{2}} \,.\, \sin^n\theta = x^n - nx^{n-1} \,.\, \frac{1}{x} + \frac{n(n-1)}{1\,.\,2} x^{n-2} \,.\, \frac{1}{x^2}$$

$$\ldots\ldots - \frac{n(n-1)}{1\,.\,2} x^2 \,.\, \frac{1}{x^{n-2}} + nx \,.\, \frac{1}{x^{n-1}} - \frac{1}{x^n}$$

$$= \left(x^n - \frac{1}{x^n}\right) - n\left(x^{n-2} - \frac{1}{x^{n-2}}\right) + \frac{n(n-1)}{1\,.\,2}\left(x^{n-4} - \frac{1}{x^{n-4}}\right) \ldots$$

$$\ldots\ldots\ldots\ldots\ldots(4).$$

Now, by Art. 288,

$$x^n - \frac{1}{x^n} = 2i \sin n\theta,$$

$$x^{n-2} - \frac{1}{x^{n-2}} = 2i \sin(n-2)\,\theta,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence (4) becomes

$$2^n \,.\, i \,.\, (-1)^{\frac{n-1}{2}} \sin^n\theta = 2i\sin n\theta - n \,.\, 2i \sin(n-2)\,\theta$$

$$+ \frac{n(n-1)}{1\,.\,2} \,.\, 2i \sin(n-4)\,\theta - \ldots,$$

so that $$2^{n-1}(-1)^{\frac{n-1}{2}} \sin^n\theta$$

$$= \sin n\theta - n\sin(n-2)\,\theta + \frac{n(n-1)}{1\,.\,2}\sin(n-4)\,\theta - \ldots\ldots$$

$$\ldots\ldots\ldots(5).$$

Since n is in this case odd, there are an even number of terms in (4), so that (4) can be divided into pairs of terms, and there is no middle term. The last term in (5) therefore contains $\sin\theta$.

292. **Ex. 1.** *Expand $\sin^6\theta$ in a series of cosines of multiples of θ.*

We have $$2^6 i^6 \sin^6\theta = \left(x - \frac{1}{x}\right)^6$$

$$= x^6 - 6x^4 + 15x^2 - 20 + 15 \cdot \frac{1}{x^2} - 6 \cdot \frac{1}{x^4} + \frac{1}{x^6},$$

so that $$-2^6 \sin^6\theta = \left(x^6 + \frac{1}{x^6}\right) - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) - 20$$

$$= 2\cos 6\theta - 6 \cdot 2\cos 4\theta + 15 \cdot 2\cos 2\theta - 20.$$

$$\therefore \quad -2^5 \sin^5\theta = \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10.$$

Ex. 2. *Expand $\sin^7\theta$ in a series of sines of multiples of θ.*

We have $$2^7 i^7 \sin^7\theta = \left(x - \frac{1}{x}\right)^7$$

$$= x^7 - 7x^5 + 21x^3 - 35x + 35 \cdot \frac{1}{x} - 21 \cdot \frac{1}{x^3} + 7 \cdot \frac{1}{x^5} - \frac{1}{x^7}$$

$$= \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right).$$

$$\therefore \quad -2^7 \cdot i \cdot \sin^7\theta = 2i\sin 7\theta - 7 \cdot 2i\sin 5\theta + 21 \cdot 2i\sin 3\theta - 35 \cdot 2i\sin\theta.$$

$$\therefore \quad -2^6 \sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta.$$

Ex. 3. *Expand $\cos^5\theta\sin^7\theta$ in a series of sines of multiples of θ.*

We have
$$2^5\cos^5\theta = \left(x + \frac{1}{x}\right)^5, \text{ and } 2^7 i^7 \sin^7\theta = \left(x - \frac{1}{x}\right)^7.$$

Hence $$2^{12} \cdot i^7 \cdot \cos^5\theta\sin^7\theta = \left(x^2 - \frac{1}{x^2}\right)^5 \left(x - \frac{1}{x}\right)^2$$

$$= \left[x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right]\left[x^2 - 2 + \frac{1}{x^2}\right]$$

$$= \left(x^{12} - \frac{1}{x^{12}}\right) - 2\left(x^{10} - \frac{1}{x^{10}}\right) - 4\left(x^8 - \frac{1}{x^8}\right) + 10\left(x^6 - \frac{1}{x^6}\right)$$
$$+ 5\left(x^4 - \frac{1}{x^4}\right) - 20\left(x^2 - \frac{1}{x^2}\right).$$

Hence, as before, we have

$$-2^{11}\cos^5\theta\sin^7\theta = \sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta + 5\sin 4\theta$$
$$- 20\sin 2\theta.$$

EXAMPLES. LI.

Prove that

1. $\sin^5\theta = \frac{1}{16}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$.

2. $\cos^9\theta = \frac{1}{256}[\cos 9\theta + 9\cos 7\theta + 36\cos 5\theta + 84\cos 3\theta + 126\cos\theta]$.

3. $\cos^{10}\theta =$
$\frac{1}{512}[\cos 10\theta + 10\cos 8\theta + 45\cos 6\theta + 120\cos 4\theta + 210\cos 2\theta + 126]$.

4. $\sin^8\theta = \frac{1}{128}[\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35]$.

5. $\sin^9\theta = \frac{1}{256}[\sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta]$.

****293.** *To express* $\frac{\sin n\theta}{\sin\theta}$ *in a series of descending powers of* $\cos\theta$.

If x be < 1, we have

$$\frac{\sin\theta}{1 - 2x\cos\theta + x^2} = \sin\theta + x\sin 2\theta + x^2\sin 3\theta + \ldots$$
$$+ x^{n-1}\sin n\theta + \ldots \text{ ad inf. } \ldots\ldots\ldots(1).$$

This may be shewn by multiplying each side by

$$1 - 2x\cos\theta + x^2,$$

when it will be found that the right-hand member will reduce to $\sin\theta$.

Another proof will be found in Art. 358.

Equating coefficients of x^{n-1} in (1), we have

$$\frac{\sin n\theta}{\sin\theta} = \text{coefficient of } x^{n-1} \text{ in } [1 - 2x\cos\theta + x^2]^{-1}$$
$$= \text{coefficient of } x^{n-1} \text{ in } [1 - x(2\cos\theta - x)]^{-1}$$
$$= \text{coefficient of } x^{n-1} \text{ in}$$
$$1 + x(2\cos\theta - x) + x^2(2\cos\theta - x)^2 + \ldots\ldots$$
$$+ x^{n-3}(2\cos\theta - x)^{n-3} + x^{n-2}(2\cos\theta - x)^{n-2}$$
$$+ x^{n-1}(2\cos\theta - x)^{n-1} + x^n(2\cos\theta - x)^n + \ldots\ldots (2).$$

Now coefficient of

$$x^{n-1} \text{ in } x^{n-1}(2\cos\theta - x)^{n-1} = (2\cos\theta)^{n-1},$$

coefficient of x^{n-1} in $\quad x^{n-2}(2\cos\theta - x)^{n-2}$

$$= \text{coefficient of } x \text{ in } (2\cos\theta - x)^{n-2}$$

$$= -(n-2)(2\cos\theta)^{n-3},$$

coefficient of x^{n-1} in $\quad x^{n-3}(2\cos\theta - x)^{n-3}$

$$= \text{coefficient of } x^2 \text{ in } (2\cos\theta - x)^{n-3}$$

$$= \frac{(n-3)(n-4)}{1.2}(2\cos\theta)^{n-5},$$

and so on.

Hence, from (2) picking out in this manner all the coefficients of x^{n-1}, we have

$$\frac{\sin n\theta}{\sin\theta} = (2\cos\theta)^{n-1} - (n-2)(2\cos\theta)^{n-3}$$
$$+ \frac{(n-3)(n-4)}{1.2}(2\cos\theta)^{n-5}$$
$$- \frac{(n-4)(n-5)(n-6)}{1.2.3}(2\cos\theta)^{n-7} + \ldots\ldots$$

If n be odd, the last term could be proved to be $(-1)^{\frac{n-1}{2}}$; if n be even, it could be shewn to be $(-1)^{\frac{n}{2}-1}(n\cos\theta)$.

**294. *To express $\cos n\theta$ in a series of descending powers of $\cos\theta$.*

If x be < 1, we have

$$\frac{1-x^2}{1-2x\cos\theta+x^2} = 1 + 2x\cos\theta + 2x^2\cos 2\theta + 2x^3\cos 3\theta + \ldots$$
$$\ldots + 2x^n\cos n\theta + \ldots \text{ ad inf. } \ldots\ldots(1).$$

This may be shewn by multiplying both sides by

$$1 - 2x\cos\theta + x^2,$$

when it will be found that all the terms on the right-hand side will reduce to $1-x^2$.

Another proof will be found in Art. 358.

Equating coefficients of x^n on the two sides of (1), we have

$$2\cos n\theta = \text{coefficient of } x^n \text{ in } (1-x^2)\,[1-2x\cos\theta+x^2]^{-1}$$

$$= \text{coefficient of } x^n - \text{coefficient of } x^{n-2} \text{ in } [1-x(2\cos\theta-x)]^{-1}$$

$$= \text{coefficient of } x^n - \text{coefficient of } x^{n-2} \text{ in } 1+x(2\cos\theta-x)+x^2(2\cos\theta-x)^2+\dots$$

$$\dots+x^{n-2}(2\cos\theta-x)^{n-2}+x^{n-1}(2\cos\theta-x)^{n-1}+x^n(2\cos\theta-x)^n+x^{n+1}(2\cos\theta-x)^{n+1}+\dots.$$

Picking out the required coefficients as in the last article, starting with the term

$$x^n(2\cos\theta-x)^n,$$

we have $2\cos n\theta$

$$=(2\cos\theta)^n-(n-1)(2\cos\theta)^{n-2}+\frac{(n-2)(n-3)}{1.2}(2\cos\theta)^{n-4}$$

$$-\frac{(n-3)(n-4)(n-5)}{1.2.3}(2\cos\theta)^{n-6}+\dots\dots$$

$$-\left[(2\cos\theta)^{n-2}-(n-3)(2\cos\theta)^{n-4}+\frac{(n-4)(n-5)}{1.2}(2\cos\theta)^{n-6}-\dots\dots\right]$$

$$=(2\cos\theta)^n-n(2\cos\theta)^{n-2}+\left[\frac{(n-2)(n-3)}{1.2}+(n-3)\right](2\cos\theta)^{n-4}$$

$$-\left[\frac{(n-3)(n-4)(n-5)}{1.2.3}+\frac{(n-4)(n-5)}{1.2}\right](2\cos\theta)^{n-6}+\dots,$$

so that, finally,

$$2\cos n\theta = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{1.2}(2\cos\theta)^{n-4} - \frac{n(n-4)(n-5)}{1.2.3}(2\cos\theta)^{n-6} + \dots \quad \dots\dots(2).$$

The last term could be shewn to be

$$(-1)^{\frac{n-1}{2}}.n.(2\cos\theta) \text{ or } (-1)^{\frac{n}{2}}.2,$$

according as n is odd or even.

**295. To expand $\frac{\sin n\theta}{\sin\theta}$ in a series of ascending powers of $\cos\theta$.

As in Art. 293, we have

$$\frac{\sin n\theta}{\sin\theta} = \text{coefficient of } x^{n-1} \text{ in } [1 - 2x\cos\theta + x^2]^{-1}$$
$$= \text{coefficient of } x^{n-1} \text{ in } [1 + x(x - 2\cos\theta)]^{-1}$$
$$= \text{coefficient of } x^{n-1} \text{ in}$$
$$1 - x(x - 2\cos\theta) + x^2(x - 2\cos\theta)^2 - \dots\dots$$
$$\dots\dots + (-1)^r x^r (x - 2\cos\theta)^r + \dots\dots(1).$$

Case I. Let n be **odd,** so that $(n-1)$ is even.

The lowest term in (1) which gives any coefficient of x^{n-1} is then that for which

$$r = \frac{n-1}{2}.$$

Hence, in this case,

$$\frac{\sin n\theta}{\sin\theta} = \text{coefficient of } x^{n-1} \text{ in } 1 - x(x - 2\cos\theta) + \dots$$
$$+ (-1)^{\frac{n-1}{2}} x^{\frac{n-1}{2}} (x - 2\cos\theta)^{\frac{n-1}{2}} + (-1)^{\frac{n+1}{2}} x^{\frac{n+1}{2}} (x - 2\cos\theta)^{\frac{n+1}{2}}$$
$$+ (-1)^{\frac{n+3}{2}} x^{\frac{n+3}{2}} (x - 2\cos\theta)^{\frac{n+3}{2}} + \dots\dots$$
$$+ (-1)^{n-1} x^{n-1} (x - 2\cos\theta)^{n-1} + \dots\dots.$$

Picking out the required coefficients as in Art. 293, we have

$$\frac{\sin n\theta}{\sin\theta} = (-1)^{\frac{n-1}{2}} + (-1)^{\frac{n+1}{2}}\left[\frac{\frac{n+1}{2}\cdot\frac{n-1}{2}}{1.2}\right](-2\cos\theta)^2$$

$$+(-1)^{\frac{n+3}{2}}\cdot\frac{\frac{n+3}{2}\cdot\frac{n+1}{2}\cdot\frac{n-1}{2}\cdot\frac{n-3}{2}}{1.2.3.4}(-2\cos\theta)^4+\dots$$

$$+(2\cos\theta)^{n-1}.$$

Hence, finally, when n is **odd,** we have

$$(-1)^{\frac{n-1}{2}}\cdot\frac{\sin n\theta}{\sin\theta} = 1 - \frac{n^2-1^2}{1.2}\cos^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{\underline{|4}}\cos^4\theta$$

$$-\frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{\underline{|6}}\cos^6\theta - \dots\dots$$

$$+(-1)^{\frac{n-1}{2}}(2\cos\theta)^{n-1}\dots\dots\dots(2).$$

Case II. Let n be **even,** so that $n-1$ is odd.

The lowest term in (1) which gives any coefficient of x^{n-1} is then that for which

$$r=\frac{n}{2}.$$

Hence, in this case,

$$\frac{\sin n\theta}{\sin\theta} = \text{coefficient of } x^{n-1} \text{ in } 1 - x(x-2\cos\theta)+\dots$$

$$+(-1)^{\frac{n}{2}}x^{\frac{n}{2}}(x-2\cos\theta)^{\frac{n}{2}}+(-1)^{\frac{n}{2}+1}x^{\frac{n}{2}+1}(x-2\cos\theta)^{\frac{n}{2}+1}$$

$$+(-1)^{\frac{n}{2}+2}x^{\frac{n}{2}+2}(x-2\cos\theta)^{\frac{n}{2}+2}+\dots$$

$$+(-1)^{n-1}x^{n-1}(x-2\cos\theta)^{n-1}+\dots\dots$$

Picking out the required coefficients, we have

$$\frac{\sin n\theta}{\sin\theta} = (-1)^{\frac{n}{2}} \cdot \frac{n}{2}(-2\cos\theta)$$

$$+(-1)^{\frac{n}{2}+1} \cdot \frac{\left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)}{1 \cdot 2 \cdot 3}(-2\cos\theta)^3$$

$$+(-1)^{\frac{n}{2}+2} \cdot \frac{\left(\frac{n}{2}+2\right)\left(\frac{n}{2}+1\right)\frac{n}{2}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(-2\cos\theta)^5$$

$$+ \dots\dots\dots\dots + (2\cos\theta)^{n-1}.$$

Hence, finally, when n is **even,** we have

$$(-1)^{\frac{n}{2}+1}\frac{\sin n\theta}{\sin\theta}$$

$$= n\cos\theta - \frac{n(n^2-2^2)}{\underline{|3}}\cos^3\theta + \frac{n(n^2-2^2)(n^2-4^2)}{\underline{|5}}\cos^5\theta$$

$$- \dots\dots + (-1)^{\frac{n}{2}+1}(2\cos\theta)^{n-1} \dots\dots\dots\dots (3).$$

N.B. It will be noted that equations (2) and (3) of this article are simply the series of Art. 293 written backwards. This is clear from the method of proof, or the statement could be easily verified independently.

296. *To expand $\cos n\theta$ in a series of ascending powers of $\cos\theta$.*

As in Art. 294, we have

$$2\cos n\theta = \text{coefficient of } x^n - \text{coefficient of } x^{n-2} \text{ in } (1 - 2x\cos\theta + x^2)^{-1}$$

$$= \text{coefficient of } x^n - \text{coefficient of } x^{n-2} \text{ in}$$

$$1 - x(x - 2\cos\theta) + x^2(x-2\cos\theta)^2 - \dots\dots + (-1)^r x^r (x-2\cos\theta)^r + \dots \quad \dots(1),$$

as in Art. 295.

Case I. Let n be **odd,** so that $n-1$ is even.

The lowest term in (1) which will give any of the coefficients we want is that for which

$$r=\frac{n-1}{2}.$$

Hence $2\cos n\theta =$ coefficient of $x^n -$ coefficient of x^{n-2} in

$$1-x(x-2\cos\theta)+\ldots+(-1)^{\frac{n-1}{2}}x^{\frac{n-1}{2}}(x-2\cos\theta)^{\frac{n-1}{2}}$$
$$+(-1)^{\frac{n+1}{2}}x^{\frac{n+1}{2}}(x-2\cos\theta)^{\frac{n+1}{2}}+(-1)^{\frac{n+3}{2}}x^{\frac{n+3}{2}}(x-2\cos\theta)^{\frac{n+3}{2}}$$
$$+\ldots\ldots\ldots+(-1)^n x^n(x-2\cos\theta)^n\ldots\ldots$$
$$=(-1)^{\frac{n-1}{2}}\left[-\frac{n-1}{2}(-2\cos\theta)\right]$$
$$+(-1)^{\frac{n+1}{2}}\left[\frac{n+1}{2}(-2\cos\theta)-\frac{\frac{n+1}{2}\cdot\frac{n-1}{2}\cdot\frac{n-3}{2}}{1.2.3}(-2\cos\theta)^3\right]$$
$$+(-1)^{\frac{n+3}{2}}\left[\frac{\frac{n+3}{2}\cdot\frac{n+1}{2}\cdot\frac{n-1}{2}}{1.2.3}(-2\cos\theta)^3\right.$$
$$\left.-\frac{\frac{n+3}{2}\frac{n+1}{2}\frac{n-1}{2}\frac{n-3}{2}\frac{n-5}{2}}{1.2.3.4.5}(-2\cos\theta)^5\right]$$
$$+\ldots\ldots\ldots\ldots\ldots+(2\cos\theta)^n.$$

$$\therefore\ (-1)^{\frac{n-1}{2}}.2\cos n\theta$$
$$=\cos\theta[(n-1)+(n+1)]-\frac{(n+1)(n-1)}{\underline{|3}}\cos^3\theta[(n-3)+(n+3)]$$
$$+\frac{(n+3)(n+1)(n-1)(n-3)}{\underline{|5}}\cos^5\theta[(n-5)+(n+5)]+\ldots.$$
$$+(-1)^{\frac{n-1}{2}}(2\cos\theta)^n.$$

Hence, finally, when n is **odd,**

$$(-1)^{\frac{n-1}{2}} \cos n\theta$$

$$= n\cos\theta - \frac{n(n^2-1^2)}{\lfloor 3}\cos^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{\lfloor 5}\cos^5\theta$$

$$- \ldots\ldots (-1)^{\frac{n-1}{2}} . 2^{n-1}\cos^n\theta \ldots\ldots (2).$$

Case II. Let n be **even.**

The lowest term in (1) which will give any of the required coefficients is that for which

$$r = \frac{n-2}{2}.$$

Hence we have

$2\cos n\theta =$ coefficient of $x^n -$ coefficient of x^{n-2} in

$$1 - x(x-2\cos\theta) + \ldots + (-1)^{\frac{n-2}{2}} x^{\frac{n-2}{2}} (x-2\cos\theta)^{\frac{n-2}{2}}$$

$$+ (-1)^{\frac{n}{2}} x^{\frac{n}{2}} (x-2\cos\theta)^{\frac{n}{2}} + (-1)^{\frac{n+2}{2}} x^{\frac{n+2}{2}} (x-2\cos\theta)^{\frac{n+2}{2}}$$

$$+ \ldots\ldots\ldots\ldots\ldots + (-1)^n x^n (x-2\cos\theta)^n + \ldots\ldots$$

$$= (-1)^{\frac{n-2}{2}}[-1] + (-1)^{\frac{n}{2}}\left[1 - \frac{\frac{n}{2}\cdot\frac{n-2}{2}}{1.2}(-2\cos\theta)^2\right]$$

$$+ (-1)^{\frac{n+2}{2}}\left[\frac{\frac{n+2}{2}\cdot\frac{n}{2}}{1.2}(-2\cos\theta)^2\right.$$

$$\left. - \frac{\frac{n+2}{2}\cdot\frac{n}{2}\cdot\frac{n-2}{2}\cdot\frac{n-4}{2}}{1.2.3.4}(-2\cos\theta)^4\right]$$

$$+(-1)^{\frac{n+4}{2}}\left[\frac{\frac{n+4}{2}\cdot\frac{n+2}{2}\cdot\frac{n}{2}\cdot\frac{n-2}{2}}{1.2.3.4}(-2\cos\theta)^4\right.$$

$$\left.-\frac{\frac{n+4}{2}\cdot\frac{n+2}{2}\cdot\frac{n}{2}\cdot\frac{n-2}{2}\cdot\frac{n-4}{2}\cdot\frac{n-6}{2}}{\underline{|6}}(-2\cos\theta)^6\right]$$

$$+\ldots\ldots\ldots\ldots\ldots+(2\cos\theta)^n.$$

$$\therefore\quad (-1)^{\frac{n}{2}}.2\cos n\theta$$

$$=[1+1]-\frac{\cos^2\theta}{\underline{|2}}[n(n-2)+(n+2).n]$$

$$+\frac{\cos^4\theta}{\underline{|4}}[(n+2).n.(n-2)(n-4)+(n+4)(n+2).n.(n-2)]$$

$$+\ldots\ldots\ldots\ldots\ldots+(-1)^{\frac{n}{2}}.(2\cos\theta)^n.$$

Hence, finally, when n is **even,**

$$(-1)^{\frac{n}{2}}\cos n\theta=1-\frac{n^2\cos^2\theta}{\underline{|2}}+\frac{n^2(n^2-2^2)}{\underline{|4}}\cos^4\theta$$

$$-\frac{n^2(n^2-2^2)(n^2-4^2)}{\underline{|6}}\cos^6\theta+\ldots$$

$$+(-1)^{\frac{n}{2}}2^{n-1}\cos^n\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

N.B. As before, the equations (2) and (3) of this article are only the series (2) of Art. 294 written backwards.

****297.** From equation (2) of Art. 295 and equation (2) of Art. 296 we have, if n be **odd,**

$$(-1)^{\frac{n-1}{2}}\frac{\sin n\theta}{\sin\theta}=1-\frac{n^2-1^2}{\underline{|2}}\cos^2\theta+\frac{(n^2-1^2)(n^2-3^2)}{\underline{|4}}\cos^4\theta$$

$$-\frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{\underline{|6}}\cos^6\theta+\ldots\ldots$$

$$+(-1)^{\frac{n-1}{2}}(2\cos\theta)^{n-1}+\ldots\ldots(1),$$

and $(-1)^{\frac{n-1}{2}} \cos n\theta = n \cos\theta - \frac{n(n^2-1^2)}{\underline{|3}} \cos^3\theta$

$$+ \frac{n(n^2-1^2)(n^2-3^2)}{\underline{|5}} \cos^5\theta + \ldots\ldots + (-1)^{\frac{n-1}{2}} 2^{n-1} \cos^n\theta \quad \ldots\ldots(2).$$

In these equations change θ into $\frac{\pi}{2} - \theta$, and therefore $\cos\theta$ into $\sin\theta$.

Then $\sin n\theta$ will become

$$\sin\left(\frac{n\pi}{2} - n\theta\right), \text{ i.e. } (-1)^{\frac{n-1}{2}} \cos n\theta,$$

and $\cos n\theta$ will become

$$\cos\left(\frac{n\pi}{2} - n\theta\right), \text{ i.e. } (-1)^{\frac{n-1}{2}} \sin n\theta.$$

On making these substitutions we shall have, if n be **odd,**

$$\cos n\theta = \cos\theta \left\{1 - \frac{n^2-1^2}{\underline{|2}} \sin^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{\underline{|4}} \sin^4\theta - \ldots + (-1)^{\frac{n-1}{2}} . 2^{n-1} \sin^{n-1}\theta\right\} \quad \ldots\ldots\ldots(3),$$

and

$$\sin n\theta = n \sin\theta - \frac{n(n^2-1^2)}{\underline{|3}} \sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{\underline{|5}} \sin^5\theta + \ldots\ldots + (-1)^{\frac{n-1}{2}} 2^{n-1} \sin^n\theta \ldots\ldots(4).$$

****298.** Again from equation (3) of Art. 295 and equation (3) of Art. 296 we have, if n be **even,**

$$(-1)^{\frac{n}{2}+1} \frac{\sin n\theta}{\sin\theta} = n\cos\theta - \frac{n(n^2-2^2)}{\underline{|3}} \cos^3\theta + \frac{n(n^2-2^2)(n^2-4^2)}{\underline{|5}} \cos^5\theta + \ldots\ldots + (-1)^{\frac{n}{2}+1} (2\cos\theta)^{n-1} \quad \ldots\ldots(1),$$

and

$$(-1)^{\frac{n}{2}} \cos n\theta = 1 - \frac{n^2}{\underline{|2}} \cos^2 \theta + \frac{n^2(n^2-2^2)}{\underline{|4}} \cos^4 \theta - \ldots\ldots$$

$$+ (-1)^{\frac{n}{2}}\, 2^{n-1} (\cos^n \theta) \ldots\ldots (2).$$

In these equations change θ into $\frac{\pi}{2} - \theta$, and therefore $\cos \theta$ into $\sin \theta$.

Then $\sin n\theta$ will become

$$\sin\left(\frac{n\pi}{2} - n\theta\right), \quad i.e. \quad (-1)^{\frac{n}{2}+1} \sin n\theta,$$

and $\cos n\theta$ will become

$$\cos\left(\frac{n\pi}{2} - n\theta\right), \quad i.e. \quad (-1)^{\frac{n}{2}} \cos n\theta.$$

On making these substitutions we have, if n be **even,**

$$\frac{\sin n\theta}{\cos\theta} = n \sin \theta - \frac{n(n^2-2^2)}{\underline{|3}} \sin^3\theta + \frac{n(n^2-2^2)(n^2-4^2)}{\underline{|5}} \sin^5\theta \ldots$$

$$+ (-1)^{\frac{n}{2}+1} (2 \sin \theta)^{n-1} \ldots\ldots (3),$$

and

$$\cos n\theta = 1 - \frac{n^2}{\underline{|2}} \sin^2\theta + \frac{n^2(n^2-2^2)}{\underline{|4}} \sin^4 \theta$$

$$+ \ldots\ldots (-1)^{\frac{n}{2}}\, 2^{n-1} \sin^n \theta \ldots\ldots (4).$$

****299.** Equations (1) and (2) of Art. 297 and equations (1) and (2) of Art. 298 give the expansions of $\sin n\theta$ and $\cos n\theta$ in ascending powers of $\cos \theta$ for the cases when n is even or odd. Equations (3) and (4) of the same two articles give the expansions of the same two quantities in terms of $\sin \theta$.

EXAMPLES. LII.

1. $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$.

2. $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$.

3. $\sin 8\theta = \sin\theta\,[128\cos^7\theta - 192\cos^5\theta + 80\cos^3\theta - 8\cos\theta]$.

4. $\cos 8\theta = 1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 128\sin^8\theta$.

5. $\sin 9\theta = \sin\theta\,[256\cos^8\theta - 448\cos^6\theta + 240\cos^4\theta - 40\cos^2\theta + 1]$.

6. Express $\cos 6\theta$ in terms of $\cos\theta$ only and verify for the cases

$$\theta = \frac{\pi}{3}, \quad \theta = \frac{\pi}{2}$$

respectively.

7. Prove the algebraic identity

$$p^n + q^n = (p+q)^n - n\,(p+q)^{n-2}pq + \frac{n\,(n-3)}{1\,.\,2}\,(p+q)^{n-4}p^2q^2 + \ldots.$$

Deduce that

$$2\cos n\theta = (2\cos\theta)^n - n\,(2\cos\theta)^{n-2} + \frac{n\,(n-3)}{1\,.\,2}\,(2\cos\theta)^{n-4} - \ldots.$$

** **300. Ex.** *Prove that the roots of the equation*

$$8x^3 - 4x^2 - 4x + 1 = 0$$

are $$\cos\frac{\pi}{7},\ \cos\frac{3\pi}{7}\ \text{and}\ \cos\frac{5\pi}{7},$$

and hence that $$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2},$$

$$\cos\frac{\pi}{7}\cos\frac{3\pi}{7} + \cos\frac{3\pi}{7}\cos\frac{5\pi}{7} + \cos\frac{5\pi}{7}\cos\frac{\pi}{7} = -\frac{1}{2},$$

and $$\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = -\frac{1}{8}.$$

On putting $n = 7$ in equation (2) of Art. 294, we have

$$2\cos 7\theta = (2\cos\theta)^7 - 7\,(2\cos\theta)^5 + \frac{7\,.\,4}{1\,.\,2}\,(2\cos\theta)^3 - \frac{7\,.\,3\,.\,2}{1\,.\,2\,.\,3}\,(2\cos\theta),$$

i.e., on reduction,

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta \ldots\ldots\ldots(1).$$

Now put $\cos 7\theta = -1$, so that

$$\theta = \frac{\pi}{7},\ \text{or}\ \frac{3\pi}{7},\ \text{or}\ \frac{5\pi}{7}\ldots\ldots$$

Equation (1) then becomes

$$64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta + 1 = 0 \quad\text{..........(2)},$$

and its roots are

$$\cos\frac{\pi}{7},\ \cos\frac{3\pi}{7},\ \cos\frac{5\pi}{7},\ \cos\frac{7\pi}{7},\ \cos\frac{9\pi}{7},\ \cos\frac{11\pi}{7}\ \text{and}\ \cos\frac{13\pi}{7}.$$

Now

$$\cos\frac{7\pi}{7} = -1,\quad \cos\frac{13\pi}{7} = \cos\frac{\pi}{7},\quad \cos\frac{11\pi}{7} = \cos\frac{3\pi}{7},\ \text{and}\ \cos\frac{9\pi}{7} = \cos\frac{5\pi}{7}.$$

The roots of (2) are therefore -1, and $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, and $\cos\frac{5\pi}{7}$, the latter three roots being twice repeated.

Writing c, for shortness, for $\cos\theta$, the equation (2) may be written

$$(c+1)(8c^3 - 4c^2 - 4c + 1)^2 = 0.$$

Hence $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, and $\cos\frac{5\pi}{7}$ are the roots of the equation

$$8c^3 - 4c^2 - 4c + 1 = 0 \quad\text{..........(3)}.$$

We therefore have

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{4}{8} = \frac{1}{2},$$

$$\cos\frac{\pi}{7}\cos\frac{3\pi}{7} + \cos\frac{3\pi}{7}\cos\frac{5\pi}{7} + \cos\frac{5\pi}{7}\cos\frac{\pi}{7} = \frac{-4}{8} = -\frac{1}{2},$$

and

$$\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = \frac{-1}{8} = -\frac{1}{8}.$$

In equation (3), putting $\frac{1}{c^2} = x$, and therefore $c = \frac{1}{\sqrt{x}}$, it follows that the quantities $\sec^2\frac{\pi}{7}$, $\sec^2\frac{3\pi}{7}$, and $\sec^2\frac{5\pi}{7}$ are the roots of the equation

$$8\cdot\frac{1}{x\sqrt{x}} - \frac{4}{x} - \frac{4}{\sqrt{x}} + 1 = 0,$$

or, on rationalizing,

$$x^3 - 24x^2 + 80x - 64 = 0 \quad\text{..........(4)}.$$

Again, putting $x = 1 + y$, then, since $\sec^2\theta = 1 + \tan^2\theta$, it follows that

$$\tan^2\frac{\pi}{7},\ \tan^2\frac{3\pi}{7},\ \text{and}\ \tan^2\frac{5\pi}{7}$$

are the roots of the equation

$$(1+y)^3 - 24(1+y)^2 + 80(1+y) - 64 = 0,$$

i.e. of

$$y^3 - 21y^2 + 35y - 7 = 0.$$

Aliter. Without assuming the series of Art. 294, which is difficult to remember, the equation (2) may be deduced directly from De Moivre's Theorem.

For the equation $(\cos\theta + i\sin\theta)^7 = -1$ ....(5),

i.e. $\cos 7\theta + i\sin 7\theta = -1$,

is clearly satisfied when θ has either of the values

$$\frac{\pi}{7},\ \frac{3\pi}{7},\ \frac{5\pi}{7},\ \frac{7\pi}{7},\ \frac{9\pi}{7},\ \frac{11\pi}{7} \text{ and } \frac{13\pi}{7} \quad\text{..............(6).}$$

Writing c for $\cos\theta$ and s for $\sin\theta$, the equation (5) on being expanded by the Binomial Theorem becomes

$$c^7 + 7ic^6 s - 21c^5 s^2 - 35ic^4 s^3 + 35c^3 s^4 + 21ic^2 s^5 - 7cs^6 - is^7 = -1.$$

Equating the real parts on each side, we have

$$c^7 - 21c^5 s^2 + 35c^3 s^4 - 7cs^6 + 1 = 0.$$

Putting $s^2 = 1 - c^2$, we see that the cosine of each of the angles (6) satisfies the equation

$$64c^7 - 112c^5 + 56c^3 - 7c + 1 = 0.$$

But this is equation (2).

**** 301. Ex.** *Find the value of*

$$\sec\theta + \sec\left(\theta + \frac{2\pi}{n}\right) + \sec\left(\theta + \frac{4\pi}{n}\right) + \ldots \text{ to } n \text{ terms,}$$

$$\sec^2\theta + \sec^2\left(\theta + \frac{2\pi}{n}\right) + \sec^2\left(\theta + \frac{4\pi}{n}\right) + \ldots \text{ to } n \text{ terms.}$$

From equations (2) and (3) of Art. 296, we know that

$$nc - \frac{n(n^2-1^2)}{\underline{|3}}c^3 + \frac{n(n^2-1^2)(n^2-3^2)}{\underline{|5}}c^5 + \ldots + (-1)^{\frac{n-1}{2}} 2^{n-1} c^n$$
$$= (-1)^{\frac{n-1}{2}} \cos n\theta \quad\text{............................(1),}$$

when n is **odd**,
and that

$$1 - \frac{n^2}{\underline{|2}}c^2 + \frac{n^2(n^2-2^2)}{\underline{|4}}c^4 + \ldots + (-1)^{\frac{n}{2}} 2^{n-1} c^n = (-1)^{\frac{n}{2}} \cos n\theta \quad\text{...... (2),}$$

when n is **even**,
where in each series c stands for $\cos\theta$.

If $\cos n\theta$ be now given, the equations (1) and (2) give $\cos\theta$.

But since $$\cos n\theta = \cos(n\theta+2\pi) = \cos(n\theta+4\pi) = \dots\dots\dots,$$

these equations would also give

$$\cos\left(\theta+\frac{2\pi}{n}\right),\ \cos\left(\theta+\frac{4\pi}{n}\right),\dots.$$

Hence, in each case, the roots are

$$\cos\theta,\ \cos\left(\theta+\frac{2\pi}{n}\right),\ \cos\left(\theta+\frac{4\pi}{n}\right),\dots\dots \text{ to } n \text{ terms.}$$

In (1) and (2) put $c=\frac{1}{y}$ and multiply by y^n.

We have then the equations

$$(-1)^{\frac{n-1}{2}}\cos n\theta\times y^n - n\,.\,y^{n-1}+\frac{n(n^2-1^2)}{\lfloor 3}y^{n-3}-\dots=0\dots\dots\dots(3),$$

when n is **odd**,

and $$\left[(-1)^{\frac{n}{2}}\cos n\theta-1\right]y^n+\frac{n^2}{\lfloor 2}y^{n-2}-\dots=0\dots\dots\dots\dots\dots\dots(4),$$

when n is **even**.

The roots of these equations are respectively

$$\sec\theta,\ \sec\left(\theta+\frac{2\pi}{n}\right),\ \sec\left(\theta+\frac{4\pi}{n}\right),\dots.$$

Call these $$y_1,\ y_2,\ \dots,\ y_n.$$

Then

$$y_1+y_2+\dots+y_n = \text{sum of the roots}$$

$$=\frac{n}{(-1)^{\frac{n-1}{2}}\cos n\theta}=(-1)^{\frac{n-1}{2}}n\sec n\theta, \text{ when } n \text{ is } \textbf{odd},$$

and $$=0, \text{ when } n \text{ is } \textbf{even}.$$

Also

$$y_1^2+y_2^2+\dots+y_n^2=(y_1+y_2+\dots+y_n)^2-2(y_1y_2+y_2y_3+\dots)$$

$$=\frac{n^2}{\cos^2 n\theta}=n^2\sec^2 n\theta, \text{ when } n \text{ is } \textbf{odd},$$

and $$=-2\,.\,\frac{\frac{n^2}{\lfloor 2}}{\overline{-1}|^{\frac{n}{2}}\cos n\theta-1}=\frac{n^2}{1-(-1)^{\frac{n}{2}}\cos n\theta}, \text{ when } n \text{ is } \textbf{even}.$$

EXAMPLES. LIII.

1. Prove that

$$\left(x-2\cos\frac{2\pi}{5}\right)\left(x-2\cos\frac{4\pi}{5}\right)\left(x-2\cos\frac{6\pi}{5}\right)\left(x-2\cos\frac{8\pi}{5}\right)$$
$$=x^4+2x^3-x^2-2x+1.$$

2. Prove that the roots of the equation

$$8x^3+4x^2-4x-1=0 \text{ are } \cos\frac{2\pi}{7},\ \cos\frac{4\pi}{7},\ \text{and } \cos\frac{6\pi}{7}.$$

3. Prove that $\sin\frac{2\pi}{7}$, $\sin\frac{4\pi}{7}$ and $\sin\frac{8\pi}{7}$ are the roots of the equation $x^3-\frac{\sqrt{7}}{2}x^2+\frac{\sqrt{7}}{8}=0.$

Prove that

4. $$\frac{1}{4-\sec^2\frac{2\pi}{7}}+\frac{1}{4-\sec^2\frac{4\pi}{7}}+\frac{1}{4-\sec^2\frac{6\pi}{7}}=1.$$

5. $$\cos^4\frac{\pi}{9}+\cos^4\frac{2\pi}{9}+\cos^4\frac{3\pi}{9}+\cos^4\frac{4\pi}{9}=\frac{19}{16}.$$

6. $$\sec^4\frac{\pi}{9}+\sec^4\frac{2\pi}{9}+\sec^4\frac{3\pi}{9}+\sec^4\frac{4\pi}{9}=1120.$$

7. $$\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}=\frac{1}{2}.$$

8. Form the equation whose roots are

$$\tan^2\frac{\pi}{11},\ \tan^2\frac{2\pi}{11},\ \tan^2\frac{3\pi}{11},\ \tan^2\frac{4\pi}{11} \text{ and } \tan^2\frac{5\pi}{11}.$$

[*Commence with equation* (3) *of Art.* 277.]

9. $$\cot^2\frac{\pi}{11}+\cot^2\frac{2\pi}{11}+\cot^2\frac{3\pi}{11}+\cot^2\frac{4\pi}{11}+\cot^2\frac{5\pi}{11}=15.$$

10. $$\sec^2\frac{\pi}{11}+\sec^2\frac{2\pi}{11}+\sec^2\frac{3\pi}{11}+\sec^2\frac{4\pi}{11}+\sec^2\frac{5\pi}{11}=60.$$

Prove that

11. $$\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{18\pi}{13}=\frac{\sqrt{13}-1}{4}.$$

12. $\cos\frac{10\pi}{13}+\cos\frac{14\pi}{13}+\cos\frac{22\pi}{13}=\frac{-\sqrt{13}-1}{4}$.

13. $\cos\frac{\pi}{15}+\cos\frac{7\pi}{15}+\cos\frac{11\pi}{15}+\cos\frac{13\pi}{15}=-\frac{1}{2}$.

14. Prove that $\sin\frac{\pi}{14}$ is a root of the equation

$$64x^6-80x^4+24x^2-1=0.$$

Find the value of

15. $\cos\theta\cos\left(\theta+\frac{2\pi}{n}\right)\cos\left(\theta+\frac{4\pi}{n}\right)\ldots\ldots\cos\left\{\theta+(n-1)\frac{2\pi}{n}\right\}$.

16. $\sin\theta\sin\left(\theta+\frac{2\pi}{n}\right)\sin\left(\theta+\frac{4\pi}{n}\right)\ldots\ldots\sin\left\{\theta+(n-1)\frac{2\pi}{n}\right\}$.

