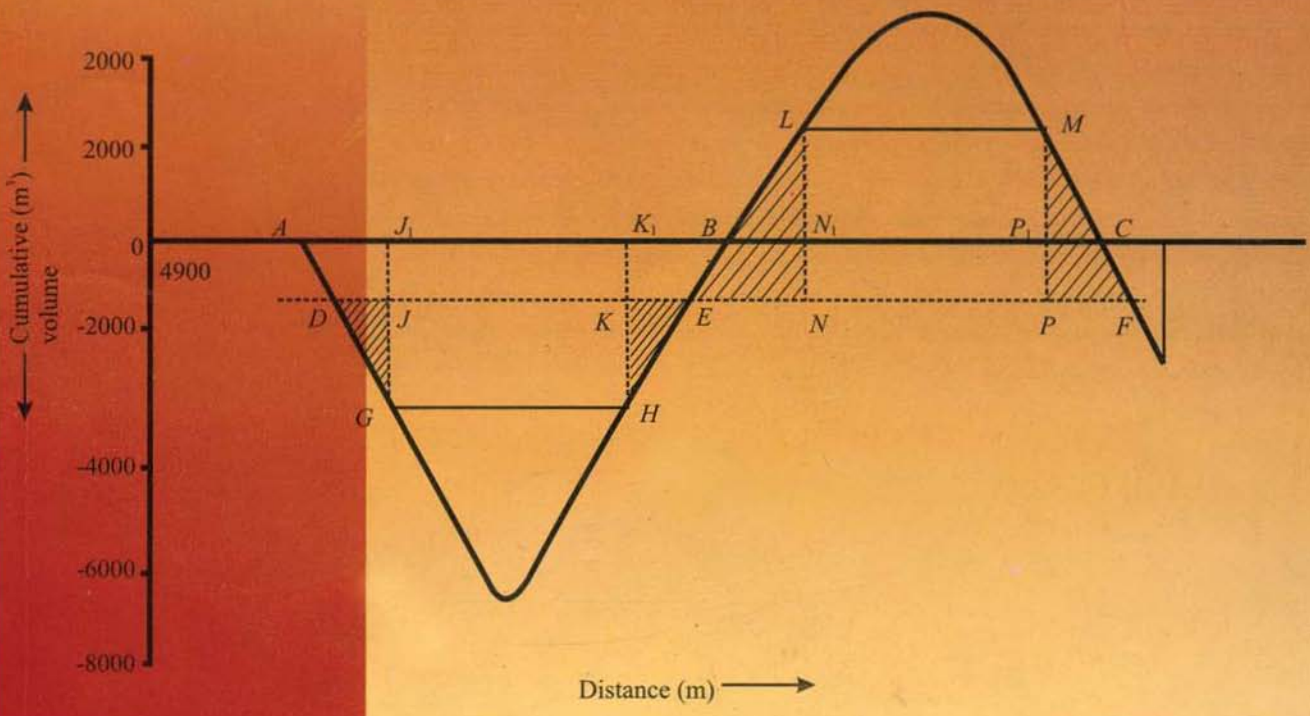
Surveying

Problem Solving with Theory and Objective Type Questions

Dr A M Chandra



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Dedicated
to
My Parents

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PREFACE

The book adopts a classical pedagogical approach by providing a vivid insight into the theory of surveying and its application through solving typical problems in the field of surveying. It aims at helping the students understand surveying more comprehensively through solving field related problems. Each chapter of the book commences with a summary of basic theory and a range of worked out examples making it very useful for all undergraduate and postgraduate courses in surveying. Alternative solutions to the problems wherever possible, have also been included for stimulating the budding minds. A number of objective type questions which are now a days commonly used in many competitive examinations, have been included on each topic to help the readers to get better score in such examinations. At the end, a number of selected unsolved problems have also been included to attain confidence on the subject by solving them. The book is also intended to help students preparing for AMIE, IS, and Diploma examinations. The practicing engineers and surveyors will also find the book very useful in their career while preparing designs and layouts of various application-oriented projects.

Constructive suggestions towards the improvement of the book in the next edition are fervently solicited.

The author expresses his gratitude to the Arba Minch University, Ethiopia, for providing him a conducive environment during his stay there from Sept. 2002 to June 2004, which made it possible for writing this book.

The author also wishes to express his thanks to all his colleagues in India and abroad who helped him directly or indirectly, in writing this book.

Roorkee —Dr A M Chandra

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1 Errors in Measurements and Their Propagation

1.1 ERROR TYPES

Gross errors are, in fact, not errors at all, but results of mistakes that are due to the carelessness of the observer. The gross errors must be detected and eliminated from the survey measurements before such measurements can be used. *Systematic errors* follow some pattern and can be expressed by functional relationships based on some deterministic system. Like the gross errors, the systematic errors must also be removed from the measurements by applying necessary corrections. After all mistakes and systematic errors have been detected and removed from the measurements, there will still remain some errors in the measurements, called the *random errors* or *accidental errors*. The random errors are treated using probability models. Theory of errors deals only with such type of observational errors.

1.2 PROBABILITY DISTRIBUTION

If a large number of masurements have been taken, the frequency distribution could be considered to be the probability distribution. The statistical analysis of survey observations has indicated that the survey measurements follow *normal distribution* or *Gaussian distribution*, being expressed by the equation,

$$dy = \frac{1}{\sigma\sqrt{2\pi}} e^{-(\mu - x_1)^2/2\sigma^2} dx \qquad ...(1.1)$$

where dy is the probability that the value will lie between the limits of x_1 and $(x_1 + dx)$, μ is the true mean of the population, and σ is the standard deviation.

1.3 MOST PROBABLE VALUE

Different conditions under which the measurements are made, cause variations in measurments and, therefore, no measured quantity is completely detrminable. A fixed value of a quantity may be concieved as its *true value*. The difference between the measured quantity and its true value τ is known as *error* ε, i.e.,

$$\varepsilon = x - \tau \qquad ...(1.2)$$

Since the true value of a measured quantity cannot be determined, the exact value of ε can never be found out. However, if a best estimate $\hat{x}$ which is known as the *most probable value* of τ, can be determined, $\hat{x}$ can be used as a reference to express the variations in x. If we define υ as *residual* then

$$\upsilon = \hat{x} - x \qquad ...(1.3)$$

The residuals express the *variations* or *deviations* in the measurements.

1.4 STANDARD DEVIATION

Standard deviation also called the *root-mean square* (R.M.S.) *error*, is a measure of spread of a distribution and for the population, assuming the observations are of equal reliability it is expressed as

$$\sigma_n = \pm\sqrt{\left[\frac{\Sigma(\mu - x)^2}{n}\right]} \qquad ...(1.4)$$

However, μ cannot be determined from a sample of observations. Instead, the arithmetic mean $\bar{x}$ is accepted as the most probable value and the population standard deviation is estimated as

$$\sigma_{n-1} = \pm\sqrt{\left[\frac{\Sigma\,(\hat{x} - x)^2}{(n-1)}\right]} \qquad ...(1.5)$$

or

$$= \pm\sqrt{\left[\frac{\Sigma\upsilon^2}{(n-1)}\right]} \qquad ...(1.6)$$

The standard deviation given by the above expression is also called the *standard error*. Henceforth in this book the symbol σ will mean σ_{n-1}.

1.5 VARIANCE

Variance of a quantity is expressed as

$$V = \frac{\Sigma\upsilon^2}{n-1} \qquad ...(1.7)$$

or

$$= \sigma_{n-1}^2$$

or

$$= \sigma^2 \qquad ...(1.8)$$

and is also used as a measure of dispersion or spread of a distribution.

1.6 STANDARD ERROR OF MEAN

The *standard error of mean* σ_m is given by

$$\sigma_m = \pm\sqrt{\left[\frac{\Sigma\upsilon^2}{n(n-1)}\right]} \qquad ...(1.9)$$

or

$$= \pm\frac{\sigma}{\sqrt{n}} \qquad ...(1.10)$$

and hence the precision of the mean is enhanced with respect to that of a single observation. There are n deviations (or residuals) from the mean of the sample and their sum will be zero. Thus, knowing $(n - 1)$ deviations the surveyor could deduce the remaining deviation and it may be said that there are $(n - 1)$ degrees of freedom. This number is used when estimating the population standard deviation.

1.7 MOST PROBABLE ERROR

The most probable error is defined as the error for which there are equal chances of the true error being less and greater than probable error. In other words, the probability of the true error being less than the probable error is 50% and the probability of the true error being greater than the probable error is also 50%. The most probable error is given by

$$e = \pm\, 0.6745\sqrt{\left[\frac{\Sigma \upsilon^2}{(n-1)}\right]} \qquad \text{...(1.11)}$$

$$= \pm\, 0.6745\sigma \qquad \text{...(1.12)}$$

1.8 CONFIDENCE LIMITS

After establishing the sample mean as estimate of the true value of the quantity, the range of values within which the true value should lie for a given probability is required. This range is called the *confidence interval*, its bounds called the *confidence limits*. Confidence limits can be established for that stated probability from the standard deviation for a set of observations. Statistical tables are available for this purpose. A figure of 95% frequently chosen implies that nineteen times out of twenty the true value will lie within the computed limits. The presence of a very large error in a set of normally distributed errors, suggests an occurance to the contrary and such an observation can be rejected if the residual error is larger than three times the standard deviation.

1.9 WEIGHT

This quantity ω is known as *weight* of the measurement indicates the reliability of a quantity. It is inversely proportional to the variance (σ^2) of the observation, and can be expressed as

$$\omega = \frac{k}{\sigma^2}$$

where k is a constant of proportionality. If the weights and the standard errors for observations x_1, x_2, ,….., etc., are respectively ω_1, ω_2,….., etc., and σ_1, σ_2,….., etc., and σ_u is the standard error for the observation having unit weight then we have

$$\omega_1\sigma_1^2 = \omega_2\sigma_2^2 = \ldots\ldots = \sigma_u^2\,. \qquad \text{...(1.13)}$$

Hence

$$\omega_1 = \frac{\sigma_u^2}{\sigma_1^2}, \qquad \omega_2 = \frac{\sigma_u^2}{\sigma_2^2}, \quad \text{etc.,}$$

and $$\frac{\omega_1}{\omega_2} = \frac{\sigma_2^2}{\sigma_1^2}, \quad \text{etc.} \qquad ...(1.14)$$

The weights are applied to the individual measurements of unequal reliability to reduce them to one standard. The most probable value is then the weighted mean $\bar{x}_m$ of the measurements. Thus

$$\hat{x}_m = \frac{\Sigma(\omega x)}{\Sigma\omega}, \qquad ...(1.15)$$

and standard error of the wieghted mean

$$\sigma_{\bar{x}_m} = \pm\sqrt{\left[\frac{\Sigma\left\{\omega(\hat{x}_m - x)^2\right\}}{(n-1)\,\Sigma\omega}\right]} \qquad ...(1.16)$$

The standard deviation of an observation of unit weight is given by

$$\sigma_u = \pm\sqrt{\left[\frac{\Sigma\left\{\omega(\hat{x}_m - x)^2\right\}}{(n-1)}\right]} \qquad ...(1.17)$$

and the standard deviation of an observation of weight ω_n is given by

$$\sigma_w = \pm\sqrt{\left[\frac{\Sigma\left\{\omega(\hat{x}_m - x)^2\right\}}{\omega_n\,(n-1)}\right]} \qquad ...(1.18)$$

1.10 PRECISION AND ACCURACY

Precision is the degree of closeness or conformity of repeated measurements of the same quantity to each other whereas the *accuracy* is the degree of conformity of a measurement to its true value.

1.11 PROPAGATION OF ERROR

The calculation of quantities such as areas, volumes, difference in height, horizontal distance, etc., using the measured quantities distances and angles, is done through mathematical relationships between the computed quantities and the measured quantities. Since the measured quantities have errors, it is inevitable that the quantities computed from them will not have errors. Evaluation of the errors in the computed quantities as the function of errors in the measurements, is called *error propagation.*

Let $y = f(x_1, x_2, ..., x_n)$ then the error in y is

$$dy = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_n}dx_n \qquad ...(1.19)$$

and the standard deviation of y is

$$\sigma_y^2 = \left(\frac{\partial f}{\partial x_1}\sigma_{x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\sigma_{x_2}\right)^2 + ... + \left(\frac{\partial f}{\partial x_n}\sigma_{x_n}\right)^2 \qquad ...(1.20)$$

where dx_1, dx_2, ..., etc., are the errors in x_1, x_2,....., etc., and $\sigma_{x_1}, \sigma_{x_2}$, ..., etc., are their standard deviations. In a similar way if

$$y = x_1 + x_2 + ... + x_n$$

then
$$\sigma_y^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + ... + \sigma_{x_n}^2 \text{, since } \frac{\partial f}{\partial x_1} \text{, etc. = 1.} \qquad ...(1.21)$$

And if

$$y = kx_1 \text{ in which } k \text{ is free of error}$$

$$\sigma_y = k\sigma_{x_1}$$

since
$$\frac{\partial f}{\partial x_1} = k.$$

In the above relationships it is assumed that x_1, x_2,......., x_n are independent implying that the probability of any single observation having a certain value does not depend on the values of other observations.

1.12 NORMAL DISTRIBUTION

The expression for the normal distribution is

$$dy = \frac{1}{\sigma\sqrt{2\pi}} e^{-(\mu - x_1)^2/2\sigma^2} dx. \qquad ...(1.23)$$

Taking $u = \dfrac{\mu - x}{\sigma}$, the expression becomes

$$dy = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du. \qquad ...(1.24)$$

Eq. (1.24) is the standardized form of the above expression, and Fig. 1.1 illustrates the relationship between *dy/du* and *u* is illustrated in Fig. 1.1.

