COMPREHENSIVE TRIGONOMETRY

with Challenging Problems & Solutions *for*

JEE Main & Advanced

Mc Graw Hill Education

Rejaul Makshud

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ABOUT THE AUTHOR

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PREFACE

This text book on *TRIGONOMETRY with Problems & Solutions for JEE Main and Advanced* is meant for aspirants preparing for the entrance examination of different technical institutions, especially NIT/IIT/BITSAT/IISc. In writing this book I have drawn heavily from my long teaching experience at National Level Institutes. After many years of teaching I have realised the need of designing a book that will help the readers to build their base, improve their level of mathematical concepts and enjoy the subject.

This book is designed keeping in view the new pattern of questions asked in JEE Main and Advanced Exams. It has eight chapters. Each chapter has a large number of worked out problems and exercise based problems as given below:

Level – I: Questions based on Fundamentals
Level – II: Mixed Problems (Objective Type Questions)
Level – III: Problems for JEE Advanced Exam
(0.......9): Integer type Questions
Passages: Comprehensive link passages
Matching: Match Matrix
Reasoning: Assertion and Reasoning
Previous years papers: Questions asked in past IIT-JEE Exams

Remember friends, no problem in mathematics is difficult. Once you understand the concept, they will become easy. So please don't jump to exercise problems before you go through the Concept Booster and the objectives. Once you are confident in the theory part, attempt the exercises. The exercise problems are arranged in a manner that they gradually require advanced thinking.

I hope this book will help you to build your base, enjoy the subject and improve your confidence to tackle any type of problem easily and skilfully.

My special thanks goes to Mr. M.P. Singh (IISc. Bangalore), Mr. Manoj Kumar (IIT, Delhi), Mr. Nazre Hussain (B. Tech.), Dr. Syed Kashan Ali (MBBS) and Mr. Shahid Iqbal, who have helped, inspired and motivated me to accomplish this task. As a matter of fact, teaching being the best learning process, I must thank all my students who inspired me most for writing this book.

I would like to convey my affectionate thanks to my wife, who helped me immensely and my children who bore with patience my neglect during the period I remained devoted to this book.

I also convey my sincere thanks to Mr Biswajit of McGraw Hill Education for publishing this book in such a beautiful format.

I owe a special debt of gratitude to my father and elder brother, who taught me the first lesson of Mathematics and to all my learned teachers— Mr. Swapan Halder, Mr. Jadunandan Mishra, Mr. Mahadev Roy and Mr. Dilip Bhattacharya, who instilled the value of quality teaching in me.

I have tried my best to keep this book error-free. I shall be grateful to the readers for their constructive suggestions toward the improvement of the book.

REJAUL MAKSHUD
M. Sc. (Calcutta University, Kolkata)

Dedicated to

My Beloved Mom and Dad

Contents

CHAPTER 1

The Ratios and Identities

1.1 INTRODUCTION

Trigonometry (from Greek trigonon 'triangle' + metron "measure") is a branch of mathematics that study triangles and the relationships between the lengths of their sides and the angles between those sides.

Trigonometry defines the trigonometric functions which describe those relationships that have applicability to cyclical phenomena, such as waves. This field was evolved during the third century BC as a branch of geometry used extensively for astronomical studies. It is also the foundation of the practical art of surveying.

Trigonometry basics are often taught in school either as a separate course or as a part of a precalculus course. The trigonometric functions are pervasive in parts of pure mathematics and applied mathematics such as Fourier analysis and the wave equation, which are in turn essential to many branches of science and technology.

1.2 APPLICATION OF TRIGONOMETRY

There are an enormous number of uses of trigonometry and trigonometric functions. For instance, the technique of triangulation is used in astronomy to measure the distance nearby stars, in geography to measure distances between landmarks, and in satellite navigation systems. The sine and cosine functions are fundamental to the theory of periodic functions that describe sound and light waves.

The fields that use trigonometry or trigonometric functions include astronomy (especially for locating apparent positions of celestial objects, in which spherical trigonometry is essential) and hence navigation (on the oceans, in aircraft, and in space), music theory, acoustics, optics, analysis of financial markets, electronics, probability theory, statistics, biology, medical imaging (CAT scans and ultrasound), pharmacy, chemistry, number theory (and hence cryptology), seismology, meteorology, oceanography, many physical sciences, land surveying and geodesy, architecture, phonetics, economics, electrical engineering, mechanical engineering, civil engineering, computer graphics, cartography, crystallography and game development.

1.3 TRIGONOMETRICAL FUNCTIONS

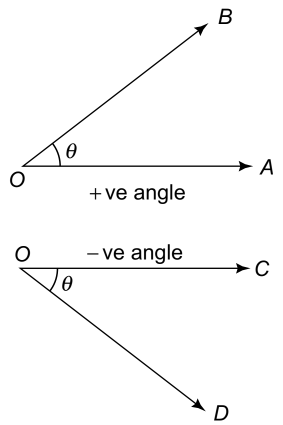
In mathematics, the trigonometric functions (also called as the **circular functions**) are functions of an angle. They are used to relate the angles of a triangle to the lengths of the sides of a triangle. Trigonometric functions are important in the study of triangles and modeling periodic phenomena, among many other applications. The most familiar trigonometric functions are sine, cosine, and tangent. In the context of the standard unit circle with radius 1, where a triangle is formed by a ray originating at the origin and making some angle with the x-axis, the sine of the angle gives the length of the y-component (rise) of the triangle, the cosine gives the length of the x-component (run), and the tangent function gives the slope (y-component divided by the x-component). More precise definitions are given below in detail. Trigonometric functions are commonly defined as ratios of two sides of a right triangle containing the angle, and can equivalently be defined as the lengths of various line segments from a unit circle. More modern definitions express them as infinite series or as solutions of certain differential equations, allowing their extension to arbitrary positive and negative values and even to complex numbers.

Trigonometric functions have a wide range of uses including computing unknown lengths and angles in triangles (often right triangles). In this case, trigonometric functions are used, for instance, in navigation, engineering, and physics. A common use in elementary physics is resolving a vector into Cartesian coordinates. The sine and cosine functions are also commonly used to model periodic function phenomena, such as sound and light waves, the position and velocity of harmonic oscillators, sunlight intensity and day length, and average temperature variations throughout the year.

In modern usage, there are six basic trigonometric functions tabulated here with equations that relate them to one another. Especially, with the last four, these relations are often taken as the definitions of those functions, but one can define them equally well geometrically, or by other means, and then derive these relations.

1.4 MEASUREMENT OF ANGLES

1. **Angle:** The measurement of an angle is the amount of rotation from the initial side to the terminal side.
2. **Sense of an Angle:** The sense of an angle is +ve or –ve according to the initial side that rotates in anti-clockwise or clockwise direction to get the terminal side.



3. **System of measuring angles:**

   There are three systems of measuring angles such as
   (i) Sexagesimal system
   (ii) Centisimal system
   (iii) Circular system

   In sexagesimal system, we have

   1 right angle = 90°

   $1° = 60'$

   $1' = 60''$

   In centasimal system, we have

   1 right angle $= 100^g$

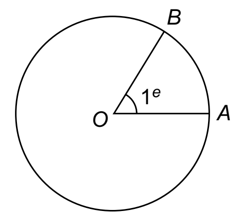
   $1^g = 100'$

   $1' = 100''$

   In circular system, the unit of measurement is radian.

   Radian: One radian is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.

   Here, $\angle AOB = 1$ radian $= 1^e$.



Notes:

(i) When an angle is expressed in radians, the word radian is omitted.

(ii) Since $180° = \pi$ radian $= \left(\dfrac{22}{7 \times 180}\right)$ radian = 0.01746 radian

(iii) 1 radian $= \dfrac{180°}{\pi} = \left(\dfrac{180}{22} \times 7\right)$ = 57° 16' 22'

(iv) The angle between two consecutive digits is $30° \left(\dfrac{\pi}{6} \text{ radians}\right)$

(v) The hour hand rotates through an angle of 30° in 1 hour (i.e., $\left(\dfrac{1}{2}\right)$ in 1 minute).

(vi) The minute hand rotates through an angle of 6° in 1 minute.

(vii) The relation amongst three systems of measurement of an angle is

$$\frac{D}{90°} = \frac{G}{100} = \frac{2R}{\pi}$$

(viii) The number of radians in an angle subtended by an arc of a circle at the centre is $\dfrac{\text{Arc}}{\text{Radius}}$

i.e., $\theta = \dfrac{s}{r}$

1.5 SOME SOLVED EXAMPLES

Ex-1. If the radius of the earth is 4900 km, what is the length of its circumference?

Soln. Given $r = 4900$ km

Circumference $= 2\pi r$

$= 2 \times \dfrac{22}{7} \times 4900$

$= 44 \times 700$

$= 30{,}800$ km

Ex-2. The angles of a triangle are in the ratio 3 : 4 : 5. Find the smallest angle in degrees and the greatest angle in radians.

Soln. Let the three angles be $3x$, $4x$ and $5x$, respectively

Thus, $3x + 4x + 5x = 180°$

$\Rightarrow \quad 12x = 180°$

$\Rightarrow \quad x = 15°$

Therefore, the smallest angle

$= 3x = 3 \times 15° = 45°$

and the greatest angle

$= 5x = 5 \times 15° = 75°$

$= \left(75 \times \frac{\pi}{180}\right)$ radians

$= \left(\frac{5\pi}{12}\right)$ radians

Ex-3. The angles of a triangle are in AP and the number of degrees in the least is to the number of radians in the greatest as 60 to π, find the angles in degrees.

Soln. Let the three angles be $a + d$, a, $a - d$

Thus, $a + d + a + a - d = 180°$

$\Rightarrow \quad 3a = 180°$

$\Rightarrow \quad a = \frac{180°}{3} = 60°$

It is given that,

$(a - d)° : (a + d) \times \frac{\pi}{180} = \frac{60}{\pi}$

$\Rightarrow \quad \frac{(a-d)}{(a+d)} \times \frac{180}{\pi} = \frac{60}{\pi}$

$\Rightarrow \quad \frac{(a-d)}{(a+d)} = \frac{1}{3}$

$\Rightarrow \quad a + d = 3a - 3d$

$\Rightarrow \quad 4d = 2a$

$\Rightarrow \quad d = \frac{a}{2} = 30°$

Hence, the three angles are 90°, 60°, and 30°.

Ex-4. The number of sides in two regular polygons are 5 : 4 and the difference between their angles is 9. Find the number of sides of the polygon.

Soln. Let the number of sides of the given polygons be $5x$ and $4x$, respectively.

It is given that,

$\left(\frac{2 \times 5x - 4}{5x} - \frac{2 \times 4x - 4}{4x}\right) \times 90 = 9$

$\Rightarrow \quad \left(\frac{10x - 4}{5x} - \frac{2x - 1}{x}\right) = \frac{1}{10}$

$\Rightarrow \quad \left(\frac{10x - 4 - 10x + 5}{5x}\right) = \frac{1}{10}$

$\Rightarrow \quad \left(\frac{1}{x}\right) = \frac{1}{2}$

$\Rightarrow \quad x = 2$

Hence, the number of sides of the polygons will be 10 and 8, respectively.

Ex-5. The angles of a quadrilateral are in A.P. and the greatest is double the least. Express the least angles in radians.

Soln. Let the angles of the quadrilateral be

$a - 3d, a - d, a + d, a + 3d$

It is given that, $\quad a + 3d = 2(a - 3d)$

$\Rightarrow \quad a + 3d = 2a - 6d$

$\Rightarrow \quad a = 9d$

Also, $a + 3d + a - d + a + d + a + 3d = 360$

$\Rightarrow \quad 4a = 360$

$\Rightarrow \quad a = 90$

and $\quad d = 10$

Hence, the smallest angle = 90° – 30°

$= 60°$

$= \left(\frac{\pi}{3}\right)^c$.

Ex-6. Find the angle between the hour hand and the minute hand in circular measure at half past 4.

Soln. Clearly, at half past 4, hour hand will be at $4\frac{1}{2}$ and minute hand will be at 6.

In 1 hour angle made by the hour hand will be 30°

In $4\frac{1}{2}$ hours angle made by the hour hand

$= \frac{9}{2} \times 30° = 135°$

In 1 minute angle made by the minute hand = 6°

In 30° minutes, angle made by the minute hand = $6 \times 30° = 180°$

Thus, the angle between the hour hand and the minute hand = $180° - 135°$

= 45°.

Ex-7. Find the length of an arc of a circle of radius 10 cm subtending an angle of 30° at the centre.

Soln. Angle subtended at the centre

$= 30° = \left(30 \times \frac{\pi}{180}\right) = \frac{\pi}{6}$

Hence, $\quad l = 10 \times \frac{\pi}{6} = \frac{5\pi}{3}$.

Ex-8. The minute hand of a watch is 35 cm long. How far does its tip move in 18 minutes?

Soln. The angle traced by a minute hand in 60 minutes

$= 360° = 2\pi$ radians

Thus, the angle traced by minute hand in 18 minutes

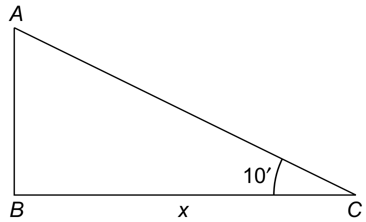
$= 2\pi \times \frac{18}{60} = \frac{3\pi}{5}$ radians

Hence, the distance moved by the tip in 18 minutes

$= l = 35 \times \frac{3\pi}{5} = 21 \times \frac{22}{7} = 66$ cm

Ex-9. At what distance does a man, whose height is 2 m subtend an angle of 10'?

Soln. Let AB be the height of the man and the required distance be x, where $BC = x$



Therefore, $\frac{2}{x} \times \frac{180}{\pi} = \frac{10}{60}$

$\Rightarrow \quad x = \frac{2}{10} \times \frac{180}{\pi} \times 60$

$\Rightarrow \quad x = \frac{12 \times 180}{\pi}$

$\Rightarrow \quad x = \frac{12 \times 180}{\frac{22}{7}} = \frac{12 \times 180 \times 7}{22}$

$\Rightarrow \quad x = \frac{42 \times 180}{11} = 687.3$

Ex-10. Find the distance at which a globe $5\frac{1}{2}$ cm in diameter, will subtend an angle of 6'.

Soln. Let the required distance be x cm

According to the question,

$$6' = \frac{11}{2 \times x} \times \frac{180}{\pi}$$

$\Rightarrow \quad \frac{6}{60} = \frac{11}{2 \times x} \times \frac{180}{\pi}$

$\Rightarrow \quad x = \frac{11}{2} \times \frac{180}{\pi} \times \frac{60}{6}$

$\Rightarrow \quad x = \frac{11}{2} \times \frac{180 \times 7}{22} \times 10$

$\Rightarrow \quad x = 45 \times 7 \times 10 = 3150$

Hence, the required distance will be 3150 cms.

Ex-11. The radius of the earth being taken as 6400 km and the distance of the moon from the earth being 60 times the radius of the earth, find the radius of the moon which subtends an angle of 16' at the earth.

Soln. Let the radius of the moon be x km

It is given that, $\frac{16}{60} = \frac{2x}{60 \times 6400} \times \frac{180}{\pi}$

$\Rightarrow \quad x = \frac{16 \times 6400 \times \pi}{180 \times 2}$

$\Rightarrow \quad x = \frac{4 \times 640 \times \pi}{9}$

$\Rightarrow \quad x = \frac{4 \times 640 \times 22}{9 \times 7}$

$\Rightarrow \quad x = 894$

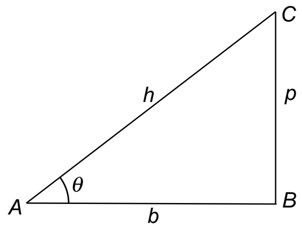
Hence, the radius of the moon be 894 km.

EXERCISE 1

1. Find the length of an arc of a circle of radius 5 cm. subtending a central angle of measuring 15°.
2. In a circle of diameter 40 cm. the length of a chord is 20 cm. Find the length of minor arc corresponding to the chord.
3. If the arcs of same length in the circles subtends angles of 60° and 75° at their centres. Find the ratio of their radii.
4. A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and discribes 88 meters when it has traced out 72° at the centre, find the length of the rope.
5. The Moon's distance from the Earth is 36,000 kms. and its diameter subtends an angle of 31° at the eye of the observer. Find the diameter of the Moon.
6. The difference between the acute angles of a right angled triangle is $\frac{2\pi}{3}$ radians. Express the angles in degrees.
7. The angles of a quadrilateral are in A.P. and the greatest angle is 120°. Find the angles in radians.
8. The angles of a triangle are in A.P. such that the greatest is 5 times the least. Find the angles in radians.
9. A wheel makes 180 revolutions per minute through how many radians does it turn in 1 second?
10. Find the distance from the eye at which a coin of 2 cm. diameter should be held so as to conceal the full moon whose angular diameter is 31'.
11. The interrior angles of a triangle are in A.P. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon.
12. A wheel makes 30 revolutions per minute. Find the circular measure of the angle described by a spoke in 1/2 second.
13. A man running along a circular track at the rate of 10 miles per hour travels in 36 seconds, an arc which subtends 56° at the centre. Find the diameter of the circle.
14. At what distance does a man $5\frac{1}{2}$ ft in height, subtends an angle of 15"?
15. Find the angle between the hour hand and minute hand in circular measure at 4 O' clock.

1.6 TRIGONOMETRICAL RATIOS

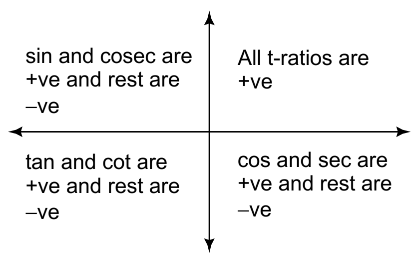
1.6.1 Definitions of Trigonometric Ratios



1. Sin $\theta = \frac{p}{h}$
2. cosec $\theta = \frac{h}{p}$
3. cos $\theta = \frac{b}{h}$
4. sec $\theta = \frac{h}{b}$
5. tan $\theta = \frac{p}{b}$
6. cot $\theta = \frac{b}{p}$

1.6.2 Signs of Trigonometrical Ratios

The signs of the trigonometrical ratios in different quadrants are remembered by the following chart.



It is also known as all, sin, tan, cos formula.

1.6.3 Relation between the Trigonometrical Ratios of an Angle

Step I (i) $\sin\theta \,.\, \text{cosec}\,\theta = 1$
(ii) $\cos\theta \,.\, \sec\theta = 1$
(iii) $\tan\theta \,.\, \cot\theta = 1$

Step II (i) $\tan\theta = \frac{\sin\theta}{\cos\theta}$
(ii) $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Step III (i) $\sin\theta \,.\, \text{cosec}\,\theta = 1$
(ii) $\cos\theta \,.\, \sec\theta = 1$
(iii) $\tan\theta \,.\, \cot\theta = 1$

Step IV (i) $\sin^2\theta + \cos^2\theta = 1$
(ii) $\sec^2\theta = 1 + \tan^2\theta$
(iii) $\text{cosec}^2\theta = 1 + \cot^2\theta$

Step V Ranges of odd power *t*-ratios.
(i) $-1 \le \sin^{2n+1}\theta, \cos^{2n+1}\theta \le 1$
(ii) $-\infty < \tan^{2n+1}\theta, \cot^{2n+1}\theta < \infty$
(iii) $\text{cosec}^{2n+1}\theta, \sec^{2n+1}\theta \ge 1$
$\text{cosec}^{2n+1}\theta, \sec^{2n+1}\theta \le -1$
where $n \in W$

Step VI Ranges of even power *t*-ratios.
(i) $0 \le \sin^{2n}\theta, \cos^{2n}\theta \le 1$
(ii) $0 \le \tan^{2n}\theta, \cot^{2n}\theta < \infty$
(iii) $1 \le \text{cosec}^{2n}\theta, \sec^{2n}\theta < \infty$
where $n \in N$

1.7 LIMITS OF THE VALUES OF TRIGONOMETRICAL FUNCTIONS

1. $-1 \le \sin\theta \le 1$
2. $-1 \le \cos\theta \le 1$
3. $-\infty < \tan\theta < \infty$
4. $-\infty < \cot\theta < \infty$
5. $\text{cosec}\,\theta \ge 1$ and $\text{cosec}\,\theta \le -1$
6. $\sec\theta \ge 1$ and $\sec\theta \le -1$

1.8 SOME SOLVED EXAMPLES

Ex-1. If $\sec\theta + \tan\theta = 3$, where θ lies in the first quadrant, then find the value of cos θ.

Soln. Given $\sec\theta + \tan\theta = 3$ (i)

$$\Rightarrow (\sec\theta - \tan\theta) = \frac{1}{(\sec\theta + \tan\theta)} = \frac{1}{3} \quad \text{(ii)}$$

Adding (i) and (ii), we get,

$$2\sec\theta = 3 + \frac{1}{3} = \frac{10}{3}$$

$$\Rightarrow \sec\theta = \frac{5}{3}$$

$$\Rightarrow \cos\theta = \frac{5}{3}$$

Ex-2. If $\text{cosec}\,\theta - \cot\theta = \frac{1}{5}$, then find the value of sin θ.

Soln. Given $\text{cosec}\,\theta - \cot\theta = \frac{1}{5}$ (i)

$$\Rightarrow \text{cosec}\,\theta - \cot\theta = \frac{1}{\text{cosec}\,\theta + \cot\theta} = 5 \quad \text{(ii)}$$

Adding (i) and (ii), we get,

$$2\,\text{cosec}\,\theta = 5 + \frac{1}{5} = \frac{26}{5}$$

$$\Rightarrow \text{cosec}\,\theta = \frac{13}{5}$$

$$\Rightarrow \sin\theta = \frac{5}{13}$$

Ex-3. If $a = c\cos\theta + d\sin\theta$ and $b = c\sin\theta - d\cos\theta$ such that $a^m + b^n = c^p + d^q$, where $m, n, p, q \in N$ then find the value of $m + n + p + q + 42$.

Soln. Given $a = c\cos\theta + d\sin\theta$ (i)

and $b = c\sin\theta - d\cos\theta$ (ii)

Squaring and adding (i) and (ii), we get,

$$a^2 + b^2 = (c\cos\theta + d\sin\theta)^2 + (c\sin\theta - d\cos\theta)^2$$

$$\Rightarrow a^2 + b^2 = \left(c^2\cos^2\theta + d^2\sin^2\theta\right) + \left(c^2\sin^2\theta + d^2\cos^2\theta\right)$$

$$\Rightarrow a^2 + b^2 = c^2 + d^2$$

$\Rightarrow m = 2, n = 2, p = 2, q = 2$

Hence, the value of $m + n + p + q + 42 = 50$.

Ex-4. If $3\sin\theta + 4\cos\theta = 5$, then find the value of $3\cos\theta - 4\sin\theta$.

Soln. Let $x = 3\cos\theta - 4\sin\theta$ (i)

and $5 = 3\sin\theta - 4\cos\theta$ (ii)

Squaring and adding (i) and (ii), we get

$$x^2 + 5^2 = (3\cos\theta + 4\sin\theta)^2 + (3\sin\theta - 4\cos\theta)^2$$

$$\Rightarrow x^2 + 5^2 = \left(9\cos^2\theta + 16\sin^2\theta + 24\sin\theta\cos\theta\right) + \left(9\sin^2\theta + 16\cos^2\theta - 24\sin\theta\cos\theta\right)$$

$$= \left(9\cos^2\theta + 16\sin^2\theta\right) + \left(9\sin^2\theta + 16\cos^2\theta\right)$$

$$= 9\left(\cos^2\theta + \sin^2\theta\right) + 16\left(\cos^2\theta + \sin^2\theta\right)$$

$\Rightarrow x^2 + 25 = 25$

$\Rightarrow x^2 = 0$

$\Rightarrow x = 0$

$\Rightarrow 3\cos\theta - 4\sin\theta = 0.$

Ex-5. If $x = r\cos\theta\sin\varphi$, $y = r\cos\theta\cos\varphi$ and $z = r\sin\theta$ such that $x^m + y^n + z^p = r^2$, where $m, n, p \in N$, then find the value of $(m + n + p - 4)^{m+n+p+4}$.

Soln. We have, $x^2 + y^2 + z^2$

$$= (r\cos\theta\cos\varphi)^2 + (r\cos\theta\sin\varphi)^2$$

$\Rightarrow x^2 + y^2 + z^2$

$$= \left(r^2\cos^2\theta\cos^2\varphi\right) + \left(r^2\cos^2\theta\sin^2\varphi\right) + \left(r^2\sin^2\theta\right)$$

$\Rightarrow x^2 + y^2 + z^2$

$$= r^2\cos^2\theta\left(\cos^2\varphi + \sin^2\varphi\right) + \left(r^2\sin^2\theta\right)$$

$$= r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2$$

$\Rightarrow x^2 + y^2 + z^2 = r^2$

$\Rightarrow m = 2, n = 2, p = 2$

Thus, the value of $(m + n + p - 4)^{(m+n+p+4)}$

$$= 2^{10} = 1024$$

Ex-6. If $x = \dfrac{2\sin\alpha}{1 + \cos\alpha + 3\sin\alpha}$, then find the value of $\dfrac{\sin\alpha - 3\cos\alpha + 3}{2 - 2\cos\alpha}$

Soln. Given $x = \dfrac{2\sin\alpha}{1 + \cos\alpha + 3\sin\alpha}$

We have, $\dfrac{\sin\alpha - 3\cos\alpha + 3}{2 - 2\cos\alpha}$

$$= \frac{\sin\alpha + 3(1 - \cos\alpha)}{2(1 - \cos\alpha)}$$

$$= \frac{\sin\alpha}{2(1 - \cos\alpha)} + \frac{3}{2}$$

$$= \frac{\sin\alpha(1 + \cos\alpha)}{2\left(1 - \cos^2\alpha\right)} + \frac{3}{2}$$

$$= \frac{\sin\alpha(1 + \cos\alpha)}{2\sin^2\alpha} + \frac{3}{2}$$

$$= \frac{(1 + \cos\alpha)}{2\sin\alpha} + \frac{3}{2}$$

$$= \frac{(1 + \cos\alpha + 3\sin\alpha)}{2\sin\alpha}$$

$$= \frac{1}{x}.$$

Ex-7. If $P = \sec^6\theta - \tan^6\theta - 3\sec^2\theta\tan^2\theta$, $Q = \operatorname{cosec}^6\theta - \cot^6\theta - 3\operatorname{cosec}^2\theta\cot^2\theta$ and $R = \sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta$, then find the value of $(P + Q + R)^{(P+Q+R)}$

Soln. We have, $P = \sec^6\theta - \tan^6\theta - 3\sec^2\theta\tan^2\theta$

$$= \left(\sec^2\theta - \tan^2\theta\right)^3 = 1$$

$$Q = \operatorname{cosec}^6\theta - \cot^6\theta - 3\operatorname{cosec}^2\theta\cot^2\theta$$

$= \left(\text{cosec}^2\theta - \cot^2\theta\right)^3 = 1$

and $R = \sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta$

$= \left(\sin^2\theta + \cos^2\theta\right)^3 = 1$

Hence, the value of $(P+Q+R)^{(P+Q+R)}$

$= 3^3 = 27.$

Ex-8. If $3\sin x + 4\cos x = 5$, for all x in $\left(0, \frac{\pi}{2}\right)$, then find the value of $2\sin x + \cos x + 4\tan x$

Soln. We have $3\sin x + 4\cos x = 5$

Let $y = 3\cos x - 4\sin x$

Now, $y^2 + 5^2 = (3\cos x - 4\sin x)^2 + (3\sin x + 4\cos x)^2$

$\Rightarrow\ y^2 + 25 = 9\cos^2 x + 16\sin^2 x - 24\sin x\cos x + 9\sin^2 x + 16\cos^2 x + 24\sin x\cos x$

$\Rightarrow\ y^2 + 25 = 25(\cos^2 x + \sin^2 x) = 25$

$\Rightarrow\ y^2 = 0$

$\Rightarrow\ y = 0$

$\Rightarrow\ 3\cos x - 4\sin x = 0$

$\Rightarrow\ 3\cos x = 4\sin x$

$\Rightarrow\ \tan x = 3/4$

Hence, the value of $2\sin x + \cos x + 4\tan x$

$= 2\left(\frac{3}{5}\right) + \left(\frac{4}{5}\right) + 4\left(\frac{3}{4}\right) = 2 + 3 = 5.$

Ex-9. If $\sin A + \sin B + \sin C + 3 = 0$, then find the value of $\cos A + \cos B + \cos C + 10$.

Soln. Given $\sin A + \sin B + \sin C = -3$

$\Rightarrow\ \sin A = -1, \sin B = -1, \sin C = -1$

$\Rightarrow\ A = -\frac{\pi}{2}, B = -\frac{\pi}{2}, C = -\frac{\pi}{2}$

Hence, the value of $\cos A + \cos B + \cos C + 10$

$= 0 + 0 + 0 + 10 = 10.$

Ex-10. If $(1+\sin\theta)(1+\cos\theta) = \frac{5}{4}$, then find the value of $(1-\sin\theta)(1-\cos\theta)$.

Soln. We have $(1+\sin\theta)(1+\cos\theta) = \frac{5}{4}$

$\Rightarrow\ 1 + \sin\theta + \cos\theta + \sin\theta\cos\theta = \frac{5}{4}$

$\Rightarrow\ 1 + t + \left(\frac{t^2-1}{2}\right) = \frac{5}{4}$ $(\sin\theta + \cos\theta = t$, say)

$\Rightarrow\ t + \left(\frac{t^2-1}{2}\right) = \frac{1}{4}$

$\Rightarrow\ t^2 + 2t - 1 = \frac{1}{2}$

$\Rightarrow\ 2t^2 + 4t - 3 = 0$

$\Rightarrow\ t = \frac{-4 \pm \sqrt{16+24}}{4}$

$= \frac{-4 \pm 2\sqrt{10}}{4} = -1 \pm \frac{1}{2}\sqrt{10}$

$\Rightarrow\ t = -1 + \frac{1}{2}\sqrt{10}$

$\Rightarrow\ \sin\theta + \cos\theta = -1 + \frac{1}{2}\sqrt{10}$

Now, $(1-\sin\theta)(1-\cos\theta)$

$= 1 - \sin\theta - \cos\theta + \sin\theta\cos\theta$

$= 1 - (\sin\theta + \cos\theta) + \sin\theta\cos\theta$

$= 1 - \left(-1 + \frac{\sqrt{10}}{2}\right) + \frac{1}{2}\left(\frac{10}{4} - \sqrt{10}\right)$

$= \left(2 + \frac{5}{4}\right) - \sqrt{10}$

$= \left(\frac{13}{4} - \sqrt{10}\right).$

Ex-11. Find the minimum value of the expression $f(x) = \frac{9x^2\sin^2 x + 4}{x\sin x}$, for all x in $(0, \pi)$.

Soln. Given $f(x) = \frac{9x^2\sin^2 x + 4}{x\sin x} = 9x\sin x + \frac{4}{x\sin x}$

Applying, A.M ≥ G.M, we get,

$\left(\frac{9x\sin x + \frac{4}{x\sin x}}{2}\right) \ge \sqrt{9x\sin x \times \frac{4}{x\sin x}}$

$\Rightarrow \left(\frac{9x\sin x + \frac{4}{x\sin x}}{2}\right) \ge 6$

$\Rightarrow \left(9x\sin x + \frac{4}{x\sin x}\right) \ge 12$

Hence, the minimum value of $f(x)$ is 12.

Ex-12. If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, than prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

Soln. We have, $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$

$\Rightarrow \quad \sin\theta = \left(\sqrt{2}-1\right)\cos\theta$

$\Rightarrow \quad \cos\theta = \dfrac{\sin\theta}{\left(\sqrt{2}-1\right)}$

$\Rightarrow \quad \cos\theta = \left(\sqrt{2}+1\right)\sin\theta$

$\Rightarrow \quad \cos\theta - \sin\theta = \sqrt{2}\sin\theta$

Ex-13. If $\tan^2\theta = 1-e^2$ then prove that $\sec\theta + \tan^3\theta.\text{cosec}\,\theta = \left(2-e^2\right)^{3/2}$

Soln. We have, $\tan^2\theta = 1-e^2$

$\Rightarrow \quad 1+\tan^2\theta = 1+1-e^2 = 2-e^2$

$\Rightarrow \quad \sec^2\theta = 2-e^2$

$\Rightarrow \quad \sec\theta = \sqrt{2-e^2}$

Now, $\sec\theta + \tan^3\theta.\text{cosec}\,\theta$

$= \sec\theta + \dfrac{\sin^3\theta}{\cos^3\theta}.\dfrac{1}{\sin\theta}$

$= \sec\theta + \dfrac{\sin^2\theta}{\cos^3\theta}$

$= \sec\theta + \dfrac{\sin^2\theta}{\cos^2\theta}.\dfrac{1}{\cos\theta}$

$= \sec\theta + \tan^2\theta.\sec\theta$

$= \sec\theta\left(1+\tan^2\theta\right)$

$= \sec^3\theta$

$= \left(2-e^2\right)^{3/2}$

Ex-14. If $\sin\theta + \sin^2\theta + \sin^3\theta = 1$, then prove that, $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4\cdot$

Soln. Given $\sin\theta + \sin^2\theta + \sin^3\theta = 1$

$\Rightarrow \quad \left(\sin\theta + \sin^3\theta\right) = 1-\sin^2\theta = \cos^2\theta$

$\Rightarrow \quad \left(\sin\theta + \sin^3\theta\right)^2 = \left(\cos^2\theta\right)^2$

$\Rightarrow \quad \left(\sin\theta + \sin^3\theta\right)^2 = \left(\cos^2\theta\right)^2$

$\Rightarrow \quad \left(1-\cos^2\theta\right)\left(2-\cos^2\theta\right)^2 = \cos^4\theta$

$\Rightarrow \quad \left(1-\cos^2\theta\right)\left(4-4\cos^2\theta+\cos^4\theta\right) = \cos^4\theta$

$\Rightarrow \quad 4-4\cos^2\theta+\cos^4\theta-4\cos^2\theta+4\cos^4\theta$

$\Rightarrow \quad -\cos^6\theta = \cos^4\theta$

$\Rightarrow \quad \cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

Ex-15. If $x = \dfrac{2\sin\theta}{1+\cos\theta+\sin\theta}$, then prove that $\dfrac{1-\cos\theta+\sin\theta}{1+\sin\theta} = x$.

Soln. Given $x = \dfrac{2\sin\theta}{1+\cos\theta+\sin\theta}$

$$= \frac{2\sin\theta}{(1+\sin\theta)+\cos\theta}$$

$$= \frac{2\sin\theta((1+\sin\theta)-\cos\theta)}{((1+\sin\theta)+\cos\theta)((1+\sin\theta)-\cos\theta)}$$

$$= \frac{2\sin\theta((1+\sin\theta)-\cos\theta)}{\left((1+\sin\theta)^2-\cos^2\theta\right)}$$

$$= \frac{2\sin\theta((1+\sin\theta)-\cos\theta)}{\left(1+\sin^2\theta+2\sin\theta-\cos^2\theta\right)}$$

$$= \frac{2\sin\theta((1+\sin\theta)-\cos\theta)}{\left(\sin^2\theta+2\sin\theta+\left(1-\cos^2\theta\right)\right)}$$

$$= \frac{2\sin\theta((1+\sin\theta)-\cos\theta)}{\left(2\sin\theta+2\sin^2\theta\right)}$$

$$= \frac{2\sin\theta((1+\sin\theta)-\cos\theta)}{2\sin\theta(1+\sin\theta)}$$

$$= \frac{((1+\sin\theta)-\cos\theta)}{(1+\sin\theta)}$$

$$= \frac{(1-\cos\theta+\sin\theta)}{(1+\sin\theta)}$$

Ex-16. If $\dfrac{\sin^4\alpha}{a} + \dfrac{\cos^4\alpha}{b} = \dfrac{1}{a+b}$, then prove that $\dfrac{\sin^8\alpha}{a^3} + \dfrac{\cos^8\alpha}{b^3} = \dfrac{1}{(a+b)^3}$

Soln. We have, $\dfrac{\sin^4\alpha}{a} + \dfrac{\cos^4\alpha}{b} = \dfrac{1}{a+b}$

$\Rightarrow \quad \left(\dfrac{a+b}{a}\right)\sin^4\alpha + \left(\dfrac{a+b}{b}\right)\cos^4\alpha = 1$

$\Rightarrow \quad \left(1+\dfrac{b}{a}\right)\sin^4\alpha + \left(1+\dfrac{a}{b}\right)\cos^4\alpha = 1$

$\Rightarrow \quad \left(\dfrac{b}{a}\sin^4\alpha + \dfrac{a}{b}\cos^4\alpha\right) + \left(\sin^4\alpha + \cos^4\alpha\right) = 1$

$\Rightarrow \quad \left(\dfrac{b}{a}\sin^4\alpha + \dfrac{a}{b}\cos^4\alpha\right) + \left(1-2\sin^2\alpha.\cos^2\alpha\right) = 1$

$\Rightarrow \quad \left(\dfrac{b}{a}\sin^4\alpha + \dfrac{a}{b}\cos^4\alpha - 2\sin^2\alpha.\cos^2\alpha\right) = 0$

$$\Rightarrow \left(\sqrt{\frac{b}{a}}\sin^2\alpha\right)^2+\left(\sqrt{\frac{a}{b}}\cos^2\alpha\right)^2$$

$$-2\sqrt{\frac{b}{a}}\sin^2\alpha.\sqrt{\frac{a}{b}}\cos^2\alpha=0$$

$$\Rightarrow \left(\sqrt{\frac{b}{a}}\sin^2\alpha-\sqrt{\frac{a}{b}}\cos^2\alpha\right)^2=0$$

$$\Rightarrow \left(\sqrt{\frac{b}{a}}\sin^2\alpha-\sqrt{\frac{a}{b}}\cos^2\alpha\right)=0$$

$$\Rightarrow \sqrt{\frac{b}{a}}\sin^2\alpha=\sqrt{\frac{a}{b}}\cos^2\alpha$$

$$\Rightarrow \frac{\sin^2\alpha}{a}=\frac{\cos^2\alpha}{b}=\frac{1}{a+b}$$

$$\Rightarrow \sin^2\alpha=\frac{a}{a+b},\ \cos^2\alpha=\frac{b}{a+b}$$

Now, $\dfrac{\sin^8\alpha}{a^3}+\dfrac{\cos^8\alpha}{b^3}$

$$=\frac{\left(\sin^2\alpha\right)^4}{a^3}+\frac{\left(\cos^2\alpha\right)^4}{b^3}$$

$$=\frac{\left(\frac{a}{a+b}\right)^4}{a^3}+\frac{\left(\frac{b}{a+b}\right)^4}{b^3}$$

$$=\frac{a^4}{a^3(a+b)^4}+\frac{b^4}{b^3(a+b)^4}$$

$$=\frac{a}{(a+b)^4}+\frac{b}{(a+b)^4}$$

$$=\frac{a+b}{(a+b)^4}$$

$$=\frac{1}{(a+b)^3}$$

Ex-17. Prove that $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2$
$+4\left(\sin^6 x+\cos^6 x\right)=13$.

Soln. We have, $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin 6x + \cos 6x)$

$= 3(\sin^4 x - 4\sin^3 x\cos x + 6\sin^2 x\cos^2 x$
$\quad - 4\sin x\cos^3 x + \cos^4 x)$
$\quad + 6(\sin^2 x + \cos^2 x + 2\sin x\cos x)$
$\quad + 4\{(\sin^2 x)^3 + (\cos^2 x)3\}$

$= 3(\sin^4 x + \cos^4 x - 4\sin x\cos x$
$\quad (\sin^2 x + \cos^2 x) + 6\sin^2 x\cos^2 x)$
$\quad + 6(1 + 2\sin x\cos x)$
$\quad + 4(\sin^2 x + \cos^2 x)^2$
$\quad - 12\sin^2 x\cos^2 x.$

$= 3 - 6\sin^2 x\cos^2 x - 12\sin x\cos x$
$\quad + 18\sin^2 x\cos^2 x + 6 + 12\sin x\cos x$
$\quad + 4 - 12\sin^2 x\cos^2 x$
$\quad = 3 + 6 + 4$
$\quad = 13.$

Ex-18. If $\sin x + \sin^2 x = 1$, then find the value of $\cos^8 x + 2\cos^6 x + \cos^4 x$.

Soln. We have, $\sin x + \sin^2 x = 1$

$\Rightarrow \sin x = 1 - \sin^2 x = \cos^2 x$

Now, $\cos^8 x + 2\cos^6 x + \cos^4 x$

$= (\cos^4 x)^2 + 2 . \cos^4 x . \cos^2 x + (\cos^2 x)^2$
$= (\cos^4 x + \cos^2 x)^2$
$= (\sin^2 x + \sin x)^2$
$= (1)^2 = 1.$

Ex-19. If $0 \le , \le 180^\circ$ and $81^{\sin^2\theta}+81^{\cos^2\theta}=30$, then find the value of θ.

Soln. We have, $81^{\sin^2\theta}+81^{\cos^2\theta}=30$

$$\Rightarrow \quad 81^{\sin^2\theta}+81^{1\text{-}\sin^2\theta}=30$$

$$\Rightarrow \quad 81^{\sin^2\theta}+\frac{81}{81^{\sin^2\theta}}=30$$

$$\Rightarrow \quad a+\frac{81}{a}=30,\quad a=81^{\sin^2\theta}$$

$$\Rightarrow \quad a^2-30a+81=0$$

$$\Rightarrow \quad (a-27)(a-3)=0$$

$$\Rightarrow \quad a=3,\ 27$$

When $a = 3$

$$\Rightarrow \quad 81^{\sin^2\theta}=3$$

$$\Rightarrow \quad 3^{4\sin^2\theta}=3$$

$$\Rightarrow \quad 4\sin^2\theta=1$$

$$\Rightarrow \quad \sin^2\theta=\left(\frac{1}{2}\right)^2$$

$$\Rightarrow \quad \sin^2\theta=\sin^2\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad \theta=\left(n\pi\pm\frac{\pi}{6}\right)$$

$$\Rightarrow \quad \theta=\frac{\pi}{6},\frac{5\pi}{6}$$

When $a = 27$

$\Rightarrow \quad 81^{\sin^2\theta} = 27$

$\Rightarrow \quad 3^{4\sin^2\theta} = 3^3$

$\Rightarrow \quad 4\sin^2\theta = 3$

$\Rightarrow \quad \sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2$

$\Rightarrow \quad \sin^2\theta = \sin^2\left(\frac{\pi}{3}\right)$

$\Rightarrow \quad \theta = \left(n\pi \pm \frac{\pi}{3}\right)$

$\Rightarrow \quad \theta = \frac{\pi}{3}, \frac{2\pi}{3}$

Hence, the values of θ are $\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$.

Ex-20. Let $f_k(\theta) = \sin^k(\theta) + \cos^k(\theta)$,

then find the value of $\frac{1}{6}f_6(\theta) - \frac{1}{4}f_4(\theta)$.

Soln. We have $f_6(\theta) = \sin^6\theta + \cos^6\theta$

$= 1 - 3\sin^2\theta\cos^2\theta$

Also, $f_4(\theta) = \sin^4\theta + \cos^4\theta$

$= 1 - 2\sin^2\theta\cos^2\theta$

Now, $\frac{1}{6}f_6(\theta) - \frac{1}{4}f_4(\theta)$

$= \frac{1}{6}\left(1 - 3\sin^2\theta\cos^2\theta\right) - \frac{1}{4}\left(1 - 2\sin^2\theta\cos^2\theta\right)$

$= \frac{1}{6} - \frac{1}{2}\sin^2\theta\cos^2\theta - \frac{1}{4} + \frac{1}{2}\sin^2\theta\cos^2\theta$

$= \frac{1}{6} - \frac{1}{4}$

$= -\frac{1}{12}$.

EXERCISE 2

1. Prove that $\frac{\cot\theta - \tan\theta}{1 - 2\sin^2\theta} = \sec\theta \operatorname{cosec}\theta$.
2. Prove that $\tan\theta + \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta$.
3. Prove that $\frac{\tan\alpha + \sec\alpha - 1}{\tan\alpha - \sec\alpha + 1} = \frac{1+\sin\alpha}{\cos\alpha}$.
4. Prove that $\sin^8 A - \cos^8 A$
$= (\sin^2 A - \cos^2 A) \times (1 - 2\sin^2 A\cos^2 A)$.
5. If $U_n = \sin^n\theta + \cos^n\theta$, prove that
$2U_6 - 3U_4 + 1 = 0$.
6. Prove that $\frac{1}{\operatorname{cosec}\theta - \cot\theta} - \frac{1}{\sin\theta}$
$= \frac{1}{\sin\theta} - \frac{1}{\operatorname{cosec}\theta + \cot\theta}$
7. Prove that the equation $\frac{(a+b)^2}{4ab} = \sin^2\theta$ is possible only when $a = b$.
8. Prove that $\sin^2\theta + \operatorname{cosec}^2\theta \geq 2$.
9. Prove that $\sec^2\theta + \operatorname{cosec}^2\theta \geq 4$.
10. If $\sec\theta + \tan\theta = 3$, then find the value of $\cos\theta$.
11. If $\operatorname{cosec}\theta - \cot\theta = \frac{1}{5}$, then find the value of $\sin\theta$.
12. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, then prove that $x^2 + y^2 = a^2 + b^2$.
13. If $3\sin\theta + 5\cos\theta = 5$, then prove that
$5\sin\theta - 3\cos\theta = \pm 3$.
14. If $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$, $z = r\sin\theta$, then prove that $x^2 + y^2 + z^2 = r^2$.
15. If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, than prove that
$\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.
16. If $\tan^2 A = 1 + 2\tan^2 B$, then
prove that $\cos^2 B = 2\cos^2 A$.
17. If $\tan^2\theta = 1 - e^2$, then prove that
$\sec\theta + \tan^3\theta.\operatorname{cosec}\theta = \left(2 - e^2\right)^{3/2}$
18. If $\sin^2\theta + \sin\theta = 1$, then
prove that $\cos^4\theta + \cos^2\theta = 1$.
19. If $\sin^2\theta + \sin\theta = 1$, then find
the value of $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta$.
20. If $\cos^4\theta + \cos^2\theta = 1$, then
prove that $\tan^4\theta + \tan^2\theta = 1$.
21. If $\sin^4\theta + \sin^2\theta = 1$, then
prove that $\cos^8\theta + 2\cos^6\theta + \cos^4\theta = 1$.
22. If $\sin\theta + \sin^2\theta + \sin^3\theta = 1$, then prove that,
$\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 1$.
23. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$,
then prove that, $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0$.

24. If $x = \dfrac{2\sin\theta}{1+\cos\theta+\sin\theta}$, then

prove that, $\dfrac{1-\cos\theta+\sin\theta}{1+\sin\theta} = x$.

25. Prove that $1 - \dfrac{\sin^2\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} - \dfrac{\sin\theta}{1-\cos\theta} = \cos\theta$.

26. Prove that,

$3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2$

$+ 4(\sin^6 x + \cos^6 x) = 13$

27. If $a^3 = \operatorname{cosec}\theta - \sin\theta$ and

$b^3 = \sec\theta - \cos\theta$, then

prove that $a^2b^2(a^2+b^2) = 1$.

28. If $x\sin^3\alpha + y\cos^3\alpha = \sin\alpha\cos\alpha$ and

$x\sin\alpha = y\cos\alpha$ then prove that $x^2+y^2 = 1$.

29. If $\tan\theta + \sin\theta = m$, $\tan\theta - \sin\theta = n$,

then prove that $m^2 - n^2 = 4\sqrt{mn}$.

30. If $\dfrac{\cos^4 x}{\cos^2 y} + \dfrac{\sin^4 x}{\sin^2 y} = 1$,

then prove that $\dfrac{\cos^4 y}{\cos^2 x} + \dfrac{\sin^4 y}{\sin^2 x} = 1$

31. If $\dfrac{\sin^4\alpha}{a} + \dfrac{\cos^4\alpha}{b} = \dfrac{1}{a+b}$, then

prove that $\dfrac{\sin^8\alpha}{a^3} + \dfrac{\cos^8\alpha}{b^3} = \dfrac{1}{(a+b)^3}$

32. If $\dfrac{\sin^4\theta}{2} + \dfrac{\cos^4\theta}{3} = \dfrac{1}{5}$, then

prove that $\dfrac{\sin^8\theta}{8} + \dfrac{\cos^8\theta}{27} = \dfrac{1}{125}$.

33. Let $f_k(\theta) = \sin^k(\theta) + \cos^k(\theta)$. Then prove

that $\dfrac{1}{6}f_6(\theta) - \dfrac{1}{4}f_4(\theta) = -\dfrac{1}{12}$.

34. If $f_n(\theta) = \sin^n\theta + \cos^n\theta$, prove that

$2f_6(\theta) - 3f_4(\theta) + 1 = 0$.

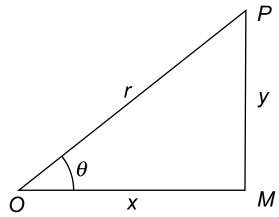
35. If $\dfrac{\sin A}{\sin B} = p$, $\dfrac{\cos A}{\cos B} = q$, prove that

$\tan A . \tan B = \dfrac{p}{q}\left(\dfrac{q^2-1}{1-p^2}\right)$.

1.9 MEASUREMENT OF THE ANGLES OF DIFFERENT T-RATIOS

1.9.1 Recognization of the quadrants

We have introdcuced six *t*-ratios. Signs of these *t*-ratios depend upon the quadrant in which the terminal side of the angle lies. We always take the length of *OP* vector denoted by '*r*', which is always positive.

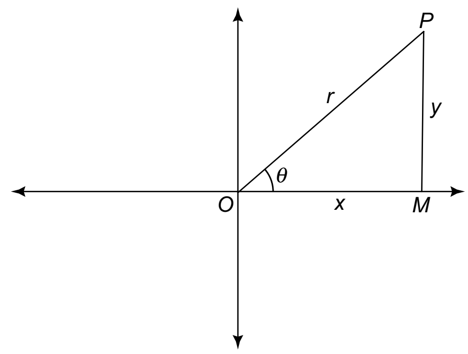


Thus, $\sin\theta = \dfrac{y}{r}$ has the sign of y, $\cos\theta = \dfrac{x}{r}$ has the sign of x

and $\tan\theta = \dfrac{y}{x}$ depends on the signs of both x and y.

Similarly, the signs of other trigonometric functions can be obtained by the signs of x and y.

(A) In first quadrant, we have $x > 0, y > 0$



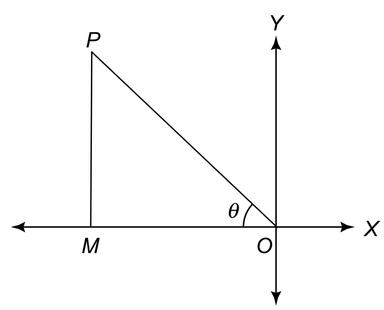
(i) $\sin\theta = \dfrac{y}{r} > 0$, $\operatorname{cosec}\theta = \dfrac{r}{y} > 0$

(ii) $\cos\theta = \dfrac{y}{r} > 0$, $\sec\theta = \dfrac{r}{y} > 0$

(iii) $\tan\theta = \dfrac{y}{x} > 0$, $\cot\theta = \dfrac{y}{x} > 0$

Thus, in the first quadrant all trigonometric ratios are positive. Due to this reason, first quadrant is represented by 'ALL'.

(B) In the second quadrant, we have $x < 0$ and $y > 0$



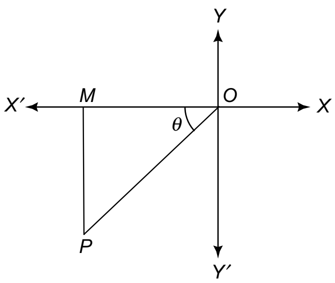
(i) $\sin\theta = \dfrac{y}{r} > 0,\ \text{cosec}\,\theta = \dfrac{y}{r} > 0$

(ii) $\cos\theta = \dfrac{x}{r} < 0,\ \sec\theta = \dfrac{r}{x} < 0$

(iii) $\tan\theta = \dfrac{y}{x} < 0,\ \cot\theta = \dfrac{y}{x} < 0$

Thus, in the second quadrant all *t*-raios are negative other than sin and cosec. Due to this reason, second quadrant is denoted by 'SIN'.

(C) In the third quadrant, we have
$x < 0$ and $y < 0$



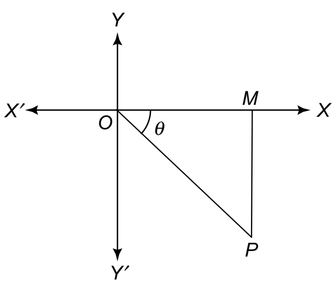
(i) $\sin\theta = \dfrac{y}{r} < 0,\ \text{cosec}\,\theta = \dfrac{r}{y} < 0$

(ii) $\cos\theta = \dfrac{x}{r} < 0,\ \sec\theta = \dfrac{r}{x} < 0$

(iii) $\tan\theta = \dfrac{y}{x} > 0,\ \cot\theta = \dfrac{y}{x} > 0$

Thus, in the third quadrant all *t*-raios are negative other than tan and cot. Due to this reason, third quadrant is denoted by 'TAN'.

(D) In the fourth quadrant, we have
$x > 0$ and $y < 0$



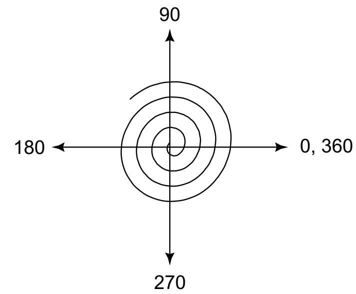
(i) $\sin\theta = \dfrac{y}{r} < 0,\ \text{cosec}\,\theta = \dfrac{r}{y} < 0$

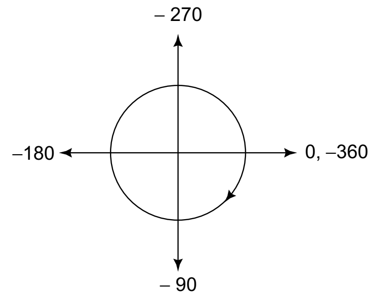
(ii) $\cos\theta = \dfrac{x}{r} > 0,\ \sec\theta = \dfrac{r}{x} > 0$

(iii) $\tan\theta = \dfrac{y}{x} < 0,\ \cot\theta = \dfrac{y}{x} < 0$

Thus, in the fourth quadrant all *t*-raios are negative other than cos and sec. Due to this reason, fourth quadrant is denoted by 'COS'.

(E) Rotation





1.9.2 *T*-ratios of the angle $(-\theta)$, in terms of θ, for all values of θ.

1. (i) $\sin(-\theta) = -\sin\theta$
   (ii) $\cos(-\theta) = \cos\theta$
   (iii) $\tan(-\theta) = -\tan\theta$
   (iv) $\text{cosec}(-\theta) = -\text{cosec}\,\theta$
   (v) $\sec(-\theta) = \sec\theta$
   (vi) $\cot(-\theta) = -\cot\theta$

1.9.3 *T*-ratios of the different angles in terms of θ, for all values of θ.

2. (i) $\sin(90-\theta) = \sin(90°\times 1-\theta) = \cos\theta$
   (ii) $\sin(90+\theta) = \sin(90°\times 1+\theta) = \cos\theta$
   (iii) $\sin(180-\theta) = \sin(90°\times 2-\theta) = \sin\theta$
   (iv) $\sin(180+\theta) = \sin(90°\times 2+\theta) = -\sin\theta$
   (v) $\sin(270-\theta) = \sin(90°\times 3-\theta) = -\cos\theta$
   (vi) $\sin(270+\theta) = \sin(90°\times 3+\theta) = -\cos\theta$
   (vii) $\sin(360-\theta) = \sin(90°\times 4-\theta) = -\sin\theta$
   (viii) $\sin(360+\theta) = \sin(90°\times 4+\theta) = \sin\theta$
3. (i) $\cos(90-\theta) = \cos(90°\times 1-\theta) = \sin\theta$
   (ii) $\cos(90+\theta) = \cos(90°\times 1+\theta) = -\sin\theta$
   (iii) $\cos(180-\theta) = \cos(90°\times 2-\theta) = -\cos\theta$
   (iv) $\cos(180+\theta) = \cos(90°\times 2+\theta) = -\cos\theta$
   (v) $\cos(270-\theta) = \cos(90°\times 3-\theta) = -\sin\theta$
   (vi) $\cos(270+\theta) = \cos(90°\times 3+\theta) = -\sin\theta$
   (vii) $\cos(360-\theta) = \cos(90°\times 4-\theta) = \cos\theta$
   (viii) $\cos(360+\theta) = \cos(90°\times 4+\theta) = \cos\theta$.

4. (i) $\tan(90 - \theta) = \tan(90° \times 1 - \theta) = \cot\theta$

(ii) $\tan(90 + \theta) = \tan(90° \times 1 + \theta) = -\cot\theta$

(iii) $\tan(180 - \theta) = \tan(90° \times 2 - \theta) = -\tan\theta$

(iv) $\tan(180 + \theta) = \tan(90° \times 2 + \theta) = \tan\theta$

(v) $\tan(270 - \theta) = \tan(90° \times 3 - \theta) = \cot\theta$

(vi) $\tan(270 + \theta) = \tan(90° \times 3 + \theta) = -\cot\theta$

(vii) $\tan(360 - \theta) = \tan(90° \times 4 - \theta) = -\tan\theta$

(viii) $\tan(360 + \theta) = \tan(90° \times 4 + \theta) = \tan\theta$.

Note: All the above results can be remembered by the following simple rule.

1. If θ be measured with an even multiple of 90° by + or – sign, then the *T*-ratios remains unaltered (i.e. sine remains sine and cosine remains cosine, etc.) and treating θ as an acute angle, the quadrant in which the associated angle lies, is determined and then the sign of the *T*-ratio is determined by the All – Sin – Tan – Cos formula.
2. If θ be associated with an odd multiple of 90 by +ve or –ve sign, then the *T*-ratios is altered in the form (i.e. sine becomes cosine and cosine becomes sine, tangent becomes cotangent and conversely, etc.) and the sign of the ratio is determined as in the previous paragraph.
3. If the multiple of 90 is more than 4, then divide it by 4 and find out remainder. If remainder is 0, then the degree lies on the right of *x*-axis, if remainder is 1, then the degree lies on the +ve *y*-axis, if remainder is 2, then the degree lies on the –ve of *x*-axis and if the remainder is 3, then the degree lies on the –ve of *y*–axis, respectively.

For examples:

(i) sin (570°)

$= \sin(90 \times 6 + 30°)$

$= -\sin 30° = -\frac{1}{2}$

(ii) tan (1950°)

$= \tan(90 \times 22 - 30°)$

$= -\tan(30°) = -\frac{1}{\sqrt{3}}$.

(iii) cos (2310°)

$= \cos(90 \times 25 + 60°)$

$= -\sin(60°) = -\frac{\sqrt{3}}{2}$.

1.10 SOME SOLVED EXAMPLES

Ex-1. Find the value of

(i) sin 120°
(ii) sin 150°
(iii) sin 210°
(iv) sin 225°
(v) sin 300°
(vi) sin 330°
(vii) sin 405°
(viii) sin 650°
(ix) sin 1500°
(x) sin 2013°

Soln.

(i) $\sin(120°) = \sin(90 \times 1 + 30°)$

$= \cos(30°) = \frac{\sqrt{3}}{2}$

(ii) $\sin(150°) = \sin(90 \times 2 - 30°)$

$= \sin(30°) = \frac{1}{2}$

(iii) $\sin(210°) = \sin(90 \times 2 + 30°)$

$= -\sin(30°) = -\frac{1}{2}$

(iv) $\sin(225°) = \sin(90 \times 2 + 45°)$

$= -\sin(45°) = -\frac{1}{\sqrt{2}}$

(v) $\sin(300°) = \sin(90 \times 3 + 30°)$

$= -\cos(30°) = -\frac{\sqrt{3}}{2}$

(vi) $\sin(330°) = \sin(90 \times 3 + 60°)$

$= -\cos(60°) = -\frac{1}{2}$

(vii) $\sin(405°) = \sin(90 \times 4 + 45°)$

$= \sin(45°) = \frac{1}{\sqrt{2}}$

(viii) $\sin(660°) = \sin(90 \times 7 + 30°)$

$= \sin(30°) = \frac{1}{2}$

(ix) $\sin(1500°) = \sin(90 \times 16 + 60°)$

$= \sin(60°) = \frac{\sqrt{3}}{2}$

(x) $\sin(2013°) = \sin(90 \times 22 + 33°)$

$= -\sin(33°)$.

Ex-2. Find the value of

$\cos(1°).\cos(2°).\cos(3°)......\cos(189°)$.

Soln. We have,

$$\cos(1°).\cos(2°).\cos(3°)..........\cos(189°).$$

$$= \cos(1°).\cos(2°).\cos(3°).........\cos(89°)$$

$\cos(90°)\cos(91°).........\cos(189°)$.

$= \cos(1°).\cos(2°).\cos(3°).....\cos(89°) \times 0 \times \cos(91°)................\cos(189°)$.

$= 0$.

Ex-3. Find the value of $\tan(1°).\tan(2°).\tan(3°).......\tan(89°)$.

Soln. We have,

$\tan(1°).\tan(2°).\tan(3°).....\tan(89°)$

$= \tan(1°).\tan(2°).\tan(3°).......\tan(44°)$ $\tan(45°)\tan(46°)...\tan(87°)\tan(88°)\tan(89°)$

$= \{\tan(1°)\times\tan(89°)\}.\{\tan(2°)\times\tan(88°)\}......\{\tan(44°)\times\tan(46°)\}.\tan(45°)$

$= 1$.

Ex-4. Find the value of $\tan 35°.\tan 40°.\tan 45°.\tan 50°.\tan 55°$

Soln. We have,

$\tan 35°.\tan 40°.\tan 45°.\tan 50°.\tan 55°$

$= \{\tan 35°\times\tan 55°\}\{\tan 40°\times\tan 50°\}\times\tan 45°$

$= \{\tan 35°\times\cot 35°\}.\{\tan 40°\times\cot 40°\}\times\tan 45°$

$= 1$.

Ex-5. Find the value of $\sin(10°)+\sin(20°)+\sin(30°)+\sin(40°)+..........+\sin(360°)$.

Soln. We have, $\sin(10°)+\sin(20°)+\sin(30°)+\sin(40°)+..........+\sin(360°)$

$= \sin(10°)+\sin(20°)+\sin(30°)+\sin(40°)+.........+\sin(150°)+\sin(340°)+\sin(350°)+\sin(360°)$

$= \sin(10°)+\sin(20°)+\sin(30°)+\sin(40°)+.........+\sin(80°)+\sin(90°)+\sin(100°)+\sin(360°-40°)+\sin(360°-30°)+\sin(360°-20°)+\sin(360°-10°)+\sin(360°)$

$= \sin(10°)+\sin(20°)+\sin(30°)+\sin(40°)+.........+\sin(80°)+\sin(90°)+\sin(100°)-\sin(40°)-\sin(30°)-\sin(20°)-\sin(10°)+\sin(180°)$

$= 0$.

Ex-6. Find the value of $\cos(10°)+\cos(20°)+\cos(30°)+\cos(40°)+.........+\cos(360°)$.

Soln. We have, $\cos(10°)+\cos(20°)+\cos(30°)+\cos(40°)+.........+\cos(360°)$

$= \cos 20° + \cos 30° + \cos 40° + ...+ \cos 140° + \cos 150° + \cos 160° + \cos 170° + \cos 180° + (\cos 190° + \cos 200° + \cos 210° + \cos 220° +.............+ \cos 360°)$

$= \cos 10° + \cos 20° + \cos 30° + \cos 40° +...... - \cos 40° - \cos 50° - \cos 60 - \cos 70° + \cos 180° + (\cos 190° + \cos 200° + \cos 210° + \cos 220° + ... + \cos 360°)$

$= \cos 180° + \cos 360°$

$= -1 + 1$

$= 0$.

Ex-7. Find the value of $\sin^2 5°+\sin^2 10°+\sin^2 15°+.......+\sin^2 90°$

Soln. We have,

$\sin^2 5°+\sin^2 10°+\sin^2 15°+....+\sin^2 90°$

$= \sin^2 5°+\sin^2 10°+\sin^2 15°+......+\sin^2 40+\sin^2 45+\sin^2 50+\sin^2 80+\sin^2 85+\sin^2 90°$

$= \left(\sin^2 5°+\sin^2 85°\right)+\left(\sin^2 10°+\sin^2 80°\right)+\left(\sin^2 15°+\sin^2 75°\right)+.........+\left(\sin^2 40°+\sin^2 50°\right)+\left(\sin^2 45°+\sin^2 90°\right)$

$= \left(\sin^2 5°+\cos^2 5°\right)+\left(\sin^2 10°+\cos^2 10°\right)+\left(\sin^2 15°+\cos^2 15°\right)+..................+\left(\sin^2 40°+\cos^2 40°\right)+\left(\sin^2 45°+\sin^2 90°\right)$

$= (1 + 1 + ... 8 \text{ times}) + \left(\frac{1}{2}+1\right)$

$= \left(8+1+\frac{1}{2}\right)$

$= 9\frac{1}{2}$.

Ex-8. Find the value of

$$\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{\pi}{9}\right)+\sin^2\left(\frac{4\pi}{9}\right)+\sin^2\left(\frac{7\pi}{18}\right).$$

Soln. We have, $\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{\pi}{9}\right)$

$$+\sin^2\left(\frac{4\pi}{9}\right)+\sin^2\left(\frac{7\pi}{18}\right)$$

$$= \sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{2\pi}{18}\right)$$

$$+\sin^2\left(\frac{8\pi}{18}\right)+\sin^2\left(\frac{7\pi}{18}\right)$$

$$= \sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{\pi}{2}-\frac{7\pi}{18}\right)$$

$$+\sin^2\left(\frac{\pi}{2}-\frac{\pi}{18}\right)+\sin^2\left(\frac{7\pi}{18}\right)$$

$$= \sin^2\left(\frac{\pi}{18}\right)+\cos^2\left(\frac{7\pi}{18}\right)$$

$$+\cos^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{7\pi}{18}\right)$$

$$= \left\{\sin^2\left(\frac{\pi}{18}\right)+\cos^2\left(\frac{\pi}{18}\right)\right\}$$

$$+\left\{\cos^2\left(\frac{7\pi}{18}\right)+\sin^2\left(\frac{7\pi}{18}\right)\right\}$$

$= 1 + 1$

$= 2.$

Ex-9. Find the value of $\tan(20°)\tan(25°)\tan(45°)\tan(65°)\tan(70°)$.

Soln. We have

$\tan(20°)\tan(25°)\tan(45°)\tan(65°)\tan(70°)$

$$= \tan(20°)\tan(25°)\tan(45°)\tan(90°-25°)\tan(90°-20°)$$

$$= \tan(20°)\tan(25°)\tan(45°)\cot(25°)\cot(20°)$$

$$= \tan(45°)$$

$= 1.$

Ex-10. Find the value of $\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)$ if $\sin(\theta_1)+\sin(\theta_2)+\sin(\theta_3)=3$.

Soln. Given $\sin(\theta_1)+\sin(\theta_2)+\sin(\theta_3)=3$

It is possible only when each term of the above equation will provide the maximum value

Thus, $\sin(\theta_1)=1, \sin(\theta_2)=1, \sin(\theta_3)=1$

$$\theta_1=\frac{\pi}{2}, \theta_2=\frac{\pi}{2}, \theta_3=\frac{\pi}{2}$$

Hence, the value of $\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3)$

$$= \cos\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{2}\right)$$

$= 0.$

EXERCISE 3

1. Find the values of
   (i) sin (135°) (ii) cos (150°)
   (iii) tan (120°) (iv) sin (225°)
   (v) sec (240°) (vi) tan (300°)
   (vii) sin (330°) (viii) tan (315°)
   (ix) cos (315°) (x) sin (405°)
2. Find the values of
   (i) sin (675°) (ii) cos (1230°)
   (iii) tan 1020° (iv) cosec (1305°)
   (v) sec (–1035°) (vi) tan (–1755°)
   (vii) sin (1410°) (viii) cos (1450°)
   (ix) tan (2010°) (x) sin (1950°)
3. Express in terms of ratios of smallest +ve angles.
   (i) sin 240° (ii) cos 780°
   (iii) sin (–1358°) (iv) cosec (–1150°)
   (v) tan (–1750°)
4. If $\theta = \frac{23\pi}{6}$, then find the value of $\sec\theta - \tan\theta$.
5. Simplify: $\frac{\sin(270+A)\cos(90-A)}{\sin(180-A)\cos(180-A)}$.
6. Simplify: tan 25° . tan 35° . tan 45° tan 55° . tan 65°
7. Prove that
   $$\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\sin^2\frac{7\pi}{8}=2.$$
8. Prove that
   $$\sin^2\frac{\pi}{4}+\sin^2\frac{3\pi}{4}+\sin^2\frac{5\pi}{4}+\sin^2\frac{7\pi}{4}=2.$$
9. Find the value of
   $\sin^2 5°+\sin^2 10°+....+\sin^2 90°$.
10. Find the value of
    $\sin^2 6°+\sin^2 12°+....+\sin^2 90°$
11. Find the value of
    $\sin^2 10°+\sin^2 20°+....+\sin^2 90°$
12. Find the value of
    $\sin^2 9°+\sin^2 18°+....+\sin^2 90°$

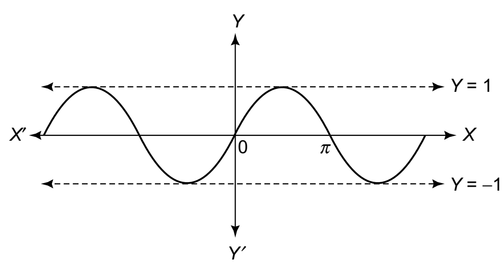
13. Find the value of
$\tan 1°.\tan 2°.\tan 3°....\tan 89°$
14. Find the value of
$\cos 1°.\cos 2°.\cos 3°......\cos 189°$
15. Solve for θ; $2\sin^2\theta + 3\cos\theta = 0$ where $0 < \theta < 360°$.
16. Solve for θ; $\cos\theta + \sqrt{3}\sin\theta = 2$, where $0 < \theta < 360°$.
17. If $4n\ \alpha = \pi$, then prove that
$\tan\alpha \tan 2\alpha \tan 3\alpha .... \tan(2n-1)\alpha = 1$.

Q. If A, B, C, D be the angles of a cyclic quadrilateral $ABCD$, then prove that

18. $\cos A + \cos B + \cos C + \cos D = 0$
19. $\tan A + \tan B + \tan C + \tan D = 0$.
20. $\sin 2A + \sin 2B + \sin 2C + \sin 2D = 0$
21. Find the value of
$\cos(18°) + \cos(234°) + \cos(162°) + \cos(306°)$.
22. Find the value of
$\cos(20°) + \cos(40°) + \cos(60°) + ... + \cos(180°)$
23. Find the value of
$\sin(20°) + \sin(40°) + \sin(60°) + ... + \sin(360°)$

Graph of Trigonometric functions:

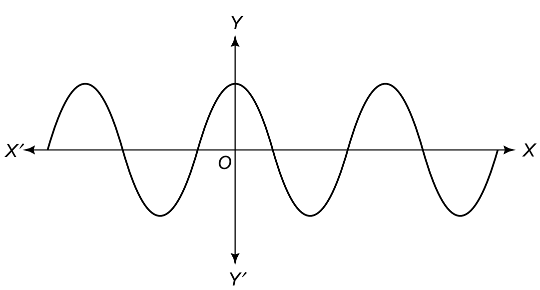
1. Graph of $f(x) = \sin x$.



Characteristics of sine function:

1. It is an odd function, since $\sin(-x) = -\sin x$
2. It is a periodic function with period 2π
3. $\sin x = 1 \Rightarrow x = (4n+1)\dfrac{\pi}{2},\ n \in I$
4. $\sin x = 0 \Rightarrow x = n\pi,\ n \in I$
5. $\sin x = -1 \Rightarrow x = (4n-1)\dfrac{\pi}{2},\ n \in I$

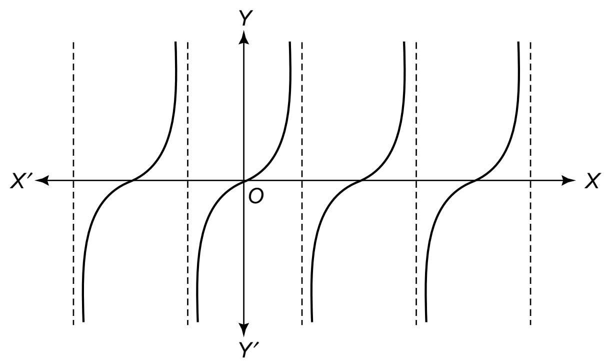
Graph of $f(x) = \cos x$:



Characteristics of cosine function:

1. It is an even function, since $\cos(-x) = \cos x$
2. It is a periodic function with period 2π.
3. $\cos x = 1 \Rightarrow x = 2n\pi,\ n \in I$
4. $\cos x = 0 \Rightarrow x = (2n+1)\dfrac{\pi}{2},\ n \in I$
5. $\cos x = -1 \Rightarrow x = (2n+1)\pi,\ n \in I$

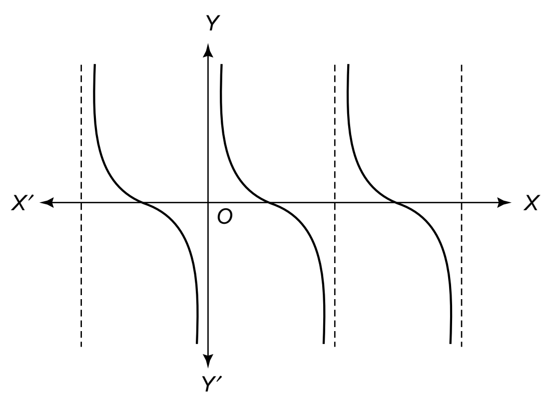
3. Graph of $f(x) = \tan x$.



Characteristics of tangent function:

1. It is an odd function, since $\tan(-x) = -\tan x$
2. It is a periodic function with period π
3. $\tan x = 1 \Rightarrow x = (4n+1)\dfrac{\pi}{4},\ n \in I$
4. $\tan x = 0 \Rightarrow x = n\pi,\ n \in I$
5. $\tan x = -1 \Rightarrow x = (4n-1)\dfrac{\pi}{4},\ n \in I$

4. Graph of $f(x) = \cot x$

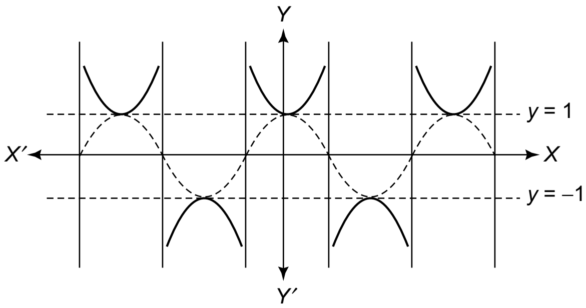


Characteristics of cotangent function:

1. It is an odd function, since $\cot(-x) = -\cot x$
2. It is a periodic function with period π
3. $\cot x = 1 \Rightarrow x = (4n+1)\dfrac{\pi}{4},\ n \in I$

4. $\cot x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2},\ n \in I$
5. $\cot x = -1 \Rightarrow x = (4n-1)\frac{\pi}{4},\ n \in I$

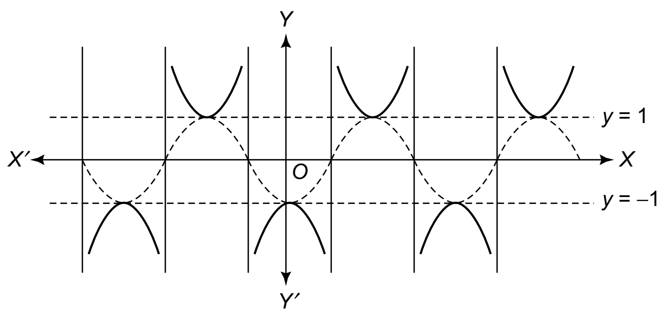
Graph of $f(x) = \sec x$



Characteristics of secant function

1. It is an even function, sec(–x) = sec x
2. It is a periodic function with period 2π
3. sec x can never be zero.
4. $\sec x = 1 \Rightarrow x = 2n\pi,\ n \in I$
5. $\sec x = -1 \Rightarrow x = (2n+1)\pi,\ n \in I$

Graph of $f(x) = \text{cosec}\ x$



Characteristics of cosecant function:

1. It is an odd function, since cosec(–x) = –cosec x
2. It is a periodic function with period 2π
3. $\text{cosec}\ x = 1 \Rightarrow x = (4n+1)\frac{\pi}{2},\ n \in I$
4. cosec x can never be zero.
5. $\text{cosecx} = -1 \Rightarrow x = (4n-1)\frac{\pi}{2}, n \in I$

EXERCISE 4

Q. Draw the graphs of
1. $f(x) = \sin x + 1$
2. $f(x) = \sin x - 1$
3. $f(x) = -\sin x$
4. $f(x) = 1 - \sin x$
5. $f(x) = -1 - \sin x$
6. $f(x) = \sin 2x, \sin 3x$
7. $f(x) = \sin^2 x$
8. $f(x) = \cos^2 x$
9. $f(x) = \max\{\sin x, \cos x\}$
10. $f(x) = \min\{\sin x, \cos x\}$
11. $f(x) = \min\left\{\sin x, \frac{1}{2}, \cos x\right\}$
12. $f(x) = \max\{\tan x, \cot x\}$
13. $f(x) = \min\{\tan x, \cot x\}$

Q. Find the number of solutions of
1. $\sin x = \frac{1}{2},\ \forall\ x \in [0, 6]$
2. $\cos x = \frac{\sqrt{3}}{2},\ \forall\ x \in [0, 10]$
3. $4\sin^2 x - 1 = 0,\ \forall\ x \in [0, 10]$
4. $\sin^2 x - 3\sin x + 2 = 0,\ \forall\ x \in [0, 10]$
5. $\cos^2 x - \cos x - 2 = 0,\ \forall x \in [0, 10]$

1.11 *T*-RATIOS OF COMPOUND ANGLES

1.11 Definition

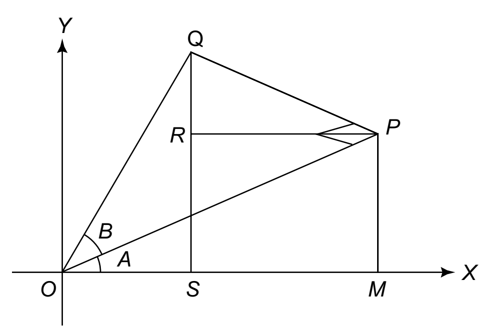
The algebraic sum or difference of two or more angles is called a compound angle such as

$A + B, A - B, A + B + C, A + B - C$ etc.

1.11.1 The Addition Formula

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B$
2. $\cos(A + B) = \cos A \cos B - \sin A \sin B$
3. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A . \tan B}$

Proof:



Let $\angle POX = A$ & $\angle POQ = B$

Draw PM and QS perpendicular on OX and PR is parallel to OX.

Clearly, $\angle PQR = A$.

1. $\sin(A+B)$

$$= \frac{QS}{OQ} = \frac{SR+QR}{OQ}$$

$$= \frac{PM+QR}{OQ}$$

$$= \frac{PM}{OQ} + \frac{RQ}{OQ}$$

$$= \frac{PM}{OP}.\frac{OP}{OQ} + \frac{QR}{PQ}.\frac{PQ}{OQ}$$

$= \sin A \cos B + \cos A.\sin$ B.

2. $\cos(A+B)$

$$= \frac{OS}{OQ} = \frac{OM-MS}{OQ}$$

$$= \frac{OM}{OQ} - \frac{MS}{OQ}$$

$$= \frac{OM}{OQ} - \frac{MS}{OQ}$$

$$= \frac{OM}{OP}.\frac{OP}{OQ} - \frac{PR}{PQ}.\frac{PQ}{OQ}$$

$= \cos A.\cos B - \sin A.\sin B$.

3. $\tan(A+B)$

$$= \frac{QS}{OS} = \frac{QR+RS}{OS}$$

$$= \frac{QR+RS}{OM-SM}$$

$$= \frac{\frac{QR}{OM}+\frac{RS}{OM}}{1-\frac{SM}{OM}}$$

$$= \frac{\frac{QR}{OM}+\frac{RS}{OM}}{1-\frac{PR}{PQ}.\frac{PQ}{OM}}$$

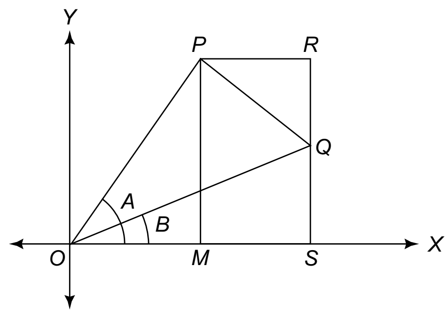
$$= \frac{\frac{PM}{OM}+\frac{PQ}{OP}}{1-\frac{PM}{OQ}.\frac{PQ}{OP}}$$

$$= \frac{\tan A+\tan B}{1-\tan A.\tan B}$$

Subtraction formulae:

1. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
2. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
3. $\tan(A-B) = \dfrac{\tan A-\tan B}{1+\tan A\tan B}$.

Proof:



Let $\angle POX = A$ and $\angle QOX = B$

Clearly, $\angle POQ = A - B$

Draw PM, RS perpendicular on OX and PR is parallel to OX.

From geometry, we can say that, $\angle PQR = A$

In ΔQOS,

1. $\sin(A-B)$

$$= \frac{QS}{OQ} = \frac{RS-RQ}{OQ} = \frac{PM-RQ}{OQ}$$

$$= \frac{PM}{OQ} - \frac{RQ}{OQ}$$

$$= \frac{PM}{OP}.\frac{OP}{OQ} - \frac{RQ}{PQ}.\frac{PQ}{OQ}$$

$= \sin A.\cos B - \cos A.\sin B$.

2. $\cos(A-B)$

$$= \frac{OQ}{OQ} = \frac{OM+MS}{OQ} = \frac{OM+PR}{OQ}$$

$$= \frac{OM}{OQ} + \frac{PR}{OQ}$$

$$= \frac{OM}{OP}.\frac{OP}{OQ} + \frac{PR}{PQ}.\frac{PQ}{OQ}$$

$= \cos A.\cos B + \sin A.\sin B$.

3. $\tan(A-B)$

$$= \frac{QS}{OS} = \frac{RS - QR}{OM + MS} = \frac{RS - QR}{OM + PR}$$

$$= \frac{\frac{PL}{OM} - \frac{QR}{OM}}{1 + \frac{PR}{OM}}$$

$$= \frac{\frac{PL}{OM} - \frac{QR}{OM}}{1 + \frac{PR}{PM} \cdot \frac{PM}{OM}}$$

$$= \frac{\frac{PM}{OM} - \frac{PQ}{OM}}{1 + \frac{PM}{OM} \cdot \frac{PQ}{OM}}$$

$$= \frac{\tan A - \tan B}{1 + \tan A . \tan B}.$$

Note:

1. $\tan\left(\frac{\pi}{4} + A\right) = \frac{1 + \tan A}{1 - \tan A}$
2. $\tan\left(\frac{\pi}{4} - A\right) = \frac{1 - \tan A}{1 + \tan A}$

1.12 SOME IMPORTANT DEDUCTIONS

Deduction 1.

$$\sin (A + B) \sin (A - B)$$
$$= \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$$

Proof: We have $\sin(A + B) \sin(A - B)$

$$= \{\sin A \cos B + \cos A \sin B\}$$
$$\times \{\sin A \cos B - \cos A \sin B\}$$
$$= \{\sin^2 A \cos^2 B - \cos^2 A \sin^2 B\}$$
$$= \{\sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B\}$$
$$= \{\sin^2 A - \sin^2 A \sin^2 B - \sin^2 B - \sin^2 A \sin^2 B\}$$
$$= \sin^2 A - \sin^2 B$$
$$= (1 - \cos^2 A) - (1 - \cos^2 B)$$
$$= \cos^2 B - \cos^2 A$$

Deduction 2.

$$\cos (A + B) \cos (A - B)$$
$$= \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$$

Proof: We have, $\cos(A + B)\cos(A - B)$

$$= \{\cos A \cos B + \sin A \sin B\}$$
$$\times \{\cos A \cos B - \sin A \sin B\}$$
$$= \{\cos^2 A \cos^2 B - \sin^2 A \sin^2 B\}$$
$$= \{\cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B\}$$
$$= \{\cos^2 A - \cos^2 A \sin^2 B - \sin^2 B + \cos^2 A \sin^2 B\}$$
$$= \cos^2 A - \sin^2 B$$

Deduction 3.

$$\cot (A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

Proof: We have, $\cot(A + B)$

$$= \frac{\cos(A + B)}{\sin(A + B)}$$
$$= \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$$
$$= \frac{\frac{\cos A \cos B}{\sin A \sin B} - \frac{\sin A \sin B}{\sin A \sin B}}{\frac{\sin A \cos B}{\sin A \sin B} + \frac{\cos A \sin B}{\sin A \sin B}}$$
$$= \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

Deduction 4.

$$\cot (A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

Proof: We have, $\cot (A - B)$

$$= \frac{\cos(A - B)}{\sin(A - B)}$$
$$= \frac{\cos A \cos B + \sin A \sin B}{\sin A \cos B - \cos A \sin B}$$
$$= \frac{\frac{\cos A \cos B}{\sin A \sin B} + \frac{\sin A \sin B}{\sin A \sin B}}{\frac{\sin A \cos B}{\sin A \sin B} - \frac{\cos A \sin B}{\sin A \sin B}}$$
$$= \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

Deduction 5.

$$\sin (A + B + C)$$
$$= \cos A \cos B \cos C[\tan A + \tan B + \tan C$$
$$- \tan A \tan B \tan C]$$

Proof: We have $\sin(A + B + C)$

$$= \sin(A + B)\cos C + \cos(A + B)\sin C$$
$$= \{\sin A . \cos B + \cos A . \sin B\}\cos C$$
$$+ \{\cos A \cos B - \sin A \sin B\}\sin C$$

$= \sin A.\cos B.\cos C + \sin B.\cos A.\cos C$

$+ \sin C.\cos A\cos B - \sin A.\sin B.\sin C$

$= \cos A.\cos B.\cos C\ [\tan A + \tan B + \tan C$

$- \tan A.\tan B.\tan C]$

Deduction 6.

$$\cos(A + B + C)$$
$$= \cos A\cos B\cos C \times [1 - \tan A\tan B - \tan B\tan C - \tan C\tan A]$$

Proof: We have, $\cos(A+B+C)$

$= \cos(A+B)\cos C - \sin(A+B)\sin C$

$= \{\cos A\cos B - \sin A\sin B\}\cos C$

$-\{\sin A\cos B + \cos A\sin B\}\sin C$

$= \cos A\cos B\cos C - \sin A\sin B\cos C$

$\{-\sin A\sin C\cos B - \cos A\sin B\sin C\}$

$= \cos A\cos B\cos C$

$\times[1 - \tan A\tan B - \tan B\tan C - \tan C\tan A]$

Deduction 7.

$$\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$$

Proof: We have, $\tan(A+B+C)$

$$= \frac{\sin(A+B+C)}{\cos(A+B+C)}$$

$$= \frac{\cos A\cos B\cos C\{\tan A + \tan B + \tan C - \tan A\tan B\tan C\}}{\cos A\cos B\cos C\{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A\}}$$

$$= \left(\frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}\right)$$

1.13 SOME SOLVED EXAMPLES

Ex-1. Find the values of

(i) $\sin(15°)$,

(ii) $\cos(15°)$,

(iii) $\tan(15°)$

Soln. We have,

(i) $\sin(15°) = \sin(45° - 30°)$

$= \sin(45°)\cos(30°) - \cos(45°)\sin(30°)$

$$= \frac{1}{2}.\frac{\sqrt{3}}{2} - \frac{1}{2}.\frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(ii) $\cos(15°) = \cos(45° - 30°)$

$= \cos(45°)\cos(30°) + \sin(45°)\sin(30°)$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(iii) $\tan(15°) = \tan(45° - 30°)$

$$= \frac{\tan 45° - \tan 30°}{1 + \tan 45°.\tan 30°}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$= \frac{\left(\sqrt{3}-1\right)^2}{3-1}$$

$$= \frac{3+1-2\sqrt{3}}{2}$$

$$= \frac{4-2\sqrt{3}}{2}$$

$$= 2-\sqrt{3}$$

Note:

(i) $\cot(15°) = \dfrac{1}{\tan(15°)} = \dfrac{1}{2-\sqrt{3}}$

$= 2+\sqrt{3}$

(ii) $\tan(105°) = -\cot(15°) = -(2+\sqrt{3})$

(iii) $\cot(105°) = -\tan(15°) = -(2-\sqrt{3})$

$= \sqrt{3}-2$.

Ex-2. Find the value of $\tan(75°) + \cot(75°)$

Soln. We have, $\tan(75°) + \cot(75°)$

$= \cot(15°) + \tan(15°)$

$= (2+\sqrt{3}) + (2-\sqrt{3})$

$= 4.$

Ex-3. Prove that $\cos(9°) + \sin(9°) = \sqrt{2}\sin(54°)$

Soln. We have, $\cos(9°) + \sin(9°)$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos(9°) + \frac{1}{\sqrt{2}}\sin(9°)\right)$$

$= \sqrt{2}\left(\sin(45°)\cos(9°) + \cos(45°)\sin(9°)\right)$

$= \sqrt{2}\left(\sin(45° + 9°)\right)$

$= \sqrt{2}\sin(54°)$.

Ex-4. Prove that $\tan(70°) = 2\tan(50°) + \tan(20°)$

Soln. We have, $\tan(70°) = \tan(50° + 20°)$

$\Rightarrow \quad \tan(70°) = \dfrac{\tan 50° + \tan 20°}{1 - \tan 50° \tan 20°}$

$\Rightarrow \quad \tan(70°) - \tan(70°)\tan(50°)\tan(20°) = \tan 50° + \tan 20°$

$\Rightarrow \quad \tan(70°) - \tan(70°)\tan(50°)\cot(70°) = \tan 50° + \tan 20°$

$\Rightarrow \quad \tan(70°) - \tan(50°) = \tan 50° + \tan 20°$

$\Rightarrow \quad \tan(70°) = 2\tan 50° + \tan 20°$

Ex-5. If $A + B = 45°$, then find the value of $(1 + \tan A)(1 + \tan B)$

Soln. We have, $A + B = 45°$

$\Rightarrow \quad \tan(A + B) = \tan(45°)$

$\Rightarrow \quad \tan(A + B) = 1$

$\Rightarrow \quad \dfrac{\tan A + \tan B}{1 - \tan A . \tan B} = 1$

$\Rightarrow \quad \tan A + \tan B = 1 - \tan A . \tan B$

$\Rightarrow \quad \tan A + \tan B + \tan A . \tan B = 1$

$\Rightarrow \quad 1 + \tan A + \tan B + \tan A . \tan B = 1 + 1 = 2$

$\Rightarrow \quad (1 + \tan A) + \tan B(1 + \tan A) = 2$

$\Rightarrow \quad (1 + \tan A)(1 + \tan B) = 2$

Ex-6. Find the value of
$(1 + \tan 20°)(1 + \tan 24°)$
$(1 + \tan 25°)(1 + \tan 21°)$

Soln. We have,

$(1 + \tan 20°)(1 + \tan 24°)(1 + \tan 25°)(1 + \tan 21°)$

$= \{(1 + \tan 20°)(1 + \tan 25°)\} \times \{(1 + \tan 24°)(1 + \tan 21°)\}$

$= 2 \times 2$

$= 4$.

Ex-7. Find the value of

$(1 + \tan 245°)(1 + \tan 250°)$
$(1 + \tan 260°)(1 - \tan 200°)$
$(1 - \tan 205°)(1 - \tan 215°)$

Soln. We have, $(1 + \tan 245°)(1 + \tan 250°)$
$(1 + \tan 260°)(1 - \tan 200°)$
$(1 - \tan 205°)(1 - \tan 215°)$

$= \{(1 + \tan 245°)(1 + \tan(-200°))\}$
$\{(1 + \tan 250°)(1 + \tan(-205°))\}$
$\{(1 + \tan 260°)(1 + \tan(-215°))\}$

$= 2 \times 2 \times 2$

$= 8$.

Ex-8. If $\tan\alpha + \tan\beta = a$, $\cot\alpha + \cot\beta = b$ and $\tan(\alpha + \beta) = c$ then find a relation in a, b and c

Soln. We have, $\tan(\alpha + \beta) = c$

$\Rightarrow \quad \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = c$

$\Rightarrow \quad \dfrac{a}{1 - \tan\alpha\tan\beta} = c$

$\Rightarrow \quad \tan\alpha\tan\beta = \dfrac{a}{c} - 1 = \dfrac{a - c}{c}$ (i)

Now, $\cot\alpha + \cot\beta = b$

$\Rightarrow \quad \dfrac{1}{\tan\alpha} + \dfrac{1}{\tan\beta} = b$

$\Rightarrow \quad \dfrac{\tan\alpha + \tan\beta}{\tan\alpha . \tan\beta} = b$

$\Rightarrow \quad \tan\alpha . \tan\beta = \dfrac{a}{b}$ (ii)

From (i) and (ii), we get, $\dfrac{a - c}{c} = \dfrac{a}{b}$

$\Rightarrow \quad ac + bc = ab$

Which is the required relation.

Ex-9. If $\tan\beta = \dfrac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$,

then prove that $\tan(\alpha - \beta) = (1 - n)\tan\alpha$ Soln.

We have, $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha . \tan\beta}$

$$= \frac{\dfrac{\sin\alpha}{\cos\alpha} - \left(\dfrac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}\right)}{1 + \dfrac{\sin\alpha}{\cos\alpha} \times \left(\dfrac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}\right)}$$

$$= \frac{\sin\alpha\left(1 - n\sin^2\alpha\right) - n\sin\alpha\cos^2\alpha}{\cos\alpha\left(1 - n\sin^2\alpha\right) + n\sin^2\alpha . \cos\alpha}$$

$$= \frac{\sin\alpha - n\sin\alpha\left(\sin^2\alpha + \cos^2\alpha\right)}{\cos\alpha - n\sin^2\alpha . \cos\alpha + n\sin^2\alpha . \cos\alpha}$$

$$= \frac{\sin\alpha - n\sin\alpha}{\cos\alpha}$$

$$= \frac{(1-n)\sin\alpha}{\cos\alpha}$$

$= (1 - n) \tan \alpha.$

Ex-10. If $x + y + z = 0$, then prove that,

$\cot(x + y - z). \cot(y + z - x)$
$+ \cot (y + z - x). \cot (z + x - y)$
$+ \cot (z + x - y). \cot (x + y - z) = 1$

Soln. Let $A = x + y - z$, $B = y + z - x$, $C = x + y - z$

Then, $A + B + C$

$= (x + y - z) + (y + z - x) + (x + y - z)$

$= (x + y + z) = 0$

$\Rightarrow A + B = - C$

$\Rightarrow \cot(A + B) = \cot(-C)$

$$\Rightarrow \frac{\cot A \cot B - 1}{\cot A + \cot B} = \cot(-C)$$

$$\Rightarrow \frac{\cot A \cot B - 1}{\cot A + \cot B} = -\cot C$$

$\Rightarrow \cot A \cot B - 1 = -\cot A \cot C - \cot B \cot C$

$\Rightarrow \cot A \cot B + \cot A \cot C + \cot B \cot C = 1$

$\Rightarrow \cot(x + y - z). \cot(y + z - x)$
$+ \cot (y + z - x). \cot (z + x - y)$
$+ \cot (z + x - y). \cot (x + y - z) = 1.$

Ex-11. If $2 \tan \alpha = 3 \tan \beta$, then show that,

$$\tan(\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}$$

Soln. We have,

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha.\ \tan\beta}$$

$$= \frac{\frac{3}{2}\tan\beta - \tan\beta}{1 + \frac{3}{2}\tan\beta\ \tan\beta}$$

$$= \frac{\tan\beta}{2 + 3\tan^2\beta}$$

$$= \frac{\frac{\sin\beta}{\cos\beta}}{2 + 3\frac{\sin^2\beta}{\cos^2\beta}}$$

$$= \frac{\sin\beta\ \cos\beta}{2\cos^2\beta + 3\sin^2\beta}$$

$$= \frac{\sin\beta\ \cos\beta}{2 + \sin^2\beta}$$

$$= \frac{2\sin\beta\ \cos\beta}{4 + 2\sin^2\beta}$$

$$= \frac{\sin 2\beta}{4 + 1 - \cos 2\beta}$$

$$= \frac{\sin 2\beta}{5 - \cos 2\beta}$$

EXERCISE 5

1. Find the values of sin15°, cos15°, sin 75°, cos 75°.
2. Find the values of tan 15°, cot 15°, tan 75°, cot 75°.
3. Prove that
   (i) tan 15° + cot 15 ° = 4
   (ii) tan 75 ° + cot 75 ° = 4
   (iii) cot 15 ° – tan 15 ° = $2\sqrt{3}$
   (iv) tan 75° – cot 75 ° = $2\sqrt{3}$.
4. Prove that cos 18° – sin 18° = $\sqrt{2}$ sin 27°.
5. Prove that $\sin(n + 1)x \sin (n + 2) x + \cos (n + 1)x \cos(n + 2)x = \cos x$.
6. Prove that $1 + \tan x.\tan\left(\frac{x}{2}\right) = \sec x$.
7. Prove that $\cot x - \cot 2x = \text{cosec}2x$.
8. Prove that $\frac{\tan 69° + \tan 66°}{1 - \tan 69° \tan 66°} = -1$.
9. Prove that tan 70° = 2 tan 50° + tan 20°.
10. If $A + B = \frac{\pi}{4}$, then prove that $(1 + \tan A)(1 + \tan B) = 2$.
11. Prove that $\frac{\cos 20° - \sin 20°}{\cos 20° + \sin 20°}$ = tan 25°.
12. Prove that $\frac{\cos 7° + \sin 7°}{\cos 7° - \sin 7°}$ = tan 52°.
13. Prove that tan20° + tan 25° + tan 20° tan 25° = 1.
14. Prove that $\tan 13A - \tan 9A - \tan 4A = \tan 4A . \tan 9A . \tan 13A$.
15. Prove that $\tan 9A - \tan 7A - \tan 2A = \tan 2A . \tan 7A. \tan 9A$.
16. Prove that $\cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot 2x = 1$.

17. If $\tan \alpha = \dfrac{m}{m+1}$, $\tan \beta = \dfrac{1}{2m+1}$,

then prove that $\alpha + \beta = \dfrac{\pi}{4}$.

18. Find the value of
 (i) $\sin^2 75° - \sin^2 15°$
 (ii) $\cos^2 75° - \sin^2 15°$.

19. Prove that $\cos^2\left(\dfrac{\pi}{4}+\theta\right) - \sin^2\left(\dfrac{\pi}{4}-\theta\right) = 0$

20. Prove that $\sin^2 B$
$= \sin^2 A + \sin^2 (A - B) - 2 \sin A \cos B \sin (A - B)$.

21. Prove that $\cos(2x + 2y)$
$= \cos 2x \cos 2y + \cos^2 (x + y) - \cos^2 (x - y)$.

22. If $\sin \beta = \dfrac{x-y}{x+y}$,

then prove that $\tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right) = +\sqrt{\dfrac{x}{y}}$.

23. If $\tan \beta = \dfrac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha}$,

prove that $\tan (\alpha - \beta) = (1 - n) \tan \alpha$.

24. If $\tan \alpha = \dfrac{Q \sin \beta}{P + Q \cos \beta}$,

prove that $\tan(\beta - \alpha) = \dfrac{P \sin \alpha}{Q + P \cos \alpha}$

25. If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\dfrac{3}{2}$

prove that $\cos\alpha + \cos\beta + \cos \gamma = 0$
and $\sin\alpha + \sin\beta + \sin\gamma = 0$.

26. If $\tan (\alpha - \theta) = n \tan (\alpha - \theta)$, show that

$\dfrac{\sin 2\theta}{\sin 2\alpha} = \dfrac{n-1}{n+1}$

27. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$,
then show that

 (i) $\cos(\alpha + \beta) = \dfrac{b^2 - a^2}{b^2 + a^2}$

 (ii) $\sin(\alpha + \beta) = \dfrac{2ab}{b^2 + a^2}$

28. If α and β are the roots of
$a \cos \theta + b \sin \theta = c$, then prove that

 (i) $\cos(\alpha + \beta) = \dfrac{a^2 - b^2}{a^2 + b^2}$

 (ii) $\cos(\alpha - \beta) = \dfrac{2c - (a^2 + b^2)}{a^2 + b^2}$

29. If α and β are the roots of
$a \tan \theta + b \tan \theta = c$, then show that
$\tan(\alpha + \beta) = \dfrac{2ac}{a^2 - c^2}$

30. If $\tan (\pi \cos \theta) = \cot(\pi \sin \theta)$, prove that
$\cos\left(\theta - \dfrac{\pi}{4}\right) = \pm \dfrac{1}{2\sqrt{2}}$.

31. If $\tan\theta = \dfrac{x \sin\varphi}{1 - x \cos\varphi}$ and $\tan\varphi = \dfrac{y \sin\theta}{1 - y \cos\theta}$,

then prove that, $x \sin \varphi = y \sin \theta$.

1.14 TRANSFORMATION FORMULAE

1.14.1 Transformation of products into sums or differences

1. $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$
2. $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$
3. $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$
4. $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

Proof: As we know that

$$\sin (A + B) = \sin A. \cos B + \cos A. \sin B \quad \text{(i)}$$
$$\sin (A - B) = \sin A. \cos B - \cos A. \sin B \quad \text{(ii)}$$

Adding (i) and (ii), we get
1. $2 \sin A. \cos B = \sin (A + B) + \sin (A - B)$
 Subtracting (i) and (ii), we get,
2. $2 \cos A. \sin B = \sin (A + B) - \sin (A - B)$
 Also, we have,
$$\cos (A + B) = \cos A. \cos B - \sin A. \sin B \quad \text{(iii)}$$
$$\cos(A - B) = \cos A. \cos B + \sin A. \sin B \quad \text{(iv)}$$
 Adding, (iii) and (iv), we get,
3. $2 \cos A. \cos B = \cos (A + B) + \cos (A - B)$
 Subtracting (iv) from (iii), we get,
4. $2 \sin A. \sin B = \cos (A - B) - \cos (A + B)$.

1.14.2 Transformations of sums or differences into Products

1. $\sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$
2. $\sin C - \sin B = 2 \cos \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$
3. $\cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$
4. $\cos C - \tan D = -2 \sin \dfrac{C-D}{2} \sin \dfrac{C-D}{2}$.

Proof: As we know that,

$$\sin (A + B) + \sin (A - B) = 2 \sin A \cos B \quad \text{(i)}$$
$$\sin (A + B) - \sin (A - B) = 2 \cos A \sin B \quad \text{(ii)}$$

$\cos(A+B)+\cos(A-B)=2\cos A\cos B$ (iii)

$\cos(A+B)-\cos(A-B)=-2\sin A\sin B$ (iv)

Put $A+B=C \,\&\, A-B=D$

$\Rightarrow$ $A=\dfrac{C+D}{2} \,\&\, B=\dfrac{C-D}{2}$

From (i), we get,

1. $\sin C+\sin D=2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

From (ii), we get,

2. $\sin C-\sin D=2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

From (iii), we get,

3 $\cos C+\cos D=2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

From (iv), we get,

4. $\cos C-\cos D=-2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

1.14.3 Some solved examples

Ex-1. Prove that $\sin(47°)+\cos(77°)=\cos(17°)$

Soln. We have, $\sin(47°)+\cos(77°)$

$=\sin(47°)+\sin(13°)$

$=2\sin\left(\dfrac{47°+13°}{2}\right)\cos\left(\dfrac{47°-13°}{2}\right)$

$=2\sin(30°)\cos(17°)$

$=2\times\dfrac{1}{2}\times\cos(17°)$

$=\cos(17°)$

Ex-2. Prove that

$\cos(80°)+\cos(40°)-\cos(20°)=0.$

Soln. We have, $\cos(80°)+\cos(40°)-\cos(20°)$

$=2\cos\left(\dfrac{80°+40°}{2}\right)\cos\left(\dfrac{80°-40°}{2}\right)-\cos(20°)$

$=2\cos(60°)\cos(20°)-\cos(20°)$

$=2\times\dfrac{1}{2}\times\cos(20°)-\cos(20°)$

$=\cos(20°)-\cos(20°)$

$=0.$

Ex-3. Prove that

$\sin(10°)+\sin(20°)+\sin(40°)+\sin(50°)$
$-\sin(70°)-\sin(80°)=0.$

Soln. We have, $\sin(10°)+\sin(20°)+\sin(40°)$
$+\sin(50°)-\sin(70°)-\sin(80°)$

$=\{\sin(50°)+\sin(10°)\}+\{\sin(40°)+\sin(20°)\}$
$-\sin(70°)-\sin(80°)$

$=2\sin\left(\dfrac{50°+10°}{2}\right)\cos\left(\dfrac{50°-10°}{2}\right)$

$+2\sin\left(\dfrac{40°+20°}{2}\right)\cos\left(\dfrac{40°-20°}{2}\right)$

$-\sin(70°)-\sin(80°)$

$=2\sin(30°)\cos(20°)+2\sin(30°)\cos(10°)$

$-\sin(70°)-\sin(80°)$

$=\cos(20°)+\cos(10°)-\sin(70°)-\sin(80°)$

$=\cos(20°)+\cos(10°)-\cos(20°)-\cos(10°)$

$=0.$

Ex-4. Prove $\dfrac{\sin A+\sin 2A+\sin 4A+\sin 5A}{\cos A+\cos 2A+\cos 4A+\cos 5A}$

$=\tan 3A$

Soln. We have, $\dfrac{\sin A+\sin 2A+\sin 4A+\sin 5A}{\cos A+\cos 2A+\cos 4A+\cos 5A}$

$=\dfrac{(\sin 5A+\sin A)+(\sin 4A+\sin 2A)}{(\cos 5A+\cos A)+(\cos 4A+\cos 2A)}$

$=\dfrac{2\sin 3A\cos 2A+2\sin 3A\cos A}{2\cos 3A\cos 2A+2\cos 3A\cos A}$

$=\dfrac{\sin 3A(\cos 2A+\cos A)}{\cos 3A(\cos 2A+\cos A)}$

$=\dfrac{\sin 3A}{\cos 3A}$

$=\tan 3A.$

Hence, the result.

Ex-5. Prove that

$\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

$=4\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\beta+\gamma}{2}\right)\cos\left(\dfrac{\gamma+\alpha}{2}\right)$

Soln. We have

$\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

$=(\cos\alpha+\cos\beta)+(\cos(\alpha+\beta+\gamma)+\cos\gamma)$

$=2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$

$+2\cos\left(\dfrac{\alpha+\beta+\gamma+\gamma}{2}\right)\cos\left(\dfrac{\alpha+\beta+\gamma-\gamma}{2}\right)$

$=2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$

$+2\cos\left(\dfrac{\alpha+\beta+2\gamma}{2}\right)\cos\left(\dfrac{\alpha+\beta}{2}\right)$

$=2\cos\left(\dfrac{\alpha+\beta}{2}\right)\times$

$\left\{\cos\left(\dfrac{\alpha-\beta}{2}\right)+2\cos\left(\dfrac{\alpha+\beta+2\gamma}{2}\right)\right\}$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\times \cos\left(\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta+2\gamma}{2}}{2}\right)$$

$$\cos\left(\frac{\frac{\alpha-\beta}{2}-\frac{\alpha+\beta+2\gamma}{2}}{2}\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\times\left\{\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\right\}$$

Ex-6. If $\sin A-\sin B=\frac{1}{2}$ and $\cos A-\cos B=\frac{1}{3}$, then find $\tan\left(\frac{A+B}{2}\right)$

Soln. Given $\sin A-\sin B=\frac{1}{2}$ ....(i)

and $\cos A-\cos B=\frac{1}{3}$ ....(ii)

Dividing (i) by (ii), we get,

$$\frac{\sin A-\sin B}{\cos A-\cos B}=\frac{1/2}{1/3}=\frac{3}{2}$$

$$\Rightarrow \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}=\frac{3}{2}$$

$$\Rightarrow \frac{\cos\left(\frac{A+B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)}=-\frac{3}{2}$$

$$\Rightarrow \cot\left(\frac{A+B}{2}\right)=-\frac{3}{2}$$

$$\Rightarrow \tan\left(\frac{A+B}{2}\right)=-\frac{2}{3}.$$

EXERCISE 6

1. Express as a sum or difference:
   (i) $2\sin 3\alpha\cos 2\beta$
   (ii) $\frac{1}{4}\cos 10\alpha\cos 20\beta$
   (iii) $2\sin 5\theta\cos\theta$
   (iv) sin 75° cos 15°
   (v) cos 75° cos 15°
2. Express as a product:
   (i) $\cos\phi-\cos 5\phi$
   (ii) cos 45° + sin 75°
   (iii) $\cos 6\theta-\cos 8\theta$
   (iv) $\cos 4\theta+\cos 8\theta$
   (v) $\sin 6\theta-\sin 2\theta$
3. Prove that:
   (i) $\frac{\sin 5A-\sin 3A}{\cos 5A+\cos 3A}=\tan A$
   (ii) $\frac{\sin A+\sin 3A}{\cos A+\cos 3A}=\tan 2A$
   (iii) $\frac{\sin A+\sin B}{\cos A+\cos B}=\tan\left(\frac{A+B}{2}\right)$
4. Prove that :
   (i) sin 38° + sin 22° = sin 82°
   (ii) sin 105° + cos 105° = cos 45°
   (iii) cos 55° + cos 65° + cos 175° = 0
   (iv) cos 20° + cos 100° + cos 140° = 0
   (v) sin 50° – sin 70 + sin 10° = 0
5. Prove that : $(\cos\alpha+\cos\beta)2+(\sin\alpha+\sin\beta)2$ $=4\cos^2\left(\frac{\alpha-\beta}{2}\right)$
6. Prove that : $(\cos\alpha-\cos\beta)2+(\sin\alpha-\sin\beta)2$ $=4\sin^2\left(\frac{\alpha-\beta}{2}\right)$
7. Prove that: $(\sin\alpha-\sin\beta)2+(\cos\alpha-\cos\beta)2$ $=4\sin^2\left(\frac{\alpha-\beta}{2}\right)$
8. Prove that cos 20 co 40 cos 80 = $\frac{1}{8}$
9. Prove that sin 20 . sin 40 sin 80 = $\frac{\sqrt{3}}{8}$
10. Prove that sin 10 sin 50 . sin 60 . sin 70 = $\frac{\sqrt{3}}{16}$
11. Prove that cos 10 . cos 30 . cos 50 . cos 70° = $\frac{3}{16}$
12. If cos A + cos B = $\frac{1}{2}$, sin A + sin B = $\frac{1}{4}$, then prove that $\tan\left(\frac{A+B}{2}\right)=\frac{1}{2}$
13. Prove that :
   (i) $\frac{\sin A+\sin 3A+\sin 5A}{\cos A+\cos 3A+\cos 5A}=\tan 3A$

(ii) $\dfrac{\cos 4x+\cos 3x+\cos 12x}{\sin 4x+\sin 3x+\sin 2x}=\cot 3x$

(iii) $\dfrac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}=\tan 4A$

14. If cosec A + sec A = cosec B + sec B, then prove that $\tan A \tan B=\cot\left(\dfrac{A+B}{2}\right)$
15. If sin $2A = \lambda$ sin $2B$, then prove that $\dfrac{\tan(A+B)}{\tan(A-B)}=\dfrac{\lambda+1}{\lambda-1}$
16. Find the value of $\sqrt{3}\cot(20°)-4\cos(20°)$.
17. If sin A + sin B = a and cos A + cos B = b, then find cos (A + B).
18. If $2\cos A=x+\dfrac{1}{x}$, $2\cos B=y+\dfrac{1}{y}$, then find cos (A – B).
19. Prove that sin(47°) + sin(61°) – sin(11°) – sin(25°) = cos(7°).
20. If $\tan\alpha=\dfrac{m}{m+1}$, and $\tan\beta=\dfrac{1}{2m+1}$, then find tan ($\alpha+\beta$).
21. Find the number of integral values of k for which 7 cos$x$ + 5 sin$x$ = 2k + 1 has a solution.

1.15 MULTIPLE ANGLES

1.15.1 Definition

An angle of the form nA, $n \in Z$ is called a multiple angle of A. Such as $2A$, $3A$, $4A$ etc. are each multiple angles of A.

1.15.2 Trigonometrical ratios of 2A in terms of t-ratio of A

1. sin $2A$ = 2 sin A cos A.
2. cos $2A$ = cos2 A – sin2 A
   = 2 cos2 A – 1 = 1 – 2 sin2 A.
3. tan $2A$ = $\dfrac{2\tan A}{1-\tan^2 A}$

Proof:

1. As we know that,

   sin (A + B) = sin$A$ cos$B$ + cos$A$ sin$B$

   Put $B = A$, we get,

   sin 2 A = sin A . cos A. + sin A. cos A

   = 2 sin A. cos A.

2. As we know that,

   cos (A + B) = cos A. cos B – sin A. sin B

   Put $B = A$, we get,

   cos 2 A = cos$A$. cos A – sin A. sin$A$

   = $\cos^2 A-\sin^2 A$.

   = $\cos^2 A-(1-\cos^2 A)$

   = 2 $\cos^2$ I – 1

   = 2 (1 – $\sin^2$ A) – 1

   = 1 – 2 $\sin^2 A$.

3. As we know that,

   $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A.\tan B}$

   Put $B = A$, we get,

   tan 2 A = $\dfrac{\tan A+\tan A}{1-\tan A.\tan A}=\dfrac{2\tan A}{1-\tan^2 A}$

1.14.3 *T*-ratios of angle 2A

4. sin2 A = $\dfrac{2\tan A}{1+\tan^2 A}$,
5. cos $2A$ = $\dfrac{1-\tan^2 A}{1+\tan^2 A}$
6. 1– cos $2A$ = 2sin2A,
7. 1 + cos $2A$ = 2 cos2A
8. tan A = $\dfrac{\sin 2A}{1+\cos 2A}$,
9. tan A = $\dfrac{1-\cos 2A}{\sin 2A}$

Proof: 4. As we know that, sin $2A$

= 2 sin A cos A

$=\dfrac{2\sin A\cos A}{1}$

$=\dfrac{2\sin A\cos A}{\cos^2 A+\sin^2 A}$

$=\dfrac{\dfrac{2\sin A\cos A}{\cos^2 A}}{1+\dfrac{\sin^2 A}{\cos^2 A}}$

$=\dfrac{2\tan A}{1+\tan^2 A}$.

5. Also, we have, cos 2 A

   = cos2 A – sin2 A

   $=\dfrac{\cos^2 A-\sin^2 A}{1}$

   $=\dfrac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}$

$$= \frac{1 - \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\sin^2 A}{\cos^2 A}}$$

$$= \frac{1 - \tan^2 A}{1 + \tan^2 A}.$$

6. we have, $1 - \cos^2 A$
$= 1 - (1 - 2\sin^2 A)$
$= 2\sin^2 A$
7. Also, we have, $1 + \cos 2A$
$= 1 + (2\cos^2 A - 1)$
$= 2\cos^2 A$.
8. $\tan A$
$$= \frac{\sin A}{\cos A} = \frac{2\sin A\cos A}{2\cos^2 A} = \frac{\sin 2A}{1 + \cos 2A}$$
9. $\tan A$
$$= \frac{\sin A}{\cos A} = \frac{2\sin^2 A}{2\sin A\cos A} = \frac{1 - \cos 2A}{\sin 2A}$$

1.14.4 Trigonometrical ratios of 3A in terms of t-ratio of A.

10. $\sin 3A = 3\sin A - 4\sin^3 A$
11. $\cos 3A = 4\cos^3 A - 3\cos A$
12. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Proof: 10. We have, $\sin 3A$
$= \sin(2A + A)$
$= \sin 2A\cos A + \cos 2A \cdot \sin A$
$= 2\sin A \cdot \cos A \cdot \cos A + (1 - 2\sin^2 A)\sin A$
$= 2\sin A \cdot \cos^2 A + (1 - 2\sin^2 A) \cdot \sin A$
$= 2\sin A(1 - \sin^2 A) + (1 - 2\sin^2 A) \cdot \sin A$
$= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A$
$= 3\sin A - 4\sin^3 A$.

11. We have, $\cos 3A$
$= \cos(2A + A)$
$= \cos 2A \cdot \cos A - \sin 2A \cdot \sin A$
$= (2\cos^2 A - 1)\cos A - 2\sin^2 A \cdot \cos A$
$= (2\cos^2 A - 1)\cos A - 2\cos A(1 - \cos^2 A)$
$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$.
$= 4\cos^3 A - 3\cos A$.

12. we have, $\tan 3A$
$= \tan(2A + A)$
$$= \frac{\tan 2A + \tan A}{1 - \tan 2A.\tan A}$$
$$= \frac{\frac{2\tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2\tan A}{1 - \tan^2 A}.\tan A}$$
$$= \frac{2\tan A + \tan A - \tan^3 A}{1 - \tan^2 A - 2\tan^2 A}$$
$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

1.16 SOME IMPORTANT DEDUCTIONS

Deduction 1

$$\sin^2 A = \frac{1}{2}(1 - \cos^2 A)$$

Proof: We have,

$$\sin^2 A = \frac{1}{2}(2\sin^2 A) = \frac{1}{2}(1 - \cos 2A)$$

Deduction 2

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$$

Proof: We have, $\cos 2A$

$$= \frac{1}{2}(2\cos^2 A) = \frac{1}{2}(1 + \cos 2A)$$

Deduction 3

$$\sin 3A = \frac{1}{4}(3\sin A - \sin 3A)$$

Proof: We have, $\sin 3A = 3\sin A - 4\sin^3 A$

$\Rightarrow \quad 4\sin^3 A = 3\sin A - \sin 3A$

$\Rightarrow \quad \sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$

Deduction 4.

$$\cos 3A = \frac{1}{4}(\cos 3A + 3\cos A)$$

Proof: We have, $\cos 3A = 4\cos 3A - 3\cos A$

$\Rightarrow \quad 4\cos 3A = \cos 3A + 3\cos A$

$\Rightarrow \quad \cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$

Deduction 5

$$\sin A\sin(60 - A) \cdot \sin(60 + A) = \frac{1}{4}\sin 3A$$

Proof: We have,

$\sin A.\sin(60° - A)\sin(60° + A)$
$= \sin A.(\sin 2\ 60° - \sin 2\ A)$
$= \sin A.\left(\frac{3}{4} - \sin^2 A\right)$
$= \frac{\sin A}{4}(3 - 4\sin^2 A)$
$= \frac{1}{4}(3\sin A - 4\sin^3 A)$
$= \frac{1}{4} \times \sin 3A$

Deduction 6

$$\cos A.\cos(60-A).\cos(60+A) = \frac{1}{4}\cos 3A$$

Proof: We have,

$$\cos A.\cos(60°-A).\cos(60°+A)$$
$$= \cos A.(\cos^2 60° - \sin^2 A)$$
$$= \cos A\left(\frac{1}{4} - 1 + \cos^2 A\right)$$
$$= \cos A\left(-\frac{3}{4} + \cos^2 A\right)$$
$$= \frac{\cos A}{4}.\left(-3 + 4\cos^2 A\right)$$
$$= \frac{1}{4}\left(-3\cos A + 4\cos^3 A\right)$$
$$= \frac{1}{4}(4\cos^3 A - 3\cos A)$$
$$= \frac{1}{4}\times\cos 3A.$$

Deduction 7

$$\tan A.\tan(60-A).\tan(60+A) = \tan 3A$$

Proof: We have,

$$\tan A.\tan(60°-A).\tan(60°+A)$$
$$= \frac{\sin A.\sin(60°-A).\sin(60°+A)}{\cos A.\cos(60°-A).\cos(60°+A)}$$
$$= \frac{\frac{1}{4}\sin 3A}{\frac{1}{4}\cos 3A}$$
$$= \frac{\sin 3A}{\cos 3A}$$
$$= \tan 3A.$$

Deduction 8

$$\sin 4A = 4\sin\cos A - 8\cos A\sin^3 A$$

Proof: We have, $\sin 4A$

$$= 2\sin 2A.\cos 2A$$
$$= 2(2\sin A\cos A)(1 - 2\sin 2A)$$
$$= 4\sin A.\cos A(1 - 2\sin 2A)$$
$$= 4\sin A.\cos A - 8\sin 3A.\cos A$$

Deduction 9

$$\cos 4A = 1 - 8\sin^2 A + 8\sin^4 A$$

Proof: We have, $\cos 4A$

$$= \cos 2(2A)$$
$$= 1 - 2\sin^2(2A)$$
$$= 1 - 2(2\sin A.\cos A)^2$$
$$= 1 - 8\sin^2 A.\cos^2 A$$
$$= 1 - 8\sin^2 A(1 - \sin^2 A)$$
$$= 1 - 8\sin^2 A + 8\sin^4 A$$

Deduction 10

$$\tan 4A = \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}$$

Proof: We have, $\tan 4A$

$$= \tan 2.(2A)$$
$$= \frac{2\tan 2A}{1+\tan^2 2A}$$
$$= \frac{\frac{4\tan A}{1-\tan^2 A}}{1-\left(\frac{2\tan A}{1-\tan^2 A}\right)^2}$$
$$= \frac{4\tan A(1-\tan^2 A)}{(1-\tan^2 A)^2 - 4\tan^2 A}$$
$$= \frac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}.$$

Deduction 11

$$\sin 5A = 16\sin^5 A - 20\sin^3 A + 5\sin A$$

Proof: We have, $\sin 5A$

$$= \sin(3A + 2A)$$
$$= \sin 3A\cos 2A + \cos 3A.\sin 2A$$
$$= (3\sin A - 4\sin^3 A)(1 - 2\sin 2A) + 2(4\cos^3 A - 3\cos A)\sin A\cos A$$
$$= (3\sin A - 4\sin^3 A)(1 - 2\sin^2 A) + 2(4\cos^2 A - 3)\sin A\cos^2 A$$
$$= (3\sin A - 4\sin^3 A)(1 - 2\sin^2 A) + 2(1 - 4\sin^2 A)(\sin A - \sin^3 A)$$
$$= (3\sin A - 4\sin^3 A - 6\sin^3 A + 8\sin^5 A) + 2(\sin A - 4\sin^3 A - \sin^3 A + 4\sin^5 A)$$
$$= 5\sin A - 20\sin^3 A + 16\sin^5 A$$
$$= 16\sin^5 A - 20\sin^3 A + 5\sin A.$$

Deduction 12

$$\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$$

Proof: We have, $\cos 5A$

$$= \cos(3A + 2A)$$
$$= \cos 3A\cos 2A - \sin 3A\sin 2A$$

$= (4\cos^3 A - 3\cos A)(2\cos^2 A - 1)$
$\quad - (3\sin A - 4\sin^3 A)(2\sin A\cos A)$
$= 8\cos^5 A - 6\cos^3 A - 4\cos^3 A$
$\quad + 3\cos A - (3 - 4\sin^2 A)\, 2\cos A\,(1 - \cos^2 A)$
$= 8\cos^5 A - 10\cos^3 A + 3\cos A -$
$\quad (4\cos^2 A - 1)(2\cos A - 2\cos^3 A)$
$= 8\cos^5 A - 10\cos^3 A + 3\cos A - 8\cos^3 A$
$\quad + 2\cos A + 8\cos^5 A - 2\cos^3 A$
$= 16\cos^5 A - 20\cos^3 A + 5\cos A.$

Deduction 13

$\sin 6A = (6\sin A - 32\sin^3 A + 32\sin^5 A)\cos A.$

Proof: We have $\sin 6A$
$= \sin 2(3A)$
$= 2\sin 3A \,.\cos 3A$
$= 2(3\sin A - 4\,\text{Sin}^3 A)(4\cos^3 A - 3\cos A)$
$= 2(3\sin A - 4\sin^3 A)(1 - 4\sin^2 A)\cos A$
$= 2(3\sin A - 4\sin^3 A - 12\sin^3 A + 16\sin 5A)\cos A$
$= 2(3\sin A - 16\sin^3 A + 16\sin^5 A)\cos A$
$= (6\sin A - 32\sin^3 A + 32\sin^5 A)\cos A.$

Deduction 14

$\cos 6A = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1$

Proof: We have, $\cos 6A$
$= \cos 2(3A)$
$= 2\cos^2(3A) - 1$
$= 2(4\cos^3 A - 3\cos A)^2 - 1$
$= 2(16\cos^6 A - 24\cos^4 A + 9\cos^2 A) - 1$
$= 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1$

1.16.1 Some solved examples

Ex-1. Prove that $\left(\dfrac{1-\cos 2\theta}{\sin 2\theta}\right) = \tan\theta$

Soln. We have,

$$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$
$$= \frac{\sin\theta}{\cos\theta} = \tan\theta$$

Ex-2. Prove that $\left(\dfrac{1+\cos 2\theta}{\sin 2\theta}\right) = \cot\theta$

Soln. We have,

$$\frac{1+\cos 2\theta}{\sin 2\theta} = \frac{2\cos^2\theta}{2\sin\theta\cos\theta}$$
$$= \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

Ex-3. Prove that $\cot\theta - \tan\theta = 2\cot(2\theta)$

Soln. We have, $(\cot\theta - \tan\theta)$

$$= \left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\right)$$
$$= \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{2\cos 2\theta}{2\sin\theta\cos\theta}$$
$$= \frac{2\cos 2\theta}{\sin 2\theta}$$
$$= 2\cot 2\theta.$$

Ex-4. If $\tan\theta = \dfrac{b}{a}$,

prove that, $a\cos(2\theta) + b\sin(2\theta) = a$

Soln. We have, $a\cos(2\theta) + b\sin(2\theta)$

$$= a\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) + b\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$
$$= a\left(\frac{1-\frac{b^2}{a^2}}{1+\frac{b^2}{a^2}}\right) + b\left(\frac{2\frac{b}{a}}{1+\frac{b^2}{a^2}}\right)$$
$$= a\left(\frac{a^2-b^2}{a^2+b^2}\right) + b\left(\frac{2ab}{a^2+b^2}\right)$$
$$= \frac{a(a^2-b^2+2b^2)}{a^2+b^2}$$
$$= \frac{a(a^2+b^2)}{a^2+b^2}$$
$= a.$

Ex-5. Prove that $\sqrt{3}\,\text{cosec}(20°) - \sec(20°) = 4$

Soln. We have, $\sqrt{3}\,\text{cosec}(20°) - \sec(20°)$

$$= \frac{\sqrt{3}}{\sin(20°)} - \frac{1}{\cos(20°)}$$
$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos\left(20^{O}\right) - \frac{1}{2}\sin\left(20^{O}\right)\right)}{2\sin\left(20^{Ÿ}\right)\cos\left(20^{O}\right)}$$
$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos(20°) - \frac{1}{2}\sin(20°)\right)}{2\sin(20°)\cos(20°)}$$
$$= \frac{4(\sin(60°)\cos(20°) - \cos(60°)\sin(20°))}{2\sin(20°)\cos(20°)}$$

$$= \frac{4(\sin(60° - 20°))}{\sin(40°)}$$

$= 4.$

Ex-6. Prove that

$\tan(9°) - \tan(27°) - \tan(63°) + \tan(81°) = 4$

Soln. We have,

$$\tan(9°) - \tan(27°) - \tan(63°) + \tan(81°)$$
$$= \{\tan(9°) + \tan(81°)\} - \{\tan(27°) + \tan(63°)\}$$
$$= \{\tan(9°) + \cot(9°)\} - \{\tan(27°) + \cot(27°)\}$$
$$= \left\{\frac{\sin(9°)}{\cos(9°)} + \frac{\cos(9°)}{\sin(9°)}\right\} - \left\{\frac{\sin(27°)}{\cos(27°)} + \frac{\cos(27°)}{\sin(27°)}\right\}$$
$$= \left\{\frac{\sin^2(9°) + \cos^2(9°)}{\sin(9°)\cos(9°)}\right\} - \left\{\frac{\sin^2(27°) + \cos^2(27°)}{\sin(27°)\cos(27°)}\right\}$$
$$= \left\{\frac{2}{2\sin(9°)\cos(9°)}\right\} - \left\{\frac{2}{2\sin(27°)\cos(27°)}\right\}$$
$$= \left\{\frac{2}{\sin(18°)}\right\} - \left\{\frac{2}{\sin(54°)}\right\}$$
$$= \left\{\frac{2}{\left(\frac{\sqrt{5}-1}{4}\right)}\right\} - \left\{\frac{2}{\frac{\sqrt{5}+1}{4}}\right\}$$
$$= \left\{\frac{8}{\sqrt{5}-1}\right\} - \left\{\frac{8}{\sqrt{5}+1}\right\}$$
$$= \left\{\frac{8\left(\sqrt{5}+1-\sqrt{5}+1\right)}{5-1}\right\}$$
$$= 4.$$

Ex-7. Prove that $\left(\frac{\sec 8A - 1}{\sec 4A - 1}\right) = \frac{\tan 8A}{\tan 2A}$

Soln. We have, $\left(\frac{\sec 8A - 1}{\sec 4A - 1}\right)$

$$= \frac{\frac{1}{\cos 8A} - 1}{\frac{1}{\cos 4A} - 1}$$
$$= \frac{1 - \cos 8A}{1 - \cos 4A} \times \frac{\cos 4A}{\cos 8A}$$
$$= \frac{2\sin^2 4A}{2\sin^2 2A} \times \frac{\cos 4A}{\cos 8A}$$
$$= \frac{2\sin 4A \cos 4A}{\cos 8A} \times \frac{\sin 4A}{2\sin^2 2A}$$
$$= \frac{\sin 8A}{\cos 8A} \times \frac{2\sin 2A \cos 2A}{2\sin^2 2A}$$
$$= \frac{\sin 8A}{\cos 8A} \times \frac{\cos 2A}{\sin 2A}$$
$$= \tan 8A \leftrightarrow \cot 2A$$
$$= \frac{\tan 8A}{\tan 2A}.$$

Ex-8. Prove that

$\tan\theta + 2\tan(2\theta) + 4\tan(4\theta) + 8\cot 8\theta = \cot\theta$

Soln. We have,

$$\tan\theta + 2\tan(2\theta) + 4\tan(4\theta) + 8\cot 8\theta$$
$$= \cot\theta - (\cot\theta - \tan\theta) + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta$$
$$= \cot\theta - 2\cot 2\theta + 2\tan 2\theta + 4\tan 4\theta + 8\cot 8\theta$$
$$= \cot\theta - 2(\cot 2\theta - \tan 2\theta) + 4\tan 4\theta + 8\cot 8\theta$$
$$= \cot\theta - 4\cot 4\theta + 4\tan 4\theta + 8\cot 8\theta$$
$$= \cot\theta - 4(\cot 4\theta - \tan 4\theta) + 8\cot 8\theta$$
$$= \cot\theta - 8\cot 8\theta + 8\cot 8\theta$$
$$= \cot\theta.$$

Ex-9. Prove that

$$\cos^2(\theta) + \cos^2\left(\frac{2\pi}{3} - \theta\right) + \cos^2\left(\frac{2\pi}{3} + \theta\right) = \frac{3}{2}$$

Soln. We have,

$$\cos^2(\theta) + \cos^2\left(\frac{2\pi}{3} - \theta\right) + \cos^2\left(\frac{2\pi}{3} + \theta\right)$$
$$= \frac{1}{2}\left(2\cos^2(\theta) + 2\cos^2\left(\frac{2\pi}{3} - \theta\right) + 2\cos^2\left(\frac{2\pi}{3} + \theta\right)\right)$$
$$= \frac{1}{2}(1 + \cos(2\theta)) + \frac{1}{2}\left(1 + \cos\left(\frac{4\pi}{3} - 2\theta\right)\right) + \frac{1}{2}\left(1 + \cos\left(\frac{4\pi}{3} + 2\theta\right)\right)$$
$$= \frac{1}{2}\left(3 + \left(\cos 2\theta + \cos\left(\frac{4\pi}{3} - 2\theta\right) + \cos\left(\frac{4\pi}{3} + 2\theta\right)\right)\right)$$
$$= \frac{1}{2}\left(3 + \left(\cos 2\theta + 2\cos\left(\frac{4\pi}{3}\right)\cos(2\theta)\right)\right)$$
$$= \frac{1}{2}\left(3 + \left(\cos 2\theta + 2\left(-\frac{1}{2}\right)\cos(2\theta)\right)\right)$$
$$= \frac{3}{2}.$$

Ex-10 Prove that

$$\sin^2\theta + \sin^2(120° + \theta) + \sin^2(120° + \theta) = \frac{3}{2}$$

Soln. We have,

$\sin^2\theta + \sin^2(120° + \theta) + \sin^2(240° + \theta)$

$$= \frac{1}{2}\left(2\sin^2\theta + 2\sin^2\left(\frac{2\pi}{3} + \theta\right)\right)$$

$$+ \frac{1}{2}\left(2\sin^2\left(\frac{4\pi}{3} + \theta\right)\right)$$

$$= \frac{1}{2}\left((1 - \cos 2\theta) + \left(1 - \cos\left(\frac{4\pi}{3} + 2\theta\right)\right)\right)$$

$$+ \frac{1}{2}\left(\left(1 - \cos\left(\frac{8\pi}{3} + 2\theta\right)\right)\right)$$

$$= \frac{3}{2} - \frac{1}{2}\cos 2\theta - \frac{1}{2}\left[\cos\left(\frac{4\pi}{3} + 2\theta\right) + \cos\left(\frac{8\pi}{3} + 2\theta\right)\right]$$

$$= \frac{3}{2} - \frac{1}{2}\cos 2\theta - \frac{1}{2}(2\cos(120°)\cos 2\theta)$$

$$= \frac{3}{2} - \frac{1}{2}\cos 2\theta + \frac{1}{2}\cos(2\theta)$$

$$= \frac{3}{2}.$$

Ex-11. Prove that

$4\sin(\theta)\sin(60° + \theta)\sin(60° - \theta) = \sin 3\theta$

Soln. We have, $4\sin(\theta)\sin\left(\frac{\pi}{3} - \theta\right)\sin\left(\frac{\pi}{3} + \theta\right)$

$$= 4\sin(\theta) \times \left(\sin\left(\frac{\pi}{3} - \theta\right)\sin\left(\frac{\pi}{3} + \theta\right)\right)$$

$$= 4\sin(\theta) \times \left(\sin^2\left(\frac{\pi}{3}\right) - \sin^2\theta\right)$$

$$= 4\sin(\theta) \times \left(\frac{3}{4} - \sin^2\theta\right)$$

$$= 4 \times \left(\frac{3}{4}\sin\theta - \sin^3\theta\right)$$

$= (3\sin\theta - 4\sin^3\theta)$

$= \sin(3\theta)$

Ex-12. Prove that $\sin(20°)\sin(40°)\sin(80°) = \frac{\sqrt{3}}{8}$

Soln. We have, $\sin(20°)\sin(40°)\sin(80°)$

$$= \frac{1}{4}(\sin(3.20°))$$

$$= \frac{1}{4}(\sin(60°))$$

$$= \frac{\sqrt{3}}{8}.$$

Ex-13. Prove that

$4\cos(\theta).\cos(60° - \theta).\cos(60° + \theta)$

$= \cos(3\theta)$

Soln. We have,

$4\cos(\theta).\cos(60° - \theta).\cos(60° + \theta)$

$= 4\cos(\theta).(\cos(60° - \theta).\cos(60° + \theta))$

$= 4\cos(\theta).(\cos^2(60°) - \sin^2\theta)$

$$= 4\cos(\theta).\left(\frac{1}{4} - 1 + \cos^2\theta\right)$$

$$= 4\cos(\theta).\left(-\frac{3}{4} + \cos^2\theta\right)$$

$= \cos(\theta).(-3 + 4\cos 2\theta)$

$= (4\cos^3\theta - 3\cos(\theta))$

$= \cos(3\theta).$

Ex-14. Prove that

$$\cos(10°).\cos(50°).\cos(70°) = \frac{\sqrt{3}}{8}$$

Soln. We have, $\cos(10°).\cos(50°).\cos(70°)$

$= \cos(10°)\cos(60° - 10°)\cos(60° + 10°)$

$$= \frac{1}{4}(4\cos(10°).\cos(60° - 10°).\cos(60° + 10°))$$

$$= \frac{1}{4}(4\cos(3.10°))$$

$$= \frac{\sqrt{3}}{8}.$$

Ex-15. Prove that

$\tan(\theta) + \tan(60° + \theta) - \tan(60° - \theta)$

$= 3\tan(3\theta)$

Soln. We have,

$\tan(\theta) - \tan(60° - \theta) + \tan(60° + \theta)$

$$= \tan(\theta) - \frac{\tan(60°) - \tan(\theta)}{1 + \tan(60°).\tan(\theta)}$$

$$+ \frac{\tan(60°) + \tan(\theta)}{1 - \tan(60°).\tan(\theta)}$$

$$= \tan(\theta) - \frac{\sqrt{3} - \tan(\theta)}{1 + \sqrt{3}.\tan(\theta)} + \frac{\sqrt{3} + \tan(\theta)}{1 - \sqrt{3}.\tan(\theta)}$$

$$= \tan(\theta)\,\frac{\begin{array}{c}-\sqrt{3}+3\tan(\theta)+\tan(\theta)-\sqrt{3}\tan^2(\theta)\\ +\sqrt{3}+3\tan(\theta)+\tan(\theta)+\sqrt{3}\tan^2(\theta)\end{array}}{\left(1-3\tan^2\theta\right)}$$

$$= \tan(\theta)+\frac{8\tan(\theta)}{1-3\tan^2(\theta)}$$

$$= \frac{\tan(\theta)-3\tan^3(\theta)+8\tan(\theta)}{1-3\tan^2(\theta)}$$

$$= 3\times\left(\frac{3\tan(\theta)-\tan^3(\theta)}{1-3\tan^2(\theta)}\right)$$

$= 3 \times \tan(3\theta)$.

Ex-16. Prove that

$$\cos(\theta)\cos(2\theta)\cos(2^2\theta)\cos(2^3\theta)\ldots\cos(2^{n-1}\theta) = \frac{\sin\left(2^n\theta\right)}{2^n\sin\theta}$$

Soln. We have,

$$\cos(\theta)\cos(2\theta)\cos(2^2\theta)\cos(2^3\theta)\ldots\cos(2^{n-1}\theta)$$

$$= \frac{1}{2\sin\theta}(2\sin\theta\cos\theta)\left(\cos 2\theta.\cos\left(2^2\theta\right)\ldots\cos\left(2^{2n-1}\theta\right)\right)$$

$$= \frac{1}{2^2\sin\theta}(2\sin 2\theta\cos 2\theta)\left(\cos\left(2^2\theta\right)\ldots\ldots\cos\left(2^{2n-1}\theta\right)\right)$$

$$= \frac{1}{2^3\sin\theta}(2\sin 4\theta\cos 4\theta)\left(\cos\left(2^3\theta\right)\ldots\ldots\cos\left(2^{2n-1}\theta\right)\right)$$

$$= \frac{1}{2^4\sin\theta}\left(2\sin 2^3\theta\cos 2^3\theta\right)\left(\cos\left(2^4\theta\right)\ldots\ldots\cos\left(2^{2n-1}\theta\right)\right)$$

....

....

$$= \frac{1}{2^n\sin\theta}\left(2\sin 2^{n-1}\theta\cos 2^{n-1}\theta\right)$$

$$= \frac{1}{2^n\sin\theta}\left(\sin 2^n\theta\right)$$

$$= \frac{\sin(2^n\theta)}{2^n\sin\theta}$$

Ex-17. Prove that $\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{8\pi}{7}\right) = \frac{1}{8}$

Soln. We have $\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{8\pi}{7}\right)$

$$= \cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\pi+\frac{\pi}{7}\right)$$

$$= -\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2\sin\left(\frac{\pi}{7}\right)}\left(2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2^2\sin\left(\frac{\pi}{7}\right)}\left(2\sin\left(\frac{2\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\right)\cos\left(\frac{4\pi}{7}\right)$$

$$= -\frac{1}{2^3\sin\left(\frac{\pi}{7}\right)}\left(2\sin\left(\frac{4\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\right)$$

$$= -\frac{\sin\left(\frac{8\pi}{7}\right)}{2^3\sin\left(\frac{\pi}{7}\right)}$$

$$= -\frac{\sin\left(\pi+\frac{\pi}{7}\right)}{2^3\sin\left(\frac{\pi}{7}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{7}\right)}{8\sin\left(\frac{\pi}{7}\right)}$$

$$= \frac{1}{8}.$$

Ex-18. Prove that $\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}$

Soln. Let $z = \cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)$

$$\Rightarrow \quad 2z\sin\left(\frac{\pi}{7}\right)$$

$$= 2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)+2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right) +2\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right)$$

$$= \sin\left(\frac{3\pi}{7}\right)-\sin\left(\frac{\pi}{7}\right)+\sin\left(\frac{5\pi}{7}\right)-\sin\left(\frac{3\pi}{7}\right) +\sin\left(\frac{7\pi}{7}\right)-\sin\left(\frac{5\pi}{7}\right)$$

$$= -\sin\left(\frac{\pi}{7}\right)$$

Thus, $\quad 2z\sin\left(\frac{\pi}{7}\right) = -\sin\left(\frac{\pi}{7}\right)$

$\Rightarrow \quad z = -\frac{1}{2}$

$$\Rightarrow \quad \cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)=-\frac{1}{2}.$$

Ex-19. If $M=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}+\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$ such that max $(M^2)=m^1$ and min $(M^2)=m^2$, then find the value of m^1-m^2.

Soln. Given

$$M=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}+\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$$

$$\Rightarrow M^2 = a^2\cos^2\theta+b^2\sin 2\theta+a^2\sin^2\theta+b^2\cos^2\theta+2\sqrt{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)\left(a^2\sin^2\theta+b^2\cos^2\theta\right)}$$

$$\Rightarrow M^2= a^2+b^2+2\sqrt{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)}\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}$$

$$\Rightarrow M^2= a^2+b^2+2\left[\left(a^4+b^4\right)\sin^2\theta\cos^2\theta\right.$$

$$\Rightarrow \quad \left.a^2b^2\left(\sin^4\theta+\cos^4\theta\right)\right]^{1/2}$$

$$M^2= a^2+b^2+2\left[\left(a^4+b^4\right)\sin^2\theta\cos^2\theta\right.$$

$$\Rightarrow \quad \left.+a^2b^2\left(1-2\sin^2\theta\cos^2\theta\right)\right]^{1/2}$$

$$\Rightarrow M^2= a^2+b^2+2\left(\left(a^4+b^4-2a^2b^2\right)\sin^2\theta\cos^2\theta+a^2b^2\right)^{1/2}$$

$$\Rightarrow M^2= a^2+b^2+\sqrt{\left(4\left(a^4+b^4-2a^2b^2\right)\sin^2\theta\cos^2\theta+4a^2b^2\right)}$$

Thus,max $(M^2)=a^2+b^2+(a^2+b^2)=2(a^2+b^2)$

and min $(M^2)=a^2+b^2+2ab=(a+b)^2$

Hence, the value of m^1-m^2

$=$ max (M^2) – min (M^2)

$=2(a^2+b^2)-(a+b)^2.$

EXERCISE 7

Q. Prove that:

1. $\dfrac{\sin 2\theta}{1+\cos 2\theta}=\tan\theta$
2. $\dfrac{\cos 2\theta}{1+\sin 2\theta}=\tan\left(\dfrac{\pi}{4}-\theta\right)$
3. $\dfrac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta}=\cot\theta$
4. $\cot\theta-\tan\theta=2\cot 2\theta$
5. $\sqrt{2+\sqrt{2+2\cos 4\theta}}=2\cos\theta$
6. Prove that: cosec 10° – $\sqrt{3}$ sec 10° = 4.
7. If $\tan\theta=\dfrac{b}{a}$, then prove that $a\cos 2\theta+b\sin 2\theta=a$

Q. Prove that:

8. $\cos^2\left(\dfrac{\pi}{8}\right)+\cos^2\left(\dfrac{3\pi}{8}\right)+\cos^2\left(\dfrac{5\pi}{8}\right)+\cos^2\left(\dfrac{7\pi}{8}\right)=2$
9. $\sin^2\left(\dfrac{\pi}{8}\right)+\sin^2\left(\dfrac{3\pi}{8}\right)+\sin^2\left(\dfrac{5\pi}{8}\right)+\sin^2\left(\dfrac{7\pi}{8}\right)=2$
10. Prove that: $\dfrac{\sec 8A-1}{\sec 4A-1}=\dfrac{\tan 8A}{\tan 2A}$
11. If cosec $2A$ + cosec $2B$ + cosec $2C$ = 0, then prove that $\tan A+\tan B+\tan C+\cot A+\cot B+\cot C=0$
12. If tan 25° = a, then prove that: $\dfrac{\tan 155°-\tan 115°}{1+\tan 155°.\tan 115°}=\dfrac{1-a^2}{2a}$
13. If $\tan^2\theta=2\tan^2\phi+1$, then prove that $\cos 2\theta+\sin^2\phi=0$
14. If $2\tan\alpha=3\tan\beta$, then prove that: $\tan(\alpha-\beta)=\dfrac{\sin 2\beta}{5-\cos 2\beta}$
15. If $a+\dfrac{1}{a}=2\cos\theta$, then prove that : $a^4+\dfrac{1}{a^4}=2\cos 4\theta.$
16. Prove that: $\tan\alpha+2\tan 2\alpha+4\tan 4\alpha+8\cot 8\alpha=\cot\alpha$
17. Prove that $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x)=13$.
18. If α and β are the roots of $3\cos x+4\sin x=5$, then find $\cot(\alpha+\beta)$
19. If α and β are the solutions of the equation $a\tan\theta+b\cot\theta=c$, then find $\tan(\alpha+\beta)$.
20. If $\tan\beta=3\tan\alpha$, then prove that $\tan(\alpha+\beta)=\dfrac{2\sin 2\beta}{1+2\cos 2\beta}$
21. Find the value of $\cot(91°)\cot(92°)\cot(93°)....\cot(179°)$.

22. Find the value of
$$\left(1+\cos\left(\frac{\pi}{8}\right)\right)\left(1+\cos\left(\frac{3\pi}{8}\right)\right)\left(1+\cos\left(\frac{5\pi}{8}\right)\right)\left(1+\cos\left(\frac{7\pi}{8}\right)\right)$$
23. If $\alpha+\beta=90°$, then find the maximum value of $\cos\alpha\cos\beta$
24. Prove that:
$$\cos^2\theta+\cos^2\left(\frac{2\pi}{3}-\theta\right)+\cos^2\left(\frac{2\pi}{3}+\theta\right)=\frac{3}{2}$$
25. Prove that:
$$\cos^2\theta+\cos^2\left(\frac{\pi}{3}-\theta\right)+\cos^2\left(\frac{\pi}{3}+\theta\right)=\frac{3}{2}$$
26. Prove that: $\sqrt{3}\,\text{cosec}20°-\sec 20°=4$
27. Prove that: $\sin\theta\cos^3\theta-\sin^3\theta\cos\theta=\frac{1}{4}\sin 3\theta$
28. Prove that:
$$\tan\theta+\tan\left(\frac{\pi}{3}+\theta\right)+\tan\left(\frac{\pi}{3}-\theta\right)=3\tan(3\theta)$$
29. Prove that:
$$\cot\theta+\cot\left(\frac{\pi}{3}+\theta\right)+\cot\left(\frac{2\pi}{3}+\theta\right)=3\cot 3\theta$$
30. Prove that: $\tan 4\theta=\dfrac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$.
31. Prove that: $\sin^3\alpha+\sin^3\left(\frac{2\pi}{3}+\alpha\right)+\sin^3\left(\frac{4\pi}{3}+\alpha\right)$
$$=-\frac{3}{4}\sin 3\alpha\cdot$$
32. Prove that:
$$2^n\cos(\theta)\cos(2\theta).\cos\left(2^2\theta\right).\cos\left(2^3\theta\right)...\cos\left(2^{n-1}\theta\right)$$
$$=\frac{\sin\left(2^n\theta\right)}{\sin\theta}$$
33. If $\theta=\dfrac{\pi}{2^n+1}$, then prove that
$$2^n\cos\theta.\cos 2\theta.\cos 2^2\theta....\cos 2^{n\text{-}1}\theta=1.$$
34. If $\cos 6\theta=A\cos^6\theta+B\cos^4\theta+C\cos^2\theta+D$, then find the vale of $A+B+C+D+2$.
35. If $\sin 5\theta=A\sin\theta+B\sin^3\theta+C\sin^5\theta$, then find the value of $A+B+C+10$.
36. Prove that:
$$\left(\frac{\sin x}{\cos 3x}+\frac{\sin 3x}{\cos 9x}+\frac{\sin 9x}{\cos 27x}\right)=\frac{1}{2}(\tan 27x-\tan x)$$
37. Prove that $\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)=-\frac{1}{2}$.
38. Prove that:
$$(1+\sec 2,)(1+\sec 2^2,)(1+\sec 2^3,)...(1+\sec 2^{\text{n}},).$$
39. Prove that:
$$(1+\sec 2\theta)\left(1+\sec 2^2\theta\right)\left(1+\sec 2^3\theta\right)...\left(1+\sec 2^n\theta\right)$$
$$=\frac{\tan^{2n}\theta}{\tan\theta}$$

1.17 THE MAXIMUM AND MINIMUM VALUES OF $f(x) = a\cos x + b\sin x + c$

We have $f(x)=a\cos x+b\sin x+c$
Let $a=r\sin\theta$ & $b=r\cos\theta$

Then $r=\sqrt{a^2+b^2}$ and $\tan(\theta)=\dfrac{a}{b}$

Now, $f(x) = a\cos x + b\sin x + c$
$= r(\sin\theta\cos x+\cos\theta\sin x)$
$= r\sin(\theta+x)$

As we know that, $-1\le\sin(\theta+x)\le 1$
$\Rightarrow\ -r+c\le r\sin(\theta+x)+c\le r+c$
$\Rightarrow\ -r+c\le f(x)\le r+c$
$\Rightarrow\ -\sqrt{a^2+b^2}+c\le f(x)\le\sqrt{a^2+b^2}+c$

Thus, the maximum value of $f(x)$ is $\sqrt{a^2+b^2}+c$
and the minimum values of $f(x)$ is $-\sqrt{a^2+b^2}+c$.

Ex-1. Find the max and min values of
$$f(x) = 3\sin x + 4\cos x + 10.$$
Soln. Here, $a = 3$, $b = 4$ and $c = 10$
Thus, the minimum values of $f(x)$
$$= -\sqrt{a^2+b^2}+c = -5+10 = 5$$
and the maximum values of
$$f(x) = \sqrt{a^2+b^2}+c = 5+10 = 15.$$
Ex-2. Find the range of $f(x) = \sin x + \cos x + 3$
Soln. $R_f=[\min f(x),\ \max f(x)]$
$$=\left[-\sqrt{2}+3,\ \sqrt{2}+3\right]$$
Ex-3. Let $A=\sin^4\theta+\cos^4\theta$, then find A
Soln. We have, $A=\sin^4\theta+\cos^4\theta$
$$=\left(\sin^2\theta\right)^2+\left(\cos^2\theta\right)^2$$

$$= 1-\frac{(\sin 2)^2}{2}$$

$$= 1-\frac{(\sin 2\theta)^2}{2}$$

$$= 1+\frac{\{-\sin^2(2\theta)\}}{2}$$

As we know that, $-1 \le \{-\sin^2(2\theta)\} \le 0$

$$\Rightarrow \quad -\frac{1}{2} \le \frac{\{-\sin^2(2\theta)\}}{2} \le 0$$

$$\Rightarrow \quad -\frac{1}{2}+1 \le \frac{\{-\sin^2(2\theta)\}}{2} \le 0+1$$

$$\Rightarrow \quad \frac{1}{2} \le A \le 1.$$

Ex-4. Find the max and min values of $f(\theta)=\sin^6\theta+\cos^6\theta$

Soln. We have, $f(\theta)=\sin^6\theta+\cos^6\theta$

$$= (\sin^2\theta)^3+(\cos^2\theta)^3$$

$$= (\sin^2\theta+\cos^2\theta)^2-3\sin^2\theta\cos^2\theta(\sin^2\theta+\cos^2\theta)$$

$$= 1-3\sin^2\theta\cos^2\theta$$

$$= 1-\frac{3}{4}(4\sin^2\theta\cos^2\theta)$$

$$= 1-\frac{3}{4}(\sin^2 2\theta)$$

$$= 1+\frac{3}{4}(-\sin^2 2\theta)$$

As we know that, $-1 \le (-\sin^2 2\theta) \le 0$

$$\Rightarrow \quad -\frac{3}{4} \le \frac{3(-\sin^2 2\theta)}{4} \le 0$$

$$\Rightarrow \quad -\frac{3}{4} \le \frac{3(-\sin^2 2\theta)}{4} \le 0$$

$$\Rightarrow \quad \frac{1}{4} \le f(\theta) \le 1$$

Hence, the maximum value = 1 and the minimum value = $\frac{1}{4}$.

Ex-5. If $A=\cos^2\theta+\sin^4\theta$ and $B=\cos^4\theta+\sin^2\theta$ such that m_1 = Max of A and m_2 = Min of B then find the value of $m_1^2+m_2^2+m_1m_2$

Soln. Now, A

$$= \cos^2\theta+\sin^4\theta$$

$$= \frac{1}{2}(2\cos^2\theta)+\frac{1}{4}(2\sin^2\theta)^2$$

$$= \frac{1}{2}(1+\cos(2\theta))+\frac{1}{4}(1-\cos(2\theta))^2$$

$$= \frac{1}{2}(1+\cos(2\theta))+\frac{1}{4}(1-2\cos(2\theta)+\cos^2(2\theta))$$

$$= \frac{1}{2}+\frac{1}{2}\cos(2\theta)+\frac{1}{4}-\frac{1}{2}\cos(2\theta)+\frac{1}{4}(\cos^2(2\theta))$$

$$= \frac{3}{4}+\frac{1}{4}(\cos^2(2\theta))$$

Max value of $A = m_1 = \frac{3}{4}+\frac{1}{4}.1=1$

Also, B

$$= \sin^2\theta+\cos^4\theta$$

$$= \frac{1}{2}(2\sin^2\theta)+\frac{1}{4}(4\cos^4\theta)$$

$$= \frac{1}{2}(2\sin^2\theta)+\frac{1}{4}(2\cos^2\theta)^2$$

$$= \frac{1}{2}(1-\cos(2\theta))+\frac{1}{4}(1+\cos(2\theta))^2$$

$$= \frac{1}{2}(1-\cos(2\theta))+\frac{1}{4}(1+2\cos(2\theta)+\cos^2(2\theta))$$

$$= \frac{1}{2}+\frac{1}{4}+\frac{1}{4}\cos^2(2\theta)$$

$$= \frac{3}{4}+\frac{1}{4}\cos^2(2\theta)$$

Thus, the minimum value of B

$$= m_2 = \frac{3}{4}+\frac{1}{4}.0=\frac{3}{4}$$

Now, the value of

$$m_1^2+m_2^2+m_1m_2$$

$$= 1+\frac{9}{16}+\frac{3}{4}$$

$$= \frac{37}{16}$$

Ex-9. Find the minimum value of

$$f(x)=\frac{x^2\sin^2 x+4}{x\sin x},$$

where $x \in \left(0, \frac{\pi}{2}\right)$

Soln. We have, $f(x) = \dfrac{x^2 \sin^2 x + 4}{x \sin x}$

$= x\sin x + \dfrac{4}{x\sin x} \geq 4$

Hence, the minimum values of $f(x)$ is 4.

Ex-10. Find the minimum value of

$$f(a,b,c,d) = \frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)}{abcd}$$

where $a,b,c,d > 0$

Soln. We have $f(a,b,c,d)$

$$= \frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)}{abcd}$$

$$= \frac{(a^2+1)}{a} \times \frac{(b^2+1)}{b} \times \frac{(c^2+1)}{c} \times \frac{(d^2+1)}{d}$$

$$= \left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right)\left(d + \frac{1}{d}\right)$$

$$\geq 2.2.2.2 = 16$$

Hence, the minmimum value is 16.

EXERCISE 8

1. Find the maximum and minimum values of
   (i) $f(x) = 3 \sin x + 4 \cos x + 5$
   (ii) $f(x) = 3 \sin (100)x + 4 \cos (100)x + 10$
   (iii) $f(x) = 3 \sin x + 4$
   (iv) $f(x) = 2 \cos x + 5$
   (v) $f(x) = \sin x + \cos x$
   (vi) $f(x) = \sin x - \cos x$
   (vii) $f(x) = \sin (\sin x)$
   (viii) $f(x) = \cos (\cos x)$
   (ix) $f(x) = \sin (\sin x) + \cos (\sin x)$
   (x) $f(x) = \cos (\sin x) + \sin (\cos x)$.
2. Find the max and min value of
   $$3 \sin 2x + 4 \cos 2x + 3$$
3. Prove that
   $$-4 \leq 5 \cos\theta + 3 \cos\left(\theta + \frac{\pi}{3}\right) + 3 \leq 10.$$
4. Find the max and min values of $f(x) = 3\sin x + 5$
5. Find the greatest and the least values of $2 \sin^2\theta + 3 \cos^2\theta$.
6. Find the least value of $\operatorname{cosec}^2 x + 25 \sec^2 x$.
7. Find the ratio of the greatest value of $2 - \cos x + \sin^2 x$ to its least value.
8. If $y = 4 \sin^2 \theta - \cos 2\theta$, then y lies in the interval............
9. If m is the minimum value of $g(x) = 3 - 2\sin x$ and n is the maximum value of $(m + n + 2)$ then find the value of $(m + n + 2)$.
10. Find the maximum and minimum values of $f(x) = \sin^4 x + \cos^4 x$.
11. Find the maximum and minimum values of $f(x) = \sin^6 x + \cos^6 x$.
12. Find the max and min values of
    $$f(x) = (\sin x + \cos x + \operatorname{cosec} 2x)^3$$
    where $x \in \left(0, \frac{\pi}{2}\right)$
13. Find the max and min values of $f(x) = \log_x y + \log_y x$.
14. Find the max and min values of
    $$f(x) = \frac{5}{\sin^2\theta - 6\sin\theta\cos\theta + 3\cos^2\theta}$$
15. Find the maximum and the minimum values of $f(x) = \sin^2 x + \cos^4 x$
16. Find the maximum and the minimum values of $f(x) = \cos^2 x + \sin^4 x$
17. Find the minimum value of
    $$f(x) = \frac{a^2}{\cos^2 x} + \frac{b^2}{\sin^2 x},\ x \in \left(0, \frac{\pi}{2}\right)$$
18. Find the minimum values of $f(x) = 2\log_{10} x - \log_x (0.01),\ x > 1$
19. Find the minimum value of
    $$f(x,y,z) = \frac{(x^2+1)(y^2+1)(z^2+1)}{xyz}, x,y,z > 0.$$
20. Find the minimum value of
    $$f(x,y,z) = \frac{(x^3+2)(y^3+2)(z^3+2)}{xyz}, x,y,z > 0.$$

1.18 SUB–MULTIPLE ANGLES

1.18.1 Definition

An angle is of the form $\frac{A}{n}$, $n \in Z$ ($\neq 0$), is called a sub-multiple angle of A. Thus, $\frac{A}{2}, \frac{A}{3}, \frac{A}{4}, \frac{A}{5}$ etc. are each a sub-multiple angle of A.

1.18.2 *T*-ratios of angle $\left(\frac{A}{2}\right)$ and $\left(\frac{A}{3}\right)$

1. $\sin A = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) = \dfrac{2\tan\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)}$

2. $\cos A = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$

   $= 2\cos^2\left(\frac{A}{2}\right) - 1$

   $= 1 - 2\sin^2\left(\frac{A}{2}\right) = \dfrac{1-\tan^2\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)}$

3. $\tan A = \dfrac{2\tan\left(\frac{A}{2}\right)}{1-\tan^2\left(\frac{A}{2}\right)}$.

4. $\sin A = 3\sin\left(\frac{A}{3}\right) - 4\sin^3\left(\frac{A}{3}\right)$

5. $\cos A = 4\cos^3\left(\frac{A}{3}\right) - 3\cos\left(\frac{A}{3}\right)$

6. $\tan A = \dfrac{3\tan\left(\frac{A}{3}\right) - \tan^3\left(\frac{A}{3}\right)}{1 - 3\tan^2\left(\frac{A}{3}\right)}$

1.18.3 Values of sin 18°, cos 18° and tan 18°

1. $\sin(18°) = \left(\dfrac{\sqrt{5}-1}{4}\right)$

   Proof: Let $A = 18°$

   $\Rightarrow \quad 5A = 90°$

   $\Rightarrow \quad 2A = 90° - 3A$

   $\Rightarrow \quad \sin 2A = \sin(90° - 3A) = \cos 3A$

   $\Rightarrow \quad 2\sin A\cos A = 4\cos^3 A - 3\cos A$

   $\Rightarrow \quad 2\sin A = 4\cos^2 A - 3$

   $\Rightarrow \quad 2\sin A = 4 - 4\sin^2 A - 3 = 1 - 4\sin^2 A$

   $\Rightarrow \quad 4\sin^2 A + 2\sin A - 1 = 0$

   $\Rightarrow \quad \sin A = \dfrac{-2\pm\sqrt{20}}{8} = \dfrac{-2\pm 2\sqrt{5}}{8}$

   $\Rightarrow \quad \sin A = \dfrac{-1\pm\sqrt{5}}{4}$

   $\Rightarrow \quad \sin A = \dfrac{\sqrt{5}-1}{4}, \dfrac{-\sqrt{5}-1}{4}$

   $\Rightarrow \quad \sin(18°) = \dfrac{\sqrt{5}-1}{4}$,

   since 18° lies on the first quadrant.

2. $\cos 18° = \dfrac{1}{4}\sqrt{10+2\sqrt{5}}$

   Proof: We have, cos(18°)

   $= \sqrt{1-\sin^2(18°)}$

   $= \sqrt{1-\left(\dfrac{\sqrt{5}-1}{4}\right)^2}$

   $= \sqrt{1-\left(\dfrac{5+1-2\sqrt{5}}{16}\right)}$

   $= \sqrt{\left(\dfrac{16-5-1+2\sqrt{5}}{16}\right)}$

   $= \dfrac{1}{4}\sqrt{10+2\sqrt{5}}$

3. $\tan 18° = \dfrac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$

   Proof: We have, tan(18°)

   $= \dfrac{\sin(18°)}{\cos(18°)}$

   $= \dfrac{\left(\dfrac{\sqrt{5}-1}{4}\right)}{\dfrac{\sqrt{10+2\sqrt{5}}}{4}}$

   $= \left(\dfrac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}\right)$

Notes:

(i) $\sin 72° = \cos 18° = \frac{1}{4}\sqrt{10+2\sqrt{5}}$

(ii) $\cos 72° = \sin 18° = \frac{\sqrt{5}+1}{4}$

1.18.4 Values of sin 36°, cos 36° and tan 36°

1. $\cos 36° = \frac{\sqrt{5}+1}{4}$

Proof: We have,

$$= \cos(36°)$$
$$= 1-2\sin^2(18°)$$
$$= 1-2\left(\frac{\sqrt{5}-1}{4}\right)^2$$
$$= 1-2\left(\frac{5+1-2\sqrt{5}}{16}\right)$$
$$= \left(\frac{8-5-1+2\sqrt{5}}{8}\right)$$
$$= \left(\frac{2+2\sqrt{5}}{8}\right)$$
$$= \left(\frac{\sqrt{5}+1}{4}\right)$$

2. $\sin 36° = \frac{1}{4}\sqrt{10-2\sqrt{5}}$

Proof: We have, $\sin(36°) = \sqrt{1-\cos^2(36°)}$

$$= \sqrt{1-\left(\frac{\sqrt{5}+1}{4}\right)^2}$$
$$= \sqrt{1-\left(\frac{5+1+2\sqrt{5}}{16}\right)}$$
$$= \sqrt{\left(\frac{16-5-1-2\sqrt{5}}{16}\right)}$$
$$= \sqrt{\left(\frac{10-2\sqrt{5}}{16}\right)}$$
$$= \frac{1}{4}\sqrt{10-2\sqrt{5}}$$

3. $\tan 36° = \frac{1}{4}\times(\sqrt{5}-1)\times\sqrt{10-2\sqrt{5}}$.

Proof: We have $\tan(36°)$

$$= \frac{\sin(36°)}{\cos(36°)}$$
$$= \frac{\frac{\sqrt{10-2\sqrt{5}}}{4}}{\frac{\sqrt{5}+1}{4}}$$
$$= \frac{\sqrt{10-2\sqrt{5}}}{\left(\sqrt{5}+1\right)}$$
$$= \frac{\left(\sqrt{5}-1\right)\times\left(\sqrt{10-2\sqrt{5}}\right)}{4}$$

Notes:

i) $\sin 54° = \cos 36° = \frac{\sqrt{5}+1}{4}$

ii) $\cos 54° = \sin 36° = \frac{1}{4}\sqrt{10-2\sqrt{5}}$.

1.18.5 Some Important Deductions

Deduction 1:

$$\tan\left(7\frac{1°}{2}\right) = \sqrt{6}-\sqrt{4}-\sqrt{3}+\sqrt{2}$$

Proof: As we know that, $\tan\theta = \frac{1-\cos(2\theta)}{\sin(2\theta)}$

Put $\theta = 7\frac{1°}{2}$, then

$$\tan\left(7\frac{1°}{2}\right) = \frac{1-\cos(15°)}{\sin(15°)}$$
$$= \frac{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$
$$= \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$$
$$= \frac{\left(2\sqrt{2}-\sqrt{3}-1\right)\left(\sqrt{3}+1\right)}{\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)}$$
$$= \frac{\left(2\sqrt{2}-\sqrt{3}-1\right)\left(\sqrt{3}+1\right)}{2}$$
$$= \frac{2\sqrt{6}-3-\sqrt{3}+2\sqrt{2}-\sqrt{3}-1}{2}$$

$$= \frac{2\left(\sqrt{6}-\sqrt{4}-\sqrt{3}+\sqrt{2}\right)}{2}$$

$$= \left(\sqrt{6}-\sqrt{4}-\sqrt{3}+\sqrt{2}\right)$$

Deduction 2:

$$\cot\left(7\frac{1^\circ}{2}\right) = \sqrt{6}+\sqrt{4}+\sqrt{3}+\sqrt{2}$$

Proof: As we know that, $\cot(\theta) = \frac{1+\cos(2\theta)}{\sin(2\theta)}$

Put $\theta = 7\frac{1^\circ}{2}$,

Now, $\cot\left(7\frac{1^\circ}{2}\right)$

$$= \frac{1+\cos(15^\circ)}{\sin(15^\circ)}$$

$$= \frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$= \frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{\left(2\sqrt{2}+\sqrt{3}+1\right)\left(\sqrt{3}+1\right)}{\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)}$$

$$= \frac{\left(2\sqrt{6}+3+\sqrt{3}+2\sqrt{2}+\sqrt{3}+1\right)}{2}$$

$$= \frac{2\left(\sqrt{6}+\sqrt{4}+\sqrt{3}+\sqrt{2}\right)}{2}$$

$$= \left(\sqrt{6}+\sqrt{4}+\sqrt{3}+\sqrt{2}\right)$$

Deduction 3:

$$\sin\left(22\frac{1^\circ}{2}\right) = \frac{1}{2}\sqrt{2-\sqrt{2}}$$

Proof: As we know that,

$$2\sin^2(\theta) = 1-\cos 2\theta$$

Put, $\theta = 22\frac{1^\circ}{2}$,

$$2\sin^2\left(22\frac{1^\circ}{2}\right) = 1-\cos(45^\circ)$$

$$= 1-\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$\Rightarrow \sin\left(22\frac{1^\circ}{2}\right) = \pm\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

$$\Rightarrow \sin\left(22\frac{1^\circ}{2}\right) = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$ since, $22\frac{1^\circ}{2}$ lies in the first quadrant.

$$\Rightarrow \sin\left(22\frac{1^\circ}{2}\right) = \frac{1}{2}\sqrt{2-\sqrt{2}}.$$

Deduction 4:

$$\cos\left(22\frac{1^\circ}{2}\right) = \frac{1}{2}\sqrt{2+\sqrt{2}}$$

Proof: As we know that, $2\cos^2(\theta) = 1-\cos 2\theta$

Put, $\theta = 22\frac{1^\circ}{2}$,

$$\Rightarrow 2\cos^2\left(22\frac{1^\circ}{2}\right) = 1+\cos(45^\circ)$$

$$= 1+\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{2}+1}{\sqrt{2}}$$

$$\Rightarrow \cos\left(22\frac{1^\circ}{2}\right) = \pm\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$$

$$\Rightarrow \cos\left(22\frac{1^\circ}{2}\right) = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

Since $22\frac{1^\circ}{2}$ lies in the first quadrant,

thus, $\cos\left(22\frac{1^\circ}{2}\right) = \frac{1}{2}\sqrt{2+\sqrt{2}}$.

Deduction 5:

$$\tan\left(22\frac{1^\circ}{2}\right) = \sqrt{2}-1$$

Proof: As we know that, $\tan\theta = \frac{1-\cos(2\theta)}{\sin(2\theta)}$

Put $\theta = 22\frac{1^\circ}{2}$, $\tan\left(22\frac{1^\circ}{2}\right) = \frac{1-\cos(45^\circ)}{\sin(45^\circ)}$

$$= \frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2}-1$$

Deduction 6:

$$\cot\left(22\frac{1^\circ}{2}\right) = \sqrt{2}+1$$

Proof: As we know that,

$$\cot(\theta) = \frac{1+\cos(2\theta)}{\sin(2\theta)}$$

Put $\theta = 22\frac{1^\circ}{2}$,

$$\Rightarrow \quad \cot\left(22\frac{1^\circ}{2}\right) = \frac{1+\cos(45^\circ)}{\sin(45^\circ)}$$

$$= \frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2}+1$$

Deduction 7:

$$\sin\left(112\frac{1^\circ}{2}\right) = \frac{1}{2}\sqrt{2+\sqrt{2}}$$

Proof: We have, $\sin\left(112\frac{1^\circ}{2}\right)$

$$= \sin\left(90^\circ\times 1 + 22\frac{1^\circ}{2}\right)$$

$$= \cos\left(22\frac{1^\circ}{2}\right)$$

$$= \frac{1}{2}\sqrt{2+\sqrt{2}}$$

Deduction 8:

$$\cos\left(112\frac{1^\circ}{2}\right) = \frac{1}{2}\sqrt{2-\sqrt{2}}$$

Proof: We have, $\cos\left(112\frac{1^\circ}{2}\right)$

$$= \cos\left(90^\circ\times 1 + 22\frac{1^\circ}{2}\right)$$

$$= -\sin\left(22\frac{1^\circ}{2}\right)$$

$$= -\frac{1}{2}\sqrt{2+\sqrt{2}}$$

Deduction 9:

$$\tan\left(112\frac{1^\circ}{2}\right) = -\left(\sqrt{2}+1\right)$$

Proof: We have, $\tan\left(112\frac{1^\circ}{2}\right)$

$$= \tan\left(90^\circ\times 1 + 22\frac{1^\circ}{2}\right)$$

$$= -\cot\left(22\frac{1^\circ}{2}\right)$$

$$= -\left(\sqrt{2}+1\right)$$

1.19 SOME SOLVED EXAMPLES

Ex-1. Prove that: $\sin^2(24^\circ) - \sin^2(6^\circ) = \frac{\left(\sqrt{5}-1\right)}{8}$

Soln. We have, $\sin^2(24^\circ) - \sin^2(6^\circ)$

$$= \sin(24^\circ+6^\circ)\times\sin(24^\circ-6^\circ)$$

$$= \sin(30^\circ)\times\sin(18^\circ)$$

$$= \frac{1}{2}\times\sin(18^\circ)$$

$$= \frac{1}{2}\times\frac{\sqrt{5}-1}{4}$$

$$= \frac{\left(\sqrt{5}-1\right)}{8}$$

Ex-2. Prove that $\sin^2(48^\circ) - \cos^2(12^\circ) = \frac{\left(\sqrt{5}+1\right)}{8}$

Soln. We have, $\sin^2(48^\circ) - \cos^2(12^\circ)$

$$= \cos(48^\circ+12^\circ)\times\cos(48^\circ-12^\circ)$$

$$= \cos(60^\circ)\times\cos(36^\circ)$$

$$= \frac{1}{2}\times\frac{\sqrt{5}+1}{4}$$

$$= \frac{\sqrt{5}+1}{8}$$

Ex-3. Prove that $\sin(12^\circ).\sin(48^\circ).\sin(54^\circ) = \frac{1}{8}$

Soln. We have, $\sin(12^\circ).\sin(48^\circ).\sin(54^\circ)$

$$= \frac{1}{\sin(72^\circ)}\left(\sin(12^\circ).\sin(48^\circ).\sin(72^\circ)\right)\left(\sin(54^\circ)\right)$$

$$= \frac{1}{4\sin(72°)}$$

$$\left(4\sin(60° - 12°).\sin(12°).\sin(60° + 12°)\right)$$

$$\left(\cos(36°)\right)$$

$$= \frac{1}{4\sin(72°)}\left(.\sin(36°).\cos(36°)\right)$$

$$= \frac{1}{8\sin(72°)}(2\sin(36°).\cos(36°))$$

$$= \frac{1}{8\sin(72°)}(\sin(72°))$$

$$= \frac{1}{8}$$

Ex-4. Prove that:

$$\sin(6°).\sin(42°).\sin(66°).\sin(78°) = \frac{1}{16}$$

Soln. We have,

$$\sin(6°).\sin(42°).\sin(66°).\sin(78°)$$

$$= \frac{1}{4\sin(54°)}$$

$$\left(4\sin(6°).\sin(60° - 6°).\sin(60° + 6°).\right)$$

$$\times(\sin(78°).\sin(42°))$$

$$= \frac{1}{4\cos(36°)}(\sin(18°)\sin(72°).\sin(42°))$$

$$= \frac{1}{16\cos(36°)}$$

$$\left(4\sin(18°)\sin(60° + 18°).\sin(60° - 18)\right)$$

$$= \frac{1}{16\cos(36°)}\left(\sin(54°)\right)$$

$$= \frac{1}{16\cos(36°)}\left(\cos(36°)\right)$$

$$= \frac{1}{16}$$

Ex-5. Prove that:

$$4\left(\sin(24°) + \cos(6°)\right) = \left(1 + \sqrt{5}\right)$$

Soln. We have, $4\left(\sin(24°) + \cos(6°)\right)$

$$= 4\left(\sin(24°) + \sin(84°)\right)$$

$$= 4\left(2\sin\left(\frac{24° + 84°}{2}\right)\cos\left(\frac{24° - 84°}{2}\right)\right)$$

$$= 8\left(\sin(54°)\cos(30°)\right)$$

$$= 8\left(\cos(36°)\cos(30°)\right)$$

$$= 8\left(\frac{\sqrt{5}+1}{4} \times \frac{1}{2}\right)$$

$$= \left(\sqrt{5} + 1\right)$$

Ex-6. Prove that:

$$\tan(6°).\tan(42°).\tan(66°).\tan(78°) = 1$$

Soln. We have,

$$\tan(6°).\tan(42°).\tan(66°).\tan(78°)$$

$$= (\tan(6°).\tan(66°)) \times (\tan(42°).\tan(78°))$$

$$= \frac{1}{\tan(54°)}\left(\tan(6°).\tan(54°).\tan(66°)\right)$$

$$\times\left(\tan(42°).\tan(78°)\right)$$

$$= \frac{1}{\tan(54°)}\left(\tan(6°).\tan(60° - 6°).\tan(60° + 6°)\right)$$

$$\times\left(\tan(42°).\tan(78°)\right)$$

$$= \frac{1}{\tan(54°)}(\tan(18°)) \times (\tan(42°).\tan(78°))$$

$$= \frac{1}{\tan(54°)}$$

$$\left(\tan(60° - 18°).\tan(18°).\tan(60° + 18°)\right)$$

$$= \frac{1}{\tan(54°)} \times \left(.\tan(54°)\right)$$

$$= 1$$

Ex-8. Prove that:

$$\left(1 + \cos\left(\frac{\pi}{8}\right)\right).\left(1 + \cos\left(\frac{3\pi}{8}\right)\right)$$

$$= \left(1 + \cos\left(\frac{5\pi}{8}\right)\right).\left(1 + \cos\left(\frac{7\pi}{8}\right)\right) = \frac{1}{8}$$

Soln. We have, $\left(1 + \cos\left(\frac{\pi}{8}\right)\right).\left(1 + \cos\left(\frac{3\pi}{8}\right)\right)$

$$.\left(1+\cos\left(\frac{5\pi}{8}\right)\right).\left(1+\cos\left(\frac{7\pi}{8}\right)\right)$$

$$= \left(1+\cos\left(\frac{\pi}{8}\right)\right).\left(1+\cos\left(\frac{7\pi}{8}\right)\right)$$

$$.\left(1+\cos\left(\frac{3\pi}{8}\right)\right).\left(1+\cos\left(\frac{5\pi}{8}\right)\right)$$

$$= \left(1+\cos\left(\frac{\pi}{8}\right)\right).\left(1-\cos\left(\frac{\pi}{8}\right)\right)$$

$$.\left(1+\cos\left(\frac{3\pi}{8}\right)\right).\left(1-\cos\left(\frac{3\pi}{8}\right)\right)$$

$$= \left(1-\cos^2\left(\frac{\pi}{8}\right)\right).\left(1-\cos^2\left(\frac{3\pi}{8}\right)\right)$$

$$= \left(\sin^2\left(\frac{\pi}{8}\right)\right).\left(\sin^2\left(\frac{3\pi}{8}\right)\right)$$

$$= \frac{1}{4}\left(2\sin^2\left(\frac{\pi}{8}\right)\right).\left(2\sin^2\left(\frac{3\pi}{8}\right)\right)$$

$$= \frac{1}{4}\left(1-\cos\left(\frac{\pi}{4}\right)\right).\left(1-\cos\left(\frac{3\pi}{4}\right)\right)$$

$$= \frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right).\left(1+\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{4}\left(1-\frac{1}{2}\right).$$

$$= \frac{1}{8}.$$

Ex-9. Prove that:

$$\sin^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{3\pi}{8}\right)+\sin^4\left(\frac{5\pi}{8}\right)+\sin^4\left(\frac{7\pi}{8}\right)=\frac{3}{2}$$

Soln. We have,

$$\sin^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{3\pi}{8}\right)+\sin^4\left(\frac{5\pi}{8}\right)+\sin^4\left(\frac{7\pi}{8}\right)$$

$$= \sin^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{3\pi}{8}\right)+\sin^4\left(\frac{3\pi}{8}\right)+\sin^4\left(\frac{\pi}{8}\right)$$

$$= 2\left(\sin^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{3\pi}{8}\right)\right)$$

$$= 2\left(\sin^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)\right)$$

$$= 2\left(1-2\sin^2\left(\frac{\pi}{8}\right).\cos^2\left(\frac{\pi}{8}\right)\right)$$

$$= \left(2-2\sin^2\left(\frac{\pi}{8}\right).2\cos^2\left(\frac{\pi}{8}\right)\right)$$

$$= \left(2-\left(2\sin^2\left(\frac{\pi}{8}\right)\right).\left(2\cos^2\left(\frac{\pi}{8}\right)\right)\right)$$

$$= \left(2-\left(1-\cos\left(\frac{\pi}{4}\right)\right).\left(1+\cos\left(\frac{\pi}{4}\right)\right)\right)$$

$$= \left(2-\left(1-\frac{1}{\sqrt{2}}\right).\left(1+\frac{1}{\sqrt{2}}\right)\right)$$

$$= \left(2-\left(1-\frac{1}{2}\right).\right)$$

$$= \frac{3}{2}$$

Ex-10. Prove that:

$$\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{5\pi}{8}\right)+\cos^4\left(\frac{7\pi}{8}\right)=\frac{3}{2}$$

Soln. We have,

$$\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{5\pi}{8}\right)+\cos^4\left(\frac{7\pi}{8}\right)$$

$$= \cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)$$

$$+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)$$

$$= 2\left(\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)\right)$$

$$= 2\left(\cos^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{\pi}{8}\right)\right)$$

$$= 2\left(1-2\cos^2\left(\frac{\pi}{8}\right).\sin^2\left(\frac{\pi}{8}\right)\right)$$

$$= \left(2-\left(2\cos^2\left(\frac{\pi}{8}\right)\right).\left(2\sin^2\left(\frac{\pi}{8}\right)\right)\right)$$

$$= \left(2-\left(1+\cos\left(\frac{\pi}{4}\right)\right).\left(1+\cos\left(\frac{\pi}{4}\right)\right)\right)$$

$$= \left(2-\left(1+\frac{1}{\sqrt{2}}\right).\left(1-\frac{1}{\sqrt{2}}\right)\right)$$

$$= \left(2-\left(1-\frac{1}{2}\right)\right)$$

$$= \frac{\theta}{2}$$

EXERCISE 9

Q Prove that:

1. $\tan\frac{\theta}{2} + \cot\frac{\theta}{2} = 2\operatorname{cosec}\theta$
2. $\operatorname{cosec}\theta - \cot\theta = \tan\frac{\theta}{2}$
3. $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\frac{\theta}{2}$
4. $\sec\theta + \tan\theta = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$
5. $(\cos A + \cos B)^2 + (\sin A + \sin B)^2$ $= 4\cos^2\left(\frac{A-B}{2}\right)$
6. Prove that: $\sin^2 24 - \sin^2 6 = \frac{\sqrt{5}-1}{8}$.
7. Prove that: $\sin^2 48° - \cos^2 12° = -\frac{\sqrt{5}+1}{8}$.
8. Prove that: $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right) = \frac{1}{8}$
9. Prove that: $\sin 12° \sin 48° \sin 54° = \frac{1}{8}$
10. Prove that: $\sin(47°)+\sin(61°)-\sin(11°)-\sin(25°)=\cos(7°)$
11. Prove that: $\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)=\frac{3}{2}$
12. Prove that: $\sin^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{3\pi}{8}\right)+\sin^4\left(\frac{5\pi}{8}\right)+\sin^4\left(\frac{7\pi}{8}\right)=\frac{3}{2}$
13. Prove that: $\tan 20° \tan 80° = \sqrt{3}\tan 50°$
14. Prove that: $\tan 10° \tan 70° = \sqrt{3}\tan 40°$
15. If $m = \tan 12° \tan 48°$ and $n = \tan 6° \tan 66°$, then find the value of $\left(\frac{m}{n}+10\right)$.
16. If $m = \sin 10° \sin 50° \sin 70°$ and $n = \cos 20° \cos 60° \cos 80°$, then find the value of $\left(\frac{m}{n}+2012.\right)$
17. If $m = \sin 6° \sin 42°$ and $n = \sin 66° \sin 72°$, then find the value of $16mn+10$.
18. If $m = \sin 8° \sin 52° \sin 66° \sin 68°$ and $8n = \cos 42°$, then prove that $m = n$.
19. If $\sin 17° \sin 43° \sin 77° = m\cos 49°$, then find the value of m.
20. If $m = \sin\left(\frac{2\pi}{7}\right)+\sin\left(\frac{4\pi}{7}\right)+\sin\left(\frac{8\pi}{7}\right)$ then find the value of $\left(2m+2-\sqrt{7}\right)$.
21. If $m = \cos 13° \cos 47° \cos 73°$ and $n = \frac{1}{4}\sin 51°$, then prove that $m = n$.
22. Prove that: $\sin 55° - \sin 19° + \sin 53° - \sin 17° = \cos 1°$
23. Prove that: $\cos\frac{2\pi}{7} . \cos\frac{4\pi}{7} . \cos\frac{6\pi}{7} = \frac{1}{8}$.
24. Prove that: $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$
25. Find the value of $\tan\left(7\frac{1°}{2}\right)+\cot\left(7\frac{1°}{2}\right)$.
26. If $\cos\left(\frac{x}{2}\right)-\sqrt{3}\sin\left(\frac{x}{2}\right)$ takes its minimum value then find its x.
27. If α and β be two different roots $a\cos\theta + b\sin\theta = c$, then prove that $\sin(\alpha+\beta) = \frac{2ab}{a^2+b^2}$.

1.20 CONDITIONAL TRIGONOMETRICAL IDENTITIES

Here, we shall deal with trigonometrical identities involving two or more angles. In establishing such identities we will be frequently using properties of supplementary and complementary angles and hence students are advised to go through all the above formulae from Ist topics to previous topic.

We have certain trigonometrical identities like, $\sin 2\theta + \cos 2\theta = 1$ and $\sec 2\theta = 1 + \tan 2\theta$, etc. Such identities are identities in the sense that they hold for all values of the angles which satisfy the given condition amongst them and they are called Conditional Identities.

If A, B, C denote the angles of a triangle ABC, then the relation $A + B + C = \pi$ enables us to establish many important identities involving trigonometric ratio of these angles.

(i) If $A + B + C = \pi$, then $A + B = \pi - C$, $B + C = \pi - A$ and $C + A = \pi - B$

(ii) If $A + B + C = \pi$, then
$\sin(A + B) = \sin(\pi - C) = \sin C$
similarly, $\sin(B - C) = \sin(\pi - A) = \sin A$
and $\sin(C + A) = \sin(\pi - B) = \sin B$

(iii) If $A + B + C = \pi$, then
$\cos(A + B) = \cos(\pi - C) = -\cos C$
Similarly, $\cos(B + C) = \cos(\pi - A) = -\cos A$
and $\cos(C + A) = \cos(\pi - B) = -\cos B$

(iv) If $A + B + C = \pi$, then
$\tan(A + B) = \tan(\pi - C) = -\tan C$
Similarly, $\tan(B + C) = \tan(\pi - A) = -\tan A$
and $\tan(C + A) = \tan(\pi - B) = -\tan B$

v) If $A + B + C = \pi$, then $\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$,

$\frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}$ and $\frac{C+A}{2} = \frac{\pi}{2} - \frac{B}{2}$

Therefore,

$$\sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\left(\frac{C}{2}\right)$$

$$\cos\left(\frac{A+B}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = \sin\left(\frac{C}{2}\right)$$

$$\tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\left(\frac{C}{2}\right).$$

Note: Dear students, please recollect the following formulae from basic trigonometry

Step I:

(i) $\sin 2A = 2\sin A\cos A$
(ii) $\cos 2A = \cos^2 A - \sin^2 A$
(iii) $\cos 2A = 2\cos^2 A - 1$
(iv) $\cos 2A = 1 - 2\sin^2 A$

Step II:

(i) $1 + \cos 2A = 2\cos^2 A$
(ii) $1 - \cos 2A = 2\sin^2 A$
(iii) $1 + \cos A = 2\cos^2\left(\frac{A}{2}\right)$
(iv) $1 - \cos A = 2\sin^2\left(\frac{A}{2}\right)$

Step III:

(i) $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$
(ii) $\cos(A - B) - \cos(A+B) = 2\sin A\sin B$.

Step IV:

(i) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(ii) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

(iii) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(iv) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$.

1.21 SOME SOLVED EXAMPLES

Ex-1. If $A + B + C = \pi$, then prove that,
$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

Soln. We have, $\sin 2A + \sin 2B + \sin 2C$

$$= (\sin 2A + \sin 2B) + \sin 2C$$
$$= 2(\sin(A+B)\cos(A-B)) + \sin 2C$$
$$= 2(\sin(\pi - C).\cos(A-B)) + 2\sin C\cos C$$
$$= 2(\sin C.\cos(A-B)) + 2\sin C\cos C$$
$$= 2\sin C(\cos(A-B) + \cos C)$$
$$= 2\sin C(\cos(A-B) + \cos(\pi - (A+B)))$$
$$= 2\sin C(\cos(A-B) - \cos(A+B))$$
$$= 2\sin C(2\sin A\sin B)$$
$$= 4\sin A.\sin B.\sin C$$

Ex-2. If $A + B + C = \pi$, then prove that,
$\cos 2A + \cos 2B + \cos 2C$
$= -1 - 4\cos A\cos B\cos C$

Soln. We have $\cos 2A + \cos 2B + \cos 2C$

$$= (\cos 2A + \cos 2B) + \cos 2C$$
$$= 2\cos(A+B)\cos(A-B) + \cos 2C$$
$$= 2\cos\{\pi - C\}\cos(A-B) + \cos 2C$$
$$= -2\cos C\cos(A-B) + 2\cos^2 C - 1$$
$$= -1 - 2\cos C(\cos(A-B) - \cos C)$$
$$= -1 - 2\cos C(\cos(A-B) + \cos(A+B))$$
$$= -1 - 2\cos C(2\cos A.\cos B)$$
$$= -1 - 4\cos A.\cos B.\cos C$$

Ex-3. If $A + B + C = \pi$, then prove that
$\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A\sin B\cos C$

Soln. We have, $\sin^2 A + \sin^2 B - \sin^2 C$

$$= \sin^2 A + (\sin^2 B - \sin^2 C)$$
$$= \sin^2 A + \sin(B+C)\sin(B-C)$$
$$= \sin^2 A + \sin(\pi - A)\sin(B-C)$$
$$= \sin^2 A + \sin A\sin(B-C)$$

$= \sin A(\sin A + \sin(B-C))$

$= \sin A(\sin(B+C) + \sin(B-C))$

$= \sin A(2\sin B\cos C)$

$= 2\sin A.\sin B.\cos C$

Ex-4. If $A+B+C=\pi$, then prove that,
$\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A\cos B\cos C$

Soln. We have, $\sin^2 A + \sin^2 B + \sin^2 C$

$= 1 - \cos^2 A + \sin^2 B + \sin^2 C$

$= 1 - (\cos^2 A - \sin^2 B) + (1 - \cos^2 C)$

$= 2 - (\cos^2 A - \sin^2 B) - \cos^2 C$

$= 2 - (\cos(A+B).\cos(A-B)) - \cos^2 C$

$= 2 - (\cos(\pi - C).\cos(A-B)) - \cos^2 C$

$= 2 + \cos C(.\cos(A-B) - \cos C)$

$= 2 + \cos C(.\cos(A-B) + \cos(A+B))$

$= 2 + \cos C(2\cos A.\cos B)$

$= 2 + 2\cos A.\cos B.\cos C$

Ex-5. In a triangle ΔABC, prove that,
$\tan A + \tan B + \tan C = \tan A.\tan B.\tan C$

Soln. We have, $A+B+C=\pi$

$\Rightarrow \quad A+B = \pi - C$

$\Rightarrow \quad \tan(A+B) = \tan(\pi - C)$

$\Rightarrow \quad \dfrac{\tan A + \tan B}{1 - \tan A.\tan B} = -\tan C$

$\Rightarrow \quad \tan A + \tan B = -\tan C(1 - \tan A.\tan B)$

$\Rightarrow \quad \tan A + \tan B = -\tan C + \tan A.\tan B.\tan C$

$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A.\tan B.\tan C$

Ex-6. In a triangle ΔABC, prove that,
$\cot A.\cot B + \cot B.\cot C + \cot C.\cot A = 1$

Soln. As we know that,
$\tan A + \tan B + \tan C = \tan A.\tan B.\tan C$

Dividing both the sides by '$\tan A.\ \tan B.\ \tan C$', we get,

$= \dfrac{\tan A}{\tan A.\tan B.\tan C} + \dfrac{\tan B}{\tan A.\tan B.\tan C}$

$+ \dfrac{\tan C}{\tan A.\tan B.\tan C} = 1$

$\Rightarrow \quad \dfrac{1}{\tan B.\tan C} + \dfrac{1}{\tan A.\tan C} + \dfrac{1}{\tan A.\tan B} = 1$

$\Rightarrow \quad \cot B.\cot C + \cot A.\cot C + \cot A.\cot B = 1$

Ex-7. If A, B, C and D be the angles of a quadrilateral, then prove that,

$\dfrac{\tan A + \tan B + \tan C + \tan D}{\cot A + \cot B + \cot C + \cot D}$
$= \tan A.\tan B.\tan C.\tan D$

Soln. We have, $A+B+C+D = 2\pi$

$\Rightarrow \quad A+B+C+D = 2\pi$

$\Rightarrow \quad A+B = 2\pi - (C+D)$

$\Rightarrow \quad \tan(A+B) = \tan\{2\pi - (C+D)\}$

$\Rightarrow \quad \tan(A+B) = -\tan(C+D)$

$\Rightarrow \quad \dfrac{\tan A + \tan B}{1 - \tan A.\tan B} = -\dfrac{\tan C + \tan D}{1 - \tan C.\tan D}$

$\Rightarrow \quad (\tan A + \tan B)(1 - \tan C.\tan D)$

$= -(\tan C + \tan D)(1 - \tan A.\tan B)$

$\Rightarrow \quad \tan A + \tan B - \tan A.\tan C.\tan D$

$- \tan B.\tan C.\tan D$

$= -\tan C - \tan D + \tan A.\ \tan B\ \tan C$
$+ \tan A.\ \tan B.\ \tan D$

$\Rightarrow \quad \tan A + \tan B + \tan C + \tan D$

$= \tan A\ .\ \tan C.\ \tan D + \tan B.\ \tan C.\ \tan D$
$+ \tan A.\ \tan B.\ \tan D + \tan A.\ \tan B.\ \tan D$

$\Rightarrow \quad \dfrac{\tan A + \tan B + \tan C + \tan D}{\tan A.\tan B.\tan C.\tan D}$

$= \dfrac{\tan A.\tan B.\tan C}{\tan A.\tan B.\tan C.\tan D}$

$+ \dfrac{\tan A.\tan C.\tan D}{\tan A.\tan B.\tan C.\tan D}$

$+ \dfrac{\tan A.\tan B.\tan D}{\tan A.\tan B.\tan C.\tan D}$

$+ \dfrac{\tan B.\tan C.\tan D}{\tan A.\tan B.\tan C.\tan D}$

$\Rightarrow \quad \dfrac{\tan A + \tan B + \tan C + \tan D}{\tan A.\tan B.\tan C.\tan D}$

$= \dfrac{1}{\tan A} + \dfrac{1}{\tan B} + \dfrac{1}{\tan C} + \dfrac{1}{\tan D}$

$\Rightarrow \quad \dfrac{\tan A + \tan B + \tan C + \tan D}{\tan A.\tan B.\tan C.\tan D}$

$= \cot A + \cot B + \cot C + \cot D$

$$\Rightarrow \quad \frac{\tan A + \tan B + \tan C + \tan D}{\cot A + \cot B + \cot C + \cot D}$$

$$\tan A . \tan B . \tan C . \tan D$$

Ex-8. In a triangle ABC, prove that,

$$(\cot A + \cot B)(\cot B + \cot C)$$

$$(\cot C + \cot A) = \text{cosec } A \text{ cosec } B \text{ cosec } C.$$

Soln. We have, $(\cot A + \cot B) = \dfrac{\cos A}{\sin A} + \dfrac{\cos B}{\sin B}$

$$= \frac{\cos A \sin B + \sin A. \ \cos B}{\sin A \sin B}$$

$$= \frac{\sin(A+B)}{\sin A \sin B} = \frac{\sin C}{\sin A \sin B}.$$

Similarly, $(\cot B + \cot C) = \dfrac{\sin A}{\sin B \sin C}$

and $(\cot C + \cot A) = \dfrac{\sin B}{\sin A \sin C}$

Thus, $(\cot A + \cot B)(\cot B + \cot C)$

$(\cot C + \cot A)$

$$= \frac{\sin C}{\sin A \sin B} \times \frac{\sin B}{\sin A \sin C} \times \frac{\sin A}{\sin B \sin C}$$

$$= \frac{1}{\sin A \sin B \ \sin C}$$

$$= \frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} = \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

Ex-9. If $x y + y z + z x = 1$, then prove that,

$$\frac{x}{1\text{-}x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} = \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

Soln. Put $x = \tan A$, $y = \tan B$ and $z = \tan C$
Given, $x y + y z + z x = 1$

$$\Rightarrow \quad \tan A.\tan B + \tan B.\tan C + \tan C.\tan A = 1$$

$$\Rightarrow \quad \tan B.\tan C + \tan C.\tan A = 1 \text{ - } \tan A.\tan B$$

$$\Rightarrow \quad \tan C(\tan B + \tan A) = 1 - \tan A.\tan B$$

$$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A.\tan B} = \frac{1}{\tan C}$$

$$\Rightarrow \quad \tan(A+B) = \cot C = \tan\left(\frac{\pi}{2} - C\right)$$

$$\Rightarrow \quad (A+B) = \left(\frac{\pi}{2} - C\right)$$

$$\Rightarrow \quad (A+B+C) = \frac{\pi}{2}$$

Now,

$$\text{L.H.S} = \frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2}$$

$$= \frac{\tan A}{1-\tan^2 A} + \frac{\tan B}{1-\tan^2 B} + \frac{\tan C}{1-\tan^2 C}$$

$$= \frac{1}{2}\left(\frac{2\tan A}{1-\tan^2 A} + \frac{2\tan B}{1-\tan^2 B} + \frac{2\tan C}{1-\tan^2 C}\right)$$

$$= \frac{1}{2}(\tan 2A + \tan 2B + \tan 2C)$$

$$= \frac{1}{2}(\tan 2A.\tan 2B.\tan 2C)$$

$$= \frac{1}{2}\left(\frac{2\tan A}{1-\tan^2 A} . \frac{2\tan B}{1-\tan^2 B} . \frac{2\tan C}{1-\tan^2 C}\right)$$

$$= \left(\frac{4\tan A.\tan B.\tan C}{(1-\tan^2 A)(1-\tan^2 B)(1-\tan^2 C)}\right)$$

$$= \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

Hence, the result.

Ex-10. If $xy + yz + zx = 1$, then prove that,

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} = \frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}}$$

Soln. Put $x = \tan A$, $y = \tan B$ and $z = \tan C$
Given, $x y + y z + z x = 1$

$$\Rightarrow \tan A.\tan B + \tan B.\tan C + \tan C.\tan A = 1$$

$$\Rightarrow \quad A + B + C = \frac{\pi}{2}$$

Now, L.H.S

$$= \frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2}$$

$$= \frac{1}{2}\left(\frac{2x}{1+x^2} + \frac{2y}{1+y^2} + \frac{2z}{1+z^2}\right)$$

$$= \frac{1}{2}\left(\frac{2\tan A}{1+\tan^2 A} + \frac{2\tan B}{1+\tan^2 B} + \frac{2\tan C}{1+\tan^2 C}\right)$$

$$= \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C)$$

$$= \frac{1}{2}(4\cos A.\cos B.\cos C)$$

$$= 2\cos A.\cos B.\cos C$$

$$= \frac{2}{\sec A.\sec B.\sec C}$$

$$= \frac{2}{\sqrt{(1+\tan^2 A)(1+\tan^2 B)(1+\tan^2 C)}}$$

$$= \frac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}}$$

Hence, the result.

EXERCISE 10

1. If $A + B + C = \pi$, then prove that: $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.
2. If $A + B + C = \pi$, then prove that: $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$.
3. If $A + B + C = \pi$, then prove that: $\cos 2A + \cos 2B + \cos 2C = 1 - 4\cos A \cos B \cos C$
4. If $A + B + C = \pi$, then prove that: $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C$.
5. If $A + B + C = \pi$, then prove that: $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
6. If $A + B + C = \pi$, then prove that: $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$
7. If $A + B + C = \pi$, then prove that: $\cos 2A + \cos 2B + \cos 2C = 1 - 2 \sin A \sin B \cos C$.
8. If $A + B + C = \pi$, then prove that $\sin 2A + \sin 2B + \sin 2C = 2 (1 + \cos A \cos B \cos C)$
9. If $A + B + C = \pi$, then prove that: $\frac{\sin 2A + \sin 2B + \sin 2C}{\cos A + \cos B + \cos C - 1} = 8 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
10. If $A + B + C = \pi$, then prove that: $\sin^2\left(\frac{A}{2}\right) + \sin^2\left(\frac{B}{2}\right) - \sin^2\left(\frac{C}{2}\right) = 1 - 2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$

Q. If $A + B + C = \frac{\pi}{2}$, then prove that:

11. $\sin 2A + \sin 2B + \sin 2C = 1 - 2 \sin A \sin B \sin C$
12. $\cos 2A + \cos 2B + \cos 2C = 2 + 2 \sin A \sin B \sin C$

Q. If $A + B + C = \pi$, then prove that:

13. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.
14. $\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$.
15. $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$
16. $\cot (A/2) \cot (B/2) + \cot (B/2) \cot (C/2) + \cot (C/2) \cot (A/2) = 1$.

Q. If $A + B + C = \frac{\pi}{2}$, then prove that:

17. $\cot A + \cot B + \cot C = \cot A \cot B \cot C$
18. $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$
19. If $xy + yz + zx = 1$, then prove that:
$$\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} = \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$
20. Prove that:
$$\tan(\alpha-\beta) + \tan(\beta-\gamma) + \tan(\gamma-\alpha) = \tan(\alpha-\beta)\tan(\beta-\gamma)\tan(\gamma-\alpha)$$
21. In a ΔABC, if $\cot A + \cot B + \cot C = \sqrt{3}$, then prove that the triangle is an equilateral.
22. If $x + y + z = x y z$, then prove that:
$$\frac{3x-x^3}{1-3x^2} + \frac{3y-y^3}{1-3y^2} + \frac{3z-z^3}{1-3z^2} = \frac{3x-x^3}{1-3x^2}.\frac{3y-y^3}{1-3y^2}.\frac{3z-z^3}{1-3z^2}$$
23. Prove that $1 + \cos 56° + \cos 58° - \cos 66° = 4\cos 28° \cos 29° \sin 33°$.

1.22 TRIGONOMETRICAL SERIES

1.21.1 Introduction

In this section, we are mainly connected with different procedures to find out the summation of trigonometrical series.

To find out the sum of diffrent trigonometrical series, first we observe the nature of the angles of the trigonometrical terms. We must observe whether the angles form any sequence or not? If they form any sequences, then we must check, what kind of sequence it is? We also bserve the sequence formed (if any) by the coefficients of erms of the series.

So, our main attempt will be

(i) to express each term as a diffrence of the two terms directly or by manupulation and then add or

(ii) to arrange the series in such a way that it follows some standard trigonometrical expansion.

1.23 DIFFERENT TYPES OF THE SUMMATION OF A TRIGONOMETRICAL SERIES

1. A trigonometrical series involved with the terms of sines or cosines.

Rule: Whenever angles are in A.P. and the trigonometrical terms involved with sines or cosines having power 1.

1. We must multiply each term by
$$2\sin\left(\frac{\text{common diffrence of angles}}{2}\right)$$
2. and then express each term as a diffrenec of two terms.
3. Finally add them.

Ex.-1 Prove that:

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)$$
$$+\sin(\alpha+3\beta)+.........+\sin(\alpha+(n-1)\beta)$$
$$=\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\times\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]$$

Proof: Let $S=\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)$
$$+\sin(\alpha+3\beta)+.........+\sin(\alpha+(n-1)\beta)$$

Now,

$$2\sin\alpha\sin\left(\frac{\beta}{2}\right)=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{\beta}{2}\right)$$
$$2\sin(\alpha+\beta).\sin\left(\frac{\beta}{2}\right)=\cos\left(\alpha+\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{3\beta}{2}\right)$$
$$2\sin(\alpha+2\beta).\sin\left(\frac{\beta}{2}\right)=\cos\left(\alpha+\frac{3\beta}{2}\right)-\cos\left(\alpha+\frac{5\beta}{2}\right)$$

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Adding, we get

$$2\sin(\alpha+(n-1)\beta).\sin\left(\frac{\beta}{2}\right)$$
$$=\cos\left(\alpha+\frac{(2n-3)\beta}{2}\right)-\cos\left(\alpha+\frac{(2n-1)\beta}{2}\right)$$
$$=2\sin\left(\alpha+\frac{(n-1)\beta}{2}\right)\times\sin\left(\frac{\beta}{2}\right)$$

Thus $S=\dfrac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\times\sin\left(\alpha+\frac{(n-1)\beta}{2}\right)$

EXERCISE 11

1. Prove that: $\sin\alpha+\sin2\alpha+\sin3\alpha+..........+\sin n\alpha$
$$=\frac{\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\times\sin(n+1)\left(\frac{\alpha}{2}\right)$$
2. Prove that:
$$\sin\theta+\sin3\theta+......+\sin(2n-1)\theta=\frac{\sin^2 n\theta}{\sin\theta}$$
3. Prove that: $\sin^2\alpha+\sin^2(\alpha+\beta)+\sin^2(\alpha+2\beta)+...$
$......+\sin^2(\alpha+(n-1)\beta)$
$$=\frac{n}{2}-\frac{1}{2}\times\frac{\sin n\beta}{\sin\beta}\times\cos[2\alpha+(n-1)\beta]$$
4. Prove that: $\sin^2\alpha+\sin^2\left(\alpha+\frac{2\pi}{n}\right)$
$$+\sin^2\left(\alpha+\frac{2\pi}{n}\right)+.....\text{ to}-n-\text{terms}=\frac{n}{2}$$
5. If $4n\theta=\pi$, then prove that:
$\sin^2\theta+\sin^2 3\theta+\sin^2 5\theta+....to-n-\text{terms}=n$
6. Prove that:
$\sin^3\theta+\sin^3 3\theta+\sin^3 5\theta+....to-n-\text{terms}$
$$=\frac{1}{4}\left[\frac{3\sin^2 n\theta}{\sin\theta}-\frac{3\sin^2 3n\theta}{\sin3\theta}\right].$$
7. Prove that: $\sin\alpha+\cos(\alpha+\beta)$
$-\sin(\alpha+2)\beta\;-\cos(\alpha+2\beta)$
$+\sin(\alpha+4\beta)+....to-n-terms$
$$=\frac{\sin\left(\frac{n(2\beta+\pi)}{4}\right)}{\sin\left(\frac{2\beta+\pi}{4}\right)}\times\sin\left\{\alpha+\frac{(n-1)(2\beta+\pi)}{4}\right\}$$

8. Prove that:

$$\sin^3\alpha+\sin^3\left(\alpha+\frac{2\pi}{n}\right)+\sin^3\left(\alpha+\frac{4\pi}{n}\right)$$
$$+.........to-n-\text{terms}=0$$

9. Prove that:

$$\sin\theta.\sin 2\theta+\sin 2\theta\ .\sin 3\theta$$
$$+\sin 3\theta\ \ .\sin 4\theta+.....to-n-\text{terms}$$
$$=\frac{n}{2}\cos\theta-\frac{1}{2}\times\frac{\sin n\theta}{\sin\theta}\times\cos(n+2)\theta$$

10. Prove that: $\sin^4 x+\sin^4\left(x+\frac{2\pi}{n}\right)$

$$+\sin^4\left(x+\frac{2\pi}{n}\right)+.....\text{to}-n-\text{terms}=\frac{3\pi}{8}$$

2. A Trigonometrical series involved with the terms of cosines:

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+\beta)$$
$$+\cos(\alpha+\beta)+....+\cos(\alpha+(n-1)\beta)$$
$$=\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\times\cos\left(\alpha+(n-1)\frac{\beta}{2}\right)$$

Proof: Let $S=\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+\beta)$
$+\cos(\alpha+\beta)+....+\cos(\alpha+(n-1)\beta)$

Now, $2\cos\alpha\ \sin\left(\frac{\beta}{2}\right)$

$$=\sin\left(\alpha+\frac{\beta}{2}\right)-\sin\left(\alpha-\frac{\beta}{2}\right)$$

$$2\cos(\alpha+\beta)\sin\left(\frac{\beta}{2}\right)$$

$$=\sin\left(\alpha+\frac{3\beta}{2}\right)-\sin\left(\alpha+\frac{\beta}{2}\right)$$

$$2\cos(\alpha+3\beta)\sin\left(\frac{\beta}{2}\right)$$

$$=\sin\left(\alpha+\frac{5\beta}{2}\right)-\sin\left(\alpha+\frac{3\beta}{2}\right)$$

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$$2\cos(\alpha+(n-1)\beta)\sin\left(\frac{\beta}{2}\right)$$

$$=\sin\left(\alpha+\frac{(2n-1)\beta}{2}\right)-\sin\left(\alpha+\frac{(2n-3)\beta}{2}\right)$$

Adding all, we get,

$$2\sin\left(\frac{\beta}{2}\right)\times S=\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin(\alpha-\beta)$$

$$\Rightarrow\quad 2\sin\left(\frac{\beta}{2}\right)\times S$$

$$=2\cos\left(\alpha+\frac{n-1}{2}\beta\right)\times\sin\left(\frac{n\beta}{2}\right)$$

$$\Rightarrow\quad S=\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\times 2\cos\left(\alpha+\frac{n-1}{2}\beta\right)$$

EXERCISE 12

1. Prove that, $\cos\alpha+\cos 2\alpha+\cos 3\alpha+.......$

$$....+\cos n\alpha=\frac{\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\times\cos\left(\frac{(n+1)\alpha}{2}\right)$$

2. Prove that, $\cos\alpha+\cos\left(\alpha+\frac{2\pi}{n}\right)$

$$+\cos\left(\alpha+\frac{2\pi}{n}\right)+....\text{to}-n-\text{terms}$$
$$=0$$

3. If B be the exist for angles of a regular polygon of n-sides and A is any constant, then prove that cos A + cos (A+B) + cos (A+2B) +.............to n terms = 0.
4. Show that, if $n > 2$,

$$\cos^2\left(\frac{\pi}{n}\right)+\cos^2\left(\frac{3\pi}{n}\right)+\cos^2\left(\frac{5\pi}{n}\right)$$
$$+.........\text{to}-n-\text{terms}=\frac{n}{2}$$

5. Prove that, $\sqrt{1-\sin 2\theta}+\sqrt{1-\sin 4\theta}$

$$+\sqrt{1-\sin 6\theta}+......\text{to}-n-\text{terms}$$
$$=\frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\times\left[\cos(n+1)\frac{\theta}{2}-\sin(n+1)\frac{\theta}{2}\right].$$

6. Prove that, $\cos^2 x+\cos^2 3x+\cos^2 5x+...$ ....to $-n-$ terms
$$=\frac{1}{2}\times\left[n+\frac{\sin 4nx}{2\sin 2x}\right].$$
7. Prove that, cos x + sin $3x$ + cos $5x$ + sin $7x$ + ...+ sin $(2n+1)x$
$$=\frac{\sin 2nx}{\sin 2x}\times[\cos(2n-1)x+\sin(2n+1)x].$$
8. Prove that, $\cos^3\theta+\cos^3\left(\theta-\frac{2\pi}{n}\right)+\cos^3\left(\theta-\frac{4\pi}{n}\right)+\cos^3\left(\theta-\frac{6\pi}{n}\right)$ +....to $-n-$ terms $=0$
9. Prove that, $\cos^2\alpha+\cos^2(\alpha+\beta)+\cos^2(\alpha+2\beta)+$.......to $-n-$ terms
$$=\frac{n}{2}+\frac{1}{2}\times\frac{\sin n\beta}{\sin\beta}\times\cos[2\alpha+(n-1)\beta]$$
10. Prove that, $\cos\theta\cos 2\theta+\cos 3\theta\ \cos 4\theta+\cos 5\theta\ \cos 6\theta+$......to $-n-$ terms
$$=\frac{1}{2}\times\left[n\cos\theta+\frac{\sin 2n\theta}{\sin 2\theta}\times\cos(2n+1)\theta\right].$$

3. A Trigonomeytrical Series Based Method of difference:

Rules: 1. Express each term of the series as a diffreence of two expressions.
2. Finally added them and we shall get the required result.

Ex-1. Find the sum of n-terms of the series
$$\frac{\sin x}{\sin 2x.\sin 3x}+\frac{\sin x}{\sin 3x.\sin 4x}+\frac{\sin x}{\sin 4x.\sin 5x}+....\text{to}-n-\text{terms}$$

Soln. Let $t_n=\frac{\sin x}{\sin(n+1)x.\sin(n+2)x}$
$$\Rightarrow\quad t_n=\frac{\sin[(n+2)x-(n+1)x]}{\sin(n+1)x.\sin(n+2)x}$$
$$=\frac{\sin(n+2)x\ \cos(n+1)x}{\sin(n+1)x.\ \sin(n+2)x}-\frac{\cos(n+2)x\ \sin(n+1)x}{\sin(n+1)x.\ \sin(n+2)x}$$
$$=\cot(n+1)x-\cot(n+2)x$$

Thus, $t1$ = cot $2x$ – cot $3x$

$t2$ = cot$3x$ – cot 4 x

$t3$ = cot $4x$ – cot 5 x

....

....

tn = cot$(n+1)x$ – cot$(n+2)x$.

Adding all,

we get, S = cot $2x$ – cot $(n+2)$ x.

EXERCISE 13

1. Prove that,
$\operatorname{cosec}\theta.\operatorname{cosec}2\theta+\operatorname{cosec}2\theta.\operatorname{cosec}3\theta+\operatorname{cosec}3\theta.\operatorname{cosec}4\theta+$....to $-n-$ terms $=\operatorname{cosec}\theta\,[\cot\theta-\cot(n+1)\theta]$
2. Prove that, $\frac{1}{\cos\theta+\cos 2\theta}+\frac{1}{\cos\theta+\cos 5\theta}+\frac{1}{\cos\theta+\cos 7\theta}+$....to $-n-$ terms
$$=\operatorname{cosec}\theta\left[\tan(n+1)\theta-\tan\theta\right]$$
3. Prove that, $\frac{\sin x}{\cos x+\cos 2x}+\frac{\sin x}{\cos x+\cos 4x}+\frac{\sin x}{\cos x+\cos 6x}+$.....to $-n-$ terms.
$$=\frac{1}{4}\times\operatorname{cosec}\left(\frac{n}{2}\right)\times\left[\sec(2n+1)\frac{x}{2}-\sec\left(\frac{x}{2}\right)\right].$$
4. Prove that, $\operatorname{cosec}\theta+\operatorname{cosec}2\theta+\operatorname{cosec}2^2\theta+\operatorname{cosec}2^4\theta+$.....to $-n-$ terms
$$=\cot\left(\frac{\theta}{2}\right)-\cot\left(2^{n-1}\theta\right).$$
5. Prove that, $\operatorname{cosec}\theta+\operatorname{cosec}\left(\frac{\theta}{2}\right)+\operatorname{cosec}\left(\frac{\theta}{2^2}\right)+\operatorname{cosec}\left(\frac{\theta}{2^3}\right)+$ to $-n-$ terms
$$=\cot\left(\frac{\theta}{2^n}\right)-\cot\theta.$$
6. Prove that,
tan x . tan $2x$ + tan $2x$. tan $3x$

+ ....+ tan nx . tan $(n + 1)x$
= cot x [tan $(n + 1)$ x – tan x] – n .

7. Prove that, $\tan^{-1}\left(\frac{x}{1+2x^2}\right)+\tan^{-1}\left(\frac{x}{1+6x^2}\right)$

$+\tan^{-1}\left(\frac{x}{1+12x^2}\right)$ +..............

..........+ $\tan^{-1}\left(\frac{x}{1+n(n+1)x^2}\right)$

$= \tan^{-1}(n+1)x-\tan^{-1}x$

8. Prove that, $\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\tan^{-1}\left(\frac{1}{19}\right)$

$+.....+\tan^{-1}\left(\frac{1}{n^2+3n+3}\right)$

$= \tan^{-1}(n+2)-\tan^{-1}2$.

9. Prove that, $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)$

$+\sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}\right)+......\text{to}\infty = \frac{\pi}{2}$.

10. Prove that, $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{2}{9}\right)$

$+\tan^{-1}\left(\frac{4}{23}\right)+......\text{to}\infty = \frac{\pi}{4}$.

PROBLEMS FOR JEE ADVANCED EXAM

Ex-1 Prove that,

$$\tan\alpha+2\tan2\alpha+4\tan4\alpha+8\cot8\alpha=\cot\alpha$$

Soln. We have

$$\tan\alpha+2\tan2\alpha+4\tan4\alpha+8\cot8\alpha$$
$$=\cot\alpha-(\cot\alpha-\tan\alpha)+2\tan2\alpha+4\tan4\alpha+8\cot8\alpha$$
$$=\cot\alpha-2(\cot2\alpha-\tan2\alpha)+4\tan4\alpha+8\cot8\alpha$$
$$=\cot\alpha-2.2\cot4\alpha+4\tan4\alpha+8\cot8\alpha$$
$$=\cot\alpha-4(\cot4\alpha-\tan4\alpha)+8\cot8\alpha$$
$$=\cot\alpha-4.2\cot8\alpha+8\cot8\alpha$$
$$=\cot\alpha-8\cot8\alpha+8\cot8\alpha$$
$$=\cot\alpha$$

Ex-2 Prove that,

$$\tan9°-\tan27°-\tan63°+\tan81°=4$$

Soln. We have $\tan9°-\tan27°-\tan63°+\tan81°$

$$=(\tan9°+\tan81°)-(\tan27°+\tan63°)$$
$$=(\tan9°+\cot9°)-(\tan27°+\cot27°)$$
$$=\left(\frac{1}{\sin9°\cos9°}\right)-\left(\frac{1}{\sin27°\cos27°}\right)$$
$$=\left(\frac{2}{2\sin9°\cos9°}\right)-\left(\frac{2}{2\sin27°\cos27°}\right)$$
$$=\left(\frac{2}{\sin18°}-\frac{2}{\sin54°}\right)$$
$$=\left(\frac{2}{\sin18°}-\frac{2}{\cos36°}\right)$$
$$=\left(\frac{8(\sqrt{5}+1-\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}\right)$$
$$=\left(\frac{8(\sqrt{5}+1-\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}\right)$$
$$=\left(\frac{8\times2}{4}\right)=4$$

Ex-3 Prove that:

$$\frac{\sin x}{\cos3x}+\frac{\sin3x}{\cos9x}+\frac{\sin9x}{\cos27x}=\frac{1}{2}(\tan27x-\tan x)$$

Soln. We have

$$\frac{\sin x}{\cos3x}+\frac{\sin3x}{\cos9x}+\frac{\sin9x}{\cos27x}$$
$$=\frac{1}{2}\left(\frac{2\sin x\cos x}{\cos3x\cos x}+\frac{2\sin3x\cos3x}{\cos3x\cos9x}+\frac{2\sin9x\cos9x}{\cos27x\cos9x}\right)$$
$$=\frac{1}{2}\left(\frac{\sin2x}{\cos3x\cos x}+\frac{\sin6x}{\cos3x\cos9x}+\frac{\sin18x}{\cos9x\cos27x}\right)$$
$$=\frac{1}{2}\left(\frac{\sin(3x-x)}{\cos3x\cos x}+\frac{\sin(9x-3x)}{\cos3x\cos9x}+\frac{\sin(27x-9x)}{\cos9x\cos27x}\right)$$
$$=\frac{1}{2}\left(\frac{\sin3x\cos x-\cos3x\sin x}{\cos3x\cos x}+\frac{\sin9x\cos3x-\cos9x\sin3x}{\cos3x\cos9x}\right.$$

$$+\frac{\sin 27x\cos 9x - \cos 27x\sin 9x}{\cos 9x\cos 27x}\Bigg)$$

$$= \frac{1}{2}(\tan 3x - \tan x + \tan 9x - \tan 3x$$

$$+\tan 27x - \tan 9x)$$

$$= \frac{1}{2}(\tan 27x - \tan x)$$

Ex-4. If $\frac{\sin x}{\sin y} = \frac{1}{2}$, $\frac{\cos x}{\cos y} = \frac{3}{2}$, where $x, y \in R$ then find the value of $\tan(x+y)$.

Soln. We have, $\frac{\sin x}{\sin y} = \frac{1}{2}$, $\frac{\cos x}{\cos y} = \frac{3}{2}$

$$\frac{\tan x}{\tan y} = \frac{1}{3}$$

Now $\tan(x+y)$

$$= \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$= \frac{\tan x + 3\tan x}{1 - \tan x . 3\tan x}$$

$$= \frac{4\tan x}{1 - 3\tan^2 x} \quad \text{......(i)}$$

Also, $\sin y = 2\sin x, \cos y = \frac{2}{3}\cos x$

$$\sin^2 y + \cos^2 y$$

$$= 4\sin^2 x + \frac{4}{9}\cos^2 x$$

$$= \frac{36\sin^2 x + 4\cos^2 x}{9}$$

$$= \frac{32\sin^2 x + 4}{9}$$

$$\Rightarrow \frac{32\sin^2 x + 4}{9} = 1$$

$$\Rightarrow 32\sin^2 x + 4 = 9$$

$$\Rightarrow 32\sin^2 x = 5$$

$$\Rightarrow \sin^2 x = \frac{5}{32}$$

$$\Rightarrow \sin x = \frac{\sqrt{5}}{4\sqrt{2}}$$

$$\Rightarrow \tan x = \frac{\sqrt{5}}{3\sqrt{3}} \quad \text{........(ii)}$$

From (i) and (ii), we get,

$$\tan(x+y) = \frac{\frac{4\sqrt{5}}{3\sqrt{3}}}{1 - \frac{15}{27}} = \frac{4\sqrt{5}\times 27}{12\times 3\sqrt{3}} = \sqrt{15}$$

Ex-5 Prove that,

$$\sin^4\left(\frac{\pi}{16}\right) + \sin^4\left(\frac{3\pi}{16}\right) + \sin^4\left(\frac{5\pi}{16}\right) + \sin^4\left(\frac{7\pi}{16}\right) = \frac{3}{2}$$

Soln. We have,

$$\sin^4\left(\frac{\pi}{16}\right) + \sin^4\left(\frac{3\pi}{16}\right) + \sin^4\left(\frac{5\pi}{16}\right) + \sin^4\left(\frac{7\pi}{16}\right)$$

$$= \sin^4\left(\frac{\pi}{16}\right) + \sin^4\left(\frac{3\pi}{16}\right)$$

$$+\sin^4\left(\frac{\pi}{2} - \frac{3\pi}{16}\right) + \sin^4\left(\frac{\pi}{2} - \frac{\pi}{16}\right)$$

$$= \sin^4\left(\frac{\pi}{16}\right) + \sin^4\left(\frac{3\pi}{16}\right) + \cos^4\left(\frac{3\pi}{16}\right) + \cos^4\left(\frac{\pi}{16}\right)$$

$$= \left(\sin^4\left(\frac{\pi}{16}\right) + \cos^4\left(\frac{\pi}{16}\right)\right) + \left(\sin^4\left(\frac{3\pi}{16}\right) + \cos^4\left(\frac{3\pi}{16}\right)\right)$$

$$= 2 - 2\sin^2\left(\frac{\pi}{16}\right).\cos^2\left(\frac{\pi}{16}\right) - 2\sin^2\left(\frac{3\pi}{16}\right).\cos^2\left(\frac{3\pi}{16}\right)$$

$$= 2 - \frac{1}{2}\left(\left(2\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{16}\right)\right)^2 + \left(2\sin\left(\frac{3\pi}{16}\right)\cos\left(\frac{3\pi}{16}\right)\right)^2\right)$$

$$= 2 - \frac{1}{2}\left(\sin^2\left(\frac{\pi}{8}\right) + \sin^2\left(\frac{3\pi}{8}\right)\right)$$

$$= 2 - \frac{1}{2}\left(\sin^2\left(\frac{\pi}{8}\right) + \sin^2\left(\frac{\pi}{2} - \frac{\pi}{8}\right)\right)$$

$$= 2 - \frac{1}{2}\left(\sin^2\left(\frac{\pi}{8}\right) + \cos^2\left(\frac{\pi}{8}\right)\right)$$

$$= 2 - \frac{1}{2}$$

$$= \frac{3}{2}.$$

Ex-6 If $\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = -\frac{3}{2}$, then prove that, $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$

Soln. We have,

$$\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = -\frac{3}{2}$$

$\Rightarrow 2(\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha))+3=0$

$\Rightarrow 2(\cos\alpha\,\cos\beta+\cos\beta\,\cos\gamma+\cos\gamma\,\cos\alpha)$

$+\ 2(\sin\alpha\sin\beta+\sin\beta\sin\gamma+\sin\gamma\sin\alpha)+3=0$

$\Rightarrow (\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)$

$+(\cos^2\gamma+\sin^2\gamma)$

$2(\cos\alpha\,\cos\beta+\cos\beta\,\cos\gamma+\cos\gamma\,\cos\alpha)$

$+\ 2(\sin\alpha\sin\beta+\sin\beta\sin\gamma+\sin\gamma\sin\alpha)=0$

$\Rightarrow \cos^2\alpha+\cos^2\beta+\cos^2\gamma$

$+2\cos\alpha\cos\beta+2\cos\beta\cos\gamma$

$2\cos\gamma\cos\alpha\ +\ \sin^2\alpha+\sin^2\beta+\sin^2\gamma$

$+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma$

$+2\sin\gamma\ \sin\alpha=0$

$\Rightarrow (\cos\alpha+\cos\beta+\cos\gamma)^2$

$+\ (\sin\alpha+\sin\beta+\sin\gamma)^2=0$

$\Rightarrow (\cos\alpha+\cos\beta+\cos\gamma)=0$

and $(\sin\alpha+\sin\beta+\sin\gamma)=0$

Ex-7. If $\sin\alpha\sin\beta-\cos\alpha\cos\beta+1=0$, then prove that $1+\cot\alpha\tan\beta=0$

Soln. We have, $\sin\alpha\sin\beta-\cos\alpha\cos\beta+1=0$

$\Rightarrow \quad \cos\alpha\cos\beta-\sin\alpha\sin\beta=1$

$\Rightarrow \quad \cos(\alpha+\beta)=1$

Therefore,

$\sin(\alpha+\beta)=\sqrt{1-\cos^2(\alpha+\beta)}=\sqrt{1-1}=0$

Now, $1+\cot\alpha\,.\tan\beta$

$=1+\dfrac{\cos\alpha\,.\sin\beta}{\sin\alpha\,.\cos\beta}$

$=\dfrac{\sin\alpha\,.\cos\beta+\cos\alpha\,.\sin\beta}{\sin\alpha\,.\cos\beta}$

$=\dfrac{\sin(\alpha+\beta)}{\sin\alpha\,.\cos\beta}$

$=0.$

Ex-8: If $\alpha+\beta=90°$ and $\beta+\gamma=\alpha$, then prove that $\tan\alpha=\tan\beta+2\tan\gamma$

Soln. Given $\beta+\gamma=\alpha$

$\Rightarrow \quad \tan(\beta+\gamma)=\tan\alpha$

$\Rightarrow \quad \tan(\beta+\gamma)=\tan(90°-\alpha)$

$\Rightarrow \quad \tan(\beta+\gamma)=\cot\alpha$

$\Rightarrow \quad \dfrac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}=\cot\alpha$

$\Rightarrow \quad \dfrac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}=\cot\beta$

$\Rightarrow \quad \tan\beta+\tan\gamma=\cot\beta-\tan\beta.\cot\beta.\tan\gamma$

$\Rightarrow \quad \tan\beta+\tan\gamma=\cot\beta-\tan\gamma$

$\Rightarrow \quad \tan\beta+\tan\gamma=\cot(90°-\alpha)-\tan\gamma$

$\Rightarrow \quad \tan\beta+\tan\gamma=\tan\alpha-\tan\gamma$

$\Rightarrow \quad \tan\alpha=\tan\beta+2\tan\gamma$

Ex-9 If $\tan\left(\dfrac{\pi}{24}\right)=(\sqrt{a}-\sqrt{b})(\sqrt{c}-\sqrt{d})$, where a, b, c, d are positive integers, then find the value of $(a+b+c+d+2)$

Soln. We have $\tan\left(\dfrac{\pi}{24}\right)=\dfrac{\sin(\pi/24)}{\cos(\pi/24)}$

$=\dfrac{2\sin(\pi/24)\cos(\pi/24)}{2\cos^2(\pi/24)}$

$=\dfrac{\sin(\pi/12)}{1+\cos(\pi/12)}$

$=\dfrac{\sqrt{3}-1}{2\sqrt{2}+\sqrt{3}+1}$

$=\dfrac{\sqrt{3}-1}{(2\sqrt{2}+(\sqrt{3}+1))}\times\dfrac{(2\sqrt{2}-(\sqrt{3}+1))}{(2\sqrt{2}-(\sqrt{3}+1))}$

$=\dfrac{(\sqrt{3}-1)(2\sqrt{2}-(\sqrt{3}+1))}{\left(8-(\sqrt{3}+1)^2\right)}$

$=\dfrac{2\sqrt{6}-3-\sqrt{3}-2\sqrt{2}+\sqrt{3}+1}{(4-2\sqrt{3})}$

$=\dfrac{2\sqrt{6}-2-2\sqrt{2}}{(4-2\sqrt{3})}$

$=\dfrac{\sqrt{6}-1-\sqrt{2}}{(2-\sqrt{3})}$

$=(\sqrt{6}-\sqrt{2}-1)(2+\sqrt{3})$

$=2\sqrt{6}-2\sqrt{2}-2+\sqrt{18}-\sqrt{6}-\sqrt{3}$

$= \sqrt{6}+\sqrt{2}-\sqrt{4}-\sqrt{3}$

$= \sqrt{6}-2-\sqrt{3}+\sqrt{2}$

$= \left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{2}-1\right)$

Thus, $a = 3$, $b = 2$, $c = 2$ and $d = 1$

Hence, the value of $(a+b+c+d+2)$.

$= 3 + 2 + 2 + 1 + 2$

$= 10.$

Ex-10 If $\dfrac{x}{\cos\theta} = \dfrac{y}{\cos\left(\theta+\dfrac{2\pi}{3}\right)} = \dfrac{z}{\cos\left(\theta-\dfrac{2\pi}{3}\right)}$, then find the value of $x + y + z$.

Soln. Given

$$\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta+\frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta-\frac{2\pi}{3}\right)} = m \text{ (say)}$$

Now, $x + y + z$

$= m\times\left(\cos\theta+\cos\left(\theta+\dfrac{2\pi}{3}\right)+\cos\left(\theta-\dfrac{2\pi}{3}\right)\right)$

$= m\times\left(\cos\theta+2\cos\theta.\cos\left(\dfrac{2\pi}{3}\right)\right)$

$= m\times\left(\cos\theta+2\cos\theta.\left(-\dfrac{1}{2}\right)\right)$

$= m\times(\cos\theta-\cos\theta)$

$= 0$

Hence, the value of $x + y + z$ is zero.

Ex-11 If $\sin(25°)\sin(35°)\sin(85°) = \dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}$ where $a, b, c \in I^+$, find the value of $(a+b+c-2)$

Soln. We have, $\sin(25°)\sin(35°)\sin(85°)$

$= \sin(35°)\sin(25°)\sin(85°)$

$= \dfrac{1}{4}\left(4\sin(60°-25)\sin(25°)\sin(60°+25°)\right)$

$= \dfrac{1}{4}\times\sin(3\times 25°)$

$= \dfrac{1}{4}\times\sin(75°)$

$= \dfrac{1}{4}\times\cos(15°)$

$= \dfrac{1}{4}\times\left(\dfrac{\sqrt{3}+1}{2\sqrt{2}}\right)$

$= \left(\dfrac{\sqrt{3}+1}{8\sqrt{2}}\right)$

$= \left(\dfrac{\sqrt{3}+1}{\sqrt{128}}\right)$

Thus, $a = 3$, $b = 1$ and $c = 128$

Hence, the value of $(a+b+c-2)$

$= 3 + 1 + 128 - 2$

$= 130.$

Ex-12 Find the value of $\sqrt{3}\cot(20°)-4\cos(20°)$.