17. $\operatorname{cosec}^2\theta+\operatorname{cosec}^2\left(\theta+\frac{2\pi}{n}\right)+\operatorname{cosec}^2\left(\theta+\frac{4\pi}{n}\right)\ldots\ldots$ to n terms.

18. $\tan^2\theta+\tan^2\left(\theta+\frac{2\pi}{n}\right)+\tan^2\left(\theta+\frac{4\pi}{n}\right)\ldots\ldots$ to n terms.

[*For the following* 5 *questions commence with equation* (5) *of Art.* 277.]

19. $\tan\theta+\tan\left(\theta+\frac{1\pi}{n}\right)+\tan\left(\theta+\frac{2\pi}{n}\right)\ldots\ldots$ to n terms.

20. $\cot\theta+\cot\left(\theta+\frac{1\pi}{n}\right)+\cot\left(\theta+\frac{2\pi}{n}\right)\ldots\ldots$ to n terms.

21. $\tan\theta\tan\left(\theta+\frac{\pi}{n}\right)\tan\left(\theta+\frac{2\pi}{n}\right)\ldots\ldots$ to n factors.

22. $\tan^2\theta+\tan^2\left(\theta+\frac{\pi}{n}\right)+\tan^2\left(\theta+\frac{2\pi}{n}\right)+\ldots\ldots$ to n terms.

23. If n be odd, prove that $S=3C=n^2-1$, where

$$S=\sec^2\frac{\pi}{n}+\sec^2\frac{2\pi}{n}+\sec^2\frac{3\pi}{n}+\ldots\ldots \text{ to } n-1 \text{ terms,}$$

and $$C=\operatorname{cosec}^2\frac{\pi}{n}+\operatorname{cosec}^2\frac{2\pi}{n}+\operatorname{cosec}^2\frac{3\pi}{n}+\ldots\ldots \text{ to } n-1 \text{ terms.}$$

24. Find the sum of the products, taken two at a time, of expressions of the form $\sec\left(\theta+\frac{2r\pi}{n}\right)$, where r has all values from zero to $n-1$.

CHAPTER XXV.

EXPONENTIAL SERIES FOR COMPLEX QUANTITIES. CIRCULAR FUNCTIONS FOR COMPLEX ANGLES. HYPERBOLIC FUNCTIONS.

302. WHEN x is a real quantity we have proved in Art. 253 that

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \text{ad inf.}\ldots\ldots\ldots\ldots(1).$$

When x is not real but is complex, *i.e.* of the form $a + b\sqrt{-1}$, the expression e^x has no meaning at present.

Let us so define it that for **all** values of x (whether real or complex) it shall mean the series

$$1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \text{ad inf.}\ldots\ldots\ldots\ldots(2).$$

303. We can easily shew that this series is convergent when x is complex.

For let $\qquad x = r(\cos\theta + \sqrt{-1}\sin\theta).$

Then
$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots \text{ad inf.}$$
$$= 1 + r(\cos\theta + i\sin\theta) + \frac{r^2(\cos 2\theta + i\sin 2\theta)}{\lfloor 2}$$
$$+ \frac{r^3(\cos 3\theta + i\sin 3\theta)}{\lfloor 3} + \ldots\ldots \text{ad inf.}$$
$$= 1 + r\cos\theta + \frac{r^2\cos 2\theta}{\lfloor 2} + \frac{r^3\cos 3\theta}{\lfloor 3} + \ldots\ldots$$
$$+ \sqrt{-1}\left[r\sin\theta + \frac{r^2\sin 2\theta}{\lfloor 2} + \frac{r^3\sin 3\theta}{\lfloor 3} + \ldots\ldots\right].$$

The quantity
$$1 + r\cos\theta + \frac{r^2}{\lfloor 2}\cos 2\theta + \frac{r^3}{\lfloor 3}\cos 3\theta + \ldots\ldots$$
is
$$< 1 + r + \frac{r^2}{\lfloor 2} + \frac{r^3}{\lfloor 3} + \ldots\ldots$$
and is therefore convergent since this series is convergent for all real values of r. (Art. 254.)

Similarly the quantity
$$r\sin\theta + \frac{r^2}{\lfloor 2}\sin 2\theta + \ldots\ldots$$
is convergent.

Hence the series for e^x is always convergent.

304. When x is a complex quantity the quantity e^x is then a *short way* of writing
$$1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots$$
Unless x be real, the e in e^x does not mean the series
$$1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots\ldots$$

When x is complex, e^x stands for a series of *the same form* as that series which, when x is real, has been proved to be equal to

$$\left(1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\ldots\right)^x.$$

Instead of e^x the expressions $E(x)$ and $\exp(x)$ are sometimes used.

305. By a proof similar to that of Art. 300, C. Smith's *Algebra*, it may be shewn that

$$e^x \cdot e^y = e^{x+y},$$

whether x and y be real or complex quantities, so that the functions e^x and e^y obey a law of the same form as the index law.

306. If x be put equal to θi, where θ is real, we then have

$$e^{\theta i} = 1 + \theta i + \frac{\theta^2 i^2}{\underline{|2}} + \frac{\theta^3 i^3}{\underline{|3}} + \ldots\ldots$$

$$= 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \ldots\ldots$$

$$+ i\left[\theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \ldots\ldots\right]$$

$$= \cos\theta + i\sin\theta. \qquad \text{(Arts. 279 and 280.)}$$

So $$e^{-\theta i} = \cos\theta - i\sin\theta.$$

Hence, by addition, we have

$$\cos\theta = \frac{e^{\theta i} + e^{-\theta i}}{2},$$

and, by subtraction,

$$\sin\theta = \frac{e^{\theta i} - e^{-\theta i}}{2i}.$$

Circular functions of complex angles.

307. When x is a complex quantity, the functions $\sin x$ and $\cos x$ have at present no meaning.

For **real** values of x we have already shewn in Arts. 279 and 280 that

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots\ldots \text{ad inf.}$$

and $$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \ldots\ldots \text{ad inf.}$$

Let us **define** $\sin x$ and $\cos x$, when x is **complex,** so that these relations may always be true, *i.e.* for all values of x let

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots\ldots \quad \ldots\ldots\ldots (1),$$

and $$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \ldots\ldots \quad \ldots\ldots\ldots (2).$$

When x is complex, the quantities $\sin x$ and $\cos x$ are then only *short ways* of writing the series on the right-hand sides of (1) and (2).

308. We have then, for all values of x, real or complex,

$$\cos x + i \sin x = 1 + xi - \frac{x^2}{\underline{|2}} - \frac{x^3 i}{\underline{|3}} + \frac{x^4}{\underline{|4}} \ldots\ldots$$

$$= 1 + xi + \frac{(xi)^2}{\underline{|2}} + \frac{(xi)^3}{\underline{|3}} + \frac{(xi)^4}{\underline{|4}} \ldots\ldots$$

$$= e^{xi}. \qquad \text{(Art. 302.)}$$

So $$\cos x - i \sin x = e^{-xi}.$$

Hence for all values of x, real or complex, we have

$$\mathbf{\cos x = \frac{e^{xi} + e^{-xi}}{2}}, \text{ and } \mathbf{\sin x = \frac{e^{xi} - e^{-xi}}{2i}}.$$

These results are known as Euler's Exponential Values.

309. We can now shew that the Addition and Subtraction Theorems hold for imaginary angles, *i.e.* that, whether x be real or complex, then

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y,$$

and $$\cos(x-y) = \cos x \cos y + \sin x \sin y.$$

Since

$$\cos x = \frac{e^{xi} + e^{-xi}}{2} \text{ and } \sin x = \frac{e^{xi} - e^{-xi}}{2i},$$

we have $$\sin x \cos y + \cos x \sin y$$

$$= \frac{e^{xi} - e^{-xi}}{2i} \; \frac{e^{yi} + e^{-yi}}{2} + \frac{e^{xi} + e^{-xi}}{2} \; \frac{e^{yi} - e^{-yi}}{2i}$$

$$= \frac{e^{xi}.2e^{yi} - e^{-xi}.2e^{-yi}}{4i} = \frac{e^{(x+y)i} - e^{-(x+y)i}}{2i} \quad \text{(Art. 305)}$$

$$= \sin(x+y).$$

Similarly the other results may be proved.

310. It follows that all formulæ which have been proved for real angles and which are founded on the Addition and Subtraction Theorems are also true when we substitute for the real angle any complex quantity.

For example, since

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$

where θ is real, it follows that

$$\cos 3(x+yi) = 4\cos^3(x+yi) - 3\cos(x+yi).$$

Again, since, by De Moivre's Theorem, we know that

$$\cos n\theta + i\sin n\theta$$

is always one of the values of

$$(\cos\theta + i\sin\theta)^n,$$

when θ is real and n has any value, it follows that

$$\cos n(x+yi) + i\sin n(x+yi)$$

is always one of the values of

$$[\cos(x+yi) + i\sin(x+yi)]^n.$$

311. Periods of complex circular functions. In equations (1) and (2) of Art. 309 let x be complex and let $y = 2\pi$.

Then $\sin(x+2\pi) = \sin x\cos 2\pi + \cos x\sin 2\pi$

$$= \sin x,$$

and $\cos(x+2\pi) = \cos x\cos 2\pi - \sin x\sin 2\pi$

$$= \cos x.$$

Hence $\sin x$ and $\cos x$ both remain the same when x is increased by 2π. Similarly they will remain the same when x is increased by

$$4\pi,\ 6\pi, \ldots\ldots\ldots\ 2n\pi.$$

Hence, when x is complex, the expressions $\sin x$ and $\cos x$ are periodic functions whose period is 2π.

This corresponds with the results we have already found for real angles. (Art. 61.)

EXAMPLES. LIV.

Assuming that $\cos x = \dfrac{e^{xi} + e^{-xi}}{2}$ and $\sin x = \dfrac{e^{xi} - e^{-xi}}{2i}$ prove that, for all values of x, real or complex,

1. $\cos^2 x + \sin^2 x = 1$.
2. $\cos(-x) = \cos x$.
3. $\sin(-x) = \sin x$.
4. $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$.
5. $\sin 3x = 3\sin x - 4\sin^3 x$.
6. $\cos x - \cos y = 2\sin\dfrac{x+y}{2}\sin\dfrac{y-x}{2}$.
7. $\sin x - \sin y = 2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$.

Prove that

8. $\{\sin(\alpha+\theta) - e^{\alpha i}\sin\theta\}^n = \sin^n\alpha\, e^{-n\theta i}$.
9. $\sin(\alpha+n\theta) - e^{\alpha i}\sin n\theta = e^{-n\theta i}\sin\alpha$.
10. $\{\sin(\alpha-\theta) + e^{\pm\alpha i}\sin\theta\}^n = \sin^{n-1}\alpha\,\{\sin(\alpha-n\theta) + e^{\pm\alpha i}\sin n\theta\}$.

312. In the formulæ of Art. 308 if x be a pure imaginary quantity and equal to yi, we have, since

$$i^2 = -1,$$

$$\cos yi = \frac{e^{yi.i} + e^{-yi.i}}{2} = \frac{e^{-y} + e^{y}}{2} = \frac{e^{y} + e^{-y}}{2},$$

and

$$\sin yi = \frac{e^{yi.i} - e^{-yi.i}}{2i} = \frac{e^{-y} - e^{y}}{2i} = i\,\frac{e^{-y} - e^{y}}{2(-1)}$$

$$= i\,\frac{e^{y} - e^{-y}}{2}.$$

313. Hyperbolic Functions. Def. The quantity

$$\frac{e^{y} - e^{-y}}{2},$$

whether y be real or complex, is called the hyperbolic sine of y and is written **sinh y.**

Similarly the quantity

$$\frac{e^y + e^{-y}}{2}$$

is called the hyperbolic cosine of y and is written

$$\cosh y.$$

[It will be observed that the values of $\sinh y$ and $\cosh y$ are obtained from the exponential expressions for $\sin y$ and $\cos y$ by simply omitting the i's.]

The hyperbolic tangent, secant, cosecant, and cotangent are obtained from the hyperbolic sine and cosine just as the ordinary tangent, secant, cosecant, and cotangent are obtained from the ordinary sine and cosine.

Thus $$\tanh y = \frac{\sinh y}{\cosh y} = \frac{e^y - e^{-y}}{e^y + e^{-y}},$$

$$\operatorname{cosech} y = \frac{1}{\sinh y} = \frac{2}{e^y - e^{-y}},$$

$$\operatorname{sech} y = \frac{1}{\cosh y} = \frac{2}{e^y + e^{-y}},$$

and $$\coth y = \frac{1}{\tanh y} = \frac{e^y + e^{-y}}{e^y - e^{-y}}.$$

The hyperbolic cosine and sine have the same relation to the curve called the rectangular hyperbola that the ordinary circular cosine and sine have to the circle. Hence the use of the word hyperbolic.

314. From Arts. 312 and 313 we clearly have

$$\cos(yi) = \cosh y,$$

and $$\sin(yi) = i \sinh y.$$

So $$\tan(yi) = i \tanh y.$$

315. Corresponding to most general trigonometrical formulæ involving the ratios of angles there are formulæ involving the hyperbolic ratios.

For example, we have, for all values of the angle x,

$$\cos^2 x + \sin^2 x = 1,$$

so that

$$\cos^2 (yi) + \sin^2 (yi) = 1,$$

and hence, by the last article,

$$\cosh^2 y - \sinh^2 y = 1.$$

[This may be deduced independently from the definition of the hyperbolic functions. For

$$\cosh^2 y - \sinh^2 y = \left(\frac{e^y + e^{-y}}{2}\right)^2 - \left(\frac{e^y - e^{-y}}{2}\right)^2$$

$$= \frac{e^{2y} + 2 + e^{-2y}}{4} - \frac{e^{2y} - 2 + e^{-2y}}{4} = 1.]$$

Again, for all values of u and v we have

$$\sin (u + v) = \sin u \cos v + \cos u \sin v.$$

Put

$$u = xi \text{ and } v = yi,$$

so that

$$\sin [(x + y) i] = \sin (xi) \cos (yi) + \cos (xi) \sin (yi).$$

The expressions of the last article then give

$$i \sinh (x + y) = i \sinh x \cosh y + \cosh x \times i \sinh y,$$

$$\therefore \sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y.$$

[Directly from the definition of the hyperbolic ratios we have

$$\sinh x \cosh y + \cosh x \sinh y$$

$$= \frac{e^x - e^{-x}}{2} \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \frac{e^y - e^{-y}}{2} = \frac{2e^{x+y} - 2e^{-(x+y)}}{4},$$

on multiplication, $= \sinh (x + y)$.]

Again, for all values of θ, we have

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

Put then $\theta = xi$, and we have

$$\tan(3xi) = \frac{3\tan(xi) - \tan^3(xi)}{1 - 3\tan^2(xi)}.$$

Hence the substitutions of Art. 314 give

$$i\tanh(3x) = \frac{3i\tanh x - i^3\tanh^3 x}{1 - 3i^2\tanh^2 x}$$

$$= \frac{3i\tanh x + i\tanh^3 x}{1 + 3\tanh^2 x};$$

so that $$\tanh(3x) = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}.$$

As before, this may be easily proved from the definition of $\tanh x$.

316. In general it follows from (1) of Art. 314 that any general formula which is true for cosines of angles is also true if instead of cos we read cosh.

From (2) of the same article, since

$$\sin^2(yi) = -\sinh^2 y,$$

it follows that any general formula involving the cosine and square of the sine of an angle is true if for cos we read cosh and for $\sin^2$ we read $-\sinh^2$.

Similarly from (3) we may turn a formula involving $\tan^2$ into another by writing for $\tan^2$ the quantity $-\tanh^2$.

In this manner formulæ and series involving the hyperbolic functions may be obtained from Arts. 241, 242, 274, 275, 277, 289, 291, and 293—298.

317. From the values in Art. 313 it follows, by Art. 302, that

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$= 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \frac{x^6}{\underline{|6}} + \ldots\ldots,$$

$$\sinh x = \frac{1}{2}[e^x - e^{-x}]$$

$$= x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \frac{x^7}{\underline{|7}} + \ldots\ldots.$$

These are the expansional values of $\cosh x$ and $\sinh x$.

***318.** *Periods of the hyperbolic functions.*

For all values of θ, real or complex, we have $\cos \theta i = \cosh \theta$.

Hence

$$\cosh(x+yi) = \cos\{(x+yi)\,i\} = \cos(xi-y) = \cos[-2\pi + xi - y] \quad \text{(Art. 311)}$$

$$= \cos[(2\pi i + x + yi)\,i] = \cosh[2\pi i + x + yi]$$

$$= \text{(similarly)} \cosh[4\pi i + x + yi] = \ldots\ldots$$

Hence the hyperbolic cosine is periodic, its period being imaginary and equal to $2\pi i$.

Again, since $\sinh \theta = -i \sin \theta i$, we have

$$\sinh(x+yi) = -i\sin\{(x+yi)\,i\} = -i\sin[xi-y]$$

$$= -i\sin[-2\pi + xi - y] = -i\sin\{[2\pi i + x + yi]\,i\}$$

$$= \sinh[2\pi i + x + yi],$$

so that the period of $\sinh(x+yi)$ is $2\pi i$.

Similarly it may be shewn that the period of $\tanh(x+yi)$ is πi.

The hyperbolic functions therefore differ from the circular functions in having no real period ; their period is imaginary.

319. Ex. 1. *Separate into its real and imaginary parts the expression sin* $(\alpha + \beta i)$.

We have $$\sin(\alpha+\beta i)=\sin\alpha\cos\beta i+\cos\alpha\sin\beta i$$

$$=\sin\alpha\frac{e^{\beta}+e^{-\beta}}{2}+\cos\alpha\frac{e^{-\beta}-e^{\beta}}{2i}$$

$$=\sin\alpha\frac{e^{\beta}+e^{-\beta}}{2}+i\cos\alpha\frac{e^{\beta}-e^{-\beta}}{2}$$

$$=\sin\alpha\cosh\beta+i\cos\alpha\sinh\beta.$$

Ex. 2. *Separate into its real and imaginary parts the expression tan* $(\alpha+\beta i)$.

We have $$\tan(\alpha+\beta i)=\frac{\sin(\alpha+\beta i)}{\cos(\alpha+\beta i)}$$

$$=\frac{2\sin(\alpha+\beta i)\cos(\alpha-\beta i)}{2\cos(\alpha+\beta i)\cos(\alpha-\beta i)}$$

$$=\frac{\sin 2\alpha+\sin 2\beta i}{\cos 2\alpha+\cos 2\beta i}$$

$$=\frac{\sin 2\alpha+i\sinh 2\beta}{\cos 2\alpha+\cosh 2\beta}. \qquad \text{(Art. 314.)}$$

Aliter. Let $\tan(\alpha+\beta i)=x+yi$, so that $\tan(\alpha-\beta i)=x-yi$.

$$\therefore\quad x=\tfrac{1}{2}[\tan(\alpha+\beta i)+\tan(\alpha-\beta i)]$$

$$=\frac{\sin(\alpha+\beta i)\cos(\alpha-\beta i)+\cos(\alpha+\beta i)\sin(\alpha-\beta i)}{2\cos(\alpha+\beta i)\,.\cos(\alpha-\beta i)}$$

$$=\frac{\sin 2\alpha}{\cos 2\alpha+\cos 2\beta i}=\frac{\sin 2\alpha}{\cos 2\alpha+\cosh 2\beta}.$$

Also $$y=\frac{1}{2i}[\tan(\alpha+\beta i)-\tan(\alpha-\beta i)]$$

$$=\frac{1}{2i}\frac{\sin(\alpha+\beta i)\cos(\alpha-\beta i)-\cos(\alpha+\beta i)\sin(\alpha-\beta i)}{\cos(\alpha+\beta i)\cos(\alpha-\beta i)}$$

$$=\frac{1}{i}\frac{\sin 2\beta i}{\cos 2\alpha+\cos 2\beta i}=\frac{\sinh 2\beta}{\cos 2\alpha+\cosh 2\beta}.$$

$$\therefore\quad \tan(\alpha+\beta i)=\frac{\sin 2\alpha+i\sinh 2\beta}{\cos 2\alpha+\cosh 2\beta}.$$

Ex. 3. *Separate into its real and imaginary parts the expression cos h* $(\alpha+\beta i)$.

We have $\cosh(\alpha+\beta i) = \dfrac{e^{\alpha+\beta i} + e^{-\alpha-\beta i}}{2}$ (Art. 313)

$$= \frac{e^{\alpha} \cdot e^{\beta i} + e^{-\alpha} \cdot e^{-\beta i}}{2} = \frac{e^{\alpha}(\cos\beta + i\sin\beta) + e^{-\alpha}(\cos\beta - i\sin\beta)}{2} \quad \text{(Art. 308)}$$

$$= \frac{\cos\beta\,(e^{\alpha}+e^{-\alpha}) + i\sin\beta\,(e^{\alpha}-e^{-\alpha})}{2} = \cos\beta\cosh\alpha + i\sin\beta\sinh\alpha.$$

Aliter. $\cosh(\alpha+\beta i) = \cos\{(\alpha+\beta i)\,i\}$ (Art. 313)

$$= \cos\{\alpha i - \beta\}$$
$$= \cos(\alpha i)\cos\beta + \sin(\alpha i)\sin\beta$$
$$= \cosh\alpha\cos\beta + i\sinh\alpha\sin\beta.$$

EXAMPLES. LV.

Prove that

1. $\cosh 2x = 1 + 2(\sinh x)^2 = 2(\cosh x)^2 - 1.$

2. $\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta.$

3. $\cosh(\alpha+\beta) - \cosh(\alpha-\beta) = 2\sinh\alpha\sinh\beta.$

4. $\tanh(\alpha+\beta) = \dfrac{\tanh\alpha + \tanh\beta}{1+\tanh\alpha\tanh\beta}.$

5. $\cosh 3x = 4\cosh^3 x - 3\cosh x.$

6. $\sinh 3x = 3\sinh x + 4\sinh^3 x.$

7. $\sinh(x+y)\cosh(x-y) = \frac{1}{2}(\sinh 2x + \sinh 2y).$

8. $\cosh 2x + \cosh 5x + \cosh 8x + \cosh 11x$

$$= 4\cosh\frac{13x}{2}\cosh 3x\cosh\frac{3x}{2}.$$

9. $\cosh x + \cosh(x+y) + \cosh(x+2y) + \ldots\ldots$ to n terms

$$= \frac{\cosh\left(x + \dfrac{n-1}{2}y\right)\sinh\dfrac{ny}{2}}{\sinh\dfrac{y}{2}}.$$

10. $\sinh x + \sinh(x+y) + \sinh(x+2y) + \ldots\ldots$ to n terms

$$= \frac{\sinh\left(x + \dfrac{n-1}{2}y\right)\sinh\dfrac{ny}{2}}{\sinh\dfrac{y}{2}}.$$

11. $\sinh x + n\sinh 2x + \frac{n(n-1)}{1.2}\sinh 3x + \dots\dots$ to $(n+1)$ terms

$$=2^n \cosh^n \frac{x}{2} \sinh\left(\frac{n}{2}+1\right)x.$$

12. $\sinh\beta \sin\alpha + i\cosh\beta\cos\alpha = i\cos(\alpha+\beta i)$.

13. $\sin 2\alpha + i\sinh 2\beta = 2\sin(\alpha+i\beta)\cos(\alpha-i\beta)$.

14. $\cos(\alpha+i\beta)+i\sin(\alpha+i\beta)=e^{-\beta}(\cos\alpha+i\sin\alpha)$.

15. If $\tan y = \tan\alpha\tanh\beta$, and $\tan z = \cot\alpha\tanh\beta$, then prove that $\tan(y+z)=\sinh 2\beta \operatorname{cosec} 2\alpha$.

16. If $u=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, prove that $\tanh\frac{u}{2}=\tan\frac{\theta}{2}$.

Separate into their real and imaginary parts the quantities

17. $\cos(\alpha+\beta i)$.

18. $\cot(\alpha+\beta i)$.

19. $\operatorname{cosec}(\alpha+\beta i)$.

20. $\sec(\alpha+\beta i)$.

21. $\sinh(\alpha+\beta i)$.

22. $\tanh(\alpha+\beta i)$.

23. $\operatorname{sech}(\alpha+\beta i)$.

24. Prove that $\tan\frac{u+iv}{2}=\frac{\sin u + i\sinh v}{\cos u+\cosh v}$.

25. If $\sin(A+iB)=x+iy$, prove that

$$\frac{x^2}{\cosh^2 B}+\frac{y^2}{\sinh^2 B}=1, \text{ and } \frac{x^2}{\sin^2 A}-\frac{y^2}{\cos^2 A}=1.$$

26. If $\sin(\theta+\phi i)=\cos\alpha+i\sin\alpha$, prove that $\cos^2\theta=\pm\sin\alpha$.

27. If $\sin(\theta+\phi i)=\rho(\cos\alpha+i\sin\alpha)$, prove that

$$\rho^2=\tfrac{1}{2}[\cosh 2\phi-\cos 2\theta] \text{ and } \tan\alpha=\tanh\phi\cot\theta.$$

28. If $\cos(\theta+\phi i)=R(\cos\alpha+i\sin\alpha)$, prove that

$$\phi=\tfrac{1}{2}\log\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}.$$

29. If $\tan(\theta+\phi i)=\tan\alpha+i\sec\alpha$, prove that $e^{2\phi}=\pm\cot\frac{\alpha}{2}$, and that $2\theta=n\pi+\frac{\pi}{2}+\alpha$.

30. If $\tan(\theta+\phi i)=\cos\alpha+i\sin\alpha$, prove that

$$\theta=\frac{n\pi}{2}+\frac{\pi}{4}, \text{ and } \phi=\tfrac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

31. If $A+iB=c\tan(x+iy)$, then

$$\tan 2x=\frac{2cA}{c^2-A^2-B^2}.$$

32. If $\tan(\theta+\phi i)=\sin(x+iy)$, then

$$\coth y \sinh 2\phi=\cot x \sin 2\theta.$$

33. If $\tan(\alpha+i\beta)=i$, α and β being real, prove that α is indeterminate and β is infinite.

Prove that

34. $\frac{1}{2}(\sinh x+\sin x)=x+\frac{x^5}{\underline{|5}}+\frac{x^9}{\underline{|9}}+\ldots\ldots \text{ad inf.}$

35. $\frac{1}{2}(\cosh x+\cos x)=1+\frac{x^4}{\underline{|4}}+\frac{x^8}{\underline{|8}}+\ldots\ldots \text{ad inf.}$

320. **Inverse Circular Functions. When α and β are real and $\alpha=\cos\beta$, we defined, in Art. 237, the inverse cosine of α to be that value of β which lies between 0 and π, and it was pointed out that β was a many-valued quantity.

If now $$x+yi=\cos(u+vi),$$

then similarly $u+vi$ is said to be an inverse cosine of $x+yi$.

But since

$$x+yi=\cos(u+vi)=\cos[2n\pi\pm(u+vi)] \quad \text{(Art. 311)}$$

it follows that $2n\pi\pm(u+vi)$ is also an inverse cosine of $x+yi$, where n is any integer.

The inverse cosine of $x+yi$ is hence a many-valued function. When the many-valuedness of the inverse cosine is considered it is written

$$\text{Cos}^{-1}(x+yi).$$

The principal value of the inverse cosine of $x+yi$ is that value of $2n\pi\pm(u+vi)$ which is such that either $2n\pi+u$ or $2n\pi-u$ lies between 0 and π.

This principal value is denoted by $\cos^{-1}(x+yi)$.

We have then

$$\text{Cos}^{-1}(x+yi) = 2n\pi \pm \cos^{-1}(x+yi).$$

****321.** Similarly if

$$x+yi = \sin(u+vi) = \sin\{n\pi + (-1)^n(u+vi)\},$$

then $n\pi + (-1)^n(u+vi)$ is an inverse sine of $x+yi$. It is a many-valued quantity and is denoted by $\text{Sin}^{-1}(x+yi)$. Its principal value is such that its real part lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and is denoted by $\text{Sin}^{-1}(x+yi)$.

We then have

$$\text{Sin}^{-1}(x+yi) = n\pi + (-1)^n \sin^{-1}(x+yi).$$

Similarly $\tan^{-1}(x+yi)$ and $\text{Tan}^{-1}(x+yi)$ are defined, so that the principal value of $\text{Tan}^{-1}(x+yi)$ is such that its real part lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, and

$$\text{Tan}^{-1}(x+yi) = n\pi + \tan^{-1}(x+yi).$$

Similarly

$$\text{Sec}^{-1}(x+yi) = 2n\pi \pm \sec^{-1}(x+yi),$$

$$\text{Cosec}^{-1}(x+yi) = n\pi + (-1)^n \text{cosec}^{-1}(x+yi),$$

and $$\text{Cot}^{-1}(x+yi) = n\pi + \cot^{-1}(x+yi).$$

****322.** We shall henceforward use $\sin^{-1}$, Sin^{-1}, $\cos^{-1}$, Cos^{-1},... with the meanings above assigned.

****323. Inverse hyperbolic functions.** If $x = \cosh y$ then similarly, as in Art. 320, we write $y = \cosh^{-1} x$.

If x be real, we have

$$x = \frac{e^y + e^{-y}}{2},$$

so that $$e^{2y} - 2xe^y + 1 = 0,$$

and hence $$e^y = x \pm \sqrt{x^2-1}$$

$$= x + \sqrt{x^2-1} \text{ or } \frac{1}{x+\sqrt{x^2-1}},$$

$$\therefore\ y = \pm \log (x + \sqrt{x^2-1}).$$

The positive value of the right-hand side is the one always taken.

Hence, when x is real, $\cosh^{-1}x$ is a single-valued function.

Similarly $\sinh^{-1}x$ and $\tanh^{-1}x$ are defined; they are single-valued functions, when x is real.

**324. If $\alpha + \beta i = \cosh(x+yi)$, then $x+yi$ is said to be an inverse hyperbolic cosine of $\alpha+\beta i$.

But $\cosh(x+yi) = \cosh\{2n\pi i \pm (x+yi)\}$, as in Art. 318.

Hence $2n\pi i \pm (x+yi)$ is an inverse hyperbolic cosine of $\alpha+\beta i$. Its principal value is that value whose imaginary part lies between 0 and πi, *i.e.* such that $2n\pi \pm y$ lies between 0 and π.

Similarly the inverse hyperbolic sine and tangent of $\alpha+\beta i$ are defined. In this case the principal values are such that the imaginary part lies between $-\frac{\pi}{2}i$ and $\frac{\pi}{2}i$.

325. **Ex. 1. *Separate into real and imaginary parts the quantity*

$\sin^{-1}(\cos\theta + i\sin\theta)$, where θ is real.

Let $$\sin^{-1}(\cos\theta + i \sin\theta) = x + yi,$$

so that $$\cos\theta + i\sin\theta = \sin(x+yi) = \sin x \cos yi + \cos x \sin yi$$

$$= \sin x \cosh y + i \cos x \sinh y.$$

Hence $$\sin x \cosh y = \cos\theta \qquad (1),$$

and $$\cos x \sinh y = \sin\theta \qquad (2).$$

Squaring and adding, we have

$$1=\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y=\sin^2 x\,(1+\sinh^2 y)+\cos^2 x\sinh^2 y$$

$$=\sin^2 x+\sinh^2 y,$$

$$\therefore\quad \sinh^2 y=\cos^2 x.$$

Hence from (2) we have $\cos^2 x=\sin\theta$, assuming $\sin\theta$ to be positive.

Therefore, since x is to lie between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ (Art. 321),

we have $\quad \cos x=+\sqrt{\sin\theta}$, and hence $x=\cos^{-1}(\sqrt{\sin\theta})$.

The equation (2) then gives

$$\sinh y=+\sqrt{\sin\theta},$$

so that $$e^{2y}-2e^y\sqrt{\sin\theta}=1.$$

i.e. $$e^y=\sqrt{\sin\theta}+\sqrt{1+\sin\theta},$$

i.e. $$y=\log\left[\sqrt{\sin\theta}+\sqrt{1+\sin\theta}\right].$$

Ex. 2. *Separate into its real and imaginary parts the quantity*

$$tan^{-1}(\alpha+\beta i).$$

Let $\quad \tan^{-1}(\alpha+\beta i)=(x+yi)$, so that $\tan(x+yi)=\alpha+\beta i$,

and $$\tan(x-yi)=\alpha-\beta i.$$

$$\therefore\quad \tan 2x=\tan\{(x+yi)+(x-yi)\}$$

$$=\frac{(\alpha+\beta i)+(\alpha-\beta i)}{1-(\alpha+\beta i)(\alpha-\beta i)}=\frac{2\alpha}{1-\alpha^2-\beta^2}.$$

$$\therefore\quad x=\tfrac{1}{2}\tan^{-1}\frac{2\alpha}{1-\alpha^2-\beta^2}.$$

Again $\quad \tan(2yi)=\tan[(x+yi)-(x-yi)]$

$$=\frac{(\alpha+\beta i)-(\alpha-\beta i)}{1+(\alpha+\beta i)(\alpha-\beta i)}=\frac{2\beta i}{1+\alpha^2+\beta^2}.$$

$$\therefore\quad i\,\frac{e^{2y}-e^{-2y}}{e^{2y}+e^{-2y}}=\frac{2\beta i}{1+\alpha^2+\beta^2}\quad\ldots\ldots\ldots\ldots(1).$$

$$\therefore\quad \frac{e^{2y}}{e^{-2y}}=\frac{1+\alpha^2+\beta^2+2\beta}{1+\alpha^2+\beta^2-2\beta}=\frac{(1+\beta)^2+\alpha^2}{(1-\beta)^2+\alpha^2}.$$

$$\therefore\quad y=\tfrac{1}{4}\log\left\{\frac{(1+\beta)^2+\alpha^2}{(1-\beta)^2+\alpha^2}\right\}.$$

Or again (1) gives $$\tanh 2y = \frac{2\beta}{1+\alpha^2+\beta^2},$$

so that $$y = \frac{1}{2}\tanh^{-1}\frac{2\beta}{1+\alpha^2+\beta^2}.$$

We should have $\text{Tan}^{-1}(\alpha+\beta i) = n\pi + \tan^{-1}(\alpha+\beta i)$

$$= n\pi + \tfrac{1}{2}\tan^{-1}\frac{2\alpha}{1-\alpha^2-\beta^2} + \frac{i}{2}\tanh^{-1}\frac{2\beta}{1+\alpha^2+\beta^2}.$$

EXAMPLES. LVI.

Separate into their real and imaginary parts the quantities

1. $\tan^{-1}(\cos\theta + i\sin\theta)$.

2. $\cos^{-1}(\cos\theta + i\sin\theta)$, where θ is a positive acute angle.

Prove that

3. $\sinh^{-1} x = \log(x + \sqrt{x^2+1})$.

4. $\tanh^{-1} x = \sinh^{-1}\dfrac{x}{\sqrt{1-x^2}}$.

5. $\cosh^{-1} x = \log(\sqrt{x^2-1} + x)$.

6. $\tanh^{-1} x = \tfrac{1}{2}\log\dfrac{1+x}{1-x}$.

7. $\text{Sin}^{-1}(\text{cosec}\,\theta) = \{2n + (-1)^n\}\dfrac{\pi}{2} + i(-1)^n \log\cot\dfrac{\theta}{2}$.

8. $\text{Tan}^{-1}(e^{\theta i}) = \dfrac{n\pi}{2} + \dfrac{\pi}{4} - \dfrac{i}{2}\log\tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$.

9. $\text{Tan}^{-1}\dfrac{\tan 2\theta + \tanh 2\phi}{\tan 2\theta - \tanh 2\phi} + \text{Tan}^{-1}\dfrac{\tan\theta - \tanh\phi}{\tan\theta + \tanh\phi} = \text{Tan}^{-1}(\cot\theta\coth\phi)$.

CHAPTER XXVI.

LOGARITHMS OF COMPLEX QUANTITIES.

326. If $\alpha = e^x$, where α and x are real quantities, we know that x is called the logarithm of α to base e and we have shewn in Art. 253 that

$$\alpha = e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \dots\dots\dots \text{ad inf.}$$

We may therefore look upon the logarithm, x, of α to base e as being derived as a root of the equation

$$\alpha = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \dots\dots \text{ad inf.} \dots\dots\dots (1).$$

As in other cases we shall now extend this result to complex quantities.

327. **Def.** *If $x + yi$ be any complex quantity and if $\alpha + \beta i$ be a quantity which is equal to e^{x+yi}, i.e. to the series*

$$1 + (x + yi) + \frac{(x + yi)^2}{\underline{|2}} + \frac{(x + yi)^3}{\underline{|3}} + \dots\dots\dots,$$

then $x + yi$ is said to be a logarithm of $\alpha + \beta i$.

We say "a" logarithm because, as we shall now shew, there are with the above definition many logarithms of a quantity.

We have $$\alpha + \beta i = e^{x+yi} \quad\text{..................(1).}$$

Now, by Art. 308, we have, for all integral values of n,

$$e^{2n\pi i} = \cos 2n\pi + i \sin 2n\pi = 1 \quad\text{............(2).}$$

Hence from (1) and (2) we have, by Art. 305,

$$\alpha + \beta i = e^{x+yi} \cdot e^{2n\pi i} = e^{x+(y+2n\pi)i}.$$

According to the above definition we see that, if $x + yi$ be a logarithm of $\alpha + \beta i$, so also is

$$x + yi + 2n\pi i, \quad \textit{i.e.} \quad x + (y + 2n\pi)\, i.$$

328. We proceed to find the logarithms of the complex quantity $\alpha + \beta i$, where α and β are real.

By Art. 267, we have

$$\alpha + \beta i = r\,[\cos(2n\pi + \theta) + i \sin(2n\pi + \theta)]$$

where n is any integer, $r = +\sqrt{\alpha^2 + \beta^2}$, and θ is that value lying between $-\pi$ and $+\pi$ such that $\cos\theta$ is $\frac{\alpha}{r}$ and $\sin\theta$ is $\frac{\beta}{r}$, *i.e.* with the restriction of Art. 267,

$$\theta = \tan^{-1}\frac{\beta}{\alpha}.$$

If $x + yi$ be a logarithm of $\alpha + \beta i$, we have then

$$r\,[\cos(2n\pi + \theta) + i \sin(2n\pi + \theta)] = e^{x+yi}$$
$$= e^x \cdot e^{yi} \qquad \text{(Art. 305)}$$
$$= e^x(\cos y + i \sin y).$$

By equating real and imaginary parts, we have

$$e^x \cos y = r \cos (2n\pi + \theta),$$

and $$e^x \sin y = r \sin (2n\pi + \theta).$$

Hence $$e^x = r, \text{ and } y = 2n\pi + \theta.$$

Since x and r are both real, x is the ordinary algebraic Napierian logarithm of r, so that

$$x = \log_e r.$$

Hence a logarithm of $\alpha + \beta i$ is

$$\log_e r + i(2n\pi + \theta),$$

i.e. $$\log_e \sqrt{\alpha^2 + \beta^2} + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right).$$

Since n is any integer we see that there are therefore an infinite number of logarithms of $\alpha + \beta i$, and that these only differ by multiples of $2\pi i$.

329. With the extended definition of a logarithm given in Art. 327, it follows by the last article that the logarithm of any number is many-valued.

When this many-valuedness is taken into consideration we write the logarithm of $\alpha + \beta i$ as $\text{Log}(\alpha + \beta i)$.

Hence

$$\text{Log}(\alpha + \beta i) = \log_e \sqrt{\alpha^2 + \beta^2} + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right).$$

If we put n equal to zero in the value of $\text{Log}(\alpha + \beta i)$ the result is called the principal value of the logarithm and is denoted by $\log(\alpha + \beta i)$, so that

$$\log(\alpha + \beta i) = \log_e \sqrt{(\alpha^2 + \beta^2)} + i \tan^{-1}\frac{\beta}{\alpha},$$

and

$$\text{Log}(\alpha + \beta i) = 2n\pi i + \log(\alpha + \beta i).$$

This distinction between log and Log is to be hereafter assumed.

330. *Any positive quantity has one real logarithm and an infinite number of imaginary ones.*

In the result of the preceding article put β equal to zero, and we have

$$\text{Log}\, \alpha = 2n\pi i + \log_e \alpha.$$

We therefore observe that, with our extended definition of a logarithm, every real quantity α has a real logarithm (which is equal to $\log_e \alpha$ as ordinarily defined) and an infinite number of imaginary logarithms, which are obtained by adding any multiple of $2\pi i$ to its real logarithm.

This might have been directly deduced from equation (1) of Art. 326. For this is an equation of infinite degree and therefore it has an infinite number of roots, of which only one is real.

It will be noted that the principal value of the logarithm (according to our extended definition) of a real number is equal to its ordinary algebraic logarithm.