The curve is symmetrical and its total area is 1, the two parts about $u = 0$ having areas of 0.5. The shaded area has the value

$$\int_{-\infty}^{+u_1} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du$$

and it gives the probability of u being lying between $-\infty$ and + u_1. The unshaded area gives the probability that u will be larger than + u_1. Since the curve is symmetrical, the probability that u takes up a value outside the range + u_1 to – u_1 is given by the two areas indicated in Fig. 1.2.

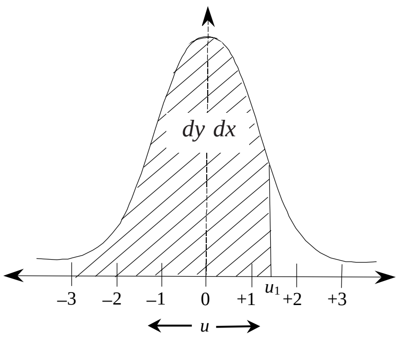


Fig. 1.1

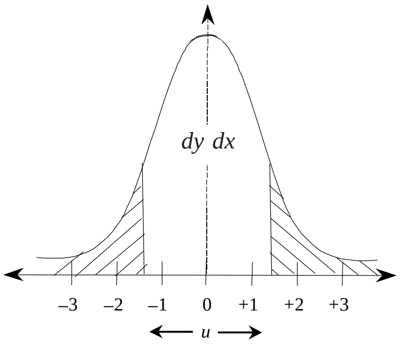


Fig. 1.2

The values of the ordinates of the standardized form of the expression for the normal distribution, and the corresponding definite integrals, have been determined for a wide range of *u* and are available in various publications. A part of such table is given in Table 1.5 and some typical values used in this example have been taken from this table.

Example 1.1. The following are the observations made on the same angle:

47°26′13″	47°26′18″
47°26′10″	47°26′15″
47°26′16″	47°26′12″
47°26′09″	47°26′15″
47°26′18″	47°26′14″

Determine

(a) the most probable value of the angle,

(b) the range,

(c) the standard deviation,

(d) the standard error of the mean, and

(e) the 95% confidence limits.

Solution:

For convenience in calculation of the required quantities let us tabulate the data as in Table 1.1. The total number of observations $n = 10$.

(a) Most probable value = $\hat{x}$ = **47°26′14″**

(b) Range = 47°26′18″ – 47°26′09″ = **9″**

(c) Standard deviation

$$\sigma = \pm\sqrt{\left[\frac{\Sigma \upsilon^2}{(n-1)}\right]}$$

$$= \pm\sqrt{\left[\frac{84}{(10-1)}\right]} = \pm \mathbf{3.1''}.$$

(d) Standard error of mean

$$\sigma_m = \pm \frac{\sigma}{\sqrt{n}}$$

$$= \pm \frac{3.1}{\sqrt{10}} = \pm \mathbf{1.0''}.$$

Table 1.1

Observed angles (x)	$(\hat{x} - x) = \upsilon$	$(\hat{x} - x)^2 = \upsilon^2$
47°26'13″	+ 1	1
10″	+ 4	16
16″	– 2	4
09″	+ 5	25
18″	– 4	16
18″	– 4	16
15″	– 1	1
12″	+ 2	4
15″	– 1	1
14″	0	0
Σ = 140″	Σ = 0	Σ = 84
$\hat{x} = \frac{\Sigma x}{n} = 47°6' \frac{140''}{10} = 47°26'14''$		

(e) 95% confidence limits

The lower confidence limit $= \hat{x} - \frac{t\sigma}{\sqrt{n}}$

The upper confidence limit $= \hat{x} + \frac{t\sigma}{\sqrt{n}}$ ...(1.22)

where t is selected from statistical tables for a given value of n. For $n = 10$, $t = 2.26$ and so

$$\frac{t\sigma}{\sqrt{n}} = \frac{2.26 \times 3.1}{\sqrt{10}} = 2.2''.$$

Hence the 95% confidence limits are **47°26′14″ ± 2.2″**.

It is a common practice in surveying to reject any observation that differs from the most probable value by more than three times the standard deviation.

Example 1.2. The length of a base line was measured using two different EDM instruments A and B under identical conditions with the following results given in Table 1.2. Determine the relative precision of the two instruments and the most probable length of the base line.

Table 1.2

A (m)	*B* (m)
1001.678	1001.677
1001.670	1001.681
1001.667	1001.675
1001.682	1001.678
1001.674	1001.677
1001.679	1001.682
	1001.679
	1001.675

Solution:

(*i*) The standard deviation of the masurements by *A*

$$\sigma_A = \pm\sqrt{\left[\frac{\Sigma\upsilon^2}{(n-1)}\right]} = \pm\sqrt{\left[\frac{164}{(6-1)}\right]}$$

$$= \pm\, 5.73 \text{ mm.}$$

Table 1.3

A			*B*		
Distance (m)	υ (mm)	υ^2 (mm^2)	Distance (m)	υ (mm)	υ^2 (mm^2)
1001.678	– 3	9	1001.677	+ 1	1
1001.670	+ 5	25	1001.681	– 3	9
1001.667	+ 8	64	1001.675	+ 3	9
1001.682	– 7	49	1001.678	0	0
1001.674	+ 1	1	1001.677	+ 1	1
1001.679	– 4	16	1001.682	– 4	16
			1001.679	– 1	1
			1001.675	+ 3	9
Σ=6001.050		Σ=164	Σ=8013.424		Σ=46
$\hat{x}_A = \frac{6001.050}{6} = 1001.675$ m			$\hat{x}_B = \frac{8013.424}{8} = 1001.678$ m		

The standard deviation of the measurements by *B*

$$\sigma_B = \pm\sqrt{\left[\frac{46}{(8-1)}\right]} = \pm\, 2.56 \text{ mm.}$$

The standard error of the mean for A

$$\sigma_{mA} = \pm\frac{5.73}{\sqrt{6}} = \pm 2.34 \text{ mm.}$$

(*ii*) The standard error of the mean for B

$$\sigma_{mB} = \pm\frac{2.56}{\sqrt{8}} = \pm 0.91 \text{ mm.}$$

(*iii*) The relative precision of the two instruments A and B is calculated as follows:

If the weights of the measurements 1001.675 m and 1001.678 m are ω_A and ω_B having the standard errors of means as ± 2.34 mm and ± 0.91 mm, repectively, then the ratio ω_A/ω_B is a measure of the relative precision of the two instruments. Thus

$$\frac{\omega_A}{\omega_B} = \frac{\sigma_B^2}{\sigma_A^2} = \frac{0.91^2}{2.34^2}$$

$$= \frac{1}{6.6}$$

Therefore, $$\omega_A = \frac{\omega_B}{6.6}.$$

(*iv*) The most probable length of the line is the weighted mean of the two observed lengths. Now

$$\bar{x}_\omega = \frac{\Sigma(\omega x)}{\Sigma\omega} = \frac{\omega_A L_A + \omega_B L_B}{\omega_A + \omega_B}$$

$$= \frac{\frac{\omega_B}{6.6} \times 1001.675 + \omega_B \times 1001.678}{\frac{\omega_B}{6.6} + \omega_B}$$

$$= \frac{1001.675 + 6.6 \times 1001.678}{1 + 6.6} = \mathbf{1001.6776\ m}.$$

In accordance with the observations, $\hat{x}_\omega$ could be written as **1001.678** m to the nearest millimetre.

Example 1.3. An angle was measured with different weights as follows:

Determine

(a) the most probable value of the angle,

(b) the standard deviation of an obsevation of unit weight,

(c) the standard deviation of an observation of weight 3, and

(d) the standard error of the weighted mean.

Angle	Weight (ω)
86°47′25″	1
86°47′28″	3
86°47′22″	1
86°47′26″	2
86°47′23″	4
86°47′30″	1
86°47′28″	3
86°47′26″	3

Solution: Tabulating the data and the weighted results working from a datum of 86°47′, we get the values as given in Table 1.4.

(a) The most probable value of the angle is the weighted mean

$$\bar{x}_\omega = \text{Datum} + \frac{\Sigma(\omega x)}{\Sigma\omega}$$

$$= 86°\,47' + \frac{467}{18} = 86°\,47'25.9''$$

(b) Standard deviation of an observation of unit weight

$$\sigma_u = \pm\sqrt{\left[\frac{\Sigma(\omega\upsilon^2)}{(n-1)}\right]} = \pm\sqrt{\frac{93}{(8-1)}} = \pm\mathbf{3.64''}.$$

Table 1.4

Observed angle	x	ω	ωx	υ	$\omega\upsilon^2$
86°47′25″	25	1	25	+ 1	1
86°47′28″	28	3	84	– 2	12
86°47′22″	22	1	22	+ 4	16
86°47′26″	26	2	52	0	0
86°47′23″	23	4	92	+ 3	36
86°47′30″	30	1	30	– 4	16
86°47′28″	28	3	84	– 2	12
86°47′26″	26	3	78	0	0
	$\Sigma=208$	$\Sigma=18$	$\Sigma=467$		$\Sigma=93$
$\hat{x} = \frac{208}{8} = 26$					

(c) Standard deviation of an observation of weight 3

$$\sigma_{\varpi} = \pm\sqrt{\left[\frac{\Sigma(\omega\upsilon^2)}{\omega_n(n-1)}\right]}$$

$$= \pm\sqrt{\left[\frac{93}{3\times(8-1)}\right]} = \pm\,\mathbf{2.10''}.$$

Alternatively,

$$\omega_1\sigma_1^2 = \omega_2\sigma_2^2 = \ldots\ldots = \sigma_u^2$$

We have $\omega_3 = 3$, therefore

$$\sigma_{\varpi}^2 = \frac{\sigma_u^2}{3} = \frac{3.64^2}{3}$$

$$\sigma_{\varpi} = \pm\frac{3.64}{\sqrt{3}} = \pm\ \mathbf{2.10''}.$$

(d) Standard error of the weighted mean

$$\sigma_{\bar{x}_m} = \pm\sqrt{\left[\frac{\Sigma(\omega\upsilon)^2}{(n-1)\,\Sigma\omega}\right]}$$

$$= \pm\sqrt{\left[\frac{93}{(8-1)\times 18}\right]} = \pm\,\mathbf{0.86''}.$$

Alternatively,

$$\omega_m s_m^2 = \sigma_u^2$$

Since $\omega_m = \Sigma\omega$, we have

$$s_m = \pm\frac{\sigma_u}{\sqrt{\Sigma\omega}}$$

$$= \pm\frac{3.64}{\sqrt{18}} = \pm\,\mathbf{0.86''}.$$

Example 1.4. If the standard deviation σ_u of a single measurement in Example 1.1 is ± 3″, calculate

(i) the magnitude of the deviation likely to occur once in every two measurements,

(ii) the probability that a single measurement may deviate from the true value by ± 6″, and

(iii) the probability that the mean of nine measurments may deviate from the true value by ± *1.5″*.

Solution:

If a deviation is to occur once in every two measurements a probability of 50% is implied. Thus in Fig. 1.2 the two shaded parts have areas of 0.25 each and the total shaded area is 0.5.

Table 1.5

u	dy/du	$\int_{-\infty}^{+u}$
0.0	0.3989	0.5000
0.6	0.3332	0.7257
0.7	0.3123	0.7580
1.5	0.1295	0.9332
2.0	0.0540	0.9772

Since $\int_{-\infty}^{+u_1}$ is the shaded area as shown in Fig. 1.2, a value of u is required such that

$1-\int_{-\infty}^{+u_1} = 0.25$, *i.e.*, $\int_{-\infty}^{+u_1} = 0.75$.

By inspection, we find in the Table 1.5 that the value 0.75 of the integral lies between the values 0.6 and 0.7 of u. The value of u is 0.6745 for $\int_{-\infty}^{+u_1} = 0.75$.

Now $$u = \frac{\mu - x}{\sigma} = 0.6745$$

therefore, deviation $$(u - x) = 0.6745\ \sigma$$
$$= \pm\ 0.6745 \times 3$$
$$= \pm\ 2.0''.$$

(b) For a deviation $(u - x)$ of ± 6″ for a single measure

$$u = \frac{\mu - x}{\sigma} = \frac{6}{3} = 2.0''.$$

For $u = +\ 2.0$ from Table 1.5, we have

$$\int_{-\infty}^{+2.0} = 0.9772.$$

Hence $$1-\int_{-\infty}^{+2.0} = 1 - 0.9772 = 0.0228.$$

For the deviation to lie at the limits of, or outside, the range + 6″ to – 6″, the probability is

= 2 × 0.0228

= **0.0456**, or **4.6%**.

(c) The standard deviation of the mean of nine observations

$$\sigma_m = \frac{\sigma}{\sqrt{n}} = \frac{3.0}{\sqrt{9}} = \pm 1.0''.$$

For a deviation of $\pm 1.5''$

$$u = \frac{\mu - x}{\sigma_m} = \frac{15''}{1.0}$$

$$= 1.5''.$$

For $u = +1.5$ from the Table 1.5, we get

$$1 - \int_{-\infty}^{+1.5} = 1 - 0.9332 = 0.0668.$$

Therefore the probability of assuming a deviation of $\pm 1.5''$

= 2 × 0.0668

= 0.1336, or 13.4%.

Example 1.5. The coordinates with standard deviations of two stations *A* and *B* were determined as given below. Calculate the length and standard deviation of *AB*.

Station	Easting	Northing
A	456.961 m ± 20 mm	573.237 m ± 30 mm
B	724.616 m ± 40 mm	702.443 m ± 50 mm

The length of *AB* was independently measured as 297.426 m ± 70 mm and its separate determination by EDM is as 297.155 m ± 15 mm. Calculate the most probable length of the line and its standard deviation.