Soln. We have, $\sqrt{3}\cot(20°)-4\cos(20°)$

$= \dfrac{\sqrt{3}\cos(20°)-4\sin(20°)\cos(20°)}{\sin(20°)}$

$= \dfrac{2\left(\dfrac{\sqrt{3}}{2}\cos(20°)-2\sin(20°)\cos(20°)\right)}{\sin(20°)}$

$= \dfrac{2(\sin(60°)\cos(20°)-\sin(40°))}{\sin(20°)}$

$= \dfrac{(2\sin(60°)\cos(20°)-2\sin(40°))}{\sin(20°)}$

$= \dfrac{(\sin(80°)+\sin(40°)-2\sin(40°))}{\sin(20°)}$

$= \dfrac{(\sin(80°)-\sin(40°))}{\sin(20°)}$

$= \dfrac{2\cos(60°)\sin(20°)}{\sin(20°)}$

$= 1.$

Ex-13. Prove that,

$\sin(2°)+\sin(4°)+\sin(6°)+$

$\sin(8°)+............+\sin(180°) = \cot(1°)$

Soln. We have, $\sin(2°)+\sin(4°)+\sin(6°)$

$+\sin(8°)+............+\sin(180°)$

$= \sin(2°)+\sin(2°+2°)+\sin(2°+2.2°)$

$+\sin(2°+3.2°)+....+\sin(2°+(90-1)2°)$

$= \dfrac{\sin\left(\dfrac{90°.2°}{2}\right)}{\sin\left(\dfrac{2°}{2}\right)}\times\sin\left(2°+(90°-1)\dfrac{2°}{2}\right)$

$$= \frac{1}{\sin(1°)} \times \sin(91°)$$

$$= \frac{\cos(1°)}{\sin(1°)}$$

$$= \cot(1°)$$

Ex-14. Find the value of

$$\sin\left(\frac{\pi}{2013}\right) + \sin\left(\frac{3\pi}{2013}\right) + \sin\left(\frac{5\pi}{2013}\right)$$

$$+\sin\left(\frac{7\pi}{2013}\right) + \text{...upto } (2013) \text{ trems}$$

Soln. We have, $+\sin\left(\frac{À}{2013} + \frac{2.2À}{2013}\right)$

$$+\sin\left(\frac{\pi}{2013} + \frac{2.2\pi}{2013}\right)$$

$$+......+\sin\left(\frac{\pi}{2013} + \frac{1006.2\pi}{2013}\right)$$

$$= \frac{\sin\left(2013.\left(\frac{\frac{2\pi}{2013}}{2}\right)\right)}{\sin\left(\frac{\frac{2\pi}{2013}}{2}\right)}$$

$$\times \sin\left(\frac{\pi}{2013} + (2012)\left(\frac{\frac{2\pi}{2013}}{2}\right)\right)$$

$= 0.$

Ex-15. If $\tan y = \left(\frac{1+\sqrt{1+y}}{1+\sqrt{1-y}}\right)$,

then prove that $\sin(4y) = y$

Soln. Put $y = \sin(4\theta)$

Then, $\tan(y) = \frac{1+\sqrt{1+\sin(4\theta)}}{1+\sqrt{1-\sin(4\theta)}}$

$$\Rightarrow \quad \tan(y) = \frac{1+\sqrt{(\cos(2\theta)+\sin(2\theta))^2}}{1+\sqrt{(\cos(2\theta)-\sin(2\theta))^2}}$$

$$\Rightarrow \quad \tan(y) = \frac{1+\cos(2\theta)+\sin(2\theta)}{1+(\cos(2\theta)-\sin(2\theta))}$$

$$\Rightarrow \quad \tan(y) = \frac{2\cos^2(\theta)+\sin(2\theta)}{2\cos^2(\theta)-\sin(2\theta)}$$

$$\Rightarrow \quad \tan(y) = \frac{2\cos^2(\theta)+2\sin(\theta)\cos(\theta)}{2\cos^2(\theta)-2\sin(\theta)\cos(\theta)}$$

$$\Rightarrow \quad \tan(y) = \frac{\cos(\theta)+\sin(\theta)}{\cos(\theta)-\sin(\theta)}$$

$$\Rightarrow \quad \tan(y) = \tan\left(\frac{\pi}{4} - \theta\right)$$

$$\Rightarrow \quad y = \frac{\pi}{4} - \theta$$

$$\Rightarrow \quad 4y = \pi - 4\theta$$

$$\Rightarrow \quad \sin(4y) = \sin(\pi - 4\theta) = \sin(4\theta) = y$$

Hence, the result.

Ex-16 If $\tan\left(\frac{\pi}{4} + \frac{y}{2}\right) = \tan^3\left(\frac{\pi}{4} + \frac{x}{2}\right)$,

prove that $\frac{\sin y}{\sin x} = \frac{3+\sin^2 x}{1+3\sin^2 x}$

Soln. We have,

$$\tan\left(\frac{\pi}{4} + \alpha\right) = \frac{1+\tan\alpha}{1-\tan\alpha} = \frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}$$

$$\Rightarrow \tan^2\left(\frac{\pi}{4} + \alpha\right) = \frac{1+\sin 2\alpha}{1-\sin 2\alpha}$$

Thus, $\tan\left(\frac{\pi}{4} + \frac{y}{2}\right) = \tan^3\left(\frac{\pi}{4} + \frac{x}{2}\right)$,

$$\Rightarrow \frac{1+\sin y}{1-\sin y} = \frac{(1+\sin x)^3}{(1-\sin x)^3}$$

Applying componendo and dividendo, we get,

$$\Rightarrow \frac{2\sin y}{2} = \frac{2(3\sin x + \sin^3 x)}{2(1+3\sin^2 x)}$$

$$\Rightarrow \sin y = \frac{(3\sin x + \sin^3 x)}{(1+3\sin^2 x)}$$

Hence, the result.

Ex-17 If $\tan\beta = \frac{\tan\alpha + \tan\gamma}{1+\tan\alpha\tan\gamma}$, then

prove that $\sin(2\beta) = \frac{\sin(2\alpha)+\sin(2\gamma)}{1+\sin(2\alpha).\sin(2\gamma)}$

Soln. Given, $\tan\beta = \dfrac{\tan\alpha + \tan\gamma}{1+\tan\alpha\tan\gamma}$

$$= \frac{\sin\alpha\cos\gamma + \cos\alpha\sin\gamma}{\cos\alpha\cos\gamma + \sin\alpha\sin\gamma}$$

$$= \frac{\sin(\alpha+\gamma)}{\cos(\alpha-y)}$$

Thus, $\sin(2\beta) = \dfrac{2\tan\beta}{1+\tan^2\beta}$

$$= \frac{2\left(\dfrac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}\right)}{1+\dfrac{\sin^2(\alpha+\gamma)}{\cos^2(\alpha-\gamma)}}$$

$$= \frac{2\sin(\alpha+\gamma)\cos(\alpha-\gamma)}{\cos^2(\alpha-\gamma)+\sin^2(\alpha+\gamma)}$$

$$= \frac{\sin((\alpha+\gamma)+(\alpha-\gamma))+\sin((\alpha+\gamma)-(\alpha-\gamma))}{1+\sin^2(\alpha+\gamma)-\sin^2(\alpha-\gamma)}$$

$$= \frac{\sin(2\alpha)+\sin(2\gamma)}{1+\sin(\alpha+\gamma+\alpha-\gamma)\sin(\alpha+\gamma-\alpha+\gamma)}$$

$$= \frac{\sin(2\alpha)+\sin(2\gamma)}{1+\sin(2\alpha)\sin(2\gamma)}$$

Hence, the result.

Ex-18 If $4\sin(27°) = \left(a+\sqrt{b}\right)^{1/2} - \left(c-\sqrt{d}\right)^{1/2}$ where $a, b, c, d \in N$, find the value of $(a+b+c+d+2)$.

Soln. We have, $16\sin^2(27°)$

$$= 8\times 2\sin^2(27°)$$

$$= 8\times(1-\cos(54°))$$

$$= 8\times\left(1-\frac{\sqrt{10-2\sqrt{5}}}{4}\right)$$

$$= \left(8-2\sqrt{10-2\sqrt{5}}\right)$$

$$= \left(8-2\sqrt{(5+\sqrt{5})(3-\sqrt{5})}\right)$$

$$= \left((5+\sqrt{5})+(3-\sqrt{5})-2\sqrt{(5+\sqrt{5})}\sqrt{(3-\sqrt{5})}\right)$$

$$= \left(\sqrt{(5+\sqrt{5})}-\sqrt{(3-\sqrt{5})}\right)^2$$

$$\Rightarrow 4\sin(27°) = \left(\sqrt{(5+\sqrt{5})}-\sqrt{(3-\sqrt{5})}\right)$$

Thus, $a = 5$, $b = 5$, $c = 3$ and $d = 5$

Now, $a+b+c+d+2$

$= 5 + 5 + 3 + 5 + 2$

$= 20$.

Ex-19 If $(1+\sin\theta)(1+\cos\theta) = \dfrac{5}{4}$, find the value of $(1-\sin\theta)(1-\cos\theta)$.

Soln. We have, $(1+\sin\theta)(1+\cos\theta) = \dfrac{5}{4}$

$$\Rightarrow \quad 1+\sin\theta+\cos\theta+\sin\theta\cos\theta = \frac{5}{4}$$

$$\Rightarrow \quad 1+t+\left(\frac{t^2-1}{2}\right) = \frac{5}{4}$$

$(\sin\theta+\cos\theta = t, \text{ say})$

$$\Rightarrow t+\left(\frac{t^2-1}{2}\right) = \frac{1}{4}$$

$$\Rightarrow t^2+2t-1 = \frac{1}{2}$$

$$\Rightarrow 2t^2+4t-3 = 0$$

$$\Rightarrow t = \frac{-4\pm\sqrt{16+24}}{4}$$

$$= \frac{-4\pm 2\sqrt{10}}{4} = -1\pm\frac{1}{2}\sqrt{10}$$

$$\Rightarrow t = -1+\frac{1}{2}\sqrt{10}$$

$$\Rightarrow \sin\theta+\cos\theta = -1+\frac{1}{2}\sqrt{10}$$

Now, $(1-\sin\theta)(1-\cos\theta)$

$$= 1-\sin\theta-\cos\theta+\sin\theta\cos\theta$$

$$= 1-(\sin\theta+\cos\theta)+\sin\theta\cos\theta$$

$$= 1-\left(-1+\frac{\sqrt{10}}{2}\right)+\frac{1}{2}\left(\frac{10}{4}-\sqrt{10}\right)$$

$$= \left(2+\frac{5}{4}\right)-\sqrt{10}$$

$$= \left(\frac{13}{4}-\sqrt{10}\right)$$

Ex-20 If $3\sin x + 4\cos x = 5$, where $x \in \left(0, \frac{\pi}{2}\right)$

then find the value of $2\sin x + \cos x + 4\tan x$

Soln. We have, 3 sin x + 4 cos x = 5

Let y = 3 cos x – 4 sin x

Now, $y^2 + 5^2$

$= (3\cos x - 4\sin x)\ 2 + (3\sin x + 4\cos x)^2$

$= 9\cos^2 x + 16\sin^2 x - 24\sin x\cos x$

$+ 9\sin^2 x + 16\cos^2 x + 24\sin x\cos x$

$\Rightarrow \quad y^2 + 25 = 25$

$\Rightarrow \quad y^2 = 0$

$\Rightarrow \quad y = 0$

$\Rightarrow \quad 3\cos x - 4\sin x = 0$

$\Rightarrow \quad 3\cos x = 4\sin x$

$\Rightarrow \quad \tan x = 3/4.$

Hence, the value of 2 sin x + cos x + 4 tan x

$= 2\left(\frac{3}{5}\right) + \left(\frac{4}{5}\right) + 4\left(\frac{3}{4}\right) = 2 + 3 = 5.$

Ex-21 If $\cos A = \tan B, \cos B = \tan C, \cos C = \tan A$, prove that $\sin A = \sin B = \sin C = 2\sin\left(18^o\right)$

Soln. We have, $\cos A = \tan B$

$\Rightarrow \cos^2 A = \tan^2 B$

$\Rightarrow \cos^2 A = \sec^2 B - 1$

$\Rightarrow 1 + \cos^2 A = \sec^2 B$

$\Rightarrow 1 + \cos^2 A = \sec^2 B = \cot^2 C$

$\Rightarrow 1 + \cos^2 A = \cot^2 C$

$\Rightarrow 2 - \sin^2 A = \frac{\cos^2 C}{\sin^2 C} = \frac{\cos^2 C}{1 - \cos^2 C}$

$\Rightarrow 2 - \sin^2 A = \frac{\tan^2 A}{1 - \tan^2 A}$

$\Rightarrow 2 - \sin^2 A = \frac{\sin^2 A}{\cos^2 A - \sin^2 A}$

$\Rightarrow 2 - \sin^2 A = \frac{\sin^2 A}{1 - 2\sin^2 A}$

$\Rightarrow 2 - 4\sin^2 A - \sin^2 A + 2\sin^4 A = \sin^2 A$

$\Rightarrow 2\sin^4 A - 6\sin^2 A + 2 = 0$

$\Rightarrow \sin^4 A - 3\sin^2 A + 1 = 0$

$\Rightarrow \sin^2 A = \frac{3 \pm \sqrt{9 - 4}}{2}$

$\Rightarrow \sin^2 A = \frac{3 \pm \sqrt{5}}{2}$

$\Rightarrow \sin^2 A = \frac{6 \pm 2\sqrt{5}}{4}$

$\Rightarrow \sin^2 A = \left(\frac{\sqrt{5} - 1}{2}\right)^2$

$\Rightarrow \sin A = \left(\frac{\sqrt{5} - 1}{2}\right)$

$\Rightarrow \sin A = 2\left(\frac{\sqrt{5} - 1}{4}\right) = 2\sin(18°)$

Similarly, we can prove that,

$\sin B = 2\sin(18°) = \sin C$.

Ex-22. Find the value of

$\tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{3\pi}{16}\right) + \tan^2\left(\frac{5\pi}{16}\right) + \tan^2\left(\frac{7\pi}{16}\right)$

Soln. We have,

$\tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{3\pi}{16}\right) + \tan^2\left(\frac{5\pi}{16}\right) + \tan^2\left(\frac{7\pi}{16}\right)$

$= \tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{3\pi}{16}\right)$

$+ \tan^2\left(\frac{\pi}{2} - \frac{3\pi}{16}\right) + \tan^2\left(\frac{\pi}{2} - \frac{\pi}{16}\right)$

$= \tan^2\left(\frac{\pi}{16}\right) + \cot^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{3\pi}{16}\right) + \cot^2\left(\frac{3\pi}{16}\right)$

$= \left(\tan\left(\frac{\pi}{16}\right) + \cot\left(\frac{\pi}{16}\right)\right)^2 + \left(\tan\left(\frac{3\pi}{16}\right) + \cot\left(\frac{3\pi}{16}\right)\right)^2 - 4$

$= \frac{1}{\sin^2\left(\frac{\pi}{16}\right)\cos^2\left(\frac{\pi}{16}\right)} + \frac{1}{\sin^2\left(\frac{3\pi}{16}\right)\cos^2\left(\frac{3\pi}{16}\right)} - 4$

$= \frac{4}{\left(2\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{16}\right)\right)^2} + \frac{4}{\left(2\sin\left(\frac{3\pi}{16}\right)\cos\left(\frac{3\pi}{16}\right)\right)^2} - 4$

$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{4}{\sin^2\left(\frac{3\pi}{8}\right)} - 4$

$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{4}{\sin^2\left(\frac{\pi}{2} - \frac{\pi}{8}\right)} - 4$

$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{4}{\cos^2\left(\frac{\pi}{8}\right)} - 4$

$$= 4\left(\frac{1}{\sin^2\left(\frac{\pi}{8}\right)}+\frac{1}{\cos^2\left(\frac{\pi}{8}\right)}\right)-4$$

$$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right)}-4$$

$$= \frac{8}{\left(2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\right)^2}-4$$

$$= \frac{8}{\sin^2\left(\frac{\pi}{4}\right)}-4$$

$$= \frac{8}{\frac{1}{2}}-4=16-4=12$$

Ex-23 If $\sin(1°).\sin(3°).\sin(5°)....\sin(89°)=\frac{1}{2^n}$ then find the value of n.

Soln. We have, $\sin(1°).\sin(3°).\sin(5°)....\sin(89°)$

$$= \sin(1°).\sin(3°).\sin(5°)....\sin(44°)\sin(45°)$$
$$\sin(46°).\sin(47°).\sin(48°)....\sin(89°)$$

$$= \sin(1°).\sin(2°).\sin(3°)....\sin(44°)\sin(45°)$$
$$\cos(44°)\cos(43°)\cos(42°)....\cos(1°)$$

$$= \sin(1°)\sin(2°)\sin(3°)...\sin(44°)\sin(45°)$$
$$\cos(1°)\cos(2°)\cos(3°)...\cos(44°)$$

$$= \frac{1}{\sqrt{2}}\times\frac{1}{2^{44}}\left(\sin(2°)\sin(4°)\sin(6°)...\sin(88°)\right)$$

$$= \frac{1}{2^{89/2}}\left(\sin\left(2^o\right)\sin\left(4^o\right)\sin\left(6^o\right)...\sin\left(88^o\right)\right)$$

Thus, $\sin(2°)\sin(4°)\sin(6°)...\sin(88°)$

$$= \frac{1}{2^{89/2}}$$

Therefore, $n=\frac{89}{2}$.

Ex-24. If $\left(1+\tan(1°)\right)\left(1+\tan(2°)\right)\left(1+\tan(3°)\right)$ ....$\left(1+\tan(45°)\right)=2^n$, then find n.

Soln. We have, $\left(1+\tan(1°)\right)\left(1+\tan(2°)\right)\left(1+\tan(3°)\right)$

.........$\left(1+\tan(45°)\right)=2^n$

$\Rightarrow$ $\left(1+\tan(1°)\right)\left(1+\tan(44°)\right)\left(1+\tan(2°)\right)\left(1+\tan(43°)\right)$

......$(1+\tan(22°))(1+\tan(23°))\times(1+\tan(24°))=2^n$

$\Rightarrow$ $2^{22}\times(1+1)=2^n$

$\Rightarrow$ $2^n=2^{23}$

$\Rightarrow$ $n=23$.

Hence, the value of n is 23.

Ex-25. Prove that:

$$\left(1+\cos\left(\frac{\pi}{10}\right)\right)\left(1+\cos\left(\frac{3\pi}{10}\right)\right)\left(1+\cos\left(\frac{7\pi}{10}\right)\right)\left(1+\cos\left(\frac{9\pi}{10}\right)\right)=\frac{1}{16}$$

Soln. We have,

$$\cos\left(\frac{7\pi}{10}\right)=\cos\left(\pi-\frac{3\pi}{10}\right)=-\cos\left(\frac{3\pi}{10}\right)$$

$$\cos\left(\frac{9\pi}{10}\right)=\cos\left(\pi-\frac{\pi}{10}\right)=-\cos\left(\frac{\pi}{10}\right)$$

Therefore,

$$\left(1+\cos\left(\frac{\pi}{10}\right)\right)\left(1+\cos\left(\frac{3\pi}{10}\right)\right)\left(1+\cos\left(\frac{7\pi}{10}\right)\right)\left(1+\cos\left(\frac{9\pi}{10}\right)\right)$$

$$= \left(1+\cos\left(\frac{\pi}{10}\right)\right)\left(1+\cos\left(\frac{3\pi}{10}\right)\right)\left(1-\cos\left(\frac{3\pi}{10}\right)\right)\left(1-\cos\left(\frac{\pi}{10}\right)\right)$$

$$= \left(1-\cos^2\left(\frac{\pi}{10}\right)\right)\left(1-\cos^2\left(\frac{3\pi}{10}\right)\right)$$

$$= \sin^2\left(\frac{\pi}{10}\right)\sin^2\left(\frac{3\pi}{10}\right)$$

$$= \left(\frac{\sqrt{5}-1}{4}\right)^2\left(\frac{\sqrt{5}+1}{4}\right)^2$$

$$= \left(\frac{\sqrt{5}-1}{4}\right)^2\left(\frac{\sqrt{5}+1}{4}\right)^2$$

$$= \left(\frac{5-1}{16}\right)^2=\frac{1}{16}$$

Ex-27 Prove that:

$$\cos(60°)\cos(36°)\cos(42°)\cos(78°)=\frac{1}{16}$$

Soln. We have,

$$\cos(60°)\cos(36°)\cos(42°)\cos(78°)$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}+1}{4}\right).\frac{1}{2}\left(2\cos 78°\cos 42°\right)$$

$$= \frac{1}{4}\left(\frac{\sqrt{5}+1}{4}\right)\left(\cos 120° + \cos 36°\right)$$

$$= \frac{1}{4}\left(\frac{\sqrt{5}+1}{4}\right)\left(-\frac{1}{2}+\frac{\sqrt{5}+1}{4}\right)$$

$$= \frac{1}{4}\left(\frac{\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}-1}{4}\right)$$

$$= \frac{(5-1)}{64} = \frac{4}{64} = \frac{1}{16}$$

Ex-28. Let $f_k(\theta) = \sin^k(\theta) + \cos^k(\theta)$.

Then find the value of $\frac{1}{6}f_6(\theta) - \frac{1}{4}f_4(\theta)$

Soln. We have, $f_6(\theta) = \sin^6\theta + \cos^6\theta$

$$= 1 - 3\sin^2\theta\cos^2\theta$$

Also, $f_4(\theta) = \sin^4\theta + \cos^4\theta$

$$= 1 - 2\sin^2\theta\cos^2\theta$$

Now, $\frac{1}{6}f_6(\theta) - \frac{1}{4}f_4(\theta)$

$$= \frac{1}{6}\left(1 - 3\sin^2\theta\cos^2\theta\right) - \frac{1}{4}\left(1 - 2\sin^2\theta\cos^2\theta\right)$$

$$= \frac{1}{6} - \frac{1}{2}\sin^2\theta\cos^2\theta - \frac{1}{4} + \frac{1}{2}\sin^2\theta\cos^2\theta$$

$$= \frac{1}{6} - \frac{1}{4}$$

$$= -\frac{1}{12}$$

Ex-29. Find the max and min values of

$$f(\theta) = \sin^2(\sin\theta) + \cos^2(\cos\theta)$$

Soln. We have, $\sin^2(\sin\theta) + \cos^2(\cos\theta)$

$$= \sin^2(\cos\theta) + \cos^2(\cos\theta)$$
$$+ \sin^2(\sin\theta) - \sin^2(\cos\theta)$$

$$= \left(\sin^2(\cos\theta) + \cos^2(\cos\theta)\right)$$
$$+ \sin^2(\sin\theta) - \sin^2(\cos\theta)$$

$$= 1 + \left(\sin^2(\sin\theta) - \sin^2(\cos\theta)\right)$$

Max value of $f(\theta)$

$$= 1 + \left(\sin^2\left(\sin\left(\frac{\pi}{2}\right)\right) - \sin^2\left(\cos\left(\frac{\pi}{2}\right)\right)\right)$$

$$= 1 + \sin^2(1)$$

Min value of $f(\theta)$

$$= 1 + \left(\sin^2(\sin(0)) - \sin^2(\cos(0))\right)$$

$$= 1 - \sin^2(1)$$

Ex-30. Find the minimum value of

$$f(\theta) = \left(3\sin(\theta) - 4\cos(\theta) - 10\right)$$
$$\left(3\sin(\theta) + 4\cos(\theta) - 10\right)$$

Soln. We have, $f(\theta) = \left(3\sin(\theta) - 4\cos(\theta) - 10\right)$

$$\left(3\sin(\theta) + 4\cos(\theta) - 10\right)$$

$$= \left(9\sin^2(\theta) - 16\cos^2(\theta)\right)$$
$$-10\left(3\sin\theta + 4\cos\theta\right) - 10\left(3\sin\theta - 4\cos\theta\right)$$

$$= \left(9\sin^2(\theta) - 16\cos^2(\theta)\right)$$
$$-10\left(3\sin\theta + 4\cos\theta + 3\sin\theta - 4\cos\theta\right)$$

$$= \left(9\sin^2(\theta) - 16\cos^2(\theta)\right) - 60\sin(\theta)$$

$$= 25\sin^2\theta - 60\sin(\theta) - 16$$

$$= (5\sin\theta - 6)^2 - 36 - 16$$

$$= (5\sin\theta - 6)^2 - 52$$

Hence, the minimum value of

$$f(\theta) = 121 - 52 = 69$$

Ex-31. Find the range of $A = \sin^{2010}\theta + \cos^{2014}\theta$

Soln. Now, $0 < \sin^{2010}\theta \le \sin^2\theta$ ....(i)

and $0 < \cos^{2014}\theta \le \cos^2\theta$ ....(ii)

Adding (i) and (ii), we get,

$$0 < \sin^{2012}\theta + \cos^{2014}\theta \le \sin^2\theta + \cos^2\theta$$

$$\Rightarrow \quad 0 < A \le 1$$

Thus, the range of A = $(0, 1]$

32. If $\frac{\sin A}{\sin B} = p$, $\frac{\cos A}{\cos B} = q$, prove that

$$\tan A.\tan B = \frac{p}{q}\left(\frac{q^2-1}{1-p^2}\right).$$

Soln. We have, $\frac{\sin A}{\sin B} = p$, $\frac{\cos A}{\cos B} = q$

$$\Rightarrow \frac{\sin A}{\cos A} / \frac{\sin B}{\cos B} = \frac{p}{q}$$

$$\Rightarrow \frac{\tan A}{\tan B} = \frac{p}{q}$$

$$\Rightarrow \frac{\tan A}{p} = \frac{\tan B}{q} = \lambda$$

Also, $\frac{\sin A}{\sin B} = p$, $\frac{\cos A}{\cos B} = q$

$$\Rightarrow \frac{\sin A \cos A}{\sin B \cos B} = pq$$

$$\Rightarrow \frac{2\sin A \cos A}{2\sin B \cos B} = pq$$

$$\Rightarrow \frac{\sin 2A}{\sin 2B} = pq$$

$$\Rightarrow \frac{2\tan A}{1+\tan^2 A} / \frac{2\tan B}{1+\tan^2 B} = pq$$

$$\Rightarrow \frac{2p\lambda}{1+p^2\lambda^2} / \frac{2q\lambda}{1+q^2\lambda^2} = pq$$

$$\Rightarrow \frac{p}{(1+p^2\lambda^2)} \times \frac{(1+q^2\lambda^2)}{q} = pq$$

$$\Rightarrow \frac{(1+q^2\lambda^2)}{(1+p^2\lambda^2)} = q^2$$

$$\Rightarrow (1+q^2\lambda^2) = q^2(1+p^2\lambda^2)$$

$$\Rightarrow \lambda^2(1-p^2)q^2 = q^2-1$$

$$\Rightarrow \lambda^2 = \frac{(q^2-1)}{(1-p^2)q^2}$$

$$\Rightarrow \lambda = \pm\frac{1}{q}\sqrt{\frac{(q^2-1)}{(1-p^2)}}$$

Therefore, $\tan A = \pm\frac{p}{q}\sqrt{\frac{(q^2-1)}{(1-p^2)}}$

and $\tan B = \pm\sqrt{\frac{(q^2-1)}{(1-p^2)}}$

Ex-33. If $\frac{\tan(\alpha-\beta)}{\tan\alpha} + \frac{\sin^2\gamma}{\sin^2\alpha} = 1$, then prove that $\tan^2\gamma = \tan\alpha.\tan\beta$

Soln. We have,

$$\frac{\tan(\alpha-\beta)}{\tan\alpha} + \frac{\sin^2\gamma}{\sin^2\alpha} = 1$$

$$\Rightarrow \frac{\sin^2\gamma}{\sin^2\alpha} = 1 - \frac{\tan(\alpha-\beta)}{\tan\alpha}$$

$$\Rightarrow \frac{\sin^2\gamma}{\sin^2\alpha} = \frac{1}{\tan\alpha}\left(\tan\alpha - \tan(\alpha-\beta)\right)$$

$$\Rightarrow \sin^2\gamma = \frac{\sin^2\alpha}{\tan\alpha}\left(\tan\alpha - \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right)$$

$$\Rightarrow \sin^2\gamma = \frac{\sin^2\alpha}{\tan\alpha}\left(\frac{\tan\beta(1+\tan^2\alpha)}{(1+\tan\alpha\tan\beta)}\right)$$

$$\Rightarrow \sin^2\gamma(1+\tan\alpha\tan\beta) = \frac{\sin^2\alpha}{\tan\alpha}\left(\frac{\tan\beta}{\cos^2\alpha}\right)$$

$$\Rightarrow \sin^2\gamma(1+\tan\alpha\tan\beta) = \frac{\sin^2\alpha}{\cos^2\alpha}\left(\frac{\tan\beta}{\tan\alpha}\right)$$

$$\Rightarrow \sin^2\gamma(1+\tan\alpha\tan\beta) = \tan^2\alpha\left(\frac{\tan\beta}{\tan\alpha}\right)$$

$$\Rightarrow \sin^2\gamma(1+\tan\alpha\tan\beta) = \tan\alpha\tan\beta$$

$$\Rightarrow \sin^2\gamma = \tan\alpha\tan\beta(1-\sin^2\gamma)$$

$$\Rightarrow \frac{\sin^2\gamma}{\cos^2\gamma} = \tan\alpha\tan\beta$$

$$\Rightarrow \tan^2\gamma = \tan\alpha\tan\beta$$

Hence, the result.

Ex-34. If $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\varphi}{2}\right)$, then prove that $\cos\varphi = \frac{\cos\theta - e}{1-e\cos\theta}$

Soln. We have, $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\varphi}{2}\right)$

$$\Rightarrow \tan^2\left(\frac{\theta}{2}\right) = \left(\frac{1-e}{1+e}\right)\tan^2\left(\frac{\varphi}{2}\right)$$

$$\Rightarrow \frac{1}{\tan^2\left(\frac{\varphi}{2}\right)} = \left(\frac{1-e}{1+e}\right)\frac{1}{\tan^2\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \frac{1-\tan^2\left(\frac{\varphi}{2}\right)}{1+\tan^2\left(\frac{\varphi}{2}\right)} = \frac{1-e-\tan^2\left(\frac{\theta}{2}\right)-e\tan^2\left(\frac{\theta}{2}\right)}{1-e+\tan^2\left(\frac{\theta}{2}\right)+e\tan^2\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \cos\varphi = \frac{\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)-e\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)}{\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)-e\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)}$$

$$\Rightarrow \cos\varphi = \frac{\dfrac{\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)}{\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)}-e}{1-e\dfrac{\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)}{\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)}}$$

$$\Rightarrow \cos\varphi = \frac{\cos\theta - e}{1+e\cos\theta}$$

Hence, the result.

Ex-35. If $\cos\theta = \dfrac{a\cos\varphi + b}{a + b\cos\varphi}$, then prove that

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{a-b}{a+b}}\tan\left(\frac{\varphi}{2}\right)$$

Soln. We have, $\cos\theta = \dfrac{a\cos\varphi + b}{a + b\cos\varphi}$

$$\Rightarrow \frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{a\left(\dfrac{1-\tan^2(\varphi/2)}{1+\tan^2(\varphi/2)}\right)+b}{a+b\left(\dfrac{1-\tan^2(\varphi/2)}{1+\tan^2(\varphi/2)}\right)}$$

$$\Rightarrow \frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{(a+b)-(a-b)\tan^2(\varphi/2)}{(a+b)+(a-b)\tan^2(\varphi/2)}$$

$$\Rightarrow \frac{2}{-2\tan^2\left(\frac{\theta}{2}\right)} = \frac{2(a+b)}{-2(a-b)\tan^2(\varphi/2)}$$

$$\Rightarrow \frac{1}{\tan^2\left(\frac{\theta}{2}\right)} = \frac{(a+b)}{(a-b)\tan^2(\varphi/2)}$$

$$\Rightarrow \frac{\tan^2(\varphi/2)}{\tan^2(\theta/2)} = \frac{(a+b)}{(a-b)}$$

$$\Rightarrow \tan^2(\theta/2) = \frac{(a+b)}{(a-b)}\tan^2(\varphi/2)$$

$$\Rightarrow \tan(\theta/2) = \sqrt{\frac{(a+b)}{(a-b)}}\tan(\varphi/2)$$

Hence, the result.

Ex-36. If $\sin x + \sin y = a, \cos x + \cos y = b$

then prove that $\tan\left(\dfrac{x-y}{2}\right) = \pm\sqrt{\dfrac{4-a^2-b^2}{a^2+b^2}}$

Soln. We have, $a^2 + b^2$

$$= (\sin x + \sin y)^2 + (\cos x + \cos y)^2$$
$$= 2 + 2(\cos x\cos y + \sin x\sin y)$$
$$= 2 + 2\cos(x-y)$$

$$\Rightarrow \cos(x-y) = \left(a^2+b^2-2\right)/2$$

As we know that,

$$\tan^2\left(\frac{A}{2}\right) = \frac{1-\cos A}{1+\cos A}$$

$$\Rightarrow \tan^2\left(\frac{x-y}{2}\right) = \frac{1-\cos(x-y)}{1+\cos(x-y)}$$

$$\Rightarrow \tan^2\left(\frac{x-y}{2}\right) = \frac{1-\left(a^2+b^2-2\right)/2}{1+\left(a^2+b^2-2\right)/2}$$

$$\Rightarrow \tan^2\left(\frac{x-y}{2}\right) = \frac{2-\left(a^2+b^2-2\right)}{2+\left(a^2+b^2-2\right)}$$

$$\Rightarrow \tan^2\left(\frac{x-y}{2}\right) = \frac{4-a^2-b^2}{a^2+b^2}$$

$$\Rightarrow \tan\left(\frac{x-y}{2}\right) = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

Hence, the result.

Ex-37. If $\tan\left(\dfrac{\pi}{16}\right) = \left(a + b\sqrt{2}\right)^{1/2} - \left(\sqrt{c} + d\right)$, where a, b, c, d are +ve integers, then find the value of $(a+b+c+d+1)$.

Soln. As we know that

$$\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta}$$

Put $\theta = \left(22\frac{1°}{2}\right)$

$$\Rightarrow \tan\left(22\frac{1°}{2}\right) = \frac{1-\cos 45°}{\sin 45°}$$

$$= \frac{1-1/\sqrt{2}}{1/\sqrt{2}} = \sqrt{2}-1$$

Let $A = 11\frac{1°}{4}$

$$\Rightarrow 2A = 22\frac{1°}{2}$$

$$\Rightarrow \tan(2A) = \tan\left(22\frac{1°}{2}\right) = \sqrt{2}-1$$

$$\Rightarrow \frac{2\tan A}{1-\tan^2 A} = \sqrt{2}-1$$

$$\Rightarrow \frac{2y}{1-y^2} = \sqrt{2}-1$$

$$\Rightarrow 1-y^2 = \frac{2}{(\sqrt{2}-1)}y$$

$$\Rightarrow 1-y^2 = 2(\sqrt{2}+1)y$$

$$\Rightarrow y^2 + 2(\sqrt{2}+1)y - 1 = 0$$

$$\Rightarrow y = \frac{-2(\sqrt{2}+1) \pm \sqrt{4(\sqrt{2}+1)^2+4}}{2}$$

$$\Rightarrow y = -(\sqrt{2}+1) + \sqrt{(\sqrt{2}+1)^2+1}$$

$$\Rightarrow y = -(\sqrt{2}+1) + \sqrt{4+2\sqrt{2}}$$

$$\Rightarrow y = \sqrt{4+2\sqrt{2}} - (\sqrt{2}+1)$$

$$\Rightarrow \tan\left(11\frac{1°}{4}\right) = \sqrt{4+2\sqrt{2}} - (\sqrt{2}+1)$$

Thus, $a = 4$, $b = 2$, $c = 2$, $d = 1$

Hence, the value of $(a+b+c+d+1)$

$= 4 + 2 + 2 + 1 + 1$

$= 10.$

Ex-38. If α and β are two values of θ satisfying the equation $\frac{\cos\theta}{a} + \frac{\sin\theta}{b} = \frac{1}{c}$ prove that $\cot\left(\frac{\alpha+\beta}{2}\right) = \frac{b}{a}$

Soln. Given equation is $\frac{\cos\theta}{a} + \frac{\sin\theta}{b} = \frac{1}{c}$

$$\Rightarrow bc\cos\theta + ac\sin\theta = ab$$

$$\Rightarrow bc\left(\frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}\right) + ac\left(\frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)}\right) = ab$$

$$\Rightarrow bc\left(1-\tan^2(\theta/2)\right) + 2ac\tan(\theta/2)$$

$$= ab\left(1+\tan^2(\theta/2)\right)$$

$$\Rightarrow (ab+bc)\tan^2(\theta/2) - 2ac\tan(\theta/2) + (ab-bc) = 0$$

Let its roots are $\tan(\alpha/2)$ and $\tan(\beta/2)$

$$\tan(\alpha/2) + \tan(\beta/2) = \frac{2ac}{b(a+c)}$$

and $\tan(\alpha/2).\tan(\beta/2) = \frac{(a-c)}{(a+c)}$

Now, $\tan\left(\frac{\alpha+\beta}{2}\right)$

$$= \frac{\tan(\alpha/2)+\tan(\beta/2)}{1-\tan(\alpha/2).\tan(\beta/2)}$$

$$= \frac{2ac/b(a+c)}{1-(a-c)/(a+c)}$$

$$= \frac{2ac/b}{a+c-a+c}$$

$$= \frac{2ac}{2bc} = \frac{a}{b}$$

Thus, $\cot\left(\frac{\alpha+\beta}{2}\right) = \frac{b}{a}$.

Ex-39. Prove that $\sin\left(\frac{\pi}{14}\right)$ is a root of

$$8x^3 - 4x^2 - 4x + 1 = 0$$

Soln. Let $\frac{\pi}{14} = \theta$

$$\Rightarrow 7\theta = \frac{\pi}{2}$$

$$\Rightarrow 4\theta = \frac{\pi}{2} - 3\theta$$

$$\Rightarrow \sin(4\theta) = \sin\left(\frac{\pi}{2} - 3\theta\right) = \cos(3\theta)$$

$$\Rightarrow \sin(4\theta) = \cos(3\theta)$$

$\Rightarrow \quad 2\sin(2\theta)\cos(2\theta) = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow \quad 4\sin\theta\cos\theta\left(1 - 2\sin^2\theta\right)$

$= \cos\theta\left(4\cos^2\theta - 3\right)$

$\Rightarrow \quad 4\sin\theta\left(1 - 2\sin^2\theta\right) = \left(1 - 4\sin^2\theta\right)$

$\Rightarrow \quad 4\sin\theta - 8\sin^3\theta = \left(1 - 4\sin^2\theta\right)$

$\Rightarrow \quad 8\sin^3\theta - 4\sin^2\theta - 4\sin\theta + 1 = 0$

put $\sin\theta = x$

Hence, the required equation is

$8x^3 - 4x^2 - 4x + 1 = 0$

Ex-40. Prove that:

$\left(\tan^2\theta + \tan^2 2\theta + \tan^2 3\theta\right)$,

$\times\left(\cot^2\theta + \cot^2 2\theta + \cot^2 3\theta\right) = 105$

where $\theta = \dfrac{\pi}{7}$

Ex-41 If $x + y + z = xyz$, prove that

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2}$$

$$= \frac{2x}{1-x^2}.\frac{2y}{1-y^2}.\frac{2z}{1-z^2}$$

[Roorkee – 1983]

Soln. Let $x = \tan A, y = \tan B, z = \tan C$

Given $\tan A + \tan B + \tan C = \tan A.\tan B.\tan C$

$\Rightarrow \quad \tan A + \tan B = -\tan C + \tan A.\tan B.\tan C$

$\Rightarrow \quad \tan A + \tan B = -\tan C\left(1 - \tan A.\tan B\right)$

$\Rightarrow \quad \dfrac{\tan A + \tan B}{\left(1 - \tan A.\tan B\right)} = -\tan C$

$\Rightarrow \quad \tan(A + B) = -\tan C$

$\Rightarrow \quad \tan(A + B) = \tan(\pi - C)$

$\Rightarrow \quad (A + B) = (\pi - C)$

$\Rightarrow \quad A + B + C = \pi$

$\Rightarrow \quad 2A + 2B + 2C = 2\pi$

$\Rightarrow \quad 2A + 2B = 2\pi - 2C$

$\Rightarrow \quad \tan(2A + 2B) = \tan(2\pi - 2C)$

$\Rightarrow \quad \tan(2A + 2B) = -\tan 2C$

$\Rightarrow \quad \dfrac{\tan 2A + \tan 2B}{1 - \tan 2A.\tan 2B} = -\tan 2C$

$\Rightarrow \quad \tan 2A + \tan 2B + \tan 2C = \tan 2A.\tan 2B.\tan 2C$

$\Rightarrow \quad \dfrac{2\tan A}{1-\tan^2 A} + \dfrac{2\tan B}{1-\tan^2 B} + \dfrac{2\tan C}{1-\tan^2 C}$

$\Rightarrow \quad \dfrac{2\tan A}{1-\tan^2 A}.\dfrac{2\tan B}{1-\tan^2 B}.\dfrac{2\tan C}{1-\tan^2 C}$

$\Rightarrow \quad \dfrac{2x}{1-x^2} + \dfrac{2y}{1-y^2} + \dfrac{2z}{1-z^2}$

$\Rightarrow \quad \dfrac{2x}{1-x^2}.\dfrac{2y}{1-y^2}.\dfrac{2z}{1-z^2}$

$= \dfrac{2x}{1-x^2}.\dfrac{2y}{1-y^2}.\dfrac{2z}{1-z^2}$

Note: No questions asked in 1984.

Ex-42. If $\cos\alpha + \cos\beta + \cos\gamma = 0$

and $\sin\alpha + \sin\beta + \sin\gamma = 0$, then prove that

$\cos(3\alpha) + \cos(3\beta) + \cos(3\gamma) = 3\cos(\alpha + \beta + \gamma)$

and $\sin(3\alpha) + \sin(3\beta) + \sin(3\gamma) = 3\sin(\alpha + \beta + \gamma)$

[Roorkee – 1985]

Soln. Let $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$

and $c = \cos\gamma + i\sin\gamma$

Then $a + b + c$

$= (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i.0 = 0$

Therefore, $a^3 + b^3 + c^3 = 3abc$

$\Rightarrow \quad (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3$

$+(\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$= 3\left(\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)\right)$

$\Rightarrow \quad (\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta)$

$+(\cos 3\gamma + i\sin 3\gamma)$

$= 3\left(\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma)\right)$

$\Rightarrow \quad (\cos 3\alpha + \cos 3\beta + \cos 3\gamma)$

$+i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$

$= 3(\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma))$

Comparing the real and imaginary parts, we get,

$\cos(3\alpha)+\cos(3\beta)+\cos(3\gamma)=3\cos(\alpha+\beta+\gamma)$

and $\sin(3\alpha)+\sin(3\beta)+\sin(3\gamma)=3\sin(\alpha+\beta+\gamma)$

Hence, the result.

Note: No questions asked in 1986, 1987, 1988.

Ex-43. Show that (without using tables)

$\tan 9°-\tan 27°-\tan 63°+\tan 81°=4$

[Roorkee – 1989]

Soln. We have $\tan 9°-\tan 27°-\tan 63°+\tan 81°$

$= (\tan 9°+\tan 81°)-(\tan 27°+\tan 63°)$

$= (\tan 9°+\cot 9°)-(\tan 27°+\cot 27°)$

$= \left(\frac{\sin 9°}{\cos 9°}+\frac{\cos 9°}{\sin 9°}\right)-\left(\frac{\sin 27°}{\cos 27°}+\frac{\cos 27°}{\sin 27°}\right)$

$= \frac{1}{\sin 9°\cos 9°}-\frac{1}{\sin 27°\cos 27°}$

$= \frac{1\times 2}{2\sin 9°\cos 9°}-\frac{1\times 2}{2\sin 27°\cos 27°}$

$= \frac{2}{\sin 18°}-\frac{2}{\sin 54°}$

$= \frac{2}{\left(\frac{\sqrt{5}-1}{4}\right)}-\frac{2}{\left(\frac{\sqrt{5}+1}{4}\right)}$

$= \frac{8}{(\sqrt{5}-1)}-\frac{8}{(\sqrt{5}+1)}$

$= 8\left(\frac{\sqrt{5}+1-\sqrt{5}+1}{5-1}\right)$

$= \frac{16}{4}$

$= 4$

Ex-44. Find 'a' and 'b' such that the inequality

$a \le \cos x+5\sin\left(x-\frac{\pi}{6}\right)\le b$ holds good for all x.

[Roorkee – 1989]

Soln. We have, $\cos x+5\sin\left(x-\frac{\pi}{6}\right)$

$= \cos x+5\left(\sin x\cos\left(\frac{\pi}{6}\right)-\cos x\sin\left(\frac{\pi}{6}\right)\right)$

$= \cos x+5\left(\frac{\sqrt{3}}{2}\sin x-\frac{1}{2}\cos x\right)$

$= \left(1-\frac{5}{2}\right)\cos x+\frac{5\sqrt{3}}{2}\sin x$

$= -\frac{3}{2}\cos x+\frac{5\sqrt{3}}{2}\sin x$

Max value $= \sqrt{\frac{9}{4}+\frac{75}{4}}=\sqrt{\frac{84}{4}}=\sqrt{21}$

Main value $= -\sqrt{\frac{9}{4}+\frac{75}{4}}=-\sqrt{\frac{84}{4}}=-\sqrt{21}$

Thus, $a=-\sqrt{21}$, $b=\sqrt{21}$

Note. No questions asked in 1990, 1991.

Ex-45. If $A=\cos^2\theta+\sin^4\theta$, then for all values of θ, find the range of A. [Roorkee – 1992]

Soln. $A=\cos^2\theta+\sin^4\theta$

$= \frac{1}{2}(2\cos^2\theta)+\frac{1}{4}(\sin^2\theta)^2$

$= \frac{1}{2}(1+\cos 2\theta)+\frac{1}{4}(1-\cos 2\theta)^2$

$= \frac{1}{2}(1+\cos 2\theta)+\frac{1}{4}(1-2\cos 2\theta+\cos^2 2\theta)$

$= \frac{1}{2}+\frac{1}{4}+\frac{1}{4}\cos^2 2\theta$

$= \frac{3}{4}+\frac{1}{4}\cos^2 2\theta$

Max value $= \frac{3}{4}+\frac{1}{4}.1=1$

Min value $= \frac{3}{4}+\frac{1}{4}.0=\frac{3}{4}$

Hence, the range of A is $\left[\frac{3}{4},1\right]$.

Ex-46. Given the product p of sines of the angles of a triangle and the product q of their cosines, find the cubic equation, whose co-efficients are functions of p and q and whose roots are the tangents of the angles of the triangle.

Ans. $qx^3-px^2+(1+q)x-p=0$

[Roorkee Main – 1992]

Soln. Given, $\sin A\ \sin B\ \sin C=p$

$\cos A\ \cos B\ \cos C=q$

Thus, $\tan A\ \tan B\ \tan C = \frac{p}{q}$

Also, $A + B + C = \pi$

$\tan A + \tan B + \tan C = \tan A\ \tan B\ \tan C$

$\tan A + \tan B + \tan C = \frac{p}{q}$

Also, $\tan A \tan B + \tan B \tan C + \tan C \tan A$

$= \frac{1+q}{q}$

Hence, the required equation is

$$x^3 - \left(\frac{p}{q}\right)x^2 + \left(\frac{1+q}{q}\right)x - \left(\frac{p}{q}\right) = 0$$

$q x^3 - p x^2 + (1+q)x - p = 0$.

Note. No questions asked in 1993, 1994.

Ex-48 If $x = \cos 10° \cos 20° \cos 40°$, then find the value of x

[Roorkee – 1995]

Soln. Given, $x = \cos 10° \cos 20° \cos 40°$

$= \frac{1}{2\sin 10°}(2\sin 10° \cos 10°)\cos 20° \cos 40°$

$= \frac{1}{4\sin 10°}(2\sin 20° \cos 20°)\cos 40°$

$= \frac{1}{8\sin 10°}(2\sin 40° \cos 40°)$

$= \frac{1}{8\sin 10°}(\sin 80°)$

$= \frac{1}{8\sin 10°}(\sin 10°)$

$= \frac{1}{8}\cot 10°$

Note. No questions asked in 1996, 1997, 1998, 1999.

Ex-49. Find the real values of x for which $27^{\cos 2x}.81^{\sin 2x}$ is minimum and also find its minimum value

[Roorkee Main – 2000]

Soln. Let $y = 27^{\cos 2x}.81^{\sin 2x}$

$= 3^{3\cos 2x + 4\sin 2x}$

Max value of $y = 3^5 = 243$

Min value of $y = 3^{-5} = \frac{1}{243}$

The expression $y = 27^{\cos 2x}.81^{\sin 2x}$ is min when

Let $f(x) = 3\cos 2x + 4\sin 2x$

$f'(x) = -6\sin 2x + 8\cos 2x$

Now, $f'(x) = 0$ gives, $\tan 2x = -\frac{8}{6} = -\frac{3}{4}$

$\tan 2x = -\frac{3}{4} = \tan\alpha$

$2x = n\pi + \alpha$, where $\alpha = \tan^{-1}\left(-\frac{3}{4}\right)$

Ex-50. If $e^{i\theta - \log\cos(x - iy)} = 1$, then find the values of x and y in terms of θ.

[Roorkee Main – 2001]

Soln. Given, $e^{i\theta - \log\cos(x - iy)} = 1$

$\Rightarrow \quad e^{i\theta} = \cos(x - iy)$

$\Rightarrow \quad (\cos\theta + i\sin\theta)$

$= \cos x \cos(iy) + \sin x \sin(iy)$

$= \cos x \cos(hy) + i \sin x \sin(hy)$

Thus, $\cos\theta = \cos x \cos(hy)$

and $\sin\theta = \sin x\ \sin(h\,y)$

Now, $\frac{\cos^2\theta}{\cos^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$

$\Rightarrow \quad \frac{\cos^2\theta}{1-p^2} - \frac{\sin^2\theta}{p^2} = 1, \sin x = p$

$\Rightarrow \quad p^2(1-p^2) = p^2\cos^2\theta + (p^2 - 1)\sin^2\theta$

$= p^2 - \sin^2\theta$

$\Rightarrow \quad p^4 = \sin^2\theta$

$\Rightarrow \quad p = \pm(\sin\theta)^{1/2}$

$\Rightarrow \quad \sin x = \pm(\sin\theta)^{1/2}$

$\Rightarrow \quad x = 2n\pi \pm (\sin\theta)^{1/2}$

Also, $\sin(hy) = \frac{\sin\theta}{\pm(\sin\theta)^{1/2}} = \pm(\sin\theta)^{1/2}$

$\Rightarrow \quad y = h^{-1}\left(\sin^{-1}\left(\pm(\sin\theta)^{1/2}\right)\right)$

LEVEL I
(PROBLEMS BASED ON FUNDAMENTALS)

1. Find the values of the expression
$$3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$$
2. If $m = \sin^6 x + \cos^6 x$, then find m.
3. If $\operatorname{cosec}\theta - \cot\theta = q$, then find cosec θ.

4. Find the value of
$\tan\frac{\pi}{20} \cdot \tan\frac{3\pi}{20} \cdot \tan\frac{5\pi}{20} \cdot \tan\frac{7\pi}{20} \cdot \tan\frac{9\pi}{20}$

5. If $x = \frac{2\sin\theta}{1+\cos\theta+\sin\theta}$, then find the values of $\frac{1-\cos\theta+\sin\theta}{1+\sin\theta}$

6. If $\sin x + \sin^2 x = 1$, then find the value of $\cos^8 x + 2\cos^6 x + \cos^4 x$

7. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then find the value of $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$.

8. If $\sin x + \sin^2 x + \sin^3 x = 1$, then find the value of $\cos^6 x - 4\cos^4 x + 8\cos^2 x$

9. If $\cos x + \cos y + \cos\alpha = 0 = \sin x + \sin y + \sin\alpha$ then find the value of $\cot\left(\frac{x+y}{2}\right)$.

10. If $f(x) = \sin^2 x + \sin^2\left(x+\frac{\pi}{3}\right) + \cos x\cos\left(x+\frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1$ then find the value of $(gof)(x)$

11. If $\sin x + \sin^2 x = 1$, then find the value of $\cos^2 x + \cos^4 x$.

12. If $A = \cos^2\theta + \sin^4\theta$, then find the range of A.

13. Find the maximum or minimum values of:
(i) $3\cos x + 4\sin x + 8$
(ii) $5\cos x + 3\cos\left(x+\frac{\pi}{3}\right) + 3$

14. If $\cos 25° + \sin 25° = p$, then find $\cos 50°$.

15. Find the maximum value of $12\sin\theta - 9\sin^2\theta$.

16. If $\operatorname{cosec}\theta - \sin\theta = a^3$ and $\sec\theta - \cos\theta = b^3$, then find the value of $a^2b^2(a^2+b^2)$.

17. If $x = \sec\theta - \tan\theta$ and $y = \operatorname{cosec}\theta + \cot\theta$, then prove that $xy + x - y + 1 = 0$.

18. If $a\cos\theta - b\sin\theta = c$, then find the value of $a\sin\theta + b\cos\theta$

19. If $\frac{x}{\cos\theta} = \frac{y}{\cos\left(\theta-\frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta+\frac{2\pi}{3}\right)}$, then find the value of $x + y + z$.

20. Find the value of $\tan 70° - \tan 20°$.

21. If $\tan\theta = \frac{b}{a}$, then find $a\sin 2\theta + b\cos 2\theta$.

22. Find the value of $\tan 9° - \tan 27° - \tan 63° + \tan 81°$.

23. If $\tan\alpha + \tan\left(\alpha+\frac{\pi}{3}\right) + \tan\left(\alpha+\frac{2\pi}{3}\right) = \lambda\tan 3\alpha$, then find λ.

24. Find the value of $\frac{1}{\sin 10°} - \frac{\sqrt{3}}{\cos 10°}$.

25. If $2\cos\theta = x + \frac{1}{x}$, $2\cos\phi = y + \frac{1}{y}$, then find the value of $\cos(\theta - \varphi)$.

26. Prove that $\frac{\sec 8\theta - 1}{\sec 4\theta - 1} = \tan 8\theta . \cot 2\theta$.

27. If α and β are the solutions of the equation $a\tan\theta + b\sec\theta = c$, then find $\tan(\alpha + \beta)$.

28. If $a\cos 2\theta + b\sin 2\theta = c$, has α and β as its solution, then find
(i) $\tan\alpha + \tan\beta$
(ii) $\tan\alpha . \tan\beta$.

29. If $\sin A + \sin B = a$ and $\cos A + \cos B = b$, then find $\cos(A + B)$

30. If $\left(1+\sqrt{1+y}\right)\tan y = \left(1+\sqrt{1-y}\right)$, then prove that $\sin 4y = y$.

31. Prove that, $(2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1) \ldots(2\cos 2^{n-1}\theta - 1) = \left(\frac{2\cos(2^n\theta)+1}{2\cos\theta+1}\right)$

32. Prove that, $\cos(9°) + \sin(9°) = \left(\frac{\sqrt{3+\sqrt{5}}}{2}\right)$

33. If $\sin x + \cos x = \frac{\sqrt{7}}{2}$, where $x \in \left[0, \frac{\pi}{4}\right]$ then prove that $\tan\left(\frac{x}{2}\right) = \left(\frac{\sqrt{7}-2}{3}\right)$

34. If $\frac{\sin x}{\sin y} = \frac{1}{2}$ and $\frac{\cos x}{\cos y} = \frac{3}{2}$, where $x, y \in \left(0, \frac{\pi}{2}\right)$, then prove that $\tan(x+y) = \sqrt{15}$.

35. If $\sec(x+y) + \sec(x-y) = 2\sec x$, where $x, y \in \left(0, \frac{\pi}{2}\right)$, then prove that $\cos x = \sqrt{2}\cos\left(\frac{y}{2}\right)$.

LEVEL II
(MIXED PROBLEMS)

1. If $\sec x = p + \frac{1}{p}$, then sec x + tan x is
(a) p (b) $2p$ (c) $\frac{1}{4p}$ (d) $\frac{4}{p}$.

2. If cosec x – sin x = a^3, sec x – cos x = b^3, then $a^2 b^2 (a^2 + b^2)$ is
(a) 0 (b) 1 (c) –1 (d) ab.

3. If sec x + cos x = 2, then the value of $\sec^3 x\,(1+ \sec^3 x) + \cos^3 x\,(1 + \cos^3 x)$ is
(a) 2 (b) 4 (c) 6 (d) 8.

4. The value of $\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)$ is
(a) 1/2 (b) –1/2 (c) 0 (d) None.

5. Which of the following is smallest ?
(a) sin 1 (b) sin 2 (c) sin 3 (d) sin 4.

6. Which of the following is greatest ?
(a) sin 1 (b) cos 1 (c) tan 1 (d) cot 1

7. If $A = \cos(\cos x) + \sin(\cos x)$, then the least and greatest value of A are
(a) 0, 2 (b) –1, 1
(c) $-\sqrt{2}$, $\sqrt{2}$ (d) 0, $\sqrt{2}$.

8. If $A+B=\frac{\pi}{3}$, A, $B > 0$ then the maximum value of tan A. tan B is
(a) 1/3 (b) 1 (c) 1/2 (d) 2/3

9. The maximum value of a sin 2 x + b cos 2 x for all real x is
(a) $a + b$ (b) $\sqrt{a^2+b^2}$
(c) Max {|a|, |b|} (d) Max {a, b}.

10. Which of the following is / are true ?
(a) sin 1 > sin ° (b) tan 1 > tan °
(c) sin 4 > sin ° (d) tan 4 > tan °.

11. If cos 5 x = $a\cos^5 x + b\cos^3 x + c\cos x + d$, then
(a) a = 16 (b) b –20 (c) c = 5 (d) d = 2.

12. If $\sin^3 x \sin 3x = c_0 + c_1 \cos x + c_2 \cos 2x + c_3 \cos 3x + \dots\dots\dots + c_n \cos n_x$, then
(a) Highest value of n is 6
(b) c_0 = 1/8
(c) $c_2 = -c_4$
(d) $c_1 = c_3 = c_5$.

13. If $f(x)=\cos[\pi]x+\sin[\pi]x$, where [,] is the greatest integer function, then $f\left(\frac{\pi}{2}\right)$ is
(a) 0 (b) cos 3 (c) cos 4 (d) None.

14. Let $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$,
then the maximum value of $f(x)$ is
(a) 0 (b) 2 (c) 6 (d) None.

15. For any real x, the maximum value of $\cos^2(\cos x)+\sin^2(\sin x)$ is
(a) 1 (b) 1 + $\sin^2 1$
(c) 1 + $\cos^2 1$ (d) None.

16. If $a = \sin\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{7\pi}{18}\right)$ and x is the solution of the equation y = 2 [x] + 2 and y = 3[x – 2], where [,] = G.I.F, then a is
(a) [x] (b) 1/[x] (c) 2[x] (d) $[x]^2$.

17. The minimum value of $\sin^8 x + \cos^8 x$ is
(a) 0 (b) 1 (c) 1/8 (d) 2.

18. If $\frac{\sin^4\theta}{a}+\frac{\cos^4\theta}{b}=\frac{1}{a+b}$, then $\frac{\sin^8\theta}{a^3}+\frac{\cos^8\theta}{b^3}$ is
(a) $\frac{1}{a^3+b^3}$ (b) $\frac{1}{(a+b)^3}$
(c) $\frac{1}{(a-b)^3}$ (d) None.

19. The value of $\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{2\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right)$ is
(a) 1 (b) $\frac{1}{\sqrt{7}}$ (c) $\sqrt{7}$ (d) None.

20. If α and β are the solutions of $\sin^2 x + a\sin x + b = 0$ as well as that of $\cos^2 x + c\cos x + d = 0$, then $\sin(\alpha + \beta)$ is
(a) $\frac{2bd}{b^2+d^2}$ (b) $\frac{a^2+c^2}{2ac}$
(c) $\frac{b^2+d^2}{2bd}$ (d) $\frac{2ac}{a^2+c^2}$.

21. If $\sec\theta+\tan\theta=1$, then one of the roots of the equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ is
(a) tan θ (b) sec θ (c) cos θ (d) sin θ.

22. If α is the common +ve root of the equation $x^2-ax+12=0$, $x^2-bx+15=0$ and $x^2-(a+b)x+36=0$ and $\cos x+\cos 2x+\cos 3x=0$, then $\sin x+\sin 2x+\sin 3x$ is

(a) 3 (b) –3 (c) 0 (d) 2

23. For any real θ, the maximum value of $\cos^2(\cos\theta)+\sin^2(\sin\theta)$ is
(a) 1 (b) $1+\sin^2 1$
(c) $1+\cos^2 1$ (d) $1-\cos^2 1$

24. If $\alpha+\beta=\pi/2$ and $\beta+\gamma=\alpha$, then α is
(a) $2(\tan\beta+\tan\gamma)$
(b) $(\tan\beta+\tan\gamma)$
(c) $(\tan\beta+2\tan\gamma)$
(d) $(2\tan\beta+\tan\gamma)$

25. The maximum value of $\cos\alpha_1.\cos\alpha_2.\cos\alpha_3.............\cos\alpha_n$ under the restriction $0\le\alpha_1,\alpha_2,\alpha_3,...............,\alpha_n\le\frac{\pi}{2}$ and $\cot\alpha_1.\cot\alpha_2.\cot\alpha_3..............\cot\alpha_n=1$, is
(a) $\frac{1}{2^{n/2}}$ (b) $\frac{1}{2^n}$ (c) $\frac{1}{2^n}$ (d) 1

26. If $A>0$ and $B>0$ and $A+B=\frac{\pi}{3}$, then the maximum value of $\tan A.\tan B$ is
(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{2}}$

27. If $\tan\beta=2\sin\alpha.\sin\gamma.\operatorname{cosec}(\alpha+\gamma)$, then $\cot\alpha$, $\cot\beta$, $\cot\gamma$ are in
(a) A.P (b) G.P (c) H.P (d) A.G.P

28. The minimum value of the expression $\sin\alpha+\sin\beta+\sin\gamma$, where α,β,γ are real +ve angles satisfying $\alpha+\beta+\gamma=\pi$, is
(a) +ve (b) –ve (c) 0 (d) –3

29. The value of $4\cos20°-\sqrt{3}\cot20°$ is
(a) 1 (b) –1 (c) –1/2 (d) 1/4

30. The maximum value of $4\sin^2 x+3\cos^2 x+\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)$ is
(a) $4+\sqrt{2}$ (b) $3+\sqrt{2}$
(c) $4-\sqrt{2}$ (d) 4

31. The value of the expression $\left(\sqrt{3}\sin75°-\cos75°\right)$ is
(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$
(c) $\sqrt{2}$ (d) 2

32. The value of $(4+\sec20°)\sin20°$ is
(a) 1 (b) $\sqrt{2}$
(c) $\sqrt{3}$ (d) $2\sqrt{3}$

33. If $(1+\tan1°)(1+\tan2°)...(1+\tan45°)=2^n$ then the value of n is
(a) 20 (b) 21
(c) 22 (d) 23

34. The number of all possible triplets (a_1,a_2,a_3) such that $a_1+a_2\cos(2x)+a_3\sin^2 x=0$ is
(a) 0 (b) 1
(c) 2 (d) infinite

35. If $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$, then the value of $\sin(2\theta)$ is
(a) –1/2 (b) –1/3
(c) –2/3 (d) –3/4

36. A real root of the equation $8x^3-6x-1=0$ is
(a) $\cos\left(\frac{\pi}{5}\right)$ (b) $\cos\left(\frac{\pi}{9}\right)$
(c) $\cos\left(\frac{\pi}{18}\right)$ (d) $\cos\left(\frac{\pi}{36}\right)$

37. The value of $\left(\sqrt{3}\cot(20°)-4\cos(20°)\right)$ is
(a) 1 (b) –1
(c) $-\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}}{2}$

38. If $\tan^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{a}{b}$, then $\sin(\theta)$ is
(a) $\left(\frac{a-b}{a+b}\right)$ (b) $-\left(\frac{a-b}{a+b}\right)$
(c) $\left(\frac{a+b}{a-b}\right)$ (d) $-\left(\frac{a+b}{a-b}\right)$

39. The least value of $\operatorname{cosec}^2 x+25\sec^2 x$ is
(a) 26 (b) 36
(c) 16 (d) 12

40. Let $y=\frac{\sin^3 x}{\cos x}-\frac{\cos^3 x}{\sin x}, 0<x<\frac{\pi}{2}$
Then the minimum value of y
(a) 0 (a) 1
(c) 3/2 (d) 2

41. The expression $\tan(55°)\tan(65°)\tan(75°)$ simplifies to $\cot(x°)$, $0<x<90$, then x is
(a) 5 (b) 8
(c) 9 (d) 10

42. If x_1 and x_2 are the roots of $x^2+(1-\sin\theta)x-\frac{1}{2}\cos^2\theta=0$, then the maximum value of $x_1^2+x_2^2$ is
(a) 2 (b) 3 (c) 9/4 (d) 4

43. The value of the expression $\cos^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{3\pi}{8}\right)+\cos^2\left(\frac{5\pi}{8}\right)+\cos^2\left(\frac{7\pi}{8}\right)$ is
(a) rational (b) integral
(c) prime (d) composite

44. If $\tan x=a$, then the value of $\cot\left(\frac{\pi}{4}-a\right)$ is
(a) $\left(\frac{a-1}{a+1}\right)$ (b) $\left(\frac{a^2-1}{a^2+1}\right)$
(c) $\left(\frac{a^2+1}{a^2-1}\right)$ (d) $\left(\frac{a+1}{a-1}\right)$

45. If $\sin\theta+\cos\theta=\frac{1}{5}$, $0\le\theta\le\pi$, then $\tan\theta$
(a) 3/4 (b) 4/3
(c) –3/4 (d) –4/3

LEVEL III
(TOUGHER PROBLEMS FOR JEE ADVANCED)

1. Prove that the sum of
$\tan x\tan 2x+\tan 2x\tan 3x+...+\tan x\tan(n+1)x$
$=\cot x\tan(n+1)x-(n+1)$

2. Prove that
$\operatorname{cosec} x+\operatorname{cosec}2x+\operatorname{cosec}4x+....$to n terms
$=\cot\left(\frac{x}{2}\right)-\cot\left(2^{n-1}x\right)$.

3. Prove that, $\cot(16°)\cot(44°)+\cot(44°)\cot(76°)-\cot(76°)\cot(16°)=3$

5. If $\theta=\frac{\pi}{2^n-1}$, prove that $2^n\cos\theta$
$\cos(2\theta).\cos(4\theta).\cos(8\theta)....\cos\left(2^{n-1}\theta\right)=-1$

6. Prove that
$\sin\left(\frac{2\pi}{7}\right)+\sin\left(\frac{4\pi}{7}\right)+\sin\left(\frac{8\pi}{7}\right)=\frac{\sqrt{7}}{2}$.

7. Prove that
$\tan^2\left(\frac{\pi}{16}\right)+\tan^2\left(\frac{2\pi}{16}\right)+....+\tan^2\left(\frac{7\pi}{16}\right)=35$

8. Prove that, $\left(\tan^2\left(\frac{\pi}{7}\right)+\tan^2\left(\frac{2\pi}{7}\right)+\tan^2\left(\frac{3\pi}{7}\right)\right)$
$\times\left(\cot^2\left(\frac{\pi}{7}\right)+\cot^2\left(\frac{2\pi}{7}\right)+\cot^2\left(\frac{3\pi}{7}\right)\right)=105$

9. Prove that, $\frac{3+\cot(76°)\cot(16°)}{\cot(76°)+\cot(16°)}=\cot(44°)$

10. If $\cos x+\cos y+\cos z=0$, then prove that $\cos(3x)+\cos(3y)+\cos(3z)=12\cos x\cos y\cos z$

11. Prove that, $\tan^6\left(\frac{\pi}{9}\right)-33\tan^4\left(\frac{\pi}{9}\right)+27\tan^2\left(\frac{\pi}{9}\right)=3$

12. If $\cos A+\cos B+\cos C=0=\sin A+\sin B+\sin C$ then prove that $\sin^2 A+\sin^2 B+\sin^2 C$
$=\cos^2 A+\cos^2 B+\cos^2 C=\frac{3}{2}$.

13. Let A, B, C be three angles such that $A=\frac{\pi}{4}$ and $\tan B.\tan C=p$. Find all possible values of p such that A, B, C are three angles of a triangle.

14. If $\frac{\tan 3A}{\tan A}=k$, show that $\frac{\sin 3A}{\sin A}=\frac{2k}{k-1}$ and k cannot lie between 1/3 and 3.

15. If $A+B+C=\pi$, then prove that
$\cot A+\cot B+\cot C-\operatorname{cosec} A\operatorname{cosec} B\operatorname{cosec} C$
$=\cot A.\cot B.\cot C$

16. If $\tan\alpha=\frac{1}{\sqrt{x(x^2+x+1)}}$ $\tan\beta=\frac{\sqrt{x}}{\sqrt{(x^2+x+1)}}$ and
$\tan\gamma=\frac{\sqrt{(x^2+x+1)}}{x\sqrt{x}}$ then prove that $\alpha+\beta=\gamma$

17. If α and β are acute angles and $\cos 2\alpha=\frac{3\cos 2\beta-1}{3-\cos 2\beta}$,
prove that $\tan\alpha:\tan\beta=\sqrt{2}:1$.

18. If $\tan^3\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)=\tan\left(\frac{\beta}{2}+\frac{\pi}{4}\right)$, then prove that
$\sin\beta=\frac{(3+\sin^2\alpha)\sin\alpha}{1+3\sin^2\alpha}$

19. If $\sin\beta=\frac{1}{5}\sin(2\alpha+\beta)$, then prove that
$\tan(\alpha+\beta)=\frac{3}{2}\tan\alpha$

20. If $\sin x+\sin y=3(\cos x-\cos y)$ then prove that $\sin(3x)+\sin(3y)=0$

21. If $\sec(\varphi-\alpha)$, $\sec\varphi$, $\sec(\varphi+\alpha)$ are in A.P then prove that $\cos(\varphi)=\sqrt{2}\cos\left(\frac{\alpha}{2}\right)$

22. If $\tan\left(\frac{x+y}{2}\right), \tan z, \tan\left(\frac{x-y}{2}\right)$ are in G.P then prove that $\cos(x)=\cos(y)\cos(2z)$

23. Prove that $\frac{\sec^2\theta-\tan\theta}{\sec^2\theta+\tan\theta}$ lies between 1/3 and 3 for all real θ.

24. If $\theta=\frac{\pi}{2^n+1}$, then find the value of $2^n\cos(\theta)\cos(2\theta)\cos(2^2\theta)...\cos(2^{n-1}\theta)$

25. Find the value of $\tan(6°)\tan(42°)\tan(66°)\tan(78°)$

26. If $\frac{\tan(\alpha+\beta-\gamma)}{\tan(\alpha-\beta+\gamma)}=\frac{\tan\gamma}{\tan\beta}$, then prove that $\sin(\beta-\gamma)=0$ or $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$.

27. If $A+B+C=\pi$, then prove that $\cot A+\frac{\sin A}{\sin B\sin C}=\cot B+\frac{\sin B}{\sin A\sin C}$ $=\cot C+\frac{\sin C}{\sin A\sin B}$

28. If $\frac{\sin(\theta+A)}{\sin(\theta+B)}=\sqrt{\frac{\sin(2A)}{\sin(2B)}}$, then prove that $\tan^2\theta=\tan A\tan B$.

29. If $\cos(x-y)=-1$, then prove that $\cos x+\cos y=0$ and $\sin x+\sin y=0$.

30. If $\sqrt{2}\cos A=\cos B+\cos^3 B$ and $\sqrt{2}\sin A=\sin B-\sin^3 B$ then prove that $\sin(A-B)=\pm\frac{1}{3}$

31. Prove that $\sin(9°)=\frac{1}{4}\left(\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}\right)$

32. Find the range of $f(x)=\sin\left(\sqrt{\frac{\pi^2}{36}-x^2}\right)$

33. Find the value of $\sum_{k=1}^{6}\left(\sin\left(\frac{2k\pi}{7}\right)-i\cos\left(\frac{2k\pi}{7}\right)\right)$ where $i=\sqrt{-1}$.

34. If $\cos\theta+\cos\varphi=a$ and $\sin\theta+\sin\varphi=b$ then find the value of $\tan\left(\frac{\theta}{2}\right)+\tan\left(\frac{\varphi}{2}\right)$

35. If $\frac{\tan\theta}{\tan\theta-\tan 3\theta}=\frac{1}{3}$, then find the value of $\frac{\cot\theta}{\cot(\theta)-\cot(3\theta)}$.

INTEGER TYPE QUESTIONS

1. If $\frac{x}{\cos\theta}=\frac{y}{\cos\left(\theta-\frac{2\pi}{3}\right)}=\frac{z}{\cos\left(\theta+\frac{2\pi}{3}\right)}$, then find the value of $(x+y+z+4)$
2. Find the numerical value of $\sum_{r=0}^{9}\sin^2\left(\frac{\pi r}{18}\right)$
3. If $\frac{\sin x}{\sin y}=\frac{1}{2}$ and $\frac{\cos x}{\cos y}=\frac{3}{2}$, where $x,y\in\left(0,\frac{\pi}{2}\right)$. then find the value of $\frac{\tan^2(x+y)}{5}$.
4. If $\cos(x-y),\cos x,\cos(x+y)$ are in H.P such that $\left|\sec x.\cos\left(\frac{y}{2}\right)\right|=m$, then find the value of (m^2+2).
5. If $\tan x+\tan\left(\frac{\pi}{3}+x\right)+\tan\left(\frac{2\pi}{3}+x\right)$ $=k\tan 3x$, then find k.
6. Let $f(\theta)=\sin^2\theta+\sin^2\left(\frac{2\pi}{3}+\theta\right)+\sin^2\left(\frac{4\pi}{3}+\theta\right)$. Then find the value of $2f\left(\frac{\pi}{15}\right)$.
7. If $m=\sqrt{3}\operatorname{cosec}(20°)-\sec(20°)$ and $n=\sin(12°)\sin(48°)\sin(54°)$ where $m,n\in N$, then find the value of $(m+8n+2)$.

8. Let $\tan(15^\circ)$ and $\tan(30^\circ)$ are the roots of $x^2+px+q=0$, then find the value of $(2+q-p)$.

9. Let $x=\dfrac{\sum_{n=1}^{44}\cos(n^\circ)}{\sum_{n=1}^{44}\sin(n^\circ)}$, then find the value $[x+3]$, where [,] = G.I.F

10. If the value of the expression $\sin(25^\circ)\sin(35^\circ)\sin(85^\circ)$ can be expressed as $\dfrac{\sqrt{a}+\sqrt{b}}{c}$ where $a,b,c\in N$ and are in their lowest form, find the value of $\left(\dfrac{c}{a+b}+2\right)$

11. Let $m=\sum_{k=1}^{17}\cos\left(\dfrac{k\pi}{9}\right)$, then find the value of (m^2+m+2).

12. If the expression $\tan(55^\circ)\tan(65^\circ)\tan(75^\circ)$ simplifies to $\cot(x^\circ)$ and m is the numerical value of the expression $\tan(27^\circ)+\tan(18^\circ)+\tan(27^\circ)\tan(18^\circ)$, then find the value of $(m+x+1)$.

LINK COMPREHENSION TYPE
(FOR JEE ADVANCED EXAM ONLY)

PASSAGE I

Increasing product with angles are in G. P

$\cos\alpha.\cos 2\alpha.\cos 2^2\alpha....\cos 2^{n-1}\alpha$

$$=\begin{cases}\dfrac{\sin 2^n\alpha}{2^n\sin\alpha} : if\ \alpha\neq n\pi\\ \dfrac{1}{2^n} : if\ \alpha=\dfrac{\pi}{2^n+1}\\ -\dfrac{1}{2^n} : if\ \alpha=\dfrac{\pi}{2^n-1}\end{cases},$$

where n is an integer.

On the basis of above information, answer the following questions:

1. The value of $\cos\left(\dfrac{2\pi}{7}\right)\cos\left(\dfrac{4\pi}{7}\right)\cos\left(\dfrac{6\pi}{7}\right)$ is
 (a) –1/2 (b) 1/2 (c) 1/4 (d) 1/8

2. If $\alpha=\dfrac{\pi}{13}$, then the value of $\prod_{r=1}^{6}(\cos(r\alpha))$ is
 (a) 1/64 (b) –1/64 (c) 1/32 (d) –1/8

3. The value of $\sin\left(\dfrac{\pi}{14}\right)\sin\left(\dfrac{3\pi}{14}\right)\sin\left(\dfrac{5\pi}{14}\right)\sin\left(\dfrac{7\pi}{14}\right)\sin\left(\dfrac{9\pi}{14}\right)\sin\left(\dfrac{11\pi}{14}\right)\sin\left(\dfrac{13\pi}{14}\right)$ is
 (a) 1 (b) 1/8 (c) 1/32 (d) 1/64.

4. The value of $\sin\left(\dfrac{\pi}{18}\right)\sin\left(\dfrac{5\pi}{18}\right)\sin\left(\dfrac{7\pi}{18}\right)$ is
 (a) 1/16 (b) 1/8 (c) –1/8 (d) –1

5. The value of $64\sqrt{3}\sin\left(\dfrac{\pi}{48}\right)\cos\left(\dfrac{\pi}{48}\right)\cos\left(\dfrac{\pi}{24}\right)\cos\left(\dfrac{\pi}{12}\right)\cos\left(\dfrac{\pi}{6}\right)$ is
 (a) 8 (b) 6 (c) 4 (d) –1.

PASSAGE II

If $\cos\left(\dfrac{\pi}{7}\right),\cos\left(\dfrac{3\pi}{7}\right),\cos\left(\dfrac{5\pi}{7}\right)$ are the roots of the equation $8x^3-4x^2-4x+1=0$.

On the basis of the above information, answer the following questions.

1. The value of $\sec\left(\dfrac{\pi}{7}\right)+\sec\left(\dfrac{3\pi}{7}\right)+\sec\left(\dfrac{5\pi}{7}\right)$ is
 (a) 2 (b) 4 (c) 8 (d) None.

2. The value of $\sin\left(\dfrac{\pi}{14}\right)\sin\left(\dfrac{3\pi}{14}\right)\sin\left(\dfrac{5\pi}{14}\right)$ is
 (a) 1/4 (b) 1/8 (c) $\dfrac{\sqrt{7}}{4}$ (d) $\dfrac{\sqrt{7}}{8}$.

3. The value of $\cos\left(\dfrac{\pi}{14}\right)\cos\left(\dfrac{3\pi}{14}\right)\cos\left(\dfrac{5\pi}{14}\right)$ is
 (a) 1/4 (b) 1/8 (c) $\dfrac{\sqrt{7}}{4}$ (d) $\dfrac{\sqrt{7}}{8}$

4. The equation whose roots are $\tan^2\left(\dfrac{\pi}{7}\right),\tan^2\left(\dfrac{3\pi}{7}\right),\tan^2\left(\dfrac{5\pi}{7}\right)$ is
 (a) $x^3-35x^2+7x-21=0$
 (b) $x^3-35x^2+21x-7=0$
 (c) $x^3-35x^2+35x-7=0$
 (d) $x^3-21x^2+7x-35=0$

5. The value of
$\sum_{r=1}^{3}\left\{\tan^2\left(\frac{2r-1}{7}\right)\pi\right\}\times\sum_{r=1}^{3}\left\{\cot^2\left(\frac{2r-1}{7}\right)\pi\right\}$ is
(a) 15 (b) 105 (c) 21 (d) 147

PASSAGE III

Let $x^2+y^2=1$ for every x, y in R.
Then

1. The value of $P=(3x-4x^3)^2+(3y-4y^3)^2$ is
(a) 2 (b) 1 (c) 0 (d) –1
2. The minimum value of $Q=x^6+y^6$ is
(a) 1 (b) 1/2 (c) 1/4 (d) –1
3. The maximum value of $R=x^2+y^4$ is
(a) 0 (b) 1 (c) 1/2 (d) 3/4

PASSAGE IV

Consider the polynomial
$P(x)=(x-\cos 36°)(x-\cos 84°)(x-\cos 156°)$
Then

1. The co-efficient of x^2 is
(a) 0 (b) 1
(c) –1/2 (d) $\left(\frac{\sqrt{5}-1}{2}\right)$
2. The co-efficient of x is
(a) 3/2 (b) –3/2
(c) –3/4 (d) 2
3. The constant term in $P(x)$ is
(a) $\left(\frac{\sqrt{5}-1}{4}\right)$ (b) $\left(\frac{\sqrt{5}-1}{16}\right)$
(c) $\left(\frac{\sqrt{5}+1}{16}\right)$ (d) $\frac{1}{16}$

PASSAGE V

If $a\sin x+b\cos x=1$ such that $a^2+b^2=1$ for all $a,b\in(0,1)$
Then

1. The value of $\sin x$ is
(a) a (b) b
(c) a/b (d) b/a
2. The value of cosx is
(a) a (b) b
(c) a/b (d) b/a
3. The value of $\tan x$ is
(a) a (b) b
(c) a/b (d) b/a

PASSAGE VI

Let $\sec x+\tan x=\frac{22}{7}$, where $0<x<\frac{\pi}{2}$
Then

1. The value of $\tan\left(\frac{x}{2}\right)$ is
(a) 15/29 (b) 13/25
(c) 14/29 (d) –15/29
2, The value of $\left(1-\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$ is
(a) 15/29 (b) 14/29
(c) 0 (d) 12/25
3. The value of $(\operatorname{cosec} x+\cot x)$ is
(a) 29/14 (b) 15/28
(c) 29/15 (d) 15/29

MATCH MATRIX

1. Match the following columns:

Column I	Column II
(A) If $\theta+\varphi=\frac{\pi}{2}$, where θ and ϕ are positive, then $(\sin\theta+\sin\varphi)\sin\left(\frac{\pi}{4}\right)$ is always less than	(P) 1
	(Q) 2
(B) If $\sin\theta-\sin\varphi=a$ and $\cos\theta+\cos\varphi=b$, then a^2+b^2 can not exceed	(R) 3
	(S) 4
(C) If $3\sin\theta+5\cos\theta=5,(\theta\neq 0)$, then the value of $5\sin\theta-3\cos\theta$ is	(T) 5

2. Match the following columns:

Column I	Column II
(A) The value of $\cos(20°).\cos(40°).\cos(80°)$ is	(P) $\sqrt{3}/8$
(B) The value of $\cos(20°).\cos(40°).\cos(60°).\cos(80°)$ is	(Q) $\sqrt{3}/16$
(C) The value of $\sin(20°).\sin(40°).\sin(80°)$ is	(R) $\sqrt{3}/32$
(D) The value of $\sin(20°).\sin(40°).\sin(60°).\sin(80°)$ is	(S) 1/16
	(T) 1/8

3. Match the following columns:

Column I	Column II
(A) If maximum and minimum values of $\frac{7+6\tan\theta-\tan^2\theta}{1+\tan^2\theta}$ For all real values of $\theta\left(\neq\frac{\pi}{2}\right)$ are λ and μ, respectively, then	(P) $\lambda+\mu=2$
	(Q) $\lambda-\mu=6$
(B) If maximum and minimum values of $5\cos\theta+3\cos\left(\theta+\frac{\pi}{3}\right)+3$ For all real values of θ are λ and μ resp., then	(R) $\lambda+\mu=6$
	(S)
(C) If maximum and minimum values of $1+\sin\left(\frac{\pi}{4}+\theta\right)+2\cos\left(\frac{\pi}{4}-\theta\right)$ For all real values of θ are λ resp., then	(T) $\lambda-\mu=14$

4. Match the following columns:

In a triangle ABC

Column I	Column II
(A) $\sin 2A+\sin 2B+\sin 2C$ is	(P) $4\sin A\,.\,\sin B\,.\,\sin C$
(B) $\cos 2A+\cos 2B+\cos 2C$ is	(Q) $-1-4\cos A\,.\,\cos B.\,\cos C$
(C) $\sin^2 A+\sin^2 B+\sin^2 C$ is	(R) $2+2\cos A\,.\,\cos B.\,\cos C$
(D) $\cos^2 A+\cos^2 B+\cos^2 C$ is	(S) $1-2\cos A\,.\,\cos B\,.\,\cos C$

5. Match the following columns:

Column I	ColumnII
(A) The value of $\cos^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{3\pi}{8}\right)+\cos^4\left(\frac{5\pi}{8}\right)+\cos^4\left(\frac{7\pi}{8}\right)$ is	(P) 1/8
	(Q) –3/2
(B) The value of $\sin(12°).\sin(48°).\sin(54°)$ is	(R) 3/2
(C) The value of $\sin(6°).\sin(42°).\sin(66°).\sin(78°)$ is	(S) 1/16
(D) The value of $\tan(6°).\tan(42°).\tan(66°).\tan(78°)$ is	(T) 1

6. Match the following columns:

If $A+B=\frac{\pi}{4}$, then the value of $(1+\tan A)(1+\tan B)$ is 2

Column I	ColumnII
(A) The value of $(1+\tan 21°).(1+\tan 22°).(1+\tan 23°).(1+\tan 24°)$ is	(P) 2
	(Q) 4
(B) The value of $(1+\tan 2058°).(1-\tan 2013°)$ is	(R) 8
(C) The value of $\left(1+\tan\left(\frac{\pi}{8}-x\right)\right).\left(1+\tan\left(x+\frac{\pi}{8}\right)\right).$ is	(S) –8
(D) The value of $(1+\tan 235°).(1-\tan 190°).$ is	(T) –4

7. Match the following columns:

Column I	Column II
(A) The value of $2\tan(50°)+\tan(20°)$ is	(P) 3
(B) The value of $\tan(40°)+2\tan(10°)$ is	(Q) 5
(C) The value of $\tan(20°).\tan(40°).\tan(60°).\tan(80°)$ is	(R) $\tan(70°)$
(D) If $3\sin x+4\cos x=5$, then the value of $2\sin x+\cos x+4\tan x$ is	(S) $\tan(50°)$

8. Match the following columns:

Column I	Column II
(A) The minimum value of $2\sin^2\theta+3\cos^2\theta$ is	(P) 1
(B) The maximum value of $\sin^2\theta+\cos^4\theta$ is	(Q) 3/4
(C) The least value of $\sin^4\theta+\cos^2\theta$ is	(R) 2
(D) The greatest value of $\sin^{2014}\theta+\cos^{2010}\theta$ is	(S) 4

9. Match the following columns:

If α and β are the solutions of $a\cos\theta+b\sin\theta=c$, then

Column I	Column II
(A) the value of $\sin\alpha+\sin\beta$ is	(P) $\frac{c^2-b^2}{a^2+b^2}$

(B) the value of $\sin\alpha.\sin\beta$ is (Q) $\frac{2ac}{a^2+b^2}$

(C) the value of $\cos\alpha+\cos\beta$ is (R) $\frac{c^2-a^2}{a^2+b^2}$

(D) the value of $\cos\alpha.\cos\beta$ is (S) $\frac{2bc}{a^2+b^2}$

10. Match the following columns:

Column I	ColumnII
(A) The value of $\cos(12°)+\cos(84°)+\cos(156°)+\cos(132°)$ is	(P) 0
	(Q) 1
(B) The value of $2\tan\left(\frac{\pi}{10}\right)+3\sec\left(\frac{\pi}{10}\right)-4\cos\left(\frac{\pi}{10}\right)$ is	(R) 2
(C) The value of $\sqrt{3}\cot(20°)-4\cos(20°)$ is	(S) –1/2
(D) The value of $\tan(20°)+2\tan(50°)-\tan(70°)$ is	(T) –1

ASSERTION & REASON

CODES

(A) Both *A* and *R* are individually true and *R* is the correct explanation of *A*.
(B) Both *A* and *R* are individually true and *R* is not the correct explanation of *A*.
(C) *A* is true but *R* is false.
(D) *A* is false but *R* is true.

1. Assertion (*A*): $\sin\theta = x+\frac{1}{x}$ is impossible if $x \in R-\{0\}$.
   Reason (*R*): $A.M \geq G.M$
   (a) *A* (b) *B* (c) *C* (d) *D*

2. Assertion (*A*): *A* is an obtuse angle in ΔABC, then $\tan B \,.\, \tan C > 1$
   Reason (*R*): In ΔABC, $\tan A = \frac{\tan B+\tan C}{\tan B\tan C-1}$
   (a) *A* (b) *B* (c) *C* (d) *D*

3. Assertion (*A*): $\sin\left(\frac{2\pi}{7}\right)+\sin\left(\frac{4\pi}{7}\right)+\sin\left(\frac{6\pi}{7}\right)=-\frac{1}{2}$
   Reason (*R*): $\cos\left(\frac{2\pi}{7}\right)+i\sin\left(\frac{2\pi}{7}\right)$ is complex 7^{th} root of unity.
   (a) *A* (b) *B* (c) *C* (d) *D*

4. Assertion (*A*): $\tan\alpha+2\tan(2\alpha)+4\tan(4\alpha)+8\tan(8\alpha)-16\cot(16\alpha)=\cot\alpha$
   Reason (*R*): $\cot\alpha-\tan\alpha=2\cot 2\alpha$
   (a) *A* (b) *B* (c) *C* (d) *D*

5. Assertion (*A*):
   $\cos^2\alpha+\cos^2\left(\alpha+\frac{\pi}{3}\right)+\cos^2\left(\alpha+\frac{4\pi}{3}\right)$
   $= 3\cos\alpha\cos\left(\alpha+\frac{2\pi}{3}\right)\cos\left(\alpha+\frac{4\pi}{3}\right)$
   Reason (R): If $a+b+c=0$, then $a^3+b^3+c^3=3abc$
   (a) *A* (b) *B* (c) *C* (d) *D*

6. Assertion (*A*): $\tan(5\theta)-\tan(3\theta)-\tan(\theta)$
   $= \tan(5\theta).\tan(3\theta).\tan(\theta)$
   Reason (*R*): If $x = y + z$, then
   $\tan x - \tan y - \tan z = \tan x.\ \tan y\ .\ \tan z$
   (a) *A* (b) *B* (c) *C* (d) *D*

7. Assertion (*A*): The maximum value of $\sin\theta+\cos\theta$ is 2
   Reason (*R*): The maximum value of $\sin\theta$ is 1 and that of $\cos\theta$ is also 1.
   (a) *A* (b) *B* (c) *C* (d) *D*

8. Assertion (*A*): The maximum value of $\prod_{i=1}^{n}\cos(\alpha_i)$ under the restriction
   $0 \leq \alpha_1,\alpha_2,\alpha_3,...,\alpha_n \leq \frac{\pi}{2}$ is $\frac{1}{2^{n/2}}$
   Reason (*R*): $\prod_{i=1}^{n}\cot(\alpha_i)=1$
   (a) *A* (b) *B* (c) *C* (d) *D*

9. Assertion (*A*): If *A* + *B* + *C* = *π*, then the maximum value of
   $\tan A.\ \tan B\ .\ \tan C$ is $3\sqrt{3}$
   Reason (*R*): $A.\mathrm{M} \geq \mathrm{G.M}$
   (a) *A* (b) *B* (c) *C* (d) *D*

10. Assertion (*A*): $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is positive for all real values of *x* and *y* only when *x* = *y*
    Reason (*R*): $\sec^2\theta \geq 1$.
    (a) *A* (b) *B* (c) *C* (d) *D*

(QUESTIONS ASKED IN PAST IIT-JEE EXAMS)

1. Prove that:
   $\sin x\sin y\sin(x-y)+\sin y\sin z\sin(y-z)$
   $+ \sin z\ \sin x\sin(z-x)$
   $+ \sin(x-y)\sin(y-z)\sin(z-x)=0$
   [IIT-JEE – 1978]

2. If $\cos(\alpha+\beta)=\frac{4}{5}$, $\sin(\alpha-\beta)=\frac{5}{13}$ and α, β lie between 0 and $\frac{\pi}{4}$, find $\tan 2\alpha$.
[IIT-JEE – 1979]

3. Given $A=\sin^2\theta+\cos^4\theta$ for all values of θ, then
(a) $1 \le A \le 2$ (b) $3/4 \le A \le 1$
(c) $13/6 \le A \le 1$ (d) $3/4 \le A \le 13/6$.
[IIT-JEE – 1980]

4. If $\tan A=\frac{1-\cos B}{\sin B}$, then $\tan 2A=\tan B$. Is it true/false?
[IIT-JEE – 1980]

5. Suppose $\sin^3 x \sin 3x=\sum_{m=0}^{n} C_m \cos(mx)$ is an identity in x, when $C_0, C_2, \ldots\ldots, C_n$ are constants and $C_n \ne 0$ the value of $n = \ldots$.
[IIT-JEE – 1981]

6. Without using the tables, prove that
$\sin 12^\circ \sin 54^\circ \sin 48^\circ=\frac{1}{8}$. [IIT-JEE – 1982]

7. If $\alpha+\beta+\gamma=\pi$, then prove that,
$\sin^2\alpha+\sin^2\beta-\sin^2\gamma=2\sin\alpha .\sin\beta .\sin\gamma$
[IIT-JEE – 1983]

8. Prove that
$$16\cos\left(\frac{2\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)\cos\left(\frac{16\pi}{15}\right)=1$$
[IIT-JEE – 1983]

9. The value of
$$\left(1+\cos\left(\frac{\pi}{8}\right)\right)\left(1+\cos\left(\frac{3\pi}{8}\right)\right)\left(1+\cos\left(\frac{5\pi}{8}\right)\right)\left(1+\cos\left(\frac{7\pi}{8}\right)\right)$$ is equal to
(a) 1/2 (b) $\cos\frac{\pi}{8}$ (c) 1/8 (d) $\frac{1+\sqrt{2}}{2\sqrt{2}}$.
[IIT-JEE – 1984]

10. No questions asked in 1985.

11. The expression
$$3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$$ is equal to
(a) 0 (b) 1
(c) 3 (d) $\sin 4\alpha+\cos\alpha$.
[IIT-JEE – 1986]

12. No questions asked in 1987.

13. The value of the expression $\sqrt{3}\,\text{cosec}(20^\circ)-\sec(20^\circ)$ is equal to
(a) 2 (b) $\frac{2\sin 20^\circ}{\sin 40^\circ}$
(c) 4 (d) $\frac{4\sin 20^\circ}{\sin 40^\circ}$.
[IIT-JEE – 1988]

14. Prove that:
$\tan\alpha+2\tan 2\alpha+4\tan 4\alpha+8\cot 8\alpha=\cot\alpha$.
[IIT-JEE – 1988]

15. No questions asked in between 1989 – 1990.

16. Find the value of
$$\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)\sin\left(\frac{7\pi}{14}\right)\sin\left(\frac{9\pi}{14}\right)\sin\left(\frac{11\pi}{14}\right)\sin\left(\frac{13\pi}{14}\right)$$
[IIT-JEE – 1991]

17. If $f(x)=\cos\left[\pi^2\right]x+\cos\left[-\pi^2\right]x$, where [,] = G.I.F, then
(a) $f\left(\frac{\pi}{2}\right)=1$ (b) $f(\pi)=1$
(c) $f(-\pi)=0$ (d) $f\left(\frac{\pi}{4}\right)=1$
[IIT-JEE-1991]

18. Match the following columns:

Column I	Column II
(i) Positive	(A) $\left(\frac{13\pi}{48},\frac{14\pi}{48}\right)$
(ii) Negative	(B) $\left(\frac{14\pi}{48},\frac{18\pi}{48}\right)$
	(C) $\left(\frac{18\pi}{48},\frac{23\pi}{48}\right)$
	(D) $\left(0,\frac{\pi}{2}\right)$

[IIT-JEE – 1992]

19. If $k=\sin\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{7\pi}{18}\right)$, then the numerical value of k is…
[IIT-JEE – 1993]

20. If $A > 0$, $B > 0$ and $A+B=\frac{\pi}{3}$, then the maximum value of $\tan A \tan B$ is ……
[IIT-JEE – 1993]

21. Let $o<x<\frac{\pi}{4}$, then $(\sec 2x-\tan 2x)$ equals

(a) $\tan\left(x-\frac{\pi}{4}\right)$ (b) $\tan\left(\frac{\pi}{4}-x\right)$

(c) $\tan\left(\frac{\pi}{4}+x\right)$ (d) $\tan^2\left(\frac{\pi}{4}+x\right)$

[IIT-JEE – 1994]

22. The value of the expression $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x)$ is

(a) 11 (b) 12 (c) 13 (d) 14

[IIT-JEE – 1995]

23. The minimum value of the expression $\sin\alpha+\sin\beta+\sin\gamma$, where α,β,γ are real numbers satisfying $\alpha+\beta+\gamma=\pi$ is

(a) positive (b) 0

(c) negative (d) –3.

[IIT-JEE – 1995]

24. $\sec^2\theta=\left(\frac{4xy}{(x+y)^2}\right)$ is true if and only if

(a) $x+y=0$ (b) $x=y, x\neq 0$

(c) $x=y$ (d) $x\neq 0, y\neq 0$.

[IIT-JEE – 1996]

25 The graph of the function $\cos x\cos(x+2)-\cos^2(x-1)$ is

(a) a straight line passing through (0, – $\sin^2$ 1) with slope 2.

(b) a straigh line passing through (0, 0)

(c) a parabola with vertex (1, – $\sin^2$ 1)

(d) a straight line passing through the point $\left(\frac{\pi}{2},-\sin^2 1\right)$ and parallel to x-axis.

[IIT-JEE-1997]

26. Which of the following numbers is/are rational?

(a) $\sin 15°$ (b) $\cos 15°$

(c) $\sin 15°\cos 15°$ (d) $\sin 15°\cos 75°$

[IIT-JEE-1998]

27. For a positive integer n, let

$f_n(\theta)=\tan\left(\frac{\theta}{2}\right)(1+\sec\theta)(1+\sec 2\theta)......$

$(1+\sec 4\theta)........(1+\sec 2^n\theta)$, then

(a) $f_2\left(\frac{\pi}{16}\right)=1$ (b) $f_3\left(\frac{\pi}{32}\right)=1$

(c) $f_4\left(\frac{\pi}{64}\right)=1$ (d) $f_5\left(\frac{\pi}{128}\right)=1$

[IIT-JEE-1998]

28 Let $f(\theta)=\sin\theta(\sin\theta+\sin 3\theta)$, then $f(\theta)$

(a) ≥ 0 when $\theta\geq 0$

(b) ≤ 0 for all real ≥ 0

(c) ≤ 0 for all real $\theta\leq 0$

(d) ≤ 0 only when $\theta\leq 0$

[IIT-JEE-2000]

29. The maximum value of $\cos\alpha_1.\cos\alpha_2......\cos\alpha_n$ under the restriction

$0\leq\alpha_1,\alpha_2,.......,\alpha_n\leq\frac{\pi}{2}$ and

$\cot\alpha_1.\cot\alpha_2......\cot\alpha_n=1$ is

(a) $\frac{1}{2^{n/2}}$ (b) $\frac{1}{2^n}$

(c) $\frac{1}{2n}$ (d) 1

[IIT-JEE-2001]

30. No questions asked in 2002.

31. If $\alpha+\beta=\frac{\pi}{2}$ and $\alpha=\beta+\gamma$, then $\tan\alpha$ is

(a) $2(\tan\beta+\tan\gamma)$ (b) $\tan\beta+\tan\gamma$

(c) $(\tan\beta+2\tan\gamma)$ (d) $2\tan\beta+\tan\gamma$

[IIT-JEE - 2003]

32. If $\alpha\in\left(0,\frac{\pi}{2}\right)$, then the expression

$y=\sqrt{x^2+x}+\frac{\tan^2\alpha}{\sqrt{x^2+x}}$ is always greater than or equal to

(a) $2\tan\alpha$ (b) 2

(c) 1 (d) $\sec^2\alpha$

[IIT-JEE-2003]

33 Given that both θ and ϕ are acute angles and $\sin\theta=\frac{1}{2}$, $\cos\varphi=\frac{1}{3}$, then the value of $(\theta+\varphi)$ belongs to the interval

(a) $\left(\frac{\pi}{3},\frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{2},\frac{2\pi}{3}\right)$

(c) $\left(\frac{2\pi}{3},\frac{5\pi}{6}\right)$ (d) $\left(\frac{5\pi}{6},\pi\right)$

[IIT-JEE – 2004]

34. Find the range of values of t for which

$2\sin t=\frac{1-2x+5x^2}{3x^2-2x-1}, t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

[IIT-JEE – 2005]

35. $\cos(\alpha-\beta)=1$ *and* $\cos(\alpha+\beta)=\frac{1}{e}$, where $\alpha,\beta\in[-\pi,\pi]$. Values of α, β which satisfy the equations is/are

(a) 0 (b) 1 (c) 2 (d) 4.

[IIT-JEE – 2005]

36. Let $\theta \in \left(0, \frac{\pi}{4}\right)$ and $t_1 = (\tan\theta)^{\tan\theta}$, $t_2 = (\tan\theta)^{\cot\theta}$

$t_3 = (\cot\theta)^{\tan\theta}$, $t_4 = (\cot\theta)^{\cot\theta}$, then

(a) $t_1 > t_2 > t_3 > t_4$ (b) $t_4 > t_3 > t_1 > t_2$

(c) $t_3 > t_1 > t_2 > t_4$ (d) $t_2 > t_3 > t_1 > t_4$

[IIT-JEE – 2006]

37 If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then

(a) $\tan^2 x = \frac{2}{3}$ (b) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

(c) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$ (d) $\tan^2 x = \frac{1}{3}$

[IIT-JEE - 2009]

38 The maximum value of the expression

$\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is ....

[IIT-JEE-2010]

39. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and

$Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets.Then

(a) $P \subset Q$ and $Q - P \neq \varphi$

(b) $Q \not\subset P$

(c) $P \not\subset Q$

(d) $P = Q$ [IIT-JEE - 2011]

40. The positive integral value of $n > 3$ satisfying the equation

$\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is ....

[IIT-JEE - 2011]

41. Let $f : (-1,1) \to R$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$

for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the values

of $f\left(\frac{1}{3}\right)$ is /are

(a) $1 - \sqrt{\frac{3}{2}}$ (b) $1 + \sqrt{\frac{3}{2}}$

(c) $1 - \sqrt{\frac{2}{3}}$ (d) $1 + \sqrt{\frac{2}{3}}$

[IIT-JEE - 2012]

42. No questions asked in between 2013 to 2015.

ANSWERS

EXERCISE 1

1. $\frac{5}{12}\pi$ cm.
2. $\frac{20\pi}{3}$ cm.
3. 5 : 4
4. 70 meters
5. 3247.62 kms.
6. 81°, 9°
7. $\frac{\pi}{3}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{2\pi}{3}$.
8. $\frac{\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{9}$
9. 6π
10. 2.217 m.
11. No. of sides = 9.
12. $\pi/2$
13. 1080 ft
14. 14.32 miles
15. $\frac{4\pi}{3}$.

EXERCISE 2

10. 3/5.
11. 5/13.

EXERCISE 3

4. $\sqrt{3}$
5. –1
6. 1
9. $9\frac{1}{2}$
10. 8
11. 5
12. $5\frac{1}{2}$
15. $\theta = 120°, 240°$
16. $\theta = \frac{5\pi}{3}$
21. 0
22. –1
23. 0

EXERCISE 4

1. 6
2. 10
3. 20
4. 5
5. 10

EXERCISE 6

16. –1.
17. $\left(\frac{b^2-a^2}{b^2+a^2}\right)$.
18. $\frac{1}{2}\left(xy+\frac{1}{xy}\right)$.
20. 1.
21. 7.

EXERCISE 7

18. 5/12
19. $\frac{2c}{1-c^2}$
21 0
22. 1/8
23. 1/2
34. 3
35. 11.

EXERCISE 8

1. (i) Max $V = 10$, Min $V = 0$
   (ii) Max $V = 15$, Min $V = 5$
   (iii) Max $V = 7$, Min $V = 1$
   (iv) Max $V = 7$, Min $V = 2$
   (v) Max $V = \sqrt{2}$, Min $V = -\sqrt{2}$
   (vi) Max $V = \sqrt{2}$, Min $V = -\sqrt{2}$
   (vii) Max $V = \sin 1$, Min $V = -\sin 1$
   (viii) Max $V = \cos 1$, Min $V = 0$
   (ix) Max $V = \sqrt{2}$, Min $V = -\sqrt{2}$
   (x) Max $V = \sqrt{2}-\sin 1$, Min $V = -\sqrt{2}+\sin 1$
2. Max $V = 8$, Min $V = -2$
4. Max $V = 4+\sqrt{10}$, Min $V = 4-\sqrt{10}$
5 Max $V = 4$, Min $V = 2$
6 Least $V = 10$
7. 13 :12
8. $[-1,5]$
9. 9
10 Max $V = 1$, Min $V = 1/2$
11. Max $V = 1$, Min $V = 1/4$
12. Min $V = 27/2$, Max V is not defined
13. Min $V = 2$, Max value is not defined
14. Max $V = \frac{5}{2+\sqrt{10}}$, Min $V = \frac{5}{2-\sqrt{10}}$
15. Max $V = 1$, Min $V = 3/4$
16. Max $V = 1$, Min $V = 3/4$
17. Min $V = (a+b)^2$
18. Min $V = 4$
19 Min $V = 8$
20 Min $V = 27$.

EXERCISE 9

15. 11
16. 2013
17. 10
19. 1/4
25. 2
26. $x = \frac{4\pi}{3}$

Level I

1. 13
2. $\left[\frac{1}{4},1\right]$
3. $\frac{1}{2}\left(q+\frac{1}{q}\right)$
4. 1
5. x
6. 1
7. 0
8. 4
9. $\cot\alpha$
10. 1
11. 1
12. $\left[\frac{3}{4},1\right]$
13. (i) 3, 13 (ii) −4, 10
14. $P\sqrt{2-p^2}$
15. 4
16. 1
18. $\pm\sqrt{(a^2+b^2-c^2)}$
19. 0

20. 2tan 50°
21. b
22. 4
23. 3
24. 4
25. $\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)$
27. $\frac{2ac}{a^2-c^2}$
28. (i) $\frac{2b}{c+a}$ (ii) $\frac{c-a}{c+a}$
29. $\frac{b^2-a^2}{b^2+a^2}$

Level II

1. (b)	2. (b)	3. (b)	4. (b)
5. (d)	6. (c)	7. (c)	8. (a, b)
9. (d)	10. (a, b, d)	11. (a, c)	12. (a, c, d)
13. (c)	14. (c)	15. (b)	16. (b)
17. (a)	18. (b)	19. (c)	20. (d)
21. (b, c)	22. (c)	23. (b)	24. (c)
25. (a)	26. (b)	27. (c)	28. (a)
29. (b)	30. (a)	31. (c)	32. (c)
33. (d)	34. (d)	35. (d)	36. (b)
37. (a)	38. (a)	39. (b)	40. (b)
41. (a)	42. (d)	43. (a, b, c)	44. (d)
45. (d)			

Level III

13. $p\in(-\infty,3-2\sqrt{2}]\cup[3+2\sqrt{2},\infty)$.
16. $qx^3-px^2+(1+q)x-p=0$
32. $R_f=\left[0,\frac{1}{2}\right]$
33. i
34. $\left(\frac{4b}{a^2+2a+b^2}\right)$
35. 2/3

Integer Type Questions

1. 4	2. 5	3. 3	4. 4
5. 3	6. 3	7. 7	8. 3
9. 5	10. 4	11. 2	12. 7

Comprehensive Link Passages

P-I:	1. (d)	2. (a)	3. (d)	4. (b)	5. (b)
P-II:	1. (b)	2. (b)	3. (d)	4. (c)	5. (b)
P-III:	1. (b)	2. (c)	3. (b)		
P-IV:	1. (a)	2. (c)	2. (b)		
P-V:	1. (a)	2. (b)	3. (c)		
P-VI:	1. (a)	2. (b)	3. (c)		

(Match Matrix)

1. (A) → (P,Q,R,S,T): (B) → (S, T); (C) → (R)
2. (A) → T; (B) → S; (C) → P; (D) → Q
3. (A) → (R, S); (B) → (R, T); (C) → (P, Q)
4. (A) → P; (B) → Q; (C) → R, (D) → S
5. (A) → (R); (B) → (P); (C) → (S), (D) → (T)
6. (A) → (Q); (B) → (P); (C) → (P), (D) → (P)
7. (A) → (R); (B) → (S); (C) → (P), (D) → (Q)
8. (A) → (R); (B) → (P); (C) → (Q); (D) → (P)
9. (A) → (S); (B) → (R); (C) → (Q), (D) → (P)
10. (A) → (S); (B) → (P); (C) → (Q), (D) → (P)

Assertion and Reason

1. (a)	2. (d)	3. (d)	4. (a)
5. (a)	6. (a)	7. (d)	8. (a)
9. (a)	10. (a)		

HINTS AND SOLUTIONS

LEVEL III

1. We have, $\tan x$

$$= \tan(2x-x)$$

$$= \frac{\tan 2x-\tan x}{1+\tan x\tan 2x}$$

$$\tan x(1+\tan x\tan 2x)=\tan 2x-\tan x$$

$$(1+\tan x\tan 2x)=\cot x(\tan 2x-\tan x)$$

$$\tan x\tan 2x=\cot x(\tan 2x-\tan x)-1$$

Similarly,

$$\tan 2x\tan 3x=\cot x(\tan 3x-\tan 2x)-1$$

$$\tan 3x\tan 4x=\cot x(\tan 4x-\tan 3x)-1$$

....

....

$$\tan nx\tan(n+1)x=\cot x(\tan(n+1)x-\tan nx)-1$$

Hence, the required sum is

$$=\cot x(\tan(n+1)x-\tan x)-n$$

$$=\cot x(\tan(n+1)x)-(n+1)$$

2. Prove that,

$$\operatorname{cosec} x+\operatorname{cosec}2x+\operatorname{cosec}4x+\text{....to } n \text{ terms}$$

$$=\cot\left(\frac{x}{2}\right)-\cot(2^{n-1}x).$$

We have,

$$\operatorname{cosec} x+\operatorname{cosec}2x+\operatorname{cosec}4x+\text{....to } n \text{ terms}$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{\sin(x/2)}{\sin x.\sin(x/2)}$$

$$= \frac{\sin\left(x - \frac{x}{2}\right)}{\sin x.\sin(x/2)}$$

$$= \frac{\sin x\cos(x/2) - \cos x\sin(x/2)}{\sin x.\sin(x/2)}$$

$$= \cot(x/2) - \cot x$$

Similarly, $\operatorname{cosec}(2x) = \cot(x) - \cot(2x)$

$\operatorname{cosec}(4x) = \cot(2x) - \cot(4x)$

....

....

$$\operatorname{cosec}(2^{n-1}x) = \cot(2^{n-2}x) - \cot(2^{n-1}x)$$

Adding all, we get,

$\operatorname{cosec} x + \operatorname{cosec} 2x + \operatorname{cosec} 4x + ...$ to n terms

$$= \cot\left(\frac{x}{2}\right) - \cot(2^{n-1}x)$$

3. Now, $\cot(A)\cos(B) - 1 = \dfrac{\cos(A \pm B)}{\sin A \sin B}$

So, $\cot(16°)\cot(44°) - 1 = \dfrac{\cos(60°)}{\sin(16°)\sin(44°)}$

$$\cot(76°)\cot(44°) - 1 = \frac{\cos(120°)}{\sin(76°)\sin(44°)}$$

$$\cot(76°)\cot(16°) - 1 = \frac{\cos(60°)}{\sin(76°)\sin(16°)}$$

Now, LHS

$$= \frac{\cos(60°)}{\sin(16°)\sin(44°)} + \frac{\cos(120°)}{\sin(76°)\sin(44°)} - \frac{\cos(60°)}{\sin(76°)\sin(16°)}$$

$$= \frac{\cos(60°)\sin(76°) + \cos(120°)\sin(16°) - \cos(60°)\sin(44°)}{\sin(16°)\sin(44°)\sin(76°)}$$

$$= \frac{1}{2}\left[\frac{\sin(76°) - \sin(16°) - \sin(44°)}{\sin(16°)\sin(44°)\sin(76°)}\right]$$

$$= \frac{1}{2}\left[\frac{(\sin(76°) - \sin(16°)) - \sin(44°)}{\sin(16°)\sin(44°)\sin(76°)}\right]$$

$$= \frac{1}{2}\left[\frac{2\sin(30°) - \cos(46°) - \sin(44°)}{\sin(16°)\sin(44°)\sin(76°)}\right]$$

$$= \frac{1}{2}\left[\frac{\cos(46°) - \sin(44°)}{\sin(16°)\sin(44°)\sin(76°)}\right]$$

$$= \frac{1}{2}\left[\frac{\cos(46°) - \cos(44°)}{\sin(16°)\sin(44°)\sin(76°)}\right]$$

$$= 0$$

Hence, the result.

5. As we know that

$$\cos\theta.\cos(2\theta).\cos(2^2\theta)\cos(2^3\theta)...\cos(2^{n-1}\theta)$$

$$= \frac{\sin(2^n\theta)}{2^n\sin(\theta)}$$

$$= \frac{\sin(\pi + \theta)}{2^n\sin(\theta)}$$

$$= -\frac{\sin(\theta)}{2^n\sin(\theta)}$$

$$= -\frac{1}{2^n}$$

Hence, the value of

$$2^n\cos\theta.\cos(2\theta).\cos(2^2\theta)\cos(2^3\theta)...\cos(2^{n-1}\theta)$$

$$= -1.$$

6. Let $y = \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)$

$$y^2 = \left(\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)\right)^2$$

$$= \sin^2\left(\frac{2\pi}{7}\right) + \sin^2\left(\frac{4\pi}{7}\right) + \sin^2\left(\frac{8\pi}{7}\right)$$

$$+ 2\left(\sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right)\sin\left(\frac{8\pi}{7}\right) + \sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{8\pi}{7}\right)\right)$$

$$= y_1 + y_2 \text{ (say)}$$

Now, $y_1 = \sin^2\left(\frac{2\pi}{7}\right) + \sin^2\left(\frac{4\pi}{7}\right) + \sin^2\left(\frac{8\pi}{7}\right)$

$$= \sin^2\left(\frac{\pi}{7}\right) + \sin^2\left(\frac{2\pi}{7}\right) + \sin^2\left(\frac{4\pi}{7}\right)$$

$$= \frac{1}{2}\left[2\sin^2\left(\frac{\pi}{7}\right) + 2\sin^2\left(\frac{2\pi}{7}\right) + 2\sin^2\left(\frac{4\pi}{7}\right)\right]$$

$$= \frac{1}{2}\left[3 - \cos\left(\frac{2\pi}{7}\right) - \cos\left(\frac{4\pi}{7}\right) - \cos\left(\frac{8\pi}{7}\right)\right]$$

$$= \frac{1}{2}\left[3 - \cos\left(\frac{2\pi}{7}\right) - \cos\left(\frac{4\pi}{7}\right) - \cos\left(\frac{6\pi}{7}\right)\right]$$

$$= \frac{1}{2}\left[3 - \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right)\right]$$

$$= \frac{1}{2}\left[3 - \left(-\frac{1}{2}\right)\right]$$

$$= \frac{7}{4}$$

Now, y_2

$$= 2\left(\sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right)\sin\left(\frac{8\pi}{7}\right) + \sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{8\pi}{7}\right)\right)$$

$$= \left[\cos\left(\frac{2\pi}{7}\right) - \cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) - \cos\left(\frac{12\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) - \cos\left(\frac{10\pi}{7}\right)\right]$$

$$= \left[\cos\left(\frac{2\pi}{7}\right) - \cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) - \cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) - \cos\left(\frac{4\pi}{7}\right)\right]$$

$$= 0$$

Thus, $y^2 = \left[\sin^2\left(\frac{2\pi}{7}\right) + \sin^2\left(\frac{4\pi}{7}\right) + \sin^2\left(\frac{8\pi}{7}\right)\right]$

$$\Rightarrow \quad y^2 = \frac{7}{4}$$

$$\Rightarrow \quad y^2 = \frac{7}{4}$$

Hence, the value of

$\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)$ is $\frac{\sqrt{7}}{2}$

7. We have

$$= \tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{2\pi}{16}\right) + ... + \tan^2\left(\frac{7\pi}{16}\right)$$

$$= \left(\tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{7\pi}{16}\right)\right) + \left(\tan^2\left(\frac{2\pi}{16}\right) + \tan^2\left(\frac{6\pi}{16}\right)\right) + \left(\tan^2\left(\frac{\pi}{16}\right) + \tan^2\left(\frac{7\pi}{16}\right)\right) + \left(\tan^2\left(\frac{4\pi}{16}\right)\right)$$

$$= \left(\tan^2\left(\frac{\pi}{16}\right) + \cot^2\left(\frac{\pi}{16}\right)\right) + \left(\tan^2\left(\frac{2\pi}{16}\right) + \cot^2\left(\frac{2\pi}{16}\right)\right) + \left(\tan^2\left(\frac{\pi}{16}\right) + \cot^2\left(\frac{\pi}{16}\right)\right) + \left(\tan^2\left(\frac{\pi}{4}\right)\right)$$

$$= \left(\tan\left(\frac{\pi}{16}\right) + \cot\left(\frac{\pi}{16}\right)\right)^2 + \left(\tan\left(\frac{2\pi}{16}\right) + \cot\left(\frac{2\pi}{16}\right)\right)^2 + \left(\tan\left(\frac{\pi}{16}\right) + \cot\left(\frac{\pi}{16}\right)\right)^2 - 2 - 2 - 2 + 1$$

$$= \left(\frac{1}{\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{16}\right)}\right)^2 + \left(\frac{1}{\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{16}\right)}\right)^2 + \left(\frac{1}{\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{16}\right)}\right)^2 - 5$$

$$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{4}{\sin^2\left(\frac{2\pi}{8}\right)} + \frac{4}{\sin^2\left(\frac{3\pi}{8}\right)} - 5$$

$$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{4}{\sin^2\left(\frac{\pi}{4}\right)} + \frac{4}{\sin^2\left(\frac{3\pi}{8}\right)} - 5$$

$$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)} + \frac{4}{\cos^2\left(\frac{\pi}{8}\right)} + 8 - 5$$

$$= \frac{4}{\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right)} + 3$$

$$= \frac{16}{4\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right)} + 3$$

$$= \frac{16}{\left(2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\right)^2} + 3$$

$$= \frac{16}{\sin^2\left(\frac{\pi}{4}\right)} + 3$$

$$= 32 + 3 = 35.$$

8. Let $\theta = \frac{\pi}{7}$

$$4\theta = \pi - 3\theta$$

$$\tan(4\theta) = \tan(\pi - 3\theta)$$

$\tan(4\theta) = -\tan(3\theta)$

$$\frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta} = -\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$$

$$\frac{4x-4x^3}{1-6x^2+x^4} = -\frac{3x-x^3}{1-3x^2}$$

On simplification, we get,

$$x^6 - 21x^4 + 35x^2 - 7 = 0$$

$$y^3 - 21y^2 + 35y - 7 = 0,\ y = x^2 = \tan^2\theta \quad \text{.......(i)}$$

Let its roots are $\tan^2\theta, \tan^2(2\theta), \tan^2(3\theta)$

Thus, $\tan^2\theta + \tan^2(2\theta) + \tan^2(3\theta) = 21$

$$\tan^2\left(\frac{\pi}{7}\right) + \tan^2\left(\frac{2\pi}{7}\right) + \tan^2\left(\frac{3\pi}{7}\right) = 21$$

Replacing y by $1/y$ in (i), we get,

$$\frac{1}{y^3} - \frac{21}{y^2} + \frac{35}{y} - 7 = 0$$

$$-7y^3 + 35y^2 - 21y + 1 = 0$$

$$7y^3 - 35y^2 + 21y - 1 = 0$$

Let its roots are $\cot^2(\theta), \cot^2(2\theta), \cot^2(3\theta)$

Thus, $\cot^2(\theta) + \cot^2(2\theta) + \cot^2(3\theta) = \frac{35}{7} = 5$

$$\cot^2\left(\frac{\pi}{7}\right) + \cot^2\left(\frac{2\pi}{7}\right) + \cot^2\left(\frac{3\pi}{7}\right) = 5$$

Hence, the value of

$$\left(\tan^2\left(\frac{\pi}{7}\right) + \tan^2\left(\frac{2\pi}{7}\right) + \tan^2\left(\frac{3\pi}{7}\right)\right)$$

$$\times\left(\cot^2\left(\frac{\pi}{7}\right) + \cot^2\left(\frac{2\pi}{7}\right) + \cot^2\left(\frac{3\pi}{7}\right)\right)$$

$= 35 \times 3 = 105$.

9. We have $\dfrac{3+\cot(76°)\cot(16°)}{\cot(76°)+\cot(16°)}$

$$= \frac{2+1+\cot(76°)\cot(16°)}{\cot(76°)+\cot(16°)}$$

$$= \frac{2+1+\dfrac{\cos(76°)\cos(16°)}{\sin(76°)\sin(16°)}}{\dfrac{\cos(76°)}{\sin(76°)}+\dfrac{\cos(16°)}{\sin(16°)}}$$

$$= \frac{2(\sin(76°)\sin(16°)) + \cos(76°-16°)}{\cos(76°)\sin(16°)+\sin(76°)\cos(16°)}$$

$$= \frac{2(\sin(76°)\sin(16°)) + \cos(76°-16°)}{\sin(76°+16°)}$$

$$= \frac{2(\sin(76°)\sin(16°)) + \cos(60°)}{\sin(92°)}$$

$$= \frac{\cos(60°) - \cos(92°) + \cos(60°)}{\sin(92°)}$$

$$= \frac{\frac{1}{2} - \cos(92°) + \frac{1}{2}}{\sin(92°)}$$

$$= \frac{1-\cos(92°)}{\sin(92°)}$$

$$= \frac{2\sin^2(46°)}{2\sin(46°)\cos(46°)}$$

$= \tan(46°)$

$= \cot(44°)$

10. Let $a = \cos x, b = \cos y,\ c = \cos z$

So, $a+b+c=0$

then $a^3+b^3+c^3 = 3abc$

Now, $\cos(3x)+\cos(3y)+\cos(3z)$.

$= 4(\cos^3 x + \cos^3 y + \cos^3 z) - 3(\cos x + \cos y + \cos z)$

$= 4(\cos^3 x + \cos^3 y + \cos^3 z)$

$= 4.3\cos x\cos y\cos z$

$= 12 \cos x \cos y \cos z$

11. Let $\theta = \frac{\pi}{9}$

$\Rightarrow \quad 3\theta = \frac{\pi}{3}$

$\Rightarrow \quad \tan(3\theta) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\Rightarrow \quad \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = \sqrt{3}$

$\Rightarrow \quad \left(3\tan\theta - \tan^3\theta\right)^2 = 3\left(1 - 3\tan^2\theta\right)^2$

$\Rightarrow \quad 9\tan^2\theta - 6\tan^4\theta + \tan^6\theta = 3\left(1 - 6\tan^2\theta + \tan^4\theta\right)$

$\Rightarrow \quad \tan^6\theta - 9\tan^4\theta + 27\tan^2\theta = 3$

Hence, the result.

12. Let $a = \cos A + i\sin A, b = \cos B + i\sin B$,
$c = \cos C + i\sin C$

Clearly, $a + b + c = 0$

Also, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

$= (\cos A - \sin A) + (\cos B - \sin B) + (\cos C - \sin C) = 0$

Now, $(a + b + c)^2 = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2(ab + bc + ca) = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 = 0$

$\Rightarrow \quad (\cos A + i\sin A)^2 + (\cos B + i\sin B)^2 + (\cos C + i\sin C)^2 = 0$

$\Rightarrow \quad (\cos 2A + i\sin 2A) + (\cos 2B + i\sin 2B) + (\cos 2C + i\sin 2C) = 0$

$\Rightarrow \quad (\cos 2A + \cos 2B + \cos 2C) + i(\sin 2A + \sin 2B + \sin 2C) = 0$

$\Rightarrow \quad (\cos 2A + \cos 2B + \cos 2C) = 0$

and $(\sin 2A + \sin 2B + \sin 2C) = 0$

$\Rightarrow \quad (\cos 2A + \cos 2B + \cos 2C) = 0$

$\Rightarrow \quad 2\cos^2 A - 1 + 2\cos^2 B - 1 + 2\cos^2 C - 1 = 0$

$\Rightarrow \quad \cos^2 A + \cos^2 B + \cos^2 C = \frac{3}{2}$

$\Rightarrow \quad 1 - \sin^2 A + 1 - \sin^2 B + 1 - \sin^2 C = \frac{3}{2}$

$\Rightarrow \quad \sin^2 A + \sin^2 B + \sin^2 C = 3 - \frac{3}{2} = \frac{3}{2}$

Hence, the result.

13. Given $A = \frac{\pi}{4}$

$\Rightarrow B + C = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Now, $p = \tan B \tan C$

$= \tan B \tan\left(\frac{3\pi}{4} - B\right)$

$= \tan B\left(\frac{-1 - \tan B}{1 - \tan B}\right)$

$= \left(\frac{-\tan B - \tan^2 B}{1 - \tan B}\right)$

$\Rightarrow \quad \tan^2 B + (1 - p)\tan B + p = 0$

since angles are real, so $D \geq 0$

$\Rightarrow \quad (1 - p)^2 - 4p \geq 0$

$\Rightarrow \quad p^2 - 6p + 1 \geq 0$

$\Rightarrow \quad (p - 3)^2 - 8 \geq 0$

$\Rightarrow \quad \left(p - 3 + 2\sqrt{2}\right)\left(p - 3 - 2\sqrt{2}\right) \geq 0$

$\Rightarrow \quad p \leq 3 - 2\sqrt{2}, \ p \geq 3 + 2\sqrt{2}$

14. Given $k = \frac{\tan 3A}{\tan A}$

$\Rightarrow \quad k - 1 = \frac{\tan 3A}{\tan A} - 1 = \frac{\tan 3A - \tan A}{\tan A}$

$\Rightarrow \quad k - 1 = \frac{\sin(2A)}{\cos 3A \sin A}$

$\Rightarrow \quad k - 1 = \frac{2\cos A}{\cos 3A}$

Now, $\frac{2k}{k-1} = \frac{2\tan 3A}{\tan A} \cdot \frac{\cos 3A}{2\cos A}$

$\Rightarrow \quad \frac{2k}{k-1} = \frac{\sin 3A}{\sin A}$

$\Rightarrow \quad \frac{\sin 3A}{\sin A} = \frac{2k}{k-1}$

Also, $\frac{3\sin A - 4\sin^3 A}{\sin A} = \frac{2k}{k-1}$

$\Rightarrow \quad 3 - 4\sin^2 A = \frac{2k}{k-1}$

$\Rightarrow \quad 4\sin^2 A = 3 - \frac{2k}{k-1} = \frac{k-3}{k-1}$

$\Rightarrow \quad \sin^2 A = \frac{k-3}{4(k-1)}$

Clearly, $0 \leq \frac{k-3}{4(k-1)} \leq 1$

On simplification, we get,

$k \le \frac{1}{3}$ and $k \ge 3$

15. Now, $\cos(A+B+C) = \cos(\pi) = -1$

$\Rightarrow \cos A \cos B \cos C[1 - \tan A \tan B - \tan B \tan C - \tan C \tan A] = -1$

$\Rightarrow \cos A \cos B \cos C + 1 = \cos A \sin B \sin C + \cos B \sin C \sin A + \cos C \sin A \sin B$

Dividing both sides by sin$A$ sin$B$ sin$C$, we get,

$\text{cosec} A \,\text{cosec} B \,\text{cosec}\, C + \cot A \cot B \cot C$

$= \cot A + \cot B + \cot C$

$\Rightarrow \cot A + \cot B + \cot C - \text{cosec}\, A \,\text{cosec}\, B \,\text{cosec}\, C$

$= \cot A \cot B \cot C$

Hence, the result.

16. Now, $\tan(\alpha+\beta)$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\dfrac{1}{\sqrt{x(x^2+x+1)}} + \dfrac{\sqrt{x}}{\sqrt{(x^2+x+1)}}}{1 - \dfrac{1}{x^2+x+1}}$$

$$= \frac{(1+x)(x^2+x+1)}{(x^2+x)\sqrt{x(x^2+x+1)}}$$

$$= \frac{\sqrt{(x^2+x+1)}}{x\sqrt{x}} = \tan\lambda$$

Thus, $\alpha + \beta = \gamma$

17. We have, $\cos 2\alpha = \dfrac{3\cos 2\beta - 1}{3 - \cos 2\beta}$

$$\Rightarrow \quad \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{3\left(\dfrac{1-\tan^2\beta}{1+\tan^2\beta}\right) - 1}{3 - \left(\dfrac{1-\tan^2\beta}{1+\tan^2\beta}\right)}$$

$$\Rightarrow \quad \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{3 - 3\tan^2\beta - 1 - \tan^2\beta}{3 + 3\tan^2\beta - 1 + \tan^2\beta}$$

$$\Rightarrow \quad \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{2 - 4\tan^2\beta}{2 + 4\tan^2\beta}$$

$$\Rightarrow \quad \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{1 - 2\tan^2\beta}{1 + 2\tan^2\beta}$$

$$\Rightarrow \quad \frac{2}{-2\tan^2\alpha} = \frac{2}{-4\tan^2\beta}$$

$$\Rightarrow \quad \frac{1}{-\tan^2\alpha} = \frac{1}{-2\tan^2\beta}$$

$$\Rightarrow \quad \tan^2\alpha = 2\tan^2\beta$$

$$\Rightarrow \quad \frac{\tan^2\alpha}{\tan^2\beta} = 2$$

$$\Rightarrow \quad \frac{\tan\alpha}{\tan\beta} = \sqrt{2} = \frac{\sqrt{2}}{1}$$

Hence, the result.

18. Now, $\tan\left(\dfrac{\pi}{4} + \alpha\right) = \dfrac{1+\tan\alpha}{1-\tan\alpha} = \dfrac{\cos\alpha + \sin\alpha}{\cos\alpha - \sin\alpha}$

$$\tan^2\left(\frac{\pi}{4} + \alpha\right) = \frac{1+\sin(2\alpha)}{1-\sin(2\alpha)}$$

So, $\tan^3\left(\dfrac{\alpha}{2} + \dfrac{\pi}{4}\right) = \tan\left(\dfrac{\beta}{2} + \dfrac{\pi}{4}\right)$

$$\Rightarrow \quad \frac{(1+\sin\alpha)^3}{(1-\sin\alpha)^3} = \frac{1+\sin\beta}{1-\sin\beta}$$

Applying componendo and dividendo, we get,

$$\frac{2\sin\beta}{2} = \frac{2(3\sin\alpha + \sin^3\alpha)}{2(1+3\sin^2\alpha)}$$

$$\sin\beta = \frac{(3\sin\alpha + \sin^3\alpha)}{(1+3\sin^2\alpha)}$$

Hence, the result.

19. Given, $\sin\beta = \dfrac{1}{5}\sin(2\alpha+\beta)$

$$\Rightarrow \quad \frac{\sin\beta}{\sin(2\alpha+\beta)} = \frac{1}{5}$$

$$\Rightarrow \quad \frac{\sin\beta + \sin(2\alpha+\beta)}{\sin\beta - \sin(2\alpha+\beta)} = \frac{1+5}{1-5}$$

$$\Rightarrow \quad \frac{2\sin(\alpha+\beta)\cos(\alpha)}{2\cos(\alpha+\beta)\sin(-\alpha)} = -\frac{3}{2}$$

$$\Rightarrow \quad \tan(\alpha+\beta)\cot\alpha = \frac{3}{2}$$

$\Rightarrow \quad \tan(\alpha+\beta)=\frac{3}{2}\tan\alpha$

20. Given, $\sin x+\sin y=3(\cos x-\cos y)$

$\Rightarrow \quad 3\cos x+\sin x=3\cos y-\sin y$

Put $3=r\cos\alpha$,$1=r\sin\alpha$

$\Rightarrow \quad r=\sqrt{10}$ and $\tan\alpha=\frac{1}{3}$

Now, $3\cos x+\sin x=3\cos y-\sin y$

$\Rightarrow \quad r\cos(x-\alpha)=r\cos(y+\alpha)$

$\Rightarrow \quad \cos(x-\alpha)=\cos(y+\alpha)$

$\Rightarrow \quad (x-\alpha)=\pm(y+\alpha)$

$\Rightarrow \quad x=-y, x=y+2\alpha$

$\Rightarrow \quad x=-y$ satiesfies the given equation

$\Rightarrow \quad 3x=-3y$

$\Rightarrow \quad \sin(3x)=\sin(-3y)$

$\Rightarrow \quad \sin(3x)=-\sin(3y)$

$\Rightarrow \quad \sin(3x)+\sin(3y)=0$

Hence, the result.

21. Given $\sec(\varphi-\alpha)$,$\sec\varphi$, $\sec(\varphi+\alpha)$ are in A.P

$\Rightarrow \quad 2\sec\varphi=\sec(\varphi-\alpha)+\sec(\varphi+\alpha)$

$\Rightarrow \quad \frac{2}{\cos\varphi}=\frac{1}{\cos(\varphi-\alpha)}+\frac{1}{\cos(\varphi+\alpha)}$

$\Rightarrow \quad \frac{2}{\cos\varphi}=\frac{\cos(\varphi+\alpha)+\cos(\varphi-\alpha)}{\cos(\varphi-\alpha)\cos(\varphi+\alpha)}$

$\Rightarrow \quad \frac{1}{\cos\varphi}=\frac{\cos\varphi\cos\alpha}{\cos^2(\varphi)-\sin^2(\alpha)}$

$\Rightarrow \quad \cos^2(\varphi)-\sin^2(\alpha)=\cos^2\varphi\cos\alpha$

$\Rightarrow \quad \cos^2(\varphi)(1-\cos\alpha)=\sin^2(\alpha)$

$\Rightarrow \quad \cos^2(\varphi)=\frac{\sin^2(\alpha)}{(1-\cos\alpha)}$

$\Rightarrow \quad \cos^2(\varphi)=\frac{4\sin^2(\alpha/2)\cos^2(\alpha/2)}{2\sin^2(\alpha/2)}$

$\Rightarrow \quad \cos^2(\varphi)=2\cos^2(\alpha/2)$

$\Rightarrow \quad \cos(\varphi)=\sqrt{2}\cos(\alpha/2)$

Hence, the result.

22. Given $\tan\left(\frac{x+y}{2}\right), \tan z, \tan\left(\frac{x-y}{2}\right)$ are in G.P

$\Rightarrow \quad \tan^2 z=\tan\left(\frac{x+y}{2}\right)\tan\left(\frac{x-y}{2}\right)$

$\Rightarrow \quad \tan^2 z=\frac{\sin^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{y}{2}\right)}{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{y}{2}\right)}$

$\Rightarrow \quad \tan^2 z=\frac{\cos y-\cos x}{\cos y+\cos x}$

$\Rightarrow \quad \frac{\tan^2 z}{1}=\frac{\cos y-\cos x}{\cos y+\cos x}$

$\Rightarrow \quad \frac{1}{\tan^2 z}=\frac{\cos y+\cos x}{\cos y-\cos x}$

Applying componendo and dividendo, we get,

$\frac{1-\tan^2 z}{1+\tan^2 z}=\frac{2\cos x}{2\cos y}$

$\Rightarrow \quad \cos(2z)=\frac{\cos x}{\cos y}$

$\Rightarrow \quad \cos x=\cos y\cos(2z)$

23. Let $y=\frac{\sec^2\theta-\tan\theta}{\sec^2\theta+\tan\theta}$

$\Rightarrow \quad y=\frac{\tan^2\theta-\tan\theta+1}{\tan^2\theta+\tan\theta+1}$

$\Rightarrow \quad y=\frac{x^2-x+1}{x^2+x+1}, x=\tan\theta$

$\Rightarrow \quad y=\frac{x^2-x+1}{x^2+x+1}$

$\Rightarrow \quad (x^2+x+1)y=(x^2-x+1)$

$\Rightarrow \quad (y-1)x^2+(y+1)x+(y-1)=0$

For all real θ, $D\geq 0$

$\Rightarrow \quad (y+1)^2-4(y-1)^2\geq 0$

$\Rightarrow \quad (y+1)^2-(2y-2)^2\geq 0$

$\Rightarrow \quad (y+1+2y-2)(y+1-2y+2)\geq 0$

$\Rightarrow \quad (3y-1)(-y+3)\geq 0$

$\Rightarrow\ (3y-1)(y-3) \le 0$

$\Rightarrow\ \frac{1}{3} \le y \le 3$

$\Rightarrow\ \frac{1}{3} \le \left(\frac{\sec^2\theta - \tan\theta}{\sec^2\theta + \tan\theta}\right) \le 3$

24. As we know that

$\cos\theta.\cos(2\theta).\cos(2^2\theta)\cos(2^3\theta)...\cos(2^{n-1}\theta)$

$= \frac{\sin(2^n\theta)}{2^n\sin(\theta)}$

$= \frac{\sin(\pi-\theta)}{2^n\sin(\theta)}$

$= \frac{\sin(\theta)}{2^n\sin(\theta)}$

$= \frac{1}{2^n}$

Hence, the value of

$2^n\cos\theta.\cos(2\theta).\cos(2^2\theta)\cos(2^3\theta)...\cos(2^{n-1}\theta)$

$= 1.$

25. We have, $\tan(6°)\tan(42°)\tan(66°)\tan(78°)$

$= \{\tan(6°)\tan(66°)\}\{\tan(42°)\tan(78°)\}$

$= \frac{1}{\tan(54°)\tan(18°)}\{\tan(54°)\tan(6°)\tan(66°)\}$

$\times\{\tan(42°)\tan(18°)\tan(78°)\}$

$= \frac{\tan(54°)\tan(18°)}{\tan(54°)\tan(18°)}$

$= 1.$

26. As we know that

$\tan A \pm \tan B = \frac{\sin(A \pm B)}{\cos A\cos B}$

So, $\frac{\tan(\alpha+\beta-\gamma)}{\tan(\alpha-\beta+\gamma)} = \frac{\tan\gamma}{\tan\beta}$

Applying, componendo and dividendo, we get

$\frac{\tan(\alpha+\beta-\gamma)+\tan(\alpha-\beta+\gamma)}{\tan(\alpha+\beta-\gamma)-\tan(\alpha-\beta+\gamma)} = \frac{\tan\gamma+\tan\beta}{\tan\gamma-\tan\beta}$

$\frac{\sin(\alpha+\beta-\gamma+\alpha-\beta+\gamma)}{\sin(\alpha+\beta-\gamma-\alpha+\beta-\gamma)} = -\frac{\sin(\beta+\gamma)}{\sin(\beta-\gamma)}$

$\frac{\sin(2\alpha)}{\sin 2(\beta-\gamma)} = -\frac{\sin(\beta+\gamma)}{\sin(\beta-\gamma)}$

$\frac{\sin(2\alpha)}{2\sin(\beta-\gamma)\cos(\beta-\gamma)} = -\frac{\sin(\beta+\gamma)}{\sin(\beta-\gamma)}$

$\sin(\beta-\gamma)[\sin(2\alpha)+2\cos(\beta-\gamma)\sin(\beta+\gamma)] = 0$

$\sin(\beta-\gamma)[\sin(2\alpha)+\sin(2\beta)+\sin(2\gamma)] = 0$

$\sin(\beta-\gamma) = 0, [\sin(2\alpha)+\sin(2\beta)+\sin(2\gamma)] = 0$

Hence, the result.

27. Now, $\cot A + \frac{\sin A}{\sin B\sin C}$

$= \cot A + \frac{\sin(\pi-(B+C))}{\sin B\sin C}$

$= \cot A + \frac{\sin(B+C)}{\sin B\sin C}$

$= \cot A + \frac{\sin B\cos C+\cos B\sin C}{\sin B\sin C}$

$= \cot A + \frac{\sin B\cos C}{\sin B\sin C} + \frac{\cos B\sin C}{\sin B\sin C}$

$= \cot A + \frac{\cos B\sin C}{\sin B\sin C} + \frac{\sin B\cos C}{\sin B\sin C}$

$= \cot A + \cot B + \cot C$

Hence, the result.

28. Given, $\frac{\sin(\theta+A)}{\sin(\theta+B)} = \sqrt{\frac{\sin(2A)}{\sin(2B)}}$

$\Rightarrow\ \frac{\sin\theta\cos A+\cos\theta\sin A}{\sin\theta\cos B+\cos\theta\sin B} = \sqrt{\frac{\sin A\cos A}{\sin B\cos B}}$

$\Rightarrow\ \frac{\tan\theta\cos A+\sin A}{\tan\theta\cos B+\sin B} = \sqrt{\frac{\sin A\cos A}{\sin B\cos B}}$

$\Rightarrow\ \tan\theta\left(\cos A\sqrt{\sin B\cos B} - \cos B\sqrt{\sin A\cos A}\right)$

$= \left(\sin B\sqrt{\sin A\cos A} - \sin A\sqrt{\sin B\cos B}\right)$

$\Rightarrow\ \tan\theta\sqrt{\cos A\cos B}\left(\sqrt{\cos A\sin B} - \sqrt{\cos B\sin A}\right)$

$= \sqrt{\sin A\sin B}\left(\sqrt{\cos A\sin B} - \sqrt{\cos B\sin A}\right)$

$\Rightarrow\ \tan\theta = \frac{\sqrt{\sin A\sin B}}{\sqrt{\cos A\cos B}} = \sqrt{\tan A\tan B}$

$\Rightarrow \quad \tan\theta = \sqrt{\tan A \tan B}$

$\Rightarrow \quad \tan^2\theta = \tan A \tan B$

Hence, the result.

29. Given, $\cos(x-y) = -1$

$\Rightarrow \quad \cos x \cos y + \sin x \sin y = -1$

$\Rightarrow \quad 2\cos x \cos y + 2\sin x \sin y = -2$

$\Rightarrow \quad (1 + 2\cos x \cos y) + (1 + 2\sin x \sin y) = 0$

$\Rightarrow \quad (\cos^2 x + \cos^2 y + 2\cos x \cos y)$

$+ (\sin^2 x + \sin^2 y + 2\sin x \sin y) = 0$

$\Rightarrow \quad (\cos x + \cos y)^2 + (\sin x + \sin y)^2 = 0$

$\Rightarrow \quad (\cos x + \cos y)^2 = 0, (\sin x + \sin y)^2 = 0$

$\Rightarrow \quad (\cos x + \cos y) = 0, (\sin x + \sin y) = 0$

Hence, the result.

30. If $\sqrt{2}\cos A = \cos B + \cos^3 B$

and $\sqrt{2}\sin A = \sin B - \sin^3 B$

then prove that $\sin(A-B) = \pm\frac{1}{3}$

30. Given, $\sqrt{2}\cos A = \cos B + \cos^3 B$

and $\sqrt{2}\sin A = \sin B - \sin^3 B$

squaring and adding, we get,

$(\cos B + \cos^3 B)^2 + (\sin B - \sin^3 B)^2 = 2$

$(\sin^6 B + \cos^6 B) - 2(\sin^4 B - \cos^4 B) + 1 = 2$

$(1 - 3\sin^2 B\cos^2 B) - 2(\sin^2 B - \cos^2 B) + 1 = 2$

$(3\sin^2 B\cos^2 B) + 2(\sin^2 B - \cos^2 B) = 0$

$(3\sin^2 B\cos^2 B) = 2\cos(2B)$

$\frac{3}{4}(\sin^2 2B) = 2\cos(2B)$

$3 - 3\cos^2 2B = 8\cos(2B)$

$3\cos^2 2B + 8\cos(2B) - 3 = 0$

$3\cos^2(2B) + 9\cos(2B) - \cos(2B) - 3 = 0$

$3\cos(2B)(\cos(2B) + 3) - (\cos(2B) + 3) = 0$

$(3\cos(2B) - 1)(\cos(2B) + 3) = 0$

$(3\cos(2B) - 1) = 0, (\cos(2B) + 3) = 0$

$(3\cos(2B) - 1) = 0$

$\cos(2B) = \frac{1}{3}$

$\sin(2B) = \pm\sqrt{1 - \frac{1}{9}} = \pm\frac{2\sqrt{2}}{3} \quad$ ....(i)

On simplification, we get,

$\sin(A-B) = -\frac{1}{2\sqrt{2}}\sin(2B) \quad$ ....(ii)

From (i) and (ii), we get,

$\sin(A-B) = \pm\frac{1}{3}$

Hence, the result.

31. We have, $16\sin^2(9°)$

$= 8(2\sin^2(9°))$

$= 8(1 - \cos(18°))$

$= 8\left(1 - \frac{\sqrt{10 + 2\sqrt{5}}}{4}\right)$

$= \left(8 - 2\sqrt{10 + 2\sqrt{5}}\right)$

$= \left(8 - 2\sqrt{(3+\sqrt{5})(5-\sqrt{5})}\right)$

$= \left(\left(\sqrt{(3+\sqrt{5})}\right)^2 + \left(\sqrt{(5-\sqrt{5})}\right)^2 - 2\sqrt{(3+\sqrt{5})(5-\sqrt{5})}\right)$

$= \left(\left(\sqrt{(3+\sqrt{5})}\right) - \left(\sqrt{(5-\sqrt{5})}\right)\right)^2$

Thus, $4\sin(9°) = \left(\left(\sqrt{(3+\sqrt{5})}\right) - \left(\sqrt{(5-\sqrt{5})}\right)\right)$

$\Rightarrow \quad \sin(9°) = \frac{1}{4}\left(\left(\sqrt{(3+\sqrt{5})}\right) - \left(\sqrt{(5-\sqrt{5})}\right)\right)$

Hence, the result.

32. The function $f(x)$ will provide us the max value if $x = 0$ and min value if $x = \frac{\pi}{6}$

Hence, the range is

$= \left[f\left(\frac{\pi}{6}\right), f(0)\right]$

$$= \left[0, \sin\left(\frac{\pi}{6}\right)\right]$$

$$= \left[0, \frac{1}{2}\right].$$

33. We have, $\sum_{k=1}^{6}\left(\sin\left(\frac{2k\pi}{7}\right) - i\cos\left(\frac{2k\pi}{7}\right)\right)$

$$= -i\sum_{k=1}^{6}\left(\cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right)\right)$$

$$= -i\sum_{k=1}^{6} e^{i\left(\frac{2k\pi}{7}\right)}$$

$$= -i\left(e^{i\frac{2\pi}{7}} + e^{i\frac{4\pi}{7}} + \dots + e^{i\frac{12\pi}{7}}\right)$$

$$= -ie^{i\frac{2\pi}{7}}\left(1 + e^{i\frac{2\pi}{7}} + e^{i\frac{4\pi}{7}} + \dots + e^{i\frac{10\pi}{7}}\right)$$

$$= -ie^{i\frac{2\pi}{7}}\left(\frac{1 - e^{i\frac{12\pi}{7}}}{1 - e^{i\frac{2\pi}{7}}}\right)$$

$$= -i\left(\frac{e^{i\frac{2\pi}{7}} - e^{i\frac{12\pi}{7} + i\frac{2\pi}{7}}}{1 - e^{i\frac{2\pi}{7}}}\right)$$

$$= -i\left(\frac{e^{i\frac{2\pi}{7}} - e^{i(2\pi)}}{1 - e^{i\frac{2\pi}{7}}}\right)$$

$$= -i\left(\frac{e^{i\frac{2\pi}{7}} - 1}{1 - e^{i\frac{2\pi}{7}}}\right)$$

$$= i\left(\frac{1 - e^{i\frac{2\pi}{7}}}{1 - e^{i\frac{2\pi}{7}}}\right) = i$$

34. Given $\cos\theta + \cos\varphi = a$

and $\sin\theta + \sin\varphi = b$

Now, $\frac{\sin\theta + \sin\varphi}{\cos\theta + \cos\varphi} = \frac{b}{a}$

$$\Rightarrow \quad \frac{2\sin\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)}{2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)} = \frac{b}{a}$$

$$\Rightarrow \quad \frac{2\sin\left(\frac{\theta+\varphi}{2}\right)}{2\cos\left(\frac{\theta+\varphi}{2}\right)} = \frac{b}{a}$$

$$\Rightarrow \quad \tan\left(\frac{\theta+\varphi}{2}\right) = \frac{b}{a}$$

On simplification, we get,

$$\cos\left(\frac{\theta-\varphi}{2}\right) = \frac{\sqrt{a^2+b^2}}{2}$$

and $\cos\left(\frac{\theta+\varphi}{2}\right) = \frac{a^2+b^2}{\sqrt{a^2+b^2}+2}$

Now, $\tan\left(\frac{\theta}{2}\right) + \tan\left(\frac{\varphi}{2}\right)$

$$= \frac{\sin\left(\frac{\theta+\varphi}{2}\right)}{\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\varphi}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\theta+\varphi}{2}\right)}{2\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\varphi}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\theta+\varphi}{2}\right)}{\cos\left(\frac{\theta+\varphi}{2}\right) + \cos\left(\frac{\theta-\varphi}{2}\right)}$$

$$= \frac{2\tan\left(\frac{\theta+\varphi}{2}\right)}{1 + \frac{\cos\left(\frac{\theta-\varphi}{2}\right)}{\cos\left(\frac{\theta+\varphi}{2}\right)}}$$

$$= 1 + \frac{\frac{\sqrt{a^2+b^2}}{2}}{\frac{a^2+b^2}{\sqrt{a^2+b^2}+2}}$$

$$= 1+\frac{\sqrt{a^2+b^2}+2}{\sqrt{a^2+b^2}}$$

$$= \left(2+\frac{2}{\sqrt{a^2+b^2}}\right)$$

35. We have, $\frac{\tan\theta}{\tan\theta-\tan 3\theta}=\frac{1}{3}$

$$\Rightarrow \quad \frac{1}{1-\frac{\tan 3\theta}{\tan\theta}}=\frac{1}{3}$$

$$\Rightarrow \quad \frac{1}{1-\frac{3-\tan^2\theta}{1-3\tan^2\theta}}=\frac{1}{3}$$

$$\Rightarrow \quad \frac{1-3\tan^2\theta}{1-3\tan^2\theta-3+\tan^2\theta}=\frac{1}{3}$$

$$\Rightarrow \quad \frac{1-3\tan^2\theta}{2+2\tan^2\theta}=-\frac{1}{3}$$

$$\Rightarrow \quad 2+2\tan^2\theta=-3+9\tan^2\theta$$

$$\Rightarrow \quad 7\tan^2\theta=5$$

$$\Rightarrow \quad \tan^2\theta=\frac{5}{7}$$

Now, $\frac{\cot\theta}{\cot\theta-\cot(3\theta)}$

$$=\frac{1}{1-\left(\frac{\cot(3\theta)}{\cot\theta}\right)}$$

$$=\frac{1}{1+\frac{1}{2}}=\frac{2}{3}$$

INTEGER TYPE QUESTIONS

1. Given $\frac{x}{\cos\theta}=\frac{y}{\cos\left(\theta-\frac{2\pi}{3}\right)}=\frac{z}{\cos\left(\theta+\frac{2\pi}{3}\right)}=k$

Now, $x+y+z$

$$=k\left[\cos\theta+\cos\left(\theta-\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{2\pi}{3}\right)\right]$$

$$=k\left[\cos\theta+2\cos\theta\cos\left(\frac{2\pi}{3}\right)\right]$$

$$=k\left[\cos\theta-2\cos\theta.\frac{1}{2}\right]$$

$$=k\left[\cos\theta-\cos\theta\right]$$

$$=0$$

Hence, the value of $(x+y+z+4)=4$.

2. We have, $\sum_{r=0}^{9}\sin^2\left(\frac{\pi r}{18}\right)$

$$=\sin^2\left(\frac{\pi}{18}\right)+\sin^2\left(\frac{2\pi}{18}\right)+\sin^2\left(\frac{3\pi}{18}\right)+....+\sin^2\left(\frac{9\pi}{18}\right)$$

$$=\sin^2(10°)+\sin^2(20°)+\sin^2(30°)+....+\sin^2(90°)$$

$$=4\times1+1=5$$

3. We have, $\frac{\tan x}{\tan y}=\frac{1}{3}$

$$\frac{\tan x}{1}=\frac{\tan y}{3}$$

Now, $\tan x=\frac{1}{3}\sqrt{\frac{1-\frac{9}{4}}{\frac{1}{4}-1}}=\frac{1}{3}\sqrt{\frac{5}{3}}$

We have, $\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$

$$=\frac{\tan x+3\tan x}{1-\tan x.3\tan x}$$

$$=\frac{4\tan x}{1-3\tan^2 x}$$

$$=\frac{\frac{4}{3}\sqrt{\frac{5}{3}}}{1-\frac{3}{9}\times\frac{5}{3}}$$

$$=\frac{4\sqrt{5}}{3\sqrt{3}}\times\frac{9}{4}$$

$$=\frac{3\sqrt{5}}{\sqrt{3}}=\sqrt{15}$$

Thus, $\frac{\tan^2(x+y)}{5}=3$.

4. Given $\cos(x-y),\cos x,\cos(x+y)$ are in H.P

$$\Rightarrow \quad \cos x=\frac{2\cos(x-y)\cos(x+y)}{\cos(x-y)+\cos(x+y)}$$

$$\Rightarrow \quad \cos x=\frac{\cos 2x+\cos 2y}{2\cos x\cos y}$$

$\Rightarrow \quad 2\cos^2 x\cos y = 2\cos^2 x + 2\cos^2 y - 2$

$\Rightarrow \quad \cos^2 x\cos y = \cos^2 x + \cos^2 y - 1$

$\Rightarrow \quad \cos^2 x(\cos y - 1) = \cos^2 y - 1$

$\Rightarrow \quad \cos^2 x = \cos y + 1$

$\Rightarrow \quad \cos^2 x = 2\cos^2\left(\frac{y}{2}\right)$

$\Rightarrow \quad \cos^2 x\sec^2\left(\frac{y}{2}\right) = 2$

$\Rightarrow \quad \left|\cos x\sec\left(\frac{y}{2}\right)\right| = \sqrt{2}$

Thus, $m = \sqrt{2}$.

Hence, the value of $(m^2 + 2) = 4$.

5. We have, $\tan x + \tan\left(\frac{\pi}{3} + x\right) + \tan\left(\frac{2\pi}{3} + x\right)$

$$= \tan x + \tan\left(\frac{\pi}{3} + x\right) + \tan\left(\pi - \frac{\pi}{3} + x\right)$$

$$= \tan x + \tan\left(\frac{\pi}{3} + x\right) - \tan\left(\frac{\pi}{3} - x\right)$$

$$= \tan x + \frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x} - \frac{\sqrt{3} - \tan x}{1 + \sqrt{3}\tan x}$$

$$= \tan x + \frac{8\tan x}{1 - 3\tan^2 x}$$

$$= \frac{\tan x - 3\tan^3 x + 8\tan x}{1 - 3\tan^2 x}$$

$$= \frac{9\tan x - 3\tan^3 x}{1 - 3\tan^2 x}$$

$$= \frac{3(3\tan x - \tan^3 x)}{1 - 3\tan^2 x}$$

$$= 3\tan(3x)$$

Thus, $k = 3$.

6. Given $f(\theta) = \sin^2\theta + \sin^2\left(\frac{2\pi}{3} + \theta\right) + \sin^2\left(\frac{4\pi}{3} + \theta\right)$

$$= \sin^2\theta + \sin^2\left(\frac{2\pi}{3} + \theta\right) + \sin^2\left(\pi + \frac{\pi}{3} + \theta\right)$$

$$= \sin^2\theta + \sin^2\left(\frac{2\pi}{3} + \theta\right) + \sin^2\left(\frac{\pi}{3} + \theta\right)$$

$$= \sin^2\theta + \sin^2\left(\frac{\pi}{3} - \theta\right) + \sin^2\left(\frac{\pi}{3} + \theta\right)$$

$$= \frac{1}{2}\left[1 - \cos(2\theta) + 1 - \cos\left(\frac{2\pi}{3} - 2\theta\right) + 1 - \cos\left(\frac{2\pi}{3} + 2\theta\right)\right]$$

$$= \frac{1}{2}\left[3 - \cos(2\theta) - \cos\left(\frac{2\pi}{3} - 2\theta\right) - \cos\left(\frac{2\pi}{3} + 2\theta\right)\right]$$

$$= \frac{1}{2}\left[3 - \cos(2\theta) - 2\cos\left(\frac{2\pi}{3}\right)\cos(2\theta)\right]$$

$$= \frac{1}{2}\left[3 - \cos(2\theta) - 2 \times -\frac{1}{2} \times \cos(2\theta)\right]$$

$$= \frac{3}{2}$$

Hence, the value of $2f\left(\frac{\pi}{15}\right) = 2 \times \frac{3}{2} = 3$.

7. We have, $m = \sqrt{3}\operatorname{cosec}(20°) - \sec(20°) = 4$

and $n = \sin(12°)\sin(48°)\sin(54°) = \frac{1}{8}$

Hence, the value of $(m + 8n + 2)$

$= 4 + 1 + 2$

$= 7.$

8. Here, $\tan(15°) + \tan(30°) = -p$

and $\tan(15°) + \tan(30°) = q$

We have, $\tan(45°) = 1$

$\Rightarrow \quad \tan(30° + 15°) = 1$

$\Rightarrow \quad \frac{\tan(30°) + \tan(15°)}{1 - \tan(30°)\tan(15°)} = 1$

$\Rightarrow \quad \frac{-p}{1 - q} = 1$

$\Rightarrow \quad -p = 1 - q$

$\Rightarrow \quad q - p = 1$

$\Rightarrow \quad 2 + q - p = 2 + 1 = 3$

Hence, the value of $(2 + q - p)$ is 3.

9. We have, $x = \dfrac{\sum_{n=1}^{44}\cos(n°)}{\sum_{n=1}^{44}\sin(n°)}$

$$= \frac{\cos\left(1° + \frac{43°}{2}\right)}{\sin\left(1° + \frac{43°}{2}\right)}$$

$$= \frac{\cos\left(\frac{45°}{2}\right)}{\sin\left(\frac{45°}{2}\right)}$$

$$= \frac{2\cos^2\left(\frac{45°}{2}\right)}{\sin(45°)}$$

$$= \frac{(1+\cos 45°)}{\sin(45°)} = (\sqrt{2}+1)$$

Hence, the value of $[x+3]$

$$= [\sqrt{2}+1+3]$$

$$= [\sqrt{2}]+4 = 1+4 = 5$$

10. We have, $\sin(25°)\sin(35°)\sin(85°)$

$$= \frac{1}{4}[4\sin(35°)\sin(25°)\sin(85°)]$$

$$= \frac{1}{4}\times\sin(75°)$$

$$= \frac{1}{4}\times\cos(15°)$$

$$= \frac{1}{4}\times\frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{16}$$

Hence, the value of $\left(\frac{c}{a+b}+2\right) = 4$.

11. We have, $m = \sum_{k=1}^{17}\cos\left(\frac{k\pi}{9}\right)$

$$= \cos\left(\frac{\pi}{9}\right)+\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{3\pi}{9}\right)+...+\cos\left(\frac{17\pi}{9}\right)$$

$$= \frac{\sin\left(\frac{17\pi}{18}\right)}{\sin\left(\frac{\pi}{18}\right)}\times\cos\left(\frac{\pi}{9}+\frac{16\pi}{18}\right)$$

$$= \cos\left(\frac{18\pi}{18}\right) = -1$$

Hence, the value of $(m^2+m+2) = 4$.

12. We have, $\tan(55°)\tan(65°)\tan(75°)$

$$= \frac{1}{\tan(5°)}\times[\tan(55°)\tan(5°)\tan(65°)]\times\tan(75°)$$

$$= \frac{1}{\tan(5°)}\times\tan(15°)\times\tan(75°)$$

$$= \frac{1}{\tan(5°)}\times\tan(15°)\times\cot(15°)$$

$$= \frac{1}{\tan(5°)}$$

$$= \cot(5°)$$

Thus, $x = 5$

Also, $\tan(27°)+\tan(18°)+\tan(27°)\tan(18°)$

$$= \tan(27°+18°)$$

$$= \tan(45°)$$

$$= 1$$

So, $m = 1$

Hence, the value of $(m+x+1)$ is 7.

QUESTIONS ASKED IN IIT-JEE EXAMS

1. Now, $\sin x\sin y\sin(x-y)$

$$= \frac{1}{2}(2\sin x\sin y)\sin(x-y)$$

$$= \frac{1}{2}(\cos(x-y)-\cos(x+y))\sin(x-y)$$

$$= \frac{1}{4}(2\cos(x-y)\sin(x-y)-2\sin(x-y)\cos(x+y))$$

$$= \frac{1}{4}(\sin(2x-2y)-\sin 2x+\sin 2y)$$

Similarly, $\sin y\ \sin z\ \sin(y-z)$

$$= \frac{1}{4}(\sin(2y-2z)-\sin 2y+\sin 2z)$$

and $\sin z\ \sin x\ \sin(z-x)$

$$= \frac{1}{4}(\sin(2z-2x)-\sin 2z+\sin 2x)$$

Therefore, the given expression reduces to

$$\frac{1}{4}(\sin 2A+\sin 2B+\sin 2C)+\sin A.\sin B.\sin C$$

Where $A = x-y, B = y-z, C = z-x$

$$= \frac{1}{4}(\sin 2A+\sin 2B+\sin 2C)+\sin A.\sin B.\sin C$$

$$= \frac{1}{4}(2\sin(A+B)\cos(A-B)+\sin 2C) + \sin A.\sin B.\sin C$$

$$= \frac{1}{4}(2\sin(-C)\cos(A-B)+\sin 2C) + \sin A.\sin B.\sin C$$

$$= \frac{1}{4}(-2\sin(C)\cos(A-B)+2\sin C\cos C)$$

$$+\sin A.\sin B.\sin C$$

$$= \frac{1}{4}\left(-2\sin(C)\cos(A-B)+2\sin C\cos C\right)$$
$$+\sin A.\sin B.\sin C$$

$$= -\frac{1}{4}\times 2\sin C\left(\cos(A-B)-\cos(A+B)\right)$$
$$+\sin A.\sin B.\sin C$$

$$= -\frac{1}{4}\times 2\sin C\times 2\sin A\sin B + \sin A.\sin B.\sin C$$

$$= -\sin A\sin B\sin C + \sin A.\sin B.\sin C$$

$$= 0$$

2. Ans. 56/33

We have $\tan 2\alpha$

$$= \tan\left((\alpha+\beta)+(\alpha-\beta)\right)$$

$$= \frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$$

$$= \frac{\left(\frac{3}{4}+\frac{5}{12}\right)}{1-\left(\frac{3}{4}.\frac{5}{12}\right)}$$

$$= \frac{36+20}{48-15}$$

$$= \frac{56}{33}$$

3. Ans. (b)

We have, $A=\sin^2\theta+\cos^4\theta$

$$= \frac{1}{2}\left(2\sin^2\theta\right)+\frac{1}{4}\left(2\cos^2\theta\right)^2$$

$$= \frac{1}{2}(1-\cos 2\theta)+\frac{1}{4}(1+\cos 2\theta)^2$$

$$= \frac{1}{2}(1-\cos 2\theta)+\frac{1}{4}\left(1+2\cos 2\theta+\cos^2 2\theta\right)$$

$$= \frac{1}{2}+\frac{1}{4}+\frac{1}{4}\cos^2 2\theta$$

$$= \frac{3}{4}+\frac{1}{4}\cos^2 2\theta$$

Max value $= \frac{3}{4}+\frac{1}{4}.1=1$

Min value $= \frac{3}{4}+\frac{1}{4}.0=\frac{3}{4}$

Therefore, $\frac{3}{4}\le A\le 1$

4. Ans. True

Given $\tan A=\dfrac{1-\cos B}{\sin B}$

$$\tan A=\frac{2\sin^2(B/2)}{2\sin(B/2)\cos(B/2)}=\tan\left(\frac{B}{2}\right)$$

Now, $\tan 2A$

$$= \frac{2\tan A}{1-\tan^2 A}$$

$$= \frac{2\tan(B/2)}{1-\tan^2(B/2)}$$

$$= \tan B$$

Hence, the result.

5. Ans. $n=6$

We have, $\sin^3 x\sin 3x$

$$= \left(\sin^2 x\right)(\sin 3x\sin x)$$

$$= \frac{1}{4}\left(2\sin^2 x\right)(2\sin 3x\sin x)$$

$$= \frac{1}{4}(1-\cos 2x)(\cos 2x-\cos 4x)$$

$$= \frac{1}{4}\left(\cos 2x-\cos^2 2x-\cos 4x+\cos 2x.\cos 4x\right)$$

$$= \frac{1}{8}\left[2\cos 2x-2\cos^2 2x-2\cos 4x\right.$$
$$\left.+2\cos 4x.\cos 2x\right]$$

$$= \frac{1}{8}\left[2\cos 2x-(1+\cos 4x)-2\cos 4x\right.$$
$$\left.+\cos 6x+\cos 2x\right]$$

$$= \frac{1}{8}(-1+3\cos 2x-3\cos 4x+\cos 6x)$$

Thus, $n = 6$.

6. We have $\sin 12°\sin 54°\sin 48°$

$$= (\sin 12°\sin 48°)\sin 54°$$

$$= \frac{1}{4\sin 72°}(4\sin 48°\sin 12°\sin 72°)\sin 54°$$

$$= \frac{1}{4\sin 72°}(\sin 36°\cos 36°)$$

$$= \frac{1}{4\times 2\sin 72°}(2\sin 36°\cos 36°)$$

$$= \frac{1}{4 \times 2\sin 72°}(\sin 72°)$$

$$= \frac{1}{8}$$

7. We have, $\sin^2 \alpha + \sin^2 \beta - \sin^2 \gamma$

$$= \sin^2 \alpha + \sin(\beta + \gamma)\sin(\beta - \gamma)$$

$$= \sin^2 \alpha + \sin(\pi - \alpha)\sin(\beta - \gamma)$$

$$= \sin^2 \alpha + \sin\alpha \sin(\beta - \gamma)$$

$$= \sin\alpha(\sin\alpha + \sin(\beta - \gamma))$$

$$= \sin\alpha(\sin(\pi - (\beta + \gamma)) + \sin(\beta - \gamma))$$

$$= \sin\alpha(\sin(\beta + \gamma) + \sin(\beta - \gamma))$$

$$= \sin\alpha \times 2\sin\beta\sin\gamma$$

$$= 2\sin\alpha\ \sin\beta\sin\gamma$$

Hence, the result.

8. We have

$$16\cos\left(\frac{2\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)\cos\left(\frac{16\pi}{15}\right)$$

$$= -16\cos\left(\frac{\pi}{15}\right)\cos\left(\frac{2\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)$$

$$= \frac{-16}{2\sin\left(\frac{\pi}{15}\right)}\sin\left(\frac{2\pi}{15}\right)\cos\left(\frac{2\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)$$

$$= \frac{-8}{2\sin\left(\frac{\pi}{15}\right)}\sin\left(\frac{4\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)$$

$$= \frac{-4}{2\sin\left(\frac{\pi}{15}\right)}\sin\left(\frac{8\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)$$

$$= \frac{-1}{\sin\left(\frac{\pi}{15}\right)}\sin\left(\frac{16\pi}{15}\right)$$

$$= \frac{-1}{\sin\left(\frac{\pi}{15}\right)}\sin\left(\pi + \frac{\pi}{15}\right)$$

$$= \frac{-1}{\sin\left(\frac{\pi}{15}\right)} \times -\sin\left(\frac{\pi}{15}\right) = 1$$

9. Ans (c)

We have $\left(1+\cos\left(\frac{\pi}{8}\right)\right)\left(1+\cos\left(\frac{3\pi}{8}\right)\right)\left(1+\cos\left(\frac{5\pi}{8}\right)\right)\left(1+\cos\left(\frac{7\pi}{8}\right)\right)$

$$= \left(1+\cos\left(\frac{\pi}{8}\right)\right)\left(1+\cos\left(\frac{3\pi}{8}\right)\right)\left(1-\cos\left(\frac{3\pi}{8}\right)\right)\left(1-\cos\left(\frac{\pi}{8}\right)\right)$$

$$= \left(1-\cos^2\left(\frac{\pi}{8}\right)\right)\left(1-\cos^2\left(\frac{3\pi}{8}\right)\right)$$

$$= \sin^2\left(\frac{\pi}{8}\right)\sin^2\left(\frac{3\pi}{8}\right)$$

$$= \frac{1}{4}\left(2\sin^2\left(\frac{\pi}{8}\right)\right)\left(2\sin^2\left(\frac{3\pi}{8}\right)\right)$$

$$= \frac{1}{4}\left(1-\cos\left(\frac{\pi}{4}\right)\right)\left(1-\cos\left(\frac{3\pi}{4}\right)\right)$$

$$= \frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{4}\left(1-\frac{1}{2}\right)$$

$$= \frac{1}{4} \times \frac{1}{2}$$

$$= \frac{1}{8}$$

10. No questions asked in 1985

11. Ans. (b)

The given expression reduces to

$$3(\cos^4\alpha + \sin^4\alpha) - 2(\cos^6\alpha + \sin^6\alpha)$$

$$= 3(1 - 2\sin^2\alpha\cos^2\alpha) - 2(1 - 3\sin^2\alpha\cos^2\alpha)$$

$$= 3 - 2$$

$$= 1$$

12. No questions asked in 1987.

13. Ans. (c)

We have, $\sqrt{3}\,\text{cosec}(20°) - \sec(20°)$

$$= \frac{\sqrt{3}}{\sin(20°)} - \frac{1}{\cos(20°)}$$

$$= \frac{\sqrt{3}\cos(20°) - \sin(20°)}{\sin(20°)\cos(20°)}$$

$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos(20°) - \frac{1}{2}\sin(20°)\right)}{2\sin(20°)\cos(20°)}$$

$$= \frac{4(\sin(60°)\cos(20°) - \cos(60°)\sin(20°))}{\sin(20°)}$$

$$= \frac{4(\sin(60° - 20°))}{\sin(40°)}$$

$$= \frac{4(\sin(40°))}{\sin(40°)}$$

$$= 4$$

14. $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha$

$= \cot\alpha - (\cot\alpha - \tan\alpha) + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha$

$= \cot\alpha - (2\cot 2\alpha - 2\tan 2\alpha) + 4\tan 4\alpha + 8\cot 8\alpha$

$= \cot\alpha - 2(\cot 2\alpha - \tan 2\alpha) + 4\tan 4\alpha + 8\cot 8\alpha$

$= \cot\alpha - 2.2\cot 4\alpha + 4\tan 4\alpha + 8\cot 8\alpha$

$= \cot\alpha - 4(\cot 4\alpha - \tan 4\alpha) + 8\cot 8\alpha$

$= \cot\alpha - 4.2\cot 8\alpha + 8\cot 8\alpha$

$= \cot\alpha - 8\cot 8\alpha + 8\cot 8\alpha$

$= \cot\alpha$

15. No questions asked in between 1989 and 1990.

16. Ans. $\frac{1}{64}$

$$\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)\sin\left(\frac{7\pi}{14}\right)$$
$$\sin\left(\frac{9\pi}{14}\right)\sin\left(\frac{11\pi}{14}\right)\sin\left(\frac{13\pi}{14}\right)$$

$$= \sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)\sin\left(\frac{\pi}{2}\right)$$
$$\sin\left(\frac{9\pi}{14}\right)\sin\left(\frac{11\pi}{14}\right)\sin\left(\frac{13\pi}{14}\right)$$

$$= \sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)\sin\left(\frac{\pi}{2}\right)$$
$$\sin\left(\pi - \frac{5\pi}{14}\right)\sin\left(\pi - \frac{3\pi}{14}\right)\sin\left(\pi - \frac{\pi}{14}\right)$$

$$= \sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)$$
$$\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)$$

$$= \left(\sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)\right)^2$$

$$= \left(\cos\left(\frac{\pi}{2} - \frac{\pi}{14}\right)\cos\left(\frac{\pi}{2} - \frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2} - \frac{5\pi}{14}\right)\right)^2$$

$$= \left(\cos\left(\frac{6\pi}{14}\right)\cos\left(\frac{4\pi}{14}\right)\cos\left(\frac{2\pi}{14}\right)\right)^2$$

$$= \left(\cos\left(\frac{3\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{\pi}{7}\right)\right)^2$$

$$= \left(\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\right)^2$$

$$= \left(\frac{1}{8}\right)^2$$

$$= \frac{1}{64}$$

17 Ans. (c)

We have $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$

$= \cos 9x + \cos 10x$

Now, $f\left(\frac{\pi}{2}\right) = \cos\left(\frac{9\pi}{2}\right) + \cos(5\pi)$

$= 0 - 1 = -1$

$f(\pi) = \cos(9\pi) + \cos(10\pi) = -1 + 1 = 0$

$f(-\pi) = \cos(-9\pi) + \cos(-10\pi)$

$= \cos(9\pi) + \cos(10\pi)$

$= -1 + 1 = 0$

$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{9\pi}{4}\right) + \cos\left(\frac{10\pi}{4}\right)$

$= \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$

18. Ans. (i) $\to$ C; (ii) $\to$ A.

19. We have, $k = \sin\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{7\pi}{18}\right)$

$= \sin(10°)\sin(50°)\sin(70°)$

$= \sin(50°)\sin(10°)\sin(70°)$

$= \frac{1}{4} \times (4\sin(60° - 10°)\sin(10°)\sin(60° + 10°))$

$= \frac{1}{4} \times (\sin(3 \times 10°))$

$= \frac{1}{4} \times \frac{1}{2}$

$= \frac{1}{8}$

20. Let $y = \tan A \tan B$

$= \tan A \tan\left(\frac{\pi}{3} - A\right)$

$= \tan A \left(\frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}\right)$

$= \left(\frac{\sqrt{3}\tan A - \tan^2 A}{1 + \sqrt{3}\tan A}\right)$

$= \left(\frac{\sqrt{3}\,t - t^2}{1 + \sqrt{3}\,t}\right)$

$\Rightarrow \quad \frac{dy}{dt} = \frac{(1 + \sqrt{3}t)(\sqrt{3} - 2t) - (\sqrt{3}t - t^2)\sqrt{3}}{(1 + \sqrt{3}t)^2}$

$= \frac{\sqrt{3} + 3t - 2t - 2\sqrt{3}t^2 - 3t + \sqrt{3}t^2}{(1 + \sqrt{3}t)^2}$

$= \frac{\sqrt{3} - 2t - \sqrt{3}t^2}{(1 + \sqrt{3}t)^2}$

$= -\frac{\sqrt{3}t^2 + 2t - \sqrt{3}}{(1 + \sqrt{3}t)^2}$

$= -\frac{\sqrt{3}t^2 + 3t - t - \sqrt{3}}{(1 + \sqrt{3}t)^2}$

$= -\frac{\sqrt{3}t(t + \sqrt{3}) - (t + \sqrt{3})}{(1 + \sqrt{3}t)^2}$

$= -\frac{(\sqrt{3}t - 1)(t + \sqrt{3})}{(1 + \sqrt{3}t)^2}$

$\frac{dy}{dt} = 0$ gives $t = \frac{1}{\sqrt{3}}, -\sqrt{3}$

Thus, $t = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \tan A = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad A = \frac{\pi}{6}$

When $A = \frac{\pi}{6}$, $B = \frac{\pi}{3} - A = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

Thus, the minimum value of $\tan A \,.\, \tan B$

$= \frac{1}{\sqrt{3}} . \frac{1}{\sqrt{3}}$

$= \frac{1}{3}$

21. Ans. (b)

We have $(\sec 2x - \tan 2x)$

$= \left(\frac{1 - \sin 2x}{\cos 2x}\right)$

$= \left(\frac{(\cos x - \sin x)^2}{\cos^2 x - \sin^2 x}\right)$

$= \left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

$= \left(\frac{1 - \tan x}{1 + \tan x}\right)$

$= \tan\left(\frac{\pi}{4} - x\right)$

22 Ans(c)

$3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2$

$+ 4(\sin^6 x + \cos^6 x)$

$= 3[\sin^4 x - 4\sin^3 x\cos x + 6\sin^2 x\cos^2 x$

$-4\sin x\cos^3 x + \cos^4 x] + 6(1 + 2\sin x\cos x)$

$+4(1 - 3\sin^2 x\cos^2 x)$

$3[(\sin^4 x + \cos^4 x) - 4\sin x\cos x(\sin^2 x + \cos^2 x)$

$+6\sin^2 x\cos^2 x] + 6(1 + 2\sin x\,\cos x)$

$+4(1 - 3\sin^2 x\cos^2 x)$

$3[1 - 2\sin^2 x\cos^2 x + 4\sin x\cos x$

$+6\sin^2 x\cos^2 x] + 6(1 + 2\sin x\cos x)$

$+4(1 - 3\sin^2 x\cos^2 x)$

$= 3 + 6 + 4$

$= 13.$

23. Ans. (c)

As we know that,

$$(\sin\alpha+\sin\beta+\sin\gamma)<\sin(\alpha+\beta+\gamma)$$

$$=\sin(\pi)=0$$

Thus, $\sin\alpha+\sin\beta+\sin\gamma<0$

24 Ans. (b)

we have $\sec^2\theta=\left(\frac{4xy}{(x+y)^2}\right)$

$\Rightarrow \quad 1+\tan^2\theta=\left(\frac{4xy}{(x+y)^2}\right)$

$\Rightarrow \quad \tan^2\theta=\left(\frac{4xy}{(x+y)^2}-1\right)$

$\Rightarrow \quad \tan^2\theta=\left(\frac{4xy-(x+y)^2}{(x+y)^2}\right)=-\left(\frac{x-y}{x+y}\right)^2$

$\Rightarrow \quad -\left(\frac{x-y}{x+y}\right)^2=\tan^2\theta\ge 0$

$\Rightarrow \quad -(x-y)^2\ge 0$

$\Rightarrow \quad (x-y)^2\le 0$

$\Rightarrow \quad (x-y)^2=0$

$\Rightarrow \quad x-y=0$

$\Rightarrow \quad x=y$

Therefore, $x=y\ne 0$

25 Ans. (d)

We have $y=\cos x\cos(x+2)-\cos^2(x+1)$

$$=\frac{1}{2}\left(2\cos x\cos(x+2)-2\cos^2(x+1)\right)$$

$$=\frac{1}{2}\left(2\cos(2x+2)+\cos(1)-(1+\cos(2x+2))\right)$$

$$=-\frac{1}{2}(1-\cos(1))$$

$$=-\frac{1}{2}\times 2\sin^2(1)$$

$$=-\sin^2(1)$$

which is a straight line passing through $\left(\frac{\pi}{2},-\sin^2 1\right)$ and parallel to x-axis.

26 Ans. (c)

we have $\sin 15^\circ\cos 15^\circ$

$$=\frac{1}{2}(2\sin 15^\circ\cos 15^\circ)$$

$$=\frac{1}{2}(\sin 30^\circ)$$

$$=\frac{1}{4}$$

27. Ans. (a, b, c, d)

We have

$$f_n(\theta)=\tan\left(\frac{\theta}{2}\right)(1+\sec\theta)(1+\sec 2\theta)\ldots\ldots(1+\sec 4\theta)\ldots\ldots(1+\sec 2^n\theta)$$

$$=\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}\times\left(\frac{1+\cos\theta}{\cos\theta}\right)(1+\sec 2\theta)(1+\sec 4\theta)\ldots\ldots(1+\sec 2^n\theta)$$

$$=\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\cos\theta}(1+\sec 2\theta)(1+\sec 4\theta)(1+\sec 8\theta)\ldots\ldots(1+\sec 2^n\theta)$$

$$=\frac{\sin\theta}{\cos\theta}\times\left(\frac{1+\cos 2\theta}{\cos 2\theta}\right)(1+\sec 4\theta)(1+\sec 8\theta)\ldots\ldots(1+\sec 2^n\theta)$$

$$=\frac{2\sin\theta\cos\theta}{\cos 2\theta}\times\left(\frac{1+\cos 4\theta}{\cos 4\theta}\right)(1+\sec 8\theta)\ldots\ldots(1+\sec 2^n\theta)$$

$$=\frac{\sin 2\theta}{\cos 2\theta}\times\frac{2\cos^2 2\theta}{\cos 4\theta}(1+\sec 8\theta)\ldots(1+\sec 2^n\theta)$$

$$=\frac{\sin 4\theta}{\cos 4\theta}\left(\frac{1+\cos 8\theta}{\cos 8\theta}\right)\ldots\ldots(1+\sec 2^n\theta)$$

$$=\frac{\sin 4\theta}{\cos 4\theta}\left(\frac{2\cos^2 4\theta}{\cos 8\theta}\right)\ldots\ldots(1+\sec 2^n\theta)$$

$$=\frac{\sin 8\theta}{\cos 8\theta}\ldots\ldots\ldots\ldots(1+\sec 2^n\theta)$$

$$=\frac{\sin(2^n\theta)}{\cos(2^n\theta)}$$

$$=\tan(2^n\theta)$$

Now, $f_2\left(\frac{\pi}{16}\right)=\tan\left(2^2\times\frac{\pi}{16}\right)=\tan\left(\frac{\pi}{4}\right)=1$

$$f_3\left(\frac{\pi}{32}\right) = \tan\left(2^3 \times \frac{\pi}{32}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

$$f_4\left(\frac{\pi}{64}\right) = \tan\left(2^4 \times \frac{\pi}{64}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

$$f_5\left(\frac{\pi}{128}\right) = \tan\left(2^5 \times \frac{\pi}{128}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

Hence, the result.

28 Ans. (c)

We have $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$

$= \sin\theta(\sin 3\theta + \sin\theta)$

$= \sin\theta \times 2\sin 2\theta\cos\theta$

$= 2\sin\theta\cos\theta \times \sin 2\theta$

$= \sin 2\theta \times \sin 2\theta$

$= \sin^2 2\theta$

≥ 0 for all real θ

29. Ans. (a)

Given $\cot\alpha_1.\cot\alpha_2......\cot\alpha_n = 1$

$\Rightarrow \quad \cos\alpha_1.\cos\alpha_2....\cos\alpha_n = \sin\alpha_1.\sin\alpha_2.....\sin\alpha_n$

$\Rightarrow \quad (\cos\alpha_1.\cos\alpha_2....\cos\alpha_n)^2$

$= (\cos\alpha_1\sin\alpha_1)(\cos\alpha_2\sin\alpha_2)...(\cos\alpha_n\sin\alpha_n)$

$= \frac{1}{2^n}(2\cos\alpha_1\sin\alpha_1)(2\cos\alpha_2\sin\alpha_2)...(2\cos\alpha_n\sin\alpha_n)$

$\leq \frac{1}{2^n}$

$\Rightarrow \quad (\cos\alpha_1.\cos\alpha_2....\cos\alpha_n) \leq \frac{1}{2^{n/2}}$

30. No questions asked in 2002.

31. Ans. (c)

Given $\alpha = \beta + \gamma$

$\Rightarrow \quad \tan\alpha = \tan(\beta + \gamma)$

$\Rightarrow \quad \tan\alpha = \frac{\tan\beta + \tan\gamma}{1 - \tan\beta\ \tan\gamma}$

$\Rightarrow \quad \tan\alpha - \tan\alpha.\tan\beta.\tan\gamma = \tan\beta + \tan\gamma$

$\Rightarrow \quad \tan\alpha - \tan\gamma = \tan\beta + \tan\gamma$

$\Rightarrow \quad \tan\alpha = \tan\beta + 2\tan\gamma$

32. Ans. (a)

As we know that $A.M \geq G.M$

$$\Rightarrow \quad \frac{\left(\sqrt{x^2+x} + \frac{\tan^2\alpha}{\sqrt{x^2+x}}\right)}{2} \geq \sqrt{\sqrt{x^2+x}.\frac{\tan^2\alpha}{\sqrt{x^2+x}}}$$

$$\Rightarrow \quad \frac{\left(\sqrt{x^2+x} + \frac{\tan^2\alpha}{\sqrt{x^2+x}}\right)}{2} \geq \tan\alpha$$

$$\Rightarrow \quad \left(\sqrt{x^2+x} + \frac{\tan^2\alpha}{\sqrt{x^2+x}}\right) \geq 2\tan\alpha$$

Hence, the result.

33. Given $\sin\theta = \frac{1}{2}\ \theta = \frac{\pi}{6}$

Also, $\quad \cos\varphi = \frac{1}{3} < \frac{1}{2}$

$\Rightarrow \quad \frac{\pi}{2} < \varphi < \frac{\pi}{3}$

Thus, $\quad \frac{\pi}{6} + \frac{\pi}{3} < \theta + \varphi < \frac{\pi}{6} + \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} < \theta + \varphi < \frac{2\pi}{3}$

34. Let $y = \frac{1 - 2x + 5x^2}{3x^2 - 2x - 1}$

$\Rightarrow \quad 3x^2y - 2xy - y = 1 - 2x + 5x^2$

$\Rightarrow \quad (3y-5)x^2 + 2(1-y)x - (1+y) = 0$

As x is real, so $(y-1)^2 + (3y-5)(y+1) \geq 0$

$\Rightarrow \quad y^2 - 2y + 1 + (3y^2 - 5y + 3y - 5) \geq 0$

$\Rightarrow \quad 4y^2 - 4y - 4 \geq 0$

$\Rightarrow \quad y^2 - y - 1 \geq 0$

$\Rightarrow \quad (2y-1)^2 \geq (\sqrt{5})^2$

$\Rightarrow \quad (2y-1) \geq \sqrt{5}\ or\ (2y-1) \leq -\sqrt{5}$

$\Rightarrow \quad y \geq \left(\frac{\sqrt{5}+1}{2}\right)\ or\ y \leq \left(\frac{1-\sqrt{5}}{2}\right)$

$\Rightarrow \quad y \geq \left(\frac{\sqrt{5}+1}{2}\right)\ or\ y \leq -\left(\frac{\sqrt{5}-1}{2}\right)$

$\Rightarrow \quad 2\sin t \geq \left(\frac{\sqrt{5}+1}{2}\right)$ or $2\sin t \leq -\left(\frac{\sqrt{5}-1}{2}\right)$

$\Rightarrow \quad \sin t \geq \left(\frac{\sqrt{5}+1}{4}\right)$ or $\sin t \leq -\left(\frac{\sqrt{5}-1}{4}\right)$

$\Rightarrow \quad \sin t \geq \sin(54°)$ or $\sin t \leq \sin(-18°)$

$\Rightarrow \quad \sin t \geq \sin\left(\frac{3\pi}{10}\right)$ or $\sin t \leq \sin\left(-\frac{\pi}{10}\right)$

$\Rightarrow \quad \dfrac{3\pi}{10} \le t \le \dfrac{\pi}{2}$ or $-\dfrac{\pi}{2} \le t \le -\dfrac{\pi}{10}$

Therefore, $t \in \left[-\dfrac{\pi}{2}, -\dfrac{\pi}{10}\right] \cup \left[\dfrac{3\pi}{10}, \dfrac{\pi}{2}\right]$

35. Ans.(d)

Given $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = \dfrac{1}{e}$

$\Rightarrow \quad \cos(\alpha - \beta) = \cos(0)$

$\Rightarrow \quad \alpha = \beta$

Also, $\cos(\alpha + \beta) = \dfrac{1}{e}$

$\Rightarrow \quad \cos(2\alpha) = \dfrac{1}{e}$

Given $-\pi < \alpha < \pi$

$-2\pi < 2\alpha < 2\pi$

Thus, there are 4 values of the ordered paiss of (α, β) satisfies the relation $\cos(2\alpha) = \dfrac{1}{e}$

36. Ans. (b)

As $0 < \theta < \dfrac{\pi}{4}$

$\Rightarrow \quad 0 < \tan\theta < 1$ and $\cot\theta > 1$

We also know that, if $0 < x < 1$ and $0 < a < b$

$\Rightarrow$ then $x^b < x^a < \left(\dfrac{1}{x}\right)^a < \left(\dfrac{1}{x}\right)^b$

$\Rightarrow \quad (\tan\theta)^{\cot\theta} < (\tan\theta)^{\tan\theta} < (\cot\theta)^{\tan\theta} < (\cot\theta)^{\cot\theta}$

$\Rightarrow \quad t_2 < t_1 < t_3 < t_4$

Hence, the result.

37. Given $\dfrac{\sin^4 x}{2} + \dfrac{\cos^4 x}{3} = \dfrac{1}{5}$

$\Rightarrow \quad \dfrac{5}{2}\sin^4 x + \dfrac{5}{3}\cos^4 x = 1$

$\Rightarrow \quad \left(1 + \dfrac{3}{2}\right)\sin^4 x + \left(1 + \dfrac{2}{3}\right)\cos^4 x = 1$

$\Rightarrow \quad (\sin^4 x + \cos^4 x) + \left(\dfrac{3}{2}\sin^4 x + \dfrac{2}{3}\cos^4 x\right) = 1$

$\Rightarrow \quad 1 - 2\sin^2 x\cos^2 x + \left(\dfrac{3}{2}\sin^4 x + \dfrac{2}{3}\cos^4 x\right) = 1$

$\Rightarrow \quad \left(\dfrac{3}{2}\sin^4 x + \dfrac{2}{3}\cos^4 x - 2\sin^2 x\cos^2 x\right) = 0$

$\Rightarrow \quad \left(\sqrt{\dfrac{3}{2}}\sin^2 x - \sqrt{\dfrac{2}{3}}\cos^2 x\right)^2 = 0$

$\Rightarrow \quad \left(\sqrt{\dfrac{3}{2}}\sin^2 x - \sqrt{\dfrac{2}{3}}\cos^2 x\right) = 0$

$\Rightarrow \quad \sqrt{\dfrac{3}{2}}\sin^2 x = \sqrt{\dfrac{2}{3}}\cos^2 x$

$\Rightarrow \quad \dfrac{\sin^2 x}{2} = \dfrac{\cos^2 x}{3} = \dfrac{1}{5}$

$\Rightarrow$ Thus, $\tan^2 x = \dfrac{2}{3}$

Now, $\dfrac{\sin^8 x}{8} + \dfrac{\cos^8 x}{27}$

$= \dfrac{(\sin^2 x)^4}{8} + \dfrac{(\cos^2 x)^4}{27}$

$= \dfrac{(2/5)^4}{8} + \dfrac{(3/5)^4}{27}$

$= \dfrac{2}{5^4} + \dfrac{3}{5^4}$

$= \dfrac{2+3}{5^4} = \dfrac{1}{5^3} = \dfrac{1}{125}$

38. Let $f(\theta) = \sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta$

$= 1 + 3\sin\theta\cos\theta + 4\cos^2\theta$

$= 1 + \dfrac{3}{2}\sin 2\theta + 2(1 + \cos 2\theta)$

$= 3 + \dfrac{3}{2}\sin 2\theta + 2\cos 2\theta$

Max value $= \sqrt{\dfrac{9}{4} + 4} + 3 = \dfrac{5}{2} + 3 = \dfrac{11}{2}$

Min value $= -\sqrt{\dfrac{9}{4} + 4} + 3 = -\dfrac{5}{2} + 3 = \dfrac{1}{2}$

Therefore, the min value of

$\dfrac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is $\dfrac{2}{11}$

39. Now, $P: \sin\theta - \cos\theta = \sqrt{2}\cos\theta$

$\Rightarrow \quad \sin\theta = (\sqrt{2} + 1)\cos\theta$

$\Rightarrow \quad \cos\theta = \dfrac{\sin\theta}{(\sqrt{2} + 1)}$

$$\Rightarrow \quad \cos\theta = \frac{\sin\theta}{(\sqrt{2}+1)} \times \frac{(\sqrt{2}-1)}{(\sqrt{2}-1)}$$

$$\Rightarrow \quad \cos\theta = (\sqrt{2}-1)\sin\theta$$

$$\Rightarrow \quad \cos\theta + \sin\theta = \sqrt{2}\sin\theta$$

$$Q : \cos\theta + \sin\theta = \sqrt{2}\sin\theta$$

Thus, $P = Q$

40. Given $\dfrac{1}{\sin\left(\dfrac{\pi}{n}\right)} = \dfrac{1}{\sin\left(\dfrac{2\pi}{n}\right)} + \dfrac{1}{\sin\left(\dfrac{3\pi}{n}\right)}$

Let $\dfrac{\pi}{n} = \theta$

Then $\dfrac{1}{\sin\theta} = \dfrac{1}{\sin 2\theta} + \dfrac{1}{\sin 3\theta}$

$$\Rightarrow \quad \frac{1}{\sin\theta} - \frac{1}{\sin 3\theta} = \frac{1}{\sin 2\theta}$$

$$\Rightarrow \quad \frac{\sin 3\theta - \sin\theta}{\sin\theta \sin 3\theta} = \frac{1}{\sin 2\theta}$$

$$\Rightarrow \quad \frac{2\cos 2\theta \sin\theta}{\sin\theta \sin 3\theta} = \frac{1}{\sin 2\theta}$$

$$\Rightarrow \quad \frac{2\cos 2\theta}{\sin 3\theta} = \frac{1}{\sin 2\theta}$$

$$\Rightarrow \quad \sin 4\theta = \sin 3\theta$$

$$\Rightarrow \quad \sin 4\theta = \sin(\pi - 3\theta)$$

$$\Rightarrow \quad 4\theta = \pi - 3\theta$$

$$\Rightarrow \quad 7\theta = \pi$$

$$\Rightarrow \quad 7.\frac{\pi}{n} = \pi$$

$$\Rightarrow \quad n = 7$$

42. Let $\cos 4\theta = \dfrac{1}{3}$

$$\Rightarrow \quad 2\cos^2(2\theta) - 1 = \frac{1}{3}$$

$$\Rightarrow \quad 2\cos^2(2\theta) = 1 + \frac{1}{3} = \frac{4}{3}$$

$$\Rightarrow \quad \cos^2(2\theta) = \frac{2}{3}$$

$$\Rightarrow \quad \cos(2\theta) = \pm\sqrt{\frac{2}{3}}$$

$$\Rightarrow \quad 2\cos^2\theta - 1 = \pm\sqrt{\frac{2}{3}}$$

$$\Rightarrow \quad 2\cos^2\theta = 1 \pm \sqrt{\frac{2}{3}}$$

Now, $f(\cos 4\theta) = \dfrac{2}{2 - \sec^2\theta} = \dfrac{2\cos^2\theta}{2\cos^2\theta - 1}$

$$\Rightarrow \quad f\left(\frac{1}{3}\right) = \frac{2\left(1 \pm \sqrt{\frac{2}{3}}\right)}{2\left(1 \pm \sqrt{\frac{2}{3}}\right) - 1}$$

$$= 1 \pm \sqrt{\frac{3}{2}}.$$

CHAPTER 2

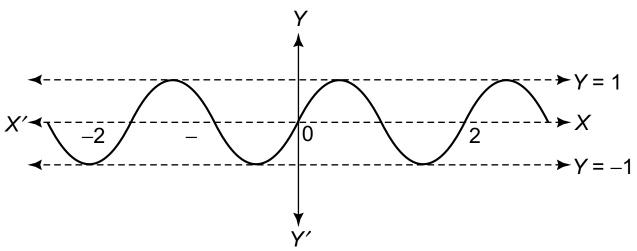
Graphs of Trigonometric Functions

2.1 INTRODUCTION

There are six trigonometric ratios. In this section, we shall describe each trigonometric function and their characteristics and graph.

1. Sine function: *A* function $f: R \to R$ is defined as $f(x) = \sin x$

   Graph of $f(x) = \sin x$:



Characteristics of sine function:

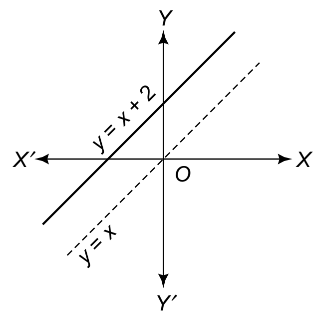
1. $D_f = R$
2. $R_f = [-1, 1]$
3. It is an odd function.
4. It is a periodic function
5. It is non-monotonic function
6. If $\sin x = 1 \Rightarrow x = (4n + 1)\dfrac{\pi}{4}, n \in I$
7. $\sin x = -1 \Rightarrow x = (4n - 1)\dfrac{\pi}{4}, n \in I$
8. $\sin x = 0 \Rightarrow x = n\pi, n \in I$
9. If $\sin x > 0 \Rightarrow x \in (2n\pi, (2n + 1)\pi), n \in I$
10. If $\sin x < 0 \Rightarrow x \in ((2n - 1)\pi, 2n\pi), n \in I.$
11. If $x > y \Rightarrow \sin x > \sin y,\ \forall\ x, y \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

2. RULE TO DRAW THE GRAPHS OF DIFFERENT TYPES OF FUNCTIONS

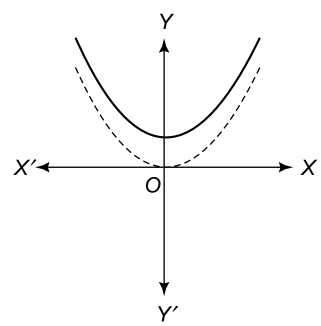
Rule I: $y = f(x)$ transforms to $y = f(x) + a$

Rule: Lift the graph of $y = f(x)$, a-units upwards.

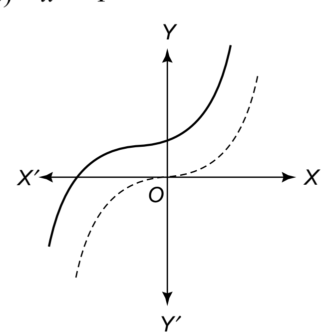
Ex-1. Let $f(x) = x + 2$



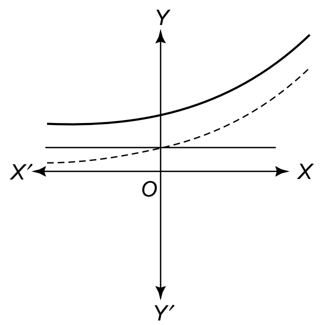
Ex-2 Let $f(x) = x^2 + 1$



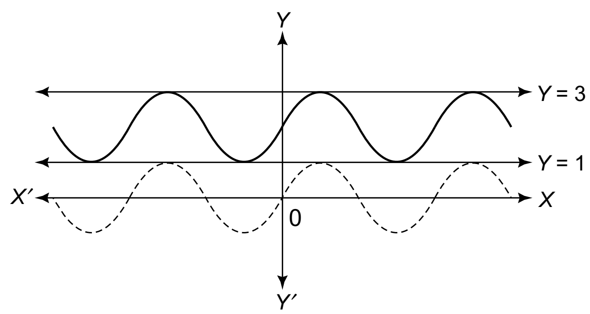
Ex-3. Let $f(x) = x^3 + 1$



Ex-4. Let $f(x) = e^x + 1$

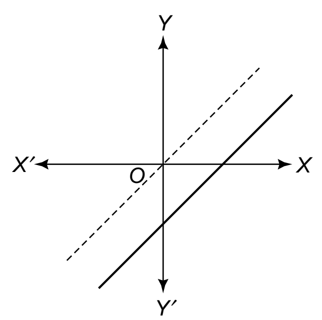


Ex-5. Let $f(x) = \sin x + 2$

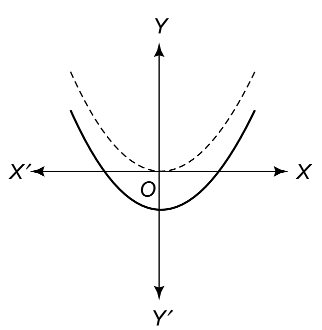


Rule II: $y = f(x)$ transforms to $y = f(x) - b$
Rule: Down the graph of $y = f(x)$, b-units downwards.

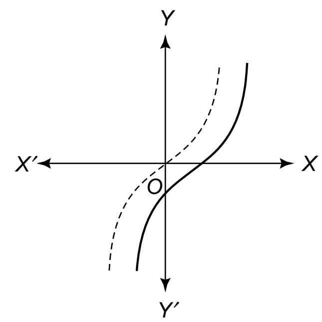
Ex-1. Let $f(x) = x - 2$



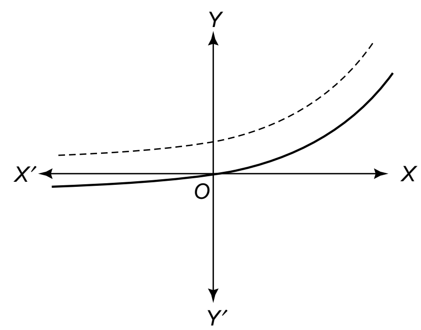
Ex-2 Let $f(x) = x^2 - 1$



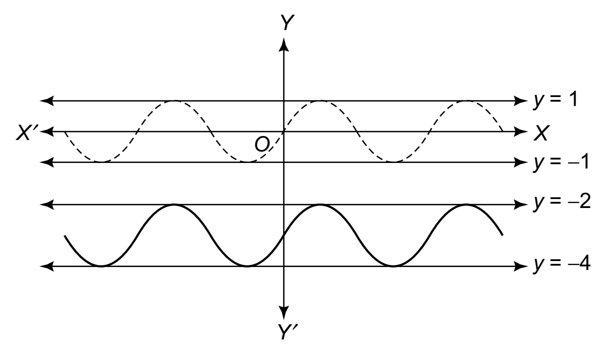
Ex-3. Let $f(x) = x^3 - 1$



Ex-4. Let $f(x) = 2^x - 1$

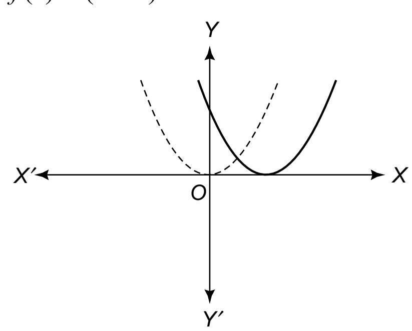


Ex-5. Let $y = \sin x - 3$

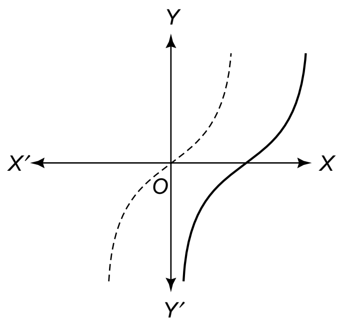


Rule III: $y = f(x)$ transforms to $y = f(x - a)$
Rule: Shift the graph of $y = f(x)$, a-units rightwards.

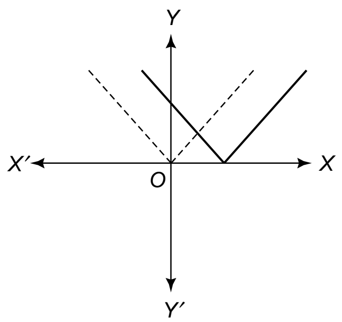
Ex-1. Let $f(x) = (x - 1)^2$



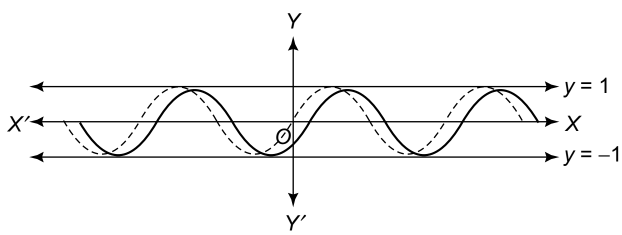
Ex-2. Let $f(x) = (x - 2)^3$



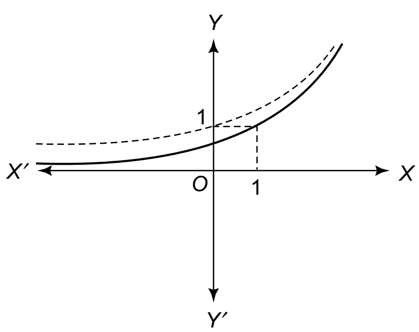
Ex-3. Let $f(x) = |x - 2|$



Ex-4. Let $f(x) = \sin\left(x - \frac{\pi}{4}\right)$

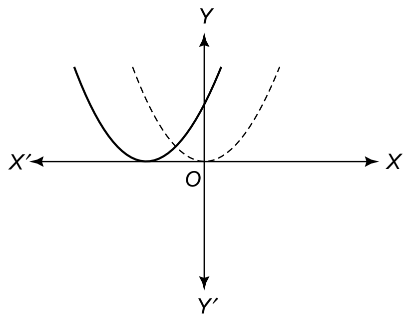


Ex-5. Let $f(x) = e^{x-1}$

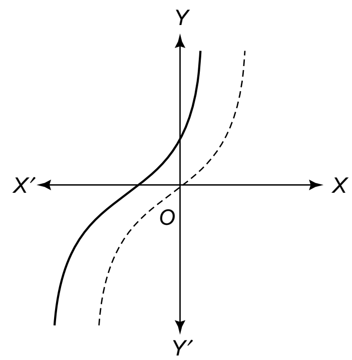


Rule IV: $y = f(x)$ transforms to $y = f(x + b)$
Rule: Shift the graph of $y = f(x)$, b units leftwards.

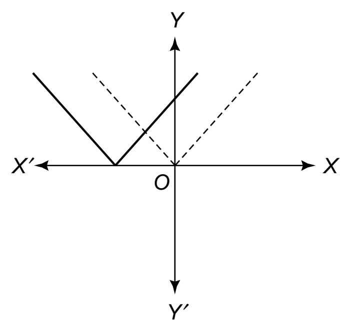
Ex-1. Let $f(x) = (x + 2)^2$



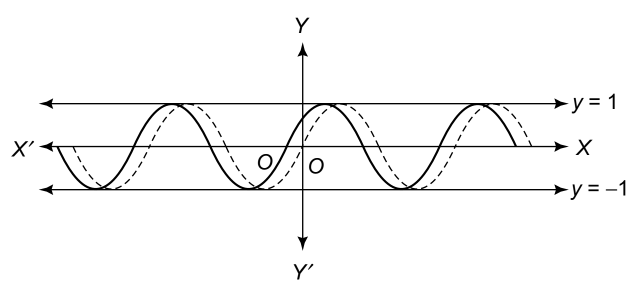
Ex-2. Let $f(x) = (x + 1)^3$



Ex-3. Let $f(x) = |x + 2|$



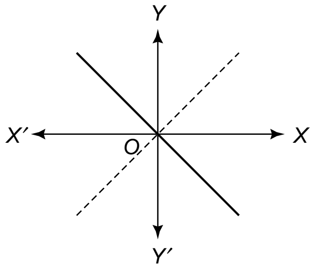
Ex-4. Let $f(x) = \sin\left(x - \frac{\pi}{4}\right)$



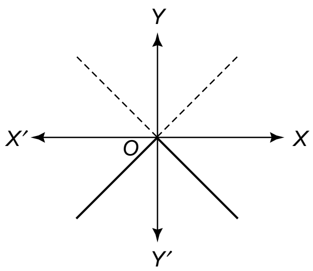
Ex-5. Let $f(x) = e^{x+1}$

Rule V: $y = f(x)$ transforms to $y = -f(x)$
Rule: Take the image of the graph of $y = f(x)$ with respect to x-axis.

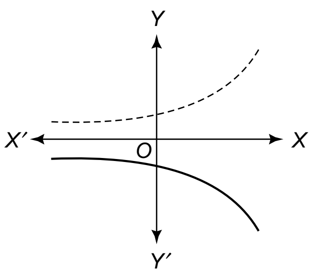
Ex-1. Let $f(x) = -x$



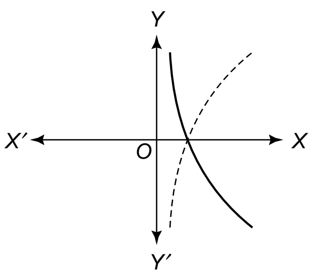
Ex-2. Let $f(x) = -|x|$



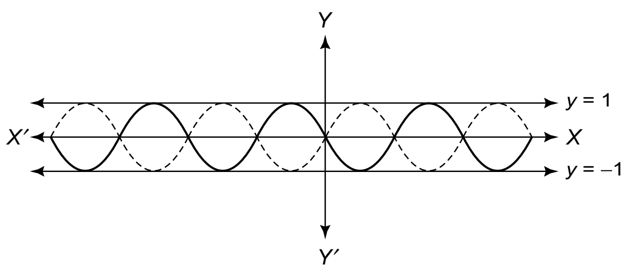
Ex-3. Let $f(x) = -e^x$



Ex-4. Let $f(x) = -\log_e x$

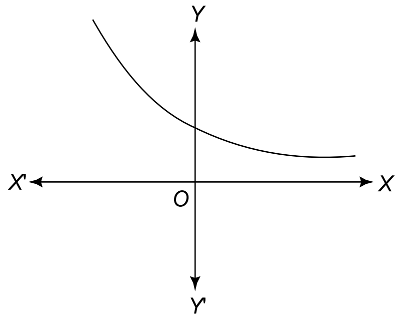


Ex-5. Let $f(x) = -\sin x$

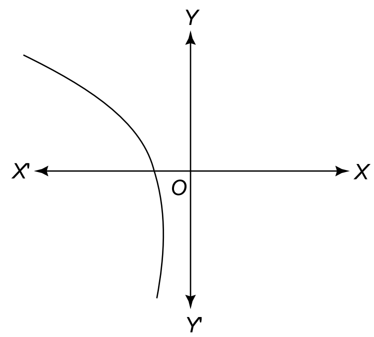


Rule VI: $y = f(x)$ transforms to $y = f(-x)$
Rule: Take the image of the graph of $y = f(x)$ with respect to y-axis.

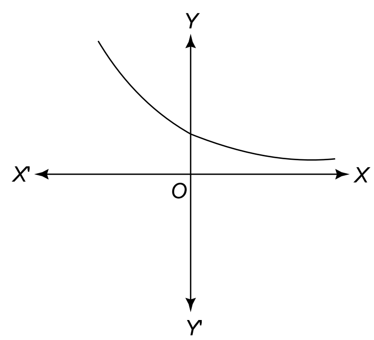
Ex-1. $f(x) = e^{-x}$



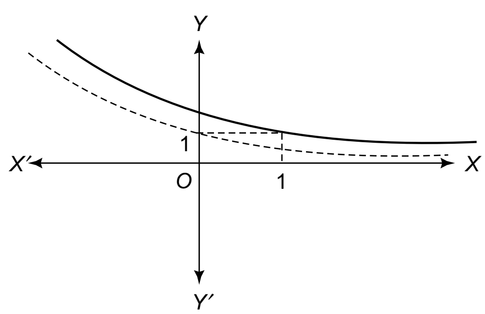
Ex-2. $f(x) = \log_e(-x)$



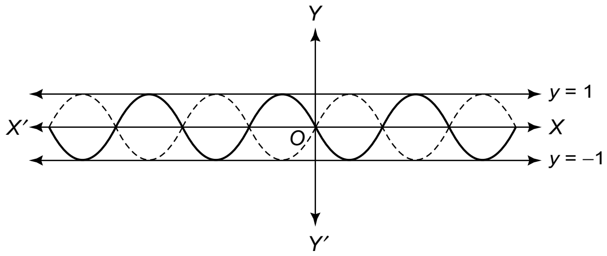
Ex-3. $f(x) = 2^{-x}$



Ex-4. $f(x) = e^{1-x}$

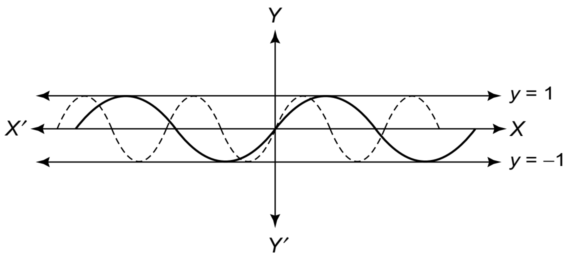


Ex-5. $f(x) = \sin(-x) = -\sin(x)$

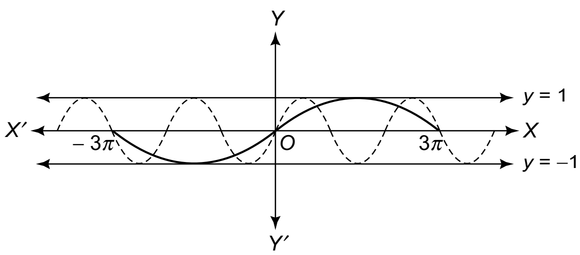


Rule VII: when $y = f(x)$ transforms to $y = f(ax)$, where $a > 1$
Rule: Shrink the graph of $y = f(x)$, a-times along the x-axis

Ex-1. Let $f(x) = \sin 2x$



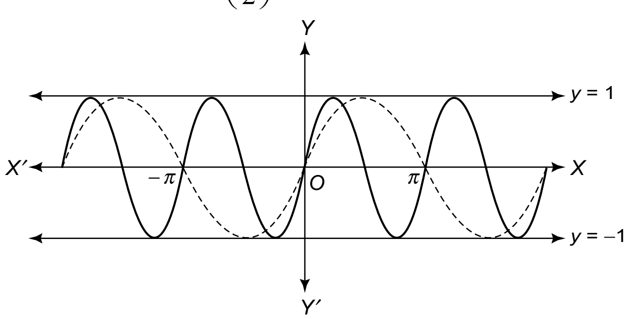
Ex-2. Let $f(x) = \sin 3x$



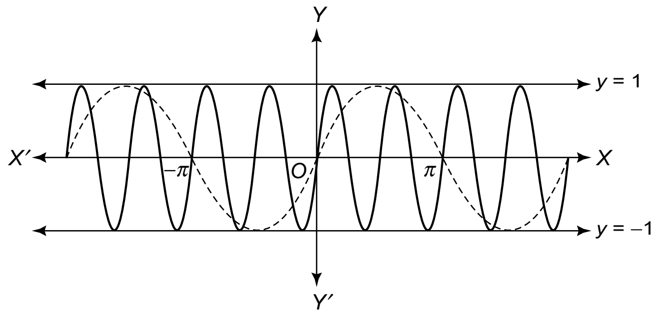
Rule VIII: when $y = f(x)$ transforms to $y = f(ax)$, where $0 < a < 1$

Rule: Stretch the graph of $y = f(x)$, $\frac{1}{a}$– times along the x-axis.

Ex-1. Let $f(x) = \sin\left(\frac{x}{2}\right)$

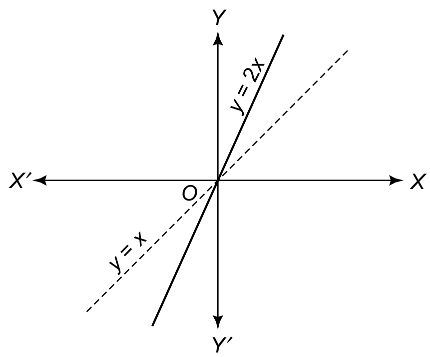


Ex-2. $f(x) = 2x^2$

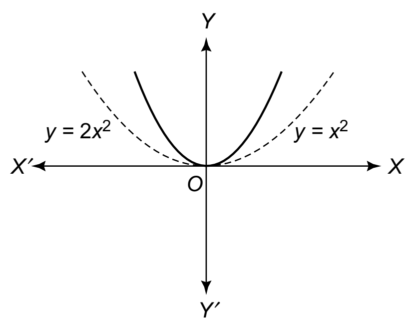


Rule IX: when $y = f(x)$ transforms to $y = af(x)$, where $a > 1$.
Rule: Stretch the graph of $y = f(x)$, a –times along the y-axis.

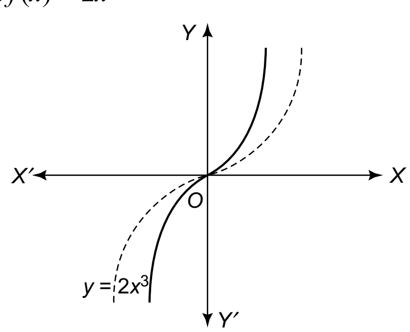
Ex-1. Let $f(x) = 2x$



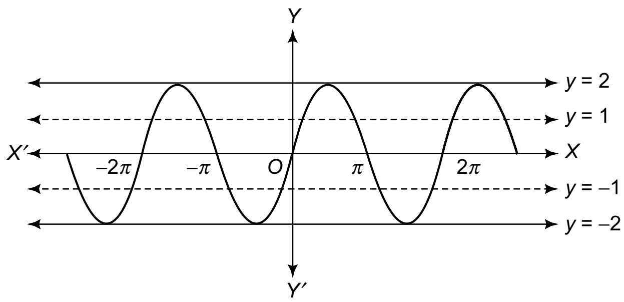
Ex-2. Let $f(x) = 2x^2$



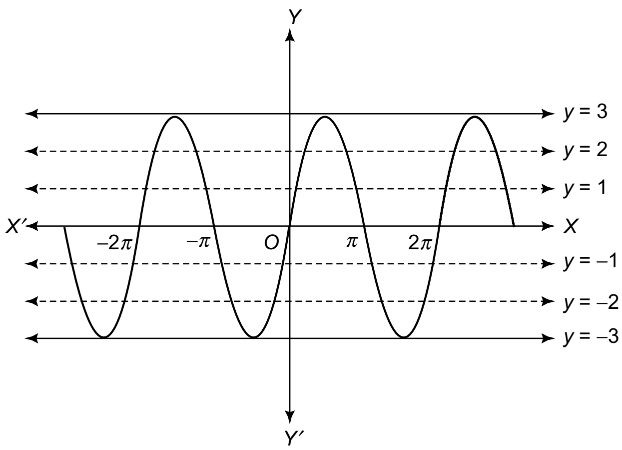
Ex-3. Let $f(x) = 2x^3$



Ex-4. Let $f(x) = 2\sin x$



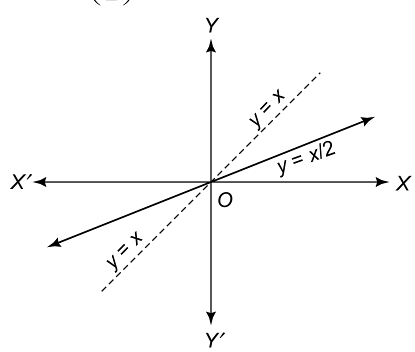
Ex-5. Let $f(x) = 3\sin x$



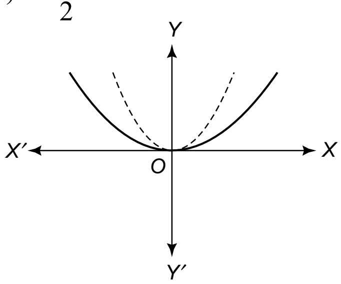
Rule X: when $y = f(x)$ transforms to $y = a\,f(x)$, where $0 < a < 1$.

Rule: Shrink the graph of $y = f(x)$, $\frac{1}{a}$ –times along the y-axis.

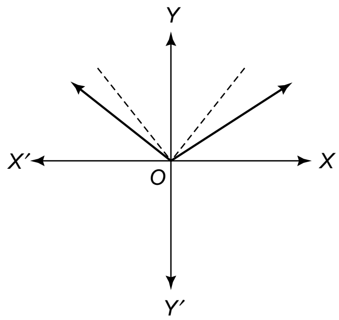
Ex-1. Let $f(x) = \left(\frac{x}{2}\right)$



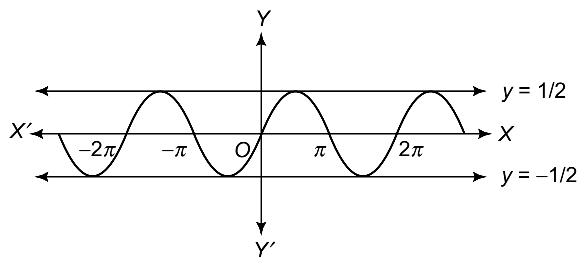
Ex-2. Let $f(x) = \frac{x^2}{2}$



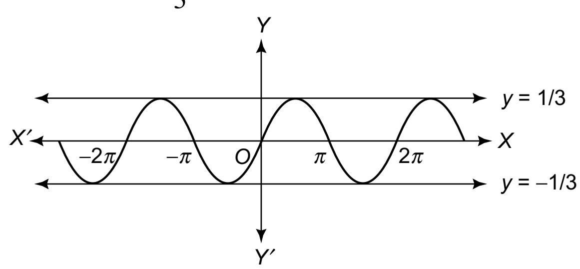
Ex-3. Let $f(x) = \frac{|x|}{2}$



Ex-4. Let $f(x) = \frac{1}{2}\sin x$

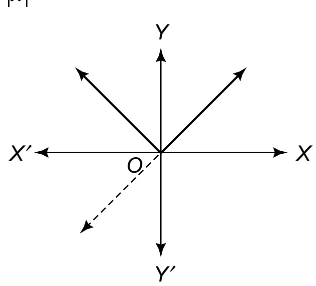


Ex-5. Let $f(x) = \frac{1}{3}\sin x$

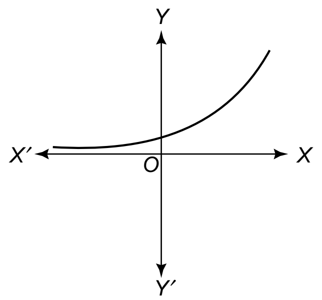


Rule XI: when $y = f(x)$ transforms to $y = |f(x)|$.
Rule : 1. First, we draw the graph of $y = f(x)$
: 2. Leave the part of $y = f(x)$ as it is which lies above x-axis
: 3. Take the image of the graph of $y = f(x)$ with respect to x-axis, which lies below x-axis.

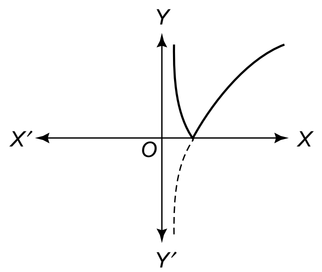
Ex-1. Let $y = |x|$



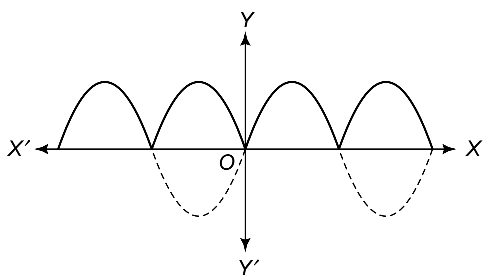
Ex-2. Let $y = |e^x|$



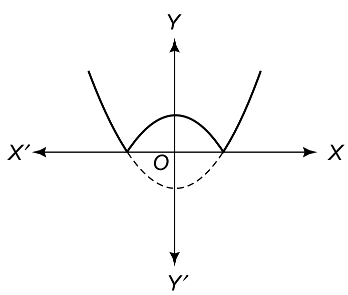
Ex-3. Let $y = |\log_e x|$



Ex-4. Let $y = |\sin x|$



Ex-5. Let $y = |x^2 - 1|$



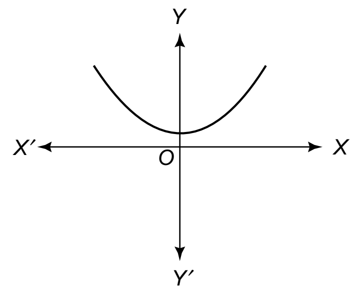
Rule XI: when $y = f(x)$ transforms to $y = f(|x|)$.

Rule : 1. First, we draw the graph of $y = f(x)$

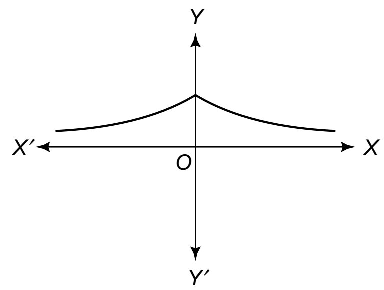
: 2. Remove the part of $y = f(x)$, which lies left of y-axis

: 3. Take the image of the part of $y = f(x)$ which lies right of y-axis with respect to y-axis.

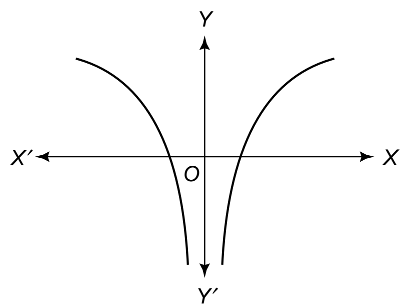
Ex-1. Let $y = e^{|x|}$



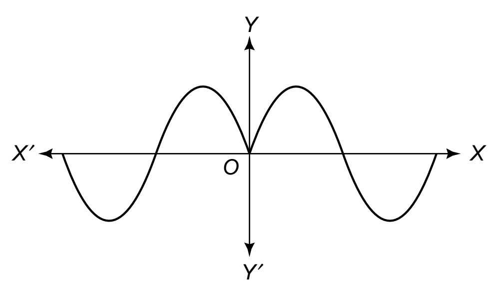
Ex-2. Let $y = e^{-|x|}$



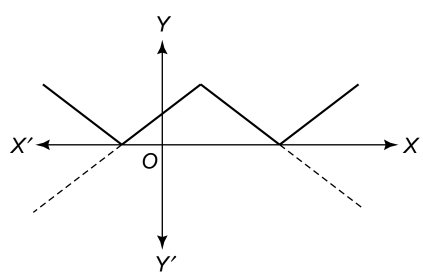
Ex-3. Let $y = \log_e|x|$



Ex-4. Let $y = \sin|x|$



Ex-5. Let $y = |2 - |x - 1||$

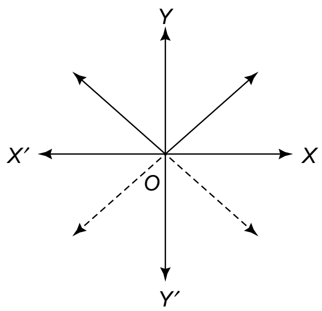


Rule XII: when $y = \max\{f(x), g(x)\}$

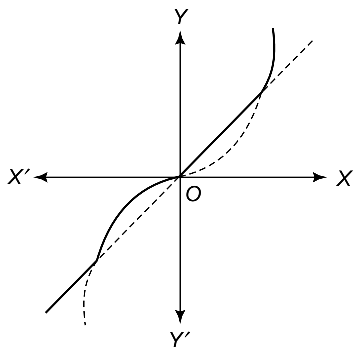
Rule: 1. First, we draw the graph of $y = f(x)$ and $y = g(x)$ independently on the same co-ordinate axes.

: 2. Take the upper part between the graphs of $y = f(x)$ and $y = g(x)$.

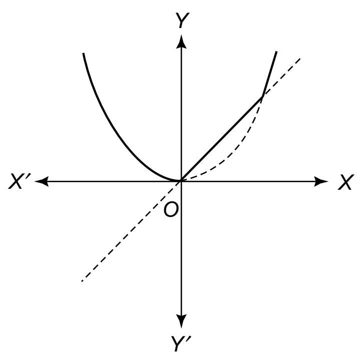
Ex-1. Let $y = \max\{-x, x\}$



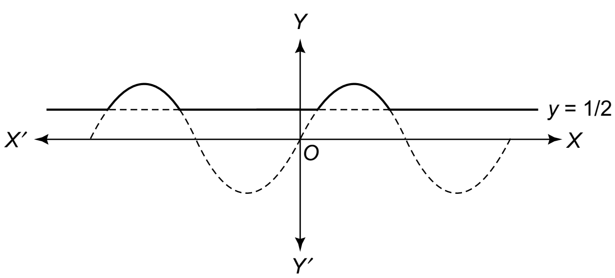
Ex-2. Let $y = \max\{x^3, x\}$



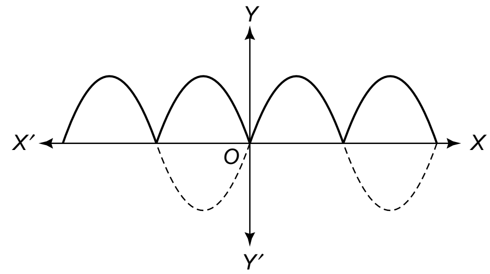
Ex-3. Let $y = \max\{x^2, x\}$



Ex-4. Let $y = \left\{\sin x, \frac{1}{2}\right\}, \forall x \in [-2\pi, 2\pi]$



Ex-5. Let $y = \max\{\sin x, -\sin x\}, \forall x \in [-2\pi, 2\pi]$

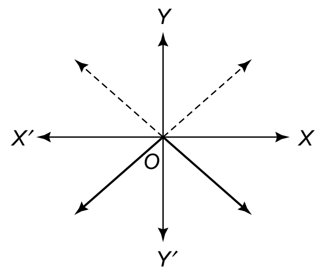


Rule XIII: when $y = \min\{f(x), g(x)\}$

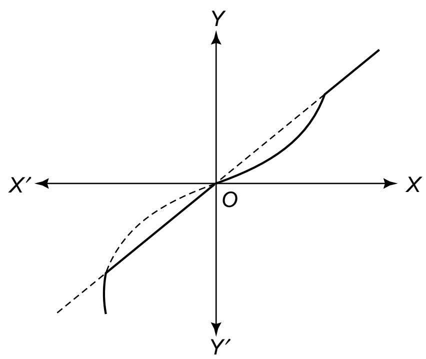
Rule: 1. First, we draw the graph of $y = f(x)$ and $y = g(x)$ independently on the same co-ordinate axes.

: 2. Take the lower part between the graphs of $y = f(x)$ and $y = g(x)$.

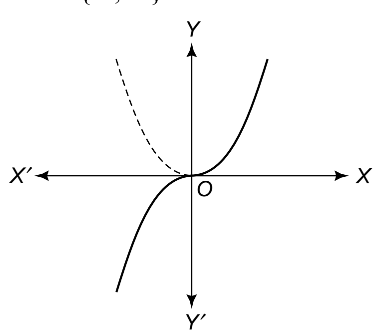
Ex-1. Let $y = \min\{-x, x\}$



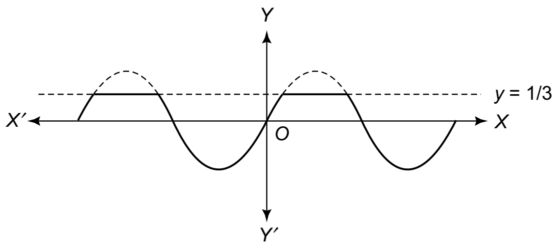
Ex-2. Let $y = \min\{x^3, x\}$



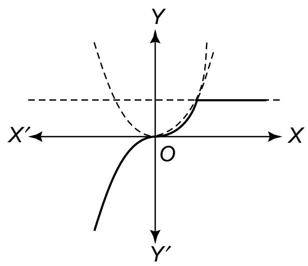
Ex-3. Let $y = \min\{x^3, x^2\}$



Ex-4. Let $y = \min\left\{\sin x, \frac{1}{3}\right\}, \forall x \in [-2\pi, 2\pi]$

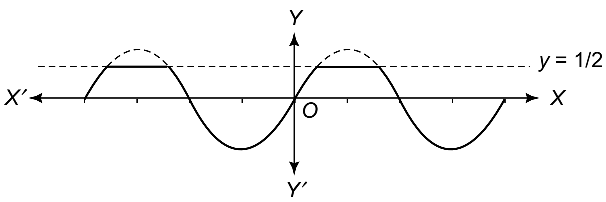


Ex-5. Let $y = \min\{x^3, 1, x^2\}$.