331. *Logarithm of a negative quantity.* In the result of Art. 329 put $\beta = 0$, and $\alpha = -x$, where x is a real positive quantity.

$$\therefore\ +\sqrt{\alpha^2+\beta^2} = +x, \text{ and } \tan^{-1}\frac{\beta}{\alpha}$$

[which is an angle such that its cosine is $\dfrac{-x}{+x}$ *i.e.* -1, and its sine zero (Art. 267)] is equal to π.

$$\therefore\ \text{Log}\,(-x) = 2n\pi i + \log_e x + \pi i,$$

and $$\log\,(-x) = \log_e x + \pi i.$$

Hence the principal value of the logarithm of a negative quantity $-x$ (with our extended definition) is equal to the ordinary algebraic logarithm of x added on to πi.

332. *Logarithm of a quantity which is wholly imaginary.* In the result of Art. 329 put $\alpha = 0$, and we have

$$\text{Log}(\beta i) = 2n\pi i + \log_e \beta + i\frac{\pi}{2}$$

$$= \log_e \beta + i\left(2n + \frac{1}{2}\right)\pi,$$

so that the logarithm of any quantity which is wholly imaginary, consists of two parts, the first of which is real, and the second of which is imaginary and many-valued.

As a particular case, put $\beta = 1$, and we have

$$\text{Log}(\sqrt{-1}) = i\left(2n + \frac{1}{2}\right)\pi,$$

so that the principal value of Log $(\sqrt{-1})$ is $\frac{\pi}{2}i$.

333. In the result of Art. 329 put

$$\alpha = \cos\theta \text{ and } \beta = \sin\theta.$$

$$\therefore \text{Log}(\cos\theta + i\sin\theta)$$

$$= \log_e 1 + i(2n\pi + \theta) = \theta i + 2n\pi i,$$

$$\therefore \text{Log}\, e^{\theta i} = \theta i + 2n\pi i.$$

The principal value of Log $e^{\theta i}$, *i.e.* log $e^{\theta i}$, is therefore that value of $(\theta + 2n\pi)i$ which is such that $\theta + 2n\pi$ lies between $-\pi$ and $+\pi$.

334. **Ex. 1.** *Resolve into its real and imaginary parts the expression*

$$\text{Log} \sin (x+yi).$$

Let $\text{Log} \sin (x+yi) = u+vi$, so that

$$e^{u+vi} = \sin (x+yi) = \sin x \cos yi + \cos x \sin yi$$

$$= \sin x \frac{e^y + e^{-y}}{2} + i \cos x \frac{e^y - e^{-y}}{2} \text{.....................(1).}$$

As in Art. 267 let the right-hand side of this expression equal

$$r [\cos (2n\pi + \theta) + i \sin (2n\pi + \theta)],$$

so that

$$r = +\sqrt{\sin^2 x \left(\frac{e^y + e^{-y}}{2}\right)^2 + \cos^2 x \left(\frac{e^y - e^{-y}}{2}\right)^2}$$

$$= \tfrac{1}{2} \sqrt{(e^{2y} + e^{-2y}) - 2 \cos 2x}$$

$$= \tfrac{1}{2} \sqrt{2 \cosh 2y - 2 \cos 2x} = \sqrt{\frac{\cosh 2y - \cos 2x}{2}},$$

and $$\theta = \tan^{-1} \left[\cot x \frac{e^y - e^{-y}}{e^y + e^{-y}}\right] = \tan^{-1} [\cot x \tanh y],$$

with the usual restriction of Art. 267.

We have then from (1)

$$e^u (\cos v + i \sin v) = r [\cos (2n\pi + \theta) + i \sin (2n\pi + \theta)].$$

Hence $e^u = r$, so that $u = \log_e r$,

and $$v = 2n\pi + \theta.$$

$$\therefore \text{ Log} \sin (x+yi) = u + vi = \log_e r + (2n\pi + \theta) i$$

$$= \frac{1}{2} \log_e \left[\frac{\cosh 2y - \cos 2x}{2}\right] + i [2n\pi + \tan^{-1} (\cot x \tanh y)].$$

By putting n equal to zero, we have the principal value of

$$\text{Log} \sin (x + iy).$$

Ex. 2. *Find the general value of Log* (-3).

Let $x + yi = \text{Log} (-3)$, so that

$$e^{x+yi} = -3.$$

Put $$-3 = r \{\cos (2n\pi + \theta) + i \sin (2n\pi + \theta)\},$$
as in Art. 267.

Then we have $r = 3$ and $\theta = \pi$.

Hence $3\{\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)\}$

$$=e^{x+yi}=e^x\,.\,e^{yi}=e^x\{\cos y+i\sin y\}.$$

Hence $e^x=3$, so that $x=\log_e 3$, and $y=2n\pi+\pi$.

$$\therefore\ \text{Log}(-3)=\log_e 3+(2n\pi+\pi)\,i.$$

The principal value, obtained by putting n equal to zero, is

$$\log_e 3+\pi i.$$

EXAMPLES. LVII.

Prove that

1. $\log(\cos\theta+i\sin\theta)=i\theta$, if $-\pi<\theta\not>\pi$.

2. $\log(-1)=\pi i$.

3. $\log(-i)=-\dfrac{\pi}{2}i$.

4. $\log(1+\cos 2\theta+i\sin 2\theta)=\log_e(2\cos\theta)+i\theta$, if $-\pi<\theta\not>\pi$.

5. $\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}i\right)=i\tan^{-1}\sin hx$.

6. $\log\cos(x+yi)=\dfrac{1}{2}\log_e\left(\dfrac{\cosh 2y+\cos 2x}{2}\right)+i\tan^{-1}(\tan x\tanh y)$.

7. $\log\dfrac{\sin(x+yi)}{\sin(x-yi)}=2i\tan^{-1}(\cot x\tanh y)$.

8. $\log\dfrac{\cos(x-yi)}{\cos(x+yi)}=2i\tan^{-1}(\tan x\tanh y)$.

9. $i\log\dfrac{x-i}{x+i}=\pi-2\tan^{-1}x$.

10. $\log(1+i\tan\alpha)=\log_e\sec\alpha+\alpha i$, where α is a positive acute angle.

11. $\log\left(\dfrac{1}{1-e^{\theta i}}\right)=\log_e\left(\dfrac{1}{2}\text{cosec}\dfrac{\theta}{2}\right)+i\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)$.

12. $\log\dfrac{a+bi}{a-bi}=2i\tan^{-1}\dfrac{b}{a}$.

13. $\text{Log}(-5)=\log_e 5+(2n\pi+\pi)\,i$.

14. $\text{Log}(1+i)=\log_e 2+i\left(2n\pi+\dfrac{\pi}{4}\right)$.

15. Find the value of $\log\log\sin(x+yi)$.

335. Definition of a^x when a and x are any quantities, complex or real. When a and x are real quantities we know that

$$a^x = e^{x \log_e a}. \quad \text{(Art. 253.)}$$

When a and x are complex the ordinary algebraic definition of a^x no longer holds.

Let us so define it that

$$a^x = e^{x \operatorname{Log} a},$$

for all values of x and a, whether real or complex.

Now, by Art. 329, Log a is many-valued and complex when a is complex. Hence a^x is many-valued and complex, so that

$$a^x = e^{x \operatorname{Log} a} = e^{x\,(2n\pi i + \log a)}.$$

From Art. 305 it now follows that $a^x \times a^y = a^{x+y}$, so that a^x obeys the ordinary algebraic law of indices.

The value of a^x obtained by putting n equal to zero is called its principal value.

Hence the principal value of a^x

$$= e^{x \log a}$$

$$= 1 + x \log a + \frac{x^2}{\underline{|2}} (\log a)^2 + \ldots \quad \text{(by Art. 304.)}$$

336. It may now be shewn that, if y be complex,

$$\log(1+y) = y - \frac{1}{2} y^2 + \frac{1}{3} y^3 - \frac{1}{4} y^4 \ldots\ldots\ldots$$

The proof is similar to the proof when y is real. (Art. 256.)

It is, in general, necessary that the modulus of y be < 1; otherwise the Binomial Theorem does not hold for complex quantities. (Art. 273.)

If the modulus of y be equal to unity, so that y may be put equal to $\cos\phi + i\sin\phi$, the expansion can be shewn to be still true, except in the cases when ϕ is equal to an odd multiple of π.

Since $\qquad \text{Log}(1+y) = 2n\pi i + \log(1+y),$
we have

$$\text{Log}(1+y) = 2n\pi i + y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \dots\dots$$

337. *To separate into its real and imaginary parts the expression* $(\alpha+\beta i)^{x+yi}$.

Let $\qquad \alpha + \beta i = r(\cos\theta + i\sin\theta),$

so that as in Art. 265,

$$r = \sqrt{\alpha^2+\beta^2}, \text{ and } \theta = \tan^{-1}\frac{\beta}{\alpha}.$$

Then, by definition,

$$(\alpha+\beta i)^{x+yi} = e^{(x+yi)\,\text{Log}\,(\alpha+\beta i)}$$

$$= e^{\{x+yi\}\,\{\log(\alpha+\beta i)+2m\pi i\}}$$

$$= e^{\{x+yi\}\,\{\log r+(\theta+2m\pi)i\}}$$

$$= e^{\{x\log r - y(\theta+2m\pi)\}+i\{y\log r+x(\theta+2m\pi)\}}$$

$$= e^{x\log r}\,.\,e^{-y(\theta+2m\pi)}\,.\,e^{i\{y\log r+x(\theta+2m\pi)\}}$$

$$= r^x\,.\,e^{-y(\theta+2m\pi)}\,[\cos\{y\log r + x(\theta+2m\pi)\}$$
$$+ i\sin\{y\log r + x(\theta+2m\pi)\}].$$

If we put m equal to zero, we obtain the principal value of the given quantity, viz.

$$r^x e^{-y\theta}[\cos(y\log r + x\theta) + i\sin(y\log r + x\theta)].$$

338. **Ex. 1.** *Find the general value of* $[\sqrt{-1}]^{\sqrt{-1}}$.

We have $\qquad [\sqrt{-1}]^{\sqrt{-1}} = e^{\sqrt{-1}\,\text{Log}\sqrt{-1}}.$

But $\quad \text{Log}\sqrt{-1} = \text{Log}\left[\cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)\right]$

$$= \text{Log}\, e^{\left(2n\pi + \frac{\pi}{2}\right)i} = \left(2n\pi + \frac{\pi}{2}\right)i.$$

$$\therefore\ [\sqrt{-1}]^{\sqrt{-1}} = e^{\left(2n\pi + \frac{\pi}{2}\right)i^2} = e^{-\left(2n\pi + \frac{\pi}{2}\right)},$$

where n has any integral value.

The principal value of $[\sqrt{-1}]^{\sqrt{-1}}$ is $e^{-\frac{\pi}{2}}$.

Ex. 2. *Find the general value of* $Log_2(-3)$.

Let $\text{Log}_2(-3) = x + yi$, so that $2^{x+yi} = -3$,

i.e. $\quad e^{(x+yi)\text{Log}\,2} = 3\{\cos(2m\pi + \pi) + i\sin(2m\pi + \pi)\}$ (Art. 265).

But $\quad \text{Log}\,2 = 2n\pi i + \log_e 2$, and $3 = e^{\log_e 3}$,

$$\therefore\ e^{(x+yi)(2n\pi i + \log_e 2)} = e^{\log_e 3} \cdot e^{(2m\pi + \pi)i}.$$

$$\therefore\ (x+yi)(2n\pi i + \log_e 2) = \log_e 3 + (2m\pi + \pi)i.$$

Equating real and imaginary parts, we have

$$x\log_e 2 - 2n\pi y = \log_e 3,$$

and

$$x \cdot 2n\pi + y\log_e 2 = 2m\pi + \pi.$$

Solving, we have

$$x = \frac{\log_e 3\log_e 2 + (2m\pi + \pi) \cdot 2n\pi}{(\log_e 2)^2 + 4n^2\pi^2},$$

and

$$y = \frac{(2m\pi + \pi)\log_e 2 - 2n\pi\log_e 3}{(\log_e 2)^2 + 4n^2\pi^2}.$$

Hence $\quad \text{Log}_2(-3)$

$$= \frac{\{\log_e 3\log_e 2 + 2n(2m+1)\pi^2\} + i\pi\{(2m+1)\log_e 2 - 2n\log_e 3\}}{(\log_e 2)^2 + 4n^2\pi^2}.$$

If $m = n = 0$, the principal value is obtained, viz.

$$\frac{\log_e 3 + \pi i}{\log_e 2}.$$

339. It could now be shewn that the general values of the logarithms of complex quantities satisfy the ordinary laws of logarithms, viz.

$$\operatorname{Log} mn = \operatorname{Log} m + \operatorname{Log} n,$$

and
$$\operatorname{Log} \frac{m}{n} = \operatorname{Log} m - \operatorname{Log} n.$$

It could also be shewn that $\operatorname{Log} m^n = n \operatorname{Log} m + 2p\pi i$, where p is some integer or zero. The proof is left as an exercise for the student.

EXAMPLES. LVIII.

Prove that

1. $a^i = e^{-2m\pi} \{\cos(\log a) + i \sin(\log a)\}$.

2. $i^a = \cos\left\{\left(2m+\frac{1}{2}\right)\pi a\right\} + i \sin\left\{\left(2m+\frac{1}{2}\right)\pi a\right\}$.

3. $i^{i^i} = \cos\theta + i\sin\theta$, where

$$\theta = \left(2m+\frac{1}{2}\right)\pi \,.\, e^{-\left(2n\pi+\frac{\pi}{2}\right)}.$$

4. If $i^{i^{i \ldots \text{ad inf.}}} = A + Bi$, principal values only being considered, prove that

$$\tan\frac{\pi A}{2} = \frac{B}{A}, \text{ and } A^2 + B^2 = e^{-\pi B}.$$

5. If $i^{\alpha+\beta i} = \alpha + \beta i$, prove that

$$\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}.$$

6. If $\dfrac{(1+i)^{p+qi}}{(1-i)^{p-qi}} = \alpha + \beta i$, prove that one value of $\tan^{-1}\dfrac{\beta}{\alpha}$ is

$$\frac{1}{2}p\pi + q\log_e 2.$$

7. If $(a+bi)^p = m^{x+yi}$, prove that one of the values of $\dfrac{y}{x}$ is

$$\frac{2\tan^{-1}\frac{b}{a}}{\log_e(a^2+b^2)}.$$

8. If $a^{\alpha+\beta i}=(x+yi)^{p+qi}$, principal values only being considered, prove that

$$\alpha=\frac{1}{2}p\log_a(x^2+y^2)-q\tan^{-1}\frac{y}{x}\log_a e,$$

and that
$$\log_a(x^2+y^2)=2\frac{\alpha p+\beta q}{p^2+q^2}.$$

9. Prove that the real part of the principal value of $(i)^{\log(1+i)}$ is

$$e^{-\frac{\pi^2}{8}}\cos\left(\frac{\pi}{4}\log 2\right).$$

10. Prove that the principal value of $(a+ib)^{\alpha+i\beta}$ is wholly real or wholly imaginary according as

$$\frac{1}{2}\beta\log(a^2+b^2)+\alpha\tan^{-1}\frac{b}{a}$$

is an even or an odd multiple of $\frac{\pi}{2}$.

11. Prove that the general value of

$$(1+i\tan\alpha)^{-i}$$

is
$$e^{\alpha+2m\pi}[\cos\{\log\cos\alpha\}+i\sin\{\log\cos\alpha\}].$$

12. If
$$\left(\frac{a+x+iy}{a-x-iy}\right)^{\lambda+\mu i}=X+iY,$$

prove that one of the values of

$$\tan^{-1}\frac{Y}{X}\text{ is }\lambda\tan^{-1}\left(\frac{2ay}{a^2-x^2-y^2}\right)+\frac{\mu}{2}\log\frac{(a+x)^2+y^2}{(a-x)^2+y^2}.$$

13. Prove that
$$\text{Log}_{\sqrt{-1}}(\sqrt{-1})=\frac{4n+1}{4m+1},$$

where m and n are any integers.

14. Prove that the general value of $\text{Log}_4(-2)$ is

$$\frac{(\log 2)^2+m\,.\,(2n+1)\,\pi^2}{2\,(\log 2)^2+2m^2\pi^2}+i\,\frac{(2n+1-m)\,\pi\log 2}{2\,(\log 2)^2+2m^2\pi^2}.$$

Explain the fallacies in the following arguments:

15. For all integral values of n we have

$$e^{2n\pi i}=\cos 2n\pi+i\sin 2n\pi=1,$$

so that
$$e^{2\pi i}=e^{4\pi i}=e^{6\pi i}=\ldots\ldots$$

Raise all these quantities to the power $\sqrt{-1}$; thus

$$e^{-2\pi}=e^{-4\pi}=e^{-6\pi}=\ldots\ldots$$

$$\therefore\ 2\pi=4\pi=6\pi=\ldots\ldots$$

16. For all values of θ we have

$$\cos(\theta-\pi)+i\sin(\theta-\pi)=\cos(\theta+\pi)+i\sin(\theta+\pi),$$

so that $$e^{i(\theta-\pi)}=e^{i(\theta-\pi)}.$$

Hence $$\theta-\pi=\theta+\pi,\ \textit{i.e.}\ \pi=0.$$

17. If θ and ϕ be the principal values of the amplitudes of two complex numbers x and y, prove that

$$\log xy=\log x+\log y+2n\pi i,$$

where n is -1, 0, or $+1$ according as $\theta+\phi$ is $>\pi$, greater than $-\pi$ and not greater than π, and not greater than $-\pi$, respectively.

CHAPTER XXVII.

GREGORY'S SERIES. CALCULATION OF THE VALUE OF π.

340. Gregory's Series. *To prove that, if θ be not less than* $-\frac{\pi}{4}$ *and be not greater than* $+\frac{\pi}{4}$, *then*

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \dots\dots$$

We have $$\cos\theta + i\sin\theta = e^{\theta i},$$

and $$\cos\theta - i\sin\theta = e^{-\theta i}.$$

$$\therefore \frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta} = \frac{e^{\theta i}}{e^{-\theta i}} = e^{2\theta i} = e^{(2\theta + 2n\pi)i},$$

where n is an integer.

$$\therefore e^{(2\theta+2n\pi)i} = \frac{1 + i\tan\theta}{1 - i\tan\theta}.$$

$$\therefore (2\theta + 2n\pi)\, i = \log(1 + i\tan\theta) - \log(1 - i\tan\theta).$$

Now $\log(1 + i\tan\theta)$ may be expanded provided that $\tan\theta$ be numerically less than unity.

Hence $(2\theta + 2n\pi) i$

$$= i \tan\theta - \frac{1}{2} i^2 \tan^2\theta + \frac{1}{3} i^3 \tan^3\theta - \ldots\ldots$$

$$- \left[- i \tan\theta - \frac{1}{2} i^2 \tan^2\theta - \frac{1}{3} i^3 \tan^3\theta - \ldots\ldots \right]$$

$$= 2 \left[i \tan\theta + \frac{1}{3} i^3 \tan^3\theta + \frac{1}{5} i^5 \tan^5\theta + \ldots\ldots \right].$$

$$\therefore\ \boldsymbol{\theta + n\pi = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \ldots\ldots} \quad \ldots(1),$$

where n is an integer.

The right-hand member of (1) may be written in the two forms

$$\tan\theta \left(1 - \frac{1}{3}\tan^2\theta\right) + \frac{1}{5}\tan^5\theta \left(1 - \frac{5}{7}\tan^2\theta\right) + \ldots\ldots \ (2),$$

and

$$\tan\theta - \frac{1}{3}\tan^3\theta \left(1 - \frac{3}{5}\tan^2\theta\right) - \frac{1}{7}\tan^7\theta \left(1 - \frac{7}{9}\tan^2\theta\right) + \ldots\ldots(3).$$

If θ lie between 0 and $\frac{\pi}{4}$, so that $\tan\theta$ is positive and less than 1, then from (2) we see that the sum of the series is positive, and from (3) that it is less than $\tan\theta$ and therefore less than unity.

In this case, therefore, n must be zero and we have

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots\ldots \text{ ad inf.} \ \ldots(4).$$

If we change the sign of θ, then every term in (4) changes its sign, so that the series must also be true for values of θ between 0 and $-\frac{\pi}{4}$.

341. When $\tan\theta$ is equal to unity, we have

$$\log(1+i\tan\theta)=\log(1+i)=\log\left[1+\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\right],$$

so that the expansion of $\log(1+i\tan\theta)$ is by Art. 336, still true.

Similarly for $\quad \log(1-i\tan\theta)$.

Hence Gregory's series is true for the extreme values

$$\theta=\frac{\pi}{4} \text{ and } \theta=-\frac{\pi}{4}.$$

342. If θ lie between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, or between

$$\frac{5\pi}{4} \text{ and } \frac{7\pi}{4}, \ldots\ldots$$

or, generally, between

$$n\pi+\frac{\pi}{4} \text{ and } n\pi+\frac{3\pi}{4},$$

$\tan\theta$ is greater than unity; in these cases the expansion of $\log(1+i\tan\theta)$ does not hold, and there is no such expansion as equation (1) of Art. 340.

343. If θ lie between $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$, let it equal $\pi+\alpha$, so that α lies between

$$-\frac{\pi}{4} \text{ and } +\frac{\pi}{4},$$

and hence, by equation (4) of Art. 340,

$$\alpha=\tan\alpha-\frac{1}{3}\tan^3\alpha+\frac{1}{5}\tan^5\alpha-\ldots\ldots$$

But $\quad \tan\alpha=\tan(\theta-\pi)=\tan\theta,$

so that this equation is

$$\theta-\pi=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\ldots\ldots$$

Comparing this with equation (1) we see that n equals -1 when θ lies between

$$\frac{3\pi}{4} \text{ and } \frac{5\pi}{4}.$$

If θ lie between $\frac{7\pi}{4}$ and $\frac{9\pi}{4}$ it may be similarly shewn, by putting θ equal to $2\pi + \alpha$, that in this case n is equal to -2.

In general if θ lie between $p\pi - \frac{\pi}{4}$ and $p\pi + \frac{\pi}{4}$, the equation (1) of Art. 340 is

$$\theta - p\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots\ldots$$

344. The series (2) of Art. 340 may be slightly transformed by writing $\tan\theta = x$, so that x must be not less than -1 and not greater than 1.

It then becomes

$$\mathbf{\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \ldots\ldots \text{ ad inf.},}$$

where $\tan^{-1} x$ is that value which lies between

$$-\frac{\pi}{4} \text{ and } +\frac{\pi}{4}.$$

345. The results of the preceding articles may be more directly connected with the preceding chapter in the following manner.

If $\tan\theta$ be numerically less than, or numerically equal to, unity we have, by Art. 336,

$$\text{Log}(1 + i\tan\theta) = 2p\pi i + i\tan\theta - \frac{1}{2}i^2\tan^2\theta + \frac{1}{3}i^3\tan^3\theta \ldots,$$

and $$\text{Log}(1 - i\tan\theta) = 2q\pi i - i\tan\theta - \frac{1}{2}i^2\tan^2\theta - \frac{1}{3}i^3\tan^3\theta \ldots,$$

where p and q are both integers.

Hence, by subtraction,

$$\text{Log}\frac{1+i\tan\theta}{1-i\tan\theta}=2(p-q)\pi i+2i\left[\tan\theta-\frac{1}{3}\tan^3\theta+\dots\right]\dots\dots(1).$$

But
$$\text{Log}\,\frac{1+i\tan\theta}{1-i\tan\theta}=\text{Log}\left[\frac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta}\right]$$
$$=\text{Log}\,(\cos\theta+i\sin\theta)^2=\text{Log}\,[\cos 2\theta+i\sin 2\theta]$$
$$=2r\pi i+i\,.\,2\theta\,\dots\dots\dots\dots\dots\dots\,(2),$$

where r is an integer.

Some one of the values of the right hand of (1) must therefore be equivalent to some one of the values on the right hand of (2).

Hence, by equating and putting $r-p+q=n$, we must have, for *some* integral value of n, the relation

$$\theta+n\pi=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\dots.$$

If we consider principal values only of the logarithms then in (1) both p and q are zero and $\tan\theta$ is numerically less than unity.

Also, by Art. 333, the value of r in (2) is zero and θ lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

Combining these two statements we see that p, q, and r are zero, and therefore n is zero, when θ lies between $-\frac{\pi}{4}$ and $+\frac{\pi}{4}$.

346. Value of π. One of the chief uses of Gregory's series is its application to find the value of π.

In Art. 344 put $x=1$, and we have

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots\dots$$
$$=1-\left(\frac{1}{3}-\frac{1}{5}\right)-\left(\frac{1}{7}-\frac{1}{9}\right)-\left(\frac{1}{11}-\frac{1}{13}\right)\dots\dots$$
$$=1-2\left[\frac{1}{3\,.\,5}+\frac{1}{7\,.\,9}+\frac{1}{11\,.\,13}+\dots\dots\right].$$

This series may be used to calculate π; its defect however is that the successive terms do not rapidly become small, so that a very large number of terms would have to be taken to obtain the value of π correct to any great degree of accuracy.

For this reason other series have been sought for.

347. Euler's Series. We can easily prove that

$$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}.$$

In Art. 344 put in succession x equal to

$$\frac{1}{2} \text{ and } \frac{1}{3},$$

and we have

$$\frac{\pi}{4}=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$$

$$=\frac{1}{2}-\frac{1}{3}\cdot\frac{1}{2^3}+\frac{1}{5}\cdot\frac{1}{2^5}-\frac{1}{7}\cdot\frac{1}{2^7}+\cdots\cdots$$

$$+\frac{1}{3}-\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}-\frac{1}{7}\cdot\frac{1}{3^7}+\cdots\cdots.$$

This series converges more quickly than the preceding series; but more than eleven terms of the series for $\tan^{-1}\frac{1}{2}$ would have to be taken to give π correct to 7 places of decimals.

348. Machin's Series. A more convergent series than the preceding is Machin's, which is derived from the expression

$$4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\frac{\pi}{4} \qquad \text{(Art. 240, Ex. 4).}$$

By substituting in succession $\frac{1}{5}$ and $\frac{1}{239}$ for x in Art. 344, we have

$$\frac{\pi}{4}=4\left[\frac{1}{5}-\frac{1}{3}\cdot\frac{1}{5^3}+\frac{1}{5}\cdot\frac{1}{5^5}-\frac{1}{7}\cdot\frac{1}{5^7}\cdots\cdots\right]$$
$$-\left[\frac{1}{239}-\frac{1}{3}\cdot\frac{1}{239^3}+\frac{1}{5}\cdot\frac{1}{239^5}-\cdots\cdots\right].$$
$$\therefore\ \pi=16\left[\frac{2}{10}-\frac{1}{3}\frac{2^3}{10^3}+\frac{1}{5}\frac{2^5}{10^5}-\frac{1}{7}\frac{2^7}{10^7}\cdots\cdots\right]$$
$$-4\left[\frac{1}{239}-\frac{1}{3}\frac{1}{239^3}+\frac{1}{5}\frac{1}{239^5}\cdots\cdots\right].$$

Now
$$16\times\frac{2}{10}=3{\cdot}2$$
$$16\times\frac{1}{5}\cdot\frac{2^5}{10^5}=\ {\cdot}001024$$
$$16\times\frac{1}{9}\ \frac{2^9}{10^9}=\ {\cdot}0000009102$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$4\times\frac{1}{3}\ \frac{1}{239^3}=\ {\cdot}0000000977$$

$$3{\cdot}2010250079$$

Also
$$16\times\frac{1}{3}\ \frac{2^3}{10^3}={\cdot}0426666666\ldots$$
$$16\times\frac{1}{7}\cdot\frac{2^7}{10^7}={\cdot}0000292571\ldots$$
$$16\times\frac{1}{11}\cdot\frac{2^{11}}{10^{11}}={\cdot}0000000298\ldots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$4\times\frac{1}{239}={\cdot}0167364017\ldots$$

$${\cdot}0594323552$$

Hence

$$
\begin{array}{r}
3{\cdot}2010250079 \\
-\ {\cdot}0594323552 \\
\hline
\pi = 3{\cdot}14159265/27
\end{array}
$$

This is the value of π correct to 8 places of decimals.

By taking the first series to 21 terms and the second series to three terms we should get π correct to sixteen places.

349. Rutherford's Series. A further simplification of Machin's formula is the expression

$$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{\pi}{4}.$$

For we have

$$\tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99} = \tan^{-1}\frac{\frac{1}{70} - \frac{1}{99}}{1 + \frac{1}{70}\cdot\frac{1}{99}} = \tan^{-1}\frac{29}{6931}$$

$$= \tan^{-1}\frac{1}{239}.$$

EXAMPLES. LIX.

Assuming that

$$\theta - n\pi = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots,$$

write down the value of n when θ lies between

1. $\frac{11\pi}{4}$ and $\frac{13\pi}{4}$.

2. $\frac{7\pi}{4}$ and $\frac{9\pi}{4}$.

3. $\frac{19\pi}{4}$ and $\frac{21\pi}{4}$.

4. $-\frac{3\pi}{4}$ and $-\frac{5\pi}{4}$.

5. $-\frac{11\pi}{4}$ and $-\frac{13\pi}{4}$.

6. Prove that

$$\pi = 2\sqrt{3}\left\{1 - \frac{1}{3^2} + \frac{1}{5\,.\,3^2} - \frac{1}{7\,.\,3^3} + \ldots\right\}$$

7. Prove that

$$\frac{\pi}{4} = \frac{2}{3} + \frac{1}{7} - \frac{1}{3}\left(\frac{2}{3^3} + \frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5} + \frac{1}{7^5}\right) - \ldots.$$

8. If x be $< \sqrt{2} - 1$, prove that

$$2\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 \ldots\ldots \text{ ad inf.}\right)$$

$$= \frac{2x}{1-x^2} - \frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3 + \frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5 - \ldots\ldots \text{ ad inf.}$$

Find the value of π to three places of decimals

9. By using Euler's Series.

10. By using Machin's Series.

11. By using Rutherford's Series.

12. To the second order of small quantities, prove that

$$\frac{1}{2}\sqrt{1 + \sin\theta}\,\log(1-\theta) + \tan^{-1}\theta\,\sin\left(\frac{\pi}{3} + \theta\right) = \frac{\sqrt{3}-1}{2}\theta.$$

13. When both θ and $\tan^{-1}(\sec\theta)$ lie between 0 and $\frac{\pi}{2}$, prove that

$$\tan^{-1}(\sec\theta) = \frac{\pi}{4} + \tan^2\frac{\theta}{2} - \frac{1}{3}\tan^6\frac{\theta}{2} + \frac{1}{5}\tan^{10}\frac{\theta}{2} - \ldots.$$

CHAPTER XXVIII.

SUMMATION OF SERIES. EXPANSIONS IN SERIES.

350. We shall now apply the results of the preceding chapters to the summation of some trigonometrical series.

The chief series may be divided into four classes;

(1) Those depending for their summation on a Geometrical Progression ultimately,

(2) Those depending ultimately on the Binomial Theorem,

(3) Those depending ultimately on the Exponential Theorem, including, as sub-cases, the Sine and Cosine Series,

and (4) Those depending ultimately on the Logarithmic Series and, as a sub-case, Gregory's Series.

351. In Arts. 352—355 we shall sum one example of each of these classes. It will generally be found more convenient in summing one of these series involving *sines* of multiple angles (such as $\sin\alpha$, $\sin 2\alpha$, $\sin 3\alpha \dots$) to also sum at the same time the companion series involving the *cosines* of the same multiple angles

$$(i.e.\ \cos\alpha,\ \cos 2\alpha,\ \cos 3\alpha \dots).$$

The method will be best seen by a careful study of the following four articles.

352. Ex. *Sum to n terms, and to infinity, the series*

$$1 + c \cos \alpha + c^2 \cos 2\alpha + \ldots\ldots,$$

where c is less than unity.

Let

$$C \equiv 1 + c\cos\alpha + c^2\cos 2\alpha + \ldots + c^{n-1}\cos(n-1)\alpha \quad \ldots\ldots(1),$$

and

$$S \equiv c\sin\alpha + c^2\sin 2\alpha + \ldots\ldots + c^{n-1}\sin(n-1)\alpha \quad \ldots\ldots\ldots(2).$$

Multiplying (2) by i and adding to (1), we have

$$C + Si = 1 + c(\cos\alpha + i\sin\alpha) + c^2(\cos 2\alpha + i\sin 2\alpha) + \ldots\ldots$$

$$= 1 + ce^{\alpha i} + c^2 e^{2\alpha i} + \ldots + c^{n-1}e^{(n-1)\alpha i} \quad \text{(Art. 308)}$$

$$= \frac{1 - c^n e^{n\alpha i}}{1 - ce^{\alpha i}}, \text{ by summing the G.P.,}$$

$$= \frac{\{1 - c^n(\cos n\alpha + i\sin n\alpha)\}}{1 - c\cos\alpha - ic\sin\alpha} \quad \text{(Art. 308)}$$

$$= \frac{\{1 - c^n\cos n\alpha - ic^n\sin n\alpha\}\{1 - c\cos\alpha + ic\sin\alpha\}}{(1 - c\cos\alpha)^2 + c^2\sin^2\alpha}$$

$$= \frac{\{(1 - c\cos\alpha)(1 - c^n\cos n\alpha) + c^{n+1}\sin n\alpha\sin\alpha\} + i\{c\sin\alpha(1 - c^n\cos n\alpha) - c^n\sin n\alpha(1 - c\cos\alpha)\}}{1 - 2c\cos\alpha + c^2}.$$

Hence, by equating real and imaginary parts, we have

$$C = \frac{(1 - c\cos\alpha)(1 - c^n\cos n\alpha) + c^{n+1}\sin n\alpha\sin\alpha}{1 - 2c\cos\alpha + c^2},$$

and $$S = \frac{c\sin\alpha(1 - c^n\cos n\alpha) - c^n\sin n\alpha(1 - c\cos\alpha)}{1 - 2c\cos\alpha + c^2},$$

i.e. $$C = \frac{1 - c\cos\alpha - c^n\cos n\alpha + c^{n+1}\cos(n-1)\alpha}{1 - 2c\cos\alpha + c^2},$$

and $$S = \frac{c\sin\alpha - c^n\sin n\alpha + c^{n+1}\sin(n-1)\alpha}{1 - 2c\cos\alpha + c^2}.$$

The sum to infinity is obtained by omitting the terms containing c^n and c^{n+1}, which become indefinitely small when n is very great.

Hence $$C_\infty = \frac{1 - c\cos\alpha}{1 - 2c\cos\alpha + c^2},$$

and $$S_\infty = \frac{c\sin\alpha}{1 - 2c\cos\alpha + c^2}.$$

From the results for C and S it is now clear that the above series might have been summed, without the use of imaginary quantities, by multiplying both sides of (1) and (2) by the quantity $1 - 2c\cos\alpha + c^2$. The coefficients of c^2, c^3......c^{n-1} would then be found to vanish and the values of C and S be easily obtained.

353. Ex. *Sum the series*

$$\frac{1}{2}\sin\alpha + \frac{1.3}{2.4}\sin 2\alpha + \frac{1.3.5}{2.4.6}\sin 3\alpha + \ldots \text{ ad inf.}$$

Let $$S = \frac{1}{2}\sin\alpha + \frac{1.3}{2.4}\sin 2\alpha + \frac{1.3.5}{2.4.6}\sin 3\alpha + \ldots,$$

and $$C = 1 + \frac{1}{2}\cos\alpha + \frac{1.3}{2.4}\cos 2\alpha + \frac{1.3.5}{2.4.6}\cos 3\alpha + \ldots.$$

Hence, multiplying the first by i and adding to the second, we have

$$C + Si = 1 + \frac{1}{2}e^{\alpha i} + \frac{1.3}{2.4}e^{2\alpha i} + \frac{1.3.5}{2.4.6}e^{3\alpha i} + \ldots$$

$$= (1 - e^{\alpha i})^{-\frac{1}{2}}, \text{ if } \alpha \neq 2n\pi,$$

by the Binomial Theorem. (Art. 273.)

$$\therefore C + Si = \{1 - \cos\alpha - i\sin\alpha\}^{-\frac{1}{2}}$$

$$= \left\{2\sin\frac{\alpha}{2}\left(\sin\frac{\alpha}{2} - i\cos\frac{\alpha}{2}\right)\right\}^{-\frac{1}{2}}$$

$$= \left\{2\sin\frac{\alpha}{2}\right\}^{-\frac{1}{2}}\left\{\cos\left(\frac{\alpha}{2} - \frac{\pi}{2}\right) + i\sin\left(\frac{\alpha}{2} - \frac{\pi}{2}\right)\right\}^{-\frac{1}{2}}$$

$$= \left\{2\sin\frac{\alpha}{2}\right\}^{-\frac{1}{2}}\left\{\cos\left(\frac{\pi - \alpha}{4}\right) + i\sin\frac{\pi - \alpha}{4}\right\}.$$

Hence, by equating real and imaginary parts, we have

$$C=\left\{2\sin\frac{\alpha}{2}\right\}^{-\frac{1}{2}}\cos\frac{\pi-\alpha}{4},$$

and
$$S=\left\{2\sin\frac{\alpha}{2}\right\}^{-\frac{1}{2}}\sin\frac{\pi-\alpha}{4}.$$

If $\alpha=2n\pi$, clearly $S=0$ and $C=\infty$.

EXAMPLES. LX.

Sum the series

1. $\sin\alpha+\frac{1}{2}\sin 2\alpha+\frac{1}{2^2}\sin 3\alpha+\ldots\ldots$ ad inf.

2. $\cos\alpha\,.\cos\alpha+\cos^2\alpha\cos 2\alpha+\cos^3\alpha\cos 3\alpha\ldots\ldots$ ad inf.

3. $\sin\alpha\,.\sin\alpha+\sin^2\alpha\sin 2\alpha+\sin^3\alpha\sin 3\alpha+\ldots\ldots$ ad inf., where $\alpha \neq \pm\frac{\pi}{2}$.

4. $\sin\alpha\,.\cos\alpha+\sin^2\alpha\,.\cos 2\alpha+\sin^3\alpha\,.\cos 3\alpha+\ldots\ldots$ ad inf., where $\alpha \neq \pm\frac{\pi}{2}$.

5. $\sin\alpha+c\sin(\alpha+\beta)+c^2\sin(\alpha+2\beta)+\ldots\ldots$ to n terms and ad inf.

6. $1+c\cosh\alpha+c^2\cosh 2\alpha+\ldots\ldots+c^{n-1}\cosh(n-1)\alpha$.

7. $c\sinh\alpha+c^2\sinh 2\alpha+\ldots\ldots+\ldots\ldots$ ad inf.

8. $1-2\cos\alpha+3\cos 2\alpha-4\cos 3\alpha+\ldots\ldots$ to n terms.

9. $3\sin\alpha+5\sin 2\alpha+7\sin 3\alpha+\ldots\ldots$ to n terms.

10. When $\alpha=\frac{\pi}{2}$, find what are the values of the series in Exs. 3 and 4.

11. $\sin\alpha+n\sin(\alpha+\beta)+\frac{n(n-1)}{1.2}\sin(\alpha+2\beta)+\ldots\ldots$ to $(n+1)$ terms, n being a positive integer.

12. $\sin\alpha+\frac{1}{2}\sin 3\alpha+\frac{1.3}{2.4}\sin 5\alpha+\ldots\ldots$ ad inf.

13. $\cos^n\alpha-n\cos^{n-1}\alpha\cos\alpha+\frac{n(n-1)}{1.2}\cos^{n-2}\alpha\cos 2\alpha\ldots$ to $(n+1)$ terms, n being a positive integer.

14. $n\sin\alpha+\frac{n(n+1)}{1.2}\sin 2\alpha+\frac{n(n+1)(n+2)}{1.2.3}\sin 3\alpha+\ldots\ldots$ ad inf.

15. $1+\frac{1}{2}\cos 2\theta-\frac{1}{2\,.\,4}\cos 4\theta+\frac{1\,.\,3}{2\,.\,4\,.\,6}\cos 6\theta-\ldots\ldots$ ad inf.

16. $\sinh u+n\sinh 2u+\frac{n(n-1)}{1\,.\,2}\sinh 3u+\ldots\ldots$ to n terms, where n is a positive integer.