Solution:

If ΔE is the difference in the eastings of *A* and *B* and ΔN is the difference in the northings then the length of the line *AB*

$$= \sqrt{\Delta E^2 + \Delta N^2}$$

$$= \sqrt{(724.616 - 456.961)^2 + (702.443 - 573.237)^2}$$

$$= \sqrt{267.655^2 + 129.206^2}$$

$$= 297.209 \text{ m}.$$

From Eq. (1.21) the standard deviation σ_E and σ_N of Δ_E and Δ_N, respectively, are

$$\sigma_E^2 = \sigma_{EA}^2 + \sigma_{EB}^2$$

$$\sigma_N^2 = \sigma_{NA}^2 + \sigma_{NB}^2$$

$\sigma_{EA} = \pm\, 20$ mm; $\sigma_{EB} = \pm\, 40$ mm; $\sigma_{NA} = \pm\, 30$ mm; $\sigma_{NB} = \pm\, 50$ mm.

Therefore,

$$\sigma_E^2 = 20^2 + 40^2, \quad \text{or} \quad \sigma_E = \pm\, 44.7 \text{ mm}$$

$$\sigma_N^2 = 30^2 + 50^2, \quad \text{or} \quad \sigma_N = \pm\, 58.3 \text{ mm}$$

Now from Eq. (1.20), the standard deviation of the computed length *AB*

$$\sigma_{AB}^2 = \left(\frac{\partial L}{\partial(\Delta E)}\sigma_E\right)^2 + \left(\frac{\partial L}{\partial(\Delta L)}\sigma_N\right)^2 \quad \text{...(a)}$$

where $$L = \sqrt{\Delta E^2 + \Delta N^2} \quad \text{...(b)}$$

Now by differentiating Eq. (b), we get

$$\frac{\partial L}{\partial(\Delta E)} = \frac{1}{2} \times 2 \times \Delta E \left(\Delta E^2 + \Delta N^2\right)^{-1/2}$$

$$= \frac{\Delta E}{\sqrt{\Delta E^2 + \Delta N^2}} = \frac{\Delta E}{L} = \frac{267.655}{297.209} = 0.901$$

Similarly,

$$\frac{\partial L}{\partial(\Delta E)} = \frac{\Delta N}{L} = \frac{129.206}{297.209} = 0.435$$

Hence from Eq. (a), we get

$$\sigma_{AB}^2 = (0.901 \times 44.7)^2 + (0.435 \times 58.3)^2 = 2265.206$$

or $$\sigma_{AB} = \pm\, 47.6 \text{ mm.}$$

Now we have three values of the length *AB* and their standard deviations as given in Table 1.6.

Table 1.6

Length (*l*) by (m)	σ (mm)	ω = 1/σ²
Tape	297.426 ± 70	1/4900
EDM	297.155 ± 15	1/225
Calculation	297.209 ± 47.6	1/2266

Since the weight of a measured quantity is inversely proportional to its variance, we can calculate the weights of the lengths obtained by different methods, and these have been given in Table 1.6.

The most probable length of AB is the weighted mean of the three values of AB. Thus

$$L = \frac{\frac{1}{4900}\times 297.426+\frac{1}{225}\times 297.155+\frac{1}{2266}\times 297.209}{\frac{1}{4900}+\frac{1}{225}+\frac{1}{2266}}$$

$$= \mathbf{297.171\ m}.$$

The weight of L is

$$\Sigma\omega = \frac{1}{4900}+\frac{1}{225}+\frac{1}{2266} = 0.00509$$

Since $$\omega = 1/\sigma^2$$

$$\omega_L = \frac{1}{\sigma_L^2}$$

$$\sigma_L = \sqrt{\frac{1}{\omega_L}} = \sqrt{\frac{1}{0.00509}}$$

$$= \pm\ \mathbf{14.0\ mm}.$$

The standard deviation of the length 297.171 m is ± 14.0 mm.

Example 1.6. A base line AB was measured accurately using a subtense bar 1 m long. From a point C near the centre of the base, the lengths AC and CB were measured as 9.375 m and 9.493 m, respectively. If the standard error in the angular mesurement was ± 1″, determine the error in the length of the line.

Solution:

In Fig. 1.3, the subtense bar PQ is at C and the angles α and β were measured at A and B, respectively.

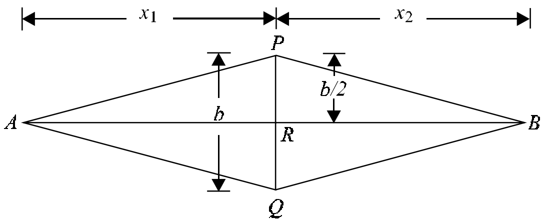


Fig. 1.3

It is given that

$$PQ = b = 1 \text{ m}$$
$$AC = x_1 = 9.375 \text{ m}$$
$$CB = x_2 = 9.493 \text{ m}$$

For subtense bar measurements, we have

$$x = \frac{b}{2\tan\frac{\theta}{2}} \qquad \text{...(a)}$$

where x = the computed distance, and

θ = the angle subtended at the station by the subtense bar.

When θ is small, Eq. (a) can be written as

$$x = \frac{b}{\theta}, \quad \text{or} \quad \theta = \frac{b}{x}$$

Therefore $$dx = -\frac{b\,d\theta}{\theta^2} = -\frac{b\,d\theta}{\left(\frac{b}{x}\right)^2} = -\frac{x^2\,d\theta}{b} \quad (d\theta \text{ in radians})$$

Writing $\sigma_{AB}, \sigma_{AC}, \sigma_{CB}, \sigma_{\alpha}$, and σ_{β} as the respective standard errors, we have

$$\sigma_{AC} = -\frac{x_1^2}{b}\sigma_{\alpha} = \frac{9.375^2}{1} \times \frac{1}{206265} = \pm 0.000426 \text{ m}$$

$$\sigma_{CB} = -\frac{x_2^2}{b}\sigma_{\beta} = \frac{9.493^2}{1} \times \frac{1}{206265} = \pm 0.000437 \text{ m}$$

$$\sigma_{AB}^2 = \sigma_{AC}^2 + \sigma_{CB}^2$$

$$\sigma_{AB} = \pm\sqrt{0.000426^2 + 0.000437^2}$$

$$= \pm \mathbf{0.61 \ mm}.$$

The ratio of the standard error to the measured length (AB = 9.375 + 9.493 = 18.868 m) is given as

$$= \frac{0.00061}{18.868} = 1 \text{ in } 30931$$

Example 1.7. The sides of a rectangular tract were measured as 82.397 m and 66.132 m with a 30 m metallic tape too short by 25 mm. Calculate the error in the area of the tract.

Solution: Let the two sides of the tract be x_1 and x_2 then the area

$$y = x_1 . x_2 \quad \text{...(a)}$$

If the errors in x_1 and x_2 are dx_1 and dx_2, respectively, then the error in y

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 \quad \text{...(b)}$$

Now from Eq. (a), we get

$$\frac{\partial y}{\partial x_1} = x_2 = 66.132 \text{ m}$$

$$\frac{\partial y}{\partial x_2} = x_1 = 82.397 \text{ m}.$$

The values of dx_1 and dx_2 are computed as

$$dx_1 = \frac{0.025}{30} \times 82.397 = 0.069 \text{ m}$$

$$dx_2 = \frac{0.025}{30} \times 66.132 = 0.055 \text{ m.}$$

Therefore from Eq. (b), we get

$$dy = 66.132 \times 0.069 + 82.397 \times 0.055 = \mathbf{9.095\ m^2}.$$

The percentage of error

$$= \frac{9.095}{82.397 \times 66.132} \times 100 = 0.17\ \%.$$

Example 1.8. Two sides and the included angle of a triangle were measured as under:

a = 757.64 ± 0.045 m

b = 946.70 ± 0.055 m

C = 54°18' ± 25"

Compute the area of the triangle and its standard error.

Solution:

(*a*) Area of a triangle $A = \frac{1}{2} ab \sin C$ ...(a)

$$= \frac{1}{2} \times 757.64 \times 946.70 \times \sin 54°18' = \mathbf{291236.62\ m^2.}$$

(*b*) Standard error in *A*

$$\sigma_A^2 = \left(\frac{\partial A}{\partial a}\sigma_a\right)^2 + \left(\frac{\partial A}{\partial b}\sigma_b\right)^2 + \left(\frac{\partial A}{\partial c}\sigma_c\right)^2 \quad ...(b)$$

Differentiating Eq. (a), we get

$$\frac{\partial A}{\partial a} = \frac{1}{2} b \sin C = \frac{1}{2} \times 946.70 \times \sin 54°18' = 384.400$$

$$\frac{\partial A}{\partial b} = \frac{1}{2} b \sin C = \frac{1}{2} \times 747.64 \times \sin 54°18' = 307.633$$

$$\frac{\partial A}{\partial C} = -\frac{1}{2} ab \cos C = -\frac{1}{2} \times 946.70 \times 757.64 \times \sin 54°18' = 209274.739$$

Now from Eq. (b), we get

$$\sigma_A = \pm\sqrt{(384.400 \times 0.045)^2 + (307.633 \times 0.055)^2 + \left(209274.739 \times \frac{25}{206265}\right)^2}$$

$$= \pm\ \mathbf{35.05\ m^2}.$$

OBJECTIVE TYPE QUESTIONS

1. Accuracy is a term which indicates the degree of conformity of a measurement to its

(*a*) most probable value. (*b*) mean value.

(*c*) true value. (*d*) standard error.

2. Precision is a term which indicates the degree of conformity of

(*a*) measured value to its true value.

(*b*) measured value to its mean value.

(*c*) measured value to its weighted mean value.

(*d*) repeated measurements of the same quantity to each other.

3. Theory of probability is applied to

(*a*) gross errors. (*b*) systematic errors.

(*c*) random errors. (*d*) all the above.

4. Residual of a measured quantity is the

(*a*) difference of the observed value from its most probable value.

(*b*) value obtained by adding the most probable value to its true value.

(*c*) remainder of the division of the true value by its most probable value.

(*d*) product of the most probable value and the observed value.

5. If the standard deviation of a quantity is ± 1″, the maximum error would be

(*a*) 2.39″. (*b*) 3.29″.

(*c*) 2.93″. (*d*) 9.23″.

6. If the standard deviation of an observation is ± 10 m, the most probable error would be

(*a*) 6.745 m. (*b*) 20 m.

(*c*) 10 m. (*d*) 0.6745 m.

7. The systematic errors

(*a*) are always positive. (*b*) are always negative.

(*c*) may be positive or negative. (*d*) have same sign as the gross errors.

8. Variance of a quantity is an indicator of

(*a*) precision. (*b*) accuracy.

(*c*) randomness. (*d*) regular nature.

9. In the case of a function $y = f(x_1, x_2)$, the error in y is computed as

(*a*) $$dy = \left(\frac{\partial f}{\partial x_1}\right) dx_1 + \left(\frac{\partial f}{\partial x_2}\right) dx_2$$

(*b*) $$dy = \left(\frac{\partial f}{\partial x_1}\right)^2 dx_1 + \left(\frac{\partial f}{\partial x_2}\right)^2 dx_2$$

(*c*) $dy = \left(\frac{\partial f}{\partial x_1}\right)(dx_1)^2 + \left(\frac{\partial f}{\partial x_2}\right)(dx_2)^2$

(*d*) $dy = \left(\frac{\partial f}{\partial x_1} dx_1\right)^2 + \left(\frac{\partial f}{\partial x_2} dx_2\right)^2$

10. The adjusted value of an observed quantity may contain

(*a*) small gross errors. (*b*) small systematic errors.

(*c*) small random errors. (*d*) all the above.

11. One of the characteristics of random errors is that

(*a*) small errors occur as frequently as the large errors.

(*b*) plus errors occur more frequently than the negative errors.

(*c*) small errors occur more frequently than the large errors.

(*d*) large errors may occur more frequently.

12. If the standard error of each tape length used to measure a length is ± 0.01 m. the standard error in 4 tape lengths will be

(*a*) 0.01 m. (*b*) 0.02 m.

(*c*) 0.04 m. (*d*) 0.16 m.

ANSWERS

1. (*c*)	**2.** (*d*)	**3.** (*c*)	**4.** (*a*)	**5.** (*b*)	**6.** (*a*)
7. (*c*)	**8.** (*a*)	**9.** (*a*)	**10.** (*c*)	**11.** (*c*)	**12.** (*b*).

2 DISTANCE MEASUREMENT

Three methods of distance measurement are briefly discussed in this chapter. They are

Direct method using a tape or wire

Tacheometric method or optical method

EDM (Electromagnetic Distance Measuring equipment) method.

2.1 DIRECT METHOD USING A TAPE

In this method, steel tapes or wires are used to measure distance very accurately. Nowadays, EDM is being used exclusively for accurate measurements but the steel tape still is of value for measuring limited lengths for setting out purposes.

Tape measurements require certain corrections to be applied to the measured distance depending upon the conditions under which the measurements have been made. These corrections are discussed below.

Correction for Absolute Length

Due to manufacturing defects the *absolute length* of the tape may be different from its *designated* or *nominal length*. Also with use the tape may stretch causing change in the length and it is imperative that the tape is regularly checked under standard conditions to determine its absolute length. The correction for absolute length or *standardization* is given by

$$c_a = \frac{c}{l} L \qquad ...(2.1)$$

where

c = the correction per tape length,

l = the designated or nominal length of the tape, and

L = the measured length of the line.

If the absolute length is more than the nominal length the sign of the correction is positive and *vice versa*.

Correction for Temperature

If the tape is used at a field temperature different from the standardization temperature then the temperature correction to the measured length is

$$c_t = \alpha (t_m - t_0) L \qquad ...(2.2)$$

where

α = the coefficient of thermal expansion of the tape material,

t_m = the mean field temperature, and

t_0 = the standardization temperature.

The sign of the correction takes the sign of $(t_m - t_0)$.