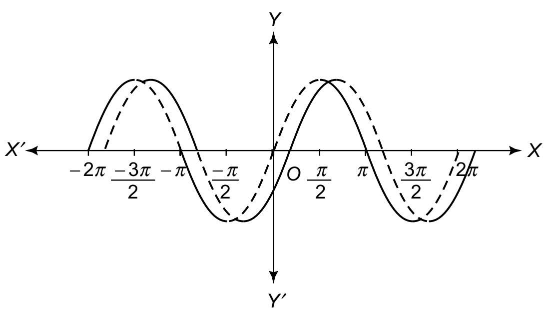


2.1.1 Some solved examples

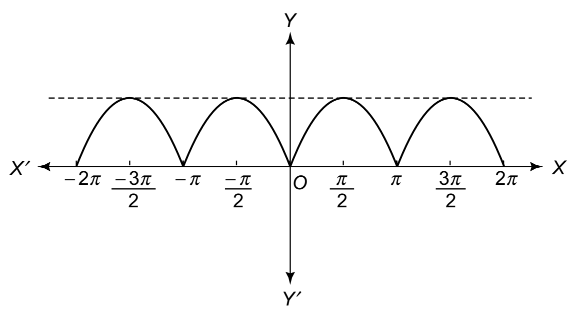
Ex. 1. Let $f(x) = \min\left\{\sin x, \frac{1}{2}\right\}, \forall x[-2\pi, 2\pi]$



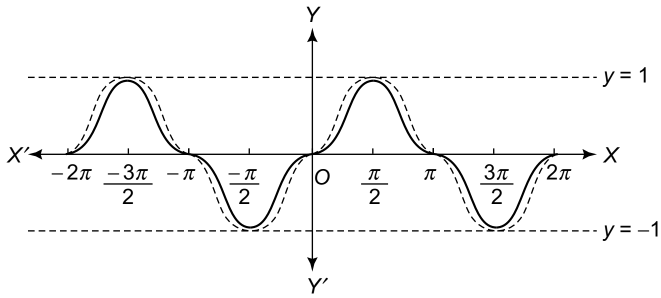
Ex. 2. Let $f(x) = \min\left\{\sin x, \sin\left(x - \frac{\pi}{4}\right)\right\}, \forall x[-2\pi, 2\pi]$



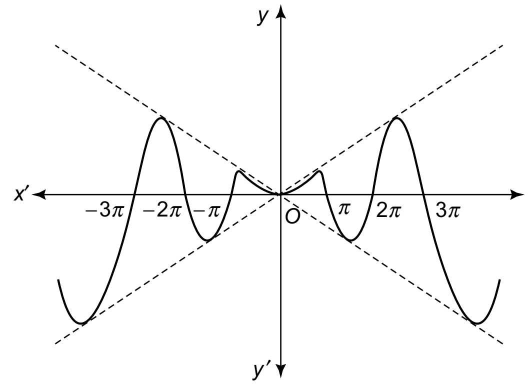
Ex. 3. Draw the graph of $f(x) = \sin^2 x$, $\forall x \in [-2\pi, 2\pi]$



Ex. 4. Draw the graph of $f(x) = \sin^3 x$, $\forall x \in [-2\pi, 2\pi]$



Ex. 5. Draw the graph of $f(x) = x\sin x$



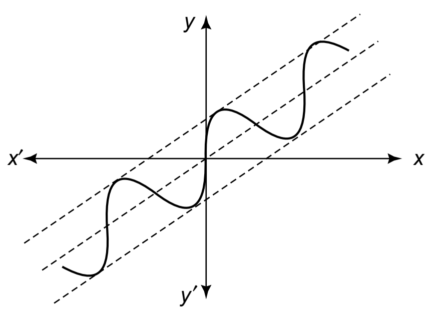
Ex. 6. Draw the graph of $f(x) = x + \sin x$

As we know that

$-1 \leq \sin x \leq 1$

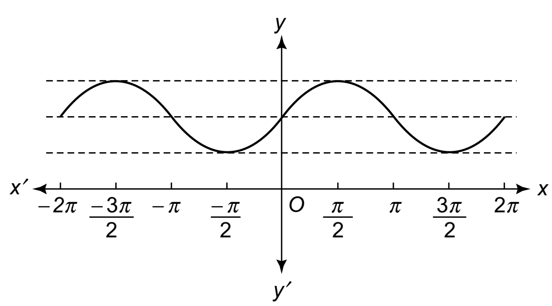
$\Rightarrow \quad x - 1 \leq x + \sin x \leq x + 1$

$\Rightarrow \quad x - 1 \leq f(x) \leq x + 1$

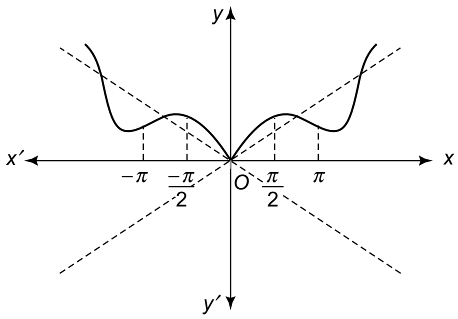


Ex. 7. Draw the graph of

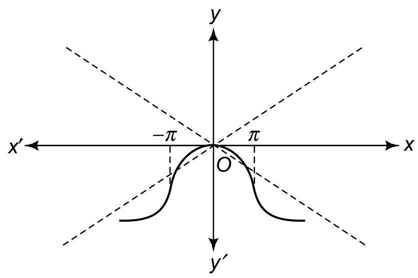
$f(x) = 2^{\sin x}, \; \forall x \in [-2\pi, 2\pi]$



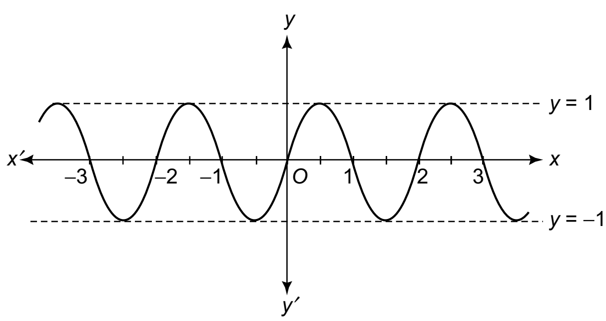
Ex. 8. Draw the graph of $f(x) = \sin|x| + |x|$



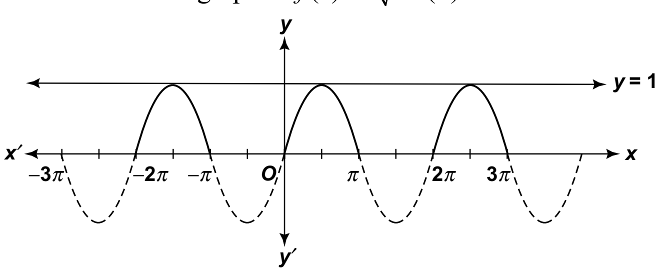
Ex. 9. Draw the graph of $f(x) = \sin|x| - |x|$



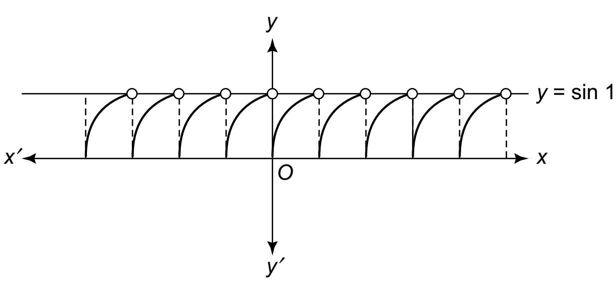
Ex. 10. Draw a right graph of $f(x) = \sin(\pi x)$



Ex. 11 Draw the graph of $f(x) = \sqrt{\sin(x)}$

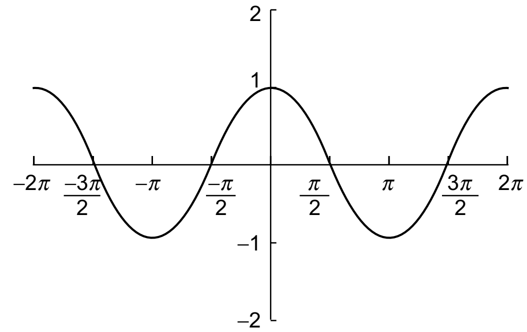


Ex. 12. Draw the graph of $f(x) = \sin(x - [x])$, [,] = G.I.F



3. Cosine function: A function $f: R \to R$ is defined as $f(x) = \cos x$.

Graph of $f(x) = \cos x$:

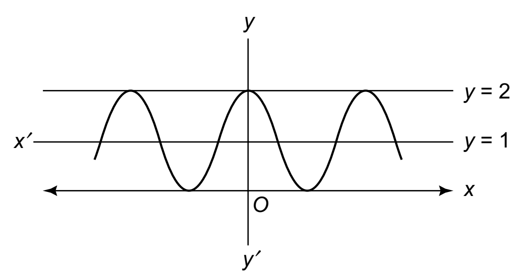


2.2 CHARACTERISTICS OF CO-SINE FUNCTION

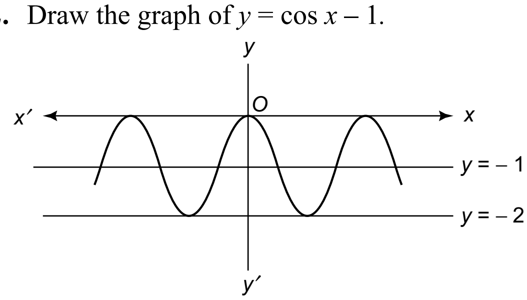
1. $D_f = R$
2. $R_f = [-1, 1]$
3. It is an even function.
4. It is a periodic function
5. It is non-monotonic function
6. If $\cos x = 1 \Rightarrow x = 2n\pi, n \in I$
7. $\cos x = -1 \Rightarrow x = (2n + 1)\pi, n \in I$
8. $\cos x = 0 \Rightarrow x = (2n + 1)\dfrac{\pi}{2}, n \in I$
9. If $\cos x > 0 \Rightarrow x = \left((2n-1)\dfrac{\pi}{2}, (2n+1)\dfrac{\pi}{2}\right), n \in I$
10. If $\cos x < 0$

    $\Rightarrow x \in \left((2n+1)\dfrac{\pi}{2}, (2n+3)\dfrac{\pi}{2}\right), n \in I$
11. If $x > y \Rightarrow \cos x < \cos y, \forall x, y \in (0, \pi)$

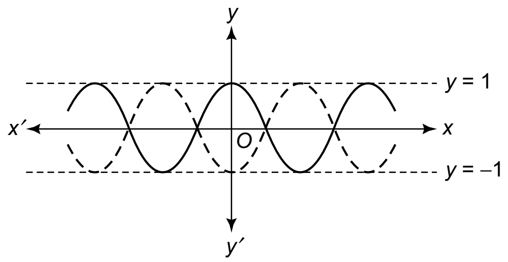
Ex. 1. Draw the graph of $y = \cos x + 1$.



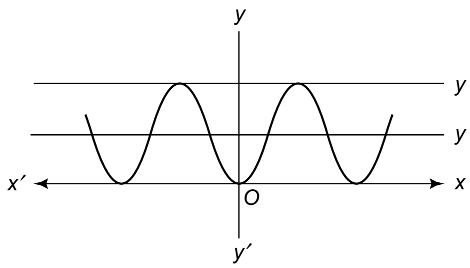
Ex. 2. Draw the graph of $y = \cos x - 1$.



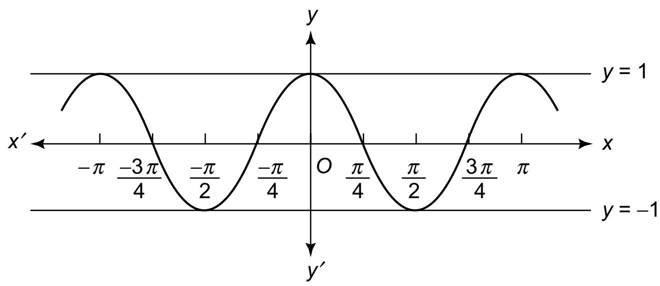
Ex. 3. Draw the graph of $y = \cos x - 1$.



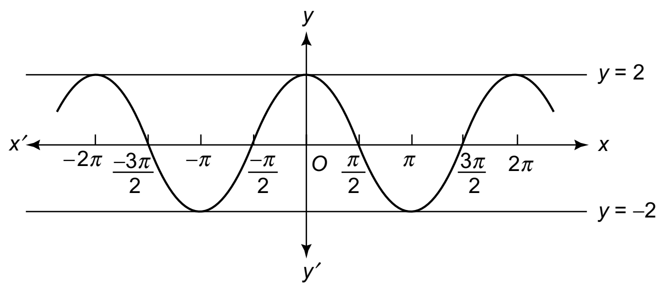
Ex. 4. Draw the graph of $y = 1 - \cos x$.



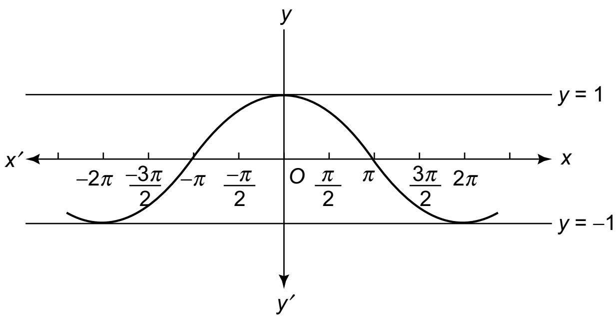
Ex. 5. Draw the graph of $y = \cos 2x$.



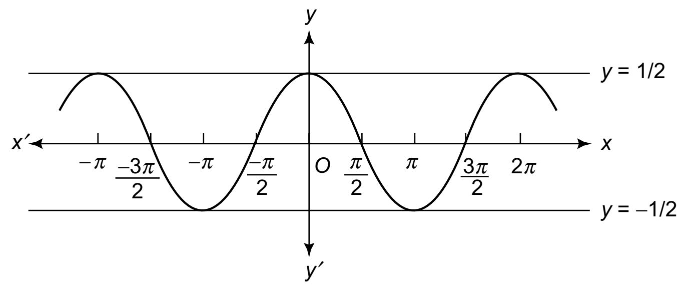
Ex. 6. Draw the graph of $y = 2\cos x$.



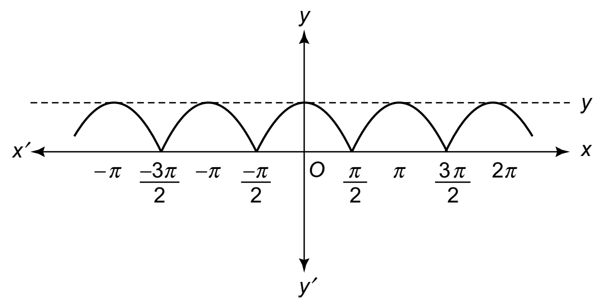
Ex. 7. Draw the graph of $y = \cos\left(\dfrac{x}{2}\right)$.



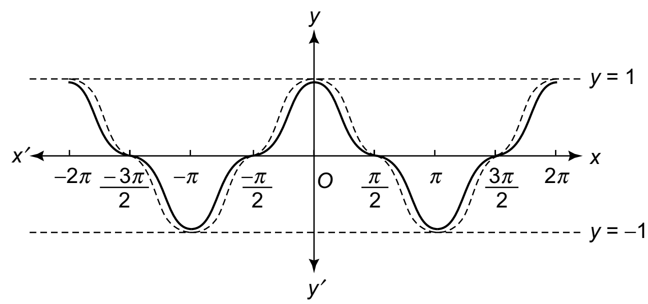
Ex. 8. Draw the graph of $y = \dfrac{1}{2}\cos x$.



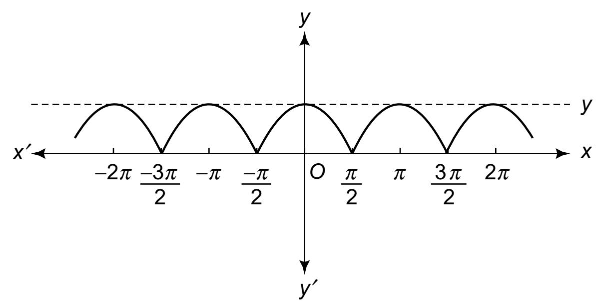
Ex. 9. Draw the graph of $y = \cos^2 x$.



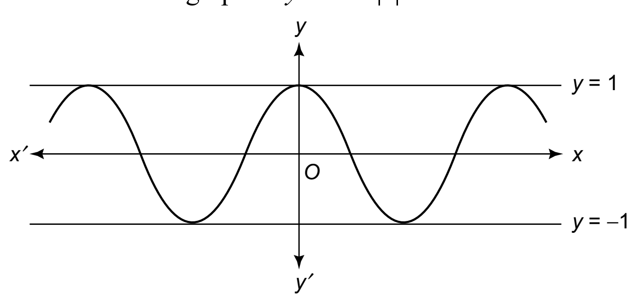
Ex. 10. Draw the graph of $y = \cos^3 x$.



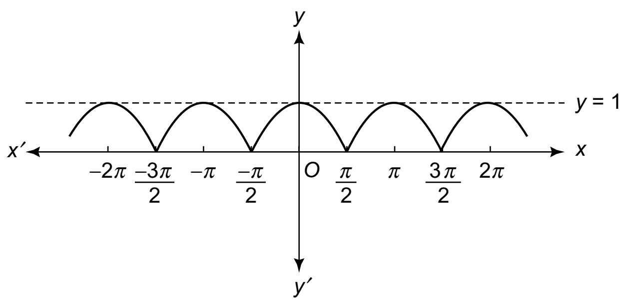
Ex. 11. Draw the graph of $y = |\cos x|$.



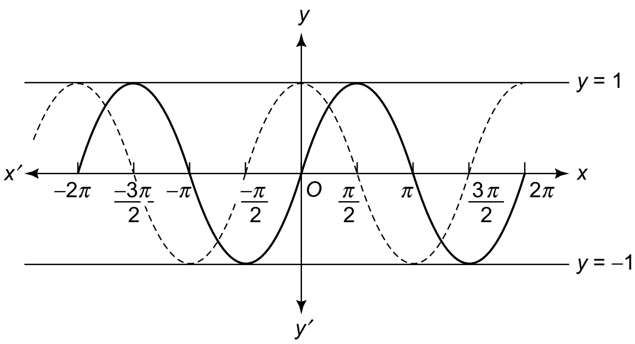
Ex. 12. Draw the graph of $y = \cos |x|$.



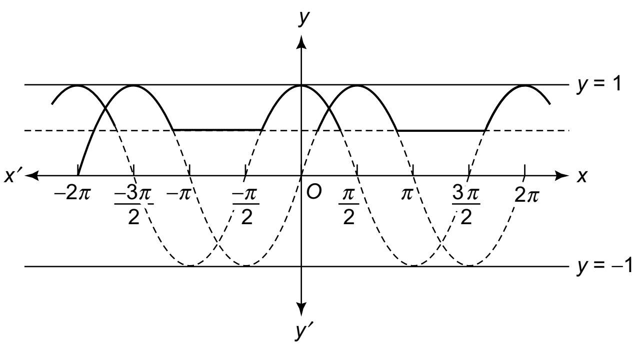
Ex. 13. Draw the graph of $y = |\cos |x||$



Ex. 14. Draw the graph of $y = \max\{\sin x, \cos x\}\ \forall\ x \in (-2\pi, 2\pi)$.

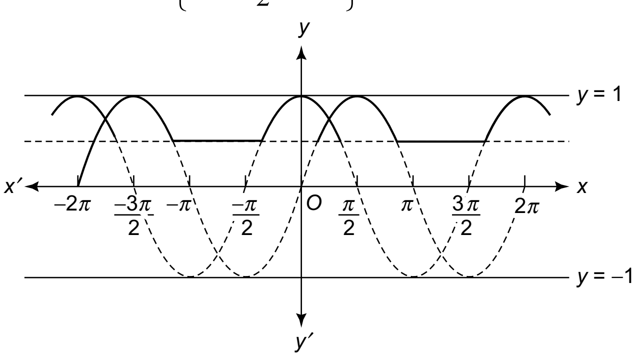


Ex. 15. Draw the graph of $y = \min\{\sin x, \cos x\}\ \forall\ x \in (-2\pi, 2\pi)$.



Ex. 16. Draw the graph of

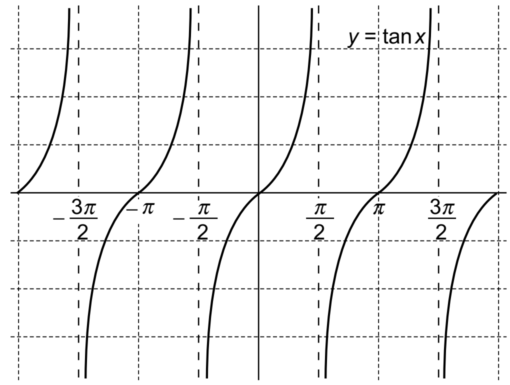
$$y = \max\left\{\sin x, \frac{1}{2}, \cos x\right\}\ \forall\ x \in (-2\pi, 2\pi).$$



4. Tangent Function

A function $f : R \to R$ is defined as $f(x) = \tan x$.

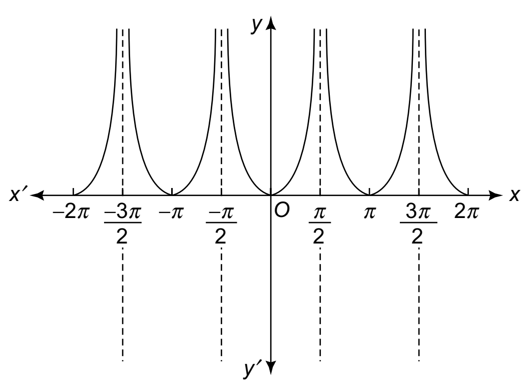
Graph of $\boldsymbol{f(x) = \tan x}$



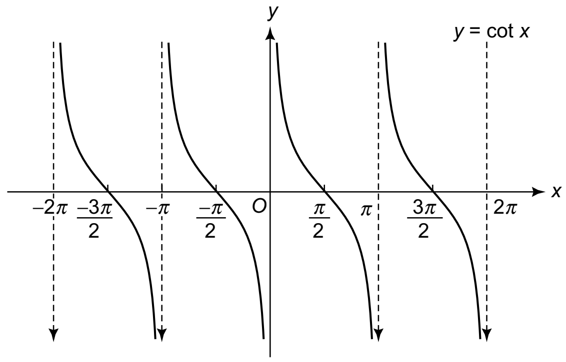
2.3 CHARACTERISTICS OF TANGENT FUNCTION

1. $D_f = R - (2n + 1)\dfrac{\pi}{2}$, $n \in I$
2. $R_f = R$
3. It is an odd function.
4. It is a periodic function
5. It is monotonic function
6. If $\tan x = 1 \Rightarrow x = (4n + 1)\dfrac{\pi}{4}$, $n \in I$
7. If $\tan x = -1 \Rightarrow x = (4n - 1)\dfrac{\pi}{4}$, $n \in I$
8. If $\tan x = 0 \Rightarrow x = n\pi$, $n \in I$
9. If $\tan x > 0$
   $\Rightarrow x \in \left(n\pi, (2n+1)\dfrac{\pi}{2}\right)$, $n \in I$
10. If $\tan x < 0$
    $\Rightarrow x \in \left((2n-1)\dfrac{\pi}{2}, n\pi\right)$, $n \in I$
11. If $x > y \Rightarrow \tan x > \tan y\ \forall\ x, y \in R - n\pi$, $n \in I$

Ex. 1. Draw the graph of $f(x) = |\tan x|$



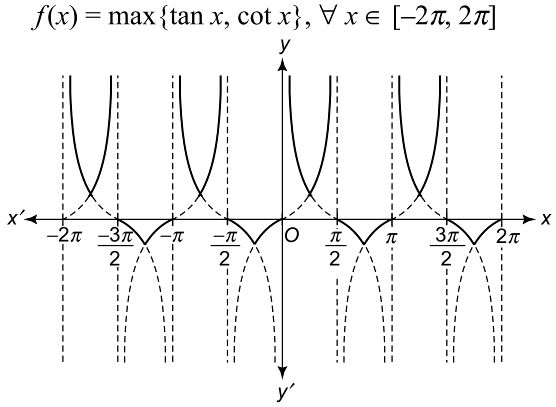
4. **Co-tangent function**: A function
$f: R \to R$ is defined as $f(x) = \cot x$
Graph of $f(x) = \cot x$



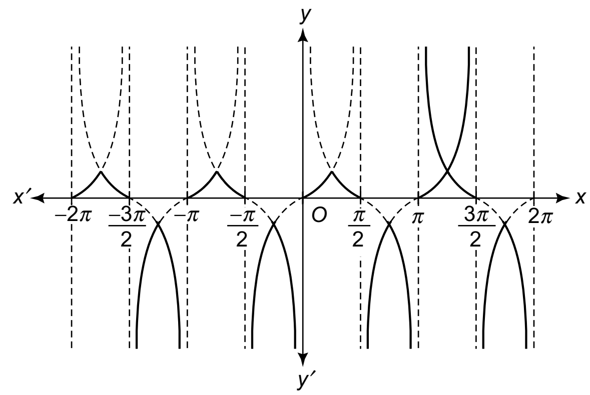
2.4 CHARACTERISTICS OF CO-TANGENT FUNCTION

1. $D_f = R - n\pi, n \in I$
2. $R_f = R$
3. It is an odd function.
4. It is a periodic function
5. It is monotonic function
6. If $\cot x = 1 \Rightarrow x = (4n + 1)\dfrac{\pi}{4}, n \in I$
7. If $\cot x = -1 \Rightarrow x = (4n - 1)\dfrac{\pi}{4}, n \in I$
8. If $\cot x = 0 \Rightarrow x = (2n + 1)\dfrac{\pi}{2}, n \in I$
9. If $\cot x > 0$
$\Rightarrow x \in \left(n\pi, (2n+1)\dfrac{\pi}{2}\right), n \in I$
10. If $\cot x < 0$
$\Rightarrow x \in \left((2n-1)\dfrac{\pi}{2}, n\pi\right), n \in I$
11 $x > y \Rightarrow \cot x < \cot y$,
$\forall\, x, y \in R - (2n + 1)\dfrac{\pi}{2}, n \in I$

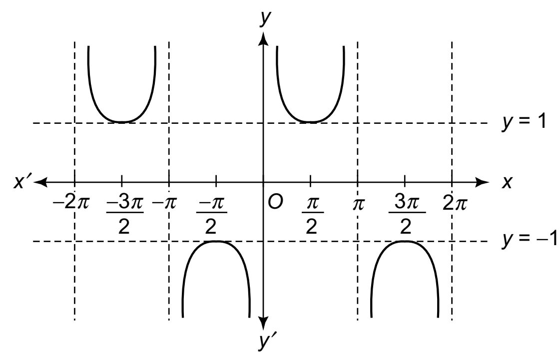
Ex. 1. Draw the graph of
$f(x) = \max\{\tan x, \cot x\}, \forall\, x \in [-2\pi, 2\pi]$



Ex. 2. Draw the graph of
$f(x) = \min\{\tan x, \cot x\}, \forall\, x \in [-2\pi, 2\pi]$



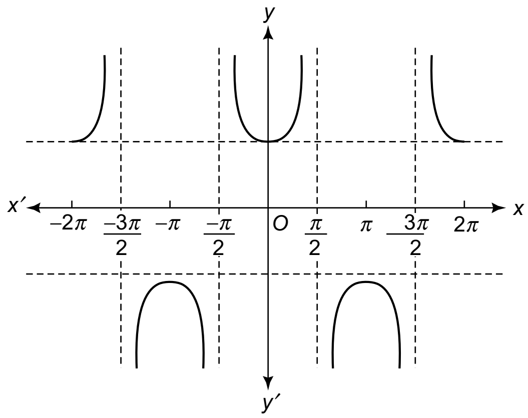
5. **Co-secant function**: A function
$f: R \to R$ is defined as $f(x) = \text{cosec } x$
Graph of $f(x) = \text{cosec } x$



2.5 CHARACTERISTICS OF CO-SECANT FUNCTION

1. $D_f = R - n\pi, n \in I$
2. $R_f = (-\infty, -1] \cup [1, \infty)$
3. It is an odd function
4. It is a periodic function
5. It is non-monotonic function
6. If $\text{cosec } x = 1 \Rightarrow x = (4n + 1)\dfrac{\pi}{4}, n \in I$
7. If $\text{cosec } x = -1 \Rightarrow x = (4n - 1)\dfrac{\pi}{4}, n \in I$
8. cosec x can never be zero
9. If $\text{cosec } x > 0 \Rightarrow x \in (2n\pi, (2n + 1)\pi), n \in I$
10. If $\text{cosec } x < 0 \Rightarrow x \in ((2n - 1)\pi, 2n\pi), n \in I.$

6. **Secant function:** A function
$f: R \to R$ is defined as $f(x) = \sec x$
Graph of $f(x) = \sec x$

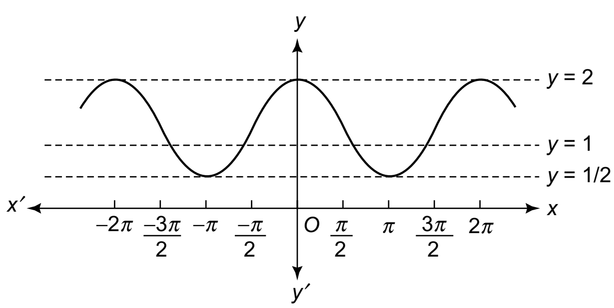


2.6 CHARACTERISTICS OF SECANT FUNCTION:

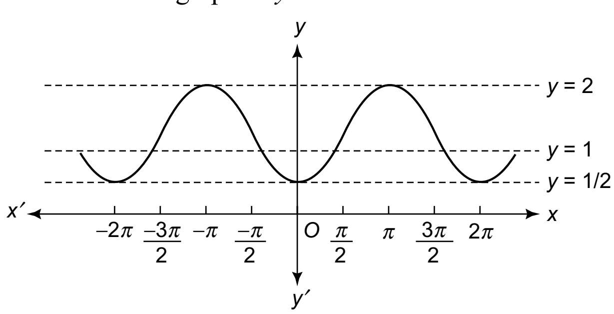
1. $D_f = R - (2n+1),\ n \in I$
2. $R_f = (-\infty, -1] \cup [1, \infty)$
3. It is an even function
4. It is a periodic function
5. It is non-monotonic function
6. If $\sec x = 1 \Rightarrow x = 2n\pi,\ n \in I$
7. If $\sec x = -1 \Rightarrow x = (2n+1)\pi,\ n \in I$
8. $\sec x$ can never be zero
9. If $\sec x > 0 \Rightarrow x \in \left((4n-1)\frac{\pi}{2}, (4n-1)\frac{\pi}{2}\right), n \in I$
10. If $\sec x < 0$

    $\Rightarrow x \in \left((4n+1)\frac{\pi}{2}, (4n+3)\frac{\pi}{2}\right),\ n \in I$
10. Some solved examples:

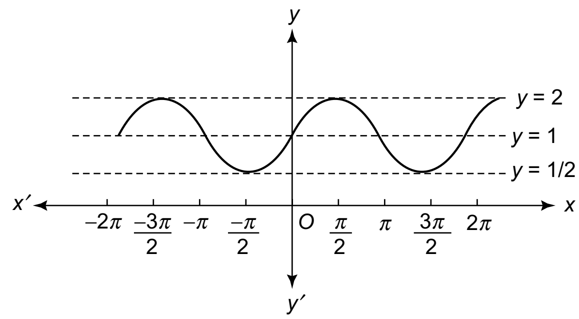
Ex. 1. Draw the graph of $y = 2^{\cos x}$



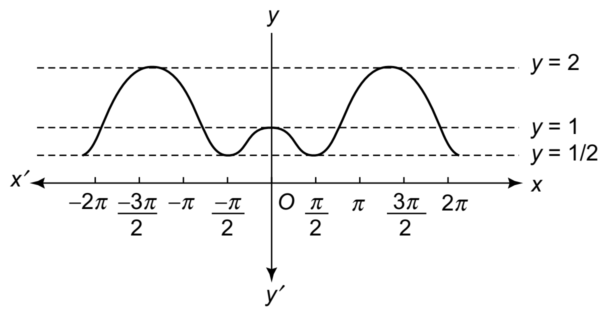
Ex. 2. Draw the graph of $y = 2^{-\cos x}$



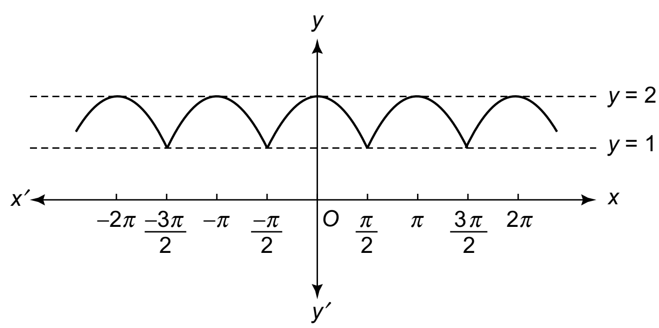
Ex. 3. Draw the graph of $y = 2^{\sin x}$



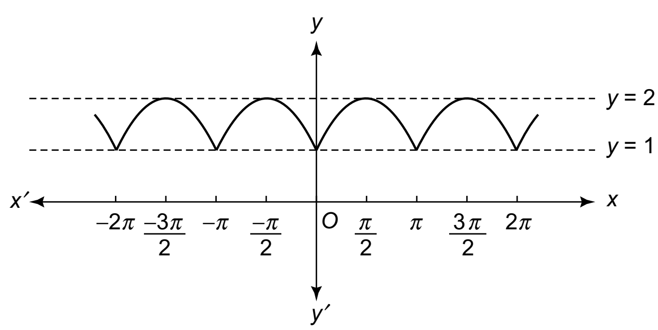
Ex. 4. Draw the graph of $y = 2^{-\sin x}$



Ex. 5. Draw the graph of $y = 2^{|\cos x|}$



Ex. 6. Draw the graph of $y = 2^{|\sin x|}$

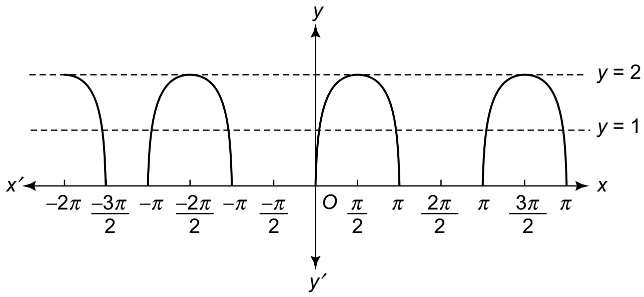


Ex. 7. Draw the graph of $y = \sin x + |\sin x|$

We have $y = \sin x + |\sin x|$

$$= \begin{cases} \sin x + \sin x : \sin x \geq 0 \\ \sin x - \sin x : \sin x < 0 \end{cases}$$

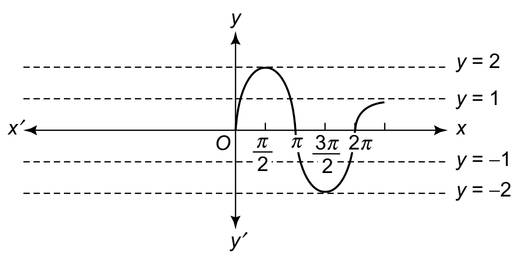
$$= \begin{cases} 2\sin x : x \in \left[2n\pi, (2n+1)\pi\right], n \in I \\ 0 \quad : x \in \left((2n-1)\pi, 2n\pi\right), n \in I \end{cases}$$



Ex. 8. Draw the graph of $y = \sin x + \sin |x|$

We have $y = \sin x + \sin |x|$

$$= \begin{cases} 2\sin x : x \geq 0 \\ 0 \quad : x < 0 \end{cases}$$

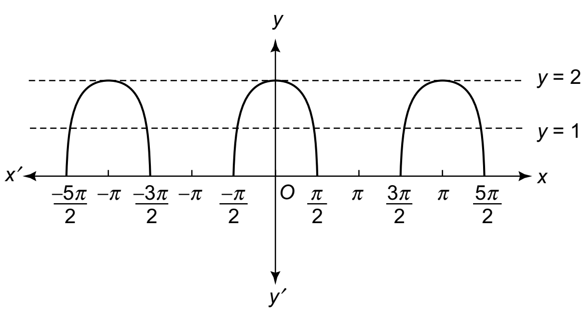


Ex. 9. Draw the graph of $y = \cos x + |\cos x|$

We have $y = \cos x + |\cos x|$

$$= \begin{cases} \cos x + \cos x : \cos x \geq 0 \\ \cos x - \cos x : \cos x < 0 \end{cases}$$

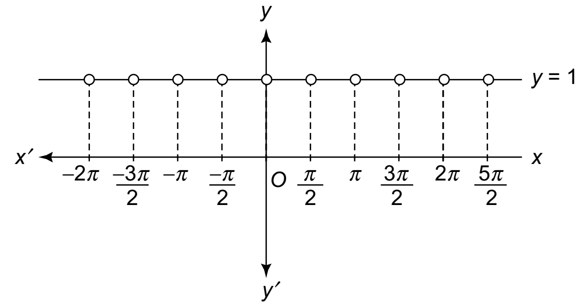
$$= \begin{cases} 2\cos x : x \in \left[(2n-1)\dfrac{\pi}{2}, (2n+1)\dfrac{\pi}{2}\right], n \in I \\ 0 \quad : x \in \left(\left((2n+1)\dfrac{\pi}{2}\right), (2n+3)\dfrac{\pi}{2}\right), n \in I \end{cases}$$



Ex. 10. Draw the graph of $y = \tan x \cdot \cot x$

We have $y = \tan x \cdot \cot x$

$$= 1, x \in \left\{(2n+1)\frac{\pi}{2} - n\pi\right\}, n \in I$$

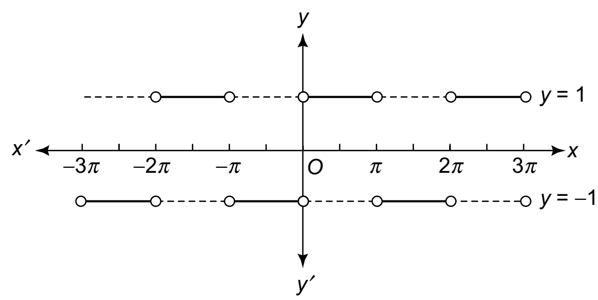


Ex. 11. Draw the graph of $y = \dfrac{|\sin x|}{\sin x}$

We have $y = \dfrac{|\sin x|}{\sin x}$

$$= \begin{cases} \dfrac{\sin x}{\sin x} : \sin x \geq 0 \\ -\dfrac{\sin x}{\sin x} : \sin x < 0 \end{cases}$$

$$= \begin{cases} 1 \quad : x \in \left[2n\pi, (2n+1)\pi\right], n \in I \\ -1 : x \in \left(2n\pi, (2n+1)\pi\right), n \in I \end{cases}$$

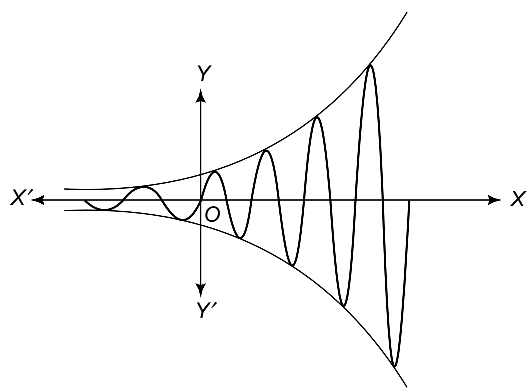


Ex. 12. Draw the graph of $y = 2^x \sin x$

We have $-1 \leq \sin x \leq 1$

$\Rightarrow -2^x \leq 2^x \sin x \leq 2^x$

$\Rightarrow -2^x \leq f(x) \leq 2^x$



Ex. 13. Draw the graph of $y = \dfrac{\sin x}{x}$

We have $-1 \le \sin x \le 1$

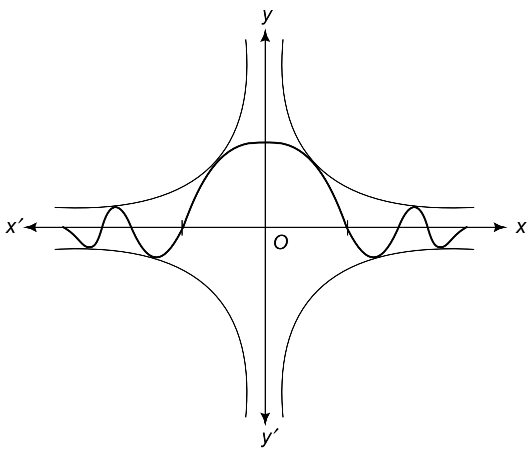
$$\Rightarrow \quad -\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

Also, $\lim_{x \to 0}\left(\dfrac{\sin x}{x}\right) = 1$

Hence, $y = \dfrac{\sin x}{x}$ lies between the curves

$y = \dfrac{1}{x}$ and $y = -\dfrac{1}{x}$

It is now periodic curve and cuts the x-axis at $x = n\pi$, $n \in I - \{0\}$ and does not cut the y-axis



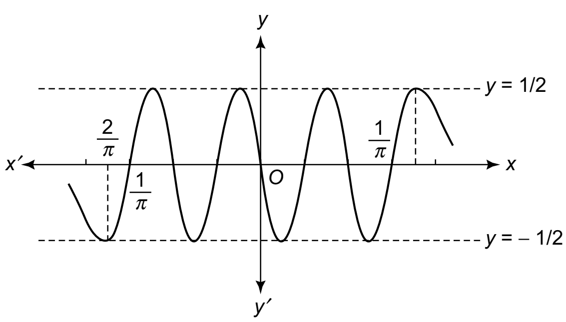
Ex. 14. Draw the graph of $y = \sin\left(\dfrac{1}{x}\right)$

We have $-1 \le \sin x \le 1$

$\Rightarrow -1 \le \sin\left(x - \dfrac{\pi}{6}\right) \le 1$

It cuts the x-axis at $x = \dfrac{1}{n\pi}$, $n \in -I\ \{0\}$

But it does not cut the y-axis.



Ex. 15. Draw the graph of $y = x \sin\left(\dfrac{1}{x}\right)$

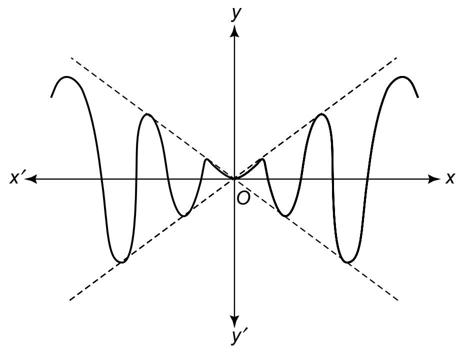
We have $-1 \le \sin\left(\dfrac{1}{x}\right) \le 1$

$$\Rightarrow \quad -x \le x\sin\left(\frac{1}{x}\right) \le x$$

Hence, the curve $y = x\sin\left(\dfrac{1}{x}\right)$ lies between $y = x$ and $y = -x$.

Also, $\operatorname*{Lt}_{x \to 0^+}\left(x\sin\left(\dfrac{1}{x}\right)\right) = 0 = \operatorname*{Lt}_{x \to 0^-}\left(x\sin\left(\dfrac{1}{x}\right)\right)$

It cuts the x-axis at $x = \dfrac{1}{n\pi}$, $n \in I - \{0\}$



LEVEL I
(PROBLEMS BASED ON FUNDAMENTALS)

Q. Draw the graphs of

1. $f(x) = \sin 2x$
2. $f(x) = \sin 3x$
3. $f(x) = \sin\left(\dfrac{x}{2}\right)$
4. $f(x) = \sin\left(\dfrac{x}{3}\right)$
5. $f(x) = \sin x \dfrac{1}{2}$
6. $f(x) = \sin x - \dfrac{1}{2}$
7. $f(x) = \sin\left(x - \dfrac{\pi}{6}\right)$
8. $f(x) = \sin\left(x + \dfrac{\pi}{6}\right)$
9. $f(x) = \sin(x - 1)$
10. $f(x) = \sin(x + 1)$
11. $f(x) = -\sin x$
12. $f(x) = \dfrac{1}{2} - \sin x$
13. $f(x) = \sin\left(x - \dfrac{\pi}{4}\right) + 1$
14. $f(x) = \sin\left(x - \dfrac{\pi}{4}\right) - 1$
15. $f(x) = \sin |x|$

16. $f(x) = \sin|x| + 1$
17. $f(x) = \sin|x| - 1$
18. $f(x) = 1 - \sin|x|$
19. $f(x) = \sin|x|$
20. $f(x) = \sin|x - 1|$
21. $f(x) = \sin|x + 1|$
22. $f(x) = \sin^2 x$
23. $f(x) = \sin^2 x + 1$
24. $f(x) = \sin^2 x - 1$
25. $f(x) = 1 - \sin^2 x$
26. $f(x) = \sin^2 x + \cos^2 x$
27. $f(x) = |\sin|x||$
28. $f(x) = |\sin|x|| + 1$
29. $f(x) = |\sin|x|| - 1$
30. $f(x) = \cos 2x$
31. $f(x) = \cos 4x$
32. $f(x) = \cos\left(\frac{x}{2}\right)$
33. $f(x) = \cos\left(\frac{x}{3}\right)$
34. $f(x) = -\cos x$
35. $f(x) = 1 - \cos x$
36. $f(x) = \cos\left(x - \frac{\pi}{4}\right)$
37. $f(x) = \cos\left(x + \frac{\pi}{4}\right)$
38. $f(x) = \cos(x - 1)$
39. $f(x) = \cos(x + 1)$
40. $f(x) = \cos\left(x - \frac{\pi}{4}\right) + 1$
41. $f(x) = \cos\left(x + \frac{\pi}{4}\right) - 1$
42. $f(x) = |\cos x|$
43. $f(x) = -|\cos x|$
44. $f(x) = 1 - |\cos x|$
45. $f(x) = ||\cos x||$
46. $f(x) = |\cos(x - 1)|$
47. $f(x) = \cos|x|$
48. $f(x) = \cos|x| + 1$
49. $f(x) = \cos^2 x$
50. $f(x) = \cos^3 x$
51. $f(x) = \cos^2 x - \sin^2 x$
52. $f(x) = \frac{1}{2}\cos^2 x$
53. $f(x) = \sin x + \cos x$
54. $f(x) = \sin x - \cos x$
55. $f(x) = \tan 2x$
56. $f(x) = \tan^2 x$
57. $f(x) = \tan^3 x$
58. $f(x) = |\tan x|$
59. $f(x) = \cot 2x$
60. $f(x) = |\cot x|$
61. $f(x) = \tan x \cdot \cot x$
62. $f(x) = \frac{1 - \cos 2x}{\sin 2x}$
63. $f(x) = \frac{1 + \cos 2x}{\sin 2x}$
64. $f(x) = \frac{\sin 2x}{1 + \cos 2x}$
65. $f(x) = \frac{\sin 2x}{1 - \cos 2x}$
66. $f(x) = \operatorname{cosec} 2x$
67. $f(x) = \sec 2x$
68. $f(x) = \tan x + \cot x$
69. $f(x) = \cot x - \tan x$
70. $f(x) = \sqrt{\sin^2 x}$
71. $f(x) = \sqrt{\cos^2 x}$
72. $f(x) = \sqrt{\sin^2 x}$
73. $f(x) = \sqrt{-\sin x}$
74. $f(x) = \sqrt{-\cos x}$
75. $f(x) = \sqrt{\sin\left(x - \frac{\pi}{4}\right)}$

LEVEL II
(PROBLEMS FOR JEE MAIN)

Q. Draw the graphs of

1. $f(x) = x \sin x$
2. $f(x) = x + \sin x$
3. $f(x) = x - \sin x$
4. $f(x) = 2^x \sin x$
5. $f(x) = e^x \sin x$
6. $f(x) = \sin x - \cos x$
7. $f(x) = \sin x + \cos x$
8. $f(x) = x \cos x$
9. $f(x) = x + \cos x$
10. $f(x) = x - \cos x$
11. $f(x) = \sqrt{\sin x}$
12. $f(x) = \sqrt{-\sin x}$
13. $f(x) = \sqrt{\sin x} + \sqrt{-\sin x}$
14. $f(x) = \sqrt{\sin x} - \sqrt{-\sin x}$
15. $f(x) = \sin x + |\sin x|$
16. $f(x) = \sin x - |\sin x|$
17. $f(x) = \sin x + \sin|x|$
18. $f(x) = \sin x - \sin|x|$

19. $f(x) = \cos x + |\cos x|$
20. $f(x) = \cos x - |\cos x|$
21. $f(x) = \cos x + \cos |x|$
22. $f(x) = \cos x - \cos |x|$
23. $f(x) = \max\left\{\sin x, \frac{1}{2}\right\}$
26. $f(x) = \max\left\{\sin\left(x - \frac{\pi}{4}\right), \sin\left(x + \frac{\pi}{4}\right)\right\}$
27. $f(x) = \tan x + \tan|x|$
28. $f(x) = \tan x - \tan|x|$
29. $f(x) = \tan x \cdot \cot x$
30. $f(x) = \sin x \cdot \operatorname{cosec} x$
31. $f(x) = \cos x \cdot \sec x$
32. Find the number of solutions of
    (i) $\sin x = x$
    (ii) $\sin x = |x|$
    (iii) $\sin x = 2|x|$
    (iv) $\sin x = \frac{1}{2}|x|$
    (v) $\sin x = x + \frac{1}{x}, x > 0$
    (vi) $\sin x = x + \frac{1}{x}, x < 0$
    (vii) $\sin x = x^2 + x + 1$
    (viii) $\sin x = -x^2 + x - 1$
    (ix) $\sin x = 2^x + 2^{-x}$
    (x) $\sin x = 2^x - 2^{-x}$
    (xi) $2^{\sin x} = \frac{3}{4}, \forall\ x \in [-2\pi, 2\pi]$
    (xii) $2^{\cos x} = \frac{3}{4}, \forall\ x \in [-2\pi, 2\pi]$
    (xiii) $2^{|\sin x|} = \frac{5}{6}, \forall\ x \in [-2\pi, 2\pi]$
    (xiv) $2^{|\cos x|} = 1, \forall\ x \in [-2\pi, 2\pi]$
    (xv) $3^{\sin x} = \frac{1}{2}, \forall\ x \in [-2\pi, 2\pi]$
    (xvi) $2^{\sin x} = |\sin x|, \forall\ x \in [-2\pi, 2\pi]$
    (xvii) $2^{\cos x} = |\sin x|, \forall\ x \in [-2\pi, 2\pi]$
    (xix) $2^{\cos x} = |\cos x|, \forall\ x \in [-2\pi, 2\pi]$
33. Find the maximum and minimum values of
    (i) $f(x) = \sin x + \cos x$
    (ii) $f(x) = \sin x - \cos x$
    (iii) $f(x) = 3\sin x + 4\cos x + 10$
    (iv) $f(x) = 5\sin x + 12\cos x + 12$
    (v) $f(x) = \sin(x - 1) + \cos x$
    (vi) $f(x) = \sin x + \cos(x - 1)$
    (vii) $f(x) = 3\sin x + 4$
    (viii) $f(x) = 2 - 4\sin x$
    (ix) $f(x) = -3 - 2\sin x$
    (x) $f(x) = 3\cos x + 5$
    (xi) $f(x) = -2\cos x + 3$
    (xii) $f(x) = 3\sin^2 x + 4$
    (xiii) $f(x) = 2 - 4\sin^2 x$
    (xiv) $f(x) = 3\sin^2 x + 4\cos^2 x$
    (xv) $f(x) = 2\sin^2 x - 3\cos^2 x$
    (xvi) $f(x) = 3\sin^2 x + 4\cos^2 x + 10\sin x \cos x$
    (xvii) $f(x) = \frac{1}{3\sin^2 x + 2\cos^2 x}$
    (xviii) $f(x) = \frac{1}{3\sin^2 x - 2\cos^2 x}$
    (xix) $f(x) = 2^{5\sin^2 x + 6\cos^2 x}$
    (xx) $f(x) = \log_2(3\sin^2 x + 1)$
34. Find the domains and ranges of
    (i) $f(x) = 2\sin x + 4$
    (ii) $f(x) = \sin x + \cos x$
    (iii) $f(x) = \sin x + \cos x + 1$
    (iv) $f(x) = 3\sin x + 4\cos x + 10$
    (v) $f(x) = \sin\left(x - \frac{\pi}{4}\right) + \cos x$
    (vi) $f(x) = \sin\left(x - \frac{\pi}{4}\right) + \cos\left(x - \frac{\pi}{6}\right)$
    (vii) $f(x) = \sin\left(x - \frac{\pi}{4}\right) + \cos\left(x - \frac{\pi}{4}\right)$
    (viii) $f(x) = 3\sin^2 x + 2$
    (ix) $f(x) = 3\sin^2 x + 2\cos^2 x$
    (x) $f(x) = 3\sin^2 x - 4\cos^2 x$

LEVEL III (PROBLEMS FOR JEE ADVANCED)

Q. Draw the graphs of

1. $f(x) = \sin\left(\frac{1}{x}\right)$
2. $f(x) = x\sin\left(\frac{1}{x}\right)$
3. $f(x) = \sin x + |x|$
4. $f(x) = \sin x - |x|$
5. $f(x) = |x| + \sin x$
6. $f(x) = |x| - \sin |x|$
7. $f(x) = |x| + \cos |x|$
8. $f(x) = |x| - \cos |x|$
9. $f(x) = \log_2(\sin x)$
10. $f(x) = \log_2(\cos x)$
11. $f(x) = [\sin x]$, where [,] = G.I.F
12. $f(x) = [2\sin x]$, [,] = G.I.F
13. $f(x) = [\sin x + \cos x]$, [,] = G.I.F
14. $f(x) = \max\{\sin x, \cos x\}, \forall\ x \in [-2\pi, 2\pi]$
15. $f(x) = \max\left\{\sin x, \frac{1}{2}, \cos x\right\}, \forall\ x \in [-2\pi, 2\pi]$

16. $f(x) = \dfrac{\sin x + \cos x}{2} - \left|\dfrac{\sin x - \cos x}{2}\right|, \forall\ x \in [-2\pi, 2\pi]$

17. $f(x) = \dfrac{\sin x + \cos x}{2} + \left|\dfrac{\sin x - \cos x}{2}\right|, \forall\ x \in [-2\pi, 2\pi]$

18. $f(x) = x \sin\left(\dfrac{1}{x}\right)$

19. $f(x) = \text{sgn}(\sin x)$

20. $f(x) = \text{sgn}(\cos x)$

21. Find the maximum and minimum values of
   (i) $f(x) = \sin^2 x + \cos^4 x$
   (ii) $f(x) = \sin^4 x + \cos^2 x$
   (iii) $f(x) = \sin^4 x + \cos^4 x$
   (iv) $f(x) = \sin^6 x + \cos^6 x$

22. Find the number of solutions of
   (i) $2^{\sin x} = \sin x, \forall\ x \in [0, 4\pi]$
   (ii) $2^{\sin x} = |\sin x|, \forall\ x \in [0, 4\pi]$
   (iii) $2^{\cos x} = |\sin x|, \forall\ x \in [0, 8\pi]$
   (iv) $2^{\cos x} = |\cos x|, \forall\ x \in [0, 10\pi]$
   (v) $2^{\cos x} = \cos x, \forall\ x \in [0, 20\pi]$
   (vi) $x\sin x - 1 = 0, \forall\ x \in [0, 2\pi]$
   (vii) $x\sin^2 x - 1 = 0, \forall\ x \in [0, 2\pi]$
   (viii) $\sin x = 2^x + 2^x, \forall\ x \in [0, 2\pi]$
   (ix) $\cos x = 2^x - 2^x, \forall\ x \in [0, 2\pi]$
   (x) $x^2\cos x - 2 = 0, \forall\ x \in [0, 2\pi]$

ANSWERS

Level II

32. (i) 1 (ii) 1 (iii) 1
   (iv) 3 (v) 0 (vi) 0
   (vii) 0 (viii) 0 (ix) 0
   (x) 3 (xi) 4 (xii) 4
   (xiii) 0 (xiv) 4 (xv) 4
   (xvi) 4 (xvii) 8 (xviii) 4

33. (i) Max $V = \sqrt{2}$ and Min $V = -\sqrt{2}$
   (ii) Max $V = \sqrt{2}$ and Min $V = -\sqrt{2}$
   (iii) Max $V = 15$ and Min $V = 5$
   (iv) Max $V = 25$ and Min $V = -1$
   (v) Max $V = \sqrt{2}\left(\cos\left(\dfrac{1}{2}\right) - \sin\left(\dfrac{1}{2}\right)\right)$
   Min $V = -\sqrt{2}\left(\cos\left(\dfrac{1}{2}\right) - \sin\left(\dfrac{1}{2}\right)\right)$
   (vi) Max $V = \sqrt{2}\left(\cos\left(\dfrac{1}{2}\right) + \sin\left(\dfrac{1}{2}\right)\right)$
   Min $V = -\sqrt{2}\left(\cos\left(\dfrac{1}{2}\right) + \sin\left(\dfrac{1}{2}\right)\right)$
   (vii) Max $V = 7$ and Min $V = 1$
   (viii) Max $V = 6$ and Min $V = -2$
   (ix) Max $V = -1$ and Min $V = -5$
   (x) Max $V = 8$ and Min $V = 2$
   (xi) Max $V = 5$ and Min $V = 1$
   (xii) Max $V = 7$ and Min $V = 4$
   (xiii) Max $V = 2$ and Min $V = -2$
   (xiv) Max $V = 4$ and Min $V = 3$
   (xv) Max $V = 2$ and Min $V = -3$
   (xvi) Max $V = \left(\dfrac{13}{2} + \dfrac{\sqrt{101}}{2}\right)$
   Min $V = \left(\dfrac{13}{2} - \dfrac{\sqrt{101}}{2}\right)$
   (xvii) Max $V = 1/2$ and Min $V = 1/3$
   (xviii) Max $V = 1/3$ and Min $V = -1/2$
   (xix) Max $V = 2^6$ and Min $V = 2^5$
   (xx) Max $V = 2$ and Min $V = 0$

33. (i) $D_f = R$ and $R_f = [2, 6]$
   (ii) $D_f = R$ and $R_f = [-\sqrt{2}, \sqrt{2}]$
   (iii) $D_f = R$ and $R_f = [1-\sqrt{2}, 1+\sqrt{2}]$
   (iv) $D_f = R$ and $R_f = [5, 15]$
   (v) $D_f = R$ and $R_f = [-\sqrt{5+2\sqrt{2}}, \sqrt{5+2\sqrt{2}}]$
   (vi) $D_f = R$ and $R_f = \left[-\sqrt{5+\sqrt{2}+\sqrt{6}}, \sqrt{5+\sqrt{2}+\sqrt{6}}\right]$
   (vii) $D_f = R$ and $R_f = [-\sqrt{2}, \sqrt{2}]$
   (viii) $D_f = R$ and $R_f = [2, 5]$
   (ix) $D_f = R$ and $R_f = [2, 3]$
   (x) $D_f = R$ and $R_f = [-4, 3]$

Level-III

21. (i) Max $V = 1$ and Min $V = 3/4$
   (ii) Max $V = 1$ and Min $V = 3/4$
   (iii) Max $V = 1$ and Min $V = 1/2$
   (iv) Max $V = 1$ and Min $V = 1/4$

22. (i) 0 (ii) 4
   (iii) 16 (iv) 10
   (v) 0 (vi) 2
   (vii) 2 (viii) 0
   (ix) 1 (x) 2

CHAPTER 3

The Trigonometric Equation

3.1 DEFINITION

An equation involving one or more trigonometrical ratios of unknown angle is called trigonometrical equation.

For examples, $\sin^2 x - \cos x - 2 = 0$, $\tan x = 1$, $\cos^2 x + \cos x - 2 = 0$ etc. are trigonomteric equations.

3.2 SOLUTION OF A TRIGONOMETRIC EQUATION

A value of the unknown angle which satisfies the given trigonometrical equation is called a solution or root of the equation

For example, $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

Types of Solutions:

(i) Principal Solution

(ii) General Solution.

PRINCIPAL SOLUTION:

The smallest numerical value of the angle which satisfies the given equation is called the principal solutions.

GENERAL SOLUTIONS:

Since trigonometric functions are periodic function, therefore solution of trigonometric equations can be genralised with the help of periodicity of a trigonometrical function. The solution consisting of all possible solution of a trigonometrical equation is called the general solution.

3.3 GENERAL SOLUTION OF TRIGONOMETRIC EQUATIONS

The general solution of the equations are as follows:

Step I

1. $\sin\theta = 0 \Rightarrow \theta = n\pi, n \in I$
2. $\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}, n \in I$
3. $\tan\theta = 0 \Rightarrow \theta = n\pi, n \in I$

Step II

1. $\sin\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{2}, n \in I$
2. $\cos\theta = 1 \Rightarrow \theta = 2n\pi, n \in I$
3. $\tan\theta = 1 \Rightarrow \theta = (4n+1)\frac{\pi}{4}, n \in I$

Step III

1. $\sin\theta = -1 \Rightarrow \theta = (4n-1)\frac{\pi}{2}, n \in I$
2. $\cos\theta = -1 \Rightarrow \theta = (2n+1)\pi, n \in I$
3. $\tan\theta = -1 \Rightarrow \theta = (4n-1)\frac{\pi}{4}, n \in I$

Step IV

1. $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n \alpha, n \in I$
2. $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha, n \in I$
3. $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha, n \in I$

Step V

1. $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$
2. $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$
3. $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha, n \in I$

3.4 RANGES OF TRIGONOMETRIC FUNCTIONS

Step I

1. $-1 \le \sin\theta, \cos\theta \le 1$
2. $-1 \le -\sin\theta, -\cos\theta \le 1$
3. $-\infty < \tan\theta, \cot\theta < \infty$
4. $-\infty < -\tan\theta, -\cot\theta < \infty$

5. $\operatorname{cosec}\theta, \sec\theta \in (-\infty, 1] \cup [1, \infty)$
6. $-\operatorname{cosec}\theta, -\sec\theta \in (-\infty, 1] \cup [1, \infty)$

Step II

1. $0 \le \sin^2\theta, \cos^2\theta \le 1$
2. $0 \le \tan^2\theta, \cot^2\theta < \infty$
3. $1 \le \operatorname{cosec}^2\theta, \sec^2\theta < \infty$

Step III

1. $-1 \le -\sin^2\theta, -\cos^2\theta \le 0$
2. $-\infty < -\tan^2\theta, -\cot^2\theta \le 0$
3. $-\infty < -\operatorname{cosec}^2\theta, -\sec^2\theta \le -1$

3.5 SOME SOLVED EXAMPLES

Ex-1. Solve for θ: $\sin 3\theta = 0$.

Soln. We have, $\sin 3\theta = 0$

$\Rightarrow \quad 3\theta = n\pi$

$\Rightarrow \quad \theta = \dfrac{n\pi}{3}$, where $n \in I$

Ex-2. Solve for θ: $\cos^2(5\theta) = 0$.

Soln. We have, $\cos^2(5\theta) = 0$

$\Rightarrow \quad \cos^2(5\theta) = \cos^2\left(\dfrac{\pi}{2}\right)$

$\Rightarrow \quad (5\theta) = n\pi \pm \left(\dfrac{\pi}{2}\right)$

$\Rightarrow \quad \theta = \dfrac{1}{5}\left(n\pi \pm \left(\dfrac{\pi}{2}\right)\right)$, where $n \in I$.

Ex-3. Solve for θ: $\tan\theta = \sqrt{3}$.

Soln. We have, $\tan\theta = \sqrt{3}$

$\Rightarrow \quad \tan\theta = \tan\left(\dfrac{\pi}{3}\right)$

$\Rightarrow \quad \theta = n\pi + \left(\dfrac{\pi}{3}\right)$, where $n \in I$.

Ex-4. Solve for θ: $\sin 2\theta = \sin\theta$.

Soln. We have, $\sin 2\theta = \sin\theta$

$\Rightarrow \quad 2\sin\theta\cos\theta = \sin\theta$

$\Rightarrow \quad \sin\theta(2\cos\theta - 1) = 0$

$\Rightarrow \quad \sin\theta = 0 \;\&\; (2\cos\theta - 1) = 0$

$\Rightarrow \quad \sin\theta = 0 \;\&\; \cos\theta = \dfrac{1}{2}$

$\Rightarrow \quad \theta = n\pi \;\&\; \theta = 2n\pi \pm \dfrac{\pi}{3}$, where $n \in I$.

Ex-5. Solve for θ: $\sin(9\theta) = \sin\theta$.

Soln. We have, $\sin(9\theta) = \sin\theta$

$\Rightarrow \quad \sin(9\theta) - \sin\theta = 0$

$\Rightarrow \quad 2\cos\left(\dfrac{9\theta + \theta}{2}\right)\sin\left(\dfrac{9\theta - \theta}{2}\right) = 0$

$\Rightarrow \quad 2\cos(5\theta)\sin(4\theta) = 0$

$\Rightarrow \quad \cos(5\theta) = 0 \;\&\; \sin(4\theta) = 0$

$\Rightarrow \quad (5\theta) = (2n+1)\dfrac{\pi}{2} \;\&\; (4\theta) = n\pi$

$\Rightarrow \quad \theta = (2n+1)\dfrac{\pi}{10} \;\&\; \theta = \left(\dfrac{n\pi}{4}\right)$.

where $n \in I$.

Ex-6. Solve for θ: $5\sin^2\theta + 3\cos^2\theta = 4$

Soln. We have, $5\sin^2\theta + 3\cos^2\theta = 4$

$\Rightarrow \quad 2\sin^2\theta + 3(\sin^2\theta + \cos^2\theta) = 4$

$\Rightarrow \quad 2\sin^2\theta + 3 = 4$

$\Rightarrow \quad 2\sin^2\theta = 1$

$\Rightarrow \quad \sin^2\theta = \dfrac{1}{2}$

$\Rightarrow \quad \sin^2\theta = \left(\dfrac{1}{\sqrt{2}}\right)^2 = \sin^2\left(\dfrac{\pi}{4}\right)$

$\Rightarrow \quad \theta = n\pi \pm \left(\dfrac{\pi}{4}\right)$, where $n \in I$

Ex-7. Solve for θ: $\tan(\theta - 15°) = 3\tan(\theta + 15°)$.

Soln. We have, $\tan(\theta - 15°) = 3\tan(\theta + 15°)$

$\Rightarrow \quad \dfrac{\tan(\theta - 15°)}{\tan(\theta + 15°)} = \dfrac{3}{1}$

$\Rightarrow \quad \dfrac{\tan(\theta - 15°) + \tan(\theta + 15°)}{\tan(\theta - 15°) - \tan(\theta + 15°)} = \dfrac{3+1}{3-1}$

$\Rightarrow \quad \dfrac{\sin(\theta + 15° + \theta - 15°)}{\sin(\theta + 15° - \theta + 15°)} = \dfrac{3+1}{3-1}$

$\Rightarrow \quad 2\sin(2\theta) = 2$

$\Rightarrow \quad \sin(2\theta) = 1$

$\theta = (4n+1)\dfrac{\pi}{4}, n \in I$.

Ex-8. Solve for θ: $\tan^2(\theta) + \cot^2(\theta) = 2$

Soln. We have, $\tan^2(\theta) + \cot^2(\theta) = 2$

$\Rightarrow \quad \tan^2(\theta) + \dfrac{1}{\tan^2(\theta)} = 2$

$\Rightarrow \quad \tan^4(\theta) - 2\tan^2(\theta) + 1 = 0$

$\Rightarrow \quad \left(\tan^2(\theta) - 1\right)^2 = 0$

$\Rightarrow \quad \left(\tan^2(\theta) - 1\right) = 0$

$\Rightarrow \quad \theta = n\pi \pm \left(\frac{\pi}{4}\right), n \in I$

Ex-9. Solve for θ: $\cos(\theta) + \cos(2\theta) + \cos(3\theta) = 0$

Soln. We have,

$\Rightarrow \quad (\cos(3\theta) + \cos(\theta)) + \cos(2\theta) = 0$

$\Rightarrow \quad \left(\cos(3\theta) + \cos(\theta)\right) + \cos(2\theta) = 0$

$\Rightarrow \quad 2\cos(2\theta)\cos(\theta) + \cos(2\theta) = 0$

$\Rightarrow \quad \cos(2\theta)\left(2\cos(\theta) + 1\right) = 0$

$\Rightarrow \quad \cos(2\theta) = 0 \ \& \ \left(2\cos(\theta) + 1\right) = 0$

$\Rightarrow \quad \cos(2\theta) = 0 \ \& \ \cos(\theta) = -\frac{1}{2}$

$\Rightarrow \quad \theta = (2n+1)\left(\frac{\pi}{4}\right) \ \& \ \theta = n\pi \pm \left(\frac{2\pi}{3}\right)$

Ex-10. Solve for θ: $\sin(2\theta) + \sin(4\theta) + \sin(6\theta) = 0$.

Soln. We have, $\sin(2\theta) + \sin(4\theta) + \sin(6\theta) = 0$

$\Rightarrow \quad \sin(6\theta) + \sin(2\theta) + \sin(4\theta) = 0$

$\Rightarrow \quad 2\sin(4\theta).\cos(2\theta) + \sin(4\theta) = 0$

$\Rightarrow \quad \sin(4\theta)\left(2\cos(2\theta) + 1\right) = 0$

$\Rightarrow \quad \sin(4\theta) = 0 \ \& \ \left(2\cos(2\theta) + 1\right) = 0$

$\Rightarrow \quad (4\theta) = n\pi \ \& \ \cos(2\theta) = -\frac{1}{2}$

$\Rightarrow \quad \theta = \left(\frac{n\pi}{4}\right) \ \& \ (2\theta) = n\pi \pm \left(\frac{2\pi}{3}\right)$

$\Rightarrow \quad \theta = \left(\frac{n\pi}{4}\right) \ \& \ \theta = \left(\frac{n\pi}{2}\right) \pm \left(\frac{\pi}{3}\right), \ n \in I$

Ex-11. Solve for θ:
$\tan(\theta) + \tan(2\theta) + \tan(\theta)\tan(2\theta) = 1$.

Soln. We have,

$\tan(\theta) + \tan(2\theta) + \tan(\theta)\tan(2\theta) = 1$

$\Rightarrow \quad \tan(2\theta) + \tan(\theta) = 1 - \tan(\theta)\tan(2\theta)$

$\Rightarrow \quad \frac{\tan(2\theta) + \tan(\theta)}{1 - \tan(\theta)\tan(2\theta)} = 1$

$\Rightarrow \quad \tan(3\theta) = 1$

$\Rightarrow \quad \tan(3\theta) = \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad (3\theta) = n\pi + \left(\frac{\pi}{4}\right)$

$\Rightarrow \quad \theta = \left(\frac{n\pi}{3}\right) + \left(\frac{\pi}{12}\right), \ n \in I$

Ex-12. Solve for θ: $\tan(\theta) + \tan(2\theta) + \tan(3\theta)$
$= \tan(\theta).\tan(2\theta).\tan(3\theta)$.

Soln. We have,

$\tan(\theta) + \tan(2\theta) + \tan(3\theta)$
$= \tan(\theta).\tan(2\theta).\tan(3\theta)$

$\Rightarrow \quad \tan(\theta) + \tan(2\theta)$
$= -\tan(3\theta) + \tan(\theta).\tan(2\theta).\tan(3\theta)$

$\Rightarrow \quad \tan(\theta) + \tan(2\theta)$
$= -\tan(3\theta)\left(1 - \tan(\theta).\tan(2\theta)\right)$

$\Rightarrow \quad \left(\frac{\tan(\theta) + \tan(2\theta)}{\left(1 - \tan(\theta).\tan(2\theta)\right)}\right) = -\tan(3\theta)$

$\Rightarrow \quad \tan(3\theta) = -\tan(3\theta)$

$\Rightarrow \quad 2\tan(3\theta) = 0$

$\Rightarrow \quad (3\theta) = n\pi$

$\Rightarrow \quad \theta = \left(\frac{n\pi}{3}\right), \ n \in I$.

EXERCISE 1

Solve the general values of θ:

1. $\sin 2\theta = 0$
2. $\cos 3\theta = 0$
3. $\tan 5\theta = 0$
4. $\sin\left(\frac{5\theta}{2}\right) = 0$
5. $\cos\left(\frac{7\theta}{2}\right) = 0$
6. $\tan\left(\frac{7\theta}{3}\right) = 0$

7. $\sin^2(7\theta)=0$
8. $\cos^2(3\theta)=0$
9. $\sin\theta=\frac{\sqrt{3}}{2}$
10. $\cos\theta=\frac{1}{2}$
11. $\tan\theta=\sqrt{2}$
12. $2\sin\theta+1=0$
13. $\cos 3\theta=-\frac{1}{\sqrt{2}}$
14. $\tan 3\theta+1=0$
15. $2\sin\theta-1=0$
16. $2\cos\theta-\sqrt{3}=0$
17. $\sqrt{3}\tan\theta=1$
18. $\sec(\theta)+2=0$
19. $\sin 2\theta=\sin\theta$
20. $\cos 3\theta=\cos\theta$
21. $\sin(3\theta)=\sin(\theta)$
22. $\sin(5\theta)=\cos(2\theta)$
23. $7\sin^2\theta+3\cos^2\theta=4$
24. $\sin\theta+\sin 3\theta+\sin 5\theta=0$
25. $\cos\theta+\cos 3\theta-2\cos 2\theta=0$
26. $\sin 2\theta+\sin 4\theta+\sin 6\theta=0$
27. $\cot^2\theta+\frac{3}{\sin\theta}+3=0$
28. $2\tan\theta-\cot\theta=-1$
29. $\tan^2\theta+(1-\sqrt{3})\tan\theta-\sqrt{3}=0$
30. $\tan\theta+\tan 2\theta+\tan\theta\,\tan 2\theta=1$
31. $\tan\theta+\tan 2\theta+\tan 3\theta=\tan\theta\tan 2\theta\tan 3\theta$
32. $\tan\theta+\tan\left(\theta+\frac{\pi}{3}\right)+\tan\left(\theta+\frac{2\pi}{3}\right)=3$
33. $3\tan(\theta-15°)=\tan(\theta+15°)$
34. $\tan\theta+\tan 2\theta+\tan 3\theta=0$
35. $\cos 2\theta\cos 4\theta=\frac{1}{2}$
36. $\cot\theta-\tan\theta=\cos\theta-\sin\theta$
37. $(1-\tan\theta)(1+\sin 2\theta)=1+\tan\theta$
38. $2\sin^2\theta+\sin^2 2\theta=2$
39. $\sin 3\theta=4\sin\theta.\sin(\theta+\alpha).\sin(\theta-\alpha)$, $\alpha\neq n\pi, \pi n\in Z$
40. $4\sin\theta\sin 2\theta\sin 4\theta=\sin 3\theta$

3.6 A TRIGONOMETRIC EQUATION IS OF THE FORM

$a\cos\theta\pm b\sin\theta=c$

Rule: 1. Divide by $\sqrt{a^2+b^2}$ on both the sides
2. Reduce the given equation into either $\sin(\theta\pm\alpha)$ or $\cos(\theta\pm\alpha)$
3. simplify the given equation.

Ex-1. Solve for θ: $\sin(\theta)+\cos(\theta)=1$

Soln. We have, $\sin(\theta)+\cos(\theta)=1$

$$\Rightarrow \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin(\theta)+\frac{1}{\sqrt{2}}\cos(\theta)\right)=1$$

$$\Rightarrow \left(\frac{1}{\sqrt{2}}\sin(\theta)+\frac{1}{\sqrt{2}}\cos(\theta)\right)=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin\left(\theta+\frac{\pi}{4}\right)=\left(\sin\left(\frac{\pi}{4}\right)\right)$$

$$\Rightarrow \left(\theta+\frac{\pi}{4}\right)=\left(n\pi+(-1)^n\left(\frac{\pi}{4}\right)\right)$$

$$\Rightarrow \theta=\left(n\pi+(-1)^n\left(\frac{\pi}{4}\right)-\frac{\pi}{4}\right),\ n\in I$$

Ex-2. Solve for θ: $\sqrt{3}\sin(\theta)+\cos(\theta)=2$

Soln. We have, $\sqrt{3}\sin(\theta)+\cos(\theta)=2$

$$\Rightarrow \frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta=1$$

$$\Rightarrow \sin\left(\theta+\frac{\pi}{6}\right)=1$$

$$\Rightarrow \sin\left(\theta+\frac{\pi}{6}\right)=\sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \left(\theta+\frac{\pi}{6}\right)=n\pi+(-1)^n\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \theta=n\pi+(-1)^n\left(\frac{\pi}{2}\right)-\frac{\pi}{6}$$

Ex-3. Solve for θ:
$\sin(2\theta)+\cos(2\theta)+\sin(\theta)+\cos(\theta)+1=0$

Soln. We have,

$$\sin(2\theta)+\cos(2\theta)+\sin(\theta)+\cos(\theta)+1=0$$

$$\Rightarrow \left(\sin(\theta)+\cos(\theta)\right)+\left(1+\sin(2\theta)\right)+\cos(2\theta)=0$$

$$\Rightarrow \left(\sin(\theta)+\cos(\theta)\right)+\left(\sin(\theta)+\cos(\theta)\right)^2+\left(\cos^2\theta-\sin^2\theta\right)=0$$

$$\Rightarrow \left(\sin(\theta)+\cos(\theta)\right)+\left(\sin(\theta)+\cos(\theta)\right)^2+\left(\cos\theta+\sin\theta\right)\left(\cos\theta-\sin\theta\right)=0$$

$$\Rightarrow \left(\sin(\theta)+\cos(\theta)\right)\left(1+\left(\sin(\theta)+\cos(\theta)\right)+\left(\cos\theta-\sin\theta\right)\right)=0$$

$$\Rightarrow \left(\sin(\theta)+\cos(\theta)\right)\left(1+2\cos\theta\right)=0$$

$$\Rightarrow \left(\sin(\theta)+\cos(\theta)\right)=0 \ \& \ \left(1+2\cos\theta\right)=0$$

$$\Rightarrow \left(\sin\left(\frac{\pi}{4}+\theta\right)\right)=0 \ \& \ \cos\theta=-\frac{1}{2}$$

$$\Rightarrow \left(\frac{\pi}{4}+\theta\right)=n\pi \ \& \ \theta=2n\pi\pm\left(\frac{2\pi}{3}\right)$$

$$\Rightarrow \theta=n\pi-\frac{\pi}{4} \ \& \ \theta=2n\pi\pm\left(\frac{2\pi}{3}\right), n\in I$$

Ex-4. Solve for θ: $\sin^3\theta+\sin\theta\cos\theta+\cos^3\theta=1$

Soln. We have, $\sin^3\theta+\sin\theta\cos\theta+\cos^3\theta=1$

$$\Rightarrow \left(\sin^3\theta+\cos^3\theta\right)+\sin\theta\cos\theta=1$$

$$\Rightarrow \left(\sin\theta+\cos\theta\right)\left(1-\sin\theta\cos\theta\right)+\sin\theta\cos\theta=1$$

$$\Rightarrow \left(\sin\theta+\cos\theta\right)\left(1-\sin\theta\cos\theta\right)=\left(1-\sin\theta\cos\theta\right)$$

$$\Rightarrow \left(\sin\theta+\cos\theta-1\right)\left(1-\sin\theta\cos\theta\right)=0$$

$$\Rightarrow \left(\sin\theta+\cos\theta-1\right)=0 \ \& \ \left(1-\sin\theta\cos\theta\right)=0$$

$$\Rightarrow \left(\sin\theta+\cos\theta\right)=1 \ \& \ \sin(2\theta)=\frac{1}{2}$$

$$\Rightarrow \left(\sin\left(\theta+\frac{\pi}{4}\right)\right)=\sin\left(\frac{\pi}{2}\right)$$

$$\& \ \sin(2\theta)=\sin\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \theta=n\pi+(-1)^n\left(\frac{\pi}{2}\right)-\left(\frac{\pi}{4}\right)$$

$$\& \ \theta=\frac{1}{2}\left(n\pi+(-1)^n\left(\frac{\pi}{6}\right)\right), \text{ where } n\in I.$$

EXERCISE 2

Q Solve for θ:

1. $\sin\theta+\sqrt{3}\cos\theta=\sqrt{2}$
2. $\sqrt{2}\sec\theta+\tan\theta=1$
3. $\cot\theta+\text{cosec}\theta=\sqrt{3}$
4. $\sin\theta+\cos\theta=\sqrt{2}$
5. $\sqrt{3}\cos\theta+\sin\theta=1$
6. $\sin\theta+\cos\theta=1$
7. $\text{cosec}\theta=1+\cot\theta$
8. $\tan\theta+\sec\theta=\sqrt{3}$
9. $\cos\theta+\sqrt{3}\sin\theta=2\cos 2\theta$
10. $\sqrt{3}(\cos\theta-\sqrt{3}\sin\theta)=4\sin 2\theta.\cos 3\theta$

3.7 PRINCIPAL VALUE

The numerically least angle is called the principal value.

For example, $\sin\theta=\frac{1}{2}$

Then,

$$\theta=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6},-\frac{11\pi}{6},-\frac{7\pi}{6}$$

Among all these values of θ, $\frac{\pi}{6}$ is the numerically smallest.

So principal value of $\sin\theta=\frac{1}{2}$ is $\theta=\frac{\pi}{6}$

3.8 METHOD TO FIND OUT THE PRINCIPAL VALUE

(i) First draw a trigonometric circle and mark the quadrant in which the angle may lie.

(ii) Select anti-clockwise direction for 1st and 2nd quadrant and select clockwise direction for 3rd and 4th quadrants.

(iii) Find the angle in the first rotation.

(iv) Select the numerically least angle among these two values. The angle thus formed will be the principal value.

(v) In case, two angles, one with +ve sign and the other with -ve sign, qualify for the numerically least angle, then it is the conventional of mathematics, to consider the angle with +ve signs as a principal value.

Ex-1. Find the principal value of $\sin(\theta) = -\frac{1}{2}$

Soln. We have, $\sin(\theta) = -\frac{1}{2}$

$\Rightarrow \quad \theta = -\frac{\pi}{6}$

Hence, the principal value of θ is $\left(-\frac{\pi}{6}\right)$

Ex-2. Find the principal value of $\sin(\theta) = \frac{1}{\sqrt{2}}$

Soln. We have, $\sin(\theta) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

Hence, the principal value of θ is $\frac{\pi}{4}$

Ex-3. Find the principal value of $\tan(\theta) = -\sqrt{3}$

Soln. We have, $\tan(\theta) = -\sqrt{3}$

$\Rightarrow \quad (\theta) = -\frac{\pi}{3}$

Hence, the principal value of θ is $-\frac{\pi}{3}$

EXERCISE 3

Find the principal values of

1. $\tan\theta = -1$
2. $\cos\theta = \frac{1}{2}$
3. $\cos\theta = -\frac{1}{2}$
4. $\tan\theta = -\sqrt{3}$
5. $\sec\theta = \sqrt{2}$

3.9 SOLUTIONS IN CASE OF TWO EQUATIONS ARE GIVEN:

Two equations are given and we have to find the value of θ which may satisfy both the given equations like $\cos\theta = \cos\alpha$, $\sin\theta = \sin\alpha$ and $\tan\theta = \tan\alpha$.

The common solution is $\theta = 2n\pi + \alpha$, where $n \in Z$

Similarly, $\sin\theta = \sin\alpha$, $\tan\theta = \tan\alpha$.

The common solution is $\theta = 2n\pi + \alpha, n \in Z$

Rule: (i) Find the common value of θ between 0 and 2π

(ii) Add $2n\pi$ to this common value

(iii) then we shall get the general value of the given two equations.

Ex-1. If $\sin(\theta) = \frac{1}{\sqrt{2}}$ and $\cos(\theta) = -\frac{1}{\sqrt{2}}$, then find the general values of θ.

Soln. Now, $\sin(\theta) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

and $\cos(\theta) = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = \frac{3\pi}{4}$

Thus, the commoon value of θ is $\frac{3\pi}{4}$

Hence, the general value of θ is $\left(2n\pi + \frac{3\pi}{4}\right)$

Ex-2. If $\sin(\theta) = \frac{1}{\sqrt{2}}$ and $\tan(\theta) = -1$, then find the general values of θ.

Soln. We have, $\sin(\theta) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

Also, $\tan(\theta) = -1$

$\Rightarrow \quad \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

Thus, the common value of θ is $\frac{3\pi}{4}$

Hence, the general values of θ is $\left(2n\pi + \frac{3\pi}{4}\right)$, where $n \in I$.

Ex-3. If $(1+\tan A)(1+\tan B) = 2$, then find all the values of $A + B$.

Soln. We have, $(1+\tan A)(1+\tan B) = 2$

$\Rightarrow \quad 1 + \tan A + \tan B + \tan A.\tan B = 2$

$\Rightarrow \quad \tan A + \tan B = 1 - \tan A.\tan B$

$\Rightarrow \quad \left(\frac{\tan A + \tan B}{1 - \tan A.\tan B}\right) = 1$

$\Rightarrow \quad \tan(A+B) = 1$

$\Rightarrow \quad \tan(A+B) = \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad (A+B) = n\pi + \left(\frac{\pi}{4}\right)$, where $n \in I$.

Ex-4. If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, then prove that,

$$\cos\left(\theta \pm \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$$

Soln. We have, $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$

$\Rightarrow \quad \sin(\pi \cos\theta) = \sin\left(\frac{\pi}{2} - \pi\sin\theta\right)$

$\Rightarrow \quad (\pi \cos\theta) = \left(\frac{\pi}{2} - \pi\sin\theta\right)$

$\Rightarrow \quad \cos\theta = \left(\frac{1}{2} - \sin\theta\right)$

$\Rightarrow \quad \cos\theta + \sin\theta = \frac{1}{2}$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{2\sqrt{2}}$

$\Rightarrow \quad \cos\left(\theta + \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

Similarly, we can prove that,

$$\cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$$

Ex-5. If $\tan(\pi \cos\theta) = \cot(\pi\sin\theta)$,

then prove that $\cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

Soln. We have, $\tan(\pi \cos\theta) = \cot(\pi\sin\theta)$

$\Rightarrow \quad \tan(\pi\cos\theta) = \tan\left(\frac{\pi}{2} - \pi\sin\theta\right)$

$\Rightarrow \quad (\pi\cos\theta) = \left(\frac{\pi}{2} - \pi\sin\theta\right)$

$\Rightarrow \quad \cos(\theta) + \sin(\theta) = \frac{1}{2}$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos(\theta) + \frac{1}{\sqrt{2}}\sin(\theta) = \frac{1}{2\sqrt{2}}$

$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

Ex-6. If $\sin A = \sin B$ and $\cos A = \cos B$, then find the values of A in terms of B.

Soln. Given $\sin A = \sin B$ ....(i)

and $\cos A = \cos B$ ....(ii)

Dividing (i) and (ii), we get,

$$\frac{\sin A}{\cos A} = \frac{\sin B}{\cos B}$$

$\Rightarrow \quad \tan A = \tan B$

$\Rightarrow \quad A = n\pi + B$, where $n \in I$

Ex-7. If A and B are acute +ve angles satisfying the equations $3\sin^2 A + 2\sin^2 B = 1$ and $3\sin 2A - 2\sin 2B = 0$, then find $A + 2B$

Soln. Given equations are

$3\sin^2 A + 2\sin^2 B = 1$ ....(i)

and $3 \sin 2A - 2 \sin 2B = 0$ ....(ii)

From (ii), we get,

$$3 \sin 2A = 2 \sin 2B$$

$\Rightarrow \quad \frac{\sin 2A}{2} = \frac{\sin 2B}{3}$

$\Rightarrow \quad \frac{\sin 2B}{\sin 2A} = \frac{3}{2}$

From (i), we get,

$$\frac{3}{2}\left(2\sin^2 A\right) + \left(2\sin^2 B\right) = 1$$

$\Rightarrow \quad \frac{3}{2}(1 - \cos 2A) + (1 - \cos 2B) = 1$

$\Rightarrow \quad \frac{3}{2}\cos 2A + \cos 2B = \frac{3}{2}$

$\Rightarrow \quad \frac{\sin 2B}{\sin 2A}\cos 2A + \cos 2B = \frac{\sin 2B}{\sin 2A}$

$\Rightarrow \quad \sin 2B \cos 2A + \sin 2A \cos 2B = \sin 2B$

$\Rightarrow \quad \sin(2A + 2B) = \sin 2B$

$\Rightarrow \quad \sin(2A + 2B) = \sin(\pi - 2B)$

$\Rightarrow \quad (2A + 2B) = (\pi - 2B)$

$\Rightarrow \quad (2A + 4B) = \pi$

$\Rightarrow \quad (A + 2B) = \frac{\pi}{2}$

Ex-8. Solve: $x + y = \frac{\pi}{4}$ and $\tan x + \tan y = 1$

Soln. Given $x + y = \frac{\pi}{4}$ and $\tan x + \tan y = 1$

$\Rightarrow \quad \tan(x + y) = \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad \frac{\tan x + \tan y}{1 - \tan x . \tan y} = 1$

$\Rightarrow \quad 1 - \tan x . \tan y = 1$

$\Rightarrow \quad \tan x . \tan y = 0$

$\Rightarrow \quad \tan x = 0$ & $\tan y = 0$

$\Rightarrow \quad x = n\pi = y$

Thus, no values of x and y satisfying the given equations.

Therefore, the given equations have no solutions.

Ex-9. Solve: $\sin x + \sin y = 1$, $\cos 2x - \cos 2y = 1$.

Soln. Given $\sin x + \sin y = 1$ ....(i)

and $\cos 2x - \cos 2y = 1$ ....(ii)

From (ii), we get, $\cos 2x - \cos 2y = 1$

$\Rightarrow \quad 1 - 2\sin^2 x - 1 + \sin^2 y = 1$

$\Rightarrow \quad \sin^2 x - \sin^2 y = -\frac{1}{2}$

$\Rightarrow \quad y = n\pi + (-1)^n \sin^{-1}\left(\frac{3}{4}\right)$,

where $n \in I$

$\Rightarrow \quad \sin x - \sin y = -\frac{1}{2}$ ....(iii)

Adding (i) and (iii), we get,

$2 \sin x = \frac{1}{2}$

$\Rightarrow \quad \sin x = \frac{1}{4}$

$\Rightarrow \quad x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{4}\right), n \in I$,

Subtracting (i) and (iii), we get,

$2\sin y = \frac{3}{2}$

$\Rightarrow \quad \sin y = \frac{3}{4}$

$\Rightarrow \quad y = n\pi + (-1)^n \sin^{-1}\left(\frac{3}{4}\right), n \in I$.

Ex-10. If $r\sin\theta = 3$ and $r = 4(1 + \sin\theta)$, where $0 \le \theta \le 2\pi$, then find the value of θ.

Soln. Given equations are

$r\sin\theta = 3$ ....(i)

and $r = 4(1 + \sin\theta)$ ....(ii)

Eliminating (i) and (ii), we get,

$4(1 + \sin\theta)\sin\theta = 3$

$\Rightarrow \quad 4\sin^2\theta + 4\sin\theta - 3 = 0$

$\Rightarrow \quad 4\sin^2\theta + 6\sin\theta - 2\sin\theta - 3 = 0$

$\Rightarrow \quad 2\sin\theta(2\sin\theta + 3) - 1(2\sin\theta + 3) = 0$

$\Rightarrow \quad (2\sin\theta + 3)(2\sin\theta - 1) = 0$

$\Rightarrow \quad \sin\theta = -\frac{3}{2}, \frac{1}{2}$

$\Rightarrow \quad \sin\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Ex-13. Find the set of values of x for which

$$\frac{\tan 3x - \tan 2x}{1 + \tan 3x . \tan 2x} = 1$$

Soln. We have, $\frac{\tan 3x - \tan 2x}{1 + \tan 3x . \tan 2x} = 1$

$\Rightarrow \quad \tan(3x - 2x) = 1$

$\Rightarrow \quad \tan x = 1$

$\Rightarrow \quad x = n\pi + \frac{\pi}{4}$, where $n \in I$

But the values of x do not satisfy the given equation.

Hence, the set of values of x is ϕ.

Ex-14. Find the number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$

Soln. Given equation is $\tan x + \sec x = 2\cos x$

$\Rightarrow \quad (1 + \sin x) = 2\cos^2 x$

$\Rightarrow \quad (1 + \sin x) = 2(1 - \sin^2 x)$

$\Rightarrow \quad (1 + \sin x) = 2(1 + \sin x).(1 - \sin x)$

$\Rightarrow \quad (1 + \sin x)(1 - 2 + 2\sin x) = 0$

$\Rightarrow \quad (1 + \sin x)(2\sin x - 1) = 0$

$\Rightarrow \quad (1 + \sin x) = 0$ & $(2\sin x - 1) = 0$

$\Rightarrow \quad \sin x = -1$ & $\sin x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

But $x = \frac{\pi}{2}$ doest not satisfy the given equation.

Thus, the values of x are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

Hence, the number of solutions is 2.

Ex-15. Find the number of values of x in the interval $[0, 3\pi]$ satisfying the equation.

$2\sin^2 x + 5\sin x - 3 = 0$

Soln. Given equation is $2\sin^2 x+5\sin x-3=0$

$\Rightarrow \quad 2\sin^2 x+6\sin x-\sin x-3=0$

$\Rightarrow \quad 2\sin x(\sin x+3)-1(\sin x+3)=0$

$\Rightarrow \quad (\sin x+3)\,(2\sin x-1)=0$

$\Rightarrow \quad \sin x=-3, \dfrac{1}{2}$

$\Rightarrow \quad \sin x=\dfrac{1}{2}$

$\Rightarrow \quad x=\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6}, \dfrac{17\pi}{6}$.

Hence, the number of values of x is 4 .

Ex-16. Find the smallest positive value of x such that

$\tan(x+20°)=\tan(x-10°).\tan x.\tan(x+10°)$.

Soln. Given,

$\tan(x+20°)=\tan(x-10°).\tan x.\tan(x+10°)$

$\Rightarrow \quad \dfrac{\tan(x+20°)}{\tan x}=\tan(x-10°).\tan(x+10°)$

$\Rightarrow \quad \dfrac{\sin(x+20°)\cos x}{\cos(x+20°)\sin x}=\dfrac{\sin(x-10°)\sin(x+10°)}{\cos(x-10°).\cos(x+10°)}$

$\Rightarrow \quad \dfrac{\sin(x+20°)\cos x+\cos(x+20°)\sin x}{\sin(x+20°)\cos x-\cos(x+20°)\sin x}$

$=\dfrac{\sin(x-10°)\sin(x+10°)+\cos(x-10°).\cos(x+10°)}{\sin(x-10°)\sin(x+10°)-\cos(x-10°).\cos(x+10°)}$

$\Rightarrow \quad \dfrac{\sin(x+20°+x)}{\sin(x+20°-x)}=-\dfrac{\cos(x+10°-x+10°)}{\cos(x+10°+x-10°)}$

$\Rightarrow \quad \dfrac{\sin(2x+20°)}{\sin(20°)}=-\dfrac{\cos(20°)}{\cos(2x)}$

$\Rightarrow \quad \sin(2x+20°)\cos(2x)=-\sin(20°)\cos(20°)$

$\Rightarrow \quad 2\sin(2x+20°)\cos(2x)=-2\sin(20°)\cos(20°)$

$\Rightarrow \quad \sin(4x+20°)+\sin(20°)=-\sin(40°)$

$\Rightarrow \quad \sin(4x+20°)=-\sin(40°)-\sin(20°)$

$\Rightarrow \quad \sin(4x+20°)=-2\sin(30°)\cos(10°)$

$\Rightarrow \quad \sin(4x+20°)=-\cos(10°)$

$\Rightarrow \quad \sin(4x+20°)=-\sin(80°)$

$\Rightarrow \quad \sin(4x+20°)=\sin(-80°)$

$\Rightarrow \quad \sin(4x+20°)=\sin(\pi-(-80°))$

$\Rightarrow \quad (4x+20°)=(\pi-(-80°))$

$\Rightarrow \quad (4x+20°)=260°$

$\Rightarrow \quad 4x=260°-20°=240°$

$\Rightarrow \quad x=60°$

Hence, the smallest positive value of x is $60°$.

EXERCISE 4

1. If $\cos\theta=\dfrac{1}{\sqrt{2}}$ and $\tan\theta=-1$, then find the genral value of θ.
2. Find the most general value of θ which satisfy the equations $\sin\theta=\dfrac{1}{2}$ and $\tan\theta=\dfrac{1}{\sqrt{3}}$.
3. If A and B are acute +ve angles satisfying the equations $3\sin^2 A+2\sin^2 B=1$ and $3\sin 2A-2\sin 2B=0$, then find $A+2B$.
4. If $\tan(A-B)=1$ and $\sec(A+B)=\dfrac{2}{\sqrt{3}}$, then find the smallest +ve values of A and B and their most genral values.
5. Solve: $x+y=\dfrac{2\pi}{3}$ and $\sin x=2\sin y$.
6. Solve: $x+y=\dfrac{2\pi}{3}$ and $\cos x+\cos y=\dfrac{3}{2}$.
7. Solve: $x+y=\dfrac{\pi}{4}$ and $\tan x+\tan y=1$.
8. Solve: $r\sin\theta=3$ and $r=4(1+\sin\theta), 0\le\theta\le 2\pi$
9. If $\sin A=\sin B$ and $\cos A=\cos B$, then find the values of A in terms of B.
10. Solve: $\sin x+\sin y=1$ $\cos 2x-\cos 2y=1$.
11. Find the co-ordinates of the point of inter section of the curves $$y=\cos x \ \& \ y=\sin 2x$$
12. Find all points of x, y that satisfying the equations $$\cos x+\cos y+\cos(x+y)=-\dfrac{3}{2}.$$
13. If $0<\theta,\varphi<\pi$ and $8\cos\theta\cos\varphi\cos(\theta+\varphi)+1=0$, then find $\theta \ \& \ \varphi$.
14. Solve: $4^{\sin x}+3^{\frac{1}{\cos y}}=11$, $5.16^{\sin x}+2.3^{\frac{1}{\cos y}}=2$. if $-\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2}$

3.10 SOME IMPORTANT REMARKS TO KEEP IN MIND WHILE SOLVING A TRIGONOMETRIC EQUATION

1. Squaring the equation at any stage should be avoided as far as possible. If squaring is necessary, check the solutions for extraneous roots.
2. Never cancel terms containing unknown terms on both the sides, which are in product form. It may cause roots loose.
3. Domain should not be changed. If it is changed, necessary correction must be made.
4. Check the denominator is non zero at any stage, while solving equations.
5. The answer should not contain such values of angles, which may be any of the terms undefined.
6. Some-times, we may find our answers differ from those in the package in their notations.This leads to the different methods of solving the same problem.

   Whenever we come across such situation, we must check authenticity. This will ensure that our answer is correct.
7. Sometimes the two soltuion set consists partly of common values. In all such cases the common part must be presented only once.

3.11 TYPES OF TRIGONOMETRIC EQUATIONS

TYPE 1

A trigonometric equation reduces to Quadratic/Higher degree Equations.

Rules: 1. Transform the terms to be a only one trigonometric ratio involving angles in same form.

2. Factorize the equation and express it in $f(x) \times g(x) = 0$

   $\Rightarrow f(x) = 0$ or $g(x) = 0$.

3. Solve both the equations one by one to get the general value of the variables.

Ex-1. Solve: $5\cos 2x + 2\cos^2\left(\frac{x}{2}\right) + 1 = 0$.

Soln. The given equation can be expressed as

$$5\left(2\cos^2 x - 1\right) + \left(1 + \cos x\right) + 1 = 0$$

$$\Rightarrow \quad 10\cos^2 x + \cos x - 3 = 0$$

$$\Rightarrow \quad (5\cos x + 3)(2\cos x - 1) = 0$$

$$\Rightarrow \quad \left(5\cos x + 3\right) = 0, \left(2\cos x - 1\right) = 0$$

$$\Rightarrow \quad \cos x = -\frac{3}{5} = \cos\alpha,$$

$$\cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \quad x = 2n\pi \pm \alpha = 2n\pi \pm \cos^{-1}\left(\frac{\pi}{3}\right),$$

$$x = 2n\pi \pm \frac{\pi}{3}, n \in Z.$$

EXERCISE 5

1. $4\sin^4 x + \cos^4 x = 1$
2. $4\cos^2 x \sin x - 2\sin^2 x = 2\sin x$
3. $\sin x - \cos x - 4\cos^2 x \sin x = 4\sin^2 x$
4. $2\cos 2x + \sqrt{2\sin x} = 2$
5. $1 + \sin^2 x + \cos^2 x = \frac{3}{2}\sin 2x$
6. $\sin^6 x + \cos^6 x = \frac{7}{16}$
7. $\sin^8 x + \cos^8 x = \frac{17}{16}\cos^2 2x$
8. $2\sin^3 x = \cos^2 3x$
9. $\cos 4x = \cos^2 3x$
10. $\cos 2x = 6\tan^2 x - 2\cos^2 x$

TYPE 2.

A trigonometric equation is solved by factorization method.

Rule: Simply reduces to a single trigonometric ratio of the unknown angles and factorize by basic algebraic method.

Ex.-1 Solve: $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$.

Soln. The given equation can be written as

$$(2\sin x - \cos x)(1 + \cos x)$$

$$= (1 - \cos x)(1 + \cos x)$$

$$\Rightarrow \quad (1 + \cos x)(2\sin x - \cos x - 1 + \cos x) = 0$$

$$\Rightarrow \quad (1 + \cos x)(2\sin x - 1) = 0.$$

$\Rightarrow \quad \cos x = -1, \sin x = 1/2$

$\Rightarrow \quad \cos x = -1 = \cos\pi\ ,\ \sin x = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right).$

$\Rightarrow \quad x = 2n\pi \pm \pi\ ,\ x = n\pi + (-1)^n \frac{\pi}{6}, n \in Z\ .$

EXERCISE 6

1. $2\sin^2 x + \sin x - 1 = 0,$ where $0 \le x \le 2\pi$
2. $5\sin^2 x + 7\sin x - 6 = 0$,where $0 \le x \le 2\pi$
3. $\sin^2 x - \cos x = \frac{1}{4}$, where $0 \le x \le 2\pi$
4. $\tan^2 x - 2\tan x - 3 = 0$
5. $2\cos^2 x - \sqrt{3}\sin x + 1 = 0$

TYPE 3.

A trigonometric equation is solved by transformation as a sum or difference into a product.

Rules:

1. The given equation is reducible to

   $f(x) \times g(x) = 0$

   $\Rightarrow \quad f(x) = 0, g(x) = 0$

2. Solve both the equations one by one to get the general value of the variable x.

Ex.-1 Solve: $\sin x + \sin 3x + \sin 5x = 0$,

$0 \le x \le \frac{\pi}{2}$.

Soln. The given equation can be written as

$(\sin x + \sin 5x) + \sin 3x = 0.$

$\Rightarrow \quad 2\sin 3x . \cos 2x + \sin 3x = 0$

$\Rightarrow \quad \sin 3x\,(2\cos 2x + 1) = 0$

$\Rightarrow \quad \sin 3x = 0, \cos 2x = -1/2$

$\Rightarrow \quad \sin 3x = 0,\ \cos 2x = \cos\left(\frac{2\pi}{3}\right)$

$\Rightarrow \quad 3x = n\pi\ ,\ 2x = 2n\pi \pm \frac{2\pi}{3}\ , n \in Z$

$\Rightarrow \quad x = \frac{n\pi}{3}, x = n\pi \pm \frac{\pi}{3}, n \in Z$

$\Rightarrow \quad x = 0, \frac{\pi}{3}$

EXERCISE 7

Q. Solve for x

1. $\cos x - \cos 2x = \sin 3x$
2. $\sin 7x + \sin 4x + \sin x = 0,\ 0 \le x \le \frac{\pi}{2}$
3. $\cos 3x + \cos 2x$

   $= \sin\left(\frac{3x}{2}\right) + \sin\left(\frac{x}{2}\right),\ 0 \le x \le 2\pi$
4. $\sin x + \sin 2x + \sin 3x$

   $= \cos x + \cos 2x + \cos 3x\ ,\ -\pi \le x \le \pi$
5. $\cos 2x + \cos 4x = 2\cos x$
6. $\sin 2x + \cos 2x + \sin x + \cos x + 1 = 0$
7. $\tan x + \tan 2x + \tan 3x = 0$
8. $\tan 3x + \tan x = 2\tan 2x$
9. $(1 - \tan x)(1 + \sin 2x) = (1 + \tan 4x)$
10. $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

TYPE 4.

A trigonometric equation is solved by transformation as a product into a sum or difference:

Rule:

1. The given equation can be reduced to $f(x) = 0, g(x) = 0$
2. Solve both the equation one-by-one to get the genral value of the variable x.

Ex.-1. Solve: $4\sin x . \sin 2x . \sin 4x = \sin 3x$

Soln. The given equation can be written as

$(2\sin 2x . \sin x) . 2\sin 4x - \sin 3x = 0$

$\Rightarrow \quad 2(\cos x - \cos 3x)\sin 4x - \sin 3x = 0$

$\Rightarrow \quad 2\sin 4x\cos x - 2\sin 4x\cos 3x - \sin 3x = 0$

$\Rightarrow \quad (\sin 5x + \sin 3x) - (\sin 7x + \sin x) - \sin 3x = 0$

$\Rightarrow \quad (\sin 7x - \sin 5x) + \sin x = 0$

$\Rightarrow \quad \sin x\,(2\cos 6x + 1) = 0$

$\Rightarrow \quad \sin x = 0, \cos 6x = -1/2,$

$\Rightarrow \quad \sin x = 0,\ \cos 6x = \cos\left(\frac{2\pi}{3}\right)$

$\Rightarrow \quad x = n\pi, \quad 6x = 2n\pi \pm \frac{2\pi}{3}, n \in Z$

$\Rightarrow \quad x = n\pi\ , x = (3n \pm 1)\frac{\pi}{9}, n \in Z$

EXERCISE 8

1. $\cos x \cdot \cos 2x \cdot \cos 3x = 1/4,\ 0 \le x \le \pi$
2. $\sin 3\alpha = 4\sin\alpha.\sin(x+\alpha).\sin(x-\alpha)$
3. $\sin 2x \cdot \sin 4x \cdot + \cos 2x = \cos 6x$
4. $\sec x \cdot \cos 5x + 1 = 0.,\ 0 \le x \le \pi$
5. $\cos x \cdot \cos 6x = -1$

TYPE 5

A trigonometric equation is of the form

$$b_0 \sin^n x + b_1 \sin^{n-1} x.\cos x +$$
$$b_2 \sin^{n-2} x.\cos^2 x + ... + b_n \cos^n x = 0$$

where, $b_0, b_1, b_2, ..., b_n \in R$ is a homogenous equation of $\sin x$ and $\cos x$, where $\cos x$ is nonzero.

Rule: 1. Divide both the sides by highest power of cos x.
2. The given equation can be reduced to
$b_0 \tan^n x + b_1 \tan^{n-1} x + ... + b_n = 0$
3. and then use the factorization method.

Ex.-1 Solve: $2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -2$

Soln. The given equation can be written as

$$2\sin^2 x - 5\sin x\cos x - 8\cos^2 x$$
$$= -2(\sin^2 x + \cos^2 x)$$

$\Rightarrow \quad 2\tan^2 x - 5\tan x - 8 = -2(\tan^2 x + 1)$

$\Rightarrow \quad 4\tan^2 x - 5\tan x - 6 = 0$

$\Rightarrow \quad (\tan x - 2)(4\tan x + 3) = 0$

$\Rightarrow \quad \tan x = -2, \tan x = -\dfrac{3}{4}$

$\Rightarrow \quad x = n\pi + \alpha, x = n\pi + \beta$,

where $\alpha = \tan^{-1}(2), \beta = \tan^{-1}\left(-\dfrac{3}{4}\right),\ n \in Z$.

EXERCISE 9

Q. Solve for x:

1. $5\sin^2 x - 7\sin x\cos x + 6\cos^2 x = 4$
2. $2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -3$
3. $\sin 3x.\cos x + \sin^2 x\cos^2 x,\ 0 \le x \le 2\pi$

Type 6

A Trigonometric equation is of the form R (sin m x, cos n x, tan p x, cot q x) = 0, where R is a rational function and $m, n, p, q \in N$, can be reduced to a rational function with respect to sin x, cos x, tan x and cot x.

Rule: 1. Use half angle formulae of tangents:

$$\sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)},$$

$$\cos x = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$$

$$\tan x = \frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)},$$

$$\cot x = \frac{1-\tan^2\left(\frac{x}{2}\right)}{2\tan\left(\frac{x}{2}\right)}$$

2. Substitute $\tan\left(\dfrac{x}{2}\right) = t$, and then solve it.

Ex.-1 Solve: $1 + 2\text{cosec}x = -\dfrac{\sec^2\left(\frac{x}{2}\right)}{2}$

Soln. The given equation can be written as

$$1 + \frac{2}{\sin x} = -\frac{1}{2}\left(1 + \tan^2\left(\frac{x}{2}\right)\right)$$

$\Rightarrow \quad 2(\sin x + 2) = -\left(1 + \tan^2\left(\dfrac{x}{2}\right)\right)\sin x$

$$\Rightarrow \quad 2\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + 2\right)$$

$$= -\left(1 + \tan^2\frac{x}{2}\right).\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)$$

$\Rightarrow \quad 2\left(\dfrac{2t}{1+t^2} + 2\right) = -(1+t^2) \times \left(\dfrac{2t}{1+t^2}\right),$

where $t = \tan(x/2)$

$\Rightarrow \quad t^3 + 2t^2 + 3t + 2 = 0$

$\Rightarrow \quad t^3 + t^2 + t^2 + t + 2t + 2 = 0$

$\Rightarrow \quad (t+1)(t^2 + t + 2) = 0$

$\Rightarrow \quad t + 1 = 0,\ t^2 + t + 2 \neq 0$

$\Rightarrow \quad \tan\left(\frac{x}{2}\right) = -1 = \tan\left(\frac{-\pi}{4}\right)$

$\Rightarrow \quad \frac{x}{2} = n\pi - \frac{\pi}{4}, n \in Z$

$\Rightarrow \quad x = 2n\pi - \frac{\pi}{2}, n \in Z \cdot$

EXERCISE 10

1. Solve: $(\cos x - \sin x)(2\tan x + \sec x) + 2 = 0$
2. Solve: $\dfrac{\sin^3 \frac{x}{2} - \cos^3 \frac{x}{2}}{2 + \sin x} = \dfrac{\cos x}{3}$
3. Solve: $\cot\left(\frac{x}{2}\right) - \operatorname{cosec}\left(\frac{x}{2}\right) = \cot x$
4. If $\theta_1, \theta_2, \theta_3, \theta_4$ be the four roots of the equation $\sin(\theta + \alpha) = k\sin 2\theta$, no two of which differ by a multiple of 2π, then prove that $\theta_1 + \theta_2 + \theta_3 + \theta_4 = (2n+1)\frac{\pi}{4}, n \in Z \cdot$

TYPE 7

A trigonometric equation is of the form $R(\sin x + \cos x, \sin x \,.\, \cos x) = 0$, where R is a rational function of the argument of $\sin x$ and $\cos x$.

Rule: 1. Put $\sin x + \cos x = t$

2. Use the identity

$(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

$\Rightarrow \quad \sin x \cos x = \frac{t^2 - 1}{2}$

3. So, the given equation reduces to $R\left(t, \frac{t^2-1}{2}\right)$ and then solve it.

Ex.-1 Solve: $\sin x + \cos x - 2\sqrt{2}\ \sin x \cos x = 0.$

Soln. Let $\sin x + \cos x = t$

$\Rightarrow \quad \sin x \,.\, \cos x = \frac{t^2 - 1}{2}$

So, the given equation can be reduced to

$t - 2\sqrt{2}\left(\frac{t^2-1}{2}\right) = 0$

$\Rightarrow \quad \sqrt{2}t^2 - t - \sqrt{2} = 0$

$\Rightarrow \quad (\sqrt{2}t + 1)(t - \sqrt{2}) = 0$

$\Rightarrow \quad t = \sqrt{2}, -\frac{1}{\sqrt{2}}$

When $\sin x + \cos x = \sqrt{2}$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 1$

$\Rightarrow \quad \sin\left(x + \frac{\pi}{4}\right) = 1 = \sin\left(\frac{\pi}{2}\right)$

$\Rightarrow \quad x + \frac{n\pi}{4} = n\pi + (-1)^n \frac{\pi}{2},\ n \in Z$

When $\sin x + \cos x = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{2}$

$\Rightarrow \quad \sin\left(x + \frac{\pi}{4}\right) = \sin\left(-\frac{\pi}{6}\right)$

$\Rightarrow \quad \left(x + \frac{\pi}{4}\right) = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$

$\Rightarrow \quad x = n\pi - (-1)^n \frac{\pi}{6} - \frac{\pi}{4}, n \in Z.$

EXERCISE 11

Q. Solve for *x:*

1. $\sin^3 x + \sin x \cos x + \cos^3 x = 1$
2. $\sin x + \cos x = 1 - \sin x \cos x$
3. $1 + \sin^3 x + \cos^3 x = \frac{3}{2}\sin 2x$
4. $\sin 2x - 12(\sin x - \cos x) + 12 = 0,\ 0 \le x \le 2\pi$

TYPE 8

A trigonometrical equation is based on extreme values of $\sin x$ and $\cos x$.

Rule: 1. Whenever terms are sin, cos in power 1 and all terms connected with plus sign and number of terms in L.H.S (with +ve or –ve sign) then each term must have in the extreme value.

2. In such problems, each term will be +1 when the value of R.H.S is +ve and each term will be (–1) when the value of R.H.S is –ve.

Ex.-1 Solve: sin 6x + cos 4x + 2 = 0.

Soln. The given equation can be written as

sin 6x + cos 4x = –2

$\Rightarrow$ sin 6x = –1 and cos 4x = –1

$$\Rightarrow \sin 6x = \sin\frac{3\pi}{2},\ \cos 4x = \cos\pi$$

$$\Rightarrow 6x = 2n\pi + \frac{3\pi}{2}, 4x = 2n\pi + \pi, n \in Z$$

$$\Rightarrow x = \frac{n\pi}{3} + \frac{\pi}{4}, x = \frac{n\pi}{2} + \frac{\pi}{4}, n \in Z$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \ldots$$

$$\ldots x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots..$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$$

Hence, the general solution will be,

$$\Rightarrow x = 2n\pi + \frac{\pi}{4}, 2n\pi + \frac{5\pi}{4}, n \in Z$$

$$\Rightarrow x = 2n\pi + \frac{\pi}{4}, (2n+1)\pi + \frac{\pi}{4}, n \in Z$$

$$\Rightarrow x = m\pi + \frac{\pi}{4}, m \in Z$$

EXERCISE 12

Q. Solve for x:

1. $\sin^6 x = 1 + \cos^4 3x$
2. $\sin^4 x = 1 + \tan^8 x$
3. $\sin^2 x + \cos^2 y = 2\sec^2 z$
4. sin 3x + cos 2x + 2 = 0
5. cos 4x + sin 5x = 2.

TYPE 9

A trigonometrical equation involving with exponential, logarithmic and modulii terms:

Rule: Whenever equation contains power term, then we should use the following method.

1. Equate the base if possible
2. If it is not possible to equate the base, take log of both the sides and make its R.H.S is zero, then we proceed further.

Ex.-1 Find the values of x in $(-\pi, \pi)$ which satisfy the equation.

$$8^{1+|\cos x|+\cos^2 x+|\cos x|^3+\cos^4 x+|\cos x|^5+\ldots\ldots\ldots\text{to}\infty} = 64$$

Soln. The given equation can be written as

$$8^{1+|\cos x|+\cos^2 x+|\cos x|^3+\cos^4 x+|\cos x|^5+\ldots\ldots\ldots\text{to}\infty} = 8^2$$

$$\Rightarrow 1 + |\cos x| + \cos^2 x + |\cos x|^3 + \cos^4 x + \ldots\text{to}\infty = 2$$

$$\Rightarrow \frac{1}{1-|\cos x|} = 2$$

$$\Rightarrow \cos x = \pm\frac{1}{2}$$

When $\cos x = \frac{1}{2}$

$$\Rightarrow \cos x = \cos\left(\frac{\pi}{3}\right)$$

$$\Rightarrow x = 2n\pi \pm \frac{\pi}{3}, n \in Z$$

When $\cos x = -\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right)$

$$\Rightarrow x = 2n\pi \pm \frac{2\pi}{3}, n \in Z$$

Hence, the values of x are $\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$.

EXERCISE 13

Q. Solve for x:

1. $2^{1+|\cos x|+\cos^2 x+|\cos x|^3+\cos^4 x+|\cos x|^5+\ldots\ldots\text{to}\ \infty} = 4$
2. $1 + \sin\theta + \sin^2\theta + \sin^3\theta + \ldots\ldots\ldots\text{to}\ \infty$ $= 4 + 2\sqrt{3}$
3. $|\cos x|^{\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}} = 1$
4. $e^{\sin x} - e^{-\sin x} - 4 = 0$
5. If $e^{[\sin^2 x+\sin^4 x+\sin^6 x+\ldots\ldots\ldots\ldots\text{to}\infty]\log_e 2}$ satisfies the equations $x^2 - 9x + 8 = 0$, then find the value of $\frac{\cos x}{\cos x + \sin x}$, $0 < x < \frac{\pi}{2}$
6. $\log_{\cos x}\tan x + \log_{\sin x}\cot x = 0$
7. $3^{\sin 2x + 2\cos^2 x} + 3^{1-\sin 2x + 2\sin^2 x} = 28$
8. $\log_{\cos x}\sin x + \log_{\sin x}\cos x = 2$, where $x > 0$

TYPE 10

A trigonometrical equation involving the terms of two sides are of different nature:

Rule: 1. Let y = each side of the equation and break the equation in two parts.

2. Find the inequality for y taking L.H.S of the equation and also for the R.H.S of the equation.
3. Use the A.M ≥ G.M. on the right hand side of the equation.
4. If there is any value of y satisfying both the inequalities, then the equation will have real solution, otherwise no solution.

EXERCISE 14

Q. Solve for x:

1. $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}$, $0 < x < \frac{\pi}{2}$
2. $2\cos^2\left(\frac{x^2+x}{6}\right) = 2^x + 2^{-x}$

PROBLEMS FOR JEE ADVANCED EXAM

Ex-1. Solve for x: $\sec x - \operatorname{cosec} x = \frac{4}{3}$

Soln. The given equation is

$$\sec x - \operatorname{cosec} x = \frac{4}{3}$$

$$\Rightarrow \frac{1}{\cos x} - \frac{1}{\sin x} = \frac{4}{3}$$

$$\Rightarrow 3(\sin x - \cos x) = 4\sin x\cos x \quad \text{....(i)}$$

Put $(\sin x - \cos x) = t$

$$\Rightarrow 1 - 2\sin x\cos x = t^2$$

$$\Rightarrow \sin x\cos x = \frac{1-t^2}{2}$$

(i) reduces to $3t = 4\left(\frac{1-t^2}{2}\right)$

$$\Rightarrow 3t = 2\left(1-t^2\right)$$

$$\Rightarrow 2t^2 + 3t - 2 = 0$$

$$\Rightarrow 2t^2 + 4t - t - 2 = 0$$

$$\Rightarrow 2t(t+2) - (t+2) = 0$$

$$\Rightarrow (t+2)(2t-1) = 0$$

$$\Rightarrow t = \frac{1}{2}, -2$$

when $t = \frac{1}{2}$

$$\Rightarrow \sin x - \cos x = \frac{1}{2}$$

$$\Rightarrow \left(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right) = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \sin\left(x - \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \left(x - \frac{\pi}{4}\right) = n\pi + (-1)^n.\sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)$$

$$\Rightarrow x = \left(\frac{n\pi}{4} + n\pi + (-1)^n.\sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)\right), n \in I$$

when $t = -2$

$$\Rightarrow \sin x - \cos x = 2$$

It is impossible, since the maximum value of $(\sin x - \cos x)$ is $\sqrt{2}$

Ex-2. Solve for x:
$\sin 2x + 12 = 12(\sin x - \cos x)$

Soln. The given equation is
$\sin 2x + 12 = 12(\sin x - \cos x)$

Put $(\sin x - \cos x) = t$

$$\Rightarrow 1 - 2\sin x\cos x = t^2$$

$$\Rightarrow 2\sin x\cos x = \left(1 - t^2\right)$$

(i) reduces to $\left(1-t^2\right) + 12 = 12t$

$$\Rightarrow \left(1-t^2\right) = 12(t-1)$$

$$\Rightarrow (t+1)(t-1) = -12(t-1)$$

$$\Rightarrow (t-1)(t+1+12) = 0$$

$$\Rightarrow (t-1)(t+13) = 0$$

$$\Rightarrow t = 1, -13$$

when $t = 1$, then $\sin x - \cos x = 1$

$$\Rightarrow \left(\frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin\left(x - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \left(x - \frac{\pi}{4}\right) = n\pi + (-1)^n\left(\frac{\pi}{4}\right)$$

$$\Rightarrow x = n\pi + \frac{\pi}{4} + (-1)^n\left(\frac{\pi}{4}\right), n \in I$$

when $t = -13$

$\sin x - \cos x = -13$

It is impossible, since the maximum value of $(\sin x - \cos x)$ is $\sqrt{2}$.

Ex-3. Solve for x: $|\sec x + \tan x| = |\sec x| + |\tan x|$ in $[0, 2\pi]$.

Soln. The given equation is

$$|\sec x+\tan x|=|\sec x|+|\tan x|$$

$$\Rightarrow \quad \sec x.\tan x\geq 0$$

$$\Rightarrow \quad \frac{\sin x}{\cos^2 x}\geq 0$$

$$\Rightarrow \quad \sin x\geq 0, \cos x\neq 0$$

$$\Rightarrow \quad x\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$$

$$\Rightarrow \quad x\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$$

Hence, the solution set is $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$.

Ex-4. Let n be a positive integer such that

$\sin\left(\frac{\pi}{2n}\right)+\cos\left(\frac{\pi}{2n}\right)=\frac{\sqrt{n}}{2}$, find n.

Soln. The given equation is

$$\sin\left(\frac{\pi}{2n}\right)+\cos\left(\frac{\pi}{2n}\right)=\frac{\sqrt{n}}{2}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin\left(\frac{\pi}{2n}\right)+\frac{1}{\sqrt{2}}\cos\left(\frac{\pi}{2n}\right)=\frac{\sqrt{n}}{2\sqrt{2}}$$

$$\Rightarrow \quad \cos\left(\frac{\pi}{4}-\frac{\pi}{2n}\right)=\frac{\sqrt{n}}{2\sqrt{2}}$$

It is satisfied for $n = 6$ only.

Ex-5. If $\cos 2x+a\sin x=2a-7$ possesses a solution then find a.

Soln. The given equation is $\cos 2x+a\sin x=2a-7$

$$\Rightarrow \quad 1-2\sin^2 x+a\sin x=2a-7$$

$$\Rightarrow \quad 2\sin^2 x-a\sin x+(2a-8)=0$$

$$\Rightarrow \quad \sin x=\frac{a\pm\sqrt{a^2-16(a-4)}}{4}$$

$$\Rightarrow \quad \sin x=\frac{a\pm\sqrt{a^2-16a+64}}{4}$$

$$\Rightarrow \quad \sin x=\frac{a\pm\sqrt{(a-8)^2}}{4}$$

$$\Rightarrow \quad \sin x=\frac{a\pm(a-8)}{4}$$

$$\Rightarrow \quad \sin x=\frac{2a-8}{4},2$$

$$\Rightarrow \quad \sin x=\frac{a-4}{2}$$

$$\Rightarrow \quad -1\leq\left(\frac{a-4}{2}\right)\leq 1$$

$$\Rightarrow \quad -2\leq(a-4)\leq 2$$

$$\Rightarrow \quad 2\leq a\leq 6$$

$$\Rightarrow \quad a\in[2,6]$$

Ex-6. Solve for x: $\sin^{100}x-\cos^{100}x=1$

Soln. The given equation is

$$\sin^{100}x-\cos^{100}x=1$$

It is possible only when $\sin x=1, \cos x=0$

Hence, the general solution is

$$x=n\pi+\frac{\pi}{2}, n\in I$$

Ex-7. Solve for x:

$$\sin^{10}x+\cos^{10}x=\frac{29}{16}\cos^4 2x$$

Soln. The given equation is

$$\sin^{10}x+\cos^{10}x=\frac{29}{16}\cos^4 2x$$

$$\Rightarrow \quad \left(\frac{1-\cos 2x}{2}\right)^5+\left(\frac{1-\cos 2x}{2}\right)^5=\frac{29}{16}\cos^4 2x$$

$$\Rightarrow \quad \frac{2}{32}\left(1+10\cos^2 2x+5\cos^4 2x\right)=\frac{29}{16}\cos^4 2x$$

$$\Rightarrow \quad \left(1+10\cos^2 2x+5\cos^4 2x\right)=29\cos^4 2x$$

$$\Rightarrow \quad 24\cos^4 2x-10\cos^2 2x-1=0$$

$$\Rightarrow \quad \left(2\cos^2 2x-1\right)\left(12\cos^2 2x+1\right)=0$$

$$\Rightarrow \quad \left(2\cos^2 2x-1\right)=0, \text{ since } \left(12\cos^2 2x+1\right)\neq 0$$

$$\Rightarrow \quad \cos^2 2x=\frac{1}{2}$$

$$\Rightarrow \quad 2\cos^2 2x-1=0$$

$$\Rightarrow \quad \cos 4x=0$$

$$\Rightarrow \quad 4x=(2n+1)\frac{\pi}{2}, n\in I$$

$$\Rightarrow \quad x=(2n+1)\frac{\pi}{8}, n\in I$$

Hence, the solution is $x=(2n+1)\frac{\pi}{8}, n\in I$

Ex-8. Solve for x:

$$|\cos x|^{\sin^2 x-\frac{3}{2}\sin x+\frac{1}{2}}=1$$

Soln. The given equation is

$$|\cos x|^{\sin^2 x-\frac{3}{2}\sin x+\frac{1}{2}}=1$$

$\Rightarrow \quad \left(\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}\right)\log|\cos x| = 0$

$\Rightarrow \quad \left(\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}\right) = 0, \log|\cos x| = 0$

when $\left(\sin^2 x - \frac{3}{2}\sin x + \frac{1}{2}\right) = 0$

$\Rightarrow \quad \left(2\sin^2 x - 3\sin x + 1\right) = 0$

$\Rightarrow \quad \left(2\sin^2 x - 2\sin x - \sin x + 1\right) = 0$

$\Rightarrow \quad 2\sin x(\sin x - 1) - (\sin x - 1) = 0$

$\Rightarrow \quad (2\sin x - 1)(\sin x - 1) = 0$

$\Rightarrow \quad (2\sin x - 1) = 0, (\sin x - 1) = 0$

$\Rightarrow \quad \sin x = \frac{1}{2}, 1$

$\Rightarrow \quad \sin x = \frac{1}{2}, \text{since} |\cos x| = 0$

$\Rightarrow \quad x = n\pi + (-1)^n\left(\frac{\pi}{6}\right), n \in I$

when $\log|\cos x| = 0$

$\Rightarrow \quad \log|\cos x| = \log 1$

$\Rightarrow \quad |\cos x| = 1$

$\Rightarrow \quad \cos x = \pm 1$

when $\cos x = 1$

$\Rightarrow \quad x = 2n\pi$

when $\cos x = -1$

$x = (2n+1)\pi$

Hence, the solution is

$x = 2n\pi, (2n+1)\pi, n\pi + (-1)^n\left(\frac{\pi}{6}\right), n \in I$

Ex-9. Find the number of solutions of $\cos\left(\pi\sqrt{x-4}\right).\cos\left(\pi\sqrt{x}\right) = 1$

Soln. It is possible only when

$\cos\left(\pi\sqrt{x-4}\right) = 1.$ and $\cos\left(\pi\sqrt{x}\right) = 1$

$x = 4$ and $x = 0$

$x = 0$ does not satisfy the equation simultaneously.
Hence, the solution is $x = 4$
Therefore, the number of solution is 1

Ex-10. Find the number of solution of

$x^4 - 2x^2\sin^2\left(\frac{\pi}{2}\right)x + 1 = 0$

Soln. The given equation is

$x^4 - 2x^2\sin^2\left(\frac{\pi}{2}\right)x + 1 = 0$

$\left(x^2 - \sin^2\left(\frac{\pi}{2}\right)x\right)^2 + 1 - \sin^4\left(\frac{\pi}{2}\right)x = 0$

It is possible only when

$\left(x^2 - \sin^2\left(\frac{\pi}{2}\right)x\right) = 0,\ 1 - \sin^4\left(\frac{\pi}{2}\right)x = 0$

when $1 - \sin^4\left(\frac{\pi}{2}\right)x = 0$

$\Rightarrow \sin^4\left(\frac{\pi}{2}\right)x = 1$

$\Rightarrow \sin^4\left(\frac{\pi}{2}\right)x = \sin^2\left(\frac{\pi}{2}\right)$

$\Rightarrow \left(\frac{\pi}{2}\right)x = n\pi \pm \left(\frac{\pi}{2}\right)$

$\Rightarrow x = (2n \pm 1), n \in I$

when $\left(x^2 - \sin^2\left(\frac{\pi}{2}\right)x\right) = 0$

$\Rightarrow x^2 = \sin^2\left(\frac{\pi}{2}\right)x$

$\Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$

Hence, the number of solutions is 2.

Ex-11. If $\cos^4 x + a\cos^2 x + 1 = 0$ has atleast one real solution, then find the value of a.

Soln. The given equation is

$\cos^4 x + a\cos^2 x + 1 = 0$ ....(i)

Let $\cos^2 x = t$

Then $t \in [0,1]$

(i) reduces to $t^2 + \text{at} + 1 = 0$

since it has at-least one real root in $[0,1]$, so

$a^2 - 4 \geq 0$ and $1 + a + 1 \leq 0$

$\Rightarrow \quad |a| \geq 2,\ a \leq -2$

$\Rightarrow \quad a \geq 2,\ a \leq -2;\ a \leq -2$

$\Rightarrow \quad a \leq -2$

$\Rightarrow \quad a \in (-\infty, -2]$.

Ex-12. If the equation $\tan^4 x - 2\sec^2 x + b^2 = 0$ has at-least one real solution, then find the value of b.

Soln. The given equation is

$\tan^4 x - 2\sec^2 x + b^2 = 0$

$\Rightarrow \quad \tan^4 x - 2(1+\tan^2 x) + b^2 = 0$

$\Rightarrow \quad \tan^4 x - 2\tan^2 x + 1 = 3 - b^2$

$\Rightarrow \quad (\tan^2 x - 1)^2 = 3 - b^2$

$\Rightarrow \quad (3 - b^2) = (\tan^2 x - 1)^2 \geq 0$

$\Rightarrow \quad (3 - b^2) \geq 0$

$\Rightarrow \quad b^2 \leq 3$

$\Rightarrow \quad |b| \leq \sqrt{3}$

Ex-13. If $a, b \in [0, 2\pi]$ and the equation $x^2 + 4 + 3\sin(ax+b) = 2x$ has at-least one solution, then find $(a+b)$.

Soln. The given equation is

$x^2 + 4 + 3\sin(ax+b) = 2x$

$\Rightarrow \quad (x^2 - 2x + 1) + 3 + 3\sin(ax+b) = 0$

$\Rightarrow \quad (x-1)^2 + 3(1 + \sin(ax+b)) = 0$

It is possible only when

$(x-1) = 0, \ (1 + \sin(ax+b)) = 0$

$\Rightarrow \quad x = 1, \ \sin(ax+b) = -1$

$\Rightarrow \quad \sin(a+b) = -1$

$\Rightarrow \quad (a+b) = (4n-1)\frac{\pi}{2}, n \in I$

$\Rightarrow \quad (a+b) = \frac{3\pi}{2}, \frac{7\pi}{2}$.

Ex-14. Find the number of ordered pairs (a, b) satisfying the equations $|x| + |y| = 4$ and $\sin\left(\frac{\pi x^2}{2}\right) = 1$.

Soln. The given equation is

$|x| + |y| = 4$

$\Rightarrow \quad -4 \leq x, y \leq 4$

$\Rightarrow \quad |x| \leq 4, |y| \leq 4$

Also, $\sin\left(\frac{\pi x^2}{2}\right) = 1$

$\Rightarrow \quad \left(\frac{\pi x^2}{2}\right) = (4n+1)\frac{\pi}{2}, n \in I$

$\Rightarrow \quad x^2 = (4n+1)$

$\Rightarrow \quad x^2 = 1$

$\Rightarrow \quad x = \pm 1$

Then $|y| = 4 - 1 = 3$

$y = \pm 3$

Thus, the possible ordered pairs are

$(1,3), (1,-3), (-1,3), (-1,-3)$

Ex-15 Find the number of values of x in $(-2\pi, 2\pi)$ and satisfying $\log_{|\cos x|}|\sin x| + \log_{|\sin x|}|\cos x| = 2$.

Soln. The given equation is

$\log_{|\cos x|}|\sin x| + \log_{|\sin x|}|\cos x| = 2$

It is possible only when

$|\sin x| = |\cos x| \neq 1$

$\Rightarrow \quad |\tan x| = 1$

$\Rightarrow \quad x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, \pm\frac{7\pi}{4}$

Hence, the number of values of x is 8.

Ex-16 The number of solutions of $\tan x + \sec x = 2\cos x$ in $[0, 2\pi)$ is

(a) 2 (b) 3

(c) 0 (d) 1.

[JEE MAIN - 2002]

Soln. Ans. (a)

The given equation is

$\tan x + \sec x = 2\cos x$

$\Rightarrow \quad \frac{\sin x + 1}{\cos x} = 2\cos x$

$\Rightarrow \quad \sin x + 1 = 2\cos^2 x$

$\Rightarrow \quad 1 + \sin x = 2(1 - \sin x)(1 + \sin x)$

$\Rightarrow \quad (1 + \sin x)(1 - 2 + 2\sin x) = 0$

$\Rightarrow \quad (1 + \sin x) = 0, (2\sin x - 1) = 0$

$\Rightarrow \quad \sin x = -1, \sin x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{3\pi}{2}; x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}$

Hence, the solutions are $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$.

Note. No questions asked in between 2003 to 2005.

Ex-17. The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$

(a) 4 (b) 6

(c) 1 (d) 2.

[JEE MAIN - 2006]

Soln. The given equation is

$$2\sin^2 x + 5\sin x - 3 = 0$$

$$\Rightarrow \quad 2\sin^2 x + 6\sin x - \sin x - 3 = 0$$

$$\Rightarrow \quad 2\sin x(\sin x + 3) - (\sin x + 3) = 0$$

$$\Rightarrow \quad (\sin x + 3)(2\sin x - 1) = 0$$

$$\Rightarrow \quad (\sin x + 3) = 0, (2\sin x - 1) = 0$$

$$\Rightarrow \quad (2\sin x - 1) = 0$$

$$\Rightarrow \quad \sin x = \frac{1}{2}$$

$$\Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}$$

Hence, the number of solutions is 4.

Note. No questions asked in between 2007 to 2015.

Ex-18. Find all the angles θ between π and $-\pi$ that satisfy the equation

$5\cos(2\theta) + 2\cos^2\left(\frac{\theta}{2}\right) + 1 = 0$. [Roorkee – 1984]

Soln. The given equation is

$$5\cos(2\theta) + 2\cos^2\left(\frac{\theta}{2}\right) + 1 = 0$$

$$\Rightarrow \quad 5\left(2\cos^2\theta - 1\right) + (1 + \cos\theta) + 1 = 0$$

$$\Rightarrow \quad 10\cos^2\theta + \cos\theta - 3 = 0$$

$$\Rightarrow \quad 10\cos^2\theta + 6\cos\theta - 5\cos\theta - 3 = 0$$

$$\Rightarrow \quad 2\cos\theta(5\cos\theta + 3) - 1(5\cos\theta + 3) = 0$$

$$\Rightarrow \quad (2\cos\theta - 1)(5\cos\theta + 3) = 0$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2}, \cos\theta = -\frac{3}{5}$$

When $\cos\theta = \frac{1}{2}$

Then $\theta = -\frac{\pi}{3}, \frac{\pi}{3}$

When $\cos\theta = -\frac{3}{5}$

Then $\theta = \frac{\pi}{2} + \cos^{-1}\left(-\frac{3}{5}\right)$

and $-\frac{\pi}{2} - \cos^{-1}\left(-\frac{3}{5}\right)$

Hence, the solutions are

$$\theta = \pm\frac{\pi}{3}, \frac{\pi}{2} + \cos^{-1}\left(-\frac{3}{5}\right), -\frac{\pi}{2} - \cos^{-1}\left(-\frac{3}{5}\right).$$

Note. No questions asked in between 1985 to 1986.

Ex-19. Find the general solution to the following equation

$2(\sin x - \cos 2x) - \sin 2x(1 + 2\sin x) + 2\cos x = 0$.

[Roorkee – 1987]

Soln. The given equation is

$$2(\sin x - \cos 2x) - \sin 2x(1 + 2\sin x) + 2\cos x = 0$$

$$\Rightarrow \quad 2\sin x - 2\cos 2x - 2\sin x\cos x - \sin 2x + 2\cos x = 0$$

$$\Rightarrow \quad 2\sin x(1 - \cos x) + 4\cos^3 x - 4\cos^2 x - 2\cos x + 2 = 0$$

$$\Rightarrow \quad 2\sin x(1 - \cos x) + 4\cos^2 x(\cos x - 1) - 2(\cos x - 1) = 0$$

$$\Rightarrow \quad (\cos x - 1)\left(4\cos^2 x - 2 - 2\sin x\right) = 0$$

$$\Rightarrow \quad (\cos x - 1)\left(2\sin^2 x + \sin x - 1\right) = 0$$

$$\Rightarrow \quad (\cos x - 1)(\sin x + 1)(2\sin x - 1) = 0$$

$$\Rightarrow \quad \cos x = 1, \sin x = -1, \sin x = \frac{1}{2}$$

$$\Rightarrow \quad x = 2n\pi,\ (4n - 1)\frac{\pi}{2},\ n\pi + (-1)^n\left(\frac{\pi}{6}\right)$$

Ex-20. Solve for x and y;

$x\cos^3 y + 3x\cos y\sin^2 y = 14$,

$x\sin^3 y + 3x\cos^2 y\sin y = 13$.

[Roorkee – 1988]

Soln. The given equations are

$$x\cos^3 y + 3x\cos y\sin^2 y = 14,$$

$$x\sin^3 y + 3x\cos^2 y\sin y = 13$$

Adding and subtracting, we get,

$$x(\cos y + \sin y)^3 = 27 \quad \text{.........(i)}$$

$$x(\cos y - \sin y)^3 = 1 \quad \text{........(ii)}$$

Dividing (ii) by (i), we get,

$$\frac{(\cos y + \sin y)^3}{(\cos y - \sin y)^3} = 27$$

$$\Rightarrow \quad \frac{(\cos y + \sin y)}{(\cos y - \sin y)} = 3$$

$$\Rightarrow \quad \frac{1 + \tan y}{1 - \tan y} = 3$$

$$\Rightarrow \quad \tan y = \frac{1}{2}$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{1}{2}\right)$$

Put the value of $y=\tan^{-1}\left(\frac{1}{2}\right)$ into (ii), we get

$$x\left(\frac{2}{\sqrt{5}}-\frac{1}{\sqrt{5}}\right)^3=1$$

$$\Rightarrow \quad x\left(\frac{1}{\sqrt{5}}\right)^3=1$$

$$\Rightarrow \quad x=5\sqrt{5}$$

Hence, the solutions are

$$x=5\sqrt{5} \text{ and } y=\tan^{-1}\left(\frac{1}{2}\right)$$

Ex-21. Solve for x; $4\sin^4 x+\cos^4 x=1$ [Roorkee – 1989]

Soln. The given equation is $4\sin^4 x+\cos^4 x=1$

$$\Rightarrow 4\sin^4 x+\cos^4 x=\left(\sin^2 x+\cos^2 x\right)^2$$

$$\Rightarrow 4\sin^4 x+\cos^4 x=\sin^4 x+\cos^4 x+2\sin^2 x\cos^2 x$$

$$\Rightarrow 3\sin^4 x-2\sin^2 x\cos^2 x=0$$

$$\Rightarrow 3\sin^4 x-2\sin^2 x+2\sin^4 x=0$$

$$\Rightarrow 5\sin^4 x-2\sin^2 x=0$$

$$\Rightarrow \sin^2 x\left(5\sin^2 x-2\right)=0$$

$$\Rightarrow \sin^2 x=0,\ \left(5\sin^2 x-2\right)=0$$

$$\Rightarrow \sin x=0,\ \sin^2 x=\frac{2}{5}$$

$$\Rightarrow x=n\pi,\ n\pi\pm\alpha, \text{ where } \alpha=\sin^{-1}\left(\sqrt{\frac{2}{5}}\right)$$

Ex-22. Find all the values of 'a' for which the equation $\sin^4 x+\cos^4 x+\sin 2x+a=0$ is valid.
Also find the general solution of the equation.

[Roorkee Main – 1990]

Soln. The given equation is
$\sin^4 x+\cos^4 x+\sin 2x+a=0$

$$\Rightarrow \quad 1-2\sin^2 x\cos^2 x+\sin 2x+a=0$$

$$\Rightarrow \quad 1-\frac{1}{2}\left(4\sin^2 x\cos^2 x\right)+\sin 2x+a=0$$

$$\Rightarrow \quad 1-\frac{1}{2}(\sin 2x)^2+\sin 2x+a=0$$

$$\Rightarrow \quad 2-(\sin 2x)^2+2\sin 2x+2a=0$$

$$\Rightarrow \quad (\sin 2x)^2-2\sin 2x-2=2a$$

$$\Rightarrow \quad (\sin 2x-1)^2-3=2a$$

$$\Rightarrow \quad (\sin 2x-1)^2=2a+3$$

$$\Rightarrow \quad 2a+3=(\sin 2x-1)^2\ge 0$$

$$\Rightarrow \quad a\ge -\frac{3}{2}$$

Also, $(\sin 2x-1)^2\ge 0$

$$\Rightarrow \quad (\sin 2x-1)\ge 0$$

$$\Rightarrow \quad \sin 2x\ge 1$$

$$\Rightarrow \quad \sin 2x=1$$

$$\Rightarrow \quad 2x=n\pi+(-1)^n\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad x=\frac{n\pi}{2}+(-1)^n\left(\frac{\pi}{4}\right),\ n\in I$$

Note. No questions asked in 1991.

Ex-23. Find the general solution of the equation
$\left(\sqrt{3}-1\right)\sin\theta+\left(\sqrt{3}+1\right)\cos\theta=2$

[Roorkee – 1992]

Soln. The given equation is
$\left(\sqrt{3}-1\right)\sin\theta+\left(\sqrt{3}+1\right)\cos\theta=2$

$$\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\sin\theta+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\cos\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\theta\cos(75°)+\cos\theta\sin(75°)=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\left(\theta+\frac{5\pi}{12}\right)=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\left(\theta+\frac{5\pi}{12}\right)=\frac{1}{\sqrt{2}}=\sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad \left(\theta+\frac{5\pi}{12}\right)=n\pi+(-1)^n\left(\frac{\pi}{4}\right),\ n\in I$$

$$\Rightarrow \quad \theta=\left(n\pi+(-1)^n\left(\frac{\pi}{4}\right)-\frac{5\pi}{12}\right),\ n\in I$$

Note. No questions asked in 1993.

Ex-24. Solve for θ $\sec\theta-\operatorname{cosec}\theta=\frac{4}{3}$

[Roorkee Main – 1994]

Soln. The given equation is $\sec\theta-\operatorname{cosec}\theta=\frac{4}{3}$

$$3(\sin\theta-\cos\theta)=4\sin\theta\cos\theta \qquad \text{......(i)}$$

Let $\sin\theta-\cos\theta=t$ '

Then $\sin\theta\cos\theta=\frac{1-t^2}{2}$

Equation (i) reduces to

$\Rightarrow\ 3t = 4\times\left(\frac{1-t^2}{2}\right) = 2\left(1-t^2\right)$

$\Rightarrow\ 2t^2+3t-2=0$

$\Rightarrow\ (2t-1)(t+2)=0$

$\Rightarrow\ t=\frac{1}{2},-2$

when $t=-2$, $(\sin\theta-\cos\theta)=-2$

It is not possible.

when $t=\frac{1}{2}$, $\sin\theta-\cos\theta=\frac{1}{2}$

$\Rightarrow\ \sin\left(\theta-\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$

$\Rightarrow\ \sin\left(\theta-\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}=\sin\alpha$

$\Rightarrow\ \left(\theta-\frac{\pi}{4}\right)=n\pi+(-1)^n\alpha$, where $\alpha=\sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

$\Rightarrow\ \theta=\left(n\pi+(-1)^n\alpha+\frac{\pi}{4}\right), n\in I$

Note. No questions asked in 1995.

Ex-25. If $32\tan^8\theta=2\cos^2\alpha-3\cos\alpha$ and $3\cos 2\theta=1$, then find the general values of α.

[Roorkee Main – 1996]

Soln. Given $3\cos 2\theta=1$

$\Rightarrow\ \cos 2\theta=\frac{1}{3}$

$\Rightarrow\ \frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1}{3}$

$\Rightarrow\ 1+\tan^2\theta=3-3\tan^2\theta$

$\Rightarrow\ 4\tan^2\theta=2$

$\Rightarrow\ \tan^2\theta=\frac{1}{2}$

Also, it is given that,

$\Rightarrow\ 32\tan^8\theta=2\cos^2\alpha-3\cos\alpha$

$\Rightarrow\ 32\left(\frac{1}{2}\right)^4=2\cos^2\alpha-3\cos\alpha$

$\Rightarrow\ 2\cos^2\alpha-3\cos\alpha=2$

$\Rightarrow\ 2\cos^2\alpha-3\cos\alpha-2=0$

$\Rightarrow\ 2\cos^2\alpha-4\cos\alpha+\cos\alpha-2=0$

$\Rightarrow\ 2\cos\alpha(\cos\alpha-2)+1(\cos\alpha-2)=0$

$\Rightarrow\ (2\cos\alpha+1)(\cos\alpha-2)=0$

$\Rightarrow\ (2\cos\alpha+1)=0,\ (\cos\alpha-2)=0$

$\Rightarrow\ \cos\alpha=-\frac{1}{2}, 2$

$\Rightarrow\ \cos\alpha=2$ is not possible.

Also, when $\cos\alpha=-\frac{1}{2}$

$\Rightarrow\ x=2n\pi\pm\frac{2\pi}{3}$

Note. No questions asked in 1997.

Ex-26. Find the general values of x and y and satisfying the equations 5 sin x cos y = 1, 4 tan x = tan y.

[Roorkee Main – 1998]

Soln. Given equations are

5 sin x cos y = 1 and 4 tan x = tan y.

$\Rightarrow$ 5 sin x cos y = 1 ....(i)

and $4\sin x\cos y=\cos x\sin y$ ....(ii)

Dividing (i) by (ii), we get,

$\cos x\sin y=\frac{4}{5}$ ....(iii)

Adding (i) and (iii), we get,

$\sin(x+y)=1$

$\Rightarrow\ (x+y)=(4n+1)\frac{\pi}{2}, n\in I$ ....(iv)

Subtracting (iii) from (i), we get,

$\sin(x-y)=-\frac{3}{5}=\sin\alpha$

$\Rightarrow\ (x-y)=m\pi+(-1)^n\alpha, m\in I$ ....(v)

From (iv) and (v), we get,

$2x=(2n+m)\pi+\frac{\pi}{2}+(-1)^m\alpha$

$\Rightarrow\ x=(2n+m)\frac{\pi}{2}+\frac{\pi}{4}+(-1)^m\frac{\alpha}{2},\ \alpha=\sin^{-1}\left(-\frac{3}{5}\right)$

and

$2y=(2n-m)\pi+\frac{\pi}{2}-(-1)^m\alpha,\ \alpha=\sin^{-1}\left(-\frac{3}{5}\right)$

$\Rightarrow\ y=(2n-m)\frac{\pi}{2}+\frac{\pi}{4}-(-1)^m\frac{\alpha}{2}$

Note. No questions asked in 1999.

Ex-27. Find the smallest positive value of x and y satisfying $(x-y)=\frac{\pi}{4}$, cot x + cot y = 2.

[Roorkee Main – 2000]

Soln. Given equations are $(x-y)=\frac{\pi}{4}$,

and $\cot x + \cot y = 2$

$\Rightarrow \cos x\sin y+\sin x\cos y=2\sin x\sin y$

$\Rightarrow \sin(x+y)=\cos(x-y)-\cos(x+y)$

$\Rightarrow \sin(x+y)+\cos(x+y)=\cos(x-y)$

$\Rightarrow \frac{1}{\sqrt{2}}\sin(x+y)+\frac{1}{\sqrt{2}}\cos(x+y)=\frac{1}{\sqrt{2}}\cos(x-y)$

$\Rightarrow \frac{1}{\sqrt{2}}\sin(x+y)+\frac{1}{\sqrt{2}}\cos(x+y)=\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}}$

$\Rightarrow \sin\left(x+y+\frac{\pi}{4}\right)=\frac{1}{2}$

$\Rightarrow \left(x+y+\frac{\pi}{4}\right)=\frac{5\pi}{6}$

$\Rightarrow (x+y)=\frac{5\pi}{6}-\frac{\pi}{4}=\frac{7\pi}{12}$

Also, $(x-y)=\frac{\pi}{4}$

Thus, $x=\frac{5\pi}{12}$ & $y=\frac{\pi}{6}$

Ex-28. Solve the following equations for x and y;

$5^{\left(\operatorname{cosec}^2 x-\sec^2 y\right)}=1$

$2^{\left(2\operatorname{cosec} x+\sqrt{3}|\sec y|\right)}=64$ [Roorkee Main – 2001]

Soln. The given equations are

$5^{\left(\operatorname{cosec}^2 x-\sec^2 y\right)}=1$ ....(i)

$2^{\left(2\operatorname{cosec} x+\sqrt{3}|\sec y|\right)}=64$ ....(ii)

From (i), we get,

$\operatorname{cosec}^2 x-3\sec^2 y=0$

$\Rightarrow \operatorname{cosec}^2 x=3\sec^2 y$

$\Rightarrow \operatorname{cosec} x=\sqrt{3}|\sec y|$ ....(iii)

Also, from (ii), we get,

$2^{\left(2\operatorname{cosec} x+\sqrt{3}|\sec y|\right)}=64=2^6$

$\Rightarrow 2\operatorname{cosec} x+\sqrt{3}|\sec y|=6$

$\Rightarrow 2\operatorname{cosec} x+\operatorname{cosec} x=6$, from (iii)

$\Rightarrow 3\operatorname{cosec} x=6$

$\Rightarrow \operatorname{cosec} x=2$

$\Rightarrow \sin x=\frac{1}{2}$

$\Rightarrow x=n\pi+(-1)^n\left(\frac{\pi}{6}\right),\ n\in I$

Again, $|\sec y|=\frac{2}{\sqrt{3}}$

$\Rightarrow |\cos y|=\frac{\sqrt{3}}{2}$

$\Rightarrow \cos^2 y=\frac{3}{4}=\cos^2\left(\frac{\pi}{6}\right)$

$\Rightarrow y=n\pi\pm\left(\frac{\pi}{6}\right),\ n\in I$

Hence, the solutions are

$$\begin{cases} x=n\pi+(-1)^n\left(\frac{\pi}{6}\right), n\in I \\ y=n\pi\pm\left(\frac{\pi}{6}\right),\ n\in I \end{cases}$$

LEVEL I
(QUESTIONS BASED ON FUNDAMENTALS)

1. Solve: $7\cos^2\theta+3\sin^2\theta=4$
2. Solve: $\sin 2\theta+\sin 4\theta+\sin 6\theta=0$
3. Solve: $\tan^2\theta+(1-\sqrt{3})\tan\theta-\sqrt{3}=0$
4. Solve: $\tan\theta+\tan 2\theta+\tan\theta\tan 2\theta=1$
5. Solve: $3\tan(\theta-15°)=\tan(\theta+15°)$
6. Solve: $\tan\theta+\tan 2\theta+\tan 3\theta=0$
7. Solve: $4\sin\theta\ \sin 2\theta\ \sin 4\theta=\sin 3\theta$
8. Solve: $\sqrt{2}\sec\theta+\tan\theta=1$
9. Solve: $\sin(2013)\theta+\cos(2013)\theta=2$
10. Solve: $\cos\theta+\sqrt{3}\sin\theta=2\cos 2\theta$
11. Solve: $x+y=\frac{2\pi}{3}$ and $\cos x+\cos y=\frac{3}{2}$
12. Solve: $x+y=\frac{\pi}{4}$ and $\tan x+\tan y=1$
13. Solve: $r\sin\theta=3$ and $r=4(1+\sin\theta)$, $0\le\theta\le 2\pi$
14. Solve: $\sin x+\sin y=1$, $\cos 2x-\cos 2y=1$
15. If A and B are acute +ve angles satisfying the equations $3\sin^2 A+2\sin^2 B=1$ and $3\sin 2A=2\sin 2B$, then find $(A+2B)$.
16. If $\tan(A-B)=1$ and $\sec(A+B)=\frac{2}{\sqrt{3}}$, then find the smallest +ve values of A and B and their most general values.

17. If $\sin A = \sin B$ and $\cos A = \cos B$, then find the values of A in terms of B.
18. Solve: $4\sin^4 x + \cos^4 x = 1$
19. Solve: $4\cos^2 x \sin x - 2\sin^2 x = 2\sin x$
20. Solve: $\sin^6 x + \cos^6 x = \frac{7}{16}$
21. Solve: $\sin 7x + \sin 4x + \sin x = 0,\ 0 \le x \le \frac{\pi}{2}$
22. Solve: $\cos 3x + \cos 2x = \sin\left(\frac{3x}{2}\right) + \sin\left(\frac{x}{2}\right),\ 0 \le x \le 2\pi$
23. Solve: $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$
24. Solve: $\cos 2x + \cos 4x = 2\cos x$
25. Solve: $\sin 2x + \sin x + \cos 2x + \cos x + 1 = 0$
26. Solve: $\cos x \cos 2x \cos 4x = \frac{1}{4}$

    $0 \le x \le \pi$
27. Solve: $\sin 3\alpha = 4\sin\alpha\ .\sin(x+\alpha).\sin(x-\alpha)$
28. Solve: $\sin 2x \sin 4x + \cos 2x = \cos 6x$
29. Solve: $\sin 3x.\cos x + \sin^2 x \cos^2 x + \sin x.\cos^3 x = 1, 0 \le x \le 2\pi$
30. If $\theta_1, \theta_2, \theta_3, \theta_4$ be the four roots of the equation $\sin(\theta + \alpha) = k\sin 2\theta$, no two of which differ by a multiple of 2π, then prove that

    $\theta_1 + \theta_2 + \theta_3 + \theta_4 = (2n+1)\frac{\pi}{4}, n \in Z$

LEVEL II
(MIXED PROBLESMS)

Solve the following equations and tick the correct one.

1. $\sin^2\theta - \cos\theta = \frac{1}{2},\ 0 \le \theta \le 2\pi$

   (a) $\frac{2\pi}{3}, \frac{\pi}{3}$ (b) $\frac{\pi}{3}, \frac{5\pi}{3}$

   (c) $-\frac{\pi}{3}, \frac{2\pi}{3}$ (d) $\frac{2\pi}{3}, \frac{5\pi}{3}$

2. If $3\tan^2\theta - 2\sin\theta = 0$, then θ is

   (a) $n\pi$ (b) $n\pi + (-1)^n \frac{\pi}{6}$

   (c) $n\pi - (-1)^n \frac{\pi}{6}$ (d) $n\pi + \frac{\pi}{3}$

3. If $\tan^2 x + (1-\sqrt{3})\tan x - \sqrt{3} = 0$, then x is

   (a) $n\pi + \frac{\pi}{3}$ (b) $n\pi - \frac{\pi}{3}$ (c) $n\pi + \frac{\pi}{4}$ (d) $n\pi - \frac{\pi}{4}$

4. If $\tan^2\theta + \cot^2\theta = 2$, then θ is

   (a) $n\pi + \frac{\pi}{6}$ (b) $n\pi - \frac{\pi}{6}$ (c) $n\pi + \frac{\pi}{4}$ (d) $n\pi - \frac{\pi}{4}$

5. If $\tan\theta + \cot\theta = 2$, then θ is

   (a) $n\pi + \frac{\pi}{4}$ (b) $n\pi - \frac{\pi}{4}$ (c) (d) $n\pi - \frac{\pi}{3}$.

6. The set of values of x for which $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$ is

   (a) ϕ (b) $\frac{\pi}{4}$ (c) $n\pi + \frac{\pi}{3}$ (d) $2n\pi + \frac{\pi}{4}$

7. If $\sin 5x + \sin 3x + \sin x = 0$, then the value of x other than zero, lying between $0 \le x \le \frac{\pi}{2}$

   (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{12}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

8. If α and β are acute positive angles satisfying the equation $3\sin^2\alpha + 2\sin^2\beta = 1$ and $3\sin 2\alpha - 2\sin 2\beta = 0$, then $\alpha + 2\beta$ is

   (a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

9. If $2\sin^2 x + \sin^2 2x = 2$, $-\pi < x < \pi$, then x is

   (a) $\pm\frac{\pi}{2}$ (b) $\pm\frac{\pi}{4}$ (c) $\pm\frac{3\pi}{4}$ (d) None

10. The real roots of the equation $\cos^7 x + \sin^4 x = 1$ in $(-\pi, \pi)$ are

    (a) $-\frac{\pi}{2}, 0$ (b) $-\frac{\pi}{2}, 0, \frac{\pi}{2}$

    (c) $0, \frac{\pi}{2}$ (d) $0, \frac{\pi}{4}, \frac{\pi}{2}$

11. The general solution of $\cos^5 x - \sin^5 x - 1 = 0$ is

    (a) $n\pi$ (b) $2n\pi$

    (c) $n\pi + \frac{\pi}{2}$ (d) $2n\pi + \frac{\pi}{2}$

12. If $4\sin^4 x + \cos^4 x = 1$, then x is

    (a) $n\pi$ (b) $n\pi \pm \sin^{-1}\sqrt{\frac{2}{5}}$

    (c) $\frac{2n\pi}{3}$ (d) $2n\pi \pm \frac{\pi}{4}$

13. The number of points of intersection of $2y = 1$ and $y = \cos x$ in $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ is
(a) 1 (b) 2
(c) 3 (d) 4

14. The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$ is
(a) 6 (b) 1 (c) 2 (d) 4

15. The number of values of x in $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is
(a) 0 (b) 5 (c) 6 (d) 10

16. The number of solution of the equation
$|\cot x| = \cot x + \frac{1}{\sin x}$, $0 < x < 2\pi$, is
(a) 1 (b) 2 (c) 3 (d) 4

17. The number of solution of $|\cos x| = \sin x$ such that $0 < x < 2\pi$ is
(a) 2 (b) 4
(c) 8 (d) None

18. The number of solution of the equation $\tan x . \tan 4x = 1$, $0 < x < \pi$, is
(a) 1 (b) 2
(c) 5 (d) 8

19. The number of solution of the equation
$12\cos^3 x - 7\cos^2 x + 4\cos x - 9 = 0$, is
(a) 0 (b) 2
(c) infinity (d) None

20. The sum of all solution of the equation
$\cos\theta . \cos\left(\frac{\pi}{3} + \theta\right) . \cos\left(\frac{\pi}{3} - \theta\right) = \frac{1}{4}$ is
(a) 15π (b) 30π (c) $\frac{100\pi}{3}$ (d) None

21. The number of solution of $16^{\sin^2 x} + 16^{\cos^2 x} = 10$, $0 \le x \le 2\pi$, is
(a) 2 (b) 4 (c) 6 (d) 8.

22. The smallest positive value of x such that
$\tan(x + 20°) = \tan(x + 10°) . \tan x . \tan(x - 10°)$, is
(a) 30° (b) 45° (c) 60° (d) 75°

23. The maximum value of
$\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$ in $\left(0, \frac{\pi}{2}\right)$ is attained at
(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

24. The minimum value of $2^{\sin x} + 2^{\cos x}$ is
(a) 1 (b) $2^{1-\frac{1}{\sqrt{2}}}$
(c) $2^{-\frac{1}{\sqrt{2}}}$ (d) $\left(2 - \frac{1}{\sqrt{2}}\right)$

25. If $\cos p\theta + \cos q\theta = 0$, then the different values of θ are in A.P., whose common difference is
(a) $\frac{\pi}{p+q}$ (b) $\frac{\pi}{p-q}$
(c) $\frac{2\pi}{p \pm q}$ (d) $\frac{3\pi}{p \pm q}$

26. If $\tan 2x . \tan x = 1$, then x is
(a) $\frac{\pi}{3}$ (b) $(6n \pm 1)\frac{\pi}{6}$
(c) $(4n \pm 1)\frac{\pi}{6}$ (d) $(2n \pm 1)\frac{\pi}{6}$

27. The maximum value of $5 \sin\theta + 3\sin(\theta - \alpha)$ is 7, then the set of all possible values of α is
(a) $\left(2n\pi \pm \frac{\pi}{3}\right)$ (b) $\left(2n\pi \pm \frac{2\pi}{3}\right)$
(c) $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ (d) None

28. If $\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\sin\theta\right)$, then $\sin\theta + \cos\theta$ is
(a) $2n - 1$ (b) $2n + 1$ (c) $2n$ (d) n

29. If $\sin\left(\frac{\pi}{4}\cot\theta\right) = \cos\left(\frac{\pi}{4}\tan\theta\right)$, then θ is
(a) $\left(n\pi + \frac{\pi}{4}\right)$ (b) $\left(2n\pi \pm \frac{\pi}{4}\right)$
(c) $\left(n\pi - \frac{\pi}{4}\right)$ (d) $\left(2n\pi \pm \frac{\pi}{6}\right)$

30. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then the values of $\cos\left(\theta - \frac{\pi}{4}\right)$ is (are)
(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\pm\frac{1}{2\sqrt{2}}$ (d) None

31. If $3\tan(\theta - 15°) = \tan(\theta + 15°)$, then θ is
(a) $\left(n\pi + \frac{\pi}{4}\right)$ (b) $\left(n\pi + \frac{\pi}{8}\right)$
(c) $\left(n\pi + \frac{\pi}{3}\right)$ (d) None

32. If $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 3$, then θ is

(a) $(2n+1)\frac{\pi}{12}$ (b) $\left(n\pi \pm \frac{\pi}{3}\right)$

(c) $(4n+1)\frac{\pi}{12}$ (d) None

33. The equation a $\sin 2x + \cos 2x = 2a - 7$ posses a solution if

(a) $a > 6$ (b) $2 \le a \le 6$

(c) $a > 2$ (d) None

34. The number of all possible triplets (a_1, a_2, a_3) such that $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$ for all x, the number of possible 5-tuplets is

(a) 0 (b) 1 (c) 2 (d) None

35. If $a_1 + a_2 \sin x + a_3 \cos x + a_4 \sin 2x + a_5 \cos 2x = 0$ holds for all x, then the number of possible 5-tuplets is

(a) 0 (b) 1 (c) 2 (d) infinity

36. The number of solution of the equation $1 + \sin x.\sin^2 \frac{x}{2} = 0$ in $[-\pi, \pi]$ is

(a) 0 (b) 1 (c) 2 (d) 3

37. The solution of $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$ is solvable for

(a) $-\frac{1}{2} \le \alpha \le \frac{1}{2}$ (b) $-3 \le \alpha \le 1$

(c) $-\frac{3}{2} \le \alpha \le \frac{1}{2}$ (d) $-1 \le \alpha \le 1$

38. The equation $\sin^4 x - 2\cos^2 x + a^2 = 0$ is solvable for

(a) $-\sqrt{3} \le a \le \sqrt{3}$ (b) $-\sqrt{2} \le a \le \sqrt{2}$

(c) $-1 \le a \le 1$ (d) None

39. The number of pairs (x, y) satisfying the equations $\sin x + \sin y = \sin(x + y)$ and $|x| + |y| = 1$, is

(a) 2 (b) 4 (c) 6 (d) infinity

40. The value of 'a' for which the equation $4\text{cosec}^2[\pi(a + x)] + a^2 - 4a = 0$, has a real solution, if

(a) $a = 1$ (b) $a = 2$ (c) $a = 3$ (d) None

41. If $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$, $x \in [0, \pi]$, then

(a) $x = \frac{\pi}{4}, y = 1$ (b) $y = 0$

(c) $y = 2$ (d) $x = \frac{3\pi}{4}$

42. $|\tan x + \sec x| = |\tan x| + |\sec x|$, $x \in [0, 2\pi]$, if x belongs to that interval

(a) $[0, \pi]$ (b) $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

(c) $\left[0, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$ (d) $(\pi, 2\pi]$.

43. The number of solutions of $\sum_{r=1}^{5} \cos(rx) = 5$ in the interval $[0, 2\pi]$ is

(a) 0 (b) 1 (c) 5 (d) 10

44. If $f(x) = \max\{\tan x, \cot x\}$. The number of roots of the equation $f(x) = \frac{1}{2 + \sqrt{3}}$ in $(0, 2\pi)$ is

(a) 0 (b) 2 (c) 4 (d) ∞

45. If $\sin x + \cos x + \tan x + \cot x + \sec x + \text{cosec } x = 7$ and $\sin 2x = a - b\sqrt{c}$, then $a - b + 2c$ is

(a) 0 (b) 14 (c) 2 (d) $\frac{3}{2}$

46. If $\sin 4x + \cos 4x + 2 = 4 \sin x \cos y$ and $0 < x, y < \frac{\pi}{2}$, then $\sin x + \cos y$ is

(a) -2 (b) 0 (c) 2 (d) $\frac{3}{2}$

47. The equation $\cos 4x - (\lambda + 2)\cos 2x - (\lambda + 3) = 0$ possesses a solution if

(a) $\lambda > -3$ (b) $\lambda < -2$

(c) $-3 < \lambda < -2$ (d) $\lambda \in z+$

48. If $0 < \theta < 2\pi$ and $2\sin^2\theta - 5\sin\theta + 2 > 0$, then the range of θ is

(a) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$

(b) $\left(0, \frac{5\pi}{6}\right) \cup (\pi, 2\pi)$

(c) $\left(0, \frac{\pi}{6}\right) \cup (\pi, 2\pi)$

(d) None.

49. The number of values of x for which $\sin 2x + \cos 4x = 2$ is

(a) 0 (b) 1 (c) 2 (d) ∞

50. The number of solutions of the equation $x^3 + x^2 + 4x + 2 \sin x = 0$ in $0 < x < 2\pi$ is

(a) 0 (b) 1 (c) 2 (d) 4

51. The number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is

(a) 0 (b) 1 (c) 2 (d) 3

52. The number of solutions of the equation $2(\sin^4 2x + \cos^4 2x) + 3 \sin 2x \cos 2x = 0$ is

(a) 0 (b) 1 (c) 2 (d) 3

53. $\cos 2x + a \sin x = 2a - 7$ possesses a solution for
(a) all a (b) $a > 6$
(c) $a < 2$ (d) $a \in [2, 6]$

54. If $0 < x < 2\pi$ and $81^{\sin^2 x} + 81^{\cos^2 x} = 30$, then x is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) π (d) $\frac{\pi}{4}$

55. If $1 + \sin\theta + \sin^2\theta + ...$ to $\infty = 4 + 2\sqrt{3}$,
$0 < \theta < \pi$, $\theta \neq \frac{\pi}{2}$, then
(a) $\theta = \frac{\pi}{6}$ (b) $\theta = \frac{\pi}{3}$
(c) $\theta = \frac{\pi}{3}$ or $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$ or $\frac{2\pi}{3}$

56. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, then the value of $\cos\left(\theta - \frac{\pi}{4}\right)$ is
(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{2\sqrt{2}}$ (d) None

57. The most general values of x for which
$\sin x + \cos x = \min_{a \in R} \{1, a^2 - 4a + 6\}$ are given by
(a) $2n\pi, n \in N$
(b) $2n\pi + \frac{\pi}{2}, n \in N$
(c) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}, n \in N$
(d) None

58. If $x \in [0, 2\pi]$ and $\sin x + \sin y = 2$ then the value of $x + y$ is
(a) π (b) $\frac{\pi}{2}$ (c) 3π (d) None

59. The number of roots of the equation
$x + 2\tan x = \frac{\pi}{2}$ in the interval $[0, 2\pi]$ is
(a) 1 (b) 2 (c) 3 (d) ∞

60. The number of solutions of the equation $\cos(\pi\sqrt{x-4})\cos(\pi\sqrt{x}) = 1$ is
(a) None (b) 1 (c) 2 (d) >2

61. The number of solutions of the equation
$\sin\left(\frac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}\,x + 4$
(a) forms an empty set (b) only one
(c) is only two (d) is greater then two

62. Number of real roots of the equation
$\sec\theta + \operatorname{cosec}\theta = \sqrt{15}$ lying between 0 and 2π is
(a) 8 (b) 4 (c) 2 (d) 0

63. The general solution of the equation
$\sin 100x - \cos 100x = 1$, is
(a) $2n\pi + \frac{\pi}{3}, n \in z$ (b) $n\pi + \frac{\pi}{2}, n \in z$
(c) $n\pi + \frac{\pi}{4}\pi, n \in z$ (d) $2n\pi\ \frac{\pi}{3}, n \in z$

64. The number of solution of the equation $2^{\cos x} = |\sin x|$ in $[-2\pi, 2\pi]$ is
(a) 1 (b) 2 (c) 3 (d) 4

65. The general solution of the equation
$2^{\cos^2 x} + 1 = 3.2^{-\sin^2 x}$ is
(a) $n\pi . n \in z$
(b) $(n + 1)\pi, n \in z$
(c) $(n - 1)\pi, n \in z$
(d) None

66. If $x \in (0, 1)$, the greatest root of the equation
$\sin 2\pi x = \sqrt{2}\cos\pi x$ is
(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) None

67. The number of solutions of $\tan(5\pi\cos\alpha)$ $= \cot(5\pi\sin\alpha)$ for $\alpha \in (0, 2\pi)$ is
(a) 7 (b) 14 (c) 21 (d) 3

68. The number of solution of the equation
$1 + \sin x . \sin^2\left(\frac{x}{2}\right) = 0$ in $[-\pi, \pi]$ is
(a) 0 (b) 1 (c) 2 (d) 3

69. The number of solution of the equation
$|\cot x| = \cot x + \frac{1}{\sin x}$, $\forall x \in [0, 2\pi]$ is
(a) 0 (b) 1 (c) 2 (d) 3

70. The real roots of the equation
$\cos^7 x + \sin^4 x = 1$ in $(-\pi, \pi)$ are
(a) $-\frac{\pi}{2}, 0$ (b) $-\frac{\pi}{2}, 0, \frac{\pi}{2}$
(c) $\frac{\pi}{2}, 0$ (d) $0, \frac{\pi}{4}, \frac{\pi}{2}$

LEVEL III
(PROBLEMS FOR JEE ADVANCED EXAM)

Solve the following trigonometric equations:

1. $\cot\left(\frac{x}{2}\right) - \operatorname{cosec}\left(\frac{x}{2}\right) = \cot x$
2. $8\cos x.\cos 2x.\cos 4x = \frac{\sin 6x}{\sin x}$
3. $\frac{\tan x}{\tan 2x} + \frac{\tan 2x}{\tan x} + 2 = 0$
4. Solve: $\cos x \cos(6x) = -1$
5. Solve: $\cos(4x) + \sin(5x) = 2$

6. $\sin 2x + 5\cos x + 5\sin x + 1 = 0$
7. $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$ in the interval $0 \le x \le \pi$
8. $\sin^2 x \tan x + \cos^2 x \cot x - \sin 2x = 1 + \tan x \cot x$
9. $\sin^2 4x + \cos^2 x = 2\sin 4x \cos^4 x$
10. $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x \cos x$
11. $\sin^4 x + \cos^4 x = 2\cos\left(2x + \frac{\pi}{6}\right)\cos\left(2x - \frac{\pi}{6}\right)$
12. $\sin^4 x + \sin^4\left(x + \frac{\pi}{4}\right) = \frac{1}{4}$
13. If $\cos\left(x + \frac{\pi}{3}\right) + \cos x = a$, then find all values of a so that the equation has a real solution.
14. Find the number of roots of $\cos x - x + \frac{1}{2} = 0$ lies in $\left(0, \frac{\pi}{2}\right)$
15. Find the number of integral ordered pairs satisfy the equations $\begin{cases}\cos(xy) = x \\ \tan(xy) = y\end{cases}$
16. Find the number of real solutions of $\sin^{2016} x - \cos^{2016} x = 1$ in $[0, 2\pi]$
17. Find the number of ordered pairs which satisfy the equation $x^2 + 2x\sin(xy) + 1 = 0$ for $y \in [0, 2\pi]$.
18. Find the number of solution of the equation $\sin 5x \cdot \cos 3x = \sin 6x \cdot \cos 2x$ in $[0, \pi]$.
19. Find the number of solution of the equation $\cos 3x \cdot \tan 5x = \sin 7x$ lying in $\left[0, \frac{\pi}{2}\right]$.
20. The angles B and C ($B > C$) of a triangle satisfying the equation $2\tan x - \lambda\left(1 + \tan^2 x\right) = 0$, then find the angle A, if $0 < \lambda < 1$.
21. Determine all values of 'a' for which the equation $\cos^4 x - (a+2)\cos^2 x - (a+3) = 0$ has a solution and find those.
22. Find all the solution of the equation $\sin x + \sin\frac{\pi}{8}\left(\sqrt{(1-\cos x)^2 + \sin^2 x}\right) = 0$ in $\left[\frac{5\pi}{2}, \frac{7\pi}{2}\right]$.
23. If the equation $\sin^4 x - (k+2)\sin^2 x - (k+3) = 0$ has a solution, then find the value of k.
24. Find the number of principal solutions of the equation $4.16^{\sin^2 x} = 2^{6\sin x}$.
25. Find the general solution of $\sec x = 1 + \cos x + \cos^2 x + \cos^3 x + ...$

INTEGER TYPE QUESTIONS

1. Find the number of values of x in $(0, 5\pi)$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$
2. Find the number of integral values of k, for which the equation $2\cos x + 3\sin x = k + 1$ has a solution.
3. Find the number of distinct real roots of $\begin{vmatrix}\sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x\end{vmatrix} = 0$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
4. Find the number of pairs (x, y) satisfying the equations $\sin x + \sin y = \sin(x + y)$ and $|x| + |y| = 1$
5. Find the maximum value of $f(x) = \begin{vmatrix}1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x\end{vmatrix}$
6. Find the number of solutions of $\sin x = \frac{|x|}{10}$
7. Find the number of solutions of $\tan x + \cot x = 2\,\mathrm{cosec}\, x$ in $[-2\pi, 2\pi]$.
8. Find the number of soltions of $\cos x.\cos 2x.\cos 4x = \frac{1}{4}$ in $[0, \pi]$.
9. If $x, y \in [0, 2\pi]$, then find the number of ordered pairs (x, y) satisfying the equation $\sin x \cdot \cos y = 1$.
10. If $x \in [0, 3\pi]$, then find the number of values of x satisfying the equation $|\cot x| = \cot x + \frac{1}{\sin x}$.
11. Find the number of solutions of $\tan x \tan(4x) = 1$, for $0 < x < \pi$
12. Find the number of integral values of n for which the equation $\sin(\sin x + \cos x) = n$ has atleast one solution.
13. Find the number of real solutions of $\sin\{x\} = \cos\{x\}$ in $[0, 2\pi]$
14. Find the number of solutions of $\left(\sqrt{3}+1\right)^{2x} + \left(\sqrt{3}-1\right)^{2x} = 2^{3x}$
15. Find the number of values of y in $[-2\pi, 2\pi]$ for which $|\sin(2x)| + |\cos(2x)| = |\sin(y)|$

LINKED COMPREHENSION TYPE (FOR JEE ADVANCED EXAMS ONLY)

PASSAGE I

An equation is of the form $f(\sin x \pm \cos x, \pm \sin x \cos x) = 0$ can be solved by changing the variable.

Let $\sin x \pm \cos x = t$

$$\Rightarrow \quad \sin^2 x + \cos^2 x \pm 2\sin x \cos x = t^2$$

$$\Rightarrow \quad 1 \pm 2\sin x \cos x = t^2$$

Thus, the given equation is reducible to

$$f\left(t, \frac{t^2-1}{2}\right) = 0$$

On the basis of above information, answer the following questions .

1. If $1 - \sin 2x = \cos x - \sin x$, then x is
   (a) $2n\pi, \left(2n\pi - \frac{\pi}{2}\right), n \in I$
   (b) $2n\pi, \left(n\pi + \frac{\pi}{4}\right), n \in I$
   (c) $\left(2n\pi - \frac{\pi}{2}\right), \left(n\pi + \frac{\pi}{4}\right), n \in I$
   (d) None
2. If $\sin x + \cos x = 1 + \sin x \cos x$, then x is
   (a) $2n\pi, \left(2n\pi + \frac{\pi}{2}\right), n \in I$
   (b) $2n\pi, \left(n\pi + \frac{\pi}{4}\right), n \in I$
   (c) $\left(2n\pi - \frac{\pi}{2}\right), \left(n\pi + \frac{\pi}{4}\right), n \in I$
   (d) None
3. If $\sin^4 x + \cos^4 x = \sin x \cos x$, then x is
   (a) $n\pi, n \in I$ (b) $(6n+1)\frac{\pi}{6}, n \in I$
   (c) $(4n+1)\frac{\pi}{4}, n \in I$ (d) None
4. If $1 + \sin^3 x + \cos^3 x = \frac{3}{2}\sin 2x$, then x is
   (a) $(2n+1)\pi, \left(2n\pi + \frac{\pi}{4}\right), n \in I$
   (b) $(2n+1)\pi, \left(2n\pi - \frac{\pi}{2}\right), n \in I$
   (c) $\left(2n\pi + \frac{\pi}{4}\right), \left(2n\pi + \frac{\pi}{2}\right), n \in I$
   (d) None
5. If $(\sin x + \cos x) = 2\sqrt{2}\sin x \cos x$, then x is
   (a) $\left(2n\pi + \frac{\pi}{4}\right), n \in I$ (b) $\left(2n\pi - \frac{\pi}{4}\right), n \in I$
   (c) $\left(n\pi + \frac{\pi}{4}\right), n \in I$ (d) $\left(n\pi - \frac{\pi}{4}\right), n \in I$

PASSAGE II

α is a root of the equation $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$ β is a root of the equation $3\cos^3 x - 10\cos x + 3 = 0$ and γ is a root of the equation $1 - \sin 2x = \cos x - \sin x$, $0 \le \alpha, \beta, \gamma \le \frac{\pi}{2}$. Then on the basis of above information, answer the following questions.

1. $\cos\alpha + \cos\beta + \cos\gamma$ is equal to
   (a) $\frac{3\sqrt{6} + 2\sqrt{2} + 6}{6\sqrt{2}}$ (b) $\frac{3\sqrt{3} + 8}{6}$
   (c) $\frac{3\sqrt{3} + 2}{6}$ (d) None
2. $\sin\alpha + \sin\beta + \sin\gamma$ is equal to
   (a) $\frac{14 + 3\sqrt{2}}{6\sqrt{2}}$ (b) $\frac{5}{6}$
   (c) $\frac{3 + 4\sqrt{2}}{6}$ (d) $\frac{1 + \sqrt{2}}{2}$
3. $\sin(\alpha - \beta)$ is equal to
   (a) 1 (b) 0
   (c) $\frac{1 - 2\sqrt{6}}{6}$ (d) $\frac{\sqrt{3} - 2\sqrt{2}}{6}$

PASSAGE III

Solutions of equations $a\sin x \pm b\cos x = c$. General value satisfying two equations.
$a\cos\theta \pm b\sin\theta = c$, where θ satisfying two equations.

(a) The equations be first converted to, where $a = r\cos\theta$, $b = r\sin\theta$.
(b) Satisfying two equations. Find the common value of lying between 0 and 2π and then add $2n\pi$.

On the basis of above information, answer the following questions.

1. The number of integral values of k for which the equation $7\cos x + 5\sin x = 2k + 1$ has a solution is
   (a) 4 (b) 8 (c) 10 (d) 12

2. If $\cos 3x+\sin\left(2x-\frac{7\pi}{6}\right)=-2$, then x is

(a) $(6n+1)\frac{\pi}{3}, n\in I$ (b) $(6n-1)\frac{\pi}{3}, n\in I$

(c) $(2n+1)\frac{\pi}{3}, n\in I$ (d) None

3. The value of x such that $-\pi<x<\pi$ and satisfying the equation $8^{1+|\cos x|+\cos^2 x+|\cos x|^3+\ldots\ldots}=4^3$, then x is

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $-\frac{\pi}{3}$ (d) $-\frac{2\pi}{3}$

4. The number of solutions of the equation $e^{\sin x}-e^{-\sin x}-4=0$ is

(a) 1 (b) 2 (c) 4 (d) 0

PASSAGE IV

Suppose equation is $f(x)-g(x)=0$ or $y=f(x)=g(x)$, say, then draw the graphs of $y=f(x)$ and $y=g(x)$.

If graphs of $y = f(x)$ and $y = g(x)$ cuts at one, two, three,………, no points, then number of solutions are one, two, three, ……, zero, respectively.

On the basis of above information, answer the following questions:

1. The number of solutions of $\sin x=\frac{|x|}{10}$ is

(a) 4 (b) 6 (c) 5 (d) None

2. Total number of solutions of the equation $3x+2\tan x=\frac{5\pi}{2}, x\in[0,2\pi]$, is

(a) 1 (b) 2 (c) 3 (d) 4

3. Total number of solutions of $\sin\{x\}=\cos\{x\}$, where { } = F.P.F, in $[0,2\pi]$ is

(a) 3 (b) 5 (c) 7 (d) None.

4. If $1-\sin x=\frac{\sqrt{3}}{2}\left|x-\frac{\pi}{2}\right|+a$ has no solution, when $a\in R^+$, then

(a) $a\in R^+$ (b) $a>\frac{3}{2}+\frac{\pi}{\sqrt{3}}$

(c) $a\in\left(0,\frac{3}{2}+\frac{\pi}{\sqrt{3}}\right)$ (d) $a\in\left(\frac{3}{2},\frac{3}{2}+\frac{\pi}{\sqrt{3}}\right)$.

5. Total number of solutions of $\cos 2x=|\sin x|$, where $-\frac{\pi}{2}<x<\pi$, is

(a) 3 (b) 4 (c) 5 (d) 6

PASSAGE V

Whenever the terms of two sides of the equation are of different nature, then equations are known as non-standard form, some of them are in the form of an ordinary equation but can not be solved by standard procedures.

Non standard problems require high degree of logic, they also require the use of graphs, inverse properties of functions, in-equalities.

On the basis of above information, answer the following questions:

1. The number of solutions of the equation $2\cos\left(\frac{x}{2}\right)=\left(3^x+3^{-x}\right)$ is

(a) 1 (b) 2 (c) 3 (d) None

2. The number of solution of the equation $2\cos^2\left(\frac{x}{2}\right)\sin^2 x=\left(x^2+x^{-2}\right), 0\le x\le\frac{\pi}{2}$ is

(a) 1 (b) > 1 (c) 0 (d) None

3. The number of real solutions of the equation $\sin\left(e^x\right)=5^x+5^{-x}$ is

(a) 0 (b) 1
(c) 2 (d) infinitely many

4. If $0\le x\le 2\pi$ and $2^{\text{cosec}^2 x}\times\sqrt{\frac{y^2}{2}-y+1}\le\sqrt{2}$, then the number of ordered pairs of (x, y) is

(a) 1 (b) 2
(c) 3 (d) infinitely many

5. The number of solutions of the equation $\sin x = x^2 + x + 1$ is

(a) 0 (b) 1 (c) 2 (d) None

MATCH MATRIX

1. Match the following columns:

Column - I	Column - II
(A) The equation $\sin x+\cos x=2$ has	(P) One solution
(B) The equation $\sqrt{3}\sin x+\cos x=4$ has	(Q) Two solution
(C) The equation $3\sin x+4\cos x=6$ has	(R) Three solution
(D) The equation $\sin x\cos x=2$ has	(S) No solution

2. Match the following columns:

Column - I	Column - II
(A) If $\begin{vmatrix}\cos(A+B) & -\sin(A+B) & \cos(2B)\\ \sin A & \cos A & \sin B\\ -\cos A & \sin A & \cos B\end{vmatrix}=0$ then B is	(P) $n\pi$

(B) If $\begin{vmatrix} \cos\theta & \sin\theta & \cos\theta \\ -\sin\theta & \cos\theta & \sin\theta \\ -\cos\theta & -\sin\theta & \cos\theta \end{vmatrix} = 0$

then θ is — (Q) $(2n+1)\frac{\pi}{2}$

(C) If $\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \sin^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$,

then θ is — (R) $(2n-1)\frac{\pi}{2}$

(S) $\frac{7\pi}{24}$

3. Match the following columns;

Column - I	Column -II
(A) If $4\sin^4 x+\cos^4 x=1$, then x is	(P) $\frac{\pi}{4}$
(B) If $\sec x.\cos(5x)+1=0$, where $0 < x < 2\pi$, then x is	(Q) $\frac{\pi}{6}$
(C) If $81^{\sin^2 x}+81^{\cos^2 x}=30$, where $0 < x < 2\pi$, then x is	(R) $-\frac{\pi}{4}$
(D) If $2\sin^2 x+\sin^2 2x=2$ where $0 < x < 2\pi$, then x is	(S) $n\pi,\ n\in I$

4. Match the following columns;

Column - I	Column -II
(A) If $\cos\theta+\cos 3\theta+\cos 5\theta$, $+\cos 7\theta=0$ then θ is	(P) $\frac{n\pi}{2}+\frac{\pi}{6},\ n\in I$
(B) If $\sin x-3\sin 2x+\sin 3x$ $=\cos x-3\cos 2x+\cos 3x$ then x is	(Q) $n\pi+\frac{\pi}{2},\ n\in I$
(C) If $\sin 4\theta-\sec 2\theta=2$, then θ is	(R) $\frac{n\pi}{5}+\frac{\pi}{10},\ n\in I$
(D) If $\tan(x+100°)$ $=\tan(x+50°).\tan x$ $.\tan(x-50°)$ then x is	(S) $\frac{n\pi}{5},\ n\in I$
	(T) $\pm\left(\frac{\pi}{3}\right)$

5. Match the following columns:

Column - I	Column - II
(A) The number of real roots of $\cos^7 x+\sin^4 x=1$ in $(-\pi,\pi)$, is	(P) 8
(B) The number of real roots of cosec x = 1 + cot x in $(-2\pi,\ 2\pi)$, is	(Q) 4
(C) The number of integral values of k for which, the equation 7 cos x + 5 sin x = 2 k + 1 has a solution is	(R) 3
(D) The number of solutions of the pair of equations $2\sin^2\theta-\cos 2\theta=0$ and $2\cos^2\theta-3\sin\theta=0$ in $[0,\ 2\pi]$ is	(S) 2
	(T) 7

6. Match the following columns:

Column - I	Column - II
(A) If $\cos 3x.\cos^3 x+\sin 3x$, $.\sin^3 x=0$ then x is	(P) $\left(n\pi\pm\frac{\pi}{3}\right), n\in I$
(B) If $\sin 3\alpha=4\sin\alpha$, $\sin(x+\alpha)\sin(x-\alpha)$ then α is, where $a\neq n\pi$	(Q) $\left(n\pi+\frac{\pi}{4}\right), n\in I$
(C) If $\lvert 2\tan x-1\rvert$, $+\lvert 2\cot x-1\rvert=2$ then x is	(R) $\left(\frac{n\pi}{4}+\frac{\pi}{8}\right),\ n\in I$
(D) If $\sin^{10} x+\cos^{10} x$, $=\frac{29}{16}\cos^4(2x)$ then x is	(S) $\left(\frac{n\pi}{2}\pm\frac{\pi}{4}\right),\ n\in I$

7. Match the following columns:
If α & β are the roots of $a\cos\theta+b\sin\theta=c$, then

Column - I	Column - II
(A) $\sin\alpha+\sin\beta$ is	(P) $\frac{2b}{a+c}$
(B) $\sin\alpha\ .\sin\beta$ is	(Q) $\frac{c-a}{c+a}$
(C) $\tan\left(\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)$ is	(R) $\frac{2bc}{a^3+b^3}$
(D) $\tan\left(\frac{\alpha}{2}\right).\tan\left(\frac{\beta}{2}\right)$ is	(S) $\frac{c^2-a^2}{a^2+b^2}$.

8. Match the following columns:

Column - I	Column - II
(A) If $\sin 5x=16\sin^5 x$, then x is	(P) $(2n+1)\frac{\pi}{4}, n\in I$

(B) If $4\cos^2 x.\sin x - 2\sin^2 x$, $= 3 \sin x$ then x is — (Q) $n\pi, n \in I$

(C) If $\tan^2(2x) + \cot^2(2x)$ $+2\tan(2x) + 2\cot(2x) = 0$ — (R) $n\pi + \frac{\pi}{6}, n \in I$

(D) If $\tan^2 x.\tan^2 3x.\tan 4x$ $= \tan^2 x - \tan^2 3x + \tan 4x$ then x is — (S) $\left(n\pi + (-1)^n \frac{\pi}{8}\right)$

9. Observe the following columns:

Column - I	Column - II
(A) If $\cos(6\theta) + \cos(4\theta)$ $+ \cos(2\theta) + 1 = 0$, then θ is	(P) $2n\pi, n \in I$
(B) If $3 - 2\cos\theta - 4\sin\theta$, $- \cos 2\theta + \sin 2\theta = 0$, then θ is	(Q) $\frac{n\pi}{3}, n \in I$
(C) If $\cos\theta.\cos 2\theta.\cos 3\theta = \frac{1}{4}$, then θ is	(R) $(4n+1)\frac{\pi}{2}, n \in I$
(D) If $\sin(5\theta) + \sin(\theta)$ $= \sin(3\theta)$, then θ is	(S) $(2n+1)\frac{\pi}{12}$

ASSERTION AND REASON

Codes:

(A) Both A and R are individually true and R is the correct explanation of A.

(B) Both A and R are individually true and R is not the correct explanantion of A.

(C) A is true but R is false.

(D) A is false but R is true.

1. Assertion (A): The number of real solutions of $\sin x = x^2 + x + 1$ is 1
Reson (R): since $|\sin x| \le 1$
(a) A (b) B (c) C (d) D

2. Assertion (A): The number of real solutions of $\cos x = 3^x + 3^{-x}$
Reason (R): since $|\cos x| \le 1$
(a) A (b) B (c) C (d) D

3. Assertion (A): The maximum value of $3 \sin x + 4 \cos x + 10$ is 15
Reason (R): The least value of $2\sin^2 x + 4$ is 4
(a) A (b) B (c) C (d) D

4. Assertion (A): The greatest value of $\sin^4 x + \cos^2 x$ is 1
Reason (R): The range of the function $f(x) = \sin^2 x + \cos^2 x$ is 1
(a) A (b) B (c) C (d) D

5. Assertion (A): $a \cos x + b \cos 3x \le 1$ for every x in R
Reason (R): since $|b| \le 1$
(a) A (b) B (c) C (d) D

6. Assertion (A): The set of values of x for which $\frac{\tan 3x - \tan 2x}{1 + \tan 3x.\tan 2x} = 1$ is ϕ
Reason (R): since tan$x$ is not defined at $x = (2n+1)\frac{\pi}{2}, n \in I$
(a) A (b) B (c) C (d) D

7. Assertion (A): The number of solutions of the equation $\tan x + \sec x = 2 \cos x$ lying in the interval $[0, 2\pi]$ is 2
Reason (R): The number of solutions of the equation $3\sin^2 x - 7\sin x + 2 = 0$ in $[0, 5\pi]$ is 6
(a) A (b) B (c) C (d) D

8. Assertion (A): The number of solutions of $\tan x.\ \tan 4x = 1$ in $(0, \pi)$ is 5
Reason (R): The number of solutions of $|\cos x| = \sin x$ in $[0, 4\pi]$ is 4
(a) A (b) B (c) C (d) D

9. Assertion (A):
If $\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$, then $\sin\theta + \cos\theta = \pm\sqrt{2}$
Reason (R): $-\sqrt{2} \le \sin\theta + \cos\theta \le \sqrt{2}$
(a) A (b) B (c) C (d) D

10. Assertion (A): $\sin A = \sin B = \sin C = 2\sin(18°)$
Reason (R): If $\cos A = \tan B$, $\cos B = \tan C$, $\cos C = \tan A$
(a) A (b) B (c) C (d) D

SELF ASSESMENT I
CH: TRIGONOMETRIC EQUATIONS

Time: 3 Hrs. **Max. Marks: 100.**

Give answers of the following questions:

1. Solve for x: sin$x$ + cos$x$ $= \min_{a \in R}\{1, a^2 - 4a + 6\}$.
2. Solve for x: $|\cos x| = \cos x - 2\sin x$.
3. Find all values of α for which the equation $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$ is valid. Also, find the general solution of the equation.
4. Find the smallest positive root of the equation $\sqrt{\sin(1-x)} = \sqrt{\cos x}$.
5. Solve for x and y: $3^{\sin x + \cos y} = 1$, $25^{\sin^2 x + \cos^2 y} = 5$.

6. Solve for x and y:
$1-2x-x^2=\tan^2(x+y)+\cos^2(x+y)$.
7. Solve for x and y: $x^2+2x\sin(xy)+1=0$.
8. Find the range of y such that the equation in x, $y+\cos x=\sin x$ has a real solution. For $y=1$, find x such that $0<x<2\pi$.
9. For what values of k, the equation
$\sin x+\cos(k+x)+\cos(k-x)=2$
has real solutions.
10. If $\sin 5\theta = a\sin^5\theta + b\sin^3\theta + c\sin\theta + d$ for every $\theta \in R$, then prove that
(i) $a+b+c+d=0$ (ii) $a+b+c=1$
(iii) $5a+4b=0$ (iv) $b+4c=0$.

QUESTIONS ASKED IN PAST IIT-JEE EXAMS

1. The genral solution of the trigonometric equation $\sin x+\cos x=1$ is given by
(a) $x=2n\pi, n\in I$
(b) $x=\left(2n\pi+\frac{\pi}{2}\right), n\in I$
(c) $x=\left(n\pi+(-1)^n\frac{\pi}{4}\right), n\in I$
(d) None of these [IIT-JEE - 1981]
2. Find the point of intersections of the curves $y=\cos x$ & $y=\sin 3x$
where $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ [IIT-JEE-1982]
3. Find all solutions of
$4\cos^2 x\sin x-2\sin^2 x=3\sin x$. [IIT-JEE-1983]
4. There exist a value of θ between 0 and 2π which satisfies the equation
$\sin^4\theta-2\sin^2\theta-1=0$? [IIT-JEE-1984]
5. No questions asked in 1985.
6. Find the solution set of
$x+y=\frac{2\pi}{3}$, $\cos x+\cos y=\frac{3}{2}$, where x and y are real. [IIT-JEE-1986]
7. Find the set of all x in the interval $[0,\pi]$ for which $2\sin^2 x-3\sin x+1\ge 0$. [IIT-JEE - 1987]
8. The smallest +ve root of the equation $\tan x-x=0$ lies in
(a) $\left(0,\frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{2},\pi\right)$
(c) $\left(\pi,\frac{3\pi}{2}\right)$ (d) $\left(\frac{3\pi}{2},2\pi\right)$
[IIT-JEE-1987]
9. The general solutions of $\sin x-3\sin 2x+\sin 3x$ $=\cos x-3\cos 2x+\cos 3x$ is
(a) $n\pi+\frac{\pi}{8}, n\in I$
(b) $\frac{n\pi}{2}+\frac{\pi}{8}, n\in I$
(c) $(-1)^n\frac{n\pi}{2}+\frac{\pi}{8}, n\in I$
(d) $2n\pi+\cos^{-1}\left(\frac{2}{3}\right), n\in I$ [IIT-JEE-1989]
10. No questions asked in between 1990 to 1992.
11. Number of solutions of the equation $\tan x+\sec x=2\cos x$ lying in the interval $[0,2\pi]$ is
(a) 0 (b) 1
(c) 2 (d) 3. [IIT-JEE-1993]
12. Determine the smallest +ve value of x (in degrees) for which
$\tan(x+100°)=\tan(x+50°)\tan x\tan(x-50°)$.
[IIT-JEE-1993]
13. Let n be a +ve integer such that
$\sin\left(\frac{\pi}{2n}\right)+\cos\left(\frac{\pi}{2n}\right)=\frac{\sqrt{n}}{2}$. Then
(a) $6\le n\le 8$ (b) $4<n\le 8$
(c) $4\le n\le 8$ (d) $4<n<8$
[IIT-JEE-1994]
14. Let $2\sin^2 x+3\sin x-2\ge 0$ and $x^2-x-2<0$ (x is measured in radians). Then x lies in the interval
(a) $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ (b) $\left(-1,\frac{5\pi}{6}\right)$
(c) $(-1,2)$ (d) $\left(\frac{\pi}{6},2\right)$
[IIT-JEE-1994]
15. Find the smallest +ve value of p for which the equation. $\cos(p\sin x)=\sin(p\cos x)$ has a solution for $x\in[0,2\pi]$ [IIT-JEE-1995]
16. Find all values of θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ satisfying the equation
$(1-\tan\theta)(1+\tan\theta)\sec^2\theta+2^{\tan^2\theta}=0$
[IIT-JEE-1996]
17. Find the general value of θ satisfying the equation $\tan^2\theta+\sec 2\theta=1$ [IIT-JEE-1997]
18. Find the real roots of the equation $\cos^7 x+\sin^4 x=1$ in the interval $(-\pi,\pi)$
[IIT-JEE-1997]
19. The number of values of x in the interval $[0,5\pi]$ satisfying the equation $3\sin^2 x-7\sin x+2=0$ is
(a) 0 (b) 5
(c) 6 (d) 10 [IIT-JEE-1998]

20. Let n be an odd integer. If $\sin(n\theta) = \sum_{r=0}^{n} b_r \sin^r \theta$ for each value of θ then
(a) $b_0 = 1, b_1 = 3$
(b) $b_0 = 0, b_1 = n$
(c) $b_0 = -1, b_1 = n$
(d) $b_0 = -1, b_1 = n^2 - 3n + 3$ [IIT-JEE-1998]

21. No questions asked in between 1999 to 2001.

22. The number of values of k for which the equation $7\cos x + 5\sin x = 2k + 1$ has a solution is
(a) 4 (b) 8
(c) 10 (d) 12 [IIT-JEE-2002]

23. No questions asked in between 2003 to 2004.

24. Let $(a,b) \in [-\pi, \pi]$ be such that $\cos(a-b) = 1$ and $\cos(a+b) = \frac{1}{e}$. The number of pairs of a, b satisfying the system of equations is
(a) 0 (b) 1
(c) 2 (d) 4 [IIT-JEE-2005]

25. Find the values of $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that
$$2\sin t = \frac{5x^2 - 2x + 1}{3x^2 - 2x - 1}, \forall x \in R - \left\{1, -\frac{1}{3}\right\}$$
[IIT-JEE-2005]

26. If $0 \le \theta \le 2\pi$, $2\sin^2\theta - 5\sin\theta + 2 > 0$, then the range of θ is
(a) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ (b) $\left(0, \frac{5\pi}{6}\right) \cup (\pi, 2\pi)$
(c) $\left(0, \frac{\pi}{6}\right) \cup (\pi, 2\pi)$ (d) None of these
[IIT-JEE- 2006]

27. The number of solutions of the pair of equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is
(a) 0 (b) 1
(c) 2 (d) 4 [IIT-JEE-2007]

28. If $\sin\theta = \cos\varphi$, then the possible values of $\frac{1}{\pi}\left(\theta \pm \varphi - \frac{\pi}{2}\right)$ are ..... [IIT-JEE-2008]

29. For $0 < \theta < \frac{\pi}{2}$, the solutions of
$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$
is (are)
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{12}$ (d) $\frac{5\pi}{12}$ [IIT-JEE-2009]

30. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\varphi \ne \frac{n\pi}{2}$ for $n \in I$ and $\tan\theta = \cot 5\theta$ as well as $\sin(2\theta) = \cos(4\theta)$ is...
[IIT-JEE-2010]

31. The +ve integer value of $n > 3$ satisfying the equation
$$\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$ is...
[IIT-JEE-2011]

32. No questions asked in between 2012 to 2013.

33. For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has
(a) infinitely many solutions
(b) three solutions
(c) one solution
(d) no solution [IIT-JEE-2014]

34. The number of distinct solutions of the equation
$$\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$$
in the interval $[0, 2\pi]$ is ...... [IIT-JEE-2015]

ANSWERS

Exercise 1

1. $\theta = \frac{n\pi}{2}, n \in I$
2. $\theta = (2n+1)\frac{\pi}{6}, n \in I$
3. $\theta = \frac{n\pi}{3}, n \in I$
4. $\theta = \frac{2n\pi}{5}, n \in I$
5. $\theta = \frac{(2n+1)\pi}{7}, n \in I$
6. $\theta = \frac{(2n+1)3\pi}{14}, n \in I$

7. $\theta = \frac{n\pi}{3}, n \in I$
8. $\theta = \frac{n\pi}{3} \pm \frac{\pi}{6}, n \in I$
9. $\theta = n\pi + (-1)^n \frac{\pi}{3}, n \in I$
10. $\theta = 2n\pi \pm \frac{\pi}{3}, n \in I$
11. $\theta = n\pi + \alpha, \alpha = \tan^{-1}\left(\sqrt{2}\right), n \in I$
12. $\theta = n\pi + (-1)^{n+1}\left(\frac{\pi}{6}\right), n \in I$
13. $\theta = \frac{2n\pi}{3} \pm \frac{\pi}{4}, n \in I$
14. $\theta = \frac{n\pi}{3} - \frac{\pi}{12}, n \in I$
15. $\theta = n\pi + (-1)^n \frac{\pi}{6}, n \in I$
16. $\theta = 2n\pi \pm \frac{\pi}{6}, n \in I$
17. $\theta = n\pi + \frac{\pi}{6}, n \in I$
18. $\theta = 2n\pi \pm \frac{2\pi}{3}, n \in I$
19. $\theta = n\pi, 2n\pi \pm \frac{\pi}{3}, n \in I$
20. $\theta = (2n+1)\frac{\pi}{2}, (2n \pm 1)\frac{\pi}{2}, n \in I$
21. $\theta = n\pi,\ n\pi \pm \frac{\pi}{3},\ n \in I$
22. $\theta = \frac{2n\pi}{7} \pm \frac{\pi}{14},\ n \in I$
23. $\theta = n\pi \pm \frac{\pi}{3},\ n \in I$
24. $\theta = \frac{n\pi}{3},\ 2n\pi \pm \frac{2\pi}{3},\ n \in I$
25. $\theta = (2n+1)\frac{\pi}{4},\ 2n\pi\ n \in I$
26. $\theta = \frac{n\pi}{4},\ n\pi \pm \frac{\pi}{3},\ n \in I$
27. $\theta = (4n-1)\frac{\pi}{2},\ n\pi + (-1)^n \frac{\pi}{6},\ n \in I$
28. $\theta = n\pi + \tan^{-1}\left(\frac{1}{2}\right),\ n\pi + \frac{3\pi}{4},\ n \in I$
29. $\theta = n\pi + \frac{\pi}{3},\ n\pi + \frac{3\pi}{4},\ n \in I$
30. $\theta = \frac{n\pi}{3} + \frac{\pi}{12},\ n \in I$
31. $\theta = \frac{n\pi}{3},\ n \in I$
32. $\theta = \frac{n\pi}{3} + \frac{\pi}{12},\ n \in I$
33. $\theta = \frac{n\pi}{2} + (-1)^n\left(\frac{\pi}{4}\right),\ n \in I$
34. $\theta = n\pi,\ n\pi \pm \frac{\pi}{3},\ n\pi \pm \tan^{-1}\left(\frac{1}{\sqrt{2}}\right),\ n \in I$
35. $\theta = \frac{n\pi}{6},\ (2n+1)\frac{\pi}{4},\ (2n+1)\frac{\pi}{12},\ n \in I$
36. $\theta = 2n\pi + \frac{\pi}{2},\ 2n\pi,\ n \in I$
37. $\theta = n\pi - \frac{\pi}{4},\ n\pi,\ n \in I$
38. $\theta = (2n+1)\frac{\pi}{2},\ n \in I$
39. $\theta = n\pi + \frac{\pi}{3},\ n \in I$
40. $\theta = n\pi + \frac{\pi}{3},\ n \in I$

Exercise 2

1. $\theta = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3}, n \in I$
2. $\theta = \left(2n\pi - \frac{\pi}{4}\right), n \in I$
3. $\theta = (4n+1)\frac{\pi}{2} - \frac{\pi}{6}, n \in I$
4. $\theta = (4n+1)\frac{\pi}{2} - \frac{\pi}{4}, n \in I$
5. $\theta = \left(2n\pi \pm \frac{\pi}{3}\right) - \frac{\pi}{6}, n \in I$
6. $\theta = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}, n \in I$
7. $\theta = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}, n \in I$
8. $\theta = \left(2n\pi \pm \frac{\pi}{3}\right) - \frac{\pi}{6}, n \in I$
9. $\theta = -\left(2n\pi + \frac{\pi}{3}\right)\left(\frac{2n\pi}{3} + \frac{\pi}{9}\right), n \in I$
10. $\theta = \frac{n\pi}{2} \pm \frac{\pi}{18}, n \in I$

Exercise 3

1. $\theta = \frac{3\pi}{4}$
2. $\theta = \frac{\pi}{3}$
3. $\theta = \frac{2\pi}{3}$
4. $\theta = \frac{2\pi}{3}$
5. $\theta = \frac{\pi}{4}$

Exercise 4

1. $\theta = 2n\pi + \frac{7\pi}{4}, n \in Z$
2. $\theta = 2n\pi + \frac{7\pi}{6}, n \in Z$
3. $\theta = 2n\pi + \frac{\pi}{6}, n \in Z$
4. $A = \frac{1}{2} \times \left[(2m+n)\pi + \frac{\pi}{4} \pm \frac{\pi}{6} \right],$

   $B = \frac{1}{2} \times \left[(2m-n)\pi - \frac{\pi}{4} \pm \frac{\pi}{6} \right], m, n \in Z$
5. $x = \frac{\pi}{2} + n\pi, y = \frac{\pi}{6} - n\pi, n \in Z$
6. No. Solution., $x = \varphi = y$
7. $x = n\pi, \ y = \frac{\pi}{4} - n\pi$ or, $x = \frac{\pi}{4} - n\pi, \ y = n\pi$
8. $r = 6, \theta = \frac{\pi}{6} or \frac{5\pi}{6}$
9. $A - B = 2n\pi, n \in Z$
10. $x = n\pi + (-1)^n \frac{\pi}{6}, y = 2m\pi \pm \frac{\pi}{2}, m, n \in Z$
11. $\left(-\frac{3\pi}{8}, \cos\frac{3\pi}{8}\right), \left(\frac{\pi}{8}, \cos\frac{\pi}{8}\right), \left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$
12. $x = 2m\pi \pm \frac{2\pi}{3}, y = 2(m-n)\pi \pm \frac{2\pi}{3}, m, n \in Z$
13. $x = \frac{\pi}{3} or \frac{2\pi}{3} = \varphi$
14. $x = n\pi + (-1)^n \frac{\pi}{6}$

    $\& y = 2m\pi \pm \frac{\pi}{3}, m, n \in Z$

Exercise 5

1. $\theta = n\pi + \alpha$, where $\sin\alpha = \sqrt{\frac{2}{5}}, n \in Z$
2. $x = n\pi, x = n\pi + (-1)^n \frac{\pi}{16}, x = n\pi - (-1)^n \frac{3\pi}{16}, n \in Z$
3. $x = n\pi + \alpha$, where $\alpha = \tan^{-1}\left(-\frac{1}{3}\right), n \in Z$
4. $x = n\pi + (-1)^n \frac{\pi}{16}, \ n \in Z$
5. $x = 2n\pi + \frac{\pi}{4} \pm \frac{3\pi}{4}, \ n \in Z$
6. $x = n\pi + (-1)^n \frac{\pi}{6}, n \in Z$
7. $x = (2n+1)\frac{\pi}{8}, n \in Z$
8. $x = n\pi + \frac{\pi}{4}, n \in Z$
9. $x = n\pi, x = \frac{n\pi}{2} \pm \frac{\pi}{2}$
10. $x = \frac{n\pi}{2} \pm \frac{\pi}{6}, n \in Z$

Exercise 6

1. $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{12}$
2. $x = \sin^{-1}(0.6), \ x = \pi - \sin^{-1}(0.6)$
3. $x = \frac{\pi}{2}, \frac{5\pi}{3}$
4. $x = n\pi \pm \frac{\pi}{3}, \ n \in Z$
5. $x = n\pi + (-1)^n \frac{\pi}{3}, \ n \in Z$

Exercise 7

1. $x = (4n+1)\frac{\pi}{2}, (4n-1)\frac{\pi}{2}, n \in Z$
2. $x = 0, \frac{\pi}{4}, \frac{\pi}{2}. \frac{2\pi}{9}, \frac{4\pi}{9}$
3. $x = \frac{\pi}{7}, \frac{5\pi}{7}. \pi, \frac{9\pi}{7}, \frac{13\pi}{7}$
4. $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{9\pi}{8}, \frac{13\pi}{8}$
5. $x = (2n+1)\frac{\pi}{2}, \frac{2n\pi}{3}, n \in Z$
6. $x = n\pi - \frac{\pi}{4}, 2n\pi \pm \frac{2\pi}{3}, n \in Z$
7. $x = n\pi \pm \alpha, n \in Z$
8. $x = n\pi, \frac{n\pi}{2}, n \in Z$

9. $x = n\pi - \frac{\pi}{4}, n \in Z$

10. $x = (4n+1)\frac{\pi}{8}, n \in Z$

Exercise 8

1. $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$
2. $x = n\pi \pm \frac{\pi}{3}, n \in Z$
3. $x = \frac{n\pi}{2}, \frac{n\pi}{8} + (-1)^n \frac{3\pi}{16}, n \in Z$
4. $x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{4}, \frac{11\pi}{6}$
5. $x = (2n+1)\pi, n \in Z$

Exercise 9

1. $x = n\pi + \tan^{-1}(3), n\pi + \tan^{-1}(4), n \in Z$
2. No Solution
3. No solution

Exercise 10

1. $x = 2n\pi \pm \frac{\pi}{3}, n \in Z$
2. $x = (4n+1)\frac{\pi}{2}, n \in Z$
3. $x = 4n\pi \pm \frac{2\pi}{3}, n \in Z$

Exercise 11

1. $x = 2n\pi,\ 2n\pi + \frac{\pi}{3}, n \in Z$
2. $x = 2n\pi,\ (4n+1)\frac{\pi}{2}, n \in Z$
3. $x = 2n\pi \pm \frac{\pi}{4} \pm \frac{3\pi}{4}, n \in Z$
4. $x = \frac{\pi}{2}, \pi$

Exercise 12

1. No Solution
2. No Solution
3. No Solution
4. $x = 2n\pi + \frac{\pi}{2}, n \in Z$
5. $x = 2n\pi + \frac{\pi}{2}, n \in Z$

Exercise 13

1. $\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$
2. $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$
3. $x = n\pi,\ n\pi + (-1)^n \frac{\pi}{6}, n \in I$
4. $\frac{1}{2}\left(\sqrt{3}-1\right)$
5. No solution
6. $x = 2n\pi + \frac{\pi}{4}, n \in I$
7. $x = \left(n\pi - \frac{\pi}{4}\right), (2n+1)\frac{\pi}{2}, n \in I$
8. $x = \frac{\pi}{4}$

Exercise 14

1. No Solution
2. $x = 0$

Level I

1. $\theta = n\pi \pm \frac{\pi}{3}, n \in I$
2. $\theta = \frac{n\pi}{4}, n\pi \pm \frac{\pi}{3}, n \in I$
3. $\theta = n\pi + \frac{\pi}{3}, n\pi - \frac{\pi}{4}, n \in I$
4. $\theta = \frac{n\pi}{3} + \frac{\pi}{12}, n \in I$
5. $\theta = (4n+1)\frac{\pi}{4}, n \in I$
6. $\theta = n\pi,\ n\pi \pm \frac{\pi}{3},\ n\pi \pm \tan^{-1}\left(\frac{1}{\sqrt{2}}\right), n \in I$
7. $\theta = \frac{n\pi}{3} + (-1)^n\left(\frac{\pi}{9}\right), n \in I$
8. $\theta = 2n\pi - \frac{\pi}{4}, n \in I$
9. $\theta = n\pi, n \in I$
10. $\theta = \frac{2n\pi}{3} + \frac{\pi}{9}, -\left(\frac{\pi}{3} + 2n\pi\right) n \in I$
11. $x = \frac{\pi}{2} + n\pi,\ y = \frac{\pi}{6} - n\pi,\ n \in Z$
12. $x = \varphi = y$
13. $r = 6, \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$
14. $x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{4}\right), n \in I$

    $y = n\pi + (-1)^n \sin^{-1}\left(\frac{3}{4}\right), n \in I$

15. $\frac{\pi}{2}$

16. $A=\frac{1}{2}\times\left[(2m+n)\pi+\frac{\pi}{4}\pm\frac{\pi}{6}\right],$

$B=\frac{1}{2}\times\left[(2m-n)\pi-\frac{\pi}{4}\pm\frac{\pi}{6}\right],m,n\in Z$

17. $A=n\pi+B$, $n\in I$

18. $\theta=n\pi+\alpha$, where $\sin\alpha=\sqrt{\frac{2}{5}},n\in Z$

19. $x=n\pi$ $,x=n\pi+(-1)^n\frac{\pi}{16}$

$,x=n\pi-(-1)^n\frac{3\pi}{16},n\in Z$

20. $x=n\pi+(-1)^n\frac{\pi}{6},n\in Z$

21. $x=0,\frac{\pi}{4},\frac{\pi}{2}.\frac{2\pi}{9},\frac{4\pi}{9}$

22. $x=\frac{\pi}{7},\frac{5\pi}{7},\pi,\frac{9\pi}{7},\frac{13\pi}{7}$

23. $x=\frac{\pi}{8},\frac{5\pi}{8},\frac{2\pi}{3},\frac{4\pi}{3},\frac{9\pi}{8},\frac{13\pi}{8}$

24. $x=(2n+1)\frac{\pi}{2},\frac{2n\pi}{3},n\in Z$

25. $x=n\pi-\frac{\pi}{4},2n\pi\pm\frac{2\pi}{3},n\in Z$

26. $x=\frac{\pi}{6},\frac{\pi}{3},\frac{3\pi}{8},\frac{5\pi}{8},\frac{2\pi}{3},\frac{7\pi}{8}$

27. $x=n\pi\pm\frac{\pi}{3},n\in Z$

28. $x=\frac{n\pi}{2},\frac{n\pi}{8}+(-1)^n\frac{3\pi}{16},n\in Z$

29. No solution

Level II

1. (b)	2. (a, b)	3. (a, d)	4. (c)
5. (a)	6. (a)	7. (c)	8. (d)
9. (a, b, c)	10. (b)	11. (a)	12. (a, b)
13. (b)	14. (d)	15. (c)	16. (b)
17. (b)	18. (c)	19. (c)	20. (b)
21. (b)	22. (c)	23. (a)	24. (b)
25. (c)	26. (b)	27. (a)	28. (b)
29. (a)	30. (c)	31. (a)	32. (c)
33. (b)	34. (d)	35. (b)	36. (a)
37. (c)	38. (b)	39. (c)	40. (b)
41. (a)	42. (b)	43. (b)	44. (a)
45. (c)	46. (c)	47. (c)	48. (a)
49. (a)	50. (b)	51. (c)	52. (a)
53. (d)	54. (a)	55. (d)	56. (c)
57. (c)	58. (a)	59. (c)	60. (b)
61. (b)	62. (b)	63. (b)	64. (d)
65. (a)	66. (c)	67. (b)	68. (a)
69. (c)	70. (b)		

Level III

1. $x=4n\pi\pm\frac{2\pi}{3},n\in Z$

2. $x=(2n+1))\frac{\pi}{14},x=n\pi,n\in I$

3. $x=n\pi\pm\frac{\pi}{3},n\in I$

4. $x=2n\pi+\pi=(2n+1)\pi,n\in I$

5. $x=\left(2n\pi+\frac{\pi}{2}\right)=(4n+1)\frac{\pi}{2},n\in I$

6. $x=n\pi-\frac{\pi}{4},n\in I$

7. $x=\frac{\pi}{8},\frac{5\pi}{8},\frac{7\pi}{8},\frac{13\pi}{8},\frac{2\pi}{3},\frac{4\pi}{3}$

8. $x=\frac{n\pi}{2}+(-1)^n\frac{\alpha}{2}$

9. $x=(2n+1)\frac{\pi}{2}$

10. $x=\frac{n\pi}{2}+(-1)^n\frac{\pi}{12},n\in I$

11. $x=\frac{n\pi}{2}+\cos^{-1}\left(\frac{1}{3}\right),n\in Z$

12. $x=n\pi,n\pi-\frac{\pi}{4},n\in Z$

13. $-\sqrt{3}\le a\le\sqrt{3}$

14. 1

15. 1

16. 1

17. $\left(1,\frac{\pi}{2}\right),\left(-1,\frac{3\pi}{2}\right)$

18. 5

19. 2

20. 90°

21. $-3\le a\le -2$

22. $x=\frac{13\pi}{4}$

23. [–3, –2]

24. 3

25. $x=2n\pi\pm\frac{\pi}{3},n\in I$

INTEGER TYPE QUESTIONS

1. 6
2. 7
3. 2
4. 4
5. 6
6. 6
7. 6
8. 6
9. 3
10. 2

COMPREHENSIVE LINK PASSAGE

Passage-I:	1. (d)	2. (a)	3. (c)	4. (b)	5. (a)
Passage-II:	1. (a, b)	2. (a, c)	3. (c)		
Passage-III:	1. (b)	2. (b)	3. (a, b, c, d)	4. (d)	
Passage-IV:	1. (b)	2. (c)	3. (b)	4. (b)	5. (b)
Passage-V:	1. (a)	2. (c)	3. (a)	4. (b)	5. (a)

(MATCH MATRIX)

1. (A) → (S), (B) → (S), (C) → (S), (D) → (S)
2. (A) → (Q), (B) → (R), (C) → (S) .
3. (A) → (S), (B) → (P), (C) → (Q),(D) → (P, Q)
4. (A) → (S), (B) → (P), (C) → (Q, R),(D)→ (T)
5. (A) → (R), (B) → (S), (C) → (P), (D) → (S)
6. (A) → (S), (B) → (P), (C) → (Q), (D) → (R)
7. (A) → (R), (B) → (S), (C) → (P), (D) → (Q)
8. (A) → (Q), (B) → (R), (C) → (S), (D) → (P)
9. (A) → (S), (B) → (R), (C) → (Q), (D) → (P).

ASSERTION AND REASON

1. (a)	2. (a)	3. (b)	4. (b)
5. (a)	6. (a)	7. (b)	8. (b)
9. (d)	10. (a)		

SELF ASSESMENT I

1. $x = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$, $n \in Z$
2. $x = 2m\pi$,$(2m+1)\pi + \frac{\pi}{4}, m \in Z$.
3. $x = \frac{n\pi}{2} + \frac{(-1)^n}{2}\sin^{-1}\left\{1-\sqrt{2\alpha+3}\right\}$, where $\alpha \in \left[-\frac{3}{2}, \frac{1}{2}\right]$
4. $x = \frac{1}{2} + \frac{7\pi}{4}$
5. $\left(\frac{7\pi}{6}, \frac{\pi}{3}\right), \left(\frac{7\pi}{6}, \frac{5\pi}{3}\right), \left(\frac{11\pi}{6}, \frac{\pi}{3}\right), \left(\frac{11\pi}{6}, \frac{5\pi}{3}\right)$
6. $x = -1$, $y = n\pi \pm \frac{\pi}{4} + 1$, $n \in Z$
7. $x = \pm 1$, $y = 2n\pi + \frac{3\pi}{2}$.
8. $x = \pi, \frac{\pi}{2}$.
9. $n\pi - \frac{\pi}{4} \le k \le n\pi + \frac{\pi}{4}, n \in Z$.

HINTS AND SOLUTIONS

EXERCISE 1

27. Given equation is

$$\cot^2\theta + \frac{3}{\sin\theta} + 3 = 0$$
$$\Rightarrow \quad \cot^2\theta + 3(1+\operatorname{cosec}\theta) = 0$$
$$\Rightarrow \quad (\operatorname{cosec}^2\theta - 1) + 3(1+\operatorname{cosec}\theta) = 0$$
$$\Rightarrow \quad (\operatorname{cosec}\theta - 1 + 3)(1+\operatorname{cosec}\theta) = 0$$
$$\Rightarrow \quad (\operatorname{cosec}\theta + 2)(1+\operatorname{cosec}\theta) = 0$$
$$\Rightarrow \quad \operatorname{cosec}\theta = -1, -2$$
$$\Rightarrow \quad \sin\theta = -1, \sin\theta = -\frac{1}{2}$$
$$\Rightarrow \quad \theta = (4n-1)\frac{\pi}{2}, \theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right), n \in I .$$

28. Given equation is

$$2\tan\theta - \cot\theta = -1$$
$$\Rightarrow \quad 2\tan\theta = \cot\theta - 1$$
$$\Rightarrow \quad 2\tan\theta = \frac{1}{\tan\theta} - 1$$
$$\Rightarrow \quad 2\tan^2\theta + \tan\theta - 1 = 0$$
$$\Rightarrow \quad 2\tan^2\theta + 2\tan\theta - \tan\theta - 1 = 0$$
$$\Rightarrow \quad 2\tan\theta(\tan\theta+1) - (\tan\theta+1) = 0$$
$$\Rightarrow \quad (2\tan\theta - 1)(\tan\theta + 1) = 0$$
$$\Rightarrow \quad (2\tan\theta - 1) = 0, (\tan\theta + 1) = 0$$
$$\Rightarrow \quad \tan\theta = -1, \frac{1}{2}$$
$$\Rightarrow \quad \theta = \left(n\pi - \frac{\pi}{4}\right), \theta = n\pi + \alpha, \alpha = \tan^{-1}\left(\frac{1}{2}\right)$$

29. Given equation is

$$\tan^2\theta + (1-\sqrt{3})\tan\theta - \sqrt{3} = 0$$
$$\Rightarrow \quad \tan^2\theta + \tan\theta - \sqrt{3}(\tan\theta + 1) = 0$$
$$\Rightarrow \quad \tan\theta(\tan\theta + 1) - \sqrt{3}(\tan\theta + 1) = 0$$

$\Rightarrow \quad (\tan\theta - \sqrt{3})(\tan\theta + 1) = 0$

$\Rightarrow \quad \tan\theta = \sqrt{3}, \tan\theta = -1$

$\Rightarrow \quad \theta = n\pi + \frac{\pi}{3}, \theta = n\pi - \frac{\pi}{4}, n \in I$

32. Given equation is

$\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 3$

$\Rightarrow \quad \tan\theta + \tan\left(\frac{\pi}{3} + \theta\right) + \tan\left(\pi - \left(\frac{\pi}{3} - \theta\right)\right) = 3$

$\Rightarrow \quad \tan\theta + \tan\left(\frac{\pi}{3} + \theta\right) - \tan\left(\frac{\pi}{3} - \theta\right) = 3$

$\Rightarrow \quad \tan\theta + \frac{\sqrt{3} + \tan\theta}{1 - \sqrt{3}\tan\theta} - \frac{\sqrt{3} - \tan\theta}{1 + \sqrt{3}\tan\theta} = 3$

$\Rightarrow \quad \tan\theta + \frac{8\tan\theta}{1 - 3\tan^2\theta} = 3$

$\Rightarrow \quad \frac{9\tan\theta - 3\tan^3\theta}{1 - 3\tan^2\theta} = 3$

$\Rightarrow \quad \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = 1$

$\Rightarrow \quad \tan(3\theta) = 1$

$\Rightarrow \quad 3\theta = n\pi + \frac{\pi}{4}$

$\Rightarrow \quad \theta = \frac{n\pi}{3} + \frac{\pi}{12}, n \in I$

34. Given equation is

$\tan\theta + \tan 2\theta + \tan 3\theta = 0$

$\Rightarrow \quad \tan\theta + \tan 2\theta + \tan(2\theta + \theta) = 0$

$\Rightarrow \quad \tan\theta + \tan 2\theta + \frac{\tan(2\theta) + \tan(\theta)}{1 - \tan(2\theta)\tan(\theta)} = 0$

$\Rightarrow \quad (\tan\theta + \tan 2\theta)\left(1 + \frac{1}{1 - \tan(2\theta)\tan(\theta)}\right) = 0$

when $\quad (\tan\theta + \tan 2\theta) = 0$

$\Rightarrow \quad \tan\theta + \frac{2\tan\theta}{1 - \tan^2\theta} = 0$

$\Rightarrow \quad \tan\theta\left(1 + \frac{2}{1 - \tan^2\theta}\right) = 0$

$\Rightarrow \quad \tan\theta = 0, \left(1 + \frac{2}{1 - \tan^2\theta}\right) = 0$

$\Rightarrow \quad \tan\theta = 0, \frac{2}{1 - \tan^2\theta} = -1$

$\Rightarrow \quad \tan\theta = 0, 1 - \tan^2\theta = -2$

$\Rightarrow \quad \tan\theta = 0, \tan^2\theta = 3$

$\Rightarrow \quad \theta = n\pi, \theta = n\pi \pm \frac{\pi}{3}, n \in I$

when $\left(1 + \frac{1}{1 - \tan(2\theta)\tan(\theta)}\right) = 0$

$\Rightarrow \quad \frac{1}{1 - \tan\theta\tan 2\theta} = -1$

$\Rightarrow \quad 1 - \tan\theta\tan 2\theta = -1$

$\Rightarrow \quad \tan\theta\tan 2\theta = 2$

$\Rightarrow \quad \tan\theta\left(\frac{2\tan\theta}{1 - \tan^2\theta}\right) = 2$

$\Rightarrow \quad \tan^2\theta = 1 - \tan^2\theta$

$\Rightarrow \quad \tan^2\theta = \frac{1}{2} = \tan^2\alpha, \alpha = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad \theta = n\pi \pm \alpha, n \in I$

36. Given equation is

$\cot\theta - \tan\theta = \cos\theta - \sin\theta$

$\Rightarrow \quad (\cos\theta - \sin\theta)\left(\frac{(\cos\theta + \sin\theta)}{\sin\theta\cos\theta} - 1\right) = 0$

$\Rightarrow \quad (\cos\theta - \sin\theta) = 0, (\cos\theta + \sin\theta) = \sin\theta\cos\theta$

$\Rightarrow \quad \tan\theta = 1, (\cos\theta + \sin\theta) = \sin\theta\cos\theta$

when $\quad \tan\theta = 1$

$\Rightarrow \quad \theta = n\pi + \frac{\pi}{4}, n \in I$

when $\quad (\cos\theta + \sin\theta) = \sin\theta\cos\theta$

No real value of θ satisfies the given equation.

37. Given equation is

$(1 - \tan\theta)(1 + \sin 2\theta) = 1 + \tan\theta$

$\Rightarrow \quad (\cos\theta - \sin\theta)(\cos\theta + \sin\theta)^2 = (\cos\theta + \sin\theta)$

$\Rightarrow \quad (\cos\theta + \sin\theta)(\cos 2\theta - 1) = 0$

$\Rightarrow \quad \tan(\theta) = -1, \sin^2\theta = 0$

$\Rightarrow \quad \tan(\theta) = -1, \sin(\theta) = 0$

$\Rightarrow \quad \theta = n\pi - \frac{\pi}{4}, \theta = n\pi, n \in I$

38. Given equation is

$2\sin^2\theta + \sin^2 2\theta = 2$

$\Rightarrow \quad 2\sin^2\theta + 4\sin^2\theta\cos^2\theta = 2$

$\Rightarrow \quad \sin^2\theta + 2\sin^2\theta\cos^2\theta = 1$

$\Rightarrow \quad 2\sin^2\theta\cos^2\theta = 1 - \sin^2\theta$

$\Rightarrow \quad 2\sin^2\theta\cos^2\theta = \cos^2\theta$

$\Rightarrow \quad (2\sin^2\theta - 1)\cos^2\theta = 0$

$\Rightarrow \quad (2\sin^2\theta - 1) = 0, \cos^2\theta = 0$

$\Rightarrow \quad \sin^2\theta = \frac{1}{2}, \cos\theta = 0$

$\Rightarrow \quad \theta = (2n+1)\frac{\pi}{2}, \theta = n\pi \pm \frac{\pi}{4}, n \in I$

39. Given equation is

$\sin(3\alpha) = 4\sin\theta\sin(\theta+\alpha)\sin(\theta-\alpha)$

$\Rightarrow \quad \sin(3\alpha) = 4\sin\theta(\sin^2\theta - \sin^2\alpha)$

$\Rightarrow \quad 3\sin\alpha - 4\sin^3\alpha = 4\sin\alpha(\sin^2\theta - \sin^2\alpha)$

It is possible only when

$\sin^2\theta = \frac{3}{4}$

$\Rightarrow \quad \sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2$

$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{3}, n \in I$

40. Given equation is

$4\sin\theta\sin 2\theta\sin 4\theta = \sin 3\theta$

$\Rightarrow \quad 4\sin\theta\sin(3\theta-\theta)\sin(3\theta+\theta) = \sin 3\theta$

$\Rightarrow \quad 4\sin\theta\left[\sin^2(3\theta) - \sin^2(\theta)\right] = \sin 3\theta$

$\Rightarrow \quad 4\sin\theta\left[\sin^2(3\theta) - \sin^2(\theta)\right] = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow \quad \sin\theta\left[4\sin^2(3\theta) - 4\sin^2(\theta) + 4\sin^2\theta - 3\right] = 0$

$\Rightarrow \quad \sin\theta\left[4\sin^2(3\theta) - 3\right] = 0$

$\Rightarrow \quad \sin\theta = 0, \left[4\sin^2(3\theta) - 3\right] = 0$

$\Rightarrow \quad \sin\theta = 0, \sin^2(3\theta) = \frac{3}{4}$

$\Rightarrow \quad \theta = n\pi, \theta = n\pi \pm \frac{\pi}{3}, n \in I$

EXERCISE 2

9. Given equation is

$\cos\theta + \sqrt{3}\sin\theta = 2\cos 2\theta$

$\Rightarrow \quad \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta = \cos 2\theta$

$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{3}\right) = \cos 2\theta$

$\Rightarrow \quad \left(\theta - \frac{\pi}{3}\right) = 2n\pi \pm 2\theta$

Taking +ve one, we get,

$\theta = -\left(2n\pi + \frac{\pi}{3}\right)$

Taking –ve one, we get,

$\Rightarrow \quad \theta = \frac{2n\pi}{3} + \frac{\pi}{9}, n \in I$

10. Given equation is

$\sqrt{3}(\cos\theta - \sqrt{3}\sin\theta) = 4\sin 2\theta .\cos 3\theta$

$\Rightarrow \quad \sqrt{3}\cos\theta - 3\sin\theta = 2(\sin 5\theta - \sin\theta)$

$\Rightarrow \quad \sqrt{3}\cos\theta - \sin\theta = 2(\sin 5\theta)$

$\Rightarrow \quad \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = (\sin 5\theta)$

$\Rightarrow \quad \sin\left(\frac{\pi}{3} - \theta\right) = \sin 5\theta$

$\Rightarrow \quad 5\theta = n\pi + (-1)^n\left(\frac{\pi}{3} - \theta\right)$

when n is even

$5\theta = 2k\pi + \left(\frac{\pi}{3} - \theta\right)$

$\Rightarrow \quad 6\theta = 2k\pi + \frac{\pi}{3}$

$\Rightarrow \quad \theta = \frac{k\pi}{3} + \frac{\pi}{18}, k \in I$

when n is odd

$5\theta = (2k+1)\pi - \left(\frac{\pi}{3} - \theta\right)$

$\Rightarrow \quad 4\theta = (2k+1)\pi - \frac{\pi}{3}$

$\Rightarrow \quad \theta = (2k+1)\frac{\pi}{4} - \frac{\pi}{12}, k \in I$

EXERCISE 4

4. Given $\tan(A-B) = 1$

$\Rightarrow \quad (A-B) = \frac{\pi}{4}, \frac{5\pi}{4}$

Also, $\sec(A+B) = \frac{2}{\sqrt{3}}$

$\Rightarrow \quad \cos(A+B) = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad (A+B) = \frac{\pi}{6}, \frac{11\pi}{6}$

Here, we observe that $A-B$ is +ve

So, $A>B$

$\Rightarrow \quad A+B>A-B$

$$\begin{cases} A+B=\dfrac{11\pi}{6} \\ A-B=\dfrac{\pi}{4} \end{cases} or \begin{cases} A+B=\dfrac{11\pi}{6} \\ A-B=\dfrac{5\pi}{4} \end{cases}$$

On solving, we get,

$$\begin{cases} A=\dfrac{25\pi}{24} \\ B=\dfrac{19\pi}{24} \end{cases} or \begin{cases} A=\dfrac{19\pi}{24} \\ B=\dfrac{7\pi}{24} \end{cases}$$

General values of $\tan(A-B)=1$

is $\quad (A-B)=n\pi+\dfrac{\pi}{4}, n\in I$ ......(i)

General values of $\sec(A+B)=\dfrac{2}{\sqrt{3}}$

is $\quad (A+B)=2n\pi+\dfrac{\pi}{6}, n\in I$ .....(ii)

On solving (i) and (ii), we get

$$\begin{cases} A=(2n+m)\dfrac{\pi}{2}+\dfrac{\pi}{24} \\ B=(2n-m)\dfrac{\pi}{2}-\dfrac{5\pi}{24} \end{cases}.$$

5. Given $\sin x=2\sin y$

$\Rightarrow \quad \sin x=2\sin\left(\dfrac{2\pi}{3}-x\right)$

$\Rightarrow \quad \sin x=2\left(\dfrac{\sqrt{3}}{2}\cos x+\dfrac{1}{2}\sin x\right)$

$\Rightarrow \quad \sin x=\sqrt{3}\cos x+\sin x$

$\Rightarrow \quad \sqrt{3}\cos x=0$

$\Rightarrow \quad \cos x=0$

$\Rightarrow \quad x=(2n+1)\dfrac{\pi}{2}$

when $x=(2n+1)\dfrac{\pi}{2}$, then $y=n\pi-\dfrac{\pi}{6}$

Hence, the solutions are

$$\begin{cases} x=(2n+1)\dfrac{\pi}{2} \\ y=n\pi-\dfrac{\pi}{6} \end{cases}, n\in I$$

6. Given $x+y=\dfrac{2\pi}{3}$ and $\cos x+\cos y=\dfrac{3}{2}$

Now $\quad \cos x+\cos y=\dfrac{3}{2}$

$\Rightarrow \quad \cos x+\cos\left(\dfrac{2\pi}{3}-x\right)=\dfrac{3}{2}$

$\Rightarrow \quad \cos x-\dfrac{1}{2}\cos x+\dfrac{\sqrt{3}}{2}\sin x=\dfrac{3}{2}$

$\Rightarrow \quad \dfrac{1}{2}\cos x+\dfrac{\sqrt{3}}{2}\sin x=\dfrac{3}{2}$

$\Rightarrow \quad \cos x+\sqrt{3}\sin x=3$

It is not possible, since the maximum value of L.H.S is 2.