354. Ex. *Sum the series*

$$1+\frac{c^2\cos 2\theta}{\underline{|2}}+\frac{c^4\cos 4\theta}{\underline{|4}}+\ldots\ldots \textit{ad inf.}$$

Let $$C\equiv 1+\frac{c^2\cos 2\theta}{\underline{|2}}+\frac{c^4\cos 4\theta}{\underline{|4}}+\ldots\ldots \text{ad inf.} \ldots(1),$$

and $$S\equiv \frac{c^2\sin 2\theta}{\underline{|2}}+\frac{c^4\sin 4\theta}{\underline{|4}}+\ldots\ldots \text{ad inf.} \ldots(2).$$

Hence

$$C+Si=1+\frac{c^2e^{2\theta i}}{\underline{|2}}+\frac{c^4e^{4\theta i}}{\underline{|4}}+\ldots\ldots \text{ad inf.}$$

$$=1+\frac{y^2}{\underline{|2}}+\frac{y^4}{\underline{|4}}+\frac{y^6}{\underline{|6}}+\ldots\ldots,$$

where $y=ce^{\theta i}=c(\cos\theta+i\sin\theta)$.

$$\therefore\ C+Si=\frac{e^y+e^{-y}}{2}$$

$$=\frac{1}{2}e^{c\cos\theta+ic\sin\theta}+\frac{1}{2}e^{-c\cos\theta-ic\sin\theta}\ \ldots\ldots\ldots\ldots(3)$$

$$=\frac{1}{2}e^{c\cos\theta}[\cos(c\sin\theta)+i\sin(c\sin\theta)]$$

$$+\frac{1}{2}e^{-c\cos\theta}[\cos(c\sin\theta)-i\sin(c\sin\theta)].\quad \text{(Art. 307.)}$$

By equating real and imaginary parts we therefore have

$$C=\frac{1}{2}\cos(c\sin\theta)[e^{c\cos\theta}+e^{-c\cos\theta}]$$

$$=\cos(c\sin\theta)\cosh(c\cos\theta),$$

and $$S = \frac{1}{2}\sin(c\sin\theta)\left[e^{c\cos\theta} - e^{-c\cos\theta}\right]$$
$$= \sin(c\sin\theta)\sinh(c\cos\theta).$$

Aliter. From (3) we have

$$C + Si = \frac{1}{2}e^{(c\sin\theta - ic\cos\theta)i} + \frac{1}{2}e^{-(c\sin\theta - ic\cos\theta)i}$$
$$= \cos(c\sin\theta - ic\cos\theta) \quad \text{(Art. 307)}$$
$$= [\cos(c\sin\theta)\cos(ic\cos\theta) + \sin(c\sin\theta)\sin(ic\cos\theta)]$$
$$= [\cos(c\sin\theta)\cosh(c\cos\theta) + i\sin(c\sin\theta)\sinh(c\cos\theta)]$$
(Art. 314).

Hence C and S as before.

355. Ex. *Sum the two series*

$$c\sin\alpha + \frac{c^2}{2}\sin 2\alpha + \frac{c^3}{3}\sin 3\alpha + \ldots\ldots \text{ ad inf.,}$$

and $$c\cos\alpha + \frac{c^2}{2}\cos 2\alpha + \frac{c^3}{3}\cos 3\alpha + \ldots\ldots \text{ ad inf.,}$$

where c is numerically not greater than unity.

Let S and C stand for these two series; then, as before, we have

$$C + Si = c(\cos\alpha + i\sin\alpha) + \frac{c^2}{2}(\cos 2\alpha + i\sin 2\alpha) + \ldots\ldots$$
$$= ce^{\alpha i} + \frac{c^2}{2}e^{2\alpha i} + \frac{c^2}{3}e^{3\alpha i} + \ldots\ldots\ldots \quad \ldots\ldots\ldots(1)$$
$$= -\log[1 - ce^{\alpha i}] \quad \text{(by Art. 336)} \ \ldots\ldots\ldots\ldots (2)$$
$$= -\log[1 - c\cos\alpha - ic\sin\alpha] \qquad \text{(Art. 308).}$$

Let $1 - c\cos\alpha = r\cos\theta$, and $-c\sin\alpha = r\sin\theta$, so that

$$r = +\sqrt{1 - 2c\cos\alpha + c^2}, \quad \cos\theta = \frac{1 - c\cos\alpha}{r},$$

and

$$\sin\theta = -\frac{c\sin\alpha}{r},\ i.e.\ \theta = \tan^{-1}\frac{-c\sin\alpha}{1-c\cos\alpha},$$

with the convention of Art. 267.

$$\therefore\ C + Si = -\log\left[\sqrt{1-2c\cos\alpha+c^2}\,(\cos\theta + i\sin\theta)\right]$$
$$= -\log\left[\sqrt{1-2c\cos\alpha+c^2}\,.\,e^{\theta i}\right]$$
$$= -\log\sqrt{1-2c\cos\alpha+c^2} - \theta i.$$

$$\therefore\ C = -\log\sqrt{(1-2c\cos\alpha+c^2)} = -\frac{1}{2}\log(1-2c\cos\alpha+c^2)$$

.........(3),

and $$S = -\theta = -\tan^{-1}\left(\frac{-c\sin\alpha}{1-c\cos\alpha}\right)\ \ldots\ldots\ldots\ldots(4).$$

Exceptional cases. When $c = 1$, the quantity (2)
$= \log[1-\cos\alpha - i\sin\alpha] = \log[1+\cos(\alpha-\pi) + i\sin(\alpha-\pi)]$.

This, by Art. 336, is always equal to the series (1) except when $\alpha - \pi$ is equal to $(2n+1)\pi$, *i.e.* except when α is a multiple of 2π.

In this case $S = 0$,

and $$C = +1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\ldots\ldots,$$

which is known to be a divergent series.

When $c = -1$, the quantity (2)

$$= \log[1+\cos\alpha + i\sin\alpha].$$

This by Art. 336 is always equal to the series (1) except when $\alpha = (2n+1)\pi$.

In this case $S = 0$, and

$$C = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$$

The results (3) and (4) give then the sum of the two series except when (1) $c = 1$ and $\alpha = 2n\pi$, (2) $c = -1$ and $\alpha = (2n+1)\pi$, and (3) when $c > 1$.

In examples depending on the logarithm series it will be often found that for some particular values of the angle there is no sum.

Particular case. Let $c = \cos\alpha$, where α lies between 0 and $\frac{\pi}{2}$, so that

$$S = \cos\alpha \,.\, \sin\alpha + \frac{1}{2}\cos^2\alpha \sin 2\alpha + \frac{1}{3}\cos^3\alpha \sin 3\alpha + \ldots.$$

In this case

$$S = -\tan^{-1}\left(\frac{-\sin\alpha\cos\alpha}{\sin^2\alpha}\right) \quad \text{by (4)}$$

$$= -\tan^{-1}(-\cot\alpha)$$

$$= -\left(\alpha - \frac{\pi}{2}\right),$$

remembering the convention mentioned above,

$$= \frac{\pi}{2} - \alpha.$$

EXAMPLES. LXI.

Sum the series

1. $\sin\alpha + c\sin(\alpha+\beta) + \frac{c^2}{\lfloor 2}\sin(\alpha+2\beta) + \ldots\ldots$ ad inf.

2. $\cos\alpha + c\cos(\alpha+\beta) + \frac{c^2}{\lfloor 2}\cos(\alpha+2\beta) + \ldots\ldots$ ad inf.

3. $1 - \cos\alpha\cos\beta + \frac{\cos^2\alpha}{\lfloor 2}\cos 2\beta - \frac{\cos^3\alpha}{\lfloor 3}\cos 3\beta \ldots\ldots$ ad inf.

4. $\sin\alpha - \frac{\sin(\alpha+2\beta)}{\lfloor 2} + \frac{\sin(\alpha+4\beta)}{\lfloor 4} - \ldots\ldots$ ad inf.

5. $\cos\alpha - \frac{\cos(\alpha+2\beta)}{\lfloor 3} + \frac{\cos(\alpha+4\beta)}{\lfloor 5} - \ldots\ldots$ ad inf.

6. $1+\cosh\alpha + \frac{\cosh 2\alpha}{\lfloor 2} + \frac{\cosh 3\alpha}{\lfloor 3} + \ldots\ldots$ ad inf.

7. $\sinh\alpha + \frac{\sinh 2\alpha}{\lfloor 2} + \frac{\sinh 3\alpha}{\lfloor 3} + \ldots\ldots$ ad inf.

8. $1 + e^{\cos\alpha}\cos(\sin\alpha) + \frac{1}{\lfloor 2} e^{2\cos\alpha}\cos(2\sin\alpha) + \ldots\ldots$ ad inf.

9. $1 + e^{\sin\alpha}\cos(\cos\alpha) + \frac{e^{2\sin\alpha}}{\lfloor 2}\cos(2\cos\alpha) + \frac{e^{3\sin\alpha}}{\lfloor 3}\cos(3\cos\alpha) + \ldots\ldots$ ad inf.

10. $\frac{5\cos\theta}{1} + \frac{7\cos 3\theta}{\lfloor 3} + \frac{9\cos 5\theta}{\lfloor 5} + \ldots\ldots$ ad inf.

[In the following examples c may be assumed to be positive and not greater than unity; when c equals unity there will be, as in Art. 355, exceptional cases for some values of the angle α.]

11. $c\sin\alpha - \frac{c^2}{2}\sin 2\alpha + \frac{c^3}{3}\sin 3\alpha - \ldots\ldots$ ad inf.

12. $c\sin\alpha + \frac{1}{3}c^3\sin 3\alpha + \frac{1}{5}c^5\sin 5\alpha + \ldots\ldots$ ad inf.

13. $c\cos\alpha + \frac{1}{3}c^3\cos 3\alpha + \frac{1}{5}c^5\cos 5\alpha + \ldots\ldots$ ad inf.

14. $c\cos\alpha - \frac{1}{3}c^3\cos 3\alpha + \frac{1}{5}c^5\cos 5\alpha + \ldots\ldots$ ad inf.

15. $c\sin\alpha - \frac{1}{3}c^3\sin 3\alpha + \frac{1}{5}c^5\sin 5\alpha + \ldots\ldots$ ad inf.

16. $\cos\alpha - \frac{1}{3}\cos 3\alpha + \frac{1}{5}\cos 5\alpha - \ldots\ldots$ ad inf.

17. $c\cos\alpha - \frac{1}{3}c^3\cos(\alpha+2\beta) + \frac{1}{5}c^5\cos(\alpha+4\beta)\ldots\ldots$ ad inf.

18. $\sin\alpha\sin\beta + \frac{1}{2}\sin 2\alpha\sin 2\beta + \frac{1}{3}\sin 3\alpha\sin 3\beta\ldots\ldots$ ad inf.

19. $m\sin^2\alpha - \frac{1}{2}m^2\sin^2 2\alpha + \frac{1}{3}m^3\sin^2 3\alpha - \ldots\ldots$ ad inf.

20. $\sinh\alpha - \frac{1}{2}\sinh 2\alpha + \frac{1}{3}\sinh 3\alpha - \ldots\ldots$ ad inf.

21. $e^{\alpha}\cos\beta - \frac{1}{3}e^{3\alpha}\cos 3\beta + \frac{1}{5}e^{5\alpha}\cos 5\beta - \ldots\ldots$ ad inf.

22. $\cos\frac{\pi}{3} + \frac{1}{3}\cos\frac{2\pi}{3} + \frac{1}{5}\cos\frac{3\pi}{3} + \frac{1}{7}\cos\frac{4\pi}{3}\ldots\ldots$ ad inf.

23. If $\theta - \alpha = \tan^2\frac{\omega}{2}\sin 2\theta - \frac{1}{2}\tan^4\frac{\omega}{2}\sin 4\theta + \frac{1}{3}\tan^6\frac{\omega}{2}\sin 6\theta - \ldots$ ad inf.

prove that $$\tan\alpha = \tan\theta \,.\, \cos\omega.$$

24. If θ and ϕ be positive acute angles prove that the sum of the series

$$\sin\theta\cos\phi + \frac{1}{3}\sin 3\theta\cos 3\phi + \frac{1}{5}\sin 5\theta\cos 5\phi + \ldots\ldots \text{ ad inf.}$$

is $\frac{\pi}{4}$ or 0, according as $\theta >$ or $< \phi$.

Prove that

25. $\tanh x + \frac{1}{3}\tanh^3 x + \frac{1}{5}\tanh^5 x + \ldots\ldots$

$= \tan x - \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x - \ldots\ldots$, where x lies between $-\frac{\pi}{4}$ and $+\frac{\pi}{4}$.

26. $2\sin^2\theta + \frac{1}{2}\,.\,4\sin^4\theta + \frac{1}{3}\,.\,8\sin^6\theta + \ldots\ldots$

$= 2\left(\tan^2\theta + \frac{1}{3}\tan^6\theta + \frac{1}{5}\tan^{10}\theta + \ldots\ldots\right)$, where θ lies between $-\frac{\pi}{4}$ and $+\frac{\pi}{4}$.

27. $\sin\theta + \frac{1}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + \ldots\ldots$

$= 2\left(\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \ldots\ldots\right)$, where $\theta \neq (2n+1)\frac{\pi}{2}$.

356. We subjoin some examples of series which come under neither of the foregoing heads nor under that of Chapter XIX. In general they are to be summed by the artifice of splitting each term into the difference of two terms. Considerable ingenuity is often required. When the answer is known the method of summation can usually be easily seen; for the answer when n is put equal to unity gives the form in which the first term of the series has to be put.

Ex. 1. *Sum to n terms the series*

$$\sin^3\frac{\theta}{3}+3\sin^3\frac{\theta}{3^2}+3^2\sin^3\frac{\theta}{3^3}+\ldots$$

Since always $\sin 3\phi=3\sin\phi-4\sin^3\phi$, we have

$$\sin^3\frac{\theta}{3}=\frac{1}{4}\left(3\sin\frac{\theta}{3}-\sin\theta\right),$$

$$3\,.\,\sin^3\frac{\theta}{3^2}=\frac{1}{4}\,.\,3\,.\left[3\sin\frac{\theta}{3^2}-\sin\frac{\theta}{3}\right]=\frac{1}{4}\left[3^2\sin\frac{\theta}{3^2}-3\sin\frac{\theta}{3}\right],$$

$$3^2\sin^3\frac{\theta}{3^3}=\frac{1}{4}\left[3^3\sin\frac{\theta}{3^3}-3^2\sin\frac{\theta}{3^2}\right],$$

....

$$3^{n-1}\sin^3\frac{\theta}{3^n}=\frac{1}{4}\left[3^n\sin\frac{\theta}{3^n}-3^{n-1}\sin\frac{\theta}{3^{n-1}}\right].$$

Hence, by addition, the required sum

$$=\frac{1}{4}\left[3^n\sin\frac{\theta}{3^n}-\sin\theta\right].$$

Also the sum to infinity

$$=\frac{1}{4}[\theta-\sin\theta]. \qquad \text{(Art. 228.)}$$

Ex. 2. *Sum the series*

$$\tan\alpha+2\tan 2\alpha+2^2\tan 2^2\alpha+\ldots\ldots+2^{n-1}\tan 2^{n-1}\alpha.$$

We have easily

$$\tan\alpha=\cot\alpha-2\cot 2\alpha,$$

$$\tan 2\alpha=\cot 2\alpha-2\cot 2^2\alpha,$$

$$\tan 2^2\alpha=\cot 2^2\alpha-2\cot 2^3\alpha,$$

and

$$\tan 2^{n-1}\alpha=\cot 2^{n-1}\alpha-2\cot 2^n\alpha.$$

By multiplying these rows in succession by 1, 2, 2^2, ...2^{n-1} we have

$$\tan\alpha+2\tan 2\alpha+2^2\tan 2^2\alpha+\ldots\ldots+2^{n-1}\tan 2^{n-1}\alpha=\cot\alpha-2^n\cot 2^n\alpha,$$

the other terms all disappearing.

The required sum therefore $=\cot\alpha-2^n\cot 2^n\alpha$.

Ex. 3. *Sum the series*

$$\tan\alpha\tan(\alpha+\beta)+\tan(\alpha+\beta)\tan(\alpha+2\beta)+\tan(\alpha+2\beta)\tan(\alpha+3\beta)+\ldots\ldots$$

to n terms.

Let $u_r \equiv$ the rth term, *i.e.*

$$\tan\{\alpha+(r-1)\beta\}\tan\{\alpha+r\beta\},$$

$\therefore\ (u_r+1)\tan\beta$

$$=[1+\tan\{\alpha+(r-1)\beta\}\tan\{\alpha+r\beta\}]\times\tan[\alpha+r\beta-(\alpha+\overline{r-1}\beta)]$$
$$=\tan\{\alpha+r\beta\}-\tan\{\alpha+\overline{r-1}\,\beta\}. \quad \text{[Art. 98.]}$$

Hence giving r in succession the values 1, 2,......n, we have

$$(1+u_1)\tan\beta=\tan(\alpha+\beta)-\tan\alpha,$$
$$(1+u_2)\tan\beta=\tan(\alpha+2\beta)-\tan(\alpha+\beta),$$
$$\dots\dots\dots\dots\dots\dots$$
$$(1+u_n)\tan\beta=\tan\{\alpha+n\beta\}-\tan\{\alpha+(n-1)\beta\}.$$

Hence by addition

$$(n+S_n)\tan\beta=\tan(\alpha+n\beta)-\tan\alpha,$$

so that $$S_n=\frac{\tan(\alpha+n\beta)-\tan\alpha-n\tan\beta}{\tan\beta}.$$

EXAMPLES. LXII.

Sum the series

1. $\operatorname{cosec}\theta+\operatorname{cosec}2\theta+\operatorname{cosec}4\theta+\dots\dots$ to n terms.

2. $\operatorname{cosec}\theta\operatorname{cosec}2\theta+\operatorname{cosec}2\theta\operatorname{cosec}3\theta+\operatorname{cosec}3\theta\operatorname{cosec}4\theta+\dots\dots$ to n terms.

3. $\sec\theta\sec2\theta+\sec2\theta\sec3\theta+\sec3\theta\sec4\theta+\dots\dots$ to n terms.

4. $\sec\theta\sec(\theta+\phi)+\sec(\theta+\phi)\sec(\theta+2\phi)+\sec(\theta+2\phi)\sec(\theta+3\phi)+\dots\dots$ to n terms.

5. $\dfrac{1}{\cos\alpha+\cos3\alpha}+\dfrac{1}{\cos\alpha+\cos5\alpha}+\dfrac{1}{\cos\alpha+\cos7\alpha}+\dots\dots$ to n terms.

6. $\tan\theta+\dfrac{1}{2}\tan\dfrac{\theta}{2}+\dfrac{1}{2^2}\tan\dfrac{\theta}{2^2}+\dfrac{1}{2^3}\tan\dfrac{\theta}{2^3}+\dots\dots$ ad inf.

7. $\tanh\theta+\dfrac{1}{2}\tanh\dfrac{\theta}{2}+\dfrac{1}{2^2}\tanh\dfrac{\theta}{2^2}+\dfrac{1}{2^3}\tanh\dfrac{\theta}{2^3}+\dots\dots$ to n terms.

8. $\tan\theta\sec2\theta+\tan2\theta\sec4\theta+\tan4\theta\sec8\theta+\dots\dots$ to n terms.

9. $\tan\dfrac{\theta}{2}\sec\theta+\tan\dfrac{\theta}{2^2}\sec\dfrac{\theta}{2}+\tan\dfrac{\theta}{2^3}\sec\dfrac{\theta}{2^2}+\dots\dots$ to n terms and to infinity.

10. $\dfrac{1}{2\cos\theta}+\dfrac{1}{2^2\cos\theta\cos2\theta}+\dfrac{1}{2^3\cos\theta\cos2\theta\cos2^2\theta}+\dots\dots$ to n terms.

11. $\sin 2\theta \cos^2\theta - \frac{1}{2}\sin 4\theta\cos^2 2\theta + \frac{1}{4}\sin 8\theta\cos^2 4\theta - \ldots\ldots$ to n terms.

12. $\sin 2\theta \sin^2\theta + \frac{1}{2}\sin 4\theta\sin^2 2\theta + \frac{1}{4}\sin 8\theta\sin^2 4\theta + \ldots\ldots$ to n terms.

13. $\dfrac{\sin\theta}{\cos\theta+\cos 2\theta} + \dfrac{\sin 2\theta}{\cos\theta+\cos 4\theta} + \dfrac{\sin 3\theta}{\cos\theta+\cos 6\theta} + \ldots\ldots$ to n terms.

14. $\tan^2\alpha\tan 2\alpha + \frac{1}{2}\tan^2 2\alpha\tan 4\alpha + \frac{1}{2^2}\tan^2 4\alpha\tan 8\alpha + \ldots\ldots$ ad inf.

15. $\cos^3\theta - \frac{1}{3}\cos^3 3\theta + \frac{1}{3^2}\cos^3 3^2\theta - \frac{1}{3^3}\cos^3 3^3\theta + \ldots\ldots$ to n terms.

16. $\sin^3\frac{\theta}{3} + 3\sin^3\frac{\theta}{3^2} + 3^2\sin^3\frac{\theta}{3^3} + \ldots\ldots$ to n terms.

17. $\dfrac{1}{\cot\theta - 3\tan\theta} + \dfrac{3}{\cot 3\theta - 3\tan 3\theta} + \dfrac{3^2}{\cot 3^2\theta - 3\tan 3^2\theta} + \ldots\ldots$ to n terms.

18. $\dfrac{\cos\theta-\cos 3\theta}{\sin 3\theta} + 3\,\dfrac{\cos 3\theta - \cos 3^2\theta}{\sin 3^2\theta} + 3^2\,\dfrac{\cos 3^2\theta-\cos 3^3\theta}{\sin 3^3\theta} + \ldots\ldots$ to n terms.

19. $\tan^{-1}\dfrac{4}{1+3.4} + \tan^{-1}\dfrac{6}{1+8.9} + \tan^{-1}\dfrac{8}{1+15.16} + \ldots\ldots$ to n terms.

20. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \tan^{-1}\frac{1}{21} + \ldots\ldots$ to n terms.

21. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{2}{9} + \ldots\ldots + \tan^{-1}\dfrac{2^{n-1}}{1+2^{2n-1}} + \ldots\ldots$ ad inf.

22. $\sin^{-1}\dfrac{1}{\sqrt{2}} + \sin^{-1}\dfrac{\sqrt{2}-1}{\sqrt{6}} + \sin^{-1}\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{12}} + \ldots\ldots$
$+ \sin^{-1}\dfrac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}} + \ldots\ldots$ ad inf.

Expansions.

357. In some branches of higher Mathematics it is desirable to be able to expand certain quantities in a series of ascending powers.

As an example we will expand

$$\log(1 - 2a\cos\theta + a^2)$$

in ascending powers of a.

Since $$2\cos\theta = e^{\theta i} + e^{-\theta i},$$
we have

$$\log(1 - 2a\cos\theta + a^2) = \log[1 - a(e^{\theta i} + e^{-\theta i}) + a^2]$$
$$= \log[(1 - ae^{\theta i})(1 - ae^{-\theta i})]$$
$$= \log(1 - ae^{\theta i}) + \log(1 - ae^{-\theta i})$$
$$= -ae^{\theta i} - \frac{1}{2}a^2e^{2\theta i} - \frac{1}{3}a^3e^{3\theta i} - \frac{1}{4}a^4e^{4\theta i} - \ldots\ldots$$
$$-ae^{-\theta i} - \frac{1}{2}a^2e^{-2\theta i} - \frac{1}{3}a^3e^{-3\theta i} - \ldots\ldots$$
$$= -a[e^{\theta i} + e^{-\theta i}] - \frac{1}{2}a^2[e^{2\theta i} + e^{-2\theta i}] - \frac{1}{3}a^3[e^{3\theta i} + e^{-3\theta i}] - \ldots\ldots\ldots$$
$$= -a \cdot 2\cos\theta - \frac{1}{2}a^2 \cdot 2\cos 2\theta - \frac{1}{3}a^3 \cdot 2\cos 3\theta \ldots\ldots$$
$$= -2\left[a\cos\theta + \frac{1}{2}a^2\cos 2\theta + \frac{1}{3}a^3\cos 3\theta + \ldots\ldots\right].$$

The expansion of $\log(1 - ae^{\theta i})$ is legitimate, by Art. 336, if the modulus of $-ae^{\theta i}$ be less than unity.

Now $$-ae^{\theta i} = a\cos(\pi + \theta) + i\sin(\pi + \theta),$$
so that its modulus is equal to a. Hence the above expansion is legitimate provided that a is less than unity.

The expansion is also legitimate if a be equal to unity provided that θ do not equal an even multiple of π.

It is also legitimate if a be equal to -1 and θ do not equal an odd multiple of π.

358. Ex. *Expand*

$$\frac{1 - a^2}{1 - 2a\cos\theta + a^2}$$

in a series of ascending powers of a.

We have

$$\frac{1-a^2}{1-2a\cos\theta+a^2}=-1+\frac{2-2a\cos\theta}{1-2a\cos\theta+a^2}$$

$$=-1+\frac{2-a(e^{\theta i}+e^{-\theta i})}{1-a(e^{\theta i}+e^{-\theta i})+a^2}$$

$$=-1+\frac{2-a(e^{\theta i}+e^{-\theta i})}{(1-ae^{\theta i})(1-ae^{-\theta i})}$$

$$=-1+\frac{1}{1-ae^{\theta i}}+\frac{1}{1-ae^{-\theta i}}$$

$$=-1+(1-ae^{\theta i})^{-1}+(1-ae^{-\theta i})^{-1}$$

$$=-1+1+ae^{\theta i}+a^2e^{2\theta i}+a^3e^{3\theta i}+\ldots\ldots$$

$$+1+ae^{-\theta i}+a^2e^{-2\theta i}+a^3e^{-3\theta i}+\ldots\ldots$$

$$=1+a(e^{\theta i}+e^{-\theta i})+a^2(e^{2\theta i}+e^{-2\theta i})+\ldots\ldots$$

$$=1+2a\cos\theta+2a^2\cos 2\theta+2a^3\cos 3\theta+\ldots\text{ad inf.}$$

The expansions of $(1-ae^{\theta i})^{-1}$ and $(1-ae^{-\theta i})^{-1}$ by the Binomial Theorem are legitimate if the modulus of $ae^{\theta i}$ be less than unity, *i.e.* if a be numerically <1, but not otherwise. (Art. 273.)

The above series is the one assumed in Art. 294.

Similarly we can deduce the series of Art. 293. For we have

$$\frac{2a\sin\theta}{1-2a\cos\theta+a^2}=\frac{1}{i}\,\frac{a(e^{\theta i}-e^{-\theta i})}{1-a(e^{\theta i}+e^{-\theta i})+a^2}$$

$$=\frac{1}{i}\,\frac{ae^{\theta i}-ae^{-\theta i}}{(1-ae^{\theta i})(1-ae^{-\theta i})}=\frac{1}{i}\left[\frac{1}{1-ae^{\theta i}}-\frac{1}{1-ae^{-\theta i}}\right]$$

$$=\frac{1}{i}\{(1+ae^{\theta i}+a^2e^{2\theta i}+\ldots)-(1+ae^{-\theta i}+a^2e^{-2\theta i}+\ldots)\}$$

$$=2a\sin\theta+2a^2\sin 2\theta+2a^3\sin 3\theta+\ldots\ldots\text{ad inf.}$$

As before this expansion is legitimate only if $a<1$.

359. Ex. *If $\sin x = n \sin (\alpha + x)$, expand x in a series of ascending powers of n, where n is less than unity.*

Since

$$\sin x = n \sin (\alpha + x) = n (\sin \alpha \cos x + \cos \alpha \sin x),$$

$$\therefore \tan x = \frac{n \sin \alpha}{1 - n \cos \alpha},$$

$$\therefore \frac{e^{xi} - e^{-xi}}{e^{xi} + e^{-xi}} = \frac{ni \sin \alpha}{1 - n \cos \alpha},$$

$$\therefore \frac{e^{xi}}{e^{-xi}} = \frac{1 - n \cos \alpha + ni \sin \alpha}{1 - n \cos \alpha - ni \sin \alpha} = \frac{1 - ne^{-\alpha i}}{1 - ne^{\alpha i}},$$

$$\therefore 2xi = \log (1 - ne^{-\alpha i}) - \log (1 - ne^{\alpha i})$$

$$= - ne^{-\alpha i} - \frac{1}{2} n^2 e^{-2\alpha i} - \frac{1}{3} n^3 e^{-3\alpha i} - \ldots\ldots$$

$$+ ne^{\alpha i} + \frac{1}{2} n^2 e^{2\alpha i} + \frac{1}{3} n^3 e^{3\alpha i} + \ldots\ldots$$

$$= n (e^{\alpha i} - e^{-\alpha i}) + \frac{1}{2} n^2 (e^{2\alpha i} - e^{-2\alpha i})$$

$$+ \frac{1}{3} n^3 (e^{3\alpha i} - e^{-3\alpha i}) \ldots\ldots \text{ad inf.}$$

$$= n \,.\, 2i \sin \alpha + \frac{1}{2} n^2 \,.\, 2i \sin 2\alpha + \frac{1}{3} n^3 \,.\, 2i \sin 3\alpha + \ldots$$

$$\therefore x = n \sin \alpha + \frac{1}{2} n^2 \sin 2\alpha + \frac{1}{3} n^3 \sin 3\alpha + \ldots \quad \ldots(1).$$

In this equation we have assumed x to lie between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$; if it do not, then, instead of $2xi$, we should read $2k\pi i + 2xi$; the left hand of equation (1) would then be $x + k\pi$, and we must choose k so that $x + k\pi$ shall lie between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

As before the expansions are legitimate if n be $<$ unity.

360. Ex. *Expand $e^{ax}\cos bx$ in a series of ascending powers of x.*

We have

$$e^{ax}\cos bx = e^{ax} . \frac{e^{bxi}+e^{-bxi}}{2}$$

$$= \frac{1}{2}e^{(a+bi)x} + \frac{1}{2}e^{(a-bi)x}$$

$$= \frac{1}{2}\left[1+(a+bi)x+\frac{(a+bi)^2x^2}{\underline{|2}}+\frac{(a+bi)^3x^3}{\underline{|3}}+\dots\dots\right]$$

$$+\frac{1}{2}\left[1+(a-bi)x+\frac{(a-bi)^2x^2}{\underline{|2}}+\dots\dots\right].$$

The coefficient of x^n

$$= \frac{(a+bi)^n+(a-bi)^n}{2\,\underline{|n}}.$$

If $a+bi = r(\cos\alpha + i\sin\alpha)$, so that

$$r = +\sqrt{a^2+b^2} \text{ and } \tan\alpha = \frac{b}{a},$$

with the convention of Art. 267, then the coefficient of x^n

$$= \frac{\{r(\cos\alpha+i\sin\alpha)\}^n+\{r(\cos\alpha-i\sin\alpha)\}^n}{2\,\underline{|n}}$$

$$= r^n\frac{\cos n\alpha}{\underline{|n}},$$

by De Moivre's Theorem.

Hence we have

$$e^{ax}\cos bx = 1 + r\cos\alpha\,.\,x + \frac{r^2\cos 2\alpha}{\underline{|2}}x^2 + \frac{r^3\cos 3\alpha}{\underline{|3}}x^3 + \dots\dots,$$

where

$$r = +\sqrt{a^2+b^2} \text{ and } \tan\alpha = \frac{b}{a}.$$

This expansion is legitimate for all values of a, b, and x. (Art. 303.)

EXAMPLES. LXIII.

Expand in an infinite series

1. $\dfrac{1+a\cos\theta}{1+2a\cos\theta+a^2}$.

2. $\dfrac{\cos\theta-a\cos(\theta-\phi)}{1-2a\cos\phi+a^2}$.

3. $\dfrac{\sin\theta-a\sin(\theta-\phi)}{1-2a\cos\phi+a^2}$.

4. $e^{a\cos\phi}\cos(\theta+a\sin\phi)$.

5. $e^{a\theta}\sin b\theta$.

Prove that

6. $$\log\frac{a^2}{a^2\cos^2\theta+b^2\sin^2\theta}=4\left[c\sin^2\theta-\frac{1}{2}c^2\sin^2 2\theta+\frac{1}{3}c^3\sin^2 3\theta-\dots\right],$$

where $$c=\frac{a-b}{a+b}.$$

7. $$\tan^{-1}\frac{a\sin\theta}{1-a\cos\theta}=a\sin\theta+\frac{1}{2}a^2\sin 2\theta+\frac{1}{3}a^3\sin 3\theta+\dots\dots \text{ ad inf.}$$

8. $$\frac{1}{2}\tan^{-1}(\sin\alpha\tan 2\beta)=\sin\alpha\tan\beta+\frac{1}{3}\sin 3\alpha\tan^3\beta$$
$$+\frac{1}{5}\sin 5\alpha\tan^5\beta+\dots\dots \text{ ad inf.}$$

9. If $\sin\theta=x\cos(\theta+\alpha)$, expand θ in a series of ascending powers of x.

10. Expand y in terms of $\cos\alpha$, where
$$2\tan y=\sin x\operatorname{cosec}\frac{x+\alpha}{2}\operatorname{cosec}\frac{x-\alpha}{2}.$$

11. If $\tan x=n\tan y$, and $m=\dfrac{1-n}{1+n}$, prove that
$$x+r\pi=y-m\sin 2y+\frac{m^2}{2}\sin 4y-\frac{m^3}{3}\sin 6y+\dots\dots\text{ad inf.},$$
where r is to be so chosen that $x+r\pi-y$ lies between $-\dfrac{\pi}{2}$ and $+\dfrac{\pi}{2}$.

12. What does the series of the preceding question become when (1) $n=\cos\alpha$, and (2) $n=\dfrac{1}{\cos 2\alpha}$?

13. Expand $\log\cos\left(\dfrac{\pi}{4}+\theta\right)$ in a series of sines and cosines of ascending multiples of θ.

14. Expand $\log \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ in a series of sines of ascending multiples of θ.

15. Prove that

$$(1+e^{\theta i}\tan\alpha)(1+e^{-\theta i}\tan\alpha)(1+e^{\theta i}\cot\alpha)(1+e^{-\theta i}\cot\alpha) = 4(\sec\beta+\cos\theta)^2,$$

where $$\beta=\frac{\pi}{2}-2\alpha.$$

Hence expand $\log(1+\cos\beta\cos\theta)$ in a series of cosines of multiples of θ.

16. Prove that

$$\frac{2a\cos\theta}{1-2a\sin\theta+a^2}=2a\cos\theta+2a^2\sin 2\theta-2a^3\cos 3\theta-2a^4\sin 4\theta+\ldots\ldots \text{ ad inf.}$$

17. Prove that

$$\log\cos\theta=-\log 2+\cos 2\theta-\frac{1}{2}\cos 4\theta+\frac{1}{3}\cos 6\theta-\ldots\ldots \text{ ad inf.},$$

if θ be an angle whose cosine is positive.

18. In any triangle where $a>b$, prove that

$$\log c=\log a-\frac{b}{a}\cos C-\frac{1}{2}\frac{b^2}{a^2}\cos 2C-\frac{1}{3}\frac{b^3}{a^3}\cos 3C-\ldots\ldots \text{ ad inf.}$$

[We have $c^2=a^2+b^2-2ab\cos C=a^2\left(1-\frac{b}{a}e^{iC}\right)\left(1-\frac{b}{a}e^{-iC}\right)$.]

19. Prove that the coefficient of x^n in the expansion of

$$e^{ax}\sin bx+e^{bx}\sin ax$$

in powers of x is

$$\frac{2(a^2+b^2)^{\frac{n}{2}}}{\underline{|n}}\sin\frac{n\pi}{4}\cos\frac{n}{2}\left[\frac{\pi}{2}-2\tan^{-1}\frac{b}{a}\right].$$

20. Prove that the coefficient of c^n in the expansion of

$$\log(a^3+b^3+c^3-3abc)$$

is $$(-1)^{n-1}\frac{1}{n}\left[\frac{1}{(a+b)^n}+\frac{2\cos n\theta}{(a^2+b^2-ab)^{\frac{n}{2}}}\right],$$

where $$\tan\theta=\frac{a-b}{a+b}\sqrt{3}.$$

CHAPTER XXIX.

RESOLUTION INTO FACTORS. INFINITE PRODUCTS FOR SIN θ AND COS θ.

361. WE know from Algebra that, if P be any expression containing x and if the value $x = \alpha$ would make P vanish, then $x - \alpha$ is a factor of P.

Hence to find the factors of any expression P we first solve the equation $P = 0$. If the roots thus found be $\alpha, \beta, \ldots$ we know that $x - \alpha$, $x - \beta, \ldots$ are factors of P.

We shall apply this method in the following articles.

362. *To resolve into factors the expression*

$$x^{2n} - 2x^n \cos n\theta + 1.$$

We have first to solve the equation

$$x^{2n} - 2x^n \cos n\theta + 1 = 0,$$

i.e. $$x^{2n} - 2x^n \cos n\theta + \cos^2 n\theta = -\sin^2 n\theta,$$

so that $$x^n - \cos n\theta = \pm\sqrt{-1} \sin n\theta,$$

and therefore

$$x = [\cos n\theta \pm \sqrt{-1} \sin n\theta]^{\frac{1}{n}}.$$

As in Art. 271 the values of this expression are the $2n$ quantities

$$\cos\theta \pm i\sin\theta,\ \cos\left(\theta+\frac{2\pi}{n}\right) \pm i\sin\left(\theta+\frac{2\pi}{n}\right),$$

$$\cos\left(\theta+\frac{4\pi}{n}\right) \pm i\sin\left(\theta+\frac{4\pi}{n}\right),\ \ldots\ldots\ldots$$

$$\cos\left\{\theta+\frac{2(n-1)\pi}{n}\right\} \pm i\sin\left\{\theta+\frac{2(n-1)\pi}{n}\right\}.$$

Taking the first pair of these quantities we have the corresponding factors

$$x-\cos\theta-i\sin\theta \text{ and } x-\cos\theta+i\sin\theta,$$

or, in one factor,

$$(x-\cos\theta)^2+\sin^2\theta,$$

i.e. the quadratic factor

$$x^2-2x\cos\theta+1.$$

Similarly the second, third, ... pairs of the above quantities give as factors respectively

$$x^2-2x\cos\left(\theta+\frac{2\pi}{n}\right)+1,$$

$$x^2-2x\cos\left(\theta+\frac{4\pi}{n}\right)+1,$$

$$\ldots\ldots\ldots$$

and $$x^2-2x\cos\left\{\theta+\frac{2n-2}{n}\pi\right\}+1.$$

Also on multiplying together these n factors we see that the coefficient of x^{2n} in their product is unity, which is also the coefficient of x^{2n} in the original expression. No other numerical factor is therefore required.

Hence

$$\mathbf{x^{2n} - 2x^n \cos n\theta + 1}$$

$$= \{\mathbf{x^2 - 2x\cos\theta + 1}\}\left\{\mathbf{x^2 - 2x\cos\left(\theta + \frac{2\pi}{n}\right) + 1}\right\}$$

$$\left\{\mathbf{x^2 - 2x\cos\left(\theta + \frac{4\pi}{n}\right) + 1}\right\}$$

$$\ldots\left\{\mathbf{x^2 - 2x\cos\left(\theta + \frac{2n-2}{n}\pi\right) + 1}\right\} \ldots\ldots (1).$$

By dividing by x^n we have

$$x^n + \frac{1}{x^n} - 2\cos n\theta = \left\{x + \frac{1}{x} - 2\cos\theta\right\}\left\{x + \frac{1}{x} - 2\cos\left(\theta + \frac{2\pi}{n}\right)\right\}$$

$$\ldots\left\{x + \frac{1}{x} - 2\cos\left(\theta + \frac{2n-2}{n}\pi\right)\right\} \ldots\ldots\ldots (2).$$

The relation (2) may be written

$$x^n + \frac{1}{x^n} - 2\cos n\theta = \prod_{r=0}^{r=n-1}\left\{x + \frac{1}{x} - 2\cos\left(\theta + \frac{2r\pi}{n}\right)\right\}$$

where $\prod_{r=0}^{r=n-1}$ stands for the product for all integral values of r from $r = 0$ to $r = n - 1$ of the expression following it.

Similarly we may shew that

$$x^{2n} - 2a^n x^n \cos n\theta + a^{2n}$$

$$= \{x^2 - 2ax\cos\theta + a^2\}\left\{x^2 - 2ax\cos\left(\theta + \frac{2\pi}{n}\right) + a^2\right\}$$

$$\left\{x^2 - 2ax\cos\left(\theta + \frac{4\pi}{n}\right) + a^2\right\}\ldots\left\{x^2 - 2ax\cos\left(\theta + \frac{2n-2}{n}\pi\right) + a^2\right\}$$

$$\ldots\ldots\ldots\ldots (3).$$

363. The proposition of the last article may also be proved by induction.

We shall first shew that $x^n+\frac{1}{x^n}-2\cos n\alpha$ is divisible by

$$x+\frac{1}{x}-2\cos\alpha.$$

Let $x^n+\frac{1}{x^n}-2\cos n\alpha$ be denoted by $\phi(n)$, and $x+\frac{1}{x}-2\cos\alpha$ by λ, so that we have to shew that $\phi(n)$ is divisible by λ, for all positive integral values of n.

Assume that this is true for $\phi(n-1)$ and $\phi(n-2)$.