Correction for Pull or Tension

If the pull applied to the tape in the field is different from the standardization pull, the pull correction is to be applied to the measured length. This correction is

$$c_p = \frac{(P - P_0)}{AE} L \qquad \text{...(2.3)}$$

where

P = the pull applied during the measurement,

P_0 = the standardization pull,

A = the area of cross-section of the tape, and

E = the Young's modulus for the tape material.

The sign of the correction is same as that of $(P - P_0)$.

Correction for Sag

For very accurate measurements the tape can be allowed to hang in catenary between two supports (Fig. 2.1a). In the case of long tape, intermediate supports as shown in Fig. 2.1b, can be used to reduce the magnitude of correction.

(a) (b)

Fig. 2.1

The tape hanging between two supports, free of ground, sags under its own weight, with maximum dip occurring at the middle of the tape. This necessitates a correction for sag if the tape has been standardized on the flat, to reduce the curved length to the chord length. The correction for the sag is

$$c_g = \frac{1}{24}\left(\frac{W}{P}\right)^2 L \qquad \text{...(2.4)}$$

where

W = the weight of the tape per span length.

The sign of this correction is always negative.

If both the ends of the tape are not at the same level, a further correction due to slope is required. It is given by

$$c_g' = c_g \cos \alpha \qquad ...(2.5)$$

where

α = the angle of slope between the end supports.

Correction for Slope

If the length L is measured on the slope as shown in Fig. 2.2, it must be reduced to its horizontal equivalent $L \cos \theta$. The required slope correction is

$$c_s = (1 - \cos \theta) L \quad (exact) \qquad ...(2.6)$$

$$= \frac{h^2}{2L} \quad (approximate) \qquad ...(2.7)$$

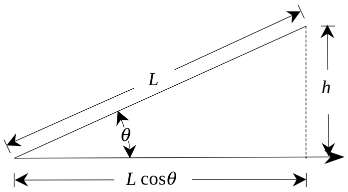


Fig. 2.2

where

θ = the angle of the slope, and

h = the difference in elevation of the ends of the tape.

The sign of this correction is always negative.

Correction for Alignment

If the intermediate points are not in correct alignment with ends of the line, a correction for alignment given below, is applied to the measured length (Fig. 2.3).

$$c_m = \frac{d^2}{2L} \quad (approximate) \qquad ...(2.8)$$

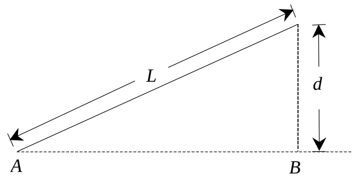


Fig. 2.3

where

d = the distance by which the other end of the tape is out of alignment.

The correction for alignment is always negative.

Reduction to Mean Sea Level (M.S.L.)

In the case of long lines in triangulation surveys the relationship between the length AB measured on the ground and the equivalent length $A'B'$ at mean sea level has to be considered (Fig. 2.4). Determination of the equivalent mean sea level length of the measured length is known as reduction to mean sea level.

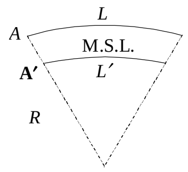


Fig. 2.4

The reduced length at mean sea level is given by

$$L' = \frac{R}{(R+H)} L \qquad ...(2.9)$$

where

R = the mean earth's radius (6372 km), and

H = the average elevation of the line.

When H is considered small compared to R, the correction to L is given as

$$c_{msl} = \frac{HL}{R} \text{ (approximate)} \quad \text{...(2.10)}$$

The sign of the correction is always negative.

The various tape corrections discussed above, are summarized in Table 2.1.

2.2 ERROR IN PULL CORRECTION DUE TO ERROR IN PULL

If the nominal applied pull is in error the required correction for pull will be in error. Let the error in the nominal applied pull P be $\pm\ \delta P$ then the

$$\text{actual pull correction} = \frac{(P \pm \delta P - P_0)}{AE} L \quad \text{...(2.11)}$$

and

$$\text{nominal pull correction} = \frac{(P - P_0)}{AE} L \quad \text{...(2.12)}$$

Therefore error = actual pull correction – nominal pull correction

$$= \frac{(P \pm \delta P - P_0)}{AE} L - \frac{(P - P_0)}{AE} L$$

$$= \pm \frac{L}{AE} \delta P \quad \text{...(2.13)}$$

From Eq. (2.12), we have

$$\frac{L}{AE} = \frac{\text{nominal pull correction}}{P - P_0}$$

Therefore from Eq. (2.13), we get

$$\text{Error in pull correction} = \pm \frac{\text{nominal pull correction}}{P - P_0} \delta P \quad \text{...(2.14)}$$

From Eq. (2.14), we find that an increase in pull increases the pull correction.

2.3 ERROR IN SAG CORRECTION DUE TO ERROR IN PULL

If the applied pull is in error the computed sag correction will be in error. Let the error in pull be $\pm\ \delta P$ then

$$\text{the actual sag correction} = -\frac{1}{24}\left[\frac{W}{P \pm \delta P}\right]^2 L$$

$$= -\frac{1}{24}\left(\frac{W}{P}\right)^2 L\left(1 \pm \frac{\delta P}{P}\right)^{-2}$$

and nominal sag correction $= -\frac{1}{24}\left(\frac{W}{P}\right)^2 L$

Therefore error $= -\frac{1}{24}\left(\frac{W}{P}\right)^2 L\left[\left(1 \pm \frac{\delta P}{P}\right)^{-2} - 1\right]$

$$= -\frac{1}{24}\left(\frac{W}{P}\right)^2 L\left(\mp 2\frac{\delta P}{P}\right)$$ neglecting the terms of higher power.

$$= \mp \text{ nominal sag correction } \left(\frac{2\delta P}{P}\right) \qquad ...(2.15)$$

Eq. (2.15) shows that an increase in pull correction reduces the sag correction.

2.4 ELONGATION OF A STEEL TAPE WHEN USED FOR MEASUREMENTS IN A VERTICAL SHAFT

Elongation in a steel tape takes place when transferring the level in a tunnel through a vertical shaft. This is required to establish a temporary bench mark so that the construction can be carried

Table 2.1

Correction	Sign	Formula
Absolute length (c_a)	±	$\frac{c}{l}L$
Temperature (c_t)	±	$\alpha(t_m - t_0)L$
Pull (c_p)	±	$\frac{(P - P_0)}{AE}L$
Sag (c_g)	–	$\frac{1}{24}\left(\frac{W}{P}\right)^2 L$
Slope (c_s)	–	$(1 - \cos\theta)L$ (*exact*)
Alignment (c_m)	–	$\frac{h^2}{2L}$ (*approximate*)
Mean sea level (c_{msl})	–	$\frac{d^2}{2L}$ (*approximate*)
		$\frac{HL}{R}$ (*approximate*)

out to correct level as well as to correct line. Levels are carried down from a known datum, may be at the side of the excavated shaft at top, using a very long tape hanging vertically and free of restrictions to carry out operation in a single stage. In the case when a very long tape is not available, the operation is carried out by marking the separate tape lengths in descending order.

The elongation in the length of the tape AC hanging vertically from a fixed point A due to its own weight as shown in Fig. 2.5, can be determined as below.

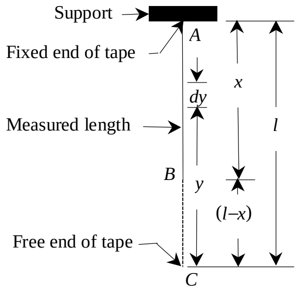


Fig. 2.5

Let s = the elongation of the tape,

g = the acceleration due to gravity,

x = the length of the suspended tape used for the measurement,

$(l - x)$ = the additional length of the tape not required in the measurements,

A = the area of cross-section of the tape,

E = the modulus of elasticity of the tape material,

m = the mass of the tape per unit length,

M = the attached mass,

l = the total length of the tape, and

P_0 = the standard pull.

The tension sustained by the vertical tape due to self-loading is maximum at A. The tension varies with y considered from free-end of the tape, i.e., it is maximum when y is maximum and, therefore, the elongations induced in the small element of length dy, are greater in magnitude in the upper regions of the tape than in the lower regions.

Considering an element dy at y,

loading on the element $dy = mgy$

and extension over the length $dy = mgy\dfrac{dy}{AE}$

Therefore, extension over length AB, $E_x = \int_{(l-x)}^{l} mgy\dfrac{dy}{AE}$

$$= \left[\frac{mg}{AE}\frac{y^2}{2}\right]_{(l-x)}^{l} + \text{constant}$$

We have $E_x = 0$ when $y = 0$, therefore the constant = 0. Thus

$$E_x = \frac{mg}{2AE}\left[l^2 - (l-x)^2\right] = \frac{mgx}{AE}\left[\frac{2l - x}{2}\right] \qquad \ldots(2.16)$$

To ensure verticality of the tape and to minimize the oscillation, a mass M may be attached to the lower end A. It will have a uniform effect over the tape in the elongation of the tape.

Additional extension due to mass M over length x

$$= Mg \frac{x}{AE}$$

If the standard pull is P_0, it should be allowed in the same way as the standard pull in the pull correction.

Therefore elongation over length x becomes

$$E_x = \frac{mgx}{AE}\left[\frac{2l - x}{2}\right] + \frac{Mgx}{AE} - \frac{P_0 x}{AE}$$

$$= \frac{gx}{AE}\left[\frac{m}{2}(2l - x) + M - \frac{P_0}{g}\right] \quad \ldots(2.17)$$

2.5 TACHEOMETRIC OR OPTICAL METHOD

In *stadia tacheometry* the line of sight of the tacheometer may be kept horizontal or inclined depending upon the field conditions. In the case of horizontal line of sight (Fig. 2.6), the horizontal distance between the instrument at A and the staff at B is

$$D = ks + c \quad \ldots(2.18)$$

where

k and c = the multiplying and additive constants of the tacheometer, and

s = the staff intercept,

= $S_T - S_B$, where S_T and S_B are the top hair and bottom hair readings, respectively.

Generally, the value of k and c are kept equal to 100 and 0 (zero), respectively, for making the computations simpler. Thus

$$D = 100\ s \quad \ldots(2.19)$$

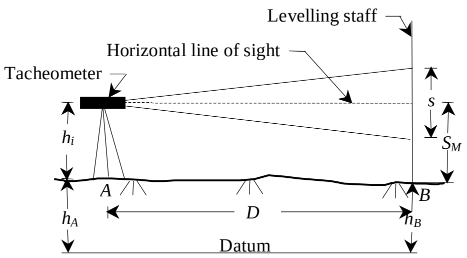


Fig. 2.6

The elevations of the points, in this case, are obtained by determining the height of instrument and taking the middle hair reading. Let

h_i = the height of the instrument axis above the ground at A,

h_A, h_B = the elevations of A and B, and

S_M = the middle hair reading

then the height of instrument is

$$\text{H.I.} = h_A + h_i$$

and $$h_B = \text{H.I.} - S_M$$
$$= h_A + h_i - S_M \quad \text{...(2.20)}$$

In the case of inclined line of sight as shown in Fig. 2.7, the vertical angle α is measured, and the horizontal and vertical distances, D and V, respectively, are determined from the following expressions.

$$D = ks\cos^2\alpha \quad \text{...(2.21)}$$

$$V = \frac{1}{2}ks\sin 2\alpha \quad \text{...(2.22)}$$

The elevation of B is computed as below.

$$h_B = h_A + h_i + V - S_M \quad \text{...(2.23)}$$

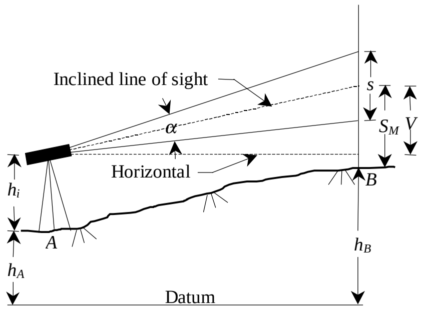


Fig. 2.7

2.6 SUBTENSE TACHEOMETRY

In *subtense tacheometry* the distance is determined by measuring the horizontal angle subtended by the subtense bar targets (Fig. 2.8a) and for heighting, a vertical angle is also measured (Fig. 2.8b).

Let b = the length of the subtense bar PQ,

θ = the horizontal angle subtended by the subtense bar targets P and Q at the station A, and

α = the vertical angle of R at O

then $$D = \frac{b}{2\tan\frac{\theta}{2}} \approx \frac{b}{\theta} \text{ (when } \theta \text{ is small)} \quad \text{...(2.24)}$$

$$V = D\tan\alpha \quad \text{...(2.25)}$$

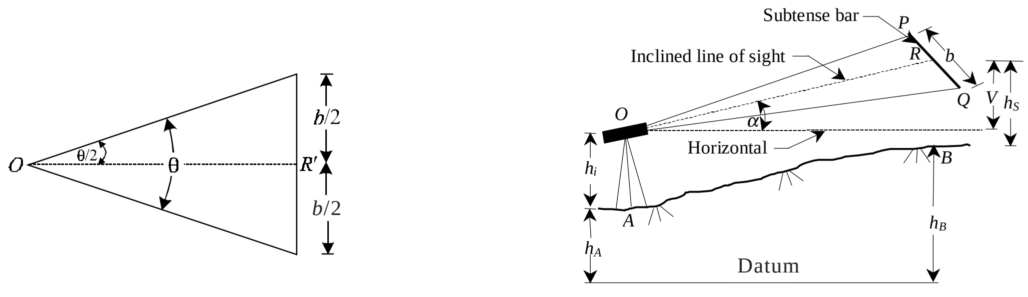
and $$h_B = h_A + h_i + V - h_s \quad \text{...(2.26)}$$

where h_s = the height of the subtense bar above the ground.

When a vertical bar with two targets is used vertical angles are required to be measured and the method is termed as *tangential system*.

2.7 EFFECT OF STAFF VERTICALITY

In Fig. 2.9, the staff is inclined through angle δ towards the instrument. The staff intercept for the inclined staff would be PQ rather than the desired value MN for the vertical staff.