So, the given system of equations has no solutions.

8. Given equations are

$r\sin\theta=3$ ....(i)

and $r=4(1+\sin\theta)$ ....(ii)

Eliminating (i) and (ii), we get,

$4(1+\sin\theta)\sin\theta=3$

$\Rightarrow \quad 4\sin^2\theta+4\sin\theta-3=0$

$\Rightarrow \quad 4\sin^2\theta+6\sin\theta-2\sin\theta-3=0$

$\Rightarrow \quad 2\sin\theta(2\sin\theta+3)-1(2\sin\theta+3)=0$

$\Rightarrow \quad (2\sin\theta+3)(2\sin\theta-1)=0$

$\Rightarrow \quad \sin\theta=-\dfrac{3}{2}, \dfrac{1}{2}$

$\Rightarrow \quad \sin\theta=\dfrac{1}{2}$

$\Rightarrow \quad \theta=\dfrac{\pi}{6}, \dfrac{5\pi}{6}$

7. Given $x+y=\dfrac{\pi}{4}$ and $\tan x+\tan y=1$

$\Rightarrow \quad \tan(x+y)=\tan\left(\dfrac{\pi}{4}\right)$

$\Rightarrow \quad \dfrac{\tan x+\tan y}{1-\tan x.\tan y}=1$

$\Rightarrow \quad 1-\tan x.\tan y=1$

$\Rightarrow \quad \tan x.\tan y=0$

$\Rightarrow \quad \tan x=0 \ \& \ \tan y=0$

$\Rightarrow \quad x=n\pi=y.$

Thus, no values of x and y satisfying the given equations.

Therefore, the given equations have no solutions.

10. Given equations are

$\sin x+\sin y=1$ ....(i)

and $\cos 2x-\cos 2y=1$

Now, $\cos 2x - \cos 2y = 1$

$\Rightarrow \quad 1 - 2\sin^2 x - 1 + 2\sin^2 y = 1$

$\Rightarrow \quad -2\sin^2 x - 1 + 2\sin^2 y = 0$

$\Rightarrow \quad 2(\sin^2 x - \sin^2 y) = -1$

$\Rightarrow \quad (\sin x + \sin y)(\sin x - \sin y) = -\frac{1}{2}$

$\Rightarrow \quad (\sin x - \sin y) = -\frac{1}{2}$ ....(ii)

On solving, we get,

$\sin x = 0, \sin y = 1$

$\Rightarrow \quad x = n\pi, y = (4n+1)\frac{\pi}{2}, n \in I$

Hence, the solutions are

$$\begin{cases} x = n\pi \\ y = (4n+1)\frac{\pi}{2}, n \in I \end{cases}$$

11. Given curves are $y = \cos x$ & $y = \sin 2x$

Thus, $\sin 2x = \cos x$

$\Rightarrow \quad 2\sin x \cos x = \cos x$

$\Rightarrow \quad (2\sin x - 1)\cos x = 0$

$\Rightarrow \quad (2\sin x - 1) = 0, \cos x = 0$

$\Rightarrow \quad \sin x = \frac{1}{2}, \cos x = 0$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}$

then $\quad y = \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0.$

Hence, the solutions are

$$\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right), \left(\frac{5\pi}{6}, -\frac{\sqrt{3}}{2}\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$$

12. Given equation is

$\Rightarrow \quad \cos x + \cos y + \cos(x+y) = -\frac{3}{2}$

$\Rightarrow \quad 2(\cos x + \cos y) + 2\cos(x+y) + 2 = 1$

$\Rightarrow \quad 4\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + 4\cos^2\left(\frac{x+y}{2}\right) = 1$

$\Rightarrow \quad 4\cos^2\left(\frac{x+y}{2}\right) + 4\cos\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right) + 1 = 0$

For real x and y

$16\cos^2\left(\frac{x-y}{2}\right) - 16 \geq 0$

$\Rightarrow \quad \cos^2\left(\frac{x-y}{2}\right) = 1$

$\Rightarrow \quad \cos^2\left(\frac{x-y}{2}\right) \geq 1$

$\Rightarrow \quad \left(\frac{x-y}{2}\right) = 0$

$\Rightarrow \quad x = y$

The given equation

$4\cos^2\left(\frac{x+y}{2}\right) + 4\cos\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right) + 1 = 0$

reduces to $4\cos^2(x) + 4\cos(x) + 1 = 0$

$\Rightarrow \quad (2\cos(x) + 1)^2 = 0$

$\Rightarrow \quad \cos(x) = -\frac{1}{2}$

$\Rightarrow \quad x = \frac{2\pi}{3} = y$

13. Given equation is

$8\cos\theta \cos\varphi \cos(\theta+\varphi) + 1 = 0$

$\Rightarrow \quad 2\cos\theta\cos\varphi\cos(\theta+\varphi) = -\frac{1}{4}$

$\Rightarrow \quad 4[\cos(\theta+\varphi) + \cos(\theta-\varphi)]\cos(\theta+\varphi) + 1 = 0$

$\Rightarrow \quad 4\cos^2(\theta+\varphi) + 4\cos(\theta-\varphi)\cos(\theta+\varphi) + 1 = 0$

For all real $0 < \theta, \varphi < \pi$,

$16\cos^2(\theta-\varphi) - 16 \geq 0$

$\Rightarrow \quad \cos^2(\theta-\varphi) \geq 1$

$\Rightarrow \quad \cos^2(\theta-\varphi) = 1$

$\Rightarrow \quad \theta - \varphi = 0$

$\Rightarrow \quad \theta = \varphi$

when $\theta = \varphi$, then the equation

$4\cos^2(\theta+\varphi) + 4\cos(\theta-\varphi)\cos(\theta+\varphi) + 1 = 0$

reduces to

$\Rightarrow \quad 4\cos^2(2\theta) + 4\cos(2\theta) + 1 = 0$

$\Rightarrow \quad (2\cos(2\theta) + 1)^2 = 0$

$\Rightarrow \quad (2\cos(2\theta) + 1) = 0$

$\Rightarrow \quad \cos(2\theta) = -\frac{1}{2}$

$\Rightarrow \quad (2\theta) = \frac{2\pi}{3}$

$\Rightarrow \quad \theta = \frac{\pi}{3} = \varphi$

EXERCISE 5

1. Given equation is

$$4\sin^4 x + \cos^4 x = 1$$
$$\Rightarrow \quad 4\sin^4 x = 1 - \cos^4 x$$
$$\Rightarrow \quad 4\sin^4 x = (1 + \cos^2 x)\sin^2 x$$
$$\Rightarrow \quad \sin^2 x(4\sin^2 x - \cos^2 x - 1) = 0$$
$$\Rightarrow \quad \sin^2 x = 0, (5\sin^2 x - 2) = 0$$
$$\Rightarrow \quad \sin x = 0, \sin^2 x = \frac{2}{5}$$
$$\Rightarrow \quad x = n\pi, x = n\pi \pm \alpha, \alpha = \sin^{-1}\left(\sqrt{\frac{2}{5}}\right)$$

2. Given equation is

$$4\cos^2 x \sin x - 2\sin^2 x = 2\sin x$$
$$\Rightarrow \quad 4(1 - \sin^2 x)\sin x - 2\sin^2 x = 2\sin x$$
$$\Rightarrow \quad 2(1 - \sin^2 x)\sin x - \sin^2 x = \sin x$$
$$\Rightarrow \quad 2\sin x - 2\sin^3 x - \sin^2 x - \sin x = 0$$
$$\Rightarrow \quad \sin x - 2\sin^3 x - \sin^2 x = 0$$
$$\Rightarrow \quad 2\sin^3 x + \sin^2 x - \sin x = 0$$
$$\Rightarrow \quad \sin x(2\sin^2 x + \sin x - 1) = 0$$
$$\Rightarrow \quad \sin x = 0, (2\sin^2 x + \sin x - 1) = 0$$
$$\Rightarrow \quad \sin x = 0, \sin x = \frac{-1 \pm 3}{2}$$
$$\Rightarrow \quad \sin x = 0, \sin x = 1, \sin x = -2$$
$$\Rightarrow \quad \sin x = 0, \sin x = 1$$
$$\Rightarrow \quad x = n\pi, x = (4n+1)\frac{\pi}{2}, n \in I$$

3. Given equation is

$$\sin 3x + \cos 2x = 1$$
$$\Rightarrow \quad \sin 3x = 1 - \cos 2x$$
$$\Rightarrow \quad \sin x(3 - 4\sin^2 x) = 2\sin^2 x$$
$$\Rightarrow \quad \sin x(3 - 4\sin^2 x - 2\sin x) = 0$$
$$\Rightarrow \quad \sin x = 0, (4\sin^2 x + 2\sin x - 3) = 0$$
$$\Rightarrow \quad \sin x = 0, \sin x = \frac{-2 \pm \sqrt{4 + 48}}{8}$$
$$\Rightarrow \quad \sin x = 0, \sin x = \frac{-1 \pm \sqrt{13}}{4}$$
$$\Rightarrow \quad \sin x = 0, \sin x = \frac{\sqrt{13} - 1}{4}$$
$$\Rightarrow \quad x = n\pi, x = n\pi + (-1)^n \alpha, \alpha = \sin^{-1}\left(\frac{\sqrt{13} - 1}{4}\right).$$

4. Given equation is

$$2\cos 2x + \sqrt{2\sin x} = 2$$
$$\Rightarrow \quad \sqrt{2}\sqrt{\sin x} = 2(1 - \cos 2x)$$
$$\Rightarrow \quad \sqrt{2}\sqrt{\sin x} = 4\sin^2 x$$
$$\Rightarrow \quad \sqrt{\sin x} = 2\sqrt{2}\sin^2 x$$
$$\Rightarrow \quad \sqrt{\sin x}(1 - 2\sqrt{2}\sin^{3/2} x) = 0$$
$$\Rightarrow \quad \sqrt{\sin x} = 0, \sin^{3/2} x = \frac{1}{2\sqrt{2}}$$
$$\Rightarrow \quad \sin x = 0, \sin x = \frac{1}{\sqrt{2}}$$
$$\Rightarrow \quad x = n\pi, n = n\pi + (-1)^n\left(\frac{\pi}{4}\right), n \in I$$

5. Given equation is

$$1 + \sin^3 x + \cos^3 x = \frac{3}{2}\sin 2x$$
$$\Rightarrow \quad 1 + (\sin^3 x + \cos^3 x) = 3\sin x \cos x$$
$$\Rightarrow \quad 1 + (\sin^3 x + \cos^3 x) - 3\sin x \cos x.1 = 0$$
$$\Rightarrow \quad (\sin x + \cos x + 1)(2 - \sin x \cos x - \sin x - \cos x) = 0$$
$$\Rightarrow \quad (\sin x + \cos x + 1) = 0$$
$$\Rightarrow \quad \sin x + \cos x = -1$$
$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{\sqrt{2}}$$
$$\Rightarrow \quad \sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$
$$\Rightarrow \quad \left(x + \frac{\pi}{4}\right) = n\pi + (-1)^n\left(-\frac{\pi}{4}\right)$$
$$\Rightarrow \quad x = n\pi + (-1)^n\left(-\frac{\pi}{4}\right) - \frac{\pi}{4}, n \in I$$

6. Given equation is

$$\sin^6 x + \cos^6 x = \frac{7}{16}$$
$$\Rightarrow \quad 1 - 3\sin^2 x \cos^2 x = \frac{7}{16}$$
$$\Rightarrow \quad 3\sin^2 x \cos^2 x = 1 - \frac{7}{16} = \frac{9}{16}$$

$\Rightarrow \quad \sin^2 x\cos^2 x = \frac{3}{16}$

$\Rightarrow \quad 4\sin^2 x\cos^2 x = \frac{3}{4}$

$\Rightarrow \quad \sin^2(2x) = \frac{3}{4}$

$\Rightarrow \quad (2x) = n\pi \pm \frac{\pi}{3}$

$\Rightarrow \quad x = \frac{n\pi}{3} \pm \frac{\pi}{6}, n \in I$

7. Given equation is

$\sin^8 x + \cos^8 x = \frac{17}{16}\cos^2 2x$

$\Rightarrow \left(\sin^4 x + \cos^4 x\right)^2 - 2\sin^4 x\cos^4 x = \frac{17}{16}\cos^2 2x$

$\Rightarrow \left(1 - 2\sin^2 x\cos^2 x\right)^2 - 2\sin^4 x\cos^4 x = \frac{17}{16}\cos^2 2x$

$\Rightarrow \left(1 - 4\sin^2 x\cos^2 x\right) + 2\sin^4 x\cos^4 x = \frac{17}{16}\cos^2 2x$

$\Rightarrow 16\left(1 - 4\sin^2 x\cos^2 x + 2\sin^4 x\cos^4 x\right)$

$= 17\left(\cos^4 x + \sin^4 x - 2\sin^2 x\cos^2 x\right)$

$= 17\left(1 - 4\sin^2 x\cos^2 x\right)$

$\Rightarrow 32\sin^4 x\cos^4 x + 4\sin^2 x\cos^2 x - 1 = 0$

$\Rightarrow 2\sin^4(2x) + \sin^2 2x - 1 = 0$

$\Rightarrow \sin^2(2x) = \frac{-1 \pm \sqrt{5}}{4}$

$\Rightarrow \sin^2(2x) = \frac{\sqrt{5} - 1}{4}$

$\Rightarrow 2x = n\pi \pm \alpha, \alpha = \sin^{-1}\left(\sqrt{\frac{\sqrt{5}-1}{4}}\right)$

$\Rightarrow x = \frac{n\pi}{2} \pm \frac{\alpha}{2}, \alpha = \sin^{-1}\left(\sqrt{\frac{\sqrt{5}-1}{4}}\right)$

8. Given equation is

$2\sin^3 x + 2 = \cos^2 3x$

$\Rightarrow \quad 2\sin^3 x + 2 = 1 - \sin^2 3x$

$\Rightarrow \quad 2\sin^3 x + \sin^2 3x + 1 = 0$

$\Rightarrow \quad 2\sin^3 x + \left(3\sin x - 4\sin^3 x\right)^2 + 1 = 0$

$\Rightarrow \quad 2\sin^3 x + 9\sin^2 x - 24\sin^4 x + 16\sin^6 x + 1 = 0$

$\Rightarrow \quad 16\sin^6 x - 24\sin^4 x + 2\sin^3 x + 9\sin^2 x + 1 = 0$

$\Rightarrow \quad \sin x = -1$

$\Rightarrow \quad x = (4n-1)\frac{\pi}{2}, n \in I$

9. Given equation is

$\cos 4x = \cos^2 3x$

$\Rightarrow \quad 2\cos^2 2x - 1 = \cos^2 3x$

$\Rightarrow \quad 2\cos^2 2x = 1 + \cos^2 3x$

It is possible only when

$2\cos^2 2x = 1 + \cos^2 3x$

It is true for $x = 0$.

Hence, the solution is $x = n\pi, n \in I$

10. Given equation is

$\cos 2x = 6\tan^2 x - 2\cos^2 x$

$\Rightarrow \quad 2\cos^2 x - 1 = 6\left(\frac{\sin^2 x}{\cos^2 x}\right) - 2\cos^2 x$

$\Rightarrow \quad 2\cos^4 x - \cos^2 x = 6 - 6\cos^2 x - 2\cos^4 x$

$\Rightarrow \quad 4\cos^4 x + 5\cos^2 x - 6 = 0$

$\Rightarrow \quad 4\cos^4 x + 8\cos^2 x - 3\cos^2 x - 6 = 0$

$\Rightarrow \quad 4\cos^2 x\left(\cos^2 x + 2\right) - 3\left(\cos^2 x + 2\right) = 0$

$\Rightarrow \quad \left(4\cos^2 x - 3\right)\left(\cos^2 x + 2\right) = 0$

$\Rightarrow \quad \left(4\cos^2 x - 3\right) = 0$

$\Rightarrow \quad \cos^2 x = \frac{3}{4}$

$\Rightarrow \quad x = n\pi \pm \frac{\pi}{6}, n \in I$

EXERCISE 6

1. Given equation is

$2\sin^2 x + \sin x - 1 = 0,$

$\Rightarrow \quad \sin x = \frac{-1 \pm 3}{4} = \frac{1}{2}, -1$

$\Rightarrow \quad \sin x = \frac{1}{2}, \sin x = -1$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

2. Given equation is

$5\sin^2 x + 7\sin x - 6 = 0$

$\Rightarrow \quad 5\sin^2 x + 10\sin x - 3\sin x - 6 = 0$

$\Rightarrow \quad 5\sin x(\sin x + 2) - 3(\sin x + 2) = 0$

$\Rightarrow \quad (5\sin x - 3)(\sin x + 2) = 0$

$\Rightarrow \quad (5\sin x-3)=0,(\sin x+2)=0$

$\Rightarrow \quad (5\sin x-3)=0$

$\Rightarrow \quad \sin x=\frac{3}{5}$

$\Rightarrow \quad x=n\pi+(-1)^n\alpha,\alpha=\sin^{-1}\left(\frac{3}{5}\right)$

Hence, the solution is

$x=\sin^{-1}\left(\frac{3}{5}\right),\pi-\sin^{-1}\left(\frac{3}{5}\right)$

3. Given equation is

$\sin^2 x-\cos x=\frac{1}{4}$

$\Rightarrow \quad 4\sin^2 x-4\cos x-1=0$

$\Rightarrow \quad 4-4\cos^2 x-4\cos x-1=0$

$\Rightarrow \quad 3-4\cos^2 x-4\cos x=0$

$\Rightarrow \quad 4\cos^2 x+4\cos x-3=0$

$\Rightarrow \quad \cos x=\frac{-4\pm 8}{8}=\frac{1}{2},-\frac{3}{2}$

$\Rightarrow \quad \cos x=\frac{1}{2}$

$\Rightarrow \quad x=\frac{\pi}{3},\frac{5\pi}{3}$

4. Given equation is

$\tan^2 x-2\tan x-3=0$

$\Rightarrow \quad (\tan x-3)(\tan x+1)=0$

$\Rightarrow \quad (\tan x-3)=0,(\tan x+1)=0$

$\Rightarrow \quad (\tan x-3)=0,(\tan x+1)=0$

$\Rightarrow \quad \tan x=-1,\tan x=3$

$\Rightarrow \quad x=n\pi-\frac{\pi}{4},x=n\pi+\alpha,\alpha=\tan^{-1}(3)\cdot$

5. Given equation is

$2\cos^2 x-\sqrt{3}\sin x+1=0$

$\Rightarrow \quad 2-2\sin^2 x-\sqrt{3}\sin x+1=0$

$\Rightarrow \quad 3-2\sin^2 x-\sqrt{3}\sin x=0$

$\Rightarrow \quad 2\sin^2 x+\sqrt{3}\sin x-3=0$

$\Rightarrow \quad \sin x=\frac{-\sqrt{3}\pm\sqrt{27}}{4}=\frac{-\sqrt{3}\pm 3\sqrt{3}}{4}$

$\Rightarrow \quad \sin x=\frac{-4\sqrt{3}}{4},\frac{2\sqrt{3}}{4}$

$\Rightarrow \quad \sin x=-\sqrt{3},\frac{\sqrt{3}}{2}$

$\Rightarrow \quad \sin x=\frac{\sqrt{3}}{2}$

$\Rightarrow \quad x=n\pi+(-1)^n\left(\frac{\pi}{3}\right),n\in I$

EXERCISE 7

1. Given equation is

$\cos x-\cos 2x=\sin 3x$

$\Rightarrow \quad 2\sin\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right)=2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)$

$\Rightarrow \quad 2\sin\left(\frac{3x}{2}\right)\left(\sin\left(\frac{x}{2}\right)-\cos\left(\frac{3x}{2}\right)\right)=0$

$\Rightarrow \quad \sin\left(\frac{3x}{2}\right)=0,\left(\sin\left(\frac{x}{2}\right)-\cos\left(\frac{3x}{2}\right)\right)=0$

when $\sin\left(\frac{3x}{2}\right)=0$

Then $\frac{3x}{2}=n\pi$

$\Rightarrow \quad x=\frac{2n\pi}{3}$

when $\left(\sin\left(\frac{x}{2}\right)-\cos\left(\frac{3x}{2}\right)\right)=0$

$\Rightarrow \quad \left(\sin\left(\frac{x}{2}\right)=\cos\left(\frac{3x}{2}\right)\right)$

$\Rightarrow \quad \sin\left(\frac{x}{2}\right)=\cos\left(\frac{3x}{2}\right)$

$\Rightarrow \quad \cos\left(\frac{3x}{2}\right)=\cos\left(\frac{\pi}{2}-\frac{x}{2}\right)$

$\Rightarrow \quad \left(\frac{3x}{2}\right)=2n\pi\pm\left(\frac{\pi}{2}-\frac{x}{2}\right)$

Taking +ve sign, we get,

$2x=2n\pi+\frac{\pi}{2}$

$\Rightarrow \quad x=n\pi+\frac{\pi}{4}$

Taking –ve sign, we get,

$x=2n\pi-\frac{\pi}{2},n\in I$

2. Given equation is

$\sin 7x+\sin 4x+\sin x=0$

$\Rightarrow \quad (\sin 7x+\sin x)+\sin 4x=0$

$\Rightarrow \quad 2\sin 4x\cos 3x+\sin 4x=0$

$\Rightarrow \quad \sin 4x(2\cos 3x+1)=0$

$\Rightarrow \quad \sin 4x=0\ ,\ (2\cos 3x+1)=0$

$\Rightarrow \quad \sin 4x=0\ ,\ \cos 3x=-\dfrac{1}{2}$

$\Rightarrow \quad 4x=n\pi\ ,\ 3x=2n\pi\pm\dfrac{2\pi}{3}$

$\Rightarrow \quad x=\dfrac{n\pi}{4}\ ,\ x=\dfrac{2n\pi}{3}\pm\dfrac{2\pi}{9}\ ,n\in I$

Hence, the solutions are

$x=0,\dfrac{2\pi}{9},\dfrac{\pi}{4},\dfrac{\pi}{2}$

3. Given equation is

$$\cos 3x+\cos 2x=\sin\left(\frac{3x}{2}\right)+\sin\left(\frac{x}{2}\right)$$

$\Rightarrow \quad 2\cos\left(\dfrac{5x}{2}\right)\cos\left(\dfrac{x}{2}\right)=2\sin x\cos\left(\dfrac{x}{2}\right)$

$\Rightarrow \quad 2\left(\cos\left(\dfrac{5x}{2}\right)-\sin x\right)\cos\left(\dfrac{x}{2}\right)=0$

$\Rightarrow \quad 2\left(\cos\left(\dfrac{5x}{2}\right)-\sin x\right)=0,\cos\left(\dfrac{x}{2}\right)=0$

$\Rightarrow \quad \cos\left(\dfrac{5x}{2}\right)=\sin x,\cos\left(\dfrac{x}{2}\right)=0$

$\Rightarrow \quad \cos\left(\dfrac{5x}{2}\right)=\cos\left(\dfrac{\pi}{2}-x\right),\cos\left(\dfrac{x}{2}\right)=0$

$\Rightarrow \quad \left(\dfrac{5x}{2}\right)=2n\pi\pm\left(\dfrac{\pi}{2}-x\right),\left(\dfrac{x}{2}\right)=(2n+1)\dfrac{\pi}{2}$

$\Rightarrow \quad x=\dfrac{4n\pi}{5}\pm(\pi-2x),x=(2n+1)\pi$

$\Rightarrow \quad x=\dfrac{4n\pi}{5}\pm(\pi-2x),x=(2n+1)\pi$

$\Rightarrow \quad x=\dfrac{\pi}{3},\dfrac{\pi}{5},\dfrac{3\pi}{5},\dfrac{13\pi}{15},\dfrac{17\pi}{15},\dfrac{7\pi}{5},\dfrac{5\pi}{3},\dfrac{29\pi}{15}$

4. Do yourself.
5. Given equation is

$\cos 2x+\cos 4x=2\cos x$

$\Rightarrow \quad 2\cos 3x\cos x=2\cos x$

$\Rightarrow \quad (2\cos 3x-1)\cos x=0$

$\Rightarrow \quad (2\cos 3x-1)=0,\cos x=0$

$\Rightarrow \quad \cos 3x=\dfrac{1}{2},\cos x=0$

$\Rightarrow \quad x=\dfrac{2n\pi}{3}\pm\dfrac{2\pi}{9},x=(2n+1)\dfrac{\pi}{2}$

6. Given equation is

$\sin 2x+\cos 2x+\sin x+\cos x+1=0$

$\Rightarrow \quad (1+\sin 2x)+(\sin x+\cos x)+\cos 2x=0$

$\Rightarrow \quad (\sin x+\cos x)^2+(\sin x+\cos x)+\left(\cos^2 x-\sin^2 x\right)=0$

$\Rightarrow \quad (\sin x+\cos x)(2\cos x+1)=0$

$\Rightarrow \quad (\sin x+\cos x)=0,(2\cos x+1)=0$

$\Rightarrow \quad \tan x=-1,\cos x=-\dfrac{1}{2}$

$\Rightarrow \quad x=n\pi-\dfrac{\pi}{4}\ ,\ x=2n\pi\pm\dfrac{2\pi}{3},n\in I$

7. Do yourself.
8. Given equation is

$\tan 3x+\tan x=2\tan 2x$

$\Rightarrow \quad \dfrac{\sin 4x}{\cos 3x\cos x}=\dfrac{2\sin 2x}{\cos 2x}$

$\Rightarrow \quad \dfrac{2\sin 2x\cos 2x}{\cos 3x\cos x}=\dfrac{2\sin 2x}{\cos 2x}$

$\Rightarrow \quad 2\sin 2x\left(\dfrac{\cos 2x}{\cos 3x\cos x}-\dfrac{1}{\cos 2x}\right)=0$

$\Rightarrow \quad 2\sin 2x=0,\left(\dfrac{\cos 2x}{\cos 3x\cos x}=\dfrac{1}{\cos 2x}\right)$

$\Rightarrow \quad \sin 2x=0,\ 2\cos^2 2x=\cos 4x+\cos 2x$

$\Rightarrow \quad \sin 2x=0,\ 2\cos^2 2x=2\cos^2 2x-1+\cos 2x$

$\Rightarrow \quad \sin 2x=0,\ \cos 2x=1$

$\Rightarrow \quad 2x=n\pi,2x=2n\pi$

$\Rightarrow \quad x=\dfrac{n\pi}{2},\ x=n\pi,n\in I$

9. Given equation is

$(1-\tan x)(1+\sin 2x)=(1+\tan x)$

$\Rightarrow \quad (1-\tan x)\left(1+\dfrac{2\tan x}{1+\tan^2 x}\right)=(1+\tan x)$

$\Rightarrow \quad (1-\tan x)(1+\tan x)^2=(1+\tan x)\left(1+\tan^2 x\right)$

$\Rightarrow \quad \left(1-\tan^2 x\right)(1+\tan x)=(1+\tan x)\left(1+\tan^2 x\right)$

$\Rightarrow \quad \left(\left(1-\tan^2 x\right)-\left(1+\tan^2 x\right)\right)(1+\tan x)=0$

$\Rightarrow \quad \tan^2 x(1+\tan x)=0$

$\Rightarrow \quad \tan^2 x=0,(1+\tan x)=0$

$\Rightarrow \quad \tan^2 x=0,\tan x=-1$

$\Rightarrow \quad x = n\pi, x = n\pi - \frac{\pi}{4}, n \in I$

10. Given equation is

$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

$\Rightarrow \quad (\sin 3x + \sin x) - 3\sin 2x = (\cos 3x + \cos x) - 3\cos 2x$

$\Rightarrow \quad 2\sin 2x\cos x - 3\sin 2x = 2\cos 2x\cos x - 3\cos 2x$

$\Rightarrow \quad \sin 2x(2\cos x - 3) = (2\cos x - 3)\cos 2x$

$\Rightarrow \quad \frac{\sin 2x}{\cos 2x}(2\cos x - 3) = (2\cos x - 3)$

$\Rightarrow \quad \frac{\sin 2x}{\cos 2x} = 1$

$\Rightarrow \quad \tan 2x = 1$

$\Rightarrow \quad 2x = n\pi + \frac{\pi}{4}$

$\Rightarrow \quad x = \frac{n\pi}{2} + \frac{\pi}{8}, n \in I$

EXERCISE 8

1. Do yourself.
2. Do yourself.
3. Given equation is

$\sin 4x\sin 2x = \cos 6x - \cos 2x$

$\Rightarrow \quad \sin 4x\sin 2x = -2\sin 4x\sin 2x$

$\Rightarrow \quad 3\sin 4x\sin 2x = 0$

$\Rightarrow \quad \sin 4x = 0, \sin 2x = 0$

$\Rightarrow \quad 4x = n\pi, 2x = n\pi, n \in I$

$\Rightarrow \quad x = \frac{n\pi}{4}, x = \frac{n\pi}{2}, n \in I$

4. Given equation is

$\sec x\cos 5x + 1 = 0$

$\Rightarrow \quad \cos 5x + \cos x = 0$

$\Rightarrow \quad 2\cos(3x)\cos(2x) = 0$

$\Rightarrow \quad 2\cos(3x) = 0, \cos(2x) = 0$

$\Rightarrow \quad \cos(3x) = 0, \cos(2x) = 0$

$\Rightarrow \quad 3x = (2n+1)\frac{\pi}{2}, 2x = (2n+1)\frac{\pi}{2}, n \in I$

$\Rightarrow \quad x = (2n+1)\frac{\pi}{6}, x = (2n+1)\frac{\pi}{4}, n \in I$

Hence, the solutions are

$x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$

5. Given equation is

$\cos(6x)\cos x = -1$

$\Rightarrow \quad 2\cos(6x)\cos x = -2$

$\Rightarrow \quad \cos 7x + \cos 5x = -2$

It is possible only when

$\cos(7x) = -1, \cos(5x) = -1$

$\Rightarrow \quad x = (2n+1)\frac{\pi}{7}, x = (2n+1)\frac{\pi}{5}, n \in I$

EXERCISE 9

1. Given equation is

$5\sin^2 x - 7\sin x\cos x + 10\cos^2 x = 4$

$\Rightarrow \quad 5\tan^2 x - 7\tan x + 10 = 4\sec^2 x$

$\Rightarrow \quad 5\tan^2 x - 7\tan x + 10 = 4 + 4\tan^2 x$

$\Rightarrow \quad \tan^2 x - 7\tan x + 6 = 0$

$\Rightarrow \quad (\tan x - 1)(\tan x - 6) = 0$

$\Rightarrow \quad \tan x = 1, 6$

$\Rightarrow \quad x = n\pi + \frac{\pi}{4}, x = n\pi + \alpha, \alpha = \tan^{-1}(5)$

2. Given equation is

$2\sin^2 x - 5\sin x\cos x - 8\cos^2 x = -3$

$\Rightarrow \quad 2\tan^2 x - 5\tan x - 8 = -3\sec^2 x$

$\Rightarrow \quad 2\tan^2 x - 5\tan x - 8 = -3 - 3\tan^2 x$

$\Rightarrow \quad 5\tan^2 x - 5\tan x - 5 = 0$

$\Rightarrow \quad \tan^2 x - \tan x - 1 = 0$

$\Rightarrow \quad \tan x = \frac{1 \pm \sqrt{5}}{2}$

$\Rightarrow \quad x = n\pi + \alpha, \alpha = \tan^{-1}\left(\frac{1 \pm \sqrt{5}}{2}\right)$

3. Given equation is

$\sin^3 x\cos x + \sin^2 x\cos^2 x + \sin x\cos^3 x = 1$

$\Rightarrow \quad \sin x\cos x\left[\sin^2 x + \sin x\cos x + \cos^2 x\right] = 1$

$\Rightarrow \quad \sin x\cos x[1 + \sin x\cos x] = 1$

$\Rightarrow \quad 2\sin x\cos x[2 + 2\sin x\cos x] = 4$

$\Rightarrow \quad \sin(2x)(2 + \sin(2x)) = 4$

$\Rightarrow \quad \sin^2(2x) + 2\sin(2x) - 4 = 0$

$\Rightarrow \quad \sin(2x) = \frac{-2 \pm \sqrt{20}}{2}$

$$\Rightarrow \quad \sin(2x) = \frac{-2 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}$$

It is not possible .
So, it has no solution.

EXERCISE 10

1. Given equation is
$(\cos x - \sin x)(2\tan x + \sec x) + 2 = 0$

$$\Rightarrow \quad (\cos x - \sin x)(2\sin x + 1) + 2\cos x = 0$$

$$\Rightarrow \quad \left(\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)} - \frac{2\tan(x/2)}{1+\tan^2(x/2)}\right)\left(\frac{4\tan(x/2)}{1+\tan^2(x/2)} + 1\right) + 2\left(\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right) = 0$$

Put $\tan\left(\frac{x}{2}\right) = t$ and then solve it.

2. Given equation is

$$\frac{\sin^3\frac{x}{2} - \cos^3\frac{x}{2}}{2+\sin x} = \frac{\cos x}{3}$$

$$\Rightarrow \quad \frac{\left(\sin\left(\frac{x}{2}\right) - \cos\left(\frac{x}{2}\right)\right)\left(1 + \frac{\sin x}{2}\right)}{2+\sin x} = \frac{\left(\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right)}{3}$$

$$\Rightarrow \quad -\frac{3}{2} = \left(\cos\left(\frac{x}{2}\right) + \sin\frac{x}{2}\right)$$

It is not possible.
So, it has no solution.

3. Given equation is

$$\cot\left(\frac{x}{2}\right) - \text{cosec}\left(\frac{x}{2}\right) = \cot x$$

$$\Rightarrow \quad \frac{\cos\left(\frac{x}{2}\right) - 1}{\sin\left(\frac{x}{2}\right)} = \cot x$$

$$\Rightarrow \quad 2\sin^2\left(\frac{x}{2}\right) = -\sin\left(\frac{x}{2}\right)\cot x$$

$$\Rightarrow \quad \left(2\sin\left(\frac{x}{2}\right) + \cot x\right)\sin\left(\frac{x}{2}\right) = 0$$

$$\Rightarrow \quad \left(2\sin\left(\frac{x}{2}\right) + \cot x\right) = 0, \sin\left(\frac{x}{2}\right) = 0$$

when $\sin\left(\frac{x}{2}\right) = 0$

$$\Rightarrow \quad x = 2n\pi, n \in I$$

when $\left(2\sin\left(\frac{x}{2}\right) + \cot x\right) = 0$

$$\Rightarrow \quad 2\sin\left(\frac{x}{2}\right) = -\frac{\cos x}{\sin x}$$

$$\Rightarrow \quad 2\sin\left(\frac{x}{2}\right) = -\frac{\left(\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$\Rightarrow \quad 4\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = 0$$

$$\Rightarrow \quad \cos^2\left(\frac{x}{2}\right) + 4\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = 0$$

For all real x,

$$16\sin^4\left(\frac{x}{2}\right) + 4\sin^2\left(\frac{x}{2}\right) \geq 0$$

$$\Rightarrow \quad 4\sin^4\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) = 0$$

$$\Rightarrow \quad \sin^2\left(\frac{x}{2}\right)\left(4\sin^2\left(\frac{x}{2}\right) + 1\right) = 0$$

$$\Rightarrow \quad \sin^2\left(\frac{x}{2}\right) = 0, \left(4\sin^2\left(\frac{x}{2}\right) + 1\right) = 0$$

$$\Rightarrow \quad \sin^2\left(\frac{x}{2}\right) = 0$$

$$\Rightarrow \quad \sin\left(\frac{x}{2}\right) = 0$$

$$\Rightarrow \quad x = 2n\pi, n \in I$$

4. Given equation is
$\sin(\theta + \alpha) = k\sin(2\theta)$

$$\Rightarrow \quad \sin\theta\cos\alpha + \cos\theta\sin\alpha = k\sin(2\theta)$$

$$\Rightarrow \quad \left(\frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)\cos\alpha + \left(\frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)\sin\alpha = 2k\left(\frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)\left(\frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)$$

$$\Rightarrow \quad \left(\frac{2t}{1+t^2}\right)\cos\alpha+\left(\frac{1-t^2}{1+t^2}\right)\sin\alpha$$

$$=2k.\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)$$

$$\Rightarrow \quad 2t\left(1+t^2\right)\cos\alpha+\left(1-t^4\right)\sin\alpha=4kt\left(1-t^2\right)$$

$$\Rightarrow \quad (\sin\alpha)t^4-(4k+2\cos\alpha)t^3 +(4k-2\cos\alpha)t-\sin\alpha=0.$$

Let t_1, t_2, t_3 and t_4 are its four roots

$$\sum t_1=\frac{4k+2\cos\alpha}{\sin\alpha}=s_1$$

$$\sum t_1t_2=0=s_2$$

$$\sum t_1t_2t_3=\frac{2\cos\alpha-4k}{\sin\alpha}=s_3$$

$$\sum t_1t_2t_3t_4=-\frac{\sin\alpha}{\sin\alpha}=-1=s_4$$

Now, $\tan\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right)=\frac{s_1-s_3}{1-s_2+s_4}$

$$\Rightarrow \quad \tan\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right)=\infty$$

$$\Rightarrow \quad \tan\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right)=\tan\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right)=n\pi+\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad (\theta_1+\theta_2+\theta_3+\theta_4)=(2n+1)\pi, n\in I.$$

EXERCISE 11

1. Given equation is

$\sin x+\cos x=1-\sin x\cos x$ .....(i)

Put $\sin x+\cos x=t$

$$\Rightarrow \sin x.\cos x=\frac{t^2-1}{2}$$

Now, equation (i) becomes

$$t=1-\frac{t^2-1}{2}$$

$$\Rightarrow \quad 2t=2-t^2+1$$

$$\Rightarrow \quad t^2+2t-3=0$$

$$\Rightarrow \quad (t+3)(t-1)=0$$

$$\Rightarrow \quad (t+3)=0,(t-1)=0$$

$$\sin x+\cos x=1, \sin x+\cos x=-3$$

$$\Rightarrow \quad \sin x+\cos x=1$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad x=n\pi+(-1)^n\left(\frac{\pi}{4}\right)-\frac{\pi}{4}, n\in I$$

2. Given equation is

$1+\sin^3 x+\cos^3 x=\frac{3}{2}\sin 2x$ ....(i)

$$\Rightarrow \quad 1+(\sin x+\cos x)(1-\sin x\cos x)=3\sin x\cos x$$

Put $\sin x+\cos x=t$

$$\Rightarrow \quad \sin x.\cos x=\frac{t^2-1}{2}$$

Now, equation (i) becomes

$$1+t\left(1-\frac{t^2-1}{2}\right)=\frac{3}{2}\left(t^2-1\right)$$

$$\Rightarrow \quad 2+t\left(3-t^2\right)=3\left(t^2-1\right)$$

$$\Rightarrow \quad 2+3t-t^3-3t^2+3=0$$

$$\Rightarrow \quad 3t-t^3-3t^2+5=0$$

$$\Rightarrow \quad t^3+3t^2-3t-5=0$$

$$\Rightarrow \quad t^3+t^2+2t^2+2t-5t-5=0$$

$$\Rightarrow \quad t^2(t+1)+2t(t+1)-5(t+1)=0$$

$$\Rightarrow \quad \left(t^2+2t-5\right)(t+1)=0$$

$$\Rightarrow \quad t=-1, t=\frac{-2\pm\sqrt{24}}{2}$$

$$\Rightarrow \quad \sin x+\cos x=-1, \frac{-2\pm\sqrt{24}}{2}$$

$$\Rightarrow \quad \sin x+\cos x=-1$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x=-\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\left(x+\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad x=n\pi+(-1)^n\left(-\frac{\pi}{4}\right)-\frac{\pi}{4}, n\in I$$

3. Given equation is

$$\sin 2x-12(\sin x-\cos x)+12=0$$

$$2\sin x\cos x-12(\sin x-\cos x)+12=0$$

$\sin x\cos x-6(\sin x-\cos x)+6=0$ ...(i)

Put $\sin x+\cos x=t$

$\Rightarrow \quad \sin x . \cos x = \dfrac{t^2 - 1}{2}$

Now, equation (i) becomes

$\Rightarrow \quad \dfrac{t^2-1}{2} - 6t + 6 = 0$

$\Rightarrow \quad t^2 - 1 - 12t + 12 = 0$

$\Rightarrow \quad t^2 - 12t + 11 = 0$

$\Rightarrow \quad (t-1)(t-11) = 0$

$\Rightarrow \quad \sin x + \cos x = 1$

$\Rightarrow \quad \sin\left(x + \dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad x = n\pi + (-1)^n \left(\dfrac{\pi}{4}\right) - \dfrac{\pi}{4}, n \in I$

$\Rightarrow \quad x = 0, \dfrac{\pi}{2}, 2\pi$

Hence, the solution is

$\sin 6x = \sin \dfrac{3\pi}{2}$ ·

4. The given equation can be written as

$\sin 6x + \cos 4x = -2$

$\Rightarrow \quad \sin 6x = -1$ and $\cos 4x = -1$

$\Rightarrow \quad \sin 6x = \sin \dfrac{3\pi}{2}, \ \cos 4x = \cos \pi$

$\Rightarrow \quad 6x = 2n\pi + \dfrac{3\pi}{2}, 4x = 2n\pi + \pi, n \in Z$

$\Rightarrow \quad x = \dfrac{n\pi}{3} + \dfrac{\pi}{4}, x = \dfrac{n\pi}{2} + \dfrac{\pi}{4}, n \in Z$

$\Rightarrow \quad x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{12}, \dfrac{11\pi}{12}, ...$

$...x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}, ...$

$\Rightarrow \quad x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

Hence, the general solution will be,

$\Rightarrow \quad x = 2n\pi + \dfrac{\pi}{4}, 2n\pi + \dfrac{5\pi}{4}, n \in Z$

$\Rightarrow \quad x = 2n\pi + \dfrac{\pi}{4}, (2n+1)\pi + \dfrac{\pi}{4}, n \in Z$

$\Rightarrow \quad x = m\pi + \dfrac{\pi}{4}, m \in Z$

EXERCISE 12

2. Given equation is

$\sin^4 x = 1 + \tan^8 x$

It is possible only when

$\sin^4 x = 1, \tan^8 x = 0$

$\Rightarrow \quad \sin^2 x = 1, \tan x = 0$

$\Rightarrow \quad x = n\pi \pm \dfrac{\pi}{2}, x = n\pi, n \in I$

There is no common value which satisfies both the above equations

Hence, the equation has no solution.

3. Given $\sin^2 x + \cos^2 y = 2\sec^2 z$

Here, L.H.S ≤ 2 and R.H.S ≥ 2

It is possible only when

$\sin^2 x = 1, \cos^2 y = 1, \sec^2 z = 1$

$\Rightarrow \quad \cos^2 x = 0, \sin^2 y = 0, \cos^2 z = 1$

$\Rightarrow \quad \cos^2 x = 0, \sin^2 y = 0, \sin^2 z = 1$

$\Rightarrow \quad \cos x = 0, \sin y = 0, \sin z = 0$

$\Rightarrow \quad x = (2n+1)\dfrac{\pi}{2}, y = m\pi, z = k\pi$

where, $n, m, k \in I$

4. Given equation is

$\sin 3x + \cos 2x + 2 = 0$

It is possible only when

$\sin 3x = -1, \cos 2x = -1$

$\Rightarrow \quad 3x = \dfrac{3\pi}{2}, \ 2x = \pi$

$\Rightarrow \quad x = \dfrac{\pi}{2}, \ x = \dfrac{\pi}{2}$

Hence, the general solution is

$x = 2n\pi + \dfrac{\pi}{2}, n \in I$

5. Given equation is

$\cos 4x + \sin 5x = 2$

It is possible only when

$\cos 4x = 1, \sin 5x = 1$

$\Rightarrow \quad 4x = 2n\pi, 5x = (4n+1)\dfrac{\pi}{2}$

$\Rightarrow \quad x = \dfrac{2n\pi}{4}, x = (4n+1)\dfrac{\pi}{10}$

Thus, $x = \dfrac{\pi}{2}$ satisfies both

Hence, the solution is

$x = 2n\pi + \dfrac{\pi}{2} = (4n+1)\dfrac{\pi}{2}, n \in I$

EXERCISE 13

1. Given equation is

$$2^{1+|\cos x|+\cos^2 x+|\cos x|^3+\cos^4 x+|\cos x|^5+........\text{to } \infty} = 4$$

$\Rightarrow \quad 2^{\frac{1}{1-|\cos x|}} = 4 = 2^2$

$\Rightarrow \quad \dfrac{1}{1-|\cos x|} = 2$

$\Rightarrow \quad 1-|\cos x| = \dfrac{1}{2}$

$\Rightarrow \quad |\cos x| = 1-\dfrac{1}{2} = \dfrac{1}{2}$

$\Rightarrow \quad \cos x = \pm\dfrac{1}{2}$

Hence, the values of x are $\pm\dfrac{\pi}{3}, \pm\dfrac{2\pi}{3}$

2. Given equation is

$1+\sin\theta+\sin^2\theta+... = 4+2\sqrt{3}$

$\Rightarrow \quad \dfrac{1}{1-\sin\theta} = 4+2\sqrt{3}$

$\Rightarrow \quad 1-\sin\theta = \dfrac{1}{4+2\sqrt{3}}$

$\Rightarrow \quad \sin\theta = 1-\dfrac{1}{4+2\sqrt{3}}$

$\Rightarrow \quad \sin\theta = 1-\dfrac{1}{4+2\sqrt{3}} = 1-\dfrac{2-\sqrt{3}}{2} = \dfrac{\sqrt{3}}{2}$

$\Rightarrow \quad \theta = n\pi+(-1)^n\left(\dfrac{\pi}{3}\right), n \in I$

3. Given equation is

$|\cos x|^{\sin^2 x-\frac{3}{2}\sin x+\frac{1}{2}} = 1$

$\Rightarrow \quad \left(\sin^2 x-\dfrac{3}{2}\sin x+\dfrac{1}{2}\right)\log|\cos x| = 0$

$\Rightarrow \quad (2\sin^2 x-3\sin x+1)\log|\cos x| = 0$

$\Rightarrow \quad (\sin x-1)(2\sin x-1)\log|\cos x| = 0$

$\Rightarrow \quad (\sin x-1) = 0, (2\sin x-1) = 0, \log|\cos x| = 0$

$\Rightarrow \quad \sin x = 1, \sin x = \dfrac{1}{2}, \log|\cos x| = 0$

$\Rightarrow \quad \sin x = \dfrac{1}{2}, |\cos x| = 1$

$\Rightarrow \quad \sin x = \dfrac{1}{2}, \cos = 1, \cos x = -1$

$\Rightarrow \quad x = n\pi+(-1)^n\left(\dfrac{\pi}{6}\right), x = 2n\pi, x = (2n+1)\pi$

4. Given equation is

$e^{\sin x}-e^{-\sin x}-4 = 0$

$\Rightarrow \quad t-\dfrac{1}{t}-4 = 0, t = e^{\sin x}$

$\Rightarrow \quad t^2-4t-1 = 0$

$\Rightarrow \quad (t-2)^2 = 5$

$\Rightarrow \quad t = 2\pm\sqrt{5}$

$\Rightarrow \quad e^{\sin x} = 2\pm\sqrt{5}$

$\Rightarrow \quad \sin x = \log_e(2\pm\sqrt{5})$

$\Rightarrow \quad \sin x = \log_e(2+\sqrt{5})$

$\Rightarrow \quad \sin x = \log_e(2+\sqrt{5}) > 1$

It is not possible.
So, it has no solution.

5. We have

$e^{[\sin^2 x+\sin^4 x+\sin^6 x+...............\text{to}\infty]\log_e 2}$

$= e^{\left(\frac{\sin^2 x}{1-\cos^2 x}\right)\log_e 2} = e^{\tan^2 x\log_e 2} = 2^{\tan^2 x}$

Let $a = 2^{\tan^2 x}$

Thus, $a^2-9a+8 = 0$

$\Rightarrow \quad (a-1)(a-8) = 0$

$\Rightarrow \quad a = 1, 8$

when $a = 1$, then $2^{\tan^2 x} = 1 = 2^0$

$\Rightarrow \quad 2^{\tan^2 x} = 1 = 2^0$

$\Rightarrow \quad \tan^2 x = 0$

$\Rightarrow \quad x = n\pi, n \in I$

when $a = 8$, then $2^{\tan^2 x} = 8 = 2^3$

$\Rightarrow \quad \tan^2 x = 3$

$\Rightarrow \quad \tan x = \sqrt{3}$

Now, $\dfrac{\cos x}{\cos x+\sin x}$

$= \dfrac{1}{1+\tan x} = \dfrac{1}{\sqrt{3}+1}$

$= \dfrac{(\sqrt{3}-1)}{2}$

6. Given equation is

$\log_{\cos x} \tan x + \log_{\sin x} \cot x = 0$

$\Rightarrow \quad \log_{\cos x}\left(\frac{\sin x}{\cos x}\right) + \log_{\sin x}\left(\frac{\cos x}{\sin x}\right) = 0$

$\Rightarrow \quad \log_{\cos x}(\sin x) + \log_{\sin x}(\cos x) = 2$

It is possible only when

$\sin x = \cos x$

$\Rightarrow \quad \tan x = 1$

$\Rightarrow \quad x = \frac{\pi}{4}$

7. Given equation is

$3^{\sin 2x + 2\cos^2 x} + 3^{1 - \sin 2x + 2\sin^2 x} = 28$

$\Rightarrow \quad 3^{\sin 2x + 2\cos^2 x} + 3^{1 - \sin 2x + 2 - 2\cos^2 x} = 28$

$\Rightarrow \quad 3^{\sin 2x + 2\cos^2 x} + \frac{3^3}{3^{\sin 2x + 2\cos^2 x}} = 28$

$\Rightarrow \quad a + \frac{27}{a} = 28, a = 3^{\sin 2x + 2\cos^2 x}$

$\Rightarrow \quad a^2 - 28a + 27 = 0$

$\Rightarrow \quad (a - 27)(a - 1) = 0$

$\Rightarrow \quad a = 27, 1$

when $a = 27$, then $3^{\sin 2x + 2\cos^2 x} = 3^3$

$\Rightarrow \quad \sin 2x + 2\cos^2 x = 3$

$\Rightarrow \quad \sin 2x + 1 + \cos 2x = 3$

$\Rightarrow \quad \sin 2x + \cos 2x = 2$

It is not possible.

when $a = 1$, then $3^{\sin 2x + 2\cos^2 x} = 3^0$

$\Rightarrow \quad \sin 2x + 2\cos^2 x = 0$

$\Rightarrow \quad \sin 2x + 1 + \cos 2x = 0$

$\Rightarrow \quad \sin 2x + \cos 2x = -1$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin 2x + \frac{1}{\sqrt{2}}\cos 2x = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \sin\left(2x + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \left(2x + \frac{\pi}{4}\right) = n\pi + (-1)^n\left(-\frac{\pi}{4}\right)$

$\Rightarrow \quad x = \frac{n\pi}{2} + (-1)^n\left(-\frac{\pi}{8}\right) - \frac{\pi}{8}, n \in I$

8. Do yourself.

EXERCISE 14

1. Given equation is

$2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}$

Here, L.H.S. < 2 for $0 < x < \frac{\pi}{2}$

and R.H.S. ≥ 2

So, it has no solutions

2. Given equation is

$2\cos^2\left(\frac{x^2 + x}{6}\right) = 2^x + 2^{-x}$

It is possible only when $x = 0$.

Hence, the solution is $x = 0$.

LEVEL III

1. Given equation is

$\cot\left(\frac{x}{2}\right) - \operatorname{cosec}\left(\frac{x}{2}\right) = \cot x$

$\Rightarrow \quad \frac{\cos(x/2) - 1}{\sin(x/2)} = \cot x$

$\Rightarrow \quad -\frac{2\sin^2(x/4)}{\sin(x/2)} = \cot x$

$\Rightarrow \quad -\frac{2\sin^2(x/4)}{2\sin(x/4)\cos(x/4)} = \cot x$

$\Rightarrow \quad \tan(x/4) + \cot x = 0$

$\Rightarrow \quad \frac{\sin(x/4)}{\cos(x/4)} + \frac{\cos x}{\sin x} = 0$

$\Rightarrow \quad \cos\left(x - \frac{x}{4}\right) = 0$

$\Rightarrow \quad \cos\left(\frac{3x}{4}\right) = 0$

$\Rightarrow \quad \frac{3x}{4} = (2n + 1)\frac{\pi}{2}$

$\Rightarrow \quad x = (4n + 2)\frac{\pi}{3}, n \in I$

2. Given equation is

$8\cos x.\cos 2x.\cos 4x = \frac{\sin 6x}{\sin x}$

$\Rightarrow \quad 4\sin 2x\cos 2x\cos 4x = \sin 6x$

$\Rightarrow \quad 2\sin 4x\cos 4x = \sin 6x$

$\Rightarrow \quad \sin 8x - \sin 6x = 0$

$\Rightarrow \quad 2\cos(7x)\sin x = 0$

$\Rightarrow \quad \cos(7x) = 0, \sin x = 0$

$\Rightarrow \quad (7x) = (2n+1)\dfrac{\pi}{2}, x = n\pi$

$\Rightarrow \quad x = (2n+1)\dfrac{\pi}{14}, x = n\pi, n \in I$

2. Given equation is

$\Rightarrow \quad \dfrac{\tan x}{\tan 2x} + \dfrac{\tan 2x}{\tan x} + 2 = 0$

$\Rightarrow \quad (\tan x + \tan 2x)^2 = 0$

$\Rightarrow \quad (\tan x + \tan 2x) = 0$

$\Rightarrow \quad \sin(2x + x) = 0$

$\Rightarrow \quad \sin(3x) = 0$

$\Rightarrow \quad 3x = n\pi$

$\Rightarrow \quad x = \dfrac{n\pi}{3}, n \in I$

4. Given equation is

$\cos x \cos(6x) = -1$

$\Rightarrow \quad 2\cos(6x)\cos x = -2$

$\Rightarrow \quad \cos(7x) + \cos(5x) = -2$

It is possible only when

$\cos(7x) = -1, \cos(5x) = -1$

$\Rightarrow \quad 7x = (2k+1)\pi, 5x = (2m+1)\pi$

$\Rightarrow \quad x = (2k+1)\dfrac{\pi}{7}, x = (2m+1)\dfrac{\pi}{5}$

when $k = 3$ and $m = 2$, then common value of x is π.
Hence, the general solution is

$x = 2n\pi + \pi = (2n+1)\pi, n \in I$

5. Given equation is

$\cos(4x) + \sin(5x) = 2$

It is possible only when

$\cos(4x) = 1, \sin(5x) = 1$

$\Rightarrow \quad 4x = 2k\pi, 5x = (4m+1)\dfrac{\pi}{2}$

$\Rightarrow \quad x = \dfrac{k\pi}{2}, x = (4m+1)\dfrac{\pi}{10}, k, m \in I$

when $k = 1$ and $m = 1$, then the common value of x is $\dfrac{\pi}{2}$

Hence, the general solution is

$x = \left(2n\pi + \dfrac{\pi}{2}\right) = (4n+1)\dfrac{\pi}{2}, n \in I$

6. Given equation is

$(1 + \sin 2x) + 5(\sin x + \cos x) = 0$

$\Rightarrow \quad (\sin x + \cos x)^2 + 5(\sin x + \cos x) = 0$

$\Rightarrow \quad ((\sin x + \cos x) + 5)(\sin x + \cos x) = 0$

$\Rightarrow \quad ((\sin x + \cos x) + 5) = 0, (\sin x + \cos x) = 0$

$\Rightarrow \quad (\sin x + \cos x) = 0$

$\Rightarrow \quad \tan x = -1$

$x = n\pi - \dfrac{\pi}{4}, n \in I$

7. Given equation is

$\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$

$\Rightarrow \quad (\sin 3x + \sin x) + \sin 2x = (\cos 3x + \cos x) + \cos 2x$

$\Rightarrow \quad 2\sin 2x \cos x + \sin 2x = 2\cos 2x \cos x + \cos 2x$

$\Rightarrow \quad \sin 2x(2\cos x + 1) = \cos 2x(2\cos x + 1)$

$\Rightarrow \quad (\sin 2x - \cos 2x)(2\cos x + 1) = 0$

$\Rightarrow \quad (\sin 2x - \cos 2x) = 0, (2\cos x + 1) = 0$

when $(\sin 2x - \cos 2x) = 0$

$\Rightarrow \quad \tan 2x = 1$

$\Rightarrow \quad 2x = n\pi + \dfrac{\pi}{4}$

$\Rightarrow \quad x = \dfrac{n\pi}{2} + \dfrac{\pi}{8}$

when $2\cos x + 1 = 0$

$\Rightarrow \quad \cos x = -\dfrac{1}{2}$

$\Rightarrow \quad x = 2n\pi \pm \dfrac{2\pi}{3}$

Hence, the solution is

$x = \dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{8}, \dfrac{13\pi}{8}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

8. Given equation is

$\dfrac{\sin^3 x}{\cos x} + \dfrac{\cos^3 x}{\sin x} - 2\sin x \cos x = 2$

$\Rightarrow \quad \dfrac{\sin^4 x + \cos^4 x}{\sin x \cos x} = 2\sin x \cos x + 2$

$\Rightarrow \quad 1 - 2\sin^2 x \cos^2 x = 2\sin^2 x \cos^2 x + \sin(2x)$

$\Rightarrow \quad 4\sin^2 x \cos^2 x + \sin(2x) - 1 = 0$

$\Rightarrow \quad \sin^2 2x + \sin(2x) - 1 = 0$

$\Rightarrow \quad \sin(2x) = \dfrac{-1 \pm \sqrt{1+4}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \quad \sin(2x) = \dfrac{-1+\sqrt{5}}{2}$

$\Rightarrow \quad \sin(2x) = \sin\alpha, \alpha = \sin^{-1}\left(\dfrac{-1+\sqrt{5}}{2}\right)$

$\Rightarrow \quad (2x) = n\pi + (-1)^n \alpha$

$\Rightarrow \quad x = \dfrac{n\pi}{2} + (-1)^n \dfrac{\alpha}{2}$

9. Given equation is

$\sin^2 4x + \cos^2 x = 2\sin 4x \cos^4 x$

$\Rightarrow \quad \sin^2 4x - 2\sin 4x\cos^4 x + \cos^2 x = 0$

$\Rightarrow \quad \left(\sin^2 4x - \cos^4 x\right)^2 + \cos^2 x - \cos^8 x = 0$

$\Rightarrow \quad \left(\sin^2 4x - \cos^4 x\right)^2 + \cos^2 x\left(1 - \cos^6 x\right) = 0$

It is possible only when

$\left(\sin^2 4x - \cos^4 x\right) = 0, \cos^2 x\left(1-\cos^6 x\right) = 0$

Now, $\quad \cos x = 0, \cos^2 x = 1$

when $\quad \cos x = 0$ then $x = (2n+1)\dfrac{\pi}{2}$

So, $\quad \sin 4\left(n + \dfrac{1}{2}\right)\pi = 0$

which is true

when $\quad \cos^2 x = 1$, then $x = n\pi$

which will not satisfy the equation

$\sin(4x) - \cos^4 x = 0$

Hence, the solution is $x = (2n+1)\dfrac{\pi}{2}$

10. Given equation is

$\sin^4 x + \cos^4 x = \dfrac{7}{2}\sin x \cos x$

$\Rightarrow \quad 1 - 2\sin^2 x\cos^2 x = \dfrac{7}{4}\sin(2x)$

$\Rightarrow \quad 4 - 8\sin^2 x\cos^2 x = 7\sin(2x)$

$\Rightarrow \quad 4 - 2\sin^2 2x - 7\sin(2x) = 0$

$\Rightarrow \quad 2\sin^2 2x + 7\sin(2x) - 4 = 0$

$\Rightarrow \quad 2\sin^2 2x + 8\sin(2x) - \sin(2x) - 3 = 0$

$\Rightarrow \quad \left(2\sin(2x)+1\right)\left(\sin(2x)+4\right) = 0$

$\Rightarrow \quad \left(2\sin(2x)+1\right) = 0, \left(\sin(2x)+4\right) = 0$

$\Rightarrow \quad \left(2\sin(2x)+1\right) = 0$

$\Rightarrow \quad \sin(2x) = -\dfrac{1}{2}$

$\Rightarrow \quad \sin(2x) = \sin\left(-\dfrac{\pi}{6}\right)$

$\Rightarrow \quad 2x = n\pi + (-1)^n\left(-\dfrac{\pi}{6}\right)$

$\Rightarrow \quad x = \dfrac{n\pi}{2} + (-1)^n\left(-\dfrac{\pi}{12}\right), n \in I$

11. Given equation is

$\sin^4 x + \cos^4 x = \cos(4x) + \dfrac{1}{2}$

$\Rightarrow \quad 1 - 2\sin^2 x\cos^2 x = \cos(4x) + \dfrac{1}{2}$

$\Rightarrow \quad 2 - 4\sin^2 x\cos^2 x = 2\cos(4x) + 1$

$\Rightarrow \quad 2 - \sin^2 2x = 2\cos(4x) + 1$

$\Rightarrow \quad 2\cos(4x) + \sin^2 2x = 1$

$\Rightarrow \quad 2\left(1 - 2\sin^2 2x\right) + \sin^2(2x) = 1$

$\Rightarrow \quad 3\sin^2(2x) = 1$

$\Rightarrow \quad \sin^2(2x) = \dfrac{1}{3}$

$\Rightarrow \quad \sin^2(2x) = \dfrac{1}{3} = \sin^2\alpha$

$\Rightarrow \quad 2x = n\pi \pm \alpha$

$\Rightarrow \quad x = \dfrac{n\pi}{2} \pm \dfrac{\alpha}{2}, n \in I, \alpha = \sin^{-1}\left(\dfrac{1}{3}\right)$

12. Given equation is

$\sin^4 x + \sin^4\left(x + \dfrac{\pi}{4}\right) = \dfrac{1}{4}$

$\Rightarrow \quad 4\sin^4 x + 4\sin^4\left(x + \dfrac{\pi}{4}\right) = 1$

$\Rightarrow \quad \left(2\sin^2 x\right)^2 + \left(2\sin^2\left(x + \dfrac{\pi}{4}\right)\right)^2 = 1$

$\Rightarrow \quad \left(1 - \cos(2x)\right)^2 + \left(1 - \cos\left(2x + \dfrac{\pi}{2}\right)\right)^2 = 1$

$\Rightarrow \quad \left(1 - \cos(2x)\right)^2 + \left(1 + \sin(2x)\right)^2 = 1$

$\Rightarrow \quad 1 - 2\cos(2x) + 1 + 2\sin(2x) = 0$

$\Rightarrow \quad 2 - 2\left(\cos(2x) - \sin(2x)\right) = 0$

$\Rightarrow \quad (\cos(2x)-\sin(2x))=1$

$\Rightarrow \quad \left(\frac{1}{\sqrt{2}}\cos(2x)-\frac{1}{\sqrt{2}}\sin(2x)\right)=\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \cos\left(2x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \left(2x+\frac{\pi}{4}\right)=2n\pi\pm\frac{\pi}{4}$

$\Rightarrow \quad x=n\pi, x=n\pi-\frac{\pi}{4}, n\in I$

13. We have $a=\cos\left(x+\frac{\pi}{3}\right)+\cos x$

$=\cos x\cos\left(\frac{\pi}{3}\right)-\sin x\sin\left(\frac{\pi}{3}\right)+\cos x$

$=\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x+\cos x$

$=\frac{3}{2}\cos x-\frac{\sqrt{3}}{2}\sin x$

The equation will provide us a real solution if

$\frac{3}{2}\cos x-\frac{\sqrt{3}}{2}\sin x$

$\Rightarrow \quad -\sqrt{3}\le a\le\sqrt{3}$

14. Let $f(x)=\cos x-x+\frac{1}{2}$

Now, $f(0)=1+\frac{1}{2}=\frac{3}{2}>0$

and $f\left(\frac{\pi}{2}\right)=0-\frac{\pi}{2}+\frac{1}{2}=\frac{1}{2}-\frac{\pi}{2}<0$

By intermediate value theorem there is a root lies in $\left(0,\frac{\pi}{2}\right)$.

Hence, the number of roots is 1.

15. Now, $\cos(xy)\tan(xy)=xy$

$\Rightarrow \quad \sin(xy)=xy$

It is possible only when $xy=0$

$\Rightarrow \quad x=1$ and $y=0$

Thus, the solution is (1, 0).

Hence, the number of integral ordered pairs is 1.

16. Given equation is

$\sin^{2016}x-\cos^{2016}x=1$

$\Rightarrow \quad \sin^{2016}x=\cos^{2016}x+1$

It is possible only when

$\sin^{2016}x=1$, $\cos^{2016}x=0$

$\Rightarrow \quad \sin x=1$, $\cos x=0$

Hence, the solution is

$x=2n\pi+\frac{\pi}{2}$, $n\in I$

Thus, the number of solutions is 1.

17. Given equation is

$x^2+2x\sin(xy)+1=0$

$\Rightarrow \quad (x+\sin(xy))^2+(1-\sin^2(xy))=0$

$\Rightarrow \quad (x+\sin(xy))^2+\cos^2(xy)=0$

It is possible only when,

$(x+\sin(xy))^2=0$, $\cos^2(xy)=0$

$\Rightarrow \quad (x+\sin(xy))=0$, $\cos(xy)=0$

$\Rightarrow \quad \cos(xy)=0$

$\Rightarrow \quad xy=(2n+1)\frac{\pi}{2}, n\in I$

when $x=1$, $n=0$, then $y=\frac{\pi}{2}$

when $x=-1$, $n=1$, then $y=\frac{3\pi}{2}$

Hence, the number of ordered pairs are

$\left(1,\frac{\pi}{2}\right),\left(-1,\frac{3\pi}{2}\right)$

18. Given equation is

$\sin 5x \cdot \cos 3x=\sin 6x \cdot \cos 2x$

$\Rightarrow \quad \sin(5x)\cos(3x)=2\sin(3x)\cos(3x)\cos(2x)$

$\Rightarrow \quad (\sin(5x)-2\sin(3x)\cos(2x))\cos(3x)=0$

$\Rightarrow \quad (\sin(5x)-\sin(5x)-\sin(x))\cos(3x)=0$

$\Rightarrow \quad \sin(x)\cos(3x)=0$

$\Rightarrow \quad \sin(x)=0, \cos(3x)=0$

$x=n\pi, x=(2n+1)\frac{\pi}{6}, n\in I$

$x=0,\pi,\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6}$

Hence, the number of solutions is 5.

19. Given equation is

$\cos 3x \cdot \tan 5x=\sin 7x$

$\Rightarrow \quad \cos(3x)\sin(5x)=\sin(7x)\cos(5x)$

$\Rightarrow \quad 2\cos(3x)\sin(5x) = 2\sin(7x)\cos(5x)$

$\Rightarrow \quad \sin(8x) + \sin(2x) = \sin(12x) + \sin(2x)$

$\Rightarrow \quad \sin(8x) = \sin(12x)$

$\Rightarrow \quad \sin(12x) - \sin(8x) = 0$

$\Rightarrow \quad 2\cos(10x)\sin(2x) = 0$

$\Rightarrow \quad \cos(10x) = 0, \sin(2x) = 0$

$\Rightarrow \quad 10x = (2n+1)\frac{\pi}{2}, 2x = n\pi$

$\Rightarrow \quad x = (2n+1)\frac{\pi}{20}, x = \frac{n\pi}{2}, n \in I$

$\Rightarrow \quad x = 0, \frac{\pi}{20}$

Hence, the number of solutions is 2.

20. Given equation is

$$2\tan x - \lambda(1 + \tan^2 x) = 0$$

$\Rightarrow \quad \lambda \tan^2 x - 2\tan x + \lambda = 0$

Let it has two roots, say, tan B and tan C

Now, $\quad \tan B + \tan C = \frac{2}{\lambda}$

$\Rightarrow \quad \tan B . \tan C = 1$

Now, $\tan(B+C) = \frac{\tan B + \tan C}{1 - \tan B \tan C}$

$\Rightarrow \quad \tan(\pi - A) = \infty$

$\Rightarrow \quad (\pi - A) = \frac{\pi}{2}$

$\Rightarrow \quad A = \frac{\pi}{2}$

21. Given equation is

$$\cos^4 x - (a+2)\cos^2 x - (a+3) = 0$$

$\Rightarrow \quad \cos^4 x - 2\cos^2 x - 3 = a(1 + \cos^2 x)$

$\Rightarrow \quad (\cos^2 x - 3)(\cos^2 x + 1) = a(1 + \cos^2 x)$

$\Rightarrow \quad (\cos^2 x - 3) = a$

$\Rightarrow \quad a + 3 = \cos^2 x$

Clearly, $0 \le a + 3 \le 1$

$\Rightarrow \quad -3 \le a \le -2$

22. Given equation is

$$\sin x + \sin\frac{\pi}{8}\left(\sqrt{(1-\cos x)^2 + \sin^2 x}\right) = 0$$

$\Rightarrow \quad \sin x + \sin\left(\frac{\pi}{8}\right)\left(\sqrt{2(1-\cos x)}\right) = 0$

$\Rightarrow \quad \sin x = -\sin\left(\frac{\pi}{8}\right)\sqrt{2(1-\cos x)}$

$\Rightarrow \quad \sin^2 x = 2\sin^2\left(\frac{\pi}{8}\right)(1-\cos x)$

$\Rightarrow \quad \sin^2 x = \left(1 - \frac{1}{\sqrt{2}}\right)(1-\cos x)$

$\Rightarrow \quad 4\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) = \left(1 - \frac{1}{\sqrt{2}}\right)2\sin^2\left(\frac{x}{2}\right)$

$\Rightarrow \quad 2\cos^2\left(\frac{x}{2}\right) = \left(1 - \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad 1 + \cos x = \left(1 - \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad \cos x = -\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \cos x = \cos\left(\frac{3\pi}{4}\right)$

$\Rightarrow \quad x = 2n\pi \pm \frac{3\pi}{4}, n \in I$

Hence, the solution is $x = \frac{13\pi}{4}$

23. Given equation is

$$\sin^4 x - (k+2)\sin^2 x - (k+3) = 0$$

$\Rightarrow \quad \sin^4 x - 2\sin^2 x - 3 = k(\sin^2 x + 1)$

$\Rightarrow \quad k = \frac{\sin^4 x - 2\sin^2 x - 3}{(\sin^2 x + 1)}$

$\Rightarrow \quad k = \frac{(\sin^2 x + 1)(\sin^2 x - 3)}{(\sin^2 x + 1)}$

$\Rightarrow \quad k = (\sin^2 x - 3)$

$\Rightarrow \quad k + 3 = \sin^2 x$

$\Rightarrow \quad 0 \le k + 3 \le 1$

$\Rightarrow \quad -3 \le k \le -2$

24. Given equation is

$$4.16^{\sin^2 x} = 2^{6\sin x}$$

$\Rightarrow \quad 4.4^{2\sin^2 x} = 4^{3\sin x}$

$\Rightarrow \quad 4^{1+2\sin^2 x} = 4^{3\sin x}$

$\Rightarrow \quad 1 + 2\sin^2 x = 3\sin x$

$\Rightarrow \quad 2\sin^2 x - 3\sin x + 1 = 0$

$\Rightarrow \quad 2\sin^2 x - 2\sin x - \sin x + 1 = 0$

$\Rightarrow \quad 2\sin x(\sin x - 1) - (\sin x - 1) = 0$

$\Rightarrow \quad (2\sin x - 1)(\sin x - 1) = 0$

$\Rightarrow \quad \sin x = \frac{1}{2}, 1$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$

Thus, the number of principal solutions is 3.

25. Given equation is

$$\sec x = 1 + \cos x + \cos^2 x + \cos^3 x + ...$$

$\Rightarrow \quad \sec x = \frac{1}{1 - \cos x}$

$\Rightarrow \quad \frac{1}{\cos x} = \frac{1}{1 - \cos x}$

$\Rightarrow \quad 2\cos x = 1$

$\Rightarrow \quad \cos x = \frac{1}{2}$

$\Rightarrow \quad x = 2n\pi \pm \frac{\pi}{3}, n \in I$

INTEGER TYPE QUESTIONS

1. Given equation is

$$3\sin^2 x - 7\sin x + 2 = 0$$

$\Rightarrow \quad 3\sin^2 x - 6\sin x - \sin x + 2 = 0$

$\Rightarrow \quad 3\sin x(\sin x - 2) - (\sin x - 2) = 0$

$\Rightarrow \quad (3\sin x - 1)(\sin x - 2) = 0$

$\Rightarrow \quad \sin x = \frac{1}{3}, 2$

$\Rightarrow \quad \sin x = \frac{1}{3}$

Hence, the number of real solutions is 6.

2. Given equation is

$2\cos x + 3\sin x = k + 1$

$\Rightarrow \quad -\sqrt{13} \le (k+1) \le \sqrt{13}$

$\Rightarrow \quad -\sqrt{13} - 1 \le k \le \sqrt{13} - 1$

$\Rightarrow \quad k = -4, -3, -2, -1, 0, 1, 2$

Hence, the number of integral values of k is 7

3. Given equation is

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$

$$\begin{vmatrix} \sin x + 2\cos x & \cos x & \cos x \\ \sin x + 2\cos x & \sin x & \cos x \\ \sin x + 2\cos x & \cos x & \sin x \end{vmatrix} = 0$$

$(C_1 \to C_1 + C_2 + C_3)$

$$(\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

$$(\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix} = 0$$

$(\sin x + 2\cos x)(\sin x - \cos x)^2 = 0$

$\tan x = 1, -2$

So, there is only one solution $x = \frac{\pi}{4}$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

4. Given equation is

$\sin x + \sin y = \sin(x + y)$

$2\sin\left(\frac{x+y}{2}\right)\left(\cos\left(\frac{x-y}{2}\right) - \cos\left(\frac{x+y}{2}\right)\right) = 0$

$4\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right) = 0$

$\sin\left(\frac{x+y}{2}\right) = 0, \sin\left(\frac{x}{2}\right) = 0, \sin\left(\frac{y}{2}\right) = 0$

$x + y = 0, x = 0, y = 0$

It is also given that $|x| + |y| = 1$

when $x = 0$, then $|y| = 1 \Rightarrow y = \pm 1$

when $y = 0$, then $|x| = 1 \Rightarrow x = \pm 1$

when $y = -x$, then $|x| = \frac{1}{2} \Rightarrow x = \pm\frac{1}{2}$

and then $y = \mp\frac{1}{2}$

Hence, the pairs of solutions are

$(0,1), (0,-1), (1,0), (-1,0), \left(\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right)$

Thus, the numbers of pairs is 6

5. Given expression is

$$f(x) = \begin{vmatrix} 1 + \sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1 + \cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1 + 4\sin 2x \end{vmatrix}$$

$$= \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

$$= \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$= 4\sin 2x + \left(1+\sin^2 x + \cos^2 x\right)$$

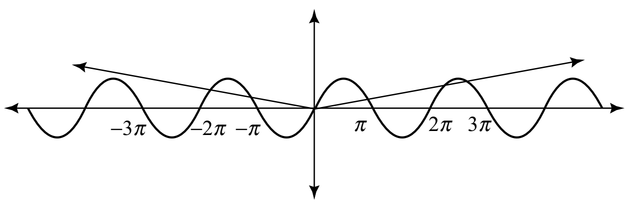
$$= 4\sin 2x + 2$$

So, the maximum value is 6.

6 We have, $-1 \le \sin x \le 1$

$$\Rightarrow \quad -1 \le \frac{|x|}{10} \le 1$$

$$\Rightarrow \quad -10 \le |x| \le 10$$



Clearly, the number of solutions is 6.

7. Given equation is

$\tan x + \cot x = 2\operatorname{cosec} x$

$$\Rightarrow \quad \frac{1}{\sin x \cos x} = \frac{2}{\sin x}$$

$$\Rightarrow \quad \cos x = \frac{1}{2}$$

$$\Rightarrow \quad x = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}$$

Thus, the number of solutions is 4.

8. Given equation is

$$\cos x.\cos 2x.\cos 3x = \frac{1}{4}$$

$$2(2\cos 3x \cos x)\cos 2x = 1$$

$$2(\cos 4x + \cos 2x)\cos 2x = 1$$

$$2(\cos 4x)\cos 2x + 2\cos^2 2x = 1$$

$$2\left(2\cos^2 2x - 1\right)\cos 2x + 2\cos^2 2x = 1$$

$$4\left(\cos^3 2x\right) + 2\cos^2 2x - 2\cos 2x = 1$$

$$2\cos^2 2x(2\cos 2x + 1) = (2\cos 2x + 1)$$

$$\left(2\cos^2 2x - 1\right)(2\cos 2x + 1) = 0$$

$$4\cos 4x(2\cos 2x + 1) = 0$$

$$4\cos 4x = 0, (2\cos 2x + 1) = 0$$

$$\cos 4x = 0, \cos 2x = -\frac{1}{2}$$

$$4x = (4n+1)\frac{\pi}{2}, 2x = 2n\pi \pm \frac{\pi}{3}$$

$$x = (4n+1)\frac{\pi}{8}, x = n\pi \pm \frac{\pi}{6}, n \in I$$

$$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Hence, the number of solutions is 4.

9. Given equation is $\sin x.\cos y = 1$
It is possible only when

$\sin x = 1, \cos y = 1$

$$\Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2} \text{ and } y = 0, 2\pi$$

Also, when $\sin x = -1, \cos y = -1$

$$\Rightarrow \quad x = \frac{3\pi}{2}, y = \pi$$

Hence, the number of ordered pairs is 5

i.e. $\left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 2\pi\right), \left(\frac{3\pi}{2}, 0\right), \left(\frac{3\pi}{2}, \pi\right), \left(\frac{3\pi}{2}, 2\pi\right)$

10. When $\cot x$ is positive
The equation becomes

$$\cot x = \cot x + \frac{1}{\sin x}$$

$$\operatorname{cosec} x = 0$$

It is not possible,
when $\cot x$ is negative.
The given equation becomes

$$\Rightarrow \quad -\cot x = \cot x + \frac{1}{\sin x}$$

$$\Rightarrow \quad 2\cot x + \frac{1}{\sin x} = 0$$

$$\Rightarrow \quad 2\cos x + 1 = 0$$

$$\Rightarrow \quad \cos x = -\frac{1}{2}$$

$\Rightarrow \quad x = 2n\pi \pm \frac{2\pi}{3}, n \in I$

$\Rightarrow \quad x = \frac{2\pi}{3}, \frac{4\pi}{3}$

Hence, the number of solutions is 2.

11. Given equation is

$\tan(4x)\tan x = 1$

$\cos(4x)\cos x - \sin(4x)\sin x = 0$

$\cos(5x) = 0$

$5x = (2n+1)\frac{\pi}{2}$

$x = (2n+1)\frac{\pi}{10}, n \in I$

$x = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$

Hence, the number of solutions is 5.

12. Given equation is

$\sin x(\sin x + \cos x) = n$

$\Rightarrow \quad n = \sin x(\sin x + \cos x)$

$= \sin^2 x + \sin x \cos x$

$= \frac{1-\cos(2x)}{2} + \frac{\sin(2x)}{2}$

$\Rightarrow \quad \sin(2x) - \cos(2x) = 2n - 1$

$\Rightarrow \quad -\sqrt{2} \le 2n - 1 \le \sqrt{2}$

$\Rightarrow \quad \frac{1-\sqrt{2}}{2} \le n \le \frac{1+\sqrt{2}}{2}$

$\Rightarrow \quad n = 0, 1$

Hence, the number of integral values of n is 2.

13. Given equation is

$\sin\{x\} = \cos\{x\}$

$\Rightarrow \quad \tan\{x\} = 1$

$\Rightarrow \quad x = \frac{\pi}{4}, 1+\frac{\pi}{4}, 2+\frac{\pi}{4}, 3+\frac{\pi}{4}, 4+\frac{\pi}{4}, 5+\frac{\pi}{4}$

Hence, the number of solutions is 6.

14. Given equation is

$(\sqrt{3}+1)^{2x} + (\sqrt{3}-1)^{2x} = 2^{3x}$

$\Rightarrow \quad (\sqrt{3}+1)^{2x} + (\sqrt{3}-1)^{2x} = (2\sqrt{2})^{2x}$

$\Rightarrow \quad \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^{2x} + \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^{2x} = 1$

$\Rightarrow \quad \left(\sin\left(\frac{5\pi}{12}\right)\right)^{2x} + \left(\cos\left(\frac{5\pi}{12}\right)\right)^{2x} = 1$

It is satified only when $x = 1$.

Thus, the number of solutions is 1.

15. Here, $1 \le |\sin(2x)| + |\cos(2x)| \le \sqrt{2}$ and $|\sin(y)| \le 1$

It is possible only when $|\sin(y)| = 1$

$\Rightarrow \quad \sin(y) = \pm 1$

$\Rightarrow \quad y = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}.$

Thus, the number of values of y is 4

PAST IIT-JEE QUESTIONS

1. Ans. (a)

The given equation is $\sin x + \cos x = 1$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \cos\left(x - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad \left(x - \frac{\pi}{4}\right) = \left(2n\pi \pm \frac{\pi}{4}\right)$

$\Rightarrow \quad x = \left(2n\pi \pm \frac{\pi}{4} + \frac{\pi}{4}\right)$

$\Rightarrow \quad x = 2n\pi, 2n\pi + \frac{\pi}{2}$

But $x = 2n\pi + \frac{\pi}{2}$ does not satisfy the given equation.

Therefore, the solution is $x = 2n\pi, n \in I$

2. We have $\cos x = \sin 3x$

$\cos x = \sin 3x = \cos\left(\frac{\pi}{2} - 3x\right)$

$\Rightarrow \quad x = 2n\pi \pm \left(\frac{\pi}{2} - 3x\right)$

Taking +ve sign, we get,

$x = 2n\pi + \left(\frac{\pi}{2} - 3x\right)$

$\Rightarrow \quad 4x = 2n\pi + \frac{\pi}{2}$

$\Rightarrow \quad x = \frac{2n\pi}{4} + \frac{\pi}{8}$

$\Rightarrow \quad x = \frac{n\pi}{2} + \frac{\pi}{8}, n \in I$

Taking –ve sign, we get,

$\Rightarrow \quad x = 2n\pi - \left(\frac{\pi}{2} - 3x\right)$

$\Rightarrow \quad -2x = 2n\pi - \frac{\pi}{2}$

$\Rightarrow \quad x = -n\pi + \frac{\pi}{4}, n \in I$

As $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $x = \frac{\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{4}$

Thus, the point of intersections of two curves are

$\left(\frac{\pi}{8}, \cos\left(\frac{\pi}{8}\right)\right), \left(-\frac{3\pi}{8}, \cos\left(\frac{3\pi}{8}\right)\right), \left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$

3. The given equation is

$4\cos^2 x \sin x - 2\sin^2 x = 3\sin x$

$\Rightarrow \quad 4\left(1-\sin^2 x\right)\sin x - 2\sin^2 x = 3\sin x$

$\Rightarrow \quad 4\sin x - 4\sin^3 x - 2\sin^2 x = 3\sin x$

$\Rightarrow \quad \sin x - 4\sin^3 x - 2\sin^2 x = 0$

$\Rightarrow \quad 4\sin^3 x + 2\sin^2 x - \sin x = 0$

$\Rightarrow \quad \sin x\left(4\sin^2 x + 2\sin x - 1\right) = 0$

$\Rightarrow \quad \sin x = 0, \; \left(4\sin^2 x + 2\sin x - 1\right) = 0$

$\Rightarrow \quad \sin x = 0, \; \sin x = \frac{-2 \pm 2\sqrt{5}}{8}$

$\Rightarrow \quad \sin x = 0, \; \sin x = \frac{-1 \pm \sqrt{5}}{4}$

$\Rightarrow \quad x = n\pi, \; x = n\pi + (-1)^n \sin^{-1}\left(\frac{-1 \pm \sqrt{5}}{4}\right)$

where $n \in I$.

4. The given equation is

$\sin^4\theta - 2\sin^2\theta - 1 = 0$

$\Rightarrow \quad \left(\sin^2\theta - 1\right)^2 = 2$

$\Rightarrow \quad \left(\sin^2\theta - 1\right) = \pm\sqrt{2}$

$\Rightarrow \quad \sin^2\theta = \left(1 \pm \sqrt{2}\right)$

$\Rightarrow \quad \sin^2\theta = \left(1+\sqrt{2}\right), \left(1-\sqrt{2}\right)$

since $\left(1+\sqrt{2}\right) > 1$ and $\left(1-\sqrt{2}\right) < 0$, so

there is no value of θ satisfying the given equation.

5. No questions asked in 1985.

6. We have $\cos x + \cos y = \frac{3}{2}$

$2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{3}{2}$

$\Rightarrow \quad 2\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{x-y}{2}\right) = \frac{3}{2}$

$\Rightarrow \quad 2 \times \frac{1}{2} \times \cos\left(\frac{x-y}{2}\right) = \frac{3}{2}$

$\Rightarrow \quad \cos\left(\frac{x-y}{2}\right) = \frac{3}{2}$.

It is not possible, so the solution set is $x = \varphi$.

7. The given inequation is

$2\sin^2 x - 3\sin x + 1 \ge 0$

$\Rightarrow \quad (2\sin x - 1)(\sin x - 1) \ge 0$

$\Rightarrow \quad \sin x \le \frac{1}{2}$ & $\sin x \ge 1$

$\Rightarrow \quad \sin x \le \frac{1}{2}$ & $\sin x = 1$

$x \in \left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, \pi\right]$ and $x = \frac{\pi}{2}$

Hence, the solution set is

$x \in \left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, \pi\right] \cup \left\{\frac{\pi}{2}\right\}$.

8. Ans. (c)

As we know that $\tan x \ge x$

So there is no root between $\left(0, \frac{\pi}{2}\right)$

$\left(\frac{\pi}{2}, \pi\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$

But there is a root in $\left(\pi, \frac{3\pi}{2}\right)$.

9. Ans. (b)

The given equation is

$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

$\Rightarrow \quad (\sin x + \sin 3x) - 3\sin 2x = (\cos x + \cos 3x) - 3\cos 2x$

$\Rightarrow \quad 2\sin 2x \cos x - 3\sin 2x = 2\cos 2x \cos x - 3\cos 2x$

$\Rightarrow \quad (2\cos x - 3)(\sin 2x - \cos 2x) = 0$

$\Rightarrow \quad (\sin 2x - \cos 2x) = 0, \; (\because 2\cos x - 3 \ne 0)$

$\tan 2x = 1$

$\Rightarrow \quad \tan 2x = 1 = \tan\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad 2x = \left(n\pi + \frac{\pi}{4}\right), n \in I$

$\Rightarrow \quad x = \left(\frac{n\pi}{2} + \frac{\pi}{8}\right), n \in I$

10. No questions asked in between 1990 to 1992.
11. Ans. (c)
The given equation is
$\tan x + \sec x = 2\cos x$
$\Rightarrow \quad \sin x + 1 = 2\cos^2 x$
$\Rightarrow \quad \sin x + 1 = 2\left(1 - \sin^2 x\right)$
$\Rightarrow \quad (\sin x + 1) = 2(1 - \sin x)(1 + \sin x)$
$\Rightarrow \quad (\sin x + 1)(1 - 2(1 - \sin x)) = 0$
$\Rightarrow \quad \sin x = -1, \sin x = \frac{1}{2}$
$\Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.
But $x = \frac{3\pi}{2}$ does not satisfy the given equation.
Hence, the number of solutions is 2.
12. The given equation can be written as
$\tan(x + 100°)\cot x = \tan(x + 50°)\tan(x - 50°)$
$$\Rightarrow \quad \frac{\sin(x+100°)\cos x}{\cos(x+100°)\sin x} = \frac{\sin(x-50°)\sin(x-50°)}{\cos(x+50°)\cos(x-50°)}$$
Applying compenendo and dividendo, we get,
$$\Rightarrow \quad \frac{\sin(x+100°)\cos x + \cos(x+100°)\sin x}{\sin(x+100°)\cos x - \cos(x+100°)\sin x} = \frac{\sin(x+50°)\sin(x-50°) + \cos(x+50°)\cos(x-50°)}{\sin(x+50°)\sin(x-50°) - \cos(x+50°)\cos(x-50°)}$$
$$\Rightarrow \quad \frac{\sin(x+100°+x)}{\sin(x+100°-x)} = \frac{\cos(x+50°-x+50°)}{-\cos(x+50°+x-50°)}$$
$\Rightarrow \quad \sin(2x + 100°)\cos 2x = -\sin(100°)\cos(100°)$
$\Rightarrow \quad 2\sin(2x + 100°)\cos 2x = -2\sin(100°)\cos(100°)$
$\Rightarrow \quad \sin(4x + 100°) + \sin(100°) = -\sin(200°)$
$\Rightarrow \quad \sin(4x + 100°) = -(\sin(200°) + \sin(100°))$
$\Rightarrow \quad \sin(4x + 100°) = -2\sin(150°)\cos(50°)$
$\Rightarrow \quad \sin(4x + 100°) = -2 \times \frac{1}{2} \times \sin(40°)$
$\Rightarrow \quad \sin(4x + 100°) = -\sin(40°) = \sin(220°)$
$\Rightarrow \quad (4x + 100°) = (220°)$
$\Rightarrow \quad 4x = 120°$
$\Rightarrow \quad x = 30°$
Hence, the result.
13. Ans. (d)
Let $\theta = \frac{\pi}{2n}$
The given equation reduces to
$$\sin\theta + \cos\theta = \frac{\sqrt{n}}{2}$$
$\Rightarrow \quad (\sin\theta + \cos\theta)^2 = \frac{n}{4}$
$\Rightarrow \quad 1 + 2\sin\theta\cos\theta = \frac{n}{4}$
$\Rightarrow \quad \sin 2\theta = \frac{n}{4} - 1 = \left(\frac{n-4}{4}\right)$
As per choices, $n \geq 4$
$\Rightarrow \quad 0 < 2\theta < \frac{\pi}{2}$
$\Rightarrow \quad 0 < \sin 2\theta < 1$
$\Rightarrow \quad 0 < \left(\frac{n-4}{4}\right) < 1$
$\Rightarrow \quad 0 < (n-4) < 4$
$\Rightarrow \quad 4 < n < 8$
14. The given in-equations are
$2\sin^2 x + 3\sin x - 2 \geq 0$ and $x^2 - x - 2 < 0$
$\Rightarrow \quad (2\sin x - 1)(\sin x + 2) \geq 0$
$\Rightarrow \quad (2\sin x - 1) \geq 0\,(\because \sin x + 2 > 0,\ \forall x \in R)$
$\Rightarrow \quad \sin x \geq \frac{1}{2}$
$\Rightarrow \quad \frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$ ....(i)
Also, $x^2 - x - 2 < 0$
$\Rightarrow \quad (x-2)(x+1) < 0$
$\Rightarrow \quad -1 < x < 2$ ......(ii)
From (i) and (ii), we get,
$$x \in \left(\frac{\pi}{6}, 2\right)$$
15. The given equation is
$\cos(p\sin x) = \sin(p\cos x)$
$\Rightarrow \quad \cos(p\sin x) = \cos\left(\frac{\pi}{2} - p\cos x\right)$
$\Rightarrow \quad (p\sin x) = 2n\pi \pm \left(\frac{\pi}{2} - p\cos x\right)$
$\Rightarrow \quad p(\sin x + \cos x) = 2n\pi \pm \frac{\pi}{2}$

$\Rightarrow \quad \sqrt{2}p\sin\left(x+\frac{\pi}{4}\right)=2n\pi\pm\frac{\pi}{2}$

As we know that, $-1\le\sin\left(x+\frac{\pi}{4}\right)\le 1$

$-\sqrt{2}p\le\sqrt{2}p\sin\left(x+\frac{\pi}{4}\right)\le\sqrt{2}p$

$\Rightarrow \quad -\sqrt{2}p\le 2n\pi\pm\frac{\pi}{2}\le\sqrt{2}p$

$\Rightarrow \quad \sqrt{2}p\ge 2n\pi\pm\frac{\pi}{2}$

$\Rightarrow \quad \sqrt{2}p\ge 2n\pi+\frac{\pi}{2},\ 2n\pi-\frac{\pi}{2}$

As we require smallest +ve value of p, so we consider,

$\sqrt{2}p=\frac{\pi}{2}$

$\Rightarrow \quad p=\frac{\pi}{2\sqrt{2}}$.

For this value of p, $x=\frac{\pi}{4}$ is a solution of the given equation.

16. The given equation can be written as

$(1-\tan^2\theta)(1+\tan^2\theta)+2^{\tan^2\theta}=0$

$\Rightarrow \quad (1-\tan^2\theta)(1+\tan^2\theta)+2^{\tan^2\theta}=0$

$\Rightarrow \quad (1-\tan^4\theta)+2^{\tan^2\theta}=0$

$\Rightarrow \quad 2^{\tan^2\theta}=(\tan^4\theta-1)$

$\Rightarrow \quad 2^x=(x^2-1)$, where $x=\tan^2\theta$

It is true for $x = 3$

Thus, $\tan^2\theta=3$

$\Rightarrow \quad \tan^2\theta=(\sqrt{3})^2$

$\Rightarrow \quad \tan\theta=\pm(\sqrt{3})$

$\Rightarrow \quad \theta=\pm\left(\frac{\pi}{3}\right)$.

17. The given equation is

$\tan^2\theta+\sec 2\theta=1$

$\Rightarrow \quad \tan^2\theta+\frac{1+\tan^2\theta}{1-\tan^2\theta}=1$

$\Rightarrow \quad x+\frac{1+x}{1-x}=1$, where $x=\tan^2\theta$

$\Rightarrow \quad x-x^2+1+x=1-x$

$\Rightarrow \quad x^2-3x=0$

$\Rightarrow \quad x=0,3$

$\Rightarrow \quad \tan^2\theta=0,3$

$\tan\theta=0$ and $\tan\theta=\pm\sqrt{3}$

$\Rightarrow \quad \theta=n\pi,\ \theta=n\pi\pm\frac{\pi}{3},\ n\in I$

18 The given equation is $\cos^7 x+\sin^4 x=1$

$\cos^7 x=1-\sin^4 x$

$\Rightarrow \quad \cos^7 x=(1-\sin^2 x)(1+\sin^2 x)$

$\Rightarrow \quad \cos^7 x=\cos^2 x(2-\cos^2 x)$

$\Rightarrow \quad \cos^2 x(\cos^5 x-(2-\cos^2 x))=0$

$\Rightarrow \quad \cos^2 x=0,\ (\cos^5 x-(2-\cos^2 x))=0$

$\Rightarrow \quad \cos^2 x=0,\ \cos^5 x+\cos^2 x=2$

$\Rightarrow \quad \cos x=0,\ \cos^5 x+\cos^2 x=2$

$\Rightarrow \quad \cos x=0,\ \cos x=1$

$\Rightarrow \quad x=\pm\frac{\pi}{2},0$.

Hence, the real roots are $\left\{\pm\frac{\pi}{2},0\right\}$.

19. The given equation is

$\Rightarrow \quad 3\sin^2 x-7\sin x+2=0$

$\Rightarrow \quad (3\sin x-1)(\sin x-2)=0$

$\Rightarrow \quad \sin x=\frac{1}{3},2$

$\Rightarrow \quad \sin x=\frac{1}{3}$

There is 2 solutions in its period 2π.
So, the number of solutions is 6.

20. We have $\sin(n\theta)=\sum_{r=0}^{n}b_r\sin^r\theta$

$\Rightarrow \quad \sin(n\theta)=[b_0+b_1\sin\theta+b_2\sin^2\theta+b_3\sin^3\theta+......+b_n\sin^n\theta]$

Put $\theta=0$, then $b_0=0$

Thus,

$\sin(n\theta)=b_1\sin\theta+b_2\sin^2\theta+......+b_n\sin^n\theta$

$\Rightarrow \quad \frac{\sin(n\theta)}{\sin\theta}=b_1+b_2\sin\theta+......+b_n\sin^{n-1}\theta$

Taking limit $\theta \to 0$, we get,

$b_1 = n$

Therefore, $b_0 = 0, b_1 = n$.

22. We have $-\sqrt{74} \le 7\cos x + 5\sin x \le \sqrt{74}$

$\Rightarrow \quad -\sqrt{74} \le (2k+1) \le \sqrt{74}$

$\Rightarrow \quad -\sqrt{74} - 1 \le 2k \le \sqrt{74} - 1$

$\Rightarrow \quad -8 - 1 \le 2k \le 8 - 1 \ (\because \sqrt{74} < 9)$

$\Rightarrow \quad -9 \le 2k \le 7$

$\Rightarrow \quad -4.5 \le k \le 3.5$

$\Rightarrow \quad k = -4, -3, -2, -1, 0, 1, 2, 3.$

Thus, the number of integral values of k is 8.

23. No questions asked in between 2003 to 2004.

24. Ans. (d).

Given $-\pi \le a, b \le \pi$

$\Rightarrow \quad -\pi \le a \le \pi, \ -\pi \le b \le \pi$

$\Rightarrow \quad -\pi \le a \le \pi, \ -\pi \le -b \le \pi$

$\Rightarrow \quad -2\pi \le a - b \le 2\pi$

Given $\cos(a-b) = 1$

$a = b$

Also, $\cos(a+b) = \frac{1}{e}$

$\Rightarrow \quad \cos(2a) = \frac{1}{e}$

It has one solution in its period π.

So it has 4 solutions in $[0, 4\pi]$.

25. Let $y = \frac{5x^2 - 2x + 1}{3x^2 - 2x - 1}$

$\Rightarrow \quad (3y-5)x^2 - 2(y-1)x - (y+1) = 0$

As x is real, so

$(y-1)^2 + (3y-5)(y+1) \ge 0$

$\Rightarrow \quad y^2 - 2y + 1 + 3y^2 - 2y - 5 \ge 0$

$\Rightarrow \quad 4y^2 - 4y - 4 \ge 0$

$\Rightarrow \quad y^2 - y - 1 \ge 0$

$\Rightarrow \quad \left(y - \left(\frac{1-\sqrt{5}}{2}\right)\right)\left(y - \left(\frac{1+\sqrt{5}}{2}\right)\right) \ge 0$

$\Rightarrow \quad y \le \left(\frac{1-\sqrt{5}}{2}\right), \ y \ge \left(\frac{1+\sqrt{5}}{2}\right)$

$\Rightarrow \quad 2\sin t \le \left(\frac{1-\sqrt{5}}{2}\right), \ 2\sin t \ge \left(\frac{1+\sqrt{5}}{2}\right)$

$\Rightarrow \quad \sin t \le \left(\frac{1-\sqrt{5}}{4}\right), \ \sin t \ge \left(\frac{1+\sqrt{5}}{4}\right)$

$\Rightarrow \quad \sin t \le \sin\left(-\frac{\pi}{10}\right), \ \sin t \ge \sin\left(\frac{3\pi}{10}\right)$

As $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, we get,

$\Rightarrow \quad -\frac{\pi}{2} \le t \le -\frac{\pi}{10}$ and $\frac{3\pi}{10} \le t \le \frac{\pi}{2}$

Thus, $t \in \left[-\frac{\pi}{2}, -\frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$.

26. The given in-equation is

$2\sin^2\theta - 5\sin\theta + 2 > 0$

$\Rightarrow \quad (2\sin\theta - 1)(\sin\theta - 2) > 0$

$\Rightarrow \quad (2\sin\theta - 1) < 0$

$\Rightarrow \quad \sin\theta < \frac{1}{2}$

$\Rightarrow \quad \theta \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$.

27. The given equations are

$2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$

Now, $2\sin^2\theta - (1 - 2\sin^2\theta) = 0$

$\Rightarrow \quad 4\sin^2\theta = 1$

$\Rightarrow \quad \sin^2\theta = \left(\frac{1}{2}\right)^2 = \sin^2\left(\frac{\pi}{6}\right)$

$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{6}$

$\Rightarrow \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

Also, $2\cos^2\theta - 3\sin\theta = 0$

$\Rightarrow \quad 2 - 2\sin^2\theta - 3\sin\theta = 0$

$\Rightarrow \quad 2\sin^2\theta + 3\sin\theta - 2 = 0$

$\Rightarrow \quad (2\sin\theta - 1)(\sin\theta + 2) = 0$

$\Rightarrow \quad \sin\theta = \frac{1}{2}, 2$

$\Rightarrow \quad \sin\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

Hence, the solutions are $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

Therefore, the number of solutions is 2.

28. The given equation is $\sin\theta = \cos\varphi$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\varphi$$

$$\Rightarrow \quad \left(\frac{\pi}{2} - \theta\right) = 2n\pi \pm \varphi,\ n \in I$$

$$\Rightarrow \quad -2n\pi = \left(\theta \pm \varphi - \frac{\pi}{2}\right)$$

$$\Rightarrow \quad \left(\theta \pm \varphi - \frac{\pi}{2}\right) = -2n\pi$$

$$\Rightarrow \quad \frac{1}{\pi}\left(\theta \pm \varphi - \frac{\pi}{2}\right) = -2n$$

Thus, $\frac{1}{\pi}\left(\theta \pm \varphi - \frac{\pi}{2}\right)$ is an even integer.

29. We have

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$

$$\Rightarrow \quad [\sum_{m=1}^{6} \frac{1}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)} = 4\sqrt{2}$$

$$\Rightarrow \quad \sum_{m=1}^{6} \frac{\sin\left(\left(\theta + \frac{m\pi}{4}\right) - \left(\theta + \frac{(m-1)\pi}{4}\right)\right)}{\sin\left(\theta + \frac{(m-1)\pi}{4}\right)\sin\left(\theta + \frac{m\pi}{4}\right)} = 4$$

$$\Rightarrow \quad \sum_{m=1}^{6} \left(\cot\left(\theta + \left(\frac{m-1}{4}\right)\pi\right) - \cot(\theta + m\pi)\right) = 4$$

$$\Rightarrow \quad \left(\cot\theta - \cot\left(\theta + \frac{6\pi}{4}\right)\right) = 4$$

$$\Rightarrow \quad (\cot\theta + \tan\theta) = 4$$

$$\Rightarrow \quad \frac{1}{\sin\theta\cos\theta} = 4$$

$$\Rightarrow \quad \frac{1}{2\sin\theta\cos\theta} = 2$$

$$\Rightarrow \quad \frac{1}{\sin 2\theta} = 2$$

$$\Rightarrow \quad \sin 2\theta = \frac{1}{2}$$

$$\Rightarrow \quad 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\Rightarrow \quad \theta = \frac{\pi}{12}, \frac{5\pi}{12}.$$

Hence, the solutions are $\left\{\frac{\pi}{12}, \frac{5\pi}{12}\right\}$.

30. We have $\tan\theta = \cot 5\theta$

$$\frac{\sin\theta}{\cos\theta} = \frac{\cos 5\theta}{\sin 5\theta}$$

$$\Rightarrow \quad 2\cos 5\theta\cos\theta = 2\sin 5\theta\sin\theta$$

$$\Rightarrow \quad \cos 6\theta + \cos 4\theta = \cos 4\theta - \cos 6\theta$$

$$\Rightarrow \quad 2\cos 6\theta = 0$$

$$\Rightarrow \quad \cos 6\theta = 0$$

$$\Rightarrow \quad 6\theta = \pm\frac{5\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{\pi}{2}$$

$$\Rightarrow \quad \theta = \pm\frac{5\pi}{12}, \pm\frac{3\pi}{12}, \pm\frac{\pi}{12}$$

$$\Rightarrow \quad \theta = \pm\frac{5\pi}{12}, \pm\frac{\pi}{4}, \pm\frac{\pi}{12}$$

Also, $\sin(2\theta) = \cos(4\theta)$

$$\Rightarrow \quad \cos(4\theta) = \cos\left(\frac{\pi}{2} - 2\theta\right)$$

$$\Rightarrow \quad 4\theta = 2n\pi \pm \left(\frac{\pi}{2} - 2\theta\right)$$

Taking +ve sign, we get,

$$4\theta = 2n\pi + \left(\frac{\pi}{2} - 2\theta\right)$$

$$\Rightarrow \quad 6\theta = 2n\pi + \frac{\pi}{2}$$

$$\Rightarrow \quad \theta = \frac{n\pi}{3} + \frac{\pi}{12}$$

$$\Rightarrow \quad \theta = \frac{\pi}{12}, \frac{5\pi}{12}, -\frac{\pi}{4}$$

Taking –ve sign, we get,

$$4\theta = 2n\pi - \left(\frac{\pi}{2} - 2\theta\right)$$

$$\Rightarrow \quad 2\theta = 2n\pi - \frac{\pi}{2}$$

$$\Rightarrow \quad \theta = n\pi - \frac{\pi}{4}$$

$$\Rightarrow \quad \theta = n\pi - \frac{\pi}{4}$$

Hence, the solutions are $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, -\frac{\pi}{4}$.

31. Let $\frac{\pi}{n} = \theta$

Then the given equation becomes

$$\frac{1}{\sin\theta}=\frac{1}{\sin(2\theta)}+\frac{1}{\sin(3\theta)}$$

$$\Rightarrow \quad \frac{1}{\sin\theta}-\frac{1}{\sin(3\theta)}=\frac{1}{\sin(2\theta)}$$

$$\Rightarrow \quad \frac{\sin 3\theta-\sin\theta}{\sin\theta\sin(3\theta)}=\frac{1}{\sin(2\theta)}$$

$$\Rightarrow \quad \frac{2\cos 2\theta\sin\theta}{\sin\theta\sin 3\theta}=\frac{1}{\sin 2\theta}$$

$$\Rightarrow \quad \frac{2\cos 2\theta}{\sin 3\theta}=\frac{1}{\sin 2\theta}$$

$$\Rightarrow \quad \sin 4\theta=\sin 3\theta$$

$$\Rightarrow \quad \sin 4\theta=\sin(\pi-3\theta)$$

$$\Rightarrow \quad 4\theta=\pi-3\theta$$

$$\Rightarrow \quad 7\theta=\pi$$

$$\Rightarrow \quad \theta=\frac{\pi}{7}$$

$$\Rightarrow \quad \frac{\pi}{n}=\frac{\pi}{7}$$

$$\Rightarrow \quad n=7.$$

Hence, the integral value of n is 7.

32. No questions asked in between 2012 to 2013.
33. Ans. (d)

The given equation is

$$\sin x+2\sin 2x-\sin 3x=3$$

$$\Rightarrow \quad \sin x+4\sin x\cos x-3\sin x+4\sin^3 x=3$$

$$\Rightarrow \quad \sin x\left[-2+4\cos x+4\left(1-\cos^2 x\right)\right]=3.$$

$$\Rightarrow \quad \sin x\left[2-\left(4\cos 2x-4\cos x+1\right)+1\right]=3$$

$$\Rightarrow \quad \sin x\left[3-\left(2\cos x-1\right)^2\right]=3$$

It is possible only when

$\sin x=1$ and $(2\cos x-1)=0$

$$\Rightarrow \quad x=\frac{\pi}{2} \text{ and } x=\frac{\pi}{3}.$$

Thus, the values of x does not saitsfy the given equation.

34. Ans. 8

The given equation can be written as

$$\frac{5}{4}\cos^2 2x+\cos^4 x+\sin^4 x+\cos^6 x+\sin^6 x=2$$

$$\Rightarrow \quad \frac{5}{4}\cos^2 2x+\left(1-2\sin^2 x\cos^2 x\right)+\left(1-3\sin^2 x\cos^2 x\right)=2$$

$$\Rightarrow \quad \frac{5}{4}\cos^2 2x+\left(1-\frac{1}{2}\sin^2 2x\right)+\left(1-\frac{3}{4}\sin^2 2x\right)=2$$

$$\Rightarrow \quad \frac{5}{4}\cos^2 2x-\frac{5}{4}\sin^2 2x=0$$

$$\Rightarrow \quad \cos^2 2x-\sin^2 2x=0$$

$$\Rightarrow \quad \cos^2 2x=\sin^2 2x$$

$$\Rightarrow \quad \tan^2 2x=1$$

$$\Rightarrow \quad 2x=n\pi\pm\frac{\pi}{4}, n\in I$$

$$\Rightarrow \quad x=\frac{n\pi}{2}\pm\frac{\pi}{8}, n\in I.$$

Hence, the solutions are

$$x=\frac{\pi}{8},\frac{3\pi}{8},\frac{5\pi}{8},\frac{7\pi}{8},\frac{9\pi}{8},\frac{11\pi}{8},\frac{13\pi}{8},\frac{15\pi}{8}.$$

CHAPTER 4

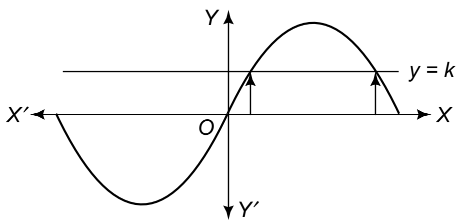
Trigonometric In-Equation

4.1 TRIGONOMETRIC INEQUALITIES

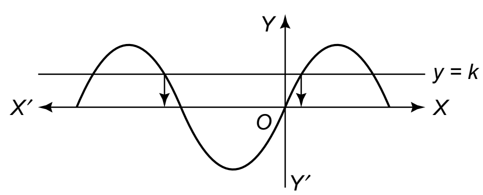
Suppose we have to solve $f(x) > k$ or $f(x) < k$.

When we solve the inequation, we often use the graphs of the functions $y = f(x)$ and $y = k$.

Then, the solution of the inequality $f(x) > k$ is the values of x for which the point $(x, f(x))$ of the graph of $y = f(x)$ lies above the striaght line $y = k$.

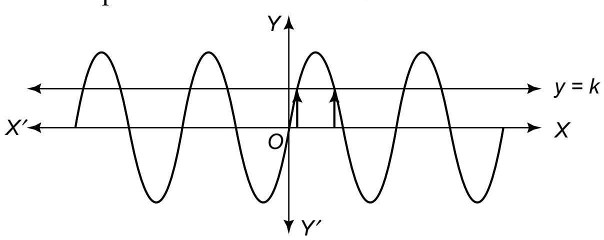


Similarly, when we solve $f(x) < k$, then the solution of the inequalition $f(x) < k$ is the values of x for which the point $(x, f(x))$ of the graph of $y = f(x)$ lies below the striaght line $y = k$.



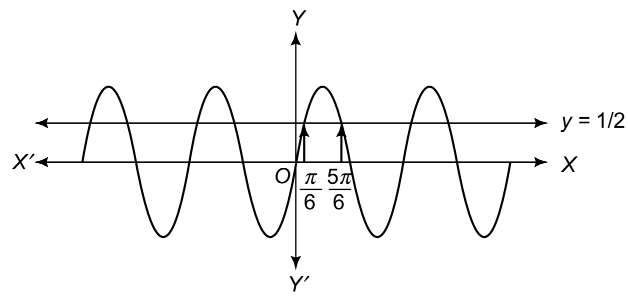
Type - I: An inequation is of the form sin $x > k$.

Rule: Find the smallest values of x that satisfies the given inequation and then add $2n\pi$ with that values of x.



Ex-1. Solve $\sin x > 1/2$.

Soln. Here, we should construct the graph of $y = \sin x$ and $y = 1/2$.

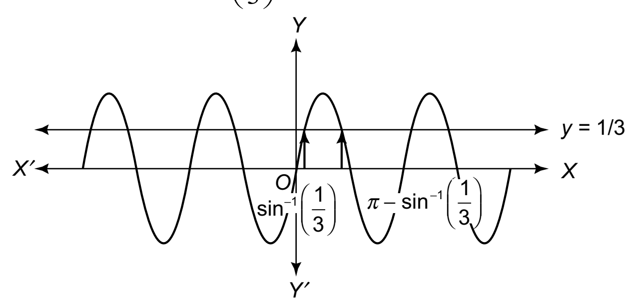


Hence, the solution set is

$$x = \bigcup_{n \in I} \left(2n\pi + \frac{\pi}{6},\ 2n\pi + \frac{5\pi}{6}\right).$$

Ex-2: Solve: $\sin x > 1/3$.

Soln. Here, we should construct the graph of $y = \sin x$ and $y = \sin^{-1}\left(\frac{1}{3}\right)$.

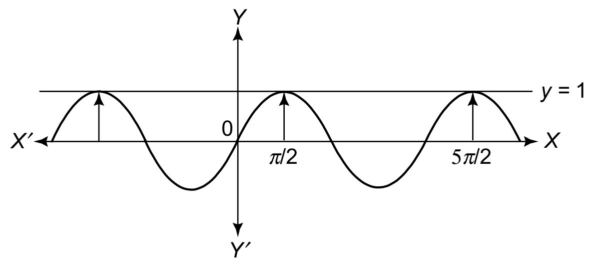


Hence, the solution set is

$$x = \bigcup_{n \in I} \left(2n\pi + \sin^{-1}\left(\frac{1}{3}\right),\ (2n+1)\pi - \sin^{-1}\left(\frac{1}{3}\right)\right)$$

Ex-3. Solve: $\sin x \geq 1$.

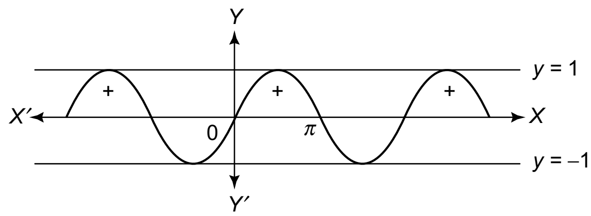
Soln. We have $\sin x \geq 1$.



$$\Rightarrow \qquad x = (4n+1)\frac{\pi}{2},\ n \in I.$$

Ex-4. Solve: $\sin x > 0$

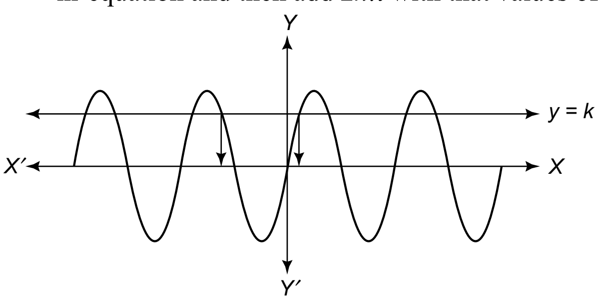
Soln. We have, $\sin x > 0$



$$\Rightarrow \qquad x = \bigcup_{n \in I} \left(2n\pi, (2n+1)\pi\right)$$

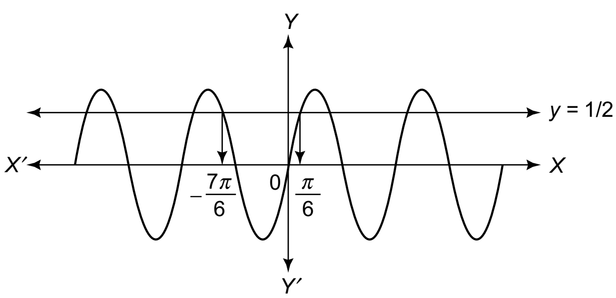
Type - II: An in-equation is of the form sin $x < k$.

Rule: Find the smallest values of x which satisfies the given in-equation and then add $2n\pi$ with that values of x.



Ex-1. Solve: $\sin x < 1/2$

Soln. Here, we should draw the graph of $y = \sin x$ and $y = 1/2$

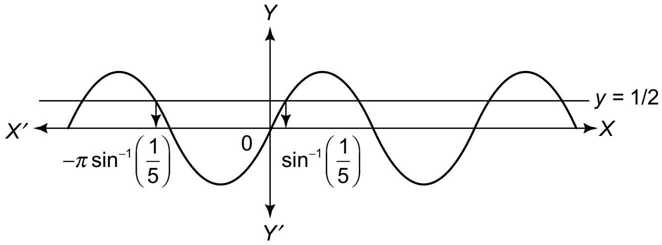


Hence, the solution set is

$$x = \bigcup_{n \in I} \left(2n\pi - \frac{7\pi}{6}, 2n\pi + \frac{\pi}{6}\right).$$

Ex-2. Solve: $\sin x < 1/5$.

Soln. Here, we should draw the graphs of $y = \sin x$ and $y = 1/5$.

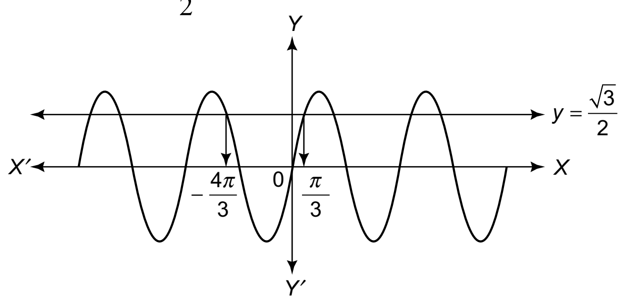


Hence, the solution set is

$$x = \bigcup_{n \in I} \left((2n-1)\pi - \sin^{-1}\left(\frac{1}{5}\right), 2n\pi + \sin^{-1}\left(\frac{1}{5}\right)\right)$$

Ex-3. Solve: $\sin x \leq \frac{\sqrt{3}}{2}$

Soln. Here, we should draw the graphs of $y = \sin x$ and $y = \frac{\sqrt{3}}{2}$

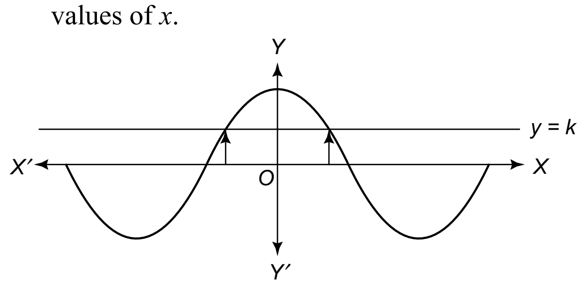


Hence, the solution set is

$$x = \bigcup_{n \in I} \left[2n\pi - \frac{4\pi}{3}, 2n\pi + \frac{\pi}{3}\right]$$

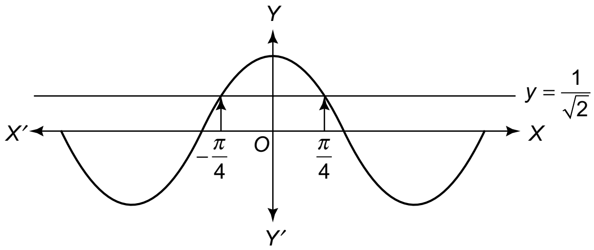
Type - III: An in-equation is of the form cos $x > k$.

Rule: First we find the smallest interval for which x satisfies the given in-equation and then add $2n\pi$ with each values of x.



Ex-1. Solve: $\cos x > \frac{1}{\sqrt{2}}$.

Soln. Here, we should draw the graphs of $y = \cos x$ and $y = \frac{1}{\sqrt{2}}$.

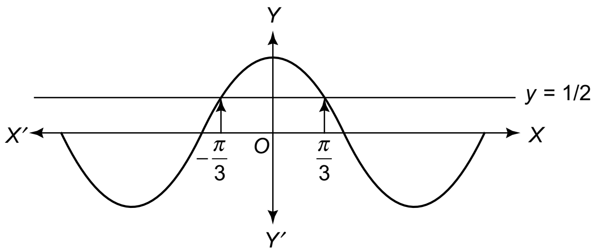


Hence, the solution set is

$$x = \bigcup_{n \in I} \left(2n\pi - \frac{\pi}{4},\ 2n\pi + \frac{\pi}{4}\right).$$

Ex-2. Solve: $\cos x \geq \frac{1}{2}$.

Soln. Here, we should draw the graphs of $y = \cos x$ and $y = 1/2$

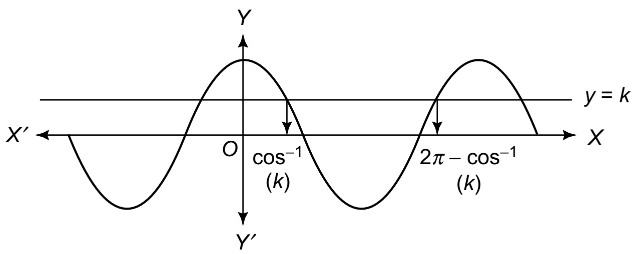


Hence, the solution set is

$$x = \bigcup_{n \in I} \left[2n\pi - \frac{\pi}{3},\ 2n\pi + \frac{\pi}{3}\right].$$

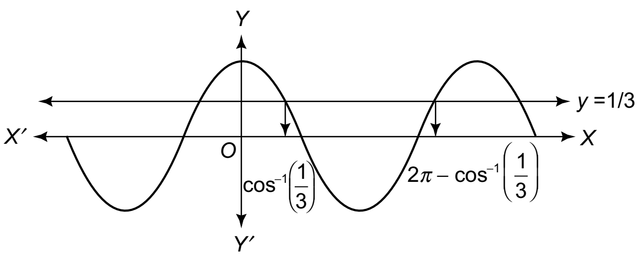
Type - IV: An in-equation is of the form cos $x < k$.

Rule: First we find the smallest interval for which x satisfies the given inequation and then add $2n\pi$ with each values of x.



Ex-1. Solve: $\cos x < \frac{1}{3}$.

Soln. Here, we should draw the graphs of $y = \cos x$ and $y = \frac{1}{3}$.

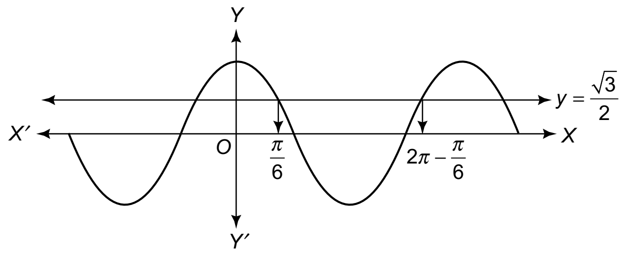


Hence, the solution set is

$$x = \bigcup_{n \in I} \left(2n\pi + \cos^{-1}\left(\frac{1}{3}\right),\ 2(n+1)\pi - \cos^{-1}\left(\frac{1}{3}\right)\right).$$

Ex-2. Solve: $\cos x \leq \frac{\sqrt{3}}{2}$.

Soln. Here, we should draw the graphs of $y = \cos x$ and $y = \frac{\sqrt{3}}{2}$.

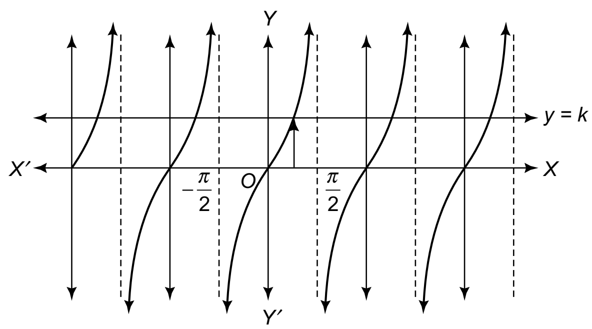


Hence, the solution set is

$$x = \bigcup_{n \in I} \left[2n\pi + \frac{\pi}{6}, 2(n+1)\pi - \frac{\pi}{6}\right].$$

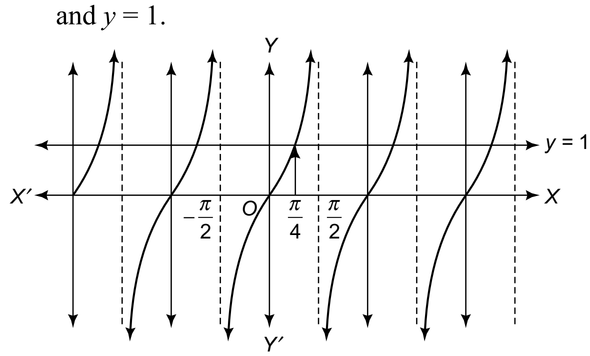
Type - V: An in-equation is of the form tan $x > k$.

Rule: First we find the smallest interval for which x satisfies the given in equation and then add $n\pi$ with each values of x.



Ex-1. Solve: $\tan x > 1$.

Soln. Here, we should draw the graphs of $y = \tan x$ and $y = 1$.

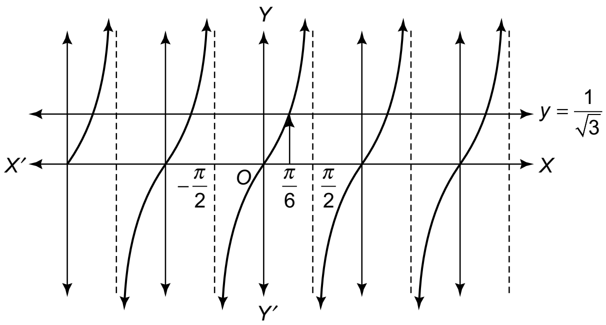


Hence, the solution set is

$$x = \bigcup_{n\in I}\left(n\pi+\frac{\pi}{4},\ n\pi+\frac{\pi}{2}\right).$$

Ex-2. Solve: $\tan x \geq \frac{1}{\sqrt{3}}$.

Soln. Here, we should draw the graphs of $y = \tan x$ and $y = \frac{1}{\sqrt{3}}$.

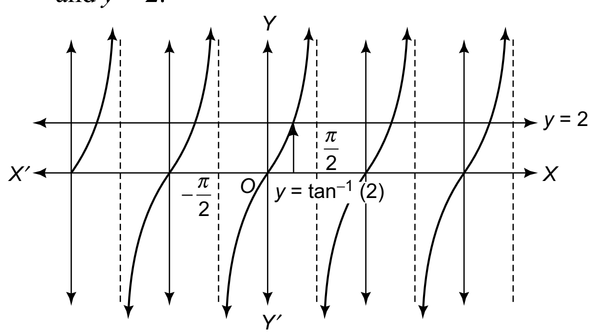


Hence, the solution set is

$$x = \bigcup_{n\in I}\left[n\pi+\frac{\pi}{6},\ n\pi+\frac{\pi}{2}\right)$$

Ex-3. Solve: $\tan x > 2$.

Soln. Here, we should draw the graphs of $y = \tan x$ and $y = 2$.

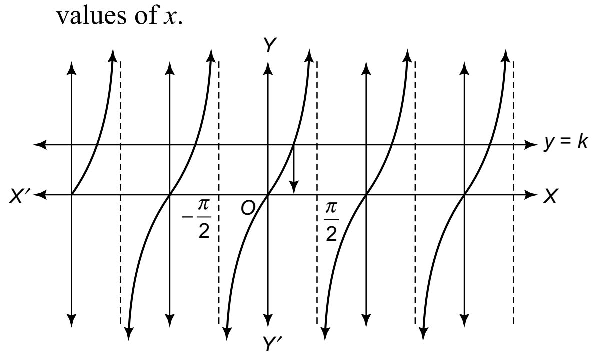


Hence, the solution set is

$$x = \bigcup_{n\in I}\left(n\pi+\tan^{-1}(2),\ n\pi+\frac{\pi}{2}\right).$$

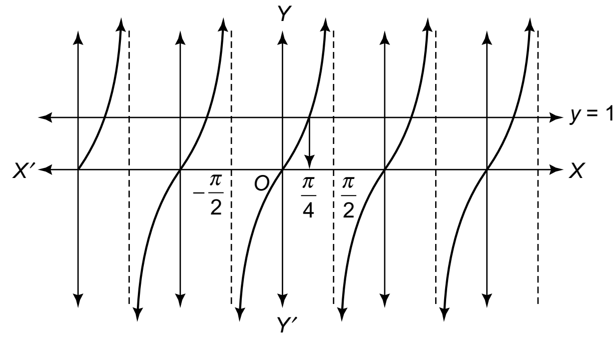
Type -VI: An inequation is of the form tan $x < k$.

Rule: First we find the smallest interval for which x satisfies the given in equation and then add $n\pi$ with each values of x.



Ex-1. Solve: $\tan x < 1$.

Soln. Here, we should draw the graphs of $y = \tan x$ and $y = 1$.

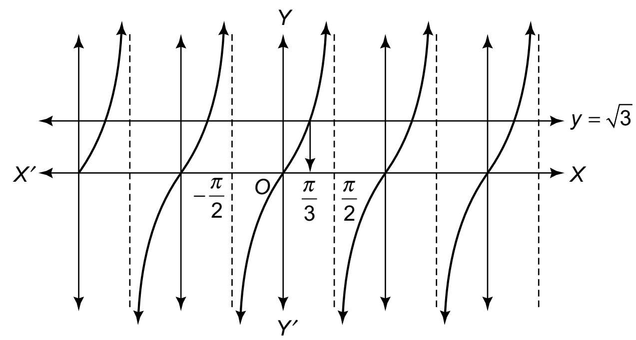


Hence, the solution set is

$$x = \bigcup_{n\in I}\left(n\pi-\frac{\pi}{2},\ n\pi+\frac{\pi}{4}\right).$$

Ex-2. Solve: $\tan x \leq \sqrt{3}$.

Soln. Here, we should draw the graphs of $y = \tan x$ and $y = \sqrt{3}$



Hence, the solution set is

$$x = \bigcup_{n\in I}\left[n\pi-\frac{\pi}{2},\ n\pi+\frac{\pi}{3}\right].$$

4.2 SOME SOLVED EXAMPLES

Ex-1. Solve: $\sin x > \cos x$.

Soln. We have $\sin x > \cos x$

$$\Rightarrow \quad \sin x - \cos x > 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x - \frac{1}{\sqrt{2}}\cos x > 0$$

$$\Rightarrow \quad \sin\left(x-\frac{\pi}{4}\right) > 0$$

$$\Rightarrow \quad x \in \left(2n\pi+\frac{\pi}{4},\ (2n+1)\pi+\frac{\pi}{4}\right)$$

Hence, the solution set is

$$x = \bigcup_{n\in I}\left(2n\pi+\frac{\pi}{4},\ (2n+1)\pi+\frac{\pi}{4}\right).$$

Ex-2. Solve: $\cos x > \sin x$.

Soln. We have $\cos x > \sin x$

$\Rightarrow \quad \cos x - \sin x > 0$

$\Rightarrow \quad \frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x > 0$

$\Rightarrow \quad \cos\left(x + \frac{\pi}{4}\right) > 0$

$\Rightarrow \quad x \in \left(2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right)$

Hence, the solution set is

$$x = \bigcup_{n \in I} \left(2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right).$$

Ex-3. Solve: $-\frac{1}{2} \le \cos x < \frac{1}{\sqrt{2}}$.

Soln. We have $\cos x < \frac{1}{\sqrt{2}}$ and $\cos x \ge -\frac{1}{2}$.

$\Rightarrow \quad x \in \left(2n\pi + \frac{\pi}{4}, 2n\pi + \frac{7\pi}{4}\right)$

and

$$x \in \left[2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right]$$

Hence, the solution set is

$$x = \bigcup_{n \in I} \left(2n\pi + \frac{\pi}{4}, 2n\pi + \frac{7\pi}{4}\right) \cup \left[2n\pi + \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right].$$

Ex-4: Solve: $|\sin x + \cos x| = |\sin x| + |\cos x|$.

Soln. We have $|\sin x + \cos x| = |\sin x| + |\cos x|$

As we know that, if

$|f(x) + g(x)| = |f(x)| + |g(x)|$

then $f(x) g(x) \ge 0$

Thus, $\sin x \cos x \ge 0$

$\Rightarrow \quad \sin 2x \ge 0$

$\Rightarrow \quad x \in \left[n\pi, n\pi + \frac{\pi}{2}\right]$

Hence, the solution set is

$$x = \bigcup_{n \in I} \left[n\pi, n\pi + \frac{\pi}{2}\right].$$

Ex-5. Solve: $\sin x \sin 2x < \sin 3x \sin 4x$, $\forall x \in \left(0, \frac{\pi}{2}\right)$.

Soln. We have, $\sin x \sin 2x < \sin 3x \sin 4x$

$\Rightarrow \quad 2 \sin x \sin 2x < 2 \sin 3x \sin 4x$

$\Rightarrow \quad \cos x - \cos 3x < \cos x - \cos 7x$

$\Rightarrow \quad \cos 3x > \cos 7x$

$\Rightarrow \quad \cos 3x - \cos 7x > 0$

$\Rightarrow \quad 2 \sin 5x \sin 2x > 0$

$\Rightarrow \quad \sin 5x > 0$ (since $\sin 2x$ is +ve for $0 < x < \pi/2$)

$\Rightarrow \quad 0 < 5x < \pi$

$\Rightarrow \quad 0 < x < \frac{\pi}{5}$

Hence, the solution set is

$$x = \left(0, \frac{\pi}{5}\right) \cup \left(0, \frac{\pi}{2}\right)$$

Ex-6. Solve: $\cos x - \sin x - \cos 2x > 0$, $\forall x \in (0, 2\pi)$

Soln. We have $\cos x - \sin x - \cos 2x > 0$

$\Rightarrow \quad (\cos x - \sin x) - (\cos^2 x - \sin^2 x) > 0$

$\Rightarrow \quad (\cos x - \sin x)(1 - \cos x - \sin x) > 0$

$\Rightarrow \quad (\sin x - \cos x)(\sin x + \cos x - 1) > 0$

$\Rightarrow \quad \sin\left(x - \frac{\pi}{4}\right)(\sin x + \cos x - 1) > 0$

Hence, the solution set is

$$x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$$

Ex-7. Solve: $\frac{5}{4}\sin^2 x + \frac{1}{4}\sin^2 2x > \cos 2x$

Soln. We have, $\frac{5}{4}\sin^2 x + \frac{1}{4}\sin^2 2x > \cos 2x$

$\Rightarrow \quad 5(2\sin^2 x) + 2(\sin^2 2x) > 8\cos 2x$

$\Rightarrow \quad 5(1 - \cos 2x) + 2(1 - \cos^2 2x) > 8\cos 2x$

$\Rightarrow \quad 5 - 5\cos 2x + 2 - 2\cos^2 2x - 8\cos 2x > 0$

$\Rightarrow \quad 2\cos^2 2x + 13\cos 2x - 7 < 0$

$\Rightarrow \quad 2\cos^2 2x + 14\cos 2x - \cos 2x - 7 < 0$

$\Rightarrow \quad 2\cos 2x(\cos 2x + 7) - (\cos 2x + 7) < 0$

$\Rightarrow \quad (\cos 2x + 7)(2\cos 2x - 1) < 0$

$\Rightarrow \quad 2\cos 2x - 1 < 0$

$\Rightarrow \quad \cos 2x < 1/2$

$\Rightarrow \quad x \in \left(n\pi + \frac{\pi}{6}, n\pi + \frac{5\pi}{6}\right)$

Hence, the solution set is

$$x = \bigcup_{n \in I} \left(n\pi + \frac{\pi}{6}, n\pi + \frac{5\pi}{6}\right)$$

Ex.-8 Solve: $6\sin^2 x - \sin x\cos x - \cos^2 x > 2$.

Soln. We have, $6\sin^2 x - \sin x\cos x - \cos^2 x > 2$

$\Rightarrow$ $6\sin^2 x - \sin x\cos x - \cos^2 x > 2(\sin^2 x + \cos^2 x)$

$\Rightarrow$ $4\sin^2 x - \sin x\cos x - 3\cos^2 x > 0$

$\Rightarrow$ $4\tan^2 x - \tan x - 3 > 0$

$\Rightarrow$ $4\tan^2 x - 4\tan x + 3\tan x - 3 > 0$

$\Rightarrow$ $4\tan x(\tan x - 1) + 3(\tan x - 1) > 0$

$\Rightarrow$ $(\tan x - 1)(4\tan x + 3) > 0$

$\Rightarrow$ $\tan x < -3/4$ and $\tan x > 1$

$\Rightarrow$ $x \in \left(n\pi - \frac{\pi}{2},\ n\pi - \tan^{-1}\left(\frac{3}{4}\right)\right)$

and $x \in \left(n\pi + \frac{\pi}{4},\ n\pi + \frac{\pi}{2}\right)$

Hence, the solution set is

$$x = \left(n\pi - \frac{\pi}{2},\ n\pi - \tan^{-1}\left(\frac{3}{4}\right)\right) \cup \left(n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{2}\right), n \in I$$

Ex-9. Solve: $\sin^6 x + \cos^6 x > \frac{13}{16}$

Soln. We have, $\sin^6 x + \cos^6 x > \frac{13}{16}$

$\Rightarrow$ $\left(1 - 3\sin^2 x\cos^2 x\right) > \frac{13}{16}$

$\Rightarrow$ $\left(1 - \frac{3}{4}\left(2\sin^2 x\right)\left(2\cos^2 x\right)\right) > \frac{13}{16}$

$\Rightarrow$ $\left(1 - \frac{3}{4}(1 - \cos 2x)(1 + \cos 2x)\right) > \frac{13}{16}$

$\Rightarrow$ $\left(1 - \frac{3}{4}\left(1 - \cos^2 2x\right)\right) > \frac{13}{16}$

$\Rightarrow$ $\left(1 - \frac{3}{4}\sin^2 2x\right) > \frac{13}{16}$

$\Rightarrow$ $\left(1 - \frac{3}{8}\left(2\sin^2 2x\right)\right) > \frac{13}{16}$

$\Rightarrow$ $\left(1 - \frac{3}{8}(1 - \cos 4x)\right) > \frac{13}{16}$

$\Rightarrow$ $\left(\frac{5}{8} + \frac{3}{8}\cos 4x\right) > \frac{13}{16}$

$\Rightarrow$ $\frac{3}{8}\cos 4x > \frac{3}{16}$

$\Rightarrow$ $\cos 4x > 1/2$

$\Rightarrow$ $4x \in \left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right)$

$\Rightarrow$ $x \in \left(\frac{n\pi}{2} - \frac{\pi}{12}, \frac{n\pi}{2} + \frac{\pi}{12}\right)$

Hence, the solution set is

$$x = \bigcup_{n \in I}\left(\frac{n\pi}{2} - \frac{\pi}{12}, \frac{n\pi}{2} + \frac{\pi}{12}\right)$$

Ex-10. Solve: $\cos^3 x.\cos 3x - \sin^3 x\sin 3x > \frac{5}{8}$

Soln. The given inequation is

$\cos^3 x.\cos 3x - \sin^3 x\sin 3x > \frac{5}{8}$

$\Rightarrow$ $(\cos 3x + 3\cos x)\cos 3x - (3\sin x - \sin 3x)\sin 3x > \frac{5}{2}$

$\Rightarrow$ $\sin^2 3x + \cos^2 3x + 3(\cos 3x\cos x - \sin 3x\sin x) > \frac{5}{2}$

$\Rightarrow$ $3\cos 4x + 1 > \frac{5}{2}$

$\Rightarrow$ $\cos 4x > \frac{1}{2}$

$\Rightarrow$ $2n\pi - \frac{\pi}{3} < 4x < 2n\pi + \frac{\pi}{3}$, $n \in I$

$\Rightarrow$ $\frac{n\pi}{2} - \frac{\pi}{12} < x < \frac{n\pi}{2} + \frac{\pi}{12}$, $n \in I$

Hence, the solution set is

$$x = \bigcup_{n \in I}\left(\left(\frac{n\pi}{2} - \frac{\pi}{12}, \frac{n\pi}{2} + \frac{\pi}{12}\right)\right)$$

Ex-11. Solve the inequality: $\sin^6 x + \cos^6 x > \frac{13}{16}$

Soln. The given inequation is

$\sin^6 x + \cos^6 x > \frac{13}{16}$

$\Rightarrow$ $\left(\sin^2 x + \cos^2 x\right)^3 - 3\sin^2 x\cos^2 x\left(\sin^2 x + \cos^2 x\right) > \frac{13}{16}$

$\Rightarrow$ $1 - 3\sin^2 x\cos^2 x > \frac{13}{16}$

$\Rightarrow$ $1 - \frac{3}{4}\left(\sin^2 2x\right) > \frac{13}{16}$

$\Rightarrow \quad 1-\frac{3}{8}\left(2\sin^2 4x\right)>\frac{13}{16}$

$\Rightarrow \quad 1-\frac{3}{8}(1-\cos 4x)>\frac{13}{16}$

$\Rightarrow \quad \frac{5}{8}+\frac{3}{8}\cos 4x>\frac{13}{16}$

$\Rightarrow \quad \frac{3}{8}\cos 4x>\frac{3}{16}$

$\Rightarrow \quad \cos 4x>\frac{1}{2}$

$\Rightarrow \quad 2n\pi-\frac{\pi}{3}<4x<2n\pi+\frac{\pi}{3},\ n\in I$

$\Rightarrow \quad \frac{n\pi}{2}-\frac{\pi}{12}<x<\frac{n\pi}{2}+\frac{\pi}{12},\ n\in I$

Hence, the solution set is

$x=\bigcup_{n\in I}\left(\frac{n\pi}{2}-\frac{\pi}{12},\frac{n\pi}{2}+\frac{\pi}{12}\right).$

Ex-12: Solve: $\frac{5}{4}\sin^2 x+\frac{1}{4}\sin^2 2x>\cos 2x$

Soln. The given inequaton is

$\frac{5}{4}\sin^2 x+\frac{1}{4}\sin^2 2x>\cos 2x$

$\Rightarrow \quad 5\sin^2 x+\sin^2 2x>4\cos 2x$

$\Rightarrow \quad 5\left(2\sin^2 x\right)+2\left(\sin^2 2x\right)>4\cos 2x$

$\Rightarrow \quad 5(1-\cos 2x)+2\left(1-\cos^2 2x\right)>8\cos 2x$

$\Rightarrow \quad 2\cos^2 2x+13\cos 2x-7<0$

$\Rightarrow \quad 2\cos^2 2x+14\cos 2x-\cos 2x-7<0$

$\Rightarrow \quad 2\cos 2x(\cos 2x+7)-1(\cos 2x+7)<0$

$\Rightarrow \quad (2\cos 2x-1)(\cos 2x+7)<0$

$\Rightarrow \quad -7<\cos 2x<\frac{1}{2}$

$\Rightarrow \quad \cos 2x<\frac{1}{2}$

$\Rightarrow \quad 2n\pi-\frac{\pi}{3}<2x<2n\pi+\frac{\pi}{3},\ n\in I$

$\Rightarrow \quad n\pi-\frac{\pi}{6}<x<n\pi+\frac{\pi}{6},\ n\in I$

Hence, the solution set is

$x=\bigcup_{n\in I}\left(n\pi-\frac{\pi}{6},\ n\pi-\frac{\pi}{6}\right).$

Ex-13: Solve: sin 3x sin 4x > sin x sin 2x $\forall\, x\in\left(0,\ \frac{\pi}{2}\right).$

Soln. The given inequation is

sin3x sin4x > sin$x$ sin2x

$\Rightarrow \quad$ 2 sin 3x sin 4x > 2 sin x sin 2x.

$\Rightarrow \quad$ cos x – cos 7x > cos x – cos 3x

$\Rightarrow \quad$ –cos 7x > – cos 3x

$\Rightarrow \quad$ cos 7x < cos 3x

$\Rightarrow \quad$ cos 7x – cos 3x < 0

$\Rightarrow \quad$ –2 sin 5x sin 2 x < 0

$\Rightarrow \quad$ sin 5x sin 2x > 0

$\Rightarrow \quad$ sin 5x > 0, since sin 2x is +ve in $\left(0,\ \frac{\pi}{2}\right)$

$\Rightarrow \quad 0<5x<\pi$

$\Rightarrow \quad 0<x<\frac{\pi}{5}$

Hence, the solution set is

$x=\left(0,\frac{\pi}{5}\right)$

Ex-14: Solve: $|\sin x+\cos x|=|\sin x|+|\cos x|$

Soln. We have, $|\sin x+\cos x|=|\sin x|+|\cos x|$

$\Rightarrow \quad \sin x\cos x\geq 0$

$\Rightarrow \quad 2\sin x\cos x\geq 0$

$\Rightarrow \quad \sin 2x\geq 0$

$\Rightarrow \quad 2n\pi\leq 2x\leq 2n\pi+\pi,\ n\in I$

$\Rightarrow \quad n\pi\leq x\leq n\pi+\frac{\pi}{2},\ n\in I$

Hence, the solution set is

$x=\bigcup_{n\in I}\left(n\pi\ ,\ n\pi+\frac{\pi}{2}\right)$

Ex-15. Solve: $|\sec x+\tan x|=|\sec x|+|\tan x|$

Soln. We have, $|\sec x+\tan x|=|\sec x|+|\tan x|$

$\Rightarrow \quad \sec x.\tan x\geq 0$

$\Rightarrow \quad \frac{\sin x}{\cos^2 x}\geq 0$

$\Rightarrow \quad \sin x\geq 0,\ \cos^2 x\neq 0$

$\Rightarrow \quad \sin x\geq 0,\ x\neq(2n+1)\frac{\pi}{2}, n\in I$

$\Rightarrow \quad 2n\pi\leq x\leq 2n\pi+\pi,\ x\neq(2n+1)\frac{\pi}{2},\ n\in I$

Hence, the solution set is

$x=\bigcup_{n\in I}\left(2n\pi\ ,\ (2n+1)\pi\right)-(2n+1)\frac{\pi}{2}$

Ex-16. Solve for x: $\sin^2 x + \sin x - 2 < 0$ and $x^2 - 3x + 2 < 0$.

Soln. We have, $\sin^2 x + \sin x - 2 < 0$

$\Rightarrow (\sin x + 2)(\sin x - 1) < 0$

$\Rightarrow -2 < \sin x < 1$

$\Rightarrow \sin x < 1$

$\Rightarrow -\frac{3\pi}{2} < x < \frac{\pi}{2}$

Also, $x^2 - 3x + 2 < 0$

$\Rightarrow (x-1)(x-2) < 0$

$\Rightarrow 1 < x < 2$

Hence, the solution set is $1 < x < \frac{\pi}{2}$

Ex-17. Solve for x: $2\sin^2 x + \sin x - 1 < 0$ and $x^2 + x - 2 < 0$

Soln. We have, $2\sin^2 x + \sin x - 1 < 0$

$\Rightarrow (2\sin x - 1)(\sin x + 1) < 0$

$\Rightarrow -1 < \sin x < 2$

$\Rightarrow -\frac{\pi}{2} < x < \frac{\pi}{6}$

Also, $x^2 + x - 2 < 0$

$\Rightarrow (x + 2)(x - 1) < 0$

$\Rightarrow -2 < x < 1$

Hence, the solution set is, $-\frac{\pi}{2} < x < \frac{\pi}{6}$

Ex-18. Solve for x: $\tan^2 x - 5\tan x + 6 > 0$ and $x^2 - 16 \le 0$

Soln. We have, $\tan^2 x - 5\tan x + 6 > 0$

$\Rightarrow (\tan x - 2)(\tan x - 3) > 0$

$\Rightarrow \tan x < 2$ and $\tan x > 3$

$\Rightarrow x < \tan^{-1}(2)$ and $x > \tan^{-1}(3)$

Also, $x^2 - 16 \le 0$

$\Rightarrow (x+4)(x-4) \le 0$

$\Rightarrow -4 \le x \le 4$

Hence, the solution set is

$x \in (-4, \tan^{-1}(2)) \cup (\tan^{-1}(3), 4)$

Ex-19. Solve for x: $\frac{x-1}{5-x} < 0$ and $\tan^2 x + \tan x - 6 < 0$.

Soln. We have, $\tan^2 x + \tan x - 6 < 0$

$\Rightarrow (\tan x + 3)(\tan x - 2) < 0$

$\Rightarrow -3 < \tan x < 2$

$\Rightarrow \tan^{-1}(-3) < x < \tan^{-1}(2)$

Also, $\frac{x-1}{5-x} < 0$

$\Rightarrow \frac{x-1}{x-5} > 0$

$\Rightarrow x < 1$ and $x > 5$

Hence, the solution set is $x \in (1, \tan^{-1}(2))$

Ex-20. Solve for x: [sin x] = 0, where [,] = G.I.F

Soln. We have, [sin x] = 0

$\Rightarrow 0 \le \sin x < 1$

Case I: When $\sin x \ge 0$

$\Rightarrow 2n\pi \le x \le (2n+1)\pi,\ n \in I$

Case II: When $\sin x < 1$

$\Rightarrow 2n\pi - \frac{3\pi}{2} < x < 2n\pi + \frac{\pi}{2}$

$\Rightarrow x \in \left((4n-3)\frac{\pi}{2}, \left((4n+1)\frac{\pi}{2}\right)\right)$

Hence, the solution set is

$$x = \bigcup_{n \in I} \left(\begin{array}{l} [2n\pi, (2n+1)\pi] \\ \cup \left((4n-3)\frac{\pi}{2}, (4n+1)\frac{\pi}{2}\right) \end{array} \right)$$

LEVEL I
(QUESTIONS BASED ON FUNDAMENTALS)

Q. Solve for x:

1. $\sin x > \frac{1}{2}$
2. $\sin x \ge 1$
3. $\sin x \le \frac{1}{\sqrt{2}}$
4. $\sin x < \frac{\sqrt{3}}{2}$
5. $\cos x > \frac{\sqrt{3}}{2}$
6. $\cos x \ge \frac{1}{2}$
7. $\cos x < \frac{1}{\sqrt{2}}$

8. $\cos x < \frac{1}{3}$
9. $\tan x > \frac{1}{\sqrt{3}}$
10. $\tan x \geq 1$
11. $\sin 2x < \frac{1}{2}$
12. $\sin 3x < \frac{\sqrt{3}}{2}$
13. $\cos 5x \geq \frac{1}{2}$
14. $\sin x + \cos x > 1$
15. $\sin x - \cos x < 1$
16. $\sqrt{3}\sin x + \cos x > 1$
17. $\sin x - \sqrt{3}\cos x < 1$
18. $\sin^2 x + \sin x - 2 < 0$
19. $\sin^2 x + 3\sin x + 2 < 0$
20. $\cos^2 x - \cos x > 0$

LEVEL II
(FOR JEE MAIN & ADVANCED EXAMS ONLY)

1. $\sin(3x-1) > 0$
2. $\cos(2x-3) < 0$
3. $|\sin x| \leq \frac{1}{2}$
4. $|\cos x| \leq \frac{1}{\sqrt{2}}$
5. $|\sin 2x + \cos 2x| = |\sin 2x| + |\cos 2x|$
6. $2\cos^2 x + \cos x < 1$
7. $4\sin^2 x - 1 \leq 0$
8. $4\cos^2 x - 3 \geq 0$
9. $|\sin x| > |\cos x|$
10. $|\cos x| > |\sin x|$
11. $\sin x + \cos x - \cos 2x > 0$
12. $x^2 + x - 2 < 0$ and $\sin x > \frac{1}{2}$
13. $x^2 - 1 \leq 0$ and $\cos x < \frac{1}{2}$
14. $4x^2 - 1 \geq 0$ and $\tan x \geq \frac{1}{\sqrt{3}}$
15. $x^2 - 3x + 2 < 0$ and $(\sin x)^2 - \sin x > 0$

LEVEL III
(FOR JEE ADVANCED EXAM ONLY)

Q. Solve for x:

1. $\frac{\sin x + \cos x}{\sin x - \cos x} > \sqrt{3}$
2. $|\sin x| > |\cos x|$
3. $\cot x + \frac{\sin x}{\cos x - 2} \geq 0$
4. $\sin x + \cos x > \sqrt{2}\cos 2x$
5. $4 \sin x \sin 2x \sin 3x > \sin 4x$
6. $\frac{\cos^2 2x}{\cos^2 x} \geq 3\tan x$
7. $\frac{\cos x + 2\cos^2 x + \cos 3x}{\cos x + 2\cos^2 x - 1} > 1$
8. $2(\sqrt{2}-1)\sin x - 2\cos 2x + \sqrt{2}(\sqrt{2}-1) < 0$
9. $\sin 2x > \sqrt{2}\sin^2 x + (2-\sqrt{2})\cos^2 x$
   , $x \in (0, 2\pi)$
10. $1 + \log_4 \sin x + 2\log_{16} \cos x > 0$

Comprehensive Link Passage

PASSAGE I

If $x_1, x_2, x_3 \in R$, then

$$\frac{f(x_1) + f(x_2) + f(x_3)}{3} \leq f\left(\frac{x_1 + x_2 + x_3}{3}\right)$$

Then

1. The value of $\sin\alpha + \sin\beta + \sin\gamma$ is, where $\alpha + \beta + \gamma = 180°$
   (a) ≤ 1 (b) ≤ 3
   (c) $\leq \frac{3\sqrt{3}}{2}$ (d) $\leq \frac{3}{2}$
2. The value of $\cos\alpha + \cos\beta + \cos\gamma$ is, where $\alpha + \beta + \gamma = 180°$
   (a) ≤ 2 (b) $\leq \frac{3}{2}$
   (c) ≤ 3 (d) $\leq \frac{\sqrt{3}}{2}$
3. The value of $\cot\alpha + \cot\beta + \cot\gamma$ is, where $\alpha + \beta + \gamma = 180°$
   (a) ≥ 1 (b) $\geq \sqrt{3}$ (c) $\geq 3/2$ (d) $\geq \frac{\sqrt{3}}{2}$
4. The value of $\cot\alpha \cot\beta \cot\gamma$ is, where $\alpha + \beta + \gamma = 180°$

(a) $\le\frac{1}{3}$ (b) $\le\frac{1}{3\sqrt{3}}$ (c) $\le\frac{1}{2\sqrt{3}}$ (d) $\le\frac{3}{2}$

5. The value of $\sin^2\alpha+\sin^2\beta+\sin^2\gamma$ is, where $\alpha+\beta+\gamma=180°$
(a) $\le 9/4$ (b) $\le 3/4$ (c) $\le 3/2$ (d) $\le 1/2$

6. The value of $\sin\alpha.\sin\beta.\sin\gamma$ is, where $\alpha+\beta+\gamma=180°$
(a) $\le\frac{\sqrt{3}}{4}$ (b) $\le\frac{3\sqrt{3}}{8}$ (c) $\le\frac{1}{2\sqrt{3}}$ (d) $\le\frac{1}{3\sqrt{2}}$

7. The value of $\cot^2\alpha+\cot^2\beta+\cot^2\gamma$ is, where $\alpha+\beta+\gamma=180°$
(a) ≥ 1 (b) ≥ 3 (c) $\ge\sqrt{3}$ (d) $\ge\sqrt{3}/2$

PASSAGE II

If $|f(x)+g(x)|=|f(x)|+|g(x)|$, then $f(x).g(x)\ge 0$

On the basis of the above information answer the following questions

1. If $|\sec x+\tan x|=|\sec x|+|\tan x|\ \ \forall\ x\in[0,2\pi]$, then x does not satisfy the equation is
(a) 0 (b) π
(c) $\frac{\pi}{2}$ (d) 2π

2. If $|x-1|+|x-3|=2$, then x is
(a) $x>1$ (b) $x>3$
(c) $x<1$ (d) $1<x<3$

3. If $|\sin x+\cos x|=|\sin x|+|\cos x|,\ \forall\ x\in[0,2\pi]$ then the solution set is
(a) $\left[0,\frac{\pi}{2}\right]$
(b) $\left[0,\frac{\pi}{2}\right]\cup\left[\pi,\frac{3\pi}{2}\right]\cup\{2\pi\}$
(c) $\left[\pi,\frac{3\pi}{2}\right]\cup\{2\pi\}$
(d) $[0,2\pi]$

MATCH MATRIX
(FOR JEE ADVANCED EXAM ONLY)

1. Match the following columns:

Column - I	Column-II
(A) The number of solutions of $\sin x>\frac{1}{2}$ in $(0,2\pi)$ is	(P) 6
(B) The number of solutions of $\lvert\tan x\rvert\le 1$ in $(-\pi,\pi)$ is	(Q) 0
(C) The number of solutions of $\lvert\cos x\rvert>1$ in $(0,2013\pi)$ is	(R) 4
(D) The number of solutions of $\lvert\sin x+\cos x\rvert=\lvert\sin x\rvert+\lvert\cos x\rvert$ in $(0,2\pi)$ is	(S) 2

2. Match the following columns:

Column - I	Column - II
(A) If $\sin x.\cos^3 x>\cos x.\sin^3 x$, $x\in[0,2\pi]$ Then x is	(P) $\left[-\pi,-\frac{3\pi}{4}\right]\cup\left[-\frac{\pi}{4},\frac{\pi}{4}\right]\cup\left[\frac{3\pi}{4},\pi\right]$
(B) If $4\sin^2 x-8\sin x+3\le 0$, $x\in[0,2\pi]$ then x is	(Q) $\left[\frac{3\pi}{2},2\pi\right]\cup\{0\}$
(C) If $\lvert\tan x\rvert\le 1,\ x\in[-\pi,\pi]$ then x is	(R) $\left(0,\frac{\pi}{4}\right)$
(D) If $\cos x-\sin x\ge 1$, $x\in[0,2\pi]$ then x is	(S) $\left[\frac{\pi}{6},\frac{5\pi}{6}\right]$

ASSERTION AND REASON
(FOR JEE ADVANCED EXAM ONLY)

Codes:
(A) Both A and R are true and R is the correct explanation of A
(B) Both A and R are true but R is not the correct explanation of A
(C) A is true but R is false
(D) A is false but R is true.

1. Assertion (A): The value of $\tan 3\alpha.\cot\alpha$ cannot lie between 3 and 1/3.
Reason (R): In a triangle ABC, the maximum value of $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$ is $\frac{1}{8}$

2. Assertion (A): The minimum value of $a^2\tan^2\theta+b^2\cot^2\theta$ is $2ab$.
Reason (R): For positive real numbers $A.M\ge G.M$

3. Assertion (A): The minimum value of $\frac{a^2}{\cos^2 x}+\frac{b^2}{\sin^2 x}$ is $(a+b)^2$

Reason (R): The maximum value of $3\sin^2 x + 4\cos^2 x$ is 4

4. Assertion (A): For all $\theta \in \left[0, \frac{\pi}{2}\right]$, $\cos(\sin\theta) > \sin(\cos\theta)$

Reason (R): In a triangle ABC, the maximum. value of $\frac{\sin A + \sin B + \sin C}{\cot A + \cot B + \cot C}$ is $\frac{3}{2}$

5. Assertion (A): $\cot^{-1} x \geq 2 \Rightarrow x \in (-\infty, 2]$

Reason (R): $\cot^{-1} x$ is a decreasing function.

ANSWERS

LEVEL I

1. $x \in \left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right), n \in I$
2. $x = (4n+1)\frac{\pi}{2}, n \in I$
3. $x \in \left(2n\pi - \frac{5\pi}{4}, 2n\pi + \frac{\pi}{4}\right), n \in I$
4. $x \in \left(2n\pi - \frac{4\pi}{3}, 2n\pi + \frac{\pi}{3}\right), n \in I$
5. $x \in \left(2n\pi - \frac{\pi}{6}, 2n\pi + \frac{\pi}{6}\right), n \in I$
6. $x \in \left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right), n \in I$
7. $x \in \left(2n\pi - \frac{7\pi}{4}, 2n\pi + \frac{\pi}{4}\right), n \in I$
8. $x \in \left(2n\pi + \cos^{-1}\left(\frac{1}{3}\right), 2(n+1)\pi - \cos^{-1}\left(\frac{1}{3}\right)\right), n \in I$
9. $x \in \left(n\pi + \frac{\pi}{6}, n\pi + \frac{\pi}{2}\right), n \in I$
10. $x \in \left(n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{2}\right), n \in I$
11. $x \in \left(n\pi + \frac{\pi}{12}, n\pi + \frac{5\pi}{12}\right), n \in I$
12. $x \in \left(\frac{2n\pi}{3} - \frac{4\pi}{9}, \frac{2n\pi}{3} + \frac{\pi}{9}\right), n \in I$
13. $x \in \left(\frac{2n\pi}{5} - \frac{\pi}{15}, \frac{2n\pi}{5} + \frac{\pi}{15}\right), n \in I$
14. $x \in (2n\pi, (2n+1)\pi), n \in I$
15. $x \in \left((2n-1)\pi, 2n\pi + \frac{\pi}{2}\right), n \in I$
16. $x \in \left(2n\pi, 2n\pi + \frac{2\pi}{3}\right), n \in I$
17. $x \in \left(2n\pi - \frac{5\pi}{6}, 2n\pi + \frac{\pi}{2}\right), n \in I$
18. $x \in \left(2n\pi - \frac{3\pi}{2}, 2n\pi + \frac{\pi}{2}\right), n \in I$
19. $x = \varphi$
20. $x \in \left(2n\pi + \frac{\pi}{12}, 2n\pi + \frac{5\pi}{12}\right), n \in I$

LEVEL II

1. $x \in \left(\frac{2n\pi + 1}{3}, \frac{(2n\pi+1)\pi + 1}{3}\right), n \in I$
2. $x \in \left((4n+1)\frac{\pi}{4} + \frac{3}{2}, (4n+3)\frac{\pi}{4} + \frac{3}{2}\right), n \in I$
3. $x \in \left(n\pi - \frac{\pi}{6}, n\pi + \frac{\pi}{6}\right), n \in I$
4. $x \in \left(n\pi - \frac{3\pi}{4}, n\pi - \frac{\pi}{4}\right), n \in I$
6. $x \in \left(n\pi - \pi, n\pi - \frac{\pi}{3}\right), n \in I$
7. $x \in \left(n\pi - \frac{\pi}{6}, n\pi + \frac{\pi}{6}\right), n \in I$
8. $x \in \left(n\pi - \frac{\pi}{6}, n\pi + \frac{\pi}{6}\right), n \in I$
9. $x \in \bigcup_{n \in I} \left\{\left(2n\pi - \frac{\pi}{2}, n\pi - \frac{\pi}{4}\right) \cup \left(n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{2}\right)\right\}$
10. $x \in \left(n\pi - \frac{\pi}{4}, n\pi + \frac{\pi}{4}\right), n \in I$
11. $x \in \left(0, \frac{3\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$
12. $\frac{\pi}{6} < x < 1$
13. $x = \varphi$
14. $x \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$

15. $x \in (\pi, 2\pi)$

LEVEL III

1. $\bigcup_{n\in I}\left(n\pi+\frac{\pi}{4},\ n\pi+\frac{\pi}{12}\right)$
2. $\bigcup_{n\in I}\left(n\pi+\frac{\pi}{4},\ n\pi+\frac{3\pi}{4}\right)$
3. $\bigcup_{n\in I}\left(2n\pi-\frac{\pi}{3},\ 2n\pi+\frac{\pi}{3}\right)$
   $\cup\left(2n\pi+\frac{\pi}{3},(2n+1)\pi\right)$
4. $\bigcup_{n\in I}\left(2n\pi+\frac{\pi}{12},\ 2n\pi+\frac{3\pi}{4}\right)$
   $\cup\left(2n\pi+\frac{17\pi}{12},\ 2n\pi+\frac{7\pi}{4}\right)$
5. $\bigcup_{n\in I}\left(n\pi-\frac{\pi}{8},\ n\pi\right)\cup\left(n\pi+\frac{\pi}{2},\ n\pi+\frac{5\pi}{8}\right)$
   $\cup\left(n\pi+\frac{\pi}{8},\ n\pi+\frac{3\pi}{8}\right)$
6. $\bigcup_{n\in I}\left(n\pi-\frac{7\pi}{12},\ n\pi-\frac{\pi}{2}\right)$
   $\cup\left(n\pi-\frac{\pi}{2},\ n\pi+\frac{\pi}{12}\right)$.
7. $\bigcup_{n\in I}\left(2n\pi-\frac{\pi}{3},\ 2n\pi+\frac{\pi}{3}\right)$
8. $\bigcup_{n\in I}\left(2n\pi-\frac{\pi}{4},\ 2n\pi+\frac{\pi}{6}\right)$
   $\cup\left(2n\pi+\frac{5\pi}{6},\ 2n\pi+\frac{5\pi}{4}\right)$
9. $\left(\tan^{-1}\left(\sqrt{2}-1\right),\ \frac{\pi}{4}\right)$
   $\cup\left(\pi+\tan^{-1}\left(\sqrt{2}-1\right),\frac{5\pi}{4}\right)$
10. $\bigcup_{n\in I}\left(2n\pi+\frac{\pi}{12},\ 2n\pi+\frac{5\pi}{12}\right)$

COMPREHENSIVE LINK PASSAGES

PASSAGE I

1. (c) 2. (b) 3. (b) 4. (b)
5. (a) 6. (b) 7. (a)

PASSAGE II

1. (c) 2. (d) 3. (b)

MATCH MATRIX

1. (A) → (S); (B) → (S); (C) → (Q); (D) → (R).
2. (A) → (R); (B) → (S); (C) → (P); (D) → (Q).

ASSERTION AND REASON

1. (B) 2. (A) 3. (B)
4. (B) 5. (A)

CHAPTER 5

Logarithm

5.1 INTRODUCTION

The first mention of the natural logarithm was by Nicholas Mercator in his work Logarithmotechnia published in 1668, although the mathematics teacher John Speidell had already in 1619 compiled a table of what in fact were effectively natural logarithms. It was formerly also called hyperbolic logarithm, as it corresponds to the area under a hyperbola. It is also sometimes referred to as the Napierian logarithm, named after John Napier, although Napier's original 'logarithms' (from which Speidell's numbers were derived) were slightly different (see Logarithm: from Napier to Euler).

NOTATIONAL CONVENTIONS

The notations 'lnx' and $\log_e x$ both refer unambigously to the natural logatithm of x.

'log x' without an explicit base may also refer to the natural logarithm. This usage is common in mathematics and some scientific contexts as well as in many programming languages.

BASIC FORMULAE ON LOGARITHM:

Step I:

1. If $a^x = n$, then $x = \log_a n$, where $n > 0$, $a > 0$ and $a \neq 1$.
2. Exponential function is always positive.
3. $\log_a a = 1$

   Pf. We have, $a^1 = a$

   $\Rightarrow \quad 1 = \log_a a$

4. $\log_a 1 = 0, a \neq 1, a > 0$

   Pf. We have, $a^0 = 1$

   $\Rightarrow \quad 0 = \log_a 1$

5. $\log_a 0 = \pm\infty$

   Pf. **Case i:** when $a > 1$

   We have, $a^{-\infty} = 0$

   $\Rightarrow \quad -\infty = \log_a 0$

   Case ii: when $0 < a < 1$

   We have $a^{\infty} = 0$

   $\Rightarrow \quad \infty = \log_a 0$

   Hence, $\log_a 0 = \pm\infty$

6. $\log_a \infty = \infty$

   Pf. **Case i:** when $a > 1$

   We have $a^{\infty} = \infty$

   $\Rightarrow \quad \log_a \infty = \infty$

Step II

1. $\log_m a + \log_m b = \log_m (ab)$.

   Pf. Let $\log_m a = x, \log_m b = y, \log_m (ab) = z$

   $m^x = a, m^y = b, m^z = ab$

   Now, $ab = m^z$

   $\Rightarrow \quad m^x.m^y = m^z$

   $\Rightarrow \quad m^{x+y} = m^z$

   $\Rightarrow \quad x + y = z$

   $\Rightarrow \quad \log_m a + \log_m b = \log_m (ab)$

2. $\log_m a - \log_m b = \log_m \left(\frac{a}{b}\right)$

   Pf. Let $\log_m a = x, \log_m b = y, \log_m \left(\frac{a}{b}\right) = z$

$\Rightarrow \quad m^x = a, m^y = b, m^z = \left(\frac{a}{b}\right)$

Now, $\left(\frac{a}{b}\right) = m^z$

$\Rightarrow \quad \frac{m^x}{m^y}. = m^z$

$\Rightarrow \quad m^{x-y} = m^z$

$\Rightarrow \quad x - y = z$

$\Rightarrow \quad \log_m a - \log_m b = \log_m \left(\frac{a}{b}\right)$

3. $\log_m a^n = n \log_m a$

Pf. Let $\log_m a^n = x, \log_m a = y$

$\Rightarrow \quad m^x = a^n, m^y = a$

$\Rightarrow \quad m^x = \left(m^y\right)^n$

$\Rightarrow \quad m^x = m^{ny}$

$\Rightarrow \quad x = ny$

$\Rightarrow \quad \log_m a^n = n \log_m a$

4. $a^{\log_a n} = n$

Pf. Let $\log_a n = x$

$\Rightarrow \quad a^x = n$

$\Rightarrow \quad a^{\log_a n} = n$

Step III

1. $\log_a b \times \log_b a = 1$

Pf. Let $\log_a b = x, \log_b a = y$

$\Rightarrow \quad a^x = b, b^y = a$

$\Rightarrow \quad \left(b^y\right)^x = b$

$\Rightarrow \quad b^{xy} = b = b^1$

$\Rightarrow \quad xy = 1$

$\Rightarrow \quad \log_a b \times \log_b a = 1$

2. $\log_a b = \frac{1}{\log_b a}$

Pf. As we know that, $\log_a b \times \log_b a = 1$

$\Rightarrow \quad \log_a b = \frac{1}{\log_b a}$

3. $\log_b a = \frac{\log_m a}{\log_m b}$

Pf. Let $\log_a b = x, \log_m b = y, \log_m a = z$

$\Rightarrow \quad a^x = b, m^y = b, m^z = a$

$\Rightarrow \quad a^x = b$

$\Rightarrow \quad \left(m^z\right)^x = m^y$

$\Rightarrow \quad m^{zx} = m^y$

$\Rightarrow \quad zx = y$

$\Rightarrow \quad x = \frac{y}{z}$

$\Rightarrow \quad \log_b a = \frac{\log_m a}{\log_m b}$

4. $a^{\log_n b} = b^{\log_n a}$

Pf. Let $\log_n b = x, \log_n a = y$

$\Rightarrow \quad n^x = b, n^y = a$

$\Rightarrow \quad n = b^{1/x}, n = a^{1/y}$

$\Rightarrow \quad b^{1/x} = a^{1/y}$

$\Rightarrow \quad b^y = a^x$

$\Rightarrow \quad a^{\log_n b} = b^{\log_n a}$

Note. 1. $\log_a b \times \log_b c \times \log_c d \times \log_d a = 1$

2. $\log_a b \times \log_b c \times \log_c d \times \ldots\ldots. \times \log_z a = 1$

Step IV

1. $\log_{a^\alpha} b = \frac{1}{\alpha} \log_a b$

Pf. We have $\log_{a^\alpha} b$

$= \frac{1}{\log_b \left(a^\alpha\right)}$

$= \frac{1}{\alpha \log_b a}$

$= \frac{1}{\alpha} \times \frac{1}{\log_b a}$

$= \frac{1}{\alpha} \times \log_a b$

2. $\log_{a^\alpha} \left(b^\beta\right) = \frac{\beta}{\alpha} \log_a b$

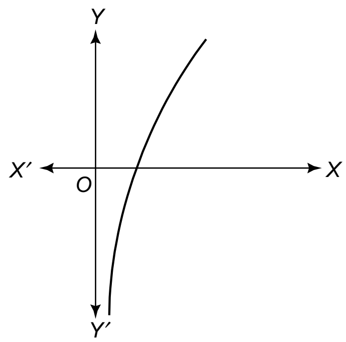
Pf. We have $\log_{a^\alpha} \left(b^\beta\right)$

$= \beta \log_{a^\alpha} (b)$

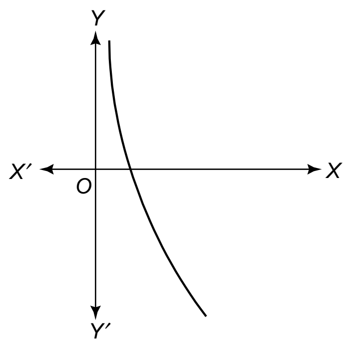
$= \frac{\beta}{\alpha} \times \log_a b$

Step V

1. If $x > y \Rightarrow \log_a x > \log_a y$, when $a > 1$.



2. If $x > y \Rightarrow \log_a x < \log_a y$, when $0 < a < 1$.



5.2 SOME SOLVED EXAMPLES

Ex-1. Find the value of $\log_2(64) + \log_4(256)$

Soln. We have $\log_2(64) + \log_4(256)$

$= \log_2(2^6) + \log_4(4^4)$

$= 6\log_2(2) + 4\log_4(4)$

$= 6 + 4$

$= 10$

Ex-2. Find the value of $\log_8 64$

Soln. We have $\log_8 64$

$= \log_{2^3}(4^3)$

$= \log_2(4)$

$= \log_2(2^2)$

$= 2$

Ex-3. Find the value of $\dfrac{1}{\log_2(36)} + \dfrac{1}{\log_3(36)}$

Soln. We have $\dfrac{1}{\log_2(36)} + \dfrac{1}{\log_3(36)}$

$= \log_{36}(2) + \log_{36}(3)$

$= \log_{36}(2.3)$

$= \log_{36}(6)$

$= \log_{6^2}(6)$

$= 1/2$

Ex-4. Find the value of $\log_2 4.\log_4 5.\log_5 10.\log_{10} 32$

Soln. We have $\log_2 4.\log_4 5.\log_5 10.\log_{10} 32$

$= \log_2(32)$

$= \log_2(2^5)$

$= 5$

Ex-6. Find the value of $2^{\log_{2\sqrt{2}}(125)}$

Soln. We have $2^{\log_{2\sqrt{2}}(125)}$

$= 2^{\log_{2\sqrt{2}}(125)}$

$= 2^{\frac{2}{3}\log_2(125)}$

$= 2^{\log_2(125)^{2/3}}$

$= (125)^{2/3}$

$= (5^3)^{2/3}$

$= 25$

Ex-7. Find the value of $3^{\log_5 2} - 2^{\log_5 3}$

Soln. We have $3^{\log_5 2} - 2^{\log_5 3}$

$= 2^{\log_5 3} - 2^{\log_5 3}$

$= 0$

Ex-8. If $\log_a(ab) = x$, then find $\log_b(ab)$.

Soln. Given $\log_a(ab) = x$

$\Rightarrow \quad \log_a a + \log_a b = x$

$\Rightarrow \quad 1 + \log_a b = x$

$\Rightarrow \quad \log_a b = x - 1$

Now, $\log_b(ab) = \log_b a + \log_b b$

$= \log_b a + 1$

$= \dfrac{1}{x-1} + 1$

$= \dfrac{x}{x-1}$

Ex-9. Find x, if $\log_2 x + \log_4 x + \log_8 x = 11$

Soln. We have $\log_2 x + \log_4 x + \log_8 x = 11$

$\Rightarrow \quad \log_2 x + \log_{2^2} x + \log_{2^3} x = 11$

$\Rightarrow \quad \left(1 + \dfrac{1}{2} + \dfrac{1}{3}\right)\log_2 x = 11$

$\Rightarrow \quad \left(\frac{11}{6}\right)\log_2 x = 11$

$\Rightarrow \quad \log_2 x = 2$

$\Rightarrow \quad x = 2^6 = 32$

Hence, the value of x is 32.

Ex-10. If $a = \log_4 5$ and $b = \log_5 6$, then find $\log_3 2$

Soln. Now, $ab = \log_4 5.\log_5 6 = \log_4 6$

$= \log_{2^2}(6)$

$= \frac{1}{2}\log_2(6)$

$= \frac{1}{2}\log_2(2.3)$

$= \frac{1}{2}(\log_2(2) + \log_2 3)$

$= \frac{1}{2}(1 + \log_2 3)$

$\Rightarrow \quad 2ab = (1 + \log_2 3)$

$\Rightarrow \quad \log_2 3 = (2ab - 1)$

$\Rightarrow \quad \frac{1}{\log_2 3} = \frac{1}{(2ab-1)}$

$\Rightarrow \quad \log_3 2 = \frac{1}{(2ab-1)}$

Ex-11. If $x = \log_a bc$, $y = \log_b ca$ and $z = \log_c ab$ then find the value of $\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z}$

Soln. Given $x = \log_a bc$

$\Rightarrow \quad a^x = bc$

$\Rightarrow \quad a.a^x = abc$

$\Rightarrow \quad a^{x+1} = abc$

$\Rightarrow \quad x + 1 = \log_a(abc)$

$\Rightarrow \quad \frac{1}{x+1} = \frac{1}{\log_a(abc)}$

Similarly, $\frac{1}{y+1} = \frac{1}{\log_b(abc)}$, $\frac{1}{z+1} = \frac{1}{\log_c(abc)}$

Thus, $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1}$

$= \frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_c(abc)}$

$= \log_{abc}(a) + \log_{abc}(b) + \log_{abc}(c)$

$= \log_{abc}(abc) = 1$

Ex-12. If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$, then prove that $a^a.b^b.c^c = 1$

Soln. Let $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b} = k$

$\Rightarrow \quad \frac{a\log a}{a(b-c)} = \frac{b\log b}{b(c-a)} = \frac{c\log c}{c(a-b)} = k$

$\Rightarrow \quad \frac{\log a^a}{a(b-c)} = \frac{\log b^b}{b(c-a)} = \frac{\log c^c}{c(a-b)} = k$

$\Rightarrow \quad \log(a^a) + \log(b^b) + \log(c^c)$

$= k(ab - bc + bc - ba + ca - cb) = 0$

$\Rightarrow \quad \log(a^a.b^b.c^c) = 0$

$\Rightarrow \quad (a^a.b^b.c^c) = e^0 = 1$

Ex-13. If $\log_2 x + \log_2 y \geq 6$, then find the least value of $x + y$.

Soln. Given $\log_2 x + \log_2 y \geq 6$

$\Rightarrow \quad \log_2(xy) \geq 6$

$\Rightarrow \quad xy \geq 2^6 = 64$

As we know that, $\frac{x+y}{2} \geq \sqrt{xy}$

$\Rightarrow \quad \frac{x+y}{2} \geq \sqrt{64} = 8$

$\Rightarrow \quad xy \geq 2.8 = 16$

Hence, the least value of $x + y$ is 16.

Ex-14. If $x^{18} = y^{21} = z^{28}$, then prove that, $3, 3\log_y x, 3\log_z y, 7\log_x z$ are in A.P.

Soln. Given $x^{18} = y^{21} = z^{28}$

$\Rightarrow \quad \log(x^{18}) = \log(y^{21}) = \log(z^{28})$

$\Rightarrow \quad 18\log x = 21\log y = 28\log z = k$ (say)

Now, $3\log_y x = 3.\frac{\log x}{\log y} = 3.\frac{21}{18} = \frac{7}{2}$

$3\log_z y = 3.\frac{\log y}{\log z} = 3.\frac{28}{21} = 4$ and

$7\log_x z = 7.\frac{\log z}{\log x} = 7.\frac{18}{28} = \frac{9}{2}$

Thus, 3, 7/2, 4, 9/2 are in A.P

Hence, the result.

Ex-15. If $\log_3 2, \log_3\left(2^x-5\right), \log_3\left(2^x-\frac{7}{2}\right)$ are in A.P., then find the value of x.

Soln. Given $\log_3 2, \log_3\left(2^x-5\right), \log_3\left(2^x-\frac{7}{2}\right)$

$\Rightarrow \quad 2\log_3\left(2^x-5\right)=\log_3 2+\log_3\left(2^x-\frac{7}{2}\right)$

$\Rightarrow \quad \log_3\left(2^x-5\right)^2=\log_3 2.\left(2^x-\frac{7}{2}\right)$

$\Rightarrow \quad \left(2^x-5\right)^2=2.2^x-7$

$\Rightarrow \quad \left(2^x\right)^2-12.2^x+32=0$

$\Rightarrow \quad a^2-12a+32=0, a=2^x$

$\Rightarrow \quad (a-4)(a-8)=0, a=2^x$

$\Rightarrow \quad a=4, 8$

when $a=4 \Rightarrow 2^x=4=2^2 \Rightarrow x=2$

when $a=8 \Rightarrow 2^x=8=2^3 \Rightarrow x=3$

But $x=2$ does not satisfy the terms.

Hence, the solution of x is 3.

Ex-16. If a, b, c are in G. P, then prove that $\frac{1}{1+\log a}, \frac{1}{1+\log b}, \frac{1}{1+\log c}$ are in H.P.

Soln. Given a, b, c are in G.P

$\Rightarrow \quad b^2=ac$

$\Rightarrow \quad \log\left(b^2\right)=\log(ac)$

$\Rightarrow \quad 2\log(b)=\log(a)+\log(c)$

$\Rightarrow \quad \log a, \log b, \log c$ are in A.P

$\Rightarrow \quad 1+\log a, 1+\log b, 1+\log c$ are in A.P

$\Rightarrow \quad \frac{1}{1+\log a}, \frac{1}{1+\log b}, \frac{1}{1+\log c}$ are in H.P.

Ex-17. If $\frac{1}{\log_3 \pi}+\frac{1}{\log_4 \pi}>x$, then find the value of x.

Soln. We have $\frac{1}{\log_3 \pi}+\frac{1}{\log_4 \pi}$

$=\log_\pi 3+\log_\pi 4$

$=\log_\pi(3.4)$

$=\log_\pi(12)>\log_\pi\left(\pi^2\right)=2$

Hence, the value of x is 2

Ex-18. Find x, if $4^{\log_9 3}+9^{\log_2 4}=10^{\log_x 83}$

Soln. Given equation is $4^{\log_9 3}+9^{\log_2 4}=10^{\log_x 83}$

$\Rightarrow \quad 4^{\frac{1}{2}\log_3 3}+9^{2\log_2 2}=10^{\log_x 83}$

$\Rightarrow \quad 2+81=10^{\log_x 83}$

$\Rightarrow \quad 83=83^{\log_x 10}$

$\Rightarrow \quad \log_x 10=1$

$\Rightarrow \quad x=10$

Hence, the solution is $x=10$.

Ex-19. If $\log_x 10=1$, $b=\log_{36} 24$ and $c=\log_{48} 36$ then prove that $\left(\frac{abc+1}{bc}\right)=2$.

Soln. We have, $abc=\log_{24} 12\times\log_{36} 24\times\log_{48} 12$

$=\log_{48} 36\times\log_{36} 24\times\log_{24} 12$

$=\log_{48} 12$

Now, $abc+1$

$=\log_{48} 12+1=\log_{48} 12+\log_{48} 48$

$=\log_{48}(12\times 48)$

Also, $bc=\log_{36} 24\times\log_{48} 36$

$=\log_{48} 36\times\log_{36} 24$

$=\log_{48} 24$

Thus, $\left(\frac{abc+1}{bc}\right)=\frac{\log_{48}(12\times 48)}{\log_{48} 24}$

$=\log_{24}(12\times 48)$

$=\log_{24}(24\times 24)$

$=\log_{24}\left(24^2\right)$

$=2$

Hence, the result.

Ex-20. If $\log_{10}\left(\sin\left(x+\frac{\pi}{4}\right)\right)=\frac{1}{2}\left(\log_{10} 6-1\right)$, then find the value of $\log_{10}\sin x+\log_{10}\cos x$

Soln. Given $\log_{10}\left(\sin\left(x+\frac{\pi}{4}\right)\right)=\frac{1}{2}\left(\log_{10} 6-1\right)$

$\Rightarrow \quad 2\log_{10}\left(\sin\left(x+\frac{\pi}{4}\right)\right)=\left(\log_{10} 6-1\right)$

$\Rightarrow \quad 2\log_{10}\left(\frac{1}{\sqrt{2}}(\sin x+\cos x)\right)=\left(\log_{10} 6-1\right)$

$$\Rightarrow\ 2\log_{10}\left(\frac{1}{\sqrt{2}}\right)+2\log_{10}(\sin x+\cos x)=(\log_{10}6-1)$$

$$\Rightarrow\ 2\log_{10}(\sin x+\cos x)=(\log_{10}6+2\log_{10}2-1)$$

$$\Rightarrow\ \log_{10}(\sin x+\cos x)^2=\log_{10}\left(\frac{24}{10}\right)$$

$$\Rightarrow\ (\sin x+\cos x)^2=\left(\frac{24}{10}\right)$$

$$\Rightarrow\ 1+\sin 2x=\frac{24}{10}$$

$$\Rightarrow\ \sin 2x=\frac{24}{10}-1=\frac{14}{10}$$

$$\Rightarrow\ \sin x.\cos x=\frac{7}{10}$$

Thus, $\log_{10}(\sin x\cos x)=\log_{10}\left(\frac{7}{10}\right)$

$$\log_{10}(\sin x)+\log_{10}(\cos x)=\log_{10}7-1$$

5.3 LOGARITHMIC EQUATION

Type 1: A logarithmic equation is of the form

$$\log_{g(x)}\ f(x)=b$$

$$\Rightarrow\ f(x)=g(x)^b,\ g(x)>0,\ g(x)\neq 1$$

Ex-1 Solve for x: $\log_x\left(3x^2+10x\right)=3$.

Soln. The given equation is $\log_x\left(3x^2+10x\right)=3$

$$\Rightarrow\ \left(3x^2+10x\right)=x^3, x>0, x\neq 1$$

$$\Rightarrow\ x^3-3x^{10}-10x=0$$

$$\Rightarrow\ x\left(x^2-3x-10\right)$$

$$\Rightarrow\ x(x-5)(x+2)$$

$$\Rightarrow\ x=0, -2, 5$$

$$\Rightarrow\ x=5$$

Hence, the solution of x is 5

Ex-2. Solve for x: $\log_{x+1}\left(x^2-3x+5\right)=2$

Soln. The given equation is $\log_{x+1}\left(x^2-3x+5\right)=2$

$$\Rightarrow\ \left(x^2-3x+5\right)=(x+1)^2, x+1>0, x\neq 0$$

$$\Rightarrow\ \left(x^2-3x+5\right)=(x+1)^2$$

$$\Rightarrow\ \left(x^2-3x+5\right)=x^2+2x+1$$

$$\Rightarrow\ 5x=4$$

$$\Rightarrow\ x=4/5$$

Hence, the solution of x is 4/5

Type 2: A logarithmic equation is of the form

$$\log_{f_1(x)}\left\{\log_{f_2(x)}\ f(x)\right\}=0$$

$$\Rightarrow\ f_2(x)=f(x):\begin{cases}f_1(x)>0,\ f_1(x)\neq 1\\ f_2(x)>0,\ f_2(x)\neq 1\end{cases}$$

Ex-3. Solve for x: $\log_{x^2+6x+6}\left\{\log_{2x^2+2x+3}\left(x^2-2x\right)\right\}=0$

Soln. Given equation is

$$\log_{x^2+6x+6}\left\{\log_{2x^2+2x+3}\left(x^2-2x\right)\right\}=0$$

$$\Rightarrow\ \left(x^2-2x\right)=2x^2+2x+3$$

$$\Rightarrow\ x^2+4x+3=0$$

$$\Rightarrow\ (x+1)(x+3)=0$$

$$\Rightarrow\ x=-1, -3$$

Also, $x^2+6x+6>0, x^2+6x+6\neq 1$

$$\Rightarrow\ x\in\left(-\infty,-3-\sqrt{3}\right)\cup\left(-3+\sqrt{3},\infty\right),\ x\neq -1,-5$$

Also, $2x^2+2x+3>0, x^2+x+1\neq 0$

$$\Rightarrow\ x\in R$$

Hence, the solution is $x=\varphi$

Ex-4. Solve for x: $\log_{x^2+x+1}\left\{\log_{2x^2+3x+5}\left(x^2+3\right)\right\}=0$.

Soln. Given equation is

$$\log_{x^2+x+1}\left\{\log_{2x^2+3x+5}\left(x^2+3\right)\right\}=0$$

$$\Rightarrow\ 2x^2+3x+5=x^2+3$$

$$\Rightarrow\ x^2+3x+2=0$$

$$\Rightarrow\ (x+1)(x+2)=0$$

$$\Rightarrow\ x=-1, -2$$

Also, $x^2+x+1>0, x^2+3\neq 1$

$$\Rightarrow\ x\in R$$

Again, $2x^2+3x+5>0, x^2+3\neq 1$

$$\Rightarrow\ x\in R$$

Thus, the solution is $x=\phi$

Type 3: A logarithmic equation is of the form

$$\log_a f_1(x)=\log_a f_2(x), a>0, a\neq 1.$$

$$\Rightarrow\ f_1(x)=f_2(x), f_1(x)>0\ \ or\ \ f_2(x)>0$$

Ex-5 Solve for x: $\log_5\left(x^2-4x+3\right)=\log_5(3x+21)$

Soln. Given equation is

$$\log_5\left(x^2-4x+3\right)=\log_5(3x+21)$$

$$\Rightarrow\quad \left(x^2-4x+3\right)=3x+21, 3x+21>0$$

$$\Rightarrow\quad \left(x^2-7x-18\right)=0, x>-7$$

$$\Rightarrow\quad (x-9)(x+2)=0, x>-7$$

$\Rightarrow\quad x=-2, 9$ and $x>-7$

Hence, the solution of x is $\{-2,9\}$

Ex-6: Solve for x: $\log_{1/3}\left(2\left(\frac{1}{2}\right)^x-1\right)=\log_{1/3}\left(\left(\frac{1}{4}\right)^x-4\right)$

Soln. Given equation is

$$\log_{1/3}\left(2\left(\frac{1}{2}\right)^x-1\right)=\log_{1/3}\left(\left(\frac{1}{4}\right)^x-4\right)$$

$$\Rightarrow\quad 2\left(\frac{1}{2}\right)^x-1=\left(\frac{1}{4}\right)^x-4$$

$$\Rightarrow\quad \left(\frac{1}{2}\right)^{2x}-2\left(\frac{1}{2}\right)^x-3=0$$

$\Rightarrow\quad a^2-2a-3=0$, where $a=\left(\frac{1}{2}\right)^x$

$$\Rightarrow\quad (a-3)(a+1)=0$$

$$\Rightarrow\quad a=3,-1$$

$\Rightarrow\quad a=3$, exponential function is always +ve

$$\Rightarrow\quad \left(\frac{1}{2}\right)^x=3$$

$$\Rightarrow\quad 2^{-x}=3$$

$$\Rightarrow\quad -x=\log_2 3$$

$$\Rightarrow\quad x=-\log_2 3=\log_2\left(\frac{1}{3}\right)$$

Hence, the solution is $x=\log_2\left(\frac{1}{3}\right)$

Type 4: A logarithmic equation is of the form

$$\log_{f_1(x)} A=\log_{f_2(x)} A$$

$$\Rightarrow\quad f_1(x)=f_2(x),\ \begin{cases} f_1(x)>0,\ f_1(x)\neq 1 \\ \quad or \\ f_2(x)>0,\ f_2(x)\neq 1\end{cases}$$

Ex-7 Solve for x: $\log_{\left(\frac{x+5}{3}\right)} 3=\log_{\left(\frac{-1}{x+1}\right)} 3$

Soln. Given equation is $\log_{\left(\frac{x+5}{3}\right)} 3=\log_{\left(\frac{-1}{x+1}\right)} 3$

$$\Rightarrow\quad \left(\frac{x+5}{3}\right)=\left(\frac{-1}{x+1}\right),\ x+5>0,\ \frac{x+5}{3}\neq 1$$

$$\Rightarrow\quad (x+1)(x+5)=-3,\quad x>-5, x\neq -2$$

$$\Rightarrow\quad x^2+6x+8=0,\quad x>-5, x\neq -2$$

$$\Rightarrow\quad (x+2)(x+4)=0,\quad x>-5, x\neq -2$$

$$\Rightarrow\quad x=-2,\ -4,\ x>-5, x\neq -2$$

Hence, the solution is $x=-4$

Ex-8 Solve for x: $\log_{\left(\frac{x+2}{3}\right)} 5=\log_{\left(\frac{4}{x-3}\right)} 5$

Soln. Given equation is $\log_{\left(\frac{x+2}{3}\right)} 5=\log_{\left(\frac{4}{x-3}\right)} 5$

$$\Rightarrow\quad \left(\frac{x+2}{3}\right)=\left(\frac{4}{x-3}\right), x>-2,\ x+2\neq 3$$

$$\Rightarrow\quad (x+2)(x-3)=12, x>-2, x\neq 1$$

$$\Rightarrow\quad x^2+5x-6=0, x>-2, x\neq 1$$

$$\Rightarrow\quad (x+6)(x-1)=0, x>-2, x\neq 1$$

$$\Rightarrow\quad x=-6,1, x>-2, x\neq 1$$

Hence, the solution is $x=1$

Type 5: A logarithmic equation is of the form

$$\log_{f(x)} g_1(x)=\log_{f(x)} g_2(x)$$

$$\Rightarrow\quad g_1(x)=g_2(x),\ \begin{cases} g_1(x)>0,\ f(x)>0,\neq 1 \\ \quad or \\ g_2(x)>0,\ f(x)>0,\neq 1\end{cases}.$$

Ex-9 Solve for x: $\log_{x^2-1}\left(x^3+6\right)=\log_{x^2-1}\left(4x^2-x\right)$

Soln. Given equation is $\log_{x^2-1}\left(x^3+6\right)=\log_{x^2-1}\left(4x^2-x\right)$

$$\Rightarrow\quad x^3+6=4x^2-x, x^3+6>0, x^2-1>0,\neq 1$$

$$\Rightarrow\quad x^3-4x^2+x+6=0, x^3>-6, x^2>1, x\neq\pm\sqrt{2}$$

Now, $x^3-4x^2+x+6=0$

$$\Rightarrow\quad x^3+x^2-5x^2-5x+6x+6=0$$

$$\Rightarrow\quad x^2(x+1)-5x(x+1)+6(x+1)=0$$

$$\Rightarrow\quad (x+1)\left(x^2-5x+6\right)=0$$

$$\Rightarrow\quad (x+1)(x-2)(x-3)=0$$

$\Rightarrow \quad x = -1, 2, 3$

Also, $x^3 > -6 \Rightarrow x > -\sqrt[3]{6}$

Again, $x^2 > 1 \Rightarrow x \in (-\infty, -1) \cup (1, \infty)$

Hence, the solution is $x = 2, 3$

Type 6: A logarithmic equation is of the form

$$\log_{g_1(x)} f(x) = \log_{g_2(x)} f(x),$$

$$\Rightarrow \quad g_1(x) = g_2(x), \begin{cases} g_1(x) > 0, \neq 1, \ f(x) > 0 \\ \quad or \\ g_2(x) > 0, \neq 1, \ f(x) > 0 \end{cases}$$

Ex-10 Solve for x: $\log_{x^3+x}(x^2-4) = \log_{4x^2-6}(x^2-4)$

Soln. Given equation is

$$\log_{x^3+x}(x^2-4) = \log_{4x^2-6}(x^2-4)$$

$\Rightarrow \quad x^3 + x = 4x^2 - 6, x^3 + x > 0, x^3 + x \neq 1, x^2 - 4 > 0$

$\Rightarrow \quad x^3 - 4x^2 + x + 6 = 0, x > 0, x^3 + x \neq 1, x^2 > 4$

$\Rightarrow \quad (x+1)(x-2)(x-3) = 0$

$\Rightarrow \quad x = -1, 2, 3$

Also, $x > 0$

Again, $x^2 - 4 > 0 \Rightarrow x \in (-\infty, -2) \cup (2, \infty)$

Hence, the solution is $x = 3$

Type 7: A logarithmic equation is of the form

$$2n \log_a f_1(x) = \log_a f_2(x), \ a > 0, a \neq 1, n \in N$$

$$\Rightarrow \quad f_1^{2n}(x) = f_2(x), \ f_1(x) > 0$$

Ex-11 Solve for x: $\log_3 2x = 2\log_3(4x - 15)$

Soln. Given equation is $\log_3 2x = 2\log_3(4x - 15)$

$$\Rightarrow \quad (4x - 15)^2 = 2x, \ x > \frac{15}{4}$$

$$\Rightarrow \quad 16x^2 - 122x + 225 = 0$$

$$\Rightarrow \quad x = \frac{122 \pm \sqrt{(122)^2 - 64 \times 225}}{32}$$

$$= \frac{122 \pm \sqrt{14884 - 14400}}{32}$$

$$= \frac{122 \pm \sqrt{484}}{32}$$

$$= \frac{122 \pm 22}{32}$$

$$= \frac{144}{32}, \frac{100}{32}$$

$$= \frac{71}{16}, \frac{25}{4}$$

Also, $x > 15/4$

Hence, the solution set is $x = 25/4$

Ex-12. Solve for x: $2\log 2x = \log(7x - 2 - 2x^2)$

Soln. Given equation is $2\log 2x = \log(7x - 2 - 2x^2)$

$$4x^2 = 7x - 2 - 2x^2, \ x > 0$$

$$6x^2 - 7x + 2 = 0, x > 0$$

$$\Rightarrow \quad 6x^2 - 3x - 4x + 2 = 0$$

$$\Rightarrow \quad 3x(2x-1) - 2(2x-1) = 0$$

$$\Rightarrow \quad (2x-1)(3x-2) = 0$$

$$\Rightarrow \quad x = \frac{1}{2}, \frac{2}{3}$$

Also, $x > 0$

Hence, the solution set is $x = \frac{1}{2}, \frac{2}{3}$

Type 8: A logarithmic equation

$$(2n+1)\log_a f_1(x) = \log_a f_2(x),$$

$a > 0, \ a \neq 1, \ n \in N$

$$\Rightarrow \quad f_1^{(2n+1)}(x) = f_2(x), \ f_1(x) > 0$$

Ex-13 Solve for x: $\log(8 - 10x - 12x^2) = 3\log(2x - 1)$

Soln. Given equation is

$$\log(8 - 10x - 12x^2) = 3\log(2x - 1)$$

$$\Rightarrow \quad (8 - 10x - 12x^2) = (2x-1)^3, x > 1/2$$

$$\Rightarrow \quad 8x^3 + 16x - 9 = 0$$

$$\Rightarrow \quad (2x-1)(4x^2 + 2x + 9) = 0$$

$$\Rightarrow \quad x = 1/2$$

Also, $x > 1/2$

Hence, the solution set is $x = \varphi$

Type 9: A logarithmic equation is of the form

$$\log_a f(x) + \log_a g(x) = \log_a m(x),$$

$a > 0, \ a \neq 1$

$$\Rightarrow \quad \begin{cases} f(x) > 0 \\ g(x) > 0 \\ f(x)\,g(x) = m(x) \end{cases}$$

Ex-14 Solve for x:

$$2\log_3 x + \log_3\left(x^2-3\right) = \log_3(0.5) + \log_3 8$$

Soln. Given equation is

$$2\log_3 x + \log_3\left(x^2-3\right) = \log_3(0.5) + \log_3 8$$

$\Rightarrow \quad x^2\left(x^2-3\right) = 4, x > 0, x^2 - 3 > 0$

$\Rightarrow \quad x^2\left(x^2-3\right) = 4$

$\Rightarrow \quad x^4 - 3x^2 - 4 = 0$

$\Rightarrow \quad \left(x^2-1\right)\left(x^2+4\right) = 0$

$\Rightarrow \quad x = \pm 1$

Also, $x > 0$

Again, $x^2 - 3 > 0 \Rightarrow x \in \left(-\infty, -\sqrt{3}\right) \cup \left(\sqrt{3}, \infty\right)$

Hence, the asolution set is $x = 1$

Type 10: A logarithmic equation is of the form

$\log_a f(x) - \log_a g(x) = \log_a h(x) - \log_a t(x)$,

where $a > 0$, $a \neq 1$

$\Rightarrow \quad \log_a f(x) + \log_a t(x) = \log_a g(x) + \log_a h(x)$

$$\Rightarrow \begin{cases} f(x) > 0, t(x) > 0, g(x) > 0, h(x) > 0 \\ f(x).t(x) = g(x).h(x) \end{cases}$$

Ex-15. Solve for x:

$$\log_2(3-x) - \log_2\left(\frac{\sin\left(\frac{3\pi}{4}\right)}{5-x}\right) = \frac{1}{2} + \log_2(x+7)$$

Soln. Given equation is

$$\log_2(3-x) - \log_2\left(\frac{\sin\left(\frac{3\pi}{4}\right)}{5-x}\right) = \frac{1}{2} + \log_2(x+7)$$

$\Rightarrow \quad \log_2(3-x) - \log_2\left(\frac{1}{\sqrt{2}(5-x)}\right) = \frac{1}{2} + \log_2(x+7)$

$\Rightarrow \quad \log_2(3-x) + \log_2\left(\sqrt{2}(5-x)\right) = \frac{1}{2} + \log_2(x+7)$

$\Rightarrow \quad \log_2(3-x) + \frac{1}{2} + \log_2(5-x) = \frac{1}{2} + \log_2(x+7)$

$\Rightarrow \quad \log_2\{(3-x)(5-x)\} = \log_2(x+7)$

$\Rightarrow \quad (x-3)(x-5) = (x+7)$

$\Rightarrow \quad x^2 - 8x + 15 - x - 7 = 0$

$\Rightarrow \quad x^2 - 9x + 8 = 0$

$\Rightarrow \quad (x-1)(x-8) = 0$

$\Rightarrow \quad x = 1, 8$

$\Rightarrow \quad x = 1$, $x = 8$ is rejected.

Hence, the solution set is $x = 1$.

5.4 LOGARITHMIC INEQUATION

Type I: A logarithmic inequation is of the form

$$\begin{cases} \log_a f(x) > \log_a g(x) \\ a > 1 \end{cases} \Rightarrow \begin{cases} g(x) > 0 \\ a > 1 \\ f(x) > g(x) \end{cases}$$

Ex-1 Solve for x:

$$\log_{(2x+1)} x^2 < \log_{(2x+1)}(2x+3),\ x > -1$$

Soln. Given equation is

$\log_{(2x+1)} x^2 < \log_{(2x+1)}(2x+3)$

$x^2 < 2x+3,\ 2x+1 > 1, 2x+3 > 0$

$x^2 - 2x - 3 < 0,\ x > -1/2, x > -3/2$

$(x-3)(x+1) < 0,\ x > -1/2, x > -3/2$

$-1 < x < 3,\ x > -1/2,\ x > -3/2$

Hence, the solution set is $x \in \left(-\frac{1}{2}, 3\right)$

Type II: A logarithmic inequation is of the form

$$\begin{cases} \log_a f(x) > \log_a g(x) \\ 0 < a < 1 \end{cases} \Rightarrow \begin{cases} f(x) > 0 \\ 0 < a < 1 \\ f(x) < g(x) \end{cases}$$

Ex-2 Solve for x: $\log_{0.3}(x-1) < \log_{0.09}(x-1)$

Soln. Given equation is $\log_{0.3}(x-1) < \log_{0.09}(x-1)$

$\Rightarrow \quad \log_{0.3}(x-1) < \log_{(0.3)^2}(x-1)$

$\Rightarrow \quad \log_{0.3}(x-1) < \frac{1}{2}\log_{(0.3)}(x-1)$

$\Rightarrow \quad 2\log_{0.3}(x-1) < \log_{(0.3)}(x-1)$

$\Rightarrow \quad \log_{0.3}(x-1)^2 < \log_{(0.3)}(x-1)$

$\Rightarrow \quad (x-1)^2 > (x-1)$

$\Rightarrow \quad (x-1)(x-2) > 0$

$\Rightarrow \quad x < 1$ and $x > 2$

$\Rightarrow \quad x > 2$

$\Rightarrow \quad x \in (2, \infty)$

Type III: A logarithmic inequation is of the form

1. $\begin{cases}\log_a x > 0 \\ a > 1\end{cases} \Rightarrow \begin{cases} x > 0 \\ a > 1\end{cases}$

2. $\begin{cases}\log_a x > 0 \\ 0 < a < 1\end{cases} \Rightarrow \begin{cases} 0 < x < 1 \\ 0 < a < 1\end{cases}$

3. $\begin{cases}\log_a x < 0 \\ a > 1\end{cases} \Rightarrow \begin{cases} 0 < x < 1 \\ a > 1\end{cases}$

4. $\begin{cases}\log_a x < 0 \\ 0 < a < 1\end{cases} \Rightarrow \begin{cases} x > 1 \\ 0 < a < 1\end{cases}.$

Ex-3 Solve for x: $\log_{1/2}(2x+3) > 0$.

Soln. Given inequation is $\log_{1/2}(2x+3) > 0$

$\Rightarrow \quad (2x+3) < \left(\frac{1}{2}\right)^0$

$\Rightarrow \quad (2x+3) < 1$

$\Rightarrow \quad 2x + 2 < 0$

$\Rightarrow \quad x + 1 < 0$

$\Rightarrow \quad x < -1$

Also, $(2x+3) > 0 \Rightarrow x > -3/2$.

Hence, the solution set is $x \in \left(-\frac{3}{2}, -1\right)$

Ex-4. Solve for x: $\log_{1/2} x > \log_{1/3} x$

Soln. Given in equation is $\log_{1/2} x > \log_{1/3} x$

$\Rightarrow \quad \log_{1/2} x > \left(\frac{\log_{1/2} x}{\log_{1/2}(1/3)}\right)$

$\Rightarrow \quad \log_{1/2} x\left(1 - \frac{1}{\log_{1/2}(1/3)}\right) > 0$

$\Rightarrow \quad \log_{1/2} x > 0 \quad \left(\because \left(1 - \frac{1}{\log_{1/2}(1/3)}\right) > 0\right)$

$\Rightarrow \quad x > 0 \text{ and } x < \left(\frac{1}{2}\right)^0$

$\Rightarrow \quad 0 < x < 1$

Ex-5. Solve for x: $\log_{1/2} x + \log_3 x > 1$

Soln. Given inequation is $\log_{1/2} x + \log_3 x > 1$

$\Rightarrow \quad -\log_2 x + \log_3 x > 1$

$\Rightarrow \quad \log_3 x - \log_2 x > 1$

$\Rightarrow \quad \frac{\log_{10} x}{\log_{10} 3} - \frac{\log_{10} x}{\log_{10} 2} > 1$

$\Rightarrow \quad \log_{10} x\left(\frac{1}{\log_{10} 3} - \frac{1}{\log_{10} 2}\right) > 1$

$\Rightarrow \quad \log_{10} x \times M > 1 \quad \left(\because M = \left(\frac{1}{\log_{10} 3} - \frac{1}{\log_{10} 2}\right)\right)$

$\Rightarrow \quad \log_{10} x > \frac{1}{M}$

$\Rightarrow \quad x > 10^{1/M}$

Hence, the solution set is $x \in \left(10^{1/M}, \infty\right)$

Ex-6. Solve for x: $\frac{1}{\log_2 x} - \frac{1}{\log_2 x - 1} < 1$

Soln. Given inequation is $\frac{1}{\log_2 x} - \frac{1}{\log_2 x - 1} < 1$

$\Rightarrow \quad \frac{1}{a} - \frac{1}{a-1} < 1, \; a = \log_2 x$

$\Rightarrow \quad \frac{1}{a} - 1 - \frac{1}{a-1} < 0,$

$\Rightarrow \quad \frac{1-a}{a} - \frac{1}{a-1} < 0,$

$\Rightarrow \quad \frac{-(1-a)^2 - a}{a(a-1)} < 0,$

$\Rightarrow \quad \frac{(1-a)^2 + a}{a(a-1)} > 0$

$\Rightarrow \quad \frac{a^2 - a + 1}{a(a-1)} > 0$

$\Rightarrow \quad \frac{1}{a(a-1)} > 0$

$\Rightarrow \quad a > 1 \text{ and } a < 0$

$\Rightarrow \quad \log_2 x > 1 \text{ and } \log_2 x < 0$

$\Rightarrow \quad x > 2 \text{ and } x < 1$

Also $\log_2 x$ is defined only when $x > 0$.

Hence, the solution set is $0 < x < 1$ and $x > 2$

i.e. $x \in (0,1) \cup (2, \infty)$

Ex-7. Solve for x: $\log_{(2x+3)} x^2 < 1$

Soln. Given inequation is $\log_{(2x+3)} x^2 < 1$

It is defined only when $x \neq 0, 2x+3 > 0, 2x+3 \neq 1$

$\Rightarrow \quad x \neq 0, x > -3/2, \; x \neq -1$.

Case I: when $0 < 2x + 3 < 1$

Then $x^2 > 2x + 3$

$\Rightarrow \quad x^2 - 2x - 3 > 0$

$\Rightarrow \quad (x-3)(x+1) > 0$

$\Rightarrow \quad x < -1 \text{ and } x > 3$ ....(i)

Also, $0 < 2x + 3 < 1$

$\Rightarrow \quad -\frac{3}{2} < x < -1$ ....(ii)

Thus, from (i) and (ii), we get,

$x \in \left(-\frac{3}{2}, -1\right)$ ....(iii)

Case II: when $2x + 3 > 1$

Then $x^2 < 2x + 3$

$\Rightarrow \quad x^2 - 2x - 3 < 0$

$\Rightarrow \quad (x-3)(x+1) < 0$

$\Rightarrow \quad -1 < x < 3$

$\Rightarrow \quad x \in (-1, 3)$. ....(iv)

Hence, the solution set from (iii) and (iv) is

$x \in \left(-\frac{3}{2} - 1\right) \cup (-1, 3) - \{0\}$

Ex-8. Solve for x: $\frac{\log^2 x - 3\log x + 3}{\log x - 1} < 1$

Soln. Given inequation is $\frac{\log^2 x - 3\log x + 3}{\log x - 1} < 1$

$\Rightarrow \quad \frac{a^2 - 3a + 3}{a - 1} < 1$, where $a = \log x$

$\Rightarrow \quad \frac{a^2 - 3a + 3}{a - 1} - 1 < 0$

$\Rightarrow \quad \frac{a^2 - 3a + 3 - a + 1}{a - 1} < 0$

$\Rightarrow \quad \frac{a^2 - 4a + 4}{a - 1} < 0$

$\Rightarrow \quad \frac{1}{a - 1} < 0$

$\Rightarrow \quad 0 < a < 1$

$\Rightarrow \quad 0 < \log x < 1$

$\Rightarrow \quad 0 < \log_{10} x < 1$

$\Rightarrow \quad 1 < x < 10$

Hence, the solution set is $x \in (1, 10)$

Ex-9. Solve for x: $\frac{\log_2 (x+1)}{(x-1)} > 0$

Soln Given inequation is $\frac{\log_2 (x+1)}{(x-1)} > 0$

It is possible only when $x > 1$, $\log_2 (x+1) > 0$ and $x > -1$

Now, $\log_2 (x+1) > 0$

$\Rightarrow \quad (x+1) > 2^0 = 1$

$\Rightarrow \quad x > 0.$

Hence, the solution set is $x \in (1, \infty)$

Ex-10. Solve for x: $\frac{1}{\log_4 \left(\frac{x+1}{x+2}\right)} \leq \frac{1}{\log_4 (x+3)}$

Soln. Given inequation is $\frac{1}{\log_4 \left(\frac{x+1}{x+2}\right)} \leq \frac{1}{\log_4 (x+3)}$

$\Rightarrow \quad \log_4 \left(\frac{x+1}{x+2}\right) \geq \log_4 (x+3)$

$\Rightarrow \quad \log_4 \left(\frac{x+1}{x+2}\right) - \log_4 (x+3) \geq 0$

$\Rightarrow \quad \log_4 \left(\frac{x+1}{(x+2)(x+3)}\right) \geq 0$

$\Rightarrow \quad \left(\frac{x+1}{(x+2)(x+3)}\right) \geq 1$

$\Rightarrow \quad \left(\frac{x+1}{(x+2)(x+3)}\right) - 1 \geq 0$

$\Rightarrow \quad \left(\frac{(x+1) - (x+2)(x+3)}{(x+2)(x+3)}\right) \geq 0$

$\Rightarrow \quad \left(\frac{(x+2)(x+3) - (x+1)}{(x+2)(x+3)}\right) \leq 0$

$\Rightarrow \quad \left(\frac{x^2 + 4x + 5}{(x+2)(x+3)}\right) \leq 0$

$\Rightarrow \quad \left(\frac{1}{(x+2)(x+3)}\right) \leq 0$

$\Rightarrow \quad -2 < x < 3$

Hence, the solution set is $x \in (-2, 3)$

Ex-11. Solve for x: $\frac{(x^2 - 4)}{\log_2 (x^2 - 1)} < 0$

Soln. Given inequation is $\frac{(x^2 - 4)}{\log_2 (x^2 - 1)} < 0$

It is possible only when $x^2 - 4 > 0$ and

$\Rightarrow \quad \frac{1}{\log_{1/2} (x^2 - 1)} < 0$

when $x^2 - 4 > 0$

$\Rightarrow \quad (x+2)(x-2) > 0$

$\Rightarrow \quad x \in (-\infty, -2) \cup (2, \infty)$ ....(i)

when $\dfrac{1}{\log_{1/2}(x^2-1)} < 0$

$\Rightarrow \quad \log_{1/2}(x^2-1) > 0$

$\Rightarrow \quad (x^2-1) < 1$

$\Rightarrow \quad x^2 - 2 < 0$

$\Rightarrow \quad -\sqrt{2} < x < \sqrt{2}$ ....(ii)

when $x^2 - 1 > 0$

$\Rightarrow \quad (x+1)(x-1) > 0$

$\Rightarrow \quad x \in (-\infty, -1) \cup (1, \infty)$ ....(iii)

From (i), (ii) and (iii), we get,

$x \in (-\infty, -2) \cup (-\sqrt{2}, -1) \cup (1, \sqrt{2}) \cup (2, \infty)$

which is the required solution set.

PROBLEMS FOR JEE ADVANCED EXAM

Ex-1. If $a^4b^5 = 1$, find the value of $\log_a(a^5b^4)$

Soln. Given $a^4b^5 = 1$

$\Rightarrow \quad \log(a^4b^5) = \log(1) = 0$

$\Rightarrow \quad 4\log a + 5\log b = 0$

$\Rightarrow \quad 4\log a = -5\log b$

$\Rightarrow \quad \dfrac{\log a}{\log b} = -\dfrac{5}{4}$

$\Rightarrow \quad \log_b a = -\dfrac{5}{4}$

Now, $\log_a(a^5b^4)$

$= 5\log_a a + 4\log_a b$

$= 5 + 4\log_a b$

$= 5 + \dfrac{4}{\log_b a}$

$= 5 - \dfrac{4\times 4}{5} = \dfrac{25-16}{5} = \dfrac{9}{5}$

Ex-2. If $x = \log_{10} 5 \times \log_{10} 20 + (\log_{10} 2)^2$ and $y = \dfrac{2\log 2 + \log 3}{\log(48) - \log(4)}$, prove that $x = y$

Soln. We have

$x = \log_{10} 5 \times \log_{10} 20 + (\log_{10} 2)^2$

$= (1 - \log_{10} 2)(1 + \log_{10} 2) + (\log_{10} 2)^2$

$= 1 - (\log_{10} 2)^2 + (\log_{10} 2)^2$

$= 1$

Also, $y = \dfrac{2\log 2 + \log 3}{\log(48) - \log(4)}$

$= \dfrac{\log(12)}{\log\left(\dfrac{48}{4}\right)} = \dfrac{\log(12)}{\log(12)} = 1$

Thus, $x = y$

Hence, the result.

Ex-3. Find the sum of all the equations $2\log x - \log(2x - 75) = 2$.

Soln. The given expression is

$2\log x - \log(2x-75) = 2$

$\Rightarrow \quad \log(x^2) - \log(2x-75) = 2$

$\Rightarrow \quad \log\left(\dfrac{x^2}{2x-75}\right) = 2$

$\Rightarrow \quad \log_{10}\left(\dfrac{x^2}{2x-75}\right) = \log_{10}(10^2)$

$\Rightarrow \quad \left(\dfrac{x^2}{2x-75}\right) = 100$

$\Rightarrow \quad x^2 = 200x - 7500$

$\Rightarrow \quad x^2 - 200x + 7500 = 0$

Hence, the sum of the roots = 200.

Ex-4. If $\log_x\left(\log_{18}\left(\sqrt{2}+\sqrt{8}\right)\right) = -\dfrac{1}{2}$, find x.

Soln. Given $\log_x\left(\log_{18}\left(\sqrt{2}+\sqrt{8}\right)\right) = -\dfrac{1}{2}$

$\Rightarrow \quad \log_x\left(\log_{18}\left(\sqrt{2}+2\sqrt{2}\right)\right) = -\dfrac{1}{2}$

$\Rightarrow \quad \log_x\left(\log_{(3\sqrt{2})^2}\left(3\sqrt{2}\right)\right) = -\dfrac{1}{2}$

$\Rightarrow \quad \log_x\left(\dfrac{1}{2}\right)\left(\log_{(3\sqrt{2})}\left(3\sqrt{2}\right)\right) = -\dfrac{1}{2}$

$\Rightarrow \quad \log_x\left(\dfrac{1}{2}\right) = -\dfrac{1}{2}$

$\Rightarrow \quad -\log_x 2 = -\dfrac{1}{2}$

$\Rightarrow \quad \log_x 2 = \dfrac{1}{2}$

$\Rightarrow \quad \sqrt{x} = 2$

$\Rightarrow \quad x = 4$

Ex-5. If $\log_6 9 - \log_9 27 + \log_8 x = \log_{64} x - \log_6 4$ then find the value of x.

Soln. Given $\log_6 9 - \log_9 27 + \log_8 x = \log_{64} x - \log_6 4$

$\Rightarrow \quad \log_8 x - \log_{64} x = \log_9 27 - \log_6 9 - \log_6 4$

$\Rightarrow \quad \log_8 x - \frac{1}{2}\log_8 x = \log_{3^2}(3)^3 - \log_6 36$

$\Rightarrow \quad \frac{1}{2}\log_8 x = \frac{3}{2} - \log_6(6^2)$

$\Rightarrow \quad \frac{1}{2}\log_8 x = \frac{3}{2} - 2 = -\frac{1}{2}$

$\Rightarrow \quad \frac{1}{2}\log_8 x = -\frac{1}{2}$

$\Rightarrow \quad \log_8 x = -1$

$\Rightarrow \quad x = \frac{1}{8}$

Ex-6. If $x = \sqrt{12+6\sqrt{3}} + \sqrt{12-6\sqrt{3}}$, then find the value of $\log_{36} x$.

Soln. We have $12+6\sqrt{3}$

$= 12 + 2.3.\sqrt{3}$

$= 3^2 + (\sqrt{3})^2 + 2.3.\sqrt{3}$

$= (3+\sqrt{3})^2$

Similarly, $12 - 6\sqrt{3} = (3-\sqrt{3})^2$.

Now, $x = \sqrt{12+6\sqrt{3}} + \sqrt{12-6\sqrt{3}}$

$= (3+\sqrt{3}) + (3-\sqrt{3})$

$= 6$

Thus, $\log_{36} x$

$= \log_{36} 6$

$= \log_{6^2} 6$

$= \frac{1}{2}\log_6 6$

$= \frac{1}{2}$

Ex-7. If $\log_a b = 2, \log_b c = 2$ and $\log_3 c = 3 + \log_3 a$ then find the value of $(a+b+c)+7$

Soln. We have $\log_a b \times \log_b c = 2 \times 2 = 4$

$\Rightarrow \quad \log_a c = 4$

$\Rightarrow \quad \frac{\log_3 c}{\log_3 a} = 4$ ....(i)

Also, $\log_3 c = 3 + \log_3 a$

$\Rightarrow \quad 4\log_3 a = 3 + \log_3 a$, from (i)

$\Rightarrow \quad 3\log_3 a = 3$

$\Rightarrow \quad \log_3 a = 1$

$\Rightarrow \quad a = 3$

Again, $\log_a b = 2$

$\Rightarrow \quad \log_3 b = 2$

$\Rightarrow \quad b = 3^2 = 9$

Further, $\log_b c = 2$

$\Rightarrow \quad \log_9 c = 2$

$\Rightarrow \quad c = 9^2 = 81$

Thus, $(a+b+c)+7$

$= 3 + 9 + 81 + 7$

$= 100$

Ex-8. If $\log_9 x + \log_4 y = \frac{7}{2}$ and $\log_9 x - \log_8 y = -\frac{3}{2}$ then find the value of $\log_4(x+y-3)$

Soln. We have $\log_9 x + \log_4 y = \frac{7}{2}$

$\Rightarrow \quad \frac{1}{2}\log_3 x + \frac{1}{2}\log_2 y = \frac{7}{2}$

$\Rightarrow \quad \log_3 x + \log_2 y = 7$ ....(i)

Also, $\log_9 x - \log_8 y = -\frac{3}{2}$

$\Rightarrow \quad \log_3 x - \log_2 y = -3$ ....(ii)

Adding (i) and (ii), we get,

$2\log_3 x = 4$

$\Rightarrow \quad \log_3 x = 2$

$\Rightarrow \quad x = 3^2 = 9$

Subtracting (i) and (ii), we get,

$2\log_2 y = 10$

$\Rightarrow \quad \log_2 y = 5$

$\Rightarrow \quad y = 2^5 = 32$

Hence, the solutions are $x = 9$ and $y = 32$

Ex-9. If $a = \log_{10} 2, b = \log_{10} 3$ such that $3^{x+2} = 45$ then find x (in terms of a and b).

Soln. Given $3^{x+2} = 45$

$\Rightarrow \quad x + 2 = \log_3(45)$

$\Rightarrow \quad x+2=\log_3(5\times 9)$

$\Rightarrow \quad x+2=\log_3 5+\log_3 9$

$\Rightarrow \quad x+2=\log_3 5+2$

$\Rightarrow \quad x=\log_3 5$

$\Rightarrow \quad x=\dfrac{\log_{10} 5}{\log_{10} 3}$

$\Rightarrow \quad x=\dfrac{\log_{10}\left(\frac{10}{2}\right)}{\log_{10} 3}=\dfrac{\log_{10} 10-\log_{10} 2}{\log_{10} 3}$

$\Rightarrow \quad x=\dfrac{1-a}{b}$

Ex-10. Let the number $N=6\log_{10} 2+\log_{10} 31$. If N lies between two successive integers, then find their sum.

Soln. We have $N=6\log_{10} 2+\log_{10} 31$

$=\log_{10} 2^6+\log_{10} 31$

$=\log_{10}(64\times 31)$

$=\log_{10}(1984)$

$<\log_{10}(1000)=3$

Also, $N=\log_{10}(1984)>\log_{10}(10000)=4$.

Thus, the sum of successive integers

$=3+4=7$

Ex-11. Let $M=\log^2_{\sqrt{2}}\left(\frac{1}{4}\right), N=\log^3_{2\sqrt{2}}(8)$ and

$P=\log_5\left(\log_3\left(\sqrt{\sqrt[5]{9}}\right)\right)$, then find the value of $\left(\frac{M}{N}+P+3\right)$

Soln. We have $M=\log^2_{\sqrt{2}}\left(\frac{1}{4}\right)=\left(\log_{\sqrt{2}}\left(2^{-2}\right)\right)^2$

$=\left(-\dfrac{2}{1/2}\log_2 2\right)^2$

$=(-4)^2=16$

and $N=\log^3_{2\sqrt{2}}(8)=\left(\log_{2\sqrt{2}}(8)\right)^3$

$=\left(\log_{2\sqrt{2}}\left(2\sqrt{2}\right)^3\right)^3$

$=\left(3\log_{2\sqrt{2}}\left(2\sqrt{2}\right)\right)^3$

$=(3)^3=27$

Also, $P=\log_5\left(\log_3\left(\sqrt{\sqrt[5]{9}}\right)\right)$

$=\log_5\left(\log_3\left(9^{1/10}\right)\right)$

$=\log_5\left(\log_3\left(3^{2/10}\right)\right)$

$=\log_5\left(\frac{1}{5}\right)\left(\log_3(3)\right)$

$=\log_5\left(5^{-1}\right)=-1$.

Thus, $\left(\frac{M}{N}+P+3\right)$

$=\dfrac{16}{27}-1+3$

$=\dfrac{16}{27}+2$

$=\dfrac{70}{27}$

Ex-12. If $\log_3(x)=a$ and $\log_7(x)=b$, then find the value of $\log_{21}(x)$

Soln. We have $\dfrac{1}{a}+\dfrac{1}{b}$

$=\log_x 3+\log_x 7$

$=\log_x(21)$

Thus, $\log_{21}(x)=\dfrac{1}{\log_x(21)}$

$=\dfrac{1}{\frac{1}{a}+\frac{1}{b}}=\dfrac{ab}{a+b}$

Ex-13. If x and y are satisfying the relations $\log_8 x+\log_4 y^2=5$ and $\log_8 y+\log_4 x^2=7$ then find the value of $2xy$.

Soln. We have $\log_8 x+\log_4 y^2=5$

$\Rightarrow \quad \frac{1}{3}\log_2 x+\frac{2}{2}\log_2 y=5$

$\Rightarrow \quad \log_2 x^{1/3}+\log_2 y=5$

$\Rightarrow \quad \log_2\left(x^{1/3}y\right)=5$

$\Rightarrow \quad \left(x^{1/3}y\right)=2^5=32 \quad$ ....(i)

Also, $\log_8 y+\log_4 x^2=7$

$\Rightarrow \quad \frac{1}{3}\log_2 y+\frac{2}{2}\log_2 x=7$

$\Rightarrow \quad \log_2 y^{1/3} + \log_2 x = 7$

$\Rightarrow \quad \log_2\left(y^{1/3}x\right) = 7$

$\Rightarrow \quad \left(y^{1/3}x\right) = 2^7 = 128$ ....(ii)

Multiplying (i) and (ii), we get,

$\left(x^{4/3}y^{4/3}\right) = 2^5.2^7 = 2^{12}$

$\Rightarrow \quad (xy)^{4/3} = 2^{12}$

$\Rightarrow \quad (xy) = 2^{12\times\frac{3}{4}}$

$\Rightarrow \quad (xy) = 2^9$

$\Rightarrow \quad (2xy) = 2^{10} = 1024$

Ex-14. If $\log_{10}\left(x^2+x\right) = \log_{10}\left(x^3-x\right)$, then find the product of all the solutions.

Soln. We have $\log_{10}\left(x^2+x\right) = \log_{10}\left(x^3-x\right)$

$\Rightarrow \quad (x+1) = \left(x^2-1\right)$

$\Rightarrow \quad (x+1) = (x+1)(x-1)$

$\Rightarrow \quad (x-1) = 1$

$\Rightarrow \quad x = 2$

Hence, the solution is 2

Thus, the product of all the solutions = 2

Ex-15. If $\log_{10}(x-2) + \log_{10} y = 0$ and $\sqrt{x} + \sqrt{y-2} = \sqrt{x+y}$, then find the value of $\left(x+y-2\sqrt{2}\right)$

Soln. We have $\log_{10}(x-2) + \log_{10} y = 0$

$\Rightarrow \quad \log_{10}\left(y(x-2)\right) = \log_{10} 1$

$\Rightarrow \quad y(x-2) = 1$ ....(i)

Also, $\sqrt{x} + \sqrt{y-2} = \sqrt{x+y}$

$\Rightarrow \quad x+y-2+2\sqrt{x(y-2)} = x+y$

$\Rightarrow \quad -2+2\sqrt{x(y-2)} = 0$

$\Rightarrow \quad -2 = -2\sqrt{x(y-2)}$

$\Rightarrow \quad x(y-2) = 1$ ....(ii)

From (i) and (ii), we get,

$x = y$

Put $x = y$ in (ii), we get,

$\Rightarrow \quad x(x-2) = 1$

$\Rightarrow \quad x^2-2x-1 = 0$

$\Rightarrow \quad x^2-2x+1 = 2$

$\Rightarrow \quad (x-1)^2 = 2$

$\Rightarrow \quad (x-1) = \pm\sqrt{2}$

$\Rightarrow \quad x = 1\pm\sqrt{2}$

$\Rightarrow \quad x = 1+\sqrt{2} = y$

Thus, the value of $x+y-2\sqrt{2}$

$= \left(1+\sqrt{2}+1+\sqrt{2}\right) - 2\sqrt{2}$

$= 2$

Ex-16. If $a, b \in R^+$ such that $\log_{27} a + \log_9 b = \frac{7}{2}$ and $\log_{27} b + \log_9 a = \frac{2}{3}$, then find ab.