We have then, by ordinary multiplication,

$$\left(x+\frac{1}{x}\right)\times\phi(n-1)=\left\{x+\frac{1}{x}\right\}\left\{x^{n-1}+\frac{1}{x^{n-1}}-2\cos(n-1)\alpha\right\}$$

$$=\left(x^n+\frac{1}{x^n}\right)+\left(x^{n-2}+\frac{1}{x^{n-2}}\right)-2\cos(n-1)\alpha\times\left(x+\frac{1}{x}\right)$$

$$=\left\{x^n+\frac{1}{x^n}-2\cos n\alpha\right\}$$

$$+\left\{x^{n-2}+\frac{1}{x^{n-2}}-2\cos(n-2)\alpha\right\}-2\cos(n-1)\alpha\left\{x+\frac{1}{x}-2\cos\alpha\right\},$$

since $\qquad 2\cos n\alpha+2\cos(n-2)\alpha=4\cos\alpha\cos(n-1)\alpha.$

Hence $\quad\left(x+\frac{1}{x}\right)\times\phi(n-1)=\phi(n)+\phi(n-2)-2\lambda\cos(n-1)\alpha,$

$$\therefore\ \phi(n)=\left(x+\frac{1}{x}\right)\phi(n-1)-\phi(n-2)+2\lambda\cos(n-1)\alpha\ \text{......}(1).$$

Now $\qquad \phi(1)=x+\frac{1}{x}-2\cos\alpha=\lambda,$

$$\phi(2)=x^2+\frac{1}{x^2}-2\cos 2\alpha=\left(x+\frac{1}{x}-2\cos\alpha\right)\left(x+\frac{1}{x}+2\cos\alpha\right)$$

$$=\lambda\left(x+\frac{1}{x}+2\cos\alpha\right),$$

so that $\phi(1)$ and $\phi(2)$ are divisible by λ.

Hence, putting $n=3$ in (1), we see that $\phi(3)$ is divisible by λ.

Similarly putting, in (1), $n=4, 5, 6$...... in succession we see that, by induction, $\phi(n)$ is divisible by λ for all values of n.

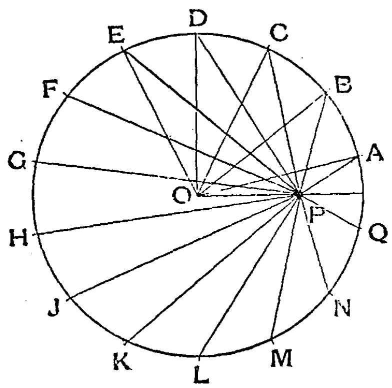
$\therefore\ x^n+\frac{1}{x^n}-2\cos n\alpha$ is divisible by $x+\frac{1}{x}-2\cos\alpha$.

Again $$x^n + \frac{1}{x^n} - 2\cos n\alpha = x^n + \frac{1}{x^n} - 2\cos n\left(\alpha + \frac{2\pi}{n}\right),$$
and is similarly divisible by
$$x + \frac{1}{x} - 2\cos\left(\alpha + \frac{2\pi}{n}\right).$$

Proceeding in this way we can shew that it is divisible by
$$x + \frac{1}{x} - 2\cos\left(\alpha + \frac{4\pi}{n}\right), \;\ldots\ldots x + \frac{1}{x} - 2\cos\left(\alpha + \frac{n-1}{n}2\pi\right),$$
and hence obtain equation (2) of Art. 362.

364. De Moivre's Property of the Circle.

A geometrical meaning may be given to the equation (3) of Art. 362.



Let $ABCD\ldots$ be the angular points of a polygon of n sides which is inscribed in a circle of radius a, so that, O being the centre, we have
$$\angle AOB = \angle BOC = \angle COD = \ldots = \frac{2\pi}{n}.$$

Let P be a point within, or without, the circle such that
$$OP = x \quad\text{and}\quad \angle POA = \theta.$$

Then
$$\angle POB = \theta + \frac{2\pi}{n}, \quad \angle POC = \theta + \frac{4\pi}{n}, \ldots$$
and we have
$$PA^2 = OP^2 + OA^2 - 2OP \,.\, OA\cos POA$$
$$= x^2 - 2ax\cos\theta + a^2,$$
$$PB^2 = OP^2 + OB^2 - 2OP \,.\, OB\cos POB$$
$$= x^2 - 2ax\cos\left(\theta + \frac{2\pi}{n}\right) + a^2,$$
$$PC^2 = x^2 - 2ax\cos\left(\theta + \frac{4\pi}{n}\right) + a^2,$$

Hence $PA^2 . PB^2 . PC^2 \ldots$ to n factors

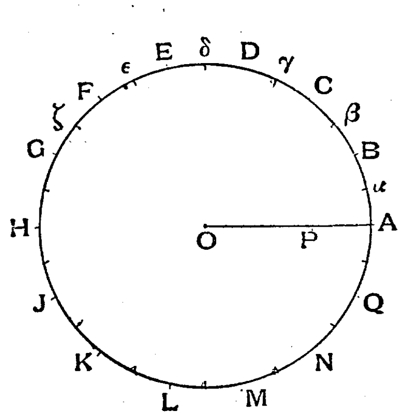
$$= \left\{x^2 - 2ax\cos\theta + a^2\right\}\left\{x^2 - 2ax\cos\left(\theta + \frac{2\pi}{n}\right) + a^2\right\}$$

$$\left\{x^2 - 2ax\cos\left(\theta + \frac{4\pi}{n}\right) + a^2\right\} \ldots \text{ to } n \text{ factors}$$

$$= x^{2n} - 2a^n x^n \cos n\theta + a^{2n}.$$

365. Cotes' Property of the Circle.

In the preceding article let the point P lie on OA, *i.e.* let it be on the line joining the centre to one of the angular points of the polygon.



In this case $\theta = 0$, and we have

$PA^2 . PB^2 . PC^2 \ldots$. to n factors

$= x^{2n} - 2a^n x^n + a^{2n}$

$= (x^n - a^n)^2$.

$\therefore\ PA . PB . PC \ldots$ to n factors

$= x^n - a^n$ or else $a^n - x^n$.

The first of these values must be taken when P is outside the circle, on OA produced, so that $x > a$.

The second must be taken when P is within the circle.

We therefore have

$$PA . PB . PC . PD \ldots \text{ to } n \text{ factors} = x^n \sim a^n \ldots(1).$$

Again let $\alpha, \beta, \gamma, \delta \ldots$ be the middle points of the arcs $AB, BC, CD, \ldots$ so that $A\alpha B\beta C\gamma \ldots$ is a polygon of $2n$ sides inscribed in the circle.

By (1) we have

$$PA . P\alpha . PB . P\beta . PC . P\gamma \ldots \text{ to } 2n \text{ factors} = x^{2n} \sim a^{2n} \ldots\ldots\ldots\ldots(2).$$

Dividing (1) by (2), we get

$$P\alpha . P\beta . P\gamma \ldots \text{ to } n \text{ factors} = x^n + a^n \ldots\ldots(3).$$

The equation (3) may also be deduced directly from equation (3) of Art. 362 by putting $\theta = \frac{\pi}{n}$. We then have

$$\left(x^2 - 2ax\cos\frac{\pi}{n} + a^2\right)\left(x^2 - 2ax\cos\frac{3\pi}{n} + a^2\right)\left(x^2 - 2ax\cos\frac{5\pi}{n} + a^2\right)$$

$$\text{...... to } n \text{ factors} = x^{2n} - 2a^n x^n \cos\pi + a^{2n}$$

$$= x^{2n} + 2a^n x^n + a^{2n} = (x^n + a^n)^2,$$

i.e. $$P\alpha^2 \,.\, P\beta^2 \,.\, P\gamma^2 \text{......to } n \text{ factors} = (x^n + a^n)^2.$$

This is relation (3).

366. *To resolve into factors the expression* $x^n - 1$.

We have first to solve the equation

$$x^n - 1 = 0,$$

i.e. $$x^n = 1 = \cos 2r\pi \pm i \sin 2r\pi,$$

where r is any integer,

so that $$x = [\cos 2r\pi \pm i\sin 2r\pi]^{\frac{1}{n}} \quad \text{...........} (1).$$

First, let n be **even.**

As in Art. 271 the values of the expression (1) are

$$\cos 0 \pm i\sin 0,\ \cos\frac{2\pi}{n} \pm i\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n} \pm i\sin\frac{4\pi}{n},$$

$$\ldots\cos\frac{n-2}{n}\pi \pm i\sin\frac{n-2}{n}\pi,\ \cos\frac{n\pi}{n} \pm i\sin\frac{n\pi}{n}.$$

But $$\cos 0^\circ \pm i\sin 0^\circ = 1,$$

and $$\cos\frac{n\pi}{n} \pm i\sin\frac{n\pi}{n} = -1.$$

Hence in this case the roots are

$$\pm 1,\ \cos\frac{2\pi}{n} \pm i\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n} \pm i\sin\frac{4\pi}{n},$$

$$\ldots\cos\frac{n-2}{n}\pi \pm i\sin\frac{n-2}{n}\pi.$$

The factors corresponding to the first of these pairs are $x-1$ and $x+1$, *i.e.* the quadratic factor x^2-1.

Those corresponding to the second pair are

$$x-\cos\frac{2\pi}{n}-i\sin\frac{2\pi}{n} \text{ and } x-\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n},$$

i.e. the quadratic factor

$$x^2-2x\cos\frac{2\pi}{n}+1.$$

Hence we get $\frac{n}{2}$ pairs of quadratic factors.

When multiplied together they give the correct coefficient for x^n, so that no constant quantity need be prefixed to their product.

Hence, finally, when n is even,

$$x^n-1=(x^2-1)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\left(x^2-2x\cos\frac{4\pi}{n}+1\right)$$
$$\dots\left(x^2-2x\cos\frac{n-2}{n}\pi+1\right)\dots\dots\dots\dots(2).$$

Secondly, let n be **odd.**

As in Art. 271 the values of the expression (1) are now

$$\cos 0\pm i\sin 0,\ \cos\frac{2\pi}{n}\pm i\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n}\pm i\sin\frac{4\pi}{n},\dots$$
$$\dots\cos\frac{n-3}{n}\pi\pm i\sin\frac{n-3}{n}\pi,\ \cos\frac{n-1}{n}\pi\pm i\sin\frac{n-1}{n}\pi.$$

The first pair reduces to the single factor $x-1$.

Taking the other pairs together, as before, we obtain, when n is odd,

$$x^n-1=(x-1)\left\{x^2-2x\cos\frac{2\pi}{n}+1\right\}\left\{x^2-2x\cos\frac{4\pi}{n}+1\right\}\dots$$
$$\dots\left\{x^2-2x\cos\frac{n-1}{n}\pi+1\right\}\dots\dots\dots\dots(3).$$

Hence we have

$$x^n - 1 = (x^2 - 1) \prod_{r=1}^{r=\frac{n}{2}-1} \left(x^2 - 2x \cos \frac{2r\pi}{n} + 1\right),$$

when n is even, and

$$x^n - 1 = (x - 1) \prod_{r=1}^{r=\frac{n-1}{2}} \left(x^2 - 2x \cos \frac{2r\pi}{n} + 1\right),$$

when n is odd.

These formulæ can also be deduced from the fundamental one of Art. 362 by putting $n\theta = 2\pi$.

367. *To resolve $x^n + 1$ into factors.*

We must solve the equation

$$x^n + 1 = 0,$$

i.e. $\qquad x^n = -1 = \cos(2r\pi + \pi) \pm i \sin(2r\pi + \pi),$

where r is any integer,

so that $\qquad x = \{\cos(2r\pi + \pi) \pm i \sin(2r\pi + \pi)\}^{\frac{1}{n}}$

$$= \cos \frac{2r\pi + \pi}{n} \pm i \sin \frac{2r\pi + \pi}{n} \quad \text{...........(1).}$$

First, let n be **even.**

As in Art. 271, the values of the expression (1) are

$$\cos \frac{\pi}{n} \pm i \sin \frac{\pi}{n}, \quad \cos \frac{3\pi}{n} \pm i \sin \frac{3\pi}{n}, \quad \cos \frac{5\pi}{n} \pm i \sin \frac{5\pi}{n}$$

$$\ldots \cos \frac{(n-1)\pi}{n} \pm i \sin \frac{(n-1)\pi}{n}.$$

The factors corresponding to the first of these pairs are

$$x - \cos \frac{\pi}{n} - i \sin \frac{\pi}{n} \quad \text{and} \quad x - \cos \frac{\pi}{n} + i \sin \frac{\pi}{n},$$

i.e. the quadratic factor

$$x^2 - 2x \cos \frac{\pi}{n} + 1.$$

The quadratic factor corresponding to the second pair is

$$x^2 - 2x \cos \frac{3\pi}{n} + 1,$$

and so on.

Hence, as in the last article, when n is even, we have

$$x^n + 1 = \left(x^2 - 2x \cos \frac{\pi}{n} + 1\right)\left(x^2 - 2x \cos \frac{3\pi}{n} + 1\right) \dots$$

$$\dots \left[x^2 - 2x \cos \frac{(n-1)\pi}{n} + 1\right].$$

Secondly, let n be **odd.**

The values of the expression (1) are in this case

$$\cos \frac{\pi}{n} \pm i \sin \frac{\pi}{n}, \quad \cos \frac{3\pi}{n} \pm i \sin \frac{3\pi}{n}, \dots$$

$$\cos \frac{(n-2)\pi}{n} \pm i \sin \frac{(n-2)\pi}{n}, \quad \cos \frac{n\pi}{n} \pm i \sin \frac{n\pi}{n}.$$

The last pair of roots reduces to the single root -1, so that $x + 1$ is one of the required factors.

The quadratic factors corresponding to the successive pairs of roots are

$$x^2 - 2x \cos \frac{\pi}{n} + 1, \quad x^2 - 2x \cos \frac{3\pi}{n} + 1, \dots$$

$$x^2 - 2x \cos \frac{n-2}{n} \pi + 1.$$

Hence finally, when n is odd, we have

$$x^n + 1 = (x+1)\left(x^2 - 2x \cos \frac{\pi}{n} + 1\right)\left(x^2 - 2x \cos \frac{3\pi}{n} + 1\right) \dots$$

$$\dots \left[x^2 - 2x \cos \frac{(n-2)\pi}{n} + 1\right].$$

We have then

$$x^n + 1 = \prod_{r=0}^{r=\frac{n-2}{2}} \left(x^2 - 2x \cos \frac{2r+1}{n} \pi + 1\right),$$

when n is even, and

$$x^n + 1 = (x+1) \prod_{r=0}^{\frac{n-3}{2}} \left(x^2 - 2x \cos \frac{2r+1}{n} \pi + 1\right),$$

when n is odd.

These formulæ can be deduced from the fundamental one of Art. 362 by putting $n\theta = \pi$.

368. **Ex. 1.** *Express as a product of n factors the quantities*

$$\cos n\phi - \cos n\theta \text{ and } \cosh n\phi - \cos n\theta.$$

In equation (2) of Art. 362 put $x = e^{\phi i}$, so that $x^{-1} = e^{-\phi i}$, and hence

$$x + x^{-1} = e^{\phi i} + e^{-\phi i} = 2 \cos \phi,$$

and $$x^n + x^{-n} = e^{n\phi i} + e^{-n\phi i} = 2 \cos n\phi.$$

We then have

$$2 \cos n\phi - 2 \cos n\theta = (2 \cos \phi - 2 \cos \theta) \left[2 \cos \phi - 2 \cos \left(\theta + \frac{2\pi}{n}\right)\right]$$

$$\left[2 \cos \phi - 2 \cos \left(\theta + \frac{4\pi}{n}\right)\right] \ldots\ldots \text{to } n \text{ factors},$$

i.e. $$\cos n\phi - \cos n\theta = 2^{n-1} \{\cos \phi - \cos \theta\} \left\{\cos \phi - \cos \left(\theta + \frac{2\pi}{n}\right)\right\} \ldots\ldots$$

$$\ldots\ldots \left\{\cos \phi - \cos \left(\theta + \frac{2n-2}{n} \pi\right)\right\}$$

$$= 2^{n-1} \prod_{r=0}^{r=n-1} \left\{\cos \phi - \cos \left(\theta + \frac{2r\pi}{n}\right)\right\}.$$

Similarly by putting $x = e^{\phi}$ we have

$\cosh n\phi - \cos n\theta$

$$= 2^{n-1} [\cosh \phi - \cos \theta] \left[\cosh \phi - \cos \left(\theta + \frac{2\pi}{n}\right)\right] \ldots\ldots$$

$$\left[\cosh \phi - \cos \left(\theta + \frac{2n-2}{n} \pi\right)\right].$$

Ex. 2. *If n be even, prove that*

$$2^{\frac{n-1}{2}} \sin\frac{2\pi}{2n} \sin\frac{4\pi}{2n} \sin\frac{6\pi}{2n} \ldots\ldots \sin\frac{n-2}{2n}\pi = \sqrt{n}.$$

In equation (2) of Art. 366 put n equal to unity.

Then, since $$\frac{x^n-1}{x^2-1} = \frac{x^{n-1}+x^{n-2}+\ldots\ldots+x+1}{x+1},$$

therefore, when x is unity, $\frac{x^n-1}{x^2-1} = \frac{n}{2}$.

Hence we have

$$\frac{n}{2} = \left(2-2\cos\frac{2\pi}{n}\right)\left(2-2\cos\frac{4\pi}{n}\right)\ldots\ldots\left(2-2\cos\frac{n-2}{n}\pi\right),$$

i.e. $$n = 2 \cdot 4\sin^2\frac{2\pi}{2n} \cdot 4\sin^2\frac{4\pi}{2n} \ldots\ldots 4\sin^2\frac{n-2}{2n}\pi,$$

there being $\frac{n}{2}-1$ factors,

$$= 2^{n-1} \cdot \sin^2\frac{2\pi}{2n} \sin^2\frac{4\pi}{2n} \ldots\ldots \sin^2\frac{n-2}{2n}\pi.$$

Hence $$\pm\sqrt{n} = 2^{\frac{n-1}{2}} \sin\frac{2\pi}{2n} \sin\frac{4\pi}{2n} \ldots\ldots \sin\frac{n-2}{2n}\pi \ldots\ldots\ldots\ldots\ldots (1).$$

Each of the angles $\frac{2\pi}{2n}, \frac{4\pi}{2n}, \ldots\ldots \frac{n-2}{2n}\pi$ is less than a right angle, so that each of the sines on the right-hand side of (1) is positive.

On the left-hand side we therefore replace the ambiguity by the positive sign and have the required result.

EXAMPLES. LXIV.

Factorize the following quantities.

1. $x^6+2x^3\cos 120°+1$.
2. $x^8-2x^4\cos 60°+1$.
3. $x^{10}-2x^5\cos\frac{\pi}{3}+1$.
4. $x^{12}+x^6+1$.
5. $x^{14}+x^7+1$.
6. x^5-1.
7. x^6+1.
8. x^7-1.
9. x^9+1.
10. $x^{10}-1$.
11. $x^{13}+1$.
12. $x^{14}-1$.
13. $x^{20}+1$.

14. If n be even, prove that

$$2^{\frac{n-1}{2}} \sin\frac{\pi}{2n} \sin\frac{3\pi}{2n} \sin\frac{5\pi}{2n} \dots\dots \sin\frac{n-1}{2n}\pi = 1$$

$$= 2^{\frac{n-1}{2}} \cos\frac{\pi}{2n} \cos\frac{3\pi}{2n} \dots\dots \cos\frac{n-1}{2n}\pi.$$

15. If n be odd, prove that

$$2^{\frac{n-1}{2}} \sin\frac{2\pi}{2n} \sin\frac{4\pi}{2n} \dots \sin\frac{n-1}{2n}\pi = \sqrt{n} = 2^{\frac{n-1}{2}} \cos\frac{\pi}{2n} \cos\frac{3\pi}{2n} \dots \cos\frac{n-2}{2n}\pi,$$

and that

$$2^{\frac{n-1}{2}} \sin\frac{\pi}{2n} \sin\frac{3\pi}{2n} \dots \sin\frac{n-2}{2n}\pi = 1 = 2^{\frac{n-1}{2}} \cos\frac{2\pi}{2n} \cos\frac{4\pi}{2n} \dots \cos\frac{n-1}{2n}\pi.$$

16. Prove that $\sin\frac{\pi}{n} \sin\frac{2\pi}{n} \dots\dots \sin\frac{n-1}{n}\pi = \frac{n}{2^{n-1}}$.

17. If n be odd, prove that

$$\tan\frac{\pi}{n} \tan\frac{2\pi}{n} \tan\frac{3\pi}{n} \dots\dots \tan\frac{\frac{1}{2}(n-1)\pi}{n} = \sqrt{n}.$$

18. Shew that $\cos n\theta$

$$= 2^{n-1}\left(\cos\theta - \cos\frac{\pi}{2n}\right)\left(\cos\theta - \cos\frac{3\pi}{2n}\right) \dots\dots \left(\cos\theta - \cos\frac{2n-1}{2n}\pi\right).$$

Prove that

19. $\sin n\phi = 2^{n-1} \sin\phi \sin\left(\phi + \frac{\pi}{n}\right) \dots\dots \sin\left(\phi + \frac{n-1}{n}\pi\right)$

$$= 2^{n-1} \prod_{r=0}^{r=n-1} \sin\left(\phi + \frac{r\pi}{n}\right).$$

[*Put* $x=1$, *and* $\theta = 2\phi$, *in the equation of Art.* 362.]

20. $\cos n\phi = 2^{n-1} \sin\left(\phi + \frac{\pi}{2n}\right) \sin\left(\phi + \frac{3\pi}{2n}\right) \dots\dots \sin\left[\phi + \frac{2n-1}{2n}\pi\right]$.

[*Change* ϕ *into* $\phi + \frac{\pi}{2n}$ *in the formula of the preceding question.*]

21. $2^{n-1} \cos\phi \cos\left(\phi + \frac{\pi}{n}\right) \cos\left(\phi + \frac{2\pi}{n}\right) \dots\dots \cos\left(\phi + \frac{n-1}{n}\pi\right)$

$$= (-1)^{\frac{n}{2}} \sin n\phi, \text{ when } n \text{ is even,}$$

and $= (-1)^{\frac{n-1}{2}} \cos n\phi$, when n is odd.

[*Change* ϕ *into* $\phi + \frac{\pi}{2}$ *in the result of Ex.* 19.]

22. $2^{n-1}\cos\frac{\pi}{2n}\cos\frac{3\pi}{2n}\cos\frac{5\pi}{2n}\ldots\ldots\cos\frac{2n-1}{2n}\pi=\cos\frac{n\pi}{2}$.

23. $2^{n-1}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\sin\frac{5\pi}{2n}\ldots\ldots\sin\frac{2n-1}{2n}\pi=1$.

24. $\cos\frac{\pi}{n}\cos\frac{2\pi}{n}\ldots\ldots\cos\frac{(2n-1)\pi}{n}=\frac{(-1)^n-1}{2^{2n-1}}$.

25. Prove that

$$\frac{x^n-a^n\cos n\theta}{x^{2n}-2a^nx^n\cos n\theta+a^{2n}}=\frac{1}{nx^{n-1}}\sum_{r=0}^{r=n-1}\frac{x-a\cos\left(\theta+\frac{2r\pi}{n}\right)}{x^2-2ax\cos\left(\theta+\frac{2r\pi}{n}\right)+a^2}.$$

[*In the expression* (3) *of Art.* (362) *change x into $x+h$, expand and equate coefficients of h.*]

26. The circumference of a circle of radius r is divided into $2n$ equal parts at points P_1, P_2, ...P_{2n}; if chords be drawn from P_1 to the other points, prove that

$$P_1P_2 \cdot P_1P_3 \ldots\ldots P_1P_n=r^{n-1}\sqrt{n}.$$

Also, if O be the middle point of the arc P_1P_{2n}, prove that

$$OP_1 \cdot OP_2 \ldots\ldots OP_n=\sqrt{2}\,r^n.$$

27. If $A_1A_2\ldots\ldots A_{2n+1}$ be a regular polygon of n sides, inscribed in a circle of radius a, and OA_{n+1} be a diameter, prove that

$$OA_1 \cdot OA_2 \ldots\ldots OA_n=a^n.$$

28. $A_1A_2\ldots\ldots A_n$ is a regular polygon of n sides. From O the centre of the polygon a line is drawn meeting the incircle in P_1 and the circum-circle in P_2.

Prove that the product of the perpendiculars on the sides drawn from P_1 is to the product of the perpendiculars from P_2 as

$$\cos^n\frac{\pi}{n}\cot^2\frac{n\theta}{2}\text{ to }1,$$

θ being the angle between OPP_1 and OA_1.

29. $ABCD\ldots\ldots$ is a regular polygon which is inscribed in a circle of radius a and centre O; prove that

$$PA^2 . PB^2 . PC^2 \ldots\ldots = r^{2n}-2a^nr^n\cos n\theta+a^{2n},$$

where OP is r and the angle AOP is θ.

Prove also that the sum of the angles that AP, BP, CP,...... make with OP is $\tan^{-1}\frac{r^n\sin n\theta}{r^n\cos n\theta-a^n}$.

Resolution of sin θ and cos θ into factors.

369. *To express sin θ as a product of an infinite series of factors.*

We have $$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2\sin\frac{\theta}{2}\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)\ \dots\dots\dots(1).$$

Similarly in (1) changing θ into $\frac{\theta}{2}$ and $\frac{\pi}{2}+\frac{\theta}{2}$ successively, we have

$$\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2^2}\sin\left(\frac{\pi}{2}+\frac{\theta}{2^2}\right) = 2\sin\frac{\theta}{2^2}\sin\left(\frac{2\pi}{2^2}+\frac{\theta}{2^2}\right),$$

and $$\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = 2\sin\left(\frac{\pi}{2^2}+\frac{\theta}{2^2}\right).\sin\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\theta}{2^2}\right)$$

$$= 2\sin\left(\frac{\pi}{2^2}+\frac{\theta}{2^2}\right).\sin\left(\frac{3\pi}{2^2}+\frac{\theta}{2^2}\right).$$

Substituting these values in the right-hand side of (1) we have, after rearranging,

$$\sin\theta = 2^3\sin\frac{\theta}{2^2}\sin\frac{\pi+\theta}{2^2}\sin\frac{2\pi+\theta}{2^2}\sin\frac{3\pi+\theta}{2^2}\dots(2).$$

Applying once more the formula (1) to each of the terms on the right hand of (2) and arranging, we have

$$\sin\theta = 2^7\sin\frac{\theta}{2^3}\sin\frac{\pi+\theta}{2^3}\sin\frac{2\pi+\theta}{2^3}\sin\frac{3\pi+\theta}{2^3}\sin\frac{4\pi+\theta}{2^3}$$

$$\sin\frac{5\pi+\theta}{2^3}\sin\frac{6\pi+\theta}{2^3}\sin\frac{7\pi+\theta}{2^3}\ \dots\dots\dots(3).$$

Continuing this process we have finally

$$\sin\theta = 2^{p-1}\sin\frac{\theta}{p}\sin\frac{\pi+\theta}{p}\sin\frac{2\pi+\theta}{p}\dots\sin\frac{(p-1)\pi+\theta}{p}$$

$$\dots\dots\dots\dots(4),$$

where p is a power of 2.

The last factor in (4)

$$=\sin\left[\pi-\frac{\pi-\theta}{p}\right]=\sin\frac{\pi-\theta}{p}.$$

The last factor but one

$$=\sin\frac{(p-2)\pi+\theta}{p}=\sin\left[\pi-\frac{2\pi-\theta}{p}\right]=\sin\frac{2\pi-\theta}{p},$$

and so on.

Hence, taking together the second and last factors, the third and next to last, and so on, the equation (4) becomes

$$\sin\theta=2^{p-1}\sin\frac{\theta}{p}\left\{\sin\frac{\pi+\theta}{p}\sin\frac{\pi-\theta}{p}\right\}\left\{\sin\frac{2\pi+\theta}{p}\sin\frac{2\pi-\theta}{p}\right\}$$

$$\ldots\ldots\ldots\ldots(5).$$

The last factor is

$$\sin\frac{\frac{p}{2}\pi+\theta}{p}$$

which $$=\sin\left(\frac{\pi}{2}+\frac{\theta}{p}\right)=\cos\frac{\theta}{p}.$$

Hence (5) is

$$\sin\theta=2^{p-1}\sin\frac{\theta}{p}\left[\sin^2\frac{\pi}{p}-\sin^2\frac{\theta}{p}\right]\left[\sin^2\frac{2\pi}{p}-\sin^2\frac{\theta}{p}\right]\ldots$$

$$\ldots\left[\sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p}-\sin^2\frac{\theta}{p}\right].\cos\frac{\theta}{p}\ldots\ldots(6).$$

Divide both sides of (6) by $\sin\frac{\theta}{p}$ and make θ zero.

Since $$\left[\frac{\sin\theta}{\sin\frac{\theta}{p}}\right]_{\theta=0}=\left[p\frac{\sin\theta}{\theta}\cdot\frac{\frac{\theta}{p}}{\sin\frac{\theta}{p}}\right]_{\theta=0}=p,$$

we have

$$p = 2^{p-1} \cdot \sin^2\frac{\pi}{p} \cdot \sin^2\frac{2\pi}{p} \sin^2\frac{3\pi}{p} \ldots \sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p} \quad \ldots(7).$$

Dividing (6) by (7), we have

$$\sin\theta = p\sin\frac{\theta}{p}\left[1 - \frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{\pi}{p}}\right]\left[1 - \frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{2\pi}{p}}\right]\left[1 - \frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{3\pi}{p}}\right]\ldots$$

$$\ldots\left[1 - \frac{\sin^2\frac{\theta}{p}}{\sin^2\left(\frac{p}{2}-1\right)\frac{\pi}{p}}\right]\cos\frac{\theta}{p} \quad \ldots\ldots(8).$$

Now make p indefinitely great.

Since

$$\left[p\sin\frac{\theta}{p}\right]_{p=\infty} = \left[\frac{\sin\frac{\theta}{p}}{\frac{\theta}{p}} \cdot \theta\right]_{p=\infty} = \theta \text{ (Art. 228)},$$

$$\left[\frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{\pi}{p}}\right]_{p=\infty} = \left[\frac{\sin^2\frac{\theta}{p}}{\frac{\theta^2}{p^2}} \frac{\frac{\pi^2}{p^2}}{\sin^2\frac{\pi}{p}} \frac{\theta^2}{\pi^2}\right] = \frac{\theta^2}{\pi^2} \text{ (Art. 228)},$$

and so on, we have

$$\boldsymbol{\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)} \ldots \text{ ad inf.}$$

This theorem may be written in the form

$$\sin\theta = \theta \prod_{r=1}^{r=\infty}\left(1 - \frac{\theta^2}{r^2\pi^2}\right).$$

370. *To express $\cos\theta$ as a product of an infinite series of factors.*

In equation (4) of Art. 369 write for θ the quantity $\frac{\pi}{2}+\theta$, and the equation becomes

$$\cos\theta = 2^{p-1}\sin\frac{\pi+2\theta}{2p}\sin\frac{3\pi+2\theta}{2p}\sin\frac{5\pi+2\theta}{2p}\dots$$
$$\sin\frac{(2p-1)\pi+2\theta}{2p}\;\dots\dots(1).$$

The last factor

$$=\sin\left[\pi-\frac{\pi-2\theta}{2p}\right]=\sin\frac{\pi-2\theta}{2p},$$

the last but one

$$=\sin\left[\frac{(2p-3)\pi+2\theta}{2p}\right]=\sin\frac{3\pi-2\theta}{2p},$$

and so on.

Hence taking the factors in pairs, as before, we have

$$\cos\theta=2^{p-1}\left[\sin\frac{\pi+2\theta}{2p}\sin\frac{\pi-2\theta}{2p}\right]\left[\sin\frac{3\pi+2\theta}{2p}\sin\frac{3\pi-2\theta}{2p}\right]\dots$$
$$=2^{p-1}\left[\sin^2\frac{\pi}{2p}-\sin^2\frac{2\theta}{2p}\right]\left[\sin^2\frac{3\pi}{2p}-\sin^2\frac{2\theta}{2p}\right]\dots(2).$$

In (2) make θ zero and we have

$$1=2^{p-1}.\sin^2\frac{\pi}{2p}.\sin^2\frac{3\pi}{2p}.\sin^2\frac{5\pi}{2p}\;\dots\dots\dots(3).$$

Dividing (2) by (3), we have

$$\cos\theta=\left[1-\frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{\pi}{2p}}\right]\left[1-\frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{3\pi}{2p}}\right]\left[1-\frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{5\pi}{2p}}\right]\dots$$
$$\dots\left[1-\frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{(p-1)\pi}{2p}}\right]\;\dots\dots(4).$$

In (4) make p infinite; then, as in the last article, we have

$$\cos\theta = \left[1 - \frac{4\theta^2}{\pi^2}\right]\left[1 - \frac{4\theta^2}{3^2\pi^2}\right]\left[1 - \frac{4\theta^2}{5^2\pi^2}\right] \dots \text{ad inf.}$$

This theorem may be written in the form

$$\cos\theta = \prod_{r=1}^{r=\infty}\left\{1 - \frac{4\theta^2}{(2r-1)^2\pi^2}\right\}.$$

Since $\cos\theta = \dfrac{\sin 2\theta}{2\sin\theta}$, the product of $\cos\theta$ may be derived from the products for $\sin 2\theta$ and $\sin\theta$.

371. The equation (4) of Art. 369 may, by means of Art. 362, be shewn to be true for all integral values of p. For we have

$$x^{2p} - 2x^p\cos p\phi + 1$$
$$= \{x^2 - 2x\cos\phi + 1\}\left\{x^2 - 2x\cos\left(\phi + \frac{2\pi}{p}\right) + 1\right\}$$
$$\left\{x^2 - 2x\cos\left(\phi + \frac{4\pi}{p}\right) + 1\right\} \text{......to } p \text{ factors.}$$

Put $x = 1$, and we have

$$2(1 - \cos p\phi) = \{2 - 2\cos\phi\}\left\{2 - 2\cos\left(\phi + \frac{2\pi}{p}\right)\right\} \text{...... to } p \text{ factors.}$$

i.e. $4\sin^2\dfrac{p\phi}{2} = 4\sin^2\dfrac{\phi}{2} \cdot 4\sin^2\left(\dfrac{\phi}{2} + \dfrac{\pi}{p}\right) \cdot 4\sin^2\left(\dfrac{\phi}{2} + \dfrac{\pi}{p}\right)$...to p factors.

Put $\dfrac{p\phi}{2} = \theta$, and extract the square root of both sides. We have then

$$\pm\sin\theta = 2^{p-1}\sin\frac{\theta}{p} \cdot \sin\frac{\pi+\theta}{p} \cdot \sin\frac{2\pi+\theta}{p} \text{......} \sin\frac{(p-1)\pi+\theta}{p} \text{....(1).}$$

If θ lie between 0 and π all the factors on the right-hand side of (1) are positive and so also is $\sin\theta$. Hence the ambiguity should be replaced by the positive sign.

If θ lie between π and 2π, all the factors on the right-hand side are positive except the last, which is negative.

Hence the product is negative and so also is $\sin\theta$, so that in this case also the positive sign is to be taken.

Similarly in any other case it may be shewn that the positive sign must be taken, and we have, for all integral values of p,

$$\sin\theta = 2^{p-1}\sin\frac{\theta}{p} \cdot \sin\frac{\pi+\theta}{p}\sin\frac{2\pi+\theta}{p} \text{......} \sin\frac{(p-1)\pi+\theta}{p}.$$

372. *Sinh θ and cosh θ in products.*

By Art. 314 we have

$$\sinh\theta = -i\sin(\theta i) \text{ and } \cosh\theta = \cos(\theta i).$$

Also the series of Arts. 369 and 370, being formed on the Addition Theorem are, by Art. 310, true when for θ we read θi.

$$\therefore \sinh\theta = -i\times\theta i\left(1-\frac{\theta^2 i^2}{\pi^2}\right)\left(1-\frac{\theta^2 i^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2 i^2}{3^2\pi^2}\right)\ldots\ldots \quad (1)$$

$$= \theta\left(1+\frac{\theta^2}{\pi^2}\right)\left(1+\frac{\theta^2}{2^2\pi^2}\right)\left(1+\frac{\theta^2}{3^2\pi^2}\right)\ldots\ldots \text{ ad inf.}$$

$$\text{and } \cosh\theta = \left(1-\frac{4\theta^2 i^2}{\pi^2}\right)\left(1-\frac{4\theta^2 i^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2 i^2}{5^2\pi^2}\right)\ldots \text{ ad inf.}$$

$$= \left(1+\frac{4\theta^2}{\pi^2}\right)\left(1+\frac{4\theta^2}{3^2\pi^2}\right)\left(1+\frac{4\theta^2}{5^2\pi^2}\right)\ldots\ldots\ldots \quad (2).$$

The products (1) and (2) are convergent. For we know (C. Smith's *Algebra*, Art. 333) that the infinite product $\Pi(1+u_n)$ is convergent if the series Σu_n be convergent.

In the case of (1), Σu_n

$$= \frac{\theta^2}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\right),$$

and the latter series is known to be convergent.

373. Sums of powers of the reciprocals of all natural numbers.

From the results of Arts. 369 and 370 we can deduce the sums of some interesting series.

From Arts. 369 and 280 we have

$$\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots\ldots \text{ ad inf.}$$

$$= \frac{\sin\theta}{\theta} = 1-\frac{\theta^2}{\lfloor 3}+\frac{\theta^4}{\lfloor 5}\ldots\ldots \text{ ad inf.}$$

Taking the logarithms of both sides, we have

$$\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\log\left(1-\frac{\theta^2}{3^2\pi^2}\right)+\ldots\ldots$$

$$=\log\left[1-\frac{\theta^2}{6}+\frac{\theta^4}{120}-\ldots\right]\ldots\ldots(1).$$

Now, by Art. 256, we have

$$\log\left(1-\frac{\theta^2}{\pi^2}\right)=-\left[\frac{\theta^2}{\pi^2}+\frac{1}{2}\frac{\theta^4}{\pi^4}+\frac{1}{3}\frac{\theta^6}{\pi^6}+\ldots\right],$$

$$\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)=-\left[\frac{\theta^2}{2^2\pi^2}+\frac{1}{2}\frac{\theta^4}{2^4\pi^4}+\frac{1}{3}\frac{\theta^6}{2^6\pi^6}+\ldots\right]$$

$$\ldots\ldots\ldots\ldots\ldots$$

so that (1) gives

$$-\frac{\theta^2}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right]-\frac{1}{2}\frac{\theta^4}{\pi^4}\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\right]$$

$$-\frac{1}{3}\frac{\theta^6}{\pi^6}\left[\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\ldots\right]\ldots\ldots$$

$$=\log\left[1-\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\ldots\right)\right]$$

$$=-\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\ldots\right)-\frac{1}{2}\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\ldots\right)^2-\ldots$$

$$=-\frac{\theta^2}{6}+\theta^4\left(\frac{1}{120}-\frac{1}{2}\cdot\frac{1}{36}\right)-\ldots\ldots$$

$$=-\frac{\theta^2}{6}-\frac{\theta^4}{180}-\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Since equation (2) is true for all values of θ the coefficients of θ^2 on both sides must be the same, and similarly those of θ^4, and so on.

Hence we have

$$-\frac{1}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots \text{ ad inf.}\right)=-\frac{1}{6},$$

$$-\frac{1}{2}\frac{1}{\pi^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots\dots\dots\right)=-\frac{1}{180},$$

....

Hence $$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots=\frac{\pi^2}{6} \quad \dots\dots\dots\dots (3),$$

and $$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots=\frac{\pi^4}{90} \quad \dots\dots\dots\dots (4),$$

....

374. By proceeding in a similar manner with the result of Art. 370 we have

$$\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\dots$$

$$=\cos\theta=1-\frac{\theta^2}{\underline{|2}}+\frac{\theta^4}{\underline{|4}}\dots,$$

so that

$$\log\left(1-\frac{4\theta^2}{\pi^2}\right)+\log\left(1-\frac{4\theta^2}{3^2\pi^2}\right)+\log\left(1-\frac{4\theta^2}{5^2\pi^2}\right)$$

$$+\dots=\log\left[1-\frac{\theta^2}{2}+\frac{\theta^4}{24}-\dots\right].$$

Hence as before

$$\frac{-4\theta^2}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)-\frac{1}{2}\frac{16\theta^4}{\pi^4}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots\right)+\dots$$

$$=\log\left[1-\left(\frac{\theta^2}{2}-\frac{\theta^4}{24}+\dots\right)\right]$$

$$=-\left(\frac{\theta^2}{2}-\frac{\theta^4}{24}+\dots\right)-\frac{1}{2}\left(\frac{\theta^2}{2}-\frac{\theta^4}{24}+\dots\right)^2+\dots$$

$$=-\frac{\theta^2}{2}+\frac{\theta^4}{24}+\dots-\frac{1}{2}\left(\frac{\theta^4}{4}-\dots\right)-=-\frac{\theta^2}{2}-\frac{\theta^4}{12}-\dots.$$

Hence, equating coefficients of θ^2 and θ^4, we have

$$-\frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)=-\frac{1}{2},$$

$$-\frac{8}{\pi^4}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots\right)=-\frac{1}{12},$$

....

and hence $$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\frac{\pi^2}{8}\ \dots\dots\dots\dots(1),$$

and $$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots=\frac{\pi^4}{96}\ \dots\dots\dots\dots(2)$$

....