(a) (b)

Fig. 2.8

Draw two lines *ab* and *cd* perpendicular to the line of sight. Since *ab* and *cd* are very close to each other, it can be assumed that $ab = cd$. Moreover,

$$\angle PBM = \delta$$

$$\angle MEa = \alpha$$

$$\angle PFc = \alpha + \delta$$

From ΔMEa, we have

$$aE = ME \cos\alpha$$

or $$ab = MN \cos\alpha = cd \qquad \ldots(2.27)$$

From DPFc, we have

$$cF = PF \cos(\alpha + \delta)$$

or $$cd = PQ \cos(\alpha + \delta) \qquad \ldots(2.28)$$

Equating the values of *cd* from Eqs. (2.27) and (2.28), we get

$$MN \cos\alpha = PQ \cos(\alpha + \delta)$$

or $$MN = \frac{PQ \cos(\alpha + \delta)}{\cos\alpha} \qquad \ldots(2.29)$$

The Eq. (2.29) holds for the case when the staff is inclined away from the instrument for angle of elevation (α).

In Fig. 2.10, the staff is inclined away from the instrument. In this case

$$\angle PFc = \alpha - \delta$$

Therefore, $$MN = \frac{PQ \cos(\alpha - \delta)}{\cos\alpha} \qquad \ldots(2.30)$$

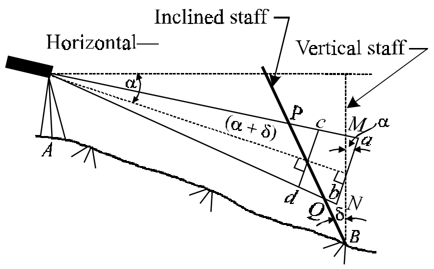


Fig. 2.9

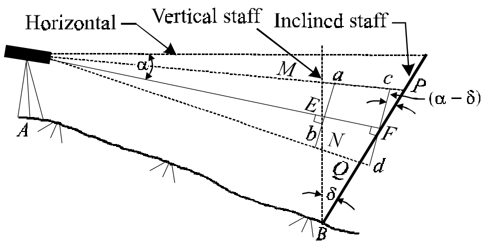


Fig. 2.10

The Eq. (2.30) holds for the case the staff is inclined towards the instrument for the angle of elevation (α).

2.8 EFFECT OF ERROR IN MEASUREMENT OF HORIZONTAL ANGLE IN SUBTENSE TACHEOMETRRY

From Eq. (2.24), the horizontal distance *RO* (Fig. 2.8a) is

$$D = \frac{b}{2\tan\frac{\theta}{2}} = \frac{b}{\theta} \quad \text{(when } \theta \text{ is small)} \qquad \ldots(2.31)$$

$$dD = -\frac{b}{\theta^2}\,d\theta$$

Substituting the value of θ from Eq. (2.31), we get

$$dD = -\frac{D^2}{b}\,d\theta \qquad \ldots(2.32)$$

The above expression gives the error in *D* for the given accuracy in θ. The negative sign shows that there is decrease in *D* for increase in θ.

The relative accuracy or fractional error in linear measurements is given by the following expression.

$$\frac{dD}{D} = -\frac{D}{b}\,d\theta \qquad \ldots(2.33)$$

2.9 EFFECT OF SUBTENSE BAR NOT BEING NORMAL TO THE LINE JOINING THE INSTRUMENT AND THE SUBTENSE BAR

Let the subtense bar *A′B′* be out from being normal to the line *OC* by an angle δ as shown in Fig. 2.11,
then

$$OC' = D' = A'C' \cot\frac{\theta}{2}$$

$$A'C' = A'C \cos\delta$$

$$D' = A'C \cos\delta \cot\frac{\theta}{2}$$

$$= \frac{b}{2}\cos\delta\cot\frac{\theta}{2} \qquad \ldots(2.34)$$

Therefore error in horizontal distance $D = D - D'$

$$= \frac{b}{2}\cot\frac{\theta}{2} - \frac{b}{2}\cos\delta\cot\frac{\theta}{2}$$

$$= \frac{b}{2}\cot\frac{\theta}{2}(1-\cos\delta) \qquad \ldots(2.35)$$

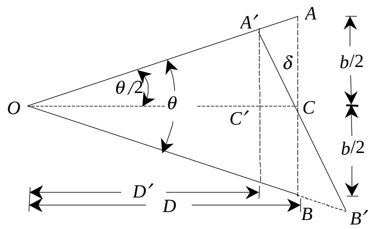


Fig. 2.11

2.10 EELECTROMAGNETIC DISTANCE MEASUREMENT (EDM)

The EDM equipments which are commonly used in land surveying are mainly *electronic* or *microwave systems* and *electro-optical instruments*. These operate on the principle that a transmitter at the master station sends modulated continuous carrier wave to a receiver at the remote station from which it is returned (Fig. 2.12). The instruments measure slope distance D between transmitter and receiver. It is done by modulating the continuous carrier wave at different frequencies and then measuring the phase difference at the master between the outgoing and incoming signals. This introduces an element of double distance is introduced. The expression for the distance D traversed by the wave is

$$2D = n\lambda + \frac{\phi}{2\pi}\lambda + k \qquad \ldots(2.36)$$

where

ϕ = the measured phase difference,

λ = the modulated wavelength,

n = the number of complete wavelength contained within the double distance (an unknown), and

k = a constant.

To evaluate n, different modulated frequencies are deployed and the phase difference of the various outgoing and measuring signals are compared.

If c_0 is the velocity of light in vacuum and f is the frequency, we have

$$\lambda = \frac{c_0}{nf} \qquad \ldots(2.37)$$

where n is the refractive index ratio of the medium through which the wave passes. Its value depends upon air temperature, atmospheric pressure, vapour pressure and relative humidity. The velocity of light c_0 in vacuum is taken as 3×10^8 m/s.

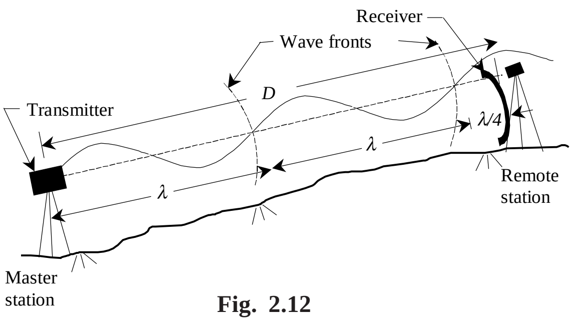


Fig. 2.12

The infrared based EDM equipments fall within the electro-optical group. Nowadays, most local survey and setting out for engineering works are being carried out using these EDM's. The infrared EDM has a passive reflector, using a retrodioptive prism to reflect the transmitted infrared wave to the master. The distances of 1-3 km can be measured with an accuracy of ± 5 mm. Many of these instruments have microprocessors to produce horizontal distance, difference in elevation, etc.

Over long ranges (up to 100 km with an accuracy of ± 50 mm) electronic or microwave instruments are generally used. The remote instrument needs an operator acting to the instructions from the master at the other end of the line. The signal is transmitted from the master station, received by the remote station and retransmitted to the master station.

Measurement of Distance from Phase Difference

The difference of the phase angle of the reflected signal and the phase angle of the transmitted signal is the *phase difference*. Thus, if ϕ_1 and ϕ_2 are the phase angles of the transmitted and reflected signals, respectively, then the phase angle difference is

$$\Delta\phi = \phi_2 - \phi_1 \qquad \ldots(2.38)$$

The phase difference is usually expressed as a fraction of the wavelength (λ). For example,

$\Delta\phi$	0°	90°	180°	270°	360°
Wavelength	0	$\lambda/4$	$\lambda/2$	$3\lambda/4$	λ

Fig. 2.13 shows a line *AB*. The wave is transmitted from the master at *A* towards the reflector at *B* and is reflected back by the reflector and received back by the master at *A*. From *A* to *B* the wave completes 2 cycles and 1/4 cycles. Thus if at *A* phase angle is 0° and at *B* it is 90° then

$$\Delta\phi = 90^\circ = \frac{\lambda}{4}$$

and the distance between *A* and *B* is

$$D = 2\lambda + \frac{\lambda}{4}$$

Again from *B* to *A*, the wave completes 2 cycles and 1/4 cycles. Thus if ϕ_1 is 90° at *B* and ϕ_2 is 180° at *A*, then

$$D\phi = 90^\circ = \frac{\lambda}{4}$$

and the distance between *A* and *B* is

$$D = 2\lambda + \frac{\lambda}{4}$$

The phase difference between the wave at *A* when transmitted and when received back is 180°, i.e., λ/2 and the number of complete cycles is 4. Thus

$$2D = 4\lambda + \frac{\lambda}{2}$$

$$D = \frac{1}{2}\left(4\lambda + \frac{\lambda}{2}\right) \quad \text{...(2.39)}$$

The above expression in a general form can be written as

$$D = \frac{1}{2}(n\lambda + \Delta\lambda)$$

where

n = the number of complete cycles of the wave in traveling from *A* to *B* and back from *B* to *A*, and

Δl = the fraction of wavelength traveled by the wave from *A* to *B* and back from *B* to *A*.

The value of Δλ depends upon the phase difference of the wave transmitted and that received back at the master. It is measured as phase angle (ϕ) at *A* by an electrical phase detector built in the master unit at *A*. Obviously,

$$\Delta\lambda = \left(\frac{\Delta\phi}{360^\circ}\right)\lambda$$

where

Δϕ = the phase difference

= $\phi_2 - \phi_1$

In Eq. (3.39), *n* is an unknown and thus the value of *D* cannot be determined. In EDM instruments the frequency can be increased in multiples of 10 and the phase difference for each frequency is determined separately. The distance is calculated by evaluating the values of *n* solving the following simultaneous equations for each frequency.

$$D = \frac{1}{2}(n_1\lambda_1 + \Delta\lambda_1) \quad \text{...(2.40)}$$

$$D = \frac{1}{2}(n_2\lambda_2 + \Delta\lambda_2) \quad \text{...(2.41)}$$

$$D = \frac{1}{2}(n_3\lambda_3 + \Delta\lambda_3) \quad \text{...(2.42)}$$

For more accurate results, three or more frequencies are used and the resulting equations are solved.

Let us take an example to explain the determination of n_1, n_2, n_3, etc. To measure a distance three frequencies f_1, f_2, and f_3 were used in the instrument and phase differences $\Delta\lambda_1$, $\Delta\lambda_2$, and $\Delta\lambda_3$ were measured. The f_2 frequency is $\frac{9}{10} f_1$ and the f_3 frequency is $\frac{99}{100} f_1$. The wavelength of f_1 is 10 m.

We know that $$\lambda \propto \frac{1}{f}$$

therefore, $$\frac{\lambda_1}{\lambda_2} = \frac{f_2}{f_1}$$

$$\lambda_2 = \frac{f_1}{f_2}\lambda_1 = \frac{f_1}{\frac{9}{10}f_1} \times 10$$

$$= \frac{100}{9} = 11.111 \text{ m.}$$

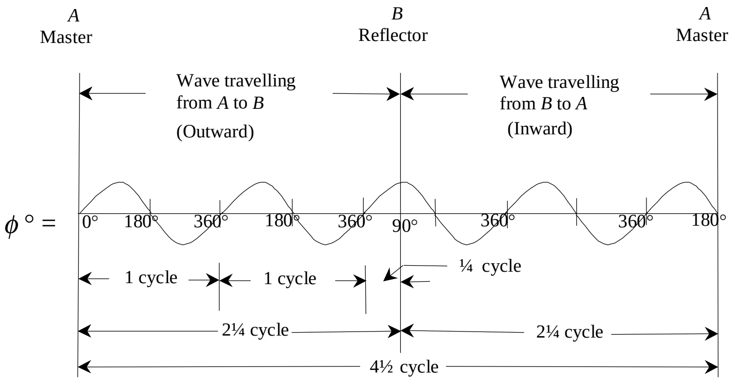


Fig. 2.13

Similarly, $$\lambda_3 = \frac{f_1}{\frac{99}{100}f_1} \times 10 = \frac{1000}{99} = 10.101 \text{ m.}$$

Let the wavelength of the frequency ($f_1 - f_2$) be λ' and that of ($f_1 - f_3$) be λ'', then

$$\lambda' = \frac{f_1\lambda_1}{(f_1 - f_2)} = \frac{f_1\lambda_1}{\frac{f_1}{10}} = 10\lambda_1 = 10 \times 10 = 100 \text{ m}$$

$$\lambda'' = \frac{f_1\lambda_1}{(f_1 - f_3)} = \frac{f_1\lambda_1}{\frac{f_1}{100}} = 100\lambda_1 = 100 \times 10 = 1000 \text{ m.}$$

Since one single wave of frequency ($f_1 - f_2$) has length of 100 m, λ_1 being 10 m and λ_2 being 11.111 m, the f_1 frequency wave has complete 10 wavelengths and the f_2 frequency wave has complete 9 wavelengths within a distance of 100 m.

To any point within the 100 m length, or stage, the phase of the ($f_1 - f_2$) frequency wave is equal to the difference in the phases of the other two waves. For example, at the 50 m point the phase of f_1 is (10/2) × 2π = 10π whilst that of f_2 is (9/2) × 2π = 9π, giving a difference of

$10\pi - 9\pi = \pi$, which is the phase of the $(f_1 - f_2)$ frequency. This relationship allows distance to be measured within 100 m. This statement applies as well when we consider a distance of 1000 m. Within distance of 500 m, the f_1 wave has phase of $(100/2) \times 2\pi = 100\pi$, the f_3 wave has $(99/2) \times 2\pi = 99\pi$, and the $(f_1 - f_3)$ wave has phase of $100\pi - 99\pi = \pi$. If in a similar manner further frequencies are applied, the measurement can be extended to a distance of 10,000 m, etc., without any ambiguity.