Soln. We have $\log_{27} a + \log_9 b = \frac{7}{2}$

$\Rightarrow \quad \log_{3^3} a + \log_{3^2} b = \frac{7}{2}$

$\Rightarrow \quad \frac{1}{3}\log_3 a + \frac{1}{2}\log_3 b = \frac{7}{2}$

$\Rightarrow \quad 2\log_3 a + 3\log_3 b = 21$ ....(i)

Also, $\log_{27} b + \log_9 a = \frac{2}{3}$

$\Rightarrow \quad \frac{1}{3}\log_3 b + \frac{1}{2}\log_3 a = \frac{2}{3}$

$\Rightarrow \quad 2\log_3 b + 3\log_3 a = 4$ ....(ii)

On solving, we get,

$\Rightarrow \quad 4\log_3 a - 9\log_3 a = 42-12$

$\Rightarrow \quad -5\log_3 a = 30$

$\Rightarrow \quad \log_3 a = -6$

$\Rightarrow \quad a = 3^{-6}$

From (ii), we get, $2\log_3 b - 18 = 4$

$\Rightarrow \quad 2\log_3 b = 22$

$\Rightarrow \quad \log_3 b = 11$

$\Rightarrow \quad b = 3^{11}$

Hence, the value of $ab = 3^{-6}\times 3^{11} = 3^5 = 243$

Ex-17. Find the number of values of x satisfying the equation $\log_{\tan x}\left(2+4\cos^2 x\right)=2\ in[0,2\pi]$

Soln. Given equation is $\log_{\tan x}\left(2+4\cos^2 x\right)=2$

$\Rightarrow\quad \left(2+4\cos^2 x\right)=\tan^2 x$

$\Rightarrow\quad \left(2+4\cos^2 x\right)=\dfrac{\sin^2 x}{\cos^2 x}$

$\Rightarrow\quad \left(4\cos^4 x+2\cos^2 x-\sin^2 x\right)=0$

$\Rightarrow\quad \left(4\cos^4 x+3\cos^2 x-1\right)=0$

$\Rightarrow\quad \left(4\cos^4 x+4\cos^2 x-\cos^2 x-1\right)=0$

$\Rightarrow\quad 4\cos^2 x\left(\cos^2 x+1\right)-1\left(\cos^2 x+1\right)=0$

$\Rightarrow\quad \left(4\cos^2 x-1\right)\left(\cos^2 x+1\right)=0$

$\Rightarrow\quad \left(4\cos^2 x-1\right)=0$

$\Rightarrow\quad \cos^2 x=\left(\dfrac{1}{2}\right)^2=\cos^2\left(\dfrac{\pi}{3}\right)$

$\Rightarrow\quad x=n\pi\pm\dfrac{\pi}{3}, n=0,1,2$

$\Rightarrow\quad x=\dfrac{\pi}{3},\dfrac{2\pi}{3},\dfrac{4\pi}{3},\dfrac{5\pi}{3}$

Hence, the number of values of x is 4

Ex-18. If cos(ln x) = 0, then find $\left(\dfrac{2}{\pi}\times\log(x)+10\right)$

Soln. Given $\cos(\ln x)=0$

$\Rightarrow\quad \ln x=\dfrac{\pi}{2}$

$\Rightarrow\quad x=e^{\frac{\pi}{2}}$

Thus, the value of $\left(\dfrac{2}{\pi}\times\log(x)+10\right)$

$=\dfrac{2}{\pi}\times\log\left(e^{\frac{\pi}{2}}\right)+10$

$=\left(\dfrac{2}{\pi}\times\dfrac{\pi}{2}\log(e)+10\right)$

$=1+10=11$

Ex-19. If $c(a-b)=a(b-c)$ such that $a\neq b\neq c$, then find the value of $\dfrac{\log(a+c)+\log(a-2b+c)}{\log(a-c)}$

Soln. Given $c(a-b)=a(b-c)$

$\Rightarrow\quad ac-bc=ab-ac$

$\Rightarrow\quad 2ac=ab+bc=b(a+c)$

$\Rightarrow\quad b=\dfrac{2ac}{(a+c)}$

Now, $\dfrac{\log(a+c)+\log(a-2b+c)}{\log(a-c)}$

$=\dfrac{\log\{(a+c)(a+c-2b)\}}{\log(a-c)}$

$=\dfrac{\log\left\{(a+c)\left(a+c-\dfrac{4ac}{(a+c)}\right)\right\}}{\log(a-c)}$

$=\dfrac{\log\left\{\left((a+c)^2-4ac\right)\right\}}{\log(a-c)}$

$=\dfrac{\log(a-c)^2}{\log(a-c)}$

$=\dfrac{2\log(a-c)}{\log(a-c)}=2$

Ex-20. If $x=\log_{10}\left(A+\sqrt{B}\right)$ is a solution of $10^x+10^{-x}=4$, then find the value of $(A+B+3)$.

Soln. Given $10^x+10^{-x}=4$

$\Rightarrow\quad 10^x+\dfrac{1}{10^x}=4$

$\Rightarrow\quad \left(10^x\right)^2-4\left(10^x\right)+1=0$

$\Rightarrow\quad \left(10^x\right)=\dfrac{4\pm\sqrt{16-4}}{2}$

$\Rightarrow\quad \left(10^x\right)=\dfrac{4\pm\sqrt{12}}{2}$

$\Rightarrow\quad \left(10^x\right)=\left(2+\sqrt{3}\right)$

$\Rightarrow\quad x=\log_{10}\left(2+\sqrt{3}\right)$

Thus $A=2$, $B=3$

Hence, the value of $A+B+3=2+3+3=8$.

Ex-21. Solve for x:

$\log_{10}\left(98+\sqrt{x^3-x^2-12x+36}\right)=2$

[Roorkee-1975]

Soln. The given equation is

$$\log_{10}\left(98+\sqrt{x^3-x^2-12x+36}\right)=2$$

$\Rightarrow\quad \left(98+\sqrt{x^3-x^2-12x+36}\right)=10^2=100$

$\Rightarrow\quad x^3-x^2-12x+36=4$

$\Rightarrow\quad x^3-x^2-12x+32=0$

$\Rightarrow\quad x^3+4x^2-5x^2-20x+8x+32=0$

$\Rightarrow\quad x^2(x+4)-5x(x+4)+8(x+4)=0$

$\Rightarrow\quad (x+4)\left(x^2-5x+8\right)=0$

$\Rightarrow\quad (x+4)=0$

$\Rightarrow\quad x=-4$

Hence, the solution set is $\{-4\}$

Ex-22. Solve for x and y:

$$\log_{10}x+\log_{10}x^{1/2}+\log_{10}x^{1/4}+....=y$$

and $\dfrac{1+3+5+.....+(2y-1)}{4+7+10+...+(3y+1)}=\dfrac{20}{7\log_{10}x}$

[Roorkee-1987]

Soln. We have $\log_{10}x+\log_{10}x^{1/2}+\log_{10}x^{1/4}+....=y$

$\Rightarrow\quad \log_{10}x+\frac{1}{2}\log_{10}x+\frac{1}{4}\log_{10}x+....=y$

$\Rightarrow\quad y=\left(1+\frac{1}{2}+\frac{1}{4}+....\right)\log_{10}x$

$\Rightarrow\quad y=\left(\dfrac{1}{1-\frac{1}{2}}\right)\log_{10}x=2\log_{10}x$

Also, $\dfrac{1+3+5+.....+(2y-1)}{4+7+10+...+(3y+1)}=\dfrac{20}{7\log_{10}x}$

$\Rightarrow\quad \dfrac{\frac{y}{2}(1+2y-1)}{\frac{y}{2}(4+3y+1)}=\dfrac{20}{7\left(\frac{y}{2}\right)}$

$\Rightarrow\quad \dfrac{2y}{(3y+5)}=\dfrac{40}{7y}$

$\Rightarrow\quad \dfrac{y}{(3y+5)}=\dfrac{20}{7y}$

$\Rightarrow\quad 7y^2=60y+100$

$\Rightarrow\quad 7y^2-60y-100=0$

$\Rightarrow\quad (y-10)(10y+7)=0$

$\Rightarrow\quad y=10,-10/7$

when $y=10$, then $2\log_{10}x=10$

$\Rightarrow\quad \log_{10}x=5$

$\Rightarrow\quad x=10^5$

when $y=-10/7$, then $2\log_{10}x=-10/7$

$\Rightarrow\quad \log_{10}x=-5/7$

$\Rightarrow\quad x=10^{-5/7}$

Ex-23. Solve for x:

$$\frac{6}{5}a^{\log_a x\log_{10}a\log_a 5-3\log_{10}\left(\frac{x}{10}\right)}=9^{\log_{100}x+\log_4 2}$$

[Roorkee-1988]

Soln. We have

$$\frac{6}{5}a^{\log_a x\log_{10}a\log_a 5-3\log_{10}\left(\frac{x}{10}\right)}=9^{\log_{100}x+\log_4 2}$$

$\Rightarrow\quad \frac{6}{5}a^{\frac{\log x}{\log a}.\frac{\log 5}{\log 10}.-3(\log_{10}x-1)}=9^{\log_{100}x+\log_{2^2}2}$

$\Rightarrow\quad \frac{6}{5}a^{\log_a x\log_{10}5.-3(\log_{10}x-1)}=9^{\log_{10^2}x+\log_{2^2}2}$

$\Rightarrow\quad \frac{6}{5}a^{\log_a x\log_{10}5.-3(\log_{10}x-1)}=9^{\frac{1}{2}(\log_{10}x+1)}$

$\Rightarrow\quad \frac{6}{5}a^{\log_a x\log_{10}5.-3(\log_{10}x-1)}=3^{(\log_{10}x+1)}$

$\Rightarrow\quad \frac{2}{5}a^{\log_a x\log_{10}5}.a^{3-3\log_{10}x}=3^{\log_{10}x}$

$\Rightarrow\quad \frac{2}{5}a^{\log_{10}x\log_a 5}.a^{3-3\log_{10}x}=3^{\log_{10}x}$

$\Rightarrow\quad \frac{2}{5}a^{b\log_a 5}.a^{3-3b}=3^b\ ,b=\log_{10}x$

It is possible only when, $b=1$

Thus, $\log_{10}x=1$

$\Rightarrow\quad x=10^1=10$

Hence, the solution is $x=10$

Ex-24. Solve for x:

$|x-1|^{\log_3 x^2-2\log_x 9}=(x-1)^7$ [Roorkee-1990]

Soln. We have

$$|x-1|^{\log_3 x^2-2\log_x 9}=(x-1)^7$$

when $x>1$, then $(x-1)^{\log_3 x^2-2\log_x 9}=(x-1)^7$

$\Rightarrow\quad \log_3 x^2-2\log_x 9=7$

$\Rightarrow \quad \log_3 x^2 - 2\log_x 3^2 = 7$

$\Rightarrow \quad 2\log_3 x - 4\log_x 3 = 7$

$\Rightarrow \quad 2\log_3 x - \dfrac{4}{\log_3 x} = 7$

$\Rightarrow \quad 2(\log_3 x)^2 - 7\log_3 x - 4 = 0$

$\Rightarrow \quad 2a^2 - 7a - 4 = 0 \ , a = \log_3 x$

$\Rightarrow \quad 2a^2 - 8a + a - 4 = 0$

$\Rightarrow \quad 2a(a-4) + 1(a-4) = 0$

$\Rightarrow \quad (a-4)(2a+1) = 0$

$\Rightarrow \quad a = 4, -\dfrac{1}{2}$

when $a = 4$, then $\log_3 x = 4$

$\Rightarrow \quad x = 3^4 = 81$

when $a = -\dfrac{1}{2}$, then $\log_3 x = -\dfrac{1}{2}$

$\Rightarrow \quad x = 3^{-1/2} = \dfrac{1}{\sqrt{3}}$

It is not possible, since $x > 1$

Hence, the solution is $\{2, 81\}$.

Ex-25. Solve for x and y

$$\log_{100}|x+y| = \frac{1}{2} \text{ and } \log_{10} y - \log_{10}|x| = \log_{100} 4$$

[Roorkee-1996]

Soln. We have $\log_{100}|x+y| = \dfrac{1}{2}$

$\Rightarrow \quad \log_{10^2}|x+y| = \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{1}{2}\log_{10}|x+y| = \dfrac{1}{2}$

$\Rightarrow \quad \log_{10}|x+y| = 1$

$\Rightarrow \quad |x+y| = 10$

$\Rightarrow \quad (x+y) = \pm 10$ ....(i)

Also, $\log_{10}\left(\dfrac{y}{|x|}\right) = \log_{100} 4$

$\Rightarrow \quad \log_{10}\left(\dfrac{y}{|x|}\right) = \log_{10^2} 2^2$

$\Rightarrow \quad \log_{10}\left(\dfrac{y}{|x|}\right) = \log_{10} 2$

$\Rightarrow \quad \left(\dfrac{y}{|x|}\right) = 2$

$\Rightarrow \quad |x| = \dfrac{y}{2}$

$\Rightarrow \quad x = \pm\dfrac{y}{2}$ ....(ii)

From (i) and (ii), we get,

$y = \pm\dfrac{20}{3}, \pm 20$

and $x = \pm\dfrac{10}{3}, \mp 10$

Hence, the solutions are

$\left\{x = \dfrac{10}{3}, y = \dfrac{20}{3}; x = 10, y = -20\right\}$

Ex-26. Find all real number x which satisfy the equation

$$2\log_2 \log_2 x + \log_{1/2}\log_2\left(2\sqrt{2}x\right) = 1$$

[Roorkee-1999]

Soln. We have $2\log_2 \log_2 x + \log_{1/2}\log_2\left(2\sqrt{2}x\right) = 1$

$\Rightarrow \quad 2\log_2\log_2 x - \log_2\log_2\left(2\sqrt{2}x\right) = 1$

$\Rightarrow \quad \log_2(\log_2 x)^2 - \log_2\left(\log_2\left(2\sqrt{2}x\right)\right) = 1$

$\Rightarrow \quad \log_2\left(\dfrac{(\log_2 x)^2}{\left(\log_2\left(2\sqrt{2}x\right)\right)}\right) = 1$

$\Rightarrow \quad \left(\dfrac{(\log_2 x)^2}{\left(\log_2\left(2\sqrt{2}x\right)\right)}\right) = 2$

$\Rightarrow \quad \left(\dfrac{(\log_2 x)^2}{\left(\log_2\left(2\sqrt{2}\right) + \log_2 x\right)}\right) = 2$

$\Rightarrow \quad \left(\dfrac{(\log_2 x)^2}{\left(\log_2\left(2^{3/2}\right) + \log_2 x\right)}\right) = 2$

$\Rightarrow \quad \left(\dfrac{(\log_2 x)^2}{\left(\dfrac{3}{2} + \log_2 x\right)}\right) = 2$

$\Rightarrow \quad \left(\dfrac{a^2}{\dfrac{3}{2} + a}\right) = 2, a = \log_2 x$

$\Rightarrow \left(\frac{2a^2}{3+2a}\right)=2$

$\Rightarrow 2a^2-4a-6=0$

$\Rightarrow a^2-2a-3=0$

$\Rightarrow (a-3)(a+1)=0$

$\Rightarrow a=-1,3$

when $a=-1 \Rightarrow \log_2 x=-1 \Rightarrow x=2^{-1}=\frac{1}{2}$

when $a=3 \Rightarrow \log_2 x=3 \Rightarrow x=2^3=8$

Hence, the solution is $x = 8$

Ex-27. Solve for x;

$$\log_{3/4}\log_8\left(x^2+7\right)+\log_{1/2}\log_{1/4}\left(x^2+7\right)^{-1}=-2$$

[Roorkee-2000]

Soln. We have

$$\log_{3/4}\log_8\left(x^2+7\right)+\log_{1/2}\log_{1/4}\left(x^2+7\right)^{-1}=-2$$

$\Rightarrow \log_{3/4}\log_{2^3}\left(x^2+7\right)+\log_{2^{-1}}\log_{2^{-2}}\left(x^2+7\right)^{-1}=-2$

$\Rightarrow \log_{3/4}\left(\frac{1}{3}\left(\log_2\left(x^2+7\right)\right)\right)-\log_2\left(\frac{1}{2}\left(\log_2\left(x^2+7\right)\right)\right)=-2$

$\Rightarrow \log_{3/4}\left(\frac{1}{3}y\right)-\log_2\left(\frac{1}{2}y\right)=-2, y=\left(\log_2\left(x^2+7\right)\right)$

$\Rightarrow \log_{3/4}\left(\frac{y}{3}\right)-\log_2\left(\frac{y}{2}\right)=-2$

$\Rightarrow y=4$, by trial

$\Rightarrow \log_2\left(x^2+7\right)=4$

$\Rightarrow \left(x^2+7\right)=2^4=16$

$\Rightarrow x^2=9$

$\Rightarrow x=\pm 3$

Hence, the solutions are $\{-3,3\}$

Ex-28. Solve for x and y:

$$\log_2 x+\log_4 x+\log_8 x+.....=y$$

$$\frac{5+9+13+.....+(4y+1)}{1+3+5+....+(2y-1)}=4\log_4 x$$

[Roorkee-2001]

Soln. We have $y=\log_2 x+\log_4 x+\log_8 x+....$

$\Rightarrow y=\log_2 x+\frac{1}{2}\log_2 x+\frac{1}{4}\log_2 x+....$

$\Rightarrow y=\left(1+\frac{1}{2}+\frac{1}{4}+....\right)\log_2 x$

$\Rightarrow y=\left(\frac{1}{1-\frac{1}{2}}\right)\log_2 x=2\log_2 x$

Also, $\frac{5+9+13+.....+(4y+1)}{1+3+5+....+(2y-1)}=4\log_4 x$

$\Rightarrow \frac{\frac{y}{2}(2.5+(y-1)4)}{\frac{y}{2}(2.+(y-1)2)}=4\log_4 x$

$\Rightarrow \frac{(10+4y-4)}{(2+2y-2)}=4\log_4 x$

$\Rightarrow \frac{(6+4y)}{(2y)}=4\log_4 x$

$\Rightarrow \frac{(3+2y)}{y}=4\log_{2^2} x=2\log_2 x=y$

$\Rightarrow y^2=2y+3$

$\Rightarrow y^2-2y-3=0$

$\Rightarrow (y-3)(y+1)=0$

$\Rightarrow y=-1, 3$

$\Rightarrow y=3$ ($y=-1$ is not possible)

when $y = 3$, then $2\log_2 x=3$

$\Rightarrow \log_2 x=\frac{3}{2}$

$\Rightarrow x=2^{3/2}$

Hence, the solutions are $x=2^{3/2}$ and $y = 3$

LEVEL I
(PROBLEMS BASED ON FUNDAMENTALS)

EXERCISE 1

1. Find the value of
   (i) $\log_2 32$
   (ii) $\log_3\left(\frac{1}{243}\right)$
   (iii) $\log_{5\sqrt{5}} 5$

(iv) $\log_{100}(0.1)$

(v) $10^{\log_{10} m+\log_{10} n}$

(vi) $\log_3 \log_5 \log_3(243)$

2. Find the value of

(i) $\log_2 \log_3 \log_4(64)$

(ii) $3^{-\frac{1}{3}\log_3 7}$

(iii) $2^{2-\log_2 5}$

(iv) $2^{\log_3 5}-5^{\log_3 2}$

(v) $\log_9 27-\log_{27} 9$

3. Find the value of

(i) $\log_{10}\tan 40°+\log_{10}\tan 41°+\log_{10}\tan 42°$ $+.......+\log_{10}\tan 50°$

(ii) $\log_{10}\tan 1°+\log_{10}\tan 2°+\log_{10}\tan 3°$ $+...........+\log_{10}\tan 89°$

(iii) $\log_3 4.\log_4 5.\log_5 6.\log_6 7.\log_7 8.\log_8 9$

4. Find the value of

(i) x, if $\log_5 a.\log_a x=2$,

(ii) x, if $\log_k x.\log_5 k=\log_k 5$, where $k\neq 1, k>0$

(iii) $A+B+10$, if $A=\log_2\log_2\log_4 256$ and $B=2\log_{\sqrt{2}} 2$,

(iv) $\log_{\sqrt{3}} 300$., if $a=\log_{\sqrt{3}} 5$, $b=\log_{\sqrt{3}} 2$

5. Find the minimum value of

(i) $\log_b a+\log_a b$

(ii) $\log_b a+\log_c b+\log_a c$.

6. Prove that $\dfrac{1}{\log_2 n}+\dfrac{1}{\log_3 n}+\dfrac{1}{\log_4 n}$ $+............+\dfrac{1}{\log_{43} n}=\dfrac{1}{\log_{43!} n}$

7. If $n=1983$, then prove that $\dfrac{1}{\log_2 n}+\dfrac{1}{\log_3 n}+\dfrac{1}{\log_4 n}+......+\dfrac{1}{\log_{1983} n}=1$

8. Determine b satisfying

(i) $\log_{\sqrt{8}} b=3\frac{1}{3}$

(ii) $\log_{\sqrt{8}} b=3^{\frac{1}{3}}$

(iii) $\log_a 2.\log_b 625=\log_{10}16.\log_a 10$

(iv) $\log_3\log_2\log_{\sqrt{5}}(625)=b$

9. If $\log_a ab=x$, then find the value of $\log_b ab$.

10. If $\log_{10} 2=x$, then find the value of $\log_{10} 5$.

11. If $a=\log_4 5$ and $b=\log_5 6$, then find $\log_3 2$.

12. Find the value of $\log_{12} 54$, where $b=\log_{12} 24$.

13. Find the value of $\dfrac{1}{\log_2 36}+\dfrac{1}{\log_3 36}$

14. Prove that $\dfrac{1}{\log_3\pi}+\dfrac{1}{\log_4\pi}>2$

15. Simplify: $7.\log\dfrac{16}{15}+5.\log\dfrac{25}{24}+3.\log\dfrac{81}{80}$

16. If $a^2+b^2=7ab$, then prove that $\log\frac{1}{3}(a+b)=\frac{1}{2}(\log a+\log b)$

17. If $a^2+b^2=11ab$, then prove that $\log\left(\dfrac{a-b}{3}\right)=\dfrac{1}{2}(\log a+\log b)$

18. Prove that $\dfrac{\log_a n}{\log_{ab} n}=1+\log_a b$.

19. If log 225 = a and log 225 = b, then prove that $\log\left(\left(\dfrac{1}{9}\right)^2\right)+\log\left(\dfrac{1}{2250}\right)=2a-3b-1$.

20. If a, b, c are in G.P, then prove that, $\log_a n$, $\log_b n$, $\log_c n$ are in H.P.

21. If $\log_3 2$, $\log_3(2^x-5)$, $\log_3\left(2^x-\dfrac{7}{2}\right)$ are in A.P., then find the value of x.

22. If $y=a^{\frac{1}{1-\log_a x}}$, $z=a^{\frac{1}{1-\log_a y}}$, then prove that, $x=a^{\frac{1}{1-\log_a z}}$.

23. If $x=\log_c b+\log_b c$, $y=\log_a c+\log_c a$ and $z=\log_b a+\log_a b$, then find the minimum value of $x^2+y^2+z^2-xyz$.

24. If $\dfrac{\log a}{b-c}=\dfrac{\log b}{c-a}=\dfrac{\log c}{a-b}$, then prove that $a^a.b^b.c^c=1$

25. Find the value of $81^{\frac{1}{\log_5 3}}+27^{\log_9 36}+3^{\frac{4}{\log_7 9}}$

26. Prove that $$\frac{1}{1+\log_b a+\log_b c}+\frac{1}{1+\log_c a+\log_c b}+\frac{1}{1+\log_a b+\log_a c}=1$$

27. Find x, if $\log_2 x+\log_4 x+\log_8 x=11$

28. Find x, if $\log_2 x+\log_4 x+\log_8 x+\log_{16} x=\frac{25}{4}$
29. If $\log_a x=\alpha$, $\log_b x=\beta$, $\log_c x=\gamma$ and $\log_d x=\delta$, then find the value of $\log_{abcd} x$.
30. If $x=\log_a bc$, $y=\log_b ca$ and $z=\log_c ab$, then find the value of $\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}$
31. If $y=2^{\frac{1}{\log_x 4}}$, then find x.
32. If $N=6^{\log_{10}40}.5^{\log_{10}36}$, then find the value of $N+10$.
33. Find the value of $(0.5)^{\log_{\sqrt{2}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\text{............to }\infty\right)}$.
34. If $x=2^{\log_{10}3}$ and $y=3^{\log_{10}2}$, then find a relation between x and y.
35. Find the value of $2^{\log_{10}3-\log_{10}5}\times 3^{\log_{10}5-\log_{10}2}\times 5^{\log_{10}2-\log_{10}3}$.
36. If $a=\log_{30}3$ and $b=\log_{30}5$, then find the value of $\log_{10}8$.
37. If $a=\log_{12}18$ and $b=\log_{24}54$, then prove that $ab+5(a-b)=1$
38. If $a=\log_7 12$ and $b=\log_{12}24$, then find the value of $\log_{54}168$.
39. If $a=\log_6 30$, $b=\log_{15}24$, then prove that $\log_{12}60=\left(\frac{2ab+2a-1}{ab+b+1}\right)$
40. Find x, if $\log_7\left(\log_5\left(\sqrt{x+5}+\sqrt{x}\right)\right)=0$
41. If $\log_2 x+\log_2 y\ \geq 6$, then find the least value of $x+y$.
42. Solve for x and y: $4^{\log x}=3^{\log y}$, $(3x)^{\log 3}=(4y)^{\log 4}$
43. If $x^{18}=y^{21}=z^{28}=k$, then prove that, $3, 3\log_y x, 3\log_z y, 7\log_x z$ are in A.P.
44. If $\frac{1}{\log_3\pi}+\frac{1}{\log_4\pi}>x$, then find the value of x.
45. If $\log_{0.3}(x-1)<\log_{0.09}(x-1)$, then find x.
46 If $\log_e\log_5\left(\sqrt{2x-2}+3\right)=0$, then find the value of x.
47. Find the least value of $2.\log_{10}x-\log_x(0.01)$ for $x>1$.
48. Find x, if $4^{\log_9 3}+9^{\log_2 4}=10^{\log_x 83}$
49. Find x, if $3^{4\log_9(x+1)}=2^{2\log_2 x}+3$
50. If $a=\log_{24}12$, $b=\log_{36}24$ and $c=\log_{48}36$ then prove that $\left(\frac{abc+1}{bc}\right)=2$.
51. If $\log_{10}\left(\sin\left(x+\frac{\pi}{4}\right)\right)=\frac{1}{2}(\log_{10}6-1)$, then find the value of $\log_{10}\sin x+\log_{10}\cos x$.
52. If a, b, c are in G.P., then prove that $\frac{1}{1+\log a},\frac{1}{1+\log b},\frac{1}{1+\log c}$ are in H.P.
53. Find x, if $5^{\log_{10}x}=50-x^{\log_{10}5}$
54. Find x, if $\log_5\left[2+\log(3+x)\right]=0$.

Q. Find the number of real solutions of

55. $\log_4(x-1)=\log_2(x-3)$
56. $\log_4(x-2)=\log_2(x-2)$
57. $\log_9(x-1)=\log_3(x-1)$.
58. $\log_2 x+\log_2(x+3)=1/4$
59. $\log_4\left(x^2+x\right)-\log_4(x+1)=2$
60. $1+2\log_{(x+2)}5=\log_5(x+2)$
61. $\log_2 x+\log_4(x+2)=2$
62. $\log_{10}(x-1)^3-3\log_{10}(x-3)=\log_{10}8$

Q Solve for x:

63. $\log_5\left(x^2-3x+3\right)>0$
64. $\log_7\left(\log_5\left(x^2-7x+15\right)\right)>0$
65. $\log_{(1/2)}\left(\log_5\left(x^2-7x+17\right)\right)>0$
66. $\log_{(1/2)}\left(\log_5\left(\log_2\left(x^2-6x+40\right)\right)\right)>0$
67. $\log_3\left(\log_5\left(\log_2\left(x^2-9x+50\right)\right)\right)>0$
68. $\log_6\left(\frac{x-2}{6-x}\right)>0$
69. $\log_{(1/2)}x>\log_{(1/3)}x$
70. $\log_{0.5}\left(x^2-5x+6\right)>-1$
71. $\log_8\left(x^2-4x+3\right)<1$
72. $\log_{(1/4)}\left(\frac{35-x^2}{x}\right)\geq-\frac{1}{2}$

Q Solve for x:

73. $\log_{(x^3+6)}(x^2-1)=\log_{(2x^2+5x)}(x^2-1)$

74. $\log(3x^2+x-3)=3\log(3x-2)$

75. $\log_{(x^2-1)}(x^3+6)=\log_{(x^2-1)}(2x^2+5x)$

76. $\log_3(x^2-3x-5)=\log_3(7-2x)$

77. $\log(\sqrt{x-1})+\frac{1}{2}\log(2x+15)=1$

78. $\log_{(3x+4)}(4x^2+4x+1)+\log_{(2x+1)}(6x^2+11x+4)=4$

79. $5^{\log_{10}x}=50-x^{\log_{10}5}$

80. $4^{\log_9 x}-6.2^{\log_9 x}+2^{\log_3 27}=0$

Q (Solve the inequality, wherever base is not given, take it as 10)

81. $(\log_2 x)^4-\left(\log_{1/2}\left(\frac{x^5}{4}\right)\right)^2-20\log_2 x+148<0$

82. $(\log 100x)^2+(\log 10x)^2+\log x\le 14$

83. $\log_{1/2}(x+1)>\log_2(2-x)$

84. $\log_{1/5}(2x^2+5x+1)<0$

85. $\log_{1/2}x+\log_3 x>1$

86. $\log_x\left(\frac{4x+5}{6-5x}\right)<-1$

87. $\log_{0.2}\left(\frac{x+2}{x}\right)\le 1$

88. $\log_{10}(x^2-16)\le\log_{10}(4x-11)$

LEVEL II
(MIXED PROBLEMS)

1. If a, b, c are in G.P. then $\log_{2016}a, \log_{2016}b, \log_{2016}c$ are in
(a) G.P. (b) A.P.
(c) H.P. (d) A.G.P.

2. If $y=3^{\frac{1}{\log_x 9}}$, then x is
(a) y (b) $\sqrt{y}$
(c) y^2 (d) y^3

3. If $\frac{1}{\log_a x}+\frac{1}{\log_c x}=\frac{2}{\log_b x}$, then a, b, c are in
(a) A.P. (b) G.P.
(c) H.P. (d) A.G.P.

4. If $x=\log_3 5$ and $y=\log_{27}25$, then
(a) $x>y$ (b) $x=y$
(c) $x<y$ (d) $x^2=y$

5. If $\log_{10}2, \log_{10}(2^x+1), \log_{10}(2^x+3)$ are in A.P. then
(a) $x=0$ (b) $x=1$
(c) $x=\log_{10}2$ (d) $x=\frac{1}{2}\log_2 5$

6. If $\log_a(ab)=x$, then $\log_b(ab)$ is
(a) $\frac{x}{1-x}$ (b) $\frac{x}{1+x}$
(c) $\frac{x}{x-2}$ (d) None

7. The value of $4^{2\log_9 3}$ is
(a) 9 (b) 2
(c) 4 (d) 3

8. If $\log_7 2=x$, then $\log_{49}(28)$ is
(a) $\left(x+\frac{1}{2}\right)$ (b) $\left(x-\frac{1}{2}\right)$
(c) $-\left(x-\frac{1}{2}\right)$ (d) $-\left(x+\frac{1}{2}\right)$

9. If $\log_{2016}\left(\log_5\left(\sqrt{2x-2}+3\right)\right)=0$, then x is
(a) 1/3 (b) 1/2
(c) 3 (d) 2

10. If $\frac{1}{\log_2\pi}+\frac{1}{\log_6\pi}>x$, then x is
(a) 2 (b) 3
(c) 4 (d) 5

11. The value of $5^{\log_2 7}-7^{\log_2 5}$ is
(a) 5 (b) 0
(c) 7 (d) 2

12. If $\log_{10}2=x$, then $\log_{10}5$ is
(a) 1 (b) $1-x$
(c) $x+1$ (d) $2x$

13. The number of real solutions of $\log_2 x+\log_4(x+2)=2$ is
(a) 1 (b) 2
(c) 3 (d) 0

14. The number of real solutions of $1+\log_2(x-1)=\log_{(x-1)}4$ is
(a) 1 (b) 2
(c) 3 (d) 0

15. The number of real solutions of $x^{\log_{\sqrt{x}}(x-2)}=9$ is
(a) 4 (b) 3
(c) 2 (d) 1

16. The value of $\left(\frac{1}{\log_3 \pi}+\frac{1}{\log_4 \pi}\right)$ lies between
    (a) $(1,2)$ (b) $(2,3)$
    (c) $(3,4)$ (d) $(0,1)$
17. The number of real roots of $x \ln x - 1 = 0$ is
    (a) 2 (b) 1
    (c) 3 (d) infinite
18. The number of real roots of $2 - x - \ln x = 0$ is
    (a) 1 (b) 2
    (c) 0 (d) infinite
19. If $3^x = 10 - \log_2 x$, then x is
    (a) 0 (b) 1
    (c) 2 (d) 3
20 If $|1-\log_{1/5} x| + 2 = |3 - \log_{1/5} x|$, then x is
    (a) 2 (b) 5
    (c) 1 (d) 3

LEVEL III
(TOUGHER PROBLEMS FOR JEE ADVANCED)

1. Solve for x: $x+\log_{10}\left(1+2^x\right) = x\log_{10}5+\log_{10}6$
2. Solve for x: $\log\left|\frac{x^2-x-1}{x^2+x-2}\right| = 0$
3. Solve for x: $|4+\log_{1/7} x| = 2+|2+\log_{1/7} x|$
4. Solve for x
   $\log^2\left(1+\frac{4}{x}\right)+\log_2\left(1-\frac{4}{x+4}\right) = 2\log^2\left(\frac{2}{x-1}-1\right)$
5. Solve the system of equations:
   $\begin{cases}\log_y x - \log_x y = \frac{8}{3} \\ xy = 16\end{cases}$.
6. Solve for x:
   $\log\left(3x^2+12x+19\right)-\log(3x+4)+\log_{32}4$
   $= 1-\log_{1/16}\left(\sqrt[5]{256}\right)$
7. Solve for x:
   $\log^2(4-x)+\log(4-x).\log\left(x+\frac{1}{2}\right)-2\log^2\left(x+\frac{1}{2}\right) = 0$
8. Solve for x:
   $\log_{3/4}\left(\log_8\left(x^2+7\right)\right)+\log_{1/2}\left(\log_{1/4}\left(x^2+7\right)^{-1}\right) = -2$
9. Solve for x:
   $\log_{10}\left(x^2-x-6\right)-x = \log_{10}(x+2)-4$
10. Solve for x:
    $\frac{1}{2}\log_5(x+5)+\log_5\left(\sqrt{x-3}\right) = \frac{1}{2}\log_5(2x+1)$
11. Solve for x:
    $\frac{3}{2}\log_4(x+2)^2+3 = \log_4(4-x)^3+\log_4(6+x)^3$
12. Solve for x:
    $\frac{1+\log_2(x-4)}{2\log_2\left(\sqrt{x+3}-\sqrt{x-3}\right)} = 1$
13. Solve for x:
    $\left(1+\frac{1}{2x}\right)\log 3 = \log\left(\frac{\sqrt[x]{3}+27}{4}\right)$
14. Solve for x:
    $4^{\log_{10} x+1}-6^{\log_{10} x}-2.3^{\log_{10} x^2+2} = 0$
15. Solve for x:
    $\log_3\left(\sqrt{x}+\left|\sqrt{x}-1\right|\right)^2 = \log_3\left(4\sqrt{3}-3+4\left|\sqrt{x}-1\right|\right)$

COMPREHENSIVE LINK PASSAGES

PASSAGE I

Let A be the sum of the roots of
$\frac{1}{5-4\log_4 x}+\frac{4}{1+\log_4 x} = 3$
B be the product of m and n, where
$2^m = 3$ and $3^n = 4$
and C be the sum of the integral roots of
$\log_{3x}\left(\frac{3}{x}\right)+\left(\log_3 x\right)^2 = 1$
Then

1 The value of $A + B$ is
  (a) 10 (b) 6
  (c) 8 (d) 4
2. The value of $B + C$ is
  (a) 6 (b) 2
  (c) 4 (d) 8
3. The value of $(A + C \div B)$ is
  (a) 5 (b) 8
  (c) 7 (d) 4

PASSAGE II

A function $f: R^+ \to R$ is defined as $f(x) = \log_a x, x > 0$, $a > 0, a \neq 1$

Then $D_f = R^+$ and $R_f = R$

1. If $f(x) = \log\left(\frac{x-3}{5-x}\right)$, then the domain of the function $f(x)$ is

(a) $(3,5)$ (b) $(-\infty, 5)$
(c) $(5,\infty)$ (d) None

2. Let $f(x)=(-x^2+3x-2)$. Then the domain of the function $f(x)$ is
(a) $(-1,2)$ (b) $(1,2)$
(c) $(-\infty,1]$ (d) $[2,\infty)$

3. Let $f(x)=\sqrt{x-2}+\sqrt{4-x}$. Then the range of the function $f(x)$ is
(a) $[\sqrt{2},2]$ (b) $[1,2]$
(c) $(2,4)$ (d) $[2,4]$

MATCHING LIST TYPE (ONLY ONE OPTION IS CORRECT)

This section contains four questions, each having two matching list. Choices for the correct combination of elements from

List-I and List-II are given as options (A), (B), (C) and (D), out of which ONE is correct.

1. Match the following lists

List I	List II
(P) The value of $\left(\dfrac{\log_2 32}{\log_3\sqrt{243}}\right)$ is	(1) 2/7
(Q) The value of $\left(\dfrac{2\log_{2016}6}{\log_{2016}12+\log_{2016}3}\right)$ is	(2) –2
(R) The value of $\left(\log_{1/4}\left(\dfrac{1}{16}\right)^{-2}\right)$ is	(3) 1
(S) The value of $\left(\dfrac{\log_5 16-\log_5 4}{\log_5 128}\right)$ is	(4) 2

codes:

	P	Q	R	S
(A)	2	3	1	4
(B)	4	2	1	3
(C)	4	3	2	1
(D)	3	1	4	2

2. Match the following lists

List-I	List-II
(P) The vaue of $\left(\dfrac{2\log 2+\log 3}{\log 48-\log 4}\right)$ is	(1) 3
(Q) The value of $\dfrac{1}{6}\log_{\frac{\sqrt{3}}{2}}\left(\dfrac{64}{27}\right)$	(2) 0.
(R) The value of $\log^2{}_{10}5+\log_{10}5.\log_{10}20+\log^2{}_{10}2-1$ is	(3) 1
(S) The value of a for which $\dfrac{\log_a 7}{\log_6 7}=\log_9 36$ holds good is	(4) –1

codes:

	P	Q	R	S
(A)	2	3	1	4
(B)	4	3	1	2
(C)	4	3	2	1
(D)	3	4	2	1

MATCH MATRIX

Column-I	Column-II
(A) If α the root of $3x^{\log_3 4}+4^{\log_3 x}=64$ then $\sqrt{\alpha}+1$ is	(P) 2
(B) The integral value of of x in $\log^2{}_2 x-\log_2 x-2=0$ is	(Q) 4
(C) The value of $4\log_2 x$ is where $x=\sqrt{3+2\sqrt{2}}+\sqrt{3-2\sqrt{2}}$	(R) 3
(D) If $a^2+b^2=1$, then the value of $\log_{ab}(a^3b^5+a^5b^3)$ is	(S) 1

INTEGER TYPE QUESTIONS

1. If $\sum_{r=0}^{n-1}\log_2\left(\dfrac{r-1}{r+1}\right)=\prod_{r=10}^{99}\log_r(r+1)$, then find the value of n.
2. Solve for x: $7^{\log_2 x}=98-x^{\log_2 7}$
3. Solve for x: $4^{\log_3 x}=32-x^{\log_3 4}$
4. If α and β are the roots of $3\log_x 4+2\log_{4x}4+3\log_{16x}4=0$, then find the value of $\dfrac{1}{2}\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right)$
5. Solve for x: $x+\log_{10}(2^x+1)=\log_{10}6+x\log_{10}5$
6. Solve for x: $\log_{x-1}(x^3-9x+8).\log_{x-1}(x+1)=3$
7. If α and β are the solutions of $|x-2|^{\log_2(x^3)-3\log_x 4}=(x-2)^3$, then find the value of $(\alpha+\beta+3)$

8. If α integral solutions of $6\left(\log_x 2-\log_4 x\right)+7=0$, then find the value of $\left(\frac{2\alpha-1}{5}\right)$.
9. Let The number $N=6\log_{10}2+\log_{10}31$. If N lies between two successive integers, then find their sum.
10. Find the value of the expession $(0.16)^{\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+......\right)}$.

Questions asked in PAST IIT-JEE EXAMS

1. For $a>0$, solve for x, the equation $2\log_x a+\log_{ax}a+3\log_{a^2x}a=0$ [IIT-JEE-1978]
2. The least value of the expression $2\log_{10}x-\log_x(0.01), x>1$ is
   (a) 10 (b) 2
   (c) –0.01 (d) None of these
   [IIT-JEE - 1980]
3. $y=10^x$ is the reflection of $y=\log_{10}x$ in the line whose equation is .... [IIT-JEE-1982]
4. For $0<a<x$, the minimum value of $\log_x a+\log_a x$ is ...... [IIT-JEE-1984]
5. If $\log_{0.3}(x-1)<\log_{0.09}(x-1)$, then x lies in
   (a) $(2,\infty)$ (b) $(1,2)$
   (c) $(-2,-1)$ (d) None of these
   [IIT-JEE-1985]
6. The solution of the equation $\log_7\left(\log_5\left(\sqrt{x+5}+\sqrt{x}\right)\right)=0$ is ...... [IIT-JEE-1986]
7. Solve for x: $\log_{(2x+3)}\left(6x^2+23x+21\right)=4-\log_{(3x+7)}\left(4x^2+12x+9\right)$ [IIT-JEE- 1987]
8. The equation $x^{(3/4)(\log_2 x)^2+\log_2 x-\frac{5}{4}}=\sqrt{2}$ has
   (a) at least one real solution
   (b) exactly three real solutions
   (c) exactly one irrational solutions
   (d) complex roots [IIT-JEE-1989]
9. If $\log_3 2, \log_3\left(2^x-5\right), \log_3\left(2^x-\frac{7}{2}\right)$ are in A.P then find x. [IIT-JEE-1990]
10. The number $\log_2 7$ is
    (a) an integer (b) a rational number
    (c) an irrational no. (d) a prime number.
    [IIT-JEE-1990]
11. The number of solutions of $\log_4(x-1)=\log_2(x-3)$ is
    (a) 3 (b) 1
    (c) 2 (d) 0
    [IIT-JEE-2001]
12. Let (x_0,y_0) be the solution of the following equations $(2x)^{\ln 2}=(3y)^{\ln 3}$, $3^{\ln x}=2^{\ln 3}$. Then x_0 is
    (a) $\frac{1}{6}$ (b) $\frac{1}{3}$
    (c) $\frac{1}{2}$ (d) 6
    [IIT-JEE-2011]
13. The value of $6+\log_{3/2}\left(\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}......to\ \infty}}}\right)$ is..... [IIT-JEE-2012]

ANSWERS

LEVEL I

1. (i) 5 (ii) –5 (iii) 2/3
   (iv) –1/2 (v) $m+n$ (vi) 0
3. (i) 0 (ii) 0 (iii) 2
4. (i) 25 (ii) 25 (iii) 12
5. (i) 2 (ii) 3
8. (i) 32 (ii) 2 (iii) 24 (iv) 21
9. $\left(\frac{x}{\lambda-1}\right)$
10. $(1-x)$
11. $\frac{1}{(2ab-1)}$
13. 1/2
15. log 2
21. $x=3$
25. 216
27. 64
28. 25/4

29. $\left(\dfrac{1}{\alpha^{-1}+\beta^{-1}+\gamma^{-1}+\delta^{-1}}\right)$
30. 1
31. $x = y^2$
32. 226
33. 4
34. $x = y$
40 $x = 4$
41. 16.
44. $x = 2$
45. $(2,\infty)$
46. $x = 3$
47. 4
48. $x = 10$
49. $x = 1$
51. –1
53 $x = 100$
54. $x = 1$
55. 1
56. 1
57. 1
58. 2
59. 1
60. 1
61. 1
62. 1
63. $(-\infty,1)\cup(2,\infty)$
64. $(-\infty,2)\cup(5,\infty)$
65. $(3,4)$
66. $(2,4)$
67. $(-\infty,3)\cup(6,\infty)$
68. $(4,6)$
69. $0 < x < 1$
70. $(1,4)$
71. $(-1,5)$
72. $(-1,0)\cup(5,\infty)$
73. $x = 3$
74. $x = 1,\ \dfrac{5\pm\sqrt{10}}{9}$
75. $x = 3$
76. $x = -3$
77. $x = 5$
78. $x = 3/4$
79. 100
80. {9, 81}
81. $x\in\left(\dfrac{1}{16},\dfrac{1}{8}\right)\cup(8,16)$
82. $x\in\left[\dfrac{1}{\sqrt{10}},10\right]$
83. $-1 < x < \dfrac{1-\sqrt{5}}{2}$ or $\dfrac{1+\sqrt{5}}{2} < x < 2$
84. $x\in(-\infty,-2.5)\cup(0,\infty)$
85. $0 < x < 3^{\frac{1}{1-\log 3}}$ (where base is 2)
86. $1/2 < x < 1$
87. $x\in\left(-\infty,-\dfrac{5}{2}\right]\cup(0,\infty)$
88. $x\in(4,5]$

LEVEL II

1. (b)	2. (c)	3. (a)	4. (c)
5. (b)	6. (a)	7. (c)	8. (a)
9. (c)	10. (a)	11. (b)	12. (b)
13. (a)	14. (b)	15. (d)	16. (b)
17. (b)	18. (a)	19. (c)	20. (b)

LEVEL III

1. 1
2. $\left\{-\sqrt{\dfrac{3}{2}},\dfrac{1}{2},\sqrt{\dfrac{3}{2}}\right\}$
3. $(0,49]$
4. $\{\sqrt{2}\text{ or }\sqrt{6}\}$
5. 5/3
6. $\{-1,7\}$
7. $\left\{0,\dfrac{7}{4},\dfrac{3+\sqrt{24}}{2}\right\}$
8. $x = 3$ or –3
9. $x\in(0,1)\cup(1983^4,\infty)$
10. $\left(\dfrac{12}{5},\infty\right)$

COMPREHENSIVE LINK PASSAGES

Passage-I: 1. (c) 2. (a) 3. (b)
Passage-II: 1. (a) 2. (b) 3. (a)

MATCHING LIST

1. (C)
2. (D)
3. (A) → (Q); (B) → (P); (C) → (Q); (D) → (R)

INTEGER TYPE QUESTIONS

1. 3
2. $x = 4$
3. $x = 9$
4. 5
5. 1
6. $x = 3$
7. 9
8. 3
9. 7
10. 4

HINTS AND SOLUTIONS

LEVEL III

1. Given equation is

$$x + \log_{10}(1+2^x) = x\log_{10} 5 + \log_{10} 6$$

$$\Rightarrow \quad x(1-\log_{10} 5) + \log_{10}\left(\frac{1+2^x}{6}\right) = 0$$

$$\Rightarrow \quad x(\log_{10} 2) + \log_{10}\left(\frac{1+2^x}{6}\right) = 0$$

$$\Rightarrow \quad \log_{10}\left(\frac{1+2^x}{6}\right) = -x(\log_{10} 2)$$

$$\Rightarrow \quad \log_{10}\left(\frac{1+2^x}{6}\right) = \log_{10} 2^{-x}$$

$$\Rightarrow \quad \left(\frac{1+2^x}{6}\right) = 2^{-x} = \frac{1}{2^x}$$

$$\Rightarrow \quad (2^x)^2 + 2^x - 6 = 0$$

$$\Rightarrow \quad a^2 + a - 6 = 0 \ , a = 2^x$$

$$\Rightarrow \quad (a+3)(a-2) = 0$$

$$\Rightarrow \quad a = 2, -3$$

$$\Rightarrow \quad 2^x = 2, -3$$

$$\Rightarrow \quad 2^x = 2$$

$$\Rightarrow \quad x = 1$$

Hence, the solution is $x = 1$.

2. Given equation is

$$\log\left|\frac{x^2-x-1}{x^2+x-2}\right| = 0$$

$$\Rightarrow \quad \left|\frac{x^2-x-1}{x^2+x-2}\right| = 1$$

$$\Rightarrow \quad |x^2-x-1| = |x^2+x-2|$$

$$\Rightarrow \quad (x^2-x-1) = \pm(x^2+x-2)$$

Taking +ve sign, we get,

$$\Rightarrow \quad (x^2-x-1) = (x^2+x-2)$$

$$\Rightarrow \quad 2x = 1$$

$$\Rightarrow \quad x = 1/2$$

Taking –ve sign, we get,

$$(x^2-x-1) = -(x^2+x-2)$$

$$\Rightarrow \quad 2x^2 = 3$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{3}{2}}$$

Hence, the solutions are

$$\left\{-\sqrt{\frac{3}{2}}, \frac{1}{2}, \sqrt{\frac{3}{2}}\right\}$$

3. Given equation is

$$|4+\log_{1/7} x| = 2 + |2+\log_{1/7} x|$$

$$\Rightarrow \quad |4+\log_{1/7} x| = |2| + |2+\log_{1/7} x|$$

$$\Rightarrow \quad 2(2+\log_{1/7} x) \geq 0$$

$$\Rightarrow \quad (2+\log_{1/7} x) \geq 0$$

$$\Rightarrow \quad \log_{1/7} x \geq -2$$

$$\Rightarrow \quad x \leq \left(\frac{1}{7}\right)^{-2}$$

$$\Rightarrow \quad x \leq 49$$

Hence, the value of x is $(0, 49]$

4. Given equation is

$$\log^2\left(1+\frac{4}{x}\right) + \log^2\left(1-\frac{4}{x+4}\right) = 2\log^2\left(\frac{2}{x-1}-1\right)$$

$$\Rightarrow \quad \log^2\left(\frac{x+4}{x}\right) + \log^2\left(\frac{x}{x+4}\right) = 2\log^2\left(\frac{2}{x-1}-1\right)$$

$$\Rightarrow \quad \log^2\left(\frac{x+4}{x}\right) + \log^2\left(\frac{x}{x+4}\right) = 2\log^2\left(\frac{3-x}{x-1}\right)$$

$$\Rightarrow \quad -2\log\left(\frac{x}{x+4}\right)\log\left(\frac{x+4}{x}\right) = 2\log^2\left(\frac{3-x}{x-1}\right)$$

$$\Rightarrow \quad -\log\left(\frac{x}{x+4}\right)\log\left(\frac{x+4}{x}\right) = \log^2\left(\frac{3-x}{x-1}\right)$$

$$\Rightarrow \quad \log^2\left(\frac{x}{x+4}\right) = \log^2\left(\frac{3-x}{x-1}\right)$$

$\Rightarrow \quad \left(\dfrac{x}{x+4}\right)=\left(\dfrac{3-x}{x-1}\right)$

$\Rightarrow \quad x^2-x=3x+12-x^2-4x$

$\Rightarrow \quad 2x^2=12$

$\Rightarrow \quad x^2=6$

$\Rightarrow \quad x=\pm\sqrt{6}$

Hence, the solution is $\{-\sqrt{6},\sqrt{6}\}$

5. We have $\log_y x-\log_x y=\dfrac{8}{3}$

$\Rightarrow \quad \dfrac{\log x}{\log y}-\dfrac{\log y}{\log x}=\dfrac{8}{3}$

$\Rightarrow \quad (\log x)^2-(\log y)^2=\dfrac{8}{3}\log x\log y$

$\Rightarrow \quad 3(\log x)^2--8\log x\log y-3(\log y)^2=0$

$\Rightarrow \quad 3a^2-8ab-3b^2=0$

where $a=\log x, b=\log y$

$\Rightarrow \quad 3a^2-9ab+ab-3b^2=0$

$\Rightarrow \quad 3a(a-3b)+b(a-3b)=0$

$\Rightarrow \quad (a-3b)(3a+b)=0$

$\Rightarrow \quad (a-3b)=0,(3a+b)=0$

$\Rightarrow \quad a=3b\ ,b=-3a$

$\Rightarrow \quad \log x=3\log y\ ,\log y=-3\log x$

$\Rightarrow \quad \log\left(\dfrac{x}{y^3}\right)=0,\log\left(yx^3\right)=0$

$\Rightarrow \quad \left(\dfrac{x}{y^3}\right)=1,\left(yx^3\right)=1$

Now, $\quad x^3y=1$

$\Rightarrow \quad x^2(xy)=1$

$\Rightarrow \quad x^2=\dfrac{1}{xy}=\dfrac{1}{16}$

$\Rightarrow \quad x=\dfrac{1}{4}$

when $x=\dfrac{1}{4}$, then $y=\dfrac{1}{x^3}=64$

Also, $\dfrac{x}{y^3}=1$

$\Rightarrow \quad y^4=xy=16$

$\Rightarrow \quad y=2$

when $y=2$, then $x=8$

Hence, the solutions set are

$\left(\dfrac{1}{4},64\right),(8,2)$

6. Given equation is

$$\log\left(\frac{3x^2+12x+19}{3x+4}\right)+\log_{2^5}4=1+\frac{1}{4}\log_2\left(\sqrt[5]{256}\right)$$

$\Rightarrow \quad \log\left(\dfrac{3x^2+12x+19}{3x+4}\right)+\dfrac{2}{5}=1+\dfrac{1}{4}\log_2\left(\sqrt[5]{256}\right)$

$\Rightarrow \quad \log\left(\dfrac{3x^2+12x+19}{3x+4}\right)=\dfrac{3}{5}+\dfrac{1}{4}\log_2\left(2^{8/5}\right)$

$\Rightarrow \quad \log\left(\dfrac{3x^2+12x+19}{3x+4}\right)=\dfrac{3}{5}+\dfrac{2}{5}\log_2(2)$

$\Rightarrow \quad \log\left(\dfrac{3x^2+12x+19}{3x+4}\right)=1$

$\Rightarrow \quad \left(\dfrac{3x^2+12x+19}{3x+4}\right)=10$

$\Rightarrow \quad 3x^2+12x+19=30x+40$

$\Rightarrow \quad 3x^2-18x-21=0$

$\Rightarrow \quad (x-7)(3x+3)=0$

$\Rightarrow \quad x=-1,7$

Hence, the solution set is $\{-1,7\}$

7. Given equation is

$$\log^2(4-x)+\log(4-x).\log\left(x+\frac{1}{2}\right)-2\log^2\left(x+\frac{1}{2}\right)=0$$

$\Rightarrow \quad a^2+ab-2b^2=0$, where

$\Rightarrow \quad a=\log(4-x), b=\log\left(x+\dfrac{1}{2}\right)$

$\Rightarrow \quad (a-b)(a+2b)=0$

$\Rightarrow \quad a=b,-2b$

when $a=b$

$\Rightarrow \quad \log(4-x)=\log\left(x+\dfrac{1}{2}\right)$

$\Rightarrow \quad (4-x)=\left(x+\dfrac{1}{2}\right)$

$\Rightarrow \quad 2x=\dfrac{7}{2}$

$\Rightarrow \quad x=\dfrac{7}{4}$

when $a=-2b$

$\Rightarrow \quad \log(4-x) = -2\log\left(x+\frac{1}{2}\right)$

$\Rightarrow \quad (4-x) = \left(x+\frac{1}{2}\right)^{-2}$

$\Rightarrow \quad (4-x) = \frac{1}{\left(x+\frac{1}{2}\right)^2}$

$\Rightarrow \quad (4-x)\left(x+\frac{1}{2}\right)^2 = 1$

$\Rightarrow \quad (4-x)\left(x^2+x+\frac{1}{4}\right) = 1$

$\Rightarrow \quad (4-x)(4x^2+4x+1) = 4$

$\Rightarrow \quad 16x^2+16x+4-4x^3-4x^2-x = 4$

$\Rightarrow \quad 12x^2+15x-4x^3 = 0$

$\Rightarrow \quad x(4x^2-12x-15) = 0$

$\Rightarrow \quad x = 0, 4x^2-12x-15 = 0$

$\Rightarrow \quad x = 0, x = \frac{12\pm\sqrt{144+240}}{8}$

$\Rightarrow \quad x = 0, x = \frac{12\pm\sqrt{384}}{8}$

$\Rightarrow \quad x = 0, x = \frac{12\pm 4\sqrt{24}}{8}$

$\Rightarrow \quad x = 0, x = \frac{3\pm\sqrt{24}}{2}$

$\Rightarrow \quad x = 0, x = \frac{3+\sqrt{24}}{2}$

Hence, the solutions are $\left\{0, \frac{7}{4}, \frac{3+\sqrt{24}}{2}\right\}$

8. $\log_{3/4}\log_8(x^2+7) + \log_{1/2}\log_{1/4}(x^2+7)^{-1} = -2$

$\Rightarrow \log_{3/4}\log_{2^3}(x^2+7) + \log_{2^{-1}}\log_{2^{-2}}(x^2+7)^{-1} = -2$

$\Rightarrow \log_{3/4}\left(\frac{1}{3}(\log_2(x^2+7))\right) - \log_2\left(\frac{1}{2}(\log_2(x^2+7))\right) = -2$

$\Rightarrow \log_{3/4}\left(\frac{1}{3}y\right) - \log_2\left(\frac{1}{2}y\right) = -2, y = (\log_2(x^2+7))$

$\Rightarrow \log_{3/4}\left(\frac{y}{3}\right) - \log_2\left(\frac{y}{2}\right) = -2$

$\Rightarrow y = 4$, by trial.

$\Rightarrow \log_2(x^2+7) = 4$

$\Rightarrow (x^2+7) = 2^4 = 16$

$\Rightarrow x^2 = 9$

$\Rightarrow x = \pm 3$

Hence, the solutions are $\{-3, 3\}$

9. Given equation is

$\log_{10}(x^2-x-6) - x = \log_{10}(x+2) - 4$

$\Rightarrow \quad \log_{10}\left(\frac{x^2-x-6}{x+2}\right) = (x-4)$

$\Rightarrow \quad \log_{10}\left(\frac{(x-3)(x+2)}{x+2}\right) = (x-4)$

$\Rightarrow \quad \log_{10}((x-3)) = (x-4)$

$\Rightarrow \quad (x-3) = 10^{(x-4)}$

Clearly $x = 4$ is the required solution by trial.

10. Given equation is

$\frac{1}{2}\log_5(x+5) + \log_5\left(\sqrt{x-3}\right) = \frac{1}{2}\log_5(2x+1)$

$\Rightarrow \quad \frac{1}{2}\log_5(x+5) + \frac{1}{2}\log_5(x-3) = \frac{1}{2}\log_5(2x+1)$

$\Rightarrow \quad \log_5(x+5) + \log_5(x-3) = \log_5(2x+1)$

$\Rightarrow \quad \log_5\{(x+5)(x-3)\} = \log_5(2x+1)$

$\Rightarrow \quad (x+5)(x-3) = (2x+1)$

$\Rightarrow \quad x^2+2x-15 = (2x+1)$

$\Rightarrow \quad x^2 = 16$

$\Rightarrow \quad x = \pm 4$

$\Rightarrow \quad x = 4$ is the required solution

11. Given equation is

$\frac{3}{2}\log_4(x+2)^2 + 3 = \log_4(4-x)^3 + \log_4(6+x)^3$

$\Rightarrow \quad 3\log_4(x+2) + 3 = 3\log_4(4-x) + 3\log_4(6+x)$

$\Rightarrow \quad \log_4(x+2) + 1 = \log_4(4-x) + \log_4(6+x)$

$\Rightarrow \quad \log_4\{4(x+2)\} = \log_4(4-x)(6+x)$

$\Rightarrow \quad 4(x+2) = (4-x)(6+x)$

$\Rightarrow \quad 4x+8 = 24-2x-x^2$

$\Rightarrow \quad x^2+6x-16 = 0$

$\Rightarrow \quad (x+8)(x-2)=0$

$\Rightarrow \quad x=2,-8$

Hence, the solution is $x = 2$.

12. The given equation is

$$\frac{1+\log_2(x-4)}{2\log_2\left(\sqrt{x+3}-\sqrt{x-3}\right)}=1$$

$\Rightarrow \quad 1+\log_2(x-4)=2\log_2\left(\sqrt{x+3}-\sqrt{x-3}\right)$

$\Rightarrow \quad \log_2\{2(x-4)\}=\log_2\left(\sqrt{x+3}-\sqrt{x-3}\right)^2$

$\Rightarrow \quad 2(x-4)=\left(\sqrt{x+3}-\sqrt{x-3}\right)^2$

$\Rightarrow \quad 2(x-4)=x+3+x-3-2\sqrt{x^2-9}$

$\Rightarrow \quad 2\sqrt{x^2-9}=-8$

$\Rightarrow \quad \sqrt{x^2-9}=-4$

$\Rightarrow \quad x^2-9=16$

$\Rightarrow \quad x^2=25$

$\Rightarrow \quad x=\pm 5$

$\Rightarrow \quad x=5$

Hence, the solution is $x = 5$

13. Given equation is

$$\left(1+\frac{1}{2x}\right)\log 3=\log\left(\frac{\sqrt[x]{3}+27}{4}\right)$$

$\Rightarrow \quad \log 3^{\left(1+\frac{1}{2x}\right)}=\log\left(\frac{\sqrt[x]{3}+27}{4}\right)$

$\Rightarrow \quad 3^{\left(1+\frac{1}{2x}\right)}=\left(\frac{\sqrt[x]{3}+27}{4}\right)$

$\Rightarrow \quad 4.3^{\left(1+\frac{1}{2x}\right)}=3^{\frac{1}{x}}+27$

$\Rightarrow \quad 4.3.3^{\frac{1}{2x}}=3^{\frac{1}{x}}+27$

$\Rightarrow \quad 12.a=a^2+27,\, a=3^{\frac{1}{2x}}$

$\Rightarrow \quad 12.a=a^2+27$

$\Rightarrow \quad a^2-12a+27=0$

$\Rightarrow \quad (a-3)(a-9)=0$

$\Rightarrow \quad a=3,9$

$\Rightarrow \quad 3^{\frac{1}{2x}}=3,3^2$

$\Rightarrow \quad x=\frac{1}{2},\frac{1}{4}$

Hence, the solutions are $\left\{\frac{1}{2},\frac{1}{4}\right\}$

14. Given equation is

$$4^{\log_{10}x+1}-6^{\log_{10}x}-2.3^{\log_{10}x^2+2}=0$$

$\Rightarrow \quad 4.4^{\log_{10}x}-6^{\log_{10}x}-2.3^2.3^{\log_{10}x^2}=0$

$\Rightarrow \quad 4.4^a-6^a-18.3^{2a}=0\ ,a=\log_{10}x$

$\Rightarrow \quad 4.4^a-6^a-18.3^{2a}=0$

$\Rightarrow \quad 4\left(2^a\right)^2-2^a.3^a-18\left(3^a\right)^2=0$

$\Rightarrow \quad 4b^2-bc-18c^2=0, b=\left(2^a\right), c=3^a$

$\Rightarrow \quad 4b^2-bc-18c^2=0$

$\Rightarrow \quad 4b^2-9bc+8bc-18c^2=0$

$\Rightarrow \quad b(4b-9c)+2c(4b-9c)=0$

$\Rightarrow \quad (4b-9c)(b+2c)=0$

$\Rightarrow \quad (4b-9c)=0,\ (b+2c)=0$

$\Rightarrow \quad 4b=9c\, ,b=-2c$

$\Rightarrow \quad 4.2^a=9.3^a\, ,2^a=-2.3^a$

$\Rightarrow \quad 4.2^a=9.3^a$

$\Rightarrow \quad 2^{a+2}=3^{a+2}$

$\Rightarrow \quad \left(\frac{2}{3}\right)^{a+2}=1=\left(\frac{2}{3}\right)^0$

$\Rightarrow \quad a+2=0$

$\Rightarrow \quad a=-2$

$\Rightarrow \quad \log_{10}x=-2$

$\Rightarrow \quad x=10^{-2}=\frac{1}{100}$

Hence, the solution is $x=10^{-2}=\frac{1}{100}$

15. Given equation is

$$\log_3\left(\sqrt{x}+\left|\sqrt{x}-1\right|\right)^2=\log_3\left(4\sqrt{3}-3+4\left|\sqrt{x}-1\right|\right)$$

$\Rightarrow \quad \left(\sqrt{x}+\left|\sqrt{x}-1\right|\right)^2=\left(4\sqrt{3}-3+4\left|\sqrt{x}-1\right|\right)$

$\Rightarrow \quad \left(\sqrt{x}+2\sqrt{x}\left|\sqrt{x}-1\right|+x-2\sqrt{x}+1\right)$

$=\left(4\sqrt{3}-3+4\left|\sqrt{x}-1\right|\right)$

$\Rightarrow \quad 2\sqrt{x}\left(\sqrt{x}-1\right)+2\sqrt{x}\left(\left|\sqrt{x}-1\right|\right)=4\left(\sqrt{x}-1\right)$

$+4\left|\sqrt{x}-1\right|$

$\Rightarrow \quad 2\sqrt{x}\left(\sqrt{x}-1\right)+2\left|\sqrt{x}-1\right|=4\left(\left(\sqrt{x}-1\right)+\left|\sqrt{x}-1\right|\right)$

$\Rightarrow \quad 2\sqrt{x}\left(\left(\sqrt{x}-1\right)+\left|\sqrt{x}-1\right|\right)=4\left(\left(\sqrt{x}-1\right)+\left|\sqrt{x}-1\right|\right)$

$\Rightarrow \quad \left(2\sqrt{x}-4\right)\left(\left(\sqrt{x}-1\right)+\left|\sqrt{x}-1\right|\right)=0$

$\Rightarrow \quad \left(2\sqrt{x}-4\right)=0, \left(\left(\sqrt{x}-1\right)+\left|\sqrt{x}-1\right|\right)=0$

$\Rightarrow \quad \sqrt{x}=2, \left(\left(\sqrt{x}-1\right)+\left|\sqrt{x}-1\right|\right)=0$

$\Rightarrow \quad \sqrt{x}=2, \left(\sqrt{x}-1\right)<0$

$\Rightarrow \quad x=4, \sqrt{x}<1$

$\Rightarrow \quad x=4, 0<x<1$

$\Rightarrow \quad x\in(0,1)\cup\{4\}$

Hence, the solution set is $(0,1)\cup\{4\}$

INTEGER TYPE QUESTIONS

1. We have $\sum_{r=0}^{n-1}\log_2\left(\frac{r+2}{r+1}\right)=\prod_{r=10}^{99}\log_r(r+1)$

$\log_2\left(\frac{2}{1}\right)+\log_2\left(\frac{3}{2}\right)+\log_2\left(\frac{4}{3}\right)+...+\log_2\left(\frac{n+1}{n}\right)$

$=\log_{10}(11).\log_{11}(12)\log_{12}(13)....\log_{99}(100)$

$=\log_{10}(100)$

Thus,

$\log_2\left(\frac{2}{1}\right)+\log_2\left(\frac{3}{2}\right)+\log_2\left(\frac{4}{3}\right)+...+\log_2\left(\frac{n+1}{n}\right)$

$=\log_{10}(100)$

$\Rightarrow \quad \log_2\left(\frac{2}{1}.\frac{3}{2}.\frac{4}{3}.....\frac{n+1}{n}\right)=2\log_{10}(10)=2$

$\Rightarrow \quad \log_2\left(\frac{n+1}{1}\right)=2$

$\Rightarrow \quad n+1=4$

$\Rightarrow \quad n=3.$

2. Given equation is $7^{\log_2 x}=98-x^{\log_2 7}$

$\Rightarrow \quad x^{\log_2 7}=98-x^{\log_2 7}$

$\Rightarrow \quad 2x^{\log_2 7}=98$

$\Rightarrow \quad x^{\log_2 7}=49$

$\Rightarrow \quad 7^{\log_2 x}=7^2$

$\Rightarrow \quad \log_2 x=2$

$\Rightarrow \quad x=2^2=4$

Hence, the solution is $x = 4$.

3. Given equation is

$4^{\log_3 x}=32-4^{\log_3 x}$

$\Rightarrow \quad 2.4^{\log_3 x}=32$

$\Rightarrow \quad 4^{\log_3 x}=16=4^2$

$\Rightarrow \quad \log_3 x=2$

$\Rightarrow \quad x=3^2=9$

Hence, the solution is $x = 3$

4. Given equation is

$3\log_x 4+2\log_{4x}4+3\log_{16x}4=0$

$\Rightarrow \quad \frac{3}{\log_4 x}+\frac{2}{\log_4(4x)}+\frac{3}{\log_4(16x)}=0$

$\Rightarrow \quad \frac{3}{\log_4 x}+\frac{2}{1+\log_4 x}+\frac{3}{2+\log_4 x}=0$

$\Rightarrow \quad \frac{3}{a}+\frac{2}{1+a}+\frac{3}{2+a}=0, a=\log_4 x$

$\Rightarrow \quad \frac{3}{a}+\frac{3}{2+a}=-\frac{2}{1+a}$

$\Rightarrow \quad \frac{6+3a+3a}{a(2+a)}=-\frac{2}{1+a}$

$\Rightarrow \quad \frac{6(1+a)}{a(2+a)}=-\frac{2}{1+a}$

$\Rightarrow \quad 6(1+a)^2+2a(a+2)=0$

$\Rightarrow \quad 6a^2+12a+6+2a^2+4a=0$

$\Rightarrow \quad 8a^2+16a+6=0$

$\Rightarrow \quad 4a^2+8a+3=0$

$\Rightarrow \quad 4a^2+6a+2a+3=0$

$\Rightarrow \quad 2a(2a+3)+1(2a+3)=0$

$\Rightarrow \quad (2a+3)(2a+1)=0$

$\Rightarrow \quad a=-\frac{3}{2},-\frac{1}{2}$

$\Rightarrow \quad \log_4 x=-\frac{3}{2},-\frac{1}{2}$

$\Rightarrow \quad x=4^{-3/2},4^{-1/2}$

$\Rightarrow \quad \alpha=\frac{1}{8}, \beta=\frac{1}{2}$

Hence, the value of $\frac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$

$=\frac{1}{2}(8+2)=5$

5. Given equation is

$$x+\log_{10}\left(2^x+1\right)=\log_{10}6+x\log_{10}5$$

$$\Rightarrow \quad x+\log_{10}\left(2^x+1\right)=\log_{10}\left(6.5^x\right)$$

$$\Rightarrow \quad \log_{10}\left(\frac{6.5^x}{2^x+1}\right)=x$$

$$\Rightarrow \quad \left(\frac{6.5^x}{2^x+1}\right)=10^x$$

$$\Rightarrow \quad 10^x\left(2^x+1\right)=6.5^x$$

$$\Rightarrow \quad 2^x\left(2^x+1\right)=6$$

$$\Rightarrow \quad \left(2^x\right)^2+\left(2^x\right)-6=0$$

$$\Rightarrow \quad a^2+a-6=0\ , a=\left(2^x\right)$$

$$\Rightarrow \quad (a+3)(a-2)=0$$

$$\Rightarrow \quad a=2,-3$$

$$\Rightarrow \quad 2^x=2,-3$$

$$\Rightarrow \quad 2^x=2$$

$$\Rightarrow \quad x=1$$

Hence, the solution is $x = 1$

6. We have

$$\log_{(x+1)}\left(x^2+x-6\right)^2=4$$

$$\left(x^2+x-6\right)^2=(x+1)^4$$

$$x^4+2x^3-11x^2-12x+36$$

$$=x^4+4x^3+6x^2+4x+1$$

$$2x^3+17x^2+16x-35=0$$

$$2x^3-2x^2+19x^2-19x+35x-35=0$$

$$(x-1)\left(2x^2+19x+35\right)=0$$

$$(x-1)=0$$

$$x=1$$

7. Given equation is

$$|x-2|^{\log_2\left(x^3\right)-3\log_x 4}=(x-2)^3$$

$$\Rightarrow \quad \log_2\left(x^3\right)-3\log_x 4=3$$

$$\Rightarrow \quad 3\log_2 x-\frac{3}{\log_4 x}=3$$

$$\Rightarrow \quad \log_2 x-\frac{2}{\log_2 x}=1$$

$$\Rightarrow \quad a-\frac{2}{a}=1\ , a=\log_2 x$$

$$\Rightarrow \quad a^2-a-2=0$$

$$\Rightarrow \quad (a-2)(a+1)=0$$

$$\Rightarrow \quad a=2,-1$$

$$\Rightarrow \quad \log_2 x=2,-1$$

$$\Rightarrow \quad x=4,\frac{1}{2}$$

Hence, the value of $(\alpha+2\beta+3)$ is 8.

8. Given equation is

$$6\left(\log_x 2-\log_4 x\right)+7=0$$

$$\Rightarrow \quad 6\left(\log_x 2-\frac{1}{\log_x 4}\right)+7=0$$

$$\Rightarrow \quad 6\left(\log_x 2-\frac{1}{2\log_x 2}\right)+7=0$$

$$\Rightarrow \quad 6\left(a-\frac{1}{2a}\right)+7=0, a=\log_x 2$$

$$\Rightarrow \quad 6\left(2a^2-1\right)+14a=0$$

$$\Rightarrow \quad 3\left(2a^2-1\right)+7a=0$$

$$\Rightarrow \quad 6a^2+7a-3=0$$

$$\Rightarrow \quad 6a^2+9a-2a-3=0$$

$$\Rightarrow \quad 3a(2a+3)-(2a+3)=0$$

$$\Rightarrow \quad (3a-1)(2a+3)=0$$

$$\Rightarrow \quad a=\frac{1}{3},-\frac{3}{2}$$

$$\Rightarrow \quad \log_x 2=\frac{1}{3},-\frac{3}{2}$$

$$\Rightarrow \quad \frac{1}{\log_2 x}=\frac{1}{3},-\frac{3}{2}$$

$$\Rightarrow \quad \log_2 x=3,-\frac{2}{3}$$

$$\Rightarrow \quad x=2^3,2^{-2/3}$$

Thus, the integral solution of x is 8.
Therefore, $\alpha=8$

Hence, the value of $\left(\frac{2\alpha-1}{5}\right)$

$$=\left(\frac{16-1}{5}\right)=3\ .$$

9. We have $N=6\log_{10}2+\log_{10}31$

$$=\log_{10}2^6+\log_{10}31$$

$= \log_{10}(64\times 31)$

$= \log_{10}(1984)$

$< \log_{10}(1000) = 3$

Also, $N = \log_{10}(1984) > \log_{10}(10000) = 4$.

Thus, the sum of successive integers

= 3 + 4 = 7.

10. Let $S = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ... = \frac{1/3}{1-(1/3)} = \frac{1}{2}$

We have

$(0.16)^{\log_{2.5}\left(\frac{1}{2}\right)}$

$= \left(\frac{4}{25}\right)^{\log_{\frac{5}{2}}\left(\frac{1}{2}\right)}$

$= \left(\frac{2}{5}\right)^{2\log_{\frac{5}{2}}\left(\frac{1}{2}\right)}$

$= \left(\frac{2}{5}\right)^{-2\log_{\frac{2}{5}}\left(\frac{1}{2}\right)}$

$= \left(\frac{2}{5}\right)^{\log_{\frac{2}{5}}\left(\frac{1}{2}\right)^{-2}} = \left(\frac{1}{2}\right)^{-2} = 4$

11. We have ab

$= \log_{12} 18 \times \log_{24} 54$

$= \frac{\log 18}{\log 12} \times \frac{\log 54}{\log 24}$

$= \frac{\log(2.3^2)}{\log(3.2^2)} \times \frac{\log(2.3^3)}{\log(3.2^3)}$

$= \frac{\log 2 + 2\log 3}{\log(3) + 2\log 2} \times \frac{\log 2 + 3\log 3}{\log 3 + 3\log 2}$

$= \frac{1 + 2\log_2 3}{\log_2(3) + 2} \times \frac{1 + 3\log_2 3}{\log_2 3 + 3}$

$= \frac{1+2x}{x+2} \times \frac{1+3x}{x+3}, x = \log_2 3$

$= \frac{1+5x+6x^2}{x^2+5x+6}$

Also, $(a-b)$

$= \log_{12} 18 - \log_{24} 54$

$= \frac{\log(2.3^2)}{\log(3.2^2)} - \frac{\log(2.3^3)}{\log(3.2^3)}$

$= \frac{\log 2 + 2\log 3}{\log(3) + 2\log 2} - \frac{\log 2 + 3\log 3}{\log 3 + 3\log 2}$

$= \frac{1 + 2\log_2 3}{\log_2(3) + 2} - \frac{1 + 3\log_2 3}{\log_2 3 + 3}$

$= \frac{1+2x}{x+2} - \frac{1+3x}{x+3}, x = \log_2 3$

$= \frac{(1-x^2)}{(2+x)(3+x)}$

Hence, the value of $5(a-b) + ab$

$= \frac{5(1-x^2)}{(2+x)(3+x)} + \frac{(1+5x+6x^2)}{(2+x)(3+x)}$

$= \frac{5(1-x^2) + (1+5x+6x^2)}{(2+x)(3+x)}$

$= \frac{x^2+5x+6}{(2+x)(3+x)}$

$= \frac{(x+2)(x+3)}{(2+x)(3+x)}$

= 1.

QUESTIONS ASKED IN PAST IIT-JEE EXAMS

1. The given equation is

$2\log_x a + \log_{ax} a + 3\log_{a^2 x} a = 0$

$\Rightarrow \quad \frac{2\log a}{\log x} + \frac{\log a}{\log(ax)} + \frac{3\log a}{\log(a^2 x)} = 0$

$\Rightarrow \quad \frac{2}{\log x} + \frac{1}{\log a + \log x} + \frac{3}{2\log a + \log x} = 0$

$\Rightarrow \quad \frac{2}{y} + \frac{1}{b+y} + \frac{3}{2b+y} = 0,$

where $\log a = b, \log x = y$

$\Rightarrow \quad 6y^2 + 11by + 4b^2 = 0$

$\Rightarrow \quad 6y^2 + 3by + 8by + 4b^2 = 0$

$\Rightarrow \quad (2y+b)(3y+4b) = 0$

$\Rightarrow \quad y = -\frac{b}{2}, -\frac{4b}{3}$

when $y = -\frac{b}{2}$, then $\log x = -\frac{\log a}{2}$

$\Rightarrow \quad \log x^2 = -\log a = \log\left(\frac{1}{a}\right)$

$\Rightarrow \quad x^2 = \frac{1}{a}$

$\Rightarrow \quad x = a^{-1/2}$

when $y = -\frac{4b}{3}$, then $\log x = -\frac{4\log a}{3}$

$\Rightarrow \quad 3\log x = 4\log\left(\frac{1}{a}\right)$

$\Rightarrow \quad \log x^3 = \log\left(\frac{1}{a}\right)^4$

$\Rightarrow \quad x^3 = \left(\frac{1}{a}\right)^4$

$\Rightarrow \quad x = a^{-4/3}$.

2. We have $2\log_{10} x - \log_x(0.01), x > 1$

$= 2\log_{10} x - \log_x (10)^{-2}$

$= 2(\log_{10} x + \log_x (10))$

$\geq 2.2 = 4$

Thus, the least value is 4.

3. The curve $y = 10^x$ is the reflection of $y = \log_{10} x$ with respect to the line $y = x$.

4. We have $\log_x a + \log_a x$

$= \left(\log_x a + \frac{1}{\log_x a}\right) \geq 2$

Thus, the minimum value of the given expression is 2.

5. We have $\log_{0.3}(x-1) < \log_{0.09}(x-1)$

$\Rightarrow \quad \log_{(0.3)}(x-1) < \log_{(0.3)^2}(x-1)$

$\Rightarrow \quad \log_{(0.3)}(x-1) < \frac{1}{2}\log_{(0.3)}(x-1)$

$\Rightarrow \quad 2\log_{(0.3)}(x-1) < \log_{(0.3)}(x-1)$

$\Rightarrow \quad \log_{(0.3)}(x-1)^2 < \log_{(0.3)}(x-1)$

$\Rightarrow \quad (x-1)^2 > (x-1)$

$\Rightarrow \quad (x-1)^2 - (x-1) > 0$

$\Rightarrow \quad (x-1)(x-1-1) > 0$

$\Rightarrow \quad (x-1)(x-2) > 0$

$\Rightarrow \quad x < 1, x > 2$

$\Rightarrow \quad x > 2$ [$x < 1$ does not satisfy the given in-equation]

Thus, $x \in (2, \infty)$.

6 The given equation is

$\log_7\left(\log_5\left(\sqrt{x+5} + \sqrt{x}\right)\right) = 0$

$\Rightarrow \quad \left(\log_5\left(\sqrt{x+5} + \sqrt{x}\right)\right) = 7^0 = 1$

$\Rightarrow \quad \left(\sqrt{x+5} + \sqrt{x}\right) = 5^1 = 5$

$\Rightarrow \quad \sqrt{x+5} = 5 - \sqrt{x}$

$\Rightarrow \quad x + 5 = 25 - 10\sqrt{x} + x$

$\Rightarrow \quad 10\sqrt{x} = 20$

$\Rightarrow \quad \sqrt{x} = 2$

$\Rightarrow \quad x = 4$.

Hence, the solution is $x = 4$

7. The given equation is

$\Rightarrow \log_{(2x+3)}(6x^2 + 23x + 21) = 4 - \log_{(3x+7)}(4x^2 + 12x + 9)$

$\Rightarrow \quad \log_{(2x+3)}(2x+3)(3x+7) = 4 - \log_{(3x+7)}(2x+3)^2$

$\Rightarrow \quad 1 + \log_{(2x+3)}(3x+7) = 4 - 2\log_{(3x+7)}(2x+3)$

$\Rightarrow \quad \log_{(2x+3)}(3x+7) = 3 - \frac{2}{\log_{(2x+3)}(3x+7)}$

$\Rightarrow \quad y = 3 - \frac{2}{y}$, where $y = \log_{(2x+3)}(3x+7)$

$\Rightarrow \quad y^2 - 3y + 2 = 0$

$\Rightarrow \quad (y-1)(y-2) = 0$

$\Rightarrow \quad y = 1, 2$

when $y = 1$, then $\log_{(2x+3)}(3x+7) = 1$

$\Rightarrow \quad (3x+7) = (2x+3)$

$\Rightarrow \quad x = -4$

when $y = 2$, then $\log_{(2x+3)}(3x+7) = 2$

$\Rightarrow \quad (3x+7) = (2x+3)^2$

$\Rightarrow \quad (3x+7) = 4x^2 + 12x + 9$

$\Rightarrow \quad 4x^2 + 9x + 2 = 0$

$\Rightarrow \quad 4x^2 + 8x + x + 2 = 0$

$\Rightarrow \quad 4x(x+2) + 1(x+2) = 0$

$\Rightarrow \quad (x+2)(4x+1) = 0$

$\Rightarrow \quad x = -2, -\frac{1}{4}$

As $x > -\frac{3}{2}$, so $x = -\frac{1}{4}$

Hence, the solution is $x = -\frac{1}{4}$

8. We have

$x^{(3/4)(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$

$\Rightarrow \left((3/4)(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right)\log x = \log\left(\sqrt{2}\right)$

$\Rightarrow \left((3/4)b^2 + b - \frac{5}{4}\right)b = \frac{1}{2},\ b = \log_2 x$

$\Rightarrow 3b^3 + 4b^2 - 5b - 2 = 0$

$\Rightarrow 3b^3 - 3b^2 + 7b^2 - 7b + 2b - 2 = 0$

$\Rightarrow 3b^2(b-1) + 7b(b-1) + 2(b-1) = 0$

$\Rightarrow (b-1)(3b^2 + 7b + 2) = 0$

$\Rightarrow (b-1)(3b^2 + 6b + b + 2) = 0$

$\Rightarrow (b-1)(3b(b+2) + 1(b+2)) = 0$

$\Rightarrow (b-1)(b+2)(3b+1) = 0$

$\Rightarrow b = 1, -2, -\frac{1}{3}$

$\Rightarrow \log_2 x = 1, -2, -\frac{1}{3}$

$\Rightarrow x = 2, 2^{-2}, 2^{-\frac{1}{3}}$

Thus, the equation has exactly three real solutions of which exactly one is irrational.

9. Given $\log_3 2, \log_3(2^x - 5), \log_3\left(2^x - \frac{7}{2}\right) \in A.P.$

$\Rightarrow 2\log_3(2^x - 5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$

$\Rightarrow \log_3(2^x - 5)^2 = \log_3 2.\left(2^x - \frac{7}{2}\right)$

$\Rightarrow (2^x - 5)^2 = 2.\left(2^x - \frac{7}{2}\right)$

$\Rightarrow (2^x)^2 - 10.2^x + 25 = 2.2^x - 7$

$\Rightarrow (2^x)^2 - 12.2^x + 32 = 0$

$\Rightarrow (a)^2 - 12.a + 32 = 0$

$\Rightarrow (a-8)(a-4) = 0$

$\Rightarrow a = 8, 4$

$\Rightarrow 2^x = 8, 4 = 2^3, 2^2$

$\Rightarrow x = 3, 2$

$\Rightarrow x = 3$, since $x = 2$ does not satisfy the logarithmic expression.

Hence, the solution is $x = 3$

10. Ans. (c)

Let $\log_2 7 = \frac{p}{q}$, where

$p, q \in N$ and $H.C.F(p,q) = 1$

$\Rightarrow 2^{\frac{p}{q}} = 7$

$\Rightarrow 2^p = 7^q$

This is not possible for any $p, q \in N$ and 7 and 2 are prime.

Thus, $\log_2 7$ is an irrational number.

11. Ans. (b)

The given equation is $\log_4(x-1) = \log_2(x-3)$

$\Rightarrow \frac{1}{2}\log_2(x-1) = \log_2(x-3)$

$\Rightarrow \log_2(x-1) = \log_2(x-3)^2$

$\Rightarrow (x-3)^2 = (x-1)$

$\Rightarrow x^2 - 6x + 9 = x - 1$

$\Rightarrow x^2 - 7x + 10 = 0$

$\Rightarrow (x-2)(x-5) = 0$

$\Rightarrow x = 2, 5$

$\Rightarrow x = 5$, since $x = 2$ does not satisfy the equation.

Thus, the number of solution is one.

12. Ans. (c)

The given equations are

$(2x)^{\ln 2} = (3y)^{\ln 3}$, $3^{\ln x} = 2^{\ln y}$

Now, $3^{\ln x} = 2^{\ln y}$

$\Rightarrow \log\left(3^{\ln x}\right) = \log\left(2^{\ln y}\right)$

$\Rightarrow \log x \log(3) = \log y \log(2)$

$\Rightarrow \frac{\log x}{\log 2} = \frac{\log(y)}{\log(3)} = \lambda(say)$

Also, $(2x)^{\ln 2} = (3y)^{\ln 3}$

$\Rightarrow \log\left((2x)^{\ln 2}\right) = \log\left((3y)^{\ln 3}\right)$

$\Rightarrow \log 2 \log(2x) = \log 3 \log(3y)$

$\Rightarrow \log 2(\log 2 + \log x) = \log 3(\log 3 + \log y)$

$\Rightarrow \log 2(\log 2 + \lambda \log 2) = \log 3(\log 3 + \lambda \log 3)$

$\Rightarrow (\log 2)^2 (1+\lambda) = (\log 3)^2 (1+\lambda)$

$\Rightarrow (1+\lambda)\left((\log 2)^2 - (\log 3)^2\right) = 0$

$\Rightarrow (1+\lambda) = 0, \ \left(\because (\log 2)^2 - (\log 3)^2 \neq 0\right)$

$\Rightarrow \lambda = -1$

Thus, $\log x = -\log 2, \ \log y = -\log 3$

$\Rightarrow \log x = \log\left(\frac{1}{2}\right), \ \log y = \log\left(\frac{1}{3}\right)$

$\Rightarrow x = \frac{1}{2}, \ y = \frac{1}{3}$

Hence, $x_0 = \frac{1}{2}$

13 Let

$$x = \sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}.......\text{to } \infty}}}$$

$\Rightarrow x = \sqrt{4 - \frac{1}{3\sqrt{2}}x}$

$\Rightarrow x^2 = \left(4 - \frac{1}{3\sqrt{2}}x\right).$

$\Rightarrow 3\sqrt{2}\,x^2 + x - 12\sqrt{2} = 0$

$\Rightarrow 3\sqrt{2}\,x^2 + 9x - 8x - 12\sqrt{2} = 0$

$\Rightarrow 3x\left(\sqrt{2}x + 3\right) - 4\sqrt{2}\left(\sqrt{2}x + 3\right) = 0$

$\Rightarrow \left(\sqrt{2}x + 3\right)\left(3x - 4\sqrt{2}\right) = 0$

$\Rightarrow x = -\frac{3}{\sqrt{2}}, \ \frac{4\sqrt{2}}{3}$

$\Rightarrow x = \frac{4\sqrt{2}}{3}$

Thus, $6 + \log_{3/2}$

$$\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}.......\text{to } \infty}}}\right)$$

$= 6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}} \times \frac{4\sqrt{2}}{3}\right)$

$= 6 + \log_{3/2}\left(\frac{4}{9}\right)$

$= 6 + \log_{\left(\frac{2}{3}\right)^{-1}}\left(\frac{2}{3}\right)^2$

$= 6 - 2\log_{\left(\frac{2}{3}\right)}\left(\frac{2}{3}\right)$

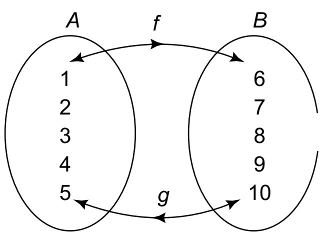
$= 6 - 2 = 4$

CHAPTER 6

Inverse Trigonometric Function

INVERSE FUNCTION

6.1 INTRODUCTION TO INVERSE FUNCTION



Let $f: X \rightarrow Y$ be a bijective function.

If we can make another function g from Y to X, then we shall say that g is the inverse of f.

i.e. $g = f^{-1} \neq \frac{1}{f}$

Thus, $f^{-1}\left(f\left(x\right)\right) = x$.

Note:

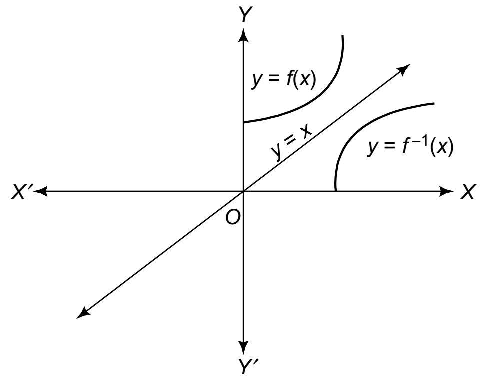
(i) The inverse of a function exists only when the function f is bijective.

(ii) If the inverse of a function exists, then it is called an invertible function.

(iii) The inverse of a bijective function is unique.

(iv) Geometrically, $f^{-1}\left(x\right)$ is the image of $f(x)$ with respect to the line $y = x$.

(v) Another way also we can say that $f^{-1}\left(x\right)$ is symmetrical with respect to the line $y = x$.



(vi) A function $f(x)$ is said to be involution if for all x for which $f(x)$ and $f(f(x))$ are defined such that $f(f(x)) = x$.

(vii) If f is an invertible function, then $\left(f^{-1}\right)^{-1} = f$.

(viii) If $f: A \rightarrow B$ be one one function, then $f^{-1} \text{o} f = I_A$ and $f \text{ o } f^{-1} = I_B$, where I_A and I_B are the identity functions of the sets A and B, respectively.

(ix) Let $f: A \rightarrow B$, $g: B \rightarrow C$ be two invertible functions, then $\left(g \text{ o } f\right)^{-1} = \left(f^{-1} \text{ o } g^{-1}\right)$ is also invertible with $f^{-1}\left(x\right)$.

RULE TO FIND OUT THE INVERSE OF A FUNCTION

(i) First, we check the given function is bijective or not.

(ii) If the function is bijective, then inverse exists, otherwise not.

(iii) Find x in terms of y.

(iv) And then replace y by x, then we get inverse of f. i.e. $f: R \rightarrow R$.

6.2 SOME SOLVED EXAMPLES

Ex-1. A function $f: R \to R$ is defined as $f(x) = 3x + 5$. Find $f^{-1}(x)$.

Soln. Given $f(x) = 3x + 5$

$\Rightarrow f'(x) = 3 > 0$

$\Rightarrow f$ is strictly increasing function.

$\Rightarrow f$ is one one function

Also, $R_f = R$ = co-domain

$\Rightarrow f$ is onto function.

Thus, f is a bijective function.

Hence, f^{-1} is exists.

Let $y = 3x + 5$

$\Rightarrow x = \dfrac{y-5}{3}$

Thus, $f^{-1}(x) = \dfrac{x-5}{3}$.

Ex-2. A function $f:(0,\infty) \to (2, \infty)$ is defined as $f(x) = x^2 + 2$. Then find $f^{-1}(x)$.

Soln. Given $f(x) = x^2 + 2$

$\Rightarrow f'(x) = 2x > 0$ for every $x > 0$

$\Rightarrow f$ is strictly increasing function.

$\Rightarrow f$ is one one function.

Also, $R_f = (2,\infty)$ = co-domain

$\Rightarrow f$ is ont function.

Thus, f is a bijective function.

Therefore, the inverse of the given function exists.

Let $y = x^2 + 2$

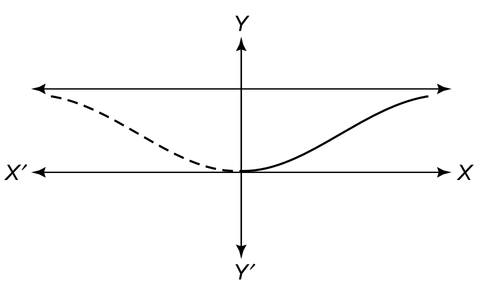
$\Rightarrow \quad x^2 = y - 2$

$\Rightarrow \quad x = \sqrt{y-2}$

Hence, $f^{-1}(x) = \sqrt{x-2}$.

Ex-3. A function $f: R^+ \to [0, 1)$ is defined as $f(x) = \dfrac{x^2}{x^2+1}$. Then find $f^{-1}(x)$.

Soln. Given $f(x) = \dfrac{x^2}{x^2+1}$



$\Rightarrow \quad f(x) = 1 - \dfrac{1}{x^2+1}$

$\Rightarrow \quad f'(x) = \dfrac{2x}{\left(x^2+1\right)^2} > 0, \ \forall \ x \in R^+$

$\Rightarrow f$ is strictly increasing function.

$\Rightarrow f$ is one one function.

Also, let $y = \dfrac{x^2}{x^2+1}$

$\Rightarrow \quad y.x^2 + y = x^2$

$\Rightarrow \quad x^2(y-1) = -y$

$\Rightarrow \quad x^2 = -\dfrac{y}{(y-1)} = \dfrac{y}{(1-y)}$

$\Rightarrow \quad x = \sqrt{\dfrac{y}{(1-y)}}$

$\Rightarrow \quad R_f = (0, 1)$ = co-domain

$\Rightarrow f$ is onto function.

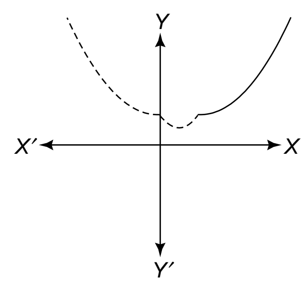
Thus, f is a bijective function.

$\Rightarrow f^{-1}(x)$ exists.

Hence, $f^{-1}(x) = \sqrt{\dfrac{x}{1-x}}$.

Ex-4. A function $f:[1, \infty) \to [1, \infty)$ is defined as $f(x) = 2^{x(x-1)}$. Find $f^{-1}(x)$.

Soln. Given $f(x) = 2^{x(x-1)}$.



$\Rightarrow \quad f'(x) = 2^{x(x-1)} \times (2x-1) \times \log_e 2 > 0$

for all x in $[1, \infty)$

$\Rightarrow f$ is strictly increasing function.

$\Rightarrow f$ is one one function.

Also, $R_f = [1, \infty)$

$\Rightarrow \quad R_f = [1, \infty)$ = co-domain

$\Rightarrow f$ is onto function.

Thus, f is a bijective function.

So its inverse exists.

Let $y = 2^{x(x-1)}$

$\Rightarrow \quad y = 2^{x^2 - x}$

$\Rightarrow \quad x^2 - x = \log_2(y)$

$\Rightarrow \quad x^2 - x - \log_2(y) = 0$

$\Rightarrow \quad x = \dfrac{1 \pm \sqrt{1 + 4\log_2(y)}}{2}$

$\Rightarrow \quad x = \dfrac{1 + \sqrt{1 + 4\log_2(y)}}{2}$

Thus, $f^{-1}(x) = \dfrac{1 + \sqrt{1 + 4\log_2(x)}}{2}$.

Ex-5. If a function f is bijective such that

$f(x) = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$, then find $f^{-1}(x)$.

Soln. Since f is a bijective function, so its inverse exists.

Let $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}} = \dfrac{10^{2x} - 1}{10^{2x} + 1}$.

$\Rightarrow \quad y.10^{2x} + y = 10^{2x} - 1$

$\Rightarrow \quad 10^{2x}(y - 1) = -y - 1$

$\Rightarrow \quad 10^{2x} = -\dfrac{y+1}{y-1} = \dfrac{y+1}{1-y}$

$\Rightarrow \quad 2x = \log_{10}\left(\dfrac{y+1}{1-y}\right)$

$\Rightarrow \quad x = \dfrac{1}{2}\log_{10}\left(\dfrac{1+y}{1-y}\right)$

Thus, $f^{-1}(x) = \dfrac{1}{2}\log_{10}\left(\dfrac{1+x}{1-x}\right)$.

Ex-6. A function $f : R \to R$ is defined as

$f(x) = x + \sin x$. Find $f^{-1}(x)$.

Soln. Given $f(x) = x + \sin x$

$\Rightarrow \quad f'(x) = 1 + \cos x \geq 0$ for all x in R.

$\Rightarrow f$ is strictly increasing function

$\Rightarrow f$ is one one function.

Also, the range of a function is R

$\Rightarrow f$ is a onto function

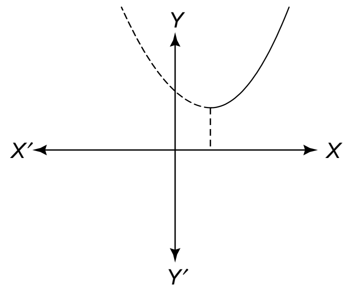
Thus, f is a bijective function.

Hence, f^{-1} exists.

Therefore, $f^{-1}(x) = x - \sin x$.

Ex-7. A function $f : [2, \infty) \to [5, \infty)$ is defined as $f(x) = x^2 - 4x + 9$. Find its inverse.

Soln. Given $f(x) = x^2 - 4x + 9$



$\Rightarrow \quad f'(x) = 2x - 4 \geq 0$ for all x in D_f

$\Rightarrow f$ is strictly increasing function.

$\Rightarrow f$ is one one function.

Also, $R_f = [5, \infty)$ = co-domain

$\Rightarrow f$ is onto function.

Therefore, f is a bijective function.

Hence, its inverse exists.

Let $y = x^2 - 4x + 5$

$\Rightarrow \quad x^2 - 4x + (5 - y) = 0$

$\Rightarrow \quad x = \dfrac{4 \pm \sqrt{16 - 4(5 - y)}}{2}$

$\Rightarrow \quad x = \dfrac{4 \pm \sqrt{4y + 16 - 20}}{2}$

$\Rightarrow \quad x = \dfrac{4 \pm \sqrt{4(y-1)}}{2} = 2 \pm \sqrt{(y-1)}$

$\Rightarrow \quad x = 2 + \sqrt{(y-1)}$, since $x \geq 2$

$\Rightarrow \quad f^{-1}(x) = 2 + \sqrt{(x-1)}$

Ex-8. Find all the real solutions to the equation

$x^2 - \dfrac{1}{4} = \sqrt{x + \dfrac{1}{4}}$

Soln. Consider the function

$f : [0, \infty) \to \left[-\dfrac{1}{4}, \infty\right)$, where

$f(x) = x^2 - \dfrac{1}{4}$

Clearly, f is one one and onto function.

So its inverse exists.

Let its inverse is $f^{-1} : \left[-\dfrac{1}{4}, \infty\right) \to [0, \infty)$.

$\Rightarrow f^{-1}(x) = \sqrt{x + \dfrac{1}{4}}$

Consequently, we can say that, the two sides of the given equation are inverse to each other.

Thus, the intersection point is the solution of the given equation, $f(x) = x$

$\Rightarrow \quad x^2 - \dfrac{1}{4} = x$

$\Rightarrow \quad x^2 - x = \frac{1}{4}$

$\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 = \frac{1}{2}$

$\Rightarrow \quad \left(x - \frac{1}{2}\right) = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow \quad x = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$

Hence, the solutions are $\left\{\frac{1}{2} + \frac{1}{\sqrt{2}}, \frac{1}{2} - \frac{1}{\sqrt{2}}\right\}$.

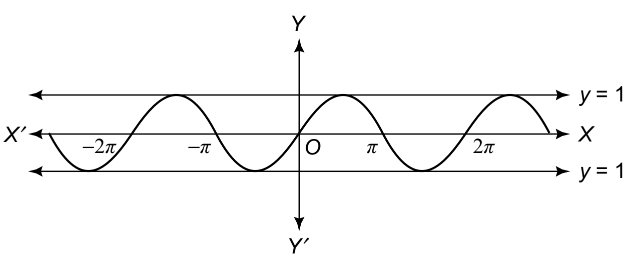
EXERCISE 1

1. A function *f* is defined as $f(x) = 3x + 5$ where $f : R \to R$, then find $f^{-1}(x)$.
2. A function *f* is defined as $f(x) = \frac{x}{x-1}$ where $f : R - \{1\} \to R - \{1\}$, then find $f^{-1}(x)$.
3. A function *f* is defined as $f(x) = \frac{1}{x^2+1}$ where $f : R^+ \cup \{0\} \to (0,1]$, find $f^{-1}(x)$.
4. A function *f* is bijective such that $f(x) = \frac{2^x - 2^{-x}}{2^x + 2^{-x}}$, then find $f^{-1}(x)$.
5. A function $f : [-1,1] \to \left[-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right]$ is defined as $f(x) = \frac{x}{x^2+1}$, then find $f^{-1}(x)$.

6.3 INVERSE TRIGONOMETRIC FUNCTIONS

As we know that sine function is defined only for every real number and the range of sine function is [–1, 1]. Thus, the graph of $f(x) = \sin(x)$ is as follows:

Graph of $f(x) = \sin x$:

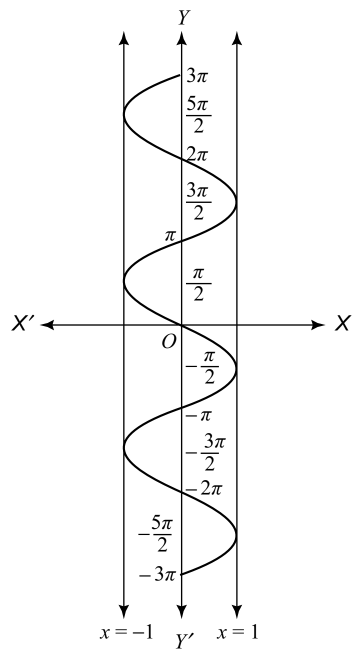


From the graph, we can say that, it will be one one and onto only when we considered it in some particular intervals like $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ and so on

If we consider the whole function, then it is not one one as well as onto.

Also, when we think the inverse function, then domain and range are interchanged.

So the graph of this function is as follows:



As a whole, inverse of this function does not exist. Its inverse exists only when, we restrict its range

So in thertvals $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$,

$\left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ and so on.

In the conventional of mathematics, we consider it in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

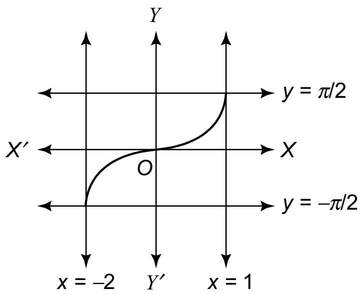
Thus, sin inverse function is defined as

$$\sin^{-1} : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Therefore, a function $f : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

is defined as $f(x) = \sin^{-1} x$.

So the graph of $f(x) = \sin^{-1} x$ is



Thus, $D_f = [-1,1]$ and $R_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

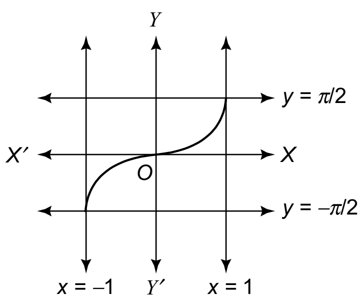
Now, we shall discuss the graphs of other inverse trigonometric functions and their characteristics.

6.4 GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

(i) $\sin^{-1} x$:

A function $f : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is defined as

$f(x) = \sin^{-1} x =$ arc $\sin x$ Graph of $f(x) = \sin^{-1} x$.



CHARACTERISTICS OF ARC SINE FUNCTION

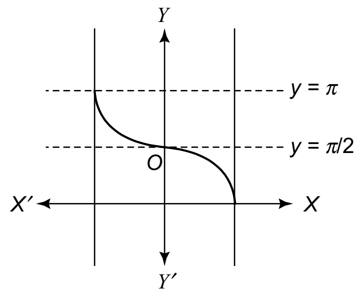
1. $D_f = [-1, 1]$
2. $R_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
3. It is not a periodic function.
4. It is an odd function.
   since, $\sin^{-1}(-x) = -\sin^{-1} x$.
5. It is strictly increasing function
6. It is one one function.
7. For $0 < x < \frac{\pi}{2}$, $\sin x < x < \sin^{-1} x$.

(ii) $\cos^{-1} x$:

A function $f : [-1,1] \to [0, \pi]$

is defined as $f(x) = \cos^{-1} x =$ arc cos x

Graph of $f(x) = \cos^{-1} x$:



CHARACTERISTICS OF ARC COSINE FUNCTION:

1. $D_f = [-1, 1]$
2. $[0, \pi]$
3. It is not a periodic function.
4. It is neither even nor odd function
   since, $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$.
5. It is strictly decreasing function.
6. It is one one function.
7. For $0 < x < \frac{\pi}{2}$,

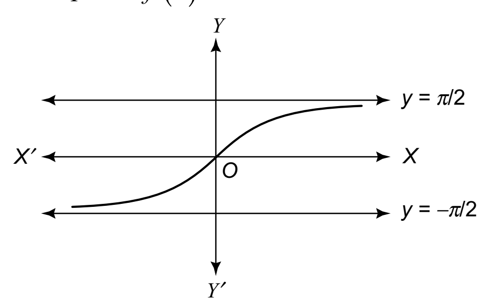
   $\cos^{-1} x < x < \cos x$

(iii) $\tan^{-1} x$:

A function $f : R \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is

defined as $f(x) = \tan^{-1} x$.

Graph of $f(x) = \tan^{-1} x$:



CHARACTERISTICS OF ARC TANGENT FUNCTION

1. $D_f = R$
2. $R_f = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
3. It is not a periodic function.
4. It is an odd function.
   since, $\tan^{-1}(-x) = -\tan^{-1} x$.
5. It is strictly increasing function.
6. It is one one function.

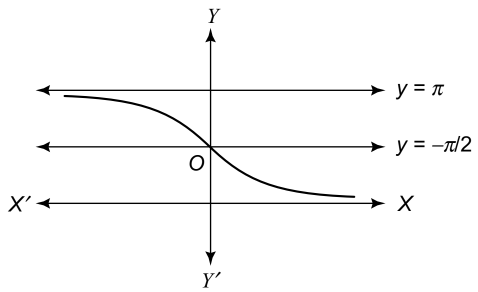
7. For $0 < x < \frac{\pi}{2}$, $\tan^{-1} x < x < \tan x$.

(iv) $\cot^{-1} x$:

A function $f : R \to (0, \pi)$ is

defined as $f(x) = \cot^{-1} x$.

Graph of $f(x) = \cot^{-1} x$



CHARACTERISTICS OF ARC CO-TANGENT FUNCTION:

1. $D_f = R$
2. $R_f = (0, \pi)$
3. It is not a periodic function.
4. It is neither even nor odd function since, $\cot^{-1}(-x) = \pi - \cot^{-1} x$.
5. It is strictly decreasing function.
6. It is one one function.
7. For $0 < x < \frac{\pi}{2}$,

   $\cot x < x < \cot^{-1} x$

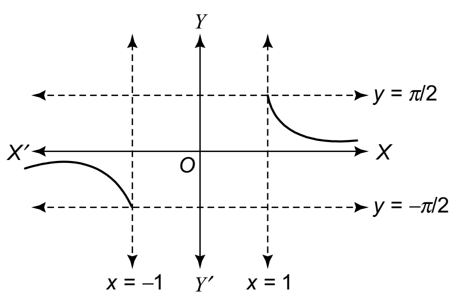
(v) $\text{cosec}^{-1} x$:

A function

$$f : (-\infty, -1] \cup [1, \infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

is defined as $f(x) = \text{cosec}^{-1} x$.

Graph of $f(x) = \text{cosec}^{-1} x$.



CHARACTERISTICS OF ARC CO-SECANT FUNCTION:

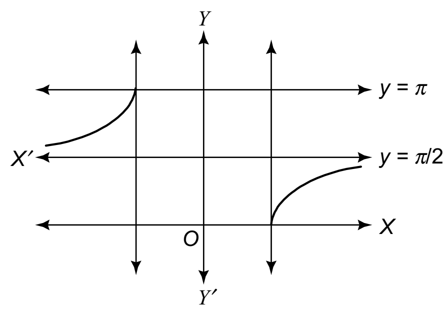
1. $D_f = (-\infty, -1] \cup [1, \infty)$
2. $R_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.
3. It is an odd function, since $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$.
4. It is non periodic function.
5. It is one one function.
6. It is strictly decreasing function with respect to its domain.
7. For $0 < x < \frac{\pi}{2}$, $\text{cosec}^{-1} x < x < \text{cosec}\, x$.

(v) $\sec^{-1} x$: A function

$$f : (-\infty, -1] \cup [1, \infty) \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$$

is defined as $f(x) = \sec^{-1} x$.

Graph of $f(x) = \sec^{-1} x$.



CHARACTERISTICS OF ARC SECANT FUNCTION:

1. $D_f = (-\infty, -1] \cup [1, \infty)$.
2. $R_f = [0, \pi] - \left\{\frac{\pi}{2}\right\}$.
3. It is neither an even function nor an odd odd function, since $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$.
4. It is non periodic function.
5. It is one one function.
6. It is strictly decreasing function with respect to its domain.
7. For $0 < x < \frac{\pi}{2}$, $\sec^{-1} x < x < \sec x$.

6.4.1 Some Solved Examples

Ex-1. Find the domain of $f(x) = \sin^{-1}(3x + 5)$.

Soln. We have $-1 \le 3x + 5 \le 1$

$\Rightarrow \quad -6 \le 3x \le -4$

$\Rightarrow \quad -2 \le x \le -\frac{4}{3}$

$\Rightarrow \quad D_f = x \in \left[-2, -\frac{4}{3}\right]$

Ex-2. Find the domain of $f(x)=\sin^{-1}\left(\frac{x}{x+1}\right)$

Soln. We have, $-1\le \frac{x}{x+1}\le 1$

Case I: when $\frac{x}{x+1}\le 1$

$\Rightarrow \quad \frac{x}{x+1}-1\le 0$

$\Rightarrow \quad \frac{-1}{x+1}\le 0$

$\Rightarrow \quad \frac{1}{x+1}\ge 0$

$\Rightarrow \quad x>-1$

Case II: When $\frac{x}{x+1}\ge -1$

$\Rightarrow \quad \frac{x}{x+1}+1\ge 0$

$\Rightarrow \quad \frac{2x+1}{x+1}\ge 0$

$\Rightarrow \quad x\in(-\infty,-1)\cup\left[-\frac{1}{2},\infty\right)$

Hence, $D_f=\left[-\frac{1}{2},\infty\right)$.

Ex-3. Find the domain of $f(x)=\sin^{-1}\left(\frac{x^2+1}{2x}\right)$

Soln. We have, $-1\le \frac{x^2+1}{2x}\le 1$

$\Rightarrow \quad \left|\frac{x^2+1}{2x}\right|\le 1$

$\Rightarrow \quad \frac{|x^2+1|}{|2x|}\le 1$

$\Rightarrow \quad \frac{|x^2+1|}{2|x|}\le 1$

$\Rightarrow \quad x^2+1\le 2|x|$

$\Rightarrow \quad |x|^2-2|x|+1\le 0$

$\Rightarrow \quad (|x|-1)^2\le 0$

$\Rightarrow \quad (|x|-1)^2=0$

$\Rightarrow \quad (|x|-1)=0$

$\Rightarrow \quad |x|=1$

$\Rightarrow \quad x=\pm 1$

Hence, $D_f=\{-1,1\}$

Ex-4. Find the domain of $f(x)=\sin^{-1}\left(\frac{|x|-1}{2}\right)$

Soln. We have, $-1\le \frac{|x|-1}{2}\le 1$

$\Rightarrow \quad -2\le |x|-1\le 2$

$\Rightarrow \quad -1\le |x|\le 3$

$\Rightarrow \quad |x|\le 3 \ (\because\ |x|\ge -1 \text{ is rejected})$

$\Rightarrow \quad -3\le x\le 3$

Hence, $D_f=[-3,3]$.

Ex-5. Find the domain of $f(x)=\sin^{-1}(\log_2 x)$

Soln. We have, $-1\le (\log_2 x)\le 1$

$\Rightarrow \quad 2^{-1}\le x\le 2^1$

$\Rightarrow \quad \frac{1}{2}\le x\le 2$

Hence, $D_f=\left[\frac{1}{2},2\right]$.

Ex-6. Find the domain of $f(x)=\sin^{-1}(\log_4 x^2)$

Soln. We have, $-1\le \log_4 x^2\le 1$

$\Rightarrow \quad 4^{-1}\le x^2\le 4^1$

$\Rightarrow \quad \frac{1}{4}\le x^2\le 4$

$\Rightarrow \quad \frac{1}{2}\le |x|\le 2$

$\Rightarrow \quad |x|\le 2 \ \&\ |x|\ge \frac{1}{2}$

$\Rightarrow \quad -2\le x\le 2$ and

$x\ge \frac{1}{2} \ \&\ x\le -\frac{1}{2}$

$\Rightarrow \quad x\in\left[-2,-\frac{1}{2}\right]\cup\left[\frac{1}{2},2\right]$

Hence, $D_f=\left[-2,-\frac{1}{2}\right]\cup\left[\frac{1}{2},2\right]$.

Ex-7. Solve for x and y: $\sin^{-1}x+\sin^{-1}y=\pi$

Soln. Given $\sin^{-1}x+\sin^{-1}y=\pi$

It is possible only when each term of the given equation provides the maximum value.

Thus, $\sin^{-1}x=\frac{\pi}{2}$ & $\sin^{-1}y=\frac{\pi}{2}$

$\Rightarrow \quad x=\sin\left(\frac{\pi}{2}\right)=1$ & $y=\sin\left(\frac{\pi}{2}\right)=1$

Hence, the solutions are $x=1$ and $y=1$

Ex-8. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then find the value of

$$x^{2013}+y^{2013}+z^{2013}-\frac{9}{x^{2014}+y^{2014}+z^{2014}}$$

Soln. Given $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$

It is possible only when each term will provide us the maximum value.

Thus, $\sin^{-1}x=\frac{\pi}{2}$, $\sin^{-1}y=\frac{\pi}{2}$

& $\sin^{-1}z=\frac{\pi}{2}$

$\Rightarrow \quad x=1,\ y=1\ \&\ z=1$.

Hence, the value of

$$x^{2013}+y^{2013}+z^{2013}-\frac{9}{x^{2014}+y^{2014}+z^{2014}}$$

$= 1 + 1 + 1 - \frac{9}{1+1+1}$

$= 3 - 3$

$= 0$.

Ex-9. Find the range of $f(x)=2\sin^{-1}(3x+5)+\frac{\pi}{4}$

Soln. We have $-\frac{\pi}{2}\le\sin^{-1}(3x+5)\le\frac{\pi}{2}$

$\Rightarrow \quad -\pi\le 2\sin^{-1}(3x+5)\le\pi$

$\Rightarrow -\pi+\frac{\pi}{4}\le 2\sin^{-1}(3x+5)+\frac{\pi}{4}\le\pi+\frac{\pi}{4}$

$\Rightarrow -\frac{3\pi}{4}\le f(x)\le\frac{5\pi}{4}$

Hence, $R_f=\left[-\frac{3\pi}{4},\frac{5\pi}{4}\right]$

Ex-10. Solve the inequality: $\sin^{-1}x>\sin^{-1}(3x-1)$

Soln. We have, $\sin^{-1}x>\sin^{-1}(3x-1)$

$\Rightarrow \quad x>(3x-1)$

$\Rightarrow \quad 2x-1<0$

$\Rightarrow \quad x<\frac{1}{2}$

$\Rightarrow \quad x\in\left[-1,\frac{1}{2}\right)$

Ex-11. Find the domain of $f(x)=\cos^{-1}(2x+4)$.

Soln. We have, $-1\le 2x+4\le 1$

$\Rightarrow \quad -5\le 2x\le -3$

$\Rightarrow \quad -\frac{5}{2}\le x\le -\frac{3}{2}$

Hence, $D_f=\left[-\frac{5}{2},-\frac{3}{2}\right]$

Ex-12. Find the range of $f(x)=2\cos^{-1}(3x+5)+\frac{\pi}{4}$

Soln. We have, $0\le\cos^{-1}(3x+5)\le\pi$

$\Rightarrow \quad 0\le 2\cos^{-1}(3x+5)\le 2\pi$

$\Rightarrow \quad \frac{\pi}{4}\le 2\cos^{-1}(3x+5)+\frac{\pi}{4}\le 2\pi+\frac{\pi}{4}$

Hence, $R_f=\left[\frac{\pi}{4},\frac{9\pi}{4}\right]$

Ex-13. Find the range of $f(x)=3\cos^{-1}\left(-x^2\right)-\frac{\pi}{2}$.

Soln. We have, $\frac{\pi}{2}\le\cos^{-1}\left(-x^2\right)\le\pi$

$\Rightarrow \quad \frac{3\pi}{2}\le 3\cos^{-1}\left(-x^2\right)\le 3\pi$

$\Rightarrow \quad \frac{3\pi}{2}-\frac{\pi}{2}\le 3\cos^{-1}\left(-x^2\right)-\frac{\pi}{2}\le 3\pi-\frac{\pi}{2}$

$\Rightarrow \quad \pi\le f(x)\le\frac{5\pi}{2}$

Hence, $R_f=\left[\pi,\frac{5\pi}{2}\right]$

Ex-14. Solve for x: $\cos^{-1}x+\cos^{-1}x^2=0$.

Soln. Given $\cos^{-1}x+\cos^{-1}x^2=0$

It is possible only when each term will provide us the minimum value.

So, $\cos^{-1}x=0$ & $\cos^{-1}x^2=0$

$\Rightarrow \quad x=1\ \&\ x^2=1$

$\Rightarrow \quad x=1\ \&\ x=\pm 1$

Hence, the solution is $x = 1$

Ex-15. Solve for x: $\left[\sin^{-1}x\right]+\left[\cos^{-1}x\right]=0$, where x is a non negative real number and [,] denotes the greatest integer function.

Soln. Given $\left[\sin^{-1}x\right]+\left[\cos^{-1}x\right]=0$ and $x\ge 0$

$\Rightarrow \quad \left[\sin^{-1}x\right]=0\ \&\ \left[\cos^{-1}x\right]=0$

$\Rightarrow \quad x\in[0,\sin 1)\ \&\ x\in(\cos 1, 1]$

$\Rightarrow \quad x\in(\cos 1,\sin 1)$

Ex-16. Find the domain of $f(x)=\cos^{-1}\left(\frac{x^2}{x^2+1}\right)$

Soln. We have, $-1\le\frac{x^2}{x^2+1}\le 1$

$\Rightarrow \quad \left|\dfrac{x^2}{x^2+1}\right| \le 1$

$\Rightarrow \quad \dfrac{|x^2|}{|x^2+1|} \le 1$

$\Rightarrow \quad \dfrac{x^2}{x^2+1} \le 1$

$\Rightarrow \quad x^2+1 \ge x^2$

$\Rightarrow \quad 1 > 0$

Hence, $x \in R$

Ex-17. Solve for x: $\cos^{-1}(x) > \cos^{-1}(x^2)$

Soln. We have, $\cos^{-1}(x) > \cos^{-1}(x^2)$

$\Rightarrow \quad x < x^2$

$\Rightarrow \quad x^2 - x > 0$

$\Rightarrow \quad x(x-1) > 0$

$\Rightarrow \quad x \in [-1, 0)$

Ex-18. Find the domain of $f(x) = \tan^{-1}\left(\sqrt{9-x^2}\right)$

Soln. Since $\tan^{-1} x$ is defined for all real values of x, so

$9 - x^2 \ge 0$

$\Rightarrow \quad x^2 - 9 \le 0$

$\Rightarrow \quad (x+3)(x-3) \le 0$

$\Rightarrow \quad -3 \le x \le 3$

Hence, $D_f = [-3, 3]$

Ex-19. Find the range of the function

$f(x) = 2\tan^{-1}(1-x^2) + \dfrac{\pi}{6}$

Soln. We have, $-\dfrac{\pi}{2} \le \tan^{-1}(1-x^2) \le \dfrac{\pi}{4}$

$\Rightarrow \quad -\pi \le 2\tan^{-1}(1-x^2) \le \dfrac{\pi}{2}$

$\Rightarrow \quad -\pi + \dfrac{\pi}{6} \le 2\tan^{-1}(1-x^2) + \dfrac{\pi}{6} \le \dfrac{\pi}{2} + \dfrac{\pi}{6}$

$\Rightarrow \quad -\dfrac{5\pi}{6} \le f(x) \le \dfrac{2\pi}{3}$

Hence, $R_f = \left[-\dfrac{5\pi}{6}, \dfrac{2\pi}{3}\right]$

Ex-20. Find the range of $f(x) = \cot^{-1}(2x - x^2)$

Soln. We have, $f(x) = \cot^{-1}(2x - x^2)$

$\Rightarrow \quad f(x) = \cot^{-1}\left(1 - (x^2 - 2x + 1)\right)$

$\Rightarrow \quad f(x) = \cot^{-1}\left(1 - (x-1)^2\right)$

since $\left(1-(x-1)^2\right) \le 1$ & $0 \le \cot^{-1} x \le \pi$

and $\cot^{-1} x$ is strictly decreasing function

so $\cot^{-1}(1) \le \cot^{-1}\left(1-(x-1)^2\right) \le \cot^{-1}(0)$

$\Rightarrow \quad \dfrac{\pi}{4} \le f(x) \le \pi$

Hence, $R_f = \left[\dfrac{\pi}{4}, \pi\right]$

Ex-21. Solve for x: $[\cot^{-1} x] + [\cos^{-1} x] = 0$,

where [,] = G.I.F.

Soln. We have, $[\cot^{-1} x] + [\cos^{-1} x] = 0$

$\Rightarrow \quad [\cot^{-1} x] = 0$ & $[\cos^{-1} x] = 0$

$\Rightarrow \quad 0 \le \cot^{-1} x < 1$ & $0 \le \cos^{-1} x < 1$

$\Rightarrow \quad x \in (\cot 1, \infty)$ & $x \in (\cos 1, 1]$

$\Rightarrow \quad x \in (\cot 1, 1]$

Ex-22. Find the number of solutions of

$\sin\{x\} = \cos\{x\}$, $\forall\ x \in [0, 2\pi]$.

Soln. We have, $\sin\{x\} = \cos\{x\}$, $\forall\ x \in [0, 2\pi]$

$\Rightarrow \quad \tan\{x\} = 1$

$\Rightarrow \quad \{x\} = \tan^{-1}(1) = \dfrac{\pi}{4}$

Hence, the number of solutions = 6.

(since $\{x\}$ is a periodic function with period 1, it has one solution between 0 to 1. So, there are six solutions between 0 and 6.28).

EXERCISE 2

Q. Find the domains of

1. $f(x) = \sin^{-1}(2x-3)$
2. $f(x) = \sin^{-1}\left(\dfrac{x}{x-1}\right)$
3. $f(x) = \cos^{-1}(3x+4)$
4. $f(x) = \sin^{-1}\left(\dfrac{|x|-1}{2}\right)$
5. $f(x) = \cos^{-1}\left(\dfrac{1-|x|}{5}\right)$
6. $f(x) = \sin^{-1}\left(\dfrac{|x|-2}{3}\right) + \cos^{-1}\left(\dfrac{1-|x|}{4}\right)$

7. $f(x)=\sin^{-1}\left(2x^2-1\right)$

8. $f(x)=\sqrt{5\pi\,\sin^{-1}x-6\left(\sin^{-1}x\right)^2}$

9. $f(x)=\log_2\left(\dfrac{3\tan^{-1}x+\pi}{\pi-4\tan^{-1}x}\right)$

10. $f(x)=\cos^{-1}\left(\dfrac{3}{2+\sin x}\right)$

11. $f(x)=\sin^{-1}\left(\dfrac{x^2+1}{2x}\right)$

12. $f(x)=\cos^{-1}\left(\dfrac{x^2+1}{x^2}\right)$

13. $f(x)=\sin^{-1}\left(\log_2\left(x^2+3x+4\right)\right)$

14. $f(x)=\sin^{-1}\left(\log_2\left(\dfrac{x^2}{2}\right)\right)$

15. $f(x)=\sin^{-1}\left[2-3x^2\right]$

16. $f(x)=\dfrac{1}{x}+3^{\sin^{-1}x}+\dfrac{1}{\sqrt{x-2}}$

17. $f(x)=\sin^{-1}\left(\log_2 x^2\right)$

18. $f(x)=e^x+\sin^{-1}\left(\dfrac{x}{2}-1\right)+\dfrac{1}{x}$

19. $f(x)=\sqrt{\sin^{-1}\left(\log_x 2\right)}$

20. $f(x)=\sqrt{\sin^{-1}\left(\log_2 x\right)}$

Q Find the range of

21. $f(x)=\sin^{-1}(2x-3)$

22. $f(x)=2\sin^{-1}(2x-1)-\dfrac{\pi}{4}$

23. $f(x)=2\cos^{-1}\left(-x^2\right)-\pi$

24. $f(x)=\dfrac{1}{2}\tan^{-1}\left(1-x^2\right)-\dfrac{\pi}{4}$

25. $f(x)=\cot^{-1}\left(2x-x^2\right)$

26. $f(x)=\sin^{-1}x+\cos^{-1}x+\tan^{-1}x$

27. $f(x)=\sin^{-1}x+\sec^{-1}x+\tan^{-1}x$

28. $f(x)=3\cot^{-1}x+2\tan^{-1}x+\dfrac{\pi}{4}$

29. $f(x)=\operatorname{cosec}^{-1}\left[1+\sin^2 x\right]$

30. $f(x)=\sin^{-1}\left(\log_2\left(x^2+3x+4\right)\right)$

6.5 CONSTANT PROPERTY

(i) $\sin^{-1}(x)+\cos^{-1}(x)=\dfrac{\pi}{2}, \forall x\in[-1,1]$

(ii) $\tan^{-1}(x)+\cot^{-1}(x)=\dfrac{\pi}{2}, \forall x\in R$

(iii) $\operatorname{cosec}^{-1}(x)+\sec^{-1}(x)=\dfrac{\pi}{2}, \forall x\in R-(-1,1)$

Proof: (i) Let $\sin^{-1}(x)=\theta$, $-\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}$

$\Rightarrow\quad x=\sin\theta$

$\Rightarrow\quad \cos\left(\dfrac{\pi}{2}-\theta\right)=x$

$\Rightarrow\quad \cos^{-1}x=\left(\dfrac{\pi}{2}-\theta\right),\ 0\le\left(\dfrac{\pi}{2}-\theta\right)\le\pi$

Thus, $\sin^{-1}x+\cos^{-1}x=\theta+\dfrac{\pi}{2}-\theta=\dfrac{\pi}{2}$.

Similarly, we can easily prove (ii) & (iii).

6.5.1 Some solved examples

Ex-1. Find the range of $f(x)=\sin^{-1}x+\cos^{-1}x+\tan^{-1}x$.

Soln. We have, $f(x)=\sin^{-1}x+\cos^{-1}x+\tan^{-1}x$ is defined only when $-1\le x\le 1$

Now, $f(1)=\sin^{-1}(1)+\cos^{-1}(1)+\tan^{-1}(1)$

$=\dfrac{\pi}{2}+0+\dfrac{\pi}{4}=\dfrac{3\pi}{4}$

and $f(-1)=\sin^{-1}(-1)+\cos^{-1}(-1)+\tan^{-1}(-1)$

$=-\dfrac{\pi}{2}+\pi-\dfrac{\pi}{4}=\pi-\dfrac{3\pi}{4}=\dfrac{\pi}{4}$

Thus, $R_f=\left[\dfrac{\pi}{4},\dfrac{3\pi}{4}\right]$

Ex-2. Solve for x: $4\sin^{-1}(x-2)+\cos^{-1}(x-2)=\pi$

Soln. We have, $4\sin^{-1}(x-2)+\cos^{-1}(x-2)=\pi$

$\Rightarrow\quad 3\sin^{-1}(x-2)+\dfrac{\pi}{2}=\pi$

$\Rightarrow\quad 3\sin^{-1}(x-2)=\dfrac{\pi}{2}$

$\Rightarrow\quad \sin^{-1}(x-2)=\dfrac{\pi}{6}$

$\Rightarrow\quad (x-2)=\sin\left(\dfrac{\pi}{6}\right)=\dfrac{1}{2}$

$\Rightarrow\quad x=2+\dfrac{1}{2}=\dfrac{5}{2}$

Hence, the solution is $x = \frac{5}{2}$.

Ex-3. Solve for x:

$\sin^{-1}\left(x^2 - 2x + 1\right) + \cos^{-1}\left(x^2 - x\right) = \frac{\pi}{2}$.

Soln. As we know that,

if $\sin^{-1}\left(f(x)\right) + \cos^{-1}\left(g(x)\right) = \frac{\pi}{2}$, then

$f(x) = g(x)$

$\Rightarrow \quad \left(x^2 - 2x + 1\right) = \left(x^2 - x\right)$

$\Rightarrow \quad 2x - x = 1$

$\Rightarrow \quad x = 1$

Hence, the solution is $x = 1$.

Ex-4. Find the number of real solutions of

$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{2}$.

Soln. We have,

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{2}$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{1}{\sqrt{x^2+x+1}}\right) + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{2}$$

$$\Rightarrow \quad \left(\frac{1}{\sqrt{x^2+x+1}}\right) = \sqrt{x^2 + x + 1}$$

$\Rightarrow \quad x^2 + x + 1 = 1$

$\Rightarrow \quad x^2 + x = 0$

$\Rightarrow \quad x(x+1) = 0$

$\Rightarrow \quad x = 0 \ \& -1$.

Hence, the number of solutions is 2.

Ex-5. If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots\ldots\right)$

$+ \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \ldots\ldots\right) = \frac{\pi}{2}$, for

$0 < |x| < \sqrt{2}$, then find x.

Soln. As we know that, if

$\sin^{-1}\left(f(x)\right) + \cos^{-1}\left(g(x)\right) = \frac{\pi}{2}$, then

$f(x) = g(x)$

$$\Rightarrow \quad \left(x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots\right) = \left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \ldots\right)$$

$$\Rightarrow \quad x\left(1 - \frac{x}{2} + \frac{x^2}{4} - \ldots\right) = x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{4} - \ldots\right)$$

$$\Rightarrow \quad x\left(\frac{1}{1+\frac{x}{2}}\right) = x^2\left(\frac{1}{1+\frac{x^2}{2}}\right)$$

$$\Rightarrow \quad \left(\frac{2x}{x+2}\right) = \left(\frac{2x^2}{x^2+2}\right)$$

$$\Rightarrow \quad x\left\{\left(\frac{1}{x+2}\right) - \left(\frac{x}{x^2+2}\right)\right\} = 0$$

$$\Rightarrow \quad x = 0 \ \& \left(\frac{1}{x+2}\right) = \left(\frac{x}{x^2+2}\right)$$

$\Rightarrow \quad x = 0 \ \& \ x = 1$

Ex-6. Solve for x: $\sin^{-1}x > \cos^{-1}x$.

Soln. We have, $\sin^{-1}x > \cos^{-1}x$

$\Rightarrow \quad 2\sin^{-1}x > \sin^{-1}x + \cos^{-1}x$

$\Rightarrow \quad 2\sin^{-1}x > \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}x > \frac{\pi}{4}$

$\Rightarrow \quad x > \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad x > \frac{1}{\sqrt{2}}$

$\Rightarrow \quad x \in \left(\frac{1}{\sqrt{2}}, 1\right]$

EXERCISE 3

Q. Solve for x:

1. $\left(\sin^{-1}x\right)^2 - 3\sin^{-1}x + 2 = 0$
2. $\sin^{-1}x + \sin^{-1}2y = \pi$
3. $\cos^{-1}x + \cos^{-1}x^2 = 2\pi$
4. $\cos^{-1}x + \cos^{-1}x^2 = 0$
5. $4\sin^{-1}(x-1) + \cos^{-1}(x-1) = \pi$
6. $\sin^{-1}x > \cos^{-1}x$
7. $\cot^{-1}\left(\frac{1}{x^2-1}\right) + \tan^{-1}\left(x^2 - 1\right) = \frac{\pi}{2}$
8. $\cot^{-1}\left(\frac{x^2-1}{2x}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$
9. $4\sin^{-1}x + \cos^{-1}x = \frac{3\pi}{4}$

10. $5\tan^{-1}x + 3\cot^{-1}x = \frac{7\pi}{4}$

11. $5\tan^{-1}x + 4\cot^{-1}x = 2\pi$

12. $\cot^{-1}x - \cot^{-1}(x+1) = \frac{\pi}{2}$

13. $[\sin^{-1}x] + [\cos^{-1}x] = 0$

14. $[\tan^{-1}x] + [\cot^{-1}x] = 0$

15. $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 0$

16. $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$

17. $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$

6.5 CONVERSION OF INVERSE TRIGONOMETRIC FUNCTIONS

Case I: When $x > 0$

Functions	Principal values
1. $\sin^{-1}x$	$\left[0, \frac{\pi}{2}\right]$
2. $\cos^{-1}x$	$\left[0, \frac{\pi}{2}\right]$
3. $\tan^{-1}x$	$\left(0, \frac{\pi}{2}\right)$
4. $\cot^{-1}x$	$\left(0, \frac{\pi}{2}\right]$
5. $\text{cosec}^{-1}x$	$\left(0, \frac{\pi}{2}\right]$
6. $\sec^{-1}x$	$\left[0, \frac{\pi}{2}\right)$

Case II: When $x < 0$

Functions	Principal values
1. $\sin^{-1}x$	$\left[-\frac{\pi}{2}, 0\right]$
2. $\text{cosec}^{-1}x$	$\left[-\frac{\pi}{2}, 0\right)$
3. $\tan^{-1}x$	$\left(-\frac{\pi}{2}, 0\right)$
4. $\cos^{-1}x$	$\left[\frac{\pi}{2}, \pi\right]$
5. $\sec^{-1}x$	$\left(\frac{\pi}{2}, \pi\right]$
6. $\cot^{-1}x$	$\left[\frac{\pi}{2}, \pi\right)$.

Here, we shall discuss, any inverse trigonometric function can be expressed in terms of any other inverse trigonometric functions.

Step I:

(i) $\sin^{-1}(x) = \text{cosec}^{-1}\left(\frac{1}{x}\right),\ x \in [-1,1] - \{0\}$

(ii) $\text{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right),\ |x| \ge 1$

(iii) $\cos^{-1}(x) = \sec^{-1}\left(\frac{1}{x}\right),\ x \in [-1,1] - \{0\}$

(iv) $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right),\ |x| \ge 1$

(v) $\tan^{-1}(x) = \begin{cases} \cot^{-1}\left(\frac{1}{x}\right) & : x > 0 \\ -\pi + \cot^{-1}\left(\frac{1}{x}\right) & : x < 0 \end{cases}$

Proof: Let $\tan^{-1}x = \theta$

Case I: When $x > 0$

Then $\tan(\theta) > 0$

$\Rightarrow \quad 0 < \theta < \frac{\pi}{2}$

Now, $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{x}$

$\Rightarrow \quad \theta = \cot^{-1}\left(\frac{1}{x}\right)$

Thus, $\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right)$.

Case II: When $x < 0$

Then $\tan\theta < 0$

$\Rightarrow \quad -\frac{\pi}{2} < \theta < 0$

$\Rightarrow \quad 0 < -\theta < \frac{\pi}{2}$

$\Rightarrow \quad -\pi < -\pi - \theta < -\pi + \frac{\pi}{2}$

$\Rightarrow \quad -\pi < -\pi - \theta < -\frac{\pi}{2}$

Now, $\cot(-\pi - \theta) = \cot\theta$

and $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{x}$

Thus, $(-\pi-\theta)=\cot^{-1}\left(\frac{1}{x}\right)$.

$$\Rightarrow \quad (-\pi-\tan^{-1}x)=\cot^{-1}\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad \tan^{-1}x=-\pi+\cot^{-1}\left(\frac{1}{x}\right).$$

(vi) $$\cot^{-1}(x)=\begin{cases}\tan^{-1}\left(\frac{1}{x}\right) & : x>0\\ \pi+\tan^{-1}\left(\frac{1}{x}\right) & : x<0\end{cases}$$

Proof: Case I: When $x > 0$

Let $\cot^{-1}(x)=\theta$

$\Rightarrow \quad \cot\theta = x$

Here, $\cot\theta>0 \Rightarrow \quad 0<\theta<\frac{\pi}{2}$

Now, $\tan\theta=\frac{1}{\cot\theta}=\frac{1}{x}$

$$\Rightarrow \quad \theta=\tan^{-1}\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad \cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right)$$

Case II: When $x < 0$

Then $\cot\theta<0$

$$\Rightarrow \quad \frac{\pi}{2}<\theta<\pi$$

Now, $\tan(-\pi+\theta)=\tan\theta=\frac{1}{x}$

$$\Rightarrow \quad (-\pi+\theta)=\tan^{-1}\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad -\pi+\cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad \cot^{-1}x=\pi+\tan^{-1}\left(\frac{1}{x}\right)$$

Step II:

(i) $$\sin^{-1}x=\begin{cases}\cos^{-1}\left(\sqrt{1-x^2}\right) & : \ 0\le x\le 1\\ -\cos^{-1}\left(\sqrt{1-x^2}\right) & : -1\le x<0\end{cases}$$

(ii) $$\sin^{-1}x=\begin{cases}\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) & : \ 0\le x\le 1\\ -\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) & : -1\le x<0\end{cases}$$

(iii) $$\cos^{-1}x=\begin{cases}\sin^{-1}\left(\sqrt{1-x^2}\right) & : \ 0\le x\le 1\\ \pi-\sin^{-1}\left(\sqrt{1-x^2}\right) & : -1\le x<0\end{cases}$$

(iv) $$\cos^{-1}x=\begin{cases}\operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) & : \ 0< x\le 1\\ \pi-\operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) & : -1\le x<0\end{cases}$$

(v) $$\sin^{-1}x=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) : -1<x<1$$

(vi) $$\sin^{-1}x=\begin{cases}\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) & : 0< x\le 1\\ -\pi+\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) & : -1\le x<0\end{cases}$$

6.5.1 Some Solved Examples

Ex-1. Find the value of $\cos\left(\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)\right)$.

Soln. Let $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)=\theta$

$$\Rightarrow \quad \cos^{-1}\left(\frac{3}{5}\right)=2\theta$$

$$\Rightarrow \quad \cos(2\theta)=\frac{3}{5}$$

$$\Rightarrow \quad 2\cos^2\theta-1=\frac{3}{5}$$

$$\Rightarrow \quad 2\cos^2\theta=1+\frac{3}{5}=\frac{8}{5}$$

$$\Rightarrow \quad \cos^2\theta=\frac{4}{5}$$

$$\Rightarrow \quad \cos\theta=\frac{2}{\sqrt{5}}$$

Ex-2. Find the value of $\sin\left(\frac{\pi}{4}+\sin^{-1}\left(\frac{1}{2}\right)\right)$.

Soln. We have, $\sin\left(\frac{\pi}{4}+\sin^{-1}\left(\frac{1}{2}\right)\right)$

$$= \sin\left(\frac{\pi}{4}+\theta\right),\ \theta=\sin^{-1}\left(\frac{1}{2}\right)$$

$$= \sin\left(\frac{\pi}{4}+\theta\right),\ \sin\theta=\frac{1}{2}$$

$= \sin\left(\frac{\pi}{4}\right)\cos(\theta)+\cos\left(\frac{\pi}{4}\right)\sin(\theta)$

$= \frac{1}{\sqrt{2}}\cos(\theta)+\frac{1}{\sqrt{2}}\sin(\theta)$

$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\times\frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

Ex-4. If m is a root of $x^2+3x+1=0$, then find the value of $\tan^{-1}(m)+\tan^{-1}\left(\frac{1}{m}\right)$.

Soln. Let m_1 and m_2 be the roots of $x^2+3x+1=0$.
Thus, $m_1+m_2=-3<0$ and $m_1.m_2=1$.
It is possible only when both are negative.
Thus, $\tan^{-1}(m)+\tan^{-1}\left(\frac{1}{m}\right)$

$= \tan^{-1}(m)-\pi+\cot^{-1}(m)$

$= \tan^{-1}(m)+\cot^{-1}(m)-\pi$

$= \frac{\pi}{2}-\pi$

$= -\frac{\pi}{2}$

Ex-5. Prove that $\cos\left(\tan^{-1}\left(\sin\left(\cot^{-1}x\right)\right)\right)=\sqrt{\frac{x^2+1}{x^2+2}}$

Soln. We have, $\cos\left(\tan^{-1}\left(\sin\left(\cot^{-1}x\right)\right)\right)$

$= \cos\left(\tan^{-1}(\sin\theta)\right),\ \cot\theta=x$

$= \cos\left(\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right)$

$= \cos\varphi,\ \tan\varphi=\left(\frac{1}{\sqrt{1+x^2}}\right)$

$= \sqrt{\frac{x^2+1}{x^2+2}}$

EXERCISE 4

Q Solve for x:

1. $6\left(\sin^{-1}x\right)^2-\pi\sin^{-1}x\le 0$
2. $\frac{2\tan^{-1}x+\pi}{4\tan^{-1}x-\pi}\le 0$
3. $\sin^{-1}x<\sin^{-1}x^2$
4. $\cos^{-1}x>\cos^{-1}x^2$
5. $\log_2\left(\tan^{-1}x\right)>1$
6. $\left(\cot^{-1}x\right)^2-5\cot^{-1}x+6>0$
7. $\sin^{-1}x<\cos^{-1}x$
8. $\sin^{-1}x>\sin^{-1}(1-x)$
9. $\sin^{-1}2x>\operatorname{cosec}^{-1}x$
10. $\tan^{-1}3x<\cot^{-1}x$
11. $\cos^{-1}2x\ge\sin^{-1}x$
12. $x^2-2x<\sin^{-1}(\sin 2)$
13. $\sin^{-1}\left(\frac{x}{2}\right)<\cos^{-1}(x+1)$
14. $\tan^{-1}2x>2\tan^{-1}x$
15. $\tan\left(\cos^{-1}x\right)\le\sin\left(\cot^{-1}\left(\frac{1}{2}\right)\right)$

6.6 COMPOSITION OF TRIGONOMETRIC FUNCTIONS AND ITS INVERSE

Let $y=\sin^{-1}x$

$\Rightarrow\quad x=\sin y$

$\Rightarrow\quad x=\sin\left(\sin^{-1}x\right)$

$\Rightarrow\quad \sin\left(\sin^{-1}x\right)=x$

Therefore, $\sin\left(\sin^{-1}x\right)$ provide us a real value which lies in [–1, 1]
Hence,

(i) $\sin\left(\sin^{-1}x\right)=x,\ |x|\le 1$

(ii) $\cos\left(\cos^{-1}x\right)=x,\ |x|\le 1$

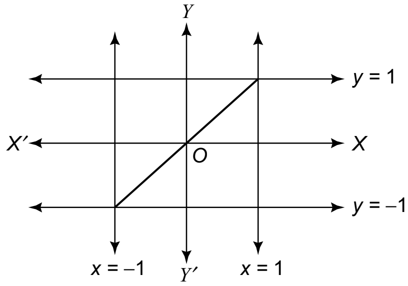
(iii) $\tan\left(\tan^{-1}x\right)=x,\ x\in R$

(iv) $\cot\left(\cot^{-1}x\right)=x,\ x\in R$

(v) $\operatorname{cosec}\left(\operatorname{cosec}^{-1}x\right)=x,\ |x|\ge 1$

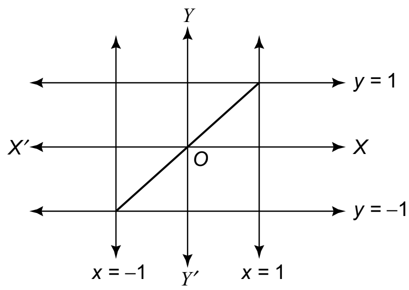
(vi) $\sec\left(\sec^{-1}x\right)=x,\ |x|\ge 1$

(i) Graph of $y = \sin\left(\sin^{-1} x\right)$



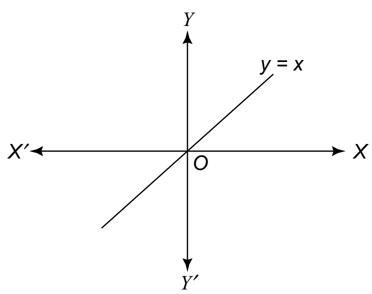
1. $D_f = [-1,1]$
2. $R_f = [-1,1]$
3. It is a non-periodic function.

(ii) Graph of $y = \cos\left(\cos^{-1} x\right)$



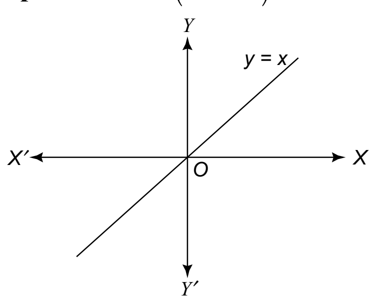
1. $D_f = [-1,1]$
2. $R_f = [-1,1]$
3. It is a non-periodic function.

(iii) Graph of $y = \tan\left(\tan^{-1} x\right)$



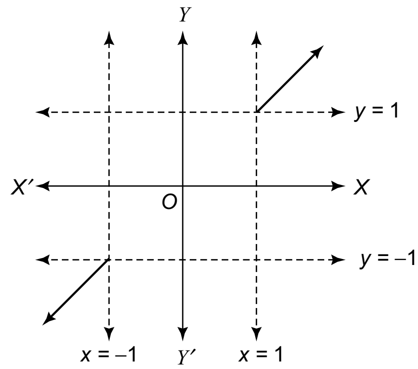
1. $D_f = R$
2. $R_f = R$
3. It is a non-periodic function.

(iv) Graph of $y = \cot\left(\cot^{-1} x\right)$



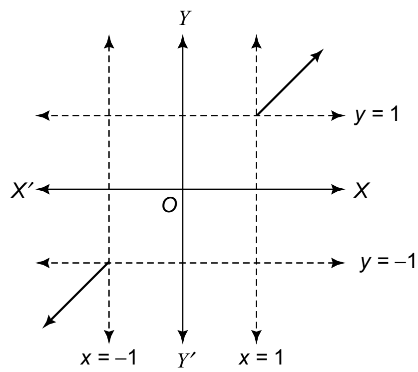
1. $D_f = R$
2. $R_f = R$
3. It is a non-periodic function.

(v) Graph of $y = \operatorname{cosec}\left(\operatorname{cosec}^{-1} x\right)$



1. $D_f = (-\infty, -1] \cup [1, \infty)$
2. $R_f = (-\infty, -1] \cup [1, \infty)$
3. It is a non-periodic function.

(vi) Graph of $y = \sec\left(\sec^{-1} x\right)$



1. $D_f = (-\infty, -1] \cup [1, \infty)$
2. $R_f = (-\infty, -1] \cup [1, \infty)$
3. It is a non-periodic function.

6.6.1 Some Solved Examples

Ex-1. Let $f(x) = \sin^{-1} x + \cos^{-1} x$, then find the value of

(i) $f\left(\dfrac{1}{m^2+1}\right)$, $m \in R$

(ii) $f\left(\dfrac{m^2}{m^2+1}\right)$, $m \in R$

(iii) $f\left(\dfrac{m}{m^2+1}\right)$, $m \in R$

(iv) $f\left(m^2 - 2m + 6\right)$, $m \in R$

(v) $f\left(m^2+1\right),\ m \in R$

Soln. As we know that, $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, for every x in [–1, 1]

(i) since $0 < \frac{1}{m^2+1} \le 1$

so, $f\left(\frac{1}{m^2+1}\right) = \frac{\pi}{2}$

(ii) Since, $0 \le \frac{m^2}{m^2+1} < 1$,

so $f\left(\frac{m^2}{m^2+1}\right) = \frac{\pi}{2}$.

(iii) Since, $-\frac{1}{2} \le \frac{m}{m^2+1} \le \frac{1}{2}$

so, $f\left(\frac{m}{m^2+1}\right) = \frac{\pi}{2}$

(iv) Since $m^2 - 2m + 6 = (m-1)^2 + 5$

Thus, $5 \le (m-1)^2 + 5 < \infty$

Hence, $f\left((m-1)^2+5\right)$ is not defined.

(v) Also, $1 \le m^2 + 1 < \infty$

So, $f\left(m^2+1\right)$ is not defined.

Ex-2. If $\cos^{-1} x + \cos^{-1} y = \frac{2\pi}{3}$, then find the value of $\sin^{-1} x + \sin^{-1} y$.

Soln. Given, $\cos^{-1} x + \cos^{-1} y = \frac{2\pi}{3}$

Now, $\sin^{-1} x + \sin^{-1} y$

$= \frac{\pi}{2} - \cos^{-1} x + \frac{\pi}{2} - \cos^{-1} y$

$= \pi - \left(\cos^{-1} x + \cos^{-1} y\right)$

$= \pi - \frac{2\pi}{3}$

$= \frac{\pi}{3}$

Ex-3. If m is the root of $x^2 + 3x + 1 = 0$, then find the value of $\tan^{-1}(m) + \tan^{-1}\left(\frac{1}{m}\right)$.

Soln. Let m_1 and m_2 are the two roots of the given equation.
Now, $m_1 + m_2 = -3$ and $m_1.m_2 = 1$
$\Rightarrow m_1$ and m_2 are two –ve roots.

Now, $\tan^{-1}(m) + \tan^{-1}\left(\frac{1}{m}\right)$

$= \tan^{-1}(m) - \pi + \cot^{-1}(m)$

$= -\pi + \tan^{-1}(m) + \cot^{-1}(m)$

$= -\pi + \frac{\pi}{2}$

$= -\frac{\pi}{2}$

Ex-4. Solve for x:

$$\sin^{-1}\left(\sin\left(\frac{2x^2+5}{x^2+2}\right)\right) > \sin^{-1}(\sin 3)$$

Soln. Let $m = \frac{2x^2+5}{x^2+2} = 2 + \frac{1}{x^2+2}$

Thus, $m \in \left[2, \frac{5}{2}\right]$

now, $\sin^{-1}\left(\sin\left(\frac{2x^2+5}{x^2+2}\right)\right) > \sin^{-1}(\sin 3)$

$\Rightarrow \quad \sin^{-1}\left(\sin\left(\pi - \frac{2x^2+5}{x^2+2}\right)\right)$

$< \sin^{-1}(\sin(\pi - 3))$

$\Rightarrow \quad \pi - \left(\frac{2x^2+5}{x^2+2}\right) > \pi - 3$

$\Rightarrow \quad \left(\frac{2x^2+5}{x^2+2}\right) < 3$

$\Rightarrow \quad \left(\frac{2x^2+5}{x^2+2} - 3\right) < 0$

$\Rightarrow \quad \left(\frac{-x^2-5}{x^2+2}\right) < 0$

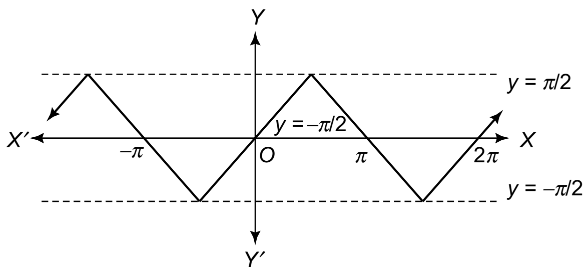
$\Rightarrow \quad \left(\frac{x^2+5}{x^2+2}\right) > 0$

$\Rightarrow \quad x \in R$

6.7 COMPOSITION OF INVERSE TRIGONO METRIC FUNCTIONS AND TRIGONOMETRIC FUNCTIONS

(i) A function $f : R \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is defined as $f(x) = \sin^{-1}(\sin x)$

Graph of $f(x) = \sin^{-1}(\sin x)$

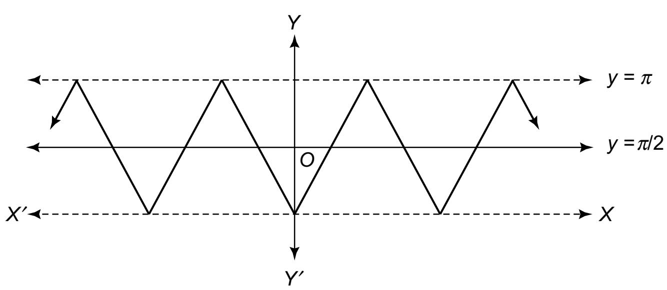


1. $D_f = R$
2. $R_f = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
3. It is an odd function.
4. It is a periodic function with period 2π
5. $$\sin^{-1}(\sin x) = \begin{cases} x & : -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} \\ \pi - x & : \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2} \\ x - 2\pi & : \dfrac{3\pi}{2} \le x \le \dfrac{5\pi}{2} \\ -\pi - x & : -\dfrac{3\pi}{2} \le x \le -\dfrac{\pi}{2} \end{cases}$$

(ii) $\cos^{-1}(\cos x)$:

A function $f : R \to [0, \pi]$ is defined as $f(x) = \cos^{-1}(\cos x)$

Graph of $f(x) = \cos^{-1}(\cos x)$:

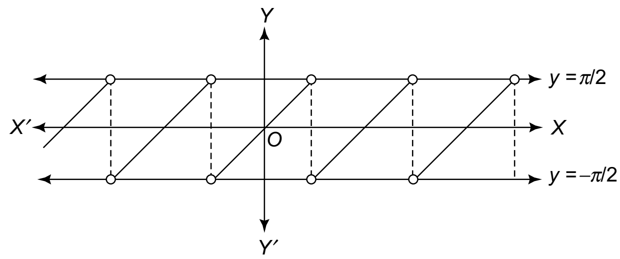


1. $D_f = R$
2. $R_f = [0, \pi]$
3. It is neither odd nor even function.
4. It is periodic function with period 2π .
5. $$\cos^{-1}(\cos x) = \begin{cases} x & : 0 \le x \le \pi \\ 2\pi - x & : \pi \le x \le 2\pi \\ x - 2\pi & : 2\pi \le x \le 3\pi \\ -x & : -\pi \le x \le 0 \end{cases}$$

(iii) $\tan^{-1}(\tan x)$:

A function $f : R - (2n+1)\dfrac{\pi}{2} \to \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is defined as $f(x) = \tan^{-1}(\tan x)$

Graph of $f(x) = \tan^{-1}(\tan x)$:

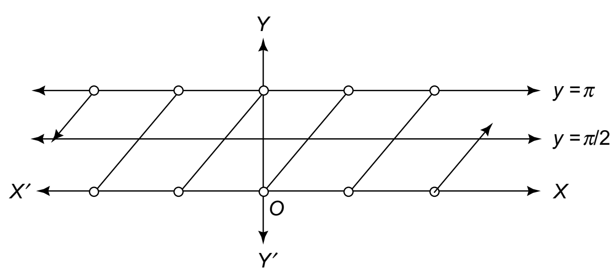


1. $D_f = R - (2n+1)\dfrac{\pi}{2},\ n \in I$
2. $R_f = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
3. It is an odd function.
4. It is a periodic function with period π
5. $$\tan^{-1}(\tan x) = \begin{cases} x & : \dfrac{-\pi}{2} < x < \dfrac{\pi}{2} \\ x - \pi & : \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \\ x - 2\pi & : \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \\ x + \pi & : \dfrac{-3\pi}{2} < x < \dfrac{-\pi}{2} \end{cases}$$

(iv) $\cot^{-1}(\cot x)$:

A function $f : R - (n\pi) \to (0, \pi)$ is defined as $f(x) = \cot^{-1}(\cot x)$

Graph of $f(x) = \cot^{-1}(\cot x)$:



1. $D_f = R - n\pi,\ \ n \in I$
2. $R_f = (0, \pi)$
3. It is neither even nor odd function.
4. It is a periodic function with period π.

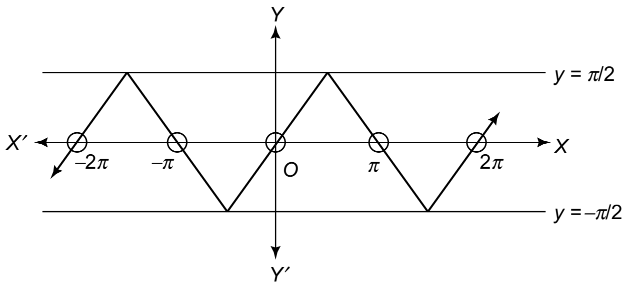
5. $\cot^{-1}(\cot x) = \begin{cases} x & : 0 < x < \pi \\ x - \pi & : \pi < x < 2\pi \\ x - 2\pi & : 2\pi < x < 3\pi \\ \pi + x & : -\pi < x < 0 \end{cases}$

(v) $\text{cosec}^{-1}(\text{cosec } x)$: A function

$f : R - (n\pi) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ is defined

as $f(x) = \text{cosec}^{-1}(\text{cosec } x)$

Graph of $f(x) = \text{cosec}^{-1}(\text{cosec } x)$



1. $D_f = R - n\pi,\ n \in I$
2. $R_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
3. It is an odd function
4. It is a periodic function with period 2π
5. $\text{cosec}^{-1}(\text{cosec } x)$

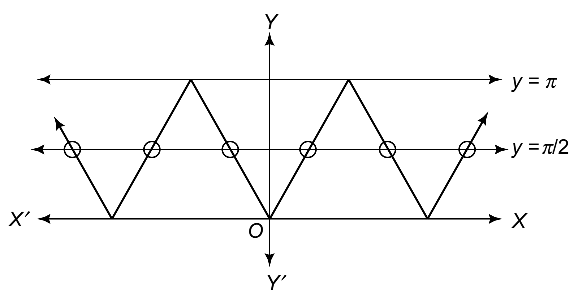
$$= \begin{cases} x & : \frac{-\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x & : \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ x - 2\pi & : \frac{3\pi}{2} \le x \le \frac{5\pi}{2} \\ -x - \pi & : \frac{-3\pi}{2} \le x \le \frac{-\pi}{2} \end{cases}$$

(vi) $\sec^{-1}(\sec x)$: A function

$f : R - (2n+1)\frac{\pi}{2} \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$ is

defined as $f(x) = \sec^{-1}(\sec x)$

Graph of $f(x) = \sec^{-1}(\sec x)$



1. $D_f = R - (2n+1)\frac{\pi}{2},\ n \in I$
2. $R_f = [0, \pi] - \left\{\frac{\pi}{2}\right\}$
3. It is neither even nor odd function.
4. It is a periodic function with period 2π
5. $\sec^{-1}(\sec x) = \begin{cases} x & : 0 \le x \le \pi \\ 2\pi - x & : \pi \le x \le 2\pi \\ x - 2\pi & : 2\pi \le x \le 3\pi \\ -x & : -\pi \le x \le 0 \end{cases}$

6.8.1 Some Solved Examples

Ex-1. Find the value of

(i) $\sin^{-1}(\sin 3)$

(ii) $\sin^{-1}(\sin 5)$

(iii) $\sin^{-1}(\sin 7)$

(iv) $\sin^{-1}(\sin 10)$

(v) $\sin^{-1}(\sin 20)$

Soln. (i) $\sin^{-1}(\sin 3)$

$= \sin^{-1}(\sin(\pi - 3))$

$= (\pi - 3)$

(ii) $\sin^{-1}(\sin 5)$

$= \sin^{-1}(\sin(5 - 2\pi))$

$= (5 - 2\pi)$

(iii) $\sin^{-1}(\sin 7)$

$= \sin^{-1}(\sin(7 - 2\pi))$

$= (7 - 2\pi)$

(iv) $\sin^{-1}(\sin 10)$

$= \sin^{-1}(\sin(3\pi - 10))$

$= (3\pi - 10)$

(v) $\sin^{-1}(\sin 20)$

$= \sin^{-1}(\sin(20 - 6\pi))$

$= (20 - 6\pi)$

Ex-2. Find the value of

(i) $\cos^{-1}(\cos 2)$

(ii) $\cos^{-1}(\cos 3)$

(iii) $\cos^{-1}(\cos 5)$

(iv) $\cos^{-1}(\cos 7)$

(v) $\cos^{-1}(\cos 10)$

Soln. (i) $\cos^{-1}(\cos 2) = 2$

(ii) $\cos^{-1}(\cos 3) = 2$

(iii) $\cos^{-1}(\cos 5)$

$= \cos^{-1}(\cos(2\pi - 5))$

$= (2\pi - 5)$

(iv) $\cos^{-1}(\cos 7)$

$= \cos^{-1}(\cos(7 - 2\pi))$

$= (7 - 2\pi)$

(v) $\cos^{-1}(\cos 10)$

$= \cos^{-1}(\cos(4\pi - 10))$

$= (4\pi - 10)$

Ex-3. Find the value of

(i) $\tan^{-1}(\tan 3)$

(ii) $\tan^{-1}(\tan 5)$

(iii) $\tan^{-1}(\tan 7)$

(iv) $\tan^{-1}(\tan 10)$

(v) $\tan^{-1}(\tan 15)$

Soln. (i) $\tan^{-1}(\tan 3)$

$= \tan^{-1}(\tan(3 - \pi))$

$= (3 - \pi)$

(ii) $\tan^{-1}(\tan 5)$

$= \tan^{-1}(\tan(5 - 2\pi))$

$= (5 - 2\pi)$

(iii) $\tan^{-1}(\tan 7)$

$= \tan^{-1}(\tan(7 - 2\pi))$

$= (7 - 2\pi)$

(iv) $\tan^{-1}(\tan 10)$

$= \tan^{-1}(\tan(10 - 3\pi))$

$= (10 - 3\pi)$

(v) $\tan^{-1}(\tan 15)$

$= \tan^{-1}(\tan(15 - 5\pi))$

$= (15 - 5\pi)$.

Ex-4. Find the value of $\cos^{-1}(\sin(-5))$.

Soln. We have, $\cos^{-1}(\sin(-5))$

$= \cos^{-1}(-\sin 5)$

$= \pi - \cos^{-1}(\sin 5)$

$= \pi - \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 5\right)\right)$

$= \pi - \left(\frac{\pi}{2} - 5\right)$

$= \frac{\pi}{2} + 5$

Ex-6. Find $f'(x)$, where $f(x) = \sin^{-1}(\sin x)$ and $-2\pi \le x \le \pi$.

Soln. We have, $f(x)$

$= \sin^{-1}(\sin x)$

$= x + 2\pi - \pi - x + x + \pi - x$

$= 2\pi$

$\Rightarrow \quad f'(x) = 0$.

Ex-7. Find $f'(x)$, where $f(x) = \cos^{-1}(\cos x)$ and $-\pi \le x \le 2\pi$.

Soln. We have $f(x)$

$= \cos^{-1}(\cos x)$

$= -x + x + 2\pi - x$

$= 2\pi - x$

$\Rightarrow \quad f'(x) = -1$

Ex-8. Solve for x: $\sin^{-1}\left(\sin\left(\frac{2x^2+5}{x^2+1}\right)\right) < \pi - 3$

Soln. We have, $\sin^{-1}\left(\sin\left(\frac{2x^2+5}{x^2+1}\right)\right) < \pi - 3$

$\Rightarrow \quad \sin^{-1}\left(\sin\left(\pi - \left(\frac{2x^2+5}{x^2+1}\right)\right)\right) > \pi - 3$

$\Rightarrow \quad \left(\pi - \left(\frac{2x^2+5}{x^2+1}\right)\right) < \pi - 3$

$\Rightarrow \quad -\left(\frac{2x^2+5}{x^2+1}\right) < -3$

$\Rightarrow \quad \left(\frac{2x^2+5}{x^2+1}\right) > 3$

$\Rightarrow \quad \left(\frac{2x^2+5}{x^2+1} - 3\right) > 0$

$\Rightarrow \quad \left(\dfrac{2x^2+5-3x^2-3}{x^2+1}\right)>0$

$\Rightarrow \quad x^2<2$

$\Rightarrow \quad -\sqrt{2}<x<\sqrt{2}$

Ex-9. Find the integral values of x satisfying the inequality, $x^2-3x<\sin^{-1}(\sin 2)$.

Soln. We have, $x^2-3x<\sin^{-1}(\sin 2)$

$\Rightarrow \quad x^2-3x<\sin^{-1}(\sin(\pi-2))$

$\Rightarrow \quad x^2-3x<(\pi-2)$

$\Rightarrow \quad x^2-3x+(2-\pi)<0$

$\Rightarrow \quad \left(x-\dfrac{3+\sqrt{1+4\pi}}{2}\right)\left(x-\dfrac{3-\sqrt{1+4\pi}}{2}\right)<0$

$\Rightarrow \quad \dfrac{3-\sqrt{1+4\pi}}{2}<x<\dfrac{3+\sqrt{1+4\pi}}{2}$

Ex-10. Find the value of

$\sin^{-1}(\sin 50)+\cos^{-1}(\cos 50)+\tan^{-1}(\tan 50)$

Soln. We have

$=\sin^{-1}(\sin 50)+\cos^{-1}(\cos 50)+\tan^{-1}(\tan 50)$

$=\sin^{-1}(\sin(50-16\pi))+\cos^{-1}(\cos(16\pi-50))+\tan^{-1}(\tan(50-16\pi))$

$=(50-16\pi)+(16\pi-50)+(50-16\pi)$

$=(50-16\pi)$

EXERCISE 5

Q. Evaluate each of the following:

1. $\sin^{-1}\left(\sin\left(\dfrac{7\pi}{3}\right)\right)$
2. $\sin^{-1}\left(\sin\left(\dfrac{11\pi}{3}\right)\right)$
3. $\sin^{-1}\left(\sin\left(\dfrac{49\pi}{8}\right)\right)$
4. $\sin^{-1}(\sin 3)$
5. $\sin^{-1}(\sin 5)$
6. $\sin^{-1}(\sin 7)$
7. $\sin^{-1}(\sin 10)$
8. $\sin^{-1}(\sin 12)$
9. $\sin^{-1}(\sin 20)$
10. $\sin^{-1}(\sin 50)$
11. $\sin^{-1}(\sin 80)$
12. $\sin^{-1}(\sin 100)$

Q. Evaluate each of the following:

13. $\cos^{-1}\left(\cos\left(\dfrac{7\pi}{6}\right)\right)$
14. $\cos^{-1}\left(\cos\left(\dfrac{31\pi}{7}\right)\right)$
15. $\cos^{-1}\left(\cos\left(\dfrac{103\pi}{7}\right)\right)$
16. $\cos^{-1}(\cos(3))$
17. $\cos^{-1}(\cos(5))$
18. $\cos^{-1}(\cos(7))$
19. $\cos^{-1}(\cos(10))$
20. $\cos^{-1}(\cos(12))$
21. $\cos^{-1}(\cos(15))$
22. $\cos^{-1}(\cos(40))$
23. $\cos^{-1}(\cos(60))$
24. $\cos^{-1}(\cos(100))$

Q. Find the value of

25. $\sin^{-1}(\sin 1)+\sin^{-1}(\sin 2)+\sin^{-1}(\sin 3)$
26. $\sin^{-1}(\sin 10)+\sin^{-1}(\sin 20)$ $+\sin^{-1}(\sin 30)+\sin^{-1}(\sin 40)$
27. $\cos^{-1}(\cos 1)+\cos^{-1}(\cos 2)$ $+\cos^{-1}(\cos 3)+\cos^{-1}(\cos 4)$
28. $\cos^{-1}(\cos 10)+\cos^{-1}(\cos 20)$ $+\cos^{-1}(\cos 30)+\cos^{-1}(\cos 40)$
29. $\sin^{-1}(\sin 10)+\cos^{-1}(\cos 10)$
30. $\sin^{-1}(\sin 50)+\cos^{-1}(\cos 50)$
31. $\sin^{-1}(\sin 100)+\cos^{-1}(\cos 100)$
32. $\cos^{-1}(\sin(-5))+\sin^{-1}(\cos(-5))$
33. Find the number of ordered pairs of (x, y) satisfying the equations $y=|\sin x|$ and $y=\cos^{-1}(\cos x)$, where $x\in[-2\pi, 2\pi]$

34. Let $f(x)=\cos^{-1}(\cos x)-\sin^{-1}(\sin x)$ in $[0, \pi]$. Find the area bounded by $f(x)$ and x-axis.

Q. Evaluate the following:

35. $\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right)$
36. $\tan^{-1}\left(\tan\left(\frac{29\pi}{5}\right)\right)$
37. $\tan^{-1}\left(\tan\left(\frac{121\pi}{12}\right)\right)$
38. $\tan^{-1}(\tan 3)$
39. $\tan^{-1}(\tan 5)$
40. $\tan^{-1}(\tan 7)$
41. $\tan^{-1}(\tan 10)$
42. $\tan^{-1}(\tan 20)$
43. $\tan^{-1}(\tan 50)$
44. $\tan^{-1}(\tan 1)+\tan^{-1}(\tan 2)$ $+\tan^{-1}(\tan 3)+\tan^{-1}(\tan 4)$
45. $\tan^{-1}(\tan 20)+\tan^{-1}(\tan 40)$ $+\tan^{-1}(\tan 60)+\tan^{-1}(\tan 80)$
46. $\sin^{-1}(\sin 15)+\cos^{-1}(\cos 15)+\tan^{-1}(\tan 15)$
47. $\sin^{-1}(\sin 50)+\cos^{-1}(\cos 50)-\tan^{-1}(\tan 50)$
48. $3x^2+8x<2\sin^{-1}(\sin 4)-\cos^{-1}(\cos 4)$
49. $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right)<\pi-3$

6.9 SUM OF ANGLES

(i) $\sin^{-1}x+\sin^{-1}y$

$$=\begin{cases}\alpha : & x^2+y^2\le 1\\ \pi-\alpha : & x>0, y>0, x^2+y^2>1\\ \alpha : & xy<0,\ x^2+y^2>1\\ -\pi-\alpha : & x<0,\ y>0,\ x^2+y^2>1\end{cases}$$

where $\alpha=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

Proof: Let $\sin^{-1}x=A$ & $\sin^{-1}y=B$

$\Rightarrow x=\sin A$ & $y=\sin B$

and $A, B\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow \cos A=\sqrt{1-x^2}$, $\cos B=\sqrt{1-y^2}$

Now, $\sin(A+B)$

(i) $\sin^{-1}x+\sin^{-1}y$

$$=\begin{cases}\alpha : & x^2+y^2\le 1\\ \pi-\alpha : & x>0, y>0, x^2+y^2>1\\ \alpha : & xy<0,\ x^2+y^2>1\\ -\pi-\alpha : & x<0,\ y>0,\ x^2+y^2>1\end{cases}$$

where $\alpha=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

Proof: Let $\sin^{-1}x=A$ & $\sin^{-1}y=B$

$\Rightarrow\quad x=\sin A$ & $y=\sin B$

and $A, B\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow\quad \cos A=\sqrt{1-x^2}$, $\cos B=\sqrt{1-y^2}$

Now, $\sin(A+B)$

$=\sin A\cos B+\cos A.\sin B$

$=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

Case I: when $-1\le x, y\le 1$ & $x^2+y^2\le 1$

In this case, $x^2+y^2\le 1$

$\Rightarrow 1-x^2\ge y^2$ & $1-y^2\ge x^2$

$\Rightarrow (1-x^2)(1-y^2)\ge x^2y^2$

$\Rightarrow \sqrt{(1-x^2)(1-y^2)}-xy\ge 0$

$\Rightarrow \cos(A+B)\ge 0$

$\Rightarrow$ $A+B$ lies either in the the first or the fourth quadrant.

$\Rightarrow A+B\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\sin(A+B)=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$\Rightarrow A+B=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\Rightarrow \sin^{-1}x+\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}$

Case II: when $xy<0$ and $x^2+y^2>1$

In this case, we have, $xy<0$

$\Rightarrow\quad$ ($x>0$ and $y<0$) or ($x<0$ and $y>0$)

$\Rightarrow\quad A\in\left(0,\frac{\pi}{2}\right]$ and $B\in\left[-\frac{\pi}{2},0\right)$

$\Rightarrow \quad -\frac{\pi}{2} \le A+B \le \frac{\pi}{2}$

Also, $x^2+y^2>1$

$\Rightarrow \quad 1-x^2>y^2$ and $1-y^2>x^2$

$\Rightarrow \quad (1-x^2)(1-y^2)>x^2y^2$

$\Rightarrow \quad \sqrt{(1-x^2)(1-y^2} < |xy|$

$\Rightarrow \quad -xy<\sqrt{(1-x^2)(1-y^2)}<xy$

$\Rightarrow \quad \sqrt{(1-x^2)(1-y^2)}-xy>0, (\because xy<0)$

$\Rightarrow \quad \cos(A+B)>0$

$\Rightarrow$ $A+B$ lies either in the first quadrant or in the fourth quadrant.

$\Rightarrow \quad A+B \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \quad \sin(A+B)=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$\Rightarrow \quad A+B=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\Rightarrow \quad \sin^{-1}x+\sin^{-1}y$

$= \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

Case III: when $0<x, y\le 1$ and $x^2+y^2<1$

In this case we have $0<x, y\le 1$

$\Rightarrow \quad A\in\left[0, \frac{\pi}{2}\right]$ and $B\in\left[0, \frac{\pi}{2}\right]$

$\Rightarrow \quad A+B\in[0, \pi]$

Also, $x^2+y^2<1$

$\Rightarrow \quad 1-x^2<y^2$ and $1-y^2<x^2$

$\Rightarrow \quad (1-x^2)(1-y^2)<x^2y^2$

$\Rightarrow \quad \sqrt{(1-x^2)(1-y^2)}<xy$

$\Rightarrow \quad 1-x^2<y^2$ and $1-y^2<x^2$

$\Rightarrow \quad (1-x^2)(1-y^2)<x^2y^2$

$\Rightarrow \quad \sqrt{(1-x^2)(1-y^2)}<xy$

$\Rightarrow \quad \sqrt{(1-x^2)(1-y^2)}-xy<0$

$\Rightarrow \quad \cos(A+B)<0$

$\Rightarrow$ $A+B$ lies either in II quadrant or in III quadrant.

$\Rightarrow \quad \frac{\pi}{2}\le A+B\le\pi$, since $A+B\in(0, \pi)$

$\Rightarrow \quad -\pi\le-(A+B)\le-\frac{\pi}{2}$

$\Rightarrow \quad 0\le\pi-(A+B)\le\frac{\pi}{2}$

Now, $\sin(A+B)=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$\Rightarrow \quad \sin(\pi-(A+B))=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$\Rightarrow \quad (\pi-(A+B))=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\Rightarrow \quad A+B=\pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\Rightarrow \quad \sin^{-1}x+\sin^{-1}y$

$=\pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

Case IV: when $-1\le x, y<0$ and $x^2+y^2>1$

In this case, we have, $-1\le x,\ y<0$

$\Rightarrow \quad A\in\left[-\frac{\pi}{2}, 0\right)$ and $B\in\left[-\frac{\pi}{2}, 0\right)$

$\Rightarrow \quad A+B\in[-\pi, 0)$

Also, $x^2+y^2>1$

$\Rightarrow \quad 1-x^2<y^2$ and $1-y^2<x^2$

$\Rightarrow \quad (1-x^2)(1-y^2)<x^2y^2$

$\Rightarrow \quad \sqrt{(1-x^2)(1-y^2)}<xy$ (since, $xy>0$)

$\Rightarrow \quad \sqrt{(1-x^2)(1-y^2)}-xy<0$

$\Rightarrow \quad \cos(A+B)<0$

$\Rightarrow$ $A+B$ lies either in the first quadrant or in the third quadrant.

$\Rightarrow \quad -\pi\le A+B\le\frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{2}\le-(A+B)\le\pi$

$\Rightarrow \quad -\frac{\pi}{2}\le-\pi-(A+B)\le 0$

$\Rightarrow \quad \sin(A+B)=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$\Rightarrow \quad -\sin\{\pi+(A+B)\}=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$\Rightarrow \quad \sin\{-\pi-(A+B)\}=\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\Rightarrow \quad -\pi-(A+B)=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\Rightarrow \quad A+B=-\pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\Rightarrow \quad \sin^{-1}x+\sin^{-1}y$

$=-\pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

(ii) $\sin^{-1}x-\sin^{-1}y$

$$=\begin{cases}\alpha : & x^2+y^2\le 1\\ \pi-\alpha : & x>0, y<0, x^2+y^2>1\\ \alpha : & xy>0,\ x^2+y^2>1\\ -\pi-\alpha : & x<0, y>0, x^2+y^2>1\end{cases}$$

where $\alpha=\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)$

Proof: Do yourself.

(iii) $\cos^{-1}x+\cos^{-1}y$

$$=\begin{cases}\alpha & : x+y\ge 0\\ 2\pi-\alpha & : x+y<0\end{cases}$$

where $\alpha=\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$

Proof: Let $\cos^{-1}x=A$ and $\cos^{-1}y=B$

Then $x=\cos A$ and $y=\cos B$

$\Rightarrow \quad A\in[0,\pi]$ and $B\in[0,\pi]$

Now,

$\sin A=\sqrt{1-x^2}$ and $\sin B=\sqrt{1-y^2}$

$\cos(A+B)=xy-\sqrt{1-x^2}\sqrt{1-y^2}$

$\cos(A-B)=xy+\sqrt{1-x^2}\sqrt{1-y^2}$

Case I:

$-1\le x,\ y\le 1$ and $x+y\ge 0$

In this case, $-1\le x,\ y\le 1$

$\Rightarrow \quad 0\le A+B\le 2\pi$ and $x+y\ge 0$

$\Rightarrow \quad 0\le A+B\le 2\pi$ and $\cos A+\cos B\ge 0$

$\Rightarrow \quad \cos A\ge -\cos B$

$\Rightarrow \quad \cos A\ge \cos(\pi-B)$

$\Rightarrow \quad A\le \pi-B$

$\Rightarrow \quad 0\le A+B\le \pi$

$\Rightarrow \quad \cos(A+B)=xy-\sqrt{1-x^2}\sqrt{1-y^2}$

$\Rightarrow \quad A+B=\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$

$\Rightarrow \quad \cos^{-1}x+\cos^{-1}y$

$=\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$

Case II:

when $-1\le x,\ y<0$ and $x+y<0$

In this case $-1\le x,\ y<0$

$\Rightarrow \quad A, B\in[0,\pi]$

$\Rightarrow \quad 0\le A+B\le 2\pi$ and $x+y\le 0$

$\Rightarrow \quad \cos A+\cos B\le 0$

$\Rightarrow \quad \cos A\le \cos(\pi-B)$

$\Rightarrow \quad A\ge \pi-B$

$\Rightarrow \quad A+B\ge \pi$

Thus, $\pi\le A+B\le 2\pi$

$\Rightarrow \quad -2\pi\le -(A+B)\le -\pi$

$\Rightarrow \quad 0\le 2\pi-(A+B)\le \pi$

$\Rightarrow \quad A+B\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow \quad \sin(A+B)=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$\Rightarrow \quad A+B=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\Rightarrow \quad \sin^{-1}x+\sin^{-1}y$

$=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

Case III:

when $0<x,y\le 1$ and $x^2+y^2<1$

In this case we have $0<x,y\le 1$

$\Rightarrow \quad A\in\left[0,\frac{\pi}{2}\right]$ and $B\in\left[0,\frac{\pi}{2}\right]$

$\Rightarrow \quad A+B\in[0,\pi]$

Also, $x^2+y^2<1$

$\Rightarrow \quad 1-x^2<y^2$ and $1-y^2<x^2$

$\Rightarrow \quad (1-x^2)(1-y^2)<x^2y^2$

$\Rightarrow \quad \sqrt{(1-x^2)(1-y^2)}<xy$

$\Rightarrow \quad \sqrt{(1-x^2)(1-y^2)}-xy<0$

$\Rightarrow \quad \cos(A+B)<0$

$\Rightarrow \quad A+B$ lies either in II quadrant or in III quadrant.

$\Rightarrow \quad \frac{\pi}{2}\le A+B\le\pi$, since $A+B\in(0,\pi)$

$\Rightarrow \quad -\pi\le -(A+B)\le -\frac{\pi}{2}$

$\Rightarrow \quad 0 \le \pi-(A+B) \le \frac{\pi}{2}$

Now, $\sin(A+B)=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$\Rightarrow \quad \sin(\pi-(A+B))=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$\Rightarrow \quad (\pi-(A+B))=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\Rightarrow \quad A+B=\pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\Rightarrow \sin^{-1}x+\sin^{-1}y=\pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

Case IV:

when $-1 \le x, y < 0$ and $x^2+y^2>1$

In this case, we have, $-1 \le x,\ y<0$

$\Rightarrow \quad A \in \left[-\frac{\pi}{2}, 0\right)$ and $B \in \left[-\frac{\pi}{2}, 0\right)$

$\Rightarrow \quad A+B \in [-\pi, 0)$

Also, $x^2+y^2>1$

$\Rightarrow \quad 1-x^2<y^2$ and $1-y^2<x^2$

$\Rightarrow \quad (1-x^2)(1-y^2)<x^2y^2$

$\Rightarrow \quad \sqrt{(1-x^2)(1-y^2)}<xy$ (since, $xy>0$)

$\Rightarrow \quad \sqrt{(1-x^2)(1-y^2)}-xy<0$

$\Rightarrow$ $\cos(A+B)<0$

$\Rightarrow$ $A+B$ lies either in the first quadrant or in the third quadrant.

$\Rightarrow \quad -\pi \le A+B \le \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{2} \le -(A+B) \le \pi$

$\Rightarrow \quad -\frac{\pi}{2} \le -\pi-(A+B) \le 0$

$\Rightarrow \quad \sin(A+B)=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$\Rightarrow \quad -\sin\{\pi+(A+B)\}=x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$\Rightarrow \quad \sin\{-\pi-(A+B)\}=\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\Rightarrow \quad -\pi-(A+B)=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\Rightarrow \quad A+B=-\pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

$\Rightarrow \quad \sin^{-1}x+\sin^{-1}y$

$=-\pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

(ii) $\sin^{-1}x-\sin^{-1}y$

$$=\begin{cases} \alpha : & x^2+y^2 \le 1 \\ \pi-\alpha : & x>0, y<0, x^2+y^2>1 \\ \alpha : & xy>0,\ x^2+y^2>1 \\ -\pi-\alpha : & x<0, y>0, x^2+y^2>1 \end{cases}$$

where $\alpha=\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)$

Proof: Do yourself.

(iii) $\cos^{-1}x+\cos^{-1}y$

$$=\begin{cases} \alpha & : x+y \ge 0 \\ 2\pi-\alpha & : x+y<0 \end{cases}$$

where $\alpha=\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$

Proof: Let $\cos^{-1}x=A$ and $\cos^{-1}y=B$

Then $x=\cos A$ and $y=\cos B$

$\Rightarrow \quad A \in [0, \pi]$ and $B \in [0, \pi]$

Now, $\sin A=\sqrt{1-x^2}$ and $\sin B=\sqrt{1-y^2}$

$\cos(A+B)=xy-\sqrt{1-x^2}\sqrt{1-y^2}$

$\cos(A-B)=xy+\sqrt{1-x^2}\sqrt{1-y^2}$

Case I:

when $-1 \le x,\ y \le 1$ and $x+y \ge 0$

In this case, $-1 \le x,\ y \le 1$

$\Rightarrow \quad 0 \le A+B \le 2\pi$ and $x+y \ge 0$

$\Rightarrow \quad 0 \le A+B \le 2\pi$ and $\cos A+\cos B \ge 0$

$\Rightarrow \quad \cos A \ge -\cos B$

$\Rightarrow \quad \cos A \ge \cos(\pi-B)$

$\Rightarrow \quad A \le \pi-B$

$\Rightarrow \quad 0 \le A+B \le \pi$

$\Rightarrow \quad \cos(A+B)=xy-\sqrt{1-x^2}\sqrt{1-y^2}$

$\Rightarrow \quad A+B=\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$

$\Rightarrow \cos^{-1}x+\cos^{-1}y=\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$

Case II:

when $-1 \le x,\ y<0$ and $x+y<0$

In this case $-1 \le x,\ y<0$

$\Rightarrow \quad A, B \in [0, \pi]$

$\Rightarrow \quad 0 \le A+B \le 2\pi$ and $x+y \le 0$

$\Rightarrow \cos A + \cos B \le 0$

$\Rightarrow \cos A \le \cos(\pi - B)$

$\Rightarrow A \ge \pi - B$

$\Rightarrow A + B \ge \pi$

Thus, $\pi \le A + B \le 2\pi$

$\Rightarrow -2\pi \le -(A+B) \le -\pi$

$\Rightarrow 0 \le 2\pi - (A+B) \le \pi$

Now, $\cos(A+B) = xy - \sqrt{1-x^2}\sqrt{1-y^2}$

$\Rightarrow \cos(2\pi - (A+B)) = xy - \sqrt{1-x^2}\sqrt{1-y^2}$

$\Rightarrow (2\pi - (A+B)) = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

$\Rightarrow A + B = 2\pi - \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

$\Rightarrow \cos^{-1} x + \cos^{-1} y$

$= 2\pi - \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

(iv) $\cos^{-1} x - \cos^{-1} y$

$$= \begin{cases} \alpha & : x \le y \\ -\alpha & : x > y \end{cases}$$

where $\alpha = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$

Proof: Do yourself.

(v) $\tan^{-1} x + \tan^{-1} y$

$$= \begin{cases} \alpha : & xy < 1 \\ \pi + \alpha & : x > 0, y > 0, xy > 1 \\ -\pi + \alpha & : x < 0, y < 0, xy > 1 \\ \frac{\pi}{2} & : x > 0, y > 0, xy = 1 \\ \frac{\pi}{2} & : x < 0, y < 0, xy = 1 \end{cases}$$

where $\alpha = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Proof: Let $\tan^{-1} x = A$ and $\tan^{-1} y = B$

Then $x = \tan A$ and $y = \tan B$.

$\Rightarrow A \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $B \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A . \tan B} = \frac{x+y}{1-xy}$$

Case I: when $x > 0, y > 0$ and $xy < 1$

In this case, $x > 0, y > 0$ and $xy < 1$

$\Rightarrow \frac{x+y}{1-xy} < 0$

$\Rightarrow \tan(A+B) > 0$

$\Rightarrow$ $A + B$ lies either in the first quadrant or in third quadrant.

$\Rightarrow 0 < A + B < \pi$

$\Rightarrow \tan(A+B) = \frac{x+y}{1-xy}$

$\Rightarrow (A+B) = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\Rightarrow \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Case II: when $x < 0, y < 0$ and $xy < 1$

In this case, $x < 0, y < 0$ and $xy < 1$

$\Rightarrow \frac{x+y}{1-xy} < 0$

$\Rightarrow \tan(A+B) < 0$

$\Rightarrow$ $A + B$ lies either in II quadrant or in IV quadrant

$\Rightarrow$ $A + B$ lies in the IV quadrant

$\Rightarrow -\pi < A + B < 0$

Now, $\tan(A+B) = \frac{x+y}{1-xy}$

$\Rightarrow A + B = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\Rightarrow \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Case III: when $(x > 0, y < 0)$ or $(x < 0$ and $y > 0)$

In this case, $x > 0$ and $y < 0$

$\Rightarrow A \in \left(0, \frac{\pi}{2}\right)$ and $B \in \left(-\frac{\pi}{2}, 0\right)$

$\Rightarrow A + B \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Now, $\tan(A+B) = \frac{x+y}{1-xy}$

$\Rightarrow \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Similarly, if $x < 0$ and $y > 0$, we have

$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

It follows from above three cases that

$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Case IV: when $x > 0$, $y > 0$ and $xy > 1$

In this case, we have $x, y > 0$ and $xy > 1$

$\Rightarrow \quad \dfrac{x+y}{1-xy} < 0$

$\Rightarrow \quad \tan(A+B) < 0$

$\Rightarrow \quad A+B$ either lies in the II quadrant or in the IV quadrant.

$\Rightarrow \quad A+B$ lies in the II quadrant.

$\Rightarrow \quad \dfrac{\pi}{2} < A+B < \pi$

$\Rightarrow \quad \dfrac{\pi}{2} - \pi < (A+B) - \pi < 0$

$\Rightarrow \quad -\dfrac{\pi}{2} < (A+B) - \pi < 0$

Now, $\tan(A+B) = \dfrac{x+y}{1-xy}$

$\Rightarrow \quad -\tan(\pi - (A+B)) = \dfrac{x+y}{1-xy}$

$\Rightarrow \quad A+B-\pi = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

$\Rightarrow \quad A+B = \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

$\Rightarrow \quad \tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

Case V: When $x < 0$, $y < 0$ and $xy > 1$

In this case $x, y < 0$ and $xy > 1$

$\Rightarrow \quad \dfrac{x+y}{1-xy} > 0$

$\Rightarrow \quad \tan(A+B) > 0$

$\Rightarrow \quad A+B$ lies either in the I quadrant or in the III quadrant

$\Rightarrow \quad A+B$ lies in the third quadrant

$\Rightarrow \quad -\pi < A+B < -\dfrac{\pi}{2}$

$\Rightarrow \quad \pi - \pi < \pi + (A+B) < \pi - \dfrac{\pi}{2}$

$\Rightarrow \quad 0 < \pi + (A+B) < \dfrac{\pi}{2}$

Now, $\quad \tan(A+B) = \dfrac{x+y}{1-xy}$

$\Rightarrow \quad \tan(\pi + (A+B)) = \dfrac{x+y}{1-xy}$

$\Rightarrow \quad (\pi + (A+B)) = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

$\Rightarrow \quad A+B = -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

$\Rightarrow \quad \tan^{-1}x + \tan^{-1}y = -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

(vi) $\tan^{-1} - \tan^{-1} y$

$$= \begin{cases} \alpha & : \quad xy > -1 \\ \pi + \alpha & : \ xy < -1, x > 0, \ y < 0 \\ -\pi + \alpha & : xy < -1, x < 0, y > 0 \\ \dfrac{\pi}{2} & : xy = -1, x > 0, \ y < 0 \\ -\dfrac{\pi}{2} & : xy = -1, \ x < 0, y > 0 \end{cases}$$

where $\alpha = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$.

Proof: Do yourself.

6.9.1 Some Solved Examples

Ex-1. Find the value of $\tan^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}\left(\dfrac{1}{3}\right)$

Soln. We have $\tan^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}\left(\dfrac{1}{3}\right)$

$= \tan^{-1}\left(\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)$

$= \tan^{-1}\left(\dfrac{5/6}{1-1/6}\right)$

$= \tan^{-1}\left(\dfrac{5/6}{5/6}\right) = \tan^{-1}(1) = \pi/4$

Ex-2. Find the value of $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3)$

Soln. We have, $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3)$

$= \dfrac{\pi}{4} + \pi + \tan^{-1}\left(\dfrac{2+3}{1-2.3}\right)$

$= \dfrac{\pi}{4} + \pi + \tan^{-1}(-1)$

$= \dfrac{\pi}{4} + \pi - \dfrac{\pi}{4}$

$= \pi$

Ex-3. Find the value of $\tan^{-1}(9)+\tan^{-1}\left(\frac{5}{4}\right)$.

Soln. We have, $\tan^{-1}(9)+\tan^{-1}\left(\frac{5}{4}\right)$

$$= \pi+\tan^{-1}\left(\frac{9+\frac{5}{4}}{1-9.\frac{5}{4}}\right)$$

$$= \pi+\tan^{-1}\left(\frac{\frac{41}{4}}{-\frac{41}{4}}\right)$$

$$= \pi+\tan^{-1}(-1)$$

$$= \pi-\frac{\pi}{4}=\frac{3\pi}{4}.$$

Ex-4. Find the value of

$\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)-\sin^{-1}\left(\frac{63}{65}\right)$.

Soln. We have, $\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)-\sin^{-1}\left(\frac{63}{65}\right)$

$$= \sin^{-1}\left(\frac{4}{5}.\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}.\sqrt{1-\left(\frac{4}{5}\right)^2}\right)-\sin^{-1}\left(\frac{63}{65}\right)$$

$$= \sin^{-1}\left(\frac{4}{5}.\frac{12}{13}+\frac{5}{13}.\frac{3}{13}\right)-\sin^{-1}\left(\frac{63}{65}\right)$$

$$= \sin^{-1}\left(\frac{48}{65}+\frac{15}{65}\right)-\sin^{-1}\left(\frac{63}{65}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\right)-\sin^{-1}\left(\frac{63}{65}\right)$$

$$= 0.$$

Ex-5. Prove that $2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\frac{\pi}{4}$.

Soln. We have, $2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$$= \tan^{-1}\left(\frac{2.\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}.\frac{1}{7}}\right)$$

$$= \tan^{-1}\left(\frac{25}{25}\right)$$

$$= \frac{\pi}{4}.$$

Ex-6. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, prove that

$x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$.

Soln. Let $\sin^{-1}x=A$, $\sin^{-1}y=B$, $\sin^{-1}z=C$

Then $x=\sin A$, $y=\sin B$, $z=\sin C$

we have, $x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}$

$$= \sin A.\cos A+\sin B.\cos B+\sin C.\cos C$$

$$= \frac{1}{2}(\sin 2A+\sin 2B+\sin 2C)$$

$$= \frac{1}{2}(4\sin A.\sin B.\sin C)$$

$$= 2\sin A.\sin B.\sin C$$

$$= 2xyz.$$

Ex-7. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$

prove that $x^2+y^2+z^2+2xyz=1$

Soln. We have, $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$

$$\Rightarrow \cos^{-1}x+\cos^{-1}y=\pi-\cos^{-1}z$$

$$\Rightarrow \cos^{-1}x+\cos^{-1}y=\cos^{-1}(-z)$$

$$\Rightarrow \cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)=\cos(-z)$$

$$\Rightarrow (xy+z)^2=(1-x^2)(1-y^2)$$

$$\Rightarrow x^2y^2+2xyz+z^2=1-x^2-y^2+x^2y^2$$

$$\Rightarrow x^2+y^2+z^2+2xyz=1$$

Ex-8. If $\cos^{-1}\left(\frac{x}{2}\right)+\cos^{-1}\left(\frac{y}{3}\right)=\theta$, prove that,

$9x^2+12xy\cos\theta+4y^2=36\sin^2\theta$

Soln. Given $\cos^{-1}\left(\frac{x}{2}\right)+\cos^{-1}\left(\frac{y}{3}\right)=\theta$

$$\Rightarrow \cos^{-1}\left(\frac{x}{2}.\frac{y}{3}-\sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right)=\theta$$

$$\Rightarrow \quad \left(\frac{xy}{6}-\sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right)=\cos\theta$$

$$\Rightarrow \quad \left(\frac{xy}{6}-\cos\theta\right)^2=\left(1-\frac{x^2}{4}\right)\left(1-\frac{y^2}{9}\right)$$

$$\Rightarrow \quad \frac{x^2y^2}{36}-\frac{xy}{3}\cos\theta+\cos^2\theta$$

$$=1-\frac{x^2}{4}-\frac{y^2}{9}+\frac{x^2y^2}{36}$$

$$\Rightarrow \quad \frac{x^2}{4}+\frac{y^2}{9}-\frac{xy}{3}\cos\theta=1-\cos^2\theta$$

$$\Rightarrow \quad \frac{x^2}{4}+\frac{y^2}{9}-\frac{xy}{3}\cos\theta=\sin^2\theta$$

$$\Rightarrow \quad 9x^2+4y^2-12\,xy\cos\theta=36\sin^2\theta$$

Ex-9. Let $m=\dfrac{\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)}{\cot^{-1}(1)+\cot^{-1}(2)+\cot^{-1}(3)}$

then find the value of $(m-1)^{2013}$.

Soln. We have, $\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)=\pi$

and $\cot^{-1}(1)+\cot^{-1}(2)+\cot^{-1}(3)$

$$=\tan^{-1}(1)+\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)$$

$$=\frac{\pi}{4}+\frac{\pi}{4}$$

$$=\frac{\pi}{2}$$

Hence, $m=\dfrac{\pi}{\pi/2}=2$.

Ex-10. Solve for x: $\tan^{-1}(2x)+\tan^{-1}(3x)=\dfrac{3\pi}{4}$.

Soln. **Case I:** When $x\le 0$

Then, $\tan^{-1}(2x)\le 0,\ \tan^{-1}(3x)\le 0,$

$$\Rightarrow \quad x\le 0$$

So, it has no solution.

Case II: When $x>0$, $2\,x.\,3\,x=6x^2<1$

$$\Rightarrow \quad x<\frac{1}{\sqrt{6}}$$

Then $\dfrac{3\pi}{4}=\tan^{-1}(2x)+\tan^{-1}(3x)<\dfrac{\pi}{2}$

So, it is not possible.

Case III: when $x>0$, $2x.3x>1$

$$\Rightarrow \quad x>\frac{1}{\sqrt{6}}$$

Then $\dfrac{3\pi}{4}=\tan^{-1}(2x)+\tan^{-1}(3x)$

$$\Rightarrow \quad \pi+\tan^{-1}\left(\frac{5x}{1-6x^2}\right)=\frac{3\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{5x}{1-6x^2}\right)=\frac{3\pi}{4}-\pi=-\frac{\pi}{4}$$

$$\Rightarrow \quad \left(\frac{5x}{1-6x^2}\right)=-1$$

$$\Rightarrow \quad 6x^2-5x-1=0$$

$$\Rightarrow \quad x=1,-1/6$$

Thus, $x=1$ is a solution.

Ex-11. Solve for x: $\sin^{-1}(x)+\sin^{-1}(2x)=\dfrac{\pi}{3}$

Soln. We have, $\sin^{-1}(x)+\sin^{-1}(2x)=\dfrac{\pi}{3}$

$$\Rightarrow \quad \sin^{-1}x+\sin^{-1}(2x)=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \quad \sin^{-1}x-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=-\sin^{-1}(2x)$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{2}-\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right)=\sin^{-1}(-2x)$$

$$\Rightarrow \quad \left(\frac{x}{2}-\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right)=-2x$$

$$\Rightarrow \quad 5x=\sqrt{3}\sqrt{1-x^2}$$

$$\Rightarrow \quad 25x^2=3(1-x^2)$$

$$\Rightarrow \quad 28x^2=3$$

$$\Rightarrow \quad x=\pm\frac{\sqrt{3}}{2\sqrt{7}}$$

$\Rightarrow \quad x=\dfrac{\sqrt{3}}{2\sqrt{7}}$, negative value of x does not satisfy the given equation.

Ex-12. Let $f(x)=\cos^{-1}(x)+\cos^{-1}\left(\dfrac{x}{2}+\dfrac{\sqrt{3-3x^2}}{2}\right)$,

for $\dfrac{1}{2}\le x\le 1$

Then find $f(2013)$.

Soln. We have $f(x)$

$= \cos^{-1}(x)+\cos^{-1}\left(\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right)$

$= \cos^{-1}(x)+\cos^{-1}\left(\frac{1}{2}\right)-\cos^{-1}(x)$

$= \frac{\pi}{3}$

Now, $f(2013)=\frac{\pi}{3}$

EXERCISE 6

1. $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$
2. $x^2-4x > \sin^{-1}\left(\sin\left[\pi^{3/2}\right]\right)+\cos^{-1}\left(\cos\left[\pi^{3/2}\right]\right)$
3. $\cos\left(\tan^{-1}x\right)=x$
4. $\sin\left(\tan^{-1}x\right)=\cos\left(\cot^{-1}(x+1)\right)$
5. $\sec^{-1}\left(\frac{x}{2}\right)-\sec^{-1}x=\sec^{-1}2$
6. $\cos\left(\tan^{-1}\left(\cot\left(\sin^{-1}\left(x+\frac{3}{2}\right)\right)\right)\right)$
   $+\tan\left(\sec^{-1}x\right)=0$
7. Find the smallest +ve integer x so that
   $\tan\left(\tan^{-1}\left(\frac{x}{10}\right)+\tan^{-1}\left(\frac{1}{x+1}\right)\right)=\tan\left(\frac{\pi}{4}\right)$
8. Find the least integral value of k for which
   $(k-2)x^2+8x+k+4>\sin^{-1}(\sin 12)+\cos^{-1}(\cos 12)$
   holds for all x in R.
9. If $\alpha=2\tan^{-1}\left(\frac{1+x}{1-x}\right)$ and
   $\beta=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for $0<x<1$,
   then prove that $\alpha+\beta=\pi$.
10. Let $f(x)=\sin^{-1}(\sin x)$, $\forall\ x\in[-\pi, 2\pi]$,
    then find $f'(x)$.
11. Let $f(x)=\cos^{-1}(\cos x)$, $\forall\ x\in[-2\pi, \pi]$,
    then find $f'(x)$.
12. Let $f(x)=\tan^{-1}(\tan x)$,
    $\forall\ x\in\left[-\frac{3\pi}{2},\frac{5\pi}{2}\right]$, then find $f'(x)$.

Q. Prove that each of the following:

13. $\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{2}{9}\right)$
14. $\tan^{-1}(9)+\tan^{-1}\left(\frac{5}{4}\right)=\frac{3\pi}{4}$
15. $\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{8}{19}\right)=\frac{\pi}{4}$
16. $\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$
17. $2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\frac{\pi}{4}$
18. $2\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$
19. $\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{8}{17}\right)=\sin^{-1}\left(\frac{77}{85}\right)$
20. $\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right)$
21. $\cos^{-1}\left(\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{7}\right)+\cos^{-1}\left(\frac{13}{14}\right)=\pi$
22. $\sin^{-1}\left(\frac{1}{5}\right)+\cot^{-1}(3)=\frac{\pi}{4}$.
23. $2\tan^{-1}\left(\frac{3}{2}\right)+\tan^{-1}\left(\frac{12}{5}\right)=\pi$.

Q. Write the simplest form of each of the following inverse trigonometric functions:

24. $\cos^{-1}\sqrt{1-x^2}$
25. $\cos^{-1}\left(1-2x^2\right)$
26. $\cos^{-1}\left(2x^2-1\right)$
27. $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$, $-\frac{\pi}{2}<x<\frac{\pi}{2}$
28. $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$, $-\frac{\pi}{2}<x<\frac{\pi}{2}$
29. $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$, $0<x<\pi$
30. $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$, $-a<x<a$
31. $\sin^{-1}\left(\sqrt{\frac{x}{1+x}}\right)$
32. $\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$

33. $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

34. $\tan^{-1}\left(\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right),\ 0<x<\pi$

35. $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right)$

36. $\cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

37. $\sin^{-1}\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right)$

38. $\sin^{-1}\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right), -\dfrac{\pi}{4}<x<\dfrac{\pi}{4}$

39. $\cos^{-1}\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right), \dfrac{\pi}{4}<x<\dfrac{5\pi}{4}$

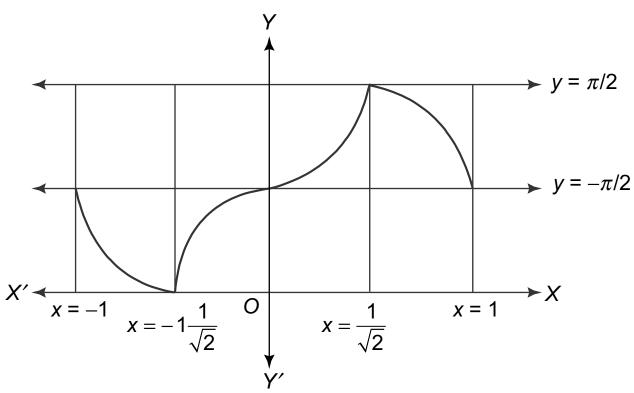
40. $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

41. $\sin^{-1}\left(\dfrac{3}{5}\cos x+\dfrac{4}{5}\sin x\right)$.

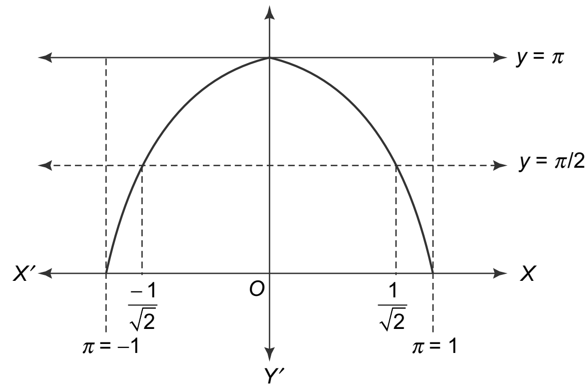
6.10 MULTIPLE ANGLES

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$$=\begin{cases} 2\sin^{-1}x & : -\dfrac{1}{\sqrt{2}}\le x\le\dfrac{1}{\sqrt{2}} \\ \pi-2\sin^{-1}x & : \dfrac{1}{\sqrt{2}}<x\le 1 \\ -\pi-2\sin^{-1}x & : -1\le x<-\dfrac{1}{\sqrt{2}} \end{cases}$$

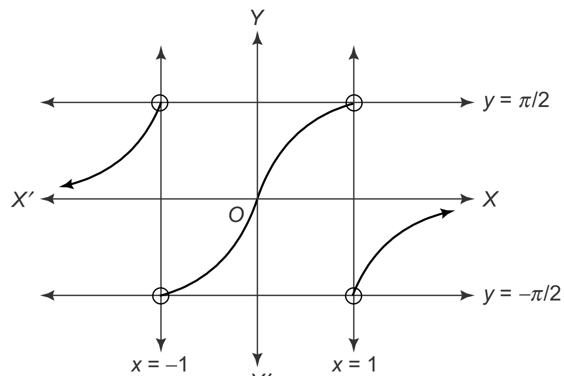


(ii) $\cos^{-1}\left(2x^2-1\right)=\begin{cases} 2\cos^{-1}x & :0\le x\le 1 \\ 2\pi-2\cos^{-1}x & :-1\le x<0 \end{cases}$



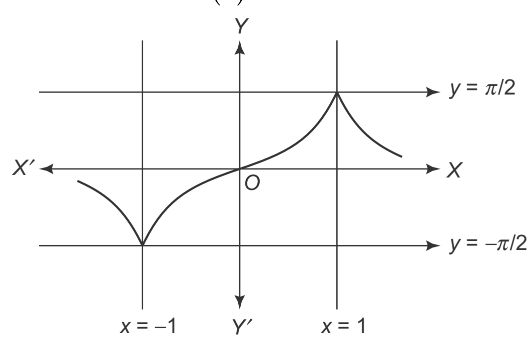
(iii) $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)=\begin{cases} \alpha & :-1<x<1 \\ -\pi+\alpha & :x>1 \\ \pi+\alpha & :x<-1 \end{cases}$

where $\alpha=2\tan^{-1}(x)$.

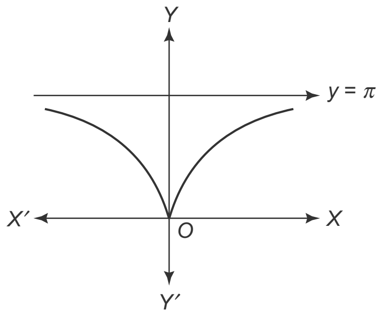


(iv) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)=\begin{cases} \alpha & :-1\le x\le 1 \\ \pi-\alpha & :x>1 \\ -\pi-\alpha & :x<-1 \end{cases}$

where $\alpha=2\tan^{-1}(x)$



(v) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)=\begin{cases} 2\tan^{-1}(x) & :x\ge 0 \\ -2\tan^{-1}(x) & :x\le 0 \end{cases}$



6.10.1 Some Solved Examples

Ex-1. Find the value of $\sin\left(2\sin^{-1}\left(\frac{1}{4}\right)\right)$

Soln. Let $\sin^{-1}\left(\frac{1}{4}\right)=\theta$

$\Rightarrow \quad \sin\theta=\frac{1}{4}$

Now, $\sin(2\theta)$

$= 2\sin\theta.\cos\theta$

$= 2\times\frac{1}{4}\times\sqrt{1-\frac{1}{16}}$

$= \frac{\sqrt{15}}{8}$

Ex-2. Find the value of $\cos\left(2\cos^{-1}\left(\frac{1}{3}\right)\right)$

Soln. Let $\cos^{-1}\left(\frac{1}{3}\right)=\theta$

$\Rightarrow \quad \cos\theta=\frac{1}{3}$

Now, $\cos(2\theta)$

$= 2\cos^2\theta-1$

$= \frac{2}{9}-1$

$= -\frac{7}{9}$

Ex-3. Find the value of $\cos\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)$

Soln. Let $\tan^{-1}\left(\frac{1}{3}\right)=\theta$

$\Rightarrow \quad \tan\theta=\frac{1}{3}$

Now, $\cos(2\theta)$

$= \frac{1-\tan^2\theta}{1+\tan^2\theta}$

$= \frac{1-\left(\frac{1}{3}\right)^2}{1+\left(\frac{1}{3}\right)^2}$

$= \frac{9-1}{9+1}$

$= \frac{8}{10}=\frac{4}{5}$

Ex-4. Find the value of $\sin\left(\frac{1}{2}\cot^{-2}\left(\frac{3}{4}\right)\right)$

Soln. Let $\cot^{-1}\left(\frac{3}{4}\right)=\theta$

Now, $\sin\left(\frac{\theta}{2}\right)$

$= \sqrt{\frac{1-\cos\theta}{2}}$

$= \sqrt{\frac{1-\frac{\cos\theta}{\sin\theta.\text{cosec}\,\theta}}{2}}$

$= \sqrt{\frac{1-\frac{\cot\theta}{\text{cosec}\,\theta}}{2}}$

$= \sqrt{\frac{1-\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}}{2}}$

$= \frac{1}{\sqrt{2}}\sqrt{1-\frac{3/4}{\sqrt{1+9/16}}}$

$= \frac{1}{\sqrt{2}}\sqrt{1-\frac{3}{5}}$

$= \frac{2}{\sqrt{5}}$

Ex-5. Find the value of $\tan^{-1}\left(\frac{3\pi}{4}-2\tan^{-1}\left(\frac{3}{4}\right)\right)$

Soln. Let $\tan^{-1}\left(\frac{3}{4}\right)=\theta$

$\Rightarrow \quad \tan\theta=\frac{3}{4}$

We have, $\tan\left(\frac{3\pi}{4}-2\theta\right)$

$$= \frac{\tan\left(\frac{3\pi}{4}\right) - \tan(2\theta)}{1 + \tan\left(\frac{3\pi}{4}\right).\tan(2\theta)}$$

$$= \frac{-1 - \tan(2\theta)}{1 - \tan(2\theta)}$$

$$= \frac{-1 - \frac{2\tan\theta}{1 - \tan^2\theta}}{1 - \frac{2\tan\theta}{1 - \tan^2\theta}}$$

$$= \frac{\tan^2\theta - 2\tan\theta - 1}{1 - \tan^2\theta - 2\tan\theta}$$

$$= \frac{\frac{9}{16} - \frac{6}{4} - 1}{1 - \frac{9}{16} - \frac{6}{4}} = \frac{41}{17}$$

EXERCISE 7

Q. Prove that each of the following inverse trigonometric functions:

1. $\sin\left(2\sin^{-1}\left(\frac{1}{2}\right)\right) = \frac{\sqrt{3}}{2}$
2. $\sin\left(3\sin^{-1}\left(\frac{1}{3}\right)\right) = \frac{23}{27}$
3. $\cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right) = \frac{3}{4}$
4. $\cos\left(\frac{1}{2}\cos^{-1}\left(-\frac{1}{10}\right)\right) = \frac{3\sqrt{5}}{10}$
5. $\sin\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{9}\right)\right) = \frac{2}{3}$
6. $\sin\left(\frac{1}{4}\tan^{-1}\left(\sqrt{63}\right)\right) = \frac{1}{2\sqrt{2}}$
7. $\cos\left(\frac{1}{4}\left(\tan^{-1}\left(\frac{24}{7}\right)\right)\right) = \frac{3}{\sqrt{10}}$
8. $\tan\left(\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right)\right) = \frac{1}{\sqrt{5}}$
9. $\tan\left(2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right) = -\frac{7}{17}$
10. $\tan\left(\frac{3\pi}{4} - \frac{1}{4}\sin^{-1}\left(-\frac{4}{5}\right)\right) = \frac{1-\sqrt{5}}{2}$
11. Find the integral values of x satisfying the inequation $x^2 - 3x < \sin^{-1}(\sin 2)$.
12. Find the value of x satisfying the inequation $3x^2 + 8x < 2\sin^{-1}(\sin 4) - \cos^{-1}(\cos 4)$.
13. For what value of x,

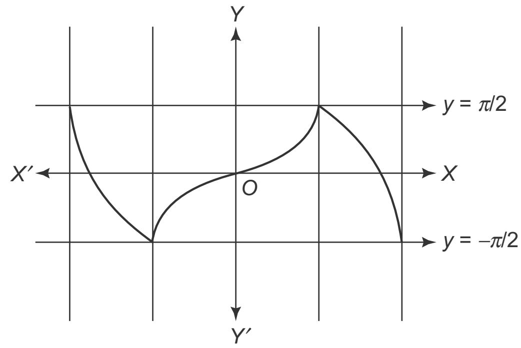
    $$f(x) = \cos^{-1} x + \cos^{-1}\left\{\frac{x}{2} + \frac{\sqrt{1-3x^2}}{2}\right\}$$

    is a constant function.

6.11 MORE MULTIPLE ANGLES

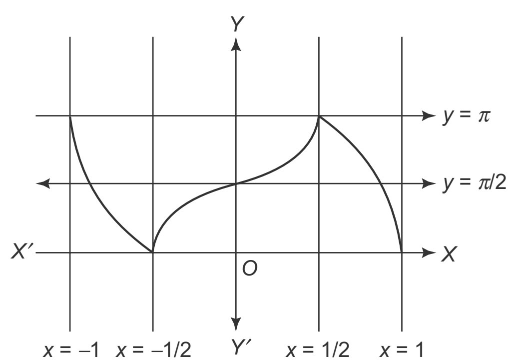
(i) $\sin^{-1}(3x - 4x^3)$

$$= \begin{cases} 3\sin^{-1} x & : -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1} x & : \frac{1}{2} < x \le 1 \\ -\pi - 3\sin^{-1} x & : -1 \le x < -\frac{1}{2} \end{cases}$$



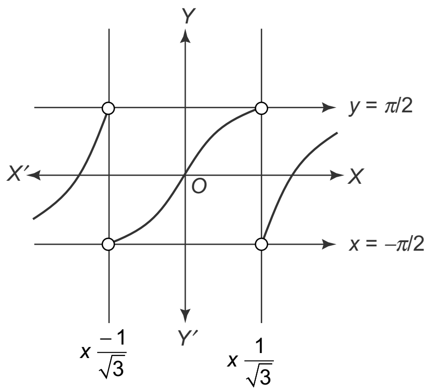
(ii) $\cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} 3\cos^{-1} x & : \frac{1}{2} \le x \le 1 \\ 2\pi - 3\cos^{-1} x & : -\frac{1}{2} \le x < \frac{1}{2} \\ -2\pi + 3\cos^{-1} x & : -1 \le x < -\frac{1}{2} \end{cases}$$



(iii) $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

$$= \begin{cases} 3\tan^{-1}x & : -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + 3\tan^{-1}x & : -\infty < x < -\dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x & : \dfrac{1}{\sqrt{3}} < x < \infty \end{cases}$$



6.11.1 Some Solved Examples

Ex-1. Let $f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) + 2\tan^{-1}(x)$, $x > 1$,

then find the value of $f(2013)$.

Soln. We have, $f(x)$

$= \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) + 2\tan^{-1}(x)$

$= \pi - 2\tan^{-1}(x) + 2\tan^{-1}(x)$

$= \pi$

Hence, the value of

$f(2013) = \pi$.

Ex-2. Let $f(x) = 2\tan^{-1}\left(\dfrac{1+x}{1-x}\right) + \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

for $0 \le x < 1$. Then find the value of $f\left(\dfrac{1}{2014}\right)$.

Soln. We have, $f(x)$

$= 2\tan^{-1}\left(\dfrac{1+x}{1-x}\right) + \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$= 2\left(\tan^{-1}(1) + \tan^{-1}(x)\right) + \dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$= 2\left(\dfrac{\pi}{4} + \tan^{-1}(x)\right) + \dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$= \left(\dfrac{\pi}{2} + 2\tan^{-1}(x)\right) + \left(\dfrac{\pi}{2} - 2\tan^{-1}(x)\right)$

$= \pi$

Hence, the value of $f\left(\dfrac{1}{2014}\right) = \pi$.

Ex-3. Let $f(x) = \sin^{-1}\left(\dfrac{6x}{x^2+9}\right) + 2\tan^{-1}\left(-\dfrac{x}{3}\right)$

is independent of x, then find the value of x.

Soln. We have, $f(x)$

$= \sin^{-1}\left(\dfrac{6x}{x^2+9}\right) + 2\tan^{-1}\left(-\dfrac{x}{3}\right)$

$= \sin^{-1}\left(\dfrac{2.\left(\dfrac{x}{3}\right)}{1+\left(\dfrac{x}{3}\right)^2}\right) - 2\tan^{-1}\left(\dfrac{x}{3}\right)$

$= 2\tan^{-1}\left(\dfrac{x}{3}\right) - 2\tan^{-1}\left(\dfrac{x}{3}\right)$

$= 0$

It will happen when $\left|\dfrac{x}{3}\right| \le 1$

$\Rightarrow \quad |x| \le 3$

$\Rightarrow \quad -3 \le x \le 3$

Ex-4. Find the interval of x for which the function

$f(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) + 2\tan^{-1}(x)$

is a constant function.

Soln. We have, $f(x)$

$= \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) + 2\tan^{-1}(x)$

$= -2\tan^{-1}(x) + 2\tan^{-1}(x)$, $x \le 0$

$= 0$

It is possible only when $x \le 0$

$\Rightarrow \quad x \in (-\infty, 0]$.

Ex-5. Find the interval of x for which the function $f(x)$

$= 3\cos^{-1}\left(2x^2-1\right) + 2\cos^{-1}\left(4x^3-3x\right)$

is independent of x.

Soln. We have $f(x)$

$= 3\cos^{-1}\left(2x^2-1\right) + 2\cos^{-1}\left(4x^3-3x\right)$

$= 3\left(2\cos^{-1}x\right) + 2\left(2\pi - 3\cos^{-1}x\right)$

(for $0 \le x < 1$) $\left(\text{for } -\frac{1}{2} \le x \le \frac{1}{2}\right)$

It is possible only when $0 \le x \le \frac{1}{2}$

$\Rightarrow \quad x \in \left[0, \frac{1}{2}\right]$

Also, $f(x)$

$= 3\cos^{-1}\left(2x^2 - 1\right) + 2\cos^{-1}\left(4x^3 - 3x\right)$

$= 3\left(2\pi - 2\cos^{-1} x\right) + 2\left(-2\pi + 3\cos^{-1} x\right)$

(for $-1 \le x \le 0$) (for $-1 \le x < -\frac{1}{2}$)

$= 2\pi$.

It is possible only when $x \in \left[-1, -\frac{1}{2}\right)$

Hence, the value of x is

$\left[-1, -\frac{1}{2}\right) \cup \left[0, \frac{1}{2}\right]$.

Ex-6. If $\tan^{-1} y : \tan^{-1} x = 4:1$, then express y as algebraic function of x. Also, prove that $\tan\left(22\frac{1^\circ}{2}\right)$ is a root of $x^4 - 6x^2 + 1 = 0$.

Soln. We have, $\tan^{-1} y = 4\tan^{-1} x$

$\Rightarrow \quad \tan^{-1} y = \tan^{-1}\left(\frac{4x - 4x^3}{1 - 6x^2 + x^4}\right)$

$\Rightarrow \quad y = \frac{4x\left(1 - x^2\right)}{1 - 6x^2 + x^4}$

Which is a function of x.

Let $\tan^{-1} x = \frac{\pi}{8}$

$\Rightarrow \quad x = \tan\left(\frac{\pi}{8}\right)$

$\Rightarrow \quad \tan^{-1} y = 4\tan^{-1} x = \frac{\pi}{2}$

$\Rightarrow \quad \frac{4x\left(1 - x^2\right)}{1 - 6x^2 + x^4} \to \infty$

$\Rightarrow \quad 1 - 6x^2 + x^4 = 0$

$\Rightarrow \quad x = \tan\left(\frac{\pi}{8}\right)$ is a root of $1 + x^4 = 6x^2$

EXERCISE 8

Q. Prove that

1. $\tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right) = \left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2\right)$

2. $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right) = \frac{2b}{a}$

3. $\tan^{-1}\left(\frac{p-q}{1+pq}\right) + \tan^{-1}\left(\frac{q-r}{1+qr}\right) + \tan^{-1}\left(\frac{r-p}{1+pr}\right) = \pi$ where $p > q > 0$ and $pr < -1 < qr$,

4. $\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right) = 0$

5. $\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right) = \left(\frac{x+y}{1-xy}\right)$, $xy < 1$

6. $\tan^{-1}\left(\frac{1-x}{1+x}\right) - \tan^{-1}\left(\frac{1-y}{1+y}\right) = \sin^{-1}\left(\frac{y-x}{\sqrt{(1+x^2)(1+y^2)}}\right)$

7. $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}\left(\cot A\right) + \tan^{-1}\left(\cot^3 A\right) = 0$

8. $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{\theta}{2}\right)\right) = \cos^{-1}\left(\frac{b + a\cos\theta}{a + b\cos\theta}\right)$

9. $\tan\left(2\tan^{-1} a\right) = 2\tan\left(\tan^{-1} a + \tan^{-1} a^3\right)$

10. $\cos^{-1} x + \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3 - 3x^2}}{2}\right) = \frac{\pi}{3}$, $\frac{1}{2} < x < 1$

11. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, then prove that $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$

12. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then prove that, $x^2 + y^2 + z^2 + 2xyz = 1$

13. If $\cos^{-1}\left(\frac{x}{2}\right) + \cos^{-1}\left(\frac{y}{3}\right) = \theta$, then prove that, $9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$.

14. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$, then prove that, $x^2 + y^2 + z^2 - 2xyz = 1$.

15. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then prove that $xy + yz + zx = 3$.
16. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then find the value of $x^{2012}+y^{2012}+z^{2012}-\frac{9}{x^{2013}+y^{2013}+z^{2013}}$
17. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, then prove that, $xy + yz + zx = 3$.
18. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, then find the value of $\left(\frac{x^{2013}+y^{2013}+z^{2013}+6}{x^{2014}+y^{2014}+z^{2014}}\right)$.
19. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, then prove that $xy + yz + zx = 1$.
20. If $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$, then prove that $x + y + xy = 1$.
21. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then prove that $x + y + z = x\,y\,z$.
22. If $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)=\alpha$, then prove that $x^2=\sin 2\alpha$.
23. Let $m = \tan^2(\sec^{-1}2) + \cot^2(\operatorname{cosec}^{-1}3)$, then find the value of (m^2+m+10).
24. If $\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)=\frac{\pi}{4}$, then find the value of $\tan\theta$.
25. Let $m=\frac{(\tan^{-1}1+\tan^{-1}2+\tan^{-1}3)}{(\cot^{-1}1+\cot^{-1}2+\cot^{-1}3)}$, then prove that $(m+2)^{m+1}=64$.

Q. Solve for x:

26. $\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{3\pi}{4}$
27. $\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x-1}{x}\right)=\tan^{-1}(-7)$
28. $\sin^{-1}(2x)+\sin^{-1}(x)=\frac{\pi}{3}$
29. $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\cos^{-1}x=\frac{\pi}{4}$
30. $\sin^{-1}(x)+\sin^{-1}(3x)=\frac{\pi}{3}$
31. $\tan^{-1}\left(\frac{1}{1+2x}\right)+\tan^{-1}\left(\frac{1}{1+4x}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$
32. $2\tan^{-1}(2x+1)=\cos^{-1}x$
33. $\cos^{-1}x-\sin^{-1}x=\cos^{-1}(x\sqrt{3})$
34. If $\tan^{-1}y:\tan^{-1}x=4:1$, express y as an algebraic function of x. Hence, prove that $\tan\left(\frac{\pi}{8}\right)$ is a root of $x^4+1=6x^2$.

PROBLEMS FOR JEE MAIN EXAM

Ex-1. Find the principal value of $\sin^{-1}(\sin 10)$.

Soln. We have $\sin^{-1}(\sin 10)$

$=\sin^{-1}(\sin(3\pi-10))$

$=(3\pi-10)$

Ex-2. Find the principal value of $\cos^{-1}(\cos 5)$.

Soln. We have $\cos^{-1}(\cos 5)$

$=\cos^{-1}(\cos(2\pi-5))$

$=(2\pi-5)$.

Ex-3. Find the value of $\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)$.

Soln. We have $\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)$

$=\tan^{-1}(1)+\pi+\tan^{-1}\left(\frac{2+3}{1-2.3}\right)$

$=\frac{\pi}{4}+\pi+\tan^{-1}\left(-\frac{5}{5}\right)$

$=\frac{\pi}{4}+\pi+\tan^{-1}(-1)$

$=\frac{\pi}{4}+\pi-\tan^{-1}(1)$

$=\frac{\pi}{4}+\pi-\frac{\pi}{4}$

$=\pi$.

Ex-4. Find x, if $\sin^{-1}x>\cos^{-1}x$.

Soln. Given $\sin^{-1}x>\cos^{-1}x$

$\Rightarrow\quad \sin^{-1}x+\sin^{-1}x>\sin^{-1}x+\cos^{-1}x$

$\Rightarrow\quad 2\sin^{-1}x>\frac{\pi}{2}$

$\Rightarrow\quad \sin^{-1}x>\frac{\pi}{4}$

$\Rightarrow \quad x > \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Also, the domain of $\sin^{-1} x$ is [–1, 1].

Thus, the solution set is $x \in \left(\frac{1}{\sqrt{2}}, 1\right]$

Ex-5. Find x, if $\sin^{-1} x < \cos^{-1} x$

Soln. Given $\sin^{-1} x < \cos^{-1} x$

$\Rightarrow \quad \sin^{-1} x + \sin^{-1} x < \sin^{-1} x + \cos^{-1} x$

$\Rightarrow \quad 2\sin^{-1} x < \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1} x < \frac{\pi}{4}$

$\Rightarrow \quad x < \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Also, the domain of $\sin^{-1} x$ is [–1, 1]

Thus, the solution set is $x \in \left[-1, \frac{1}{\sqrt{2}}\right)$

Ex-6. Find x, if $2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Soln. Given $2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Now, the range of $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Thus, $-\frac{\pi}{2} \le 2\sin^{-1} x \le \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{4} \le \sin^{-1} x \le \frac{\pi}{4}$

$\Rightarrow \quad \sin\left(-\frac{\pi}{4}\right) \le x \le \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad -\sin\left(\frac{\pi}{4}\right) \le x \le \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

Hence, the solution set is $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

Ex-7. Find x, if $3\sin^{-1} x = \pi + \sin^{-1}\left(3x - 4x^3\right)$

Soln. Given $3\sin^{-1} x = \pi + \sin^{-1}\left(3x - 4x^3\right)$

Now, the range of $\pi + \sin^{-1}\left(3x - 4x^3\right)$

is $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Thus, $\frac{\pi}{2} \le 3\sin^{-1} x \le \frac{3\pi}{2}$

$\Rightarrow \quad \frac{\pi}{6} \le \sin^{-1} x \le \frac{\pi}{2}$

$\Rightarrow \quad \sin\left(\frac{\pi}{6}\right) \le x \le \sin\left(\frac{\pi}{2}\right)$

$\Rightarrow \quad \frac{1}{2} \le x \le 1.$

Hence, the solution set is $\left[\frac{1}{2}, 1\right]$

Ex-8. Find x, if $2\tan^{-1} x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Soln. Given $2\tan^{-1} x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Now, the range of $\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

is $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Thus, $\frac{\pi}{2} \le 2\tan^{-1} x \le \frac{3\pi}{2}$

$\Rightarrow \quad \frac{\pi}{4} \le \tan^{-1} x \le \frac{3\pi}{4}$

$\Rightarrow \quad \tan\left(\frac{\pi}{4}\right) \le x < \tan\left(\frac{\pi}{2}\right)$

and $\tan\left(\frac{\pi}{2}\right) < x \le \tan\left(\frac{3\pi}{4}\right)$

$\Rightarrow \quad 1 \le x < \infty \;\; \& \; -\infty < x \le -1$

Therefore, the solution set is

$x \in (-\infty, -1] \cup [1, \infty)$.