375. *Wallis' Formula.*

In the expression of Art. 369 put $\theta=\frac{\pi}{2}$, and we have

$$1=\frac{\pi}{2}\left[1-\frac{1}{2^2}\right]\left[1-\frac{1}{4^2}\right]\left[1-\frac{1}{6^2}\right]\dots\dots \text{ ad inf.}$$

$$=\frac{\pi}{2}\frac{1.3}{2^2}\cdot\frac{3.5}{4^2}\cdot\frac{5.7}{6^2}\dots\dots\frac{(2n-3)(2n-1)}{(2n-2)^2}\cdot\frac{(2n-1)(2n+1)}{(2n)^2},$$

where n is infinite,

i.e. $$\frac{2}{\pi}=\frac{1^2.3^2.5^2.7^2\dots\dots(2n-1)^2.(2n+1)}{2^2.4^2.6^2\dots\dots(2n)^2},$$

i.e. $$\frac{2.4.6\dots\dots2n}{1.3.5\dots\dots(2n-1)}=\sqrt{\frac{\pi}{2}(2n+1)}, \text{ where } n \text{ is infinite.}$$

It follows that when n is very great (but not necessarily infinite) then

$$\frac{2.4.6\dots\dots2n}{1.3.5\dots\dots(2n-1)}=\sqrt{\frac{\pi}{2}(2n+1)} \text{ very nearly}$$

$$=\sqrt{n\pi}, \text{ ultimately.}$$

This is called Wallis' Formula, and gives in a simple form a very near approach to the product of the first n even numbers divided by the first n odd numbers when n is very great.

376. Ex. *Prove that*

$$\tan\theta = 8\theta\left[\frac{1}{\pi^2-4\theta^2}+\frac{1}{3^2\pi^2-4\theta^2}+\frac{1}{5^2\pi^2-4\theta^2}+\dots\dots\right].$$

From Art. 370 we have

$$\log\cos\theta = \log\left(1-\frac{4\theta^2}{\pi^2}\right)+\log\left(1-\frac{4\theta^2}{3^2\pi^2}\right)+\log\left(1-\frac{4\theta^2}{5^2\pi^2}\right)+\dots\dots(1).$$

In this equation substituting $\theta+h$ for θ we have

$$\log\cos(\theta+h) = \log\left[1-\frac{4}{\pi^2}(\theta+h)^2\right]+\log\left[1-\frac{4}{3^2\pi^2}(\theta+h)^2\right]+\dots\dots\dots(2).$$

Now $\log\cos(\theta+h)=\log[\cos\theta(\cos h-\tan\theta\sin h)]$

$$=\log\cos\theta+\log\left[1-\frac{h^2}{\underline{|2}}+\dots\dots-\tan\theta\left(h-\frac{h^3}{\underline{|3}}+\dots\dots\right)\right] \text{ (Art. 280)}$$

$$=\log\cos\theta+\log[1-h\tan\theta+\text{ higher powers of } h]$$

$$=\log\cos\theta-h\tan\theta+\text{powers of } h. \quad \text{(Art. 256.)}$$

Also $\log\left[1-\frac{4}{\pi^2}(\theta+h)^2\right]=\log\frac{\pi^2-4\theta^2}{\pi^2}+\log\left[1-\frac{8\theta h}{\pi^2-4\theta^2}+\dots\dots\right]$

$$=\log\left[1-\frac{4\theta^2}{\pi^2}\right]-\frac{8\theta h}{\pi^2-4\theta^2}+\text{powers of } h,$$

and $\log\left[1-\frac{4}{3^2\pi^2}(\theta+h)^2\right]$

$$=\log\left[1-\frac{4\theta^2}{3^2\pi^2}\right]-\frac{8\theta h}{3^2\pi^2-4\theta^2}+\text{powers of } h.$$

....

Substituting these values in (2) and equating on each side the coefficients of $-h$ we have

$$\tan\theta=\frac{8\theta}{\pi^2-4\theta^2}+\frac{8\theta}{3^2\pi^2-4\theta^2}+\frac{8\theta}{5^2\pi^2-4\theta^2}+\dots\dots\dots\dots(3)$$

$$=\sum_{r=0}^{r=\infty}\frac{8\theta}{(2r+1)^2\pi^2-4\theta^2}.$$

The series (3) may also be written

$$\tan\theta=\frac{2}{\pi-2\theta}-\frac{2}{\pi+2\theta}+\frac{2}{3\pi-2\theta}-\frac{2}{3\pi+2\theta}+\dots\dots$$

[The student who is acquainted with the Differential Calculus will observe that equation (3) is obtained by differentiating (1) with respect to θ.]

377. Ex. *Prove that*

$$\cosh 2\alpha - \cos 2\theta$$

$$= 2 \sin^2\theta \left[1 + \frac{\alpha^2}{\theta^2}\right]\left[1 + \left(\frac{\alpha}{\pi+\theta}\right)^2\right]$$

$$\left[1 + \left(\frac{\alpha}{\pi-\theta}\right)^2\right]\left[1 + \left(\frac{\alpha}{2\pi+\theta}\right)^2\right]\left[1 + \left(\frac{\alpha}{2\pi-\theta}\right)^2\right] \ldots\ldots \textit{ad inf.}$$

$$= 2 \sin^2\theta\, \Pi\left[1 + \left(\frac{\alpha}{\theta + r\pi}\right)^2\right],$$

where r is zero or any positive or any negative integer.

We have

$$\cosh 2\alpha - \cos 2\theta = \cos 2\alpha i - \cos 2\theta = 2 \sin(\theta + \alpha i) \sin(\theta - \alpha i)$$

$$= 2(\theta + \alpha i)\left[1 - \frac{(\theta+\alpha i)^2}{\pi^2}\right]\left[1 - \frac{(\theta+\alpha i)^2}{2^2\pi^2}\right]\ldots\ldots$$

$$\times (\theta - \alpha i)\left[1 - \frac{(\theta-\alpha i)^2}{\pi^2}\right]\left[1 - \frac{(\theta-\alpha i)^2}{2^2\pi^2}\right]\ldots\ldots\ldots(1).$$

Now $\left[1 - \frac{(\theta+\alpha i)^2}{\pi^2}\right]\left[1 - \frac{(\theta-\alpha i)^2}{\pi^2}\right]$

$$= \left[\frac{(\pi+\theta+\alpha i)(\pi-\theta-\alpha i)}{\pi^2}\right]\left[\frac{(\pi+\theta-\alpha i)(\pi-\theta+\alpha i)}{\pi^2}\right]$$

$$= \frac{(\pi+\theta)^2+\alpha^2}{\pi^2} \cdot \frac{(\pi-\theta)^2+\alpha^2}{\pi^2}.$$

Hence (1) gives

$$\cosh 2\alpha - \cos 2\theta = 2(\theta^2+\alpha^2)\left[\frac{(\pi+\theta)^2+\alpha^2}{\pi^2}\right]\left[\frac{(\pi-\theta)^2+\alpha^2}{\pi^2}\right]\left[\frac{(2\pi+\theta)^2+\alpha^2}{\pi^2}\right]$$

$$\left[\frac{(2\pi-\theta)^2+\alpha^2}{\pi^2}\right] \text{ad inf.}\ldots\ldots\ldots\ldots(2).$$

In (2) put $\alpha = 0$ and we have

$$2\sin^2\theta = 2\theta^2 \cdot \frac{(\pi+\theta)^2}{\pi^2} \cdot \frac{(\pi-\theta)^2}{\pi^2} \cdot \frac{(2\pi+\theta)^2}{\pi^2} \cdot \frac{(2\pi-\theta)^2}{\pi^2} \text{ ad inf.}\ldots\ldots\ldots\ldots(3).$$

Dividing (2) by (3) we have

$$\cosh 2\alpha - \cos 2\theta$$

$$= 2\sin^2\theta\left[1 + \frac{\alpha^2}{\theta^2}\right]\left[1 + \left(\frac{\alpha}{\pi-\theta}\right)^2\right]\left[1 + \left(\frac{\alpha}{\pi+\theta}\right)^2\right]\left[1 + \left(\frac{\alpha}{2\pi-\theta}\right)^2\right]$$

$$\left[1 + \left(\frac{\alpha}{2\pi+\theta}\right)^2\right]\ldots\ldots \text{ ad inf.}$$

The factors of $\cosh 2\alpha + \cos 2\theta$ may now be obtained by changing θ into $\theta + \frac{\pi}{2}$ and they are found to be $2\cos^2\theta\, \Pi \left\{1 + \left(\frac{\alpha}{\theta + r\pi}\right)^2\right\}$ where r is any odd integer, positive or negative.

EXAMPLES. LXV.

Prove that

1. $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$ ad inf. $= \frac{\pi^2}{12}$.

2. $\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots\ldots$ ad inf. $= 6\,\frac{(2\pi)^6}{\lfloor 9}$.

3. $\frac{1}{1\,.\,2} + \frac{1}{2\,.\,4} + \frac{1}{3\,.\,6} + \frac{1}{4\,.\,8} + \ldots\ldots$ ad inf. $= \frac{\pi^2}{12}$.

4. $\frac{1}{3^4} + \frac{3}{5^4} + \frac{6}{7^4} + \frac{10}{9^4} + \ldots\ldots$ ad inf. $= \frac{\pi^2}{64}\left(1 - \frac{\pi^2}{12}\right)$.

5. Prove that the sum of the products, taken two and two together, of the reciprocals of the squares of all odd numbers is $\frac{\pi^4}{384}$.

6. Prove that the sum of the products, taken two and two together, of the reciprocals of the squares of all even numbers is $\frac{\pi^4}{120}$.

Prove that

7. $\cot\theta = \frac{1}{\theta} - \frac{2\theta}{\pi^2 - \theta^2} - \frac{2\theta}{2^2\pi^2 - \theta^2} - \ldots\ldots$

$$= \frac{1}{\theta} + \frac{1}{\theta - \pi} + \frac{1}{\theta + \pi} + \frac{1}{\theta - 2\pi} + \frac{1}{\theta + 2\pi} \ldots\ldots \text{ ad inf.}$$

8. $\operatorname{cosec}\theta = \frac{1}{\theta} - \frac{1}{\theta - \pi} - \frac{1}{\theta + \pi} + \frac{1}{\theta - 2\pi} + \frac{1}{\theta + 2\pi} - \frac{1}{\theta - 3\pi} - \frac{1}{\theta + 3\pi} \ldots\ldots$

$$= \frac{1}{\theta} + 2\theta \sum_{n=1}^{n=\infty} \frac{(-1)^n}{\theta^2 - n^2\pi^2},$$

and hence that

$$\frac{1 + \theta\operatorname{cosec}\theta}{2\theta^2} = \frac{1}{\theta^2} - \frac{1}{\theta^2 - \pi^2} + \frac{1}{\theta^2 - 2^2\pi^2} - \ldots\ldots \text{ ad inf.}$$

[*Use the relation* $\operatorname{cosec}\theta = \frac{1}{2}\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)$.]

9. $\frac{1}{4\pi}\sec\theta = \frac{1}{\pi^2 - 4\theta^2} - \frac{3}{3^2\pi^2 - 4\theta^2} + \frac{5}{5^2\pi^2 - 4\theta^2} - \ldots\ldots$ ad inf.

[*Use the relation* $2\sec\theta = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$.]

10. $\frac{1}{4}\sec^2\theta = \frac{1}{(\pi-2\theta)^2} + \frac{1}{(\pi+2\theta)^2} + \frac{1}{(3\pi-2\theta)^2} + \frac{1}{(3\pi+2\theta)^2} + \dots$ ad inf.

[*Apply the process of Art.* 376 *to the result obtained.*]

11. $\operatorname{cosec}^2\theta = \frac{1}{\theta^2} + \frac{1}{(\theta-\pi)^2} + \frac{1}{(\theta+\pi)^2} + \frac{1}{(\theta-2\pi)^2} + \frac{1}{(\theta+2\pi)^2} + \dots$ ad inf.

Prove that

12. $$\frac{\sin(\alpha-\theta)}{\sin\alpha} = \left(1-\frac{\theta}{\alpha}\right)\left(1+\frac{\theta}{\pi-\alpha}\right)\left(1-\frac{\theta}{\pi+\alpha}\right)\left(1+\frac{\theta}{2\pi-\alpha}\right)\left(1-\frac{\theta}{2\pi+\alpha}\right)\dots\dots$$

$=\Pi\left(1-\frac{\theta}{\alpha+r\pi}\right)$, where r is any positive or negative integer or zero.

13. $\frac{\sin(\alpha+\theta)}{\sin\alpha} = \Pi\left(1+\frac{\theta}{\alpha+r\pi}\right)$, where r is any positive or negative integer, including zero.

14. $$\frac{\cos(\alpha+\theta)}{\cos\alpha} = \left(1+\frac{2\theta}{\pi+2\alpha}\right)\left(1-\frac{2\theta}{\pi-2\alpha}\right)\left(1+\frac{2\theta}{3\pi+2\alpha}\right)\left(1-\frac{2\theta}{3\pi-2\alpha}\right)\dots\dots$$

$=\Pi\left[1+\frac{2\theta}{2\alpha+r\pi}\right]$, where r is any odd integer positive or negative.

15. $\frac{\cos(\alpha-\theta)}{\cos\alpha} = \Pi\left[1-\frac{2\theta}{2\alpha+r\pi}\right]$, where r is any odd integer, positive or negative.

16. $$\frac{\cos\theta+\cos\alpha}{1+\cos\alpha} = \left[1-\frac{\theta^2}{(\pi+\alpha)^2}\right]\left[1-\frac{\theta^2}{(\pi-\alpha)^2}\right]\left[1-\frac{\theta^2}{(3\pi+\alpha)^2}\right]\left[1-\frac{\theta^2}{(3\pi-\alpha)^2}\right]\dots\dots$$

$$= \Pi\left[1-\frac{\theta^2}{(r\pi+\alpha)^2}\right],$$

where r is any odd integer positive or negative.

[*Multiply together the results of Exs.* 14 *and* 15 *and then change* 2θ *and* 2α *into* θ *and* α.]

17. $$\frac{\cos\theta-\cos\alpha}{1-\cos\alpha} = \left\{1-\frac{\theta^2}{\alpha^2}\right\}\left\{1-\frac{\theta^2}{(2\pi+\alpha)^2}\right\}\left\{1-\frac{\theta^2}{(2\pi-\alpha)^2}\right\}\left\{1-\frac{\theta^2}{(4\pi+\alpha)^2}\right\}\dots\dots$$

$$= \Pi\left[1-\frac{\theta^2}{(\alpha+r\pi)^2}\right],$$

where r is any even positive or negative integer, including zero.

Hence deduce the factors of $\cosh x - \cos\alpha$.

18. $$\frac{\sin\alpha-\sin\theta}{\sin\alpha}=\left(1-\frac{\theta}{\alpha}\right)\left(1-\frac{\theta}{\pi-\alpha}\right)\left(1+\frac{\theta}{\pi+\alpha}\right)\left(1+\frac{\theta}{2\pi-\alpha}\right)\left(1-\frac{\theta}{2\pi+\alpha}\right)\ldots\ldots$$

19. $2\cosh\theta+2\cos\alpha$

$$=4\cos^2\frac{\alpha}{2}\left[1+\frac{\theta^2}{(\alpha+\pi)^2}\right]\left[1+\frac{\theta^2}{(\alpha-\pi)^2}\right]\ldots\ldots$$

$$=4\cos^2\frac{\alpha}{2}\Pi\left[1+\frac{\theta^2}{(\alpha+r\pi)^2}\right],$$

where r is any integer positive or negative.

20. Prove that

$$\sinh nu=n\sinh u\prod_{r=1}^{r=n-1}\left[1+\frac{\sinh^2\frac{u}{2}}{\sin^2\frac{r\pi}{2n}}\right],$$

and deduce the expression for $\sinh u$ in the form of an infinite product of quadratic factors in u.

[*Start with the result, when θ is zero, of Ex.* 1, *Art.* 368. *In this result put ϕ equal to zero and divide.*]

21. Prove that the value of the infinite product

$$\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\ldots\ldots\text{ad inf.}$$

is $$\frac{1}{\pi}\sinh\pi.$$

22. A semicircle is divided into m equal parts and a concentric and similarly situated semicircle is divided into n equal parts. Every point of section of one semicircle is joined to every point of section of the other. Find the arithmetic mean of the squares of the joining lines and prove that when m and n are indefinitely increased the result is $a^2+b^2-\frac{8ab}{\pi^2}$, where a and b are the radii of the semicircles.

23. The radii of an infinite series of concentric circles are $a, \frac{a}{2}, \frac{a}{3}, \ldots$ From a point at a distance c ($>a$) from their common centre a tangent is drawn to each circle. Prove that

$$\sin\theta_1\sin\theta_2\sin\theta_3\ldots\ldots=\sqrt{\frac{c}{\pi a}\sin\frac{\pi a}{c}},$$

where θ_1, θ_2, θ_3...... are the angles that the tangents subtend at the common centre.

24. An infinite straight line is divided by an infinite number of points into portions each of length a. If any point P be taken so that y is its distance from the straight line and x is its distance measured along the straight line from one of the points of division, prove that the sum of the squares of the reciprocals of the distances of the point P from all the points of division is

$$\frac{\pi}{ay}\,\frac{\sinh\dfrac{2\pi y}{a}}{\cosh\dfrac{2\pi y}{a}-\cos\dfrac{2\pi x}{a}}.$$

[*Use the result of Ex.* 7.]

25. If a, b, c...... denote all the prime numbers 2, 3, 5...... prove that

$$\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{b^2}\right)\left(1-\frac{1}{c^2}\right)\ldots\ldots=\frac{6}{\pi^2},$$

and

$$\left(1+\frac{1}{a^2}\right)\left(1+\frac{1}{b^2}\right)\left(1+\frac{1}{c^2}\right)\ldots\ldots=\frac{15}{\pi^2}.$$

CHAPTER XXX.

PRINCIPLE OF PROPORTIONAL PARTS.

378. In the present chapter we shall consider the Principle of Proportional Parts, the truth of which we assumed in Chapter XI.

We then assumed that if n be any number and $n+1$ the next number, whose logarithms were given in our tables, and if h be any fraction, then, to 7 places of decimals, it is true that

$$\frac{\log(n+h)-\log n}{\log(n+1)-\log n}=h.$$

The truth of this statement we shall now consider.

379. Common Logarithms. We have, by Art. 260,

$$\log_{10}(n+h)-\log_{10}n=\log_{10}\frac{n+h}{n}=\mu\log_e\left(1+\frac{h}{n}\right),$$

where $$\mu\equiv{\cdot}43429448\ldots$$

Hence, by Art. 256, we have

$$\log_{10}(n+h)-\log_{10}n=\frac{\mu h}{n}-\frac{\mu}{2}\frac{h^2}{n^2}+\frac{\mu}{3}\frac{h^3}{n^3}-\ldots\ldots(1).$$

Now in our ordinary logarithm tables n contains 5 digits, *i.e.* n is not less than 10000. Hence, if h be less than unity, we have $\frac{\mu}{2}\frac{h^2}{n^2}$ less than

$$\frac{1}{2}(\cdot 43429448\ldots) \times \frac{1}{10^8},$$

i.e. less than $\frac{\cdot 21714724\ldots}{10^8}$, *i.e.* $< \cdot 0000000021\ldots.$

Also $\frac{\mu}{3}\frac{h^3}{n^3}$ is less than one-ten thousandth part of this.

Hence in (1) the omission of all the terms on the right-hand side after the first will make no difference at least as far as the *seventh* place of decimals. To seven places we therefore have

$$\log_{10}(n+h) - \log_{10} n = \frac{\mu h}{10}.$$

So $$\log_{10}(n+1) - \log_{10} n = \frac{\mu \,.\, 1}{10}.$$

Hence, by division,

$$\frac{\log_{10}(n+h) - \log_{10} n}{\log_{10}(n+1) - \log_{10} n} = h.$$

The principle assumed is therefore always true for the logarithms of ordinary numbers as given in our tables.

380. We may enquire what is the smallest number in the tables to which we can safely apply the principle of proportional parts. We must find that value of n which makes $\frac{\mu h^2}{2n^2} < \frac{1}{10^7}$, so that $n^2 > \frac{\mu}{2} \,.\, 10^7 \,.\, h^2$.

The greatest value of h being unity, we then have

$$n^2 > \frac{\mu}{2} \,.\, 10^7, \text{ i.e.} > 2171472\cdot 4\ldots\ldots$$

$$\therefore\ n > 1473.$$

The number 1473 is therefore the required least number.

381. Natural Sines. Suppose we have a table calculated for successive differences of angles, such that the number of radians in these successive differences is h.

[In the case of our ordinary tables h = number of radians in 1′

$$=\frac{\pi}{60\times 180}=\cdot 000290888\ldots,\quad i.e.\ h<\cdot 0003.]$$

Also let k be less than h. Then our principle was that

$$\frac{\sin(\theta+k)-\sin\theta}{\sin(\theta+h)-\sin\theta}=\frac{k}{h}.$$

We shall examine this assumption.

We have

$$\sin(\theta+k)-\sin\theta=\sin\theta\cos k+\cos\theta\sin k-\sin\theta$$

$$=\sin\theta\left[1-\frac{k^2}{\underline{|2}}+\frac{k^4}{\underline{|4}}-\ldots\right]+\cos\theta\left[k-\frac{k^3}{\underline{|3}}+\ldots\right]-\sin\theta$$

(Arts. 279 and 280)

$$=k\cos\theta-\frac{k^2}{\underline{|2}}\sin\theta-\frac{k^3}{\underline{|3}}\cos\theta\ldots$$

The ratio of the third term to the first $=\frac{1}{6}k^2$ and this is always less than $\frac{1}{6}(\cdot 0003)^2$, *i.e.* always less than $\cdot 00000002$. The third and higher terms may therefore be safely neglected, and we have

$$\sin(\theta+k)-\sin\theta=k\cos\theta-\frac{k^2}{\underline{|2}}\sin\theta\ \ldots\ldots\ (1).$$

The numerical ratio of the second term to the first term

$$=\frac{1}{2}k\tan\theta\ \ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

This ratio is small, *except when θ is nearly equal to* $\frac{\pi}{2}$.
Hence, except when the angle is nearly a right angle, the second term in (1) may be neglected, and we have

$$\sin(\theta + k) - \sin\theta = k\cos\theta.$$

So $$\sin(\theta + h) - \sin\theta = h\cos\theta,$$

and hence $$\frac{\sin(\theta + k) - \sin\theta}{\sin(\theta + h) - \sin\theta} = \frac{k}{h} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

When θ is very nearly a right angle we cannot say that

$$\sin(\theta + k) - \sin\theta = k\cos\theta,$$

and hence in this case the relation (3) does not hold and the difference in the sine is not proportional to the difference in the angle. In this case then the differences are **irregular.** At the same time the differences are **insensible**; for, when θ is nearly $\frac{\pi}{2}$, $k\cos\theta$ is very small. In fact $k\cos\theta$ has nothing but ciphers as far as the seventh place of decimals, so long as θ is within a few minutes of a right angle. Also

$$\frac{k^2}{\underline{|2}}\sin\theta \text{ is always} < \frac{(\cdot 0003)^2}{2}, \textit{ i.e.} < \cdot 00000005\ldots$$

Hence when the angle is nearly a right angle a comparatively small change in the sine will correspond to a comparatively large change in the angle; also at the same time these changes are irregular.

382. Natural Cosines. Since the cosine of an angle is equal to the sine of its complement this case reduces to that of the sine. The principle is therefore true except when the angle is nearly zero, in which case the differences are insensible and irregular.

383. Natural Tangents. With the same notation as before we have

$$\tan(\theta+k)-\tan\theta=\frac{\tan\theta+\tan k}{1-\tan\theta\tan k}-\tan\theta=\frac{\tan k\sec^2\theta}{1-\tan\theta\tan k}$$

$$=\tan k\sec^2\theta\,(1+\tan\theta\tan k+\tan^2\theta\tan^2 k\ldots)$$

$$=\sec^2\theta\left[k+\frac{k^3}{3}+\ldots\right]\left[1+\tan\theta\left(k+\frac{k^3}{3}+\ldots\right)\right.$$
$$\left.+\tan^2\theta\,(k^2+\ldots)\right] \quad \text{(Art. 281)}$$

$$=k\sec^2\theta+k^2\frac{\sin\theta}{\cos^3\theta}+k^3\sec^2\theta\left[\frac{1}{3}+\tan^2\theta\right]+\ldots\ldots \quad (1).$$

The third and higher terms may be omitted as before, except when θ is nearly a right angle.

Unless the quantity $k^2\dfrac{\sin\theta}{\cos^3\theta}$ be large we shall then have

$$\tan(\theta+k)-\tan\theta=k\sec^2\theta \ldots\ldots\ldots\ldots (2),$$

and the rule is approximately true.

When θ is $>\dfrac{\pi}{4}$ the second term of the equation (1) is $>2k^2$, so that taking the greatest value of k, viz. about ·0003, this would give a significant figure in the seventh place. The principle is therefore not true for angles greater than $\dfrac{\pi}{4}$, when the differences of the tabulated angles are 1′.

384. Natural Cotangents. As in the last article it can be shewn that the principle must not be relied upon for angles between 0 and 45°.

385. Natural Secant. We have $\sec(\theta+k)-\sec\theta$

$$=\frac{1}{\cos\theta\cos k-\sin\theta\sin k}-\frac{1}{\cos\theta}$$

$$=\sec\theta\left[\frac{1}{1-k\tan\theta-\frac{1}{2}k^2\ldots}-1\right]$$

$$=\sec\theta\left[k\tan\theta+k^2\left(\frac{1}{2}+\tan^2\theta\right)+\ldots\right]$$

$$=k\sec\theta\tan\theta+k^2\sec\theta\left(\frac{1}{2}+\tan^2\theta\right)+\ldots\ldots\ldots \quad (1).$$

The ratio of the second to the first term

$$=k\frac{\frac{1}{2}+\tan^2\theta}{\tan\theta}=k\left[\frac{1}{2}\cot\theta+\tan\theta\right].$$

This is small except when θ is nearly zero or $\frac{\pi}{2}$. Hence, except in these two cases, we have

$$\sec(\theta+k)-\sec\theta=k\tan\theta\sec\theta$$

and the rule is proved.

When θ is small the term $k\sec\theta\tan\theta$ is very small, so that the differences are insensible besides being irregular.

When θ is nearly $\frac{\pi}{2}$ this term is great, so that the differences are not insensible.

386. Natural Cosecant. Just as in the case of the secant it may be shewn that the differences are insensible and irregular when θ is nearly 90°, and irregular when θ is nearly zero. Otherwise the principle holds.

387. Tabular Logarithmic Sine. We have

$$L_{10}\sin(\theta+k)-L_{10}\sin\theta=\log_{10}\frac{\sin(\theta+k)}{\sin\theta}$$

$$=\log_{10}[\cos k+\cot\theta\sin k]=\log_{10}\left[1+k\cot\theta-\frac{k^2}{2}\dots\right]$$

(Arts. 279 and 280)

$$=\mu\left[k\cot\theta-\frac{k^2}{2}-\frac{1}{2}k^2\cot^2\theta+\dots\right]\quad\text{(Arts. 256 and 260)}$$

$$=\mu k\cot\theta-\frac{\mu k^2}{2}\operatorname{cosec}^2\theta\dots.$$

The numerical ratio of the second term to the first

$$=\frac{1}{2}k\cdot\frac{1}{\sin\theta\cos\theta}=\frac{k}{\sin 2\theta}.$$

This is small except when θ is near zero or a right angle.

Hence, with the exception of these two cases, we have

$$L\sin(\theta+k)-L\sin\theta=\mu\cot\theta\times k,$$

so that the rule holds in general.

If θ be small the term $\mu k\cot\theta$ is large, so that the differences are large as well as irregular. We cannot therefore apply the principle to small angles in the case of tables constructed with difference of 1′.

Even if the tables were constructed for differences of 10″ we are not sure of being free from error in the 7th place of decimals unless θ be $> 5°$.

If θ be nearly $\frac{\pi}{2}$ the terms $\mu k \cot\theta$ and $\frac{\mu k^2}{2}\operatorname{cosec}^2\theta$ are both small, so that if the angle be nearly a right angle the differences are insensible as well as irregular.

388. Tabular Logarithmic Cosine. The rule holds approximately, since the cosine is the complement of the sine, except when the angle is small, in which case the differences are insensible as well as irregular, and except when the angle is nearly a right angle, in which case the differences are large.

389. Tabular Logarithmic Tangent. Here

$$L\tan(\theta+k) - L\tan\theta = \log_{10}\frac{\tan(\theta+k)}{\tan\theta}$$

$$= \log_{10}\frac{1+\cot\theta\tan k}{1-\tan\theta\tan k} = \log_{10}\left[\frac{1+k\cot\theta}{1-k\tan\theta}\right]$$

$$= \log_{10}[(1+k\cot\theta)(1+k\tan\theta+k^2\tan^2\theta+\ldots)]$$

$$= \log_{10}\left[1+\frac{k}{\sin\theta\cos\theta}+\frac{k^2}{\cos^2\theta}+\ldots\right]$$

$$= \left[\frac{k}{\sin\theta\cos\theta}+\frac{k^2}{\cos^2\theta}-\frac{1}{2}\frac{k^2}{\sin^2\theta\cos^2\theta}+\ldots\right]$$

(Arts. 256 and 260)

$$= \frac{\mu k}{\sin\theta\cos\theta} - 2\mu k^2\frac{\cos 2\theta}{\sin^2 2\theta}+\ldots.$$

The numerical ratio of the second term to the first $= k\cot 2\theta$. This is small except when θ is near zero or a right angle.

Hence, with the exception of these two cases, we have

$$L\tan(\theta+k) - L\tan\theta = \frac{2\mu}{\sin 2\theta}\cdot k,$$

so that the principle is in general true.

In each of the exceptional cases $\frac{k}{\sin 2\theta}$ is not small, so that the differences are then irregular but not insensible.

The same statements are true for the tabular logarithmic cotangent.

390. Tabular Logarithmic Secant and Cosecant. We have

$$L \sec(\theta + k) - L \sec \theta = L \cos \theta - L \cos(\theta + k)$$

and $$L \operatorname{cosec}(\theta + k) - L \operatorname{cosec} \theta = L \sin \theta - L \sin(\theta + k).$$

Hence the results for the L sin and L cos are also true for the L cosec and L sec.

CHAPTER XXXI.

ERRORS OF OBSERVATION.

391. We have up to the present assumed that it is possible to observe any angles perfectly accurately. In practice this is by no means the case. Our observations are liable to two classes of errors, one due to the instruments themselves, which are hardly ever in *perfect* adjustment, and the other class due to mistakes on the part of the observer.

392. An error in any of our observations will clearly, in general, cause an error in the value of any quantity calculated from that observation. For example, if in Art. 192 there be a small error in the value of α, there will be a consequent error in the value of x which, as we see from the result of that article, depends on α.

393. The importance of an error in a length depends, in general, upon its ratio to that length. For example in measuring a piece of wood, about six feet long, a mistake of one inch would be a very serious error; in measuring a mile racecourse a mistake of one inch would be not worth considering; whilst in measuring the distance from the Earth to the Moon an error of one inch would be absolutely inappreciable.

394. We shall assume that the errors we have to consider are so small that their squares (when measured in radians if they be angles) may be neglected and we shall give some examples of finding the errors in derived quantities.

We shall assume that our tables and calculations are correct, so that we have not to deal with mistakes in calculation but only with errors in the original observation.

395. **Ex. 1.** *MP (Fig. Art. 42) is a vertical pole; at a point O distant a from its foot its angular elevation is found to be θ and its height then calculated; if there be an error δ in the observation of θ find the consequent error in the height.*

The calculated height $h=a\tan\theta$, clearly.

If the error δ be in excess, the real elevation is $\theta-\delta$, and hence the real height $h'=a\tan(\theta-\delta)$.

Hence the error $h-h'=a\tan\theta-a\tan(\theta-\delta)$

$$=a\frac{\sin\delta}{\cos\theta\cos(\theta-\delta)}=a\sec^2\theta\,.\,\delta,$$

if we neglect squares and higher powers of δ.

The ratio of the error to the calculated height

$$=\delta\sec^2\theta\div\tan\theta=\frac{2\delta}{\sin 2\theta}.$$

Except when $\sin 2\theta$ is small this ratio is small since δ is small. It is least when $\sin 2\theta$ is greatest, *i.e.* when θ is $\frac{\pi}{4}$.

The ratio is large when θ is near zero and when it is near $\frac{\pi}{2}$.

Hence a *small* mistake in the angle makes a relatively large mistake in the calculated result when the angle subtended is very small or when it is very nearly $\frac{\pi}{2}$.

When θ is small, both the calculated height and the absolute error, viz. $a \tan\theta$ and $a \sec^2\theta \, . \, \delta$, are small, but the latter is great *compared with the former.*

When θ is nearly 90°, both these quantities are great.

Ex. 2. *The height of a tower is found as in Art.* 192*; if there be an error θ in excess in the angle α, find the corresponding correction to be made in the height.*

The real value of α is $\alpha - \theta$; hence the real value of the height is found by substituting $\alpha - \theta$ for α in the obtained answer, and therefore

$$= a \frac{\sin(\alpha - \theta)\sin\beta}{\sin(\beta - \alpha + \theta)} = a \sin\beta \frac{\sin\alpha\cos\theta - \cos\alpha\sin\theta}{\sin(\beta-\alpha)\cos\theta + \cos(\beta-\alpha)\sin\theta}$$

$$= \frac{a\sin\alpha\sin\beta}{\sin(\beta-\alpha)} \cdot \frac{1-\theta\cot\alpha}{1+\theta\cot(\beta-\alpha)} \qquad \text{(Art. 280.)}$$

$$= \frac{a\sin\alpha\sin\beta}{\sin(\beta-\alpha)} [1-\theta\cot\alpha][1-\theta\cot(\beta-\alpha)+\ldots\ldots]$$

$$= \frac{a\sin\alpha\sin\beta}{\sin(\beta-\alpha)} [1-\theta\{\cot(\beta-\alpha)+\cot\alpha\}]$$

$$= \frac{a\sin\alpha\sin\beta}{\sin(\beta-\alpha)} - \theta \frac{a\sin^2\beta}{\sin^2(\beta-\alpha)}.$$

The error in the calculated height is therefore $\theta \, . \, \dfrac{a\sin^2\beta}{\sin^2(\beta-\alpha)}$, and is one of excess.

Also the ratio of the error to the calculated height

$$= \frac{\theta\sin\beta}{\sin\alpha\sin(\beta-\alpha)}.$$

Ex. 3. *The angles of a triangle are calculated from the sides $a = 2$, $b = 3$, and $c = 4$, but it is found that the side c is overestimated by a small quantity δ; find the consequent errors in the angles.*

From the given values of the sides we easily have

$$\cos A = \frac{7}{8}, \qquad \cos B = \frac{11}{16}, \qquad \cos C = -\frac{1}{4},$$

$$\sin A = \frac{2\sqrt{15}}{16}, \quad \sin B = \frac{3\sqrt{15}}{16}, \quad \text{and} \quad \sin C = \frac{4\sqrt{15}}{16}.$$

Corresponding to the value $4-\delta$, let the values of the angles be $A-\theta_1$, $B-\theta_2$, and $C-\theta_3$.

Then $$\cos(A-\theta_1)=\frac{3^2+(4-\delta)^2-2^2}{2(4-\delta).3}=\frac{21-8\delta}{24}\left(1-\frac{\delta}{4}\right)^{-1},$$

i.e. $$\cos A+\sin A.\theta_1=\frac{1}{24}[21-8\delta]\left[1+\frac{\delta}{4}\right]=\frac{1}{24}\left[21-\frac{11}{4}\delta\right],$$

[Arts. 279 and 280]

i.e. $$\frac{7}{8}+\frac{2\sqrt{15}}{16}\theta_1=\frac{7}{8}-\frac{11}{96}\delta,$$

so that $$\theta_1=-\frac{11\sqrt{15}}{180}\delta \quad\text{.........(1).}$$

Also $$\cos(B-\theta_2)=\frac{(4-\delta)^2+2^2-3^2}{2(4-\delta).2}=\frac{11-8\delta}{16}\left(1-\frac{\delta}{4}\right)^{-1},$$

i.e. $$\frac{11}{16}+\sin B.\theta_2=\frac{1}{16}[11-8\delta]\left[1+\frac{\delta}{4}\right]=\frac{1}{16}\left[11-\frac{21}{4}\delta\right],$$

i.e. $$\frac{3\sqrt{15}}{16}\theta_2=-\frac{21}{64}\delta,$$

so that $$\theta_2=-\frac{7\sqrt{15}}{60}\delta \quad\text{.........(2).}$$

Also $$\cos(C-\theta_3)=\frac{2^2+3^2-(4-\delta)^2}{2.2.3}=\frac{-3+8\delta}{12},$$

i.e. $$-\frac{1}{4}+\frac{4\sqrt{15}}{16}\theta_3=-\frac{1}{4}+\frac{2\delta}{3},$$

so that $$\theta_3=\frac{8\sqrt{15}}{45}\delta.$$

The errors in the angles are therefore

$$\frac{-11\sqrt{15}}{180}\delta,\quad \frac{-21\sqrt{15}}{180}\delta,\quad\text{and}\quad \frac{32\sqrt{15}}{180}\delta \text{ radians,}$$

so that the smallest angle has the least error.

We note, as might have been assumed *a priori*, that the sum of the errors in the three angles is zero. This is necessarily so, since the sum of the angles of any triangle is always two right angles.

EXAMPLES. LXVI.

1. The height of a hill is found by measuring the angles of elevation α and β of the top and bottom of a tower of height b on the top of the hill. Prove that the error in the height h caused by an error θ in the measurement of the angle α is $\theta \, . \cos\beta \sec\alpha \operatorname{cosec}(\alpha-\beta)$ times the calculated height of the hill.

2. At a distance of 100 feet from the foot of a tower the elevation of its top is found to be 30°; find the greatest and least errors in its calculated height due to errors of 1′ and 6 inches in the elevation and distance respectively.

3. In the example of Art. 196 find the errors in the calculated values of the flagstaff and tower due to an error δ in the observed value of α.

If $a=1000$ feet, $\alpha=30°$, $\beta=15°$, and there be an error of 1′ in the value of α, calculate the numerical value of these errors.

4. AB is a vertical pole, and CD a horizontal line which when produced passes through B the foot of the pole. The tangents of the angles of elevation at C and D of the top of the pole are found to be $\frac{4}{3}$ and $\frac{3}{4}$ respectively. Find the height of the pole having given that $CD=35$ feet.

Prove that an error of 1′ in the determination of the elevation at D will cause an error of approximately 1 inch in the calculated height of the pole.

5. The elevation of the summit of a tower is observed to be α at a station A and β at a station B, which is at a distance c from A in the direct horizontal line from the foot of the tower, and its height is thus found to be $\frac{c\sin\alpha\sin\beta}{\sin(\alpha-\beta)}$ feet.

If AB be measured not directly from the tower but horizontally and in a direction inclined at a small angle θ to the direct line shew that, to correct the height of the tower to the second order of small quantities, the quantity $\frac{c\cos\alpha\sin^2\beta}{\cos\beta\sin(\alpha-\beta)}\frac{\theta^2}{2}$ must be subtracted.

6. A, B, and C are three given points on a straight line; D is another point whose distance from B is found by observing that the

angles ADB and CDB are equal and of an observed magnitude θ; prove that the error in the calculated length of DB consequent on a small error δ in the observed magnitude of θ, is

$$-\frac{2ab\,(a+b)^2 \sin\theta}{(a^2+b^2-2ab\cos 2\theta)^{\frac{3}{2}}}\,\delta$$

approximately, where $AB=a$ and $BC=b$.

7. In measuring the three sides of a triangle small errors x and y are made in two of them, a and b; prove that the error in the angle C will be $-\frac{y}{b}\cot A - \frac{x}{a}\cot B$, and find the errors in the other angles.

8. In a triangle ABC we have given that approximately $a=36$ feet, $b=50$ feet, and $C=\tan^{-1}\frac{3}{4}$; find what error in the given value of a will cause an error in the calculated value of c equal to that caused by an error of $5''$ in the measurement of C.