The term *fine* frequency can be assigned to f_1 which appear in all the frequency difference values, i.e. $(f_1 - f_2)$ whilst the other frequencies needed to make up the stages, or measurements of distance 100 m, 1000 m, etc., are termed as *coarse* frequencies. The f_1 phase difference measured at the master station covers the length for 0 m to 10 m. The electronics involved in modern EDM instruments automatically takes care of the whole procedure.

On inspection of Fig. 2.14, it will be seen that two important facts arise:

(a) When $\Delta\lambda_1 < \Delta\lambda_2$, $n_1 = n_2 + 1$ $(n_1 = 7,\ n_2 = 6)$

(b) When $\Delta\lambda_1 > \Delta\lambda_2$, $n_1 = n_2$ $(n_1 = 5,\ n_2 = 5)$

These facts are important when evaluating overall phase differences.

Now from Eqs. (2.40) and (2.41), we get

$$n_1\lambda_1 + \Delta\lambda_1 = n_2\lambda_2 + \Delta\lambda_2$$

$$n_1\lambda_1 + \Delta\lambda_1 = (n_1 - 1)\lambda_2 + \Delta\lambda_2 \qquad \text{...(2.43)}$$

From Eq. (3.43) the value of n_1 can be determined.

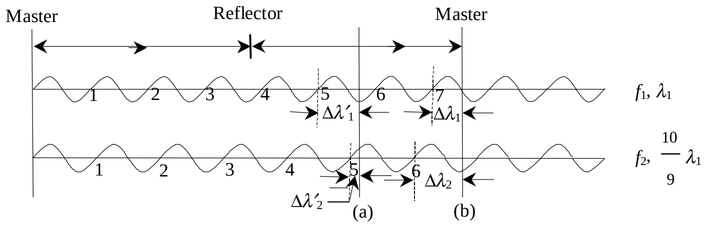


Fig 2.14

Effect of Atmospheric Conditions

All electromagnetic waves travel with the same velocity in a vacuum. The velocity of the waves is reduced when travelling through atmosphere due to retarding effect of atmosphere. Moreover, the velocity does not remain constant due to changes in the atmospheric conditions. The wavelength λ of a wave of frequency f has the following relationship with its velocity V.

$$\lambda = \frac{V}{f}$$

EDM instruments use electromagnetic waves, any change in V will affect λ and thus the measurement of the distance is also affected because the distance is measured in terms of wavelengths.

Refractive Index Ratio

The changes in velocity are determined from the changes in the *refractive index ratio* (n). The refractive index ratio is the ratio of the velocity of electromagnetic waves in vacuum to that in atmosphere. Thus

$$n = \frac{c_0}{V}$$

or
$$V = \frac{c_0}{n}$$

The value of n is equal to or greater than unity. The value depends upon air temperature, atmospheric pressure and the vapour pressure.

For the instruments using carrier waves of wavelength in or near visible range of electromagnetic spectrum, the value of n is given by

$$(n-1) = (n_0 - 1)\left(\frac{273}{T}\right)\left(\frac{p}{760}\right) \quad \ldots(2.44)$$

$$N = N_0\left(\frac{273}{T}\right)\left(\frac{p}{760}\right) \quad \ldots(2.45)$$

where

p = the atmospheric pressure in millimetre of mercury,

T = the absolute temperature in degrees Kelvin (T = 273° + t°C),

n_0 = the refractive index ratio of air at 0°C and 760 mm of mercury,

N = (n – 1) and,

N_0 = (n_0 – 1).

The value of n_0 is given by

$$(n_0 - 1) = 287.604 + \left(\frac{4.8864}{\lambda^2}\right) + \left(\frac{0.068}{\lambda^4}\right) \times 10^{-6} \quad \ldots(2.46)$$

where λ is the wavelength of the carrier wave in µm.

The instruments that use microwaves, the value of n for them is obtained from

$$(n-1)\times 10^6 = \frac{103.49}{T}(p-e) + \frac{86.26}{T}\left(1 + \frac{5748}{T}\right)e \quad \ldots(2.47)$$

where e is the water vapour pressure in millimetre of mercury.

Determination of Correct Distance

If the distance D' has not been measured under the standard conditions, it has to be corrected. The correct distance D is given by

$$D = D'\left(\frac{n_s}{n}\right) \quad \ldots(2.48)$$

where

n_s = the standardizing refractive index,

n = the refractive index at the time of measurement.

The values of n_s and n are obtained from Eq. (2.44) taking the appropriate values of p, T, n_0, and e.

Slope and Height Corrections

The measured lengths using EDM instruments are generally slope lengths. The following corrections are applied to get their horizontal equivalent and then the equivalent mean sea level length.

The correction for slope is given by Eqs. (2.6) and (2.7) and that for the mean sea level by Eq. (2.10). The sign of both the corrections is negative. Thus if the measured length is L', the correct length is

$$L = L' + c_g + c_{msl}$$

$$= L' - (1 - \cos\theta)L' - \frac{HL'}{R}$$

$$= \left(\cos\theta - \frac{H}{R}\right)L' \qquad \ldots(2.49)$$

where

θ = the slope angle of the line,

H = the average elevation of the line, and

R = the mean radius of the earth (≈ 6370 km).

2.11 ACCURACY IN VERTICAL ANGLE MEASUREMENTS

The accuracy, with which a vertical angle must be measured in order to reduce a slope distance to the corresponding horizontal distance, can be determined. Accuracy in distances can be expressed as absolute or relative. If a distance of 10,000 m is measured with an accuracy of 1 m then the absolute accuracy with which the distance has been measured is 1 m. For this case, the relative accuracy is 1 m/10,000 m or 1/10,000 or 1:10,000. The relative accuracy is preferred as it does not involve the length of lines.

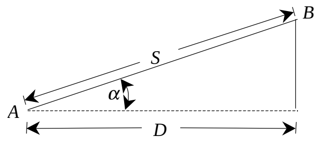


Fig 2.15

In Fig 2.15, let the slope distance AB be S and the corresponding horizontal distance be D. If α is the slope angle, we can write

$$D = S \cos \alpha \qquad \ldots(2.50)$$

Differentiating Eq. (2.50), we get

$$dD = -S \cos \alpha \, d\alpha \qquad \ldots(2.51)$$

Dividing Eq. (2.51) by Eq. (2.50) and disregarding the negative sign, the relative accuracy is given as

$$\frac{dD}{D} = \frac{S \sin\alpha \, d\alpha}{S \cos\alpha}$$

$$= \tan\alpha \, d\alpha \qquad \ldots(2.52)$$

where $d\alpha$ is *a* the desired accuracy in measurement of the slope angle α for an accuracy of $\frac{dD}{D}$ in linear measurements.

Example 2.1. A line AB between the stations *A* and *B* was measured as 348.28 using a 20 m tape, too short by 0.05 m. Determine

the correct length of *AB*,

the reduced horizontal length of *AB* if *AB* lay on a slope of 1 in 25, and

the reading required to produce a horizontal distance of 22.86 m between two pegs, one being 0.56 m above the other.

Solution:

(a) Since the tape is too short by 0.05 m, actual length of *AB* will be less than the measured length. The correction required to the measured length is

$$c_a = \frac{c}{l} L$$

It is given that

$$c = 0.05 \text{ m}$$

$$l = 20 \text{ m}$$

$$L = 348.28 \text{ m}$$

$$c_a = \frac{0.05}{20} \times 348.28 = 0.87 \text{ m}$$

The correct length of the line

$$= 348.28 - 0.87 = \mathbf{347.41 \ m}$$

(b) A slope of 1 in 25 implies that there is a rise of 1 m for every 25 m horizontal distance. If the angle of slope is α (Fig. 2.16) then

$$\tan\alpha = \frac{1}{25}$$

$$\alpha = \tan^{-1}\frac{1}{25} = 2°17'26''$$

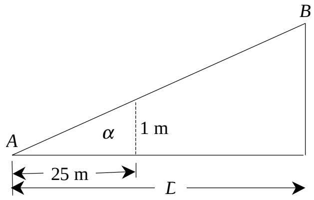


Fig. 2.16

Thus the horizontal equivalent of the corrected slope length 347.41 m is

$$D = AB \cos \alpha$$

$$= 347.41 \times \cos(2°17'26'') = \mathbf{347.13\ m.}$$

Alternatively, for small angles, $\alpha = \dfrac{1}{25}$ radians = $2°17'31''$, which gives the same value of D as above.

From Fig. 2.17, we have

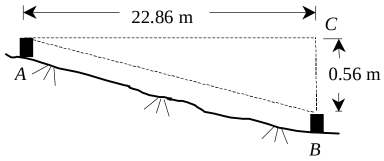


Fig. 2.17

$$AB = \sqrt{AC^2 + CB^2}$$

$$= \sqrt{22.86^2 + 0.56^2} = 22.87\ \text{m}$$

Therefore the reading required

$$= 22.87 + \frac{0.05}{20.0} \times 22.87 = \mathbf{22.93\ m}$$

Example 2.2. A tape of standard length 20 m at 85°F was used to measure a base line. The measured distance was 882.50 m. The following being the slopes for the various segments of the line:

Segment length (m)	Slope
100	2°20′
150	4°12′
50	1°06′
200	7°48′
300	3°00′
82.5	5°10′

Calculate the true length of the line if the mean temperature during measurement was 63°F and the coefficient of thermal expansion of the tape material is 6.5 × 10^{-6} per °F.

Solution:

Correction for temperature

$$c_t = \alpha(t_m - t_0)L$$

$$= 6.5 \times 10^{-6} \times (63 - 85) \times 882.50 = -\ 0.126\ \text{m}$$

Correction for slope

$$c_s = \Sigma[(1 - \cos\alpha)L]$$

$$= (1 - \cos 2°20') \times 100 + (1 - \cos 4°12') \times 150 + (1 - \cos 1°06') \times 50 + (1 - \cos 7°48') \times 200 + (1 - \cos 3°00') \times 300 + (1 - \cos 5°10') \times 82.5$$

$$= -3.092\ \text{m}$$

Total correction = $c_t + c_s$ = –0.126 + (– 3.092) = – 3.218 m

Correct length = 882.50 – 3.218 = **879.282 m**

Example 2.3. A base line was measured by tape suspended in catenary under a pull of 145 N, the mean temperature being 14°C. The lengths of various segments of the tape and the difference in level of the two ends of a segment are given in Table 2.2.

Table 2.2

Bay/Span	Length (m)	Difference in level (m)
1	29.988	+ 0.346
2	29.895	– 0.214
3	29.838	+ 0.309
4	29.910	– 0.106

If the tape was standardized on the flat under a pull of 95 N at 18°C determine the correct length of the line. Take

Cross-sectional area of the tape = 3.35 mm^2

Mass of the tape = 0.025 kg/m

Coefficient of linear expansion = 0.9×10^{-6} per °C

Young's modulus = 14.8×10^4 MN/m^2

Mean height of the line above M.S.L. = 51.76 m

Radius of earth = 6370 km

Solution:

It is given that

$$P_0 = 95 \text{ N}, \; P = 145 \text{ N}$$

$$t_0 = 18°\text{C}, \; t_m = 14°\text{C}$$

$$A = 3.35 \text{ mm}^2, \; \alpha = 0.9 \times 10^{-6} \text{ per °C}$$

$$w = mg = 0.025 \times 9.81 \text{ kg/m}$$

$$E = 14.8 \times 10^4 \text{ MN/m}^2 = \frac{14.8 \times 10^4 \times 10^6}{10^6} \text{ N/mm}^2 = 14.8 \times 10^4 \text{ N/mm}^2$$

H = 51.76 m, R = 6370 km

Total length of the tape L = 29.988 + 29.895 + 29.838 + 29.910 = 119.631 m

Temperature correction

$$c_t = \alpha(t_m - t_0)L$$

$$= 0.9 \times 10^{-6} \times (14 - 18) \times 119.631 = -0.0004 \text{ m}$$

Pull correction

$$c_p = \frac{(P - P_0)}{AE} L$$

$$= \frac{(145-95)\times 119.631}{3.35\times 14.8\times 10^4} = 0.0121 \text{ m}$$

Sag correction

$$c_g = -\frac{1}{24}\left(\frac{W}{P}\right)^2 L$$

$$= -\left[\frac{1}{24}\left(\frac{wl_1}{P}\right)^2 l_1 + \frac{1}{24}\left(\frac{wl_2}{P}\right)^2 l_2 + \frac{1}{24}\left(\frac{wl_3}{P}\right)^2 l_3 + \frac{1}{24}\left(\frac{wl_4}{P}\right)^2 l_4\right]$$

$$= -\frac{w^2}{24P^2}\left(l_1^3 + l_2^3 + l_3^3 + l_4^3\right)$$

$$= -\frac{(0.025\times 9.81)^2}{24\times 145^2}\left(29.988^3 + 29.895^3 + 29.838^3 + 29.910^3\right)$$

$$= -\ 0.0128 \text{ m}$$

Slope correction

$$c_s = -\frac{h^2}{2L}$$

$$= \frac{1}{2}\times\left[\frac{0.346^2}{29.988} + \frac{0.214^2}{29.895} + \frac{0.309^2}{29.838} + \frac{0.106^2}{29.910}\right]$$

$$= -\ 0.0045 \text{ m}$$

M.S.L. correction

$$c_{msl} = -\frac{HL}{R}$$

$$= -\frac{51.76\times 119.631}{6370\times 1000} = -\ 0.0010 \text{ m}$$

Total correction $= c_t + c_p + c_g + c_s + c_{msl}$

$= -\ 0.0004 + 0.0121 - 0.0128 - 0.0045 - 0.0010 = -\ 0.0066$ m

Correct length $= 119.631 - 0.0066 =$ **119.624 m**

Example 2.4. It is proposed to widen a highway by increasing the gradient of the side slope to 1 in 1.5. The difference in level between the bottom and top of the embankment at a critical section was measured as 15.0 m. The length of the embankment along the side slope was measured as 29.872 m using a steel tape under a pull of 151 N at a temperature of 27°C. Determine the additional road width which will be available with the new slope.