Ex-9. Find the value of $\cos\left(\frac{\pi}{6} + \cos^{-1}\left(-\frac{1}{2}\right)\right)$

Soln. We have $\cos\left(\frac{\pi}{6} + \cos^{-1}\left(-\frac{1}{2}\right)\right)$

$= \cos\left(\frac{\pi}{6} + \pi - \cos^{-1}\left(\frac{1}{2}\right)\right)$

$= \cos\left(\frac{\pi}{6} + \pi - \frac{\pi}{3}\right)$

$= \cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)$

$= \cos\left(\frac{5\pi}{6}\right)$

$$= \cos\left(\pi - \frac{\pi}{6}\right)$$

$$= -\cos\left(\frac{\pi}{6}\right)$$

$$= -\frac{1}{2}$$

Ex-10. Find the value of $\cos^{-1}\left(\cos\left(2\cot^{-1}\left(\sqrt{2}-1\right)\right)\right)$

Soln. We have $\cos^{-1}\left(\cos\left(2\cot^{-1}\left(\sqrt{2}-1\right)\right)\right)$

$$= \cos^{-1}\left(\cos\left(2\cos^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}}\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(\frac{2\left(\sqrt{2}-1\right)^2}{\left(4-2\sqrt{2}\right)}-1\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(\frac{2\left(3-2\sqrt{2}\right)-4+2\sqrt{2}}{\left(4-2\sqrt{2}\right)}\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(\frac{2-2\sqrt{2}}{\left(4-2\sqrt{2}\right)}\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(\frac{2\left(1-\sqrt{2}\right)}{2\sqrt{2}\left(\sqrt{2}-1\right)}\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\pi - \frac{\pi}{4}\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right)$$

$$= \left(\frac{3\pi}{4}\right)$$

Ex-11. Find the value of $\sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)$

Soln. We have $\sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)$

$$= \sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r(r+1)}\right)$$

$$= \sum_{r=0}^{\infty} \tan^{-1}\left(\frac{(r+1)-r}{1+r(r+1)}\right)$$

$$= \sum_{r=0}^{\infty} \left(\tan^{-1}(r+1) - \tan^{-1}(r)\right)$$

$$= \sum_{r=0}^{n} \left(\tan^{-1}(r+1) - \tan^{-1}(r)\right),\ n \to \infty$$

$$= \left(\tan^{-1}(2) - \tan^{-1}(1)\right) + \left(\tan^{-1}(3) - \tan^{-1}(2)\right) + \left(\tan^{-1}(4) - \tan^{-1}(3)\right) + \left(\tan^{-1}(5) - \tan^{-1}(4)\right) + \dots\dots\dots + \left(\tan^{-1}(n+1) - \tan^{-1}(n)\right)$$

$$= \left(\tan^{-1}(n+1) - \tan^{-1}(1)\right),\ n \to \infty$$

$$= \tan^{-1}\left(\frac{(n+1)-1}{1+(n+1).1}\right),\ n \to \infty$$

$$= \tan^{-1}\left(\frac{n}{n+2}\right),\ n \to \infty$$

$$= \tan^{-1}(1), \text{when } n \to \infty$$

$$= \frac{\pi}{4}$$

Ex-12. Find the value of $\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right)$

Soln. we have $\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right)$

$$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^{r-1}}{1+2^r.2^{r-1}}\right)$$

$$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^r - 2^{r-1}}{1+2^r.2^{r-1}}\right)$$

$$= \sum_{r=1}^{n} \left(\tan^{-1}\left(2^r\right) - \tan^{-1}\left(2^{r-1}\right)\right)$$

$$= \left(\tan^{-1}(2) - \tan^{-1}(1)\right) + \left(\tan^{-1}\left(2^2\right) - \tan^{-1}(2)\right) + \left(\tan^{-1}\left(2^3\right) - \tan^{-1}\left(2^2\right)\right) + \dots\dots + \left(\tan^{-1}\left(2^n\right) - \tan^{-1}\left(2^{n-1}\right)\right)$$

$$= \tan^{-1}\left(2^n\right) - \tan^{-1}(1)$$

$$= \tan^{-1}\left(2^n\right) - \frac{\pi}{4}$$

Ex-13. Find the value of $\sum_{r=1}^{n} \sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right)$

Soln. We have $\sum_{r=1}^{n}\sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right)$

$= \sum_{r=1}^{n}\tan^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r}.\sqrt{r-1}}\right)$

$= \sum_{r=1}^{n}\left(\tan^{-1}\left(\sqrt{r}\right)-\tan^{-1}\left(\sqrt{r-1}\right)\right)$

$= \left(\tan^{-1}(1)-\tan^{-1}(0)\right)+\left(\tan^{-1}(2)-\tan^{-1}(1)\right)$

$+\left(\tan^{-1}(3)-\tan^{-1}(2)\right)+\left(\tan^{-1}(4)-\tan^{-1}(3)\right)$

$+.......+\left(\tan^{-1}(n)-\tan^{-1}(n-1)\right)$

$= \tan^{-1}(n)$

Ex-14. Find the value of

$\tan^{-1}\left(\frac{a_1x-y}{a_1y+x}\right)+\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)$

$+\tan^{-1}\left(\frac{a_3-a_2}{1+a_3a_2}\right)+....+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right)$

$+\tan^{-1}\left(\frac{1}{a_n}\right)$, where $x, y, a_1, a_2, ..., a_n \in R^+$

Soln. We have $\tan^{-1}\left(\frac{a_1x-y}{a_1y+x}\right)+\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)$

$+\tan^{-1}\left(\frac{a_3-a_2}{1+a_3a_2}\right)+....+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right)$

$+\tan^{-1}\left(\frac{1}{a_n}\right)$

$= \tan^{-1}\left(\frac{a_1-\frac{y}{x}}{1+a_1.\frac{y}{x}}\right)+\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)$

$+\tan^{-1}\left(\frac{a_3-a_2}{1+a_3a_2}\right)+....+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right)$

$+\tan^{-1}\left(\frac{1}{a_n}\right)$

$= \tan^{-1}(a_1)-\tan^{-1}\left(\frac{y}{x}\right)+\tan^{-1}(a_2)-\tan^{-1}(a_1)+$

$\tan^{-1}(a_3)-\tan^{-1}(a_2)+...+\tan^{-1}(a_n)-\tan^{-1}(a_{n-1})$

$+\cot^{-1}(a_n)$

$= \tan^{-1}(a_n)+\cot^{-1}(a_n)-\tan^{-1}\left(\frac{y}{x}\right)$

$= \frac{\pi}{2}-\tan^{-1}\left(\frac{y}{x}\right)$

$= \cot^{-1}\left(\frac{y}{x}\right)$

$= \tan^{-1}\left(\frac{x}{y}\right)$

Ex-15. Find x, if $\left(\tan^{-1}x\right)^2+\left(\cot^{-1}x\right)^2=\frac{\pi^2}{8}$

Soln. We have $\left(\tan^{-1}x\right)^2+\left(\cot^{-1}x\right)^2=\frac{\pi^2}{8}$

$\Rightarrow \left(\tan^{-1}x+\cot^{-1}x\right)^2-2\tan^{-1}x.\cot^{-1}x=\frac{\pi^2}{8}$

$\Rightarrow \left(\frac{\pi}{2}\right)^2-2\tan^{-1}x.\cot^{-1}x=\frac{\pi^2}{8}$

$\Rightarrow \frac{\pi^2}{4}-2\tan^{-1}x.\cot^{-1}x=\frac{\pi^2}{8}$

$\Rightarrow 2\tan^{-1}x.\cot^{-1}x=\frac{\pi^2}{8}$

$\Rightarrow \tan^{-1}x.\cot^{-1}x=\frac{\pi^2}{16}$

$\Rightarrow a\left(\frac{\pi}{2}-a\right)=\frac{\pi^2}{16}$, where $a=\tan^{-1}x$

$\Rightarrow 16a\left(\frac{\pi}{2}-a\right)=\pi^2$

$\Rightarrow 16a^2-8a\pi+\pi^2=0$

$\Rightarrow (4a-\pi)^2=0$

$\Rightarrow (4a-\pi)=0$

$\Rightarrow a=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}x=\frac{\pi}{4}$

$\Rightarrow x=1.$

Hence, the solution is $x = 1$.

Ex-16. Find the maximum value of $f(x)$, if

$f(x)=\left(\sec^{-1}x\right)^2+\left(\operatorname{cosec}^{-1}x\right)^2$

Soln. Given $f(x)=\left(\sec^{-1}x\right)^2+\left(\operatorname{cosec}^{-1}x\right)^2$

$= \left(\sec^{-1}x+\operatorname{cosec}^{-1}x\right)^2-2\sec^{-1}x\,.\operatorname{cosec}^{-1}x$

$= \left(\frac{\pi}{2}\right)^2 - 2\sec^{-1}x\left(\frac{\pi}{2} - \sec^{-1}x\right)$

$= \frac{\pi^2}{4} - \pi.\sec^{-1}x + 2\left(\sec^{-1}x\right)^2$

$= 2\left(\left(\sec^{-1}x\right)^2 - \frac{\pi}{2}.\sec^{-1}x + \frac{\pi^2}{8}\right)$

$= 2\left(\left(\sec^{-1}x\right)^2 - 2.\sec^{-1}x.\frac{\pi}{4} + \left(\frac{\pi}{4}\right)^2 + \frac{\pi^2}{8} - \frac{\pi^2}{16}\right)$

$= 2\left(\sec^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{8}$

Max value of $f(x)$ is $\frac{\pi^2}{4}$ at $x = 1$.

Ex-17. Find the minimum value of $f(x)$, if

$f(x) = \left(\sin^{-1}x\right)^3 + \left(\cos^{-1}x\right)^3$

Soln. Given $f(x) = \left(\sin^{-1}x\right)^3 + \left(\cos^{-1}x\right)^3$

$= \left(\sin^{-1}x + \cos^{-1}x\right)$
$\left(\left(\sin^{-1}x\right)^2 + \left(\cos^{-1}x\right)^2 - \sin^{-1}x\cos^{-1}x\right)$

$= \left(\sin^{-1}x + \cos^{-1}x\right)$
$\left(\left(\sin^{-1}x + \cos^{-1}x\right)^2 - 3\sin^{-1}x\cos^{-1}x\right)$

$= \frac{\pi}{2}\left(\left(\frac{\pi}{2}\right)^2 - 3\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)\right)$

$= \frac{\pi}{2}\left(\left(\frac{\pi}{2}\right)^2 - 3a\left(\frac{\pi}{2} - a\right)\right)$

$= \frac{\pi}{2}\left(\frac{\pi^2}{4} - \frac{3a\pi}{2} + 3a^2\right)$

$= \frac{\pi}{2}\left(3\left(a^2 - \frac{a\pi}{2}\right) + \frac{\pi^2}{4}\right)$

$= \frac{\pi}{2}\left(3\left(a^2 - 2.\frac{\pi}{4}.a + \left(\frac{\pi}{6}\right)^2\right) + \frac{\pi^2}{4} - \frac{3\pi^2}{16}\right)$

$= \frac{\pi}{2}\left(3\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{4} - \frac{3\pi^2}{16}\right)$

$= \frac{\pi}{2}\left(3\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right)$

$= \left(\frac{3\pi}{2}\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^3}{32}\right)$

$= \frac{3\pi}{2}\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^3}{32} \geq \frac{\pi^3}{32}$

Hence, the minimum value of $f(x)$ is $\frac{\pi^3}{32}$.

Ex-18. Find x, if $\left[\cot^{-1}x\right] + \left[\cos^{-1}x\right] = 0$

Soln. Given $\left[\cot^{-1}x\right] + \left[\cos^{-1}x\right] = 0$

It is possible only when

$\Rightarrow \quad \cot^{-1}x = 0 \;\&\; \cos^{-1}x = 0$

$\Rightarrow \quad 0 \leq \cot^{-1}x < 1 \;\&\; 0 \leq \cos^{-1}x < 1$

$\Rightarrow \quad x \in (\cot 1, \infty) \;\&\; x \in (\cos 1, 1]$

Thus, the solution is $x \in (\cot 1, 1]$

Ex-19. Find x, if $\left[\sin^{-1}x\right] + \left[\cos^{-1}x\right] = 0$

Soln. Given $\left[\sin^{-1}x\right] + \left[\cos^{-1}x\right] = 0$

It is possible only when

$\Rightarrow \quad \left[\sin^{-1}x\right] = 0 \;\&\; \left[\cos^{-1}x\right] = 0$

$\Rightarrow \quad 0 \leq \sin^{-1}x < 1 \;\&\; 0 \leq \cos^{-1}x < 1$

$\Rightarrow \quad x \in [0, \sin 1) \;\&\; x \in (\cos 1, 1]$

Thus, the solution is $x \in (\cos 1, \sin 1)$.

Ex-20. Find x, if $\left[\tan^{-1}x\right] + \left[\cot^{-1}x\right] = 2$.

Soln. Given $\left[\tan^{-1}x\right] + \left[\cot^{-1}x\right] = 2$

The range of $\left[\tan^{-1}x\right]$ is $\{-2, -1, 0, 1\}$

and $\left[\cot^{-1}x\right]$ is $\{0, 1, 2, 3\}$

Case I: when $\left[\cot^{-1}x\right] = 1 \;\&\; \left[\tan^{-1}x\right] = 1$

$\Rightarrow \quad 1 \leq \cot^{-1}x < 2 \;\&\; 1 \leq \tan^{-1}x < 2$

$\Rightarrow \quad x \in (\cot 2, \cot 1] \;\&\; x \in [\tan 1, \tan 2)$

$\Rightarrow \quad x \in \varphi \quad (\because \cot 1 < \tan 1)$

Case II: when $\left[\cot^{-1}x\right] = 3 \;\&\; \left[\tan^{-1}x\right] = -1$

$\Rightarrow \quad 3 \leq \cot^{-1}x < 4 \;\&\; -1 \leq \tan^{-1}x < 0$

$\Rightarrow \quad x \in (\cot 4, \cot 3] \;\&\; x \in [-\tan 1, 0)$

$\Rightarrow \quad x \in \varphi, \; (\because \cot 3 < -\tan 1)$

Case III: when $\left[\cot^{-1}x\right] = 2 \;\&\; \left[\tan^{-1}x\right] = 0$

$\Rightarrow \quad 2 \leq \cot^{-1}x < 3 \;\&\; 0 \leq \tan^{-1}x < 1$

$\Rightarrow \quad x \in (3, \cot 2] \;\&\; x \in [0, \tan 1)$

$\Rightarrow \quad x \in \varphi, \quad (\because \cot 2 < \tan 1)$

Thus, no such value of x, where the equation is valid.

Ex-21. Find x, if $\left[\sin^{-1}\left(\cos^{-1}\left(\sin^{-1}\left(\tan^{-1}x\right)\right)\right)\right]=1$

Soln. Given $\left[\sin^{-1}\left(\cos^{-1}\left(\sin^{-1}\left(\tan^{-1}x\right)\right)\right)\right]=1$

$\Rightarrow \quad 0\le\sin^{-1}\left(\cos^{-1}\left(\sin^{-1}\left(\tan^{-1}x\right)\right)\right)<1$

$\Rightarrow \quad 0\le\left(\cos^{-1}\left(\sin^{-1}\left(\tan^{-1}x\right)\right)\right)<\sin 1$

$\Rightarrow \quad \cos(\sin 1)<\left(\sin^{-1}\left(\tan^{-1}x\right)\right)\le 1$

$\Rightarrow \quad \sin(\cos(\sin 1))<\left(\tan^{-1}x\right)\le\sin 1$

$\Rightarrow \quad \tan(\sin(\cos(\sin 1)))<x\le\tan(\sin 1)$

$\Rightarrow \quad x\in\left(\tan(\sin(\cos(\sin 1))),\tan(\sin 1)\right]$

Ex-22. Find the range of
$f(x)=\sin^{-1}x+\tan^{-1}x+\cot^{-1}x$

Soln. Given $f(x)=\sin^{-1}x+\tan^{-1}x+\cot^{-1}x$

It is defined for $-1\le x\le 1$

Thus, $f(-1)$

$=\sin^{-1}(-1)+\tan^{-1}(-1)+\cot^{-1}(-1)$

$=-\frac{\pi}{2}-\frac{\pi}{4}+\pi-\frac{\pi}{4}$

$=-\pi+\pi=0$

and $f(1)$

$=\sin^{-1}(1)+\tan^{-1}(1)+\cot^{-1}(1)$

$=\frac{\pi}{2}+\frac{\pi}{4}+\frac{\pi}{4}$

$=\pi.$

Thus, range of $f(x)$ is $=[f(-1),f(1)]=[0,\pi]$

Ex-23. Find the range of
$f(x)=\sin^{-1}x+\cos^{-1}x+\tan^{-1}x$

Soln. Given $f(x)=\sin^{-1}x+\cos^{-1}x+\tan^{-1}x$

$=\frac{\pi}{2}+\tan^{-1}x$

As we know that , the range of
$\tan^{-1}x$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Thus, the range of $f(x)$ is $(0,\pi)$

Ex-24. Find the range of
$f(x)=\sin^{-1}x+\sec^{-1}x+\tan^{-1}x$

Soln. Given $f(x)=\sin^{-1}x+\sec^{-1}x+\tan^{-1}x$

The domain of $f(x)$ is $\{-1,1\}$

Now, $f(1)=\sin^{-1}(1)+\sec^{-1}(1)+\tan^{-1}(1)$

$=\frac{\pi}{2}+0+\frac{\pi}{4}=\frac{3\pi}{4}$

and $f(-1)=\sin^{-1}(-1)+\sec^{-1}(-1)+\tan^{-1}(-1)$

$=-\frac{\pi}{2}+\pi-0-\frac{\pi}{4}$

$=\frac{\pi}{4}$

Thus, the range of $f(x)$ is $\left\{\frac{\pi}{4},\frac{3\pi}{4}\right\}$.

Ex-25. If $\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{\pi}{4}$, then find x.

Soln Given $\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x+3x}{1-2x.3x}\right)=\frac{\pi}{4}$

$\Rightarrow \quad \left(\frac{5x}{1-6x^2}\right)=1$

$\Rightarrow \quad 6x^2+5x-1=0$

$\Rightarrow \quad 6x^2+6x-x-1=0$

$\Rightarrow \quad (6x-1)(x+1)=0$

$\Rightarrow \quad x=-1,\frac{1}{6}$

Hence, the solution set is $\left\{-1,\frac{1}{6}\right\}$

Ex-26 If $\cos^{-1}x=\cot^{-1}\left(\frac{4}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)$, then find x.

Soln. Given $\cos^{-1}x=\cot^{-1}\left(\frac{4}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$\Rightarrow \quad \cos^{-1}x=\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$\Rightarrow \quad \cos^{-1}x=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}.\frac{1}{7}}\right)$

$\Rightarrow \quad \cos^{-1}x=\tan^{-1}\left(\frac{25}{25}\right)=\tan^{-1}(1)$

$\Rightarrow \quad \cos^{-1}x=\frac{\pi}{4}$

$\Rightarrow \quad x=\frac{1}{\sqrt{2}}$

Hence, the solution is $x=\frac{1}{\sqrt{2}}$

Ex-27. If $\cot^{-1}\left(\frac{n}{\pi}\right) > \frac{\pi}{6}$, where $n \in N$, then find the maximum value of n.

Soln. Given $\cot^{-1}\left(\frac{n}{\pi}\right) > \frac{\pi}{6}$

$\Rightarrow \frac{n}{\pi} < \cot\left(\frac{\pi}{6}\right)$

$\Rightarrow \frac{n}{\pi} < \sqrt{3}$

$\Rightarrow n < \pi\sqrt{3}$

$\Rightarrow n < \pi\sqrt{3} = 3.14 \times 1.732 = 5.43848$

Hence, the max value of n is 5

Ex-28. If $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$, then find x.

Soln. Given $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$

$\sin^{-1}\left(\frac{5}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{12}{x}\right)$

$\Rightarrow \sin^{-1}\left(\frac{5}{x}\right) = \cos^{-1}\left(\frac{12}{x}\right)$

$\Rightarrow \sin^{-1}\left(\frac{5}{x}\right) = \sin^{-1}\sqrt{1 - \left(\frac{12}{x}\right)^2}$

$\Rightarrow \left(\frac{5}{x}\right)^2 = 1 - \left(\frac{12}{x}\right)^2$

$\Rightarrow \left(\frac{169}{x^2}\right) = 1$

$\Rightarrow x^2 = 169$

$\Rightarrow x = \pm 13$

Ex-29. If $x = \sin^{-1}(a^6 + 1) + \cos^{-1}(a^4 + 1) + \tan^{-1}(a^2 + 1)$ then find the value of $\sin\left(x + \frac{\pi}{4}\right) + \cos\left(x + \frac{\pi}{4}\right)$

Soln. We have

$x = \sin^{-1}(b^6 + 1) + \cos^{-1}(b^4 + 1) + \tan^{-1}(a^2 + 1)$

It is possible only when $a = 0$

Thus, $x = \sin^{-1}(1) + \cos^{-1}(1) + \tan^{-1}(1)$

$= \frac{\pi}{2} + 0 + \frac{\pi}{4} = \frac{3\pi}{4}$

Therefore, $\sin\left(x + \frac{\pi}{4}\right) + \cos\left(x + \frac{\pi}{4}\right)$

$= \sin\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{4} + \frac{\pi}{4}\right)$

$= \sin\pi + \cos\pi$

$= 0 - 1 = -1$

Ex-30. Find the number of integral values of k for which the equation $\sin^{-1}x + \tan^{-1}x = 2k + 1$ has a solution.

Soln. Given $\sin^{-1}x + \tan^{-1}x = 2k + 1$

Let $g(x) = \sin^{-1}x + \tan^{-1}x$

Domain of g = [–1, 1]

Now, $g(-1) = \sin^{-1}(-1) + \tan^{-1}(-1)$

$= -\frac{\pi}{2} - \frac{\pi}{4} = -\frac{3\pi}{4}$

$g(1) = \sin^{-1}(1) + \tan^{-1}(1)$

$= \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

Range of $g = \left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right]$

Thus, $-\frac{3\pi}{4} \le 2k + 1 \le \frac{3\pi}{4}$

$\Rightarrow -\frac{3\pi}{4} - 1 \le 2k \le \frac{3\pi}{4} - 1$

$\Rightarrow -\frac{3 \times 3.14}{4} - 1 \le 2k \le \frac{3 \times 3.14}{4} - 1$

$\Rightarrow -2.35 - 1 \le 2k \le 2.35 - 1$

$\Rightarrow -3.35 \le 2k \le 1.35$

$\Rightarrow -\frac{3.35}{2} \le k \le \frac{1.35}{2}$

$\Rightarrow -1.67 \le k \le 0.67$

Thus, the integral values of k are –1 and 0.

Ex-31. If $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$, then find the value of $\left(\frac{1 + x^4 + y^4}{x^2 - x^2y^2 + y^2}\right)$

Soln. Given $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}x = \frac{\pi}{2} - \sin^{-1}y$

$\Rightarrow \sin^{-1}x = \cos^{-1}y$

$\Rightarrow \sin^{-1}x = \sin^{-1}\sqrt{1 - y^2}$

$\Rightarrow \quad x = \sqrt{1-y^2}$

$\Rightarrow \quad x^2 = 1-y^2$

$\Rightarrow \quad x^2 + y^2 = 1$

Now, $\left(\dfrac{1+x^4+y^4}{x^2-x^2y^2+y^2}\right)$

$= \left(\dfrac{1+\left(x^2+y^2\right)^2-2x^2y^2}{\left(x^2+y^2-x^2y^2\right)}\right)$

$= \left(\dfrac{1+1-2x^2y^2}{\left(1-x^2y^2\right)}\right)$

$= \dfrac{2\left(1-x^2y^2\right)}{\left(1-x^2y^2\right)}$

= 2.

Ex-32. If $\cos^{-1}x+\cos^{-1}(2x)+\cos^{-1}(3x)=\pi$ and x satisfies the equation $ax^3+bx^2+cx=1$ then find the value of $a^2+b^2+c^2+10$

Soln. Given $\cos^{-1}x+\cos^{-1}(2x)+\cos^{-1}(3x)=\pi$

$\Rightarrow \quad \cos^{-1}2x+\cos^{-1}3x=\pi-\cos^{-1}x$

$\Rightarrow \quad \cos^{-1}2x+\cos^{-1}3x=\cos^{-1}(-x)$

$\Rightarrow \quad \cos^{-1}\left(2x.3x-\sqrt{1-4x^2}\sqrt{1-9x^2}\right)=\cos^{-1}(-x)$

$\Rightarrow \quad \left(6x^2-\sqrt{1-4x^2}\sqrt{1-9x^2}\right)=-x$

$\Rightarrow \quad \left(6x^2+x\right)^2=\left(\sqrt{1-4x^2}\sqrt{1-9x^2}\right)^2$

$\Rightarrow \quad 36x^4+12x^3+x^2=1-4x^2-9x^2+36x^4$

$\Rightarrow \quad 12x^3+14x^2=1$

Also, $ax^3+bx^2+cx=1$

Thus, $a = 12$, $b = 14$ and $c = 0$.

Hence, the value of $a^2+b^2+c^2+10$

= 144 + 196 + 10

= 350.

Ex-33. If $f(x)=\sin^{-1}x+\tan^{-1}x+x^2+4x+5$ such that $R_f=[a,b]$, find the value of $a + b + 5$.

Soln. The domain of $\sin^{-1}x+\tan^{-1}x$ is [–1, 1]

Now, $f(1)=\sin^{-1}(1)+\tan^{-1}(1)+1+4+5$

$= \dfrac{3\pi}{4}+10$

and $f(-1)=\sin^{-1}(-1)+\tan^{-1}(-1)+1-4+5$

$= -\dfrac{3\pi}{4}+2$

Therefore, $a + b + 5 = 10 + 2 + 5 = 17$

QUESTIONS WITH SOLUTIONS OF PAST JEE MAIN EXAMS

1. If $\cot^{-1}\left(\sqrt{\cos\alpha}\right)+\tan^{-1}\left(\sqrt{\cos\alpha}\right)=x$, then $\sin x$ is

(a) 1 (b) $\cot^2(\alpha/2)$

(c) $\tan\alpha$ (d) $\cot\left(\dfrac{\alpha}{2}\right)$

[JEE Main – 2002]

Soln. Clearly, $x=\dfrac{\pi}{2}$

Thus, $\sin x = 1$

Ans. (a)

2. The domain of $\sin^{-1}\left(\log_3\left(\dfrac{x}{3}\right)\right)$ is

(a) [1, 9] (b) [–1, 9]

(c) [–9, 1] (d) [–9, –1].

[JEE Main - 2002]

Soln. As we know that, domian of $\sin^{-1}x$ is [–1, 1]

Therefore, $-1\le\log_3\left(\dfrac{x}{3}\right)\le 1$

$\Rightarrow \quad 3^{-1}\le\left(\dfrac{x}{3}\right)\le 3$

$\Rightarrow \quad \dfrac{1}{3}\le\left(\dfrac{x}{3}\right)\le 3$

$\Rightarrow \quad 1\le x\le 9$

Thus the domain of $f(x)$ is [1, 9].

Ans. (a)

3. The trigonometric equation $\sin^{-1}x=2\sin^{-1}a$ has a solution for

(a) all real values (b) $|a|<\dfrac{1}{2}$

(c) $|a|\le\dfrac{1}{\sqrt{2}}$ (d) $\dfrac{1}{2}<|a|<\dfrac{1}{\sqrt{2}}$.

[JEE Main - 2003]

Soln. As we know that, the range of $\sin^{-1}x$ is $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$

Therefore, $-\frac{\pi}{2} \le 2\sin^{-1} a \le \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{4} \le \sin^{-1} a \le \frac{\pi}{4}$

$\Rightarrow \quad -\frac{1}{\sqrt{2}} \le a \le \frac{1}{\sqrt{2}}$

$\Rightarrow \quad |a| \le \frac{1}{\sqrt{2}}$

Ans. (c)

4. The domain of the function $f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is

(a) [1, 2] (b) [2, 3)
(c) [2, 3] (d) [1, 2)

[JEE Main - 2004]

Soln. Now, $\sin^{-1}(x-3)$ is defined for

$-1 \le (x-3) \le 1 \Rightarrow 2 \le x \le 4$

Also, the function $\frac{1}{\sqrt{9-x^2}}$ is defined for

$\Rightarrow \quad 9 - x^2 > 0$

$\Rightarrow \quad x^2 - 9 < 0$

$\Rightarrow \quad (x+3)(x-3) < 0$

$\Rightarrow \quad -3 < x < 3$

Thus, the solution is $x \in [2, 3)$

Hence, the domain is $[2, 3)$

Ans. (b)

5. Let $f: (-1,1) \to B$ be a function defined as

$f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then f is both one one and onto, when B lies in

(a) $\left[0, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{\pi}{2}\right)$

(c) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (d) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

[JEE Main - 2005]

Soln. Since the f is ont, so the range of f is co-domain.
i.e., Range = B

Clearly, range of f is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Thus, $B = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Ans. (c)

6. If $\cos^{-1} x - \cos^{-1}\left(\frac{y}{2}\right) = \alpha$, then

$4x^2 - 4xy\cos\alpha + y^2$ is

(a) 4 (b) $2\sin 2\alpha$

(c) $-4\sin^2\alpha$ (d) $4\sin^2\alpha$

[JEE Main - 2005]

Soln. Given $\cos^{-1} x - \cos^{-1}\left(\frac{y}{2}\right) = \alpha$

$\Rightarrow \quad \cos^{-1}\left(x.\frac{y}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right) = \alpha$

$\Rightarrow \quad \left(\frac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right) = \cos\alpha$

$\Rightarrow \quad \left(\cos\alpha - \frac{xy}{2}\right)^2 = \left(\sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right)^2$

$\Rightarrow \quad \left(\cos^2\alpha - xy\cos\alpha + \left(\frac{xy}{2}\right)^2\right)$

$\Rightarrow \quad 1 - x^2 - \frac{y^2}{4} + \frac{x^2y^2}{4}$

$\Rightarrow \quad x^2 - xy\cos\alpha + \frac{y^2}{4} = 1 - \cos^2\alpha$

$\Rightarrow \quad x^2 - xy\cos\alpha + \frac{y^2}{4} = \sin^2\alpha$

$\Rightarrow \quad 4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha$

Ans. (d)

7. No questions asked in 2006.

8. If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then x is

(a) 4 (b) 5 (c) 1 (d) 3

[JEE Main - 2007]

Soln. Given $\quad \sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$

$\Rightarrow \quad x = 3$

Ans. (d)

9. Find the value of

$\cot\left(\operatorname{cosec}^{-1}\left(\frac{5}{3}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$ is

(a) $\frac{5}{17}$ (b) $\frac{6}{17}$

(c) $\frac{3}{17}$ (d) $\frac{4}{17}$

[JEE Main - 2008]

Soln. Given $\cot\left(\operatorname{cosec}^{-1}\left(\frac{5}{3}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right)$

$= \cot\left(\sin^{-1}\left(\frac{3}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right)$

$= \cot\left(\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right)$

$= \cot\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}.\frac{2}{3}}\right)\right)$

$= \cot\left(\tan^{-1}\left(\frac{17}{6}\right)\right)$

$= \cot\left(\cot^{-1}\left(\frac{6}{17}\right)\right)$

$= \frac{6}{17}$

Ans. (b).

10. No questions asked in 2009 to 2014.

PROBLEMS FOR JEE ADVANCED EXAM

Ex-1. Find the domain of

$$f(x)=\sin^{-1}\left(\frac{|x|-2}{3}\right)+\cos^{-1}\left(\frac{1-|x|}{4}\right)$$

Soln. Let D_1: $-1\le\left(\frac{|x|-2}{3}\right)\le 1$

$\Rightarrow \quad -3\le(|x|-2)\le 3$

$\Rightarrow \quad -1\le|x|\le 5$

$\Rightarrow \quad -5\le x\le 5$

and D_2: $-1\le\left(\frac{1-|x|}{4}\right)\le 1$

$\Rightarrow \quad -4\le 1-|x|\le 4$

$\Rightarrow \quad -5\le -|x|\le 3$

$\Rightarrow \quad -3\le|x|\le 5$

$\Rightarrow \quad -5\le x\le 5$

Thus, $D_f = D_1\cap D_2 = [-5,\ 5]$

Ex-2 Find the domain of

$$f(x)=\sqrt{5\pi\ \sin^{-1}x-6\left(\sin^{-1}x\right)^2}$$

Soln. The function f is defined for

$\Rightarrow \quad 5\pi\sin^{-1}x-6\left(\sin^{-1}x\right)^2\ge 0$

$\Rightarrow \quad 6\left(\sin^{-1}x\right)^2-5\pi\ \sin^{-1}x\le 0$

$\Rightarrow \quad \left(\sin^{-1}x\right)\left(6\left(\sin^{-1}x\right)-5\pi\right)\le 0$

$\Rightarrow \quad 0\le\sin^{-1}x\le\frac{5\pi}{6}$

$\Rightarrow \quad 0\le x\le\frac{1}{2}$

Also, $\sin^{-1}x$ is defined for [–1, 1]

Thus, $D_f=\left[0,\ \frac{1}{2}\right]$.

Ex-3 Find the domain of

$$f(x)=\sin^{-1}\left(\log_2\left(x^2+3x+4\right)\right)$$

Soln. f is defined for

$-1\le\log_2\left(x^2+3x+4\right)\le 1$

$\Rightarrow \quad 2^{-1}\le\left(x^2+3x+4\right)\le 2$

when $\quad \left(x^2+3x+4\right)\le 2$

$\Rightarrow \quad \left(x^2+3x+2\right)\le 0$

$\Rightarrow \quad (x+1)(x+2)\le 0$

$\Rightarrow \quad -2\le x\le -1$

when $\quad x^2+3x+4\ge\frac{1}{2}$

$\Rightarrow \quad 2x^2+6x+7\ge 0$

$\Rightarrow \quad x\in R$

Thus, $D_f=[-2,-1]$.

Ex-4 Solve for x: $\cos^{-1}x+\cos^{-1}x^2=2\pi$

Soln. Given $\cos^{-1}x+\cos^{-1}x^2=2\pi$

It is possible only when

$\cos^{-1}x=\pi\ \&\cos^{-1}x^2=\pi$

$x=\cos\pi=-1$ and $x^2=-1$

Thus, no such value of x is exist.

Ex-5 Solve for x:

$$\cot^{-1}\left(\frac{1}{x^2-1}\right)+\tan^{-1}\left(x^2-1\right)=\frac{\pi}{2}$$

Soln. It is true only when $\frac{1}{x^2-1}=x^2-1$

$\Rightarrow \quad (x^2-1)^2 = 1$

$\Rightarrow \quad (x^2-1) = \pm 1$

$\Rightarrow \quad x^2 = 1 \pm 1 = 2, 0$

$\Rightarrow \quad x = 0, \pm\sqrt{2}$

Ex-6. Solve for x:

$$\cot^{-1}\left(\frac{x^2-1}{2x}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$

Soln. Given $\cot^{-1}\left(\frac{x^2-1}{2x}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{2\pi}{3}$

$\Rightarrow \quad 2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{2\pi}{3}$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$

$\Rightarrow \quad \left(\frac{2x}{1-x^2}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\Rightarrow \quad \frac{2x}{\sqrt{3}} = 1 - x^2$

$\Rightarrow \quad x^2 + \frac{2}{\sqrt{3}}x - 1 = 0$

$\Rightarrow \quad \left(x + \frac{1}{\sqrt{3}}\right)^2 = 1 + \frac{1}{3} = \frac{4}{3}$

$\Rightarrow \quad \left(x + \frac{1}{\sqrt{3}}\right) = \pm\frac{2}{\sqrt{3}}$

$\Rightarrow \quad x = -\frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{3}}$

$\Rightarrow \quad x = \frac{1}{\sqrt{3}}, -\sqrt{3}$

Hence, the solution set is $\left\{-\sqrt{3}, \frac{1}{\sqrt{3}}\right\}$

Ex-7. Solve for x: $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right) < \pi - 3$

Soln. Given $\quad \sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right) < \pi - 3$

$\Rightarrow \quad \sin^{-1}\left(\sin\left(\pi - \frac{2x^2+4}{x^2+1}\right)\right) < \pi - 3$

$\Rightarrow \quad \left(\pi - \frac{2x^2+4}{x^2+1}\right) < \pi - 3$

$\Rightarrow \quad \frac{2x^2+4}{x^2+1} > 3$

$\Rightarrow \quad \frac{2x^2+4}{x^2+1} - 3 > 0$

$\Rightarrow \quad \frac{2x^2+4-3x^2-3}{x^2+1} > 0$

$\Rightarrow \quad \frac{-x^2+1}{x^2+1} > 0$

$\Rightarrow \quad \frac{x^2-1}{x^2+1} < 0$

$\Rightarrow \quad \frac{(x-1)(x+1)}{x^2+1} < 0$

$\Rightarrow \quad -1 < x < 1$

Ex-8. Solve for x:

$$x^2 - 4x > \sin^{-1}\left(\sin\left[\pi^{3/2}\right]\right) + \cos^{-1}\left(\cos\left[\pi^{3/2}\right]\right)$$

Soln. we have $\sin^{-1}\left(\sin\left[\pi^{3/2}\right]\right) + \cos^{-1}\left(\cos\left[\pi^{3/2}\right]\right)$

$= \sin^{-1}\left(\sin\left[\pi\sqrt{\pi}\right]\right) + \cos^{-1}\left(\cos\left[\pi\sqrt{\pi}\right]\right)$

$= \sin^{-1}\left(\sin[5.56]\right) + \cos^{-1}\left(\cos[5.56]\right)$

$= \sin^{-1}(\sin 5) + \cos^{-1}(\cos 5)$

$= \sin^{-1}(\sin(5-2\pi)) + \cos^{-1}(\cos(2\pi-5))$

$= (5-2\pi) + (2\pi-5) = 0$

Thus, the given expression reduces to

$x^2 - 4x < 0$

$x(x-4) < 0$

$0 < x < 4$

Ex-9. Solve for x: $\cos\left(\tan^{-1}\left(\cot\left(\sin^{-1}\left(x+\frac{3}{2}\right)\right)\right)\right)$

$$+\tan\left(\sec^{-1}x\right) = 0$$

Soln. Now, $\cos\left(\tan^{-1}\left(\cot\left(\sin^{-1}\left(x+\frac{3}{2}\right)\right)\right)\right)$

$$= \cos\left(\tan^{-1}\left(\cot\left(\cot^{-1}\left(\frac{\sqrt{1-\left(x+\frac{3}{2}\right)^2}}{\left(x+\frac{3}{2}\right)}\right)\right)\right)\right)$$

$$= \cos\left(\tan^{-1}\left(\frac{\sqrt{1-\left(x+\frac{3}{2}\right)^2}}{\left(x+\frac{3}{2}\right)}\right)\right)$$

$$= \cos\left(\cos^{-1}\left(\frac{x+\frac{3}{2}}{1}\right)\right)$$

$$= \left(x+\frac{3}{2}\right)$$

Thus, the given equation reduces to

$$\left(x+\frac{3}{2}\right)+\tan\left(\sec^{-1}x\right)=0$$

$$\left(x+\frac{3}{2}\right)+\tan\left(\tan^{-1}\left(\frac{\sqrt{x^2-1}}{1}\right)\right)=0$$

$$\Rightarrow \quad \left(x+\frac{3}{2}\right)+\left(\frac{\sqrt{x^2-1}}{1}\right)=0$$

$$\Rightarrow \quad \left(x+\frac{3}{2}\right)^2=\left(-\sqrt{x^2-1}\right)^2$$

$$\Rightarrow \quad \left(x+\frac{3}{2}\right)^2=x^2-1$$

$$\Rightarrow \quad x^2+3x+\frac{9}{4}=x^2-1$$

$$\Rightarrow \quad 3x+\frac{9}{4}=-1$$

$$\Rightarrow \quad 3x=-1-\frac{9}{4}=-\frac{13}{4}$$

$$\Rightarrow \quad x=-\frac{13}{12}$$

Hence, the solution is $x=-\frac{13}{12}$

Ex-10. Solve for x:

$$\tan\left(\tan^{-1}\left(\frac{x}{10}\right)+\tan^{-1}\left(\frac{1}{x+1}\right)\right)=\tan\left(\frac{\pi}{4}\right)$$

Soln. We have

$$\Rightarrow \quad \tan\left(\tan^{-1}\left(\frac{\frac{x}{10}+\frac{1}{x+1}}{1-\frac{x}{10}.\frac{1}{x+1}}\right)\right)=1$$

$$\Rightarrow \quad \left(\frac{\frac{x}{10}+\frac{1}{x+1}}{1-\frac{x}{10}.\frac{1}{x+1}}\right)=1$$

$$\Rightarrow \quad \left(\frac{x}{10}+\frac{1}{x+1}\right)=\left(1-\frac{x}{10}.\frac{1}{x+1}\right)$$

$$\Rightarrow \quad x(x+1)+10=10(x+1)-x$$

$$\Rightarrow \quad x^2+x+10=10x+10-x$$

$$\Rightarrow \quad x^2-8x=0$$

$$\Rightarrow \quad x=0,\ 8$$

Hence, the solution set is $\{0, 8\}$

Ex-11. If $\alpha=2\tan^{-1}\left(\frac{1+x}{1-x}\right)$ and

$\beta=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$, then prove that

$\alpha+\beta=\pi$.

Soln. Given $\alpha=2\tan^{-1}\left(\frac{1+x}{1-x}\right)$

$$= 2\left(\tan^{-1}1+\tan^{-1}x\right)$$

$$= 2\left(\frac{\pi}{4}+\tan^{-1}x\right)$$

$$= \frac{\pi}{2}+2\tan^{-1}x$$

Also, $\beta=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$=\frac{\pi}{2}-\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$=\frac{\pi}{2}-2\tan^{-1}x$$

Thus, $\alpha+\beta$

$$=\frac{\pi}{2}+2\tan^{-1}x+\frac{\pi}{2}-2\tan^{-1}x$$

$$=\pi$$

Ex-12. Find the range of $f(x)=2\sin^{-1}(2x-3)$

Soln. As we know that,

$$-\frac{\pi}{2}\le\sin^{-1}(2x-3)\le\frac{\pi}{2}$$

$$\Rightarrow \quad -\frac{2\pi}{2}\le 2\sin^{-1}(2x-3)\le\frac{2\pi}{2}$$

$\Rightarrow \quad -\pi \le f(x) \le \pi$

Thus, $R_f = [-\pi, \pi]$

Ex-13. Find the range of. $f(x) = 2\sin^{-1}(2x-1) - \frac{\pi}{4}$

Soln. We have $-\frac{\pi}{2} \le \sin^{-1}(2x-1) \le \frac{\pi}{2}$

$\Rightarrow \quad -\pi \le 2\sin^{-1}(2x-1) \le \pi$

$\Rightarrow \quad -\pi - \frac{\pi}{4} \le 2\sin^{-1}(2x-1) - \frac{\pi}{4} \le \pi - \frac{\pi}{4}$

$\Rightarrow \quad -\frac{5\pi}{4} \le f(x) \le \frac{3\pi}{4}$

Thus, $R_f = \left[-\frac{5\pi}{4}, \frac{3\pi}{4}\right]$

Ex-14 Find the range of $f(x) = 2\cos^{-1}(-x^2) - \pi$

Soln. We have $f(x) = 2\cos^{-1}(-x^2) - \pi$

$= 2(\pi - \cos^{-1}(x^2)) - \pi$

$= \pi - 2\cos^{-1}(x^2)$

As we know that, $0 \le \cos^{-1}(x^2) \le \pi$

$\Rightarrow \quad 0 \le 2\cos^{-1}(x^2) \le 2\pi$

$\Rightarrow \quad -2\pi \le -2\cos^{-1}(x^2) \le 0$

$\Rightarrow \quad -2\pi + \pi \le \pi - 2\cos^{-1}(x^2) \le \pi$

$\Rightarrow \quad -\pi \le f(x) \le \pi$

Thus, $R_f = [-\pi, \pi]$.

Ex-15. Find the range of $f(x) = \frac{1}{2}\tan^{-1}(1-x^2) - \frac{\pi}{4}$

Soln. We have

$-\infty < 1 - x^2 \le 1$

$\Rightarrow \quad \tan^{-1}(-\infty) < \tan^{-1}(1-x^2) \le \tan^{-1}(1)$

$\Rightarrow \quad -\frac{\pi}{2} < \tan^{-1}(1-x^2) \le \frac{\pi}{4}$

$\Rightarrow \quad -\frac{\pi}{4} < \frac{1}{2}\tan^{-1}(1-x^2) \le \frac{\pi}{8}$

$\Rightarrow \quad -\frac{\pi}{4} - \frac{\pi}{4} < \frac{1}{2}\tan^{-1}(1-x^2) - \frac{\pi}{4} \le \frac{\pi}{8} - \frac{\pi}{4}$

$\Rightarrow \quad -\frac{\pi}{2} < \frac{1}{2}\tan^{-1}(1-x^2) - \frac{\pi}{4} \le -\frac{\pi}{8}$

Hence, the range of the function

$= \left[-\frac{\pi}{2}, -\frac{\pi}{8}\right]$

Ex-16. Find the range of $f(x) = \cot^{-1}(2x - x^2)$

Soln. We have $2x - x^2$

$= -(x^2 - 2x + 1) + 1$

$= 1 - (x-1)^2$.

Thus, $\quad -\infty < 1 - (x-1)^2 \le 1$

$\Rightarrow \quad \cot^{-1}(-\infty) < \cot^{-1}\left(1 - (x-1)^2\right) \le \cot^{-1}(1)$

$\Rightarrow \quad 0 < \cot^{-1}\left(1 - (x-1)^2\right) \le \frac{\pi}{4}$

Hence, the range of the function

$= \left(0, \frac{\pi}{4}\right]$.

Ex-17. Find the range of

$f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$

Soln. The function *f* is defined for $-1 \le x \le 1$

we have $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$

$= \frac{\pi}{2} + \tan^{-1}x$

Thus, $R_f = [f(-1), f(1)]$

$= \left[\frac{\pi}{2} - \frac{\pi}{4}, \frac{\pi}{2} + \frac{\pi}{4}\right] = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

Ex-18. Find the range of

$f(x) = \sin^{-1}x + \sec^{-1}x + \tan^{-1}x$

Soln. The function *f* is defined for $x = \pm 1$

Now, $f(1) = \sin^{-1}(1) + \sec^{-1}(1) + \tan^{-1}(1)$

$= \frac{\pi}{2} + 0 + \frac{\pi}{4} = \frac{3\pi}{4}$

Also, $f(-1) = \sin^{-1}(-1) + \sec^{-1}(-1) + \tan^{-1}(-1)$

$= -\frac{\pi}{2} + \pi - 0 - \frac{\pi}{4}$

$= \frac{3\pi}{4}$

Thus, $R_f = \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$

Ex-19. Find the range of

$$f(x)=3\cot^{-1}x+2\tan^{-1}x+\frac{\pi}{4}$$

Soln. We have $f(x)$

$$=2\left(\tan^{-1}x+\cot^{-1}x\right)+\cot^{-1}x+\frac{\pi}{4}$$

$$=2.\frac{\pi}{2}+\cot^{-1}x+\frac{\pi}{4}$$

$$=\cot^{-1}x+\frac{5\pi}{4}$$

Also, $0<\cot^{-1}x<\pi$

$$\Rightarrow \frac{5\pi}{4}<\cot^{-1}x+\frac{5\pi}{4}<\pi+\frac{5\pi}{4}$$

$$\Rightarrow \frac{5\pi}{4}<f(x)<\frac{9\pi}{4}$$

Thus, $R_f=\left(\frac{5\pi}{4},\frac{9\pi}{4}\right)$.

Ex-20 Prove that $\sin\left(\cot^{-1}\left(\tan\left(\cos^{-1}x\right)\right)\right)=x$, $\forall x\in(0,1]$

Soln. We have $\sin\left(\cot^{-1}\left(\tan\left(\cos^{-1}x\right)\right)\right)$

$$=\sin\left(\cot^{-1}\left(\tan\left(\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right)\right)\right)$$

$$=\sin\left(\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right)$$

$$=\sin\left(\sin^{-1}\left(\frac{x}{1}\right)\right)$$

$$=x$$

Ex-21. Provethat $\sin\left(\operatorname{cosec}^{-1}\left(\cot\left(\tan^{-1}x\right)\right)\right)=x$ $\forall x\in(0,1]$

Soln. We have $\sin\left(\operatorname{cosec}^{-1}\left(\cot\left(\tan^{-1}x\right)\right)\right)$

$$=\sin\left(\operatorname{cosec}^{-1}\left(\cot\left(\cot^{-1}\left(\frac{1}{x}\right)\right)\right)\right)$$

$$=\sin\left(\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)\right)$$

$$=\sin\left(\sin^{-1}(x)\right)$$

$$=x$$

Ex-22 Find the value of $\sin^{-1}(\sin 5)$ + $\cos^{-1}(\cos 10)+\tan^{-1}(\tan(-6))$ $+\cot^{-1}(\cot(-10))$

Soln. We have $\sin^{-1}(\sin 5)+\cos^{-1}(\cos 10)$

$$-\tan^{-1}(\tan 6)+\pi-\cot^{-1}(\cot 10)$$

$$=\sin^{-1}(\sin(5-2\pi))+\cos^{-1}(\cos(4\pi-10))$$

$$+\pi-\tan^{-1}(\tan(6-2\pi))-\cot^{-1}(\cot(10-4\pi))$$

$$=(5-2\pi)+(4\pi-10)+\pi+(6-2\pi)-(10-4\pi)$$

$$=5\pi-9$$

Ex-23. If $U=\cot^{-1}\left(\sqrt{\cos 2\theta}\right)-\tan^{-1}\left(\sqrt{\cos 2\theta}\right)$

then prove that $\sin U=\tan^2\theta$.

Soln. Given $U=\cot^{-1}\left(\sqrt{\cos 2\theta}\right)-\tan^{-1}\left(\sqrt{\cos 2\theta}\right)$

$$\Rightarrow U=\tan^{-1}\left(\frac{1}{\sqrt{\cos 2\theta}}\right)-\tan^{-1}\left(\sqrt{\cos 2\theta}\right)$$

$$\Rightarrow U=\tan^{-1}\left(\frac{\frac{1}{\sqrt{\cos 2\theta}}-\sqrt{\cos 2\theta}}{1+\frac{1}{\sqrt{\cos 2\theta}}.\sqrt{\cos 2\theta}}\right)$$

$$\Rightarrow U=\tan^{-1}\left(\frac{1-\cos 2\theta}{2\sqrt{\cos 2\theta}}\right)$$

$$\Rightarrow U=\tan^{-1}\left(\frac{2\sin^2\theta}{2\sqrt{\cos 2\theta}}\right)$$

$$\Rightarrow U=\tan^{-1}\left(\frac{\sin^2\theta}{\sqrt{\cos 2\theta}}\right)$$

$$\Rightarrow U=\sin^{-1}\left(\frac{\sin^2\theta}{\sqrt{\sin^4\theta+1-2\sin^2\theta}}\right)$$

$$\Rightarrow \sin U=\sin\left(\sin^{-1}\left(\frac{\sin^2\theta}{\sqrt{\sin^4\theta+1-2\sin^2\theta}}\right)\right)$$

$$\Rightarrow \sin U=\left(\frac{\sin^2\theta}{\sqrt{\sin^4\theta+1-2\sin^2\theta}}\right)$$

$$\Rightarrow \sin U=\left(\frac{\sin^2\theta}{\sqrt{\left(\sin^2\theta-1\right)^2}}\right)=\frac{\sin^2\theta}{\cos^2\theta}=\tan^2\theta$$

Hence, the result.

Ex-24. Prove that $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)$

$$+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)=\frac{2b}{a}$$

Soln. Let $\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)=\theta$

$\cos(2\theta)=\frac{a}{b}$

The given expression reduces to

$$\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)$$

$$=\frac{1+\tan\theta}{1-\tan\theta}+\frac{1-\tan\theta}{1+\tan\theta}$$

$$=\frac{(1+\tan\theta)^2+(1-\tan\theta)^2}{(1-\tan^2\theta)}$$

$$=\frac{2(1+\tan^2\theta)}{(1-\tan^2\theta)}$$

$$=\frac{2}{\cos 2\theta}$$

$$=\frac{2b}{a}$$

Ex-25 Prove that $\cos^{-1}\left(\frac{\cos x+\cos y}{1+\cos x\cos y}\right)$

$$=2\tan^{-1}\left(\tan\left(\frac{x}{2}\right)\tan\left(\frac{y}{2}\right)\right)$$

Soln. R.H.S $=2\tan^{-1}\left(\tan\left(\frac{x}{2}\right)\tan\left(\frac{y}{2}\right)\right)$

$$=\cos^{-1}\left(\frac{1-\tan^2\left(\frac{x}{2}\right)\tan^2\left(\frac{y}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)\tan^2\left(\frac{y}{2}\right)}\right)$$

$$=\cos^{-1}\left(\frac{\cos^2\left(\frac{x}{2}\right)\cos^2\left(\frac{y}{2}\right)-\sin^2\left(\frac{x}{2}\right)\sin^2\left(\frac{y}{2}\right)}{\cos^2\left(\frac{x}{2}\right)\cos^2\left(\frac{y}{2}\right)+\sin^2\left(\frac{x}{2}\right)\sin^2\left(\frac{y}{2}\right)}\right)$$

$$=\cos^{-1}\left(\frac{(1+\cos x)(1+\cos y)-(1-\cos x)(1-\cos y)}{(1+\cos x)(1+\cos y)+(1-\cos x)(1-\cos y)}\right)$$

$$=\cos^{-1}\left(\frac{(2\cos x+2\cos y)}{(2+2\cos x\cos y)}\right)$$

$$=\cos^{-1}\left(\frac{(\cos x+\cos y)}{(1+1\cos x\cos y)}\right)$$

Hence, the result.

Ex-26. Prove that $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{x}{2}\right)\right)$

$$=\cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right)$$

Soln. We have $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{x}{2}\right)\right)$

$$=\cos^{-1}\left(\frac{1-\left(\frac{a-b}{a+b}\right)\tan^2\left(\frac{x}{2}\right)}{1+\left(\frac{a-b}{a+b}\right)\tan^2\left(\frac{x}{2}\right)}\right)$$

$$=\cos^{-1}\left(\frac{(a+b)-(a-b)\tan^2\left(\frac{x}{2}\right)}{(a+b)+(a-b)\tan^2\left(\frac{x}{2}\right)}\right)$$

$$=\cos^{-1}\left(\frac{a\left(1-\tan^2\left(\frac{x}{2}\right)\right)+b\left(1+\tan^2\left(\frac{x}{2}\right)\right)}{a\left(1+\tan^2\left(\frac{x}{2}\right)\right)+b\left(1-\tan^2\left(\frac{x}{2}\right)\right)}\right)$$

$$=\cos^{-1}\left(\frac{a\left(\frac{1-\tan^2\left(\frac{x}{2}\right)}{\left(1+\tan^2\left(\frac{x}{2}\right)\right)}\right)+b}{a+b\left(\frac{1-\tan^2\left(\frac{x}{2}\right)}{\left(1+\tan^2\left(\frac{x}{2}\right)\right)}\right)}\right)$$

$$=\cos^{-1}\left(\frac{a\cos x+b}{a+b\cos x}\right)$$

Ex-27 If $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ are in A.P. then prove that $(x+z)y^2+2y(1-xz)=x+z$, where $y\in(0,1)$, $xz<1, x>0$ and $z>0$.

Soln. Given $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$

$\Rightarrow \tan^{-1}x+\tan^{-1}z=2\tan^{-1}y$

$\Rightarrow \tan^{-1}\left(\frac{x+z}{1-xz}\right)=\tan^{-1}\left(\frac{2y}{1-y^2}\right)$

$\Rightarrow \left(\frac{x+z}{1-xz}\right)=\left(\frac{2y}{1-y^2}\right)$

$\Rightarrow (x+z)(1-y^2)=2y(1-xz)$

$$\Rightarrow \quad y^2(x+z)+2y(1-xz)=(x+z)$$

Ex-28. Prove that

$$\sin^{-1}\left(\sin\frac{33\pi}{7}\right)+\cos^{-1}\left(\cos\frac{46\pi}{7}\right)$$

$$+\tan^{-1}\left(-\tan\frac{13\pi}{8}\right)+\cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right)$$

$$=\frac{13\pi}{7}$$

Soln. We have $\sin^{-1}\left(\sin\frac{33\pi}{7}\right)+\cos^{-1}\left(\cos\frac{46\pi}{7}\right)$

$$+\tan^{-1}\left(-\tan\frac{13\pi}{8}\right)+\cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right)$$

$$=\sin^{-1}\left(\sin\left(4\pi+\frac{5\pi}{7}\right)\right)+\cos^{-1}\left(\cos\left(6\pi+\frac{4\pi}{7}\right)\right)$$

$$+\tan^{-1}\left(-\tan\left(2\pi-\frac{3\pi}{8}\right)\right)$$

$$+\cot^{-1}\left(-\cot\left(2\pi+\frac{3\pi}{8}\right)\right)$$

$$=\sin^{-1}\left(\sin\left(\frac{5\pi}{7}\right)\right)+\cos^{-1}\left(\cos\left(\frac{4\pi}{7}\right)\right)$$

$$+\tan^{-1}\left(\tan\left(\frac{3\pi}{8}\right)\right)+\cot^{-1}\left(-\cot\left(\frac{3\pi}{8}\right)\right)$$

$$=\sin^{-1}\left(\sin\left(\pi-\frac{2\pi}{7}\right)\right)+\cos^{-1}\left(\cos\left(\frac{4\pi}{7}\right)\right)$$

$$+\tan^{-1}\left(\tan\left(\frac{3\pi}{8}\right)\right)+\cot^{-1}\left(\cot\left(\frac{5\pi}{8}\right)\right)$$

$$=\frac{2\pi}{7}+\frac{4\pi}{7}+\frac{3\pi}{8}+\frac{5\pi}{8}$$

$$=\frac{6\pi}{7}+\pi$$

$$=\frac{13\pi}{7}$$

Ex-29. Solve for x and y:

$$\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3},\ \cos^{-1}x-\cos^{-1}y=\frac{\pi}{3}$$

Soln. Given equations are

$$\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}, \qquad \text{.......(i)}$$

$$\cos^{-1}x-\cos^{-1}y=\frac{\pi}{3} \qquad \text{......(ii)}$$

(i) reduces to

$$\frac{\pi}{2}-\cos^{-1}x+\frac{\pi}{2}-\cos^{-1}y=\frac{2\pi}{3}$$

$$\Rightarrow \quad \pi-\left(\cos^{-1}x+\cos^{-1}y\right)=\frac{2\pi}{3}$$

$$\Rightarrow \quad \left(\cos^{-1}x+\cos^{-1}y\right)=\pi-\frac{2\pi}{3}=\frac{\pi}{3} \qquad \text{......(iii)}$$

Adding (ii) and (iii), we get,

$$2\cos^{-1}x=\frac{2\pi}{3}$$

$$\Rightarrow \quad \cos^{-1}x=\frac{\pi}{3}$$

$$\Rightarrow \quad x=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$$

when $x=\frac{1}{2}$, $y=0$.

Hence, the solutions are $x = 1/2$ and $y = 0$.

Ex-30. If $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$, then prove that

$$x^2=\sin(2y)$$

Soln. Given $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$

Put $x^2=\cos(2\theta)$

Thus, $y=\tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$

$$=\tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right)$$

$$=\left(\frac{\pi}{4}-\theta\right)$$

$$\Rightarrow \quad \theta=\frac{\pi}{4}-y$$

$$\Rightarrow \quad \frac{1}{2}\cos^{-1}\left(x^2\right)=\frac{\pi}{4}-y$$

$$\Rightarrow \quad \cos^{-1}\left(x^2\right)=\frac{\pi}{2}-2y$$

$$\Rightarrow \quad \left(x^2\right)=\cos\left(\frac{\pi}{2}-2y\right)$$

$$\Rightarrow \quad x^2 = \sin(2y)$$

Ex-31. Prove that $\frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right)$

$$+\frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\left(\frac{\alpha}{\beta}\right)\right) = (\alpha+\beta)\left(\alpha^2+\beta^2\right).$$

Soln. We have

$$\frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right)+\frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\left(\frac{\alpha}{\beta}\right)\right)$$

$$= \frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{\tan^{-1}\left(\frac{\beta}{\alpha}\right)}{2}\right)+\frac{\alpha^3}{2}\sec^2\left(\frac{\tan^{-1}\left(\frac{\alpha}{\beta}\right)}{2}\right)$$

$$= \frac{\beta^3}{2\sin^2\left(\frac{\tan^{-1}\left(\frac{\beta}{\alpha}\right)}{2}\right)}+\frac{\alpha^3}{2\cos^2\left(\frac{\tan^{-1}\left(\frac{\alpha}{\beta}\right)}{2}\right)}$$

$$= \frac{\beta^3}{1-\cos\left(\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right)}+\frac{\alpha^3}{1+\cos\left(\tan^{-1}\left(\frac{\alpha}{\beta}\right)\right)}$$

$$= \frac{\beta^3}{1-\cos\left(\cos^{-1}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)\right)}$$

$$+\frac{\alpha^3}{1+\cos\left(\cos^{-1}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)\right)}$$

$$= \frac{\beta^3}{1-\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)}+\frac{\alpha^3}{1+\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)}$$

$$= \frac{\beta^3\left(\sqrt{\alpha^2+\beta^2}\right)}{\sqrt{\alpha^2+\beta^2}-\alpha}+\frac{\alpha^3\left(\sqrt{\alpha^2+\beta^2}\right)}{\sqrt{\alpha^2+\beta^2}+\beta}$$

$$= \left(\sqrt{\alpha^2+\beta^2}\right)\left(\frac{\beta^3}{\sqrt{\alpha^2+\beta^2}-\alpha}+\frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}+\beta}\right)$$

$$= \left(\sqrt{\alpha^2+\beta^2}\right)\left(\frac{\beta^3\left(\sqrt{\alpha^2+\beta^2}+\alpha\right)}{\beta^2}+\frac{\alpha^3\left(\sqrt{\alpha^2+\beta^2}-\beta\right)}{\alpha^2}\right)$$

$$= \left(\sqrt{\alpha^2+\beta^2}\right)\left(\beta\left(\sqrt{\alpha^2+\beta^2}+\alpha\right)+\alpha\left(\sqrt{\alpha^2+\beta^2}-\beta\right)\right)$$

$$= \left(\sqrt{\alpha^2+\beta^2}\right)\sqrt{\alpha^2+\beta^2}\,(\alpha+\beta)$$

$$= \left(\alpha^2+\beta^2\right)(\alpha+\beta)$$

Hence, the result.

Ex-32. Find the minimum value of n, if

$$\cot^{-1}\left(\frac{n^2-10n+21.6}{\pi}\right) > \frac{\pi}{6}, n \in N$$

Soln. Given $\cot^{-1}\left(\frac{n^2-10n+21.6}{\pi}\right) > \frac{\pi}{6}, n \in N$

$$\Rightarrow \quad \left(\frac{n^2-10n+21.6}{\pi}\right) < \cot\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad (n^2-10n+21.6 < \pi\sqrt{3}$$

$$\Rightarrow \quad (n^2-10n+21.6 < 5.6$$

$$\Rightarrow \quad (n^2-10n+16 < 0$$

$$\Rightarrow \quad (n-2)(n-8) < 0$$

$$\Rightarrow \quad 2 < n < 8$$

Thus, the minimum value of n is 2.

Ex-33. Prove that

$$\sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right)+\cos^{-1}\left(\frac{\sqrt{12}}{4}\right)+\sec^{-1}\left(\sqrt{2}\right)\right\}\right\} = 0$$

Soln. We have

$$\sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right)+\cos^{-1}\left(\frac{\sqrt{12}}{4}\right)+\sec^{-1}\left(\sqrt{2}\right)\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{4-2\sqrt{3}}{8}}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\sec^{-1}\left(\sqrt{2}\right)\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{\left(\sqrt{3}-1\right)^2}{8}}\right)+\frac{\pi}{6}+\frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)+\frac{\pi}{6}+\frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\frac{\pi}{12}+\frac{\pi}{6}+\frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left(\frac{\pi}{2}\right)\right\}$$

$= \sin^{-1}(0)$

$= 0.$

Ex-34. Solve for x ; $\left[\sin^{-1}\left(\cos^{-1}\left(\sin^{-1}\left(\tan^{-1}x\right)\right)\right)\right]=1$, where [,] = G.I.F

Soln. Given $\left[\sin^{-1}\left(\cos^{-1}\left(\sin^{-1}\left(\tan^{-1}x\right)\right)\right)\right]=1$

$\Rightarrow \quad 1 \le \sin^{-1}\left(\cos^{-1}\left(\sin^{-1}\left(\tan^{-1}x\right)\right)\right) < 2$

$\Rightarrow \quad \sin(1) \le \cos^{-1}\left(\sin^{-1}\left(\tan^{-1}x\right)\right) < \sin(2)$

$\Rightarrow \quad \cos(\sin(2)) < \sin^{-1}\left(\tan^{-1}x\right) \le \cos(\sin(1))$

$\Rightarrow \quad \sin(\cos(\sin(2))) < \left(\tan^{-1}x\right) \le \sin(\cos(\sin(1)))$

$\Rightarrow \quad \tan(\sin(\cos(\sin(2)))) < x \le \tan(\sin(\cos(\sin(1))))$

Ex-35 Find the interval for which

$2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is independent of x.

Soln. Given $2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$= 2\tan^{-1}x + \pi - 2\tan^{-1}x,\ x > 1$

$= \pi,\ x > 1$

Thus, $x \in (1, \infty)$

Ex-36. If $x = \operatorname{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\sec\left(\sin^{-1}a\right)\right)\right)\right)\right)$

and $y = \sec\left(\cot^{-1}\left(\sin\left(\tan^{-1}\left(\operatorname{cosec}\left(\cos^{-1}a\right)\right)\right)\right)\right)$,

where $a \in [0,1]$, then find the relation between x and y.

Soln. Given $x = \operatorname{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\sec\left(\sin^{-1}a\right)\right)\right)\right)\right)$

$= \operatorname{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$

$= \operatorname{cosec}\left(\tan^{-1}\left(\cos\left(\cos^{-1}\left(\frac{1}{a}\right)\right)\right)\right)$

$= \operatorname{cosec}\left(\tan^{-1}\left(\frac{1}{a}\right)\right)$

$= \operatorname{cosec}\left(\operatorname{cosec}^{-1}\left(\sqrt{a^2+1}\right)\right)$

$= \left(\sqrt{a^2+1}\right)$

and $y = \sec\left(\cot^{-1}\left(\sin\left(\tan^{-1}\left(\operatorname{cosec}\left(\cos^{-1}a\right)\right)\right)\right)\right)$

$= \sec\left(\cot^{-1}\left(\sin\left(\tan^{-1}\left(\frac{1}{\sqrt{a^2-1}}\right)\right)\right)\right)$

$= \sec\left(\cot^{-1}\left(\frac{1}{a}\right)\right)$

$= \sec\left(\sec^{-1}\left(\sqrt{a^2+1}\right)\right)$

$= \left(\sqrt{a^2+1}\right)$

Thus, $x = y$

Ex-37. Find the sum of the infinite series.

$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + ...$

Soln. We have

$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + ...$

$= \tan^{-1}\left(\frac{1}{1+2}\right) + \tan^{-1}\left(\frac{1}{1+6}\right) + \tan^{-1}\left(\frac{1}{1+12}\right)$

$+ ....$

$= \tan^{-1}\left(\frac{2-1}{1+2.1}\right) + \tan^{-1}\left(\frac{3-2}{1+3.2}\right) + \tan^{-1}\left(\frac{4-3}{1+4.3}\right)$

$+ ...... + \tan^{-1}\left(\frac{n+1-n}{1+(n+1).n}\right)$

$= \tan^{-1}(2) - \tan^{-1}(1) + \tan^{-1}(3) - \tan^{-1}(2)$

$+ \tan^{-1}(4) - \tan^{-1}(3) + ... + \tan^{-1}(n+1) - \tan^{-1}(n)$

$= \tan^{-1}(n+1) - \tan^{-1}(1)$

$= \tan^{-1}\left(\frac{n+1-1}{1+(n+1).1}\right)$

$= \tan^{-1}\left(\frac{n}{n+2}\right),\ n \to \infty$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$

Ex-38. Find the sum of

$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{6}}\right) + \sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}\right)$

$+ ...... + \sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}\times\sqrt{n+1}}\right) + ...$ to ∞

Soln. Let $t_n = \sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}\times\sqrt{n+1}}\right)$

$= \tan^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r(r-1)}}\right)$

$= \tan^{-1}\left(\sqrt{r}\right)-\tan^{-1}\left(\sqrt{r-1}\right)$

Now, $\sum_{r=1}^{n}\sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}\sqrt{n+1}}\right)$

$= \sum_{r=1}^{n}\tan^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r}\sqrt{r-1}}\right)$

$= \sum_{r=1}^{n}\left(\tan^{-1}\sqrt{r}-\tan^{-1}\sqrt{r-1}\right)$

$= \left(\tan^{-1}1-\tan^{-1}0\right)+\left(\tan^{-1}\sqrt{2}-\tan^{-1}1\right)$

$+\left(\tan^{-1}\sqrt{3}-\tan^{-1}\sqrt{2}\right)+\left(\tan^{-1}\sqrt{4}-\tan^{-1}\sqrt{3}\right)$

$+ \ldots\ldots\ldots\ldots + \left(\tan^{-1}\sqrt{n}-\tan^{-1}\sqrt{n-1}\right)$

$= \tan^{-1}\sqrt{n}-\tan^{-1}0$

$= \tan^{-1}\sqrt{n}$

When $n\to\infty$, the sum is $\tan^{-1}(\infty)=\frac{\pi}{2}$

Ex-39. Find the sum of infinite series:

$\cot^{-1}\left(2.1^2\right)+\cot^{-1}\left(2.2^2\right)+\cot^{-1}\left(2.3^2\right)+\ldots\ldots.$

Soln. We have

$\cot^{-1}\left(2.1^2\right)+\cot^{-1}\left(2.2^2\right)+\cot^{-1}\left(2.3^2\right)+\ldots\ldots.$

$= \tan^{-1}\left(\frac{1}{2.1^2}\right)+\tan^{-1}\left(\frac{1}{2.2^2}\right)+\tan^{-1}\left(\frac{1}{2.3^2}\right)+\ldots$

$= \tan^{-1}\left(\frac{2}{4}\right)+\tan^{-1}\left(\frac{2}{16}\right)+\tan^{-1}\left(\frac{2}{36}\right)+\ldots$

$= \tan^{-1}\left(\frac{2}{1+3}\right)+\tan^{-1}\left(\frac{2}{1+15}\right)+\tan^{-1}\left(\frac{2}{1+35}\right)+\ldots$

$= \tan^{-1}\left(\frac{3-1}{1+3.1}\right)+\tan^{-1}\left(\frac{5-3}{1+5.3}\right)$

$+\tan^{-1}\left(\frac{7-5}{1+7.5}\right)+\ldots+\tan^{-1}\left(\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}\right)$

$= \left(\tan^{-1}3-\tan^{-1}1\right)+\left(\tan^{-1}5-\tan^{-1}3\right)$

$+\left(\tan^{-1}7-\tan^{-1}5\right)+\left(\tan^{-1}9-\tan^{-1}7\right)$

$+\ldots\ldots+\left(\tan^{-1}(2n+1)-\tan^{-1}(2n-1)\right)$

$= \left(\tan^{-1}(2n+1)-\tan^{-1}1\right)$

$= \left(\tan^{-1}\frac{(2n+1-1)}{1+(2n+1).1}\right)$

$= \left(\tan^{-1}\left(\frac{n}{n+1}\right)\right)=\frac{\pi}{4}$, when $n\to\infty$.

Ex-40. If $\cos^{-1}\left(\frac{x}{2}\right)+\cos^{-1}\left(\frac{y}{3}\right)=\theta$, then

prove that $9x^2-12xy\cos\theta+4y^2=36\sin^2\theta$

[Roorkee - 1984]

Soln. Given $\cos^{-1}\left(\frac{x}{2}\right)+\cos^{-1}\left(\frac{y}{3}\right)=\theta$

$\Rightarrow \quad \cos^{-1}\left(\frac{x}{2}.\frac{y}{3}-\sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right)=\theta$

$\Rightarrow \quad \cos\theta=\left(\frac{x}{2}.\frac{y}{3}-\sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right)$

$\Rightarrow \quad \left(\cos\theta-\frac{xy}{6}\right)^2=\left(1-\frac{x^2}{4}\right)\left(1-\frac{x^2}{9}\right)$

$\Rightarrow \quad \left(\cos^2\theta-2.\frac{xy}{6}.\cos\theta+\frac{x^2y^2}{36}\right)$

$= 1-\frac{x^2}{4}-\frac{y^2}{9}+\frac{x^2y^2}{36}$

$\Rightarrow \quad \cos^2\theta-\frac{xy}{3}\cos\theta=1-\frac{x^2}{4}-\frac{y^2}{9}$

$\Rightarrow \quad \frac{x^2}{4}-\frac{xy}{3}.\cos\theta+\frac{y^2}{9}=1-\cos^2\theta$

$\Rightarrow \quad 9x^2-12xy\cos\theta+4y^2=12\sin^2\theta$

Hence, the result.

Note. No questions asked in 1985.

Ex-41. Evaluate: $\tan\left(\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)\right)$. [Roorkee-1986]

Soln. Let $\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)=\theta$

$\Rightarrow \quad \cos^{-1}\left(\frac{\sqrt{5}}{3}\right)=2\theta$

$\Rightarrow \quad \cos 2\theta=\frac{\sqrt{5}}{3}$

$$\Rightarrow \quad \frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{\sqrt{5}}{3}$$

$$\Rightarrow \quad \frac{1-\tan^2\theta+1+\tan^2\theta}{1-\tan^2\theta-1-\tan^2\theta}=\frac{\sqrt{5}+3}{\sqrt{5}-3}$$

$$\Rightarrow \quad \frac{2}{-2\tan^2\theta}=\frac{\sqrt{5}+3}{\sqrt{5}-3}$$

$$\Rightarrow \quad \frac{1}{\tan^2\theta}=\frac{\sqrt{5}+3}{3-\sqrt{5}}$$

$$\Rightarrow \quad \tan^2\theta=\frac{3-\sqrt{5}}{3+\sqrt{5}}=\frac{\left(3-\sqrt{5}\right)^2}{4}$$

$$\Rightarrow \quad \tan\theta=\pm\sqrt{\frac{\left(3-\sqrt{5}\right)^2}{4}}=\pm\frac{\left(3-\sqrt{5}\right)}{2}$$

Note. No questions asked in 1987, 1988, 1989, 1990 and 1991

Ex-42. Solve for x: $\sin\left[2\cos^{-1}\left\{\cot\left(2\tan^{-1}x\right)\right\}\right]=0$

[Roorkee - 1992]

Soln. Given $\sin\left[2\cos^{-1}\left\{\cot\left(2\tan^{-1}x\right)\right\}\right]=0$

Let $\tan^{-1}x=\theta \Rightarrow x=\tan\theta$

$$\Rightarrow \quad \sin\left(2\cos^{-1}\left(\cot\left(2\theta\right)\right)\right)=0$$

$$\Rightarrow \quad \sin\left(2\cos^{-1}\left(\frac{1-\tan^2\theta}{2\tan\theta}\right)\right)=0$$

$$\Rightarrow \quad \sin\left(2\cos^{-1}\left(\frac{1-x^2}{2x}\right)\right)=0$$

$$\Rightarrow \quad \sin\left(\cos^{-1}\left(2\left(\frac{1-x^2}{2x}\right)^2-1\right)\right)=0$$

$$\Rightarrow \quad \sin\left(\cos^{-1}\left(2\left(\frac{1-x^2}{2x}\right)^2-1\right)\right)=0$$

$$\Rightarrow \quad \sin\left(\sin^{-1}\left(\sqrt{1-\left(2\left(\frac{1-x^2}{2x}\right)^2-1\right)^2}\right)\right)=0$$

$$\Rightarrow \quad \sqrt{1-\left(2\left(\frac{1-x^2}{2x}\right)^2-1\right)^2}=0$$

$$\Rightarrow \quad 1-\left(2\left(\frac{1-x^2}{2x}\right)^2-1\right)^2=0$$

$$\Rightarrow \quad \left(2\left(\frac{1-x^2}{2x}\right)^2-1\right)^2=1$$

$$\Rightarrow \quad \left(2\left(\frac{1-x^2}{2x}\right)^2-1\right)=\pm 1$$

$$\Rightarrow \quad 2\left(\frac{1-x^2}{2x}\right)^2=1\pm 1=2,0$$

$$\Rightarrow \quad 2\left(\frac{1-x^2}{2x}\right)^2=2\ \&\ 2\left(\frac{1-x^2}{2x}\right)^2=0$$

$$\Rightarrow \quad \left(\frac{1-x^2}{2x}\right)^2=1\ \&\ 1-x^2=0$$

$$\Rightarrow \quad \left(\frac{1-x^2}{2x}\right)=\pm 1\ \&\ x=\pm 1$$

$$\Rightarrow \quad x^2+2x-1=0,\ x^2-2x-1=0\ \&\ x=\pm 1$$

$$\Rightarrow \quad (x+1)^2=\left(\sqrt{2}\right)^2,\ (x-1)^2=\left(\sqrt{2}\right)^2\ \&\ x=\pm 1$$

$$\Rightarrow \quad x=-1\pm\sqrt{2},\ 1\pm\sqrt{2},\pm 1$$

Ex-43. Find all positive integral solutions of

$$\tan^{-1}x+\cos^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$$

[Roorkee Main-1993]

Soln. Given $\tan^{-1}x+\cos^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$

$$\Rightarrow \quad \tan^{-1}x+\tan^{-1}\left(\frac{1}{y}\right)=\tan^{-1}3$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x+\frac{1}{y}}{1-x.\frac{1}{y}}\right)=\tan^{-1}(3)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{xy+1}{y-x}\right)=\tan^{-1}(3)$$

$$\Rightarrow \quad \left(\frac{xy+1}{y-x}\right)=3$$

$$\Rightarrow \quad xy+1=3y-3x$$

$\Rightarrow \quad 3x+1=y(3-x)$

$\Rightarrow \quad y=\dfrac{3x+1}{3-x}$

when $x=1, y=2$

Also, when $x=2, y=7$

Hence, the +ve integral solutions are 2.

Ex-44. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, then find the value of $x^2+y^2+z^2+2xyz$ [Roorkee Main - 1994]

Soln. We have, $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$

$\Rightarrow \quad \cos^{-1}x+\cos^{-1}y=\pi-\cos^{-1}z$

$\Rightarrow \quad \cos^{-1}x+\cos^{-1}y=\cos^{-1}(-z)$

$\Rightarrow \quad \cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)=\cos(-z)$

$\Rightarrow \quad (xy+z)^2=(1-x^2)(1-y^2)$

$\Rightarrow \quad x^2y^2+2xyz+z^2=1-x^2-y^2+x^2y^2$

$\Rightarrow \quad x^2+y^2+z^2+2xyz=1$

Ex-45. Convert the trigonometric function $\sin\left[2\cos^{-1}\left\{\cot\left(2\tan^{-1}x\right)\right\}\right]$ into an algebraic function $f(x)$. Then from the algebraic function $f(x)$, find all values of x for which $f(x)$ is zero.

Also, express the values of x in the form of $a\pm\sqrt{b}$, where a and b are rational numbers.

[Roorkee Main - 1995]

Soln. See solutions of Ex-42.

Note. No questions asked in 1996.

Ex-46. If $\theta=\tan^{-1}(2\tan^2\theta)-\frac{1}{2}\sin^{-1}\left(\dfrac{3\sin 2\theta}{5+4\cos 2\theta}\right)$, then find the general value of θ.

[Roorkee Main - 1997]

Soln. Given $\theta=\tan^{-1}(2\tan^2\theta)-\frac{1}{2}\sin^{-1}\left(\dfrac{3\sin 2\theta}{5+4\cos 2\theta}\right)$

Now, $\left(\dfrac{3\sin 2\theta}{5+4\cos 2\theta}\right)$

$$=\left(\frac{\frac{6\tan\theta}{1+\tan^2\theta}}{5+4\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)}\right)$$

$$=\left(\frac{6\tan\theta}{5(1+\tan^2\theta)+4(1-\tan^2\theta)}\right)$$

$$=\left(\frac{6\tan\theta}{9+\tan^2\theta}\right)$$

$$=\left(\frac{\frac{2}{3}\tan\theta}{1+\left(\frac{\tan\theta}{3}\right)^2}\right)$$

$$=\left(\frac{2.\frac{\tan\theta}{3}}{1+\left(\frac{\tan\theta}{3}\right)^2}\right)$$

Also, $\theta=\tan^{-1}(2\tan^2\theta)-\frac{1}{2}\sin^{-1}\left(\dfrac{3\sin 2\theta}{5+4\cos 2\theta}\right)$

$\Rightarrow \quad \theta=\tan^{-1}(2\tan^2\theta)-\frac{1}{2}.2\tan^{-1}\left(\dfrac{\tan\theta}{3}\right)$

$\Rightarrow \quad \theta=\tan^{-1}(2\tan^2\theta)-\tan^{-1}\left(\dfrac{\tan\theta}{3}\right)$

$\Rightarrow \quad \theta=\tan^{-1}\left(\dfrac{2\tan^2\theta-\frac{\tan\theta}{3}}{1+2\tan^2\theta.\frac{\tan\theta}{3}}\right)$

$\Rightarrow \quad \tan\theta=\left(\dfrac{6\tan^2\theta-\tan\theta}{3+2\tan^3\theta}\right)$

$\Rightarrow \quad 3\tan\theta+2\tan^4\theta-6\tan^2\theta+\tan\theta=0$

$\Rightarrow \quad 2\tan^4\theta-6\tan^2\theta+4\tan\theta=0$

$\Rightarrow \quad \tan^4\theta-3\tan^2\theta+2\tan\theta=0$

$\Rightarrow \quad \tan\theta(\tan^3\theta-3\tan\theta+2)=0$

$\Rightarrow \quad \tan\theta(\tan\theta-1)^2(\tan\theta+2)=0$

$\Rightarrow \quad \tan\theta=0,1,-2$

when $\tan\theta=0\Rightarrow\theta=n\pi,\ n\notin I$

when $\tan\theta=1\Rightarrow\theta=m\pi+\frac{\pi}{4},\ m\in I$

when $\tan\theta=-2\Rightarrow\theta=p\pi+\tan^{-1}(-2),\ p\in I$

Note. No questions asked in 1998.

Ex-47. Using the principal values, express the following expression as a single angle.

$3\tan^{-1}\left(\frac{1}{2}\right)+2\tan^{-1}\left(\frac{1}{5}\right)+\sin^{-1}\left(\frac{142}{65\sqrt{5}}\right)$

[Roorkee Main - 1999]

Soln. We have

$3\tan^{-1}\left(\frac{1}{2}\right)+2\tan^{-1}\left(\frac{1}{5}\right)+\sin^{-1}\left(\frac{142}{65\sqrt{5}}\right)$

Now, $3\tan^{-1}\left(\frac{1}{2}\right)$

$= \tan^{-1}\left(\dfrac{3.\frac{1}{2}-\left(\frac{1}{2}\right)^3}{1-3\left(\frac{1}{2}\right)^2}\right)$

$= \tan^{-1}\left(\dfrac{\frac{3}{2}-\frac{1}{8}}{1-\frac{3}{4}}\right)$

$= \tan^{-1}\left(\dfrac{12-1}{2}\right)$

$= \tan^{-1}\left(\dfrac{11}{2}\right)$

Also, $2\tan^{-1}\left(\frac{1}{5}\right)$

$= \tan^{-1}\left(\dfrac{2.\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}\right)$

$= \tan^{-1}\left(\dfrac{\frac{2}{5}}{\frac{25-1}{25}}\right)$

$= \tan^{-1}\left(\dfrac{2}{5}\times\dfrac{25}{24}\right)$

$= \tan^{-1}\left(\dfrac{5}{12}\right)$

Also, $\sin^{-1}\left(\dfrac{142}{65\sqrt{5}}\right)$

$= \tan^{-1}\left(\dfrac{142}{31}\right)$

Therefore,

$3\tan^{-1}\left(\frac{1}{2}\right)+2\tan^{-1}\left(\frac{1}{5}\right)+\sin^{-1}\left(\dfrac{142}{65\sqrt{5}}\right)$

$= \tan^{-1}\left(\dfrac{11}{2}\right)+\tan^{-1}\left(\dfrac{5}{12}\right)+\tan^{-1}\left(\dfrac{142}{31}\right)$

$= \tan^{-1}\left(\dfrac{\frac{11}{2}+\frac{5}{12}}{1-\frac{11}{2}.\frac{5}{12}}\right)+\tan^{-1}\left(\dfrac{142}{31}\right)$

$= \tan^{-1}\left(\dfrac{132+10}{24-55}\right)+\tan^{-1}\left(\dfrac{142}{31}\right)$

$= \tan^{-1}\left(-\dfrac{142}{31}\right)+\tan^{-1}\left(\dfrac{142}{31}\right)$

$= -\tan^{-1}\left(\dfrac{142}{31}\right)+\tan^{-1}\left(\dfrac{142}{31}\right)$

$= 0$

Ex-48. Solve for x:

$\sin^{-1}\left(\dfrac{ax}{c}\right)+\sin^{-1}\left(\dfrac{bx}{c}\right)=\sin^{-1}x$ where

$a^2+b^2=c^2,\ c\neq 0$. [Roorkee Main - 2000]

Soln. Given $\sin^{-1}\left(\dfrac{ax}{c}\right)+\sin^{-1}\left(\dfrac{bx}{c}\right)=\sin^{-1}x$

$\Rightarrow \sin^{-1}\left(\dfrac{ax}{c}\right)=\sin^{-1}x-\sin^{-1}\left(\dfrac{bx}{c}\right)$

$\Rightarrow \sin^{-1}\left(\dfrac{ax}{c}\right)=\sin^{-1}\left(x\sqrt{1-\dfrac{b^2x^2}{c^2}}-\dfrac{bx}{c}\sqrt{1-x^2}\right)$

$\Rightarrow \left(\dfrac{ax}{c}\right)=\left(x\sqrt{1-\dfrac{b^2x^2}{c^2}}-\dfrac{bx}{c}\sqrt{1-x^2}\right)$

$\Rightarrow \left(\dfrac{ax}{c}\right)=\left(\dfrac{x}{c}\sqrt{c^2-b^2x^2}-\dfrac{bx}{c}\sqrt{1-x^2}\right)$

$\Rightarrow x\left(\left(\sqrt{c^2-b^2x^2}-b\sqrt{1-x^2}\right)-a\right)=0$

$\Rightarrow x=0, \sqrt{c^2-b^2x^2}=b\sqrt{1-x^2}+a$

Thus, $x = 0$ and

$\sqrt{c^2-b^2x^2}=b\sqrt{1-x^2}+a$

$\Rightarrow c^2-b^2x^2=2ab\sqrt{1-x^2}+b^2\left(1-x^2\right)+a^2$

$\Rightarrow c^2-b^2x^2=2ab\sqrt{1-x^2}+b^2-b^2x^2+a^2$

$\Rightarrow c^2=2ab\sqrt{1-x^2}+b^2+a^2$

$\Rightarrow c^2=2ab\sqrt{1-x^2}+c^2$

$\Rightarrow 2ab\sqrt{1-x^2}=0$

$\Rightarrow \left(1-x^2\right)=0$

$\Rightarrow x=\pm 1$

Hence, the solution sets is $\{-1,0,1\}$

Ex-49 Solve for x:

$\cos^{-1}\left(x\sqrt{6}\right)+\cos^{-1}\left(3\sqrt{3}\ x^2\right)=\dfrac{\pi}{2}$

[Roorkee Main - 2001]

Soln. Given $\cos^{-1}\left(x\sqrt{6}\right)+\cos^{-1}\left(3\sqrt{3}\ x^2\right)=\frac{\pi}{2}$

$\Rightarrow\ \sin^{-1}\left(\sqrt{1-6x^2}\right)+\cos^{-1}\left(3\sqrt{3}\ x^2\right)=\frac{\pi}{2}$

It is possible only when, $\sqrt{1-6x^2}=3\sqrt{3}\ x^2$

$\Rightarrow\ 1-6x^2=27x^4$

$\Rightarrow\ 27x^4+6x^2-1=0$

$\Rightarrow\ 27x^4-9x^3+9x^3-3x^2+9x^2-1=0$

$\Rightarrow\ 9x^3(3x-1)+3x^2(3x-1)+(3x-1)(3x+1)=0$

$\Rightarrow\ ((3x-1))\left(9x^3+3x^2+(3x+1)\right)=0$

$\Rightarrow\ (3x-1)\left(3x^2(3x+1)+(3x+1)\right)=0$

$\Rightarrow\ (3x-1)(3x+1)\left(3x^2+1\right)=0$

$\Rightarrow\ x=-\frac{1}{3},\frac{1}{3},\pm\frac{i}{\sqrt{3}}$

Hence, the solutions are $\left\{\pm\frac{1}{3},\pm\frac{i}{\sqrt{3}}\right\}$.

Ex-50. Let x_1,x_2,x_3,x_4 be four non zero numbers satisfying the equation

$$\tan^{-1}\left(\frac{a}{x}\right)+\tan^{-1}\left(\frac{b}{x}\right)+\tan^{-1}\left(\frac{c}{x}\right)+\tan^{-1}\left(\frac{d}{x}\right)=\frac{\pi}{2}$$

then prove that

(i) $\sum_{i=1}^{4}x_i=0$

(ii) $\sum_{i=1}^{4}\left(\frac{1}{x_i}\right)=0$

(iii) $\prod_{i=1}^{4}(x_i)=abcd$

(iv) $\prod(x_1+x_2+x_3)=abcd$

Soln. Given equation is

$$\tan^{-1}\left(\frac{a}{x}\right)+\tan^{-1}\left(\frac{b}{x}\right)+\tan^{-1}\left(\frac{c}{x}\right)+\tan^{-1}\left(\frac{d}{x}\right)=\frac{\pi}{2}$$

$\Rightarrow\ \tan^{-1}\left(\dfrac{\frac{a}{x}+\frac{b}{x}}{1-\frac{a}{x}.\frac{b}{x}}\right)+\tan^{-1}\left(\dfrac{\frac{c}{x}+\frac{d}{x}}{1-\frac{c}{x}\frac{d}{x}}\right)=\frac{\pi}{2}$

$\Rightarrow\ \tan^{-1}\left(\frac{(a+b)x}{x^2-ab}\right)+\tan^{-1}\left(\frac{(c+d)x}{x^2-cd}\right)=\frac{\pi}{2}$

$\Rightarrow\ \tan^{-1}\left(\frac{(a+b)x}{x^2-ab}\right)=\frac{\pi}{2}-\tan^{-1}\left(\frac{(c+d)x}{x^2-cd}\right)$

$\Rightarrow\ \tan^{-1}\left(\frac{(a+b)x}{x^2-ab}\right)=\cot^{-1}\left(\frac{(c+d)x}{x^2-cd}\right)$

$\Rightarrow\ \left(\frac{(a+b)x}{x^2-ab}\right)=\left(\frac{x^2-cd}{(c+d)x}\right)$

$\Rightarrow\ \left(x^2-ab\right)\left(x^2-cd\right)=(a+b)(c+d)x^2$

$\Rightarrow\ x^4-(a+b+cd)x^2+abcd=(a+b)(c+d)x^2$

$\Rightarrow\ x^4-\left(a+b+cd+(a+b)(c+d)\right)x^2+abcd=0$

since x_1,x_2,x_3,x_4 are the values of the above equation, we have

$x_1+x_2+x_3+x_4=0$

$\sum x_1x_2=\left(ab+cd+(a+b)(c+d)\right)$

$\sum x_1x_2x_3=0$

$\sum x_1x_2x_3x_4=abcd$

(i) $\sum_{i=1}^{4}x_i=x_1+x_2+x_3+x_4=0$

(ii) $\sum_{i=1}^{4}\left(\frac{1}{x_i}\right)$

$=\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}$

$=\frac{x_2x_3x_4+x_1x_3x_4+x_1x_2x_4+x_1x_2x_3}{x_1x_2x_3x_4}$

$=\frac{0}{abcd}=0$

(iii) $\prod_{i=1}^{4}(x_i)$

$=x_1x_2x_3x_4$

$=abcd$

(iv) $\prod(x_1+x_2+x_3)$

$=(x_1+x_2+x_3)(x_1+x_2+x_4)(x_1+x_3+x_4)(x_2+x_3+x_4)$

$=(-x_4)(-x_3)(-x_2)(-x_1)$

$=x_1x_2x_3x_4$

$=abcd.$

Ex-51. Let $\cos^{-1}(x)+\cos^{-1}(2x)+\cos^{-1}(3x)=\pi$ If x satisfies the cubic equation $ax^3+bx^2+cx-1=0$, then find the value of $(a+b+c+2)$.

Soln. We have $\cos^{-1}(x)+\cos^{-1}(2x)+\cos^{-1}(3x)=\pi$

$\cos^{-1}(2x)+\cos^{-1}(3x)=\pi-\cos^{-1}(x)$

$\cos^{-1}(2x)+\cos^{-1}(3x)=\cos^{-1}(-x)$

$\cos^{-1}\left(2x.3x-\sqrt{(1-4x^2)(1-9x^2)}\right)=\cos^{-1}(-x)$

$6x^2-\sqrt{(1-4x^2)(1-9x^2)}=-x$

$(6x^2+x)=\sqrt{(1-4x^2)(1-9x^2)}$

$(6x^2+x)^2=(1-4x^2)(1-9x^2)$

$36x^4+12x^3+x^2=1-13x^2+36x^4$

$12x^3+14x^2-1=0$

Thus, $a=12, b=14, c=0$

Hence, the value of $(a+b+c+2)$
$= 28$.

Ex-52. If $x=\sin\left(2\tan^{-1}2\right), y=\sin\left(\frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right)$ then prove that $y^2=1-x$.

Soln. We have $x=\sin\left(2\tan^{-1}2\right)$

$\Rightarrow \quad x=\sin(2\theta), \tan^{-1}2=\theta$

$\Rightarrow \quad x=\frac{2\tan\theta}{1+\tan^2\theta}, \tan\theta=2$

$\Rightarrow \quad x=\frac{4}{5}$

Also, $\quad y=\sin\left(\frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right)$

$\Rightarrow \quad y=\sin(\theta), \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)=\theta$

$\Rightarrow \quad y=\sin(\theta), \tan(2\theta)=\frac{4}{3}$

$\Rightarrow \quad y=\sin(\theta), \tan(\theta)=\frac{1}{2}$

$\Rightarrow \quad y=\frac{1}{\sqrt{5}}$

Hence, $\quad y^2=\frac{1}{5}=1-\frac{4}{5}=1-x$

LEVEL II
(MIXED PROBLEMS)

1. The set of values of k for which $x^2 - kx + \sin^{-1}(\sin 4) > 0$ for all real x is
   (a) $\{0\}$ (b) $(-2,2)$
   (c) R (d) None of these.

2. If $x < 0$ then value of $\tan^{-1}(x)+\tan^{-1}\left(\frac{1}{x}\right)=$
   (a) $\frac{\pi}{2}$ (b) $-\frac{\pi}{2}$
   (c) 0 (d) None of these.

3. If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, then $\cos^{-1}x + \cos^{-1}y$ is
   (a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$
   (c) $\frac{\pi}{6}$ (d) π

4. Let $f(x) = \sin^{-1}x + \cos^{-1}x$. Then $\frac{\pi}{2}$ is equal to
   (a) $f\left(\frac{1}{2}\right)$
   (b) $f(k^2 - 2k + 3), k \in R$
   (c) $f\left(\frac{1}{1+k^2}\right), k \in R$
   (d) $f(-2)$

5. Which one of the following is correct ?
   (a) $\tan 1 > \tan^{-1} 1$ (b) $\tan 1 < \tan^{-1} 1$
   (c) $\tan 1 = \tan^{-1} 1$ (d) None.

6. If a $\sin^{-1}x - b\cos^{-1}x = c$, then the value of a $\sin^{-1}x + b\cos^{-1}x$ is
   (a) 0 (b) $\frac{\pi ab+c(b-a)}{a+b}$
   (c) $\frac{\pi ab-c(b-a)}{a+b}$ (d) $\frac{\pi}{2}$

7. The number of solutions of the equation $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$
   (a) 0 (b) 1
   (c) 2 (d) More than two

8. The smallest and the largest values of $\tan^{-1}\left(\frac{1-x}{1+x}\right), 0 \le x \le 1$ are
   (a) $0, \pi$ (b) $0, \frac{\pi}{4}$
   (c) $-\frac{\pi}{4}, \frac{\pi}{4}$ (d) $\frac{\pi}{4}, \frac{\pi}{2}$

9. The equation $\sin^{-1}x - \cos^{-1}x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ has
   (a) No solution (b) Unique Solution
   (c) Infinite No of soln (d) None

10. If $-\pi \le x \le 2\pi$, then $\cos^{-1}(\cos x)$ is
(a) x (b) $\pi - x$
(c) $2\pi + x$ (d) $2\pi - x$

11. If $\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is equal to
(a) 0 (b) $\frac{1}{\sqrt{5}}$
(c) $\frac{2}{\sqrt{5}}$ (d) $\frac{\sqrt{3}}{2}$

12. If $\cos [\tan^{-1}\{\sin (\cot^{-1} \sqrt{3})\}] = y$,
then the value of y is
(a) $y = \frac{4}{5}$ (b) $y = \frac{2}{\sqrt{5}}$
(c) $y = -\frac{2}{\sqrt{5}}$ (d) $y = \frac{\sqrt{3}}{2}$

13. If $x = \frac{1}{5}$, then the value of $\cos (\cos^{-1}x + 2 \sin^{-1} x)$ is
(a) $\sqrt{\frac{24}{25}}$ (b) $-\sqrt{\frac{24}{25}}$
(c) $\frac{1}{5}$ (d) $-\frac{1}{5}$

14. $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$ is equal to
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{3}$ (d) None of these

15. $\tan^{-1} a + \tan^{-1} b$, where $a > 0$, $b > 0$, $ab > 1$ is equal to
(a) $\tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (b) $\tan^{-1}\left(\frac{a+b}{1-ab}\right) - \pi$
(c) $\pi + \tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (d) $\pi - \tan^{-1}\left(\frac{a+b}{1-ab}\right)$

16. A solution to the equation
$\tan^{-1}(1+x) + \tan^{-1}(1-x) = \frac{\pi}{2}$ is
(a) $x = 1$ (b) $x = -1$
(c) $x = 0$ (d) $x = \pi$

17. All possible values of p and q for which
$\cos^{-1}\left(\sqrt{p}\right) + \cos^{-1}\left(\sqrt{1-p}\right) + \cos^{-1}\left(\sqrt{1-q}\right) = \frac{3\pi}{4}$
holds, is
(a) $p = 1, q = 1/2$ (b) $q > 1, p = 1/2$
(c) $0 < p < 1, q = 1/2$ (d) None

18. $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1} x\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1} x\right)$,
$x \ne 0$, is equal to
(a) x (b) $2x$
(c) $\frac{2}{x}$ (d) $\frac{x}{2}$

19. The value of $\cot^{-1}(3) + \text{cosec}^{-1}\left(\sqrt{5}\right)$ is
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

20. If $\sum_{i=1}^{2n} \sin^{-1} x_i = n\pi$, then $\sum_{i=1}^{2n} x_i$ is
(a) n (b) $2n$
(c) $\frac{n(n+1)}{2}$ (d) $\frac{n(n-1)}{2}$

21. If $u = \cot^{-1}\left(\sqrt{\tan\alpha}\right) - \tan^{-1}\left(\sqrt{\tan\alpha}\right)$,
then $\tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$ is equal to
(a) $\sqrt{\tan\alpha}$ (b) $\sqrt{\cot\alpha}$
(c) $\tan\alpha$ (d) $\cot\alpha$

22. The value of $\tan^{-1}\left(\frac{a}{b+c}\right) + \tan^{-1}\left(\frac{b}{a+c}\right)$,
if $\angle C = 90$, in triangle ABC is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{2}$ (d) π

23. If $\cot^{-1}\left(\frac{n}{\pi}\right) > \frac{\pi}{6}$, $n \,\varepsilon\, N$, then the maximum value of
'n' is
(a) 1 (b) 5
(c) 9 (d) None of these

24. $\sin^{-1}x > \cos^{-1} x$ holds for
(a) all values of x (b) $x \,\varepsilon \left(0, \frac{1}{\sqrt{2}}\right)$
(c) $\left(\frac{1}{\sqrt{2}}, 1\right)$ (d) $x = 0.75$

25. The value of $\cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right)$ is equal to
(a) $\frac{3}{4}$ (b) $-\frac{3}{4}$

(c) $\frac{1}{16}$ (d) 4

26. The values of x satisfying

$\tan\left(\sec^{-1}x\right)=\sin\left(\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right)$ is

(a) $\pm\frac{\sqrt{5}}{3}$ (b) $\pm\frac{3}{\sqrt{5}}$

(c) $\pm\frac{\sqrt{3}}{5}$ (d) $\pm\frac{3}{5}$

27. If $\left(\tan^{-1}x\right)^2+\left(\cot^{-1}x\right)^2=\frac{5\pi^2}{8}$, then the value of x is

(a) 0 (b) –1 (c) –2 (d) –3

28. The number of real solutions of

$\cos^{-1}x+\cos^{-1}2x=-\pi$ is

(a) 0 (b) 1

(c) 2 (d) infinitely many

29. Let a, b, c be positive real numbers and

$\theta=\tan^{-1}\left(\sqrt{\frac{a(a+b+c)}{bc}}\right)+\tan^{-1}\left(\sqrt{\frac{b(a+b+c)}{ac}}\right)+\tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ba}}\right)$,

then the value of $\tan\theta$ is

(a) 0 (b) 1

(c) –1 (d) None

30. The set of values of x satisfying the inequation $\tan^2\left(\sin^{-1}x\right)>1$ is

(a) [–1, 1] (b) $\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$

(c) $(-1,1)-\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$ (d) $[-1,1]-\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

31. The value of a for which

$ax^2+\sin^{-1}\left(x^2-2x+2\right)+\cos^{-1}\left(x^2-2x+2\right)=0$

has a real solution, is

(a) $\pi/2$ (b) $-\pi/2$

(c) $2/\pi$ (d) $-2/\pi$

32. The value of

$\sin^{-1}\left[\cot\left(\sin^{-1}\sqrt{\left(\frac{2-\sqrt{3}}{4}\right)}+\cos^{-1}\left(\frac{\sqrt{12}}{4}\right)+\sec^{-1}\sqrt{2}\right)\right]$

is

(a) 0 (b) $\pi/4$

(c) $\pi/6$ (d) $\pi/2$

33. The number of positive integral solutions of

$\tan^{-1}x+\cot^{-1}\left(\frac{1}{y}\right)=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$ is

(a) 0 (b) 1

(c) 2 (d) 3

34. The value of $\cos\left[\frac{1}{2}\cos\left\{\cos\left(\sin^{-1}\left(\frac{\sqrt{63}}{8}\right)\right)\right\}\right]$ is

(a) $\frac{3}{16}$ (b) $\frac{3}{8}$

(c) $\frac{3}{4}$ (d) $\frac{3}{2}$

35. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then the value of $\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}$ is

(a) 0 (b) 1

(c) $\frac{1}{xyz}$ (d) xyz

36. If $x<0$, then $\tan^{-1}\left(\frac{1}{x}\right)$ is

(a) $\cot^{-1}(x)$ (b) $-\cot^{-1}(x)$

(c) $-\pi+\cot^{-1}(x)$ (d) None

37. The number of triplets satisfying

$\sin^{-1}x+\cos^{-1}y+\sin^{-1}z=2\pi$, is

(a) 0 (b) 2

(c) 1 (d) infinite

38. If $x^2+y^2+z^2=r^2$, then

$\tan^{-1}\left(\frac{xy}{zr}\right)+\tan^{-1}\left(\frac{zy}{xr}\right)+\tan^{-1}\left(\frac{zx}{yr}\right)$ is equal to....

(a) π (b) $\pi/2$ (c) 0 (d) None

39. If $\tan^{-1}x+\tan^{-1}2x+\tan^{-1}3x=\pi$, then the value of x is

(a) 0 (b) –1 (c) 1 (d) ϕ

40. The number of solutions of the equation

$1+x^2+2x\sin\left(\cos^{-1}y\right)=0$ is

(a) 1 (b) 2

(c) 3 (d) 4.

41. If α is the only real root of the equation $x^3+bx^2+cx+1=0\ (b<c)$, then the value of $\tan^{-1}\alpha+\tan^{-1}\left(\frac{1}{\alpha}\right)$ is equal to

(a) $\alpha/2$ (b) $-\pi/2$ (c) 0 (d) None.

42. If α, β, γ are the roots of $x^3+px^2+2x+p=0$, the general value of $\tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma$ is

(a) $n\pi$ (b) $n\pi/2$
(c) $(2n+1)\pi/2$ (d) depend on p.

43. If $\left[\sin^{-1}\left(\cos^{-1}\left(\sin^{-1}\left(\tan^{-1}x\right)\right)\right)\right]=1$,
where [,] = G. I. F, the value of x lies in
(a) [tan sin cos 1, tan sin cos sin 1]
(b) (tan sin cos 1, tan sin cos sin 1)
(c) [–1, 1]
(d) [sin cos tan 1, sin cos sin tan 1].

LEVEL III
(PROBLEMS FOR JEE ADVANCED)

1. Prove that
$\sin^{-1}\left(\cos\left(\sin^{-1}x\right)\right)+\cos^{-1}\left(\sin\left(\cos^{-1}x\right)\right)$

2. Prove that
$\tan^{-1}\left\{\operatorname{cosec}\left(\tan^{-1}x\right)-\tan\left(\cot^{-1}x\right)\right\}=\frac{1}{2}\tan^{-1}x$
where $x\neq 0$.

3. Prove that
$\tan\left(\tan^{-1}x+\tan^{-1}y+\tan^{-1}z\right)$
$=\cot\left(\cot^{-1}x+\cot^{-1}y+\cot^{-1}z\right)$.

4. Prove that $\sin\left(\cot^{-1}\left(\tan\left(\cos^{-1}x\right)\right)\right)=x \quad \forall x\in(0,1]$.

5. Prove that $\sin\left(\operatorname{cosec}^{-1}\left(\cot\left(\tan^{-1}x\right)\right)\right)=x$
$\forall x\in(0,1]$

6. Find the value of
$\sin^{-1}(\sin 5)+\cos^{-1}(\cos 10)+\tan^{-1}(\tan(-6))$
$+\cot^{-1}(\cot(-10))$

7. Find the simplest value of
$\cos^{-1}x+\cos^{-1}\left(\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right), \forall x\in\left(\frac{1}{2},1\right)$

8. Find the value of
$\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-\tan^{-1}\left(\frac{\sqrt{5-2\sqrt{6}}}{1+\sqrt{6}}\right)$

9. Let $m=\sin^{-1}\left(a^6+1\right)+\cos^{-1}\left(a^4+1\right)-\tan^{-1}\left(a^2+1\right)$, then find the image of the line $x+y=m$ about the y-axis.

10. If $\left(\sin^{-1}x\right)^3+\left(\sin^{-1}y\right)^3+\left(\sin^{-1}z\right)^3=\frac{(3\pi)^3}{8}$
then find the value of $(3x+4y-5z+2)$.

11. Let $S=\sum_{r=1}^{n}\cot^{-1}\left(2^{r+1}+\frac{1}{2^r}\right)$,
then find $\lim_{n\to\infty}(S)$.

12. Find the value of
$\lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{4}{4r^2+3}\right)\right)\right)$.

13. Find the number of solution of the equation
$2\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\pi x^3$.

14. If $\cos^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{y}{b}\right)=\alpha$, then prove that,
$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$.

15. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$,
then prove that,
$x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$.

16. Find the greatest and least value of the function
$f(x)=\left(\sin^{-1}x\right)^3+\left(\cos^{-1}x\right)^3$.

17. Solve for x: $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{2}$.

18. Solve for x:
$\tan^{-1}\left(\frac{1}{1+2x}\right)+\tan^{-1}\left(\frac{1}{1+4x}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$

19. Solve for x:
$\tan^{-1}(x-1)+\tan^{-1}(x)+\tan^{-1}(x+1)$
$=\tan^{-1}(3x)$

20. Solve for x: $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\cos^{-1}x=\frac{\pi}{4}$.

21. Solve for x: $\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}$.

22. Solve for x:
$2\tan^{-1}x=\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)-\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right)$,
$a>0, b>0$.

23. Solve for x:
$\cot^{-1} x + \cot^{-1}\left(n^2 - x + 1\right) = \cot^{-1}(n-1)$.

24. Solve for x:
$\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\left(\frac{23}{36}\right)$

25. Solve for x:
$\sec^{-1}\left(\frac{x}{a}\right) - \sec^{-1}\left(\frac{x}{b}\right) = \sec^{-1} b - \sec^{-1} a$.

26. Find the sum of
$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{8n}{n^4 - 2n^2 + 5}\right)$.

27. Find the number of real solutions of the equation
$\sin^{-1}\left(e^x\right) + \cos^{-1}\left(x^2\right) = \frac{\pi}{2}$.

28. Find the number of real roots of
$\sqrt{\sin(x)} = \cos^{-1}(\cos x)$ in $(0, 2\pi)$.

29. If $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$
where $n \in N$, then find n.

30. If α the real root of $x^3 + bx^2 + cx + 1 = 0$
where $b < c$, then find the value of
$\tan^{-1}(\alpha) + \tan^{-1}\left(\frac{1}{\alpha}\right)$.

31. If the equation $x^3 + bx^2 + cx + 1 = 0$ has only one root α, then find the value of
$2\tan^{-1}(\operatorname{cosec}\alpha) + \tan^{-1}\left(2\sin\alpha\sec^2\alpha\right)$.

Q. Solve the following inequalities:

32. $\sin^{-1} x > \cos^{-1} x$

33. $\cos^{-1} x > \sin^{-1} x$

34. $\left(\cot^{-1} x\right)^2 - 5\left(\cot^{-1} x\right) + 6 > 0$

35. $\tan^2\left(\sin^{-1} x\right) > 1$

36. $4\left(\tan^{-1} x\right)^2 - 8\left(\tan^{-1} x\right) + 3 < 0$

37. $4\cot^{-1} x - \left(\cot^{-1} x\right)^2 - 3 \ge 0$

38. $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 2$

39. Find the maximum value of
$f(x) = \left\{\sin^{-1}(\sin x)\right\}^2 - \sin^{-1}(\sin x)$

40. Find the minimum value of
$f(x) = 8^{\sin^{-1} x} + 8^{\cos^{-1} x}$

41. Find the set of values of k for which
$x^2 - kx + \sin^{-1}(\sin 4) > 0$, for all real x.

42. If $A = 2\tan^{-1}\left(2\sqrt{2} - 1\right)$ and
$B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$,
then prove that $A > B$.

43. Prove that
$$\sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right) + \cos^{-1}\left(\frac{\sqrt{12}}{4}\right) + \sec^{-1}\left(\sqrt{2}\right)\right\}\right\} = 0.$$

44. Find the domain of the function
$f(x) = \sin^{-1}\left(\cos^{-1} x + \tan^{-1} x + \cot^{-1} x\right)$

45. If $\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sin^{-1}\left(\sqrt{1 - \frac{y}{4}}\right) + \tan^{-1} y = \frac{2\pi}{3}$
then find the maximum value of $\left(x^2 + y^2 + 1\right)$.

46. Find the number of integral ordered pairs (x, y) satisfying the equation
$\tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{1}{10}\right)$.

47. Let $\left[\cot\left(\sum_{k=1}^{10} \cot^{-1}\left(k^2 + k + 1\right)\right)\right] = \frac{a}{b}$
where a and b are co-prime, then find the value of $(a + b + 10)$.

48. If $p > q > 0$, $pr < -1 < qr$, then prove that
$\tan^{-1}\left(\frac{p-q}{1+pq}\right) + \tan^{-1}\left(\frac{q-r}{1+qr}\right) + \tan^{-1}\left(\frac{r-p}{1+rp}\right) = \pi$.

49. Consider the equation
$\left(\sin^{-1} x\right)^3 + \left(\cos^{-1} x\right)^3 = a\pi^3$
find the values of 'a' so that the given equation has a solution.

50. If the range of the function $f(x) = \cot^{-1}\left(\frac{x^2}{x^2+1}\right)$
is (a, b), find the value of $\left(\frac{b}{a} + 2\right)$.

51. If $\tan^{-1} y = 4\tan^{-1} x$, $\left(|x| < \tan\left(\frac{\pi}{8}\right)\right)$, find y as an algebraix function of x and hence prove that $\tan\left(\frac{\pi}{8}\right)$ is a root of the equation $x^4 - 6x^2 + 1 = 0$.

52. Prove that

$$\tan^{-1}\left(\sqrt{\frac{a(a+b+c)}{bc}}\right)+\tan^{-1}\left(\sqrt{\frac{b(a+b+c)}{ac}}\right)$$
$$+\tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ab}}\right)=\pi \text{, where } a, b, c>0.$$

53. Solve:

$$\theta=\tan^{-1}\left(2\tan^2\theta\right)-\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right).$$

54. Simplify:

$$\tan^{-1}\left(\frac{x\cos\theta}{1-x\sin\theta}\right)-\cot^{-1}\left(\frac{\cos\theta}{x-\sin\theta}\right).$$

55. Solve:

$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\sin^{-1}\left(\frac{2x}{x^2+1}\right)+\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}.$$

56. Prove that

$$\tan^{-1}\left(\frac{yz}{xr}\right)+\tan^{-1}\left(\frac{xz}{yr}\right)+\tan^{-1}\left(\frac{xy}{zr}\right)=\frac{\pi}{2}.$$

where $x^2+y^2+z^2=r^2$.

57. If $\sum_{r=1}^{10}\tan^{-1}\left(\frac{3}{9r^2+3r-1}\right)=\cot^{-1}\left(\frac{m}{n}\right)$ where m and n are co-prime, find the value of $(2m+n+4)$.

58. If the sum $\sum_{b=1}^{10}\sum_{a=1}^{10}\tan^{-1}\left(\frac{a}{b}\right)=m\pi$, then find the value of $(m+4)$.

59. Let

$$f(x)=\frac{1}{\pi}\left(\sin^{-1}x+\cos^{-1}x+\tan^{-1}x\right)+\frac{(x+1)}{x^2+2x+10}$$

such that the maximum value of $f(x)$ is m, then find the value of $(104m-90)$.

60. Let m be the number of solutions of $\sin(2x)+\cos(2x)+\cos x+1=0$ in $0<x<\frac{\pi}{2}$ and

$$n=\sin\left[\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)+\cos^{-1}\left(\cos\left(\frac{7\pi}{3}\right)\right)\right]$$

then find the value of $\left(m^2+n^2+m+n+4\right)$.

61. Let $f(n)=\sum_{k=-n}^{n}\left(\cot^{-1}\left(\frac{1}{k}\right)-\tan^{-1}(k)\right)$ such that $\sum_{n=2}^{10}\left(f(n)+f(n-1)\right)=a\pi$ then find the value of $(a+1)$.

INTEGER TYPE QUESTIONS

1. If the solution set of $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right)<\pi-3$ is (a,b), where $a,b\in I$, then find $(b-a+5)$.

2. If $a\sin^{-1}x-b\cos^{-1}x=c$, such that the value of $a\sin^{-1}x+b\cos^{-1}x$ is $\frac{m\pi ab+c(a-b)}{a+b}, m\in N$, then find the value of $\left(m^2+m+2\right)$.

3. If m is a root of $x^2+3x+1=0$, such that the value of $\tan^{-1}(m)+\tan^{-1}\left(\frac{1}{m}\right)$ is $\frac{k\pi}{2}, k\in I$, then find the value of $(k+4)$.

4. Find the number of real solutions of $\sin^{-1}\left(x^2-2x+1\right)+\cos^{-1}\left(x^2-x\right)=\frac{\pi}{2}$.

5. Let $\cos^{-1}(x)+\cos^{-1}(2x)+\cos^{-1}(3x)=\pi$. If x satisfies the cubic equation $ax^3+bx^2+cx+d=0$, then find the value of $(b+c)-(a+d)$.

6. Consider α, β, γ are the roots of $x^3-x^2-3x+4=0$ such that $\tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma=\theta$. If the positive value of $\tan(\theta)$ is $\frac{p}{q}$, where p and q are natural numbers, then find the value of $(p+q)$.

7. If M is the number of real solution of $\cos^{-1}x+\cos^{-1}(2x)+\pi=0$ and N is the number of values of x satisfying the equation $\sin^{-1}\left(\frac{5}{x}\right)+\sin^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}$, then find the value of $M+N+4$.

8. Find the value of

$$4\cos\left[\cos^{-1}\left(\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)\right)-\cos^{-1}\left(\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)\right)\right].$$

9. Find the value of

$$5\cot\left(\sum_{k=1}^{5}\cot^{-1}\left(k^2+k+1\right)\right)$$

10. Let $3\sin^{-1}(\log_2 x)+\cos^{-1}(\log_2 y)=\frac{\pi}{2}$

and $\sin^{-1}(\log_2 x)+2\cos^{-1}(\log_2 y)=\frac{11\pi}{6}$

then find the value of $\left(\frac{1}{x^2}+\frac{1}{y^2}+2\right)$

11. If α and β are the roots of $x^2+5x-44=0$,

then find the value of $\cot\left(\cot^{-1}\alpha+\cot^{-1}\beta\right)$.

12. If x and y are +ve integers satisfying

$\tan^{-1}\left(\frac{1}{x}\right)+\tan^{-1}\left(\frac{1}{y}\right)=\tan^{-1}\left(\frac{1}{7}\right)$, then find

the number of ordered pairs of (x, y).

COMPREHENSIVE LINK PASSAGE

In these questions, a passage (paragraph) has been given followed by questions based on each of the passage. You have to answer the questions based on the passage given.

PASSAGE 1

Function	Domain	Co-domain
$\sin^{-1}x$	[–1, 1]	$\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$
$\tan^{-1}x$	R	$\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$
$\cos^{-1}x$	[–1, 1]	$[\pi, 2\pi]$
$\cot^{-1}x$	R	$[\pi, 2\pi]$

1. $\sin^{-1}(-x)$ is

(a) $-\sin^{-1}x$

(b) $\pi+\sin^{-1}x$

(c) $2\pi-\sin^{-1}x$

(d) $2\pi-\cos^{-1}\sqrt{1-x^2}, x>0$

2. If $f(x)=3\sin^{-1}x-2\cos^{-1}x$, then $f(x)$ is

(a) even function
(b) odd function
(c) Neither even nor odd
(d) even as well as odd function

3. The minimum value of $\left(\sin^{-1}x\right)^3-\left(\cos^{-1}x\right)^3$ is

(a) $-\frac{63\pi^3}{8}$ (b) $\frac{63\pi^3}{8}$

(c) $\frac{125\pi^3}{32}$ (d) $-\frac{125\pi^3}{32}$

4. The value of $\sin^{-1}x+\cos^{-1}x$ is

(a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$

(c) $\frac{5\pi}{2}$ (d) $\frac{7\pi}{2}$

5. If the co-domain of $\sin^{-1}x$ is $\left[-\frac{5\pi}{2},-\frac{3\pi}{2}\right]$ such that $\sin^{-1}x+\cos^{-1}x=\frac{5\pi}{2}$, then the co-domain of $\cos^{-1}x$ is

(a) $[4\pi, 5\pi]$ (b) $[3\pi, 4\pi]$

(c) $[6\pi, 7\pi]$ (d) $[5\pi, 6\pi]$

PASSAGE II

We know that corresponding to every bijection function $f: A \to B$, there exist a bijection $g: B \to A$ defined by $g(y) = x$ if and only if $f(x) = y$. The function $g: B \to A$ is called the inverse of function $f: A \to B$ and is denoted by f^{-1}. Thus, we have $f(x) = y \Rightarrow f^{-1}(y) = x$. We know that trigonometric functions are periodic functions and hence, in general all trigonometric functions are not bijectives. Consequently, their inverse do not exist. However, if we restrict their domains and co-domains, they we can make then bijectives and also we can find their inverse.

Now, answer the following questions.

1. $\sin^{-1}(\sin\theta) = \theta$, for all θ belonging to

(a) $[0, \pi]$ (b) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(c) $\left[-\frac{\pi}{2},0\right]$ (d) None of these

2. $\cos^{-1}(\cos\theta) = \theta$, for all θ belonging to

(a) $[0, \pi]$ (b) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(c) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (d) None of these

3. $\tan^{-1}(\tan\theta) = \theta$, for all θ belonging to

(a) $[0, \pi]$ (b) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

(c) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (d) None of these

4. $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$, for all θ belonging to

(a) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (b) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)-\{0\}$

(c) $[0, \pi]$ (d) $(0, \pi)$

5. $\sec^{-1}(\sec\theta) = \theta$, for all θ belonging to
   (a) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ (b) $(0, \theta) - \left\{\frac{\pi}{2}\right\}$
   (c) $(0, \pi)$ (d) None of these

6. $\sin^{-1}(\sin x) = x$, for all x belonging to
   (a) R (B) φ
   (c) $[-1, 1]$ (d) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

7. The value of $\sin^{-1}(\sin 2) + \cos^{-1}(\cos 2)$ is
   (a) 0 (b) $\frac{\pi}{2}$
   (c) $-\frac{\pi}{2}$ (d) None of these.

PASSAGE III

Let $f(x) = \sin\{\cot^{-1}(x+1)\} - \cos(\tan^{-1} x)$ and

$a = \cos(\tan^{-1}(\sin(\cot^{-1} x)))$ & $b = \cos(2\cos^{-1} x + \sin^{-1} x)$

1. The value of x for which $f(x) = 0$ is
   (a) $-1/2$ (b) 0
   (c) 1/2 (d) 1

2. If $f(x) = 0$, then a^2 is equal to
   (a) 1/2 (b) 2/3
   (c) 5/9 (d) 9/5

3. If $a^2 = \frac{26}{51}$, then b^2 is equal to
   (a) 1/25 (b) 24/25
   (c) 25/26 (d) 50/51

PASSAGE IV

Every bijective (one - one onto function) $f: A \to B$ there exists a bijection $g: B \to A$ is defined by $g(y) = x$ if and only if $f(x) = y$.

The function g: $B \to A$ is called the inverse of function $f: A \to B$ and is denoted by f^{-1}. If no branch of an inverse trigonometric function is mentioned, then it means the principal value branch of that function.

1. The value of $\cos\{\tan^{-1}(\tan 2)\}$ is
   (a) $1/\sqrt{5}$ (b) $-1/\sqrt{5}$
   (c) $\cos 2$ (d) $-\cos 2$

2. If x takes negative permissible value then $\sin^{-1} x$ is
   (a) $\cos^{-1}\left(\sqrt{1-x^2}\right)$
   (b) $-\cos^{-1}\left(\sqrt{1-x^2}\right)$
   (c) $\cos^{-1}\left(\sqrt{x^2-1}\right)$
   (d) $\pi - \cos^{-1}\left(\sqrt{1-x^2}\right)$

3. If $x + \frac{1}{x} = 2$, then the value of $\sin^{-1} x$ is
   (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
   (c) π (d) $\frac{3\pi}{2}$

PASSAGE V

Let $\cos^{-1} x + (\sin^{-1} y)^2 = \frac{a\pi^2}{4}$ ....(i)

and $\cos^{-1} x.(\sin^{-1} y)^2 = \frac{\pi^2}{16}$ ....(ii)

where $-1 \le x, y \le 1$. Then

1. The set of values of 'a' for which the equation (i) holds good is
   (a) $\left(0, 2+\frac{4}{\pi}\right)$ (b) $\left[0, 1+\frac{4}{\pi}\right)$
   (c) R (d) $\left[0, -1+\frac{4}{\pi}\right)$

2. The set of values of 'a' for which equations (i) and (ii) posses solutions
   (a) $(-\infty, 2] \cup [2, \infty)$ (b) $(-2, 2)$
   (c) $\left[2, 1+\frac{4}{\pi}\right]$ (d) R

3. The values of x and y, the system of equations (i) and (ii) posses solutions for integral values of 'a'
   (a) $\left\{\cos\left(\frac{\pi^2}{4}\right), 1\right\}$ (b) $\left\{\cos\left(\frac{\pi^2}{4}\right), -1\right\}$
   (c) $\left\{\cos\left(\frac{\pi^2}{4}\right), \pm 1\right\}$ (d) $\{(x, y) : x \in R, y \in R\}$

MATCH-MATRIX

Given below are matching type questions, with two columns (each having some items) each.

Each item of column I has to be matched with the items of Column II, by encircling the correct match(es).

Note: An item of column I can be matched with more than one items of column II. All the items of column II have to be matched.

1. Match the following columns:

Column - I	Column - II
(A) The principal value of $\sin^{-1}(\sin 20)$ is	(P) $(20-6\pi)$
(B) The principal value of $\sin^{-1}(\sin 10)$ is	(Q) $(3\pi-10)$
(C) The principal value of $\cos^{-1}(\cos 10)$ is	(R) $(4\pi-10)$
(D) The principal value of $\cos^{-1}(\cos 20)$ is	(S) $(5\pi-20)$

2. Match the following columns:

Column - I	Column - II
(A) The range of $f(x)=3\sin^{-1}x+2\cos^{-1}x$ is	(P) $(0, \pi)$
(B) The range of $f(x)=\sin^{-1}x+\cos^{-1}x+\tan^{-1}x$ is	(Q) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
(C) The range of $f(x)=\sqrt{\sin^{-1}x+\pi}$ is	(R) $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$
(D) The range of $f(x)=2\tan^{-1}x+\sin^{-1}x+\sec^{-1}\left(\frac{1}{x}\right)$ is	(S) $[0, \pi]$

3. Match the following columns:

Column - I	Column - II
(A) $\sin(\sin^{-1}x)=\sin^{-1}(\sin x)$, if	(P) $-1\le x\le 1$
(B) $\cos(\cos^{-1}x)=\cos^{-1}(\cos x)$, if	(Q) $0\le x\le 1$
(C) $\tan(\tan^{-1}x)=\tan^{-1}(\tan x)$, if	(R) $0<x<\pi$
(D) $\cot(\cot^{-1}x)=\cot^{-1}(\cot x)$, if	(R) $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$.

4. Match the following columns:

Column - I	Column - II
The value of	
(A) $\sin^{-1}\left(\frac{1}{2}\right)+\cos^{-1}\left(-\frac{1}{2}\right)$ is	(P) $\frac{7\pi}{6}$
(B) $\sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\left(\frac{1}{2}\right)$ is	(Q) $\frac{5\pi}{6}$
(C) $\tan^{-1}(\sqrt{3})+\cot^{-1}(-\sqrt{3})$ is	(R) $\frac{\pi}{6}$
(D) $\sin^{-1}\left(\frac{1}{2013}\right)+\cos^{-1}\left(\frac{1}{2013}\right)$ is	(S) $\frac{\pi}{2}$

5. Match the following columns:

Column - I	Column - II
(A) The value of $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3$ is	(P) $\frac{3\pi}{4}$
(B) The value of $\tan^{-1}1+\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)$ is	(Q) $\frac{\pi}{2}$
(C) The value of $\tan^{-1}(9)+\tan^{-1}\left(\frac{5}{4}\right)$ is	(R) π
(D) The value of $2\tan^{-1}x-\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, $x>1$ is	(S) $-\frac{\pi}{2}$

6. Match the following columns:

Column - I	Column - II
(A) $(\sin^{-1}x)^3+(\cos^{-1}x)^3$ is maximum at	(P) $x=\frac{1}{\sqrt{2}}$
(B) $(\sin^{-1}x)^2+(\cos^{-1}x)^2$ is minimum at	(Q) $x=1$
(C) $(\sin^{-1}x)-(\cos^{-1}x)$ is minimum at	(R) $x=-1$
(D) $(\tan^{-1}x)^2+(\cot^{-1}x)^2$ is minimum at	(S) $x=0$.

ASSERTION AND REASON

Codes:

(A) Both A and R are individually true and R is the correct explanation of A.

(B) Both A and R are individually true and R is not the correct explanantion of A.

(C) A is true but R is false.

(D) A is false but R is true.

1. Assertion (A):

If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$,

then the value of $(x^{2013}+y^{2013}+z^{2013})$

$-\frac{9}{(x^{2014}+y^{2014}+z^{2014})}$ is zero.

Reason (R): Maximum value of $\sin^{-1}x$ is $\frac{\pi}{2}$

(a) A (b) B

(c) C (d) D

2. Assertion (A): The value of

$2\tan^{-1}x-\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ is π

Reason (R): $x>1$

(a) A (b) B
(c) C (d) D

3. Assertion (A): The value of

$\tan^{-1}(p)+\tan^{-1}\left(\frac{1}{p}\right)$ is $-\frac{\pi}{2}$

Reason (R): P is the root of $x^2+2013x+2014=0$.

(a) A (b) B (c) C (d) D

4. Assertion (A):

If $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$, then the value of

$\cos^{-1}x+\cos^{-1}y$ is $\frac{\pi}{3}$

Reason (R): $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$, when $x\in[-1,1]$

(a) A (b) B (c) C (d) D

5. Assertion (A):

The value of $\cos^{-1}(\cos 10)$ is $(2\pi-5)$

Reason(R):

The range of $\cos^{-1}x$ is $[0,\pi]$

(a) A (b) B (c) C (d) D

6. Assertion (A):

If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$,

then $x^2+y^2+z^2+2xyz=1$

Reason(R): For $-1\le x, y, z\le 1$.

(a) A (b) B (c) C (d) D

7. Assertion (A):

If $\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{\pi}{4}$, then

x is $\frac{1}{6}$.

Reason (R): For $0<2x, 3x<1$

(a) A (b) B (c) C (d) D

8. Assertion (A):

$\cos\left(\frac{1-x^2}{1+x^2}\right)=2\tan^{-1}x$

Reason (R): for $x\ge 0$

(a) A (b) B (c) C (d) D

9. Assertion (A): $\sin^{-1}(3x-4x^3)=\pi-3\sin^{-1}x$

Reason(R): for $\frac{1}{2}<x\le 1$

(a) A (b) B (c) C (d) D

10. Assertion (A): $\cos^{-1}(4x^3-3x)=2\pi-3\cos^{-1}(x)$

Reason (R): For $-\frac{1}{2}\le x<\frac{1}{2}$

(a) A (b) B (c) C (d) D

11. Assertion (A): $\cot^{-1}(x)=\tan^{-1}\left(\frac{1}{x}\right)$, $x>0$

Reason (R): $\cot^{-1}(x)=\pi+\tan^{-1}\left(\frac{1}{x}\right)$, $x<0$

(a) A (b) B (c) C (d) D

12. Assertion (A):

If α, β are the roots of $x^2-3x+2=0$, then $\sin^{-1}\alpha$ exist but not $\sin^{-1}\beta$, where $\alpha>\beta$

Reason (R): Domain of $\sin^{-1}x$ is $[-1,1]$

(a) A (b) B (c) C (d) D

QUESTIONS ASKED IN PAST IIT-JEE EXAMS

1. Let a, b, c positive real numbers such that

$\theta=\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$.

Then $\tan\theta$ is equal to. [IIT-JEE-1981]

2. The numerical value of

$\tan^{-1}\left\{2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right\}$ [IIT-JEE -1981]

3. Find the value of $\cos(2\cos^{-1}x+\sin^{-1}x)$ at $x = 1/5$, where

$0\le\cos^{-1}x\le\pi$ and $-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}$.

[IIT-JEE-1981]

4. The value of $\tan\left[\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$ is

(a) 6/17 (b) 17/6 (c) –17/6 (d) –6/17

[IIT-JEE-1983]

5. No questions asked in between 1984 to 1985.

6. The principal value of $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is

(a) $-\frac{2\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{5\pi}{3}$

[IIT-JEE - 1986]

7. No questions asked in between 1987 to 1988

8. The greater of the two angles $A = 2\tan^{-1}\left(2\sqrt{2}-1\right)$ and $B = 3\sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{3}{5}\right)$ is....... [IIT-JEE-1989]

9. No questions asked in between 1990 to 1998.

10. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$ is

(a) 0 (b) 1

(c) 2 (d) ∞ [IIT-JEE-1999]

11. No questions asked in 2000.

12. If $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-.......\right)$

$+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-....\right)=\frac{\pi}{2}$,

for $0<|x|<\sqrt{2}$, then x is

(a) 1/2 (b) 1

(c) –1/2 (d) –1 [IIT-JEE-2001]

13. Prove that $\cos\left(\tan^{-1}\left(\sin\left(\cot^{-1}x\right)\right)\right)=\sqrt{\frac{x^2+1}{x^2+2}}$. [IIT-JEE-2002]

14. The domain of $f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$ is

(a) $\left(-\frac{1}{2},\frac{1}{2}\right]$ (b) $\left[-\frac{1}{4},\frac{3}{4}\right)$

(c) $\left[-\frac{1}{4},\frac{1}{4}\right]$ (d) $\left[-\frac{1}{4},\frac{1}{2}\right]$

[IIT-JEE-2003]

15. If $\sin\left(\cot^{-1}(x+1)\right)=\cos\left(\tan^{-1}x\right)$, then the value of x is

(a) –1/2 (b) 1/2 (c) 0 (d) 9/4

[IIT-JEE-2004]

16. No questions asked in between 2005 to 2006.

17. Match the following columns:
Let (x, y) be such that

$\sin^{-1}(ax)+\cos^{-1}(y)+\cos^{-1}(bxy)=\frac{\pi}{2}$

Column - I	Column - II
(A) If $a = 1$ and $b = 0$, then (x, y)	(P) lies on the circle $x^2+y^2=1$
(B) If $a = 1$ and $b = 1$, then (x, y)	(Q) lies on $\left(x^2-1\right)\left(y^2-1\right)=0$
(C) If $a = 1$ and $b = 2$, then (x, y)	(R) lies on the line $y = x$
(D) If $a = 2$ and $b = 2$, then (x, y)	(S) lies on $\left(4x^2-1\right)\left(y^2-1\right)=0$

[IIT-JEE-2007]

18. If $0 < x < 1$, then

$\sqrt{1+x^2}\times\left[\left\{x\cos\left(\cot^{-1}x\right)+\sin\left(\cot^{-1}x\right)\right\}^2-1\right]^{1/2}$

equals

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) x

(c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

[IIT-JEE-2008]

19. No questions asked in between 2009 to 2010.

20 Let $f(\theta)=\sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$

where $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$. Then the value of

$\frac{d}{d(\tan\theta)}\left(f(\theta)\right)$ is....... [IIT-JEE-2011]

21. No questions asked in 2012.

22. The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is

(a) 23/25 (b) 25/23

(c) 23/24 (d) 24/23

[IIT-JEE - 2013]

23. The value of

$\left(\frac{1}{y^2}\left(\frac{\cos\left(\tan^{-1}y\right)+y\sin\left(\tan^{-1}y\right)}{\cot\left(\sin^{-1}y\right)+\tan\left(\sin^{-1}y\right)}\right)^2+y^4\right)^{1/2}$

is......... [IIT-JEE - 2013]

24. If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right)=\sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right)$, $x\neq 0$

then the value of x is....... [IIT-JEE - 2013]

25. Find the number of positive solutions satisfying the equation

$\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$

[IIT-JEE-2014]

26. No questions asked in 2015.

ANSWERS

EXERCISE 1

1. $f^{-1}(x)=\frac{x-5}{3}$
2. $f^{-1}(x)=\frac{x}{x-1}$
3. $f^{-1}(x)=\sqrt{\frac{1-x}{x}}$
4. $f^{-1}(x)=\frac{1}{2}\log_2\left(\frac{1+x}{1-x}\right)$
5. $f^{-1}(x)=\frac{-1+\sqrt{1-4x^2}}{2x}$

EXERCISE 2

1. $[1, 2]$
2. $\left(-\infty, \frac{1}{2}\right]$
3. $\left[-\frac{5}{3}, -1\right]$
4. $[-3, 3]$
5. $[-6, 6]$
6. $[-5, 5]$
7. $[-1, 1]$
8. $\left[0, \frac{1}{2}\right]$
9. $[-\sqrt{3}, 1]$
10. $x=(4n+1)\frac{\pi}{2}, n\in I$
11. $x=\{-1, 1\}$
12. $x=\varphi$
13. $[-2, -1]$
14. $[-2, -1]\cup[1, 2]$
15. $\left[-1, -\frac{1}{\sqrt{3}}\right]\cup\left[\frac{1}{\sqrt{3}}, 1\right]$
16. $x=\varphi$
17. $\left[-\frac{1}{\sqrt{2}}, -\sqrt{2}\right]\cup\left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$
18. $(0, 4]$
19. $(0,1)\cup(1, \infty)$
20. $[1, 2]$
21. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
22. $\left[-\frac{5\pi}{4}, \frac{3\pi}{4}\right]$
23. $[-\pi, \pi]$
24. $\left(-\frac{\pi}{2}, 0\right)$
25. $(0, \pi)$
26. $(0, \pi)$
27. $(0, \pi)$
28. $\left(\frac{5\pi}{4}, \frac{9\pi}{4}\right)$
29. [1. 2]
30. $\{\text{cosec}^{-1}(1)\}$

EXERCISE 3

1. $x=\frac{\pi}{2}$
2. $x=1, y=1/2$
3. $x=1$
4. $x=1$
5. $x=3/2$
6. $x=\left(\frac{1}{\sqrt{2}}, 1\right]$
7. $x=\pm\sqrt{2}$
8. $x=\sqrt{3}, -\frac{1}{\sqrt{3}}$
9. $x=\frac{\sqrt{3}-1}{2\sqrt{2}}$
10. $x=\sqrt{2}-1$
11. $x=0$
12. $x=\frac{-1\pm\sqrt{5}}{2}$
13. $(\cos 1, \sin 1)$
14. $(\cos 1, \tan 1)$
15. $[\tan(\sin(\sin 1)), \tan(\sin 1))$

16. $\left[\tan(\sin(\cos(\sin 1))), \tan(\sin(\cos(\sin 1)))\right]$
17. $x = 1/2$

EXERCISE 4

1. $\left[1, \frac{1}{2}\right]$
2. $(-\infty, 1]$
3. $[-1, 0)$
4. $[-1, 0)$
5. $(\tan 2, \infty)$
6. $(\tan 3, \cot 2)$
7. $\left[-1, -\frac{1}{\sqrt{2}}\right)$
8. $\left(\frac{1}{2}, 1\right]$
9. $\left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)$
10. $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(0, \frac{1}{\sqrt{3}}\right)$
11. $\left[-1, \frac{1}{2}\right]$
12. $\left(1-\sqrt{\pi-1}, 1+\sqrt{\pi-1}\right)$
13. $\left(-\frac{8}{5}, 0\right)$
14. $(-\infty, -1) \cup (1, \infty)$
15. $-\sqrt{\frac{5}{6}} \le x \le \sqrt{\frac{5}{6}}$

EXERCISE 5

1. $-\frac{\pi}{3}$
2. $-\frac{\pi}{3}$
3. $\frac{\pi}{8}$
4. $(\pi - 3)$
5. $(5 - 2\pi)$
6. $(7 - 2\pi)$
7. $(3\pi - 10)$
8. $(12 - 4\pi)$
9. $(20 - 6\pi)$
10. $(50 - 16\pi)$
11. $(25\pi - 80)$
12. $(32\pi - 100)$
13. $\frac{5\pi}{6}$
14. $\frac{3\pi}{7}$
15. $\frac{5\pi}{7}$
16. 3
17. $(2\pi - 5)$
18. $(7 - 2\pi)$
19. $(4\pi - 10)$
20. $(4\pi - 12)$
21. $(15 - 4\pi)$
22. $(40 - 12\pi)$
23. $(32\pi - 100)$
24. $(32\pi - 100)$
25. $(2\pi - 4)$
26. 0
27. $(10 - \pi)$
28. $20 - 4\pi$
29. $7\pi - 20$
30. 0
31. 0
32. $-\pi$
33. 3
34. π^2.
35. $-\frac{\pi}{3}$
36. $-\frac{\pi}{5}$
37. $\frac{\pi}{12}$
38. $(3 - \pi)$
39. $(5 - 2\pi)$
40. $(7 - 2\pi)$
41. $(10 - 3\pi)$

42. $(20-6\pi)$
43. $(50-16\pi)$
44. $(10-3\pi)$
45. $(200-63\pi)$
46. $(15-4\pi)$
46. $(16\pi-50)$
47. 2.
48. $\left(-2, -\frac{3}{2}\right)$
49. $(-1, 1)$

EXERCISE 6

1. $x = 1/2$
2. $(-\infty, 0)\cup(4, \infty)$
3. $x = \pm\sqrt{\frac{\sqrt{5}-1}{2}}$
4. No solution
5.
6. $x = \pm 2$
7. $x = -13/12$
8. $x = 8$
9. $k = 5$
10. 0
11. 1
12. 4
13. 23
24. $\sin^{-1}(x)$
25. $2\sin^{-1}(x)$
26. $2\cos^{-1}(x)$
27. $\left(\frac{\pi}{4}-\frac{x}{2}\right)$
28. $\left(\frac{\pi}{4}+\frac{x}{2}\right)$
29. $\left(\frac{\pi}{4}-x\right)$
30. $\sin^{-1}\left(\frac{x}{a}\right)$
31. $\tan^{-1}\left(\sqrt{x}\right)$
32. $\tan^{-1}\left(\frac{a}{b}\right)-x$
33. $\frac{1}{2}\tan^{-1}(x)$
34. $\frac{x}{2}$
35. $3\tan^{-1}\left(\frac{x}{a}\right)$
36. $-\frac{x}{2}, \frac{x}{2}$
37. $\sin^{-1}(x)-\sin^{-1}\left(\sqrt{x}\right)$
38. $\left(\frac{\pi}{4}+x\right)$
39. $\left(x-\frac{\pi}{4}\right)$
40. $\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\left(x^2\right)$
41. $(x+\alpha)$, where $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$

EXERCISE 7

11. $x \in \left(\frac{3-\sqrt{1+4\pi}}{2}, \frac{3+\sqrt{1+4\pi}}{2}\right)$
12. $x \in \left(-2, -\frac{2}{3}\right)$
13. $x \in \left[\frac{1}{2}, 1\right]$

EXERCISE 8

16. 0
18. 1
23. 21
24. 3
26. $x = 1$
27. $x = 2$
28. $x = 2\sqrt{\frac{3}{7}}$
29. $\frac{3}{\sqrt{10}}$
30. $\frac{1}{2}\sqrt{\frac{3}{13}}$
31. $x = 3$
32 $x = 0$
33. $x = 0, \pm\left(\frac{1}{2}\right)$

LEVEL II

1. (d)	2. (b)	3. (b)	4. (b)
5. (d)	6. (b)	7. (b)	8. (d)

9. (a)	10. (c)	11. (c)	12. (c)
13. (a, c)	14. (a)	15. (d)	16. (b)
17. (b)	18. (c)	19. (c)	20. (b)
21. (a)	22. (a)	23. (b)	24. (c, d)
25. (a)	26. (b)	27. (b)	28. (a)
29. (a)	30. (c)	31. (b)	32. (a)
33. (a)	34. (c)	35. (b)	36. (c)
37. (c)	38. (b)	39. (c)	40. (a)
41. (b)	42. (a)	43. (a)	

LEVEL III

6. $(8\pi - 21)$
7. $\frac{\pi}{3}$
16. Max value $\frac{7\pi^3}{8}$, when $x = -1$

    and Min value = $\frac{\pi^3}{32}$, when $x = \frac{1}{\sqrt{2}}$.
17 $x = \frac{1}{2}\sqrt{\frac{3}{7}}$
18. $x = 3$
19. $x = 0, 1/2, -1/2$
20. $x = \frac{3}{\sqrt{10}}$
21. $x = 2-\sqrt{3}, \sqrt{3}$
22. $x = \frac{a-b}{1+ab}$
23. $x = n^2 - n + 1, n$
24. $x = 4/3$
25. $x = ab$,
26. $x = 1/2, y = 1$.
27. $x = 1, y = 2$; $x = 2, y = 7$
29. $-\pi$
32. $x \in \left(\frac{1}{\sqrt{2}}, 1\right]$
33. $x \in \left(-1, \frac{1}{\sqrt{2}}\right]$
34. $x \in (-\infty, \cot 3) \cup (\cot 2, \infty)$
35. $x \in \left(\frac{1}{\sqrt{2}}, 1\right) \cup \left(-1, -\frac{1}{\sqrt{2}}\right)$
36 $\tan\left(\frac{1}{2}\right) < x < \tan\left(\frac{3}{2}\right)$
37. $\cot(3) \le x \le \cot(1)$
38. $x \in (-1, 1)$
39. $n = 5$
40. $n = 8$
41. $k = \phi$.
44. $\tan(\sin(\cos(\sin 1))) \le x < \tan(\sin(1))$.
45. $[1, \infty)$
49. $a \in \left[\frac{1}{32}, \frac{7}{8}\right]$
50. $x = y = \sqrt{a^2 + 1}$
53. $\theta \in n\pi + \tan^{-1}(-2), n \in Z$
54. θ
55. $x = \frac{1}{\sqrt{3}}$
56. $\frac{\pi}{2}$
57. 36
58. 29
59. 4
60. 6
61. 100

INTEGER TYPE QUESTIONS

1. 7
2. 4
3. 3
4. $x = 1$
5. 3
6. 9
7. 5, where M = 0, N = 1
8. 2
9. 7
10. 8.
11, 9
12. 6

COMPREHENSIVE LINK PASSAGES

Passage - I :	1. (c)	2. (b)	3. (a)	4. (c)	5. (a)
Passage - II :	1. (b)	2. (a)	3. (c)	4. (a)	5. (a)
	6. (c)	7. (a)			
Passage - III :	1. (a)	2. (c)	3. (b)		
Passage - IV :	1. (d)	2. (b)	3. (b)		
Passage - V :	1. (b)	2. (c)	3. (c)		

MATCH MATRIX

1. (A) (P); (B) (Q); (C) (R); (D) (P)
2. (A) (S); (B) (P); (C) (Q); (D) (R)
3. (A) (P); (B) (Q); (C) (S); (D) (R).
4. (A) (Q); (B) (R); (C) (P);(D) (S).
5. (A) (R); (B) (Q); (C) (P); (D) (R).
6. (A) (R); (B) (P); (C) (Q); (D) (Q).

ASSERTION AND REASON

1. (a)	2. (a)	3. (a)	4. (a)
5. (a)	6. (a)	7. (a)	8. (a)
9. (a)	10. (a)	11. (b)	12. (a)

HINTS AND SOLUTION

LEVEL - III

1. We have

$$\sin^{-1}(\cos(\sin^{-1} x)) + \cos^{-1}(\sin(\cos^{-1} x))$$

$$= \sin^{-1}(\sqrt{1-x^2}) + \cos^{-1}(\sqrt{1-x^2})$$

$$= \cos^{-1}(x) + \sin^{-1}(x)$$

$$= \frac{\pi}{2}$$

2. We have

$$\tan^{-1}\{\operatorname{cosec}(\tan^{-1} x) - \tan(\cot^{-1} x)\}$$

$$= \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right), x = \tan\theta$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\theta}{2}\right)\right)$$

$$= \frac{1}{2}\tan^{-1} x$$

3. We have $\tan(\tan^{-1} x + \tan^{-1} y + \tan^{-1} z)$

$$= \tan\left(\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)\right)$$

$$= \left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

Also, $\cot(\cot^{-1} x + \cot^{-1} y + \cot^{-1} z)$

$$= \cot\left(\tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{y}\right) + \tan^{-1}\left(\frac{1}{z}\right)\right)$$

$$= \cot\left(\tan^{-1}\left(\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{xyz}}{1-\left(\frac{1}{xy}+\frac{1}{xz}+\frac{1}{yz}\right)}\right)\right)$$

$$= \cot\left(\tan^{-1}\left(\frac{xy+yz+zx-1}{xyz-(x+y+z)}\right)\right)$$

$$= \cot\left(\cot^{-1}\left(\frac{xyz-(x+y+z)}{xy+yz+zx-1}\right)\right)$$

$$= \cot\left(\cot^{-1}\left(\frac{(x+y+z)-xyz}{1-(xy+yz+zx)}\right)\right)$$

$$= \left(\frac{(x+y+z)-xyz}{1-(xy+yz+zx)}\right)$$

Hence, the result.

4. Given expression is

$$\sin(\cot^{-1}(\tan(\cos^{-1} x)))$$

$$= \sin(\cot^{-1}(\tan\theta)), \theta = \cos^{-1} x$$

$$= \sin(\cot^{-1}(\tan\theta)), \cos\theta = x$$

$$= \sin\left(\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right)$$

$$= \sin\varphi, \text{where } \cot\varphi = \frac{\sqrt{1-x^2}}{x}$$

$$= x$$

5. Given expression is

$$\sin\left(\operatorname{cosec}^{-1}\left(\cot\left(\tan^{-1} x\right)\right)\right)$$

$$= \sin\left(\operatorname{cosec}^{-1}(\cot\theta)\right), \text{ where } \tan\theta = x$$

$$= \sin\left(\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)\right)$$

$$= \sin\varphi, \text{ where } \operatorname{cosec}\varphi = \frac{1}{x}$$

$$= x$$

6. Given expression is

$$\sin^{-1}(\sin 5) + \cos^{-1}(\cos 10) + \tan^{-1}(\tan(-6)) + \cot^{-1}(\cot(-10))$$

$$= (5-2\pi) + (4\pi-10) - (6-2\pi) + \pi - (10-3\pi)$$

$$= (5-2\pi) + (4\pi-10) + (2\pi-6) + \pi + (3\pi-10)$$

$$= 8\pi - 21$$

7. We have

$$\cos^{-1}x+\cos^{-1}\left(\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right),\forall x\in\left(\frac{1}{2},1\right)$$

$$=\cos^{-1}(x)+\cos^{-1}\left(x\cdot\frac{1}{2}+\sqrt{1-\frac{1}{4}}\sqrt{1-x^2}\right)$$

$$=\cos^{-1}(x)+\cos^{-1}\left(\frac{1}{2}\right)-\cos^{-1}(x)$$

$$=\cos^{-1}\left(\frac{1}{2}\right)$$

$$=\frac{\pi}{3}$$

8. We have

$$\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-\tan^{-1}\left(\frac{\sqrt{5-2\sqrt{6}}}{1+\sqrt{6}}\right)$$

$$=\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-\tan^{-1}\left(\frac{\sqrt{(\sqrt{3}-\sqrt{2})^2}}{1+\sqrt{6}}\right)$$

$$=\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-\tan^{-1}\left(\frac{(\sqrt{3}-\sqrt{2})}{1+\sqrt{3}\sqrt{2}}\right)$$

$$=\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-(\tan^{-1}(\sqrt{3})-\tan^{-1}(\sqrt{2}))$$

$$=\cot^{-1}(\sqrt{2})+\tan^{-1}(\sqrt{2})-\tan^{-1}(\sqrt{3})$$

$$=\frac{\pi}{2}-\frac{\pi}{3}=\frac{\pi}{6}$$

9. We have

$$m=\sin^{-1}(a^6+1)+\cos^{-1}(a^4+1)-\tan^{-1}(a^2+1)$$

since $\sin^{-1}(\ )+\cos^{-1}(\ )$ is defined for [–1, 1]

Put $a=0$, then

$$m=\sin^{-1}(1)+\cos^{-1}(1)-\tan^{-1}(1)$$

$$=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

Hence, the image of the line $x+y=\frac{\pi}{4}$

w.r.t to the y-axis is $x-y+\frac{\pi}{4}=0$

10. Given equation is

$$(\sin^{-1}x)^3+(\sin^{-1}y)^3+(\sin^{-1}z)^3=\frac{(3\pi)^3}{8}$$

It is possible only when

$$\sin^{-1}x=\frac{\pi}{2}=\sin^{-1}y=\sin^{-1}z$$

So, $x = 1$, $y = 1$ and $z = 1$

Hence, the value of $(3x+4y-5z+2)$

$= 3 + 4 – 5 + 2$

$= 4$

11. We have $S=\sum_{r=1}^{n}\cot^{-1}\left(2^{r+1}+\frac{1}{2^r}\right)$

$$=\sum_{r=1}^{n}\cot^{-1}\left(\frac{2^{2r+1}+1}{2^r}\right)$$

$$=\sum_{r=1}^{n}\tan^{-1}\left(\frac{2^r}{1+2^{2r+1}}\right)$$

$$=\sum_{r=1}^{n}\tan^{-1}\left(\frac{2^{r+1}-2^r}{1+2^{r+1}\cdot 2^r}\right)$$

$$=\sum_{r=1}^{n}[\tan^{-1}(2^{r+1})-\tan^{-1}(2^r)]$$

$$=[\tan^{-1}(2^{n+1})-\tan^{-1}(2)]$$

Thus, $\lim_{n\to\infty}(S)$

$$=\lim_{n\to\infty}[\tan^{-1}(2^{n+1})-\tan^{-1}(2)]$$

$$=[\tan^{-1}(\infty)-\tan^{-1}(2)]$$

$$=\frac{\pi}{2}-\tan^{-1}(2)$$

$$=\cot^{-1}(2)$$

12. We have $\lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{4}{4r^2+3}\right)\right)\right)$

$$=\lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{r^2+\frac{3}{4}}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{1+\left(r^2-\frac{1}{4}\right)}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{\left(r+\frac{1}{2}\right)-\left(r-\frac{1}{2}\right)}{1+\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(n+\frac{1}{2}\right)-\tan^{-1}\left(n-\frac{1}{2}\right)\right)\right)$$

$$\lim_{n\to\infty}\left(\tan\left(\tan^{-1}\left(n+\frac{1}{2}\right)-\tan^{-1}\left(\frac{1}{2}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\tan^{-1}\left(\frac{n+\frac{1}{2}-\frac{1}{2}}{1+\frac{1}{2}\left(n+\frac{1}{2}\right)}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{1+\frac{1}{2}\left(n+\frac{1}{2}\right)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{\frac{1}{n}+\frac{1}{2}\left(1+\frac{1}{2n}\right)}\right)$$

$= 2.$

13 Given equation is

$$2\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\pi x^3$$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\frac{\pi x^3}{2}$$

$$2\tan^{-1}x=\frac{\pi x^3}{2}$$

$$\tan^{-1}x=\frac{\pi x^3}{4}$$

Clearly , there are 3 solutions at $x = -1, 0, 1$

14. Given $\cos^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{y}{b}\right)=\alpha$

$$\Rightarrow \cos^{-1}\left(\frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right)=\alpha$$

$$\Rightarrow \left(\frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right)=\cos\alpha$$

$$\Rightarrow \left(\frac{xy}{ab}-\cos\alpha\right)^2=\left(\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right)^2$$

$$\Rightarrow \left(\frac{xy}{ab}\right)^2-2\left(\frac{xy}{ab}\right)\cos\alpha+\cos^2\alpha$$

$$=1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)+\left(\frac{xy}{ab}\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}-2\left(\frac{xy}{ab}\right)\cos\alpha+\frac{y^2}{b^2}=1-\cos^2\alpha$$

$$\Rightarrow \frac{x^2}{a^2}-2\left(\frac{xy}{ab}\right)\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha$$

15. Put $A=\sin^{-1}x, B=\sin^{-1}y, C=\sin^{-1}z$

$\Rightarrow x=\sin A, y=\sin B, z=\sin C$

$\Rightarrow \cos A=\sqrt{1-x^2}, \cos B=\sqrt{1-y^2}, \cos C=\sqrt{1-z^2}$

Thus, $A+B+C=\pi$

Now, L.H.S

$= x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}$

$= \sin A\cos A+\sin B\cos B+\sin C\cos C$

$= \frac{1}{2}[\sin 2A+\sin 2B+\sin 2C]$

$= \frac{1}{2}(4\sin A\sin B\sin C)$

$= 2\sin A\sin B\sin C$

$= 2xyz$.

Hence, the result.

16. Given $f(x)=(\sin^{-1}x)^3+(\cos^{-1}x)^3$

$$= (\sin^{-1}x+\cos^{-1}x)\{(\sin^{-1}x)^2+(\cos^{-1}x)^2-\sin^{-1}x\cos^{-1}x\}$$

$$= \frac{\pi}{2}\left\{\left(\frac{\pi}{2}\right)^2-3\sin^{-1}x\cos^{-1}x\right\}$$

$$= \frac{\pi}{2}\left\{\frac{\pi^2}{4}-3\sin^{-1}x\left(\frac{\pi}{2}-\sin^{-1}x\right)\right\}$$

$$= \frac{\pi}{2}\left\{\frac{\pi^2}{4} - 3a\left(\frac{\pi}{2} - a\right)\right\} \text{where } a = \sin^{-1} x$$

$$= \frac{\pi}{2}\left\{3a^2 - \frac{3a\pi}{2} + \frac{\pi^2}{4}\right\}$$

$$= \frac{\pi}{8}\{12a^2 - 6\pi a + \pi^2\}$$

$$= \frac{12\pi}{8}\left\{a^2 - \frac{1}{2}\pi a + \frac{\pi^2}{12}\right\}$$

$$= \frac{12\pi}{8}\left\{\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{48}\right\}$$

Min value $= \dfrac{\pi^3}{32}$ at $x = \dfrac{1}{\sqrt{2}}$

and Max value $= \dfrac{7\pi^3}{8}$ at $x = -1$

17. Given equation is

$$\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}(2x) = \frac{\pi}{2} - \sin^{-1} x = \cos^{-1} x$$

$$\Rightarrow \quad \sin^{-1}(2x) = \sin^{-1}(\sqrt{1 - x^2})$$

$$\Rightarrow \quad 2x = \sqrt{1 - x^2}$$

$$\Rightarrow \quad 4x^2 = 1 - x^2$$

$$\Rightarrow \quad 5x^2 = 1$$

$$\Rightarrow \quad x^2 = \frac{1}{5}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{1}{5}}$$

18. Given equation is

$$\tan^{-1}\left(\frac{1}{1+2x}\right) + \tan^{-1}\left(\frac{1}{1+4x}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{1}{1+2x} + \frac{1}{1+4x}}{1 - \frac{1}{1+2x} \times \frac{1}{1+4x}}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \frac{1 + 4x + 1 + 2x}{1 + 6x + 8x^2 - 1} = \frac{2}{x^2}$$

$$\Rightarrow \quad \frac{2 + 6x}{6x + 8x^2} = \frac{2}{x^2}$$

$$\Rightarrow \quad \frac{1 + 3x}{3x + 4x^2} = \frac{2}{x^2}$$

$$\Rightarrow \quad 3x^3 + x^2 + 8x^2 + 6x = 0$$

$$\Rightarrow \quad 3x^3 + 9x^2 + 6x = 0$$

$$\Rightarrow \quad x^3 + 3x^2 + 2x = 0$$

$$\Rightarrow \quad x(x^2 + 3x + 2) = 0$$

$$\Rightarrow \quad x(x+1)(x+2) = 0$$

$$\Rightarrow \quad x = 0, -1, -2$$

19 Given equation is

$$\tan^{-1}(x-1) + \tan^{-1}(x) + \tan^{-1}(x+1) = \tan^{-1}(3x)$$

$$\Rightarrow \quad \tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}(3x) - \tan^{-1}(x)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x - 1 + x + 1}{1 - (x^2 - 1)}\right) = \tan^{-1}\left(\frac{3x - x}{1 + 3x \cdot x}\right)$$

$$\Rightarrow \quad \left(-\frac{2x}{x^2}\right) = \left(\frac{2x}{1 + 3x^2}\right)$$

$$\Rightarrow \quad 2x(1 + 3x^2 + x^2) = 0$$

$$\Rightarrow \quad 2x = 0, (1 + 4x^2) = 0$$

$$\Rightarrow \quad x = 0$$

Hence, the solution is x = 0.

20. Given equation is

$$\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cos^{-1} x = \frac{\pi}{4}$$

$$\Rightarrow \quad \cos^{-1}\left(\sqrt{1 - \frac{1}{5}}\right) + \cos^{-1} x = \frac{\pi}{4}$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{2}{\sqrt{5}}\right) + \cos^{-1} x = \frac{\pi}{4}$$

$$\Rightarrow \quad \cos^{-1} x = \frac{\pi}{4} - \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow \quad x = \cos\left(\frac{\pi}{4} - \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$$

$$\Rightarrow \quad x = \cos\left(\frac{\pi}{4}\right)\cos\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$$

$\Rightarrow \quad x = \frac{1}{\sqrt{2}} \times \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{5}} = \frac{3}{\sqrt{10}}$

21. Given equation is

$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$

$\Rightarrow \quad \cos^{-1}\left(-\frac{1-x^2}{x^2+1}\right) + \tan^{-1}\left(-\frac{2x}{1-x^2}\right) = \frac{2\pi}{3}$

$\Rightarrow \quad \pi - \cos^{-1}\left(\frac{1-x^2}{x^2+1}\right) - \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{2\pi}{3}$

$\Rightarrow \quad \pi - 2\tan^{-1}(x) - 2\tan^{-1}(x) = \frac{2\pi}{3}$

$\Rightarrow \quad \pi - 4\tan^{-1}(x) = \frac{2\pi}{3}$

$\Rightarrow \quad 4\tan^{-1}(x) = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

$\Rightarrow \quad \tan^{-1}(x) = \frac{\pi}{12}$

$\Rightarrow \quad x = \tan\left(\frac{\pi}{12}\right) = (2-\sqrt{3})$

22. Given equation is

$$2\tan^{-1} x = \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) - \cos^{-1}\left(\frac{1-b^2}{1+b^2}\right)$$

$\Rightarrow \quad 2\tan^{-1} x = 2\tan^{-1}(a) - 2\tan^{-1}(b)$

$\Rightarrow \quad \tan^{-1} x = \tan^{-1}(a) - \tan^{-1}(b)$

$\Rightarrow \quad \tan^{-1} x = \tan^{-1}\left(\frac{a-b}{1+ab}\right)$

$\Rightarrow \quad x = \left(\frac{a-b}{1+ab}\right)$

23. Given equation is

$$\cot^{-1} x + \cot^{-1}(n^2 - x + 1) = \cot^{-1}(n-1)$$

$\Rightarrow \quad \tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{n^2-x+1}\right) = \tan^{-1}\left(\frac{1}{n-1}\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{1}{n^2-x+1}\right) = \tan^{-1}\left(\frac{1}{n-1}\right) - \tan^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{1}{n^2-x+1}\right) = \tan^{-1}\left(\frac{\frac{1}{n-1} - \frac{1}{x}}{1 + \frac{1}{x(n-1)}}\right)$

$\Rightarrow \quad n^2 x - x^2 + x - n^3 + nx - n + n^2 - x + 1$

$\Rightarrow \quad n^2 x - x^2 + x - n^3 + nx - n + n^2 - x + 1$

$= nx - x + 1$

$\Rightarrow \quad n^2 x - x^2 + x - n^3 - n + n^2 = 0$

$\Rightarrow \quad (n^2+1)x - (n^2+1)n - x^2 + n^2 = 0$

$\Rightarrow \quad (n^2+1)(x-n) - (x^2-n^2) = 0$

$\Rightarrow \quad (x-n)(n^2+1-x-n) = 0$

$\Rightarrow \quad (x-n) = 0, (n^2+1-x-n) = 0$

$\Rightarrow \quad x = n,\ n^2 - n + 1$

Hence, the solutions are

$$x = n,\ n^2 - n + 1$$

24. Given equation is

$$\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\left(\frac{23}{36}\right)$$

$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{x-1}{x+1} + \frac{2x-1}{2x+1}}{1 - \left(\frac{x-1}{x+1} \cdot \frac{2x-1}{2x+1}\right)}\right) = \tan^{-1}\left(\frac{23}{36}\right)$

$\Rightarrow \quad \frac{2x^2 - x - 1 + 2x^2 + x - 1}{2x^2 + 3x + 1 - 2x^2 + 3x - 1} = \frac{23}{26}$

$\Rightarrow \quad \frac{4x^2 - 2}{6x} = \frac{23}{36}$

$\Rightarrow \quad \frac{2x^2 - 1}{x} = \frac{23}{12}$

$\Rightarrow \quad 24x^2 - 23x - 12 = 0$

$\Rightarrow \quad 24x^2 - 32x + 9x - 12 = 0$

$\Rightarrow \quad 8x(3x-4) + 3(3x-4) = 0$

$\Rightarrow \quad (8x+3)(3x-4) = 0$

$\Rightarrow \quad x = \frac{4}{3}, -\frac{3}{8}$

Hence, the solutions are $x = \frac{4}{3}, -\frac{3}{8}$

25. Given equation is

$$\sec^{-1}\left(\frac{x}{a}\right) - \sec^{-1}\left(\frac{x}{b}\right) = \sec^{-1} b - \sec^{-1} a$$

$$\Rightarrow \quad \sec^{-1}\left(\frac{x}{a}\right)+\sec^{-1}(a)=\sec^{-1}\left(\frac{x}{b}\right)+\sec^{-1}(b)$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{a}{x}\right)+\cos^{-1}\left(\frac{1}{a}\right)=\cos^{-1}\left(\frac{b}{x}\right)+\cos^{-1}\left(\frac{1}{b}\right)$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{a}{x}\cdot\frac{1}{a}-\sqrt{1-\left(\frac{a}{x}\right)^2}\sqrt{1-\frac{1}{a^2}}\right)$$

$$=\cos^{-1}\left(\frac{b}{x}\cdot\frac{1}{b}-\sqrt{1-\left(\frac{b}{x}\right)^2}\sqrt{1-\frac{1}{b^2}}\right)$$

$$\Rightarrow \quad \frac{1}{x}-\frac{\sqrt{(x^2-a^2)(a^2-1)}}{ax}=\frac{1}{x}-\frac{\sqrt{(x^2-b^2)(b^2-1)}}{bx}$$

$$\Rightarrow \quad \frac{\sqrt{(x^2-a^2)(a^2-1)}}{a}=\frac{\sqrt{(x^2-b^2)(b^2-1)}}{b}$$

$$\Rightarrow \quad \frac{(x^2-a^2)(a^2-1)}{a^2}=\frac{(x^2-b^2)(b^2-1)}{b^2}$$

$$\Rightarrow \quad (x^2-a^2)(a^2-1)b^2=(x^2-b^2)(b^2-1)a^2$$

$$\Rightarrow \quad x^2((a^2-1)b^2-a^2(b^2-1))$$

$$=a^2b^2(a^2-1)-a^2b^2(b^2-1)$$

$$\Rightarrow \quad x^2[a^2-b^2]=a^2b^2[(a^2-1)-(b^2-1)]$$

$$\Rightarrow \quad x^2[(a^2-b^2)]=a^2b^2[(a^2-b^2)]$$

$$\Rightarrow \quad x^2=a^2b^2$$

$$\Rightarrow \quad x=ab$$

26. We have $\displaystyle\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{8n}{n^4-2n^2+5}\right)$

$$=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{2\{(n+1)^2-(n-1)^2\}}{4+(n-1)^2}\right)$$

$$=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\left\{\left(\frac{n+1}{2}\right)^2-\left(\frac{n-1}{2}\right)^2\right\}}{1+\left(\frac{n+1}{2}\right)^2\cdot\left(\frac{n-1}{2}\right)^2}\right)$$

$$=\sum_{n=1}^{\infty}\tan^{-1}\left\{\left(\frac{n+1}{2}\right)^2\right\}-\tan^{-1}\left\{\left(\frac{n+1}{2}\right)^2\right\}$$

$$=\left[\tan^{-1}(1^2)-0+\tan^{-1}\left(\frac{3}{2}\right)^2-\tan^{-1}(1^2)\right.$$

$$+\tan^{-1}(2^2)-\tan^{-1}\left(\frac{3}{2}\right)^2+\ldots\ldots+$$

$$=\left.\tan^{-1}\left(\frac{n+1}{2}\right)^2-\tan^{-1}\left(\frac{n-1}{2}\right)^2\right],\ n\to\infty$$

$$=\lim_{n\to\infty}\left\{\tan^{-1}\left(n+\frac{1}{2}\right)^2-0\right\}$$

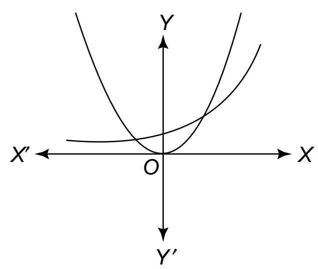
$$=\tan^{-1}(\infty)=\frac{\pi}{2}$$

27. Given equation is

$$\sin^{-1}(e^x)+\cos^{-1}(x^2)=\frac{\pi}{2}$$

It solutions exist only when

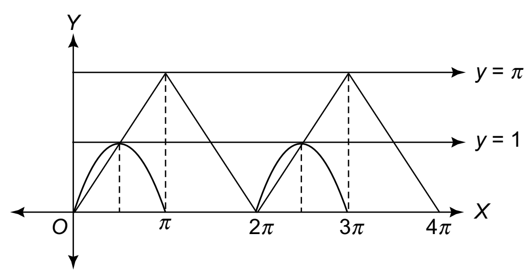
$$e^x=x^2$$



Hence, the number of solution is 2.

28. Given equation is

$$\sqrt{\sin(x)}=\cos^{-1}(\cos x)\ \text{in}\ (0,2\pi)$$



From the graph , it is clear that , the number of real solution is 1 at $x=\dfrac{\pi}{2}$

29. Given equation is

$$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{n}\right)=\frac{\pi}{4}$$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{\frac{1}{3}+\frac{1}{4}}{1-\frac{1}{3}\cdot\frac{1}{4}}\right)+\tan^{-1}\left(\dfrac{1}{5}\right)+\tan^{-1}\left(\dfrac{1}{n}\right)=\dfrac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{7}{11}\right)+\tan^{-1}\left(\dfrac{1}{5}\right)+\tan^{-1}\left(\dfrac{1}{n}\right)=\dfrac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{\frac{7}{11}+\frac{1}{5}}{1-\frac{7}{11}\cdot\frac{1}{5}}\right)+\tan^{-1}\left(\dfrac{1}{n}\right)=\dfrac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{23}{24}\right)+\tan^{-1}\left(\dfrac{1}{n}\right)=\dfrac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{1}{n}\right)=\dfrac{\pi}{4}-\tan^{-1}\left(\dfrac{23}{24}\right)$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{1}{n}\right)=\tan^{-1}(1)-\tan^{-1}\left(\dfrac{23}{24}\right)$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{1}{n}\right)=\tan^{-1}\left(\dfrac{1-\frac{23}{24}}{1+\frac{23}{24}}\right)$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{1}{n}\right)=\tan^{-1}\left(\dfrac{1}{47}\right)$

$\Rightarrow \quad n=47$

30. Given equation is

$x^3+bx^2+cx+1=0$

Let $f(x)=x^3+bx^2+cx+1$

Now, $f(0)=1>0, f(-1)=b-c<0$

So, the function $f(x)$ has a root in $-1<\alpha<0$

Now, $\tan^{-1}(\alpha)+\tan^{-1}\left(\dfrac{1}{\alpha}\right)$

$=\tan^{-1}\alpha-\pi+\cot^{-1}\alpha$

$=-\pi+(\tan^{-1}\alpha+\cot^{-1}\alpha)$

$=-\pi+\dfrac{\pi}{2}=-\dfrac{\pi}{2}$

31. Given equation is

$x^3+bx^2+cx+1=0$

Let $f(x)=x^3+bx^2+cx+1$

Now, $f(0)=1>0, f(-1)=b-c<0$

So, the function $f(x)$ has a root in $-1<\alpha<0$

Now, $2\tan^{-1}(\operatorname{cosec}\alpha)+\tan^{-1}(2\sin\alpha\sec^2\alpha)$

$=2\tan^{-1}\left(\dfrac{1}{\sin\alpha}\right)+\tan^{-1}\left(\dfrac{2\sin\alpha}{\cos^2\alpha}\right)$

$=2\tan^{-1}\left(\dfrac{1}{\sin\alpha}\right)+\tan^{-1}\left(\dfrac{2\sin\alpha}{1-\sin^2\alpha}\right)$

$=2\tan^{-1}\left(\dfrac{1}{\sin\alpha}\right)+2\tan^{-1}(\sin\alpha)$

$=2\left[\tan^{-1}\left(\dfrac{1}{\sin\alpha}\right)+\tan^{-1}(\sin\alpha)\right]$

$=2\left(-\dfrac{\pi}{2}\right)$, as $\sin\alpha<0$ $=\pi$

32. Given $\sin^{-1}x>\cos^{-1}x$

$\Rightarrow \quad 2\sin^{-1}x>\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}x>\dfrac{\pi}{4}$

$\Rightarrow \quad x>\sin\left(\dfrac{\pi}{4}\right)=\dfrac{1}{\sqrt{2}}$

Hence, the solution is $x\in\left(\dfrac{1}{\sqrt{2}},1\right]$

33. Given $\cos^{-1}x>\sin^{-1}x$

$\Rightarrow \quad \sin^{-1}x+\cos^{-1}x>2\sin^{-1}x$

$\Rightarrow \quad 2\sin^{-1}x<\dfrac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}x<\dfrac{\pi}{4}$

$\Rightarrow \quad x<\sin\left(\dfrac{\pi}{4}\right)=\dfrac{1}{\sqrt{2}}$

Hence, the solution set is $x\in\left[-1,\dfrac{1}{\sqrt{2}}\right)$

34. Let $a=\cot^{-1}x$

The given in-equation reduces to

$a^2-5a+6>0$

$\Rightarrow \quad (a-2)(a-3)>0$

$\Rightarrow \quad a<2$ and $a>3$

$\Rightarrow \quad \cot^{-1}x<2$ and $\cot^{-1}x>3$

$\Rightarrow \quad x > \cot(2)$ and $x > \cot(3)$

Hence, the solution set is $(-\infty, \cot(2)) \cup (\cot 3, \infty)$

35. $\tan^2\left(\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right) > 1$

$\Rightarrow \quad \left\{\tan\left(\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)\right\}^2 > 1$

$\Rightarrow \quad \frac{x^2}{(1-x^2)} > 1$

$\Rightarrow \quad \frac{x^2}{(1-x^2)} - 1 > 0$

$\Rightarrow \quad \frac{x^2 - 1 + x^2}{(1-x^2)} > 0$

$\Rightarrow \quad \frac{2x^2 - 1}{(x^2 - 1)} < 0$

$\Rightarrow \quad \frac{(\sqrt{2}\,x + 1)(\sqrt{2}x - 1)}{(x+1)(x-1)} < 0$

$\Rightarrow \quad x \in \left(-1, -\frac{1}{\sqrt{2}}\right) \cup \left(1, \frac{1}{\sqrt{2}}\right)$

36. Given $4(\tan^{-1} x)^2 - 8(\tan^{-1} x) + 3 < 0$

$\Rightarrow \quad 4a^2 - 8a + 3 < 0$, $a = \tan^{-1} x$

$\Rightarrow \quad 4a^2 - 6a - 2a + 3 < 0$

$\Rightarrow \quad 2a(2a-3) - 1(2a-3) < 0$

$\Rightarrow \quad (2a-1)(2a-3) < 0$

$\Rightarrow \quad \frac{1}{2} < a < \frac{3}{2}$

$\Rightarrow \quad \frac{1}{2} < \tan^{-1} x < \frac{3}{2}$

$\Rightarrow \quad \tan\left(\frac{1}{2}\right) < x < \tan\left(\frac{3}{2}\right)$

37. Given $4\cot^{-1} x - (\cot^{-1} x)^2 - 3 \geq 0$

$\Rightarrow \quad 4a - a^2 - 3 \geq 0$, $a = \cot^{-1} x$

$\Rightarrow \quad a^2 - 4a + 3 \leq 0$

$\Rightarrow \quad (a-1)(a-3) \leq 0$

$\Rightarrow \quad 1 \leq a \leq 3$

$\Rightarrow \quad 1 \leq \cot^{-1} x \leq 3$

$\Rightarrow \quad \cot(3) \leq x \leq \cot(1)$

38. Given in equation is

$\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 2$

$\Rightarrow \quad \sin^{-1}\left(\sin\left(\frac{2(x^2+1)+2}{x^2+1}\right)\right) < \pi - 2$

$\Rightarrow \quad \sin^{-1}\left(\sin\left(2 + \frac{2}{x^2+1}\right)\right) < \pi - 2$

$\Rightarrow \quad \sin^{-1}\left(\sin\left(\pi - \frac{2x^2+4}{x^2+1}\right)\right) < \pi - 2$

$\Rightarrow \quad \left(\pi - \frac{2x^2+4}{x^2+1}\right) < \pi - 2$

$\Rightarrow \quad \left(\frac{2x^2+4}{x^2+1}\right) > 2$

$\Rightarrow \quad \left(\frac{x^2+2}{x^2+1}\right) > 1$

which is a true statement.

Hence, $x \in R$

39. We have

$f(x) = \{\sin^{-1}(\sin x)\}^2 - \sin^{-1}(\sin x)$

$= \left\{\sin^{-1}(\sin x) - \frac{1}{2}\right\}^2 - \frac{1}{4}$

$= \left\{\frac{\pi}{2} + \frac{1}{2}\right\}^2 - \frac{1}{4}$

$= \frac{\pi}{4}(\pi + 2)$, since the maximum value

of $\sin^{-1}(\sin x)$ is $-\frac{\pi}{2}$

40. Clearly, both terms are positive.
Applying, $A.M \geq G.M$, we get,

$\Rightarrow \quad \frac{8^{\sin^{-1} x} + 8^{\cos^{-1} x}}{2} \geq \sqrt{8^{\sin^{1-} x} . 8^{\cos^{-1} x}}$

$\Rightarrow \quad \frac{f(x)}{2} \geq \sqrt{8^{\sin^{-1} x + \cos^{-1} x}}$

$\Rightarrow \quad \dfrac{f(x)}{2} \geq \sqrt{8^{\frac{\pi}{2}}}$

$\Rightarrow \quad f(x) \geq 2\sqrt{8^{\frac{\pi}{2}}} = 2.8^{\frac{\pi}{4}} = 2.2^{\frac{3\pi}{4}} = 2^{1+\frac{3\pi}{4}}$

Hence, the minimum value of $2^{1+\frac{3\pi}{4}}$

41. Given in equation is

$$x^2 - kx + \sin^{-1}(\sin 4) > 0$$

$\Rightarrow \quad x^2 - k\,x + \sin^{-1}(\sin(\pi - 4)) > 0$

$\Rightarrow \quad x^2 - k\,x + (\pi - 4) > 0$

$\Rightarrow$ For , all x in R, $D \geq 0$

$\Rightarrow \quad k^2 - 4(\pi - 4) \geq 0$

$\Rightarrow \quad k^2 \geq 4(\pi - 4)$

So, no real values of k satsfies the above in equ.

Hence, the solution is $k = \varphi$

42. Given

$$A = 2\tan^{-1}(2\sqrt{2} - 1)$$

$\Rightarrow A = 2\tan^{-1}(2.8 - 1) = 2\tan^{-1}(1.4)$

$\Rightarrow \quad A > \dfrac{2\pi}{3}$

and $B = 3\sin^{-1}\left(\dfrac{1}{3}\right) + \sin^{-1}\left(\dfrac{3}{5}\right)$

$\Rightarrow B = \sin^{-1}\left(\dfrac{3}{3} - \dfrac{4}{27}\right) + \sin^{-1}\left(\dfrac{3}{5}\right)$

$\Rightarrow B = \sin^{-1}\left(\dfrac{23}{27}\right) + \sin^{-1}\left(\dfrac{3}{5}\right)$

$\Rightarrow B < \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) + \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

$\Rightarrow B < \dfrac{\pi}{3} + \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

Hence, $A > B$

43. We have

$$\sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right) + \cos^{-1}\left(\frac{\sqrt{12}}{4}\right) + \sec^{-1}(\sqrt{2})\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \cos^{-1}(\sqrt{2})\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sin\left(\frac{\pi}{12}\right)\right) + \frac{\pi}{6} + \frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\frac{\pi}{12} + \frac{\pi}{6} + \frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left(\frac{6\pi}{12}\right)\right\}$$

$$= \sin^{-1}(0) = 0$$

44. Given $f(x) = \sin^{-1}(\cos^{-1}x + \tan^{-1}x + \cot^{-1}x)$

$= \sin^{-1}\left(\dfrac{\pi}{2} + \cos^{-1}x\right)$

Thus, $-1 \leq \left(\dfrac{\pi}{2} + \cos^{-1}x\right) \leq 1$

$\Rightarrow -1 - \dfrac{\pi}{2} \leq \cos^{-1}x \leq 1 - \dfrac{\pi}{2}$

But the ranges of $\cos^{-1}x$ is $[0, \pi]$

So it has no solution

Therefore, $D_f = \varphi$

45. Clearly, $0 \leq x \leq 4$

We have

$$\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sin^{-1}\left(\sqrt{1 - \frac{x}{4}}\right) + \tan^{-1}y = \frac{2\pi}{3}$$

$\Rightarrow \quad \sin^{-1}\left(\dfrac{\sqrt{x}}{2}\right) + \cos^{-1}\left(\dfrac{\sqrt{x}}{2}\right) + \tan^{-1}(y) = \dfrac{2\pi}{3}$

$\Rightarrow \quad \dfrac{\pi}{2} + \tan^{-1}(y) = \dfrac{2\pi}{3}$

$\Rightarrow \quad \tan^{-1}(y) = \dfrac{2\pi}{3} - \dfrac{\pi}{2} = \dfrac{\pi}{6}$

$\Rightarrow \quad y = \tan\left(\dfrac{\pi}{6}\right) = \dfrac{1}{\sqrt{3}}$

Hence, the maximum value of $(x^2 + y^2 + 1)$ is $= \left(16 + \dfrac{1}{3} + 1\right) = \dfrac{52}{3}$

46. Given $\tan^{-1}\left(\dfrac{1}{x}\right) + \tan^{-1}\left(\dfrac{1}{y}\right) = \tan^{-1}\left(\dfrac{1}{10}\right)$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{\frac{1}{x} + \frac{1}{y}}{1 - \frac{1}{x}.\frac{1}{y}}\right) = \tan^{-1}\left(\dfrac{1}{10}\right)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{xy-1}\right)=\tan^{-1}\left(\frac{1}{10}\right)$$

$$\Rightarrow \quad \left(\frac{x+y}{xy-1}\right)=\left(\frac{1}{10}\right)$$

$$\Rightarrow \quad 10(x+y)=xy-1$$

$$\Rightarrow \quad y(10-x)=-1-10x$$

$$\Rightarrow \quad y=\left(\frac{1+10x}{10-x}\right)$$

Thus, there are four ordered pairs

$(11,111),(111,11),(9,-91),(-91,9)$

satisfying the above equations.

47. Given $\left[\cot\left(\sum_{k=1}^{10}\cot^{-1}(k^2+k+1)\right)\right]=\frac{a}{b}$

$$\Rightarrow \quad \frac{a}{b}=\left[\cot\left(\sum_{k=1}^{10}\cot^{-1}(k^2+k+1)\right)\right]$$

$$=\left[\cot\left(\sum_{k=1}^{10}\tan^{-1}\left(\frac{1}{1+k+k^2}\right)\right)\right]$$

$$=\left[\cot\left(\sum_{k=1}^{10}\tan^{-1}\left(\frac{(k+1)-k}{1+(k+1)k}\right)\right)\right]$$

$$=\left[\cot\left(\sum_{k=1}^{10}[\tan^{-1}(k+1)-\tan^{-1}(k)]\right)\right]$$

$$=\left[\cot\left(\tan^{-1}(11)-\tan^{-1}(1)\right)\right]$$

$$=\left[\cot\left(\tan^{-1}\left(\frac{11-1}{1+11.}\right)\right)\right]$$

$$=\left[\cot\left(\tan^{-1}\left(\frac{5}{6}\right)\right)\right]$$

$$=\left[\cot\left(\cot^{-1}\left(\frac{6}{5}\right)\right)\right]$$

$$=\frac{6}{5}$$

Hence, the value of $(a+b+10)$ is 21.

48. We have

$$\tan^{-1}\left(\frac{p-q}{1+pq}\right)+\tan^{-1}\left(\frac{q-r}{1+qr}\right)+\tan^{-1}\left(\frac{r-p}{1+rp}\right)$$

$$=(\tan^{-1}p-\tan^{-1}q)+(\tan^{-1}q-\tan^{-1}r)+\pi+(\tan^{-1}r-\tan^{-1}p)$$

$$=\pi$$

49. Given equation is

$$(\sin^{-1}x)^3+(\cos^{-1}x)^3=\pi a^3$$

$$\Rightarrow \quad \frac{\pi}{2}[(\sin^{-1}x)^2+(\cos^{-1}x)^2-2\sin^{-1}x\cos^{-1}x]=\pi a^3$$

$$\Rightarrow \quad \left[\frac{\pi^2}{4}-3\sin^{-1}x\cos^{-1}x\right]=2a\pi^2$$

$$\Rightarrow \quad \left[\pi^2-12b\left(\frac{\pi}{2}-b\right)\right]=8a\pi^2$$

$$\Rightarrow \quad [\pi^2-6b\pi+12b^2]=8a\pi^2$$

$$\Rightarrow \quad \left(b^2-\frac{b\pi}{2}+\frac{\pi^2}{12}\right)=\frac{8a\pi^2}{12}=\frac{2a\pi^2}{3}$$

$$\Rightarrow \quad \left(b-\frac{\pi}{4}\right)^2=\frac{\pi^2}{12}(8a-1)+\frac{\pi^2}{16}$$

$$\Rightarrow \quad \left(b-\frac{\pi}{4}\right)^2=\frac{\pi^2}{48}(32a-1)$$

$$\Rightarrow \quad \left(\sin^{-1}x-\frac{\pi}{4}\right)^2=\frac{\pi^2}{48}(32a-1)$$

As we know that ,

$$-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}$$

$$\Rightarrow \quad -\frac{\pi}{2}-\frac{\pi}{4}\le\sin^{-1}x-\frac{\pi}{4}\le\frac{\pi}{2}-\frac{\pi}{4}$$

$$\Rightarrow \quad -\frac{3\pi}{4}\le\left(\sin^{-1}x-\frac{\pi}{4}\right)\le\frac{\pi}{4}$$

$$\Rightarrow \quad 0\le\left(\sin^{-1}x-\frac{\pi}{4}\right)^2\le\frac{9\pi^2}{16}$$

$$\Rightarrow \quad 0\le\frac{\pi^2}{48}(32a-1)\le\frac{9\pi^2}{16}$$

$$\Rightarrow \quad 0\le(32a-1)\le 27$$

$$\Rightarrow \quad 1\le 32a\le 28$$

$$\Rightarrow \quad \frac{1}{32}\le a\le\frac{7}{8}$$

50. Let $g(x)=\frac{x^2}{x^2+1}=1-\frac{1}{1+x^2}$

Clearly, $R_g=[0,1)$

Now, $R_f=\left(f(1),f(0)\right]=\left(\cot^{-1}(1),\cot^{-1}(0)\right]$

$$= \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$$

Hence, the value of $\left(\frac{b}{a}+2\right)$

$$= \left(\frac{\frac{\pi}{2}}{\frac{\pi}{4}}+2\right) = 2+2 = 4$$

51. Given $\tan^{-1} y = 4\tan^{-1} x$

$$\Rightarrow \tan^{-1} y = \tan^{-1}\left(\frac{4x-4x^3}{1-6x^2+x^4}\right)$$

$$\Rightarrow y = \left(\frac{4x-4x^3}{1-6x^2+x^4}\right)$$

$$\Rightarrow \frac{1}{y} = \left(\frac{1-6x^2+x^4}{4x-4x^3}\right)$$

$$\Rightarrow \frac{1}{y} = \cot(4\theta), \text{where}, x = \tan\theta$$

$\Rightarrow$Clearly, $\frac{1}{y} = 0$ is zero only when $\theta = \frac{\pi}{8}$

Hnece, $x^4 - 6x^2 + 1 = 0$

52. Let $x = \sqrt{\frac{a(a+b+c)}{bc}}, y = \sqrt{\frac{b(a+b+c)}{ac}}$

and $z = \sqrt{\frac{c(a+b+c)}{ab}}$

Now, $x+y+z-xyz$

$$= \sqrt{(a+b+c)}\left(\sqrt{\frac{a}{bc}}+\sqrt{\frac{b}{ca}}+\sqrt{\frac{c}{ab}}\right) - \frac{(a+b+c)^{3/2}}{\sqrt{abc}}$$

$$= \sqrt{(a+b+c)}\left(\frac{a+b+c}{\sqrt{abc}}\right) - \frac{(a+b+c)^{3/2}}{\sqrt{abc}}$$

$$= \frac{(a+b+c)^{3/2}}{\sqrt{abc}} - \frac{(a+b+c)^{3/2}}{\sqrt{abc}}$$

$$= 0$$

Now, $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$

$$= \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right) = \tan^{-1}(0) = \pi$$

Hence, the result.

53. We have

$$\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{6\tan\theta}{9+\tan^2\theta}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{2\left(\frac{\tan\theta}{3}\right)}{1+\frac{1}{3}\tan^2\theta}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\times 2\tan^{-1}\left(\frac{\tan\theta}{3}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}\left(\frac{\tan\theta}{3}\right)$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{2\tan^2\theta - \frac{\tan\theta}{3}}{1+2\tan^2\theta\cdot\frac{\tan\theta}{3}}\right)$$

$$\Rightarrow \quad \tan(\theta) = \frac{6\tan^2\theta - \tan\theta}{3+2\tan^3\theta}$$

$$\Rightarrow \quad 3\tan(\theta) + 2\tan^4\theta - 6\tan^2\theta + 2\tan\theta = 0$$

$$\Rightarrow \quad 2\tan^4\theta - 6\tan^2\theta + 5\tan\theta = 0$$

$$\Rightarrow \quad \tan\theta(2\tan^3\theta - 6\tan\theta + 5) = 0$$

$$\Rightarrow \quad \tan\theta = 0, (2\tan^3\theta - 6\tan\theta + 5) = 0$$

$$\Rightarrow \quad \theta = n\pi, n \in I$$

54. We have

$$\tan^{-1}\left(\frac{x\cos\theta}{1-x\sin\theta}\right) - \cot^{-1}\left(\frac{\cos\theta}{x-\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{x\cos\theta}{1-x\sin\theta}\right) - \tan^{-1}\left(\frac{x-\sin\theta}{\cos\theta}\right)$$

$$= \tan^{-1}\left(\frac{\frac{x\cos\theta}{1-x\sin\theta} - \frac{x-\sin\theta}{\cos\theta}}{1+\frac{x\cos\theta}{1-x\sin\theta}\times\frac{x-\sin\theta}{\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{x\cos^2\theta - x + x^2\sin\theta + \sin\theta - x\sin^2\theta}{\cos\theta(1-x\sin\theta) + x\cos\theta(x-\sin\theta)}\right)$$

$$= \tan^{-1}\left(\frac{(x^2+1)\sin\theta - 2x\sin^2\theta}{(x^2+1)\cos\theta - x\sin 2\theta}\right)$$

$$= \tan^{-1}\left(\frac{(x^2 - 2x\sin\theta + 1)\sin\theta}{(x^2 - 2x\sin\theta + 1)\cos\theta}\right)$$

$$= \tan^{-1}(\tan\theta) = \theta$$

55. Given equation is

$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \sin^{-1}\left(\frac{2x}{x^2+1}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \quad \pi - 2\tan^{-1}x + 2\tan^{-1}x - 2\tan^{-1}x = \frac{2\pi}{3}$$

$$\Rightarrow \quad \pi - 2\tan^{-1}x = \frac{2\pi}{3}$$

$$\Rightarrow \quad 2\tan^{-1}x = \frac{\pi}{3}$$

$$\Rightarrow \quad \tan^{-1}x = \frac{\pi}{6}$$

$$\Rightarrow \quad x = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

Hence, the solution is $x = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

56. We have

$$\tan^{-1}\left(\frac{yz}{xr}\right) + \tan^{-1}\left(\frac{zx}{yr}\right) + \tan^{-1}\left(\frac{xy}{zr}\right)$$

$$= \tan^{-1}\left(\frac{\frac{yz}{xr} + \frac{zx}{yr} + \frac{xy}{zr} - \frac{xyz}{r^3}}{1 - \left(\frac{z^2}{r^2} + \frac{x^2}{r^2} + \frac{y^2}{r^2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{yz}{xr} + \frac{zx}{yr} + \frac{xy}{zr} - \frac{xyz}{r^3}}{1 - \left(\frac{x^2+y^2+z^2}{r^2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{yz}{xr} + \frac{zx}{yr} + \frac{xy}{zr} - \frac{xyz}{r^3}}{1-1}\right)$$

$$= \tan^{-1}(\infty) = \frac{\pi}{2}$$

57. Given $\sum_{r=1}^{10} \tan^{-1}\left(\frac{3}{9r^2 + 3r - 1}\right)$

$$= \sum_{r=1}^{10} \tan^{-1}\left(\frac{3}{1 + (3r+2)(3r-1)}\right)$$

$$= \sum_{r=1}^{10} \tan^{-1}\left(\frac{(3r+2)-(3r-1)}{1 + (3r+2)(3r-1)}\right)$$

$$= \sum_{r=1}^{10} [\tan^{-1}(3r+2) - \tan^{-1}(3r-1)]$$

$$= \tan^{-1}(32) - \tan^{-1}(2)$$

$$= \tan^{-1}\left(\frac{32-2}{1+32.2}\right)$$

$$= \tan^{-1}\left(\frac{30}{65}\right) = \tan^{-1}\left(\frac{6}{13}\right)$$

$$= \cot^{-1}\left(\frac{13}{6}\right)$$

Hence, the value of $(2m + n + 4)$

$= 26 + 6 + 4$

$= 36.$

58. We have Σ

$$= \sum_{b=1}^{10}\sum_{a=1}^{10} \tan^{-1}\left(\frac{a}{b}\right)$$

$$= \sum_{b=1}^{10}\left[\tan^{-1}\left(\frac{1}{b}\right) + \tan^{-1}\left(\frac{2}{b}\right) + \tan^{-1}\left(\frac{3}{b}\right) + \ldots + \tan^{-1}\left(\frac{10}{b}\right)\right]$$

$$= \tan^{-1}\left(\frac{1}{1}\right) + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \ldots + \tan^{-1}\left(\frac{1}{10}\right)$$

$$+ \tan^{-1}\left(\frac{2}{1}\right) + \tan^{-1}\left(\frac{2}{2}\right) + \tan^{-1}\left(\frac{2}{3}\right) + \ldots + \tan^{-1}\left(\frac{2}{10}\right)$$

$$+ \tan^{-1}\left(\frac{3}{1}\right) + \tan^{-1}\left(\frac{3}{2}\right) + \tan^{-1}\left(\frac{3}{3}\right) + \ldots + \tan^{-1}\left(\frac{3}{10}\right)$$

....

....

....

$$+ \tan^{-1}\left(\frac{10}{1}\right) + \tan^{-1}\left(\frac{10}{2}\right) + \tan^{-1}\left(\frac{10}{3}\right) + \ldots + \tan^{-1}\left(\frac{10}{10}\right)$$

$$= 10 \times \frac{\pi}{4} + 45 \times \frac{\pi}{2}$$

$$= 25\pi$$

Hence, the value of $(m + 4)$ is 29.

59. We have

$$f(x)=\frac{1}{\pi}(\sin^{-1}x+\cos^{-1}x+\tan^{-1}x)+\frac{(x+1)}{x^2+2x+10}$$

It will provide us the max value at $x = 1$

$$f(1)=\frac{1}{\pi}\left(\frac{\pi}{2}+\tan^{-1}(1)\right)+\frac{2}{13}$$

$$=\frac{1}{\pi}\times\frac{3\pi}{4}+\frac{2}{13}$$

$$==\frac{3}{4}+\frac{2}{13}=\frac{39+8}{52}=\frac{47}{52}$$

Hence, the value of $(104m-90)$ is 4.

60. We have

$$\sin(2x)+\cos(2x)+\cos x+1=0$$

$$\sin 2x+(1+\cos 2x)+\cos x=0$$

Each term of the above equation is positive in $\left(0,\frac{\pi}{2}\right)$

So it has no solution

Thus, $m = 0$

Also, $n=\sin\left[\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)+\cos^{-1}\left(\cos\left(\frac{7\pi}{3}\right)\right)\right]$

$$=\sin\left(\frac{\pi}{6}+\frac{\pi}{3}\right)=\sin\left(\frac{\pi}{2}\right)=1$$

Hence, the value of

$$(m^2+n^2+m+n+4)$$

$= 0 +1 + 0 + 1 + 4$

$= 6.$

61. We have $f(n)=\sum_{k=-n}^{n}\left(\cot^{-1}\left(\frac{1}{k}\right)-\tan^{-1}(k)\right)$

$$=\sum_{k=-n}^{-1}\left(\cot^{-1}\left(\frac{1}{k}\right)-\tan^{-1}(k)\right)+\sum_{k=1}^{n}\left(\cot^{-1}\left(\frac{1}{k}\right)-\tan^{-1}(k)\right)$$

$$=\sum_{k=-n}^{-1}(\tan^{-1}(k)+\pi-\tan^{-1}(k))+\sum_{k=1}^{n}(\tan^{-1}(k)-\tan^{-1}(k))$$

$$=\sum_{k=-n}^{-1}(\pi)+0$$

$$=n\pi$$

Now, $\sum_{n=2}^{10}(f(n)+f(n-1))$

$$=\sum_{n=2}^{10}(n\pi+(n-1)\pi)$$

$$=\sum_{n=2}^{10}((2n-1)\pi)$$

$$=(3+5+7+9+\ldots+19)\pi$$

$$=(1+3+5+7+9+\ldots+19)\pi-\pi$$

$$=(10^2)\pi-\pi$$

$$=99\pi$$

Hence, the value of $(a+1)$ is 100.

INTEGER TYPE QUESTIONS

1. Given $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right)<\pi-3$

$$\Rightarrow \frac{2x^2+4}{x^2+1}>3$$

$$\Rightarrow \frac{2x^2+4}{x^2+1}-3>0$$

$$\Rightarrow \frac{2x^2+4-3x^2-3}{x^2+1}>0$$

$$\Rightarrow \frac{1-x^2}{x^2+1}>0$$

$$\Rightarrow \frac{x^2-1}{x^2+1}<0$$

$$\Rightarrow x\in(-1,1)$$

Hence, the value of $(b-a+5)$ = 1 + 1 + 5 = 7.

2. Given $a\sin^{-1}x-b\cos^{-1}x=c$

$$\Rightarrow a\sin^{-1}x-b\left(\frac{\pi}{2}-\sin^{-1}x\right)=c$$

$$\Rightarrow (a+b)\sin^{-1}x=c+\frac{b\pi}{2}$$

$$\Rightarrow \sin^{-1}x=\frac{2c+b\pi}{2(a+b)}$$

Now, $\cos^{-1}x=\frac{\pi}{2}-\frac{2c+b\pi}{2(a+b)}$

$$\Rightarrow \cos^{-1}x=\frac{(a+b)\pi-2c-b\pi}{2(a+b)}$$

$\Rightarrow \quad \cos^{-1} x = \dfrac{a\pi - 2c}{2(a+b)}$

Now, $a\sin^{-1} x + b\cos^{-1} x$

$= a\left(\dfrac{2c + b\pi}{2(a+b)}\right) + b\left(\dfrac{a\pi - 2c}{2(a+b)}\right)$

$= \left(\dfrac{2ac + ab\pi}{2(a+b)}\right) + \left(\dfrac{ab\pi - 2bc}{2(a+b)}\right)$

$= \left(\dfrac{2ac + ab\pi + ab\pi - 2bc}{2(a+b)}\right)$

$= \left(\dfrac{ab\pi + c(a-b)}{(a+b)}\right)$

Clearly, $m = 1$

Hence, the value of $(m^2 + m + 2)$ is 4

3. Since sum of the roots is –ve and product of the roots is positive
So , both roots are negative
Thus m is a negative root

Now, $\tan^{-1}(m) + \tan^{-1}\left(\dfrac{1}{m}\right)$

$= \tan^{-1}(m) - \pi + \cot^{-1}(m)$

$= -\pi + (\tan^{-1}(m) + \cot^{-1}(m))$

$= -\pi + \dfrac{\pi}{2} = -\dfrac{\pi}{2}$

Clearly, $k = -1$
Hence, the value of $(k+4)$ is 3.

4. Given equation is

$\sin^{-1}(x^2 - 2x + 1) + \cos^{-1}(x^2 - x) = \dfrac{\pi}{2}$

It is true only when

$(x^2 - 2x + 1) = x^2 - x$

$\Rightarrow \quad x = 1$

Thus, the number of solutions is 1.

5. Given $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$

$\Rightarrow \quad \cos^{-1}(3x) + \cos^{-1}(2x) = \pi - \cos^{-1}(x)$

$\Rightarrow \quad \cos^{-1}(3x \cdot 2x - \sqrt{1 - 9x^2}\sqrt{1 - 4x^2}) = \cos^{-1}(-x)$

$\Rightarrow \quad (3x \cdot 2x - \sqrt{10 - 9x^2}\sqrt{1 - 4x^2}) = (-x)$

$\Rightarrow \quad (6x^2 + x)^2 = (-\sqrt{1 - 9x^2}\sqrt{1 - 4x^2})^2$

$\Rightarrow \quad 36x^4 + 12x^3 + x^2 = 1 - 13x^2 + 36x^4$

$\Rightarrow \quad 12x^3 + 14x^2 - 1 = 0$

Thus, $a = 12$, $b = 14$, $c = 0$, $d = -1$

Hence, the value of $(b + c) - (a + d)$

= 14 – 11 = 3.

6. Given equation is

$x^3 - x^2 - 3x + 4 = 0$

$\Rightarrow \quad \alpha + \beta + \gamma = 1, \alpha\beta + \beta\gamma + \gamma\alpha = -3, \alpha\beta\gamma = -4$

It is given that ,

$\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = \theta$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{\alpha + \beta + \gamma - \alpha\beta\gamma}{1 - (\alpha\beta + \beta\gamma + \gamma\alpha)}\right) = \theta$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{1+4}{1+3}\right) = \theta$

$\Rightarrow \quad \tan\theta = \dfrac{5}{4}$

Hence, the value of $(p + q)$ is 9

7. Given equation is

$\cos^{-1} x + \cos^{-1}(2x) + \pi = 0$

$\Rightarrow \quad \cos^{-1}(2x) + \cos^{-1}(x) = -\pi$

$\Rightarrow \quad \cos^{-1}(2x \cdot x - \sqrt{1 - x^2}\sqrt{1 - 4x^2}) = -\pi$

$\Rightarrow \quad (2x^2 - \sqrt{1 - x^2}\sqrt{1 - 4x^2}) = \cos(-\pi) = -1$

$\Rightarrow \quad (2x^2 + 1)^2 = (1 - x^2)(1 - 4x^2)$

$\Rightarrow \quad 4x^4 + 4x^2 + 1 = 1 - 5x^2 + 4x^4$

$\Rightarrow \quad 9x^2 = 0$

$\Rightarrow \quad x = 0$

But $x = 0$ does not satisfy the equation
So it has no solution
Therefore $M = 0$

Again, $\sin^{-1}\left(\dfrac{5}{x}\right) + \sin^{-1}\left(\dfrac{12}{x}\right) = \dfrac{\pi}{2}$,

$\Rightarrow \quad \sin^{-1}\left(\dfrac{5}{x}\right) + \cos^{-1}\left(\sqrt{1 - \dfrac{144}{x^2}}\right) = \dfrac{\pi}{2}$

It is true only when,

$\left(\dfrac{5}{x}\right) = \sqrt{1 - \dfrac{144}{x^2}}$

$\Rightarrow \quad \frac{25}{x^2} = 1 - \frac{144}{x^2}$

$\Rightarrow \quad \frac{169}{x^2} = 1$

$\Rightarrow \quad x^2 = 169$

$\Rightarrow \quad x = \pm 13$

Claerly, $x = 13$ only satisfies the equation
Thus, $N = 1$
Hence, the value of $M + N + 4 = 5$.

8. We have

$$4\cos\left[\cos^{-1}\left(\frac{1}{4}(\sqrt{6}-\sqrt{2})\right) - \cos^{-1}\left(\frac{1}{4}(\sqrt{6}+\sqrt{2})\right)\right]$$

$$= 4\cos\left[\cos^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) - \cos^{-1}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\right]$$

$$= 4\cos[\cos^{-1}(\cos(75°)) - \cos^{-1}(\cos(15°))]$$

$$= 4\cos(75° - 15°)$$

$$= 4\cos(60°)$$

$$= 2.$$

9. We have

$$5\cot\left(\sum_{k=1}^{5} \cot^{-1}(k^2+k+1)\right)$$

$$= 5\cot\left(\sum_{k=1}^{5} \tan^{-1}\left(\frac{1}{1+k+k^2}\right)\right)$$

$$= 5\cot\left(\sum_{k=1}^{5} \tan^{-1}\left(\frac{(k+1)-k}{1+(k+1)k}\right)\right)$$

$$= 5\cot\left(\sum_{k=1}^{5} [\tan^{-1}(k+1) - \tan^{-1} k]\right)$$

$$= 5\cot(\tan^{-1}(6) - \tan^{-1}(1))$$

$$= 5\cot\left(\tan^{-1}\left(\frac{6-1}{1+6.1}\right)\right)$$

$$= 5\cot\left(\tan^{-1}\left(\frac{5}{7}\right)\right)$$

$$= 5\cot\left(\cot^{-1}\left(\frac{7}{5}\right)\right)$$

$$= 7.$$

10. $a = \sin^{-1}(\log_2 x)$ and $b = \cos^{-1}(\log_2 x)$
The given equations reduces to

$$\begin{cases} 3a + b = \frac{\pi}{2} \\ a + 2b = \frac{11\pi}{6} \end{cases}$$

On solving, we get,

$$a = -\frac{\pi}{6} \text{ and } b = \pi$$

$\Rightarrow \quad \sin^{-1}(\log_2 x) = -\frac{\pi}{6}$ and $\cos^{-1}(\log_2 y) = \pi$

$\Rightarrow \quad (\log_2 x) = -\frac{1}{2}$ and $(\log_2 y) = -1$

$\Rightarrow \quad x = 2^{-\frac{1}{2}}$ and $y = 2^{-1}$

$\Rightarrow \quad \frac{1}{x} = \sqrt{2}$ and $\frac{1}{y} = 2$

$\Rightarrow \quad \frac{1}{x^2} + \frac{1}{y^2} + 2 = 2 + 4 + 2 = 8$

11. Here, both roots are negative

Now, $\cot(\cot^{-1}\alpha + \cot^{-1}\beta)$

$$= \cot\left(\pi + \tan^{-1}\left(\frac{1}{\alpha}\right) + \pi + \tan^{-1}\left(\frac{1}{\beta}\right)\right)$$

$$= \cot\left(2\pi + \tan^{-1}\left(\frac{1}{\alpha}\right) + \tan^{-1}\left(\frac{1}{\beta}\right)\right)$$

$$= \cot\left(\tan^{-1}\left(\frac{1}{\alpha}\right) + \tan^{-1}\left(\frac{1}{\beta}\right)\right)$$

$$= \cot\left(\tan^{-1}\left(\frac{\alpha+\beta}{\alpha\beta-1}\right)\right)$$

$$= \cot\left(\cot^{-1}\left(\frac{\alpha\beta-1}{\alpha+\beta}\right)\right)$$

$$= \cot\left(\cot^{-1}\left(\frac{\alpha\beta-1}{\alpha+\beta}\right)\right)$$

$$= \left(\frac{44+1}{5}\right) = 9$$

12. Given equation is

$$\tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{1}{7}\right)$$

$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{xy-1}\right) = \tan^{-1}\left(\frac{1}{7}\right)$

$\Rightarrow \quad \left(\frac{x+y}{xy-1}\right)=\left(\frac{1}{7}\right)$

$\Rightarrow \quad 7x+7y=xy-1$

$\Rightarrow \quad y=\left(\frac{7x+1}{x-7}\right)$

Hence, the possible ordered pairs are $(8,57),(9,32),(12,17),(17,12),(32,9),(57,8)$

Thus, the number of ordered pairs is 6.

QUESTION ASKED IN PART IIT-JEE EXAMS

1. Now,

$$\tan^{-1}\left(\sqrt{\frac{a(a+b+c}{bc}}\right)+\tan^{-1}\left(\sqrt{\frac{b(a+b+c)}{ca}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{\frac{a(a+b+c}{bc}}+\sqrt{\frac{b(a+b+c)}{ca}}}{1-\sqrt{\frac{a(a+b+c)}{bc}}\sqrt{\frac{b(a+b+c)}{ca}}}\right)$$

$$=\tan^{-1}\left(\frac{\left(\frac{(a+b+c)}{\sqrt{c}}\right)+\left(\frac{\sqrt{a}}{\sqrt{b}}+\frac{\sqrt{b}}{\sqrt{a}}\right)}{1-\sqrt{\frac{(a+b+c)}{c}}\sqrt{\frac{(a+b+c)}{c}}}\right)$$

$$=\tan^{-1}\left(\frac{\left(\frac{(a+b+c)}{\sqrt{c}}\right)+\left(\frac{a+b}{\sqrt{ab}}\right)}{\left(\frac{c-a-b-c}{\sqrt{c}}\right)}\right)$$

$$=\tan^{-1}\left(\frac{\left(\frac{(a+b+c)}{\sqrt{c}}\right)+\left(\frac{a+b}{\sqrt{ab}}\right)}{\left(-\frac{a+b}{\sqrt{c}}\right)}\right)$$

$$=\tan^{-1}\left(-\sqrt{\frac{c(a+b+c)}{ab}}\right)$$

$$=-\tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ab}}\right)$$

Therefore, θ

$$=-\tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ab}}\right)+\tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ab}}\right)$$

$= 0$

Thus, $\tan\theta = 0$

2. Given $\left\{2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right\}$

$$=\tan\left(\tan^{-1}\left(\frac{\frac{2}{5}}{1-\left(\frac{1}{5}\right)^2}\right)-\frac{\pi}{4}\right)$$

$$=\tan\left(\tan^{-1}\left(\frac{5}{12}\right)-\tan^{-1}(1)\right)$$

$$=\tan\left(\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}.1}\right)\right)$$

$$=\tan\left(\tan^{-1}\left(-\frac{7}{17}\right)\right)$$

$$=\left(-\frac{7}{17}\right)$$

3. Now, $\cos(2\cos^{-1}x+\sin^{-1}x)$

$=\cos(\cos^{-1}x+\sin^{-1}x+\cos^{-1}x)$

$$=\cos\left(\frac{\pi}{2}+\cos^{-1}x\right)$$

$=-\sin(\cos^{-1}x)$

$$=-\sin\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$=-\sqrt{1-x^2}$$

When $x = 1/5$, then the value of the given expression is $-\left(\sqrt{1-\frac{1}{25}}\right)=-\frac{2\sqrt{4}}{5}$.

4. Given $\tan\left[\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$

$$=\tan\left(\tan^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right)$$

$$=\tan\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)\right)$$

$$=\left(\frac{9+8}{12-6}\right)$$

$$=-\frac{17}{6}$$

5. No questions asked in between 1984 to 1985

6. Given $\sin\ 1\left(\sin\left(\frac{2\pi}{3}\right)\right)$

$$= \sin\ 1\left(\sin\left(\pi - \frac{\pi}{3}\right)\right)$$

$$= \sin\ 1\left(\sin\left(\frac{\pi}{3}\right)\right)$$

$$= \left(\frac{\pi}{3}\right)$$

7. No questions asked in between 1987 to 1988

8. We have $A = 2\tan^{-1}(2\sqrt{2}-1) \ => 2\tan^{-1}(\sqrt{3}) = \frac{2\pi}{3}$

and $B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

$$= \sin^{-1}\left(3\left(\frac{1}{3}\right) - 4\left(\frac{1}{3}\right)^3\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$= \sin^{-1}\left(\frac{23}{27}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$< \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{2\pi}{3}$$

Thus, $A > B$

10. Ans. (c)

Given

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{1}{\sqrt{x^2+x+1}}\right) + \sin^{-1}\left(\sqrt{x^2+x+1}\right) = \frac{\pi}{2}$$

Thus, $\left(\frac{1}{\sqrt{x^2+x+1}}\right) = \left(\sqrt{x^2+x+1}\right)$

$$\Rightarrow \quad \left(\sqrt{x^2+x+1}\right)^2 = 1$$

$\Rightarrow \quad x^2 + x + 1 = 1$

$\Rightarrow \quad x^2 + x = 0$

$\Rightarrow \quad x(x + 1) = 0$

$\Rightarrow \quad x = 0, -1$

11. No questions asked in 2000.

12. We have

$$\left(x - \frac{x^2}{2} + \frac{x^3}{4} - ..\right) = \left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - ..\right)$$

$$\Rightarrow \quad x\left(1 - \frac{x}{2} + \frac{x^2}{4} - ..\right) = x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{4} - ..\right)$$

$$\Rightarrow \quad x\left(\frac{1}{1-\left(-\frac{x}{2}\right)}\right) = x^2\left(\frac{1}{1-\left(-\frac{x^2}{2}\right)}\right)$$

$$\Rightarrow \quad x\left(\frac{1}{1+\left(\frac{x}{2}\right)}\right) = x^2\left(\frac{1}{1+\left(\frac{x^2}{2}\right)}\right)$$

$$\Rightarrow \quad x\left(1 + \frac{x^2}{2}\right) = x^2\left(1 + \frac{x}{2}\right)$$

$\Rightarrow \quad x(2 + x^2) = x^2(2 + x)$

$\Rightarrow \quad (2x + x^3) = (2x^2 + x^3)$

$\Rightarrow \quad x^2 = x$

$\Rightarrow \quad x(x - 1) = 0$

$\Rightarrow \quad x = 0, 1$

Hence, the solution set is $\{0, 1\}$

13. We have $\cos(\tan^{-1}(\sin(\cot^{-1}x)))$

$$= \cos\left(\tan^{-1}\left(\sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right)\right)$$

$$= \cos\left(\tan^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)$$

$$= \cos\left(\cos^{-1}\left(\sqrt{\frac{x^2+1}{x^2+2}}\right)\right)$$

$$= \left(\sqrt{\frac{x^2+1}{x^2+2}}\right)$$

14. Ans. (d)

Given $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$

It is defined for, $\sin^{-1}(2x) + \frac{\pi}{6} \geq 0$

$\sin^{-1}(2x) \geq -\frac{\pi}{6}$

$(2x) \geq \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

$x \geq -\frac{1}{4}$

Also, $-1 \leq 2x \leq 1$

$-\frac{1}{2} \leq x \leq \frac{1}{2}$

Thus, the domain of the given function

$= \left[-\frac{1}{4}, \frac{1}{2}\right]$

15. Given $\sin(\cot^{-1}(x + 1)) = \cos(\tan^{-1}x)$

$\Rightarrow \quad \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+2x+2}}\right)\right)$

$= \cos\left(\cos^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)$

$\Rightarrow \quad \frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{x^2+1}}$

$\Rightarrow \quad \sqrt{x^2+2x+2} = \sqrt{x^2+1}$

$\Rightarrow \quad x^2 + 2x + 2 = x^2 + 1$

$\Rightarrow \quad 2x + 2 = 1$

$\Rightarrow \quad x = -\frac{1}{2}$

16. No questions asked in between 2005 to 2006.

17. Given $\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$

(A) when $a = 1$, $b = 0$, then

$\sin^{-1}(x) + \cos^{-1}(y) + \cos^{-1}(0) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}(x) + \cos^{-1}(y) + \frac{\pi}{2} = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}(x) + \cos^{-1}(y) = 0$

$\Rightarrow \quad \sin^{-1}(x) = -\cos^{-1}(y) = -\sin^{-1}\left(\sqrt{1-y^2}\right)$

$\Rightarrow \quad x = -\sqrt{1-y^2}$

$\Rightarrow \quad x^2 = 1 - y^2$

$\Rightarrow \quad x^2 + y^2 = 1$

Ans. (P)

(B) When $a = 1$, $b = 1$, then

$\sin^{-1}x + \cos^{-1}y + \cos^{-1}(xy) = \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} - \cos^{-1}x + \cos^{-1}y + \cos^{-1}(xy) = \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1}x = \cos^{-1}y + \cos^{-1}(xy)$

$\Rightarrow \quad \cos^{-1}x - \cos^{-1}y = \cos^{-1}(xy)$

$\Rightarrow \quad \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \cos^{-1}(xy)$

$\Rightarrow \quad \left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = (xy)$

$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2} = 0$

$\Rightarrow \quad (1 - x^2)(1 - y^2) = 0$

Ans. (Q)

(C) When $a = 1$, $b = 2$, then

$\Rightarrow \quad \sin^{-1}x + \cos^{-1}y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} - \cos^{-1}x + \cos^{-1}y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \cos^{-1}(2xy)$

$\Rightarrow \quad \left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = (2xy)$

$\Rightarrow \quad \left(\sqrt{1-x^2}\sqrt{1-y^2}\right)^2 = (-xy)^2$

$\Rightarrow \quad (1 - x^2)(1 - y^2) = x^2y^2$

$\Rightarrow \quad 1 - x^2 - y^2 + x^2y^2 = x^2y^2$

$\Rightarrow \quad x^2 + y^2 = 1$

Ans. (P)

(D) When $a = 2$, $b = 2$, then

$\Rightarrow \quad \sin^{-1}(2x) + \cos^{-1}y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} - \cos^{-1}(2x) + \cos^{-1}y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1}(2x) = \cos^{-1}y + \cos^{-1}(2xy)$

$\Rightarrow \quad \cos^{-1}(2x) - \cos^{-1}y = \cos^{-1}(2xy)$

$\Rightarrow \quad \cos^{1}\left(2x.y - \sqrt{1-4x^2}\sqrt{1-y^2}\right) = \cos^{-1}(2xy)$

$\Rightarrow \quad \left(2x.y - \sqrt{1-4x^2}\sqrt{1-y^2}\right) = (2xy)$

$\Rightarrow \quad \left(\sqrt{1-4x^2}\sqrt{1-y^2}\right) = 0$

$\Rightarrow \quad (1-4x^2)(1-y^2) = 0$

Ans. (S)

18. We have

$\sqrt{1+x^2} \times [\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)^2 - 1\}]^{1/2}$

$= \sqrt{1+x^2}\left[\left\{x\cos\left(\cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)\right.\right.$

$\left.\left. + \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right\}^2 - 1\right]^{1/2}$

$= \sqrt{1+x^2}\left[\left(\frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}\right)^2 - 1\right]^{1/2}$

$= \sqrt{1+x^2}\left[\left(\frac{x^2+1}{\sqrt{x^2+1}}\right)^2 - 1\right]^{1/2}$

$= \sqrt{1+x^2}\left[\left(\sqrt{x^2+1}\right)^2 - 1\right]^{1/2}$

$= \sqrt{1+x^2}\left[\left(x^2+1\right) - 1\right]^{1/2}$

$= x\sqrt{1+x^2}$

Ans. (c)

19. No questions asked in between 2009 to 2010.

20. Given $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$

$= \sin\left(\sin^{-1}\left(\frac{\sin\theta}{\sqrt{\sin^2\theta + \cos 2\theta}}\right)\right)$

$= \left(\frac{\sin\theta}{\sqrt{\sin^2\theta + \cos^2\theta - \sin^2\theta}}\right)$

$= \left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)$

$= \left(\frac{\sin\theta}{\cos\theta}\right) = \tan\theta$

Now, $\frac{d}{d(\tan\theta)}(\tan\theta) = 1$

21. No questions asked in 2010

22. Now, $\sum_{k-1}^{n}(2k)$

$= 2(1 + 2 + 3 + \ldots + n)$

$= 2\left(\frac{n(n+1)}{2}\right)$

$= (n^2 + n)$

Therefore, $\sum_{n-1}^{23}\cot^{-1}(1+n+n^2)$

$= \sum_{n-1}^{23}\tan^{-1}\left(\frac{1}{1+n+n^2}\right)$

$= \sum_{n-1}^{23}\tan^{-1}\left(\frac{1}{1+(n+1)n}\right)$

$= \sum_{n-1}^{23}\tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)$

$= \sum_{n-1}^{23}(\tan^{-1}(n+1) - \tan^{-1}(n))$

$= (\tan^{-1}(2) - \tan^{-1}(1)) + (\tan^{-1}(3) - \tan^{-1}(2))$
$+ \ldots + (\tan^{-1}(24) - \tan^{-1}(23))$

$= \tan^{-1}(24) - \tan^{-1}(1)$

$= \tan^{-1}\left(\frac{24-1}{1+24}\right) = \tan^{-1}\left(\frac{23}{25}\right)$

Thus, the given expression reduces to

$= \cot\left(\tan^{-1}\left(\frac{23}{25}\right)\right)$

$= \cot\left(\cot^{-1}\left(\frac{25}{23}\right)\right)$

$= \frac{25}{23}$

23. We have $\left(\frac{\cos(\tan^{-1}y) + y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y) + \tan(\sin^{-1}y)}\right)$

$$= \left(\frac{\cos\left(\cos^{-1}\left(\frac{1}{\sqrt{y^2+1}}\right)\right) + y\sin\left(\sin^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)\right)}{\cot\left(\cot^{-1}\left(\frac{\sqrt{1-y^2}}{y}\right)\right) + \tan\left(\tan^{-1}\left(\frac{y}{\sqrt{1-y^2}}\right)\right)}\right)$$

$$= \left(\frac{\left(\frac{1}{\sqrt{y^2+1}}\right) + y\left(\frac{y}{\sqrt{1+y^2}}\right)}{\left(\frac{\sqrt{1-y^2}}{y}\right) + \left(\frac{y}{\sqrt{1-y^2}}\right)} \right)$$

$$= \left(\frac{\left(\frac{1+y^2}{\sqrt{y^2+1}}\right)}{\left(\frac{1-y^2+y^2}{y\sqrt{1-y^2}}\right)} \right)$$

$$= y\left(\sqrt{1-y^4}\right)$$

Thus, the given expression reduces to

$$= \left(\frac{1}{y^2}\left(y\sqrt{1-y^4}\right)^2 + y^4 \right)^{1/2}$$

$$= \left(\frac{y^2(1-y^4)}{y^2} + y^4 \right)^{1/2}$$

$$= ((1 - y^4) + y^4)^{1/2}$$

$$= 1$$

24. Given $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1}(x\sqrt{6})\right)$

$$\Rightarrow \quad \cot\left(\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)$$

$$= \sin\left(\sin^{-1}\left(\frac{x\sqrt{6}}{\sqrt{6x^2+1}}\right)\right)$$

$$\Rightarrow \quad \left(\frac{x}{\sqrt{1-x^2}}\right) = \left(\frac{x\sqrt{6}}{\sqrt{6x^2+1}}\right)$$

$$\Rightarrow \quad \left(\frac{1}{\sqrt{1-x^2}}\right) = \left(\frac{\sqrt{6}}{\sqrt{6x^2+1}}\right)$$

$$\Rightarrow \quad 6(1 - x^2) = (6x^2 + 1)$$

$$\Rightarrow \quad 6 - 6x^2 = 6x^2 + 1$$

$$\Rightarrow \quad 12x^2 = 5$$

$$\Rightarrow \quad x^2 = \frac{5}{12}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{12}{5}}$$

QUESTIONS WITH SOLUTIONS OF PAST IIT-JEE EXAMS FROM 1981 TO 2015

1. Let a, b, c positive real numbers such that

$$\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}.$$

Then $\tan\theta$ is equal to [IIT-JEE-1981]

Soln. Now,

Let $x = \sqrt{\frac{a(a+b+c)}{bc}}$, $y = \sqrt{\frac{b(a+b+c)}{ca}}$

and $z = \sqrt{\frac{c(a+b+c)}{ab}}$

The given equation reduces to

$$\theta = \tan^{-1} x + \tan^{-1} y + \tan^{-1} z$$

$$\theta = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

$$\tan\theta = \left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

Now, $x + y + z$

$$= \sqrt{\frac{a(a+b+c)}{bc}} + \sqrt{\frac{b(a+b+c)}{ca}} + \sqrt{\frac{c(a+b+c)}{ab}}$$

$$= \sqrt{\frac{a^2(a+b+c)}{abc}} + \sqrt{\frac{b^2(a+b+c)}{abc}} + \sqrt{\frac{c^2(a+b+c)}{abc}}$$

$$= \sqrt{\frac{a+b+c}{abc}}(a+b+c)$$

Also, $x\,y\,z$

$$= \sqrt{\frac{a(a+b+c)}{bc}}.\sqrt{\frac{b(a+b+c)}{ca}}.\sqrt{\frac{c(a+b+c)}{ab}}$$

$$= \frac{(a+b+c)\sqrt{(a+b+c)}}{\sqrt{(abc)^2}} \times \sqrt{abc}$$

$$= \frac{(a+b+c)\sqrt{(a+b+c)}}{\sqrt{(abc)}}$$

Thus, $x + y + z - xyz = 0$

Therefore, $\tan\theta = 0$.

2. The numerical value of

$\tan^{-1}\left\{2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right\}$. [IIT-JEE -1981]

Soln. Given $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right\}$

$$= \tan\left(\tan^{-1}\left(\frac{\frac{2}{5}}{1-\left(\frac{1}{5}\right)^2}\right)-\frac{\pi}{4}\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{5}{12}\right)-\tan^{-1}(1)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}.1}\right)\right)$$

$$= \tan\left(\tan^{-1}\left(-\frac{7}{17}\right)\right)$$

$$= \left(-\frac{7}{17}\right)$$

3. Find the value of $\cos\left(2\cos^{-1}x+\sin^{-1}x\right)$ at $x = 1/5$, where $0 \le \cos^{-1}x \le \pi$ and $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$. [IIT-JEE-1981]

Soln. Now, $\cos\left(2\cos^{-1}x+\sin^{-1}x\right)$

$$= \cos\left(\cos^{-1}x+\sin^{-1}x+\cos^{-1}x\right)$$

$$= \cos\left(\frac{\pi}{2}+\cos^{-1}x\right)$$

$$= -\sin\left(\cos^{-1}x\right)$$

$$= -\sin\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$= -\sqrt{1-x^2}$$

When $x = 1/5$, then the value of the given expression is $-\left(\sqrt{1-\frac{1}{25}}\right) = -\frac{2\sqrt{4}}{5}$.

4. The value of $\tan\left[\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$ is

(a) 6/17 (b) 17/6 (c) –17/6 (d) –6/17

[IIT-JEE-1983]

Soln. Given $\tan\left[\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$

$$= \tan\left(\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}.\frac{2}{3}}\right)\right)$$

$$= \left(\frac{9+8}{12-6}\right)$$

$$= -\frac{17}{6}$$

5. The principal value of $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is

(a) $-\frac{2\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{5\pi}{3}$

[IIT-JEE - 1986]

Soln. Given $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

$$= \sin^{-1}\left(\sin\left(\pi-\frac{\pi}{3}\right)\right)$$

$$= \sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$$

$$= \left(\frac{\pi}{3}\right)$$

6. The greater of the two angles

$A = 2\tan^{-1}\left(2\sqrt{2}-1\right)$ and

$B = 3\sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{3}{5}\right)$ is....... [IIT-JEE-1989]

Soln. We have $A = 2\tan^{-1}\left(2\sqrt{2}-1\right)$

$$> 2\tan^{-1}\left(\sqrt{3}\right) = \frac{2\pi}{3}$$

and $B = 3\sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{3}{5}\right)$

$$= \sin^{-1}\left(3\left(\frac{1}{3}\right)-4\left(\frac{1}{3}\right)^3\right)+\sin^{-1}\left(\frac{3}{5}\right)$$

$$= \sin^{-1}\left(\frac{23}{27}\right)+\sin^{-1}\left(\frac{3}{5}\right)$$

$$< \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{2\pi}{3}$$

Thus, $A > B$.

7. The number of real solutions of

$$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2} \text{ is}$$

(a) 0 (b) 1
(c) 2 (d) ∞ [IIT-JEE-1999]

Soln. Given

$$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$$

$$\Rightarrow \cos^{-1}\left(\frac{1}{\sqrt{x^2+x+1}}\right)+\sin^{-1}\left(\sqrt{x^2+x+1}\right)=\frac{\pi}{2}$$

Thus, $\left(\frac{1}{\sqrt{x^2+x+1}}\right)=\left(\sqrt{x^2+x+1}\right)$

$$\Rightarrow \left(\sqrt{x^2+x+1}\right)^2=1$$

$$\Rightarrow x^2+x+1=1$$

$$\Rightarrow x^2+x=0$$

$$\Rightarrow x(x+1)=0$$

$$\Rightarrow x=0,-1$$

8. If $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-.......\right)$

$$+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-....\right)=\frac{\pi}{2},$$

for $0<|x|<\sqrt{2}$, then x is

(a) 1/2 (b) 1
(c) –1/2 (d) –1

[IIT-JEE-2001]

Soln. We have

$$\left(x-\frac{x^2}{2}+\frac{x^3}{4}-..\right)=\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-....\right)$$

$$\Rightarrow x\left(1-\frac{x}{2}+\frac{x^2}{4}-..\right)=x^2\left(1-\frac{x^2}{2}+\frac{x^4}{4}-....\right)$$

$$\Rightarrow x\left(\frac{1}{1-\left(-\frac{x}{2}\right)}\right)=x^2\left(\frac{1}{1-\left(-\frac{x^2}{2}\right)}\right)$$

$$\Rightarrow x\left(\frac{1}{1+\left(\frac{x}{2}\right)}\right)=x^2\left(\frac{1}{1+\left(\frac{x^2}{2}\right)}\right)$$

$$\Rightarrow x\left(1+\frac{x^2}{2}\right)=x^2\left(1+\frac{x}{2}\right)$$

$$\Rightarrow x\left(2+x^2\right)=x^2(2+x)$$

$$\Rightarrow \left(2x+x^3\right)=\left(2x^2+x^3\right)$$

$$\Rightarrow x^2=x$$

$$\Rightarrow x(x-1)=0$$

$$\Rightarrow x=0,1$$

Hence, the solution set is $\{0, 1\}$.

9. Prove that $\cos\left(\tan^{-1}\left(\sin\left(\cot^{-1}x\right)\right)\right)=\sqrt{\frac{x^2+1}{x^2+2}}$.

[IIT-JEE-2002]

Soln. We have $\cos\left(\tan^{-1}\left(\sin\left(\cot^{-1}x\right)\right)\right)$

$$= \cos\left(\tan^{-1}\left(\sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right)\right)$$

$$= \cos\left(\tan^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)$$

$$= \cos\left(\cos^{-1}\left(\sqrt{\frac{x^2+1}{x^2+2}}\right)\right)$$

$$= \left(\sqrt{\frac{x^2+1}{x^2+2}}\right)$$

10. The domain of $f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$ is

(a) $\left(-\frac{1}{2},\frac{1}{2}\right]$ (b) $\left[-\frac{1}{4},\frac{3}{4}\right)$

(c) $\left[-\frac{1}{4},\frac{1}{4}\right]$ (d) $\left[-\frac{1}{4},\frac{1}{2}\right]$

[IIT-JEE-2003]

Soln. Given $f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$

It is defined for, $\sin^{-1}(2x)+\frac{\pi}{6}\geq 0$

$\sin^{-1}(2x)\geq -\frac{\pi}{6}$

$(2x)\geq \sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}$

$x\geq -\frac{1}{4}$

Also, $-1\leq 2x\leq 1$

$-\frac{1}{2}\leq x\leq \frac{1}{2}$

Thus, the domain of the given function

$=\left[-\frac{1}{4},\frac{1}{2}\right]$

11. If $\sin\left(\cot^{-1}(x+1)\right)=\cos\left(\tan^{-1}x\right)$, then the value of x is
(a) –1/2 (b) 1/2 (c) 0 (d) 9/4
[IIT-JEE-2004]

Soln. Given $\sin\left(\cot^{-1}(x+1)\right)=\cos\left(\tan^{-1}x\right)$

$$\Rightarrow \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+2x+2}}\right)\right)=\cos\left(\cos^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)$$

$$\Rightarrow \frac{1}{\sqrt{x^2+2x+2}}=\frac{1}{\sqrt{x^2+1}}$$

$$\Rightarrow \sqrt{x^2+2x+2}=\sqrt{x^2+1}$$

$$\Rightarrow x^2+2x+2=x^2+1$$

$$\Rightarrow 2x+2=1$$

$$\Rightarrow x=-\frac{1}{2}$$

12. Match the following columns:
Let (x, y) be such that

$$\sin^{-1}(ax)+\cos^{-1}(y)+\cos^{-1}(bxy)=\frac{\pi}{2}$$

Column - I	**Column - II**
(A) If $a = 1$ and $b = 0$, then (x, y)	(P) lies on the circle $x^2+y^2=1$
(B) If $a = 1$ and $b = 1$, then (x, y)	(Q) lies on $(x^2-1)(y^2-1)=0$
(C) If $a = 1$ and $b = 2$, then (x, y)	(R) lies on the line $y = x$
(D) If $a = 2$ and $b = 2$, then (x, y)	(S) lies on $(4x^2-1)(y^2-1)=0$

[IIT-JEE-2007]

Soln. Given $\sin^{-1}(ax)+\cos^{-1}(y)+\cos^{-1}(bxy)=\frac{\pi}{2}$.

(A) When $a = 1$, $b = 0$, then

$$\sin^{-1}(x)+\cos^{-1}(y)+\cos^{-1}(0)=\frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}(x)+\cos^{-1}(y)+\frac{\pi}{2}=\frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}(x)+\cos^{-1}(y)=0$$

$$\Rightarrow \sin^{-1}(x)=-\cos^{-1}(y)=-\sin^{-1}\left(\sqrt{1-y^2}\right)$$

$$\Rightarrow x=-\sqrt{1-y^2}$$

$$\Rightarrow x^2=1-y^2$$

$$\Rightarrow x^2+y^2=1$$

Ans. (P).