9. A triangle is solved from the parts $C=15^\circ$, $a=\sqrt{6}$, and $b=2$; prove that an error of $10''$ in the value of C would cause an error of about $13{\cdot}66''$ in the calculated value of B.

10. Two sides b and c and the included angle A of a given triangle are supposed to be known; if there be a small error θ in the value of the angle A, prove that

(1) the consequent error in the calculated value of B is

$$-\theta\sin B\cos C\operatorname{cosec} A \text{ radians,}$$

(2) the consequent error in the calculated value of a is $c\sin B\,.\,\theta$,

and (3) the consequent error in the calculated area of the triangle is $\theta\cot A$ times that area.

11. There are errors in the sides a, b, and c of a triangle equal to x, y, and z respectively; prove that the consequent error in the calculated value of the circum-radius is

$$\frac{1}{2}\cot A\cot B\cot C\,[x\sec A+y\sec B+z\sec C].$$

12. The area of a triangle is found by measuring the lengths of the sides and the limit of error possible, either in excess or defect, in measuring any length is n times that length, where n is small. Prove that in the case of the triangle whose sides are measured as 110, 81, and 59 yards, the limit to the error in the deduced area of the triangle is about $3{\cdot}1433n$ times that area.

13. The three sides of a triangle are measured and found to be nearly equal. If the measurements can be wrong one per cent. in excess or defect, prove that the greatest error that can arise in calculating one of the angles is $80'$ nearly.

14. It is observed that the elevation of the summit of a mountain at each corner of a plane horizontal equilateral triangle is α; prove that the height of the mountain is

$$\frac{1}{\sqrt{3}} a \tan \alpha,$$

where a is the side of the triangle. If there be a small error n'' in the elevation at C, shew that the true height is

$$\frac{1}{\sqrt{3}} a \tan \alpha \left[1 + \frac{\sin n''}{3 \sin \alpha \cos \alpha}\right].$$

CHAPTER XXXII.

MISCELLANEOUS PROPOSITIONS.

Solution of a Cubic Equation.

396. The standard form of a cubic equation is

$$y^3 + 3ay^2 + 3by + c = 0.$$

Put $y = x - a$, and this equation becomes

$$x^3 - 3(a^2 - b)x + (2a^3 - 3ab + c) = 0,$$

i.e. it becomes of the form

$$x^3 - 3px + q = 0 \text{ .... (1).}$$

Hence any cubic equation can be reduced to the form (1), which has no term containing x^2.

397. *To solve the equation* $x^3 - 3px + q = 0$.

Put $x = \frac{z}{n}$, and we have

$$z^3 - 3pn^2z + qn^3 = 0 \text{ .... (2).}$$

Now, by Art. 107, we always have

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$

so that $$\cos^3\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta = 0 \text{............(3).}$$

Now (2) and (3) are the same equation if

$$z = \cos\theta,\ 3pn^2 = \frac{3}{4},\ \text{and}\ -\frac{1}{4}\cos 3\theta = qn^3.$$

Hence $$n = \left(\frac{1}{4p}\right)^{\frac{1}{2}},$$

and therefore $$\cos 3\theta = -4q\left(\frac{1}{4p}\right)^{\frac{3}{2}} \dots\dots\dots\dots\dots\dots (4).$$

The equation (4) can always be solved (by means of the tables if necessary) if

p be positive, and $$4q\left(\frac{1}{4p}\right)^{\frac{3}{2}} < 1,$$

i.e. if $$q^2 < 4p^3.$$

[The student who is acquainted with the Theory of Equations will notice that is the case which cannot be solved by Cardan's Method. It is the case when the roots of the original cubic are all real.]

If θ be the smallest angle satisfying equation (4), then

the values $$\theta + \frac{2\pi}{3} \text{ and } \theta + \frac{4\pi}{3}$$

also satisfy it, so that the roots of the equation

$$x^3 - 3px + q = 0$$

are $$\frac{1}{n}\cos\theta,\ \frac{1}{n}\cos\left(\theta + \frac{2\pi}{3}\right),\ \text{and}\ \frac{1}{n}\cos\left(\theta + \frac{4\pi}{3}\right),$$

i.e. $$2\sqrt{p}\cos\theta,\ 2\sqrt{p}\cos\left(\theta + \frac{2\pi}{3}\right),\ \text{and}\ 2\sqrt{p}\cos\left(\theta + \frac{4\pi}{3}\right).$$

398. **Ex.** *Solve the equation*

$$x^3 + 6x^2 + 9x + 3 = 0.$$

Put $x = y - 2$, and the equation becomes

$$y^3 - 3y + 1 = 0.$$

Put $y=\frac{z}{n}$, and the equation is

$$z^3-3n^2z+n^3=0 \quad\text{.............................}(1).$$

Now $$\cos^3\theta-\frac{3}{4}\cos\theta-\frac{1}{4}\cos 3\theta=0 \quad\text{.......................}(2).$$

Equations (1) and (2) are the same if

$$z=\cos\theta,\ n^2=\frac{1}{4},\ \text{and}\ -\frac{1}{4}\cos 3\theta=n^3,$$

i.e. if $$n=\frac{1}{2},$$

and $$\cos 3\theta=-\frac{1}{2}=\cos 120° \quad\text{.......................}(3).$$

The roots of (3) are clearly

$$40°,\ 40°+120°,\ \text{and}\ 40°+240°,$$

so that $$z=\cos 40°,\ \text{or}\ \cos 160°,\ \text{or}\ \cos 280°.$$

$$\therefore\ y=2\cos 40°,\ \text{or}\ 2\cos 160°,\ \text{or}\ 2\cos 280°.$$

$$\therefore\ x=y-2=-2+2\cos 40°,\ \text{or}\ -2-2\cos 20°,\ \text{or}\ -2+2\cos 80°.$$

On referring to the tables we then have the values of x.

EXAMPLES. LXVII.

Solve the equations

1. $2x^3-3x-1=0$.
2. $x^3+3x^2-1=0$.
3. $x^3-24x-32=0$,
4. $x^3-6x^2+6x+8=0$.
5. $x^3-21x+7=0$.
6. $x^3+4x^2+2x-1=0$.
7. $x^3-7x+5=0$.

Maximum and Minimum Values.

399. In Art. 133 we have given one example of the maximum value of a trigonometrical expression.

We add another example.

If x and y be two positive angles whose sum is a constant angle α ($\not> \pi$), find when $\sin x \sin y$ is a maximum, and extend the theorem to more than two angles.

We have $\quad 2\sin x \sin y = 2\sin x \sin(\alpha - x)$

$$= \cos(\alpha - 2x) - \cos\alpha.$$

Hence $2\sin x \sin y$ is greatest when $\cos(\alpha - 2x)$ is greatest, *i.e.* when $\alpha = 2x$, and therefore

$$x = y = \frac{\alpha}{2}.$$

The product is therefore greatest when the angles x and y are equal.

Let there be three angles x, y, and z whose sum is equal to a constant angle β. If, in the product

$$\sin x \sin y \sin z,$$

any two of the angles x and y be unequal, we can, by the preceding part of the article, increase the product by substituting for both x and y half their sum without increasing or diminishing the sum of the angles.

Hence so long as the angles x, y, and z are unequal, we can increase the given product by thus making the angles approach to equality.

The maximum value will therefore be obtained when the angles x, y, and z are equal.

This argument can clearly be applied whatever be the number of the angles $x, y, z \ldots$

400. We can now shew that *the maximum triangle that can be inscribed in a given circle is equilateral.*

For, if R be the radius of the circle, we have (as in Ex. XXXVI. 10) the area of the triangle

$$= 2R^2 \sin A \sin B \sin C,$$

where $A + B + C = 2\pi$, a constant angle. By the preceding article it follows that the triangle is greatest when

$$A = B = C.$$

EXAMPLES. LXVIII.

1. If $x+y$ be a given angle, less than π, prove that

(1) $\sin x+\sin y$, (2) $\cos^2 x+\cos^2 y$, and (3) $\cos x \cos y$

all have their greatest values when $x=y$.

2. Find the minimum value of

$$a^2 \tan x + b^2 \cot x.$$

Find the minimum values of

3. $\dfrac{2\cos\theta}{\sqrt{3}} + \dfrac{\sqrt{3}}{2\cos\theta}$.

4. $a^2 \sin^2\theta + b^2 \operatorname{cosec}^2\theta$.

5. If $x+y=a$, where a is $\not> \frac{\pi}{2}$, find when $\tan x \tan y$ is a maximum.

[We have $1-\tan x \tan y = \dfrac{2\cos a}{\cos a - \cos(a-2x)}$.]

6. Prove that the maximum triangle having a given perimeter is equilateral.

[The area of a triangle can be proved to equal $s^2 \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.]

7. Prove that the area of the pedal triangle of an acute-angled triangle is never greater than one quarter of the area of the latter.

8. If ABC be a triangle, prove that the least value of

$$\cos 2A + \cos 2B + \cos 2C \text{ is } -\frac{3}{2}.$$

Prove also that $\cos A + \cos B + \cos C$ is always >1 and not greater than $\frac{3}{2}$.

On the geometrical representation of complex quantities.

401. In Chap. IV. we pointed out that if a distance in any direction (say, horizontally towards the right) be represented by a, then $-a$ represents the same distance drawn in an opposite direction, *i.e.* horizontally towards the left.

The effect of prefixing $-$ to a is therefore (Fig. Art. 48) to rotate OA in the positive direction through two right angles. The operation -1 performed on a therefore means turning a through two right angles.

402. Now $\sqrt{-1} \times \sqrt{-1} = -1$; hence whatever meaning we give to the operation $\sqrt{-1}$ it must be such that *performing that operation twice shall be the same thing as performing the operation* -1.

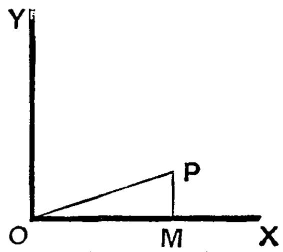
Let us therefore assign to the operation $\sqrt{-1}$ the turning any length through one right angle in the positive direction. Performing the operation $\sqrt{-1}$ on a twice will therefore, as it should do, turn a through *two* right angles.

Hence, with this interpretation, $\sqrt{-1}\, a$ means a line drawn at right angles to the line denoted by a.

403. We can now shew what is denoted by

$$x + \sqrt{-1}\, y.$$

Draw OX and OY two lines at right angles. Measure along OX a distance OM equal to x and then draw MP parallel to OY and equal to y, so that MP represents $\sqrt{-1}\, y$. Then P is the point that represents the quantity $x + \sqrt{-1}\, y$, or, again, we may say that OP is the line representing this quantity.



We have $OP = \sqrt{OM^2 + MP^2} = \sqrt{x^2 + y^2}$,

and $$\angle MOP = \tan^{-1} \frac{MP}{OM} = \tan^{-1} \frac{y}{x}.$$

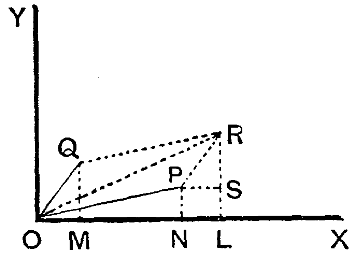
Hence the length of OP represents the modulus and MOP the principal value of the Amplitude of $x+iy$. (Art. 265.)

404. Addition of two complex quantities.

Let OP represent the complex quantity $x+iy$ and OQ represent $u+iv$, so that

$$ON=x,\ NP=y,\ OM=u,$$

and $$MQ=v.$$



Complete the parallelogram $OPRQ$, and draw RL perpendicular to OX and PS perpendicular to RL.

Since PR is equal and parallel to OQ, we have

$$NL=PS=OM, \text{ and } SR=MQ.$$

Hence $$OL=ON+NL=x+u,$$

and $$LR=LS+SR=y+v.$$

Therefore OR represents the complex quantity

$$x+u+i(y+v),$$

so that the sum of two complex quantities is represented by the diagonal of the parallelogram whose two adjacent sides represent the two given complex quantities.

405. Let

$$x+iy=r(\cos\theta+i\sin\theta),$$

as in Art. 265.

Then

$$(\cos\alpha+i\sin\alpha)(x+iy)=r(\cos\alpha+i\sin\alpha)(\cos\theta+i\sin\theta)$$
$$=r[\cos(\alpha+\theta)+i\sin(\alpha+\theta)]\ \ldots\ldots\ldots (1).$$

Now $$r[\cos\theta+i\sin\theta]$$

means, with our interpretation, a line of length r drawn at an angle θ with OX.

Also $$r\,[\cos(\alpha+\theta)+i\sin(\alpha+\theta)]$$
means a line of the same length r drawn at an angle $\alpha+\theta$ with OX (Art. 403).

Hence, by (1), the effect of multiplying $x+iy$ by $\cos\alpha+i\sin\alpha$ is to turn through an angle α the line that represents $x+iy$.

406. *Geometrical meaning of De Moivre's Theorem.*

The quantity
$$(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)(\cos\delta+i\sin\delta),$$
means the line represented by $\cos\delta+i\sin\delta$ turned first through an angle γ, then through β, and finally through α, *i.e.* altogether turned through $\alpha+\beta+\gamma$.

But this total operation gives the same line as
$$[\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)]\,[\cos\delta+i\sin\delta].$$

Similarly for any number of factors.

Hence De Moivre's Theorem expresses algebraically the geometrical fact that to turn a line through a number of angles successively has the same effect as turning the line through an angle equal to the sum of the angles.

Ex. The three cube roots of unity are easily found to be
$$\cos 0+i\sin 0,\quad \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},$$
and
$$\cos\frac{4\pi}{3}+i\sin\frac{2\pi}{3},$$
so that we have
$$(\cos 0+i\sin 0)(\cos 0+i\sin 0)(\cos 0+i\sin 0)=1,$$
$$\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=1,$$
and
$$\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)=1.$$

The first of these equations states that turning a line three times in succession through a zero angle gives the original line.

The second states that turning it three times in succession through an angle $\frac{2\pi}{3}$, (*i.e.* altogether through 2π) gives the original line.

The third states that turning it three times in succession through an angle $\frac{4\pi}{3}$, (*i.e.* altogether through 4π) gives the original line.

These statements are all clearly true.

407. Multiplication of two complex quantities.

If $$x + iy = r(\cos\theta + i\sin\theta),$$
and $$u + iv = \rho(\cos\phi + i\sin\phi),$$
we have
$$(u + iv)(x + iy) = r\rho\,[\cos(\theta + \phi) + i\sin(\theta + \phi)].$$

The effect of multiplying a complex quantity $x + iy$ by another $u + iv$ is therefore to turn the line representing $x + iy$ through an angle
$$\phi\left[i.e.\ \tan^{-1}\frac{v}{u}\right],$$
and to alter its length in the ratio
$$1 : \rho,\ i.e.\ 1 : \sqrt{u^2 + v^2}.$$

Hence the multiplying of one complex quantity by another is represented by "a turning and a stretching."

MISCELLANEOUS EXAMPLES. LXIX.

1. Prove that the equation $\tan x = kx$ has an infinite number of roots.

2. If A, B and C be the angles of a triangle, prove that

$$1 - 8\cos A\cos B\cos C$$

is always positive.

3. If α and β be the imaginary cube roots of unity prove that

$$\alpha e^{\alpha x} + \beta e^{\beta x} = e^{-\frac{x}{2}}\left[\sin\frac{\sqrt{3}x}{2} - \cos\frac{\sqrt{3}x}{2}\right].$$

4. If x be less than a radian prove that $x = 2\sqrt{\dfrac{2-3\cos x}{5+\cos x}}$ very nearly, the error in the left-hand member being nearly $\dfrac{x^5}{480}$ radians.

5. If $\cos(\theta + i\phi) = \sec(\alpha + i\beta)$, where α, β, θ, and ϕ are all real, prove that

$$\tanh^2\phi\cosh^2\beta = \sin^2\alpha \text{ and } \tanh^2\beta\cosh^2\phi = \sin^2\theta.$$

6. If $\qquad x = 2\cos\alpha\cosh\beta$ and $y = 2\sin\alpha\sinh\beta$,

prove that

$$\sec(\alpha + i\beta) + \sec(\alpha - i\beta) = \frac{4x}{x^2+y^2},$$

and

$$\sec(\alpha + i\beta) - \sec(\alpha - i\beta) = \frac{4iy}{x^2+y^2}.$$

7. Prove that

$$\sin^n\phi\cos n\theta + n\sin^{n-1}\phi\cos(n-1)\theta\sin(\theta-\phi)$$
$$+\frac{n(n-1)}{1.2}\sin^{n-2}\phi\cos(n-2)\theta\sin^2(\theta-\phi) + \ldots\ldots + \sin^n(\theta-\phi)$$
$$= \sin^n\theta\cos n\phi.$$

8. Prove that the roots of the equation

$$x^n\sin n\theta - nx^{n-1}\sin(n\theta+\phi) + \frac{n(n-1)}{1.2}x^{n-2}\sin(n\theta+2\phi)$$
$$- \ldots\ldots \text{ to } (n+1) \text{ terms} = 0,$$

are given by $\qquad x = \sin\left(\theta + \phi - k\dfrac{\pi}{n}\right)\operatorname{cosec}\left(\theta - k\dfrac{\pi}{n}\right),$

where n is an integer and k has any integral value from 0 to $n-1$.

9. Prove that the sum to infinity of the series

$$\sin\theta + \frac{1}{2}\frac{\sin^3\theta}{3} + \frac{1.3}{2.4}\frac{\sin^5\theta}{5} + \ldots\ldots$$

is θ, if θ be acute, and, generally, is $n\pi + (-1)^n\theta$, where n is so chosen that $n\pi + (-1)^n\theta$ lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

10. If the arc of a circle of radius unity be divided into n equal arcs, and right-angled isosceles triangles be described on the chords of these arcs as hypothenuses and have their vertices outwards, prove that when n is indefinitely increased the limit of the product of the distances of the vertices from the centre is $e^{\frac{\alpha}{2}}$, where α is the angle subtended by the arc at the centre.

11. The sides of a regular polygon of n sides, which is inscribed in a circle, meet the tangent at any point P of the circle in A, B, C, D...... Prove that the product $OA \,.\, OB \,.\, OC \,.\, OD \ldots\ldots = a^n \tan n\theta$ or $a^n \tan^2 n\theta$, according as n is odd or even, where a is the radius of the circle and θ is the angle which the line joining P to an angular point subtends at the circumference.

12. A regular polygon of n sides is inscribed in a circle and from any point in the circumference chords are drawn to the angular points; if these chords be denoted by $c_1, c_2, \ldots c_n$, beginning with the chord drawn to the nearest angular point and taking the rest in order, prove that the quantity

$$c_1c_2 + c_2c_3 + \ldots + c_{n-1}c_n - c_nc_1$$

is independent of the position of the point from which the chords are drawn.

13. A series of radii divide the circumference of a circle into $2n$ equal parts; prove that the product of the perpendiculars let fall from any point of the circumference upon n successive radii is

$$\frac{r^n}{2^{n-1}} \sin n\theta,$$

where r is the radius of the circle and θ is the angle between one of the extreme of these radii and the radius to the given point.

14. If a regular polygon of n sides be inscribed in a circle, and l be the length of the chord joining any fixed point on the circle to one of the angular points of the polygon, prove that

$$\Sigma l^{2m} = na^{2m} \frac{\underline{|2m}}{2^m \{\underline{|m}\}^2}.$$

15. $ABCD...$ is a regular polygon of n sides which is inscribed in a circle, whose radius is a and whose centre is O; prove that the product of the distances of its angular points from a straight line at right angles to OA and at a distance $b\,(>a)$ from the centre is

$$b^n\left[\cos^n\left(\frac{1}{2}\sin^{-1}\frac{a}{b}\right)-\sin^n\left(\frac{1}{2}\sin^{-1}\frac{a}{b}\right)\right]^2.$$

16. Prove that there is one, and only one, solution of the equation $\theta=\cos\theta$ and that it is less than $\frac{\pi}{4}$.

17. Prove that the general value of θ which satisfies the equation

$$(\cos\theta+i\sin\theta)(\cos 2\theta+i\sin 2\theta)\ \ldots\ldots \text{ to } n \text{ factors}=1$$

is

$$\frac{4m\pi}{n(n+1)}, \text{ where } m \text{ is any integer.}$$

18. Prove that

$$e^{\pi}+e^{-\pi}=2\{1+2^2\}\left\{1+\left(\frac{2}{3}\right)^2\right\}\left\{1+\left(\frac{2}{5}\right)^2\right\}\ldots\ldots \text{ ad inf.}$$

19. Prove that

$$1+\frac{x^3}{\lfloor 3}+\frac{x^6}{\lfloor 6}+\frac{x^9}{\lfloor 9}+\ ..\text{ad inf.}=\frac{1}{3}\left[e^x+2e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}x}{2}\right)\right].$$

20. Prove that

$$\cos\frac{2\pi}{17}+\cos\frac{4\pi}{17}+\cos\frac{6\pi}{17}+\ldots+\cos\frac{14\pi}{17}+\cos\frac{16\pi}{17}=-\frac{1}{2},$$

and

$$\sec\frac{2\pi}{17}+\sec\frac{4\pi}{17}+\ldots+\sec\frac{14\pi}{17}+\sec\frac{16\pi}{17}=8.$$

21. From the sum of the series

$$\sin 2\theta-\frac{1}{2}\sin 4\theta+\frac{1}{3}\sin 6\theta-\ldots \text{ ad inf.},$$

or otherwise, shew that

$$\frac{\pi\sqrt{2}}{4}=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\ldots \text{ ad inf.}$$

22. Prove that

$$\frac{1}{2x}\,\frac{\sinh x}{\cosh x-\cos\alpha}-\frac{1}{\alpha^2+x^2}=\sum_{n=1}^{n=\infty}\left\{\frac{1}{(2n\pi-\alpha)^2+x^2}+\frac{1}{(2n\pi+\alpha)^2+x^2}\right\}.$$

23. Prove that the general value of $\sinh^{-1}x$ is

$$ik\pi + (-1)^k \log[x + \sqrt{1+x^2}],$$

where k is any integer.

24. If $\rho_1, \rho_2, \ldots\ldots\rho_n$ be the distances of the vertices of a regular polygon of n sides from any point P in its plane, prove that

$$\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} + \ldots\ldots + \frac{1}{\rho_n^2} = \frac{n}{r^2 - a^2} \frac{r^{2n} - a^{2n}}{r^{2n} - 2a^n r^n \cos n\theta + a^{2n}},$$

where a is the radius of the circumcircle of the polygon, r is the distance of P from its centre O, and θ is the angle that OP makes with the radius to any angular point of the polygon.

25. If $\theta + \phi + \psi = 2\pi$, prove that

$$\cos^2\theta + \cos^2\phi + \cos^2\psi - 2\cos\theta\cos\phi\cos\psi = 1.$$

Hence deduce the relation between the six straight lines joining 4 points which are in one plane.

ANSWERS.

I. (Page 5.)

1. $\frac{2}{3}$. 2. $\frac{301}{360}$. 3. $\frac{45569}{64800}$.

4. $1\frac{9}{20}$. 5. $2\frac{3661}{10800}$. 6. $4\frac{388}{3375}$.

7. $33^g\,33^{`}\,33{\cdot}3^{``}$. 8. $90^g$. 9. $153^g\,88\;\;88{\cdot}8^{``}$.

10. $39^g\,76^{`}\,38{\cdot}8^{``}$. 11. $261^g\,34^{`}\,44{\cdot}4^{``}$.

12. $528^g\,3^{`}\,33{\cdot}3^{``}$. 13. $1\frac{1}{5}$ rt. $\angle$; $108°$.

14. ·453524 rt. $\angle$; $40°\,49'\,1{\cdot}776''$.

15. ·394536 rt. $\angle$; $35°\,30'\,29{\cdot}664''$.

16. 2·554881 rt. $\angle$; $229°\,56'\,21{\cdot}444''$.

17. 7·59456 rt. $\angle$; $683°\,30'\,37{\cdot}44''$.

28. $66°\,40'$; $5°\,33'\,20''$. 29. $47\frac{7}{19}°$; $42\frac{12}{19}°$

31. $33°\,20'$; $10°\,48'$.

II. (Page 10.)

1. 25132·74 miles nearly.
2. 19·28 miles per hour nearly.
3. 12·85 miles per hour nearly.
4. 3·14159... inches. 5. 581,184,640 miles nearly.
6. 14·994 miles nearly.

III. (Pages 13, 14.)

1. $60°$. 2. $240°$. 3. $1800°$.

4. $57°\,17'\,44{\cdot}8''$. 5. $458°\,21'\,58{\cdot}4''$. 6. 160^g.

7. $233\frac{1}{3}^g$. 8. 2000^g. 9. $\frac{\pi}{3}$. 10. $\frac{221}{360}\pi$.

11. $\frac{703}{720}\pi$. 12. $\frac{3557}{13500}\pi$. 13. $\frac{79}{36}\pi$.

14. $\frac{3\pi}{10}$. 15. $\frac{1103}{2000}\pi$. 16. $1{\cdot}726268\pi$.

17. 81°; 9°. 18. 24°, 60°, and 96°.

19. 132° 15′ 12·6″. 20. 30°, 60°, and 90°.

21. $\frac{1}{2}$, $\frac{\pi}{3}$, and $\frac{2\pi}{3} - \frac{1}{2}$ radians.

22. (1) $\frac{3\pi}{5}$; 108°. (2) $\frac{5\pi}{7}$; $128\frac{4}{7}°$.

(3) $\frac{3\pi}{4}$; 135°. (4) $\frac{5\pi}{6}$; 150°. (5) $\frac{15\pi}{17}$; $158\frac{14}{17}°$.

23. 8 and 4. 24. 10 and 8. 25. 6 and 8.

26. $\frac{\pi}{3}$. 27. (1) $\frac{5\pi^c}{12} = 75° = 83\frac{1}{3}^g$;

(2) $\frac{7\pi^c}{18} = 70° = 77\frac{7}{9}^g$; (3) $\frac{5\pi^c}{8} = 112\frac{1}{2}° = 125^g$.

IV. (Pages 17, 18.)

[Take $\pi = 3{\cdot}14159...$ and $\frac{1}{\pi} = {\cdot}31831$.]

1. $20\frac{5}{11}°$. 2. $\frac{3}{5}$ radian; 34° 22′ 38·9″.

3. 68·75 inches nearly. 4. ·05236 inch nearly.

5. 24·555 inches nearly. 6. 1° 25′ 57″ nearly.

7. 3959·8 miles nearly. 8. π ft. = 3·14159 ft.

9. 5 : 4. 10. 3·1416.

11. $\frac{4\pi}{35}$, $\frac{9\pi}{35}$, $\frac{14\pi}{35}$, $\frac{19\pi}{35}$, and $\frac{24\pi}{35}$ radians.

12. 65° 24′ 30·6″. 13. 2062·65 ft. nearly.

14. 1·5359 ft. nearly. 15. 262·6 ft. nearly.

16. 32142·9 ft. nearly. 17. 17188·7 ft. nearly.

18. 19·099′.
19. 238,833 miles.
20. 478×10^{11} miles.
21. 1105·8 miles.
22. 21600; 3437·75 nearly.

VI. (Page 31.)

6. $\frac{12}{5}$; $\frac{8}{13}$.
7. $\frac{11}{60}$; $\frac{60}{61}$; $\frac{61}{60}$.
8. $\frac{3}{5}$; $\frac{4}{3}$.
9. $\frac{40}{9}$; $\frac{41}{40}$.
10. $\frac{3}{5}$; $\frac{4}{5}$; $\frac{1}{5}$; $\frac{5}{3}$.
11. $\frac{3}{4}$.
12. $\frac{15}{17}$; $\frac{17}{8}$.
13. $\frac{1}{2}\sqrt{5}$; $\frac{3}{5}\sqrt{5}$.
14. 1 or $\frac{3}{5}$.
15. $\frac{3}{5}$ or $\frac{5}{13}$.
16. $\frac{5}{13}$.
17. $\frac{12}{13}$.
18. $\frac{1}{\sqrt{3}}$ or 1.
19. $\frac{1}{2}$.
20. $\frac{1}{\sqrt{2}}$.
21. $1+\sqrt{2}$.
22. $\frac{2x(x+1)}{2x^2+2x+1}$; $\frac{2x+1}{2x^2+2x+1}$.

VIII. (Pages 44—46.)

1. 34·64... ft.; 20 ft.
2. 160 ft.
3. 225 ft.
4. 136·6 ft.
5. 146·4... ft.
6. 367·8 yards; 453·9 yards.
7. 86·6... ft.
8. 115·359... ft.
9. 87·846... ft.
10. 43·3... ft.; 75 ft. from one of the pillars.
11. 94·641... ft.; 54·641... ft.
12. 1·366... miles.
13. 30°.
15. 13·8564 miles per hour.
16. 25·98... ft.; 70·98... ft.; 85·98... ft.
17. $32\sqrt{5} = 71·55$... ft.
19. 10 miles per hour.
20. 86·6... yards.
21. 692·8... yards.

IX. (Page 63.)

1. $\frac{2500}{6289}\pi$, $\frac{2250}{6289}\pi$ and $\frac{1539}{6289}\pi$ radians.

2. $68° 45' 17{\cdot}8''$.
4. $\dfrac{2xy}{x^2+y^2}$; $\dfrac{2xy}{x^2-y^2}$.
8. $\dfrac{1}{\tan^4 A} - \tan^4 A$.
9. $\theta = 60°$.
10. In $1\frac{1}{2}$ minutes.

X. (Pages 74, 75.)

4. $-{\cdot}366\ldots$; $2{\cdot}3094\ldots$.
5. $-1{\cdot}366\ldots$; $-2{\cdot}3094$.
6. 0; 2.
7. $1{\cdot}4142\ldots$; -2.
8. $1{\cdot}366\ldots$; $-2{\cdot}3094\ldots$.
9. 45° and 135°.
10. 120° and 240°.
11. 135° and 315°.
12. 150° and 330°.
13. 150° and 210°.
14. 210° and 330°.
15. $-\cos 25°$.
16. $\sin 6°$.
17. $-\tan 43°$.
18. $\sin 12°$.
19. $\sin 17°$.
20. $-\cot 24°$.
21. $\cos 33°$.
22. $-\cos 28°$.
23. $\cot 25°$.
24. $\cos 30°$.
25. $\cot 26°$.
26. $-\operatorname{cosec} 23°$.
27. $\operatorname{cosec} 36°$.
28. negative.
29. negative.
30. positive.
31. zero.
32. positive.
33. positive.
34. positive.
35. negative.
36. $\dfrac{1}{\sqrt{3}}$ and $\dfrac{-\sqrt{2}}{\sqrt{3}}$; $\dfrac{-1}{\sqrt{3}}$ and $\dfrac{\sqrt{2}}{\sqrt{3}}$.

XI. (Pages 83, 84.)

1. $n\pi + (-1)^n \dfrac{\pi}{6}$.
2. $n\pi - (-1)^n \dfrac{\pi}{3}$.
3. $n\pi + (-1)^n \dfrac{\pi}{4}$.
4. $2n\pi \pm \dfrac{2\pi}{3}$.
5. $2n\pi \pm \dfrac{\pi}{6}$.
6. $2n\pi \pm \dfrac{3\pi}{4}$.
7. $n\pi + \dfrac{\pi}{3}$.
8. $n\pi + \dfrac{3\pi}{4}$.
9. $n\pi + \dfrac{\pi}{4}$.
10. $2n\pi \pm \dfrac{\pi}{3}$.
11. $n\pi + (-1)^n \dfrac{\pi}{3}$.
12. $n\pi \pm \dfrac{\pi}{2}$.
13. $n\pi \pm \dfrac{\pi}{3}$.

14. $n\pi \pm \frac{\pi}{6}$. 15. $n\pi \pm \frac{\pi}{3}$. 16. $n\pi \pm \frac{\pi}{4}$.

17. $n\pi \pm \frac{\pi}{6}$. 18. $(2n+1)\pi + \frac{\pi}{4}$. 19. $2n\pi - \frac{\pi}{6}$.

20. 105° and 45°; $\left(n + \frac{m}{2}\right)\pi \pm \frac{\pi}{6} + (-1)^m \frac{\pi}{12}$, and

$$\left(\frac{m}{2} - n\right)\pi \mp \frac{\pi}{6} + (-1)^m \frac{\pi}{12},$$

where m and n are any integers.

21. $187\frac{1}{2}°$ and $142\frac{1}{2}°$;

$$\left(n + \frac{m}{2}\right)\pi + \frac{\pi}{8} \pm \frac{\pi}{12} \text{ and } \left(n - \frac{m}{2}\right)\pi - \frac{\pi}{8} \pm \frac{\pi}{12}.$$

22. (1) 60° and 120°; (2) 120° and 240°; (3) 30° and 210°.

23. (1) 2; (2) 1; (3) 1; (4) 1; (5) 1.

XII. (Pages 85, 86.)

1. $n\pi + (-1)^n \frac{\pi}{6}$. 2. $2n\pi \pm \frac{2\pi}{3}$.

3. $n\pi + (-1)^n \frac{\pi}{3}$. 4. $\cos\theta = \frac{\sqrt{5}-1}{2}$.

5. $\sin\theta = \pm\frac{\sqrt{5}-1}{4} = \sin 18°$ or $\sin(-54°)$ (Art. 120).

6. $\theta = 2n\pi \pm \frac{\pi}{3}$. 7. $\theta = n\pi + \frac{\pi}{4}$ or $n\pi + \frac{\pi}{3}$.

8. $\theta = n\pi + \frac{2\pi}{3}$ or $n\pi + \frac{5\pi}{6}$. 9. $\tan\theta = \frac{1}{a}$ or $-\frac{1}{b}$.

10. $\theta = n\pi \pm \frac{\pi}{4}$. 11. $\theta = 2n\pi$ or $2n\pi + \frac{\pi}{4}$.

12. $\frac{n\pi}{5} + (-1)^n \frac{\pi}{20}$. 13. $\frac{n\pi}{4}$ or $\frac{(2n+1)\pi}{10}$.

14. $2n\pi$ or $\frac{(2n+1)\pi}{5}$. 15. $\frac{2r\pi}{m-n}$ or $\frac{2r\pi}{m+n}$.

16. $\left(2n+\frac{1}{2}\right)\frac{\pi}{5}$ or $2n\pi-\frac{\pi}{2}$. 17. $2n\pi$ or $\frac{2n\pi}{9}$.

18. $\left(2n+\frac{1}{2}\right)\frac{\pi}{m+n}$ or $\left(2n-\frac{1}{2}\right)\frac{\pi}{m-n}$.

19. $\left(n+\frac{1}{2}\right)\frac{\pi}{9}$. 20. $\left(m+\frac{1}{2}\right)\frac{\pi}{n+1}$.

21. $\frac{n\pi}{4}\pm\sqrt{1+\frac{n^2\pi^2}{16}}$. 22. $\left(n+\frac{1}{2}\right)\frac{\pi}{3}$.

23. $\left(n+\frac{1}{2}\right)\frac{\pi}{3}\pm\frac{\alpha}{3}$. 24. $\left(n+\frac{1}{2}\right)\frac{\pi}{4}$.

25. $\frac{n\pi}{3}\pm\frac{\alpha}{3}$. 26. $n\pi\pm\frac{\pi}{6}$. 27. $\left(r+\frac{1}{2}\right)\frac{\pi}{m-n}$.

28. $\sin 2\theta=\frac{4}{2n+1}$.

29. $\theta=\left(m+\frac{n}{2}\right)\pi\pm\frac{\pi}{6}+(-1)^n\frac{\pi}{12}$;

$\phi=\left(m-\frac{n}{2}\right)\pi\pm\frac{\pi}{6}-(-1)^n\frac{\pi}{12}$.

30. $\frac{1}{5}\left[(6m-4n)\pi\pm\frac{\pi}{2}\mp\frac{2\pi}{3}\right]$; $\frac{1}{5}\left[(6n-4m)\pi\pm\pi\mp\frac{\pi}{3}\right]$.

31. 45° and 60°. 32. $\frac{1}{3}$ or $\frac{5}{3}$.

33. $\pm\frac{1}{3}\sqrt{5}$; $\pm\frac{1}{2}\sqrt{5}$.

XIII. (Pages 91, 92.)

1. $-\frac{133}{205}$; $-\frac{84}{205}$. 2. $\frac{1596}{3445}$; $\frac{3444}{3445}$.

3. $\frac{220}{221}$; $\frac{171}{221}$; $\frac{220}{21}$.

XIV. (Pages 96, 97.)

30. $2\sin(\theta+n\phi)\sin\frac{3\phi}{2}$. 31. $2\sin(\theta+n\phi)\cos\frac{\phi}{2}$.

XV. (Pages 98, 99.)

1. $\cos 2\theta - \cos 12\theta$.
2. $\sin 12\theta - \sin 2\theta$.
3. $\cos 14\theta + \cos 8\theta$.
4. $\cos 12° - \cos 120°$.

XVI. (Page 102.)

1. 3; $\frac{9}{13}$.
3. 1.

XVII. (Pages 109, 110.)

1. (1) $\pm\frac{24}{25}$; (2) $\pm\frac{120}{169}$; (3) $\frac{2016}{4225}$.
2. (1) $\frac{161}{289}$; (2) $-\frac{7}{25}$; (3) $\frac{119}{169}$.
3. a.

XVIII. (Pages 123—125.)

1. $\pm\frac{2\sqrt{2}+\sqrt{3}}{6}$; $\pm\frac{7\sqrt{3}+4\sqrt{2}}{18}$.
2. $\pm\frac{13}{12}$; $\pm\frac{\sqrt{13}}{2}$ or $\pm\frac{\sqrt{13}}{3}$; $\frac{169}{120}$.
3. $\frac{16}{305}$; $\frac{49}{305}$.
4. $\frac{7}{5\sqrt{2}}$.
5. $\pm\frac{1}{3}$; $\pm\frac{3}{4}$.
6. $\pm\frac{3}{4}$.
7. $\frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}}$; $\frac{\sqrt{4+\sqrt{2}+\sqrt{6}}}{2\sqrt{2}}$; $\sqrt{2}-1$;
$-(\sqrt{2}+1)+\sqrt{4+2\sqrt{2}}$.
8. $\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$.
23. $-$ and $+$.
24. $-$ and $-$.
25. $-$ and $-$.
29. (1) $2n\pi+\frac{\pi}{4}$ and $2n\pi+\frac{3\pi}{4}$; (2) $2n\pi+\frac{3\pi}{4}$ and $2n\pi+\frac{5\pi}{4}$;
(3) $2n\pi-\frac{\pi}{4}$ and $2n\pi+\frac{\pi}{4}$; (4) $2n\pi+\frac{\pi}{4}$ and $2n\pi+\frac{3\pi}{4}$.

30. (1) $2n\pi - \frac{\pi}{4}$ and $2n\pi + \frac{\pi}{4}$;

(2) $2n\pi + \frac{3\pi}{4}$ and $2n\pi + \frac{5\pi}{4}$;

(3) $2n\pi + \frac{5\pi}{4}$ and $2n\pi + \frac{7\pi}{4}$.

XIX. (Pages 129, 130.)

12. The sine of the angle is equal to $2 \sin 18°$.

13. $2n\frac{\pi}{5} + \frac{\pi}{10}$.

XXI. (Pages 142, 143.)

1. $\frac{n\pi}{4}$ or $\frac{1}{3}\left(2n\pi \pm \frac{\pi}{3}\right)$. 2. $\left(2n \pm \frac{1}{2}\right)\frac{\pi}{4}$ or $\left(2n \pm \frac{1}{3}\right)\frac{\pi}{3}$.

3. $\left(2n \pm \frac{1}{2}\right)\frac{\pi}{2}$ or $2n\pi$.

4. $\left(2n \pm \frac{1}{2}\right)\frac{\pi}{3}$ or $n\pi + (-1)^n\frac{\pi}{6}$.

5. $\frac{2n\pi}{3}$ or $\left(n + \frac{1}{4}\right)\pi$ or $\left(2n - \frac{1}{2}\right)\pi$.

6. $\frac{n\pi}{3}$ or $\left(2n \pm \frac{1}{3}\right)\frac{\pi}{4}$. 7. $\left(n \pm \frac{1}{4}\right)\pi$ or $2n\pi \pm \frac{2\pi}{3}$.

8. $n\frac{\pi}{3}$ or $\left(n \pm \frac{1}{3}\right)\pi$. 9. $2n\pi$; $\left(\frac{2n}{3} + \frac{1}{2}\right)\pi$.

10. $n\pi + (-1)^n\frac{\pi}{6}$; $n\pi + (-1)^n\frac{\pi}{10}$; $n\pi - (-1)^n\frac{3\pi}{10}$.

11. $\left(2n \pm \frac{1}{2}\right)\frac{\pi}{8}$; $\left(2m \pm \frac{1}{2}\right)\frac{\pi}{2}$.

12. $m\pi$; $\frac{1}{n-1}\left[m\pi - (-1)^m\frac{\pi}{6}\right]$. 13. $2m\pi$; $\frac{4m\pi}{n \pm 1}$.