The tape was standardized on the flat at 18°C under a pull of 47 N. The cross-sectional area of the tape is 6.5 mm^2, E = 20.8 × 10^4 MN/m^2 and α = 1.1 × 10^{-5} per °C.

Solution:

Temperature correction

$$c_t = \alpha(t_m - t_0)L$$

$$= 1.1 \times 10^{-5} \times (27 - 18) \times 29.872 = 0.0030 \text{ m}$$

Pull correction

$$c_p = \frac{(P - P_0)}{AE} L$$

$$= \frac{(151 - 47) \times 29.872}{6.5 \times 20.8 \times 10^4 \times \frac{10^6}{10^6}} = 0.0023 \text{ m}$$

Total correction to the measured slope length L = 0.0030 + 0.0023 = 0.0053 m

Correct slope length L' = 29.872 + 0.0053 = 29.877 m.

To determine the equivalent horizontal distance x (Fig. 2.18), the approximate formula $\frac{h^2}{2L}$ given by Eq. (2.7), should not be used. This will induce a very significant error on the steep slopes for small values of h. Instead directly the Pythagoras's theorem should be used.

$$x = \sqrt{L'^2 - h^2} = \sqrt{29.877^2 - 15^2}$$

$$= 25.839 \text{ m}$$

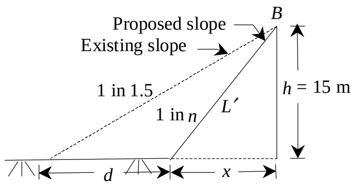


Fig. 2.18

The existing slope

$$= \frac{15}{29.877} = \frac{1}{\left(\frac{29.877}{15}\right)}$$

$$= \frac{1}{2} \quad \text{or} \quad 1 \text{ in } 2 \quad (\text{i.e., } n = 2)$$

Now the additional road width d is obtained as below.

$$\frac{d + x}{15} = \frac{1.5}{1}$$

$$d = 15 \times 1.5 - 25.839 = \mathbf{3.34 \text{ m}}$$

Example 2.5. A tape of 30 m length suspended in catenary measured the length of a base line. After applying all corrections the deduced length of the base line was 1462.36 m. Later on it was found that the actual pull applied was 155 N and not the 165 N as recorded in the field book. Correct the deduced length for the incorrect pull.

The tape was standardized on the flat under a pull of 85 N having a mass of 0.024 kg/m and cross-sectional area of 4.12 mm^2. The Young's modulus of the tape material is 152000 MN/ m^2 and the acceleration due to gravity is 9.806 m/s^2.

Solution:

It is given that

$$P_0 = 85 \text{ N}, \; P = 165 \text{ N}, \; \delta P = 155 - 165 = -10 \text{ N}$$

$$E = \frac{152000 \times 10^6}{10^6} \text{ N/mm}^2 = 152000 \text{ N/mm}^2$$

$$W = 0.024 \times 9.806 \times 30 = 7.060 \text{ kg per 30 m}$$

$$A = 4.12 \text{ mm}^2$$

$$g = 9.806 \text{ m/s}^2$$

$$\text{Nominal pull correction} = \frac{(P - P_0)}{AE} L$$

$$= \frac{(165 - 85) \times 30}{4.12 \times 152000} = 0.0038 \text{ m per 30 m}$$

$$\text{Nominal sag correction} = -\frac{1}{24}\left(\frac{W}{P}\right)^2 L$$

$$= -\frac{1}{24} \times \left(\frac{7.060}{165}\right)^2 \times 30$$

$$= -0.0023 \text{ m per 30 m}$$

From Eq. (2.14), we have

$$\text{Error in pull correction} = \frac{\text{nominal pull correction}}{P - P_0} \delta P$$

$$= \frac{0.0038}{165 - 85} \times 10$$

$$= 0.0005 \text{ m per 30 m}$$

Since the sign of the correction is same as that of δP, the correction will be

$$= -0.0005 \text{ m per 30 m}$$

$$= -0.0005 \times \frac{1462.36}{30} \text{ for the whole length}$$

$$= -0.0244 \text{ m}$$

From Eq. (2.15), we have

$$\text{Error in sag correction} = \text{nominal sag correction} \times \left(\frac{2\delta P}{P}\right)$$

Since the sign of this correction is opposite of the sign of δP, the correction of pull error in sag

$$= +(-0.0023) \times \frac{2 \times 10}{165} \text{ per 30 m}$$

$$= -0.0023 \times \frac{2 \times 10}{165} \times \frac{1462.36}{30} \text{ for the whole length}$$

$$= -0.0136 \text{ m}$$

Thus the total sag correction was too small by 0.0136 m. The length of the line has, therefore, been overestimated because the pull correction (too large) would have been added to the measured length, since ($P > P_0$), whilst the sag correction (too small) would have been subtracted.

Therefore, overestimation = 0.0244 + 0.0136

= 0.0380 m

The correct length of the line

= 1462.36 – 0.0380 = **1462.322 m**

Example 2.6. The depth of a mine shaft was measured as 834.66 m using a 1000 m steel tape having a cross-section of 10 mm^2 and a mass of 0.08 kg/m. Calculate the correct depth of the mine shaft if the tape was standardized at a tension of 182 N. The Young's modulus of elasticity of the tape material is 21×10^4 N/mm^2 and g = 9.806 m/s^2.

Solution:

The elongation in the tape hanging vertically is given by Eq. (2.17), is

$$E_x = \frac{gx}{AE}\left[\frac{m}{2}(2l - x) + M - \frac{P_0}{g}\right]$$

It is given that

P_0 = 182 N, l = 1000 m, x = 834.66 m

g = 9.806 m/s^2, E = 21 × 104 N/mm^2

A = 10 mm^2, m = 0.08 kg/m, M = 0

$$E_x = \frac{9.806 \times 834.66}{10 \times 21 \times 10^4}\left[\frac{0.08}{2}(2 \times 1000 - 834.66) + 0 - \frac{182}{9.806}\right]$$

= 0.109 m

Therefore the correct depth of the shaft

= 834.66 + 0.109

= **834.77 m**.

Example 2.7. To determine the distance between two points A and B, a tacheometer was set up at P and the following observations were recorded.

(a) Staff at A

Staff readings = 2.225, 2.605, 2.985

Vertical angle = + 7°54′

(b) Staff at B

Staff readings = 1.640, 1.920, 2.200

Vertical angle = – 1°46′

(c) Horizontal angle APB = + 68°32′30″

Elevation of A = 315.600 m

k = 100 m

c = 0.00 m

Determine the distance AB and the elevation of B.

Solution:

The horizontal distance is given by $D = ks\cos^2\alpha$

If the horizontal distances PA and PB are D_A and D_B, respectively, then

$$D_A = 100 \times (2.985 - 2.225) \times \cos^2(7°54') = 74.564 \text{ m}$$

$$D_B = 100 \times (2.200 - 1.640) \times \cos^2(1°46') = 55.947 \text{ m}$$

Now in ΔAPB if $\angle APB$ is θ and the distance AB is D then

$$D^2 = D_A^2 + D_B^2 - 2D_AD_B\cos\theta$$

$$= 74.564^2 + 55.947^2 - 2 \times 74.564 \times 55.947 \times \cos(68°32'30'')$$

$$= 5637.686$$

or $D =$ **75.085 m**

(ii) The vertical distance V is given by

$$V = \frac{1}{2} ks \sin 2\alpha$$

If the vertical distances for the points A and B are V_A and V_B, respectively, then

$$V_A = \frac{1}{2} \times 100 \times 0.760 \times \sin(2 \times 7°54') = 10.347 \text{ m}$$

$$V_B = \frac{1}{2} \times 100 \times 0.560 \times \sin(2 \times 1°46') = 1.726 \text{ m}$$

The elevation of the line of sight for A is

H.I. = Elevation of A + middle hair reading at $A - V_A$

= 315.600 + 2.605 – 10.347

= 307.858 m.

The elevation of B is

$$h_B = \text{H.I.} - V_B - \text{middle hair reading at } B$$
$$= 307.858 - 1.726 - 1.920$$
$$= \mathbf{304.212\ m}.$$

Example 2.8. The following tacheometric observations were made on two points P and Q from station A.

Table 2.3

Staff at	Vertical angle	Staff reading		
		Upper	Middle	Lower
P	– 5°12′	1.388	0.978	0.610
Q	+ 27°35′	1.604	1.286	0.997

The height of the tacheometer at A above the ground was 1.55 m. Determine the elevations of P and Q if the elevation of A is 75.500 m. The stadia constant k and c are respectively 100 and 0.00 m. Assuming that the standard error in stadia reading is ± 1.6 mm and of vertical angle ± 1.5′, also calculate the standard errors of the horizontal distances and height differences.

Solution:

Since the vertical angles are given, the line of sights are inclined for both the points. From Eqs. (2.21) and (2.22), we have

$$H = ks\cos^2\alpha \qquad \text{...(a)}$$

$$V = \frac{1}{2}ks\sin 2\alpha \qquad \text{...(b)}$$

The given data are

$$s_1 = (1.388 - 0.610) = 0.778\,\text{m}, \quad \alpha_1 = 5^\circ 12'$$

$$s_2 = (1.604 - 0.997) = 0.607\,\text{m}, \quad \alpha_2 = 27^\circ 35'$$

Therefore the distances

$$H_{AP} = 100 \times 0.778 \times \cos^2(5^\circ 12') = 77.161\,\text{m}$$

$$V_{AP} = \frac{1}{2} \times 100 \times 0.778 \times \sin(2 \times 5^\circ 12') = 7.022\ \text{m}$$

$$H_{AQ} = 100 \times 0.607 \times \cos^2(27^\circ 35') = 47.686\,\text{m}$$

$$V_{AQ} = \frac{1}{2} \times 100 \times 0.607 \times \sin(2 \times 27^\circ 35') = 24.912\,\text{m}$$

The height of the instrument

$$\text{H.I.} = \text{Elevation of } A + \text{instrument height}$$
$$= 75.500 + 1.55 = 77.050\ \text{m}$$

Elevation of P

$$h_P = \text{H.I.} - V_{AP} - \text{middle hair reading at } P$$
$$= 77.050 - 7.022 - 0.978$$
$$= \mathbf{69.050\ m}$$

Elevation Q

$$H_Q = \text{H.I.} + V_{AQ} - \text{middle hair reading at } Q$$
$$= 77.050 + 24.912 - 1.286$$
$$= \mathbf{100.676\ m.}$$

To find the standard errors of horizontal and vertical distances, the expressions (a) and (b) are differentiated with respect to staff intercept (s) and vertical angle (α), as the distances are influenced by these two quantities. It is assumed that the multiplying constant k is unchanged.

$$\frac{\partial H}{\partial s} = k\cos^2\alpha$$

$$\frac{\partial H}{\partial \alpha} = -2ks\cos\alpha\sin\alpha = -ks\sin 2\alpha$$

$$\frac{\partial V}{\partial s} = \frac{1}{2}k\sin 2\alpha \qquad \ldots(c)$$

$$\frac{\partial V}{\partial \alpha} = \frac{1}{2}ks(\cos 2\alpha)\times 2 = ks\cos 2\alpha \qquad \ldots(d)$$

Thus the standard error in horizontal distance

$$\sigma_H^2 = \left(\frac{\partial H}{\partial s}\sigma_s\right)^2 + \left(\frac{\partial H}{\partial \alpha}\sigma_\alpha\right)^2 \qquad \ldots(e)$$

and the standard error in vertical distance

$$\sigma_V^2 = \left(\frac{\partial V}{\partial s}\sigma_s\right)^2 + \left(\frac{\partial V}{\partial \alpha}\sigma_\alpha\right)^2 \qquad \ldots(f)$$

where σ_s and σ_α are the standard errors of stadia reading and vertical angle, respectively.

In obtaining the staff intercept s at a station, two readings are involved and the standard error in one stadia reading is $\pm$ 1.6 mm, thus the standard error σ_s of s is calculated as

$$\sigma_s^2 = \sigma_{s_1}^2 + \sigma_{s_2}^2 = 1.6^2 + 1.6^2 = 5.12\ \text{mm}^2$$

The standard error σ_1 of vertical angle measurement at a station is $\pm$ 1.5′, therefore

$$\sigma_\alpha^2 = (1.5)^2 = \left(\frac{1.5}{60}\times\frac{\pi}{180}\right)^2 = \left(4.3633\times 10^{-4}\right)^2$$

Now from Fig. 2.19, the difference in elevation between two points, say, A and P, is

$$h_P = h_A + h_i - V - S_M$$

or $$h_A - h_P = h = V + S_M - h_i \qquad \text{...(g)}$$

where

S_M = the middle hair reading, and

h_i = the height of the instrument above the ground.