(B) When $a = 1$, $b = 1$, then

$$\sin^{-1}x+\cos^{-1}y+\cos^{-1}(xy)=\frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2}-\cos^{-1}x+\cos^{-1}y+\cos^{-1}(xy)=\frac{\pi}{2}$$

$$\Rightarrow \cos^{-1}x=\cos^{-1}y+\cos^{-1}(xy)$$

$$\Rightarrow \cos^{-1}x-\cos^{-1}y=\cos^{-1}(xy)$$

$$\Rightarrow \cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)=\cos^{-1}(xy)$$

$$\Rightarrow \left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)=(xy)$$

$$\Rightarrow \sqrt{1-x^2}\sqrt{1-y^2}=0$$

$$\Rightarrow (1-x^2)(1-y^2)=0$$

Ans. (Q).

(C) When $a = 1$, $b = 2$, then

$$\Rightarrow \sin^{-1}x+\cos^{-1}y+\cos^{-1}(2xy)=\frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2}-\cos^{-1}x+\cos^{-1}y+\cos^{-1}(2xy)=\frac{\pi}{2}$$

$$\Rightarrow \cos^{-1}x=\cos^{-1}y+\cos^{-1}(2xy)$$

$\Rightarrow \quad \cos^{-1}x - \cos^{-1}y = \cos^{-1}(2xy)$

$\Rightarrow \quad \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \cos^{-1}(2xy)$

$\Rightarrow \quad \left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = (2xy)$

$\Rightarrow \quad \left(\sqrt{1-x^2}\sqrt{1-y^2}\right)^2 = (-xy)^2$

$\Rightarrow \quad (1-x^2)(1-y^2) = x^2y^2$

$\Rightarrow \quad 1 - x^2 - y^2 + x^2y^2 = x^2y^2$

$\Rightarrow \quad x^2 + y^2 = 1$

Ans. (P).

(D) When $a = 2$, $b = 2$, then

$\Rightarrow \quad \sin^{-1}(2x) + \cos^{-1}y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} - \cos^{-1}(2x) + \cos^{-1}y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1}(2x) = \cos^{-1}y + \cos^{-1}(2xy)$

$\Rightarrow \quad \cos^{-1}(2x) - \cos^{-1}y = \cos^{-1}(2xy)$

$\Rightarrow \quad \cos^{-1}\left(2x.y - \sqrt{1-4x^2}\sqrt{1-y^2}\right) = \cos^{-1}(2xy)$

$\Rightarrow \quad \left(2x.y - \sqrt{1-4x^2}\sqrt{1-y^2}\right) = (2xy)$

$\Rightarrow \quad \left(\sqrt{1-4x^2}\sqrt{1-y^2}\right) = 0$

$\Rightarrow \quad (1-4x^2)(1-y^2) = 0$

Ans. (S).

13. If $0 < x < 1$, then

$$\sqrt{1+x^2} \times \left[\left\{x\cos\left(\cot^{-1}x\right) + \sin\left(\cot^{-1}x\right)\right\}^2 - 1\right]^{1/2}$$

equals

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) x

(c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

[IIT-JEE-2008]

Soln. We have

$$\sqrt{1+x^2} \times \left[\left\{x\cos\left(\cot^{-1}x\right) + \sin\left(\cot^{-1}x\right)\right\}^2 - 1\right]^{1/2}$$

$$= \sqrt{1+x^2}\left[\left\{x\cos\left(\cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)\right.\right.$$

$$\left.\left.+\sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right\}^2 - 1\right]^{1/2}$$

$$= \sqrt{1+x^2}\left[\left(\frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}\right)^2 - 1\right]^{1/2}$$

$$= \sqrt{1+x^2}\left[\left(\frac{x^2+1}{\sqrt{x^2+1}}\right)^2 - 1\right]^{1/2}$$

$$= \sqrt{1+x^2}\left[\left(\sqrt{x^2+1}\right)^2 - 1\right]^{1/2}$$

$$= \sqrt{1+x^2}\left[(x^2+1) - 1\right]^{1/2}$$

$$= x\sqrt{1+x^2}$$

Ans. (c).

14. No questions asked between 2009 and 2010.

15. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$

where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of

$\frac{d}{d(\tan\theta)}(f(\theta))$ is....... [IIT-JEE-2011]

Soln. Given $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$

$$= \sin\left(\sin^{-1}\left(\frac{\sin\theta}{\sqrt{\sin^2\theta + \cos 2\theta}}\right)\right)$$

$$= \left(\frac{\sin\theta}{\sqrt{\sin^2\theta + \cos^2\theta - \sin^2\theta}}\right)$$

$$= \left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)$$

$$= \left(\frac{\sin\theta}{\cos\theta}\right) = \tan\theta$$

Now, $\frac{d}{d(\tan\theta)}(\tan\theta) = 1$

16. No questions asked in 2012.

17. The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1 + \sum_{k=1}^{n}2k\right)\right)$ is

(a) 23/25 (b) 25/23

(c) 23/24 (d) 24/23

[IIT-JEE - 2013]

Soln. Now, $\sum_{k=1}^{n}(2k)$

$$= 2(1 + 2 + 3 + \ldots.. + n)$$

$= 2\left(\frac{n(n+1)}{2}\right)$

$= (n^2 + n)$

Therefore, $\sum_{n=1}^{23} \cot^{-1}(1+n+n^2)$

$= \sum_{n=1}^{23} \tan^{-1}\left(\frac{1}{1+n+n^2}\right)$

$= \sum_{n=1}^{23} \tan^{-1}\left(\frac{1}{1+(n+1)n}\right)$

$= \sum_{n=1}^{23} \tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)$

$= \sum_{n=1}^{23}\left(\tan^{-1}(n+1) - \tan^{-1}(n)\right)$

$= \left(\tan^{-1}(2) - \tan^{-1}(1)\right) + \left(\tan^{-1}(3) - \tan^{-1}(2)\right) + ... + \left(\tan^{-1}(24) - \tan^{-1}(23)\right)$

$= \tan^{-1}(24) - \tan^{-1}(1)$

$= \tan^{-1}\left(\frac{24-1}{1+24}\right) = \tan^{-1}\left(\frac{23}{25}\right)$

Thus, the given expression reduces to

$= \cot\left(\tan^{-1}\left(\frac{23}{25}\right)\right)$

$= \cot\left(\cot^{-1}\left(\frac{25}{23}\right)\right)$

$= \frac{25}{23}$

18. The value of

$$\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1} y) + y\sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)}\right)^2 + y^4\right)^{1/2}$$

is......... [IIT-JEE - 2013]

Soln. We have $\left(\frac{\cos(\tan^{-1} y) + y\sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)}\right)$

$$= \left(\frac{\cos\left(\cos^{-1}\left(\frac{1}{\sqrt{y^2+1}}\right)\right) + y\sin\left(\sin^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)\right)}{\cot\left(\cot^{-1}\left(\frac{\sqrt{1-y^2}}{y}\right)\right) + \tan\left(\tan^{-1}\left(\frac{y}{\sqrt{1-y^2}}\right)\right)}\right)$$

$$= \left(\frac{\left(\frac{1}{\sqrt{y^2+1}}\right) + y\left(\frac{y}{\sqrt{1+y^2}}\right)}{\left(\frac{\sqrt{1-y^2}}{y}\right) + \left(\frac{y}{\sqrt{1-y^2}}\right)}\right)$$

$$= \left(\frac{\left(\frac{1+y^2}{\sqrt{y^2+1}}\right)}{\left(\frac{1-y^2+y^2}{y\sqrt{1-y^2}}\right)}\right)$$

$= y\left(\sqrt{1-y^4}\right)$

Thus, the given expression reduces to

$= \left(\frac{1}{y^2}\left(y\sqrt{1-y^4}\right)^2 + y^4\right)^{1/2}$

$= \left(\frac{y^2(1-y^4)}{y^2} + y^4\right)^{1/2}$

$= \left((1-y^4) + y^4\right)^{1/2}$

$= 1$

19. If $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right)$, $x \neq 0$ then the value of x is....... [IIT-JEE - 2013]

Soln. Given $\cot\left(\sin^{-1}\sqrt{1-x^2}\right) = \sin\left(\tan^{-1}\left(x\sqrt{6}\right)\right)$

$\Rightarrow \cot\left(\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right) = \sin\left(\sin^{-1}\left(\frac{x\sqrt{6}}{\sqrt{6x^2+1}}\right)\right)$

$\Rightarrow \left(\frac{x}{\sqrt{1-x^2}}\right) = \left(\frac{x\sqrt{6}}{\sqrt{6x^2+1}}\right)$

$\Rightarrow \left(\frac{1}{\sqrt{1-x^2}}\right) = \left(\frac{\sqrt{6}}{\sqrt{6x^2+1}}\right)$

$\Rightarrow 6(1-x^2) = (6x^2+1)$

$\Rightarrow 6 - 6x^2 = 6x^2 + 1$

$\Rightarrow 12x^2 = 5$

$\Rightarrow x^2 = \frac{5}{12}$

$\Rightarrow x = \pm\sqrt{\frac{12}{5}}$

20. The number of positive solutions satisfying the equation

$$\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right) \text{ is....}$$

[IIT-JEE-2014]

Soln. We have

$$\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{1}{2x+1}+\frac{1}{4x+1}}{1-\left(\frac{1}{2x+1}\times\frac{1}{4x+1}\right)}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{4x+1+2x+1}{(2x+1)(4x+1)-1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \left(\frac{6x+2}{\left(8x^2+6x+1-1\right)}\right)=\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \left(\frac{3x+1}{\left(8x^2+6x+1-1\right)}\right)=\left(\frac{1}{x^2}\right)$$

$$\Rightarrow \quad 3x^3+x^2=8x^2+6x$$

$$\Rightarrow \quad 3x^3-7x^2-6x=0$$

$$\Rightarrow \quad x\left(3x^2-7x-6\right)=0$$

$$\Rightarrow \quad x\left(3x^2-9x+2x-6\right)=0$$

$$\Rightarrow \quad x(x-3)(3x+2)=0$$

$$\Rightarrow \quad x=0,\,3,-\frac{2}{3}$$

Thus, the number of positive solution is one.

21. No questions asked in 2015.

CHAPTER 7

Properties of Triangles

7.1 INTRODUCTION

In any Δ, the three sides and the three angles are generally called the elements of the triangle.

A triangle which does not contain a right angle is called an oblique triangle.

In any ΔABC, the measures of the angles $\angle BAC$, $\angle CBA$ and $\angle ACB$ are denoted by the letters.

A, B and C respectively, and the sides BC, CA and AB opposite to the angles A, B and C are respectively denoted by a, b and c. These six elements of a triangle are not independent and are connected by the relations.

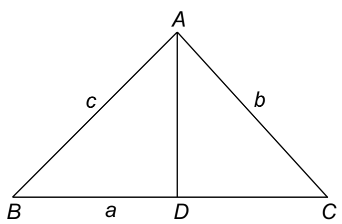
(i) $A + B + C = \pi$

(ii) $a + b > c$; $b + c > a$; $c + a > b$

7.1.1 Sine Rule Statement

The sides of a triangle are proportion to the sines of the angles opposite to them, i.e. in a ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. The above rule may also be expressed as

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



Proof: Let ABC be a triangle such that

$AB = c$, $BC = a$, $CA = b$

Draw CD is perpendicular on AB.

From ΔACD, $\sin A = \frac{CD}{AC} = \frac{CD}{b}$

$\Rightarrow \quad CD = b \sin A$ ....(i)

From ΔBCD, $\sin B = \frac{CD}{BC} = \frac{CD}{a}$

$\Rightarrow \quad CD = a \sin B$ ....(ii)

From (i) and (ii), we get,

$b \sin A = a \sin B$

$\Rightarrow \quad \frac{\sin A}{a} = \frac{\sin B}{b}$ ....(iii)

Similarly, we can prove that,

$\frac{\sin A}{a} = \frac{\sin C}{c}$ ....(iv)

From (iii) and (iv), we get,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\Rightarrow \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

7.1.2 Some Solved Examples

Ex-1. If in a ΔABC, $\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A-B)}{\sin(A+B)}$, then the triangle is right angled or isosceles.

Soln. We have $\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A-B)}{\sin(A+B)}$

$$\Rightarrow \quad \frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A-B)\sin(A+B)}{\sin^2(A+B)}$$

$$\Rightarrow \quad \frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin^2 A - \sin^2 B}{\sin^2(\pi - C)}$$

$$\Rightarrow \quad \frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin^2 A - \sin^2 B}{\sin^2 C}$$

$$\Rightarrow \quad \frac{a^2-b^2}{a^2+b^2}=\frac{k^2\left(a^2-b^2\right)}{k^2c^2}$$

$$\Rightarrow \quad \frac{a^2-b^2}{a^2+b^2}=\frac{\left(a^2-b^2\right)}{c^2}$$

$$\Rightarrow \quad \left(a^2-b^2\right)\left(\frac{1}{a^2+b^2}-\frac{1}{c^2}\right)=0$$

$$\Rightarrow \quad a^2=b^2, \frac{1}{a^2+b^2}=\frac{1}{c^2}$$

$$\Rightarrow \quad a=b, a^2+b^2=c^2$$

Thus, the triangle is isosceles or right angled.

Ex-2. In a ΔABC such that $\angle A = 45°$ and $\angle B = 75°$ then find $a+c\sqrt{2}$.

Soln. We have $\angle C = 180° - (A+B)$

$= 180° - (45° + 75°)$

$= 180° - 120°$

$= 60°$

From the sine rule, we can write

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{a}{\sin(45°)}=\frac{b}{\sin(75°)}=\frac{c}{\sin(60°)}$$

$$\Rightarrow \quad \frac{a}{\frac{1}{\sqrt{2}}}=\frac{b}{\frac{\sqrt{3}+1}{2\sqrt{2}}}=\frac{c}{\frac{\sqrt{3}}{2}}=k(say)$$

Now, $a+c\sqrt{2}$

$$=\frac{k}{\sqrt{2}}+\left(\frac{k\sqrt{3}}{2}\right)\sqrt{2}$$

$$=\frac{k}{\sqrt{2}}+\left(\frac{k\sqrt{3}}{\sqrt{2}}\right)$$

$$=\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)k$$

$$=2\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)k$$

$= 2b$

Ex-3. In a triangle ΔABC, if a^2, b^2, c^2 are in A.P. then prove that $\cot A, \cot B, \cot C$ are in A.P.

Soln. Given a^2, b^2, c^2 are in A.P.

$$\Rightarrow \quad b^2-a^2=c^2-b^2$$

$$\Rightarrow \quad \sin^2 B-\sin^2 A=\sin^2 C-\sin^2 B$$

$$\Rightarrow \quad \sin(B+A)\sin(B-A)$$

$$=\sin(C+B)\sin(C-B)$$

$$\Rightarrow \quad \sin C(\sin B\cos A-\cos B\sin A)$$

$$=\sin A(\sin C\cos B-\cos C\sin B)$$

Dividing both the sides by $\sin A \sin B \sin C$, we get,

$$\Rightarrow \quad \cot A-\cot B=\cot B-\cot C$$

$$\Rightarrow \quad \cot A, \cot B, \cot C \text{ are in A.P.}$$

Ex-4. If $\cot \frac{A}{2}=\frac{b+c}{a}$, then prove that ΔABC is right angled.

Soln. We have

$$\cot\left(\frac{A}{2}\right)=\frac{b+c}{a}$$

$$\Rightarrow \quad \cot\left(\frac{A}{2}\right)=\frac{\sin B+\sin C}{\sin A}$$

$$\Rightarrow \quad \cot\left(\frac{A}{2}\right)=\frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{\sin A}$$

$$\Rightarrow \quad \cot\left(\frac{A}{2}\right)=\frac{2\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$\Rightarrow \quad \cot\left(\frac{A}{2}\right)=\frac{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$\Rightarrow \quad \cot\left(\frac{A}{2}\right)=\frac{\cos\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$\Rightarrow \quad \frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)}=\frac{\cos\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$\Rightarrow \quad \cos\left(\frac{A}{2}\right)=\cos\left(\frac{B-C}{2}\right)$$

$\Rightarrow \quad \frac{A}{2} = \frac{B-C}{2}$

$\Rightarrow \quad A + C = B$

$\Rightarrow \quad 2B = A + B + C = 180°$

$\Rightarrow \quad B = 90°$.

Thus, the triangle is right angled.

Ex-5. In any triangle ABC, prove that

$$\prod\left(\frac{\sin^2 A + \sin A + 1}{\sin A}\right) > 27.$$

Soln. We have $\prod\left(\frac{\sin^2 A + \sin A + 1}{\sin A}\right)$

$$= \left(\sin A + \frac{1}{\sin A} + 1\right)\left(\sin B + \frac{1}{\sin B} + 1\right)$$

$$\left(\sin C + \frac{1}{\sin C} + 1\right)$$

$$> (2+1)(2+1)(2+1) = 27$$

$(\text{applying } A.M \geq G.M)$

Ex-6: In a triangle ABC, prove that,

$$a\sin\left(\frac{A}{2} + B\right) = (b+c)\sin\left(\frac{A}{2}\right).$$

Soln. Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$

We have, $\frac{b+c}{a}$

$$= \frac{k(\sin B + \sin C)}{k \sin A}$$

$$= \frac{(\sin B + \sin C)}{\sin A}$$

$$= \frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$= \frac{\sin\left(\frac{A}{2} + B\right)}{\sin\left(\frac{A}{2}\right)}$$

Ex-7. In a triangle ABC, prove that,

$$\frac{a\sin(B-C)}{b^2 - c^2} = \frac{b\sin(C-A)}{c^2 - a^2} = \frac{c\sin(A-B)}{a^2 - b^2}.$$

Soln. Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$

We have, $\frac{a\sin(B-C)}{b^2 - c^2}$

$$= \frac{k\sin A.\sin(C-A)}{k^2\left(\sin^2 C - \sin^2 A\right)}$$

$$= \frac{\sin(B+C).\sin(B-C)}{k\left(\sin^2 B - \sin^2 C\right)}$$

$$= \frac{\left(\sin^2 B - \sin^2 C\right)}{k\left(\sin^2 B - \sin^2 C\right)} = \frac{1}{k}$$

$\frac{b\sin(C-A)}{c^2 - a^2}$

$$= \frac{k\sin(C+A).\sin(C-A)}{k^2\left(\sin^2 C - \sin^2 A\right)}$$

$$= \frac{\sin^2 C - \sin^2 A}{k\left(\sin^2 C - \sin^2 A\right)} = \frac{1}{k}$$

Also, $\frac{c\sin(A-B)}{a^2 - b^2}$

$$= \frac{k\sin C\sin(A-B)}{k^2\left(\sin^2 A - \sin^2 B\right)}$$

$$= \frac{\sin(A+B)\sin(A-B)}{k\left(\sin^2 A - \sin^2 B\right)}$$

$$= \frac{\sin^2 A - \sin^2 B}{k\left(\sin^2 A - \sin^2 B\right)} = \frac{1}{k}$$

Hence, the result.

Ex-8. In a ΔABC, if $\cos A + 2\cos B + \cos C = 2$ then prove that the sides of a triangle are in A.P.

Soln. Given $\cos A + 2\cos B + \cos C = 2$

$\Rightarrow \quad \cos A + \cos C = 2(1-\cos B)$

$\Rightarrow \quad 2\cos\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right) = 4\sin^2\left(\frac{B}{2}\right)$

$\Rightarrow \quad \cos\left(\frac{\pi}{2}-\frac{B}{2}\right)\cos\left(\frac{A-C}{2}\right) = 2\sin^2\left(\frac{B}{2}\right)$

$\Rightarrow \quad \sin\left(\frac{B}{2}\right)\cos\left(\frac{A-C}{2}\right) = 2\sin^2\left(\frac{B}{2}\right)$

$\Rightarrow \quad \cos\left(\frac{A-C}{2}\right) = 2\sin\left(\frac{B}{2}\right)$

Multiplying both sides by $2\cos\left(\frac{B}{2}\right)$, we get,

$\Rightarrow \; 2\cos\left(\frac{B}{2}\right)\cos\left(\frac{A-C}{2}\right) = 2\left(2\cos\left(\frac{B}{2}\right)\sin\left(\frac{B}{2}\right)\right)$

$\Rightarrow \quad 2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right) = 2\sin B$

$\Rightarrow \quad \sin A + \sin C = 2\sin B$

$\Rightarrow \quad a + c = 2b$

$\Rightarrow \quad a, b, c$ are in A.P.

Ex-9. In a ΔABC, if $\cos A\cos B + \sin A\sin B\sin C = 1$. then prove that $a:b:c = 1:1:\sqrt{2}$

Soln. We have $\cos A \cos B + \sin A \sin B \sin C = 1$

$\Rightarrow \quad \frac{1-\cos A\cos B}{\sin A\sin B} = \sin C$

$\Rightarrow \quad \frac{1-\cos A\cos B}{\sin A\sin B} = \sin C \le 1$

$\Rightarrow \quad 1-\cos A\cos B \le \sin A\sin B$

$\Rightarrow \quad 1 \le \sin A\sin B + \cos A\cos B$

$\Rightarrow \quad \cos(A-B) \ge 1$

$\Rightarrow \quad \cos(A-B) = 1$

$\Rightarrow \quad \cos(A-B) = \cos(0)$

$\Rightarrow \quad A-B = 0$

$\Rightarrow \quad A = B$

Therefore, $\sin C = \frac{1-\cos A\cos B}{\sin A\sin B}$

$= \frac{1-\cos A\cos A}{\sin A\sin A} = \frac{1-\cos^2 A}{\sin^2 A} = \frac{\sin^2 A}{\sin^2 A} = 1$

$\Rightarrow \quad C = 90°$

Hence, $\quad A = 45° = B, C = 90°$

Now, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow \quad \frac{a}{\sin 45°} = \frac{b}{\sin 45°} = \frac{c}{\sin 90°}$

$\Rightarrow \quad \frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{1}{\sqrt{2}}} = \frac{c}{1}$

$\Rightarrow \quad \frac{a}{1} = \frac{b}{1} = \frac{c}{\sqrt{2}}$

$\Rightarrow \quad a:b:c = 1:1:\sqrt{2}$

Hence, the result.

EXERCISE 1

Q. In a ΔABC,

1. Prove that $\frac{\sin(B-C)}{\sin(B+C)} = \frac{b^2-c^2}{a^2}$.
2. Prove that $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$.
3. Prove that $\frac{a^2\sin(B-C)}{\sin B+\sin C} + \frac{b^2\sin(C-A)}{\sin C+\sin A} + \frac{c^2\sin(A-B)}{\sin A+\sin B} = 0$.
4. Prove that $\frac{b^2-c^2}{\cos B+\cos C} + \frac{c^2-a^2}{\cos C+\cos A} + \frac{a^2-b^2}{\cos A+\cos B} = 0$.
5. Prove that $\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B} = \frac{a^2+b^2}{a^2+c^2}$.
6. Prove that a^2, b^2, c^2 are in A.P., if $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$.

7.2 COSINE RULE

Statement

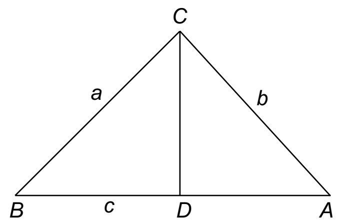
In any triangle ABC

(i) $\cos A = \frac{b^2+c^2-a^2}{2bc}$

(ii) $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

(iii) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Proof:



(i) From ΔBCD, $\sin A = \dfrac{CD}{AC} = \dfrac{CD}{b}$

$\Rightarrow \quad CD = b \sin A$

From ΔACD, $\cos A = \dfrac{AD}{AC} = \dfrac{AD}{b}$

$\Rightarrow \quad AD = b\cos A$

$\Rightarrow \quad AB - BD = b\cos A$

$\Rightarrow \quad c - BD = b\cos A$

$\Rightarrow \quad BD = c - b\cos A$

From ΔBCD,

$BC^2 = BD^2 + CD^2$

$\Rightarrow \quad a^2 = (c - b\cos A)^2 + (b \sin A)^2$

$\Rightarrow \quad a^2 = c^2 - 2bc\cos A + b^2\cos^2 A + b^2 \sin^2 A$

$= c^2 - 2bc\cos A + b^2$

$\Rightarrow \quad 2bc\cos A = b^2 + c^2 - a^2$

$\Rightarrow \quad \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Similarly, we can easily proved that, (ii) and (iii).

7.2.1 Some Solved Examples

Ex-1. In a triangle ABC, prove that

$a(b\cos C - c\cos B) = b^2 - c^2$.

Soln. We have $a(b\cos C - c\cos B)$

$= (ab\cos C - ac\cos B)$

$= ab\left(\dfrac{a^2 + b^2 - c^2}{2ab}\right) - ac\left(\dfrac{a^2 + c^2 - b^2}{2ac}\right)$

$= \left(\dfrac{a^2 + b^2 - c^2}{2}\right) - \left(\dfrac{a^2 + c^2 - b^2}{2}\right)$

$= \dfrac{1}{2}\left(a^2 + b^2 - c^2 - a^2 - c^2 + b^2\right)$

$= \dfrac{1}{2}\left(b^2 - c^2 - c^2 + b^2\right)$

$= \dfrac{1}{2}\left(2b^2 - 2c^2\right)$

$= \left(b^2 - c^2\right)$

Hence, the result.

Ex-2. In a triangle ABC, if

$(a + b + c)(a - b + c) = 3ac$, then find $\angle B$.

Soln. We have

$(a + b + c)(a - b + c) = 3ac$

$\Rightarrow \quad (a + c)^2 - b^2 = 3ac$

$\Rightarrow \quad a^2 + c^2 - b^2 = 3ac - 2ac = ac$

$\Rightarrow \quad \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{ac}{2ac} = \dfrac{1}{2}$

$\Rightarrow \quad \cos B = \dfrac{1}{2}$

$\Rightarrow \quad B = \dfrac{\pi}{3}$

Hence, the angle B is $60°$

Ex-3. In any ΔABC, if $2\cos B = \dfrac{a}{c}$ prove that the triangle is isoceles.

Soln. We have $2\cos B = \dfrac{a}{c}$

$\Rightarrow \quad 2\left(\dfrac{a^2 + c^2 - b^2}{2ac}\right) = \dfrac{a}{c}$

$\Rightarrow \quad \left(\dfrac{a^2 + c^2 - b^2}{a}\right) = a$

$\Rightarrow \quad \left(a^2 + c^2 - b^2\right) = a^2$

$\Rightarrow \quad \left(c^2 - b^2\right) = 0$

$\Rightarrow \quad c^2 = b^2$

$\Rightarrow \quad c = b$

Thus, the triangle is isosecles.

Ex-4. In a triangle ABC, if $a\cos A = b\cos B$, then prove that Δ is right angled isoceles.

Soln. We have $a\cos A = b\cos B$

$a\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right) = b\left(\dfrac{a^2 + c^2 - b^2}{2ac}\right)$

$$\Rightarrow \quad a\left(\frac{b^2+c^2-a^2}{b}\right)=b\left(\frac{a^2+c^2-b^2}{a}\right)$$

$$\Rightarrow \quad a^2\left(b^2+c^2-a^2\right)=b^2\left(a^2+c^2-b^2\right)$$

$$\Rightarrow \quad a^2\left(c^2-a^2\right)=b^2\left(c^2-b^2\right)$$

$$\Rightarrow \quad c^2\left(a^2-b^2\right)=\left(a^4-b^4\right)$$

$$\Rightarrow \quad c^2\left(a^2-b^2\right)=\left(a^2-b^2\right)\left(a^2+b^2\right)$$

$$\Rightarrow \quad \left(a^2-b^2\right)\left(\left(a^2+b^2\right)-c^2\right)=0$$

$$\Rightarrow \quad \left(a^2-b^2\right)=0,\left(a^2+b^2\right)=c^2$$

$$\Rightarrow \quad a=b,\left(a^2+b^2\right)=c^2$$

Thus, the triangle is right angled isosceles.

Ex-5. In a triangle *ABC*, the angles are in A.P, then prove that, $2\cos\left(\frac{A-C}{2}\right)=\frac{a+c}{\sqrt{a^2-ac+c^2}}$

Soln. Since the angles are in A.P, so $A + C = 2B$

$$\Rightarrow \quad A+B+C=3B$$

$$\Rightarrow \quad 3B=180°$$

$$\Rightarrow \quad B=60°$$

$$\Rightarrow \quad \cos B=\cos(60°)=\frac{1}{2}$$

$$\Rightarrow \quad \frac{a^2+c^2-b^2}{2ac}=\frac{1}{2}$$

$$\Rightarrow \quad a^2+c^2-b^2=ac$$

$$\Rightarrow \quad a^2+c^2-ac=b^2$$

Now, R.H.S

$$=\frac{a+c}{\sqrt{a^2-ac+c^2}}$$

$$=\frac{a+c}{b}$$

$$=\frac{k\left(\sin A+\sin C\right)}{k\ \sin B}$$

$$=\frac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)}{2\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)}$$

$$=\frac{\cos\left(\frac{B}{2}\right)\cos\left(\frac{A-C}{2}\right)}{\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)}$$

$$=\frac{\cos\left(\frac{A-C}{2}\right)}{\sin\left(\frac{B}{2}\right)}$$

$$=\frac{\cos\left(\frac{A-C}{2}\right)}{\sin\left(\frac{60°}{2}\right)}$$

$$=2\cos\left(\frac{A-C}{2}\right).$$

Ex-6. In a triangle *ABC*, prove that

$$\left(\frac{b^2-c^2}{a^2}\right)\sin 2A+\left(\frac{c^2-a^2}{b^2}\right)\sin 2B+\left(\frac{a^2-b^2}{c^2}\right)$$

$$\sin 2C=0$$

Soln. We have, $\left(\frac{b^2-c^2}{a^2}\right)\sin 2A$

$$=\left(\frac{b^2-c^2}{a^2}\right)\left(2\sin A.\cos A\right)$$

$$=\left(\frac{b^2-c^2}{a^2}\right)\left(2ka\left(\frac{b^2+c^2-a^2}{2bc}\right)\right)$$

$$=\left(\frac{\left(b^2-c^2\right)\left(b^2+c^2-a^2\right)}{kabc}\right)$$

$$=\frac{1}{kabc}\times\left\{\left(b^4-c^4\right)-a^2\left(b^2-c^2\right)\right\}$$

Similarly, $\left(\frac{c^2-a^2}{b^2}\right)\times\sin 2B$

$$=\frac{1}{kabc}\times\left\{\left(c^4-a^4\right)-b^2\left(c^2-a^2\right)\right\}$$

$\left(\frac{a^2-b^2}{c^2}\right)\sin 2C$ and

$$= \frac{1}{kabc} \times \left\{ \left(a^4 - b^4\right) - c^2\left(a^2 - b^2\right) \right\}$$

Thus, $\left(\frac{b^2 - c^2}{a^2}\right)\sin 2A + \left(\frac{c^2 - a^2}{b^2}\right)\sin 2B$

$$+\left(\frac{a^2 - b^2}{c^2}\right)\sin 2C$$

$$= \frac{1}{kabc} \times \left\{ \left(b^4 - c^4\right) - a^2\left(b^2 - c^2\right) \right\}$$

$$+ \frac{1}{kabc} \times \left\{ \left(c^4 - a^4\right) - b^2\left(c^2 - a^2\right) \right\}$$

$$+ \frac{1}{kabc} \times \left\{ \left(a^4 - b^4\right) - c^2\left(a^2 - b^2\right) \right\}$$

$$= \frac{1}{kabc} \times \left[\left(b^4 - c^4 + c^4 - a^4 + a^4 - b^4\right) \right.$$

$$\left. + a^2\left(c^2 - b^2\right) + b^2\left(a^2 - c^2\right) + c^2\left(b^2 - a^2\right) \right]$$

$= 0$

Ex-7. In a triangle ABC, if $\angle A = 60°$, then find the value of $\left(1 + \frac{a}{c} + \frac{b}{c}\right)\left(1 + \frac{c}{b} - \frac{a}{b}\right)$.

Soln. Given $\angle A = 60°$

$\Rightarrow \quad \cos A = \cos(60°)$

$\Rightarrow \quad \cos A = \frac{1}{2}$

$\Rightarrow \quad \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{2}$

$\Rightarrow \quad \left(b^2 + c^2 - a^2\right) = bc$ .....(i)

Now, $\left(1 + \frac{a}{c} + \frac{b}{c}\right)\left(1 + \frac{c}{b} - \frac{a}{b}\right)$

$$= \left(\frac{c + a + b}{c}\right)\left(\frac{b + c - a}{b}\right)$$

$$= \left(\frac{b + c + a}{c}\right)\left(\frac{b + c - a}{b}\right)$$

$$= \left(\frac{(b + c)^2 - a^2}{bc}\right)$$

$$= \left(\frac{b^2 + c^2 - a^2 + 2bc}{bc}\right)$$

$= \left(\frac{bc + 2bc}{bc}\right)$, from (i)

$$= \left(\frac{3bc}{bc}\right)$$

$= 3$

Ex-8 If $\frac{1}{a + c} + \frac{1}{b + c} = \frac{3}{a + b + c}$, then find $\angle C$.

Soln. Given $\frac{1}{a + c} + \frac{1}{b + c} = \frac{3}{a + b + c}$

$\Rightarrow \quad \frac{a + b + c}{a + c} + \frac{a + b + c}{b + c} = 3$

$\Rightarrow \quad 1 + \frac{b}{a + c} + 1 + \frac{a}{b + c} = 3$

$\Rightarrow \quad \frac{b}{a + c} + \frac{a}{b + c} = 1$

$\Rightarrow \quad b(b + c) + a(a + c) = (a + c)(b + c)$

$\Rightarrow \quad b^2 + bc + a^2 + ac = ab + ac + bc + c^2$

$\Rightarrow \quad a^2 + b^2 - c^2 = ab$

Now, $\cos(C) = \left(\frac{a^2 + b^2 - c^2}{2ab}\right)$

$$= \frac{ab}{2ab} = \frac{1}{2}$$

$$\Rightarrow C = \frac{\pi}{3}$$

Ex-9. If in a triangle ABC,

$$\frac{2\cos A}{a} + \frac{2\cos B}{b} + \frac{2\cos C}{c} = \frac{1}{bc} + \frac{b}{ca},$$

then find the angle A in degrees.

Soln. We have

$$\frac{2\cos A}{a} + \frac{2\cos B}{b} + \frac{2\cos C}{c} = \frac{1}{bc} + \frac{b}{ca}$$

$$\Rightarrow \quad \frac{2bc\cos A}{abc} + \frac{ac\cos B}{abc} + \frac{2bc\cos C}{abc}$$

$$= \frac{a^2}{abc} + \frac{b^2}{abc}$$

$\Rightarrow \quad 2bc\cos A + ac\cos B + 2bc\cos C = a^2 + b^2$

$$\Rightarrow \quad \left(b^2 + c^2 - a^2\right) + \frac{1}{2}\left(a^2 + c^2 - b^2\right)$$

$$+\left(a^2 + b^2 - c^2\right) = a^2 + b^2$$

$\Rightarrow \quad 2b^2 - 2a^2 + c^2 + a^2 - b^2 = 0$

$\Rightarrow \quad a^2 = b^2 + c^2$

ΔABC is a right angled triangle at A

Thus, $\angle A = 90°$

EXERCISE 2

Q. In any ΔABC,

1. Prove that $b(c\cos A - a\cos C) = (c^2 - a^2)$
2. Prove that
$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$$
3 Prove that
$$(a-b)^2\cos^2\left(\frac{C}{2}\right) + (a+b)^2\sin^2\left(\frac{C}{2}\right) = c^2$$
4. Prove that
$$(b^2-c^2)\cot A + (c^2-a^2)\cot B + (a^2-b^2)\cot C = 0$$
5. In a ΔABC, $a = 4, b = 3, \angle A = 60°$
Then prove that c is a root of the equation
$c^2 - 3c - 7 = 0$
6. In a ΔABC, if $(a+b+c)(b+c-a) = \lambda bc$
then find the value of λ.
7. If the angles A, B, C of a triangle are in A.P. and its sides a, b, c are in G.P., prove that a^2, b^2, c^2 are in A.P.
8. If the line segment joining the points $P(a_1, b_1)$ and $Q(a_2, b_2)$ subtends an angle θ at the origin,
prove that $\cos\theta = \left(\dfrac{a_1a_2 + b_1b_2}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}\right)$
9. In a triangle ABC, if $\cot A, \cot B, \cot C$ are in A.P., prove that a^2, b^2, c^2 are in A.P.
10. If the sides of a triangle are a, b and $\sqrt{a^2+ab+b^2}$ then find its greatest angle.

7.3 PROJECTION FORMULAE

Statement:

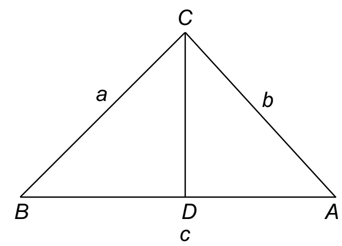
In any triangle ABC

(i) $a = b\cos C + c\cos B$

(ii) $b = c\cos A + a\cos C$

(iii) $c = a\cos B + b\cos A$

Proof:



From ΔABD, we have,

$a = BC = BD + DC$

$= c\cos B + b\cos C$

$\Rightarrow \quad a = b\cos C + c\cos B$

Similarly, we can eaisly proved that (ii) and (iii).

7.3.1 Some Solved Examples

Ex-1. In a triangle ABC, prove that,
$$\frac{\sin B}{\sin C} = \frac{c - a\cos B}{b - a\cos C}$$

Soln. We have, $\dfrac{c - a\cos B}{b - a\cos C}$

$$= \frac{a\cos B + b\cos A - a\cos B}{c\cos A + a\cos C - a\cos C}$$

$$= \frac{b\cos A}{c\cos A}$$

$$= \frac{b}{c}$$

$$= \frac{k\sin B}{k\sin C}$$

$$= \frac{\sin B}{\sin C}$$

Ex-2. In any triangle ABC, prove that,
$$2\left(a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right) = a + c - b$$

Soln. We have, $2\left(a\sin^2\dfrac{C}{2} + c\sin^2\dfrac{A}{2}\right)$

$$= a\left(2\sin^2\left(\frac{C}{2}\right)\right) + c\left(2\sin^2\left(\frac{A}{2}\right)\right)$$

$$= a(1-\cos C) + c(1-\cos A)$$

$$= a + c - (a\cos C + c\cos A)$$

$$= a + c - b$$

Ex-3. In any triangle ABC, prove that,

$$\frac{\cos A}{b\cos C + c\cos B} + \frac{\cos B}{c\cos A + a\cos C} + \frac{\cos C}{a\cos B + b\cos A} = \frac{a^2+b^2+c^2}{2abc}$$

Soln. We have,

$$\frac{\cos A}{b\cos C + c\cos B} + \frac{\cos B}{c\cos A + a\cos C} + \frac{\cos C}{a\cos B + b\cos A}$$

$$= \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \left(\frac{b^2+c^2-a^2}{2abc}\right) + \left(\frac{c^2+a^2-b^2}{2abc}\right) + \left(\frac{a^2+b^2-c^2}{2abc}\right)$$

$$= \frac{1}{2abc}\left\{(b^2+c^2-a^2) + (c^2+a^2-b^2) + (a^2+b^2-c^2)\right\}$$

$$= \frac{(a^2+b^2+c^2)}{2abc}$$

EXERCISE 3

Q. In any ΔABC, prove that

1. $2\left(a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right) = a + c - b$

2. $2\left(b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}\right) = a + b + c$

Q. In any ΔABC,

3. Prove that $\dfrac{\sin B}{\sin C} = \dfrac{c - a\cos B}{b - a\cos C}$

4. Prove that $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C$ $= (a+b+c)$

Q. In any ΔABC, prove that

5. $\dfrac{\cos A}{b\cos C + c\cos B} + \dfrac{\cos B}{c\cos A + a\cos C} + \dfrac{\cos C}{a\cos B + b\cos A} = \dfrac{a^2+b^2+c^2}{2abc}$

6. $2(bc\cos A + ca\cos B + ab\cos C) = (a^2+b^2+c^2)$

7.4 NAPIER'S ANALOGY (LAW OF TANGENTS)

Statement:

In any ΔABC, prove that

(i) $\tan\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{b+c}\cot\left(\dfrac{A}{2}\right)$

(ii) $\tan\left(\dfrac{C-A}{2}\right) = \left(\dfrac{c-a}{c+a}\right)\cot\dfrac{B}{2}$

(iii) $\tan\left(\dfrac{A-B}{2}\right) = \left(\dfrac{a-b}{a+b}\right)\cot\dfrac{C}{2}$

Proof: (i) Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$

We have, $\left(\dfrac{b-c}{b+c}\right)\cot\left(\dfrac{A}{2}\right)$

$$= \left(\frac{\sin B - \sin C}{\sin B + \sin C}\right)\times\cot\left(\frac{A}{2}\right)$$

$$= \left(\frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}\right)\times\cot\left(\frac{A}{2}\right)$$

$$= \left(\frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}\right)\times\cot\left(\frac{A}{2}\right)$$

$$= \left(\frac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}\right)\times\cot\left(\frac{A}{2}\right)$$

$$= \left(\frac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}\right)\times\frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$= \tan\left(\frac{B-C}{2}\right)$

Similarly, we can easily proved that (ii) and (iii)

7.4.1 Some Solved Examples

Ex-1. In any ΔABC, $b=\sqrt{3}+1, c=\sqrt{3}-1$ and $\angle A = 60°$, then find the value of $\tan\left(\frac{B-C}{2}\right)$.

Soln. As we know that,

$$\tan\left(\frac{B-C}{2}\right)=\left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$$

$$=\left(\frac{(\sqrt{3}+1)-(\sqrt{3}-1)}{(\sqrt{3}+1)+(\sqrt{3}-1)}\right)\cot\left(\frac{60°}{2}\right)$$

$$=\frac{1}{\sqrt{3}}\cot(30°)$$

$$=\frac{1}{\sqrt{3}}\times\sqrt{3}$$

$$=1$$

Ex-2. In any ΔABC, $b=\sqrt{3}, c=1, B-C=90°$ then find $\angle A$.

Soln. As we know that,

$$\tan\left(\frac{B-C}{2}\right)=\left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$$

$$\Rightarrow \tan\left(\frac{90°}{2}\right)=\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)\cot\left(\frac{A}{2}\right)$$

$$\Rightarrow \tan(45°)=\left(2-\sqrt{3}\right)\cot\left(\frac{A}{2}\right)$$

$$\Rightarrow \cot\left(\frac{A}{2}\right)=\frac{1}{\left(2-\sqrt{3}\right)}=\left(2+\sqrt{3}\right)$$

$$\Rightarrow \cot\left(\frac{A}{2}\right)=\cot(15°)$$

$$\Rightarrow A=30°$$

Hence, the angle A is 30°.

Ex-3. If in a triangle ABC, $a = 6$, $b = 3$ and $\cos(A-B)=4/5$, then find the angle C.

Soln. Given $\cos(A-B)=\frac{4}{5}$

$$\Rightarrow 2\cos^2\left(\frac{A-B}{2}\right)-1=\frac{4}{5}$$

$$\Rightarrow 2\cos^2\left(\frac{A-B}{2}\right)=1+\frac{4}{5}=\frac{9}{5}$$

$$\Rightarrow \cos^2\left(\frac{A-B}{2}\right)=\frac{9}{10}$$

$$\Rightarrow \cos\left(\frac{A-B}{2}\right)=\frac{3}{\sqrt{10}}$$

$$\Rightarrow \tan\left(\frac{A-B}{2}\right)=\frac{1}{3}$$

$$\Rightarrow \left(\frac{a-b}{a+b}\right)\cot\left(\frac{C}{2}\right)=\frac{1}{3}$$

$$\Rightarrow \left(\frac{6-3}{6+3}\right)\cot\left(\frac{C}{2}\right)=\frac{1}{3}$$

$$\Rightarrow \frac{1}{3}\cot\left(\frac{C}{2}\right)=\frac{1}{3}$$

$$\Rightarrow \cot\left(\frac{C}{2}\right)=1$$

$$\Rightarrow \frac{C}{2}=\frac{\pi}{4}$$

$$\Rightarrow C=\frac{\pi}{2}$$

Hence, the value of C is $\frac{\pi}{2}$.

EXERCISE 4

1. If in a triangle ABC, $a = 4$, $b = 2$ and $\cos(A-B) = 4/5$, then prove that angle C is 90°.
2. In a triangle ABC, if

   $x=\tan\left(\frac{B-C}{2}\right)\tan\left(\frac{A}{2}\right)$, $y=\tan\left(\frac{C-A}{2}\right)\tan\left(\frac{B}{2}\right)$

   and

   $z=\tan\left(\frac{A-B}{2}\right)\tan\left(\frac{C}{2}\right)$, then prove that

   $x+y+z+xyz=0$

3. In a triangle ABC, if $a = 5$, $b = 4$ and $\cos(A-B)=\frac{31}{32}$, then prove that $c = 6$.

7.5 HALF-ANGLED FORMULAE

In this section, we shall desire to the formulas for the sine, cosine and tangent to the half of the angles of any triangle in terms of its sides. The perimeter of ΔABC will be denoted by 2s

i.e. $a + b + c = 2s$ and its area is denoted by Δ.

i.e. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

1. Formulae for $\sin\left(\frac{A}{2}\right), \sin\left(\frac{B}{2}\right), \sin\left(\frac{C}{2}\right)$:

   In any ΔABC,

   (i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

   (ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

   (iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

Proof: (i) We have,

$$2\sin^2\left(\frac{A}{2}\right)$$

$$= 1 - \cos A$$

$$= 2\sin^2\left(\frac{A}{2}\right)$$

$$= 1 - \left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$= \left(\frac{2bc - b^2 - c^2 + a^2}{2bc}\right)$$

$$= \left(\frac{a^2 - (b^2 + c^2 - 2bc)}{2bc}\right)$$

$$= \left(\frac{a^2 - (b-c)^2}{2bc}\right)$$

$$= \left(\frac{\{a+(b-c)\}\times\{a-(b-c)\}}{2bc}\right)$$

$$= \left(\frac{(a+b-c)\times(a+c-b)}{2bc}\right)$$

$$= \left(\frac{(a+b+c-2c)\times(a+b+c-2b)}{2bc}\right)$$

$$= \left(\frac{(2s-2c)\times(2s-2b)}{2bc}\right)$$

$$= \left(\frac{2(s-c)\times(s-b)}{bc}\right)$$

$$\Rightarrow \quad 2\sin^2\left(\frac{A}{2}\right) = \left(\frac{2(s-c)\times(s-b)}{bc}\right)$$

$$\Rightarrow \quad \sin^2\left(\frac{A}{2}\right) = \left(\frac{(s-c)\times(s-b)}{bc}\right)$$

$$\Rightarrow \quad \sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-c)\times(s-b)}{bc}}$$

Similarly, we can easily prove that (ii) and (iii)

2. Formulae for $\cos\left(\frac{A}{2}\right), \cos\left(\frac{B}{2}\right), \cos\left(\frac{C}{2}\right)$:

   In any ΔABC,

   (i) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

   (ii) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$

   (iii) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

Proof: (i) We have,

$$2\cos^2\left(\frac{A}{2}\right)$$

$$= 1 + \cos A$$

$$= 1 + \left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$= \left(\frac{2bc + b^2 + c^2 - a^2}{2bc}\right)$$

$$= \left(\frac{(b+c)^2 - a^2}{2bc}\right)$$

$$= \left(\frac{(a+b+c)(b+c-a)}{2bc}\right)$$

$$= \left(\frac{(a+b+c)(b+c+a-2a)}{2bc}\right)$$

$$= \left(\frac{2s(2s-2a)}{2bc}\right)$$

$$= \left(\frac{2s(s-a)}{bc}\right)$$

$$\Rightarrow \quad 2\cos^2\left(\frac{A}{2}\right) = \left(\frac{2s(s-a)}{bc}\right)$$

$$\Rightarrow \quad \cos^2\left(\frac{A}{2}\right) = \left(\frac{s(s-a)}{bc}\right)$$

$$\Rightarrow \quad \cos\left(\frac{A}{2}\right) = \sqrt{\left(\frac{s(s-a)}{bc}\right)}$$

Similarly, we can easily proved that (ii) and (iii).

3. Formulae for $\tan\left(\frac{A}{2}\right), \tan\left(\frac{B}{2}\right), \tan\left(\frac{C}{2}\right)$:

In any ΔABC,

(i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

(ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

(iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Proof: (i) We have,

$$\tan\left(\frac{A}{2}\right)$$

$$= \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)}.$$

$$= \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$\Rightarrow \quad \tan\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Similarly, we can easily proved that (ii) and (iii)

7.5.1 Some Solved Examples

Ex-1. In a ΔABC, if $a = 13$, $b = 14$ and $c = 15$, then find the value of

(i) $\sin\frac{A}{2}$

(ii) $\cos\frac{B}{2}$

(iii) $\cos A$

Soln. We have $2s = a+b+c = 13+14+15$

$$s = \frac{13+14+15}{2} = \frac{42}{2} = 21$$

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

$$= \sqrt{\frac{(21-14)(21-15)}{14.15}}$$

$$= \sqrt{\frac{7.6}{14.15}} = \frac{1}{\sqrt{5}}$$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

$$= \sqrt{\frac{21(21-13)}{14.15}}$$

$$= \sqrt{\frac{21.8}{14.15}}$$

$$= \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

(iii) $\cos A$

$$= 2\cos^2\left(\frac{A}{2}\right) - 1$$

$$= 2\left(\frac{4}{5}\right) - 1$$

$$= \frac{8-5}{5} = \frac{3}{5}$$

Ex-2. In a ΔABC, if $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{b+c}{2c}}$,

prove that ΔABC is right angled at C.

Soln. We have $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{b+c}{2c}}$

$$\Rightarrow \quad \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{b+c}{2c}}$$

$$\Rightarrow \quad \frac{s(s-a)}{bc} = \frac{b+c}{2c}$$

$$\Rightarrow \quad 2s(2s-2a) = 2b(b+c)$$

$$\Rightarrow \quad (a+b+c)(b+c-a) = 2b(b+c)$$

$\Rightarrow \quad \left((b+c)^2 - a^2\right) = 2b(b+c)$

$\Rightarrow \quad b^2 + c^2 - a^2 = 2b^2$

$\Rightarrow \quad a^2 + b^2 = c^2$

Thus, the triangle ABC is right angled at C.

Ex-3. In a ΔABC, prove that,

$$b\cos^2\left(\frac{C}{2}\right) + c\cos^2\left(\frac{B}{2}\right) = s$$

Soln. We have, $b\cos^2\left(\frac{C}{2}\right) + c\cos^2\left(\frac{B}{2}\right)$

$$= b\left(\frac{s(s-c)}{ab}\right) + c\left(\frac{s(s-b)}{ac}\right)$$

$$= \frac{s}{a}(s-c+s-b)$$

$$= \frac{s}{a}(2s-c-b)$$

$$= \frac{s}{a}(a+b+c-c-b)$$

$$= s$$

Ex-4. In a ΔABC, prove that

$$bc\cos^2\left(\frac{A}{2}\right) + ca\cos^2\left(\frac{B}{2}\right) + ab\cos^2\left(\frac{C}{2}\right) = s^2$$

Soln. We have

$$bc\cos^2\left(\frac{A}{2}\right) + ca\cos^2\left(\frac{B}{2}\right) + ab\cos^2\left(\frac{C}{2}\right)$$

$$= bc\left(\frac{s(s-a)}{bc}\right) + ca\left(\frac{s(s-b)}{ca}\right) + ab\left(\frac{s(s-c)}{ab}\right)$$

$$= s(s-a) + s(s-b) + s(s-c)$$

$$= s(3s - (a+b+c))$$

$$= s(3s-2s)$$

$$= s \times s$$

$$= s^2$$

Ex.-5 In a ΔABC, prove that $2ac\sin\left(\frac{A-B+C}{2}\right)$

$$= \left(a^2 + c^2 - b^2\right).$$

Soln. We have $2ac\sin\left(\frac{A-B+C}{2}\right)$

$$= 2ac\sin\left(\frac{A+C-B}{2}\right)$$

$$= 2ac\sin\left(\frac{\pi - B - B}{2}\right)$$

$$= 2ac\sin\left(\frac{\pi}{2} - B\right)$$

$$= 2ac\cos B$$

$$= 2ac\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$$= \left(a^2+c^2-b^2\right)$$

Ex-6 . In a ΔABC, $3a = b + c$, then find the value of $\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$.

Soln. We have, $\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$

$$= \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \times \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \sqrt{\frac{s(s-b)}{(s-a)(s-c)} \times \frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \sqrt{\frac{s^2}{(s-a)^2}}$$

$$= \frac{s}{s-a}$$

$$= \frac{2s}{2s-2a}$$

$$= \frac{a+b+c}{a+b+c-2a}$$

$$= \frac{4a}{4a-2a}$$

$$= 2.$$

Ex-7. In a ΔABC, prove that

$$1 - \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right) = \frac{2c}{(a+b+c)}$$

Soln. We have $1 - \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)$

$$= 1 - \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$= 1 - \sqrt{\frac{(s-c)^2}{s^2}}$$

$= 1 - \frac{(s-c)}{s}$

$= \frac{c}{s}$

$= \frac{2c}{2s}$

$= \frac{2c}{(a+b+c)}$

Hence, the result.

Ex-8. In a ΔABC, prove that:

$$\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right) = \left(\frac{a+b+c}{b+c-a}\right)\cot\left(\frac{A}{2}\right)$$

Soln. We have, $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \sqrt{\frac{s^2(s-a)^2}{s(s-a)(s-b)(s-c)}} + \sqrt{\frac{s^2(s-b)^2}{s(s-b)(s-a)(s-c)}} + \sqrt{\frac{s^2(s-c)^2}{s(s-c)(s-a)(s-b)}}$$

$$= \sqrt{\frac{s^2(s-a)^2}{\Delta^2}} + \sqrt{\frac{s^2(s-b)^2}{\Delta^2}} + \sqrt{\frac{s^2(s-c)^2}{\Delta^2}}$$

$$= \frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta} + \frac{s(s-c)}{\Delta}$$

$$= \frac{s}{\Delta}(s-a+s-b+s-c)$$

$$= \frac{s}{\Delta}(3s-(a+b+c))$$

$$= \frac{s}{\Delta}(3s-2s)$$

$$= \frac{s^2}{\Delta}$$

$$= \frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \frac{s}{(s-a)} \times \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= \frac{2s}{(2s-2a)} \times \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= \frac{(a+b+c)}{(b+c-a)} \times \cot\left(\frac{A}{2}\right)$$

Ex-9 In a ΔABC, if $\cot\left(\frac{A}{2}\right), \cot\left(\frac{B}{2}\right), \cot\left(\frac{C}{2}\right)$ are in A.P., then prove that a, b, c are in A.P.

Soln. Given $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are in A.P.

$$\Rightarrow \quad 2\cot\left(\frac{B}{2}\right) = \cot\left(\frac{A}{2}\right) + \cot\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad 2\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$\Rightarrow \quad 2\sqrt{\frac{(s-b)}{(s-a)(s-c)}} = \sqrt{\frac{(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{(s-c)}{(s-a)(s-b)}}$$

$$\Rightarrow \quad 2\sqrt{\frac{(s-b)^2}{(s-b)(s-a)(s-c)}} = \sqrt{\frac{(s-a)^2}{(s-a)(s-b)(s-c)}} + \sqrt{\frac{(s-c)^2}{(s-a)(s-b)(s-c)}}$$

$$\Rightarrow \quad 2(s-b) = (s-a) + (s-c)$$

$$\Rightarrow \quad 2(s-b) = (2s-a-c)$$

$$\Rightarrow \quad 2b = a+c$$

$$\Rightarrow \quad a, b, c \in A.P$$

Ex-10 In a ΔABC, $c(a+b)\cos\frac{B}{2} = b(a+c)\cos\frac{C}{2}$, then prove that the triangle is isosceles.

Soln. We have, $c(a+b)\cos\frac{B}{2} = b(a+c)\cos\frac{C}{2}$

$$\Rightarrow \quad c(a+b) \times \sqrt{\frac{s(s-b)}{ac}} = b(a+c) \times \sqrt{\frac{s(s-c)}{ab}}$$

$$\Rightarrow \quad c(a+b) \times \sqrt{\frac{(s-b)}{c}} = b(a+c) \times \sqrt{\frac{(s-c)}{b}}$$

$\Rightarrow \quad c^2(a+b)^2 \times \frac{(s-b)}{c} = b^2(a+c)^2 \times \frac{(s-c)}{b}$

$\Rightarrow \quad c(a+b)^2 \times (s-b) = b(a+c)^2 \times (s-c)$

$\Rightarrow \quad c(a+b)^2 \times (s-b) = b(a+c)^2 \times (s-c)$

$\Rightarrow \quad c(a+b)^2 \times (a+c-b) = b(a+c)^2 \times (a+b-c)$

$\Rightarrow \quad \frac{(a+c-b)}{b(a+c)^2} = \frac{(a+b-c)}{c(a+b)^2}$

$\Rightarrow \quad \frac{1}{b(a+c)} - \frac{1}{(a+c)^2} = \frac{1}{c(a+b)} - \frac{1}{(a+b)^2}$

$\Rightarrow \quad \frac{1}{b(a+c)} - \frac{1}{c(a+b)} = \frac{1}{(a+c)^2} - \frac{1}{(a+b)^2}$

$\Rightarrow \quad \frac{ac+c^2-ab-b^2}{bc(a+b)(a+c)} = \frac{(a+b)^2-(a+c)^2}{(a+c)^2(a+b)^2}$

$\Rightarrow \quad \frac{a(c-b)+(c^2-b^2)}{bc} = \frac{2a(b-c)+(b^2-c^2)}{(a+c)(a+b)}$

$\Rightarrow \quad \frac{(c-b)(a+b+c)}{bc} = \frac{(b-c)(2a+b+c)}{(a+c)(a+b)}$

$\Rightarrow \quad (c-b)\left(\frac{(a+b+c)}{bc} + \frac{(2a+b+c)}{(a+c)(a+b)}\right) = 0$

$\Rightarrow \quad (c-b) = 0$

$\left(\because \left(\frac{(a+b+c)}{bc} + \frac{(2a+b+c)}{(a+c)(a+b)}\right) \neq 0\right)$

$\Rightarrow \quad b = c$

$\Rightarrow \quad \Delta$ is isoceles.

EXERCISE 5

1. In a ΔABC, $3a = b + c$, then prove that
$\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right) = 2$

2. In a ΔABC, if $\cos A + \cos C = 4\sin^2\left(\frac{B}{2}\right)$, then prove that a, b, c are in A.P.

3. In a ΔABC, if $(s-a)(s-b) = s(s-c)$ then prove that angle C is 90°.

4. In a ΔABC, if $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ anr in A.P., then prove that a, b, c are in A.P.

5. In a ΔABC, $\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right) = \left(\frac{s-c}{s}\right)$

6. In a ΔABC, prove that
$$\left(\frac{b-c}{a}\right)\cos^2\left(\frac{A}{2}\right) + \left(\frac{c-a}{b}\right)\cos^2\left(\frac{B}{2}\right) + \left(\frac{a-b}{c}\right)\cos^2\left(\frac{C}{2}\right) = 0$$

7. In a ΔABC, prove that
$$\frac{\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)}{\cot A + \cot B + \cot C} = \frac{(a+b+c)^2}{a^2+b^2+c^2}$$

8. In a ΔABC, prove that
$$(b-c)\cot\left(\frac{A}{2}\right) + (c-a)\cot\left(\frac{B}{2}\right) + (a-b)\cot\left(\frac{C}{2}\right) = 0$$

9. In a ΔABC, if
$\tan\left(\frac{A}{2}\right) = \frac{5}{6}, \tan\left(\frac{B}{2}\right) = \frac{20}{37}$, then prove that a, b, c are in A.P.

10. In a ΔABC, if $c(a+b)\cos\left(\frac{B}{2}\right) = b(a+c)\cos\left(\frac{C}{2}\right)$ then prove that the triangle ABC is isosceles.

7.6 AREA OF TRIANGLE

Statement:
Prove that the area of ΔABC is given by

(i) $\Delta = \frac{1}{2}\, bc \sin A$

(ii) $\Delta = \frac{1}{2}\, ca \sin B$

(iii) $\Delta = \frac{1}{2}\, ab \sin C$

Proof: (i) We have, $\Delta = \frac{1}{2} \times AB \times CD$

$\Rightarrow \quad \Delta = \frac{1}{2} \times c \times b \sin A$

$\Rightarrow \quad \Delta = \frac{1}{2} bc \sin A$

Similarly, we can easily proved that (ii) and (iii).

1. Area of a triangle (Hero's Formula)
In any ΔABC: $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Proof: As, we know that $\Delta = \frac{1}{2}bc\sin A$

$$= \frac{1}{2}\times bc\times 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$$

$$= bc\times\sqrt{\frac{(s-b)(s-c)}{bc}}\times\sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

7.6.1 Some Solved Examples

Ex-1. In any ΔABC, if $a=\sqrt{2}, b=\sqrt{3}$ and $c=\sqrt{5}$, then find the area of the ΔABC.

Soln. $\Delta ABC = \frac{1}{2}\times\sqrt{2}\times\sqrt{3} = \sqrt{\frac{3}{2}}$ s.u.

Ex-2. In any ΔABC, prove that

$$\Delta = \frac{a^2-b^2}{2}\times\frac{\sin A\sin B}{\sin(A-B)}.$$

Soln. We have, $\frac{a^2-b^2}{2}\times\frac{\sin A\sin B}{\sin(A-B)}$

$$= \frac{k^2\left(\sin^2 A-\sin^2 B\right)}{2}\times\frac{\sin A.\sin B}{\sin(A-B)}$$

$$= \frac{k^2\times\sin(A+B)\times\sin(A-B)}{2}\times\frac{\sin A.\sin B}{\sin(A-B)}$$

$$= \frac{k^2\times\sin(A+B)\times\sin A.\sin B}{2}$$

$$= \frac{k^2\times\sin(\pi-C)\times\sin A.\sin B}{2}$$

$$= \frac{k^2\times\sin C\times\frac{a}{k}\times\frac{b}{k}}{2}$$

$$= \frac{1}{2}\times ab\sin C$$

$= \Delta$.

Ex-3. If the angles of triangle and 30° and 45° and the included side is ($\sqrt{3}$ + 1)cm., then prove that the area of the triangle is $\frac{1}{2}$ ($\sqrt{3}$ + 1) cm^2.

Soln. As we know that, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

$$\Rightarrow \frac{\left(\sqrt{3}+1\right)}{\sin(105°)}=\frac{b}{\sin 45°}=\frac{c}{\sin 30°}$$

$$\Rightarrow \frac{\left(\sqrt{3}+1\right)}{\cos(105°)}=\frac{b}{\sin 45°}=\frac{c}{\sin 30°}$$

$$\Rightarrow \frac{\left(\sqrt{3}+1\right)}{\frac{\sqrt{3}+1}{2\sqrt{2}}}=\frac{b}{\frac{1}{\sqrt{2}}}=\frac{c}{\frac{1}{2}}$$

$$\Rightarrow 2\sqrt{2}=b\sqrt{2}=2c$$

$$\Rightarrow b=2 \ \& \ c=\sqrt{2}$$

Hence, the area of the triangle is $= \frac{1}{2}bc\sin A$.

$$= \frac{1}{2}\times 2\times\sqrt{2}\times\sin(105°)$$

$$= \frac{1}{2}\times 2\times\sqrt{2}\times\frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= \frac{\left(\sqrt{3}+1\right)}{2} \text{ sq. u.}$$

Ex-4. In a ΔABC, prove that:

$$\frac{\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)}{\cot A+\cot B+\cot C} = \frac{(a+b+c)^2}{a^2+b^2+c^2}$$

Soln. We have, $\cot A+\cot B+\cot C$

$$= \frac{\cos A}{\sin A}+\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}$$

$$= \frac{\frac{b^2+c^2-a^2}{2bc}}{ak}+\frac{\frac{c^2+a^2-b^2}{2ac}}{bk}+\frac{\frac{a^2+b^2-c^2}{2ab}}{ck}$$

$$= \frac{b^2+c^2-a^2}{2abck}+\frac{c^2+a^2-b^2}{2abck}+\frac{a^2+b^2-c^2}{2abck}$$

$$= \frac{a^2+b^2+c^2}{2abck}$$

$$= \frac{\left(a^2+b^2+c^2\right)}{4\times\left(\frac{1}{2}ab\right)\times\sin C}$$

$$= \frac{\left(a^2+b^2+c^2\right)}{4\times\Delta} \text{ ....(i)}$$

Also, $\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)$

$$= \frac{(a+b+c)}{(b+c-a)} \times \cot\left(\frac{A}{2}\right)$$

$$= \frac{s^2}{\Delta} \text{ ....(ii)}$$

Dividing (ii) by (i), we get,

$$\frac{\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)}{\cot(A)+\cot(B)+\cot(C)} = \frac{s^2}{4\left(a^2+b^2+c^2\right)}$$

$$\Rightarrow \frac{\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)}{\cot(A)+\cot(B)+\cot(C)} = \frac{(a+b+c)^2}{\left(a^2+b^2+c^2\right)}$$

Ex.-5. If in a ΔABC, prove that $\Delta < \frac{s^2}{4}$.

Soln. Let a, b, and c are the sides of a triangle and s be the semi perimeter.
Let the four quantities are s, $(s-a)$, $(s-b)$ and $(s-c)$
Applying, A.M. ≥ G.M., we get,

$$\Rightarrow \frac{s+(s-a)+(s-b)+(s-c)}{4}$$

$$\geq \sqrt[4]{s(s-a)(s-b)(s-c)}$$

$$\Rightarrow \frac{4s-(a+b+c)}{4} \geq \sqrt[4]{\Delta^2}$$

$$\Rightarrow \frac{4s-2s}{4} \geq (\Delta)^{\frac{1}{2}}$$

$$\Rightarrow \frac{s}{2} \geq (\Delta)^{\frac{1}{2}}$$

$$\Rightarrow \Delta < \frac{s^2}{4}$$

Ex-6. If α, β, γ are the lengths of the altitudes of a triangle ABC, then prove that

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{1}{\Delta}(\cot A+\cot B+\cot C)$$

Soln. Let $AD = \alpha$, $BE = \beta$ and $CF = \gamma$

Then,

$$\Delta = \frac{1}{2}\times a\times AD = \frac{1}{2}\times b\times BE = \frac{1}{2}\times c\times CF$$

$$\Rightarrow AD = \frac{2\Delta}{a},\ BE = \frac{2\Delta}{b},\ CF = \frac{2\Delta}{c}$$

$$\Rightarrow \alpha = \frac{2\Delta}{a},\ \beta = \frac{2\Delta}{b},\ \gamma = \frac{2\Delta}{c}$$

Now, $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$

$$= \frac{a^2}{4\Delta^2}+\frac{b^2}{4\Delta^2}+\frac{c^2}{4\Delta^2}$$

$$= \frac{\left(a^2+b^2+c^2\right)}{4\Delta^2}$$

$$= \frac{1}{\Delta}\times\frac{\left(a^2+b^2+c^2\right)}{4\Delta}$$

$$= \frac{1}{\Delta}\times(\cot A+\cot B+\cot C)$$

$$= \frac{(\cot A+\cot B+\cot C)}{\Delta}$$

Hence, the result.

Ex.-7. If p_1, p_2, p_3 are the altitudes of a triangle from the vertices A, B, C and Δ be the area of the triangle ABC, prove that

$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3} = \frac{2ab}{(a+b+c)\times\Delta}\times\cos^2\left(\frac{C}{2}\right).$$

Soln. Let $AD = p_1$, $BE = p_2$ and $CF = p_3$

Then, $\Delta = \frac{1}{2}\times a\times p_1 = \frac{1}{2}\times b\times p_2 = \frac{1}{2}\times c\times p_3$

$$\Rightarrow p_1 = \frac{2\Delta}{a},\ p_2 = \frac{2\Delta}{b},\ p_3 = \frac{2\Delta}{c}$$

Now, $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}$

$$= \frac{a}{2\Delta}+\frac{b}{2\Delta}-\frac{c}{2\Delta}$$

$$= \frac{(a+b-c)}{2\Delta}$$

$$= \frac{(a+b+c-2c)}{2\Delta}$$

$$= \frac{(2s-2c)}{2\Delta}$$

$$= \frac{(s-c)}{\Delta}$$

$$= \frac{2ab\times s(s-c)}{\Delta\times s}\times\frac{1}{2ab}$$

$$= \frac{2ab}{\Delta\times s}\times\frac{s(s-c)}{2ab}$$

$$= \frac{2\,ab}{(a+b+c)\Delta} \times \cos^2\left(\frac{C}{2}\right)$$

Ex-8. If a, b, c and d are the sides of a quadrilateral, then find the minimum value of $\left(\frac{a^2+b^2+c^2}{d^2}\right)$

Soln. We have, $(a-b)^2+(b-c)^2+(c-d)^2 \geq 0$

$$\Rightarrow \quad 2\left(a^2+b^2+c^2\right) \geq 2(ab+bc+ca)$$

$$\Rightarrow \quad 3\left(a^2+b^2+c^2\right) \geq \left(a^2+b^2+c^2\right) + 2(ab+bc+ca)$$

$$\Rightarrow \quad 3\left(a^2+b^2+c^2\right) > (a+b+c)^2 > d^2$$

$$\Rightarrow \quad \frac{3\left(a^2+b^2+c^2\right)}{d^2} > 1$$

$$\Rightarrow \quad \frac{\left(a^2+b^2+c^2\right)}{d^2} > \frac{1}{3}$$

Thus, the minimum value of $\frac{\left(a^2+b^2+c^2\right)}{d^2}$ is $\frac{1}{3}$.

Ex-9. In a triangle ABC, if $\cos A + \cos B + \cos C = \frac{3}{2}$, then the triangle is equilateral.

Soln. We have, $\cos A + \cos B + \cos C = \frac{3}{2}$

$$\Rightarrow \quad 1+4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) = \frac{3}{2}$$

$$\Rightarrow \quad \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) = \frac{1}{8}$$

It is possible only when

$$\sin\left(\frac{A}{2}\right) = \frac{1}{2},\ \sin\left(\frac{B}{2}\right) = \frac{1}{2},\ \sin\left(\frac{C}{2}\right) = \frac{1}{2}$$

$$\Rightarrow \quad \frac{A}{2} = \frac{\pi}{6},\ \frac{B}{2} = \frac{\pi}{6},\ \frac{C}{2} = \frac{\pi}{6}$$

$$\Rightarrow \quad A = \frac{\pi}{3},\ B = \frac{\pi}{3},\ C = \frac{\pi}{3}$$

Δ is an equilateral .

Ex-10. In a ΔPQR, if $\sin P$, $\sin Q$, $\sin R$ are in A.P. then prove that its altitude are in H.P.

Soln. Here,

$$\Delta = \frac{1}{2} \times p \times p_1 = \frac{1}{2} \times q \times p_2 = \frac{1}{2} \times r \times p_3$$

$$\Rightarrow \quad p = \frac{2\Delta}{p_1}, q = \frac{2\Delta}{p_2}, r = \frac{2\Delta}{p_3}$$

From sine rule of a triangle,

$$\frac{\sin P}{p} = \frac{\sin Q}{q} = \frac{\sin R}{r}$$

Given $\sin P$, $\sin Q$, $\sin R$ are in A.P.

$$\Rightarrow \quad p, q, r \in A.P$$

$$\Rightarrow \quad \frac{2\Delta}{p_1}, \frac{2\Delta}{p_2}, \frac{2\Delta}{p_3} \in A.P$$

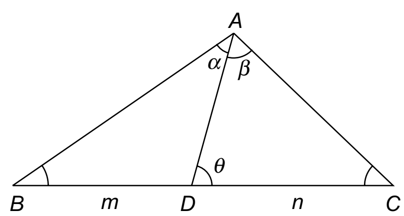
7.6.2 *m-n* Theorem

Statement: If D is a point on the side BC such that $BD : DC = m : n$ and $\angle ADC = \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$, then

(i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(ii) $(m+n)\cot\theta = n\cot B - m\cot C$

Soln. Given $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta$



So, $\angle ABD = 180° - (\alpha + 180° - \theta) = (\theta - \alpha)$

and $\angle ACD = 180° - (\theta + \beta)$

From, ΔABD, $\frac{BD}{\sin\alpha} = \frac{AD}{\sin(\theta-\alpha)}$ ......(i)

From ΔADC, $\frac{DC}{\sin\beta} = \frac{AD}{\sin(180° - (\theta+\beta))}$

$$\Rightarrow \quad \frac{DC}{\sin\beta} = \frac{AD}{\sin(\theta+\beta)} \quad ....(ii)$$

Dividing (i) and (ii), we get,

$$\frac{BD\sin\beta}{DC\sin\alpha} = \frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)}$$

$\Rightarrow \quad \dfrac{m\sin\beta}{n\sin\alpha} = \dfrac{\sin\theta\cos\beta + \cos\theta\sin\beta}{\sin\theta\cos\alpha - \cos\theta\sin\alpha}$

$\Rightarrow \quad \dfrac{m}{n} = \dfrac{(\sin\theta\cos\beta + \cos\theta\sin\beta)\sin\alpha}{(\sin\theta\cos\alpha - \cos\theta\sin\alpha)\sin\beta}$

Dividing the R.H.S by $\sin\alpha\sin\beta\sin\theta$, we get,

$$\frac{m}{n} = \frac{\cot\beta + \cot\theta}{\cot\alpha - \cot\theta}$$

$\Rightarrow \quad n(\cot\beta + \cot\theta) = m(\cot\alpha - \cot\theta)$

$\Rightarrow \quad (m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(ii) Given $\dfrac{BD}{DC} = \dfrac{m}{n}$ and $\angle ADC = \theta$

Thus, $\angle ADB = 180^o - \theta$

Here, $\angle ABD = B$ and $\angle ACD = C$

So, $\angle BAD = 180° - (180° - \theta + B) = \theta - B$

and $\angle DAC = (180° - (\theta + C))$

Now, from ΔABD, $\dfrac{BD}{\sin(\theta - B)} = \dfrac{AD}{\sin B}$ ......(i)

and from ΔADC,

$$\frac{DC}{\sin(180° - (\theta + C))} = \frac{AD}{\sin C}$$

$\Rightarrow \quad \dfrac{DC}{\sin(\theta + C)} = \dfrac{AD}{\sin C}$ ....(ii)

Dividing (i) by (ii), we get,

$$\frac{BD}{DC} \cdot \frac{\sin(\theta + C)}{\sin(\theta - B)} = \frac{\sin C}{\sin B}$$

$\Rightarrow \quad \dfrac{m}{n} \cdot \dfrac{\sin(\theta + C)}{\sin(\theta - B)} = \dfrac{\sin C}{\sin B}$

$\Rightarrow \quad \dfrac{m}{n} = \dfrac{\sin C.\sin(\theta - B)}{\sin B.\sin(\theta + C)}$

$\Rightarrow \quad \dfrac{m}{n} = \dfrac{\sin C(\sin\theta\ \cos B - \cos\theta\sin B)}{\sin B(\sin\theta\ \cos C + \cos\theta\sin C)}$

Dividing the numerator and denominator on the right hand side by $\sin B\sin C\sin\theta$, we get,

$$\frac{m}{n} = \frac{\cot B - \cot\theta}{\cot C + \cot\theta}$$

$\Rightarrow \quad (m+n)\cot\theta = n\cot B - m\cot C$

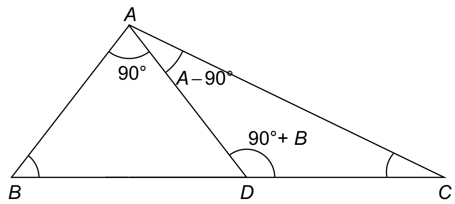
Hence, the result.

This completes the proof of the statement.

7.6.3 Some Solved Examples

Ex-1. The median AD of a triangle ABC is perpendicular **to** AB**. Prove that** $\tan A + 2\tan B = 0$

Soln. Since AD is the median, so $BD : DC = 1 : 1$



Clearly, $\angle ADC = 90° + B$.

Now, applying $m : n$ rule, we get,

$(1+1)\cot(90° + B) = 1.\cot(90°) - 1.\cot(A - 90°)$

$\Rightarrow \quad -2\tan B = 0 - (-\tan A)$

$\Rightarrow \quad -2\tan B = \tan A$

$\Rightarrow \quad \tan A + 2\tan B = 0$

Hence, the result.

$\Rightarrow \quad \dfrac{1}{p_1}, \dfrac{1}{p_2}, \dfrac{1}{p_3} \in A.P$

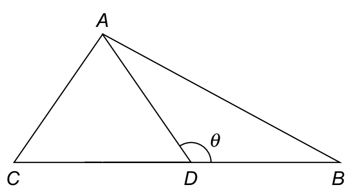
$\Rightarrow \quad p_1, p_2, p_3 \in H.P$

Thus, the altitudes are in H.P.

Ex-2. If D be the mid point of the side BC of the triangle ABC and Δ be its area, then prove that

$$\cot\theta = \frac{b^2 - c^2}{4\Delta}, \text{where } \angle ADB = \theta$$

Soln.



By $m : n$ rule, we get

$(1+1)\cot\theta = 1.\cot C - 1.\cot B$

$\Rightarrow \quad 2\cot\theta = \cot C - \cot B$

$\Rightarrow \quad 2\cot\theta = \dfrac{a^2 + b^2 - c^2}{2ab\sin C} - \dfrac{a^2 + c^2 - b^2}{2ab\sin B}$

$\Rightarrow \quad 2\cot\theta = \dfrac{a^2 + b^2 - c^2}{4\Delta} - \dfrac{a^2 + c^2 - b^2}{4\Delta}$

$$\Rightarrow \quad 2\cot\theta = \frac{2\left(b^2-c^2\right)}{4\Delta}$$

$$\Rightarrow \quad \cot\theta = \frac{\left(b^2-c^2\right)}{4\Delta}$$

Hence, the result.

EXERCISE 6

1. In a ΔABC, if $\angle A = 60°$, $b = 4$ cm. and $c = \sqrt{3}$ cm. then prove that the area of the ΔABC is 3 cm^2.
2. In a ΔABC, If $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ and the side $a = 2$, then prove that the area of the triangle ABC is $\sqrt{3}$ sq. u.
3. If in a triangle ABC, $a = 6$, $b = 3$ and $\cos(A - B) = 4/5$, then prove that $\text{ar}(\Delta ABC) = 9\, sq.u.$
4. If $\cos A + \cos B + \cos C = 3/2$, then prove that the triangle ABC is equilateral.
5. In a triangle ABC, $\Delta = \left(6+2\sqrt{3}\right) sq.u$ and $\angle B = 45°, a = 2\left(\sqrt{3}+1\right)$, then prove that the side b is 4.
6. If the angles of a triangle are $30°$ and $45°$ and the included side is $\left(\sqrt{3}+1\right)$, then prove that $\text{ar}(\Delta ABC) = \frac{1}{2}\left(\sqrt{3}+1\right) sq.u.$
7. The two adjacent sides of a cyclic quadrilateral are 2 and 3 and the angle between them is 60. If the area of the quadrilateral is $4\sqrt{3}$, then prove that the remaining two sides are 2 and 3, respectively.

7.6 RADII OF CIRCLE CONNECTED WITH A TRIANGLE

7.7 Introduction

On this topic, we shall discuss various circles connected with a triangle and the formula for their radii in terms of elements of the triangle.

7.7.1 Circumcircle of a triangle and its radius: The circle which passess through the angular pointsof a Δ is called the circumcircle. The centre of this circle is the point of intersection of perpendicular bisectors of the sides and is called the circumcentre and its radius is always denoted by R.

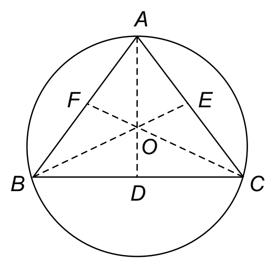
The circumcentre may lie within outside or upon one of the sides of the Δ. In a right angled triangle the circumcentre is the vertex whose right angle is formed.

7.7.2 Circum-Radius: Statement Prove that the circum-radius R of a ΔABC is given by

(i) $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$

(ii) $R = \frac{abc}{4\Delta}$

Proof:



Let ABC be a Δ and let the perpendicular bisectors of its sides BC, CA and AB intersect at O. Then O is the circumcentre such that $OA = OB = OC = R$

(i) We have $\angle BOC = 2\angle BAC = 2A$

$$\Rightarrow \quad \angle BOD = \angle COD = A$$

In ΔBOD, $\sin A = \frac{BD}{OB} = \frac{\frac{a}{2}}{R} = \frac{a}{2R}$

$$\Rightarrow \quad R = \frac{a}{2\sin A}$$

Similarly, it can be shown that

$R = \frac{b}{2\sin B}$ and $R = \frac{c}{2\sin C}$

Hence, $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$

(ii) As we know that,

$$\Delta = \frac{1}{2}\, bc \sin A.$$

$$\Rightarrow \quad \sin A = \frac{2\Delta}{bc}$$

$$\Rightarrow \quad \frac{a}{2R} = \frac{2\Delta}{bc}$$

$$\Rightarrow \quad 4\Delta R = abc$$

$\Rightarrow \quad R = \dfrac{abc}{4\Delta}$

Hence, the result.

6.7.3 Some solved examples

Ex-1. In a ΔABC, if $a = 18$ cm., $b = 24$ cm. and $c = 30$ cm., then find its circum-radius

Soln. Clearly , the triangle is right angled.
($\because\ 18^2 + 24^2 = 30^2$)
Thus, the area of the triangle

$= \dfrac{1}{2} \times 24 \times 18 = 12 \times 18$

Therefore, the circum-radius
$= R$

$= \dfrac{abc}{4\Delta}$

$= \dfrac{18 \times 24 \times 30}{4 \times 12 \times 18} = 15.$

Ex-2. In an equilateral triangle of side $2\sqrt{3}$ cm., then find the circum-radius

Soln. As we know that,

$\dfrac{a}{\sin A} = 2R$

$\Rightarrow 2R = \dfrac{2\sqrt{3}}{\sin(60°)}$

$\Rightarrow R = \dfrac{\sqrt{3}}{\frac{\sqrt{3}}{2}} = 2$

Hence, the circum-radius is 2.

Ex-3. If the length of the sides of a triangle are 3, 4 and 5 units, then find its circum-radius R.

Soln. Let $a = 3$, $b = 4$ and $c = 5$. Clearly, it is a right angled triangle

Thus , $\Delta = \dfrac{1}{2} \times 4 \times 3 \times 6$ sq.u

Hence, the circum radius R

$= \dfrac{abc}{4\Delta}$

$= \dfrac{3 \times 4 \times 5}{6} = 10.$

Ex-4. If $8R^2 = a^2 + b^2 + c^2$, then prove that the triangle is right angled

Soln. We have $8R^2 = a^2 + b^2 + c^2$
$8R^2 = (2R\sin A)^2 + (2R\sin B)^2 + (2R\sin C)^2$
$\Rightarrow \sin^2 A + \sin^2 B + \sin^2 C = 2$
$\Rightarrow 1 - \cos^2 A + 1 - \cos^2 B + \sin^2 C = 2$

Ex-5. In any ΔABC, prove that
$A \cos a + b \cos b + c \cos c = 4R \sin A \sin B \sin C$

Soln. We have, $a \cos a + b \cos b + c \cos C$

$= 2R(\sin A.\cos A + \sin B.\cos B + \sin C.\cos C)$

$= \dfrac{2R}{2}[2\sin A.\cos A + 2\sin B.\cos B + 2\sin C \cos C]$

$= R(\sin 2A + \sin 2B + \sin 2C)$

$= R(\ 4 \sin A.\ \sin B.\sin C)$

$= 4R.\sin A.\sin B.\sin C$

Ex-6. In any ΔABC, prove that
$\Delta = 2R^2 \sin A \sin B \sin C$.

Soln. We have, $\Delta = \dfrac{1}{2} \times a \times b \times \sin C$

$\Rightarrow \quad \Delta = \dfrac{1}{2} \times 2R \sin A \times 2R \sin B \times \sin C$

$\Rightarrow \quad \Delta = 2R^2 \times \sin A.\sin B.\sin C$

Ex.-7 In any triangle ABC, prove that,

$\dfrac{\sin A}{a} + \dfrac{\sin B}{b} + \dfrac{\sin C}{c} = \dfrac{3}{2R}$

Soln. We have, $\dfrac{\sin A}{a} + \dfrac{\sin B}{b} + \dfrac{\sin C}{c}$

$= \dfrac{\sin A}{2R\sin A} + \dfrac{\sin B}{2R\sin B} + \dfrac{\sin C}{2R\sin C}$

$= \dfrac{1}{2R} + \dfrac{1}{2R} + \dfrac{1}{2R} = \dfrac{3}{2R}$

Ex-8. In any triangle ABC, a, b, c are in A.P. and p_1, p_2 and p_3 are the altitudes of the given triangle, then prove that, $\dfrac{1}{p_1} + \dfrac{1}{p_2} + \dfrac{1}{p_3} \le \dfrac{3R}{\Delta}$

Soln. Let $AD = p_1$, $BE = p_2$ and $CF = p_3$.
Then.

$\Delta = \dfrac{1}{2} \times a \times p_1 = \dfrac{1}{2} \times b \times p_2 = \dfrac{1}{2} \times c \times p_3$

$\Rightarrow \quad p_1 = \dfrac{2\Delta}{a},\ p_2 = \dfrac{2\Delta}{b},\ p_3 = \dfrac{2\Delta}{c}$

Now, $\dfrac{1}{p_1} + \dfrac{1}{p_2} + \dfrac{1}{p_3}$

$= \dfrac{a + b + c}{2\Delta}$

$$= \frac{2R(\sin A + \sin B + \sin C)}{2\Delta}$$

$$\leq \frac{3R}{\Delta} \quad (\because \quad \sin A \leq 1,\ \sin B \leq 1,\ \sin C \leq 1)$$

Ex-9. If p_1, p_2 and p_3 are the altitudes of a triangle ABC from the vertices a, b and c, respectively, then prove that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}$

Soln. Here, $\Delta = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3$

$$\Rightarrow \quad p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$$

Now, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$

$$= \frac{a+b+c}{2\Delta}$$

$$= \frac{2s}{2\Delta}$$

$$= \frac{s}{\Delta}$$

$$= \frac{1}{r}$$

Ex-10. If p_1, p_2 and p_3 are the altitudes of a triangle ABC from the vertices a, b and c, respectively, then prove that $\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{1}{R}$

Soln. We have

$$\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3}$$

$$= \frac{1}{2\Delta}(a\cos A + b\cos B + c\cos C)$$

$$= \frac{1}{2\Delta}[2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C]$$

$$= \frac{R}{2\Delta}(\sin 2A + \sin 2B + \sin 2C)$$

$$= \frac{R}{2\Delta}(4\sin A\sin B\sin C)$$

$$= \frac{2R}{\Delta}(\sin A\sin B\sin C)$$

$$= \frac{2R}{\Delta} \times \left(\frac{a}{2R}\right) \times \left(\frac{b}{2R}\right) \times (\sin C)$$

$$= \frac{1}{\Delta R}\left(\frac{1}{2}ab\sin C\right)$$

$$= \frac{1}{\Delta R} \times \Delta$$

$$= \frac{1}{R}$$

EXERCISE 7

1 In a triangle ABC, the sides are 6, 8 and 10, respectively. Then find its circum radius.

2 If twice the square on the diameter of the circle is equal to sum of the squares on the sides of the inscribed triangle ABC, then prove that
$\sin^2 A + \sin^2 B + \sin^2 C = 2$

3 In an acute angled triangle ABC, prove that
$\frac{\cos A}{\sqrt{4R^2 - a^2}} = \frac{1}{R}$

4 In an acute angled triangle ABC,
prove that $\frac{\cos B}{\sqrt{4R^2 - b^2}} = \frac{1}{R}$

5 In an acute angled triangle ABC,
prove that $\frac{\cos C}{\sqrt{4R^2 - c^2}} = \frac{1}{R}$

6 If p_1, p_2 and p_3 are the altitudes of a triangle ABC from the vertices a, b and c, respectively, then
prove that $p_1 p_2 p_3 = \frac{a^2b^2c^2}{8R^3}$

7. If p_1, p_2 and p_3 are the altitudes of a triangle ABC from the vertices a, b and c, respectively, then prove that
$\frac{bp_1}{c} + \frac{cp_2}{a} + \frac{ap_3}{b} = \frac{a^2 + b^2 + c^2}{2R}$

8 O is the circum centre of R_1, R_2 and R_3 and $\Delta OBC, \Delta OCA$ are respectively the radii of the circum-centre of the triangles and ΔOAB and ΔOAB, prove that
$\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{abc}{R^3}$

9. In an acute angled triangle ABC, prove that
$\frac{a\sec A + b\sec B + c\sec C}{\tan A\tan B\tan C} = 2$

10. In any triangle ABC, prove that
$(a\cos A + b\cos B + c\cos C) = 4R\sin A\sin B\sin C$

7.8 INSCRIBED CIRCLE AND ITS RADIUS

On this topic, we shall discuss various circles connected with a triangle and the formula for the circle which can be inscribed within a Δ and touch each of the sides is called its inscribed circle or incircle.

The centre of this circle is the point of intersection of the bisecter of the angle of the triangle. The radius of this circle is always denoted by '*r*' and is equal to the length of the perpendicular from its centre to any of the sides of the Δ.

7.8.1 In-radius

Statement:

Prove that the in-radius *r* of the inscribed circle of a triangle *ABC* is given by

(i) $r = \frac{\Delta}{s}$.

(ii) $r = (s-a)\frac{A}{2}$, $r = (s-b)\frac{B}{2}$, $r = (s-c)\frac{C}{2}$;

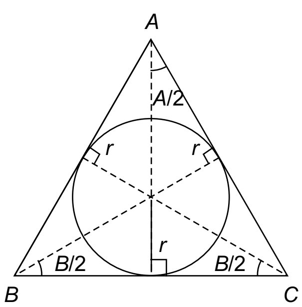
(iii) $r = \frac{a\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$;

$r = \frac{b\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{B}{2}\right)}$;

$r = \frac{c\sin\left(\frac{B}{2}\right)\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{C}{2}\right)}$

(iv) $r = 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

Proof:



Let *ABC* be a triangle such that the internal bisectors of the angles of the triangle intersect at I. Suppose the incircle touches the sides *BC*, C*A* and *AB* at *D*, *E* and F respectively then *ID*, *IE* and *IF* are perpendicular to these sides and *ID* = *IE* = *IF* = *r*.

(i) We have, *ar*(Δ*ABC*)

$$= ar(\Delta IBC) + ar(\Delta ICA) + ar(\Delta IAB)$$

$$\Rightarrow \quad \Delta = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$$

$$\Rightarrow \quad \Delta = \frac{1}{2}r(a+b+c)$$

$$= \frac{1}{2}r \times 2s$$

$$= r\,s$$

$$\Rightarrow \quad r = \frac{\Delta}{s}$$

(ii) We know that, the length of the tangents to a circle from a given point is equal.

$\Rightarrow AE = AF$, $bd = BF$ and $CD = CE$

Now, $2s = a + b + c$

$= BC + CA + AB$

$= (BD + DC) + (CE + EA) + (AF + FB)$

$= 2(BD + AE + CD)$

$= 2(BC + AE)$

$= 2(a + AE)$

$\Rightarrow s = a + AE$.

$\Rightarrow AE = s - a$

Now, in Δ*IAE*, $\tan\left(\frac{A}{2}\right) = \frac{r}{AE}$

$\Rightarrow r = AE\tan\left(\frac{A}{2}\right) = (s-a)\tan\left(\frac{A}{2}\right)$.

Similarly $r = (s-b)\tan\left(\frac{B}{2}\right)$,

$r = (s-c)\tan\left(\frac{C}{2}\right)$.

Hence, the result.

(iii) In a ΔIBD and ΔICD, we have

$\tan\left(\frac{B}{2}\right) = \frac{r}{BD}$ and $\tan\left(\frac{C}{2}\right) = \frac{r}{CD}$

$$\Rightarrow BD = \frac{r}{\tan\left(\frac{B}{2}\right)},\ CD = \frac{r}{\tan\left(\frac{C}{2}\right)}$$

Now, $a = bd + CD$

$$= \frac{r}{\tan\left(\frac{B}{2}\right)} + \frac{r}{\tan\left(\frac{C}{2}\right)}$$

$$= r\left[\frac{\cos\left(\frac{B}{2}\right)}{\sin\left(\frac{B}{2}\right)} + \frac{\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)}\right]$$

$$= r\left[\frac{\sin\left(\frac{C}{2}\right)\cos\left(\frac{B}{2}\right)+\cos\left(\frac{C}{2}\right)\sin\left(\frac{B}{2}\right)}{\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}\right]$$

$$= r\,\frac{\sin\left(\frac{C}{2}+\frac{B}{2}\right)}{\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}$$

$$= r\,\frac{\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)}{\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}$$

$$= r\,\frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}$$

$$\Rightarrow r = \text{a}\,\frac{\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$

Similarly, $r = \dfrac{b\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{B}{2}\right)}$,

$$r = \frac{c\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)}{\cos\left(\frac{C}{2}\right)}$$

(iv) We have, $r = \dfrac{a\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$

$$=2\,R\sin \text{a}\times\frac{\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$

$$= \frac{4R\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$

$$= 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

Hence, the result.

7.8.2 Some Solved Examples

Ex-1. In any ΔABC, prove that

$a\cos \text{a} + b\cos B + c\cos C = 4R\sin \text{a}\sin B\sin C$.

Soln. We have, $a\cos \text{a} + b\cos B + c\cos C$

$$= 2R(\sin A\cos A+\sin B.\cos B+\sin C.\cos C)$$

$$= R(2\sin A\cos A+2\sin B.\cos B+2\sin C.\cos C)$$

$$= R(\sin 2A+\sin 2B+\sin 2C)$$

$$= R(4\sin A.\sin B.\sin C)$$

$$= 4(R\sin A.\sin B.\sin C)$$

Ex-2. In a ΔABC, prove that

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{1}{2rR}$$

Soln. We have, $\dfrac{1}{ab}+\dfrac{1}{bc}+\dfrac{1}{ca}$

$$= \frac{a+b+c}{abc}$$

$$= \frac{2s}{4\Delta R}$$

$$= \frac{\frac{\Delta}{r}}{2\Delta R}$$

$$= \frac{1}{2rR}$$

Ex-3. In a ΔABC, prove that

$$\cos \text{a} + \cos b + \cos c = \left(1+\frac{r}{R}\right)$$

Soln. We have, $\cos \text{a} + \cos B + \cos C$

$$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C$$

$$= 2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C$$

$$= 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C$$

$$= 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\left(\frac{C}{2}\right)$$

$$= 1+2\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right)-\sin\left(\frac{C}{2}\right)\right)$$

$$= 1+2\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right)$$

$$= 1+2\sin\left(\frac{C}{2}\right)\left(2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\right)$$

$$= 1+4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

$$= \left(1+\frac{r}{R}\right)$$

Ex-4. In a ΔABC, prove that

$$\sin A + \sin B + \sin C = \frac{s}{R} = \frac{\Delta}{Rr}$$

Soln. We have, $\sin A + \sin B + \sin C$

$$= \frac{a}{2R}+\frac{b}{2R}+\frac{c}{2R}$$

$$= \frac{a+b+c}{2R}$$

$$= \frac{2s}{2R}$$

$$= \frac{s}{R}$$

$$= \frac{\frac{\Delta}{r}}{R}$$

$$= \frac{\Delta}{rR}$$

Ex-5. In any ΔABC, prove that

$a \cot a + b \cot B + c \cot C = 2(r + R)$

Soln. We have, $a \cot a + b \cot B + c \cot C$

$$= \left[2R\sin A\times\frac{\cos A}{\sin A}+2R\sin B\times\frac{\cos B}{\sin B}+2R\sin C\times\frac{\cos C}{\sin C}\right]$$

$$= 2R(\cos A+\cos B+\cos C)$$

$$= 2R\left(1+\frac{r}{R}\right)$$

$$= 2(R+r)$$

Ex-6. In any ΔABC, prove that

$$\cos^2\left(\frac{A}{2}\right)+\cos^2\left(\frac{B}{2}\right)+\cos^2\left(\frac{C}{2}\right)=2+\frac{r}{2R}$$

Soln. We have, $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}$

$$= \frac{1}{2}\left(2\cos^2\left(\frac{A}{2}\right)+2\cos^2\left(\frac{B}{2}\right)+2\cos^2\left(\frac{C}{2}\right)\right)$$

$$= \frac{1}{2}\left(1+\cos(A)+1+\cos B+1+\cos C\right)$$

$$= \frac{1}{2}\left(3+\cos(A)+\cos B+\cos C\right)$$

$$= \frac{1}{2}\left(3+\left(1+\frac{r}{R}\right)\right)$$

$$= \frac{1}{2}\left(4+\frac{r}{R}\right)$$

$$= \left(2+\frac{r}{2R}\right)$$

Ex-7. If the distances of the sides of a triangle ABC from a circum-center be x, y and z, respectively, then prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$.

Soln. Let O is the circum-centre and $OD = x$, $OE = y$, $OF = z$, respectively.
Also, $OA = R = OB = OC$
We have, $x = OD = R \cos A$

$$= \frac{a}{2\sin A}.\cos A = \frac{a}{2\tan A}$$

$$\Rightarrow \quad \tan A=\frac{a}{2x}$$

Similarly, $\tan B=\frac{b}{2y}$ & $\tan C=\frac{c}{2z}$

As we know that, in a triangle ABC,

$\tan A+\tan B+\tan C=\tan A.\tan B.\tan C$

$$\Rightarrow \quad \frac{a}{2x}+\frac{b}{2y}+\frac{c}{2z}=\frac{a}{2x}.\frac{b}{2y}.\frac{c}{2z}$$

$$\Rightarrow \quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{a.b.c}{4.x.y.z}.$$

Ex-8. If in a triangle ΔABC, O is the circum-centre and R is the circum - radius and R_1, R_2, R_3 are the circum radii of the traingles $\Delta OBC, \Delta OCA$ and ΔOAB, respectively, then prove that

$$\frac{a}{R_1}+\frac{b}{R_2}+\frac{c}{R_3}=\frac{abc}{R^3}.$$

Soln. We have,

$$R_1=\frac{OB.OC.BC}{4\Delta OBC}=\frac{R.R.a}{4\Delta_1} = \frac{R^2.a}{4\Delta_1}$$

$$\Rightarrow \quad \frac{a}{R_1}=\frac{4\Delta_1}{R^2}$$

Similarly, $\dfrac{b}{R_2} = \dfrac{4\Delta_2}{R^2}$ and $\dfrac{c}{R_3} = \dfrac{4\Delta_3}{R^2}$

Thus, $\dfrac{a}{R_1} + \dfrac{b}{R_2} + \dfrac{c}{R_3}$

$= \dfrac{4\Delta_1}{R^2} + \dfrac{4\Delta_2}{R^2} + \dfrac{4\Delta_3}{R^2}$

$= \dfrac{4(\Delta_1 + \Delta_2 + \Delta_3)}{R^2}$

$= \dfrac{4\Delta}{R^2}$

$= \dfrac{4\Delta}{R^2}$

Ex-9. If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a Δ to the opposite sides, then prove that $p_1 \cdot p_2 \cdot p_3 = \dfrac{(abc)^2}{8R^3}$.

Soln. Let $AD = p_1$, $BE = p_2$ and $CF = p_3$

Then, $\Delta = \dfrac{1}{2} a\, p_1 = \dfrac{1}{2} b\, p_2 = \dfrac{1}{2} c\, p_3$

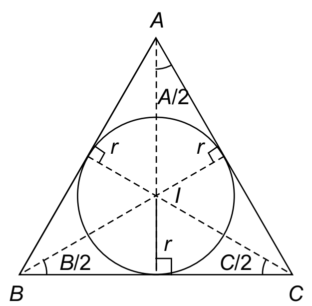
$\Rightarrow \quad p_1 = \dfrac{2\Delta}{a}, p_2 = \dfrac{2\Delta}{b}, p_3 = \dfrac{2\Delta}{c}$

We have, $p_1.p_2.p_3 = \dfrac{2\Delta}{a}.\dfrac{2\Delta}{b}.\dfrac{2\Delta}{c}$

$= \dfrac{8\Delta^3}{abc}$

$= \dfrac{8\left(\dfrac{abc}{4R}\right)^3}{abc}$

Ex-10 Find the bisectors of the angles of a ΔABC



Since IA is the internal angle bisctor of $\angle A$, so we can write

$\dfrac{AB}{AC} = \dfrac{BD}{DC}$

$\Rightarrow \quad \dfrac{BD}{DC} = \dfrac{AB}{AC} = \dfrac{c}{b}$

$\Rightarrow \quad \dfrac{DC}{BD} + 1 = \dfrac{b}{c} + 1$

$\Rightarrow \quad \dfrac{BC}{BD} = \dfrac{b+c}{c}$

$\Rightarrow \quad BD = \dfrac{ac}{b+c}$

In, ΔABD, $\dfrac{BD}{\sin\left(\dfrac{A}{2}\right)} = \dfrac{AD}{\sin B}$

$\Rightarrow \quad AD = \dfrac{BD \sin B}{\sin\left(\dfrac{A}{2}\right)}$

$= \dfrac{ac \sin B}{(b+c)\sin\left(\dfrac{A}{2}\right)} = \dfrac{2\Delta}{(b+c)\sin(A/2)}$

Similarly, $BE = \dfrac{2\Delta}{(c+a)\sin\left(\dfrac{B}{2}\right)}$

& $CF = \dfrac{2\Delta}{(a+b)\sin\left(\dfrac{C}{2}\right)}$

EXERCISE 8

1. In a ΔABC, if $a = 4$ cm., $b = 6$ cm. and $c = 8$ cm., then find its in-radius.
2. If the sides of a triangle be 18, 24, 30 cms, then find its in-radius.
3. If the sides of a triangle are 3 : 7 : 8, then find $R : r$.
4. Two sides of a triangle are 2 and $\sqrt{3}$ and the included angle is 30°, then prove that its in-radius is $\dfrac{1}{2}(\sqrt{3} - 1)$.
5. In an equilateral triangle, prove that $R = 4r$.
6. If a, b, c are sides of a triangle ABC, then prove that $\dfrac{1}{ab} + \dfrac{1}{bc} + \dfrac{1}{ca} = \dfrac{1}{2rR}$.
7. In a triangle ABC, prove that $\dfrac{a \sec A + b \sec B + c \sec C}{2 \tan A.\tan B.\tan C} = R.$
8. In a ΔABC, prove that $(b+c)\tan\left(\dfrac{A}{2}\right) + (c+a)\tan\left(\dfrac{B}{2}\right) + (a+b)\tan\left(\dfrac{C}{2}\right)$

$= 4(r+R)$

9. In a ΔABC, if $C = 90°$, prove that

$\frac{1}{2}(a+b) = R + r$

7.9 ESCRIBED CIRCLE OF A TRIANGLE AND THEIR RADII

7.9.1 In circle which touches the side BC and two sides AB and AC produced of a triangle ABC is called the escribed opposite to the angle A.

Its radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of escribed circle opposite to the angle b and c, respectively.

The centres of the escribed circle are called the ex-centres. The centre of the escribed circle opposite to the angle a is the point of intersection of the external bisectors of angles B and C.

The internal bisector of angle a also passes through the same point.

The centre is generally denoted by I_1.

7.9.2 Radii of Ex-circles

Statement:

In any ΔABC, prove that the ex-radii are given by

(i) $r_1 = \dfrac{\Delta}{s-a}$; $r_2 = \dfrac{\Delta}{s-b}$; $r_3 = \dfrac{\Delta}{s-c}$

(ii) $r_1 = s\tan\dfrac{A}{2}$; $r_2 = s\tan\dfrac{B}{2}$; $r_3 = s\tan\dfrac{C}{2}$

(iii) $r_1 = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$;

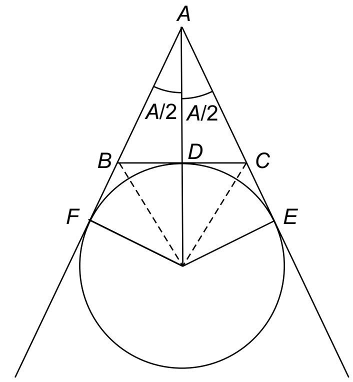
$r_2 = \dfrac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}$;

and $r_3 = \dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

(iv) $r_1 = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

$r_2 = 4R\sin\dfrac{B}{2}\cos\dfrac{A}{2}\cos\dfrac{C}{2}$

$r_3 = 4R\sin\dfrac{C}{2}\cos\dfrac{A}{2}\cos\dfrac{B}{2}$



Let I_1 be the point of intrersection of external bisectors of angles B and C of ΔABC. Suppose the circle touches the side BC at D and sides AB and AC produced at F and E, respectively.

Clearly $I_1D = I_1E = I_1F = r_1$.

(i) $ar(\Delta ABC)$

$= ar(\Delta I_1AC) + ar(\Delta I_1AB) - ar(\Delta I_1BC)$

$\Rightarrow \Delta = \frac{1}{2}r_1 b + \frac{1}{2}r_1 c - \frac{1}{2}r_1 a$

$= \frac{1}{2}r_1(b+c-a)$

$\Rightarrow \Delta = \frac{r_1}{2}(a+b+c-2a)$

$= \frac{r_1}{2}(2s-2a)$

$= r_1(s-a)$

$\Rightarrow r_1 = \dfrac{\Delta}{s-a}$

Similarly, it can be shown that

$r_2 = \dfrac{\Delta}{s-b}$ and $r_3 = \dfrac{\Delta}{s-c}$

(ii) We know that the lengths of the tangents to a circle from an external point are equal.

$\Rightarrow AE = AF$, $BD = BF$ and $CD = CE$

Now, $AE + AF = (AC + CE) + (AB + BF)$

$= (AC + CD) + (AB + BD)$

$= AC + AB + (BD + CD)$

$= AC + AB + BC$

$= b + c + a$

$\Rightarrow 2AF = 2s$

$\Rightarrow AF = s$

Now, ΔI_1AF, $\tan\left(\dfrac{A}{2}\right) = \dfrac{I_1F}{AF} = \dfrac{r_1}{AF} = \dfrac{r_1}{s}$

$\Rightarrow r_1 = s\tan\dfrac{A}{2}$

Similarly, it can be shown that

$r_2 = s \tan \frac{B}{2}$ and $r_3 = s \tan \frac{C}{2}$

Hence, the result.

(iii) In ΔI_1BD, we have

$$\tan\left(\frac{\pi - B}{2}\right) = \frac{I_1D}{BD} = \frac{r_1}{BD}$$

$$\Rightarrow \tan\left(\frac{\pi}{2} - \frac{B}{2}\right) = \frac{r_1}{BD}$$

$$\Rightarrow \cot\frac{B}{2} = \frac{r_1}{BD}$$

$$\Rightarrow BD = r_1 \tan\frac{B}{2}$$

Similarly, in ΔI_1CD, we have, $CD = r_1 \tan\frac{C}{2}$.

Now, $a = BC$

$= BD + DC$

$$= r_1 \tan\frac{B}{2} + r_1 \tan\frac{C}{2}$$

$$= r_1\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)$$

$$= r_1 \frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$= r_1 \frac{\cos\frac{A}{2}}{\cos\frac{B}{2}.\cos\frac{C}{2}}$$

$$\Rightarrow r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

Similarly, $r_2 = \frac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$

and $r_3 = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

(iv) We have, $r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

and $R = \frac{a}{2\sin A}$

$$\Rightarrow r_1 = \frac{2R\sin A\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$= R\frac{\sin\frac{A}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$= 4R \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Similarly, $r_2 = 4R \sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{C}{2}$

and $r_3 = 4R \sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$.

Hence, the result

This completes the proof of the statement.

7.9.3 Some Solved Examples

Ex-1. In a ΔABC, if $a = 18$ cm., $b = 24$ cm., and $c = 30$ cm., then find the value of r_1, r_2 and r_3.

Soln. Now $2s = a + b + c = 18 + 24 + 30 = 72$

$\Rightarrow 2s = 72$

$\Rightarrow s = 36$

We have, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{36(36-18)(36-24)(36-30)}$

$= \sqrt{36\times18\times12\times6}$

$= \sqrt{36\times9\times12\times12}$

$= 6\times3\times12$

$= 216$

Thus, $r_1 = \frac{\Delta}{s-a} = \frac{216}{36-18} = \frac{216}{18} = 12$

$r_2 = \frac{\Delta}{s-b} = \frac{216}{36-24} = \frac{216}{12} = 18$

and $r_3 = \frac{\Delta}{s-c} = \frac{216}{36-30} = \frac{216}{6} = 36$

Ex-2. In a triangle ABC if $\frac{s-c}{s-a} = \frac{b-c}{a-b}$, then prove that a, b, c are in A.P.

Soln. Given $\frac{s-c}{s-a} = \frac{b-c}{a-b}$

$\Rightarrow \quad \dfrac{2s-2c}{2s-2a} = \dfrac{b-c}{a-b}$

$\Rightarrow \quad \dfrac{a+b-c}{b+c-a} = \dfrac{b-c}{a-b}$

$\Rightarrow \quad \dfrac{a+b-c}{b-c} = \dfrac{b+c-a}{a-b}$

$\Rightarrow \quad \dfrac{a}{b-c} + 1 = \dfrac{c}{a-b} - 1$

$\Rightarrow \quad \dfrac{c}{a-b} + \dfrac{a}{b-c} = 2$

$\Rightarrow \quad \dfrac{c(b-c)+a(a-b)}{(a-b)(b-c)} = 2$

$\Rightarrow \quad \dfrac{bc-c^2+a^2-ab}{(a-b)(b-c)} = 2$

$\Rightarrow \quad bc-c^2+a^2-ab = 2(ab-ac-b^2+bc)$

$\Rightarrow \quad 2b^2-bc-c^2+a^2-3ab+2ac = 0$

Ex-3. If r_1, r_2, r_3 are in H.P., then prove that a, b, c are in A,P

Soln. Given r_1, r_2, r_3 are in H.P.

$\Rightarrow \quad r_2 = \dfrac{2r_1r_3}{r_1+r_3}$

$\Rightarrow \quad \dfrac{\Delta}{s-b} = \dfrac{2.\dfrac{\Delta}{s-a}.\dfrac{\Delta}{s-c}}{\dfrac{\Delta}{s-a}+\dfrac{\Delta}{s-c}}$

$\Rightarrow \quad \dfrac{1}{s-b} = \dfrac{2.\dfrac{1}{s-a}.\dfrac{1}{s-c}}{\dfrac{1}{s-a}+\dfrac{1}{s-c}}$

$\Rightarrow \quad \dfrac{1}{s-b}\left(\dfrac{1}{s-a}+\dfrac{1}{s-c}\right) = 2.\dfrac{1}{s-a}.\dfrac{1}{s-c}$

$\Rightarrow \quad \dfrac{1}{s-b}\left(\dfrac{s-c+s-a}{(s-a)(s-c)}\right) = \dfrac{2}{(s-a)(s-c)}$

$\Rightarrow \quad (2s-a-c) = 2(s-b)$

$\Rightarrow \quad a+c = 2b$

$\Rightarrow \quad a,b,c \in A.P$

Ex-4. In a triangle ABC, if a, b, c are in A.P. as well as in G.P. then prove that the value of $\left(\dfrac{r_1}{r_2} - \dfrac{r_2}{r_3} + 10\right)$ is 10.

Soln. Since a, b, c are in A.P. as well as in G.P., so

$a = b = c$

Now, $r_1 = \dfrac{\Delta}{s-a} = r_2 = r_3$

Thus, $\left(\dfrac{r_1}{r_2} - \dfrac{r_2}{r_3} + 10\right)$

$= 1 - 1 + 10 = 10$

Ex-5. If $r_1 < r_2 < r_3$ and the ex-radii of a right angled triangle and $r_1 = 1, r_2 = 2$, then prove that $r_3 = \dfrac{3+\sqrt{17}}{2}$.

Soln. We have, $r_1 = \dfrac{\Delta}{s-a} = 1$, $r_2 = \dfrac{\Delta}{s-b} = 2$

and $r_3 = \dfrac{\Delta}{s-c}$

$\Rightarrow \quad s-a = \Delta$, $s-b = \dfrac{\Delta}{2}$

and $s-c = \dfrac{\Delta}{r_3}$

$\Rightarrow \quad c = \Delta\left(1+\dfrac{1}{2}\right)$, $a = \Delta\left(\dfrac{1}{2}+\dfrac{1}{r_3}\right)$,

$b = \Delta\left(1+\dfrac{1}{r_3}\right)$

Since triangle is right angled, so

$a^2+b^2 = c^2$

$\Rightarrow \quad \Delta^2\left(\dfrac{3}{2}\right)^2 = \dfrac{\Delta^2(r_3+2)^2}{4(r_3)^2} + \Delta^2\left(\dfrac{r_3+1}{r_3}\right)^2$

$\Rightarrow \quad \left(\dfrac{3}{2}\right)^2 = \dfrac{(r_3+2)^2}{4(r_3)^2} + \left(\dfrac{r_3+1}{r_3}\right)^2$

$\Rightarrow \quad 9r_3^2 = (r_3+2)^2 + 4(r_3+1)^2$

$\Rightarrow \quad 4r_3^2 - 12r_3 - 8 = 0$

$\Rightarrow \quad r_3^2 - 3r_3 - 2 = 0$

$\Rightarrow \quad r_3 = \dfrac{3 \pm \sqrt{17}}{2} = \dfrac{3+\sqrt{17}}{2}$.

as r_3 is positive.

Ex-6. Two sides of a triangle are the roots of $x^2 - 5x + 3 = 0$. If the angle between the sides is $\dfrac{\pi}{3}$. then prove that the value of $r.R$ is 2/3.

Soln. Let a, b be the sides of a triangle.
Then a + b = 5 and ab = 3

Now, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

$\Rightarrow \quad \cos\left(\dfrac{\pi}{3}\right) = \dfrac{19-c^2}{6}$

$\Rightarrow \quad \dfrac{1}{2} = \dfrac{19-c^2}{6}$

$\Rightarrow \quad 19 - c^2 = 3$

$\Rightarrow \quad c = 4$

Thus, $r.R = \dfrac{\Delta}{s} \times \dfrac{abc}{4\Delta}$

$= \dfrac{abc}{4s} = \dfrac{abc}{2(a+b+c)} = \dfrac{3.4}{2(5+4)} = \dfrac{12}{18} = \dfrac{2}{3}$

Ex-7. In an isosceles triangle of which one angle is 120°, circle of radius $\sqrt{3}$ is inscribed, then prove that the area of the triangle is $\left(12+7\sqrt{3}\right)$ sq. u.

Soln. By sine rule, $\dfrac{a}{\sin(120°)} = \dfrac{b}{\sin(30°)}$

$\Rightarrow \quad \dfrac{a}{\sqrt{3}/2} = \dfrac{b}{1/2}$

$\Rightarrow \quad a = b\sqrt{3}$

Also, from the above figure, $r = \sqrt{3}$

$\Rightarrow \quad \dfrac{\sqrt{3}}{a/2} = \tan(15°)$

$\Rightarrow \quad \dfrac{2\sqrt{3}}{a} = \left(2-\sqrt{3}\right)$

$\Rightarrow \quad a = \dfrac{2\sqrt{3}}{\left(2-\sqrt{3}\right)}$

Now, $b = \dfrac{a}{\sqrt{3}} = \dfrac{2\sqrt{3}}{\left(2-\sqrt{3}\right)} \times \dfrac{1}{\sqrt{3}} = \dfrac{2}{\left(2-\sqrt{3}\right)}$

Thus, the required area

$= \dfrac{1}{2} \times ab \times \sin(30°)$

$= \dfrac{1}{2} \times \dfrac{2\sqrt{3}}{\left(2-\sqrt{3}\right)} \times \dfrac{2}{\left(2-\sqrt{3}\right)} \times \dfrac{1}{2}$

$= \sqrt{3} \times \left(2+\sqrt{3}\right)^2$

$= \sqrt{3} \times \left(7+4\sqrt{3}\right)$

$= \left(12+7\sqrt{3}\right)$ sq. u.

Ex-9. If in a triangle $r = r_1 - r_2 - r_3$, then prove that the triangle is right angled.

Soln Given $r = r_1 - r_2 - r_3$

$\Rightarrow \quad r_1 - r = r_2 + r_3$

$\Rightarrow \quad \dfrac{\Delta}{s-a} - \dfrac{\Delta}{s} = \dfrac{\Delta}{s-b} + \dfrac{\Delta}{s-c}$

$\Rightarrow \quad \dfrac{1}{s-a} - \dfrac{1}{s} = \dfrac{1}{s-b} + \dfrac{1}{s-c}$

$\Rightarrow \quad \dfrac{(s-a-s)}{(s-a)} = \dfrac{(s-c+s-b)}{(s-b)(s-c)}$

$\Rightarrow \quad \dfrac{a}{(s-a)} = \dfrac{a}{(s-b)(s-c)}$

$\Rightarrow \quad \dfrac{(s-b)(s-c)}{(s-a)} = 1$

$\Rightarrow \quad \tan^2\left(\dfrac{A}{2}\right) = 1$

$\Rightarrow \quad \tan\left(\dfrac{A}{2}\right) = 1$

$\Rightarrow \quad \tan\left(\dfrac{A}{2}\right) = \tan\left(\dfrac{\pi}{4}\right)$

$\Rightarrow \quad A = \dfrac{\pi}{2}$

Thus, the triangle is right angled.

Ex-10. In a ΔABC, prove that $r.r_1.r_2.r_3 = \Delta^2$.

Soln. We have, $r.r_1.r_2.r_3$

$= \dfrac{\Delta}{s}.\dfrac{\Delta}{(s-a)}.\dfrac{\Delta}{(s-b)}.\dfrac{\Delta}{(s-c)}$

$= \dfrac{\Delta^4}{s(s-a)(s-b)(s-c)}$

$= \dfrac{\Delta^4}{\Delta^2}$

$= \Delta^2$

Ex-14. Prove that $\dfrac{(r_1+r_2)}{1+\cos C} = \dfrac{(r_2+r_3)}{1+\cos A} = \dfrac{(r_3+r_1)}{1+\cos B}$.

Soln. We have, $\dfrac{r_1+r_2}{1+\cos C}$

$$= \frac{\frac{\Delta}{s-a}+\frac{\Delta}{s-b}}{2\cos^2\left(\frac{C}{2}\right)}$$

$$= \frac{\frac{\Delta(s-a+s-b)}{(s-a)(s-b)}}{2\cos^2\left(\frac{C}{2}\right)}$$

$$= \frac{\Delta(2s-a-b)}{(s-a)(s-b)\times 2\times\frac{s(s-c)}{ab}}$$

$$= \frac{\Delta\times abc}{s(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta\times abc}{\Delta^2} = \frac{abc}{\Delta}$$

Similarly, we can easily proved that,

$$\frac{(r_2+r_3)}{1+\cos A} = \frac{abc}{\Delta} \ \& \ \frac{(r_3+r_1)}{1+\cos B} = \frac{abc}{\Delta}$$

Thus, $\frac{(r_2+r_3)}{1+\cos A} = \frac{abc}{\Delta}$ & $\frac{(r_3+r_1)}{1+\cos B} = \frac{abc}{\Delta}$

Ex-15 Prove that $\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)$

$$= \frac{16R}{r^2(a+b+c)^2}$$

Soln. We have, $\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)$

$$= \left(\frac{s}{\Delta}-\frac{(s-a)}{\Delta}\right)\left(\frac{s}{\Delta}-\frac{(s-b)}{\Delta}\right)\left(\frac{s}{\Delta}-\frac{(s-c)}{\Delta}\right)$$

$$= \frac{1}{\Delta^3}(s-s+a)(s-s+b)(s-s+c)$$

$$= \frac{1}{\Delta^3}\times abc$$

$$= 16\times\frac{abc}{4\Delta}\times\frac{s^2}{\Delta^2(2s)^2}.$$

$$= \frac{16R}{r^2(a+b+c)^2}$$

Ex-16. Prove that $\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)\left(\frac{1}{r_3}+\frac{1}{r_1}\right) = \frac{64R^3}{(abc)^2}$

Soln. We have, $\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)\left(\frac{1}{r_3}+\frac{1}{r_1}\right)$

$$= \left(\frac{s-a}{\Delta}+\frac{s-b}{\Delta}\right)\left(\frac{s-b}{\Delta}+\frac{s-c}{\Delta}\right)\left(\frac{s-c}{\Delta}+\frac{s-a}{\Delta}\right)$$

$$= \frac{1}{\Delta^3}\times abc$$

$$= \frac{1}{\left(\frac{abc}{4R}\right)^3}\times abc$$

$$= \frac{64R^3}{(abc)^2}$$

EXERCISE 9

1. In a triangle ΔABC, prove that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$, where r is inradius and r_1, r_2, r_3 are exradii.
2. In a triangle ΔABC, prove that $r_1+r_2+r_3-r=4R$
3. In a triangle ΔABC, prove that $r_1r_2+r_2r_3+r_3r_1=s^2$
4. In a triangle ΔABC, prove that $r_1+r_2-r_3+r=4R\cos C$
5. In a triangle if $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_2}{r_3}\right)=2$, prove that the triangle is right angled.
6. In a triangle ΔABC, prove that $\frac{1}{r_1^2}+\frac{1}{r_1^2}+\frac{1}{r_1^2}+\frac{1}{r_1^2}=\frac{a^2+b^2+c^2}{\Delta^2}$
7. In a triangle ΔABC, prove that $(r_1-r)(r_2-r)(r_3-r)=4r^2R$
8. In a triangle ΔABC, prove that $r^2+r_1^2+r_2^2+r_3^2=16R^2-(a^2+b^2+c^2)$
9. In a triangle ΔABC, prove that $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$
10. In a triangle ΔABC, prove that $\frac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{(r_1r_2+r_2r_3+r_3r_1)}=4R$

7.10 REGULAR POLYGON

If a polygon has all its sides equal in length and also all its angles equal then the polygon is called a regular polygon.

The circle inscribed in the regular polygon and touching all the sides of the regular polygon is called inscribed circle.

The circle which passes through all the vertices of the regular polygon is called its circumscribed circle.

If the polygon has n-sides, then the sum of the interrior angles is $(n-2)\times\pi$ and each angle is $\dfrac{(n-2)\times\pi}{n}$.

7.10.1 Statement

In a regular polygon of $A_1A_2A_3....A_n$ of n-sides of each length a is given by

(i) $R = \dfrac{a}{2}\text{cosec}\left(\dfrac{a}{n}\right)$

where R = circum-radius.

(ii) $r = \dfrac{a}{2}\cot\left(\dfrac{a}{n}\right)$, where r = in-radius.

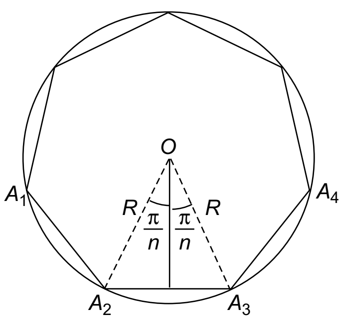
(iii) $\Delta = \dfrac{a^2 n}{4}\cot\left(\dfrac{\pi}{n}\right)$

where Δ = area of the regular polygon

(iv) $\Delta = \dfrac{nR^2}{2}\sin\left(\dfrac{2\pi}{n}\right)$

(v) $\Delta = nr^2\tan\left(\dfrac{2\pi}{n}\right)$

Proof:



(i) Let A_1A_2, A_2A_3 and A_3A_4 be three consecu tive sides of one regular polygon.
$A_1A_2 = A_2A_3 = A_3A_4 = a$.
Let the bisectors OA_2 and OA_3 of the angles $\angle A_1A_2A_3$ and $\angle A_2A_3A_4$ respectively meet at O.
Clearly, O is the centre of both circumscribed circle and inscribed circle.
Let R and r be the radii of circumcribed and inscribed circle, then clearly
$OA_2 = OA_3 = R$ and
$OM = r$, where OM is perpendicular from O on A_2A_3.

$\therefore\ A_2M = \dfrac{1}{2}a = A_3M.$

We know that, the whole angle subtended at the centre 'O' by the regular polygon is 2π radian

$\therefore\ \angle A_2\,O\,A_3 = \dfrac{2\pi}{n}$

$\Rightarrow \angle A_2\,O\,M = \angle A_3\,OM = \dfrac{\pi}{n}.$

From ΔA_2OM, we have,

$$\sin\left(\frac{\pi}{n}\right) = \frac{A_2M}{OA_2} = \frac{\frac{1}{2}a}{R} = \frac{a}{2R}$$

$$\Rightarrow R = \frac{a}{2\sin\left(\frac{\pi}{n}\right)} = \frac{a}{2}\ \text{cosec}\left(\frac{\pi}{n}\right)$$

$$\text{again, } \tan\left(\frac{\pi}{n}\right) = \frac{A_2M}{OM} = \frac{a}{2r}$$

$$\Rightarrow r = \frac{a}{2}\cot\left(\frac{a}{n}\right)$$

ii) Let the area of the regular polygon be denoted by Δ.
Join O to the n-vertices $A_1, A_2,....., A_n$ of the regular polygon.
The regular polygon is divided into n-isosceles triangle of equal area.
Therefore, the area Δ of regular polygon is given by
Δ = n-times the area of isosceles $\Delta A_2\,OA_3$

$$\Rightarrow \Delta = n\times\frac{1}{2}\times A_2\,A_3\times OM$$

$$\Rightarrow \Delta = n\times\frac{1}{2}\times a\times\frac{1}{2}\ OM$$

$$\Rightarrow \Delta = n\times\frac{1}{2}\times a\times\frac{a}{2}\cot\left(\frac{\pi}{n}\right)$$

$$\Rightarrow \Delta = \frac{na^2}{4}\cot\left(\frac{\pi}{n}\right)$$

(iii) In ΔOA_2M, we have,

$$\cos\left(\frac{\pi}{n}\right) = \frac{OM}{OA_2}$$

$$\Rightarrow OM = R\cos\left(\frac{\pi}{n}\right)$$

$$\text{Also, } \sin\left(\frac{\pi}{n}\right) = \frac{A_2M}{OA_2} = \frac{\frac{1}{2}a}{R}$$

$$\Rightarrow a = 2R\sin\left(\frac{\pi}{n}\right)$$

$\Rightarrow \Delta = \dfrac{na}{2} \times OM$

$= \dfrac{n}{2} \times 2R \sin\left(\dfrac{\pi}{n}\right) \times R\cos\left(\dfrac{\pi}{n}\right)$

$= \dfrac{nR^2}{2} \sin\left(\dfrac{2\pi}{n}\right)$

(iv) In ΔOA_2M, we have $OM = r$

$\therefore \tan\left(\dfrac{\pi}{n}\right) = \dfrac{A_2M}{OM}$

$\Rightarrow \tan\left(\dfrac{\pi}{n}\right) = \dfrac{\frac{a}{2}}{r} = \dfrac{a}{2r}$

$\Rightarrow a = 2r\tan\left(\dfrac{\pi}{n}\right)$

$\Rightarrow \Delta = \dfrac{na}{2} \times OM$

$= \dfrac{n}{2} \times 2r\tan\left(\dfrac{n}{2}\right) \times r$

$= nr^2 \tan\left(\dfrac{n}{2}\right)$

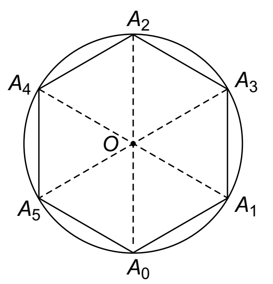
Hence, the result.

This completes the proof of the statement.

7.10.2 Some Solved Examples

Ex-1. If A_0, A_1, ...A_5 be the consecutive vertices of a regular hexagon inscribed in a unit circle, then find the product of length of A_0A_1, A_0A_2 and A_0A_4.

Soln.



Here, $OA_0 = OA_1 = OA_2 = \text{...........} = OA_5 = 1$.

and

$\angle A_0OA_1 = \dfrac{2\pi}{6} = \angle A_1OA_2 = \text{...........} = \angle A_4OA_5$

Now, $\cos\left(\dfrac{\pi}{3}\right) = \dfrac{OA_0^2 + OA_1^2 - A_1A_2}{2OA_0.OA_1}$

$\Rightarrow \quad \dfrac{1}{2} = \dfrac{1+1-A_1A_2}{2.1.1}$

$\Rightarrow \quad \dfrac{1}{2} = \dfrac{2-A_0A_1^2}{2}$

$\Rightarrow \quad A_0A_1^2 = 1$

$\Rightarrow \quad A_0A_1 = 1$

Also, $\cos\left(\dfrac{2\pi}{3}\right) = \dfrac{OA_0^2 + OA_1^2 - A_0A_2^2}{2.OA_0.OA_1}$

$\Rightarrow \quad -\dfrac{1}{2} = \dfrac{1+1-A_0A_2^2}{2.1.1}$

$\Rightarrow \quad -\dfrac{1}{2} = \dfrac{2-A_0A_2^2}{2}$

$\Rightarrow \quad A_0A_2 = \sqrt{3}$

Again, $\cos\left(\dfrac{4\pi}{3}\right) = \dfrac{OA_0^2 + OA_4^2 - A_0A_4^2}{2.OA_0.OA_4}$

$= \dfrac{1+1-A_0A_4^2}{2.1.1}$

$= \dfrac{2-A_0A_4^2}{2}$

$\Rightarrow \quad -\dfrac{1}{2} = \dfrac{2-A_0A_4^2}{2}$

$\Rightarrow \quad A_0A_4 = \sqrt{3}$

Hence, the value of

$A_0A_1.A_0A_2.A_0A_4 = 1.\sqrt{3}.\sqrt{3} = 3$.

Ex-2. If the Area of circle is A_1 and area of regular pentagon inscribed in the circle is A_2.
Find the ratio of area of two.

Soln. Let R be the radius of the circle .

Then, $A_1 = \pi R^2$

$\Delta = \dfrac{nR^2}{2}\sin\left(\dfrac{2\pi}{n}\right)$

and $A_2 = \dfrac{5.R^2}{2}\sin\left(\dfrac{360°}{5}\right)$

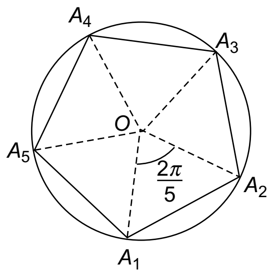
$= \dfrac{5}{2}R^2\sin(72°) = \dfrac{5R^2}{2} \times \cos(18°)$

Now, $\dfrac{A_1}{A_2}$

$= \dfrac{\pi R^2}{\frac{5}{2}R^2\cos(18°)} = \dfrac{2\pi}{5} \times \sec(18°)$

Ex-3. Let A_1, A_2, A_3, A_4 and A_5 be the vertices of a regular pentagon inscribed in a unit circle taken in order. Show that $A_1A_2 \times A_1A_3 = \sqrt{5}$.

Soln.



Here, $OA_1 = OA_2 = OA_3 = OA_4 = OA_5 = 1$

and

$$\angle A_1OA_2 = \frac{2\pi}{5} = \angle A_2OA_3 = ... = \angle A_4OA_5$$

Now, $\cos\left(\frac{2\pi}{5}\right) = \frac{OA_1^2 + OA_2^2 - A_1A_2^2}{2.OA_1.OA_2}$

$$\Rightarrow \quad \sin(18°) = \frac{1+1-A_1A_2^2}{2.1.1}$$

$$\Rightarrow \quad \frac{\sqrt{5}-1}{4} = \frac{2-A_1A_2^2}{2}$$

$$\Rightarrow \quad A_1A_2^2 = 2 - \frac{\sqrt{5}-1}{2} = \frac{5-\sqrt{5}}{2}$$

$$\Rightarrow \quad A_1A_2 = \sqrt{\frac{5-\sqrt{5}}{2}}$$

Similarly, $A_1A_3 = \sqrt{\frac{5+\sqrt{5}}{2}}$.

Thus, $A_1A_2 \times A_1A_3$

$$= \sqrt{\left(\frac{5-\sqrt{5}}{2}\right)\times\left(\frac{5+\sqrt{5}}{2}\right)} = \sqrt{\frac{25-5}{4}} = \sqrt{\frac{20}{4}} = \sqrt{5}$$

Ex-4. The sides of a regular do-decagon is 2 ft . Find the radius of the circumscribed circle.

Soln. $R = \frac{a}{2}\ \text{cosec}\left(\frac{\pi}{n}\right)$, As we know that, the circum-radius of n sided regular polygon

$= \frac{a}{2}\text{cosec}\left(\frac{\pi}{n}\right)$, where a = side and n = number of sides

$$= 6 \times \text{cosec}\left(\frac{\pi}{12}\right)$$

$$= 6 \times \text{cosec}(15°)$$

$$= \frac{6\times 2\sqrt{2}}{\sqrt{3}-1}$$

$$= 6\sqrt{2}\left(\sqrt{3}+1\right)$$

Ex-5. A regular pentagon and a regular decagon have the same perimiter. Find the ratio of its area.

Soln. Let the perimeter of the pentagon and the decagon be 10 x.

Then each side of the pentagon is $2x$ and the decagon is x.

Let A_1 = the area of the pentagon

$$= 5x^2 \cot\left(\frac{\pi}{5}\right)$$

and A_2 = the area of the decagon

$$= \frac{5}{2}x^2 \cot\left(\frac{\pi}{10}\right)$$

Now, $\frac{A_1}{A_2} = \frac{5x^2\cot\left(\frac{\pi}{5}\right)}{\frac{5}{2}x^2\cot\left(\frac{\pi}{10}\right)} = \frac{2\cot\left(\frac{\pi}{5}\right)}{\cot\left(\frac{\pi}{10}\right)}$

$$= \frac{2\cot(36°)}{\cot(18°)} = \frac{2\cos(36°)\sin(18°)}{\sin(36°)\cos(18°)}$$

$$= \frac{2\cos(36°)\sin(18°)}{2\sin(18°)\cos^2(18°)}$$

$$= \frac{2\cos(36°)}{(1+\cos(36°))}$$

$$= \frac{2\left(\frac{\sqrt{5}+1}{4}\right)}{\left(1+\frac{\sqrt{5}+1}{4}\right)}$$

$$= \frac{2\left(\sqrt{5}+1\right)}{\left(\sqrt{5}+5\right)}$$

$$= \frac{2\left(\sqrt{5}+1\right)}{\sqrt{5}\left(\sqrt{5}+1\right)}$$

$$= \frac{2}{\sqrt{5}}$$

Ex-6. If $2a$ be the sides of a regular polygon of n-sides. R and r be the circum radius and inradius, then prove that $r + R = a\cot\left(\frac{\pi}{2n}\right)$.

Soln. We have, $R = \frac{2a}{2}\text{cosec}\left(\frac{\pi}{n}\right) = a\,\text{cosec}\left(\frac{\pi}{n}\right)$

and $r = \frac{2a}{2}\cot\left(\frac{\pi}{n}\right) = a\cot\left(\frac{\pi}{n}\right)$

Now, $r + R$

$$= a\cot\left(\frac{\pi}{n}\right) + a\,\text{cosec}\left(\frac{\pi}{n}\right)$$

$$= a\left(\cot\left(\frac{\pi}{n}\right) + \text{cosec}\left(\frac{\pi}{n}\right)\right)$$

$$= a\left(\frac{1+\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}\right)$$

$$= a\left(\frac{2\cos^2\left(\frac{\pi}{2n}\right)}{2\sin\left(\frac{\pi}{2n}\right)\cos\left(\frac{\pi}{2n}\right)}\right)$$

$$= a\cot\left(\frac{\pi}{2n}\right)$$

Ex-7. A regular pentagon and a regular decagon have the same area, then find the ratio of their perimeter.

Soln. Let A_1 be the area of the regular pentagon A_2 be the area of the regular decagon.
Therefore, $A_1 = A_2$

$$\Rightarrow \quad \frac{5a^2}{4}\cot\left(\frac{\pi}{5}\right) = \frac{6b^2}{4}\cot\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad 5a^2\cot\left(\frac{\pi}{5}\right) = 6b^2\cot\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad 5a^2\cot(36°) = 6b^2\cot(30°)$$

$$\Rightarrow \quad 5a^2\cot(36°) = 6\sqrt{3}\,b^2$$

$$\Rightarrow \quad \frac{a^2}{6\sqrt{3}} = \frac{b^2}{5\cot(36°)} = \lambda$$

Hence, the ratio of their perimeters

$$= \frac{5a}{6b}$$

$$= \frac{5}{6}\times\frac{\sqrt{\lambda 6\sqrt{3}}}{\sqrt{5\lambda\cot(36°)}} = \sqrt{\frac{5\sqrt{3}}{6}\tan(36°)}$$

Ex-8. If the number of sides of two regular polygon having the same perimeter be n and $2n$, respectively, prove that their areas are in the ratio
$2\cos(\pi/n) : (1+\cos(\pi/n))$.

Soln. Let the perimeter of the two polygons are $n\,x$ and $2\,n\,x$, respectively .
Then each side of the polygons are $2x$ and x.
Let A_1 = the area of the polygon of n sides

$$= n x^2 \cot\left(\frac{\pi}{n}\right)$$

and A_2 = the area of the decagon

$$= \frac{5}{2}n^2\cot\left(\frac{\pi}{2n}\right)$$

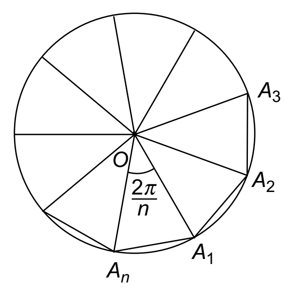
Thus, $\frac{A_1}{A_2} = \frac{5n^2\cot\left(\frac{\pi}{n}\right)}{\frac{5}{2}n^2\cot\left(\frac{\pi}{2n}\right)} = \frac{2\cot\left(\frac{\pi}{n}\right)}{\cot\left(\frac{\pi}{2n}\right)}$

$$= \frac{2\cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{2n}\right)}{2\sin\left(\frac{\pi}{2n}\right)\cos^2\left(\frac{\pi}{2n}\right)}$$

$$= \frac{2\cos\left(\frac{\pi}{n}\right)}{1+\cos\left(\frac{\pi}{n}\right)}$$

Ex-9. Let A_1, A_2, A_3, ..., A_n be the vertices of an n-sided regular polygon such that
$\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, then find the value of n.

Soln. Let O be the centre and A_1A_2.......A_n be the regular polygon of n-sides.



Let $OA_1 = OA_2 = \ldots\ldots\ldots\ldots\ldots\ldots = OA_n = r$

and $\angle A_1OA_2 = \angle A_2OA_3$

$$= \ldots\ldots\ldots = \angle A_nOA_1 = \frac{2\pi}{n}$$

From the triangle OA_1A_2,

$$\cos\left(\frac{2\pi}{n}\right) = \frac{OA_1^2 + OA_2^2 - A_1A_2^2}{2.OA_1.OA_2}$$

$$= \frac{r^2 + r^2 - A_1A_2^2}{2.r.r}$$

$$\Rightarrow \quad A_1A_2^2 = 2r^2 - 2r^2 \cos\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow \quad A_1A_2^2 = 2r^2\left(1 - \cos\left(\frac{2\pi}{n}\right)\right)$$

$$\Rightarrow \quad A_1A_2^2 = 2r^2.2\sin^2\left(\frac{2\pi}{n}\right)$$

$$= 4r^2.\sin^2\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow \quad A_1A_2 = 2r.\sin\left(\frac{2\pi}{n}\right).$$

Similarly, $A_1A_3 = 2r.\sin\left(\frac{4\pi}{n}\right)$

& $A_1A_4 = 2r.\sin\left(\frac{6\pi}{n}\right)$

Given, $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$

$$\Rightarrow \quad \frac{1}{2r.\sin\left(\frac{2\pi}{n}\right)} = \frac{1}{2r.\sin\left(\frac{4\pi}{n}\right)} + \frac{1}{2r.\sin\left(\frac{6\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{1}{\sin\left(\frac{\pi}{n}\right)} - \frac{1}{\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{\sin\left(\frac{3\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{2\cos\left(\frac{2\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \quad 2\cos\left(\frac{2\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \quad \sin\left(\frac{4\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \quad \sin\left(\frac{4\pi}{n}\right) = \sin\left(\pi - \frac{3\pi}{n}\right)$$

$$\Rightarrow \quad \left(\frac{4\pi}{n}\right) = \left(\pi - \frac{3\pi}{n}\right)$$

$$\Rightarrow \quad \left(\frac{7\pi}{n}\right) = \pi$$

$$\Rightarrow \quad n = 7$$

Ex-10. If A, A_1, A_2, A_3 are the areas of incircle and the ex-circles of a triangle, then prove that

$$\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} = \frac{1}{\sqrt{A}}.$$

Soln. Let r be the radius of the in-circle and r_1, r_2 and r_3 are the ex-radii of the given triangle

Then $\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$

$$= \frac{1}{\sqrt{\pi r_1^2}} + \frac{1}{\sqrt{\pi r_1^2}} + \frac{1}{\sqrt{\pi r_1^2}}$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)$$

$$= \frac{1}{\sqrt{\pi}} \times \frac{1}{r}$$

$$= \frac{1}{\sqrt{\pi r^2}}$$

$$= \frac{1}{\sqrt{A}}$$

Hence, the result.

EXERCISE 12

1. If R is the radius of the circum-scribing circle of a regular polygon of n-sides with side length 'a', then R is

(a) $\frac{a}{2}\sin\left(\frac{\pi}{n}\right)$ (b) $\frac{a}{2}\sin\left(\frac{\pi}{2n}\right)$

(c) $\frac{a}{2}\operatorname{cosec}\left(\frac{\pi}{2n}\right)$ (d) $\frac{a}{2}\operatorname{cosec}\left(\frac{\pi}{n}\right)$

2 If a is the area and s be the semi perimeter of a triangle, then

(a) $A \le \frac{s^2}{3\sqrt{3}}$ (b) $A \le \frac{s^2}{2}$

(c) $A > \frac{s^2}{3}$ (d) $A > \frac{s^2}{4}$

3 If A_1, A_2, A_3 are the areas of an inscribed polygon of $2n$ sides and n-sides and circum-scribed polygon of n-sides, then A_2, A_1, A_3 are in

(a) A.P. (b) G.P.
(c) H.P. (d) A.G.P.

4 If the perimeter of the circle and the perimeter of the polygon of n-sides are same, then the ratio of the area of the circle and the area of the polygon of n-sides is

(a) $\tan\left(\frac{\pi}{n}\right):\frac{\pi}{n}$ (b) $\cos\left(\frac{\pi}{n}\right):\frac{\pi}{n}$

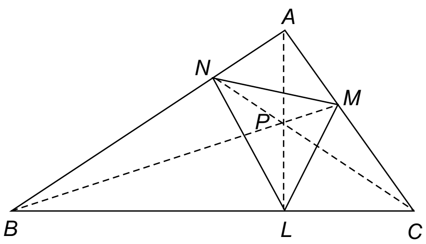
(c) $\sin\left(\frac{\pi}{n}\right):\frac{\pi}{n}$ (d) $\cot\left(\frac{\pi}{n}\right):\frac{\pi}{n}$

5. The sum of the radii of the circle, which are respectively inscribed in and circum-scribed about a regular polygon of n-sides, is

(a) $\frac{a}{2}\cot\left(\frac{\pi}{n}\right)$ (b) $\frac{a}{2}\cot\left(\frac{\pi}{2n}\right)$

(c) $\frac{a}{2}\tan\left(\frac{\pi}{2n}\right)$ (d) $\frac{a}{2}\tan\left(\frac{\pi}{n}\right)$

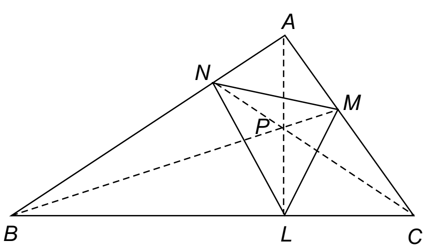
7.11 ORTHOCENTRE AND PEDAL TRIANGLE OF ANY TRIANGLE



Let ABC be any triangle and let AL, BM and CN be the perpendiculars from a, b and c upon the opposite sides of the triangle. They meet at a point P. This point P is called the orthocentre of the triangle.

The triangle LMN, which is formed by joining the feet of the perpendiculars, is called the pedal triangle.

7.11.1 Distances of the Orthocentre from the angular points of the pedal triangle



We have, $PL = LB \tan(PBL)$

$= LB\ \tan(90° - C)$

$= AB \cos B \cot C$

$= \frac{c}{\sin C} \times \cos B \cos C$

$= 2R\cos B \cos C$.

Similarly, $PM = 2R \cos A \cos C$,

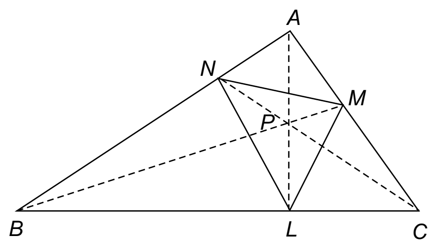
$PN = 2R \cos A \cos B$

Again, $AP = AM \sec(LAC) = c \cos A \operatorname{cosec} C$

$= \frac{c}{\sin C} \times \cos A = 2R \cos A$.

Similarly, $BP = 2R \cos B$ and $CP = 2R\cos C$.

7.11.2 The sides and the angles of the pedal triangle



Since the angles PLC, PMC are right angles, so the points P, L, C and M lie on a circle.
Thus, $\angle PLM = \angle PCM = 90° - A$

Similarly, P, L, B and M lie on a circle and therefore $\angle PLM = \angle PBN = 90° - A$

Hence, $\angle NLM = 180° - 2A$

= the supplement of $2A$.

So, $\angle LMN = 180° - 2B$ and

$\angle MNL = 180° - 2C$

Hence, its angles are the supplements of twice the angles of the triangle.

Again, from the triangle AMN, we have,

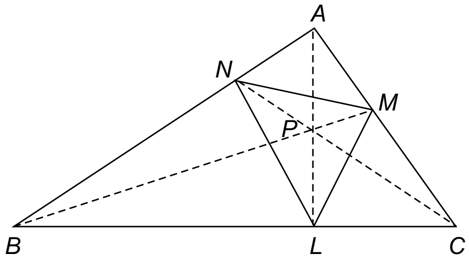
$$\frac{MN}{\sin A} = \frac{AM}{\sin(ANM)} = \frac{AB\cos A}{\cos(PNM)}$$

$$= \frac{c\cos A}{\cos(PAM)} = \frac{c\cos A}{\sin C}$$

$$\Rightarrow \quad MN = \frac{c}{\sin C} \sin A \cos A = a \cos A$$

So, $NL = b \cos B$ and $LM = c \cos C$. Hence, the sides of the pedal triangles are
$a \cos A$, $b \cos B$ and $c \cos C$, respectively.

7.11.3 Area of a pedal triangle LMN of a triangle *ABC* is $2\Delta \cos A \cos B \cos C$



Here, $PL = 2R \cos B \cos C$,
$OM = 2R \cos C \,.\, \cos A$,
$ON = 2R \cos A .\cos B$

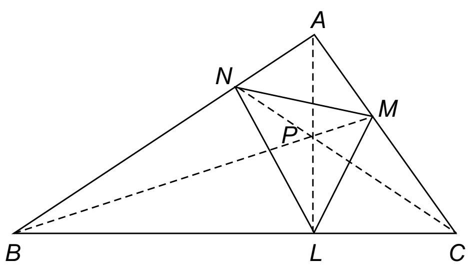
$$ar(\Delta LMN)$$
$$= \frac{1}{2} \times (R \sin 2A) \times (R \sin 2B) \times (\sin 2C)$$
$$= \frac{1}{2} \times R^2 \times (\sin 2A.\sin 2B.\sin 2C)$$
$$= \frac{1}{2} \times R^2 \times (8 \sin A.\sin B.\sin C) \times (\cos A.\cos B.\cos C)$$
$$= \frac{1}{2} \times R^2 \times \left(8.\frac{a}{2R}.\frac{b}{2R}.\frac{c}{2R}\right) \times (\cos A.\cos B.\cos C)$$
$$= \frac{1}{2} \times \frac{abc}{R} \times (\cos A.\cos B.\cos C)$$
$$= 2\Delta \times (\cos A.\cos B.\cos C)$$

7.11.4 The circum-radius of a pedal triangle *LMN* of a triangle *ABC* is $\frac{R}{2}$

Circum-radius

$$= \frac{MN}{2 \sin(MLN)} = \frac{R \sin 2A}{2 \sin(180° - 2A)} = \frac{R \sin 2A}{2 \sin 2A} = \frac{R}{2}$$

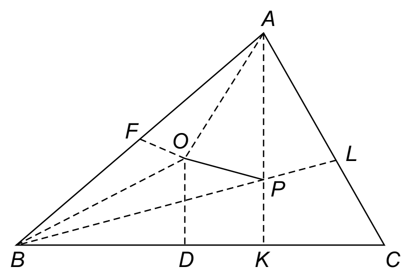
7.11.5 The in-radius of a pedal triangle *LMN* of a triangle *ABC* is $2R \cos A \cos B \cos C$



In-radius

$$= \frac{ar(\Delta LMN)}{\text{semi} - \text{perimeter}(\Delta LMN)}$$
$$= \frac{\frac{1}{2} R^2.\sin\ 2A.\sin\ 2B.\sin\ 2C}{2R.\sin\ A.\sin\ B.\sin\ C}$$
$$= 2R.\cos\ A.\cos\ B.\cos\ C$$

7.12 DISTANCE BETWEEN THE CIRCUMCENTRE AND ORTHOCENTRE



If OF perpendicular to AB, we have,

$\angle OAF = 90° - \angle AOF = 90° - C$

Also, $\angle PAL = 90° - C$

Thus, $\angle OAP$

$$= A - \angle OAF - \angle PAL$$
$$= A - 2(90° - C)$$
$$= A + 2C - 180°$$
$$= A + 2C - (A + B + C) = C - B$$

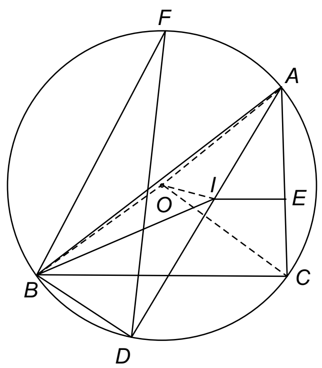
Also, $OA = R$ and $PA = 2R \cos A$

Thus, $OP^2 = OA^2 + PA^2 - 2.OA.PA.\cos(OAP)$

$$= R^2 + 4R^2 \cos^2 A - 4R^2 \cos A \cos(C - B)$$
$$= R^2 + 4R^2 \cos A[\cos A - \cos(C - B)]$$
$$= R^2 - 4R^2 \cos A[\cos(B + C) - \cos(B - C)]$$
$$= R^2 - 8R^2 \cos A \cos B\ \cos C$$
$$= R^2 (1 - 8 \cos A \cos B\ \cos C)$$

$$\Rightarrow \quad OP = R\sqrt{(1 - 8 \cos A \cos B \cos C)}$$

7.13 DISTANCE BETWEEN THE CIRCUMCENTRE AND THE INCENTRE



Let O be the circumcentre and OF be perpendicular to AB. Let I be the incentre and IE perpendicular to AC.

Then $\angle OAF = 90° - C$

$$\angle OAI = \angle IAF - \angle OAF$$

$$= \frac{A}{2} - (90° - C)$$

$$= \frac{A}{2} + C - \frac{A+B+C}{2}$$

$$= \frac{C-B}{2}$$

Also, $AI = \dfrac{IE}{\sin\left(\frac{A}{2}\right)} = \dfrac{r}{\sin\left(\frac{A}{2}\right)}$

$$= 4R\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

$$= 1 - 8\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{B}{2}+\frac{C}{2}\right)\right)$$

$$= 1 - 8\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad OI = R\sqrt{1 - 8\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)}$$

Also, OI^2

$$= R^2 - 2R \times 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

$$= R^2 - 2Rr$$

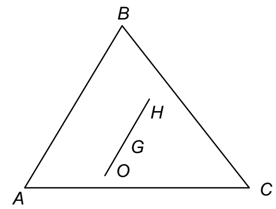
$$\Rightarrow \quad OI = \sqrt{R^2 - 2Rr}$$

Hence, the result.

7.14 DISTANCE BETWEEN THE CIRCUMCENTRE AND CENTROID

i.e. $OG^2 = R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right)$.

Soln:



As we know that, centroid divides the orthocentre and circumcentre in the ratio 2:1

Thus, $OG = \frac{1}{3}.OH$

$$\Rightarrow \quad OG^2 = \frac{1}{9}.OH^2$$

$$= \frac{1}{9}\left(R^2 - 8R^2\cos A.\cos B.\cos C\right)$$

$$\Rightarrow \quad OG^2 = \frac{R^2}{9} \times \left(1 - 4\left\{\cos(A+B) + \cos(A-B)\right\}.\cos C\right)$$

$$\Rightarrow \quad OG^2 = \frac{R^2}{9} \times \left(1 + 4\cos^2 C + 4\cos(A-B).\cos(A+B)\right)$$

$$\Rightarrow \quad OG^2 = \frac{R^2}{9}\left(1 + 4\cos^2 C + 4\cos^2 A - 4\sin^2 B\right)$$

$$\Rightarrow \quad OG^2 = \frac{R^2}{9}\left(1 + 4 - 4\sin^2 C + 4 - 4\sin^2 A - 4\sin^2 B\right)$$

$$\Rightarrow \quad OG^2 = \frac{R^2}{9}\left(9 - 4\left(\sin^2 A + \sin^2 B + \sin^2 C\right)\right)$$

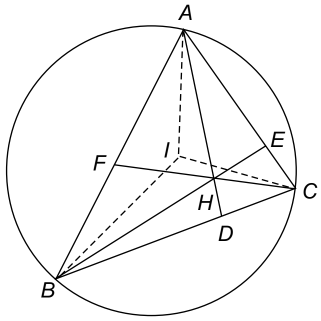
$$\Rightarrow \quad OG^2 = R^2 - \frac{1}{9} \times \left\{(2R\sin A)^2 + (2R\sin A)^2 + (2R\sin A)^2\right\}$$

$$\Rightarrow \quad OG^2 = R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right)$$

7.15 DISTANCE BETWEEN THE INCENTRE AND ORTHOCENTRE

i.e., $OH^2 = 2r^2 - 4R^2\cos A\cos B\cos C$

Soln. Let ABC be a triangle, H is the orthocentre and I is the incentre.



Join AH, AI and IH.

In triangle AIH,

$$IH^2 = AH^2 + AI^2 - 2.AH.AI.\cos(\angle IAH)$$

$$\angle IAH = \frac{A}{2} - \angle HAC$$

$$= \frac{A}{2} - (90° - C) = \frac{1}{2}(C - B)$$

Thus, $4R^2\left[\cos^2 A + 4\sin^2\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right.$

$$\left.-4\cos A\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)\cos\left(\frac{B}{2}\right)\right]$$

$$-\left[-16R^2\cos A.\sin^2\left(\frac{B}{2}\right).\sin^2\left(\frac{C}{2}\right)\right]$$

$$= 4R^2\left[\cos^2 A + 4\sin^2\left(\frac{B}{2}\right)\sin^2\left(\frac{C}{2}\right)(1 - \cos A)\right.$$

$$\left.-\cos A.\cos B.\sin C\right]$$

$$= 4R^2\left[\cos^2 A + 8\sin^2\left(\frac{A}{2}\right)\sin^2\left(\frac{B}{2}\right)\sin^2\left(\frac{C}{2}\right)\right.$$

$$\left.-\cos A.\sin B.\sin C\right]$$

$$= 32R^2\sin^2\left(\frac{A}{2}\right)\sin^2\left(\frac{B}{2}\right)\sin^2\left(\frac{C}{2}\right)$$

$$+4R^2\cos A.(\cos A - \sin B.\sin C)$$

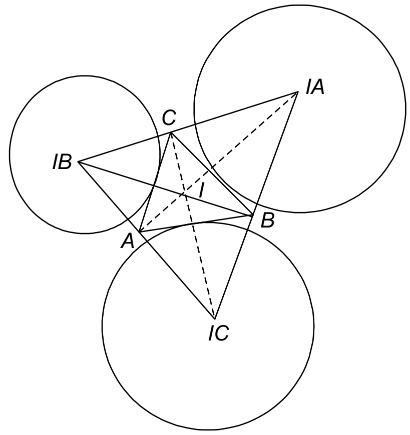
$$= 2\left[16R^2\sin^2\left(\frac{A}{2}\right)\sin^2\left(\frac{B}{2}\right)\sin^2\left(\frac{C}{2}\right)\right]$$

$$-4R^2\cos A.\left[\cos(B + C) + \sin B.\sin C\right]$$

$$= 2r^2 - 4R^2\cos A.\cos B.\cos C.$$

Hence, the result.

7.16 EXCENTRAL TRIANGLE



The triangle formed by joining the three excentres I_A, I_B, I_C of a ΔABC is called an excentral of excentric triangle

(i) ΔABC is the pedal triangle of $\Delta\ I_A I_B I_C$

(ii) Its angles are $\left(\frac{\pi}{2} - \frac{A}{2}\right), \left(\frac{\pi}{2} - \frac{B}{2}\right), \left(\frac{\pi}{2} - \frac{C}{2}\right)$

(iii) Its sides are $I_B I_C = 4R\cos\left(\frac{A}{2}\right)$,

$I_A I_C = 4R\cos\left(\frac{B}{2}\right)$,

and $I_A I_B = 4R\cos\left(\frac{C}{2}\right)$

(iv) $II_A = 4R\sin\left(\frac{A}{2}\right)$; $II_B = 4R\sin\left(\frac{B}{2}\right)$

$II_C = 4R\sin\left(\frac{C}{2}\right)$

(v) Incentre I of ΔABC is the orthocentre of the excentric triangle $\Delta\ I_A I_B I_C$.

(vi) $ar(I_A I_B I_C)$

$$= \frac{1}{2} \times \left(4R\cos\left(\frac{B}{2}\right)\right) \times \left(4R\cos\left(\frac{B}{2}\right)\right) \times \sin\left(90° - \frac{A}{2}\right)$$

$$= 8.R^2.\cos\left(\frac{A}{2}\right).\cos\left(\frac{B}{2}\right).\cos\left(\frac{C}{2}\right)$$

(vii) Circum-radius $= \dfrac{I_2 I_3}{2\sin(\angle I_2 I_1 I_3)}$

$$= \frac{I_2 I_3}{2\sin\left(90° - \frac{A}{2}\right)}$$

$$= \frac{4R\cos\left(\frac{A}{2}\right)}{2\cos\left(\frac{A}{2}\right)} = 2R$$

Soln: We know that,

$$II_A = a\sec\left(\frac{A}{2}\right),\ II_B = a\sec\left(\frac{B}{2}\right), II_C = a\sec\left(\frac{C}{2}\right)$$

Also,

$$I_A I_B = c\,\text{cosec}\left(\frac{A}{2}\right),\ \ I_B I_C = c\,\text{cosec}\left(\frac{B}{2}\right),$$

$$I_C I_A = c\,\text{cosec}\left(\frac{C}{2}\right)$$

Thus, $II_A . II_B . II_C$

$$= abc \times \sec\left(\frac{A}{2}\right)\sec\left(\frac{B}{2}\right)\sec\left(\frac{C}{2}\right) \text{ ....(i)}$$

Also, $a = 2R \sin A$, $b = 2R \sin b$ and $c = 2R \sin C$
From (i), we get, $II_A . II_B . II_C$

$$= (2R\sin A)(2R\sin B)(2R\sin C)$$

$$\times\sec\left(\frac{A}{2}\right)\sec\left(\frac{B}{2}\right)\sec\left(\frac{C}{2}\right)$$

$$= 8R^3 \times \frac{\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$= 64R^3 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= 16R^2 \times 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= 16R^2 r$$

Hence, the result.

(vi) If I is the incentre and I_A, I_B, I_C are the excentres of the triangle ΔABC, then prove that

$$II_A . II_B . II_C = 16 r R^2$$

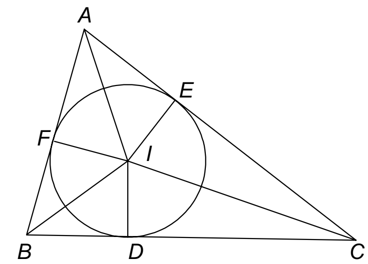
7.17 The distance between the incentre and the angular poins of *A* ΔABC

i.e., $IA = 4R\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$,

$IB = 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)$,

and $IC = 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)$

Soln.



We have, $\sin\left(\frac{A}{2}\right) = \frac{IF}{IA}$

$$\Rightarrow r = IA.\ \sin\left(\frac{A}{2}\right)$$

$$\Rightarrow 4R\ \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) = IA.\ \sin\left(\frac{A}{2}\right)$$

$$\Rightarrow\ IA = 4R\ \sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right),$$

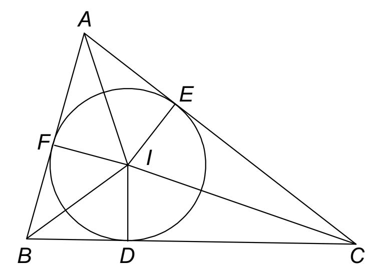
Similarly, $IB = 4R\ \sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)$

and $IC = 4R\ \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)$

7.18 If *I* be the incentre of a ΔABC, then

$$\boldsymbol{IA.IB.IC = abc \times \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)}$$

Soln.



From the above diagram,

$$IA = \frac{r}{\sin\left(\frac{A}{2}\right)},\ IB = \frac{r}{\sin\left(\frac{B}{2}\right)},\ IC = \frac{r}{\sin\left(\frac{C}{2}\right)}$$

Thus,

$IA.IB.IC$

$$= \frac{r^3}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}$$

$$= \frac{r^3 \times 4R}{4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}$$

$$= \frac{r^3.4R}{r} = 4r^2.R = 4.\frac{\Delta^2}{s^2}.\frac{abc}{4\Delta} = abc.\frac{\Delta}{s^2}$$

$$= abc \times \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s^2}$$

$$= abc \times \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^4}}$$

$$= abc \times \sqrt{\frac{(s-a)}{s}.\frac{(s-b)}{s}.\frac{(s-c)}{s}}$$

$$= abc \times \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \cdot \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\cdot \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}.$$

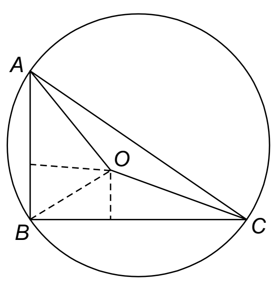
$$= abc \times \tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)$$

$$= abc \times \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)$$

Hence, the result.

7.19. *O* is the circumcentre of a triangle ΔABC and $R_1, R_2 \,\&\, R_3$ are respectively the radii of the circumference of the Δ's O*BC*, *OCA* and *OAB*, respectively, then $\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{abc}{R^3}$.

Soln.



As we know that, $R = \frac{abc}{4\Delta}$

Let $\Delta OBC = \Delta_1$, $\Delta OCA = \Delta_2$, $\Delta OAB = \Delta_3$

In $R_1 = \frac{0B.0C.BC}{4\Delta_1} = \frac{R.R.a}{4\Delta_1} = \frac{R^2.a}{4\Delta_1}$,

$$\Rightarrow \quad \frac{a}{R_1} = \frac{4\Delta_1}{R^2}$$

Similarly, $\frac{b}{R_2} = \frac{4\Delta_2}{R^2}$ & $\frac{c}{R_3} = \frac{4\Delta_3}{R^2}$

Similarly, $\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3}$

Thus, $\frac{4}{R^2}(\Delta_1 + \Delta_2 + \Delta_3)$

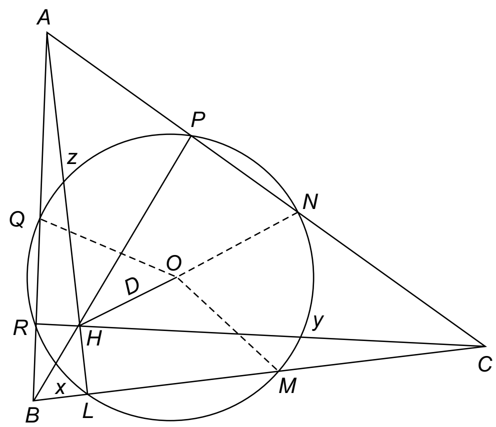
$$= \frac{4}{R^2} \times \Delta = \frac{4\Delta}{R^2} = \frac{abc}{R^3}$$

Hence, the result.

7.20 The distance between the centre of the nine - point circle from the Angle a is

$$= \frac{R}{2}\sqrt{(1 + 8\cos A.\cos B.\cos C)}$$

Soln.



Let *ABC* be a triangle, *H* = orthocentre,
O = circumcentre, *D* = nine-point centre
As we know that, nine point centre is the mid-point of the orthocentre and circumcentre of a triangle.
Thus, *AOH* be a triangle, where, *AD* is the median.
In ΔAOH, $2(AD^2 + DO^2) = AH^2 + AO^2$

$$\Rightarrow \quad 2\,AD^2 = AH^2 + AO^2 - \frac{1}{2}OH^2 \quad ....(i)$$

Now from the triangle *ABC*, we can write,
$AH = 2R\cos A$, $OA = R$,

$$OH = R\sqrt{(1 - 8\ \cos A.\cos B.\cos C)}.$$

From (i), we get, $2AD^2$

$$= R^2\left(4\cos^2 A + 1\right) - \frac{R^2}{2}(1 - 8\cos A.\cos B.\cos C)$$

$$= \frac{R^2}{2}\left[1 + 8\cos A.\{\cos(180° - B + C) + \cos B.\cos C\}\right]$$

$$\Rightarrow AD^2 = \frac{R^2}{4} \times \left[1 + 8\cos A\{-\cos B.\cos C\right.$$

$+\sin B.\sin C+\cos B.\cos C\}]$

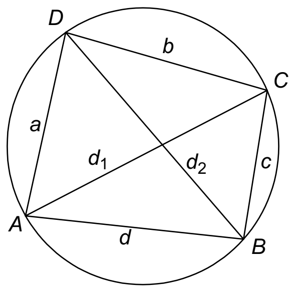
$\Rightarrow \quad AD^2 = \frac{R^2}{4}(1+8\cos A.\sin B.\sin C)$

$\Rightarrow \quad AD = \frac{R}{2}\sqrt{(1+8\cos A.\sin B.\sin C)}$

Hence, the result.

7.17 QUADRILATERAL

7.17.1 Area of a quadrilateral, which is inscribed in a circle.



Let *ABCD* be a cyclic quadrilateral such that *AB* = *a*, *BC* = *b*, *CD* = *c* and *AD* = *d*.

$ar(ABCD) = ar(\Delta ABC) + ar(\Delta ADC)$

$= \frac{1}{2}ab\sin\ B + \frac{1}{2}cd\ \sin\ D$

$= \frac{1}{2}ab\sin\ B + \frac{1}{2}cd\ \sin(\pi - B)$

$= \frac{1}{2}ab\sin\ B + \frac{1}{2}cd\ \sin B$

$= \frac{1}{2}(ab+cd)\sin\ B$

From Δ's, *BAC* and *BCD*, we have,

$a^2+b^2-2\ ab\ \cos\ B = c^2+d^2-2\ cd\ \cos\ D$

$\Rightarrow \quad a^2+b^2-2\ ab\ \cos\ B = c^2+d^2+2\ cd\ \cos\ B$

$\Rightarrow \quad \cos\ B = \left(\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right)$

Now, $\sin^2 B = 1-\cos^2 B$

$= 1-\frac{(a^2+b^2-c^2-d^2)^2}{(2(ab+cd))^2}$

$= \frac{\left[\left\{2(ab+cd)^2-(a^2+b^2-c^2-d^2)^2\right\}\right]}{4(ab+cd)^2}$

$= \frac{1}{4(ab+cd)^2}\times$

$\left[\left\{2(ab+cd)+(a^2+b^2-c^2-d^2)^2\right\}\right.$

$\left.\times\left\{2(ab+cd)+(a^2+b^2-c^2-d^2)^2\right\}\right]$

$= \frac{1}{4(ab+cd)^2}\times$

$\left[\left\{(a^2+b^2+2ab)-(c^2-2cd+d^2)\right\}\right.$

$\left.\times\left\{(c^2+2cd+d^2)-(a^2+b^2-2ab)\right\}\right]$

$= \frac{\left\{(a+b)^2-(c-d)^2\right\}\times\left\{(c+d)^2-(a-b)^2\right\}}{4(ab+cd)^2}$

$= \frac{1}{4(ab+cd)^2}\times\left[\left\{(a+b+c-d)(a+b-c+d)\right\}\right.$

$\left.\times(c+d+b-a)(c+d+a-b)\right]$

Let $a+b+c+d = 2\ s$

Thus, $(a+b+c-d)$

$= (a+b+c+d-2d) = 2(s-d)$

Similarly, $(a+b+d-c) = 2(s-c)$,

$(a+c+d-b) = 2(s-b)$,

$(b+c+d-a) = 2(s-a)$.

$\Rightarrow \quad \sin^2 B$

$= \frac{2(s-d)\times 2(s-c)\times 2(s-b)\times 2(s-a)}{4(ab+cd)^2}$

$\Rightarrow \quad \sin^2 B$

$= \frac{16\times(s-a)\times(s-b)\times(s-c)\times(s-d)}{4(ab+cd)^2}$

$\Rightarrow \quad (ab+cd)\sin\ B$

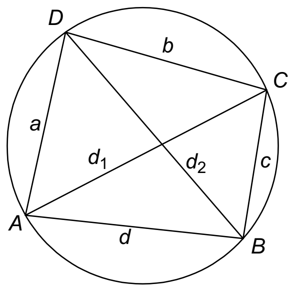
$= 2\times\sqrt{(s-a)(s-b)(s-c)(s-d)}$

Hence, the area of the quadrilateral

$= \frac{1}{2}(ab+cd)\sin\ B$

$= \sqrt{(s-a)(s-b)(s-c)(s-d)}$

7.17.2 The radius of the circle circumscribing the quadrilateral *ABCD*.



We have, $\cos B = \dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}$

and $AC^2 = a^2+b^2-2\,ab\cos\,B$

$= a^2+b^2-2\,ab\times\left(\dfrac{a^2+b^2-c^2-d^2}{ab+cd}\right)$

$= \dfrac{\left(a^2+b^2\right)cd+\left(c^2+d^2\right)ab}{(ab+cd)}$

$= \dfrac{(ac+bd)(ad+bc)}{(ab+cd)}$

In ΔABC

In $\dfrac{1}{2}\times\dfrac{AC}{\sin B}, R$

$= \sqrt{\left(\dfrac{(ac+bd)(ad+bc)}{(ab+cd)}\right)}$

$= \div 4\sqrt{\dfrac{(s-a)(s-b)(s-c)(s-d)}{(ab+cd)^2}}$

$\dfrac{1}{2}\times\dfrac{AC}{\sin B} = \dfrac{1}{4}\sqrt{\left(\dfrac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}\right)}$

$= ar\ (ABCD)$

7.17.3 Area of a quadrilateral *ABCD*, when it is not inscribed.

ar (*ABCD*)

$= ar(\Delta ABC) + ar(\Delta ACD)$

$= \dfrac{1}{2}ab\sin B + \dfrac{1}{2}cd\sin D$

$\Rightarrow \quad 4\,\Delta = 2\,ab\ \ \sin B + 2\,cd\sin D$ ....(i)

Also,

$a^2+b^2-2ab\ \cos B = c^2+d^2-2cd\ \cos D$

$2ab\ \cos B - 2cd\ \cos D = a^2+b^2-\ c^2-d^2$ ...(ii)

Squaring (i) and (ii) and adding, we get,

$16\Delta^2 + \left(a^2+b^2-c^2-d^2\right)^2$

$= 4a^2b^2 + 4c^2d^2$

$\quad -8\ abcd\ \left(\cos B\ \cos D - \sin B\ \sin D\right)$

$= 4(a^2b^2+c^2d^2) - 8\ abcd\ \cos(B+D)$

$= 4(a^2b^2+c^2d^2) - 8\ abcd\ \cos 2\alpha$

$= 4(a^2b^2+c^2d^2) - 8\ abcd\ \left(2\cos^2\alpha - 1\right)$

$= 4(ab+cd)^2 - 16\ abcd\ \cos^2\alpha$

$\Rightarrow 16\Delta^2$

$= 4(ab+cd)^2$

$\quad -\left(a^2+b^2-c^2-d^2\right)^2 - 16\ abcd\ \cos^2\alpha$

..................(iii)

Thus, $16\Delta^2$

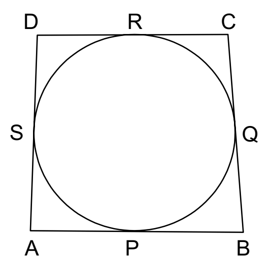
$= 2(s-a).2(s-b).2(s-c).2(s-d)$

$-16abcd\ \cos^2\alpha$

$\Rightarrow \quad \Delta^2 = (s-a)(s-b)(s-c)(s-d) - abcd\ \cos^2\alpha$

$\Rightarrow \quad \Delta = \sqrt{(s-a).(s-b).(s-c).(s-d) - abcd\ \cos^2\alpha}$

7.17.4. Area of a quadilateral which can have a circle inscribed in it.



We have, $AP = AS$, $BP = BQ$, $CQ = CR$ and $DR = RS$

$AP + BP + CR + DR = AS + BQ + CQ + DS$

$\Rightarrow \quad AB + CD = BC + DA$

$\Rightarrow \quad a + c = b + d$

Hence, $s = \dfrac{a+b+c+d}{2} = a + c = b + d$

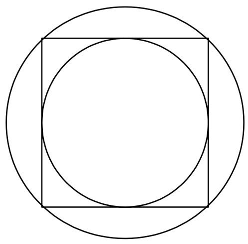
Thus, $s - a = c$, $s - b = d$, $s - c = a$, $s - d = b$

As we know that,

$\Delta^2 = abcd - abcd\ \cos^2\alpha = abcd\ \sin^2\alpha$

$\Rightarrow \quad \Delta = \sqrt{abcd}\ \sin\alpha$

7.17.5. The area of a quadrilateral, which can be both inscribed in a circle and circumscribed about another circle and the radius of the later circle is $\dfrac{2\sqrt{abcd}}{a+b+c+d}$



In a quadrilateral *ABCD*,

$\angle B + \angle D = 180°$

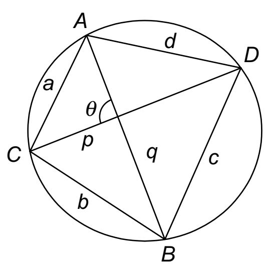
$\Rightarrow \quad 2\alpha = 180°$

$\Rightarrow \quad \alpha = 90°$

Hence, the area of the quadrilateral, which is inscribed in a circle and circumscribed anothe circle is $\sqrt{a\ b\ c\ d}$.

We have, $r = \dfrac{\Delta}{s} = \dfrac{2\Delta}{2s} = \dfrac{2\sqrt{abcd}}{a+b+c+d}$

7.17.6 *a*, *b*, *c* and *d* are the sides of a quadrilateral taken in order, and *θ* is the angle between the diagonals opposite to *b* or *d*, then the area of the quadrilateral is $\dfrac{1}{4}\left(a^2+c^2-b^2-d^2\right)\tan\theta$



Area of a quadrilateral *ABCD*

$= \dfrac{1}{2} \times AC \times BD \times \sin\theta$ ....(i)

Now, $a^2 = OA^2 + OB^2 - 2.OA.OB.\cos(180° - \theta)$

$b^2 = OC^2 + OB^2 - 2.OC.OB.\cos\theta$,

$c^2 = OC^2 + OD^2 - 2.OC.OD.\cos(180° - \theta)$

$d^2 = OA^2 + OD^2 - 2.OA.OD.\cos\theta$.

We have, $a^2 - b^2 + c^2 - d^2$

$= 2\cos\theta(OA.OB + OB.OC + OC.OD + OA.OD)$

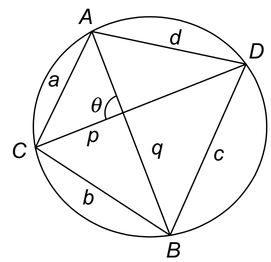
$= 2.\cos\theta.AC.BD$

From (i), we get, area of the quadrilateral *ABCD*

$= \dfrac{1}{2}\left(a^2 - b^2 + c^2 - d^2\right)\ \tan\ \theta$.

7.17.7. *a*, *b*, *c* and *d* are the sides of a quadrilateral and ***p* and *q* be its diagonals, then its area is**

$\dfrac{1}{4} \times \sqrt{\left(4p^2q^2 - \left(a^2+c^2-b^2-d^2\right)^2\right)}$



As we know that,

$a^2 - b^2 + c^2 - d^2 = 2pq\cos\ \theta$

Area of a quadrilateral *ABCD*

$= \dfrac{1}{2}pq\sin\theta$

$= \sqrt{\dfrac{1}{4}p^2q^2\sin^2\theta}$

$= \sqrt{\dfrac{1}{4}p^2q^2\left(1-\cos^2\theta\right)}$

$= \sqrt{\dfrac{1}{4}\left(4p^2q^2 - (2pq\cos\theta)^2\right)}$

$= \dfrac{1}{2}\sqrt{\left(4p^2q^2 - \left(a^2-b^2+c^2-d^2\right)^2\right)}$

Hence, the result.

7.17.8. If a quadrilateral can be inscribed in a circle, then the angle between its diagonals is

$$\sin^{-1}\left(\frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(ac+bd)}\right)$$

Soln. Let *ABCD* be a quadrilateral, whose sides are *a*, *b*, *c* and *d* respectively and its diagonals are *p* and *q*. Let the angle between the diagonals be *θ*.

Then, $pq = AC + bd$.

Area of a quadrilateral $ABCD = \frac{1}{2} \times pq \sin\theta$

$$\sin\theta = \frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(ac+bd)}$$

$$\theta = \sin^{-1}\left(\frac{2\sqrt{(s-a)(s-b)(s-c)(s-d)}}{(ac+bd)}\right)$$

Hence, the result.

7.17.9 If a quadrilateral can be inscribed in a circle as well as circumscribed about another circle, then the angle between its diagonal is

$\cos^{-1}\left(\frac{ac-bd}{ac+bd}\right)$.

Since the quadrilateral be circumscribed, then we can write, $\frac{1}{2} pq \sin\theta = \sqrt{abcd}$

$$\Rightarrow \quad \sin\theta = \left(\frac{2\sqrt{abcd}}{pq}\right)$$

Therefore, $\cos\theta = \sqrt{1-\sin^2\theta}$

$$= \cos\theta = \sqrt{1-\left(\frac{4abcd}{(ac+bd)^2}\right)} = \left(\frac{ac-bd}{ac+bd}\right)$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{ac-bd}{ac+bd}\right)$$

Hence, the result.

PROBLEMS FOR JEE ADVANCED EXAM

Ex-1. The sides of a triangle are $x^2+x+1, 2x+1$ and x^2-1, prove that the greatest angle is 120°.

Soln. Let $a = x^2+x+1, b = 2x+1$ and $c = x^2-1$

First we have to shown that, which one is greatest amongst the sides a, b and c.

Clearly, $a > 0, b > 0$ and $c > 0$

$\Rightarrow x > 1$

Now, $a - b$

$= (x^2+x+1)-(2x+1)$

$= x^2 - x$

$= x(x-1) > 0$ as $x > 1$

and $a - c$

$= (x^2+x+1)-(x^2-1)$

$= x+2 > 0, as\, x > 1$

Therefore, a is the greast side

$\Rightarrow$ A is the greatest angle

Now, $\cos A = \frac{b^2+c^2-a^2}{2bc}$

$$= \frac{(2x+1)^2+(x^2-1)^2-(x^2+x+1)^2}{2(2x+1)(x^2-1)}$$

$$= \frac{(2x+1)^2+(2x^2+x)(-x-2)}{2(2x+1)(x^2-1)}$$

$$= \frac{(2x+1)^2+x(2x+1)(-x-2)}{2(2x+1)(x^2-1)}$$

$$= \frac{(2x+1)(2x+1-x^2-2x)}{2(2x+1)(x^2-1)}$$

$$= \frac{(1-x^2)}{2(x^2-1)} = -\frac{1}{2}$$

$\Rightarrow \cos A = -\frac{1}{2}$

$\Rightarrow A = 120°$

Ex-2. In any triangle ABC, $\cos\theta = \frac{a}{b+c}, \cos\varphi = \frac{b}{a+c}$, $\cos\psi = \frac{c}{a+b}$ where φ, θ and ψ lie between 0 and π, prove that $\tan^2\left(\frac{\theta}{2}\right)+\tan^2\left(\frac{\varphi}{2}\right)+\tan^2\left(\frac{\psi}{2}\right)=1$

Soln. Given $\cos\theta = \frac{a}{b+c}$

$$1+\cos\theta = 1+\frac{a}{b+c} = \frac{a+b+c}{b+c}$$

$$\Rightarrow \quad 2\cos^2\left(\frac{\theta}{2}\right) = \frac{a+b+c}{b+c}$$

$$\Rightarrow \quad \sec^2\left(\frac{\theta}{2}\right) = \frac{2(b+c)}{(a+b+c)}$$

$$\Rightarrow \quad 1+\tan^2\left(\frac{\theta}{2}\right) = \frac{2(b+c)}{(a+b+c)}$$

Similarly, we can easily proved that

$$1+\tan^2\left(\frac{\varphi}{2}\right)=\frac{2(c+a)}{(a+b+c)}$$

$$1+\tan^2\left(\frac{\psi}{2}\right)=\frac{2(a+b)}{(a+b+c)}$$

Adding, we get,

$$3+\tan^2\left(\frac{\theta}{2}\right)+\tan^2\left(\frac{\varphi}{2}\right)+\tan^2\left(\frac{\psi}{2}\right)$$

$$=\frac{4(a+b+c)}{(a+b+c)}$$

$$\Rightarrow \quad 3+\tan^2\left(\frac{\theta}{2}\right)+\tan^2\left(\frac{\varphi}{2}\right)+\tan^2\left(\frac{\psi}{2}\right)=4$$

$$\Rightarrow \quad \tan^2\left(\frac{\theta}{2}\right)+\tan^2\left(\frac{\varphi}{2}\right)+\tan^2\left(\frac{\psi}{2}\right)=1$$

Hence, the result.

Ex-3. Given the product p of sines of the angles of a triangle and the product q of their cosines, find the cubic equation, whose co-efficients are functions of p and q and whose roots are the tangents of the angles of the triangle.

Soln. Given $\sin A\ \sin B\ \sin C=p$

$\cos A\ \cos B\ \cos C=q$

Thus, $\tan A\ \tan B\ \tan C=\dfrac{p}{q}$

Also, $A+B+C=\pi$

$\tan A+\tan B+\tan C=\tan A\ \tan B\ \tan C$

$\tan A+\tan B+\tan C=\dfrac{p}{q}$

Also, $\tan A\tan B+\tan B\tan C+\tan C\tan A$

$=\dfrac{1+q}{q}$

Hence, the required equation is

$$x^3-\left(\frac{p}{q}\right)x^2+\left(\frac{1+q}{q}\right)x-\left(\frac{p}{q}\right)=0$$

$$qx^3-px^2+(1+q)x-p=0.$$

Ex-4. In a triangle ABC, if

$$\sin^3\theta=\sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$$

prove that $\cot\theta=\cot A+\cot B+\cot C$

Soln. We have

$$\sin^3\theta=\sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$$

$$\Rightarrow \quad 2\sin^3\theta=\sin(A-\theta)\{2\sin(B-\theta)\sin(C-\theta)\}$$

$$=\sin(A-\theta)\{\cos(B-C)-\cos(B+C-2\theta)\}$$

$$\Rightarrow \quad 4\sin^3\theta$$

$$=2\sin(A-\theta)\{\cos(B-C)-\cos(B+C-2\theta)\}$$

$$=2\sin(A-\theta)\cos(B-C)$$

$$-2\sin(A-\theta)\cos(B+C-2\theta)$$

$$=(\sin(A+B-\theta-C)-\sin(A+C-\theta-B))$$

$$-(\sin(A+B+C-3\theta)-\sin(B+C-A-\theta))$$

$$=\sin(\pi-(2C+\theta))+\sin(\theta-(2B+\theta))$$

$$-\sin 3\theta+\sin(\pi-(2A+\theta))$$

$$\Rightarrow \quad \sin 3\theta+4\sin^3\theta$$

$$=\sin(2A+\theta)+\sin(2B+\theta)+\sin(2C+\theta)$$

$$\Rightarrow \quad 3\sin\theta=(\sin 2A+\sin 2B+\sin 2C)\cos\theta$$

$$+(\cos 2A+\cos 2B+\cos 2C)\sin\theta$$

$$\Rightarrow \quad (3-\cos 2A-\cos 2B-\cos 2C)\sin\theta$$

$$=(\sin 2A+\sin 2B+\sin 2C)\cos\theta$$

$$\Rightarrow \quad \{(1-\cos 2A)+(1-\cos 2B)+(1-\cos 2C)\}\sin\theta$$

$$=4\sin A\sin B\sin C\cos\theta$$

$$\Rightarrow \quad (2\sin^2 A+2\sin^2 B+2\sin^2 C)\sin\theta$$

$$=4\sin A\sin B\sin C\cos\theta$$

$$\Rightarrow \quad 2\left[(\sin^2 A+\sin^2 B-\sin^2 C)\right.$$

$$+(\sin^2 B+\sin^2 C-\sin^2 A)$$

$$\left.+(\sin^2 C+\sin^2 A-\sin^2 B)\right]\sin\theta$$

$$=4\sin A\sin B\sin C\cos\theta$$

$$\Rightarrow \quad 4[\sin A\sin B\cos C+\sin B\sin C\cos A$$

$$+\sin C\sin A\cos B]\sin\theta$$

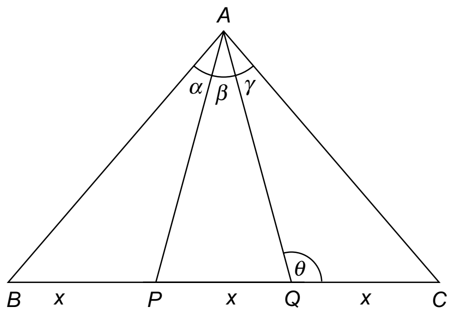
$$=4\sin A\sin B\sin C\cos\theta$$

$$\Rightarrow \quad \cot\theta=\cot A+\cot B+\cot C.$$

Ex-5. The base of a triangle is divided into three parts. If t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex, prove that

$$\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right)=4\left(1+\frac{1}{t_2^2}\right)$$

Soln.



Let $BP = PQ = QC = x$ and also

let $\angle BAP=\alpha, \angle PAQ=\beta, \angle QAC=\gamma$

It is given that $\tan\alpha = t_1, \tan\beta = t_2$

and $\tan\gamma = t_3$

Applying $m : n$ rule in triangle ABC, we get,

$(2x+x)\cot\theta = 2x\cot(\alpha+\beta) - x\cot\gamma$ .....(i)

From ΔAPC, we get,

$(x+x)\cot\theta = x\cot\beta - x\cot\gamma$ ....(ii)

Dividing (i) by (ii), we get,

$$\frac{2\cot(\alpha+\beta)-\cot\gamma}{\cot\beta-\cot\gamma}=\frac{3}{2}$$

$$\Rightarrow \quad 4\cot(\alpha+\beta)-2\cot\gamma = 3(\cot\beta-\cot\gamma)$$

$$\Rightarrow \quad 4\cot(\alpha+\beta) = 3\cot\beta-\cot\gamma$$

$$\Rightarrow \quad \frac{1}{4\cot(\alpha+\beta)}=\frac{1}{3\cot\beta-\cot\gamma}$$

$$\Rightarrow \quad \frac{\tan(\alpha+\beta)}{4}=\frac{1}{\frac{3}{\tan\beta}-\frac{1}{\tan\gamma}}$$

$$\Rightarrow \quad \frac{\tan\alpha+\tan\beta}{4(1-\tan\alpha\tan\beta)}=\frac{\tan\beta\tan\gamma}{3\tan\gamma-\tan\beta}$$

$$\Rightarrow \quad \frac{t_1+t_2}{4(1-t_1t_2)}=\frac{t_2t_3}{3t_3-t_2}$$

$$\Rightarrow \quad 4\left(t_2t_3-t_1t_2^2t_3\right)=3t_1t_3-t_1t_2+3t_2t_3-t_2^2$$

$$\Rightarrow \quad \left(t_2t_3+t_1t_2+t_1t_3+t_2^2\right)=4t_1t_3+4t_1t_2^2t_3$$

$$\Rightarrow \quad \left(t_3(t_2+t_1)+t_2(t_1+t_2)\right)=4t_1t_3\left(1+t_2^2\right)$$

$$\Rightarrow \quad (t_1+t_2)(t_3+t_2)=4t_1t_3t_2^2\left(1+\frac{1}{t_2^2}\right)$$

$$\Rightarrow \quad \left(\frac{t_1+t_2}{t_1t_2}\right)\left(\frac{t_3+t_2}{t_2t_3}\right)=4\left(1+\frac{1}{t_2^2}\right)$$

$$\Rightarrow \quad \left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right)=4\left(1+\frac{1}{t_2^2}\right)$$

Ex-6. In a ΔABC, prove that

$$\cos A+\cos B+\cos C=\left(1+\frac{r}{R}\right)$$

Soln. We have, $\cos A + \cos B + \cos C$

$$= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C$$

$$= 2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C$$

$$= 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C$$

$$= 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\left(\frac{C}{2}\right)$$

$$= 1+2\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right)-\sin\left(\frac{C}{2}\right)\right)$$

$$= 1+2\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right)$$

$$= 1+2\sin\left(\frac{C}{2}\right)\left(2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\right)$$

$$= 1+4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

$$= \left(1+\frac{r}{R}\right)$$

Ex-7. In a ΔABC, prove that

$$\sin A+\sin B+\sin C=\frac{s}{R}=\frac{\Delta}{Rr}$$

Soln. We have, $\sin A + \sin B + \sin C$

$$= \frac{a}{2R}+\frac{b}{2R}+\frac{c}{2R}$$

$$= \frac{a+b+c}{2R}$$

$$= \frac{2s}{2R}$$

$$= \frac{s}{R}$$

$$= \frac{\frac{\Delta}{r}}{R}$$

$$= \frac{\Delta}{rR}$$

Ex-8. In any ΔABC, prove that
$a \cot A + b \cot B + c \cot C = 2(r + R)$

Soln. We have, $a \cot A + b \cot B + c \cot C$

$$= \left[2R\sin A \times \frac{\cos A}{\sin A} + 2R\sin B \times \frac{\cos B}{\sin B}\right.$$
$$\left. +2R\sin C \times \frac{\cos C}{\sin C}\right]$$
$$= 2R(\cos A + \cos B + \cos C)$$
$$= 2R\left(1 + \frac{r}{R}\right)$$
$$= 2(R + r).$$

Ex-9. In any ΔABC, prove that

$$\cos^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{B}{2}\right) + \cos^2\left(\frac{C}{2}\right) = 2 + \frac{r}{2R}$$

Soln. We have, $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}$

$$= \frac{1}{2}\left(2\cos^2\left(\frac{A}{2}\right) + 2\cos^2\left(\frac{B}{2}\right) + 2\cos^2\left(\frac{C}{2}\right)\right)$$
$$= \frac{1}{2}(1 + \cos(A) + 1 + \cos B + 1 + \cos C)$$
$$= \frac{1}{2}(3 + \cos(A) + \cos B + \cos C)$$
$$= \frac{1}{2}\left(3 + \left(1 + \frac{r}{R}\right)\right)$$
$$= \frac{1}{2}\left(4 + \frac{r}{R}\right)$$
$$= \left(2 + \frac{r}{2R}\right)$$

Ex-10. If p_1, p_2 and p_3 are the altitudes of a triangle ABC from the vertices A, B and C, respectively, then prove that $\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{1}{R}$

Soln. We have

$$\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3}$$
$$= \frac{1}{2\Delta}(a\cos A + b\cos B + c\cos C)$$
$$= \frac{1}{2\Delta}[2R\sin A\cos A + 2R\sin B\cos B$$
$$+2R\sin C\cos C]$$
$$= \frac{R}{2\Delta}(\sin 2A + \sin 2B + \sin 2C)$$
$$= \frac{R}{2\Delta}(4\sin A\sin B\sin C)$$
$$= \frac{2R}{\Delta}(\sin A\sin B\sin C)$$
$$= \frac{2R}{\Delta} \times \left(\frac{a}{2R}\right) \times \left(\frac{b}{2R}\right) \times (\sin C)$$
$$= \frac{1}{\Delta R}\left(\frac{1}{2}ab\sin C\right)$$
$$= \frac{1}{\Delta R} \times \Delta$$
$$= \frac{1}{R}$$

Ex-11. If the distances of the sides of a triangle ABC from a circum-center be x, y and z, respectively, then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$.

Soln. Let O is the circum-centre and $OD = x$, $OE = y$, $OF = z$, respectively.

Also, $OA = R = OB = OC$

We have, $x = OD = R \cos A$

$$= \frac{a}{2\sin A}.\cos A = \frac{a}{2\tan A}$$

$$\Rightarrow \quad \tan A = \frac{a}{2x}$$

Similarly, $\tan B = \frac{b}{2y}$ & $\tan C = \frac{c}{2z}$

As we know that, in a triangle ABC,

$$\tan A + \tan B + \tan C = \tan A.\tan B.\tan C$$
$$\Rightarrow \quad \frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z} = \frac{a}{2x}.\frac{b}{2y}.\frac{c}{2z}$$
$$\Rightarrow \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{a.b.c}{4.x.y.z}.$$

Ex-12. If in a triangle ΔABC, O is the circum-center and R is the circum-radius and R_1, R_2, R_3 are the circum radii of the traingles ΔOBC, ΔOCA and

ΔOAB, respectively, then prove that

$$\frac{a}{R_1}+\frac{b}{R_2}+\frac{c}{R_3}=\frac{abc}{R^3}.$$

Soln. We have,

$$R_1=\frac{OB.OC.BC}{4\Delta OBC}=\frac{R.R.a}{4\Delta_1}=\frac{R^2.a}{4\Delta_1}$$

$$\Rightarrow \quad \frac{a}{R_1}=\frac{4\Delta_1}{R^2}$$

Similarly, $\frac{b}{R_2}=\frac{4\Delta_2}{R^2}$ and $\frac{c}{R_3}=\frac{4\Delta_3}{R^2}$

Thus, $\frac{a}{R_1}+\frac{b}{R_2}+\frac{c}{R_3}$

$$=\frac{4\Delta_1}{R^2}+\frac{4\Delta_2}{R^2}+\frac{4\Delta_3}{R^2}$$

$$=\frac{4(\Delta_1+\Delta_2+\Delta_3)}{R^2}$$

$$=\frac{4\Delta}{R^2}$$

$$=\frac{4\Delta}{R^2}$$

$$=\frac{abc}{4R^3}$$

Hence, the result.

Ex-13. In any ΔABC, prove that

$$\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}-\frac{1}{2R}$$

Soln. We have $\frac{r_1}{bc}$

$$=\frac{4R\sin\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)}{(2R\sin B)(2R\sin C)}$$

$$=\frac{\sin\left(\frac{A}{2}\right)}{4R\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}$$

$$=\frac{\sin^2\left(\frac{A}{2}\right)}{4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}$$

$$=\frac{\sin^2\left(\frac{A}{2}\right)}{r}$$

Similarly, $\frac{r_2}{ca}=\frac{\sin^2\left(\frac{B}{2}\right)}{r}$

and $\frac{r_3}{ab}=\frac{\sin^2\left(\frac{C}{2}\right)}{r}$

Now, $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}$

$$=\frac{1}{r}\left(\sin^2\left(\frac{A}{2}\right)+\sin^2\left(\frac{B}{2}\right)+\sin^2\left(\frac{C}{2}\right)\right)$$

$$=\frac{1}{2r}\left(2\sin^2\left(\frac{A}{2}\right)+2\sin^2\left(\frac{B}{2}\right)+2\sin^2\left(\frac{C}{2}\right)\right)$$

$$=\frac{1}{2r}(1-\cos(A)+1-\cos(B)+1-\cos(C))$$

$$=\frac{1}{2r}(3-(\cos(A)+\cos(B)+\cos(C)))$$

$$=\frac{1}{2r}\left(3-\left(1+\frac{r}{R}\right)\right)$$

$$=\frac{1}{2r}\left(2-\frac{r}{R}\right)$$

$$=\left(\frac{1}{r}-\frac{1}{2R}\right)$$

Ex-14. In any ΔABC prove that $\frac{bc}{r_1}+\frac{ca}{r_2}+\frac{ab}{r_3}$

$$=2R\left[\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)-3\right]$$

Soln. We have, $\frac{bc}{r_1}+\frac{ca}{r_2}+\frac{ab}{r_3}$

$$=abc\left(\frac{1}{ar_1}+\frac{1}{br_2}+\frac{1}{cr_3}\right)$$

$$=abc\left(\frac{s-a}{a\Delta}+\frac{s-b}{b\Delta}+\frac{s-c}{c\Delta}\right)$$

$$=\frac{abc}{\Delta}\left(\frac{s-a}{a}+\frac{s-b}{b}+\frac{s-c}{c}\right)$$

$$=\frac{abc}{2\Delta}\left(\frac{2s-2a}{a}+\frac{2s-2b}{b}+\frac{2s-2c}{c}\right)$$

$$=\frac{abc}{2\Delta}\left(\frac{b+c}{a}-1+\frac{c+a}{b}-1+\frac{a+b}{c}-1\right)$$

$$=\frac{abc}{2\Delta}\left(\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)-3\right)$$

$$= 2R\left(\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)-3\right)$$

Ex-15. In a ΔABC, prove that $(r + r_1)\tan\left(\frac{B-C}{2}\right) + (r + r_2)\tan\left(\frac{C-A}{2}\right) + (r + r_3)\tan\left(\frac{A-B}{2}\right) = 0.$

Soln. We have, $(r + r_1)\tan\left(\frac{B-C}{2}\right)$

$$= \left(\frac{\Delta}{s}+\frac{\Delta}{s-a}\right)\times\left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$$

$$= \Delta\left(\frac{s-a+s}{s(s-a)}\right)\times\left(\frac{b-c}{b+c}\right)\times\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= \Delta\left(\frac{b+c}{\sqrt{s(s-a)}}\right)\times\left(\frac{b-c}{b+c}\right)\times\sqrt{\frac{1}{(s-b)(s-c)}}$$

$$= \Delta\times(b+c)\times\left(\frac{b-c}{b+c}\right)\times\frac{1}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \Delta\times(b-c)\times\frac{1}{\Delta}$$

$= b - c$

Similarly, $(r + r_2)\tan\left(\frac{C-A}{2}\right) = c - a$

and $(r + r_3)\tan\left(\frac{A-B}{2}\right) = a - b$

Now, $(r + r_1)\tan\left(\frac{B-C}{2}\right) + (r + r_2)\tan\left(\frac{C-A}{2}\right) + (r + r_3)\tan\left(\frac{A-B}{2}\right)$

$= b - c + c - a + a - b$

$= 0$

Ex-16. If a triangle of maximum area is inscribed within a circle of radius R, then prove that

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{\sqrt{2}+1}{R}$$

Soln. Let ABC be a right angled triangle in a circle of radius R

Therefore, $BC = 2R$ = diameter.

Now, $\Delta = \frac{1}{2}\times AB\times AC\times\sin(90°)$

$= \frac{1}{2}\times AB\times AC$

It will maximum, when $AB = AC$

Thus, $AB^2 + AC^2 = BC^2 = 4R^2$

$\Rightarrow \quad 2AB^2 = 4R^2$

$\Rightarrow \quad AB^2 = 2R^2$

Now, $2s = AB + BC + CA = 2AB + BC$

$= 2R\left(1+\sqrt{2}\right)$

and $r = \frac{\Delta}{s} = \frac{R^2}{R\left(\sqrt{2}+1\right)} = \frac{R}{\left(\sqrt{2}+1\right)}$

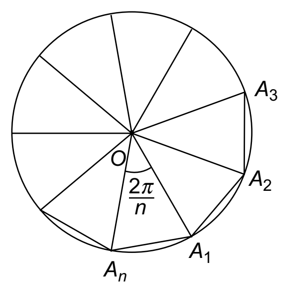
Therefore, $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}=\frac{\left(\sqrt{2}+1\right)}{R}$

Ex-17 Let $A_1, A_2, A_3, \ldots\ldots\ldots, A_n$ be the vertices of an n-sided regular polygon such that

$$\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4},$$

then find the value of n.

Soln. Let O be the centre and $A_1A_2\ldots\ldots A_n$ be the regular polygon of n-sides.



Let $OA_1 = OA_2 = \ldots\ldots\ldots\ldots\ldots\ldots = OA_n = $ r

and $\angle A_1OA_2 = \angle A_2OA_3$

$= \ldots\ldots\ldots = \angle A_nOA_1 = \frac{2\pi}{n}$

From the triangle OA_1A_2,

$$\cos\left(\frac{2\pi}{n}\right)=\frac{OA_1^2+OA_2^2-A_1A_2^2}{2.OA_1.OA_2}$$

$$=\frac{r^2+r^2-A_1A_2^2}{2.r.r}$$

$$\Rightarrow \quad A_1A_2^2 = 2r^2 - 2r^2\cos\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow \quad A_1A_2^2 = 2r^2\left(1-\cos\left(\frac{2\pi}{n}\right)\right)$$

$$\Rightarrow \quad A_1A_2^2 = 2r^2.2\sin^2\left(\frac{2\pi}{n}\right)$$

$$= 4r^2.\sin^2\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow \quad A_1A_2 = 2r.\sin\left(\frac{2\pi}{n}\right).$$

Similarly, $A_1A_3 = 2r.\sin\left(\frac{4\pi}{n}\right)$

& $A_1A_4 = 2r.\sin\left(\frac{6\pi}{n}\right)$

Given, $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$

$$\Rightarrow \frac{1}{2r.\sin\left(\frac{2\pi}{n}\right)} = \frac{1}{2r.\sin\left(\frac{4\pi}{n}\right)} + \frac{1}{2r.\sin\left(\frac{6\pi}{n}\right)}$$

$$\Rightarrow \frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$

$$\Rightarrow \frac{1}{\sin\left(\frac{\pi}{n}\right)} - \frac{1}{\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \frac{\sin\left(\frac{3\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \frac{2\cos\left(\frac{2\pi}{n}\right)\sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow 2\cos\left(\frac{2\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{4\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{4\pi}{n}\right) = \sin\left(\pi - \frac{3\pi}{n}\right)$$

$$\Rightarrow \left(\frac{4\pi}{n}\right) = \left(\pi - \frac{3\pi}{n}\right)$$

$$\Rightarrow \left(\frac{7\pi}{n}\right) = \pi$$

$$\Rightarrow n = 7$$

Ex-18. If A, A_1, A_2, A_3 are the areas of incircle and the ex-circles of a triangle, then prove that

$$\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} = \frac{1}{\sqrt{A}}.$$

Soln. Let r be the radius of the in-circle and r_1, r_2 and r_3 are the ex-radii of the given triangle

Then $\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$

$$= \frac{1}{\sqrt{\pi r_1^2}} + \frac{1}{\sqrt{\pi r_1^2}} + \frac{1}{\sqrt{\pi r_1^2}}$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)$$

$$= \frac{1}{\sqrt{\pi}} \times \frac{1}{r}$$

$$= \frac{1}{\sqrt{\pi r^2}}$$

$$= \frac{1}{\sqrt{A}}$$

Hence, the result.

Ex-19. The sides of a triangle are in A.P. and the greatest and the least angles are θ and φ, then prove that

$$4(1+\cos\theta)(1-\cos\varphi) = \cos\theta + \cos\varphi$$

Soln. Let a, b, c be the sides of a triangle such that a and c are the least and the greatest side of ΔABC

It is given that a, b, c are in A.P.

$$\Rightarrow \quad 2b = a + c$$

Now, $\cos\theta = \frac{a^2+b^2-c^2}{2ab}$

$$= \frac{a^2-c^2}{2ab} + \frac{b^2}{2ab}$$

$$= \frac{(a+c)(a-c)}{a(a+c)} + \frac{b}{2a}$$

$$= \frac{(a-c)}{a} + \frac{b}{2a}$$

$$= \frac{2a-2c+b}{2a}$$

$$= \frac{4a-4c+a+c}{4a}$$

$$= \frac{5a-3c}{4a}$$

and $\cos\varphi = \dfrac{b^2+c^2-a^2}{2bc}$

$= \dfrac{b^2}{2bc} + \dfrac{c^2-a^2}{2bc}$

$= \dfrac{b}{2c} + \dfrac{(c-a)(c+a)}{(c+a)c}$

$= \dfrac{b}{2c} + \dfrac{(c-a)}{c}$

$= \dfrac{b+2c-2a}{2c}$

$= \dfrac{2b+4c-4a}{4c}$

$= \dfrac{a+c+4c-4a}{4c}$

$= \dfrac{5c-3a}{4c}$

Now, $4(1-\cos\theta)(1-\cos\varphi)$

$= 4\left(1-\dfrac{5a-3c}{4a}\right)\left(1-\dfrac{5c-3a}{4c}\right)$

$= 4\left(\dfrac{4a-5a+3c}{4a}\right)\left(\dfrac{4c-5c+3a}{4c}\right)$

$= 4\left(\dfrac{3c-a}{4a}\right)\left(\dfrac{3a-c}{4c}\right)$

$= \dfrac{1}{4}\left(\dfrac{3c-a}{a}\right)\left(\dfrac{3a-c}{c}\right)$

$= \dfrac{1}{4}\left(\dfrac{(3c-a)(3a-c)}{ac}\right)$

$= \dfrac{9ac-3c^2+ac-3a^2}{4ac}$

$= \dfrac{10ac-3c^2-3a^2}{4ac}$

$= \dfrac{5ac-3c^2+5ac-3a^2}{4ac}$

$= \dfrac{c(5a-3c)+a(5c-3a)}{4ac}$

$= \left(\dfrac{(5c-3a)}{4a} + \dfrac{(5a-3c)}{4c}\right)$

$= \cos\theta + \cos\varphi$

Hence, the result.

Ex-20. If α, β, γ are the lengths of the altitudes of a triangle ABC, then prove that

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{1}{\Delta}(\cot A+\cot B+\cot C)$$

Soln. Let $AD = \alpha$, $BE = \beta$ and $CF = \gamma$

Then,

$$\Delta = \frac{1}{2}\times a\times AD = \frac{1}{2}\times b\times BE = \frac{1}{2}\times c\times CF$$

$$\Rightarrow \quad AD = \frac{2\Delta}{a},\ BE = \frac{2\Delta}{b},\ CF = \frac{2\Delta}{c}$$

$$\Rightarrow \quad \alpha = \frac{2\Delta}{a},\ \beta = \frac{2\Delta}{b},\ \gamma = \frac{2\Delta}{c}$$

Now, $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}+\dfrac{1}{\gamma^2}$

$= \dfrac{a^2}{4\Delta^2}+\dfrac{b^2}{4\Delta^2}+\dfrac{c^2}{4\Delta^2}$

$= \dfrac{(a^2+b^2+c^2)}{4\Delta^2}$

$= \dfrac{1}{\Delta}\times\dfrac{(a^2+b^2+c^2)}{4\Delta}$

$= \dfrac{1}{\Delta}\times(\cot A+\cot B+\cot C)$

$= \dfrac{(\cot A+\cot B+\cot C)}{\Delta}$

Hence, the result.

Ex.-21. If p_1, p_2, p_3 are the altitudes of a triangle from the vertices a, b, c and Δ be the area of the triangle ABC, prove that

$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3} = \frac{2ab}{(a+b+c)\times\Delta}\times\cos^2\left(\frac{C}{2}\right).$$

Soln. Let $AD = p_1$, $BE = p_2$ and $CF = p_3$

Then, $\Delta = \dfrac{1}{2}\times a\times p_1 = \dfrac{1}{2}\times b\times p_2 = \dfrac{1}{2}\times c\times p_3$

$$\Rightarrow \quad p_1 = \frac{2\Delta}{a},\ p_2 = \frac{2\Delta}{b},\ p_3 = \frac{2\Delta}{c}$$

Now, $\dfrac{1}{p_1}+\dfrac{1}{p_2}-\dfrac{1}{p_3}$

$= \dfrac{a}{2\Delta}+\dfrac{b}{2\Delta}-\dfrac{c}{2\Delta}$

$= \dfrac{(a+b-c)}{2\Delta}$

$$= \frac{(a+b+c-2c)}{2\Delta}$$

$$= \frac{(2s-2c)}{2\Delta}$$

$$= \frac{(s-c)}{\Delta}$$

$$= \frac{2ab \times s(s-c)}{\Delta \times s} \times \frac{1}{2ab}$$

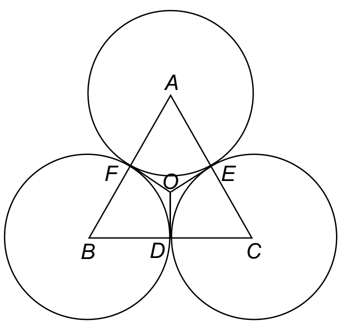
$$= \frac{2ab}{\Delta \times s} \times \frac{s(s-c)}{2ab}$$

$$= \frac{2\,ab}{(a+b+c)\Delta} \times \cos^2\left(\frac{C}{2}\right)$$

Ex-22. Three circles whose radii are a, b, c touch one another externally and the tangents at their points of contact meet in a point, prove that the distance of this point from either of their points of contact is

$$\left(\frac{abc}{a+b+c}\right)^{1/2}$$

Soln.



Let a, b, c be the centres of three circles whose radii are a, b and c, respectively.
Clearly, $OD = OF = OE$
So, O will be its in-centre.
Let $OD = OF = OE = r$
Let s be the semi-parameter of ΔABC

Thus, $s = \frac{a+b+b+c+c+a}{2} = a+b+c$

Area of a triangle ABC

$$= \sqrt{s(s-a-b)(s-b-c)(s-c-a)}$$

$$= \sqrt{(a+b+c)abc}$$

Now, $OD = r = \frac{\Delta}{s}$

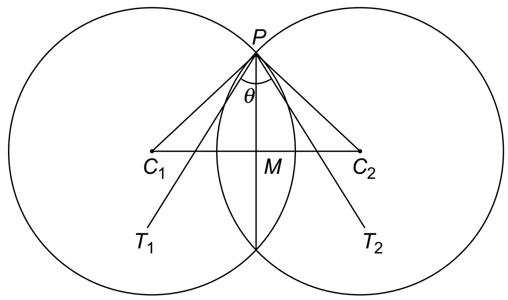
$$= \frac{\sqrt{(a+b+c)abc}}{(a+b+c)}$$

$$= \sqrt{\frac{abc}{(a+b+c)}}$$

Hence, the result.

Ex-23. Two circles of radii a and b cut each other at an angle θ, then prove that the length of the common chord is $\frac{2ab\sin\theta}{\sqrt{(a^2+b^2+2ab\cos\theta)}}$.

Soln.



Let $\angle C_1PC_2 = \theta$

Thus, $\cos(180° - \theta) = \frac{a^2+b^2-(C_1C_2)^2}{2ab}$

$$\Rightarrow \quad (C_1C_2)^2 = a^2+b^2+2ab\cos\theta$$

$$\Rightarrow \quad (C_1C_2) = \sqrt{a^2+b^2+2ab\cos\theta}$$

Area of $\Delta C_1PC_2 = \frac{1}{2}ab\sin\theta$

So, area of $\Delta C_1PC_2 = \frac{1}{2}.C_1C_2.PM$

$$\Rightarrow \quad \frac{1}{2}C_1C_2.PM = \frac{1}{2}ab\sin\theta$$

$$\Rightarrow \quad PM = \frac{ab\sin\theta}{C_1C_2}$$

$$\Rightarrow \quad PM = \frac{ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}$$

Hence, the length of the common chord

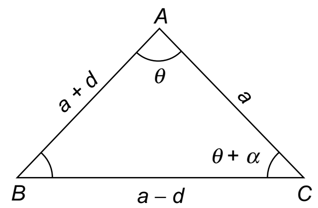
$= PQ = 2PM$

$$= \frac{2ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}$$

Ex-24. If the sides of a triangle are in A.P. and if the greatest angle exceeds the least angle α, then show that the sides are in the ratio $(1-x):x:(1+x)$

where $x=\sqrt{\dfrac{1-\cos\alpha}{7-\cos\alpha}}$

Soln. Let the sides be $a-d, a, a+d$



Consider $d>0$.

Thus, the greatest side is a + d and the smallest side is $a-d$.

Let $\angle A=\theta, \angle C=\theta+\alpha$ and

$\angle B=180°-(2\theta+\alpha)$

Applying sine rule, we get,

$$\frac{a-d}{\sin\theta}=\frac{a}{\sin(180°-(2\theta+\alpha))}=\frac{a+d}{\sin(\theta+\alpha)}$$

$$\Rightarrow \frac{a-d}{\sin\theta}=\frac{a}{\sin(2\theta+\alpha)}=\frac{a+d}{\sin(\theta+\alpha)}$$

$$=\frac{2a}{\sin\theta+\sin(\theta+\alpha)}$$

Now, $\dfrac{a-d}{\sin\theta}=\dfrac{a+d}{\sin(\theta+\alpha)}$

$$\Rightarrow \frac{a-d}{a+d}=\frac{\sin\theta}{\sin(\theta+\alpha)}$$

$$\Rightarrow \frac{2a}{2d}=\frac{\sin\theta+\sin(\theta+\alpha)}{\sin\theta-\sin(\theta+\alpha)}$$

$$\Rightarrow \frac{2a}{2d}=\frac{2\sin\left(\theta+\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{2\cos\left(\theta+\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)}$$

$$\Rightarrow \frac{a}{d}=\frac{\tan\left(\theta+\frac{\alpha}{2}\right)}{\tan\left(\frac{\alpha}{2}\right)}$$

$$\Rightarrow \frac{d}{a}=\frac{\tan\left(\frac{\alpha}{2}\right)}{\tan\left(\theta+\frac{\alpha}{2}\right)} \quad ....(i)$$

Also, $\dfrac{a}{\sin(2\theta+\alpha)}=\dfrac{2a}{\sin\theta+\sin(\theta+\alpha)}$

$$\Rightarrow \frac{\sin\theta+\sin(\theta+\alpha)}{\sin(2\theta+\alpha)}=2$$

$$\Rightarrow \frac{2\sin\left(\theta+\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{2\sin\left(\theta+\frac{\alpha}{2}\right)\cos\left(\theta+\frac{\alpha}{2}\right)}=2$$

$$\Rightarrow \frac{\cos\left(\frac{\alpha}{2}\right)}{\cos\left(\theta+\frac{\alpha}{2}\right)}=2$$

$$\Rightarrow \cos\left(\theta+\frac{\alpha}{2}\right)=\frac{\cos\left(\frac{\alpha}{2}\right)}{2}$$

$$\Rightarrow \tan\left(\theta+\frac{\alpha}{2}\right)=\frac{\sqrt{4-\cos^2\left(\frac{\alpha}{2}\right)}}{\cos\left(\frac{\alpha}{2}\right)} \quad ....(ii)$$

From (i) and (ii), we get,

$$\frac{d}{a}=\frac{\tan\left(\frac{\alpha}{2}\right)}{\dfrac{\sqrt{4-\cos^2\left(\frac{\alpha}{2}\right)}}{\cos\left(\frac{\alpha}{2}\right)}}=\frac{\sin\left(\frac{\alpha}{2}\right)}{\sqrt{4-\cos^2\left(\frac{\alpha}{2}\right)}}$$

$$=\sqrt{\frac{\sin^2\left(\frac{\alpha}{2}\right)}{4-\cos^2\left(\frac{\alpha}{2}\right)}}$$

$$=\sqrt{\frac{\frac{1-\cos\alpha}{2}}{4-\frac{1+\cos\alpha}{2}}}$$

$$=\sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}=x$$

Hence, the required ratio is

$= a-d:a:a+d$

$=1-\dfrac{d}{a}:1:1+\dfrac{d}{a}$

$=1-x:1:1+x$.

Hence, the result.

Ex-25. The sides a, b, c of a triangle ABC are the roots of $x^3 - px^2 + qx - r = 0$, then prove that its area is $\frac{1}{4}\sqrt{p\left(4pq - p^3 - 8r\right)}$.

Soln. Given equation is $x^3 - px^2 + qx - r = 0$

Thus, $a+b+c = p, ab+bc+ca = q, abc = r$

Now, Δ^2

$= s(s-a)(s-b)(s-c)$

$= \frac{p}{2}\left(\frac{p}{2}-a\right)\left(\frac{p}{2}-b\right)\left(\frac{p}{2}-c\right)$

$= \frac{p}{2}\left[\left(\frac{p}{2}\right)^3 - (a+b+c)\left(\frac{p}{2}\right)^2 + (ab+bc+ca)\left(\frac{p}{2}\right) - abc\right]$

$= \frac{p}{2}\left[\left(\frac{p}{2}\right)^3 - \left(\frac{p}{2}\right)^2 p + \left(\frac{p}{2}\right)q - r\right]$

$= \frac{p}{2}\left(\frac{p^3 - 2p^3 + 4pq - 8r}{8}\right)$

$= \frac{p}{16}\left(4pq - p^3 - 8r\right)$

$\Delta = \frac{1}{4}\sqrt{p\left(4pq - p^3 - 8r\right)}$

Thus, area of a triangle = Δ

$= \frac{1}{4}\sqrt{p\left(4pq - p^3 - 8r\right)}$

Hence, the result.

Ex-26. Let O be a point inside a triangle ABC such that $\angle OAB = \angle OBC = \angle OCA = \omega$ then prove that

(i) $\cot\omega = \cot A + \cot B + \cot C$

(ii) $\operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$

Soln. Let $\angle OCB = C - \omega$

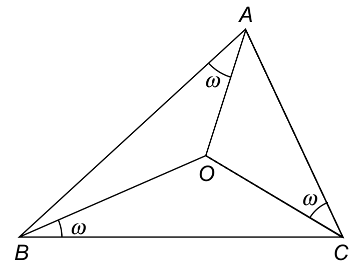
and $\angle BOC = 180° - \omega - (C - \omega) = 180° - C$

Similarly, $\angle AOB = 180° - B$

Now from ΔOAB, we have

$\frac{OB}{\sin\omega} = \frac{AB}{\sin(180° - B)}$

$\Rightarrow \quad \frac{OB}{\sin\omega} = \frac{c}{\sin B}$



$\Rightarrow \quad OB = \frac{c\sin\omega}{\sin B}$ ....(i)

Now, from ΔOBC, we get,

$\frac{OB}{\sin(C-\omega)} = \frac{BC}{\sin(180° - C)}$

$\Rightarrow \quad \frac{OB}{\sin(C-\omega)} = \frac{a}{\sin C}$

$\Rightarrow \quad OB = \frac{a\sin(C-\omega)}{\sin C}$ ....(ii)

From (i) and (ii), we get,

$\frac{c\sin\omega}{\sin B} = \frac{a\sin(C-\omega)}{\sin C}$

$\Rightarrow \quad \frac{k\sin C\sin\omega}{\sin B} = \frac{k\sin A\sin(C-\omega)}{\sin C}$

$\Rightarrow \quad \frac{\sin C\sin\omega}{\sin B} = \frac{\sin A\sin(C-\omega)}{\sin(A+B)}$

$\Rightarrow \quad \frac{\sin C\sin\omega}{\sin A\sin B} = \frac{\sin(C-\omega)}{\sin A\cos B + \cos A\sin B}$

$\Rightarrow \quad \frac{\sin C\sin\omega}{\sin A\sin B} = \frac{\sin C\cos\omega - \cos C\sin\omega}{\sin A\cos B + \cos A\sin B}$

$\Rightarrow \quad \sin C\sin A\cos B\sin\omega + \sin C\cos A\sin B\sin\omega$

$= \sin A\sin B\sin C\cos\omega - \sin A\sin B\cos C\sin\omega$

Dividing both the sides by $\sin A\sin B\sin C\sin\omega$ we get

$\cot B + \cot A = \cot\omega - \cot C$

$\Rightarrow \quad \cot A + \cot B + \cot C = \cot\omega$

(ii) We have

$\cot A + \cot B + \cot C = \cot\omega$

$\Rightarrow \quad (\cot A + \cot B + + \cot C)^2 = \cot^2\omega$

$\Rightarrow \quad \left(\cot^2 A + \cot^2 B + \cot^2 C\right) + 2 = \cot^2\omega$

$\Rightarrow \quad \operatorname{cosec}^2 A - 1 + \operatorname{cosec}^2 B - 1$

$+\text{cosec}^2 C - 1 + 2 = \text{cosec}^2\omega - 1$

$\Rightarrow \quad \text{cosec}^2 A + \text{cosec}^2 B + \text{cosec}^2 C = \text{cosec}^2\omega$

$\Rightarrow \quad \text{cosec}^2\omega = \text{cosec}^2 A + \text{cosec}^2 B + \text{cosec}^2 C$

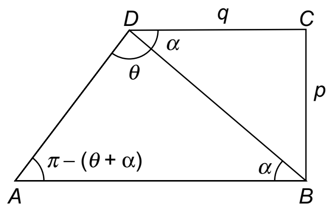
Hence, the result.

Ex-27. *ABCD* is a trapezium such that *AB* is parallel to *CD* and *CB* is perpendicular to them.

If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then

prove that $AB = \dfrac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$.

Soln.



From ΔBCD, we get, $BD = \sqrt{p^2+q^2}$

Let $\angle ABD = \angle BDC = \alpha$

then $\angle DAB = \pi - (\theta + \alpha)$.

Now, from ΔABD, we have

$$\frac{AB}{\sin\theta} = \frac{BD}{\sin(\pi - (\theta+\alpha))}$$

$$\Rightarrow \quad \frac{AB}{\sin\theta} = \frac{BD}{\sin(\theta+\alpha)}$$

$$\Rightarrow \quad AB = \frac{BD\sin\theta}{\sin(\theta+\alpha)}$$

$$\Rightarrow \quad AB = \frac{BD^2\sin\theta}{BD\sin(\theta+\alpha)}$$

$$= \frac{BD^2\sin\theta}{BD\sin\theta\cos\alpha + BD\cos\theta\sin\alpha}$$

$$\Rightarrow \quad AB = \frac{(p^2+q^2)\sin\theta}{q\sin\theta + p\cos\theta}$$

Hence, the result.

Ex-28. In an triangle *ABC*, if θ any angle, then prove that $b\cos\theta = c\cos(A-\theta) + a\cos(C+\theta)$

Soln. We have

$c\cos(A-\theta) + a\cos(C+\theta)$

$= k\sin C\cos(A-\theta) + k\sin A\cos(C+\theta)$

$= k[\sin C\cos(A-\theta) + \sin A\cos(C+\theta)]$

$= k[\sin C\cos A\cos\theta + \sin C\sin A\sin\theta$

$+ \sin A\cos C\cos\theta - \sin A\sin C\sin\theta]$

$= k[\cos\theta(\sin A\cos C + \cos A\sin C)]$

$= k\cos\theta\sin(A+C)$

$= k\cos\theta\sin(\pi - B)$

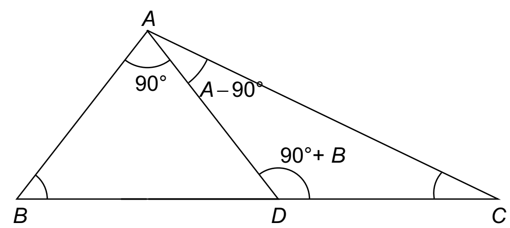
$= k\cos\theta\sin B$

$= (k\sin B)\cos\theta$

$= b\cos\theta$

Ex-29. If the median of a ΔABC through a is perpen dicular to *AB*, prove that $\tan A + 2\tan B = 0$.

Soln. Since *AD* is the median, so $BD : DC = 1 : 1$



Clearly, $\angle ADC = 90° + B$.

Now, applying $m : n$ rule, we get,

$(1+1)\cot(90° + B) = 1.\cot(90°) - 1.\cot(A - 90°)$

$\Rightarrow \quad -2\tan B = 0 - (-\tan A)$

$\Rightarrow \quad -2\tan B = \tan A$

$\Rightarrow \quad \tan A + 2\tan B = 0$

Hence, the result.

Ex-30. In a ΔABC, prove that $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \le \frac{1}{8}$

Soln. Let $u = \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

$$\Rightarrow \quad 2u = \left(2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\right)\sin\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad 2u = \left(\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right)\sin\left(\frac{C}{2}\right)$$

$$= \left(\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{C}{2}\right)\right)\sin\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad \sin^2\left(\frac{C}{2}\right) - \cos\left(\frac{A-B}{2}\right).\sin\left(\frac{C}{2}\right) + 2u = 0$$

since $\sin\left(\frac{C}{2}\right)$ is real, so $D \geq 0$

$$\Rightarrow \quad \cos^2\left(\frac{A-B}{2}\right) - 8u \geq 0$$

$$\Rightarrow \quad \cos^2\left(\frac{A-B}{2}\right) \geq 8u$$

$$\Rightarrow \quad u \leq \frac{1}{8}\cos^2\left(\frac{A-B}{2}\right)$$

$$\Rightarrow \quad u \leq \frac{1}{8}$$

Hence, $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \leq \frac{1}{8}$.

Ex-31. In a ΔABC, prove that

$$\tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right) \geq 1$$

Soln. Let $x = \tan\left(\frac{A}{2}\right), y = \tan\left(\frac{B}{2}\right), z = \tan\left(\frac{C}{2}\right)$

To prove, $x^2 + y^2 + z^2 \geq 1$

Now, $x^2 + y^2 + z^2 - xy - yz - zx$

$$= \frac{1}{2}\left[(x-y)^2 + (y-z)^2 + (z-x)^2\right] \geq 0$$

Thus, $x^2 + y^2 + z^2 \geq xy + yz + zx$ ......(i)

Since in ΔABC,

$$A + B + C = \pi$$

$$\Rightarrow \quad \frac{A}{2} + \frac{B}{2} = \frac{\pi}{2} - \frac{C}{2}$$

$$\Rightarrow \quad \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right)$$

$$\Rightarrow \quad \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \cot\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad \frac{\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right)}{1 - \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)} = \frac{1}{\tan\left(\frac{C}{2}\right)}$$

$$\Rightarrow \tan\left(\frac{A}{2}\right)\tan\left(\frac{C}{2}\right) + \tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right) + \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right) = 1$$

$$\Rightarrow \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right) + \tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right) + \tan\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right) = 1$$

$$\Rightarrow \quad xy + yz + zx = 1$$

Therefore, from (i), we get,

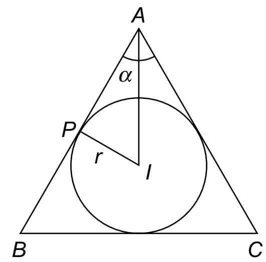
$$x^2 + y^2 + z^2 \geq 1$$

$$\Rightarrow \quad \tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right) \geq 1$$

Ex-32. If the distances of the vertices of a triangle from the points of contact of the incircle with the sides be α, β, γ, then prove that

$r^2 = \frac{\alpha\beta\gamma}{\alpha + \beta + \gamma}$ where r is in-radius.

Soln. Let the in-circle touches the side AB at p where $AP = \alpha$



Let I be its in-centre and AI bisects $\angle BAC$

Now, from ΔIPA,

$$\tan\left(\frac{A}{2}\right) = \frac{r}{\alpha}$$

$$\Rightarrow \quad \alpha = r\cot\left(\frac{A}{2}\right)$$

Similarly, $\beta = r\cot\left(\frac{B}{2}\right)$, $\gamma = r\cot\left(\frac{C}{2}\right)$

In a triangle ABC, we have

$$\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right) = \cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad \frac{\alpha}{r} + \frac{\beta}{r} + \frac{\gamma}{r} = \frac{\alpha}{r}.\frac{\beta}{r}.\frac{\gamma}{r}$$

$$\Rightarrow \quad \frac{\alpha + \beta + \gamma}{r} = \frac{\alpha\beta\gamma}{r^3}$$

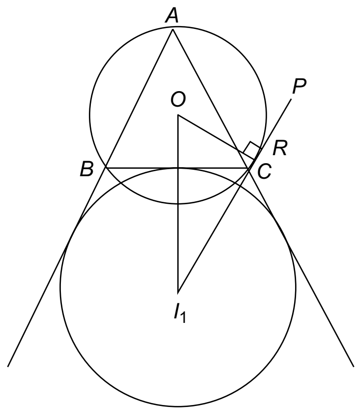
$$\Rightarrow \quad r^2 = \frac{\alpha\beta\gamma}{\alpha + \beta + \gamma}$$

Hence, the result.

Ex-33. If t_1, t_2 and t_3 are the lengths of the tangents drawn from the centre of the ex-circle to the circum-circle of ΔABC, prove that

$$\frac{1}{t_1^2} + \frac{1}{t_2^2} + \frac{1}{t_3^2} = \frac{abc}{a + b + c}$$

Soln.



Let O and I_1 be respectively the centres of the circum-circle and the ex-circle touching the line BC

Clearly, $OI_1 = \sqrt{R^2 + 2Rr_1}$

$$\Rightarrow \quad \sqrt{R^2 + t_1^2} = \sqrt{R^2 + 2Rr_1}$$

$$\Rightarrow \quad \left(R^2 + t_1^2\right) = \left(R^2 + 2Rr_1\right)$$

$$\Rightarrow \quad t_1^2 = 2Rr_1$$

$$\Rightarrow \quad \frac{1}{t_1^2} = \frac{1}{2Rr_1}$$

Similarly, $\dfrac{1}{t_2^2} = \dfrac{1}{2Rr_2}, \dfrac{1}{t_3^2} = \dfrac{1}{2Rr_3}$

Now, $\dfrac{1}{t_1^2} + \dfrac{1}{t_2^2} + \dfrac{1}{t_3^2}$

$$= \frac{1}{2Rr_1} + \frac{1}{2Rr_2} + \frac{1}{2Rr_3}$$

$$= \frac{1}{2R}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)$$

$$= \frac{1}{2R}.\frac{1}{r}$$

$$= \frac{1}{2R}.\frac{s}{\Delta}$$

$$= \frac{1}{2.\dfrac{abc}{4\Delta}}.\frac{(a+b+c)}{2\Delta}$$

$$= \frac{(a+b+c)}{abc}$$

Hence, the result.