14. $\frac{2r\pi}{m+n}$; $\frac{2}{m-n}\left(2r\pi \pm \frac{\pi}{2}\right)$. 15. $\frac{1}{m \pm n}\left[2r\pi \pm \frac{\pi}{2}\right]$

16. $m\pi$; $\frac{m\pi}{n-1}$; $\frac{1}{n}\left(2m\pi \pm \frac{\pi}{2}\right)$.

17. $2n\pi - \frac{\pi}{2}$; $\frac{1}{5}\left(2n\pi - \frac{\pi}{2}\right)$.

18. $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3}$. 19. $n\pi + (-1)^n \frac{\pi}{2} - \frac{\pi}{4}$.

20. $n\pi + \frac{\pi}{6} + (-1)^n \frac{\pi}{4}$. 21. $2n\pi + \frac{\pi}{4} \pm A$.

22. $-21°48' + n\pi + (-1)^n [68°12']$.

23. $2n\pi + 78°58'$; $2n\pi + 27°18'$. [N.B. $\cos 25° 50' = \cdot 9$.]

24. $n\pi + 45°$; $n\pi + 26°34'$. 25. $2n\pi$; $2n\pi + \frac{2\pi}{3}$.

26. $2n\pi$; $2n\pi + \frac{\pi}{2}$. 27. $2n\pi + \frac{\pi}{2}$; $2n\pi + \frac{\pi}{3}$.

28. $2n\pi + \frac{\pi}{6}$; $2n\pi - \frac{\pi}{2}$. 29. $n\pi$.

30. $\sin\theta = \frac{\pm\sqrt{17} - 1}{8}$. 31. $\cos\theta = \frac{\pm\sqrt{17} - 3}{4}$.

32. $n\pi \pm \frac{\pi}{3}$; $2n\pi \pm \frac{\pi}{2}$. 33. $2n\pi \pm \frac{\pi}{3}$; $2n\pi \pm \frac{\pi}{4}$.

34. $\left(n + \frac{1}{4}\right)\frac{\pi}{2}$. 35. $n\pi \pm \frac{\pi}{4}$. 36. $n\pi + \frac{\pi}{4}$.

37. $\theta = \frac{n\pi}{2}$ or $n\pi \pm \frac{\pi}{3}$; also $\theta = n\pi \pm \frac{\alpha}{2}$, where $\cos\alpha = \frac{1}{3}$.

38. $\left(n + \frac{1}{3}\right)\frac{\pi}{3}$. 39. $n\pi + \frac{\pi}{3}$.

XXIII. (Pages 157, 158.)

1. $\bar{1}$·90309; $\bar{3}$·4771213; $\bar{2}$·0334239; $\bar{1}$·4650389.
2. ·1553361; 2·1241781; ·5388340; $\bar{3}$·0759623.
3. 2; $\bar{2}$; 0; $\bar{4}$; $\bar{2}$; 0; 3. 4. ·312936.
5. 1·32057; 5·88453; ·461791.
6. (1) 21; (2) 13; (3) 30; (4) the 7th; (5) the 21st (6) the 32nd.

7. $\dfrac{4\log 3}{\log 7+4\log 3-\log 2}$; $\dfrac{\log 2+2\log 3}{4\log 7-3\log 3-2\log 2}$; $\dfrac{7\log 3+4\log 2}{3\log 3+\log 2-2\log 7}$.

8. ·22221. 9. 8·6414. 10. 9·6192.

11. 1·6389. 12. 4·7161. 13. ·41432.

XXIV. (Pages 167—169.)

1. 4·5527375; 1·5527394. [N.B. log 35706 = 4·5527412.]

2. 4·7689529; $\bar{3}$·7689502.

3. 478·475; ·004784777. 4. 2·583674; ·0258362.

5. (1) 4·7204815; (2) 2·7220462; (3) $\bar{2}$·7240079;
(4) 5273·63; (5) ·05296726; (6) 5·26064.

6. ·6870417. 7. 43° 23′ 45″.

8. ·8455104; ·8454509. 9. 32° 16′ 35″; 32° 16′ 21″.

10. 4·1203060; 4·1218748.

11. 4·3993263; 4·3976823. 12. 13° 8′ 47″.

13. 9·9147334. 14. 34° 44′ 27″.

15. 9·5254497; 71° 27′ 43″. 16. 10·0229414.

17. 18° 27′ 17″. 18. 36° 52′ 12″.

XXV. (Pages 171, 172.)

1. 13° 27′ 31″. 2. 22° 1′ 28″.

3. 1·0997340; 65° 24′ 12·5″.

4. 9·6198509; 22° 36′ 28″.

5. 10° 15′ 34″. 6. 44° 55′ 55″.

7. (1) 9·7279043; (2) 9·9270857; (3) 10·1958917;
(4) 10·0757907; (5) 10·2001337;
(6) 10·0725027; (7) 9·7245162.

8. (1) 57° 30′ 24″; (2) 57° 31′ 58″; (3) 32° 29′ 15″;
(4) 57° 6′ 39″.

9. ·53736037.

10. (1) $\cos(x-y)\sec x\sec y$; (2) $\cos(x+y)\sec x\sec y$;
(3) $\cos(x-y)\operatorname{cosec} x\sec y$;
(4) $\cos(x+y)\operatorname{cosec} x\sec y$;
(5) $\tan^2 x$; (6) $\tan x\tan y$.

XXVI. (Pages 179, 180.)

1. $\frac{1}{5}$, $\frac{1}{2}$, and $\frac{9}{7}$.
2. $\frac{4}{\sqrt{41}}$, $\frac{3}{5}$, and $\frac{8}{5\sqrt{41}}$; $\frac{40}{41}$, $\frac{24}{25}$, and $\frac{496}{1025}$.
3. $\frac{3}{5}$, $\frac{4}{5}$, and 1.
4. $\frac{5}{12}$, $\frac{12}{5}$, and ∞.
5. $\frac{4}{5}$, $\frac{56}{65}$ and $\frac{12}{13}$.
6. $\frac{7}{41}$ and $\frac{287}{816}$.
7. 60°, 45°, and 75°.

XXVII. (Pages 185—187.)

17. $16\frac{4}{5}$ ft. 19. $\frac{2}{5}$. 22. $\frac{313}{338}$.

XXVIII. (Page 190.)

1. 186·60... and 193·18.
2. 26° 33′ 54″; 63° 26′ 6″; $10\sqrt{5}$ ft.
3. 48° 35′ 25″, 36° 52′ 12″ and 94° 32′ 23″.
4. 75° and 15°.

XXIX. (Page 193.)

1. 90°. 2. 30°. 4. 120°.
5. 45°, 120° and 15°. 6. 45°, 60°, and 75°.
7. 58° 59′ 33″. 8. 77° 19′ 11″. 9. 76° 39′ 9″.
10. 104° 28′ 39″.
11. 56° 15′ 4″, 59° 51′ 11″ and 63° 53′ 45″.
12. 38° 56′ 33″, 47° 41′ 7″ and 93° 22′ 20″.
13. 130° 42′ 20·5″, 23° 27′ 8·5″, and 25° 50′ 31″.

XXX. (Pages 197—200.)

1. $63°13'2''$; $43°58'28''$. 2. $117°38'45''$; $27°38'45''$.
3. $8\sqrt{7}$; $79°6'24''$; $40°53'36''$.
4. $87°27'25{\cdot}5''$; $32°32'34{\cdot}5''$.
5. $40°53'26''$; $19°6'24''$; $\sqrt{7}:2$.
6. $71°44'30''$: $48°15'30''$. 7. $78°17'41''$; $48°36'19''$.
8. $108°12'25{\cdot}5''$; $49°27'34{\cdot}5''$.
9. $A=45°$; $B=75°$; $c=\sqrt{6}$. 10. $\sqrt{6}$; $15°$; $105°$.
11. ·8965. 14. 40 yds.; 120°; 30°.
15. 7·589467; $108°26'6''$; $18°26'6''$; $53°7'48''$.
16. 226·87; $73°34'50''$; $39°45'10''$. 17. 2·529823.
18. $A=83°7'39''$; $B=42°16'21''$; $c=199{\cdot}099$.
19. $B=110°48'15''$; $C=26°56'15''$; $a=93{\cdot}5192$.
20. $73°1'51''$ and $48°41'9''$.
21. $88°30'1''$ and $33°30'59''$.

XXXI. (Pages 205—207.)

1. There is no triangle.

2. $B_1=30°$, $C_1=105°$, and $b_1=\sqrt{2}$; $B_2=60°$, $C_2=75°$, and $b_2=\sqrt{6}$.

3. $B_1=30°$, $C_1=120°$, and $b_1=100$; $B_2=90°$, $C_2=60°$, and $b_2=200$.

5. $4\sqrt{3}\pm 2\sqrt{5}$.
6. $100\sqrt{3}$; the triangle is right-angled.
8. $33°29'30''$ and $101°30'30''$. 9. 17·1 or 3·68.
10. (1) The triangle is right-angled and $B=60°$.
(2) $B_1=8°41'$ and $C_1=141°19'$; $B_2=111°19'$ and $38°41'$.
11. $65°54'$ and $42°1'12''$.
12. 5·988... and 2·6718... miles per hour.
13. $63°2'12''$ or $116°57'48''$.
14. $62°31'23''$ and $102°17'37''$, or $117°28'37''$ and $47°20'23''$.
15 59266·1.

XXXII. (Page 208.)

1. 7 : 9 : 11. 4. 79·063.
5. 1 mile; 1·219714... miles. 7. 20·97615... ft.
8. 6·85673... and 5·4378468... miles. 9. 404·4352 ft.
10. 233·2883 yards. 11. 2229·02 yards.

XXXIII. (Pages 213—216.)

1. 100 ft. high and 50 ft. broad; 25 feet.
2. 25·783414 yds. 3. 33·07... ft.; $17\frac{1}{2}$ ft.
4. 18·3... ft. 5. 120 ft. 6. $h \tan \alpha \cot \beta$.
7. 1939·2... ft. 8. 100 ft. 9. 61·224... ft.
10. $100\sqrt[4]{2}$ ft.
15. $PQ = BP = BQ = 1000$ ft.; $AP = 500(\sqrt{6} - \sqrt{2})$ ft; $AQ = 1000\sqrt{2}$ ft.
16. ·32119 miles.
17. ·1736482 miles; ·9848078 miles.
18. 119·2862 ft. 19. 132·266 ft.
20. 141·682 yds. 21. 1·42771 miles.
22. 125·3167 ft.

XXXIV. (Pages 220—225.)

3. 20 ft; 40 ft.
4. $l \operatorname{cosec} \gamma$, where γ is the sun's altitude; $\sin \gamma = \frac{2}{7}$.
5. 3·732... miles; 12·342... miles per hour at an angle, whose tangent is $\sqrt{3} + 1$, S. of E.
6. 10·2426... miles per hour.
7. 16·3923... miles; 17·394... miles.
8. 2·39 miles; 1·366 miles.
9. It makes an angle whose tangent is $\frac{2}{3}$; $\frac{9}{52}$ hour.
13. $c \sin \beta \operatorname{cosec}(\alpha + \beta)$; $c \sin \alpha \sin \beta \operatorname{cosec}(\alpha + \beta)$.
14. 9 yds.; 2 yds. 17. $\frac{a}{3}$; $\frac{2a}{3}$.

21. $75\sqrt{3}$ ft. 22. $c(1-\sin\alpha)\sec\alpha$.
23. 114·122 ft. 25. 1069·745645 ft.
27. The angle whose tangent is $\frac{1}{2}$. 30. 45°.
33. 18° 24′ 6″. 35. $\tan\alpha\sec\beta : 1$.
38. 91·896 ft. 39. 1960·95 yds.
40. 2·45832 miles. 41. 333·4932 ft.

XXXV. (Pages 227, 228.)

1. 84. 2. 216. 3. 630. 4. 3720.
5. 270. 6. 117096. 7. 1470.
8. 1·183.... 12. 35 yds. and 26 yds. nearly.
13. 14·941... inch. 14. 5, 7, and 8 ft. 15. 120°.
17. 45° and 105°; 135° and 15°.
18. 17·1064... sq. ins.

XXXVI. (Pages 235, 236.)

3. $8\frac{1}{8}$, $1\frac{1}{2}$, 8, 2, and 24 respectively.

XXXVII. (Pages 244—248.)

35. 2·1547... or ·1547 times the radius of each circle.
39. $A_n = \frac{\pi}{3} + (-1)^n \cdot 2^n \cdot \left(A - \frac{\pi}{3}\right)$ ....

XXXVIII. (Pages 253, 254.)

1. (1) $3\sqrt{105}$ ft.; (2) $10\sqrt{7}$ ft. 3. $1\frac{5}{7}$ and $2\frac{1}{2}$ ft.

XXXIX. (Pages 257—259.)

1. 77·98 ins. 2. 3·215.
3. (1) 1·720... sq. ft.; (2) 2·598... sq. ft.;
(3) 4·8284... sq. ft.; (4) 7·694... sq. ft.;
(5) 11·196... sq. ft.
4. 1·8866... sq. ft. 5. 3·3136... sq. ft.
6. $2+\sqrt{2} : 4$; $\sqrt{2+\sqrt{2}} : 2$. 12. 3.
14. 6. 15. 9 or 16. 16. 20 and 10.

17. and 5, 12 and 8, 18 and 10, 22 and 11, 27 and 12, 42 and 14, 54 and 15, 72 and 16, 102 and 17, 162 and 18, 342 and 19 sides respectively. 19. $\frac{2}{3}\sqrt{3}$; $\sqrt{6}$.

XL. (Pages 264, 265.)

1. ·00204. 2. ·00007. 3. ·00029.
4. ·99999. 5. 25783·100... 6. 1·0000011.
7. 34′23″. 8. 28°41′7″. 9. 39′34″.
10. 2°26′15″. 11. 114·59... inches.

XLI. (Pages 267, 268.)

1. 435·77 sq. ft. 2. 4·9087... sq. ft.
3. 127°19′26″. 4. 6 sq. ft.
5. 11·0004 inches. 6. ·00044625 inch.
7. $\frac{2}{3}\pi r$.

XLII. (Pages 269, 270.)

1. 1°10′22″. 2. 17·14 miles.
3. ·61 miles; 1°48′ nearly.
5. About 61800 metres = about $38\frac{1}{2}$ miles.
6. 3960 miles.

XLIV. (Pages 285—287.)

1. $\frac{1}{2}\sin 2n\theta \operatorname{cosec}\theta$.
2. $\cos\frac{3n-1}{4}A\sin\frac{3n}{4}A\operatorname{cosec}\frac{3}{4}A$.
5. $\frac{1}{2}\sin\frac{2n\pi}{n+1}\cdot\operatorname{cosec}\frac{2\pi}{n+1}$.
6. $\sin\left[\alpha+\left(n-\frac{1}{2}\right)\beta\right]\sin n\beta\sec\frac{\beta}{2}$. 8 $-\sin\frac{n\theta}{n-2}$.
9. $\sin 2nx(\cos 2nx+\sin 2nx)(\cos x+\sin x)\operatorname{cosec}2x$.
10. $\frac{1}{4}[(n+1)\sin 2\alpha-\sin(2n+2)\alpha]\operatorname{cosec}\alpha$.

11. $\frac{1}{2}\sin(n+2)\alpha \,.\, \sin n\alpha \operatorname{cosec}\alpha.$

12. $\frac{n}{2}\cos 2\alpha - \frac{1}{2}\cos(n+3)\alpha\sin n\alpha\operatorname{cosec}\alpha.$

13. $\dfrac{\cos(2n\alpha-\alpha)\cos(n+1)\beta - \cos(2n\alpha+\alpha)\cos n\beta + \cos\alpha(1-\cos\beta)}{2(\cos\beta-\cos 2\alpha)}.$

14. $\frac{1}{4}[(2n+1)\sin\alpha - \sin(2n+1)\alpha]\operatorname{cosec}\alpha.$

15. $\frac{n}{2} - \frac{1}{2}\cos[2\theta+(n-1)\alpha]\sin n\alpha\operatorname{cosec}\alpha.$

16. $\frac{3}{4}\sin\frac{n+1}{2}\alpha\sin\frac{n\alpha}{2}\operatorname{cosec}\frac{\alpha}{2} - \frac{1}{4}\sin 3\frac{n+1}{2}\alpha \,.\, \sin\frac{3n\alpha}{2}\sin\frac{3\alpha}{2}.$

17. $\frac{1}{8}[3n - 4\cos(n+1)\alpha\sin n\alpha\operatorname{cosec}\alpha + \cos(2n+2)\alpha\sin 2n\alpha\operatorname{cosec}2\alpha].$

18. $\frac{1}{8}[3n + 4\cos(n+1)\alpha\sin n\alpha\operatorname{cosec}\alpha + \cos(2n+2)\alpha\sin 2n\alpha\operatorname{cosec}2\alpha].$

19. $\frac{1}{4}\sin\frac{n\theta}{2}\left[\cos\frac{n-1}{2}\theta + \cos\frac{n+3}{2}\theta + \cos\frac{n+7}{2}\theta\right]\operatorname{cosec}\frac{\theta}{2}$
$+\frac{1}{4}\sin\frac{3n\theta}{2}\cos\frac{3n+9}{2}\theta\operatorname{cosec}\frac{3\theta}{2}.$

20. $-\frac{1}{2}\sin(2\alpha+2n\beta)\sin 2n\beta\sec\beta.$

XLV. (Pages 291, 292.)

1. $a^2+b^2=c^2+d^2.$

2. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos(\alpha-\beta)=\sin^2(\alpha-\beta).$

3. $a(2c^2-d^2)=bdc.$ 4. $a\sin\alpha+b\cos\alpha=\sqrt{2b(a+b)}.$

5. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ 6. $\frac{x^2}{a}+\frac{y^2}{b}=a+b.$

7. $(p^2+1)^2+2q(p^2+1)(p+q)=4(p+q)^2.$

10. $a^2(a-b)(a-c)=b^2(b-c)(b-d).$

11. $8bc=a\{4b^2+(b^2-c^2)^2\}.$

12. $y\sqrt{(a+b+c)(-a+b+c)}=x(c^2-a^2-b^2)\sqrt{(a-b+c)(a+b-c)}.$

13. $b^2[x(b^2-a^2)+a(a^2+b^2)]^2=4c^4[b^2x^2+a^2y^2].$

XLVI. (Pages 303—305.)

8. $\log_e 2$. 9. $\log_e 3 - \log_e 2$.

XLVII. (Pages 316—318.)

1. $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.
2. $\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$.
3. $2\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$. 4. $5\left[\frac{3}{5} + i\,.\,\frac{4}{5}\right]$.
5. $\sqrt{4+2\sqrt{2}}\left[\frac{\sqrt{2}+1}{\sqrt{4+2\sqrt{2}}} + i\,\frac{1}{\sqrt{4+2\sqrt{2}}}\right]$.
6. $(\sqrt{6}-\sqrt{2})\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right]$.
7. $\cos(10\theta + 12\alpha) - i\sin(10\theta + 12\alpha)$.
8. $\cos(\alpha + \beta - \gamma - \delta) + i\sin(\alpha + \beta - \gamma - \delta)$.
9. $\cos 107\theta - i\sin 107\theta$. 10. -1.
11. $\sin(4\alpha + 5\beta) - i\cos(4\alpha + 5\beta)$.
12. $2^{n+1}\sin^n\frac{\theta-\phi}{2}\cos n\,\frac{\pi+\theta+\phi}{2}$.
23. $\cos\frac{\pi}{5} \pm i\sin\frac{5\pi}{5}$; $\cos\frac{3\pi}{5} \pm i\sin\frac{3\pi}{5}$.

XLVIII. (Pages 321, 322.)

1. 1; $\frac{-1 \pm i\sqrt{3}}{2}$. 2. $\pm i$; $\frac{\sqrt{3} \pm i}{2}$; $\frac{-\sqrt{3} \pm i}{2}$.
3. $\pm\left(\cos\frac{r\pi}{12} + i\sin\frac{r\pi}{12}\right)$, where $r = 3$, 7, or 11.
4. $\pm i$, and $\pm\left(\cos\frac{r\pi}{10} \pm i\sin\frac{r\pi}{10}\right)$, where $r = 1$ or 3.
5. $\pm\sqrt[12]{2}\left(\cos\frac{r\pi}{24} + i\sin\frac{r\pi}{24}\right)$, where $r = 1$, 9, or 17.

6. $\sqrt[3]{2048}\left[\cos\frac{r\pi}{9}+i\sin\frac{r\pi}{9}\right]$, where $r=11$, 15, or 19.

7. $\pm\sqrt[4]{2}\left[\cos\frac{r\pi}{12}-i\sin\frac{r\pi}{12}\right]$, where $r=1$ or 7.

8. $\sqrt[3]{2}\left[\cos\frac{r\pi}{18}+i\sin\frac{r\pi}{18}\right]$, where $r=1$, 13, or 25.

9. $\sqrt{4}\left[\cos\frac{r\pi}{15}+i\sin\frac{r\pi}{15}\right]$, where $r=-1$, 5, 11, 17, or 23.

10. ± 2 and $\pm 2i$.

11. 2, and $2\left[\cos\frac{r\pi}{5}\pm i\sin\frac{r\pi}{5}\right]$, where $r=2$ or 4.

12. -1024. 13. $\pm\frac{i+\sqrt{3}}{2}$ and $\pm\frac{i\sqrt{3}-1}{2}$. 14. 1.

16. $\pm 1, \pm i, \pm\left(\cos\frac{\pi}{6}\pm i\sin\frac{\pi}{6}\right)$, and $\pm\left(\cos\frac{\pi}{3}\pm i\sin\frac{\pi}{3}\right)$.

The last four values.

17. $2\sqrt[3]{2}\cos\frac{r\pi}{9}$, where $r=1$, 7, or 13.

XLIX. (Pages 327, 328.)

6. $\dfrac{5\tan\theta-10\tan^3\theta+\tan^5\theta}{1-10\tan^2\theta+5\tan^4\theta}$.

7. $\dfrac{7\tan\theta-35\tan^3\theta+21\tan^5\theta-\tan^7\theta}{1-21\tan^2\theta+35\tan^4\theta-7\tan^6\theta}$.

8. $\dfrac{9\tan\theta-84\tan^3\theta+126\tan^5\theta-36\tan^7\theta+\tan^9\theta}{1-36\tan^2\theta+126\tan^4\theta-84\tan^6\theta+9\tan^8\theta}$.

L. (Pages 337—339.)

6. $3°48'51''$. 7. $\frac{1}{6}$. 8. $\frac{2}{m^2}$. 9. $\frac{a}{b}$. 10. $\frac{1}{2}$.

11. 3. 12. $\frac{a^2}{b^2}$. 13. 0. 14. $-\frac{a^2+ab+b^2}{ab}$.

15. $-\frac{1}{2}$. 16. 2. 17. $-\frac{1}{6}$. 18. $-\frac{25}{14}$.

19. ∞. 20. $2\frac{n^2-m^2}{p^2}$. 21. $\frac{1}{60}$.

22. $\frac{2}{3}\frac{(m-n)^2}{mn}$. 23. 24. 24. 0.

25. $\log\frac{a}{b}$. 26. 1. 27. e^3. 28. 9.

29. 1. 30. 0. 31. 1. 32. $e^{-\frac{x^2}{2}}$.

33. ∞. 37. $\frac{8}{6}$; $-\frac{1}{6}$.

LIII. (Pages 361, 362.)

8. $x^5-55x^4+330x^3-462x^2+165x-11=0$.

15. $\frac{1}{2^{n-1}}\cos n\theta$, ($n$ odd); $\frac{1}{2^{n-1}}[(-1)^{\frac{n}{2}}-\cos n\theta]$, ($n$ even).

16. $(-1)^{\frac{n-1}{2}}\frac{1}{2^{n-1}}\sin n\theta$, ($n$ odd); $(-1)^{\frac{n}{2}}\frac{1}{2^{n-1}}(1-\cos n\theta)$, ($n$ even).

17. $n^2\operatorname{cosec}^2 n\theta$, ($n$ odd); $\frac{1}{2}n^2\operatorname{cosec}^2\frac{n\theta}{2}$, ($n$ even).

18. $n^2\sec^2 n\theta-n$, (n odd); $n^2\div[1-(-1)^{\frac{n}{2}}\cos n\theta]-n$, ($n$ even).

19. $-n\cot\left(\frac{n\pi}{2}+n\theta\right)$. 20. $n\cot n\theta$.

21. $(-1)^{\frac{n-1}{2}}\tan n\theta$, ($n$ odd); $(-1)^{\frac{n}{2}}$, (n even).

22. $n^2\cot^2\left(\frac{n\pi}{2}+n\theta\right)+n(n-1)$.

24. 0 or $\frac{1}{2}\frac{n^2}{(-1)^{\frac{n}{2}}\cos n\theta-1}$, according as n is odd or even.

LV. (Pages 375—377.)

17. $\cos\alpha\cosh\beta-i\sin\alpha\sinh\beta$.

18. $\frac{\sin 2\alpha-i\sinh 2\beta}{\cosh 2\beta-\cos 2\alpha}$.

19. $2\dfrac{\sin\alpha\cosh\beta - i\cos\alpha\sinh\beta}{\cosh 2\beta - \cos 2\alpha}$.

20. $2\dfrac{\cos\alpha\cosh\beta + i\sin\alpha\sinh\beta}{\cos 2\alpha + \cosh 2\beta}$.

21. $\sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$.

22. $\dfrac{\sinh 2\alpha + i\sin 2\beta}{\cosh 2\alpha + \cos 2\beta}$.

23. $2\dfrac{\cosh\alpha\cos\beta - i\sinh\alpha\sin\beta}{\cosh 2\alpha + \cos 2\beta}$.

LVI. (Page 381.)

1. $\pm\dfrac{\pi}{4} + \dfrac{i}{4}\log\dfrac{1+\sin\theta}{1-\sin\theta}$, according as $\cos\theta$ is positive or negative.

2. $\sin^{-1}(\sqrt{\sin\theta}) + i\log[\sqrt{1+\sin\theta} - \sqrt{\sin\theta}]$.

LVII. (Page 388.)

15. $\dfrac{1}{2}\log(u^2+v^2) + i\tan^{-1}\dfrac{v}{u}$, where

$$u = \frac{1}{2}\log\frac{\cosh 2y - \cos 2x}{2}, \text{ and } v = \tan^{-1}(\cot x\tanh y).$$

LIX. (Pages 402, 403.)

1. 3. 2. 2. 3. 5. 4. -1. 5. -3.

LX. (Pages 407, 408.)

1. $\dfrac{4\sin\alpha}{5-4\cos\alpha}$.

2. 0, provided α does not equal a multiple of π.

3. $\dfrac{\sin^2\alpha}{1-\sin 2\alpha+\sin^2\alpha}$. 4. $\dfrac{\sin\alpha(\cos\alpha-\sin\alpha)}{1-\sin 2\alpha+\sin^2\alpha}$.

5. $\dfrac{\sin\alpha - c\sin(\alpha-\beta) - c^n\sin(\alpha+n\beta) + c^{n+1}\sin\{\alpha+(n-1)\beta\}}{1-2c\cos\beta+c^2}$;

$$\frac{\sin\alpha - c\sin(\alpha-\beta)}{1-2c\cos\beta+c^2}.$$

6. $\dfrac{1 - c\cosh\alpha - c^n\cosh n\alpha + c^{n+1}\cosh(n-1)\alpha}{1 - 2c\cosh\alpha + c^2}$.

7. $\dfrac{c\sinh\alpha}{1 - 2c\cosh\alpha + c^2}$.

8. $\dfrac{\cos\alpha + (-1)^{n-1}\{(n+1)\cos(n-1)\alpha + n\cos n\alpha\}}{2(1+\cos\alpha)}$.

9. $\dfrac{\sin\alpha + (2n+3)\sin n\alpha - (2n+1)\sin(n+1)\alpha}{2(1-\cos\alpha)}$.

10. 0, if $n = 4m$ or $4m-1$, and 1, if $n = 4m-2$ or $4m-3$;
0, if $n = 4m$ or $4m-3$, and -1, if $n = 4m-1$ or $4m-2$.

11. $\left(2\cos\dfrac{\beta}{2}\right)^n . \sin\left(\alpha + \dfrac{n\beta}{2}\right)$.

12. $(2\sin\alpha)^{-\frac{1}{2}}\sin\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)$, except when $\alpha = n\pi$.

13. 0, if n be odd; $(-1)^{\frac{n}{2}}\sin^n\alpha$, if n be even.

14. $\left(2\sin\dfrac{\alpha}{2}\right)^{-n} . \sin\left(\dfrac{n\pi}{2} - \dfrac{n\alpha}{2}\right)$, if n be < 1.

15. $\sqrt{\cos\theta(1+\cos\theta)}$, if θ be between $-\dfrac{\pi}{2}$ and $+\dfrac{\pi}{2}$.

16. $\left(2\cosh\dfrac{u}{2}\right)^n . \sinh\dfrac{n+2}{2}u$.

LXI. (Pages 411—413.)

1. $e^{c\cos\beta}\sin(\alpha + c\sin\beta)$. 2. $e^{c\cos\beta}\cos(\alpha + c\sin\beta)$.

3. $e^{-\cos\alpha\cos\beta}\cos(\cos\alpha\sin\beta)$.

4. $\sin\alpha\cos(\cos\beta)\cosh(\sin\beta)$
$- \cos\alpha\sin(\cos\beta)\sinh(\sin\beta)$.

5. $\sin(\cos\beta)\cosh(\sin\beta)\cos(\alpha-\beta)$
$- \cos(\cos\beta)\sinh(\sin\beta)\sin(\alpha-\beta)$.

6. $\dfrac{1}{2}(e^{e^\alpha} + e^{e^{-\alpha}})$. 7. $\dfrac{1}{2}(e^{e^\alpha} - e^{e^{-\alpha}})$.

8. $e^{y\cos(\sin\alpha)}\cos\{y\sin(\sin\alpha)\}$, where $y = e^{\cos\alpha}$.

9. $e^{y\cos(\cos\alpha)}\cdot\cos\{y\sin(\cos\alpha)\}$, where $y=e^{\sin\alpha}$.

10. $\frac{1}{2}e^{\cos\theta}\{\cos(\theta+\sin\theta)+4\cos(\sin\theta)\}$

$$+\frac{1}{2}e^{-\cos\theta}\{\cos(\theta-\sin\theta)-4\cos(\sin\theta)\}.$$

11. $\tan^{-1}\dfrac{c\sin\alpha}{1+c\cos\alpha}$, except when $c=1$ and $\alpha=(2n+1)\pi$.

12. $\frac{1}{2}\tan^{-1}\dfrac{2c\sin\alpha}{1-c^2}$, except when $c=1$ and $\alpha=n\pi$.

13. $\frac{1}{4}\log\dfrac{1+2c\cos\alpha+c^2}{1-2c\cos\alpha+c^2}$.

14. $\frac{1}{2}\tan^{-1}\dfrac{2c\cos\alpha}{1-c^2}$.

15. $\frac{1}{4}\log\dfrac{1+2c\sin\alpha+c^2}{1-2c\sin\alpha+c^2}$.

16. $+\frac{\pi}{4}$, $-\frac{\pi}{4}$, or 0 according as $\cos\alpha$ is positive, negative, or zero.

17. $\frac{1}{2}\cos(\alpha-\beta)\tan^{-1}\dfrac{2c\cos\beta}{1-c^2}-\frac{1}{2}\sin(\alpha-\beta)\tanh^{-1}\dfrac{2c\sin\beta}{1+c^2}$.

18. $\log\left(\sin\dfrac{\alpha+\beta}{2}\operatorname{cosec}\dfrac{\alpha-\beta}{2}\right)$, except when $\alpha\pm\beta$ is a multiple of 2π.

19. $\log[(1+m)\div\sqrt{1+2m\cos 2\alpha+m^2}]$.

20. $\frac{\alpha}{2}$.

21. $-\frac{1}{2}\tan^{-1}(\cos\beta\operatorname{cosech}\alpha)$.

22. $\frac{1}{8}[2\sqrt{3}\log_e(2+\sqrt{3})-\pi]$.

LXII. (Pages 415, 416.)

1. $\cot\frac{\theta}{2}-\cot 2^{n-1}\theta$.

2. $\operatorname{cosec}\theta\{\cot\theta-\cot(n+1)\theta\}$.

3. $\operatorname{cosec}\theta\{\tan(n+1)\theta-\tan\theta\}$.

4. $\operatorname{cosec}\phi\{\tan(\theta+n\phi)-\tan\theta\}$.

5. $\frac{1}{2}\operatorname{cosec}\theta\{\tan(n+1)\theta-\tan\theta\}$.

6. $S_n = \frac{1}{2^{n-1}} \cot \frac{\theta}{2^{n-1}} - 2 \cot 2\theta$; $S_\infty = \frac{1}{\theta} - 2 \cot 2\theta$.

7. $2 \coth 2\theta - \frac{1}{2^{n-1}} \coth \frac{\theta}{2^{n-1}}$.

8. $\tan 2^n \theta - \tan \theta$.

9. $\tan \theta - \tan \frac{\theta}{2^n}$; $\tan \theta$.

10. $\sin \theta (\cot \theta - \cot 2^n \theta)$.

11. $\frac{1}{2} \sin 2\theta + (-1)^{n+1} \frac{1}{2^{n+1}} \sin 2^{n+1} \theta$.

12. $\frac{1}{2} \sin 2\theta - \frac{1}{2^{n+1}} \sin 2^{n+1} \theta$.

13. $\frac{1}{4} \operatorname{cosec} \frac{\theta}{2} \left(\sec \frac{2n+1}{2} \theta - \sec \frac{\theta}{2} \right)$.

14. $S_n = \frac{1}{2^{n-1}} \tan 2^n \alpha - 2 \tan \alpha$; $S_\infty = 2\alpha - 2 \tan \alpha$.

15. $\frac{1}{4} \left\{ 3 \cos \theta + \left(\frac{-1}{3} \right)^{n-1} \cos 3^n \theta \right\}$.

16. $\frac{1}{4} \left\{ 3^n \sin \frac{\theta}{3^n} - \sin \theta \right\}$.

17. $\frac{1}{8} [3^n \tan 3^n \theta - \tan \theta]$.

18. $\frac{1}{2} [\cot \theta - 3^n \cot 3^n \theta]$.

19. $\tan^{-1} \{(n+1)(n+2)\} - \tan^{-1} 2$.

20. $\tan^{-1}(n+1) - \tan^{-1} 1$, *i.e.* $\tan^{-1} \frac{n}{n+2}$.

21. $S_n = \tan^{-1} 2^n - \tan^{-1} 1$; $S_\infty = \frac{\pi}{4}$.

22. $S_n = \sin^{-1} 1 - \sin^{-1} \frac{1}{\sqrt{n+1}}$; $S_\infty = \frac{\pi}{2}$.

LXIII. (Pages 421, 422.)

1. $1 - a \cos \theta + a^2 \cos 2\theta - a^3 \cos 3\theta + \dots$ ad inf.

2. $\cos \theta + a \cos (\theta + \phi) + a^2 \cos (\theta + 2\phi) + \dots$ ad inf.

3. $\sin\theta + a\sin(\theta+\phi) + a^2\sin(\theta+2\phi) + \dots$ ad inf.

4. $\cos\theta + a\cos(\theta+\phi) + \frac{a^2}{\underline{|2}}\cos(\theta+2\phi) + \frac{a^3}{\underline{|3}}\cos(\theta+3\phi)$

$+ \dots$ ad inf.

5. $r\theta\sin\phi + \frac{r^2\theta^2}{\underline{|2}}\sin 2\phi + \frac{r^3\theta^3}{\underline{|3}}\sin 3\phi + \dots$ ad inf.,

where $r = +\sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\frac{b}{a}$.

9. $x\cos\alpha - \frac{1}{2}x^2\sin 2\alpha - \frac{1}{3}x^3\cos 3\alpha + \frac{1}{4}x^4\sin 4\alpha$

$+ \frac{1}{5}x^5\cos 5\alpha - \dots$ ad inf.

10. $x + y - r\pi = -\cos\alpha\sin x - \frac{1}{2}\cos^2\alpha\sin 2x - \frac{1}{3}\cos^3\alpha\sin 3x$

$- \dots$ ad inf.

12. (1) $m = \tan^2\frac{\alpha}{2}$; (2) $m = -\tan^2\alpha$.

13. $-\log 2 - \sin 2\theta + \frac{1}{2}\cos 4\theta + \frac{1}{3}\sin 6\theta - \frac{1}{4}\cos 8\theta$

$- \frac{1}{5}\sin 10\theta + \dots$ ad inf.

14. $2\left[\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \dots \text{ ad inf.}\right]$.

15. $\log\left(\frac{1}{2}\cos\beta\right) + (\tan\alpha + \cot\alpha)\cos\theta$

$-\frac{1}{2}(\tan^2\alpha + \cot^2\alpha)\cos 2\theta + \frac{1}{3}(\tan^3\alpha + \cot^3\alpha)\cos 3\theta - \dots$ ad inf.

LXIV. (Pages 434—436.)

1. $\Pi\left[x^2 + 2x\cos(3r+1)\frac{2\pi}{9} + 1\right]$, where $r = 0$, 1, or 2.

2. $\Pi\left[x^2 - 2x\cos(6r+1)\frac{\pi}{12} + 1\right]$, where $r = 0$, 1, 2, or 3.

3. $\Pi\left[x^2 - 2x\cos(6r+1)\frac{\pi}{15} + 1\right]$,

where $r = 0, 1, 2, 3,$ or 4.

4. $\Pi\left[x^2 - 2r\cos(3r+1)\frac{\pi}{9} + 1\right]$,

where $r = 0, 1, 2, 3, 4,$ or 5.

5. $\Pi\left[x^2 - 2x\cos(6r+2)\frac{\pi}{21} + 1\right]$,

where $r = 0, 1, 2, 3, 4, 5,$ or 6.

6. $(x-1)\,\Pi\left[x^2 - 2x\cos\frac{2r\pi}{5} + 1\right]$, where $r = 1$ or 2.

7. $\Pi\left[x^2 - 2x\cos(2r+1)\frac{\pi}{6} + 1\right]$, where $r = 0, 1,$ or 2.

8. $(x-1)\,\Pi\left[x^2 - 2x\cos\frac{2r\pi}{7} + 1\right]$, where $r = 1, 2,$ or 3.

9. $(x+1)\,\Pi\left[x^2 - 2x\cos(2r+1)\frac{\pi}{9} + 1\right]$,

where $r = 0, 1, 2,$ or 3.

10. $(x^2-1)\,\Pi\left[x^2 - 2x\cos\frac{r\pi}{5} + 1\right]$, where $r = 1, 2, 3,$ or 4.

11. $(x+1)\,\Pi\left[x^2 - 2x\cos(2r+1)\frac{\pi}{13} + 1\right]$,

where $r = 0, 1, \ldots 5$.

12. $(x^2-1)\,\Pi\left[x^2 - 2x\cos\frac{r\pi}{7} + 1\right]$, where $r = 1, 2, \ldots 6$.

13. $\Pi\left[x^2 - 2x\cos(2r+1)\frac{\pi}{20} + 1\right]$, where $r = 0, 1, 2, \ldots 9$.

LXVI. (Pages 465, 466.)

2. $\pm \cdot 32746\ldots$ ft.

3. $\dfrac{a\cos(\alpha+\beta)}{\cos^2(\alpha+2\beta)}\,\delta$ and $\dfrac{a\sin^2\beta}{\cos^2(\alpha+2\beta)}\,\delta$;

$\dfrac{10\pi\sqrt{2}}{54}$ and $\dfrac{5(2-\sqrt{3})\pi}{54}$ feet.

7. $\dfrac{x - y\cos C}{c\sin B}$ and $\dfrac{y - x\cos C}{c\sin A}$ radians.

8. $-\dfrac{\pi}{40}$ inches.

LXVII. (Page 470.)

1. -1, and $\dfrac{1 \pm \sqrt{3}}{2}$.
2. $-1 + 2\cos 40°$, $-1 + 2\cos 160°$, and $-1 + 2\cos 280°$.
3. -4, and $2 \pm 2\sqrt{3}$.
4. 4, and $1 \pm \sqrt{3}$.
5. $2\sqrt{7}\cos\theta$, where $\theta = 33° 37' 52''$, $153° 37' 52''$, and $273° 37' 52''$.
6. $-\dfrac{4}{3} + \dfrac{2\sqrt{10}}{3}\cos\theta$, where $\theta = 39° 5' 51''$, $159° 5' 51''$, and $279° 5' 51''$.
7. $\dfrac{2}{3}\sqrt{21}\cos\theta$, where $\theta = 44° 50' 49''$, $164° 50' 49''$, and $284° 50' 49''$.

LXVIII. (Page 472.)

2. $2ab$.
3. 2.
4. $2ab$.

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