Ex-34. In ΔABC, find the min value of

$$\frac{\sum \cot^2\left(\frac{A}{2}\right)\cot^2\left(\frac{B}{2}\right)}{\prod \cot^2\left(\frac{A}{2}\right)}$$

Soln. We have $\dfrac{\sum \cot^2\left(\frac{A}{2}\right)\cot^2\left(\frac{B}{2}\right)}{\prod \cot^2\left(\frac{A}{2}\right)}$

$$= \frac{\left(\cot^2\left(\frac{A}{2}\right)\cot^2\left(\frac{B}{2}\right) + \cot^2\left(\frac{B}{2}\right)\cot^2\left(\frac{C}{2}\right) + \cot^2\left(\frac{C}{2}\right)\cot^2\left(\frac{A}{2}\right)\right)}{\cot^2\left(\frac{A}{2}\right)\cot^2\left(\frac{B}{2}\right)\cot^2\left(\frac{C}{2}\right)}$$

$$= \frac{1}{\cot^2\left(\frac{A}{2}\right)} + \frac{1}{\cot^2\left(\frac{B}{2}\right)} + \frac{1}{\cot^2\left(\frac{C}{2}\right)}$$

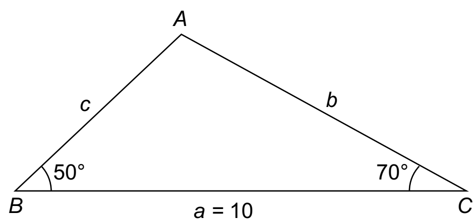
$$= \tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right)$$

$$\geq 1$$

Hence, the minimum value is 1.

Ex-35. A triangle has base 10 cm long and the base angles are $50°$ and $70°$. If the perimeter of the triangle is $x + y\cos(z°)$ where $z \in (0, 90°)$, then find the value of $(x + y + z)$.

Soln. Let $BC = a = 10$



From sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{a}{\sin(60°)} = \frac{b}{\sin(50°)} = \frac{c}{\sin(70°)}$$

$$= \frac{a+b+c}{\sin(60°) + \sin(50°) + \sin(70°)}$$

$$= \frac{a+b+c}{\frac{\sqrt{3}}{2} + 2\sin(60°)\cos(10°)}$$

$$= \frac{a+b+c}{\frac{\sqrt{3}}{2}+\sqrt{3}\cos(10°)}$$

$$\Rightarrow \quad \frac{a}{\sin(60°)} = \frac{a+b+c}{\frac{\sqrt{3}}{2}+\sqrt{3}\cos(10°)}$$

$$\Rightarrow \quad \frac{10}{\frac{\sqrt{3}}{2}} = \frac{a+b+c}{\frac{\sqrt{3}}{2}+\sqrt{3}\cos(10°)}$$

$$\Rightarrow \quad \frac{10}{1} = \frac{a+b+c}{1+2\cos(10°)}$$

$$\Rightarrow \quad a+b+c = 10+20\cos(10°)$$

$$\Rightarrow \quad a+b+c = x+y\cos(z°)$$

$$\Rightarrow \quad x=10, y=20, z=10$$

Hence, the value of $(x+y+z)$

$= 10+20+10 = 40$.

Ex-36 In a ΔABC, find the value of $\dfrac{a\cos A + b\cos B + c\cos C}{a+b+c}$

Soln. We have $\dfrac{a\cos A + b\cos B + c\cos C}{a+b+c}$.

$$= \frac{1}{2}\left(\frac{2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C}{\sin A + \sin B + \sin C}\right)$$

$$= \frac{1}{2}\left(\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C}\right)$$

$$= \frac{1}{2}\left(\frac{4\sin A\sin B\sin C}{\sin A + \sin B + \sin C}\right)$$

$$= \left(\frac{2\sin A\sin B\sin C}{\sin A + \sin B + \sin C}\right)$$

$$= \left(\frac{2\sin A\sin B\sin C}{4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)}\right)$$

$$= \left(\frac{16\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)}{4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)}\right)$$

$$= 4\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right)$$

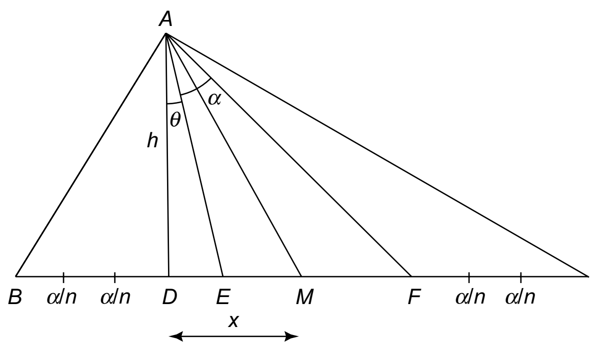
$$= \frac{1}{R}\times 4R\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right)$$

$$= \frac{r}{R}$$

Ex-37. In a right angled triangle ABC, the hypotenuse BC of length 'a' is divided into n equal parts (n an odd positive integers). Let α be the acute angle subtending from a by that segment which contains the mid-point of the hypotenuse of the triangle, prove that

$$\tan\alpha = \frac{4nh}{(n^2-1)a} \qquad \text{[Roorkee – 1983]}$$

Soln.



Each part of the base BC be $\dfrac{\alpha}{n}$

Let $AD = h$ is the altitude.

The nth part EF which contains the middle point M subtends an angle α at A.

Let $\angle DAE = \theta$ and $\angle EAF = \alpha$

$\therefore \angle DAF = \theta + \alpha$

Also, let $DM = x$

Then $DE = x - \dfrac{\alpha}{2n}$ and $DF = x + \dfrac{\alpha}{2n}$

In ΔADF,

$$\tan\theta = \frac{DE}{AD} = \frac{x-\frac{a}{2n}}{h} = \frac{2nx-a}{2nh}$$

$$2nh\tan\theta = (2nx-a) \quad ....(i)$$

Also, in ΔADF

$$\tan(\alpha+\theta) = \frac{x+\frac{a}{2n}}{h} = \frac{2nx+a}{2nh} \quad ....(ii)$$

Elimination θ from (i) and (ii), we get,

$$2nh\tan\alpha + 2nx - a$$

$$= 2nx + a - (2nx + a)\left(\frac{(2nx - a)\tan\alpha}{2nx}\right)$$

$$\Rightarrow \quad 2nh\tan\alpha = 2a - \frac{(4n^2x^2 - a^2)}{2nh}\tan\alpha$$

$$\Rightarrow \quad \left(2nh + \frac{(4n^2x^2 - a^2)}{2nh}\right)\tan\alpha = 2a$$

$$\Rightarrow \quad \left(\frac{4n^2h^2 + (4n^2x^2 - a^2)}{2nh}\right)\tan\alpha = 2a$$

$$\Rightarrow \quad \tan\alpha = \frac{4anh}{4n^2h^2 + (4n^2x^2 - a^2)}$$

$$\Rightarrow \quad \tan\alpha = \frac{4anh}{4n^2(h^2 + x^2) - a^2}$$

$$\Rightarrow \quad \tan\alpha = \frac{4anh}{4n^2\left(\frac{a^2}{4}\right) - a^2}$$

$$\left(\because AM = BM = \frac{a}{2}\right)$$

$$\Rightarrow \quad \tan\alpha = \frac{4a\,nh}{(n^2 - 1)a^2}$$

$$\Rightarrow \quad \tan\alpha = \frac{4nh}{(n^2 - 1)a}$$

Note. No questions asked in 1984, 1985.

Ex-38. If a, b, c are the sides of a triangle ABC and $3a = b + c$, then prove that $\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right) = 2.$

[Roorkee – 1986]

Soln. We have $\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$

$$= \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \times \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \sqrt{\frac{s(s-b)s(s-c)}{(s-a)(s-c)(s-a)(s-b)}}$$

$$= \sqrt{\frac{s^2}{(s-a)^2}}$$

$$= \frac{s}{s-a}$$

$$= \frac{2s}{2s-2a}$$

$$= \frac{(a+b+c)}{(a+b+c)-2a}$$

$$= \frac{(a+b+c)}{(b+c-a)}$$

$$= \frac{(a+3a)}{(3a-a)} = \frac{4a}{2a} = 2.$$

Ex-39. If in a triangle, $\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A-B)}{\sin(A+B)}$.

Prove that it is either a right angled triangle or an isosceles triangle. [Roorkee – 1987]

Soln. We have $\frac{a^2 - b^2}{a^2 + b^2} = \frac{\sin(A-B)}{\sin(A+B)}$.

$$\Rightarrow \quad \frac{\sin^2 A - \sin^2 B}{\sin^2 A + \sin^2 B} = \frac{\sin(A-B)}{\sin(A+B)}$$

$$\Rightarrow \quad \frac{\sin(A+B)\sin(A-B)}{\sin^2 A + \sin^2 B} = \frac{\sin(A-B)}{\sin(A+B)}$$

$$\Rightarrow \quad \sin(A-B) = 0,$$

and $\left(\frac{\sin(A+B)}{\sin^2 A + \sin^2 B} - \frac{1}{\sin(A+B)}\right) = 0$

Now, $\sin(A-B) = 0$

$\Rightarrow \quad A = B$

$\Rightarrow \quad \Delta$ is isosceles

and $\left(\frac{\sin(A+B)}{\sin^2 A + \sin^2 B} - \frac{1}{\sin(A+B)}\right) = 0$

$$\Rightarrow \quad \sin^2 A + \sin^2 B = \sin^2(A+B)$$

$$\Rightarrow \quad \sin^2 A + \sin^2 B = \sin^2 A\cos^2 B + \cos^2 A\sin^2 B + 2\sin A\sin B\cos A\cos B$$

$$\Rightarrow \quad -2\sin^2 A\sin^2 B + 2\sin A\sin B\cos A\cos B = 0$$

$$\Rightarrow \quad \sin A\sin B = \cos A\cos B$$

$$\Rightarrow \quad \tan A\tan B = 1$$

$\Rightarrow \quad \tan A = \cot B$

$\Rightarrow \quad \tan A = \tan\left(\frac{\pi}{2} - B\right)$

$\Rightarrow \quad A = \frac{\pi}{2} - B$

$\Rightarrow \quad A + B = \frac{\pi}{2}$

$\Rightarrow \quad C = \pi - (A+B) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

Thus, the triangle ΔABC is right angled.

Ex-40. In any triangle ABC, show that

$$\left(\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right)\right)\left(a\sin^2\left(\frac{B}{2}\right) + b\sin^2\left(\frac{A}{2}\right)\right)$$

$= c\cot\left(\frac{C}{2}\right)$. [Roorkee – 1988]

Soln. We have

$$\left(\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right)\right)\left(a\sin^2\left(\frac{B}{2}\right) + b\sin^2\left(\frac{A}{2}\right)\right)$$

$$= \left(\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\right)$$

$$\times\left(a.\frac{(s-a)(s-c)}{ac} + b.\frac{(s-b)(s-c)}{bc}\right)$$

$$= \sqrt{\frac{s}{(s-c)}}\left[\sqrt{\frac{(s-a)}{(s-b)}} + \sqrt{\frac{(s-b)}{(s-a)}}\right]$$

$$\times\left(\frac{(s-a)(s-c)}{c} + \frac{(s-b)(s-c)}{c}\right)$$

$$= \sqrt{\frac{s}{(s-c)}}\left(\frac{s-a+s-b}{\sqrt{(s-a)(s-b)}}\right)$$

$$\times\frac{s-c}{c}\left((s-a)+(s-b)\right)$$

$$= \left(\frac{c\sqrt{s}}{\sqrt{(s-c)(s-a)(s-b)}}\right) \times \frac{s-c}{c}\left(2s-(a+b)\right)$$

$$= \left(\frac{c\sqrt{s}}{\sqrt{(s-c)(s-a)(s-b)}}\right) \times (s-c)$$

$$= c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = c\cot\left(\frac{C}{2}\right).$$

Note. No questions asked in 1989.

Ex-41. If x, y, z are the perpendicular distances of the vertices of a triangle ABC from the opposite sides and Δ be the area of the triangle, then prove that

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{\Delta}(\cot A + \cot B + \cot C).$$

[Roorkee Main – 1990]

Soln. From ΔABC,

$$\Delta = \frac{1}{2}a\,x = \frac{1}{2}a\,y = \frac{1}{2}a\,z$$

$$\Rightarrow \quad \frac{1}{x} = \frac{a}{2\Delta}, \frac{1}{y} = \frac{b}{2\Delta}, \frac{1}{z} = \frac{c}{2\Delta}$$

Now, L.H.S

$$= \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$$

$$= \frac{a^2+b^2+c^2}{4\Delta^2}$$

and R.H.S

$$= \frac{1}{\Delta}(\cot A + \cot B + \cot C)$$

$$= \frac{1}{\Delta k}\left(\frac{b^2+c^2-a^2}{2abc} + \frac{a^2+c^2-b^2}{2bac} + \frac{a^2+b^2-c^2}{2abc}\right)$$

$$= \frac{1}{\Delta k}\left(\frac{a^2+b^2+c^2}{2abc}\right)$$

$$= \frac{1}{\Delta}.\left(\frac{a^2+b^2+c^2}{2.2\Delta}\right)$$

$$\left(\because \Delta = \frac{1}{2}ab\sin C = \frac{1}{2}(abc\,k)\right)$$

$$= \left(\frac{a^2+b^2+c^2}{4\Delta^2}\right)$$

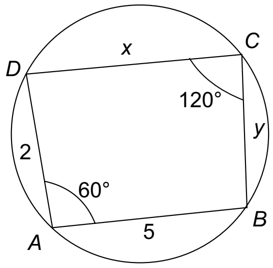
Hence, the result.

Ex-42. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, find the remaining two sides. [Roorkee Main – 1991]

Soln. Suppose $AC = 2$, $AB = 5$, $BC = x$, $CD = y$ and $\angle BAD = 60°$

Area of $\Delta ABC = \frac{1}{2}.5.2.\sin(60°)$

$$= \frac{5\sqrt{3}}{2}$$



Also, from ΔABC,

$$\cos(60°) = \frac{25+4-BD^2}{2.5.2}$$

$$= \frac{29-BD^2}{20}$$

$$\Rightarrow \quad \frac{29-BD^2}{20} = \frac{1}{2}$$

$$\Rightarrow \quad BD^2 = 19$$

$$\Rightarrow \quad BD = \sqrt{19}$$

Since A, B, C, D are concyclic, so

$$\angle BCD = 180° - 60° = 120°$$

Then from ΔBCD,

$$\cos(120°) = \frac{x^2+y^2-\left(\sqrt{19}\right)^2}{2xy}$$

$$\Rightarrow \quad \frac{x^2+y^2-\left(\sqrt{19}\right)^2}{2xy} = -\frac{1}{2}$$

$$\Rightarrow \quad \frac{x^2+y^2-\left(\sqrt{19}\right)^2}{xy} = -1$$

$$\Rightarrow \quad x^2+y^2+xy = 19 \text{ ....(i)}$$

Again, area of ΔBCD

$$= \frac{1}{2}.x.y.\frac{\sqrt{3}}{2} = \frac{\sqrt{3}\,xy}{4}.$$

Thus, area of quad. $ABCD = 4\sqrt{3}$

$$\Rightarrow \quad \frac{5\sqrt{3}}{2} + \frac{\sqrt{3}\,xy}{4} = 4\sqrt{3}$$

$$\Rightarrow \quad \frac{5}{2} + \frac{xy}{4} = 4$$

$$\Rightarrow \quad \frac{xy}{4} = 4 - \frac{5}{2} = \frac{3}{2}$$

$$\Rightarrow \quad xy = 6$$

From (i), we get,

$$x^2 + y^2 = 13$$

$$\Rightarrow \quad x = 3, y = 2$$

Note. No questions asked in 1992.

Ex-43. In a triangle ABC, R is the circum-radius and $8R^2 = a^2 + b^2 + c^2$. The the triangle ABC is

(a) Acute angled
(b) Right angled
(c) Obtuse angled
(d) None of these.

Soln. As we know that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Now we have,

$$8R^2 = a^2 + b^2 + c^2$$

$$\Rightarrow \quad 8R^2 = 4R^2\left(\sin^2 A + \sin^2 B + \sin^2 C\right)$$

$$\Rightarrow \quad \left(\sin^2 A + \sin^2 B + \sin^2 C\right) = 2$$

$$\Rightarrow \quad \left(\cos^2 A - \sin^2 C\right) + \cos^2 B = 0$$

$$\Rightarrow \quad \cos(A+C)\cos(A-C) + \cos^2 B = 0$$

$$\Rightarrow \quad \cos(\pi - B)\cos(A-C) + \cos^2 B = 0$$

$$\Rightarrow \quad -\cos B\cos(A-C) + \cos^2 B = 0$$

$$\Rightarrow \quad \cos B\left(\cos(A-C) - \cos B\right) = 0$$

$$\Rightarrow \quad \cos B.2\cos A\cos C = 0$$

$$\Rightarrow \quad \cos A = 0, \cos B = 0, \cos C = 0$$

$$\Rightarrow \quad A = \frac{\pi}{2} = B = C$$

Thus, the triangle is right angled.

Ex-44. If the sides a, b, c of a triangle are in A.P., then find the value of $\tan\left(\frac{A}{2}\right) + \tan\left(\frac{C}{2}\right)$ in terms of $\cot\left(\frac{B}{2}\right)$. [Roorkee Main – 1993]

Soln. We have $\tan\left(\frac{A}{2}\right) + \tan\left(\frac{C}{2}\right)$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \sqrt{\frac{(s-b)}{s}}\left(\sqrt{\frac{(s-c)}{(s-a)}} + \sqrt{\frac{(s-a)}{(s-c)}}\right)$$

$$= \sqrt{\frac{(s-b)}{s}}\left(\frac{s-c+s-a}{\sqrt{(s-a)(s-c)}}\right)$$

$$= \sqrt{\frac{(s-b)}{s}}\left(\frac{2s-c-a}{\sqrt{(s-a)(s-c)}}\right)$$

$$= \sqrt{\frac{(s-b)}{s(s-a)(s-c)}} \times (a+b+c-c-a)$$

$$= b \times \sqrt{\frac{(s-b)}{s(s-a)(s-c)}}$$

$$= \frac{2s}{3} \times \sqrt{\frac{(s-b)}{s(s-a)(s-c)}}$$

$$= \frac{2}{3} \times \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}$$

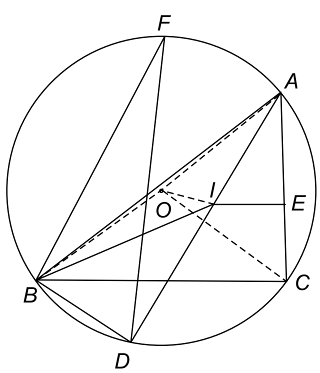
$$= \frac{2}{3} \times \cot\left(\frac{B}{2}\right)$$

Hence, the result.

Note. No questions asked in 1994.

Ex-45. A cyclic quadrilateral $ABCD$ of area $\frac{3\sqrt{3}}{4}$ is inscribed in a unit circle. If one of its sides $AB = 1$ and the diagonal $BD = \sqrt{3}$. Find the length of the other sides. [Roorkee Main – 1995]

Soln.



In ΔAOB,

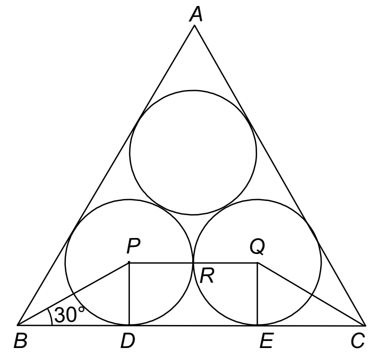
$OA = OB = 1$

and $AB = 1$ (given)

Thus, $\angle OAB = \angle OBA = \angle AOB = 60°$

In ΔBOD, $OA = 1$, $OB = 1$, $OB = 1$

and $BD = \sqrt{3}$



Let $\angle BOD = \theta$

Then $\cos\theta = \frac{OB^2 + OD^2 - BD^2}{2.OB.OD}$

$\Rightarrow \quad \cos\theta = \frac{1+1-3}{2.1.1} = -\frac{1}{2}$

$\Rightarrow \quad \theta = 120°$

Therefore, $\angle AOD = \angle AOB + \angle BOD$

$= 60° + 120° = 180°$

$\Rightarrow \quad AOD$ is a straight line

$\Rightarrow \quad AD$ = diameter = 2

If $\angle BOC = \varphi$, then $\angle COD = \frac{2\pi}{3} - \varphi$

Area of the cyclic parallelogram $ABCD$

$= ar\ of\left(\Delta AOB + \Delta BOC + \Delta BOD\right)$

$$\Rightarrow \quad \frac{3\sqrt{3}}{4} = \frac{1}{2}.1.1.\sin(60°) + \frac{1}{2}.1.1.\sin\varphi + \frac{1}{2}.1.1.\sin\left(\frac{2\pi}{3} - \varphi\right)$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \sin\varphi + \sin\left(\frac{2\pi}{3} - \varphi\right)\right) = \frac{3\sqrt{3}}{4}$$

$$\Rightarrow \quad \frac{1}{2}\left(\sin\varphi + \sin\left(\frac{2\pi}{3} - \varphi\right)\right) = \frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{4}$$

$$\Rightarrow \quad \frac{1}{2}\left(\sin\varphi + \sin\left(\frac{2\pi}{3} - \varphi\right)\right) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \left(\sin\varphi + \sin\left(\frac{2\pi}{3} - \varphi\right)\right) = \sqrt{3}$$

$$\Rightarrow \quad 2\sin\left(\frac{\pi}{3}\right)\cos\left(\varphi - \frac{\pi}{3}\right) = \sqrt{3}$$

$$\Rightarrow \quad 2 \times \frac{\sqrt{3}}{2} \times \cos\left(\varphi - \frac{\pi}{3}\right) = \sqrt{3}$$

$\Rightarrow \quad \cos\left(\varphi - \frac{\pi}{3}\right) = 1$

$\Rightarrow \quad \left(\varphi - \frac{\pi}{3}\right) = 0$

$\Rightarrow \quad \varphi = \frac{\pi}{3}$

Thus $\angle BOC = 60°$

and $\angle COD = 120° - 60° = 60°$

In $\frac{BC}{\sin 60°} = \frac{OB}{\sin 60°}$

$BC = OB = 1$

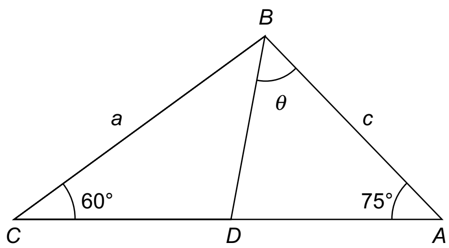
$\Rightarrow \quad \angle C = 60°$ and $\angle A = 75°$

Similarly, we can prove that $CD = 1$

Hence, $AB = 1$, $BC = 1$, $CD = 1$ and $AD = 2$.

Ex-46. In a triangle ABC, $\angle C = 60°$ and $\angle A = 75°$. If D is a point on AC such that the area of the triangle BAD is $\sqrt{3}$ times the area of the triangle BCD, find the angle $\angle ABD$. [Roorkee Main – 1996]

Soln.



Here, $\angle B = 180° - (60° + 75°) = 45°$

Let $\angle ABD = \theta$

It is given that, $\frac{ar(\Delta BAD)}{ar(\Delta BCD)} = \sqrt{3}$

$\Rightarrow \quad \frac{\frac{1}{2} cx \sin\theta}{\frac{1}{2} ax \sin(45° - \theta)} = \sqrt{3}$

$\Rightarrow \quad \frac{c \sin\theta}{a \sin(45° - \theta)} = \sqrt{3}$

$\Rightarrow \quad \frac{\sin C \sin\theta}{\sin A \sin(45° - \theta)} = \sqrt{3}$

$\Rightarrow \quad \frac{\sin 60° \sin\theta}{\sin 75° \sin(45° - \theta)} = \sqrt{3}$

$\Rightarrow \quad \frac{\frac{\sqrt{3}}{2}\sin\theta}{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\sin(45° - \theta)} = \sqrt{3}$

$\Rightarrow \quad \sqrt{2}\sin\theta = \left(\sqrt{3}+1\right)\sin\left(45° - \theta\right)$

$\Rightarrow \quad \sqrt{2}\sin\theta = \left(\sqrt{3}+1\right)\frac{1}{\sqrt{2}}\left(\cos\theta - \sin\theta\right)$

$\Rightarrow \quad 2\sin\theta = \left(\sqrt{3}+1\right)\left(\cos\theta - \sin\theta\right)$

$\Rightarrow \quad \left(3+\sqrt{3}\right)\sin\theta = \left(\sqrt{3}+1\right)\cos\theta$

$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{\left(\sqrt{3}+1\right)}{\left(3+\sqrt{3}\right)} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \tan\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta = \frac{\pi}{6}$

Hence, $\angle ABD = \frac{\pi}{6}$.

Ex-47 If in a triangle ABC, $a = 6$, $b = 3$ and $\cos(A - B) = \frac{4}{5}$, then find its area. [Roorkee Main 1997]

Soln. Given $\cos(A - B) = \frac{4}{5}$

$\Rightarrow \quad 2\cos^2\left(\frac{A-B}{2}\right) - 1 = \frac{4}{5}$

$\Rightarrow \quad 2\cos^2\left(\frac{A-B}{2}\right) = 1 + \frac{4}{5}$

$\Rightarrow \quad \cos^2\left(\frac{A-B}{2}\right) = \frac{9}{10}$

$\Rightarrow \quad \cos\left(\frac{A-B}{2}\right) = \frac{3}{\sqrt{10}}$

$\Rightarrow \quad \tan\left(\frac{A-B}{2}\right) = \frac{1}{3}$

$\Rightarrow \quad \left(\frac{a-b}{a+b}\right)\cot\left(\frac{C}{2}\right) = \frac{1}{3}$

$\Rightarrow \quad \left(\frac{6-3}{6+3}\right)\cot\left(\frac{C}{2}\right) = \frac{1}{3}$

$\Rightarrow \quad \frac{1}{3}\cot\left(\frac{C}{2}\right) = \frac{1}{3}$

$\Rightarrow \quad \cot\left(\frac{C}{2}\right) = 1$

$\Rightarrow \quad \cot\left(\frac{C}{2}\right) = \cot\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad C = \frac{\pi}{2}$

Thus, ar(ΔABC)

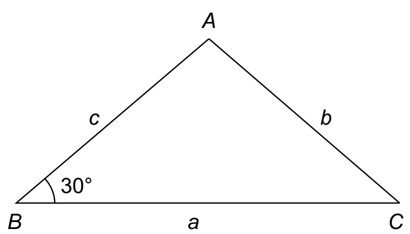
$= \frac{1}{2} ab \sin C$

$= \frac{1}{2} \times 6 \times 3 \times \sin(90°)$

= 9 sq units.

Ex-48. Two sides of a triangle are of lengths $\sqrt{6}$ and 4 and the angle opposite to smaller side is 30°. How many such triangles are possible?
Find the length of their third side and area.
[Roorkee Main – 1998]

Soln.



We have

$\frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow \quad \frac{\sqrt{6}}{\sin(30°)} = \frac{4}{\sin C}$

$\Rightarrow \quad \sin C = \frac{4}{\sqrt{6}} \times \frac{1}{2} = \frac{2}{\sqrt{6}} < 1$

C may be acute or obtuse

Also, we observe that $b < c$

$\Rightarrow B < C$

i.e., if C is obtuse, then B should be acute

$\Rightarrow$ It is possible.

Thus, two triangles are possible in this case.

Now, applying cosine rule,

$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$

$\Rightarrow \quad \cos(30°) = \frac{16 + a^2 - 6}{2.4.a}$

$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{16 + a^2 - 6}{2.4.a}$

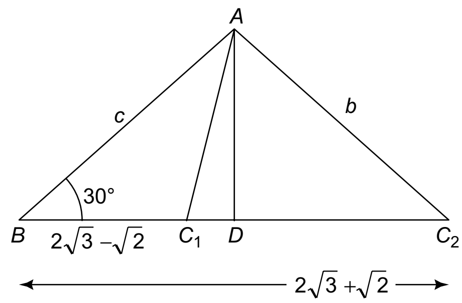
$\Rightarrow \quad \frac{a^2 + 10}{8a} = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad a^2 - 4\sqrt{3}\,a + 10 = 0$

$\Rightarrow \quad \left(a - 2\sqrt{3}\right)^2 = 12 - 10 = 2$

$\Rightarrow \quad \left(a - 2\sqrt{3}\right) = \pm\sqrt{2}$

$\Rightarrow \quad a = 2\sqrt{3} \pm \sqrt{2}$



Now, $ar\left(\Delta ABC_1\right)$

$= \frac{1}{2} \times 4 \times \left(2\sqrt{3} - \sqrt{2}\right) \times \sin(30°)$

$= \left(2\sqrt{3} - \sqrt{2}\right)$ sq units

and $ar\left(\Delta ABC_2\right)$

$= \frac{1}{2} \times 4 \times \left(2\sqrt{3} + \sqrt{2}\right) \times \sin(30°)$

$= \left(2\sqrt{3} + \sqrt{2}\right)$ sq. units.

Ex-49. The radii r_1, r_2, r_3 of escribed circles of a triangle ABC are in the harmonic progression. If its area is 24sq. cm. and its perimeter is 24 cm, then find the lengths of its sides. [Roorkee Main – 1999]

Soln. Given $2\,s = 24$

$\Rightarrow \quad s = 12$

Also, $r_1, r_2, r_3 \in H.P$

$\Rightarrow \quad \frac{\Delta}{s-a}, \frac{\Delta}{s-b}, \frac{\Delta}{s-c} \in H.P$

$\Rightarrow \quad \frac{\Delta}{s-a}, \frac{\Delta}{s-b}, \frac{\Delta}{s-c} \in H.P$

$\Rightarrow \quad \frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c} \in H.P$

$\Rightarrow \quad (s-a), (s-b), (s-c) \in A.P$

$\Rightarrow \quad a, b, c \in A.P$

$\Rightarrow \quad 2b = a + c$

$\Rightarrow \quad 3b = a + b + c = 24$

$\Rightarrow \quad b = 8$

Again, $r_1, r_2, r_3 \in H.P$

$\Rightarrow \quad \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3} \in A.P$

$\Rightarrow \quad \frac{2}{r_2} = \frac{1}{r_1} + \frac{1}{r_3}$

$\Rightarrow \quad r_2 = \frac{2r_1r_3}{r_1 + r_3}$

$\Rightarrow \quad r_1r_2 + r_2r_3 = 2r_1r_3$

$\Rightarrow \quad r_1r_2 + r_1r_3 + r_2r_3 = 2r_1r_3 + r_1r_3 = 3r_1r_3$

$\Rightarrow \quad 3r_1r_3 = s^2 = 144$

$\Rightarrow \quad r_1r_3 = \frac{144}{3} = 48$

$\Rightarrow \quad \frac{\Delta}{(s-a)} \times \frac{\Delta}{(s-c)} = 48$

$\Rightarrow \quad \frac{24}{(12-a)} \times \frac{24}{(12-c)} = 48$

$\Rightarrow \quad (12-a)(12-c) = 12$

$\Rightarrow \quad 144 - 12(a+c) + ac = 12$

$\Rightarrow \quad 144 - 12.16 + c(16-c) = 12$

$\Rightarrow \quad 16c - c^2 - 64 = 0$

$\Rightarrow \quad c^2 - 16c + 64 = 0$

$\Rightarrow \quad (c-8)^2 = 0$

$\Rightarrow \quad c = 8$

So, the lengths of the sides are 8 cm, 8 cm, 8 cm.

Note. No questions asked in 2000, 2001.

Ex-50. In a triangle *ABC*, let the sides *a*, *b*, *c* are the roots of $x^3 - 11x^2 + 38x - 40 = 0$. If the value of $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{m}{n}$, where *m* and *n* are least +ve integers, then find $(m+n)$.

Soln. Given equation is $x^3 - 11x^2 + 38x - 40 = 0$
It is also given that *a*, *b* and *c* are its roots

Thus, $a + b + c = 11$,

$ab + bc + ca = 38$

and $abc = 40$

Now, $\frac{m}{n} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$

$= \frac{b^2 + c^2 - a^2}{2abc} + \frac{c^2 + a^2 - b^2}{2abc} + \frac{a^2 + b^2 - c^2}{2abc}$

$= \frac{a^2 + b^2 + c^2}{2abc}$

$= \frac{(a+b+c)^2 - 2(ab+bc+ca)}{2abc}$

$= \frac{(11)^2 - 2.38}{2.40}$

$= \frac{121 - 76}{80}$

$= \frac{45}{80} = \frac{9}{16}$

Thus, $m = 9$ and $n = 16$

Hence, the value of $m + n$

$= 9 + 16 = 25.$

LEVEL I
(QUESTIONS BASED ON FUNDAMENTALS)

1. In any triangle *ABC*, prove that
$(b^2 - c^2)\cot A + (c^2 - a^2)\cot B + (a^2 - b^2)\cot C = 0$

2. In a tringle ΔABC, prove that
$a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$

3. In a tringle ΔABC, prove that
$\frac{a^2\sin(B-C)}{\sin A} + \frac{b^2\sin(C-A)}{\sin B} + \frac{c^2\sin(C-A)}{\sin C} = 0$

4. In a triangle *ABC*, prove that for any angle θ,
$b\cos(A-\theta) + c\cos(B+\theta) = c\cos\theta$

5. If $\cos A + \cos B = 4\sin^2\left(\frac{C}{2}\right)$, prove that the sides, *a*, c, *b* of the triangle *ABC* are in A.P.

6. If in a ΔABC $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, then prove that, a^2, b^2, c^2 are in A.P.

7. In a triangle *ABC*, prove that,
$2(bc\cos A + ca\cos B + ab\cos C) = a^2 + b^2 + c^2$

8. In a tringle ΔABC, prove that

$$(a-b)^2\cos^2\left(\frac{C}{2}\right)+(a+b)^2\sin^2\left(\frac{C}{2}\right)=c^2$$

9. With usual notation, if in a triangle ABC,

$\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$, then prove that,

$$\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$$

10. Let a, b and c be the sides of a ΔABC. If a^2, b^2 & c^2 are the roots of the equation $x^3-Px^2+Qx-R=0$, where p, q and R are constants, then find the value of $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$, in terms of P, Q and R.

11. In a triangle ABC, prove that,

$$b^2\sin 2C+c^2\sin 2B=2bc\sin A$$

12. In a tringle ΔABC, prove that $\frac{\sin B}{\sin C}=\frac{c-a\cos B}{b-a\cos C}$

13. In a ΔABC, prove that,

$$2\left[a\sin^2\left(\frac{C}{2}\right)+c\sin^2\left(\frac{A}{2}\right)\right]=c+a-b$$

14. In a tringle ΔABC, prove that

$$\frac{\cos^2\left(\frac{A}{2}\right)}{a}+\frac{\cos^2\left(\frac{B}{2}\right)}{b}+\frac{\cos^2\left(\frac{C}{2}\right)}{c}=\frac{s^2}{abc}$$

15. In a ΔABC, prove that

$$\left(bc\cos^2\left(\frac{A}{2}\right)+ca\cos^2\left(\frac{B}{2}\right)+ab\cos^2\left(\frac{C}{2}\right)\right)$$
$$=\frac{1}{4}(a+b+c)^2$$

16. In a ΔABC, prove that,

$$(b-c)\cot\left(\frac{A}{2}\right)+(c-a)\cot\left(\frac{B}{2}\right)+(a-b)\cot\left(\frac{C}{2}\right)=0$$

17. If the sides a, b, c of a triangle are in A.P., then find the value of $\tan\left(\frac{A}{2}\right)+\tan\left(\frac{C}{2}\right)$ in terms of $\cot\left(\frac{B}{2}\right)$

18. If D is mid-point of CA in triangle ABC and Δ is the area of triangle, then prove that

$$\tan(\angle ADB)=\frac{4\Delta}{a^2-c^2}$$

19. In any triangle ABC, prove that,

$$4\Delta(\cot A+\cot B+\cot C)=a^2+b^2+c^2$$

20. In any triangle ABC, prove that,

$$\left(\frac{2abc}{a+b+c}\right).\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)=\Delta$$

21. If in a triangle ABC, $a=6$, $b=3$ and $\cos(A-B)=\frac{4}{5}$, then find its area.

22. If in a triangle ABC, $\angle A=30°$ and the area of the triangle is $\frac{a^2\sqrt{3}}{4}$, then prove that either $B=4\,C$ or $C=4\,B$

23. In any triangle ΔABC, prove that,

$$Rr(\sin A+\sin B+\sin C)=\Delta.$$

24. In any triangle ΔABC, prove that,

$$a\cos B\cos C+b\cos C\cos A+c\cos A\cos B=\frac{\Delta}{R}$$

25. In any triangle ΔABC, prove that,

$$\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{1}{2Rr}$$

26. In any triangle ΔABC, prove that,

$$\cos^2\left(\frac{A}{2}\right)+\cos^2\left(\frac{B}{2}\right)+\cos^2\left(\frac{C}{2}\right)=2+\frac{r}{2R}$$

27. In any triangle ΔABC, prove that,

$$a\cot A+b\cot B+c\cot C=2(R+r)$$

28. In any triangle ΔABC, prove that,

$$r_1+r_2-r_3+r=4R\cos C$$

29. In any triangle ΔABC, prove that,

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2}=\frac{a^2+b^2+c^2}{\Delta^2}$$

30. In any triangle ΔABC, prove that, prove that,

$$\left(\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)^2=\frac{4}{r}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)$$

31. In any triangle ΔABC, prove that,

$$\frac{bc-r_2r_3}{r_1}=\frac{ca-r_1r_3}{r_2}=\frac{ab-r_1r_2}{r_3}$$

32. Prove that the radii of the three escribed circles of a triangle are the roots of

$$x^3-(r+4R)x^2+s^2x-rs^2=0$$

33. If $\alpha, \beta \& \gamma$ are the respective altitudes of a triangle ABC, prove that

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{\cot A+\cot B+\cot C}{\Delta}$$

34. If $\alpha, \beta \& \gamma$ are the respective altitudes of a triangle ABC, prove that

$$\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{\gamma}=\left(\frac{2ab}{(a+b+c)\Delta}\times\cos^2\left(\frac{C}{2}\right)\right)$$

35. If in a triangle ΔABC, $a^2+b^2+c^2=8R^2$, then prove that the triangle ABC is right angled triangle.

36. The sides of a triangle are x^2+x+1, $2x + 1$ and $x^2 - 1$, prove that the greatest angle is 120°.

37. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the side of the triangle.

38 ABC is a triangle. Its area is 12 sq. cm and base is 6 cm. The difference of the angles is 60°. Prove that the angle A opposite to the base is given by 8 sin A – 6 cos$A$ = 3.

39. In any triangle ABC, if $\cos\theta=\frac{a}{b+c}$, $\cos\varphi=\frac{b}{a+c}$, $\cos\psi=\frac{c}{a+b}$, where $\theta, \varphi \& \psi$ lie between 0 and π, prove that,

$$\tan^2\left(\frac{\theta}{2}\right)+\tan^2\left(\frac{\varphi}{2}\right)+\tan^2\left(\frac{\psi}{2}\right)=1$$

40. The sides of a triangle are ABC are in A.P. If the angles a and c are the greatest and the smallest angles respectively, then prove that,

$$4(1-\cos A)(1-\cos C)=\cos A+\cos C$$

41. In triangle ABC, if a, b, c are in H.P., prove that $\sin^2\left(\frac{A}{2}\right), \sin^2\left(\frac{B}{2}\right), \sin^2\left(\frac{C}{2}\right)$ are also in H.P.

42. In a triangle ABC, if the sides a, b, c be in A.P., prove that $\cos A.\ \cot\left(\frac{A}{2}\right)$, $\cos B.\ \cot\left(\frac{B}{2}\right)$, $\cos C.\ \cot\left(\frac{C}{2}\right)$ are also in A.P.

43. The sides of a triangle are in A.P. and its area is 3/5 th of an equilateral triangle of the same perimeter. Prove that the sides are in the ratio 3 : 5 : 7.

44. The ex-radii r_1, r_2, r_3 of a triangle ABC are in H.P., prove that the sides a, b, c are in A.P.

45. In usual notation, if $r_1=r_2+r_3+r$, then prove that the triangle is right angled.

46. If A, B, C are the angles of a triangle, then prove that $\cos A+\cos B+\cos C=1+\frac{r}{R}$, where r = in-radius and R = circum - radius.

47. In triangle ABC, if $8R^2=a^2+b^2+c^2$, prove that the triangle is right angled.

48. Let $A_1, A_2, A_3, ..., A_n$ be the vertices of an n-sided regular polygon such that $\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}$, then prove that the value of n is 7.

49. If $A, A_1, A_2 \& A_3$ are the area of the inscribed and escribed circles of a triangle, then prove that

$$\frac{1}{\sqrt{A}}=\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}.$$

50. The sides of a quadrilateral are 3, 4, 5 and 6 cms. The sum of a pair of opposite angles is 120°. prove that the area of the quadrilateral is $3\sqrt{30}$ sq. cm.

51. In triangle ABC, prove that,

$$r^2+r_1^2+r_2^2+r_3^2=16R^2-a^2-b^2-c^2$$

52. In triangle ABC, prove that,

$$IA.IB.IC=abc\tan\left(\frac{A}{2}\right).\tan\left(\frac{B}{2}\right).\tan\left(\frac{C}{2}\right)$$

53. In a triangle ABC, if $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)=2$, prove that the triangle is right angled.

54. In triangle ABC, prove that, $\cot^2 A+\cot^2 B+\cot^2 C\geq 1$

55. If cos$A$=tan B, cos$B$ = tan C and cos$C$ = tan A, then prove that, sin A = sin B = sin C = 2 sin 18.

LEVEL II
MIXED PROBLEMS

1. In ΔABC, $a\geq b\geq c$, if $\frac{a^3+b^3+c^3}{\sin^3 A+\sin^3 B+\sin^3 C}=8$, then the maximum value of a is

(a) $\frac{1}{2}$ (b) 2

(c) 8 (d) 64

2. Sides of a triangle ABC are in A.P. If $a < \min\{b, c\}$, then cos A may be equal to

(a) $\frac{3c-4b}{2a}$ (b) $\frac{3c-4b}{2c}$

(c) $\frac{4c-3b}{2b}$ (d) $\frac{4c-3b}{2c}$

3. If a ΔABC, $a^4 + b^4 + c^4 = 2a^2b^2 + b^2c^2 + 2c^2a^2$, then $\sin A$ is
(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$
(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}+1}{2\sqrt{2}}$

4. In a triangle ABC, $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, then the numerical value of $\cos B$ is
(a) 0 (b) $\frac{3}{8}$
(c) $\frac{5}{8}$ (d) $\frac{7}{8}$

5. If a, b, c be the sides of ΔABC and if roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal then $\sin^2\left(\frac{A}{2}\right).\sin^2\left(\frac{B}{2}\right).\sin^2\left(\frac{C}{2}\right)$ are in
(a) A.P. (b) G.P.
(c) H.P. (d) A.G.P.

6. In a triangle ABC, $(a + b + c)(b + c - a) = k$ bc if
(a) $k < 0$ (b) $k > 6$
(c) $0 < k < 4$ (d) $k > 4$

7. $a^3 \cos(B-C) + b^3 \cos(C-A) + c^3 \cos(A-B)$ is equal
(a) $3\,abc$ (b) $(a + b + c)$
(c) $abc\,(a + b + c)$ (d) 0

8. If $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ and $a = 2$, then the area of the triangle is
(a) 1 (b) 2
(c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{3}$

9. If in a ΔABC, $\cos A + 2\cos B + \cos C = 2$, then a, b, c are in
(a) A.P. (b) G.P.
(c) H.P. (d) None

10. If in a ΔABC, $\sin^3 A + \sin^3 B + \sin^3 C = 3\sin A \sin B \sin C$, then the value of

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

(a) 0
(b) $(a + b + c)^2$
(c) $(a + b + c)(ab + bc + ca)$
(d) None

11. If $b + c = 3a$, then the value of $\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$ is
(a) 1 (b) 2
(c) $\sqrt{3}$ (d) $\sqrt{2}$

12. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. The product of length of the line segment A_0A_1, A_0A_2, A_0A_4 is
(a) $\frac{3}{4}$ (b) $3\sqrt{3}$
(c) 3 (d) $\frac{3\sqrt{3}}{2}$

13. In a ΔABC, the value of $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is
(a) $\frac{R}{r}$ (b) $\frac{R}{2r}$
(c) $\frac{r}{R}$ (d) $\frac{2r}{R}$

14. In a ΔABC, the sides a, b, c are the roots of the equation $x^3 - 11x^2 + 38x - 40 = 0$. Then $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ is
(a) 1 (b) $\frac{3}{4}$
(c) $\frac{9}{16}$ (d) None

15. The ex-radii of a Δ r_1, r_2, r_3 are in A.P., then the sides a, b, c are in
(a) A.P. (b) G.P.
(c) H.P. (d) A.G.P.

16. In any ΔABC, $\sum \frac{\sin^2 A + \sin A + 1}{\sin A}$ is always greater than
(a) 9 (b) 3
(c) 27 (d) 36

17. In a triangle $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right) = 2$, then the triangle is
(a) right angled (b) isosceles
(c) equilateral (d) None

18. In a ΔABC, $a = 2b$ and $|a - b| = \frac{\pi}{3}$, then $\angle C$ is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{6}$ (d) None

19. If the median of ΔABC, through A is perpendicular to AB, then
(a) $\tan A + \tan B = 0$
(b) $2\tan A + \tan B = 0$
(c) $\tan A + 2\tan B = 0$
(d) None

20. In a ΔABC, $\cos A + \cos B + \cos C = \frac{3}{2}$, then the Δ
(a) Isosceles (b) right angled
(c) equilateral (d) None

21. If $A_1 .... A_n$ be a regular polygon of n-sides and $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, then
(a) $n = 5$ (b) $n = 6$
(c) $n = 7$ (c) None

22. In a ΔABC, $\tan\left(\frac{A}{2}\right) = \frac{5}{6}$ and $\tan\left(\frac{C}{2}\right) = s\frac{2}{5}$, then
(a) $a, c, b \in$ A.P. (b) $a, b, c \in$ A.P.
(c) $b, a, c \in$ A.P. (d) $a, b, c \in$ G.P.

23. If the angles of a triangle are in the ratio 1 : 2 : 3, then the corresponding sides are in the ratio
(a) 2 : 3 : 1 (b) $\sqrt{3}$: 2 : 1
(c) 2 : $\sqrt{3}$: 1 (d) 1 : $\sqrt{3}$: 2

24. In a ΔABC, $a\cot A + b\cot B + c\cot C$ is
(a) $r + R$ (b) $r - R$
(c) $2(r + R)$ (d) $2(r - R)$

25. If A, A_1, A_2, A_3 are the areas of in circle and the ex circles of a triangle, then $\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$ is
(a) $\frac{2}{\sqrt{A}}$ (b) $\frac{1}{\sqrt{A}}$
(c) $\frac{1}{2\sqrt{A}}$ (d) $\frac{3}{\sqrt{A}}$

26. In any ΔABC, $\Pi\left(\frac{\sin^2 A + \sin A + 1}{\sin A}\right)$ is always greater than
(a) 9 (b) 3
(c) 27 (d) 36

27. In an equilateral triangle, $R : r : r_2$ is
(a) 1 : 1 : 1 (b) 1 : 2 : 3
(c) 2 : 1 : 3 (d) 3 : 2 : 4

28. In a ΔABC, $\tan A \tan B \tan C = 9$. For such triangles, if $\tan^2 A + \tan^2 B + \tan^2 C = \lambda$, 7 then
(a) $9.\sqrt[3]{3} < \lambda < 27$ (b) $\lambda \nleq 27$
(c) $\lambda < 9.\sqrt[3]{3}$ (d) $\lambda > 27$

29. In a ΔABC, $a^2\cos^2 A = b^2 + c^2$, then
(a) $A < \frac{\pi}{4}$ (b) $\frac{\pi}{4} < A < \frac{\pi}{2}$
(c) $A > \frac{\pi}{2}$ (d) $A = \frac{\pi}{2}$

30. In a ΔABC, $A : B : C = 3 : 5 : 4$, then $a + b + c\sqrt{2}$ is
(a) $2b$ (b) $2c$
(c) $3b$ (d) $3a$

31. If A, B, C are angles of a triangle such that the angle A is obtuse, then $\tan B \tan C <$
(a) 0 (b) 1
(c) 2 (d) 3

32. In a triangle, if $r_1 > r_2 > r_3$, then
(a) $a > b > c$ (b) $a < b <$ c
(c) $a > b$ and $b < c$ (d) $a < b$ and $b > c$

33. $4rR\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ is
(a) s (b) s^2
(c) Δ^2 (d) Δ

34. If $(a - b)(s - c) = (b - c)(s - a)$, then r_1, r_2, r_3 are in
(a) H.P. (b) G.P.
(c) A.P. (d) A.G.P.

35. If $c^2 = a^2 + b^2$, $2s = a + b + c$, then $4s(s - a)(s - b)(s - c)$ is
(a) s^4 (b) b^2c^2
(c) c^2a^2 (d) a^2b^2

36. $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2}$ is
(a) $\frac{a^2 + b^2 + c^2}{s^2}$ (b) $\frac{\Sigma a^2}{\Delta^2}$
(c) $4R$ (d) $4r$

37. $(r_1 - r)(r_2 - r)(r_3 - r)$ is
(a) $\frac{R}{r}$ (b) $4R^2r$
(c) $4Rr^2$ (d) $4R$

38. If the sides be 13, 14, 15, then $\frac{b - c}{r_1} + \frac{c - a}{r_2} + \frac{a - b}{r_3}$ is

(a) 5 (b) 4
(c) 0 (d) 1

39. $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab}$ is

(a) $\frac{1}{2R} - \frac{1}{r}$ (b) $2R - r$

(c) $r - 2R$ (d) $\frac{1}{r} - \frac{1}{2R}$

40. $r_1 + r_2 =$

(a) $c \tan\left(\frac{C}{2}\right)$ (b) $c \cot\left(\frac{C}{2}\right)$

(c) $c \sin\left(\frac{C}{2}\right)$ (d) $c \cos\left(\frac{C}{2}\right)$

41. $16R^2\, r\, r_1\, r_2\, r_3$ is
(a) abc (b) $a^3b^3c^3$
(c) $a^2b^2c^2$ (d) $a^2b^3c^4$

42. If $\frac{r}{r_1} = \frac{r_2}{r_3}$, then

(a) $A = 90°$ (b) $B = 90°$
(c) $C = 90°$ (d) None

43. In a ΔABC, the value of $r\, r_1 r_2 r_3$ is
(a) Δ (b) Δ^2
(c) Δ^3 (d) Δ^4

44. If $r_1 = r_2 + r_3 + r$, then the Δ is
(a) Equilateral (b) Isosceles
(c) Right angled (d) None

45. $(r_1 + r_2)\,(r_2 + r_3)\,(r_3 + r_1)$ is
(a) Rs^2 (b) $2Rs^2$
(c) $3Rs^2$ (d) $4Rs^2$

46. The diameter of the circum-circle of a triangle with sides 5 cm, 6 cm., 7 cm is

(a) $\frac{3\sqrt{6}}{2}$ cm. (b) $2\sqrt{6}$ cm

(c) $\frac{35}{48}$ cm (d) $\frac{35}{2\sqrt{6}}$

47. In a ΔABC, the sides are in the ratio 4 : 5 : 6. The ratio of the circum-radius and the in-radius is
(a) 8 : 7 (b) 3 : 2
(c) 7: 3 (d) 16 : 7

48. If in a triangle, R and r are the circum radius and in radius, respectively, then the H.M. of the ex-radii of the triangle is
(a) $3r$ (b) $2R$
(c) $R + r$ (d) None

49. If a, b and c are the sides of a triangle ABC and $3a = b + c$, the value of Cot $(B/2)$ cot $(C/2)$ is
(a) 3 (b) 2
(c) 4 (d) 1.

50. In a triangle of ABC, if $\cos A + \cos B = 4 \sin2\,(C/2)$, then a, b and c are in
(a) A.P. (b) G.P.
(c) H.P. (d) None.

LEVEL III
(FOR JEE ADVANCED EXAM ONLY)

1 If in a triangle ABC, $\frac{\cos A + 2\cos C}{\cos A + 2c\cos B} = \frac{\sin B}{\sin C}$, prove that the triangle ABC is either isoceles or right angled.

2. In a triangle ABC, if
$$a \tan A + b \tan B = (a+b)\tan\left(\frac{A+B}{2}\right),$$
prove that the triangle is isosceles.

3. In any triangle ABC, prove that,
$$(r_2 + r_1)(r_3 + r_2)\sin C = 2r_3\sqrt{r_2 r_3 + r_1 r_3 + r_1 r_2}$$

4. In any triangle ABC, prove that,
$$r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - (a^2 + b^2 + c^2)$$

5. In any triangle ABC, prove that,
$$\left((r + r_1)\tan\left(\frac{B-C}{2}\right)\right) + \left((r + r_2)\tan\left(\frac{C-A}{2}\right)\right) + \left((r + r_3)\tan\left(\frac{C-A}{2}\right)\right) = 0$$

6. In any triangle ABC, prove that,
$$\frac{\tan\left(\frac{A}{2}\right)}{(a-b)(a-c)} + \frac{\tan\left(\frac{B}{2}\right)}{(b-a)(b-c)} + \frac{\tan\left(\frac{C}{2}\right)}{(c-a)(c-b)} = \frac{1}{\Delta}$$

7. If $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$, prove that the triangle is right angled triangle.

8. In triangle ABC, prove that,
$$\frac{\text{area of the in-circle}}{\text{area of triangle } ABC} = \frac{\pi}{\cot\left(\frac{A}{2}\right).\cot\left(\frac{B}{2}\right).\cot\left(\frac{C}{2}\right)}$$

9. If a, b, c are in A.P., prove that, $\cos A.\cot\left(\frac{A}{2}\right)$, $\cos B.\cot\left(\frac{B}{2}\right)$, $\cos C.\cot\left(\frac{C}{2}\right)$ are in A.P.

10. If the circumference of the ΔABC lies on its incircle, then prove that, $\cos A + \cos B + \cos C = \sqrt{2}$

11. $ABCD$ is a trapezium such that AB, DC are parallel and BC is perpendicular to them. If angle $\angle ADB = \theta$, $BC = p$ and $CD = q$, prove that, $AB = \dfrac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$

12. Let O be the circumcenter and H be the orthocenter of ΔABC. If Q is the mid-point of OH, then show that $AQ = \dfrac{R}{2}\sqrt{1+8\cos A\cos B\cos C}$.

13. If I_1, I_2 & I_3 are the centres of escribed circles of ΔABC, prove that the area of $\Delta I_1 I_2 I_3 = \dfrac{abc}{2r}$.

14. In ΔABC, prove that,
$$a^2(s-a)+b^2(s-b)+c^2(s-c)$$
$$= 4R\Delta\left(1-4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right)$$

15. Let O be a point inside a triangle ABC such that $\angle OAB = \angle OBC = \angle OCA = \omega$, then prove that
    (i) $\cot A + \cot B + \cot C = \cot\omega$
    (ii) $\operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C = \operatorname{cosec}^2\omega$

16. Find the distance between the circum-center and the mid-points of the sides of a triangle.

17. Find the distance between the in-center and the angular points of a triangle.

18. Prove that the distance between the circum-centre (O) and the in-center (I) is
$$OI = R\times\sqrt{\left(1-8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right)}$$

19. Prove that the ratio of circum-radius and in-radius of an equilateral triangle is 1/2.

20. Prove that the ratio of the area of the in-circle to the area of a triangle is $\pi : \tan\left(\frac{A}{2}\right)\cdot\tan\left(\frac{B}{2}\right)\cdot\tan\left(\frac{C}{2}\right)$.

21. Prove that the distance of the orthocenter from the sides and angular points of a triangle is $2R\cos A$, $2R\cos B$ and $2R\cos C$.

22. Prove that the distance between the circum-center and the orthocenter of a triangle is $OH = R\sqrt{1-8\cos A.\cos B.\cos C}$.

23. Prove that the area of an ex-central triangle is $8R^2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$.

24. Prove that the circum-radius of an ex-central triangle is $\dfrac{I_2I_3}{2\sin(I_1I_2I_3)} = 2R$.

25. Prove that the distance between the in-center and the ex-centers are
$$II_1 = 4R\sin\left(\frac{A}{2}\right),\ II_2 = 4R\sin\left(\frac{B}{2}\right),$$
$$II_3 = 4R\sin\left(\frac{C}{2}\right).$$

26. If a^2, b^2, c^2 are in A.P., then prove that $\cot a$. $\cot b$, $\cot c$ are in A.P.

27. In any triangle ABC, prove that,
$$a^3\cos(B-C)+b^3\cos(C-A)$$
$$+c^3\cos(A-B) = 3abc$$

28. The sides of a triangle are in A.P. and the greatest and least angles are θ and ϕ, respectively, then prove that $4(1-\cos\theta)(1-\cos\varphi) = \cos\theta + \cos\varphi$.

28. If in a triangle, the bisector of the side c be perpendicular to the side d, prove that 2 tan$A$ + tan c = 0.

29. In any triangle, if θ be any angle, then prove that, $b\cos\theta = c\cos(A-\theta) + a\cos(C+\theta)$.

30. If AD, BE and CF are the internal bisectors of the angles of a triangle ABC, prove that
$$\frac{1}{AD}\cos\left(\frac{A}{2}\right)+\frac{1}{BE}\cos\left(\frac{B}{2}\right)+\frac{1}{CF}\cos\left(\frac{C}{2}\right)$$
$$= \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

31. Prove that the triangle having sides $3x + 4y$, $4x + 3y$ and $5x + 5y$ units, respectively, where $x, y > 0$, is obtuse angled.

32. If Δ be the area and 's' be the semi perimeter of a triangle, then prove that, $\Delta \le \dfrac{s^2}{3\sqrt{3}}$

33. Let ABC be a triangle having altitudes h_1, h_2 & h_3 from the vertices A, B, C, respectively, and r be the in-radius, prove that, $\dfrac{h_1+r}{h_1-r}+\dfrac{h_2+r}{h_2-r}+\dfrac{h_3+r}{h_3-r} \ge 6$.

34. Two circles of radii a and b cut each other at an angle θ. Prove that the length of the common chord is
$$\frac{2ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}$$

35. If α, β, γ are the distances of the vertices of a triangle from the corresponding points of contact with the in-circle, prove that $r^2 = \frac{\alpha\beta\gamma}{\alpha+\beta+\gamma}$

36. Tangents are drawn to the in-circle of a triangle ABC which are parallel to its sides . If x, y, z be the lengths of the tangents and a, b, c be the sides of a triangle, then prove that, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

37. If t_1, t_2 & t_3 be the lengths of the tangents from the e-centers of escribed circles to the circum-circles, prove that $\frac{1}{t_1^2}+\frac{1}{t_1^2}+\frac{1}{t_1^2}=\frac{2s}{abc}$.

38. if x, y, z be the lengths of the perpendiculars from the circumcentre on the sides BC, CA, AB of a triangle ABC, prove that,

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{4xyz}$$

39. If p_1, p_2, p_3 are the altitudes of the triangle ABC from the vertices a, b and c, respectively, prove that

$$\frac{\cos A}{p_1}+\frac{\cos B}{p_2}+\frac{\cos C}{p_3}=\frac{1}{R}$$

40. The product of the sines of the angles of *a* triangle is p and the product of their cosines is q. Prove that the tangents of the angles are the roots of

$$qx^3-px^2+(1+q)x-p=0.$$

41. In a triangle ABC, if $\cos A \cdot \cos B + \sin A \sin B \sin C = 1$, prove that the sides are in the ratio $1:1:\sqrt{2}$.

42. The base of a triangle is divided into three equal parts. If t_1, t_2 & t_3 be the tangents of the angles subtended by these parts at the opposite vertices, prove that, $\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right)=4\left(1+\frac{1}{t_2{}^2}\right)$.

43. The three medians of a triangle ABC make angle α, β, γ with each other, prove that, $\cot\alpha+\cot\beta+\cot\gamma$

$$+\cot A+\cot B+\cot C=0$$

44. Perpendiculars are drawn from the angles A, B, C of an acute angled triangle on the opposite sides and produced to meet the circumscribing circle. If these parts be α, β, γ, respectively, then prove that

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=2(\tan A+\tan B+\tan C)$$

45. In a triangle ABC, the vertices A, B, C are at distances of p, q, r from the orthocentre, respectively. Prove that

$$\left(\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right)=\frac{abc}{pqr}$$

46. The internal bisectors of the angles of a triangle ABC meet the sides in D, E and F. Prove that the area of the triangle DEF is $\frac{2\Delta abc}{(a+b)(b+c)(c+a)}$.

47. In a triangle ABC, the measures of the angles A, B, C are 3α, 3β *and* 3γ, respectively. P, Q, and R are the points within the triangle such that

$\angle BAR = \angle RAQ = \angle QAC = \alpha$,

$\angle CBP = \angle PBR = \angle RBA = \beta$ and

$\angle ACQ = \angle QCP = \angle PCB = \gamma$, then prove that

$AR = 8R\sin\beta\,\sin\gamma\,\cos(30°-\gamma)$

48. If in a triangle ABC, the median AD and the perpendicular AE from the vertex A to the side BC divides the angle A into three equal parts, show that

$$\cos\left(\frac{A}{3}\right).\sin^2\left(\frac{A}{3}\right)=\frac{3a^2}{32bc}.$$

49. If the sides of a triangle are in A.P. and if its greatest angle exceeds the least angle by α, show that the sides are in the ratio $(1-x):1:(1+x)$, where $x=\sqrt{\frac{1-\cos\alpha}{7-\cos\alpha}}$.

50. Let A, B, C be three angles such that $A=\frac{\pi}{4}$ and $\tan B \tan C = p$. Find all possible values of p such that A, B, C are the angles of a triangle.

Ans. $p<0$ & $p \geq 3+2\sqrt{2}$.

INTEGER TYPE QUESTIONS

1. In any right angled triangle, find the value of $\frac{a^2+b^2+c^2}{R^2}$.

2. In any triangle ABC, $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$ find the value of $\left(\frac{r_1+r_2+r_3}{r}\right)$.

3. In any triangle ABC, find the minimum value of $\left(\frac{r_1+r_2+r_3}{r}\right)$.

4. In any triangle ABC, find the minimum value of $\left(\frac{\sin^2 A + \sin A + 1}{\sin A}\right)$.
5. In a triangle ABC, find the value of $\frac{1}{8R^2}\left(r^2 + r_1^2 + r_2^2 + r_3^2 + \left(a^2 + b^2 + c^2\right)\right)$.
   where r = in-radius, R = circum - radius and r_1, r_2, r_3 are ex-radii.
6. In a triangle ABC, the median $AD = \frac{1}{\sqrt{11 - 6\sqrt{3}}}$.
   and it divides the Angle a into angles $30°$ and $45°$ Find the length of the side BC.
7. In a triangle ABC, find the value of $\frac{(r_1 + r_2)(r_2 + r_3)(r_1 + r_3)}{R s^2}$, where
   R = circum - radius, r_1, r_2, r_3 are ex-radii.
   and s is the semi perimeter.
8 If in a triangle ABC, a = 6, b = 3 and $\cos(A - B) = \frac{4}{5}$, then find its area.
9 A triangle has base 10 cm long and the base angles are $50°$ and $70°$. If the perimeter of the triangle is $x + y\cos(z°)$ where $z \in (0, 90°)$, then find the value of $\left(\frac{x + y + z}{y}\right)$.
10 In any ΔABC, find the value of $\frac{a\cot A + b\cot B + c\cot C}{(r + R)}$.

SELF ASSESMENT I

CH: PROPERTIES OF TRIANGLES

Time : 3 Hrs. **Max. Marks : 100.**

Give answer of the following questions.

1. If in a triangle ABC, $\tan A$ = 2, $\tan B$ = 3/2 and $c = 2\sqrt{65}$, then find the circumradius of the triangle.
2. If in a triangle ABC, $\cos 3A + \cos 3B + \cos 3C = 1$ then prove that the measure of one of the angles must be $\frac{2\pi}{3}$.
3. If in a triangle ABC, the sides a, b, c are in A.P., then prove that
   $$\cot\left(\frac{A}{2}\right) + \cot\left(\frac{C}{2}\right) = 2\cot\left(\frac{B}{2}\right).$$
4. If in a triangle of ABC, A = 3 B, then prove that
   $$\sin b = \frac{1}{2}\sqrt{\frac{3b - a}{b}}.$$
5. Let Δ_1, Δ_2 denote the areas of a triangle and that of the incircle . prove that
   $$\frac{\Delta_1}{\Delta_2} = \left(\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}\right) : \pi$$
6. If the ex-radii r_1, r_2, r_3 of a triangle ABC are in H.P, then prove that the sides a, b, c are in A.P.
7. The lenghts of the sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.
8. Let O be a point in the triangle ABC such that $\angle OAC = \angle OCB = \angle OBA = \alpha$,
   then prove that $\cot\alpha = \cot A + \cot B + \cot C$
9. If in a triangle ABC, two sides are a = 6, b = 3 and $\cos(A - B) = \frac{4}{5}$, then prove that the area of the triangle is 9.
10. In a triangle ABC, the angles A, B, C are in A.P., then prove that
   $$2\cos\left(\frac{A - C}{2}\right) = \frac{a + c}{\sqrt{a^2 - ac + c^2}}$$

(QUESTIONS ASKED IN IIT-JEE EXAMS WITH THEIR SOLUTIONS)

1. If p_1, p_2, p_3 are the altitudes of a triangle from the vertices A, B, C, respectively and Δ be the area of a triangle, prove that
   $$\frac{1}{p_1} + \frac{1}{p_1} + \frac{1}{p_1} = \frac{2ab}{(a + b + c)\Delta}\cos^2\left(\frac{C}{2}\right).$$
   [IIT-JEE-1978]
2. A quadrilateral $ABCD$ is inscribed in a circle S and A, B, C, D are the points of contacts with S of an other quadrilateral which is circumscribed about S. If this quadrilateral is also cyclic, prove that
   $AB^2 + CD^2 = BC^2 + AD^2$. [IIT-JEE-1978]
3. If two sides of a triangle and the included angle are given by $a = \left(\sqrt{3} + 1\right)$ cm, $b = 2$ cm and $C = 60°$, find the other two angles and the third side.
   [IIT-JEE-1979]

4. If a circle is inscribed in a right angled triangle ABC with right angled at B, show that the diameter of the circle is equal to $AB + BC - AC$. [IIT-JEE-1979]

5. ABC is a triangle, d is the middle point of BC. If AD is perpendicular to AC, prove that

$\cos A \cos C = \frac{2(c^2 - a^2)}{3ac}$. [IIT-JEE-1980]

6. Let the angles A, B, C of a triangle ABC be in A.P. and let $b : c = \sqrt{3} : \sqrt{2}$. Find the angle A. [IIT-JEE-1981]

7. No questions asked in 1982.

8. The ex-radii r_1, r_2, r_3 of ΔABC are in H.P.

Show that the sides a, b, c are in A.P. [IIT-JEE-1983]

9. For a triangle ABC, it is given that

$\cos A + \cos B + \cos C = \frac{3}{2}$, prove that thetriangle is equilateral. [IIT-JEE-1984]

10. With usual notation, if in a triangle ABC

$\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, prove that

$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$ [IIT-JEE-1984]

11. In a triangle ABC, the median to the side BC is at length $\frac{1}{\sqrt{11 - 6\sqrt{3}}}$ and it divides the angle a into angles of $30°$ and $45°$. Find the length of the side BC.

[IIT-JEE 1985]

12. In a triangle ABC, if $\cot A, \cot B, \cot C$ are in A.P., then prove that a^2, b^2, c^2 are in A.P.

[IIT-JEE-1985]

13. The set of all real numbers a such that

$a^2 + 2a, 2a + 3, a^2 + 3a + 8$ are the sides of

A triangle. is .... [IIT-JEE-1985]

14. The sides of a triangle inscribed in a given circle subtends angle $\alpha, \beta \& \gamma$ at the centre. The minimum value of the arithmetic mean of

$\cos\left(\alpha + \frac{\pi}{2}\right), \cos\left(\beta + \frac{\pi}{2}\right), \cos\left(\gamma + \frac{\pi}{2}\right)$ is ...

[IIT-JEE-1986]

15 In a triangle ABC,
$\cos A \cos B + \sin A \sin B \sin C = 1$,

show that $a : b : c = 1 : 1 : \sqrt{2}$ [IIT-JEE-1986]

16. There exists a triangle ABC satisfying the conditions

(i) $b \sin A = a,\ A < \frac{\pi}{2}$

(ii) $b \sin A > a,\ A > \frac{\pi}{2}$

(iii) $b \sin A > a,\ A > \frac{\pi}{2}$

(iv) $b \sin A < a,\ A < \frac{\pi}{2}, b > a$

[IIT-JEE-1986]

17. In a triangle, the lengths of the two larger sides are 10 and 9, respectively. If the angles are in A.P., then the length of the third side can be

(a) $5 - \sqrt{6}$ (b) $3\sqrt{3}$

(c) 5 (d) $5 + \sqrt{6}$

[IIT-JEE-1987]

18. If the angles of a triangle are $30°$ and $45°$ and the included side is $(\sqrt{3} + 1)$, then the area of the triangle is ...... [IIT-JEE-1988]

19. ABC is an isosceles triangle inscribed in a circle of radius r. If $AB = AC$ and h is the altitude from A to BC, then the triangle ABC has perimeter
P = ... and area Δ = ... and

also $\lim_{x \to 0}\left(\frac{\Delta}{p^3}\right)$ = ... [IIT-JEE - 1989]

20. In a triangle ABC, a is greater than angle B. If the measures of angles A and B satisfy the equation

$3\sin x - 4\sin^3 x - k = 0$, then the measure of angle C is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$

(c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

[IIT-JEE-1990]

21. ABC is a triangle such that

$\sin(2A + B) = \sin(C - A) = -\sin(B + C) = \frac{1}{2}$.

If A, B and C are in A.P., determine the values of A, B and C. [IIT-JEE-1990]

22. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of a triangle .

[IIT-JEE-1991]

23. In a triangle of base a, the ratio of the other two sides is r (< 1). Show that the altitude of the triangle is less than or equal to $\frac{ar}{1-r^2}$. [IIT-JEE-1991]

24. Three circles touch one another externally.
The tangents at their points of contact meet a point whose distance from a point of a contact is 4. Find the ratio of the product of the radii to the sum of the radii of the circles. [IIT-JEE-1992]

25. If in a triangle ABC,
$$\frac{2\cos A}{a}+\frac{2\cos B}{b}+\frac{2\cos C}{c}=\frac{1}{bc}+\frac{b}{ca},$$
then find the angle A in degrees. [IIT-JEE-1993]

26. Let $A_1, A_2, ..., A_n$ be the vertices of n sided regular polgon such that
$$\frac{1}{A_1A_2}+\frac{1}{A_1A_3}=\frac{1}{A_1A_2}$$, then find n. [IIT-JEE-1994].

27. Consider the following statement concerning a triangle ABC
(i) The sides a, b, c and area of Δ are rational
(ii) $a, \tan\left(\frac{B}{2}\right), \tan\left(\frac{C}{2}\right)$ are rational
(iii) $a, \sin A, \sin B, \sin C$ are rational
Then prove that $(\text{i}) \Rightarrow (\text{ii}) \Rightarrow (\text{iii}) \Rightarrow (\text{i})$
[IIT-JEE-1994].

28. In a ΔABC, AD is an altitude from A, Given $b > c$, $\angle C = 23°$ and $AD = \frac{abc}{a^2-c^2}$, then find $\angle B$.
[IIT-JEE-1994].

29. If the length of the sides of a triangle are 3, 5, 7, then the largest angle of the triangle is
(a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{6}$
(c) $\frac{2\pi}{3}$ (d) $\frac{3\pi}{4}$
[IIT-JEE-1994]

30. In a triangle ABC, $\angle B = \frac{\pi}{3}$ & $\angle C = \frac{\pi}{4}$. Let D divide BC internally in the ratio 1:3, then $\frac{\sin(\angle BAD)}{\sin(\angle CAD)}$ is equal to
(a) $\frac{1}{\sqrt{6}}$ (b) $\frac{1}{3}$
(c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{\frac{2}{3}}$
[IIT-JEE-1995]

31. In a triangle ABC, a:b:c = 4:5:6. The ratio of the radius of the circum-circle to that of in-circle is.....
[IIT-JEE-1996]

32. Let ABC be three angles such that $A = \frac{\pi}{4}$ and $\tan(B)\tan(C) = p$. Find all possible values of p such that a, b, c are the angles of atriangle.
[IIT-JEE-1997].

33. Prove that a triangle ABC is equilateral if and only if $\tan(A)+\tan(B)+\tan(C)=3\sqrt{3}$. [IIT-JEE-1998]

34. If in a triangle PQR, sin$P$, sin$Q$, sin$R$ are in A.P. then
(a) the altitude are in A.P.
(b) the altitude are in H.P.
(c) the medians are in G.P.
(d) the medians are in A.P. [IIT-JEE-1998]

35. Let ABC be atriangle having O and I as its circum-centre and in-centre, respectively. If R and r are the circum-radius and in-radius, respectively, then prove that $(IO)^2 = R^2 - 2Rr$. Further show that the triangle BIO is a right angled triangle if and only if b is the arithmetic mean of a and c. [IIT-JEE-1999]

36. In any triangle ABC, prove that
$$\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)=\cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right).$$
[IIT-JEE-2000]

37. Let ABC be a triangle with incentre I and in-radius r Let D, E, F be the feet of the perpendicular from I to the sides BC, CA and AB, respectively, If r_1, r_2, r_3 are the radii of the circles inscribed in the quadrilaterals $AFIE$, $BDIF$ and $CEIF$, respectively,
prove that,
$$\frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}=\frac{r_1r_2r_3}{(r_1-r_2)(r_2-r_3)(r_3-r_1)}$$
[IIT-JEE-2000]

38. In a triangle ABC, $2ac\sin\left(\frac{A-B+C}{2}\right) =$
(a) $a^2+b^2-c^2$ (b) $c^2+a^2-b^2$
(c) $b^2-c^2-a^2$ (d) $c^2-a^2-b^2$
[IIT-JEE-2000]

39. In a triangle ABC, let $\angle C = \frac{\pi}{2}$. If r is the in-radius and R is the circum-radius of the triangle, then

$2(R+r)$ is equal to

(a) $a+b$ (b) $b+c$

(c) $c+a$ (d) $a+b+c$

[IIT-JEE-2000]

40. If Δ is the area of a triangle with side lengths a, b and c, then show that $\Delta \le \frac{1}{4}\sqrt{(a+b+c)abc}$.

Also show that the equality occurs in the above inequality if and only if $a=b=c$ [IIT-JEE-2001]

41. Which of the following pieces of data does not uniquely determine an acute angled triangle ABC (R being the radius of the circum-circle)?

(a) a, $\sin A$, $\sin B$ (b) a, b, c

(c) a, $\sin B$, R (d) a, $\sin A$, R

[IIT-JEE-2002]

42. If the angles of a triangle are in the ratio 4:1:1, then ratio of the longest side to the perimeter is

(a) $\frac{\sqrt{3}}{(2+\sqrt{3})}$ (b) $\frac{1}{6}$

(c) $\frac{1}{(2+\sqrt{3})}$ (d) $\frac{2}{3}$

[IIT-JEE-2003]

43 If I_n is the area of n-sided regular polygon inscribed in a circle of unit radius and O_n be the area of the polygon circumscribing the given circle, prove that

$$I_n = \frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right).$$ [IIT-JEE-2003]

44. The side of a triangle are in the ratio $1:\sqrt{3}:2$, then the angles of the triangle are in the ratio is

(a) 1:3:5 (b) 2:3:4

(c) 3:2:1 (d) 1:2::3

[IIT-JEE-2004]

45. In an equilateral triangle, three coins of radii 1 unit each are kept so that they touch each other and also the sides of the triangle. Area of the triangle is

(a) $4+2\sqrt{3}$ (b) $6+4\sqrt{3}$

(c) $\left(12+\frac{7\sqrt{3}}{4}\right)$ (d) $\left(3+\frac{7\sqrt{3}}{4}\right)$

[IIT-JEE-2005]

46. In a triangle ABC, a, b, c are the lengths of its sides and A, B, C are the angles of a triangle ABC. The correct relation is given by

(a) $(b-c)\sin\left(\frac{B-C}{2}\right) = a\cos\left(\frac{A}{2}\right)$

(b) $(b-c)\cos\left(\frac{A}{2}\right) = a\sin\left(\frac{B-C}{2}\right)$

(c) $(b+c)\sin\left(\frac{B+C}{2}\right) = a\cos\left(\frac{A}{2}\right)$

(d) $(b-c)\cos\left(\frac{A}{2}\right) = 2a\sin\left(\frac{B+C}{2}\right)$

[IIT-JEE-2005]

47. One angle of an isosceles triangle is $120°$ and radius of its incircle is $\sqrt{3}$. Then the area of the triangle in sq. units is

(a) $(7+12\sqrt{3})$ (b) $(12-7\sqrt{3})$

(c) $(12+7\sqrt{3})$ (d) 4π

[IIT-JEE-2006]

48. In a triangle ABC, internal angle bisector of $\angle A$ meets side BC in D, $DE \perp AD$ meets AC in E and AB in F. Then

(a) AE is H.M. of b and c

(b) $AD = \left(\frac{2bc}{b+c}\right)\cos\left(\frac{A}{2}\right)$

(c) $EF = \left(\frac{4bc}{b+c}\right)\sin\left(\frac{A}{2}\right)$

(d) ΔAEF is isosceles.

[IIT-JEE-2006]

49. Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $AB = 2\ CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its area is

(a) 3 (b) 2

(c) 3/2 (d) 1

[IIT-JEE-2007]

50 A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the cirum-circle of the triangle PQR at the point T. If S is not the centre of the circum-circle, then

(a) $\frac{1}{PS}+\frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$

(b) $\frac{1}{PS}+\frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$

(c) $\frac{1}{PS}+\frac{1}{ST} < \frac{4}{QR}$

(d) $\frac{1}{PS}+\frac{1}{ST} > \frac{4}{QR}$ [IIT-JEE-2008]

51. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2\left(\frac{A}{2}\right)$

If a, b and c denote the lengths of the sides of the triangle opposite to the angles A, B and C respectively, then

(a) $b + c = 4a$ (b) $b + c = 2a$
(c) locus of points A is an ellipse (d) locus of points A is a pair of straight lines

[IIT-JEE-2009]

52. Two parallel chords of a circle of radius 2 are at a distance $\left(\sqrt{3}+1\right)$ apart. If the chord subtends angles $\frac{\pi}{k}$ and $\frac{2\pi}{k}$ at the centre, where $k > 0$, then find the value of $[k]$., [,] = G.I.F [IIT-JEE-2010]

53. Consider a triangle ABC and let a, b, c denote the lengths of the sides opposite to vertices A, B, and C, respectively. Suppose $a = 6$, $b = 10$ and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the in-circle of the triangle, then find the value of r^2. [IIT-JEE-2010]

54. Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the side opposite to A, B and C, respectively. The values of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)

(a) $-\left(2+\sqrt{3}\right)$ (b) $\left(1+\sqrt{3}\right)$
(c) $\left(2+\sqrt{3}\right)$ (d) $4\sqrt{3}$

[IIT-JEE-2010]

53. Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the side opposite to A, B and C, respectively. The values of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)

(a) $-\left(2+\sqrt{3}\right)$ (b) $\left(1+\sqrt{3}\right)$
(c) $\left(2+\sqrt{3}\right)$ (d) $4\sqrt{3}$

[IIT-JEE-2010]

54. No questions asked in 2011.

55. Let PQR be a triangle of area Δ with a = 2, $b = 7/2$ and $c = 5/2$, where a, b, c are the lengths of the triangle opposite to the angles at P, Q, and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals

(a) $\frac{3}{4\Delta}$ (b) $\frac{45}{4\Delta}$
(c) $\left(\frac{3}{4\Delta}\right)^2$ (d) $\left(\frac{45}{4\Delta}\right)^2$

[IIT-JEE-2012]

56. No questions asked in 2013.

57. In a triangle, the sum of two sides is x and the product of the same two sides is y.
If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is

(a) $\frac{3y}{2x(x+c)}$ (b) $\frac{3y}{2c(x+c)}$
(c) $\frac{3y}{4x(x+c)}$ (d) $\frac{3y}{4c(x+c)}$

[IIT-JEE-2014]

58. No questions asked in 2015.

COMPREHENSIVE LINK PASSAGE (FOR JEE ADVANCED EXAM ONLY)

In these questions, a passage (paragraph) has been given followed by questions based on each of the passage. You have to answer the questions based on the passage given.

PASSAGE 1

If p_1, p_2, p_3 are the altitudes of a triangle ABC, from the vertices a, b, c, respectively and Δ is the area of the triangle and 's' is the semi perimeter of the triangle .

On the basis of the above information, answer the following questions.

1. If $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{2}$, then the least value of $p_1\,p_2\,p_3$ is

(a) 8 (b) 27
(c) 125 (d) 216

2. The value of $\frac{\cos A}{p_1}+\frac{\cos B}{p_2}+\frac{\cos C}{p_3}$ is

(a) $\frac{1}{r}$ (b) $\frac{1}{R}$
(c) $\frac{a^2+b^2+c^2}{2R}$ (d) $\frac{1}{\Delta}$

3. The minimum value of $\frac{b^2 p_1}{c}+\frac{c^2 p_1}{a}+\frac{a^2 p_1}{b}$ is

(a) Δ (b) 2Δ
(c) 3Δ (d) 4Δ

4. The value of $\frac{1}{p_1^3}+\frac{1}{p_2^3}+\frac{1}{p_3^3}$ is
   (a) $\frac{(\Sigma a)^2}{4\Delta^2}$ (b) $\frac{(\Pi a)^3}{8\Delta^3}$
   (c) $\frac{\Sigma a^2}{4\Delta^2}$ (d) $\frac{\Pi a^2}{8\Delta^2}$
5. In the triangle ABC, the altitudes are in A.P., then
   (a) a, b, c, are in A.P.
   (b) a, b, c are in H.P.
   (c) a, b, c are in G.P.
   (d) angles A, B, C are in A.P.

PASSAGE II

$ABCD$ be a cyclic quadrilateral inscribed in a circle of radius R.

Then $\cos B = \frac{(a^2+b^2-c^2-d^2)}{2(ab+cd)}$ and the area of the quadrilateral $= \frac{1}{2}(ab+cd)\sin B$

$= \sqrt{(s-a)(s-b)(s-c)(s-d)}$,

where, $s = \frac{a+b+c+d}{2}$.

Also, $AC^2 . BD^2 = (ac+bd)^2$

i.e., $AC. BD = AB . CD + BC . AD$

and $R = \frac{AC}{2\sin B}$

On the basis of the above information, answer the following questions.

1. The side of a quadrilateral which can be inscribed in a circle are 6, 6, 8 and 8 cm.
   Then the circum radius is
   (a) 5/2 cm (b) 24/ 7 cm
   (c) 11/7 cm (d) None.
2. The sides of a quadrilateral, which can be inscribed in a circle are 5, 5, 12 and 12 cm . Then the in-radius is
   (a) 15/17 cm (b) 30/17 cm
   (c) 60/17 cm (d) None.
3. If a quadrilateral with sides a, b, c, d can be inscribed in one circle and circumscribed about an other circle, then its area is
   (a) $\sqrt{abcd}$ (b) $\sqrt{2(abcd)}$
   (c) $2\times\sqrt{(abcd)}$ (d) None.

PASSAGE III

G is the centroid of the triangle ABC. perpendiculars from vertices A, B, C meet the sides BC, CA, AB at D, E, F respectively. P, Q, R are the feet of perpendiculars from G on sides.

BC, CA, AB respectively, L, M, N are the mid points of the sides BC, CA, AB respectively.

On the basis of the above information, answer the following questions.

1. Length of the side PG is
   (a) $\frac{1}{2}b\sin C$ (b) $\frac{1}{2}c\sin C$
   (c) $\frac{2}{3}b\sin C$ (d) $\frac{1}{3}c\sin B$
2. $ar(\Delta GPL):ar(\Delta ALD)$ is
   (a) 1/3 (b) 1/9
   (c) 2/3 (d) 4/9
3. Area of ΔPQR is
   (a) $\frac{1}{9}(a^2+b^2+c^2)\sin A.\sin B.\sin C$
   (b) $\frac{1}{18}(a^2+b^2+c^2)\sin A.\sin B.\sin C$
   (c) $\frac{2}{9}(a^2+b^2+c^2)\sin A.\sin B.\sin C$
   (d) $\frac{1}{3}(a^2+b^2+c^2)\sin A.\sin B.\sin C$

PASSAGE IV

In a triangle ABC, R be the circum radius such that $R = \frac{abc}{4\Delta}$ and r_1, r_2 & r_3 are the ex-radii, where $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-a}$ & $r_3 = \frac{\Delta}{s-a}$ and r be in-radius such that $r = \frac{\Delta}{s}$.

On the basis of above information, answer the following questions.

1. If $r_1 = r + r_2 + r_3$, then the triangle is
   (a) equilateral (b) isosceles
   (c) right angled (d) None
2. The value of cos A + cos B + cos C is
   (a) $1+\frac{r}{R}$ (b) $2\left(1+\frac{r}{R}\right)$
   (c) $\frac{1}{2}\left(1+\frac{r}{R}\right)$ (d) $\frac{1}{4}\left(1+\frac{r}{R}\right)$

3. If $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)=2$, then the triangle is
   (a) right angled (b) equilateral
   (c) isosceles (d) None.

4. The value of $r_1+r_2+r_3-4R$ is
   (a) $2r$ (b) $3r$
   (c) r (d) $4r$.

5. The value of $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$ is
   (a) $\frac{1}{r}$ (b) $\frac{2}{r}$
   (c) $\frac{1}{2r}$ (d) $\frac{3}{2r}$

6. The value of $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}$ is
   (a) $\frac{1}{r}-\frac{1}{R}$ (b) $\frac{1}{2r}-\frac{1}{R}$
   (c) $\frac{1}{r}-\frac{1}{2R}$ (d) $\frac{1}{r}-\frac{1}{3R}$.

PASSAGE V

In any triangle ABC, $\cos A=\frac{b^2+c^2-a^2}{2bc}$,

$\cos B=\frac{a^2+c^2-b^2}{2ac}$ and $\cos C=\frac{a^2+b^2-c^2}{2ab}$,

where $a = BC$, $b = CA$ and $c = AB$, respectively.

On the basis of the above information, answer the following questions.

1. If the angles A, B, C are in A.P., then $\frac{a+c}{\sqrt{a^2+c^2-ac}}$ is
   (a) $2\cos\left(\frac{A-C}{2}\right)$ (b) $2\sin\left(\frac{A-C}{2}\right)$
   (c) $\sin\left(\frac{A-C}{2}\right)$ (d) $\cos\left(\frac{A-C}{2}\right)$

2. If $\frac{1}{a+c}+\frac{1}{b+c}=\frac{1}{a+b+c}$, then
   (a) $\angle C=75°$ (b) $\angle A=75°$
   (c) $\angle A=60°$ (d) $\angle C=60°$

3. The value of $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$ is
   (a) $\frac{a^2+b^2+c^2}{2abc}$ (b) $\frac{a^2+b^2+c^2}{abc}$
   (c) $\frac{a^2-b^2+c^2}{abc}$ (d) $\frac{a^2+b^2-c^2}{abc}$

4. If $c^4-2\left(a^2+b^2\right)c^2+a^4+a^2b^2+b^4=0$,
   then the angle c is
   (a) 60° (b) 30°
   (c) 75° (d) 45°

5. The value of $\frac{\cos 2A}{a^2}-\frac{\cos 2B}{b^2}$ is
   (a) $\frac{1}{a^2}+\frac{1}{b^2}$ (b) $\frac{1}{a^2}-\frac{1}{b^2}$
   (c) $\frac{2}{a^2}-\frac{2}{b^2}$ (d) $\frac{2}{a^2}+\frac{2}{b^2}$

MATCH MATRIX
(FOR JEE ADVANCED EXAM ONLY)

1. Match the following columns:
   In any triangle ABC, then

Column-I	Column-II
(A) $b\cos C + c\cos B$	(P) c
(B) $c\cos A + a\cos C$	(Q) b
(C) $a\cos B + b\cos A$	(R) a
(D) $(b + c)\cos A + (c + a)\cos B + (a + b)\cos C$	(S) $a + b + c$

2. Match the following columns:
   In any triangle ABC, the value of

Column-I	Column-II
(A) $a\sin(B - C) + b\sin(C - A) + c\sin(A - B)$ is	(P) $3abc$
(B) $a^2\left(\cos^2 B-\cos^2 C\right)+b^2\left(\cos^2 C-\cos^2 A\right)+c^2\left(\cos^2 A-\cos^2 B\right)$ is	(Q) abc
(C) $\left(\frac{b^2-c^2}{a^2}\right)\sin 2A+\left(\frac{c^2-a^2}{b^2}\right)\sin 2B+\left(\frac{a^2-b^2}{c^2}\right)\sin 2C$ is	(R) 0
(D) $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)$ is	(S) $2abc$.

3. Match the following columns:
In any triangle ABC, the value of

Column- I	Column - II
(A) $\tan B \cot C$ is	(P) $\left(a^2+b^2+c^2\right)$
(B) 2 ($bc \cos A + ca \cos B + ab \cos C$) is	(Q) $\frac{\left(a^2+b^2-c^2\right)}{\left(a^2+b^2+c^2\right)}$
(C) $\frac{\cot A+\cot B+\cot C}{\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)}$ is	(R) $\frac{a^2+b^2+c^2}{(a+b+c)^2}$
(D) $\frac{\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)}{\cot\left(\frac{A}{2}\right)}$ is	(S) $\left(\frac{2a}{b+c-a}\right)$

4. Match the following columns:
In any triangle ABC, if

Column- I	Column - II
(A) $\cot A$, $\cot B$, $\cot C$ are in A.P., then	(P) a, b, c are in A.P.
(B) $\cos A.\cot\left(\frac{A}{2}\right)$, $\cos B.\cot\left(\frac{B}{2}\right)$, $\cos C.\cot\left(\frac{C}{2}\right)$ are in A.P., then	(Q) a^2, b^2, c^2 are in A.P.
(C) $\sin^2\left(\frac{A}{2}\right)$, $\sin^2\left(\frac{B}{2}\right)$, $\sin^2\left(\frac{C}{2}\right)$ are in A.P., then	(R) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.
(D) $\tan\left(\frac{A}{2}\right)$, $\tan\left(\frac{B}{2}\right)$, $\tan\left(\frac{C}{2}\right)$ are in A.P., then	(S) a, b, c are in G.P.

5. Match the following columns:
In any triangle ABC, if

Column- I	Column - II
(A) $\cot A+\cot B+\cot C=\sqrt{3}$, then Δ is	(P) Isoceles.
(B) $\left(a^2+b^2\right)\sin(A-B)=\left(a^2-b^2\right)\sin(A+B)$, then Δ is	(Q) Right angled.
(C) $2\cos A=\frac{\sin B}{\sin C}$, then Δ is	(R) Equilateral.
(D) $a\tan A+b\tan B=(a+b)\tan\left(\frac{A+B}{2}\right)$, then Δ is	(S) Acute angled.

6. Match the following columns:
In any triangle ABC, if

Column - I	Column - II
(A) $\sin A+\sin B+\sin C=\frac{3\sqrt{3}}{2}$, then Δ is	(P) Isoceles.
(B) $\tan A+\tan B$ \ $+\tan C=3\sqrt{3}$, then Δ is	(Q) Right angled.
(C) $8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)=1$, then Δ is	(R) Equilateral.
(D) $a^2+b^2+c^2=8R^2$, then Δ is	(S) Obtuse angled.

7. Match the following columns:
In any triangle ABC, if r be the in-radius and r_1, r_2 and r_3 be the ex-radii of the given ΔABC, then the value of

Column - I	Column - II
(A) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$ is	(P) r
(B) $\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2}$ is	(Q) $\frac{\Delta^2}{a^2+b^2+c^2}$
(C) $r_1+r_2+r_3-4R$ is	(R) $\frac{1}{r}$
(D) $\frac{r_1}{bc}+\frac{r_2}{ac}+\frac{r_3}{ba}+\frac{1}{2R}$ is	(S) $\frac{a^2+b^2+c^2}{\Delta^2}$

8. Match the following columns:
In any triangle ABC, if r be the inradius and R be the circum-radius of the given ΔABC, then the value of

Column - I	Column - II
(A) $\cos A+\cos B+\cos C$ is	(P) $\frac{\Delta}{r.R}$
(B) $a\cot A+b\cot B+c\cot C$ is	(Q) $\left(1+\frac{r}{R}\right)$

(C) $\sin A + \sin B + \sin C$ is — (*R*) $2(r+R)$

(D) $\cos^2\left(\frac{A}{2}\right)+\cos^2\left(\frac{B}{2}\right)+\cos^2\left(\frac{C}{2}\right)$

(*S*) $\left(2+\frac{r}{2R}\right)$

9. Match the following columns:
In any triangle *ABC*, if *r* be the inradius and r_1, r_2 and r_3 be the ex-radii of the given ΔABC, then the value of

Column - I	Column - II
(A) $r - r_1 + r_2 + r_3$ is	(P) 4 *R* cos *C*
(B) $r - r_2 + r_1 + r_3$ is	(Q) 4 *R* cos *B*
(C) $r - r_3 + r_1 + r_2$ is	(R) 4 *R* cos *A*
(D) $r_1 + r_2 + r_3 - r$ is	(S)

10. Match the following columns:
In any triangle *ABC*, the minimum value of

Column - I	Column - II
(A) $\cot^2 A + \cot^2 B + \cot^2 C$ is	(P) 9
(B) $\tan^2 A + \tan^2 B + \tan^2 C$ is	(Q) 1
(C) $\text{cosec}\left(\frac{A}{2}\right)+\text{cosec}\left(\frac{B}{2}\right)$ is $+\text{cosec}\left(\frac{C}{2}\right)$	(*R*) 6
(D) cos *A* + cos *B* + cos *C* is	(S) 3

11. Match the following columns:
In any triangle *ABC*, then the maximum value of

Column - I	Column - II
(A) cos *A* + cos *B* + cos *C* is	(*P*) 1
(B) cos *A* . cos *B* . cos *C* is	(*Q*) 3/2
(C) $\tan^2\left(\frac{A}{2}\right)+\tan^2\left(\frac{B}{2}\right)$ $+\tan^2\left(\frac{C}{2}\right)$ is	(*R*) 1/4
(D) $\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right)$ is	(*S*) 1/8.

ASSERTION AND REASON

Codes:

(A) Both *A* and *R* are individually true and *R* is the correct explanantion of *A*.
(B) Both *A* and *R* are individually true and *R* is not the correct explanantion of *A*.
(C) *A* is true and *R* is false.
(D) *A* is false and *R* is true.

1. Assertion (*A*) : If Δ be the area of a triangle and *s* be the semi - perimeter, then $\Delta^2 \le \frac{s}{4}$
Reson (*R*): $A.M \ge G.M$
(a) *A* (b) *B*
(c) *C* (d) *D*

2. Assertion (*A*): In a triangle *ABC*, if cos *A* + 2 cos *B* + cos *C* = 2, then *a*, *b*, *c* are in A.P.
Reason (*R*): In a triangle *ABC*, $\cos A+\cos B+\cos C$ $=1+4\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right)$
(a) *A* (b) *B*
(c) *C* (d) *D*

3. Assertion (*A*): In a right angled triangle, $a^2+b^2+c^2=8R^2$, where *R* is the circum-radius
Reason (*R*): $a^2=b^2+c^2$
(a) *A* (b) *B*
(c) *C* (d) *D*

4. Assertion (*A*): If *A*, *B*, *C* and *D* are the angles of a cyclic quadrilateral, then
sin *A* + sin *B* + sin *C* + sin *D* = 0
Reason (*R*): If *A*, *B*, *C* and *D* are the angles of a cyclic quadrilateral, then
cos *A* + cos *B* + cos *C* + cos *D* = 0
(a) *A* (b) *B*
(c) *C* (d) *D*

5. Assertion (*A*): In any triangle, *a* cos *A* + *b* cos *B* + *c* cos *C* ≤ *s*
Reason (*R*): In any triangle,
$\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\le\frac{1}{8}$
(a) *A* (b) *B*
(c) *C* (d) *D*

6. Assertion (*A*): In any triangle *ABC*, the min value of $\frac{r_1+r_2+r_3}{r}$ is 9
Reason (*R*): In a triangle *ABC*, if $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$, then $\frac{r_1+r_2+r_3}{r}=9$
(a) *A* (b) *B*
(c) *C* (d) *D*

7. Assertion (*A*): In a triangle *ABC*, the harmonic mean of the ex-radii is three times the in-radius.
Reason (*R*): In any triangle, *ABC*, $r_1+r_2+r_3=4R$
(a) *A* (b) *B*
(c) *C* (d) *D*

8. Assertion (A): If A, A_1, A_2, A_3 are the areas of in-circle and ex-circles of a triangle, then

$$\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=\frac{1}{\sqrt{A}}$$

Reason (R): In a triangle, $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$

(a) A (b) B
(c) C (d) D

9. Assertion (A): If x, y and z are respectively the distances of the vertices of a triangle ABC from its orthocentre,

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$$

Reason (R): In a triangle ABC,
$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$

(a) A (b) B
(c) C (d) D

10. Assertion (A): If x, y and z are respectively the distances of the vertices of a triangle ABC from its orthocentre, $x + y + z = 2(R + r)$

Reason (R): In a triangle ABC,

$$r = 4R\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right)$$

(a) A (b) B
(c) C (d) D

11. Assertion (A): In a triangle ABC, $r_1 + r_2 + r_3 - r = 4R$

Reason (R): In a triangle ABC, $R=\frac{abc}{4\Delta}$

(a) A (b) B
(c) C (d) D

12. Assertion (A): In any triangle ABC,

$$2\left(a\sin^2\left(\frac{C}{2}\right)+c\sin^2\left(\frac{A}{2}\right)\right)=a+c-b$$

Reason (R): In any triangle ABC,
$b = c\cos A + a\cos C$.

(a) A (b) B
(c) C (d) D

ANSWERS

EXERCISE 2

6. $0 \le \lambda \le 4$ 7. 120°.

EXERCISE 6

1. (d) 2. (a) 3. (b) 4. (a)
6. (b)

LEVEL I

37. 4, 5 and 6.

LEVEL II

1. (b)	2. (d)	3. (b)	4. (d)
5. (c)	6. (c)	7. (a)	8. (d)
9. (a)	10. (a)	11. (b)	12. (c)
13. (c)	14. (c)	15. (a)	16. (a)
17. (a)	18. (b)	19. (c)	20. (c)
21. (c)	22. (b)	23. (d)	24. (c)
25. (b)	26. (c)	27. (c)	28. (b)
29. (c)	30. (c)		
31. (b)	32. (a)	33. (d)	34. (c)
35. (d)	36. (b)	37. (c)	38. (c)
39. (d)	40. (b)	41. (c)	42. (c)
43. (b)	44. (c)	45. (d)	46. (d)
47. (d)	48. (a)	49. (b)	50. (a).

1. (d) 2. (a) 3. (b) 4. (a) 5. (b)

INTEGER TYPE QUESTIONS

1. 8
2 1
3 9
4 3
5. 2
6. 5
7. 1
8. 9
9. 2
10. 2

COMPREHENSIVE LINK PASSAGES:

Passage-I : 1. (d) 2. (b) 3. (d) 4. (c) 5. (b)
Passage-II : 1. (d) 2. (a) 3. (b)
Passage-III :
Passage-IV : 1. (c) 2. (a) 3. (a) 4. (c) 5. (a) 6. (c)
Passage-V : 1. (a) 2. (d) 3. (a) 4. (a) 5. (b)

MATCH MATRIX

1. (A) → (R); (B) →(Q); (C) →(P); (D) → (P)
2. (A) → (R); (B) → (R); (C) → (R); (D) → (P)
3. (A) → (Q); (B) → (P); (C) → (R); (D) → (S)
4. (A) → (Q); (B) → (P);(C) → (R); (D) → (P)
5. (A) → (R); (B) → (P, Q); (C) → (P);(D) → (P, Q).
6. (A) → (R); (B) → (R); (C) → (R); (D) → (Q)

7. (A) → (R); (B) → (S); (C) → (P); (D) → (*R*)
8. (A) → (Q); (B) → (R); (C) → (P); (D) → (S) .
9. (A) → (R); (B) → (Q); (C) → (P); (D) → (S)
10. (A) → (Q); (B) → (P); (C) → (R); (D) → (Q).
11. (A) → (Q); (B) → (S); (C) → (P); (D) → (S).

ASSERTION AND REASON

1. (a)	2. (b)	3. (a)	4. (d)
5. (a)	6. (b)	7. (c)	8. (a)
9. (a)	10. (a)	11. (a)	12. (a)

SELF ASSESMENT TEST

1.
7. (4, 5, 6.)

LEVEL III (PROBLEMS FOR JEE ADVANCED)

1. Given $\dfrac{\cos A + 2\cos C}{\cos A + 2\cos B} = \dfrac{\sin B}{\sin C}$

$\cos A\,(\sin B - \sin C) + (\sin 2B - \sin 2C) = 0$

$\cos A\,(\sin B - \sin C) + 2\cos(B + C)\sin(B - C) = 0$

$\cos A\,(\sin B - \sin C) + 2\cos(\pi + C)\sin(B - C) = 0$

$\cos A\,(\sin B - \sin C) - 2\cos A\sin(B - C) = 0$

$\cos A\,[(\sin B - \sin C) - 2\sin(B - C)] = 0$

$\cos A = 0,\ [(\sin B - \sin C) - 2\sin(B - C)] = 0$

when $\cos A = 0$

$\angle A = 90°$

Δ is right angled.

When $[(\sin B - \sin C) - 2\sin(B - C)] = 0$

$[(\sin B - \sin C) - 2(\sin B\cos C - \cos B\sin C)] = 0$

$$\left[(b-c) - 2\left(b\frac{a^2+b^2-c^2}{2ab} - c\frac{a^2+c^2-b^2}{2ac}\right)\right] = 0$$

$$\left[(b-c) - \left(\frac{a^2+b^2-c^2}{a} - \frac{a^2+c^2-b^2}{a}\right)\right] = 0$$

$$\left[(b-c) - \frac{1}{a}(a^2+b^2-c^2-a^2-c^2+b^2)\right] = 0$$

$$\left[(b-c) - \frac{2}{a}(b^2-c^2)\right] = 0$$

$[a(b-c) - 2(b^2-c^2)] = 0$

$(b - c)[a - 2(b + c)] = 0$

$(b - c) = 0$

$b = c$

Δ is isosceles.

2. We have

$$a\tan A + b\tan B = (a+b)\tan\left(\frac{A+B}{2}\right)$$

$$a\tan A + b\tan B = (a+b)\cot\left(\frac{C}{2}\right)$$

$$a\left(\tan A - \cot\left(\frac{C}{2}\right)\right) = b\left(\cot\left(\frac{C}{2}\right) - \tan B\right)$$

$$a\left(\frac{\sin A}{\cos A} - \frac{\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)}\right) = b\left(\frac{\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)} - \frac{\sin B}{\cos B}\right)$$

$$a\left(\frac{\sin A\sin\left(\frac{C}{2}\right) - \cos A\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)\cos A}\right)$$

$$= b\left(\frac{\cos B\cos\left(\frac{C}{2}\right) - \sin B\sin\left(\frac{C}{2}\right)}{\sin\left(\frac{C}{2}\right)\cos B}\right)$$

$$-\frac{a\cos\left(A+\frac{C}{2}\right)}{\cos A\sin\left(\frac{C}{2}\right)} = \frac{b\cos\left(B+\frac{C}{2}\right)}{\cos B\sin\left(\frac{C}{2}\right)}$$

$$-\frac{a\cos\left(A+\frac{C}{2}\right)}{\cos A} = \frac{b\cos\left(B+\frac{C}{2}\right)}{\cos B}$$

$$-\frac{\sin A\cos\left(A+\frac{C}{2}\right)}{\cos A} = \frac{\sin B\cos\left(B+\frac{C}{2}\right)}{\cos B}$$

$$-2\sin A\cos B\cos\left(A+\frac{C}{2}\right) = 2\sin B\cos A\cos\left(B+\frac{C}{2}\right)$$

$$-(\sin(A+B) + \sin(A-B))\cos\left(A+\frac{C}{2}\right)$$

$$= (\sin(A+B) - \sin(A-B))\cos\left(B+\frac{C}{2}\right)$$

$$\sin(A+B)\left\{\cos\left(B+\frac{C}{2}\right)+\cos\left(A+\frac{C}{2}\right)\right\}$$

$$=\sin(A-B)\left\{\cos\left(A+\frac{C}{2}\right)-\cos\left(B+\frac{C}{2}\right)\right\}$$

$$\sin(A+B)\left\{2\cos\left(\frac{A+B+C}{2}\right)\cos\left(\frac{B-A}{2}\right)\right\}$$

$$=\sin(A-B)\left\{2\sin\left(\frac{A+B+C}{2}\right)\sin\left(\frac{B-A}{2}\right)\right\}$$

$$\sin(A-B)\left\{2\sin\left(\frac{A+B+C}{2}\right)\sin\left(\frac{B-A}{2}\right)\right\}=0$$

$$\sin(A-B)=0,\ \left\{2\sin\left(\frac{A+B+C}{2}\right)\sin\left(\frac{B-A}{2}\right)\right\}=0$$

$\sin(A-B)=0$

$(A-B)=0$

$A=B$

Thus, the triangle is isosceles.

3. We have $a(rr_1+r_2r_3)$

$$=a\left(\frac{\Delta}{s}\cdot\frac{\Delta}{(s-a)}+\frac{\Delta}{(s-b)}\cdot\frac{\Delta}{(s-c)}\right)$$

$$=a\Delta^2\left(\frac{1}{\mathrm{s}(s-a)}+\frac{1}{(s-b)(\mathrm{s}-\mathrm{c})}\right)$$

$$=a\Delta^2\left(\frac{(s-b)(s-c)+s(s-a)}{s(s-a)(s-b)(s-c)}\right)$$

$$=a\Delta^2\left(\frac{2s^2-(a+b+c)s+bc}{s(s-a)(s-b)(s-c)}\right)$$

$$=a\Delta^2\left(\frac{2s^2-2s\cdot s+bc}{s(s-a)(s-b)(s-c)}\right)$$

$$=a\Delta^2\left(\frac{2s^2-2s^2+bc}{\Delta^2}\right)$$

$=abc$

Also, $b(rr_2+r_1r_3)$

$$=b\left(\frac{\Delta}{s}\cdot\frac{\Delta}{(s-b)}+\frac{\Delta}{(s-a)}\cdot\frac{\Delta}{(s-c)}\right)$$

$$=b\Delta\left(\frac{1}{s}\cdot\frac{1}{(s-b)}+\frac{1}{(s-a)}\cdot\frac{1}{(s-c)}\right)$$

$$=b\Delta^2\left(\frac{1}{\mathrm{s}(s-b)}+\frac{1}{(s-a)(\mathrm{s}-\mathrm{c})}\right)$$

$$=b\Delta^2\left(\frac{(s-a)(s-c)+s(s-b)}{\mathrm{s}(s-a)(\mathrm{s}-\mathrm{c})(\mathrm{s}-\mathrm{c})}\right)$$

$$=b\Delta^2\left(\frac{2s^2-2s\cdot s+ac}{\mathrm{s}(s-a)(\mathrm{s}-\mathrm{c})(\mathrm{s}-\mathrm{c})}\right)$$

$$=b\Delta^2\left(\frac{2s^2-2s\cdot s+ac}{\mathrm{s}(s-a)(\mathrm{s}-\mathrm{c})(\mathrm{s}-\mathrm{c})}\right)$$

$$=b\Delta^2\left(\frac{2s^2-2s^2+ac}{\Delta^2}\right)$$

$$=b\Delta^2\left(\frac{ac}{\Delta^2}\right)$$

$=abc$

Similarly, $c(rr_3+r_1r_2)=abc$

Hence, the result.

4. Now, $(r+r_1)\tan\left(\frac{B-C}{2}\right)$

$$=\left(\frac{\Delta}{s}+\frac{\Delta}{(s-a)}\right)\left(\frac{b-c}{b+c}\right)\cot\left(\frac{C}{2}\right)$$

$$=\Delta\left(\frac{1}{s}+\frac{1}{(s-a)}\right)\left(\frac{b-c}{b+c}\right)\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$=\Delta\left(\frac{s-a+s}{\mathrm{s}(s-a)}\right)\left(\frac{b-c}{b+c}\right)\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$=\Delta(2s-a)\left(\frac{b-c}{b+c}\right)\sqrt{\frac{1}{s(s-a)(s-b)(s-c)}}$$

$$=\Delta(a+b+c-a)\left(\frac{b-c}{b+c}\right)\times\frac{1}{\Delta}$$

$$=(b+c)\left(\frac{b-c}{b+c}\right)$$

$=(b-c)$

Thus, L.H.S.

$=(b-c)+(c-a)+(a-b)$

$=0.$

5. We have

$$\frac{\tan\left(\frac{A}{2}\right)}{(a-b)(a-c)}+\frac{\tan\left(\frac{B}{2}\right)}{(b-a)(b-c)}+\frac{\tan\left(\frac{C}{2}\right)}{(c-a)(c-b)}$$

Now, $\dfrac{\tan\left(\dfrac{A}{2}\right)}{(a-b)(a-c)} = \dfrac{\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}}{(a-b)(a-c)}$

$= \dfrac{\Delta}{s(s-a)(a-b)(a-c)}$

So, LHS

$= \dfrac{\Delta}{s}\left[\dfrac{1}{(s-a)(a-b)(a-c)} - \dfrac{1}{(s-b)(a-b)(b-c)} - \dfrac{1}{(s-c)(c-a)(b-c)}\right]$

$= \dfrac{\Delta}{s(a-b)(b-c)(c-a)}\left[\dfrac{(b-c)}{(s-a)} + \dfrac{(c-a)}{(s-b)} + \dfrac{(a-b)}{(s-b)}\right]$

$= \dfrac{\Delta}{s(a-b)(b-c)(c-a)}\left[\dfrac{\sum(b-c)\{s^2-(b+c)s+bc\}}{(s-a)(s-b)(s-c)}\right]$

$= \dfrac{\Delta}{s(a-b)(b-c)(c-a)}$

$\times\left[\dfrac{s^2\sum(b-c) - s\sum(b^2-c^2) + \sum bc(b-c)}{(s-a)(s-b)(s-c)}\right]$

$= \dfrac{\Delta}{s(a-b)(b-c)(c-a)} \times \left[\dfrac{\sum bc(b-c)}{(s-a)(s-b)(s-c)}\right]$

$= \left[\dfrac{\Delta}{s(a-b)(b-c)(c-a)}\dfrac{(a-b)(b-c)(c-a)}{(s-a)(s-b)(s-c)}\right]$

$= \dfrac{\Delta}{s(a-b)(b-c)(c-a)}$

$= \left[\dfrac{\Delta}{\Delta^2}\right]$

$= \dfrac{1}{\Delta}$

Hence, the result.

6. We have $\dfrac{\text{area of the incircle}}{\text{area of triangle } ABC}$

$\dfrac{\pi r^2}{\Delta}$

$\dfrac{\pi r^2}{\Delta}$

$= \dfrac{\pi r^2}{\Delta} = \dfrac{\pi}{\Delta} \times \dfrac{\Delta^2}{s^2} = \dfrac{\pi\Delta}{s^2}$

Now, $\cot\left(\dfrac{A}{2}\right)\cdot\cot\left(\dfrac{B}{2}\right)\cdot\cot\left(\dfrac{C}{2}\right)$

$= \left[\dfrac{s(s-a)}{(s-b)(s-c)}\cdot\dfrac{s(s-b)}{(s-a)(s-c)}\cdot\dfrac{s(s-c)}{(s-a)(s-b)}\right]^{1/2}$

$= \left[\dfrac{s^3}{(s-a)(s-b)(s-c)}\right]^{1/2}$

$= \left[\dfrac{s^4}{s(s-a)(s-b)(s-c)}\right]^{1/2}$

$= \left[\dfrac{s^4}{\Delta^2}\right]^{1/2}$

$= \dfrac{s^2}{\Delta}$

Hence, the area

$= \dfrac{\pi}{\cot\left(\dfrac{A}{2}\right)\cot\left(\dfrac{B}{2}\right)\cot\left(\dfrac{C}{2}\right)}$

7. We have $\cos A\cdot\cot\left(\dfrac{A}{2}\right)$

$= \left(1-2\sin^2\left(\dfrac{A}{2}\right)\right)\dfrac{\cos\left(\dfrac{A}{2}\right)}{\sin\left(\dfrac{A}{2}\right)}$

$= \cos\left(\dfrac{A}{2}\right) - \sin A$

Now, a, b, c are in A.P.

$\sin A$, $\sin B$, $\sin C$ are also in A.P.

and $\cot\left(\dfrac{A}{2}\right)\cot\left(\dfrac{B}{2}\right)\cot\left(\dfrac{C}{2}\right) \in$ A.P.

Thus, their differences are also in A.P.

8. Let $BC = a = c$, $CA = b = 10c$ and $AB = c$

Clearly, $a^2 + b^2 = 101c^2$

Applying, sine rule, we get,

$\dfrac{\sin A}{c} = \dfrac{\sin B}{10c} = \dfrac{\sin C}{c}$

$\dfrac{\sin A}{c} = \dfrac{\sin B}{10c} = \dfrac{\sin C}{c} = k$ (say)

Now, $\dfrac{\cot C}{\cot A + \cot B}$

$$= \frac{\frac{\cos C}{\sin C}}{\frac{\sin(A+B)}{\sin A \sin B}}$$

$$= \frac{\frac{\cos C}{\sin C}}{\frac{\sin C}{\sin A \sin B}}$$

$$= \frac{\cos C}{\sin C} \times \sin B$$

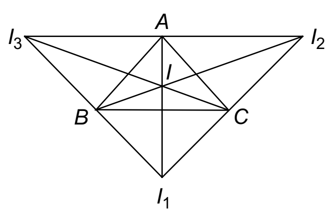
$$= \cos C \times \frac{\sin B}{\sin C}$$

$$= \left(\frac{a^2 + b^2 - c^2}{2ab}\right) \times \frac{\sin B}{\sin C}$$

$$= k^2 \left(\frac{c^2 + 100c^2 - c^2}{2 \cdot ck \cdot 10ck}\right) \times \frac{10ck}{ck}$$

$= 5 \times 10 = 50$

9. Do yourself.
10. We have



Area of the triangle $\Delta I_1 I_2 I_3$

$= \frac{1}{2} \times$ (product of two sides) $\times$ (sine of included angles)

$$= \frac{1}{2} \times \left(4R\cos\left(\frac{B}{2}\right)\right) \times \left(4R\cos\left(\frac{C}{2}\right)\right) \times \sin\left(90° - \frac{A}{2}\right)$$

$$= 8R^2 \times \cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$$

$$= 8R^2 \times \sqrt{\frac{s(s-a)}{bc}} \times \sqrt{\frac{s(s-b)}{ca}} \times \sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{8R^2}{abc} \times s\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \frac{8R^2 s}{abc} \times \Delta$$

$$= \frac{8}{abc} \times \left(\frac{abc}{4\Delta}\right)^2 \times \Delta s$$

$$= \frac{8abc}{16} \times \frac{s}{\Delta}$$

$$= \frac{abc}{2r}$$

11. Clearly, $r_1 + r_2 + r_3 = r + 4R$

and $r_1 r_2 + r_2 r_3 + r_3 r_1 = s_2$

and $r_1 r_2 r_3 = \dfrac{\Delta^2}{(s-a)(s-b)(s-c)} = \text{D}s = s^2 r$

Hence, the required equation is

$x^3 - (r + 4R)x^2 + s^2 x - rs^2 = 0$

12. Let r be the in-radius and R be the circum radius of and equilateral triangle

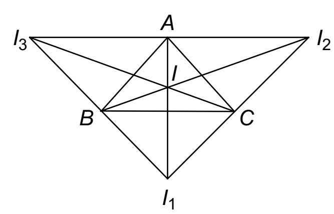
Now, $r = = \dfrac{\Delta}{s} = \dfrac{\sqrt{3}a^2}{4} \times \dfrac{2}{3a} = \dfrac{a}{2\sqrt{3}}$

and $R = \dfrac{abc}{4\Delta} = \dfrac{a^2}{4 \times \frac{\sqrt{3}a^2}{4}} = \dfrac{a}{\sqrt{3}}$.

Thus, $\dfrac{r}{R} = \dfrac{\frac{a}{2\sqrt{3}}}{\frac{a}{\sqrt{3}}} = \dfrac{1}{2}$

Hence, the result.

13. We have



Hence, the area

$= \frac{1}{2} \times$ (product of two sides) $\times$ (sine of including angles)

$$= \frac{1}{2} \times \left(4R\cos\left(\frac{B}{2}\right)\right) \times \left(4R\cos\left(\frac{C}{2}\right)\right) \times \sin\left(90° - \frac{A}{2}\right)$$

$$= \frac{1}{2} \times \left(4R\cos\left(\frac{B}{2}\right)\right) \times \left(4R\cos\left(\frac{C}{2}\right)\right) \times \cos\left(\frac{A}{2}\right)$$

$$= 8R^2 \times \cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$$

14. Hence, the circum-radius

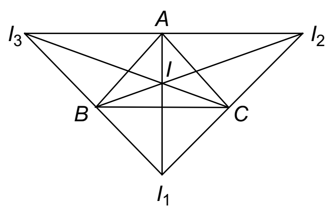
$$= \frac{I_2 I_3}{2\sin(I_2 I_1 I_3)}$$

$$= \frac{4R\cos\left(\frac{A}{2}\right)}{2\sin\left(90° - \frac{A}{2}\right)}$$

$$= \frac{4R\cos\left(\frac{A}{2}\right)}{2\cos\left(\frac{A}{2}\right)}$$

$= 2R$

15. From the figure



$\angle IBI_2$, $\angle ICI_1$ are right angles

Here, II_1 is the diameter of the circum-circle of the triangle ΔBCI_1

Thus, $II_1 = \frac{BC}{\sin(\angle BI_1C)} = \frac{a}{\sin\left(90° - \frac{A}{2}\right)}$

$$= \frac{a}{\cos\left(\frac{A}{2}\right)}$$

$$= \frac{2R\sin A}{\cos\left(\frac{A}{2}\right)}$$

$$= \frac{2R + 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$

$$= 4R\sin\left(\frac{A}{2}\right)$$

Similarly, $II_2 = 4R\sin\left(\frac{B}{2}\right)$,

$II_3 = 4R\sin\left(\frac{C}{2}\right)$

16. We have $a^3 \cos(B - C)$

$= a^2 k \sin A \cos(B - C)$

$= a^2 k \sin(B + C) \cos(B - C)$

$= \frac{a^2 k}{2}$ [2sin $(B + C)$ cos $(B - C)$]

$= \frac{a^2 k}{2}$ [sin2B + sin2C]

$= \frac{k^3}{2}$ [sin2A sin2B + $\sin^2 A$ sin2C]

$= k^3 [\sin^2 A \sin B \cos B + \sin^2 A \sin C \cos C]$

Similarly, $b^3 \cos(C - A)$

$= k^3[\sin^2 B \sin C \cos C + \sin^2 B \sin A \cos A]$

and $c^3 \cos(A - B)$

$= k^3[\sin^2 C \sin A \cos A + \sin^2 C \sin B \cos B]$

Adding all we get,

$k^3[\sin A \sin B(\sin A \cos B + \cos A \sin B)$

$+ \sin B \sin C (\sin B \cos C + \cos B \sin C)$

$+ \sin C \sin A (\sin C \cos A + \cos C \sin A)]$

$= k^3 [\sin A \sin B \sin(A + B) + \sin B \sin C$

$\sin(B + C) + \sin C \sin A \sin(C + A)]$

$= k^3[\sin A \sin B \sin C + \sin A \sin B \sin C + \sin A \sin B \sin C]$

$= k^3[3\sin A \sin B \sin C]$

$= 3(k \sin A)(k \sin B)(k \sin C)$

$= 3abc$

17. Do yourself

18. We have $c \cos(A + \theta) + a \cos(C + \theta)$

$= c(\cos A \cos \theta + \sin A \sin \theta)$

$+ a(\cos C \cos \theta - \sin C \sin \theta)$

$= \cos \theta(c \cos A + a \cos C)$

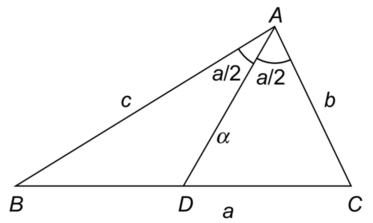
$+ \sin \theta(c \sin A - a \sin C)$

$= b \cos \theta + \sin \theta(k \sin C \sin A - k \sin C \sin A)$

$= b \cos \theta + \sin \theta \times 0$

$= b$ cos q

19. Let $BC = a$, $AC = b$, $AB = c$



Clearly, $(\Delta\, ABC) = ar(\Delta\, ABD) + \text{ar}(\Delta\, ACD)$

$$\frac{1}{2}bc\sin A = \frac{1}{2}c\alpha\sin\left(\frac{A}{2}\right) + \frac{1}{2}b\alpha\sin\left(\frac{A}{2}\right)$$

$$bc\sin A = c\alpha\sin\left(\frac{A}{2}\right) + b\alpha\sin\left(\frac{A}{2}\right)$$

$$\frac{1}{a}\cos\left(\frac{A}{2}\right) = \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right)$$

Similarly, $\frac{1}{\beta}\cos\left(\frac{B}{2}\right) = \frac{1}{2}\left(\frac{1}{c} + \frac{1}{a}\right)$

And $\frac{1}{\gamma}\cos\left(\frac{C}{2}\right) = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$

Adding, we get,

$$\frac{1}{\alpha}\cos\left(\frac{A}{2}\right) + \frac{1}{\beta}\cos\left(\frac{B}{2}\right) + \frac{1}{\gamma}\cos\left(\frac{C}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right) + \frac{1}{2}\left(\frac{1}{c} + \frac{1}{a}\right)$$

$$= \frac{1}{2}\left(\frac{2}{a} + \frac{2}{b} + \frac{2}{c}\right)$$

$$= \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

Hence, $\frac{1}{AD}\cos\left(\frac{A}{2}\right) + \frac{1}{BE}\cos\left(\frac{B}{2}\right) + \frac{1}{CF}\cos\left(\frac{C}{2}\right)$

$$= \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

20. Do yourself
21. Clearly, $2s = a + b + c$

As we know that, A.M $\geq$ G.M

$$\frac{(s-a) + (s-b) + (s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$$

$$\frac{3s - (a+b+c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$$

$$\frac{3s - 2s}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$$

$$\frac{s}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}$$

$$\left(\frac{s}{3}\right)^3 \geq (s-a)(s-b)(s-c)$$

$$s^4 \geq 27 \times s(s-a)(s-b)(s-c)$$

$$s^4 \geq 27 \times \text{D}^2$$

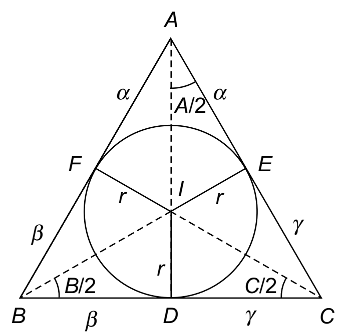
$$s^2 \geq 3\sqrt{3} \times \text{D}$$

$$\Delta \leq \frac{s^2}{3\sqrt{3}}$$

22. Do yourself.
23. We have $2\alpha + 2\beta + 2\gamma = a + b + c = 2s$

$\alpha + \beta + \gamma = s$

and $\alpha = s - a$, $\beta = s - b$, $\gamma = s - c$

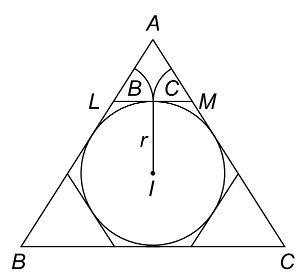


Now, $r^2 = \frac{\Delta^2}{s^2} = \frac{s(s-a)(s-b)(s-c)}{s^2}$

$$= \frac{(s-a)(s-b)(s-c)}{s}$$

$$= \frac{\alpha \cdot \beta \cdot \gamma}{\alpha + \beta + \gamma}$$

24. In triangle ALM, we have



$$AL = x \cdot \frac{x}{\sin A} = \frac{AM}{\sin B} = \frac{AL}{\sin C}$$

$AM = x \cdot \dfrac{\sin B}{\sin A} = \dfrac{bx}{a}$

From the figure, it is clear that

$r = ex$ – radius of ΔALM

$$r = \left(\frac{x + AL + AM}{2}\right)\tan\left(\frac{A}{2}\right)$$

$$r = \left(\frac{x + \frac{cx}{a} + \frac{bx}{2}}{2}\right)\tan\left(\frac{A}{2}\right)$$

$$r = \left(\frac{a+b+c}{2a}\right)x\tan\left(\frac{A}{2}\right) = \frac{sx}{a}\tan\left(\frac{A}{2}\right)$$

$$(s-a)\tan\left(\frac{A}{2}\right) = \frac{sx}{a}\tan\left(\frac{A}{2}\right)$$

$$(s-a) = \frac{sx}{a}$$

Similarly, $(s-b) = \dfrac{sx}{b}$, $(s-c) = \dfrac{sx}{c}$

Adding, we get

$$(s-a) + (s-b) + (s-c) = \frac{sx}{a} + \frac{sx}{b} + \frac{sx}{c}$$

$$3s - (a+b+c) = \frac{sx}{a} + \frac{sx}{b} + \frac{sx}{c}$$

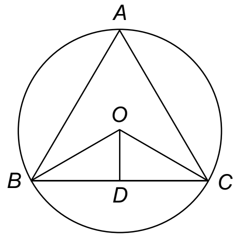
$$3s - 2s = \frac{sx}{a} + \frac{sx}{b} + \frac{sx}{c}$$

$$\frac{sx}{a} + \frac{sx}{b} + \frac{sx}{c} = s$$

$$\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = 1$$

Hence, the result.

25. It is given that, $x = OD = R\cos A$



$$= \frac{a}{2\sin A}\cdot\cos A$$

$$= \frac{a}{2\tan A}$$

$$\frac{a}{x} = 2\tan A$$

Similarly, $\dfrac{b}{y} = 2\tan B$, $\dfrac{c}{z} = 2\tan C$

As we know that,

Tan A + tan B + tan C = tan A tan B tan C

$$\frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z} = \frac{a}{2x}\cdot\frac{b}{2y}\cdot\frac{c}{2z}$$

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$$

Hence, the result.

26. Given $\sin A \sin B \sin C = p$

And $\cos A \cos B \cos C = q$

Here, $A + B + C =$ p

$\tan(A + B + C) = \tan(\pi) = 0$

$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C = \dfrac{p}{q}$

Also, $(1 - \tan A\tan B - \tan B\tan C - \tan C\tan A)$

$$= \frac{\cos(A+B+C)}{\cos A\cos B\cos C} = -\frac{1}{q}$$

Thus, $\tan A\tan B + \tan B\tan C + \tan C\tan A$

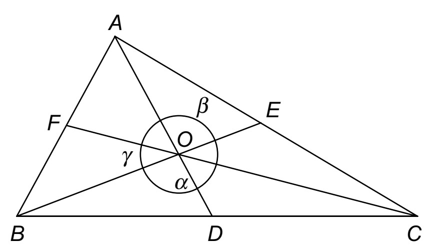
$$= 1 + \frac{1}{q} = \left(\frac{q+1}{q}\right)$$

Hence, the required equation is

$$x^3 - \left(\frac{p}{q}\right)x^2 + \left(\frac{q+1}{q}\right)x - \left(\frac{p}{q}\right) = 0$$

$$qx^3 - px^2 + (q+1)x - p = 0$$

27. Let the medians AD, BE and CF meet at O such that $\angle BOC = \alpha$, $\angle AOC = \beta$, $\angle AOB =$ g



Let $AD = p_1$, $BE = p_2$, $CF = p_3$

Clearly, $OA = \dfrac{2}{3}p_1$, $OB = \dfrac{2}{3}p_2$, $OC = \dfrac{2}{3}p_3$

From ΔAOC, we get,

$$\cos\beta = \frac{OA^2 + OC^2 - AC^2}{2OA\cdot OC}$$

$$\Rightarrow \cos\beta = \frac{\frac{4}{9}p_1^2 + \frac{4}{9}p_3^2 - b^2}{2\cdot\frac{2}{3}p_1\cdot\frac{2}{3}p_3}$$

$$\Rightarrow \cos\beta = \frac{4p_1^2 + 4p_3^2 - 9b^2}{8p_1p_3} \quad \ldots(1)$$

Also, $ar(\Delta AOC) = \frac{1}{2}\ OA\cdot OC\cdot \sin b$

$$\Rightarrow \frac{1}{3}\Delta = \frac{1}{2}\ OA\cdot OC\cdot\sin b$$

$$\Rightarrow \frac{1}{3}\Delta = \frac{1}{2}\frac{2}{3}p_1\cdot\frac{2}{3}p_3\cdot\sin\beta$$

$$\Rightarrow \sin\beta = \frac{3\Delta}{2p_1p_3} \quad \ldots(ii)$$

Dividing (i) by (ii), we get,

$$\cot\beta = \frac{4p_1^2 + 4p_3^2 - 9b^2}{12\Delta} \quad \ldots(iii)$$

Again AD is the median of ΔABC

So, $AB^2 + AC^2 + 2BD^2 + 2AD^2$

$$\Rightarrow b^2 + c^2 = 2\cdot\left(\frac{a}{2}\right)^2 + 2p_1^2$$

$$\Rightarrow b^2 + c^2 = = \frac{a^2}{2} + 2p_1^2$$

$$\Rightarrow p_1^2 = \frac{2b^2 + 2c^2 - a^2}{4}$$

similarly, $p_2^2 = \dfrac{2a^2 + 2c^2 - b^2}{4}$

and $p_3^2 = \dfrac{2a^2 + 2b^2 - c^2}{4}$

Now, from (iii), we get,

$$\cot\beta = \frac{4p_1^2 + 4p_3^2 - 9b^2}{12\Delta}$$

$$= \frac{(2b^2 + 2c^2 - a^2) + (2a^2 + 2c^2 - b^2) - 9b^2}{12\Delta}$$

$$= \frac{a^2 + c^2 - 5b^2}{12\Delta}$$

Similarly, $\cot\alpha = \dfrac{b^2 + c^2 - 5a^2}{12\Delta}$

and $\cot\gamma = \dfrac{b^2 + a^2 - 5c^2}{12\Delta}$

Now, $\cot\alpha + \cot\beta + \cot\gamma$

$$= -\frac{3(a^2 + b^2 + c^2)}{12\Delta}$$

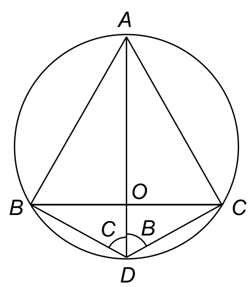
$$= -\frac{(a^2 + b^2 + c^2)}{4\Delta}$$

Also, $\cot A + \cot B + \cot C = \dfrac{(a^2 + b^2 + c^2)}{4\Delta}$

Hence,

Cot A + cot B + cot C + $\cot\alpha + \cot\beta + \cot\gamma = 0$

28. Let AO be the perpendicular from A on BC. When AO is produced, it meets the circumscribing circle at D such that OD = a



Since angles in the same segment are equal.

Thus, $\angle ADB = \angle ACB = \angle C$

and $\angle ADC = \angle ABC = \angle B$

From ΔBOD, $\tan(C) = \dfrac{OB}{OD}$ ….(i)

From ΔCOD, $\tan(B) = \dfrac{OC}{OD}$ ….(ii)

Adding (i) and (ii), we get

$$\tan B + \tan C = \frac{OB + OC}{OD} = \frac{BC}{OD} = \frac{a}{\alpha} \quad \ldots.(iii)$$

similarly, $\tan C + \tan A = \dfrac{b}{\beta}$ ….(iv)

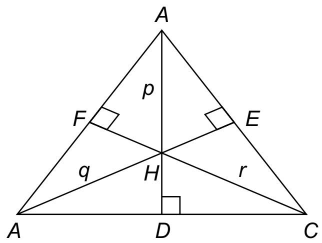
and $\tan A + \tan B = \dfrac{c}{\gamma}$ ….(v)

Adding (iii), (iv) and (v), we get,

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma} = 2(\tan A + \tan B + \tan C)$$

Hence, the result.

29. Let H be the othocentre of the triangle ABC such that $HA = p$, $HB = q$, $HC = r$



From the figure,

$\angle HBD = \angle EBC = 90° - C$

$\angle HCD = \angle FCB = 90° - B$

$\angle BHC = 180° - (\angle HBD + \angle HCD)$

$= 180° - (90° - C + 90° - B)$

$= (B + C) = 180° - A$

Similarly, $\angle AHC = 180° - B$

and $\angle AHB = 180° - C$

Now,

$ar(\Delta BHC) + ar(\Delta CHA) + ar(\Delta AHB) = ar(\Delta ABC)$

$$\Rightarrow \frac{1}{2}\, qr \sin(\angle BHC) + \frac{1}{2}\, pr \sin(\angle AHC)$$

$$+\frac{1}{2}\, pq \sin(\angle AHB) = \text{D}$$

$\Rightarrow qr \sin(180° - A) + pr \sin(180° - B)$

$+ pq \sin(180° - C) = 2\text{D}$

$\Rightarrow qr \sin A + rp \sin B + pq \sin C = \text{D}$

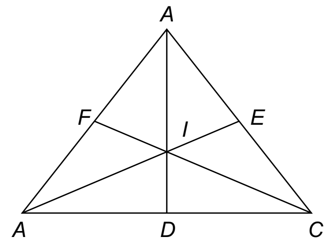
$$\Rightarrow qr\left(\frac{a}{2R}\right)+rp\left(\frac{b}{2R}\right)+pq\left(\frac{c}{2R}\right)=\frac{abc}{4R}$$

$\Rightarrow aqr + brp + cpq = abc$

$$\Rightarrow \frac{aqr}{pqr}+\frac{brp}{pqr}+\frac{cpq}{pqr}=\frac{abc}{pqr}$$

$$\Rightarrow \frac{a}{p}+\frac{b}{q}+\frac{c}{r}=\frac{abc}{pqr}$$

30. Here, AD is the internal bisector of the angle A



Clearly, $\dfrac{BD}{DC}=\dfrac{c}{b}$

$$\Rightarrow \frac{DC}{BD}=\frac{b}{c}$$

$$\Rightarrow \frac{DC}{BD}+1=\frac{b}{c}+1=\frac{b+c}{c}$$

$$\Rightarrow \frac{DC+BD}{BD}=\frac{b+c}{c}$$

$$\Rightarrow \frac{a}{BD}=\frac{b+c}{c}$$

$$\Rightarrow \frac{BD}{c}=\frac{a}{b+c}$$

Similarly, $\dfrac{BF}{a}=\dfrac{c}{a+b}$

Now, $\dfrac{ar(\Delta BDF)}{ar(\Delta ABC)}=\dfrac{\frac{1}{2}\cdot BD\cdot BF\cdot \sin B}{\frac{1}{2}a\,c\sin B}$

$$=\frac{BD\cdot BF}{ac}=\frac{BD}{a}\cdot\frac{BF}{c}=\frac{ac}{(a+b)(b+c)}$$

Similarly, $\dfrac{ar(\Delta CDE)}{ar(\Delta ABC)}=\dfrac{ac}{(a+b)(b+c)}$

and $\dfrac{ar(\Delta AEF)}{ar(\Delta\,\text{ABC})}=\dfrac{bc}{(b+c)(a+c)}$

Thus, $\dfrac{ar(\Delta DEF)}{ar(\Delta ABC)}$

$$=\frac{\Delta ABC-(\Delta BDF+\Delta CDE+\Delta AEF)}{\Delta\,\text{ABC}}$$

$$=1-\frac{\Delta BDF}{\Delta ABC}-\frac{\Delta CDE}{\Delta ABC}-\frac{\Delta AEF}{\Delta ABC}$$

$$=1-\frac{ac}{(a+b)(b+c)}-\frac{ab}{(a+c)(b+c)}-\frac{bc}{(a+b)(a+c)}$$

$$= \frac{(a+b)(b+c)(c+a) - \{ac(a+c) + ab(a+b) + bc(b+c)\}}{(a+b)(b+c)(c+a)}$$

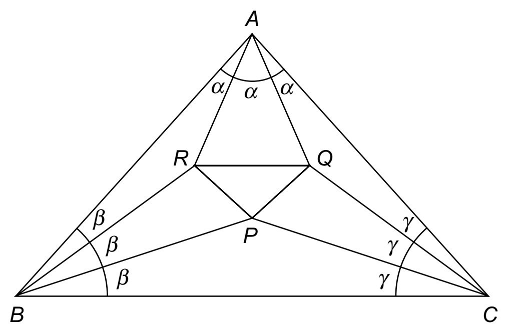
$$= \frac{2abc}{(a+b)(b+c)(c+a)}$$

$$\frac{ar(\Delta DEF)}{ar(\Delta ABC)} = \frac{2abc}{(a+b)(b+c)(c+a)}$$

$$ar\,(\Delta DEF) = \frac{2\Delta abc}{(a+b)(b+c)(c+a)}$$

Hence, the result.

31. Since $A + B + C = 180°$



$3\alpha + 3\beta + 3\gamma = 180°$

$\alpha + \beta + \gamma = 60°$

Clearly, $\angle ARB = 180° - (\alpha + \beta)$

Applying sine rule in triangle ARB

$$\frac{AR}{\sin\beta} = \frac{c}{\sin(180° - (\alpha+\beta))}$$

$$\frac{AR}{\sin\beta} = \frac{c}{\sin(\alpha+\beta)}$$

$$\frac{c}{\sin(\alpha+\beta)}$$

$$= \frac{2R\sin C\sin\beta}{\sin(\alpha+\beta)}$$

$$= \frac{2R\sin(3\gamma)\sin\beta}{\sin(\alpha+\beta)}$$

$$= \frac{2R\sin(3\gamma)\sin\beta}{\sin(60° - \gamma)}$$

$$= \frac{2R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)}{\sin(60° - \gamma)}$$

$$= \frac{2R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)}{\sin(60° - \gamma)} \cdot \frac{\cos(30° - \gamma)}{\cos(30° - \gamma)}$$

$$= \frac{4R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)\cos(30° - \gamma)}{2\sin(60° - \gamma)\cos(30° - \gamma)}$$

$$= \frac{4R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)\cos(30° - \gamma)}{\sin(90° - 2\gamma) + \sin(30°)}$$

$$= \frac{4R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)\cos(30° - \gamma)}{\cos(2\gamma) + \frac{1}{2}}$$

$$= \frac{8R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)\cos(30° - \gamma)}{2\cos(2\gamma) + 1}$$

$$= \frac{8R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)\cos(30° - \gamma)}{2(1 - 2\sin^2\gamma) + 1}$$

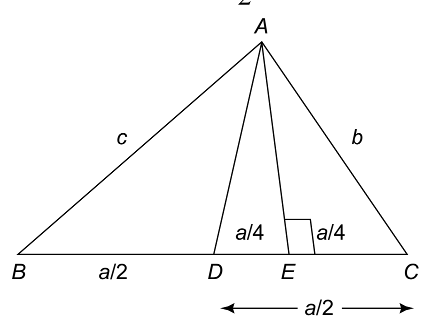
$$= \frac{8R\sin\beta\sin\gamma(3 - 4\sin^2\gamma)\cos(30° - \gamma)}{(3 - 4\sin^2\gamma)}$$

$= 8R \sin\beta \sin\gamma \text{in } \gamma(30° - \gamma)$

Hence, the result.

32. Since AD is the median, $BD = DC = \frac{a}{2}$

Also, $\angle DAE = \angle CAE = \frac{A}{2}$



Applying cosine rule in triangle ABD, we get,

$$\cos\left(\frac{A}{3}\right) = \frac{AB^2 + AD^2 - BD^2}{2AB \cdot AD}$$

$$\cos\left(\frac{A}{3}\right) = \frac{c^2 + b^2 - \left(\frac{a}{2}\right)^2}{2bc} = \frac{4b^2 + 4c^2 - a^2}{8bc} \quad \text{....(i)}$$

Applying cosine rule in triangle ABC, we get,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$4\cos^3\left(\frac{A}{3}\right) - 3\cos\left(\frac{A}{3}\right) = \frac{b^2 + c^2 - a^2}{2bc} \quad \text{.... (ii)}$$

Subtracting (ii) from (i), we get,

$$4\cos\left(\frac{A}{3}\right) - 4\cos^3\left(\frac{A}{3}\right)$$

$$= \frac{4b^2 + 4c^2 - a^2}{8bc} - \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{4b^2 + 4c^2 - a^2 - 4b^2 - 4c^2 + 4a^2}{8bc}$$

$$= \frac{3a^2}{8bc}$$

$$4\cos\left(\frac{A}{3}\right)\left(1 - \cos^2\left(\frac{A}{3}\right)\right) = \frac{3a^2}{8bc}$$

$$4\cos\left(\frac{A}{3}\right)\sin^2\left(\frac{A}{3}\right) = \frac{3a^2}{8bc}$$

$$\cos\left(\frac{A}{3}\right)\sin^2\left(\frac{A}{3}\right) = \frac{3a^2}{32bc}$$

INTEGER TYPE QUESTIONS

1. As we know that, in a right angled triangle $a^2 + b^2 + c^2 = 8R^2$

$$\left(\frac{a^2 + b^2 + c^2}{R^2}\right) = 8 = 8$$

2. We have $\left(\frac{r_1 + r_2 + r_3}{r}\right)$

$$4R\cos\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$$

$$= \frac{\left(\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right)\right)}{4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}$$

$$= \frac{\left(\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right)\right)}{\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)}$$

$= 1$

3. As we know that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$$

Using, A.M ≥ G.M

$$\left(\frac{r_1 + r_2 + r_3}{3}\right) \geq \frac{3}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}}$$

$$\left(\frac{r_1 + r_2 + r_3}{3}\right) \geq \frac{3}{\frac{1}{r}} = 3r$$

$$\left(\frac{r_1 + r_2 + r_3}{r}\right) \geq 9$$

4. We have $\left(\frac{\sin^2 A + \sin A + 1}{\sin A}\right)$

$$= \left(\sin A + \frac{1}{\sin A} + 1\right)$$

$$= \left(\sin A + \frac{1}{\sin A}\right) + 1$$

$\geq 2 + 1 = 3$

Hence, the minimum value is 3.

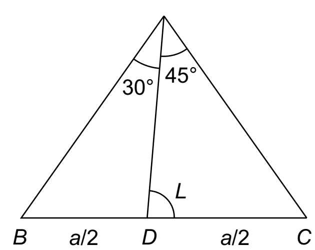
5. We have

$$\frac{1}{8R^2}\{(r^2 + r_1^2 + r_2^2 + r_3^2) + (a^2 + b^2 + c^2)\}$$

$$= \frac{1}{8R^2} \times 16R^2$$

$= 2.$

6. By *m*-*n* theorem $\cot\theta = \frac{\sqrt{3} - 1}{2}$



So, $\tan\theta = \frac{2}{(\sqrt{3} - 1)}$

$\sin\theta = \frac{2}{\sqrt{(8 - 2\sqrt{3})}}$ and $\cos\theta = \frac{(\sqrt{3} - 1)}{\sqrt{(8 - 2\sqrt{3})}}$

From ΔADC,

$$\frac{a/c}{\sin(45°)} = \frac{AD}{\sin(\pi - (\theta + 45°)} = \frac{AD}{\frac{1}{\sqrt{2}}(\sin\theta + \cos\theta)}$$

$$\frac{a/c}{\sin(45°)} = \frac{AD}{\frac{1}{\sqrt{2}}(\sin\theta + \cos\theta)}$$

$$\frac{a}{2} = \frac{AD}{(\sin\theta + \cos\theta)}$$

$$\frac{a}{2} = \frac{\sqrt{(8-2\sqrt{3})}}{(\sqrt{3}+1)} \times \frac{1}{\sqrt{11-6\sqrt{3}}}$$

$$\frac{a}{2} = \frac{\sqrt{(8-2\sqrt{3})}}{\sqrt{(4-2\sqrt{3})}} \times \frac{1}{\sqrt{11-6\sqrt{3}}}$$

$$\frac{a}{2} = \frac{\sqrt{(8-2\sqrt{3})}}{\sqrt{(8-2\sqrt{3})}} = 1$$

$a = 2$

Therefore, the length of the side BC is 2.

7. We have

$(r_1 + r_2)(r_1 + r_2)(r_1 + r_2) = 4\,R\,s^2$

Thus, $\frac{(r_1 + r_2)(r_1 + r_2)(r_1 + r_2)}{R\,s^2} = 4 \;= 4$

8. We have $\cos(A - B) = \frac{4}{5}$

$$\frac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)} = \frac{4}{5}$$

$$4\left(1+\tan^2\left(\frac{A-B}{2}\right)\right) = 5\left(1-\tan^2\left(\frac{A-B}{2}\right)\right)$$

$$9\tan^2\left(\frac{A-B}{2}\right) = 1 \;= 1$$

$$\tan^2\left(\frac{A-B}{2}\right) = \frac{1}{9}$$

$$\tan\left(\frac{A-B}{2}\right) = \frac{1}{3}$$

$$\left(\frac{a-b}{a+b}\right)\cot\left(\frac{C}{2}\right) = \frac{1}{3}$$

$$\frac{1}{3}\cot\left(\frac{C}{2}\right) = \frac{1}{3}$$

$$\cot\left(\frac{C}{2}\right) = 1$$

$C = 90°$

Hence, the area of the triangle

$= \frac{1}{2}\; a\,b\,\sin(90°)$

$= \frac{1}{2} \times 6 \times 3 \;= 9$

9. We have

$$\frac{10}{\sqrt{3}} = \frac{b}{\sin(50°)} = \frac{c}{\sin(70°)}$$

$$\frac{20}{\sqrt{3}} = \frac{b}{\sin(50°)} = \frac{c}{\sin(70°)}$$

Now, perimeter

$= 10 + b + c$

$= 10 + \frac{20}{\sqrt{3}}\sin(50°) + \frac{20}{\sqrt{3}}\sin(70°)$

$= 10 + \frac{20}{\sqrt{3}}\;[\sin(50°) + \sin(70°)]$

$= 10 + \frac{20}{\sqrt{3}}\;[\cos(40°) + \cos(20°)]$

$= 10 + 10 + \frac{20}{\sqrt{3}} \times 2\cos(30°)\cos(10°)$

$= 10 + 10 + \frac{20}{\sqrt{3}} \times 2 \times \frac{\sqrt{3}}{2} \times \cos(10°)$

$= 10 + 20\cos(10°)$

Thus, $x = 10$, $y = 20$, $z = 10$

Hence, the value of $\left(\frac{x+y+z}{y}\right)$ is 2

10. We have

$$\frac{a\cot A + b\cot B + c\cot C}{(r+R)}$$

$$= \frac{\cot A + \cot B + \cot C}{(r+R)}$$

$$= \frac{2(r+R)}{(r+R)}$$

$= 2$

HINTS & SOLUTIONS OF PAST IIT-JEEQUESTIONS

1. We have, $\Delta = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3$

$\Rightarrow \quad \frac{1}{p_1} = \frac{a}{2\Delta}, \frac{1}{p_2} = \frac{b}{2\Delta}, \frac{1}{p_3} = \frac{c}{2\Delta}$

Now, $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$

$= \frac{1}{2\Delta}(a+b-c)$

$= \frac{(a+b)^2 - c^2}{2\Delta(a+b+c)}$

$= \frac{a^2+b^2-c^2+2ab}{2\Delta(a+b+c)}$

$= \frac{2ab\cos C + 2ab}{2(a+b+c)\Delta}$

$= \frac{ab(\cos C + 1)}{(a+b+c)\Delta}$

$= \frac{2ab}{(a+b+c)\Delta} \times \cos^2\left(\frac{C}{2}\right)$

2. Let $\angle APD = \theta$, as $\angle PAO = \angle PDO = \frac{\pi}{2}$

$\angle AOD = \pi - \theta$

By the law of cosines,

$OA^2 + OD^2 - AD^2 = 2(OA)(OD)\cos(\pi - \theta)$

$\Rightarrow \quad AD^2 = 2r^2 + 2r^2\cos\theta$

$AD^2 = 2r^2(1+\cos\theta) = 4r^2\cos^2\left(\frac{\theta}{2}\right)$

Since $ABCD$ is a cyclic quadrilateral, so

$\angle QRS = \pi - \theta$

Thus, $BC^2 = 4r^2\cos^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = 4r^2\sin^2\left(\frac{\theta}{2}\right)$

Therefore,

$AD^2 + BC^2 = 4r^2\left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\right) = 4r^2$

Similarly, we can easily prove that,

$AB^2 + CD^2 = 4r^2$

Hence, $AB^2 + CD^2 = AD^2 + BC^2$

3. By the law of cosines,

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$\Rightarrow \quad \cos(60°) = \frac{a^2+b^2-c^2}{2ab}$

$\Rightarrow \quad \frac{1}{2} = \frac{a^2+b^2-c^2}{2ab}$

$\Rightarrow \quad ab = a^2+b^2-c^2$

$\Rightarrow \quad c^2 = a^2+b^2-ab$

$\Rightarrow \quad c^2 = (1+\sqrt{3})^2 + 2^2 - 2(1+\sqrt{3})$

$\Rightarrow \quad c^2 = 1+3+2\sqrt{3}+4-2-2\sqrt{3} = 6$

$\Rightarrow \quad c = \sqrt{6}$

From sine law, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$\Rightarrow \quad \sin A = \frac{a}{c}.\sin C$

$\Rightarrow \quad \sin A = \frac{(\sqrt{3}+1)}{\sqrt{6}} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin(105°)$

$\Rightarrow \quad A = 105°,\ B = 180 - (A+C) = 15°, c = \sqrt{6}$

4. We have, $r = \frac{\Delta}{s} = \frac{ac}{2s} = \frac{ac}{a+b+c}$

$= \frac{ac(a+c-b)}{(a+c)^2 - b^2}$

$= \frac{ac(a+c-b)}{a^2+c^2+2ac-b^2}$

$= \frac{1}{2}(a+c-b)\ (\because a^2+c^2 = b^2)$

$\Rightarrow \quad r = \frac{1}{2}(a+c-b)$

$\Rightarrow \quad 2r = (a+c-b) = AB + BC - AC$.

Hence, the result.

5. We have, $\cos A = \frac{b^2+c^2-a^2}{2bc}$ ....(i)

$\Rightarrow \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$ ....(ii)

From ΔADC, $\cos C = \frac{AC}{DC} = \frac{b}{a/2} = \frac{2b}{a}$ ......(iii)

From (ii) and (iii), we get,

$$\frac{a^2+b^2-c^2}{2ab}=\frac{2b}{a}$$

$\Rightarrow \quad a^2-c^2=3b^2$

From (i), we get,

$$\cos A=\frac{b^2-3b^2}{2bc}=-\frac{2b^2}{2bc}=-\frac{b}{c}$$

Thus, $\cos A.\cos C=\left(\frac{2b}{a}\right)\left(-\frac{b}{c}\right)=\frac{2\left(c^2-a^2\right)}{3ac}$

6. Let $\angle A=\angle B-\alpha$ and $\angle C=\angle B+\alpha$

We have $\angle A+\angle B+\angle C=180°$

$\Rightarrow \quad \angle B=60°$

From sine laws, $\frac{b}{\sin B}=\frac{c}{\sin C}$

$\Rightarrow \quad \sin C=\frac{c}{b}.\sin B=\left(\sqrt{\frac{2}{3}}.\right)\left(\frac{\sqrt{3}}{2}\right)=\frac{1}{\sqrt{2}}$

$\Rightarrow \quad C=45°$

Thus, $\angle A=180°-\left(B+C\right)$

$=180°-\left(60°+45°\right)=75°$.

7. No questions asked in 1982.

8. We have, $r_1=\frac{\Delta}{s-a}, r_2=\frac{\Delta}{s-b}, r_3=\frac{\Delta}{s-c}$

Given $r_1, r_2, r_3 \in H.P$

$\Rightarrow \quad \frac{\Delta}{s-a},\frac{\Delta}{s-b},\frac{\Delta}{s-c}\in H.P$

$\Rightarrow \quad \frac{1}{s-a},\frac{1}{s-b},\frac{1}{s-c}\in H.P$

$\Rightarrow \quad s-a, s-b, s-c\in A.P$

$\Rightarrow \quad -a,-b,-c\in A.P$

$\Rightarrow \quad a,b,c\in A.P$

Hence, the result.

9. We have, $\cos A+\cos B+\cos C=\frac{3}{2}$

$\Rightarrow 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C=\frac{3}{2}$

$\Rightarrow 2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\left(\frac{C}{2}\right)=\frac{3}{2}$

$\Rightarrow 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)-2\sin^2\left(\frac{C}{2}\right)=\frac{3}{2}-1=\frac{1}{2}$

$\Rightarrow 2\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right)-\sin\left(\frac{C}{2}\right)\right)=\frac{1}{2}$

$\Rightarrow 2\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right)=\frac{1}{2}$

$\Rightarrow 2\sin\left(\frac{C}{2}\right)\times 2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)=\frac{1}{2}$

$\Rightarrow \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)=\frac{1}{8}$

It is possible only when,

$\sin\left(\frac{A}{2}\right)=\frac{1}{2},\sin\left(\frac{B}{2}\right)=\frac{1}{2},\sin\left(\frac{C}{2}\right)=\frac{1}{2}$

$\Rightarrow \left(\frac{A}{2}\right)=\frac{\pi}{6},\left(\frac{B}{2}\right)=\frac{\pi}{6},\left(\frac{C}{2}\right)=\frac{\pi}{6}$

$\Rightarrow A=\frac{\pi}{3}, B=\frac{\pi}{3}, C=\frac{\pi}{3}$

Thus, Δ is equilateral.

10. We have, $\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}=\frac{2\left(a+b+c\right)}{36}$

$=\frac{\left(a+b+c\right)}{18}=\lambda$

Thus, $a=7\lambda$, $b=6\lambda$ & $c=5\lambda$

$\Rightarrow \quad \cos A=\frac{b^2+c^2-a^2}{2bc}$

$=\frac{36\lambda^2+25\lambda^2-49\lambda^2}{2(6\lambda)(5\lambda)}=\frac{1}{5}$

Similarly, $\cos B=\frac{19}{35}, \cos C=\frac{5}{7}$

$\Rightarrow \quad \cos A:\cos B:\cos C=\frac{1}{5}:\frac{19}{35}:\frac{5}{7}$

$\Rightarrow \quad \cos A:\cos B:\cos C=7:19:25$

Thus, $\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$.

11.

12. Given $\cot A, \cot B, \cot C\in A.P$

$\Rightarrow \quad \frac{\cos A}{\sin A},\frac{\cos B}{\sin B},\frac{\cos C}{\sin C}\in A.P$

$\Rightarrow \frac{\left(b^2+c^2-a^2\right)}{2abc},\frac{\left(c^2+a^2-b^2\right)}{2abc},\frac{\left(a^2+b^2-c^2\right)}{2abc}\in A.P$

$\Rightarrow \quad \frac{\left(b^2+c^2+a^2-2a^2\right)}{2abc},\frac{\left(c^2+a^2+b^2-2b^2\right)}{2abc},$

$$\frac{\left(a^2+b^2+c^2-2c^2\right)}{2abc} \in A.P$$

$\Rightarrow \quad \left(b^2+c^2+a^2-2a^2\right),\left(c^2+a^2+b^2-2b^2\right),$

$\left(a^2+b^2+c^2-2c^2\right) \in A.P$

$\Rightarrow \quad \left(-2a^2\right),\left(-2b^2\right),\left(-2c^2\right) \in A.P$

$\Rightarrow \quad a^2, b^2, c^2 \in A.P$

Hence, the result.

13. Let $x = a^2 + 2a, y = 2a + 3, z = a^2 + 3a + 8$

Here, $x > 0$, $y > 0$ and $z > 0$

since $z = a^2 + 3a + 8 > 0$ for every a in R

$\Rightarrow \quad a > 0 \quad \left(\because a < -2, a > 0 \text{ and } a > -\frac{3}{2}\right)$

Also, $\quad z - x = a + 8 > 0, z - y = a^2 + a + 5 > 0$

Thus, $\quad x + y > z$

$\Rightarrow \quad a^2 + 3a + 8 < \left(a^2 + 2a\right) + (2a + 3)$

$\Rightarrow \quad a > 5$

$\Rightarrow \quad a \in (5, \infty)$

14. Let $M = \frac{1}{3}\left(\cos\left(\alpha + \frac{\pi}{2}\right) + \cos\left(\beta + \frac{\pi}{2}\right) + \cos\left(\gamma + \frac{\pi}{2}\right)\right)$

$= -\frac{1}{3}(\sin\alpha + \sin\beta + \sin\gamma)$

M will be least, when $(\sin\alpha + \sin\beta + \sin\gamma)$ is providing us the greatest value.

Let $z = \sin\alpha + \sin\beta + \sin\gamma$

$= \sin\alpha + \sin\beta + \sin(2\pi - (\alpha + \beta))$

$= \sin\alpha + \sin\beta - \sin(\alpha + \beta)$

$= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) - 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$

$= 2\sin\left(\frac{\alpha+\beta}{2}\right)\left(\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\left(\frac{\alpha+\beta}{2}\right)\right)$

$= 2\sin\left(\frac{\alpha+\beta}{2}\right) \times 2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)$

$= 4\sin\left(\pi - \frac{\gamma}{2}\right)\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)$

$= 4\sin\left(\frac{\gamma}{2}\right)\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)$

$= 4\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$

It will be greatest, when

$\frac{\alpha}{2} = \frac{\beta}{2} = \frac{\gamma}{2} \Rightarrow \alpha = \beta = \gamma$

Thus, $\alpha = \frac{2\pi}{3} = \beta = \gamma$

Therefore, the least value of M is

$= -\frac{1}{3}\left(3\sin\left(\frac{2\pi}{3}\right)\right) = -\frac{\sqrt{3}}{2}$

15. We have $\cos A \cos B + \sin A \sin B \sin C = 1$

$\Rightarrow \quad \frac{1 - \cos A\cos B}{\sin A\sin B} = \sin C$

$\Rightarrow \quad \frac{1 - \cos A\cos B}{\sin A\sin B} = \sin C \le 1$

$\Rightarrow \quad 1 - \cos A\cos B \le \sin A\sin B$

$\Rightarrow \quad 1 \le \sin A\sin B + \cos A\cos B$

$\Rightarrow \quad \cos(A - B) \ge 1$

$\Rightarrow \quad \cos(A - B) = 1$

$\Rightarrow \quad \cos(A - B) = \cos(0)$

$\Rightarrow \quad A - B = 0$

$\Rightarrow \quad A = B$

Therefore, $\sin C = \frac{1 - \cos A\cos B}{\sin A\sin B}$

$= \frac{1 - \cos A\cos A}{\sin A\sin A} = \frac{1 - \cos^2 A}{\sin^2 A} = \frac{\sin^2 A}{\sin^2 A} = 1$

$\Rightarrow \quad C = 90°$

Hence, $A = 45° = B, C = 90°$

Now, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow \quad \frac{a}{\sin 45°} = \frac{b}{\sin 45°} = \frac{c}{\sin 90°}$

$\Rightarrow \quad \frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{1}{\sqrt{2}}} = \frac{c}{1}$

$\Rightarrow \quad \frac{a}{1} = \frac{b}{1} = \frac{c}{\sqrt{2}}$

$\Rightarrow \quad a : b : c = 1 : 1 : \sqrt{2}$

Hence, the result.

16. Ans. (a, d)

From sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B}$

$b \sin A = a \sin B$

$b \sin A = a \sin B \le a$, since $\sin B \le 1$

In this case, $B = \frac{\pi}{2}, A < \frac{\pi}{2}$

Also $b \sin A < a, \text{if } 0 < B < \pi, B \ne \frac{\pi}{2}$

If $A < \frac{\pi}{2}, B > A, B \ne \frac{\pi}{2}$,

then $b \sin A < a$, $A < \frac{\pi}{2}, b > a$

17. Ans. (a, d)

Let $\angle A = \alpha - \theta, \angle B = \theta, \angle C = \pi + \theta$

since $\angle A + \angle B + \angle C = \pi$, $\theta = 60°$

Thus, the largest angle of ΔABC is a and the smallest angle is C

Let x be the smallest side of the triangle .

Therefore, $\cos(60°) = \frac{x^2 + 10^2 - 9^2}{2.x.10}$

$\Rightarrow \quad x^2 + 19 = 10x$

$\Rightarrow \quad x^2 - 10x + 19 = 0$

$\Rightarrow \quad (x-5)^2 = 6$

$\Rightarrow \quad (x-5) = \pm\sqrt{6}$

$\Rightarrow \quad x = 5 \pm \sqrt{6}$

18. Let $\angle B = 30°, \angle C = 45°$ and $a = \left(\sqrt{3}+1\right)$

Then $\angle A = \left(\pi - (\angle B + \angle C)\right)$

$= \left(\pi - (30° + 45°)\right) = 105°$

From sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{a}{\sin(105°)} = \frac{b}{\sin(30°)} = \frac{c}{\sin(45°)}$$

$$\Rightarrow \quad \frac{\left(\sqrt{3}+1\right)}{\sin(105°)} = \frac{b}{\sin(30°)} = \frac{c}{\sin(45°)}$$

$$\Rightarrow \quad \frac{\left(\sqrt{3}+1\right)}{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)} = \frac{b}{\frac{1}{2}} = \frac{c}{\frac{1}{\sqrt{2}}}$$

$\Rightarrow \quad 2\sqrt{2} = 2b = \sqrt{2}c$ \

$\Rightarrow \quad b = \sqrt{2}, c = 2$

Thus, the area of the given triangle

$= \frac{1}{2} \times bc \times \sin A$

$= \frac{1}{2} \times \sqrt{2} \times 2 \times \frac{\left(\sqrt{3}+1\right)}{2\sqrt{2}}$

$= \frac{\left(\sqrt{3}+1\right)}{2}$ sq cm.

19. We have, $BC = 2BD$, $AD = h$, $OD = h - r$, so that $BC = 2\sqrt{r^2 - (h-r)^2} = 2\sqrt{2rh - h^2}$

Thus, $P = 2AB + BC$

$\Rightarrow \quad 2AB = P - BC$

$\Rightarrow \quad 2AB = 2\left(\sqrt{2hr - h^2} + \sqrt{2hr}\right) - 2\sqrt{2hr - h^2}$

$\Rightarrow \quad AB = \sqrt{2hr}$

The area of the $\Delta ABC = A$

$= BD \times AD$

$= h\sqrt{2hr - h^2}$

Now, $\frac{A}{P^3} = \frac{h\sqrt{2hr - h^2}}{8\left(\sqrt{2hr - h^2} + \sqrt{2hr}\right)^3}$

$= \frac{\sqrt{2r - h}}{8\left(\sqrt{2r - h} + \sqrt{2r}\right)^3}$

Thus, $\lim_{h \to 0}\left(\frac{A}{P^3}\right) = \frac{\sqrt{2r}}{8 \times \left(2\sqrt{2r}\right)^3} = \frac{1}{128r}$

20. Ans. (c)

The given equation is $k = 3\sin x - 4\sin^3 x = \sin 3x$

Thus, $k = \sin 3A, k = \sin 3B$

$\Rightarrow \quad \sin 3A = \sin 3B = \sin(\pi - 3B)$

$\Rightarrow \quad 3A = (\pi - 3B)$

$\Rightarrow \quad 3(A+B)=\pi$

$\Rightarrow \quad (A+B)=\dfrac{\pi}{3}$

Therefore, $\angle C=\pi-(A+B)=\pi-\dfrac{\pi}{3}=\dfrac{2\pi}{3}$

21. Given A, B and C are in A.P.

$\Rightarrow \quad 2B = A + C$

$\Rightarrow \quad 3B = A + B + C = \pi$

$\Rightarrow \quad B=\dfrac{\pi}{3}$

Now, $\sin(2A+B)=\dfrac{1}{2}=\sin\left(\dfrac{5\pi}{6}\right)$

$\Rightarrow \quad (2A+B)=\left(\dfrac{5\pi}{6}\right)$

$\Rightarrow \quad 2A=\left(\dfrac{5\pi}{6}-B\right)=\dfrac{5\pi}{6}-\dfrac{\pi}{3}=\dfrac{3\pi}{6}=\dfrac{\pi}{2}$

$\Rightarrow \quad A=\dfrac{\pi}{4}$

Also, $\sin(C-A)=\dfrac{1}{2}=\sin\left(\dfrac{\pi}{6}\right)$

$\Rightarrow \quad (C-A)=\left(\dfrac{\pi}{6}\right)$

$\Rightarrow \quad C=\left(\dfrac{\pi}{6}+A\right)=\left(\dfrac{\pi}{6}+\dfrac{\pi}{4}\right)=\dfrac{5\pi}{12}=75°$

Thus, $A=45°, B=60°, C=75°$.

22. Let the sides of a triangle are a – 1, a, a + 1, where $a\in I^{+}-\{1\}$

Let θ is the smallest angle and 2θ is the greatest angle of the triangle.
By the sine rule,

$$\frac{\sin\theta}{a-1}=\frac{\sin(2\theta)}{a+1}$$

$\Rightarrow \quad \dfrac{a+1}{a-1}=\dfrac{\sin(2\theta)}{\sin\theta}=2\cos\theta$

$\Rightarrow \quad \cos\theta=\dfrac{a+1}{2(a-1)}$

Again by the cosine rule,

$$\cos\theta=\frac{(a+1)^2+a^2-(a-1)^2}{2.a.(a+1)}$$

$\Rightarrow \quad \cos\theta=\dfrac{a^2+4a}{2a(1+a)}=\dfrac{a+4}{2(a+1)}$

Therefore, $\dfrac{a+1}{2(a-1)}=\dfrac{a+4}{2(a+1)}$

$\Rightarrow \quad (a+1)^2=(a+4)(a-1)$

$\Rightarrow \quad a^2+2a+1=a^2+3a-4$

$\Rightarrow \quad A = 5$

Hence, the sides of the triangle are 4, 5, 6.

23. Let $a = BC$, $b = CA$, $c = AB$ and $p = AD$.

$$\Delta ABC=\frac{1}{2}ap=\frac{1}{2}bc\sin A$$

$\Rightarrow \quad p=\dfrac{bc}{a}\sin A$

$\Rightarrow \quad p=\dfrac{abc}{a^2}\sin A$

$\Rightarrow \quad p=\dfrac{abc(\sin^2 B-\sin^2 C)}{a^2(\sin^2 B-\sin^2 C)}\sin A$

$\Rightarrow \quad p=\dfrac{abc\sin(B+C)\sin(B-C)}{(b^2-c^2)\sin^2 A}\sin A$

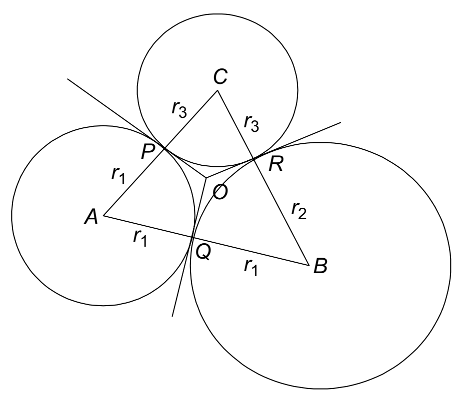
$\Rightarrow \quad p=\dfrac{abc\ \sin(B-C)}{(b^2-c^2)}$

$\Rightarrow \quad p=\dfrac{ab^2r\ \sin(B-C)}{(b^2-b^2r^2)}$, where $c=br$

$\Rightarrow \quad p=\left(\dfrac{a\,r}{1-r^2}\right)\sin(B-C)$

$\Rightarrow \quad p\le\left(\dfrac{a\,r}{1-r^2}\right)$, since $\sin(B-C)\le 1$

24.



Consider three circles with centres at A, B and C with radii r_1, r_2, r_3, respectively, which touch each other externally at P, Q, R.

Let the common tangents at P, Q, R meet each other at O.

Then $OP = OQ = QR = 4$

Also, $OP \perp AB, OQ \perp AC, OR \perp BC$

Here, O is the in-centre of the triangle ABC

For ΔABC,

$$s = \frac{(r_1 + r_2) + (r_3 + r_2) + (r_1 + r_3)}{2} = r_1 + r_2 + r_3$$

and $\Delta = \sqrt{(r_1 + r_2 + r_3) r_1 r_2 r_3}$

Now, from the relation $r = \frac{\Delta}{s}$, we get,

$$\frac{\sqrt{(r_1 + r_2 + r_3) r_1 r_2 r_3}}{r_1 + r_2 + r_3} = 4$$

$$\Rightarrow \quad \sqrt{\frac{r_1 r_2 r_3}{r_1 + r_2 + r_3}} = 4$$

$$\Rightarrow \quad \frac{r_1 r_2 r_3}{r_1 + r_2 + r_3} = 16 = \frac{16}{1}$$

$$\Rightarrow \quad (r_1 r_2 r_3) : (r_1 + r_2 + r_3) = 16 : 1$$

25. We have

$$\frac{2\cos A}{a} + \frac{2\cos B}{b} + \frac{2\cos C}{c} = \frac{1}{bc} + \frac{b}{ca}$$

$$\Rightarrow \quad \frac{2bc\cos A}{abc} + \frac{ac\cos B}{abc} + \frac{2bc\cos C}{abc} = \frac{a^2}{abc} + \frac{b^2}{abc}$$

$$\Rightarrow \quad 2bc\cos A + ac\cos B + 2bc\cos C = a^2 + b^2$$

$$\Rightarrow \quad (b^2 + c^2 - a^2) + \frac{1}{2}(a^2 + c^2 - b^2) + (a^2 + b^2 - c^2) = a^2 + b^2$$

$$\Rightarrow \quad 2b^2 - 2a^2 + c^2 + a^2 - b^2 = 0$$

$$\Rightarrow \quad a^2 = b^2 + c^2$$

ΔABC is a right angled triangle at A

Thus, $\angle A = 90°$.

26. Let O be the centre r be the radius of the circle passing through A_i, where $i = 1, 2, 3, \ldots, n$

Here, $\angle A_i O A_{i+1} = \frac{2\pi}{n}$, where $i = 1, 2, 3, \ldots, n$

Now, $\frac{1}{A_1A_2} + \frac{1}{A_1A_3} = \frac{1}{A_1A_2}$

$$\Rightarrow \quad \frac{1}{2r\sin\left(\frac{\pi}{n}\right)} = \frac{1}{2r\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{2r\sin\left(\frac{3\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$

$$\Rightarrow \quad \frac{1}{\sin\theta} = \frac{1}{\sin 2\theta} + \frac{1}{\sin 3\theta}, \left(\frac{\pi}{n}\right) = \theta$$

$$\Rightarrow \quad \frac{1}{\sin\theta} - \frac{1}{\sin 3\theta} = \frac{1}{\sin 2\theta}$$

$$\Rightarrow \quad \frac{\sin 3\theta - \sin\theta}{\sin\theta \sin 3\theta} = \frac{1}{\sin 2\theta}$$

$$\Rightarrow \quad \frac{2\cos 2\theta \sin\theta}{\sin\theta \sin 3\theta} = \frac{1}{\sin 2\theta}$$

$$\Rightarrow \quad 2\sin 2\theta \cos 2\theta = \sin 3\theta$$

$$\Rightarrow \quad \sin(4\theta) = \sin(3\theta)$$

$$\Rightarrow \quad \sin(4\theta) = \sin(\pi - 3\theta)$$

$$\Rightarrow \quad (4\theta) = (\pi - 3\theta)$$

$$\Rightarrow \quad \theta = \frac{\pi}{7}$$

$$\Rightarrow \quad \frac{\pi}{n} = \frac{\pi}{7}$$

$$\Rightarrow \quad n = 7$$

27. $(\text{i}) \Rightarrow (\text{ii})$

Suppose a, b, c and area Δ are rational.

Thus, $s = \frac{a+b+c}{2}$ = rational

Since $\tan\left(\frac{B}{2}\right) = \frac{\Delta}{s(s-b)}$

and $\tan\left(\frac{C}{2}\right) = \frac{\Delta}{s(s-c)}$

and $\Delta, a, s, s-a, s-b$ all are rational.

Therefore $a, \tan\left(\frac{B}{2}\right), \tan\left(\frac{C}{2}\right)$ are rational.

$(\text{ii}) \Rightarrow (\text{iii})$

Consider $a, \tan\left(\frac{B}{2}\right), \tan\left(\frac{C}{2}\right)$ are rational.

$$\sin B = \frac{2\tan(B/2)}{1 + \tan^2(B/2)} = \text{rational}$$

and $\sin C = \dfrac{2\tan(C/2)}{1+\tan^2(C/2)}$ = rational

Also, $\tan\left(\dfrac{A}{2}\right) = \tan\left(\dfrac{\pi}{2} - \left(\dfrac{B+C}{2}\right)\right)$

$= \cot\left(\dfrac{B+C}{2}\right)$

$= \dfrac{1-\tan\left(\dfrac{B}{2}\right)\tan\left(\dfrac{C}{2}\right)}{\tan\left(\dfrac{B}{2}\right)+\tan\left(\dfrac{C}{2}\right)}$ = rational

Thus, sin A = rational.
Hence, a, sin A, sin B, sin C are rational.

(iii) ⇒ (i)

Suppose a, sin A, sin B, sin C are rational
By the sine rules,

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$\Rightarrow b = a.\dfrac{\sin B}{\sin A},\ c = a.\dfrac{\sin C}{\sin A}$

b, c = rational

Also, $\Delta = \dfrac{1}{2}bc\sin A$ = rational

This completes the proof.

28. Figure

From the figure, $AD = b\sin(23°)$

$\Rightarrow \dfrac{abc}{b^2-c^2} = b\sin(23°)$

$\Rightarrow \dfrac{a}{b^2-c^2} = \dfrac{\sin(23°)}{c}$

$\Rightarrow \dfrac{a}{b^2-c^2} = \dfrac{\sin A}{a}$

$\Rightarrow \sin A = \dfrac{a^2}{b^2-c^2}$

$\Rightarrow \sin A = \dfrac{\sin^2 A}{\sin^2 B - \sin^2 C}$

$\Rightarrow \sin A = \sin^2 B - \sin^2 C$

$\Rightarrow \sin A = \sin(B+C)\sin(B-C)$

$\Rightarrow \sin A = \sin(\pi - A)\sin(B-C)$

$\Rightarrow \sin A = \sin(A)\sin(B-C)$

$\Rightarrow \sin(B-C) = 1 = \sin(90°)$

$\Rightarrow (B-C) = (90°)$

$\Rightarrow (B-23°) = (90°)$

$\Rightarrow B = 113°$

29. As we know that, the largest angle is the opposite to the largest side.
Let $a = BC = 3$, $b = CA = 5$, $c = AB = 7$

Thus, $\cos C = \dfrac{3^2+5^2-7^2}{2.3.5} = -\dfrac{15}{30} = -\dfrac{1}{2}$

$\Rightarrow \cos C = -\dfrac{1}{2} = \cos\left(\dfrac{2\pi}{3}\right)$

$\Rightarrow C = \dfrac{2\pi}{3}$

30. Let $\angle B = \dfrac{\pi}{3}, \angle C = \dfrac{\pi}{4}, \angle BAD = \theta, \angle CAD = \varphi$

By the sine rule,

$\dfrac{\sin\theta}{BD} = \dfrac{\sin(\pi/3)}{AD}$

$\Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2}.\dfrac{BD}{AD}$

and $\dfrac{\sin\varphi}{DC} = \dfrac{\sin(\pi/4)}{AD}$

$\Rightarrow \sin\varphi = \dfrac{1}{\sqrt{2}}.\dfrac{DC}{AD}$

Now, $\dfrac{\sin\angle BAD}{\sin\angle CAD} = \dfrac{\sin\theta}{\sin\varphi}$

$= \dfrac{\sqrt{3}}{2}.\dfrac{BD}{AD}.\sqrt{2}.\dfrac{AD}{DC}$

$= \dfrac{BD}{DC}.\sqrt{\dfrac{3}{2}}$

$= \dfrac{1}{3}\times\sqrt{\dfrac{3}{2}} = \dfrac{1}{\sqrt{6}}$

31 Let $a = 4k, b = 5k, c = 6k$

$s = \dfrac{1}{2}(a+b+c) = \dfrac{15}{2}k$

$s - a = \dfrac{15}{2}k - 4k = \dfrac{7}{2}k$

$s - b = \dfrac{15}{2}k - 5k = \dfrac{5}{2}k$

$s - c = \dfrac{15}{2}k - 6k = \dfrac{3}{2}k$

Now, $\frac{R}{r} = \frac{abc}{4\Delta} \times \frac{s}{\Delta}$

$= \frac{abcs}{4\Delta^2}$

$= \frac{abcs}{4s(s-a)(s-b)(s-c)}$

$= \frac{abc}{4(s-a)(s-b)(s-c)}$

$= \frac{(4k)(5k)(6k)}{4\left(\frac{7}{2}k\right)\left(\frac{5}{2}k\right)\left(\frac{3}{2}k\right)}$

$= \frac{16}{7}$

32. Here, $B + C = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\Rightarrow \quad \tan(B+C) = \tan\left(\frac{3\pi}{4}\right) = -1$

$\Rightarrow \quad \frac{\tan B + \tan C}{1 - \tan B \tan C} = -1$

$\Rightarrow \quad \tan B + \tan C = -1 + \tan B \tan C = -1 + p$

Let tan B and tan C are the roots, then

$x^2 - (\tan B + \tan C)x + \tan B.\tan C = 0$

$\Rightarrow \quad x^2 - (p-1)x + p = 0$

It has real roots.

So, $D \geq 0$

$\Rightarrow \quad (p-1)^2 - 4p \geq 0$

$\Rightarrow \quad p^2 - 2p + 1 - 4p \geq 0$

$\Rightarrow \quad p^2 - 6p + 1 \geq 0$

$\Rightarrow \quad (p-3)^2 - 8 \geq 0$

$\Rightarrow \quad (p-3)^2 - (2\sqrt{2})^2 \geq 0$

$\Rightarrow \quad (p - 3 + 2\sqrt{2})(p - 3 - 2\sqrt{2}) \geq 0$

$\Rightarrow \quad p \leq (3 - 2\sqrt{2}),\ p \geq (3 + 2\sqrt{2})$

$\Rightarrow \quad p \in (-\infty, (3 - 2\sqrt{2})) \cup ((3 + 2\sqrt{2}), \infty)$

33. Let ABC be an equilateral triangle

Then $\angle A = \frac{\pi}{3} = \angle B = \angle C$

Therefore, tan A + tan B + tan C

$= \tan\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)$

$= 3\tan\left(\frac{\pi}{3}\right)$

$= 3\sqrt{3}$

Conversely, let $\tan A + \tan B + \tan C = 3\sqrt{3}$

Here, $A + B + C = \pi$

$\Rightarrow \quad A + B = \pi - C$

$\Rightarrow \quad \tan(A+B) = \tan(\pi - C) = -\tan C$

$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A.\tan B} = -\tan C$

$\Rightarrow \quad \tan A + \tan B = -\tan C + \tan A.\tan B.\tan C$

$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A.\tan B.\tan C$

Thus,

$\Rightarrow \quad \tan A + \tan B + \tan C = \tan A.\tan B.\tan C = 3\sqrt{3}$

It is possible only when, $A = B = C = \frac{\pi}{3}$.

Thus, the triangle is equilateral.

34. Ans. (b)

Here, $\Delta = \frac{1}{2} \times p \times p_1 = \frac{1}{2} \times q \times p_2 = \frac{1}{2} \times r \times p_3$

$\Rightarrow \quad p = \frac{2\Delta}{p_1}, q = \frac{2\Delta}{p_2}, r = \frac{2\Delta}{p_3}$

From sine rule of a triangle,

$\frac{\sin P}{p} = \frac{\sin Q}{q} = \frac{\sin R}{r}$

Given $\sin P, \sin Q, \sin R$ are in A.P.

$\Rightarrow \quad p, q, r \in A.P$

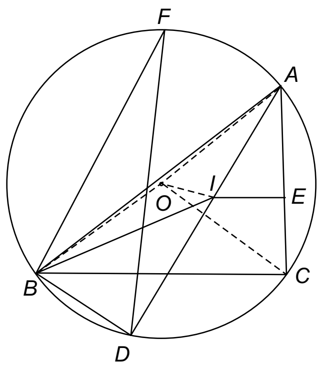
$\Rightarrow \quad \frac{2\Delta}{p_1}, \frac{2\Delta}{p_2}, \frac{2\Delta}{p_3} \in A.P$

$\Rightarrow \quad \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3} \in A.P$

$\Rightarrow \quad p_1, p_2, p_3 \in H.P$

Thus, the altitudes are in H.P.

35.



Let O be the circumcentre and OF be perpendicular to AB.

Let I be the in-centre and IE perpendicular to AC.

Then $\angle OAF = 90° - C$

$\angle OAI = \angle IAF - \angle OAF$

$= \frac{A}{2} - (90° - C)$

$= \frac{A}{2} + C - \frac{A+B+C}{2}$

$= \frac{C-B}{2}$

Also, $AI = \frac{IE}{\sin\left(\frac{A}{2}\right)} = \frac{r}{\sin\left(\frac{A}{2}\right)}$

$= 4R\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

$= 1 - 8\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\left(\cos\left(\frac{B}{2}+\frac{C}{2}\right)\right)$

$= 1 - 8\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)$

$\Rightarrow \quad OI = R\sqrt{1 - 8\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)}$

Also, OI^2

$= R^2 - 2R \times 4R\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

$= R^2 - 2Rr$

$\Rightarrow \quad OI = \sqrt{R^2 - 2Rr}$

Hence, the result.

36. We have, $A + B + C = \pi$

$\Rightarrow \quad \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \pi$

$\Rightarrow \quad \frac{A}{2} + \frac{B}{2} = \pi - \frac{C}{2}$

$\Rightarrow \quad \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(\pi - \frac{C}{2}\right)$

$\Rightarrow \quad \frac{\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right)}{1 - \tan\left(\frac{A}{2}\right).\tan\left(\frac{B}{2}\right)} = \cot\left(\frac{C}{2}\right)$

$\Rightarrow \quad \frac{\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right)}{1 - \tan\left(\frac{A}{2}\right).\tan\left(\frac{B}{2}\right)} = \frac{1}{\tan\left(\frac{C}{2}\right)}$

$\Rightarrow \quad \tan\left(\frac{A}{2}\right)\tan\left(\frac{C}{2}\right) + \tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)$

$= 1 - \tan\left(\frac{A}{2}\right).\tan\left(\frac{B}{2}\right)$

$\Rightarrow \quad \tan\left(\frac{A}{2}\right)\tan\left(\frac{C}{2}\right) + \tan\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right)$

$+ \tan\left(\frac{A}{2}\right).\tan\left(\frac{B}{2}\right) = 1$

Dividing both sides by $\tan\left(\frac{A}{2}\right).\tan\left(\frac{B}{2}\right).\tan\left(\frac{C}{2}\right)$,

we get,

$\Rightarrow \quad \frac{1}{\tan\left(\frac{C}{2}\right)} + \frac{1}{\tan\left(\frac{C}{2}\right)} + \frac{1}{\tan\left(\frac{C}{2}\right)}$

$= \frac{1}{\tan\left(\frac{C}{2}\right)}.\frac{1}{\tan\left(\frac{C}{2}\right)}.\frac{1}{\tan\left(\frac{C}{2}\right)}$

$\Rightarrow \quad \cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)$

$= \cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$

37. Figure

Here, $HE = JK = r_1$

But $IE = r$

So, $IH = r - r_1$

In a right triangle IHJ, $\angle JIH = \left(\frac{\pi}{2} - \frac{A}{2}\right)$

$\Rightarrow \quad \tan\left(\frac{\pi}{2} - \frac{A}{2}\right) = \frac{r_1}{r - r_1}$

$\Rightarrow \quad \cot\left(\frac{A}{2}\right) = \frac{r_1}{r - r_1}$

Similarly, $\cot\left(\frac{B}{2}\right) = \frac{r_2}{r - r_2}$, $\cot\left(\frac{C}{2}\right) = \frac{r_3}{r - r_3}$.

In a triangle *ABC*, we have

$$\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right) = \cot\left(\frac{A}{2}\right)\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$$

$\Rightarrow \quad \frac{r_1}{r - r_1} + \frac{r_2}{r - r_2} + \frac{r_3}{r - r_3} = \frac{r_1}{r - r_1} \cdot \frac{r_2}{r - r_2} \cdot \frac{r_3}{r - r_3}$

$\Rightarrow \quad \frac{r_1}{r - r_1} + \frac{r_2}{r - r_2} + \frac{r_3}{r - r_3} = \frac{r_1 r_2 r_3}{(r - r_1)(r - r_2)(r - r_3)}$

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38. We have, $2ac\sin\left(\frac{A - B + C}{2}\right)$

$= 2ac\sin\left(\frac{A + C}{2} - \frac{B}{2}\right)$

$= 2ac\sin\left(\frac{\pi}{2} - \frac{B}{2} - \frac{B}{2}\right)$

$= 2ac\sin\left(\frac{\pi}{2} - B\right)$

$= 2ac\cos B$

$= 2ac\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$

$= (a^2 + c^2 - b^2)$

39. Ans. (a)
As *ABC* be a right angled triangle, circum radius of this triangle is half of its hypotemuse.

Thus, $R = \frac{1}{2}\sqrt{a^2 + b^2}$

Also, $r = (s - c)\tan\left(\frac{C}{2}\right) = (s - c)\tan\left(\frac{\pi}{4}\right) = (s - c)$

Now, $2(R + r)$

$= \sqrt{a^2 + b^2} + 2s - 2c$

$= c + 2s - 2c$

$= 2s - c$

$= a + b + c - c$

$= a + b$

40.

41. By the sine rule,

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin(\pi - (A + B))}$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin(A + B)}$

Here, we can easily find out *b*. *c* and *C*,
if we know *a*, sin*A*, sin *B*.
Also, we can find the values of *A*, *B* and *C* by using the half angle formulae, if we know the values of *a*, *b* and *c*.

By using, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

We can find, $\angle B, \angle C$ and the sides *a*, *b* and *c*,
if we know *a*, sin*B* and *R*
We cannot find $\angle B, \angle C$ and the sides *b* and *c*,
if we just know $a, \sin A, R$, since

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ gives

$\frac{b}{\sin B}$ & $\frac{c}{\sin C}$ from which we cannot obtain

b, c, $\angle B$ & $\angle C$.

42. Let the angles of the triangle *ABC* are 4θ, θ *and* θ

Also $\quad 4\theta + \theta + \theta = 180°$

$\Rightarrow \quad 6\theta = 180°$

$\Rightarrow \quad \theta = \frac{180°}{6} = 30°$

By sine rules,

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow \quad \frac{a}{\sin(120°)} = \frac{b}{\sin(30°)} = \frac{c}{\sin(30°)}$

$\Rightarrow \quad \frac{a}{\sin(60°)} = \frac{b}{\sin(30°)} = \frac{c}{\sin(30°)}$

$\Rightarrow \quad \frac{a}{\frac{\sqrt{3}}{2}} = \frac{b}{\frac{1}{2}} = \frac{c}{\frac{1}{2}}$

$\Rightarrow \quad \frac{a}{\sqrt{3}} = \frac{b}{1} = \frac{c}{1}$

Hence, the required ratio is

$= \frac{a}{a+b+c} = \frac{\sqrt{3}}{2+\sqrt{3}}$

43. From figure (i),

$I_n = n.\frac{1}{2}.(OA_1).(OA_1)\sin\left(\frac{2\pi}{n}\right)$

$= \frac{\pi}{2}\sin\left(\frac{2\pi}{n}\right)$

From figure (ii)

$B_1B_2 = 2(B_1L) = 2(OL)\tan\left(\frac{\pi}{n}\right)$

$= 2.1.\tan\left(\frac{\pi}{n}\right)$

$= 2\tan\left(\frac{\pi}{n}\right)$

Thus, $O_n = n\left(\frac{1}{2}(B_1B_2)(OL)\right) = n\tan\left(\frac{\pi}{n}\right)$

Now, $\frac{I_n}{O_n}$

$= \frac{(n/2)\sin(2\theta)}{n\tan\theta}$, where $\theta = \frac{\pi}{n}$

$= \frac{2\tan\theta}{(1+\tan^2\theta)}.\frac{1}{2\tan\theta}$

$= \cos^2\theta$

$= \frac{1}{2}(2\cos^2\theta)$

$= \frac{1}{2}(1+\cos 2\theta)$

$= \frac{1}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right)$

$I_n = \frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right)$

44. Ans. (d)

Let the sides of the triangle be $\lambda, \sqrt{3}\lambda, 2\lambda$

By the cosine rule,

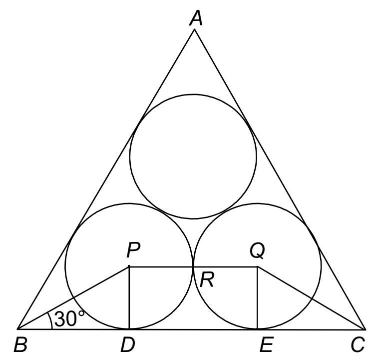
$\cos A = \frac{(\sqrt{3}\lambda)^2 + (2\lambda)^2 - (\lambda)^2}{2.(\sqrt{3}\lambda).(2\lambda)} = \frac{6\lambda^2}{4\sqrt{3}\lambda^2} = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad A = \frac{\pi}{6}$

Similarly, $B = \frac{\pi}{3}, C = \frac{\pi}{2}$

$A : B : C = 30° : 60° : 90° = 1 : 2 : 3$

45.



We have, $\frac{BD}{PD} = \cot(30°) = \sqrt{3}$

$BD = \sqrt{3}\,PD$

$DE = PQ = PR + RQ = 2$

$BC = BD + DE + EC = \sqrt{3} + 2 + \sqrt{3} = 2(\sqrt{3}+1)$

Area of ΔABC

$= \frac{\sqrt{3}}{4} \times (BC)^2$

$= \frac{\sqrt{3}}{4} \times 4(\sqrt{3}+1)^2$

$= \sqrt{3}(\sqrt{3}+1)^2$

$= \sqrt{3}(3+1+2\sqrt{3})$

$= \sqrt{3}(4+2\sqrt{3})$

$= (6+4\sqrt{3})$

46. We have, $\frac{b-c}{a}$

$= \frac{\sin B - \sin C}{\sin A}$

$$= \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)\sin\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$

Thus, $\frac{b-c}{a} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$

$$(b-c)\cos\left(\frac{A}{2}\right) = a\sin\left(\frac{B-C}{2}\right)$$

47. We have, from sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{a}{\sin(120°)} = \frac{b}{\sin(30°)} = \frac{c}{\sin(30°)}$$

$$\Rightarrow \quad \frac{a}{\frac{\sqrt{3}}{2}} = \frac{b}{\frac{1}{2}} = \frac{c}{\frac{1}{2}}$$

$$\Rightarrow \quad \frac{a}{\sqrt{3}} = \frac{b}{1} = \frac{c}{1} = \lambda(say)$$

Now, $r = \frac{\Delta}{s} = \frac{bc\sin A}{2s} = \frac{bc\sin A}{(a+b+c)}$

$$\Rightarrow \quad \sqrt{3} = \frac{\left(\frac{\sqrt{3}}{2}\right)\lambda^2}{\left(\sqrt{3}+1+1\right)\lambda}$$

$$\Rightarrow \quad \sqrt{3}\left(2+\sqrt{3}\right) = \left(\frac{\sqrt{3}}{2}\right)\lambda$$

$$\Rightarrow \quad \lambda = 2\left(2+\sqrt{3}\right)$$

Thus, the area of the ΔABC

$$= \frac{1}{2}.bc.\sin A$$

$$= \frac{1}{2}.\lambda.\lambda.\sin(120°)$$

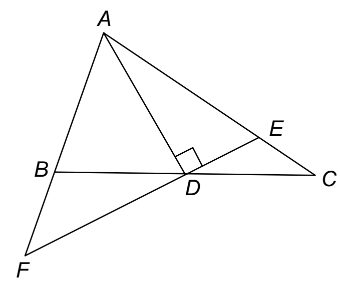
$$= \frac{\sqrt{3}}{4}\times\lambda^2$$

$$= \frac{\sqrt{3}}{4}\times 4\left(2+\sqrt{3}\right)^2$$

$$= \sqrt{3}\left(7+4\sqrt{3}\right)$$

$$= \left(12+7\sqrt{3}\right)$$

48. Ans. (a, *b*, c)



Let $AD = p$

$$ar(\Delta ABC) = ar(\Delta ABD) + (\Delta ADC)$$

$$\frac{1}{2}bc\sin A = \frac{1}{2}bp\sin\left(\frac{A}{2}\right) + \frac{1}{2}cp\sin\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad bc\sin A = p(b+c)\sin\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad bc\,2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right) = p(b+c)\sin\left(\frac{A}{2}\right)$$

$$\Rightarrow \quad p = \frac{2bc}{(b+c)}\cos\left(\frac{A}{2}\right)$$

$$AD = \frac{2bc}{(b+c)}\cos\left(\frac{A}{2}\right)$$

Also, $\frac{AD}{AE} = \cos\left(\frac{A}{2}\right)$

$$AE = AD\sec\left(\frac{A}{2}\right) = \frac{2bc}{b+c} = \frac{2}{\frac{1}{b}+\frac{1}{c}}$$

Thus, AE is the H.M. of b and c

Again, $\frac{DE}{AD} = \sin\left(\frac{A}{2}\right)$, $\frac{FD}{AD} = \sin\left(\frac{A}{2}\right)$

$$EF = DE + FD = 2AD\sin\left(\frac{A}{2}\right)$$

$$= \frac{4bc}{b+c}\cos\left(\frac{A}{2}\right)\sin\left(\frac{A}{2}\right)$$

$$= \frac{2bc}{b+c} \times \sin A$$

49. Ans. (b)

Given $AB \parallel CD$, $CD = 2\,AB$.

Let $AB = a$, $CD = 2a$
and the radius of the circle be r.
Let the circle touches at P, BC at Q, AD at R and CD at S.
Then AR $= AP = r$, $BP = BQ = a - r$,
$DR = DS = r$ and $CQ = CS = 2a - r$
In triangle BEC, $BC^2 = BE^2 + EC^2$

$\Rightarrow \quad (a - r + 2a - r)^2 = (2r)^2 + a^2$

$\Rightarrow \quad (3a - 2r)^2 = (2r)^2 + a^2$

$\Rightarrow \quad 9a^2 + 4r^2 - 12ar = 4r^2 + a^2$

$\Rightarrow \quad a = \frac{3}{2}r$

Also, $ar(\text{Quad.}ABCD) = 18$

$\Rightarrow \quad ar(\text{Quad.}ABED) + ar(\Delta BCE) = 18$

$\Rightarrow \quad a.2r + \frac{1}{2}.a.2r = 18$

$\Rightarrow \quad 3ar = 18$

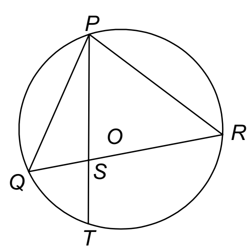
$\Rightarrow \quad 3 \times \frac{3}{2} \times r^2 = 18$

$\Rightarrow \quad r^2 = 4$

$\Rightarrow \quad r = 2$

Thus, the radius is $r = 2$.

50. Ans. (b, d)



Here, $PS \times ST = QS \times SR$

Now, $A.M > G.M$

$\Rightarrow \quad \dfrac{\frac{1}{PS} + \frac{1}{ST}}{2} > \sqrt{\frac{1}{PS}.\frac{1}{ST}}$

$\Rightarrow \quad \frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$

Also, $\frac{QS + SR}{2} > \sqrt{QS \times SR}$

$\Rightarrow \quad \frac{1}{\sqrt{QS \times SR}} > \frac{2}{QR}$

$\Rightarrow \quad \frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$

51. We have, $\cos B + \cos C = 4\sin^2\left(\frac{A}{2}\right)$

$\Rightarrow \quad 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) = 4\sin^2\left(\frac{A}{2}\right)$

$\Rightarrow \quad \cos\left(\frac{\pi}{2} - \frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right) = 2\sin^2\left(\frac{A}{2}\right)$

$\Rightarrow \quad \sin\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right) = 2\sin^2\left(\frac{A}{2}\right)$

$\Rightarrow \quad \cos\left(\frac{B-C}{2}\right) = 2\sin\left(\frac{A}{2}\right)$

$\Rightarrow \quad \cos\left(\frac{B-C}{2}\right) = 2\sin\left(\frac{\pi}{2} - \frac{B+C}{2}\right)$

$\Rightarrow \quad \cos\left(\frac{B-C}{2}\right) = 2\cos\left(\frac{B+C}{2}\right)$

$\Rightarrow \quad \dfrac{\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)} = \frac{2}{1}$

$\Rightarrow \quad \dfrac{\cos\left(\frac{B-C}{2}\right) + \cos\left(\frac{B+C}{2}\right)}{\cos\left(\frac{B-C}{2}\right) - \cos\left(\frac{B+C}{2}\right)} = \frac{2+1}{2-1}$

$\Rightarrow \quad \dfrac{2\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)}{2\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)} = 3$

$\Rightarrow \quad \cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right) = 3$

$\Rightarrow \quad \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}.\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = 3$

$\Rightarrow \quad \frac{s}{s-a} = 3$

$\Rightarrow \quad 3s - 3a = s$

$\Rightarrow \quad 2s = 3a$

$\Rightarrow \quad a + b + c = 3a$

$\Rightarrow \quad b+c=2a$

Hence, the result.

52. We have $2\cos\left(\frac{\pi}{2k}\right)+2\cos\left(\frac{\pi}{k}\right)=\sqrt{3}+1$

$\Rightarrow \quad \cos\left(\frac{\pi}{2k}\right)+\cos\left(\frac{\pi}{k}\right)=\frac{\sqrt{3}+1}{2}$

$\Rightarrow \quad \cos\left(\frac{\theta}{2}\right)+\cos(\theta)=\frac{\sqrt{3}+1}{2}$, where $\frac{\pi}{k}=\theta$

$\Rightarrow \quad 2\cos^2\left(\frac{\theta}{2}\right)-1+\cos\left(\frac{\theta}{2}\right)=\frac{\sqrt{3}+1}{2}$

$\Rightarrow \quad 2\cos^2\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)=\frac{\sqrt{3}+1}{2}+1=\frac{\sqrt{3}+3}{2}$

$\Rightarrow \quad 2t^2+t-\frac{\sqrt{3}+3}{2}=0$, where $t=\cos\left(\frac{\theta}{2}\right)$

$\Rightarrow \quad t=\frac{-1\pm\sqrt{1+4\left(3+\sqrt{3}\right)}}{4}=\frac{-1\pm\left(2\sqrt{3}+1\right)}{4}$

$\Rightarrow \quad t=\frac{-2-2\sqrt{3}}{4},\frac{\sqrt{3}}{2}$

$\Rightarrow \quad t=\frac{\sqrt{3}}{2}$

$\Rightarrow \quad \cos\left(\frac{\theta}{2}\right)=\frac{\sqrt{3}}{2}=\cos\left(\frac{\pi}{6}\right)$

$\Rightarrow \quad \left(\frac{\theta}{2}\right)=\left(\frac{\pi}{6}\right)$

$\Rightarrow \quad \theta=\left(\frac{\pi}{3}\right)$

$\Rightarrow \quad \frac{\pi}{k}=\left(\frac{\pi}{3}\right)\Rightarrow k=3$

53. We have, $ar(\Delta ABC)=\frac{1}{2}ab\sin C$

$\Rightarrow \quad 15\sqrt{3}=\frac{1}{2}.6.10.\sin C$

$\Rightarrow \quad \sin C=\frac{\sqrt{3}}{2}=\sin\left(\frac{2\pi}{3}\right)$,

since C is obtuse.

$\Rightarrow \quad C=\left(\frac{2\pi}{3}\right)$

Also, $c^2=a^2+b^2-2ab\cos C$

$=10^2+6^2-2.10.6.\cos\left(\frac{2\pi}{3}\right)$

$=100+36+60$

$=196.$

$\Rightarrow \quad C=14$

Now, $2s=a+b+c=6+10+14=30$

$\Rightarrow \quad s=15$

Therefore, $r=\frac{\Delta}{s}=\frac{15\sqrt{3}}{15}=\sqrt{3}$

$\Rightarrow \quad r^2=3$

53. Ans. (b)

We have $a^2+b^2-c^2$

$=\left(x^2+x+1\right)^2+\left(x^2-1\right)^2-(2x+1)^2$

$=x^4+x^2+1+2x^3+2x^2+2+x^4-2x^2+1$

$-4x^2-4x-1$

$=2x^4+2x^3-3x^2-2x+1$

$=(x-1)(x+1)\left(2x^2+2x-1\right)$

Now, $\cos\left(\frac{\pi}{6}\right)=\frac{a^2+b^2-c^2}{2ab}$

$\Rightarrow \quad \frac{\sqrt{3}}{2}=\frac{\left(x^2-1\right)\left(2x^2+2x-1\right)}{2\left(x^2+x+1\right)\left(x^2-1\right)}$

$\Rightarrow \quad \sqrt{3}\left(x^2+x+1\right)=2\left(2x^2+2x-1\right)$

$\Rightarrow \quad \left(2-\sqrt{3}\right)x^2+\left(2-\sqrt{3}\right)x-\left(1+\sqrt{3}\right)=0$

$\Rightarrow \quad x=-\left(2+\sqrt{3}\right),\left(\sqrt{3}+1\right)$

Thus, $x=\left(\sqrt{3}+1\right)$.

54. Ans. (c)

We have $\frac{2\sin P-\sin 2P}{2\sin P+\sin 2P}$

$=\frac{2\sin P-2\sin P\cos P}{2\sin P+2\sin P\cos P}$

$=\frac{1-\cos P}{1+\cos P}$

$=\frac{2\sin^2(P/2)}{2\cos^2(P/2)}$

$=\tan^2\left(\frac{P}{2}\right)$

$$= \frac{(s-b)(s-c)}{s(s-a)}$$

$$= \frac{(s-b)^2(s-c)^2}{s(s-a)(s-b)(s-c)}$$

$$= \frac{((s-b)(s-c))^2}{\Delta^2}$$

$$= \frac{\left(\left(4-\frac{7}{2}\right)\left(4-\frac{5}{2}\right)\right)^2}{\Delta^2}, \text{where } s = 4$$

$$= \frac{\left(\frac{1}{2}\left(\frac{3}{2}\right)\right)^2}{\Delta^2}$$

$$= \left(\frac{3}{4\Delta}\right)^2$$

57. Ans. (b)

Given $a+b=x, ab=y$

Also, $x^2-c^2=y$

$\Rightarrow \quad (a+b)^2-c^2=ab$

$\Rightarrow \quad a^2+b^2-c^2=-ab$

$\Rightarrow \quad \frac{a^2+b^2-c^2}{2ab} = -\frac{ab}{2ab} = -\frac{1}{2}$

$\Rightarrow \quad \cos C = -\frac{1}{2}$

$\Rightarrow \quad C = 120°$

Now, $R = \frac{abc}{4\Delta}, r = \frac{\Delta}{s}$

Thus, $\frac{r}{R} = \frac{4\Delta^2}{s(abc)}$

$$= \frac{4\left(\frac{1}{2}ab\sin 120°\right)^2}{\frac{x+c}{2}.y.c}$$

$$= \frac{4\left(\frac{1}{2}.y.\frac{\sqrt{3}}{2}\right)^2}{\frac{x+c}{2}.y.c}$$

$$= \frac{3y}{2(x+c)c}$$

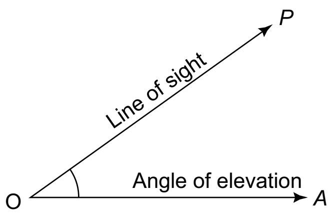
CHAPTER 8

The Heights and Distances

8.1 INTRODUCTION

In this chapter we shall study how to measure the height of the object and distance the points with the help of trigonometric relations.

8.2 ANGLE OF ELEVATION, ANGLE OF DEPRESSION AND THE LINE OF SIGHT

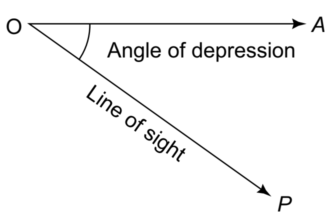


Let the point O is the observer and the point P is the object under observation.

The line OP is called the line of sight of the point P.

Let OA be the horizontal line in the same vertical plane with OP.

The acute angle $\angle AOP$, between the line of sight and the horizontal line known as the angle of elevation.

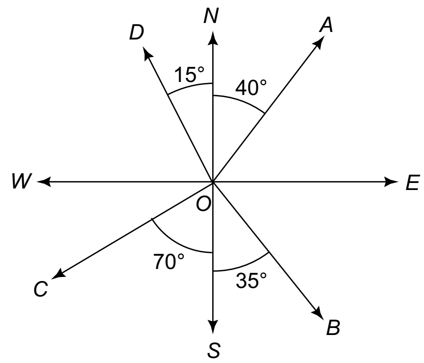


The acute angle $\angle AOP$, between the line of sight and the horizontal line known as the angle of depression, where the object P below the horizontal line OA.

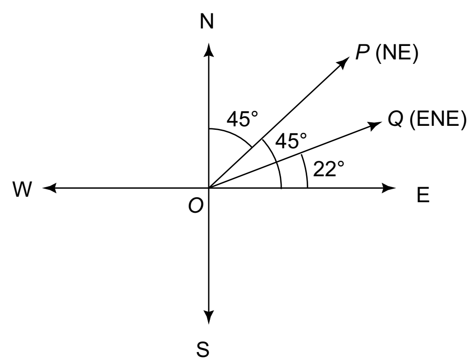
8.3 BEARING OF A LINE

The bearing of a horizontal line, i.e. a line in a horizontal plane is the positive acute angle made by this line with the north - south line in the same horizontal plane.

If a line is said to bear 20° west of north, we mean that it is inclined to the north direction at angle of 20°, this angle is measured from the north towards the west.



Here, the bearings of the lines OA, OB, OC and OD are, respectively, $N40°E$, $S35°$, $S75°W$ and $N15°W$



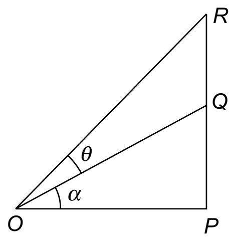
North East means equally inclined to north and east. South - East means equally inclined to south and east. *E-N-E* means equally inclined to east and north east.

8.4 SOME SOLVED EXAMPLES

Ex-1. A chimney of 20 m high standing vertically on the top of a building, subtends an angle of $\tan^{-1}\left(\frac{1}{6}\right)$ at a distance of 70 m from the foot of the building. Find the height of the building.

Soln. Let QR be the chimney, PQ be the vertical tower and O be the point of observation.

We have $OP = 70$, $QR = 20$ and $\tan\theta = \frac{1}{6}$



Let $\angle POQ = \alpha$ and $\angle QOR = \theta$ and $PQ = h$

Then $\tan(\theta+\alpha) = \frac{h+20}{70}$

$\Rightarrow \quad \frac{\tan\theta + \tan\alpha}{1-\tan\theta\tan\alpha} = \frac{h+20}{70}$

$\Rightarrow \quad \frac{\frac{1}{6}+\frac{h}{70}}{1-\frac{1}{6}\cdot\frac{h}{70}} = \frac{h+20}{70}$

$\Rightarrow \quad \frac{70+6h}{420-h} = \frac{h+20}{70}$

$\Rightarrow \quad 4900 + 420h = 420h - h^2 + 8400 - 20h$

$\Rightarrow \quad h^2 + 20h = 3500$

$\Rightarrow \quad (h+10)^2 = 3500 + 100 = 3600$

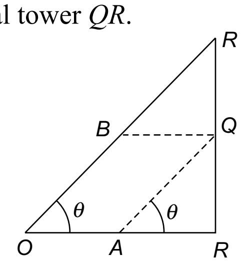
$\Rightarrow \quad h + 10 = 60$

$\Rightarrow \quad h = 50$

Hence, the height of the building is 50 m.

Ex-2. If a flag stuff on 6 m high, placed on the top of a tower costs a shadow of $2\sqrt{3}$ m along the ground, then find the angle of elevation of the sun.

Soln. Let PQ be the flag staff placed on the top of a vertcal tower QR.



Let the angle of elevation of the sun be θ.
Then it shadow on the ground $OA = BQ$

Given $OA = 2\sqrt{3} = BQ, PQ = 6$

So, $\quad \tan\theta = \frac{2\sqrt{3}}{6} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \tan\theta = \tan\left(\frac{\pi}{6}\right)$

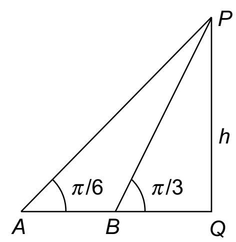
$\Rightarrow \quad \theta = \frac{\pi}{6}$

Hence, the angle of elevation is 30°.

Ex-3. The angle of elevation of the top of a tower at any point on the ground is $\frac{\pi}{6}$ and after moving 20 m towards the tower it becomes $\frac{\pi}{3}$.

Find the height of the tower.

Soln. Let PQ be the vertical tower of height h and A and B are the point of observations.
We have $AB = 20$ m



Let $BQ = x\,m$

In ΔPBQ, $\tan\left(\frac{\pi}{3}\right) = \frac{h}{x}$

$\Rightarrow \quad h = x\sqrt{3} \quad$ ......(i)

In ΔPAQ, $\tan\left(\frac{\pi}{6}\right) = \frac{h}{20+x}$

$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{20+x}$

$\Rightarrow \quad 20 + x = h\sqrt{3}$

$\Rightarrow \quad 20 + x = (x\sqrt{3})\sqrt{3}$

$\Rightarrow \quad 20 + x = 3x$

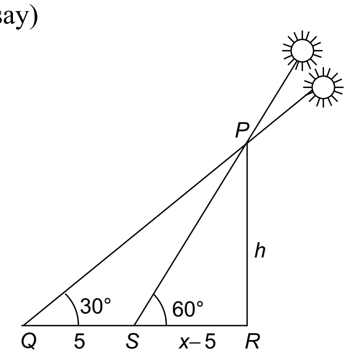
$\Rightarrow \quad 2x = 20$

$\Rightarrow \quad x = 10$

Hence, the height of the vertical tower = $10\sqrt{3}$.

Ex-4. When the suns altitude increases from 30° to 60°, the length of the shadow of a tower decrease by 5 m. Find the height of the tower.

Soln. Let PR be the height of the tower when the angle of elevation is (x — 5), the length of the shadow is $QR = x$ (say)



When the angle of elevation is 60°, then the length of the shadow is (x — 5).

In ΔPRQ, $\tan(30°) = \dfrac{h}{x}$

$\Rightarrow \quad \dfrac{1}{\sqrt{3}} = \dfrac{h}{x}$

$\Rightarrow \quad x = h\sqrt{3}$ ......(i)

In ΔPRS, $\tan(60°) = \dfrac{h}{x-5}$

$\Rightarrow \quad \sqrt{3} = \dfrac{h}{x-5}$

$\Rightarrow \quad \sqrt{3} = \dfrac{h}{h\sqrt{3}-5}$

$\Rightarrow \quad 3h - 5\sqrt{3} = h$

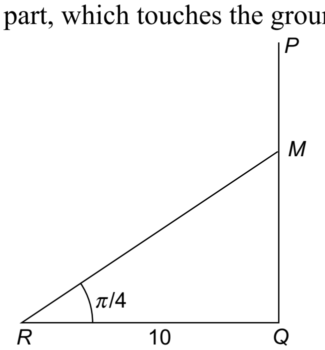
$\Rightarrow \quad 2h = 5\sqrt{3}$

$\Rightarrow \quad h = \dfrac{5\sqrt{3}}{2}$

Hence, the height of the tower is $\dfrac{5\sqrt{3}}{2}$.

Ex-5. A tree is broken by wind, its upper part touches the ground at a point 10 m from the foot of the tree and makes an angle $\dfrac{\pi}{4}$ with the ground. Find the whole length of the tree.

Soln. Let PQ be the whole length of the tree and PM its broken part, which touches the ground at R



We have $QR = 10$ and $\angle MRQ = \dfrac{\pi}{4}$

In ΔMRQ, $\tan\left(\dfrac{\pi}{4}\right) = \dfrac{QM}{10}$

$\Rightarrow \quad 1 = \dfrac{QM}{10}$

$\Rightarrow \quad QM = 10$

Also, $\quad \cos\left(\dfrac{\pi}{4}\right) = \dfrac{10}{RM}$

$\Rightarrow \quad \dfrac{1}{\sqrt{2}} = \dfrac{10}{RM}$

$\Rightarrow \quad RM = 10\sqrt{2}$

$\Rightarrow \quad PM = 10\sqrt{2}$

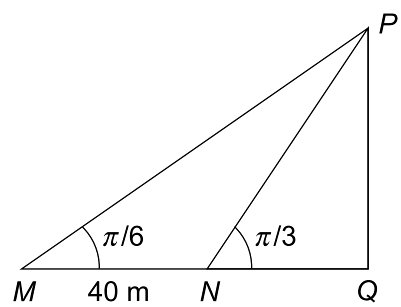
Thus, $\quad PQ = QM + PM$

$= 10 + 10\sqrt{2} = 10\left(\sqrt{2}+1\right)$

Ex-6. A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is $\dfrac{\pi}{3}$, when he retreats back 40 m from the bank he find that the angle to be $\dfrac{\pi}{6}$. Then find the breadth of the river.

Soln. Let PQ be the tree of height h and M, N be the point of observations.

We have $MN = 40$ m



Let the breadth of the river be $QN = x$

In ΔPQM, $\tan\left(\dfrac{\pi}{6}\right) = \dfrac{PQ}{x+40} = \dfrac{h}{x+40}$

$\Rightarrow \quad \dfrac{1}{\sqrt{3}} = \dfrac{h}{x+40}$

$\Rightarrow \quad h = \dfrac{x+40}{\sqrt{3}}$ ....(i)

In ΔPQN, $\tan\left(\dfrac{\pi}{3}\right) = \dfrac{h}{x}$

$\Rightarrow \quad \sqrt{3} = \dfrac{h}{x}$

$\Rightarrow \quad h = x\sqrt{3}$ ....(ii)

From (i) and (ii), we get,

$$x\sqrt{3} = \frac{x+40}{\sqrt{3}}$$

$\Rightarrow \quad 3x = x + 40$

$\Rightarrow \quad 2x = 40$

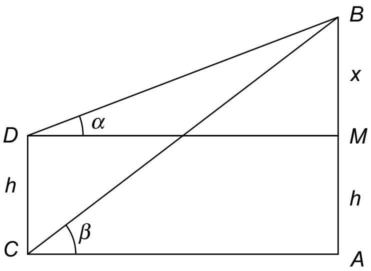
$\Rightarrow \quad x = 20$

Hence, the breadth of the river is 20 m.

Ex-7. Let the angle of elevation of the top of a hill observed from the top and bottom of a building of height h are α and β, respectively. Then find the height of the hill.

Soln. Let AB be the hill, B being its top and CD be the building of height h.

Thus, $CD = h$ and consider $BM = x$



In ΔBDM, $\tan(\alpha) = \frac{x}{DM}$

$\Rightarrow \quad DM = \frac{x}{\tan(\alpha)} = x\cot(\alpha)$ ....(i)

In ΔBCA, $\tan(\beta) = \frac{x+h}{AC}$

$\Rightarrow \quad AC = \frac{x+h}{\tan(\beta)}$

$\Rightarrow \quad DM = \frac{x+h}{\tan(\beta)}$ ....(ii)

From (i) and (ii), we get,

$$\frac{x}{\tan(\alpha)} = \frac{x+h}{\tan(\beta)}$$

$\Rightarrow \quad \frac{x}{\tan(\alpha)} - \frac{x}{\tan\beta} = \frac{h}{\tan(\beta)}$

$\Rightarrow \quad x\left(\frac{1}{\tan(\alpha)} - \frac{1}{\tan\beta}\right) = \frac{h}{\tan(\beta)}$

$\Rightarrow \quad x(\cot\alpha - \cot\beta) = h\cot\beta$

$\Rightarrow \quad x = \frac{h\cot\beta}{(\cot\alpha - \cot\beta)}$

Hence, the height of the hill

$= (x+h)$

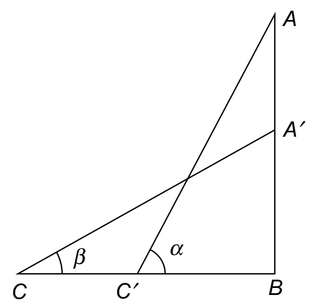
$= \frac{h\cot\beta}{\cot\alpha - \cot\beta} + h$

$= \frac{h\cot\alpha}{(\cot\alpha - \cot\beta)}$

Ex-8. A ladder rests against a vertical wall an angle α to the horizontal. Its foot is pulled away from the wall through a distance 'a' so that it slides a distance 'b' down the wall making an angle β with the horizontal. Prove that $a = b\tan\left(\frac{\alpha+\beta}{2}\right)$.

Soln. Let AC represents the ladder of length l

After being pulled through a distance 'a' away from the wall, the new position of the ladder is $A'C'$.



Clearly, $AC = A'C' = l$ and $CC' = a, AA' = b$

$\angle ACB = \alpha$ and $\angle A'C'B = \beta$

Thus, $a = BC' - BC$

$= l\cos\beta - l\cos\alpha = l(\cos\beta - \cos\alpha)$

and $\quad b = AB - A'B'$

$= l\sin\alpha - l\sin\beta = l(\sin\alpha - \sin\beta)$

Therefore, $\frac{a}{b} = \frac{l(\cos\beta - \cos\alpha)}{l(\sin\alpha - \sin\beta)}$

$\Rightarrow \quad \frac{a}{b} = \frac{(\cos\beta - \cos\alpha)}{(\sin\alpha - \sin\beta)}$

$\Rightarrow \quad \frac{a}{b} = \frac{2\sin\left(\frac{\beta+\alpha}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}$

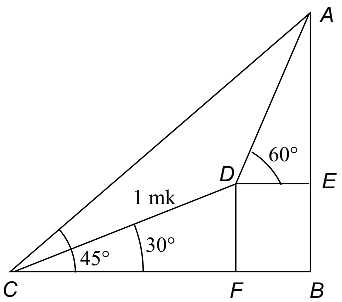
$\Rightarrow \quad \frac{a}{b} = \tan\left(\frac{\alpha+\beta}{2}\right)$

$\Rightarrow \quad a = b\tan\left(\frac{\alpha+\beta}{2}\right)$

Hence, the result.

Ex-9. At the foot of a mountain the elevation of its summit is 45°, after ascending 1 km towards the mountain upon an incline of 30°, the elevation changes to 60°. Find the height of the mountain.

Soln. Let C be the foot and A be the top of the mountain. and $AB = h$



$$CF = CD\cos(30°) = 1.\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}km$$

$$DF = CD\sin(30°) = 1.\frac{1}{2} = \frac{1}{2}km$$

In ΔABC, $\tan(45°) = \frac{AB}{BC} = \frac{h}{BC}$

$$\Rightarrow \quad BC = h$$

Now, $DE = BF = BC — CF = h - \frac{\sqrt{3}}{2}$.

and $AE = AB — BE = AB — DF = h - \frac{1}{2}$

From ΔAED, $\tan(60°) = \frac{AE}{DE}$

$$\Rightarrow \quad \sqrt{3} = \frac{h - \frac{1}{2}}{h - \frac{\sqrt{3}}{2}}$$

$$\Rightarrow \quad \sqrt{3}h - \frac{3}{2} = h - \frac{1}{2}$$

$$\Rightarrow \quad \left(\sqrt{3} - 1\right)h = \frac{3}{2} - \frac{1}{2} = 1$$

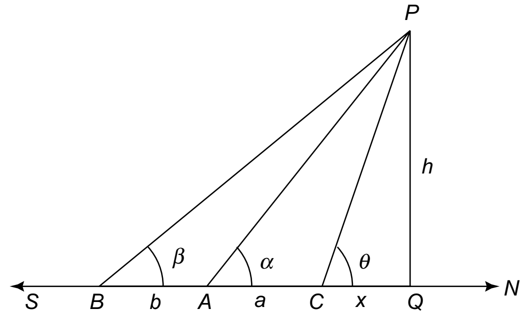
$$\Rightarrow \quad h = \frac{1}{\left(\sqrt{3} - 1\right)} = \frac{\left(\sqrt{3} + 1\right)}{2}$$

Hence, the height of the mountain is $\left(\frac{\left(\sqrt{3}+1\right)}{2}\right)$km.

Ex-10. Two stations due south of a tower, which leans towards north are at a distances a and b from its foot. If α and β are the elevations of the top of the tower from these stations, prove that its inclination θ to the horizontal is given by

$$\cot(\theta) = \frac{b\cot\alpha - a\cot\beta}{b - a}$$

Soln. Let A and B be two stations due south of the tower OP which leans towards north.



Let $AC = a$, $BC = b$ and $CQ = x$

Clearly, $\tan\alpha = \frac{h}{a+x}$

$$\Rightarrow \quad \cot\alpha = \frac{a+x}{h} \quad ....(i)$$

and $\tan(\beta) = \frac{h}{b+x}$

$$\Rightarrow \quad \cot(\beta) = \frac{b+x}{h} \quad ....(ii)$$

From (i) and (ii), we get,

$$b\cot(\alpha) - a\cot(\beta)$$

$$= \frac{ab+bx}{h} - \frac{ab+ax}{h} = \frac{(b-a)x}{h}$$

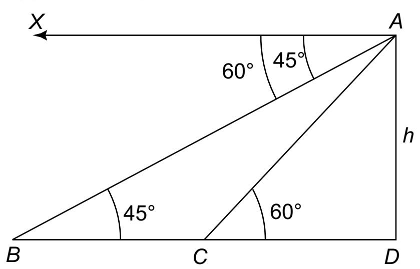
$$\Rightarrow \quad \frac{x}{h} = \frac{b\cot\alpha - a\cot\beta}{(b-a)}$$

$$\Rightarrow \quad \cot(\theta) = \frac{b\cot\alpha - a\cot\beta}{(b-a)}$$

Hence, the result.

Ex-11. From an aeroplane vertically over a straight road the angles of depression of two consecutive kilometer stones on the same side are 45° and 60°. Find the height of the aeroplane.

Soln. Let B and C be two consecutive kilometer stones. Then $BC = 1$ km and A be the position of the plane at a certain time.



Let $AD = h$ and $CD = x$

In ΔABD, $\tan(45°) = \frac{h}{1+x}$

$\Rightarrow \quad x + 1 = h$

$\Rightarrow \quad x = h - 1 \quad$ ......(i)

In ΔACD, $\tan(60°) = \frac{h}{x}$

$\Rightarrow \quad \frac{h}{x} = \sqrt{3}$

$\Rightarrow \quad \frac{h}{h-1} = \sqrt{3}$

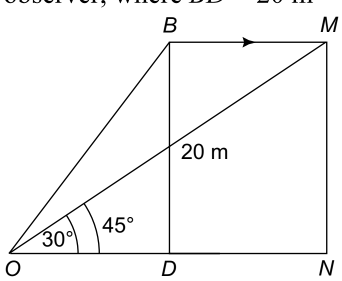
$\Rightarrow \quad (\sqrt{3} - 1)h = \sqrt{3}$

$\Rightarrow \quad h = \frac{\sqrt{3}}{(\sqrt{3}-1)} = \frac{\sqrt{3}(\sqrt{3}+1)}{2}$

Hence, the height of the plane is $\left(\frac{3+\sqrt{3}}{2}\right)$km.

Ex-12. A bird is perched on the top of a tree 20 m high and its elevation from a point on the ground is 45°. It flies off horizontally straight away from the observer and in one second the elevation of the bird is reduced to 30°. Find its speed.

Soln. Let the bird alight at B, the top of the tree BD and O be the observer, where $BD = 20$ m



Thus, $BD = MN = 20$ m

In ΔBOD, $\tan(45°) = \frac{20}{OD}$

$\Rightarrow \quad OD = 20$

In ΔMON, $\tan(30°) = \frac{MN}{ON} = \frac{20}{20 + DN}$

$\Rightarrow \quad \frac{20}{20+DN} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad 20 + DN = 20\sqrt{3}$

$\Rightarrow \quad DN = 20(\sqrt{3} - 1)$

Thus, $\quad$ speed $= \frac{\text{distance}}{\text{time}}$

$= \frac{20(\sqrt{3}-1)}{1} m/\sec$

$= 20(\sqrt{3}-1)\ m/\sec$

LEVEL I
(PROBLEMS BASED ON FUNDAMENTALS)

1. A tower subtends an angle θ at a point A in the plane of its base. The angle of depression of the foot of the tower at a point 'h' m just above 'A' is α. Find the height of the tower.
2. The angle of elevation of the top of an incomplete vertical pillar at a horizontal distance of 100 m from its base is $\frac{\pi}{4}$. If the angle of elevation of the top of the complete pillar at the same point is to be $\frac{\pi}{3}$ such that the height of the pillar is increased by h m, then find h.
3. A person walking along a straight road observes that at two points 1000 m apart, then angle of elevation of a vertical tower in front of him are $\frac{\pi}{6}$ and $\frac{5\pi}{12}$. Find the height of the tower.
4. A man in a boat rowing uniformly away from a cliff 150 m high takes 2 minutes to change the angle of elevation of the top of the hill from $\frac{\pi}{3}$ to $\frac{\pi}{4}$. Find the speed of the boat.
5. A flagstaff of 5 m high stands on a building of 25 m high. The flagstaff and the building subtends equal angles at a point P, 30 m high above the ground. Find the distance of P from the top of flagstaff.
6. AB is a vertical tower 'A' being its foot standing on a horizontal ground. 'C' is the mid-point of AB. Portion CB subtends an angle θ at the point P on the ground. If $AP = 2AB$, then find $\tan(\theta)$.
7. At the foot of a mountain the elevation of its peak is found to be $\frac{\pi}{4}$, after ascending 10 m toward the mountain up a slope of $\frac{\pi}{6}$ inclination, the elevation is found to be $\frac{\pi}{3}$. Find the height of the mountain.
8. A man finds that at a point due south of a vertical tower the angle of elevation of the tower is $\frac{\pi}{3}$. He then walks due west $10\sqrt{6}$ m on the horizontal plane and find the angle of elevation of the tower to be $\frac{\pi}{6}$. Find the original distance of the man from the tower.
9. The angle of elevation of the top of vertical tower from a point A on the horizontal ground is found to

be $\frac{\pi}{4}$. From 'A' a man walks 10 m up a path sloping at an angle $\pi/6$. After this the slope becomes steeper and after walking up another 10 m, the man reaches the top of the tower. Find the distance of 'A' from the foot of the tower.

10. A vertical tower erected at the focus of the parabola $y^2 = 40x$ subtends an angle $\frac{\pi}{3}$ at the vertex of the parabola. If the tower subtends an angle $\frac{\pi}{6}$ at a point P lying on the parabola.
Then find the possible co-ordinates of point P.

LEVEL II
(MIXED PROBLEMS)

1. On level ground the angle of elevation of the top of the tower is 30°. On moving 20 meters near then the angle of elevation is 60°. The height of the tower is
   (a) $20\sqrt{3}\,m$ (b) $10\sqrt{3}\,m$
   (c) $10\left(\sqrt{3}-1\right)m$ (d) None
2. From the top of a light house 60 meters high with its base at sea level, the angle of depression is 15°. The distance of the boat from the foot of the light house is
   (a) $60\times\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)m$ (b) $60\times\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)m$
   (c) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)m$ (d) None
3. Three poles whose feet A, B, C lie on a circle subtend angles α, β, γ and respectively, at the centre of the circle. If the height of the poles are in A.P, then $\cot(\alpha), \cot(\beta), \cot(\gamma)$ are in
   (a) A.P (b) G.P
   (c) H.P (d) None
4. A ladder 20 ft long reaches a point 20 ft below the top of a flag. The angle of elevation of the top of the flag at the foot of the ladder is 60°. Then the height of the flag is
   (a) 25 ft (b) 30 ft
   (c) 35 ft (d) 40 ft
5. From an aeroplane vertically over a straight horizontal road, the angles of depression of two consecutive mile-stones on opposite sides of the aeroplane are observed to 45° and 60°. Then the height in miles of aeroplane above the road is
   (a) $\frac{\sqrt{3}}{\sqrt{3}+1}$ (b) $\frac{\sqrt{3}}{\sqrt{3}-1}$
   (c) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ (d) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

LEVEL III
(PROBLEMS FOR JEE MAIN)

1. A man on a cliff observes a ship at an angle of depression 30° approaching the shore just beneath him. Three minutes later the angle of depression of the ship is 60°. How soon will it reach the shore?
2. A vertical tower subtends an angle of 60° at a point on the same level as the foot of the tower. On moving 100 m further from the first point in line with the tower, it subtends an angle of 30° at the point. Find the height of the tower.
3. Find the height of a tower when it is found that on walking 80 m towards it along a horizontal line through its base, the angular elevation of its top changes from 30° and 60°.
4. A vertical pole on one side of a street subtends a right angle at a window exactly on the opposite side. If the angle of elevation of the window from the base of the pole be 60° and the width of the street be 30 m, find the heights of the window and top of the pole.
5. An object is observed from three points A, B, C lying in a horizontal straight line which passes directly underneath the object. The angular elevation at B is twice that at A and at C three times at A. If $AB = a$, $BC = b$, find the height of the object.
6. A man notices two objects in a striaght line due west of him. After walking a distance c due north he observes that the objects subtend an angle α at his eye and after walking a further distance c due north, an angle β. Find the distance between the objects.
7. The angle of elevation of an aeroplane from a point 200 meters above a lake is 45° and the angle of depression of its replection is 75°. Find the height of the aeroplane above the surface of the lake.
8. From the bottom of a pole of height h, the angle of elevation of the top of a tower is α. The pole subtends an angle β at the top of the tower. Find the height of the tower.
9. The angle of elevation of a cloud from a point x feet above a lake is θ and the angle of depression of its reflection in the lake is φ. Find its height.
10. A train is moving at a constant speed at an angle θ East of North. Observations of the train are made from a fixed point. It is due north at some instant. Ten minutes earlier its bearing was α West of North, where as 10 minutes afterwards its bearing is β East of North. Find $\tan(\theta)$.

ANSWERS

LEVEL I

1. $h \tan\theta \cot\alpha$
2. $h = 100\left(\sqrt{3}-1\right)m$
3. $250\left(\sqrt{3}+1\right)$
4. $25\left(3-\sqrt{3}\right)\ m/\min$
5. $5\times\sqrt{\dfrac{3}{2}}$
6. 2/9
7. $5\left(\sqrt{3}+1\right)$
8. $5\sqrt{3}\ m$
9. $5\left(\sqrt{3}+1\right)$
10. $\left(20, 20\sqrt{2}\right)$

LEVEL II

1. (b) 2. (b) 3. (c) 4. (b) 5. (a)

LEVEL III

1. $\frac{3}{2}\min$
2. $50\sqrt{3}\,m$
3. $40\sqrt{3}\,m$
4. $30\sqrt{3}\,m, 40\sqrt{3}\,m$
5. $\dfrac{a\sqrt{(a+b)(3b-a)}}{2b}$
6. $\dfrac{3c}{2\cot\beta-\cot\alpha}$
7. $200\sqrt{3}\ m$
8. $h\sin\alpha\cos(\alpha-\beta)\operatorname{cosec}\beta$
9. $\dfrac{x\sin(\varphi+\theta)}{\sin(\varphi-\theta)}$
10. $\dfrac{2\sin\alpha\sin\beta}{\sin(\alpha-\beta)}$