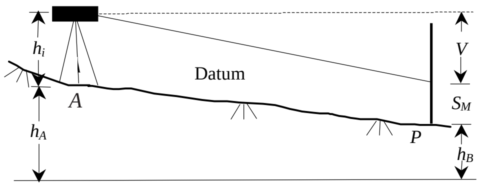


Fig. 2.19

Assuming h_i as constant, the value of h will be affected due to errors in V and S_M. Thus the standard error of h is given by

$$\sigma_h^2 = \left(\frac{\partial h}{\partial V}\sigma_V\right)^2 + \left(\frac{\partial h}{\partial S_M}\sigma_{S_M}\right)^2 \qquad \text{...(h)}$$

From Eq. (g), we get

$$\frac{\partial h}{\partial V} = 1 \quad \text{and} \quad \frac{\partial h}{\partial S_M} = 1$$

From Eq. (f), we get

$$\sigma_V^2 = \left(\frac{\partial V}{\partial s}\right)^2 \sigma_s^2 + \left(\frac{\partial V}{\partial \alpha}\right)^2 \sigma_\alpha^2$$

Substituting the value of $\frac{\partial V}{\partial s}$ and $\frac{\partial V}{\partial \alpha}$ from Eq. (c) and (d), we get

$$\sigma_V^2 = \left(\frac{1}{2}k\sin 2\alpha\right)^2 \sigma_s^2 + (ks\cos 2\alpha)^2 \sigma_\alpha^2$$

$$\sigma_{S_M}^2 = 1.6^2 = 2.56 \text{ mm}^2$$

Thus $$\sigma_h^2 = \left(\frac{1}{2}k\sin 2\alpha\right)^2 \sigma_s^2 + (ks\cos 2\alpha)^2 \sigma_\alpha^2 + \sigma_{S_M}^2$$

and from Eq. (e), we have

$$\sigma_H^2 = (k\cos^2\alpha)^2 \sigma_s^2 + (ks\sin 2\alpha)^2 \sigma_\alpha^2$$

Between A and P

$$\sigma_H^2 = (100\times\cos^2 5^\circ 12')^2 \times 5.12 + (100\times 0.778\times 1000\times \sin(2\times 5^\circ 12'))^2 \times (4.3633\times 10^{-4})^2$$

$$= 50399.9 \text{ mm}^2$$

$$\sigma_H = \pm\ 224.5 \text{ mm} = \pm\ \mathbf{0.22}$$

$$\sigma_h^2 = \left(\frac{1}{2} \times 100 \times \sin\left(2 \times 5°12'\right)\right)^2 \times 5.12 + \left(100 + 0.778 \times 1000 \times \cos\left(2 \times 5°12'\right)\right)^2 \times \left(4.3633 \times 10^{-4}\right)^2 + 2.56$$

$$= 1534.5 \text{ mm}^2$$

$$\sigma_h = \pm\ 39.2 \text{ mm} = \pm\ \mathbf{0.039\ mm}$$

Between A and Q

$$\sigma_H^2 = \left(100 \times \cos^2 27°35'\right)^2 \times 5.12 + \left(100 \times 0.607 \times 1000 \times \sin\left(2 \times 27°35'\right)\right)^2 \times \left(4.3633 \times 10^{-4}\right)^2$$

$$= 32071.2 \text{ mm}^2$$

$$\sigma_H = \pm\ 179.1 \text{ mm} = \pm\ \mathbf{0.18\ m}$$

$$\sigma_h^2 = \left(\frac{1}{2} \times 100 \times \sin\left(2 \times 27°35'\right)\right)^2 \times 5.12 + \left(100 + 0.607 \times 1000 \times \cos\left(2 \times 27°35'\right)\right)^2 \times \left(4.3633 \times 10^{-4}\right)^2 + 2.56)$$

$$= 8855.3 \text{ mm}^2$$

$$\sigma_h = \pm\ 94.1 \text{ mm} = \pm\ \mathbf{0.09\ m.}$$

Example 2.9. The following tacheometric observations were made from station A to stations 1 and 2.

Table 2.4

Instrument at station	Staff at	Zenith angle	Staff reading		
			Upper	Middle	Lower
A	1	96° 55′	1.388	0.899	0.409
	2	122°18′	1.665	1.350	1.032

Calculate the errors in horizontal and vertical distances if the staff was inclined by 1° to the vertical in the following cases:

(a) Staff inclined towards the instrument for the line A–1.

(b) Staff inclined away from the instrument for the line A–2.

The height of the instrument above ground was 1.52 m. Take stadia constants as 100 and 0.0 m.

Solution:

Let us first calculate the vertical angles from the zenith angles and the staff intercepts.

For the line A–1

$$\alpha_1 = 90° - 96°55' = -\ 6°55'$$

$$s_1 = 1.388 - 0.409 = 0.979 \text{ m}$$

For the line A–2

$$\alpha_2 = 90° - 122°18'' = -32°18'$$

$$s_2 = 1.665 - 1.032 = 0.633 \text{ m}$$

For the line A–1

The apparent horizontal distance from Eq. (2.21) when the staff is truly vertical, is

$$D' = ks_1 \cos^2 \alpha = 100 \times 0.979 \times \cos^2 6°55' = 96.48 \text{ m}$$

The correct horizontal distance D when the staff is inclined, is obtained from Eq. (2.29) by replacing in the above expression s_1 by $\dfrac{s_1 \cos(\alpha + \delta)}{\cos \alpha}$. Thus

$$D = ks_1 \frac{\cos(\alpha + \delta)}{\cos \alpha} \cos^2 \alpha = ks_1 \cos(\alpha + \delta) \cos \alpha$$

$$= 100 \times 0.979 \times \cos(6°55' + 1°) \times \cos 6°55'$$

$$= 96.26 \text{ m}$$

Therefore error $= D' - D$

$= 96.48 - 96.26 = 0.22$ m = **1 in 436** (*approximately*).

The apparent vertical distance from Eq. (2.22) is

$$V' = \frac{1}{2} ks_1 \sin(2\alpha) = \frac{1}{2} \times 100 \times 0.979 \times \sin(2 \times 6°55') = 11.70 \text{ m}$$

The correct vertical distance is

$$V = \frac{1}{2} ks_1 \frac{\cos(\alpha + \delta)}{\cos \alpha} \sin(2\alpha) = \frac{1}{2} \times \frac{100 \times 0.979 \times \cos(6°55' + 1°) \times \sin(2 \times 6°55')}{\cos(6°55')}$$

$$= 11.68 \text{ m}$$

Therefore, error $= V' - V$

$= 11.70 - 11.68 = 0.02$ m = **1 in 584** (*approximately*).

For the line A–2

The apparent horizontal distance is

$$D' = 100 \times 0.633 \times \cos^2(32°18') = 45.23 \text{ m}$$

When the staff is inclined away from the instrument the correct horizontal distance from Eq. (2.30), is

$$D = 100 \times 0.633 \times \cos(32°18' - 1°) \times \cos(32°18')$$

$$= 45.72 \text{ m}$$

Therefore error $= D' - D$

$= 45.23 - 45.72 = 0.49$ m = **1 in 93** (*approximately*).

The apparent vertical distance is

$$V' = \frac{1}{2} \times 100 \times 0.633 \times \sin\left(2 \times 32°18'\right) = 28.59 \text{ m}$$

The correct vertical distance is

$$V = \frac{1}{2} \times \frac{100 \times 0.633 \times \cos\left(32°18' - 1°\right) \times \sin\left(2 \times 32°18'\right)}{\cos\left(32°18'\right)}$$

$$= 28.90 \text{ m}$$

Therefore error = $V' - V$

$= 28.59 - 28.90 = 0.31$ m = **1 in 93** (*approximately*).

Example 2.10. In Example 2.9, if the elevation of the point 1 is 115.673 m, determine the correct elevation of the point 2. The height of the instrument above ground is 1.52 m.

Solution:

Let h_1, h_2 = the elevations of the points 1 and 2,

S_1, S_2 = the middle hair readings at the points 1 and 2,

V_1, V_2 = the vertical distances for the points 1 and 2, and

h_i = the height of the instrument above ground.

The height of the instrument above the mean sea level at A is

$$\text{H.I.} = h_1 + h_i + S_1 + V_1$$

$$= 115.673 + 1.52 + 0.899 + 11.68$$

$$= 129.772 \text{ m}$$

The elevation of the point 2 is

$$h_2 = \text{H.I.} - V_2 + S_2$$

$$= 129.772 - 28.90 - 1.350$$

$$= \mathbf{99.522 \text{ m}}.$$

Example 2.11. To measure a line *AB*, a theodolite was set up at *A* and a subtense bar of length 2 m was set up at *B*. The horizontal angle measured at *A* for the subtense bar targets was 4°02′26.4″.

Determine the length of *AB*,

the fractional error in the length *AB* if the horizontal angle was measured with an accuracy of ± 1.5″, and the error in *AB* if the subtense bar was out by 1° from being normal to *AB*.

Solution:

Horizontal distance in subtense tacheometry is given by

$$AB = \frac{b}{2} \cot \frac{\theta}{2}$$

$$= \frac{2.0}{2} \times \cot\left(\frac{4°02'26.4''}{2}\right) = \mathbf{28.348 \text{ m.}}$$

The error in distance *AB* is

$$dD = -\frac{D^2}{b} d\theta$$

$$= -\frac{28.348^2}{2} \times \frac{1.5''}{206265}$$

$$= \mathbf{0.003\ m} \qquad \textit{(neglecting the sign)}.$$

The fractional error is

$$= \frac{dD}{D} = \frac{1}{\frac{dD}{D}} = \mathbf{1\ in\ 9449}$$

The error in horizontal distance due to the bar not being normal to *AB*, is given by

$$\Delta D = \frac{b}{2} \cot \frac{\theta}{2} (1 - \cos \delta)$$

$$= \frac{2}{2} \times \cot \left(\frac{4°02'26.4''}{2} \right) \times (1 - \cos 1°) = \mathbf{0.004\ m}.$$

Example 2.12. A line *AB* was measured using *EDM*. The instrument was set up at *P* in line with *AB* on the side of *A* remote from *B*. The wavelength of frequency 1 (f_1) is 10 m exactly. Frequency 2 (f_2) is (9/10) f_1 and that of frequency 3 (f_3) is (99/100) f_1. Calculate the accurate length of *AB* that is known to be less than 200 m, from the phase difference readings given in Table 2.5.

Table 2.5

Line	Phase difference (m)		
	f_1	f_2	f_3
PA	4.337	7.670	0.600
PB	7.386	1.830	9.911

Solution:

In Sec. 2.11.1, we have found out that

$$\lambda_2 = 11.111 \text{ m}$$
$$\lambda_3 = 10.101 \text{ m}$$

From Eq. (2.43) for the line *PA*, we have

$$\lambda_1 n_1 + \Delta\lambda_1 = \lambda_2 n_2 + \Delta\lambda_2$$

$$10 n_1 + 4.337 = 11.111 n_2 + 7.670$$

Since $\Delta\lambda_1 < \Delta\lambda_2$

$$n_1 = n_2 + 1$$

or $$n_2 = n_1 - 1$$

Thus $$10n_1 + 4.337 = 11.111(n_1 - 1) + 7.670$$

or $$n_1 = 7$$

This calculation has removed any ambiguity in the number of complete wavelengths of f_1 frequency lying within the 0 to 100 m stage.

Now considering f_3 frequency which when related to f_1 gives the 0 to 1000 m stage. Let n_1' be the number of complete wavelengths of f_1 and n_3 be that of f_3, then

$$10n_1' + 4.337 = 10.101n_3 + 0.600$$

Since $$\Delta\lambda_1 > \Delta\lambda_2$$

$$n_1' = n_3$$

$$10n_1' + 4.337 = 10.101n_1' + 0.600$$

or $$n_1' = 37$$

Thus there are 37 complete wavelengths of f_1 within the distance of PA.

Therefore from Eq. (2.40), we get

$$2PA = 37 \times 10 + 4.337 = 374.337 \text{ m}$$

For the line PB, we have

$$10n_1 + 7.386 = 11.111n_2 + 1.830$$

Since $$\Delta\lambda_1 > \Delta\lambda_2$$

$$n_1 = n_2$$

or $$n_1 = 5$$

Again $$10n_1' + 7.386 = 10.101n_3 + 9.911$$

Since $$\Delta\lambda_1 < \Delta\lambda_2$$

$$n_1' = n_3 + 1$$

or $$n_1' = 75$$

Therefore, $$2PB = 75 \times 10 + 7.386 = 757.386 \text{ m}$$

Thus $$AB = \frac{2PB - 2PA}{2}$$

$$= \frac{757.386 - 374.337}{2} = \mathbf{191.525 \text{ m}}.$$

Example 2.13. The following observations were made to measure the length of a base line AB using an EDM instrument.

Measured distance = 1556.315 m

Elevation of instrument at A = 188.28 m

Elevation of reflector at B = 206.35 m

Temperature = 24°C

Pressure = 750 mm of mercury

Calculate the correct length of AB and its reduced length at mean sea level.

Take n_s = 1.0002851, n_0 = 296 × 10^{-6} and R = 6370 km.

Solution:

From Eq. (2.44), we have

$$(n-1) = (n_0 - 1)\left(\frac{273}{T}\right)\left(\frac{p}{760}\right)$$

$$(n-1) = 296 \times 10^{-6} \times \left(\frac{273}{273+24}\right)\left(\frac{750}{760}\right)$$

or n = 1.0002685

From Eq. (2.48), we have

$$D = D'\left(\frac{n_s}{n}\right)$$

$$D = 1556.315 \times \left(\frac{1.0002851}{1.0002685}\right) = 1556.341 \text{ m}$$

From Eq. (2.49), the correct length of AB is

$$= \left(\cos\theta - \frac{H}{R}\right) L'$$

$$\theta = \sin^{-1}\left(\frac{206.35 - 188.28}{1556.341}\right)$$

$$= 0^\circ 39' 54.9''$$

Therefore,

$$= \left(\cos(39'54.9'') - \frac{(188.28 + 206.35)}{2 \times 6370 \times 1000}\right) \times 1556.341$$

= **1556.188 m**.

Example 2.14. The slope distance between two stations A and B measured with EDM when corrected for meteorological conditions and instrument constants, is 113.893 m. The heights of the instrument and reflector are 1.740 m and 1.844 m, respectively, above the ground. To measure the vertical angle a theodolite was set up at *A*, 1.660 m above the ground and a target at *B* having height above the ground as 1.634 m. The measured angle above the horizontal was + 4°23′18″. Determine

(a) The horizontal length of the line *AB*.

(b) To what precision the slope angle be measured

(c) if the relative precision of the reduced horizontal distance is to be 1/100000.

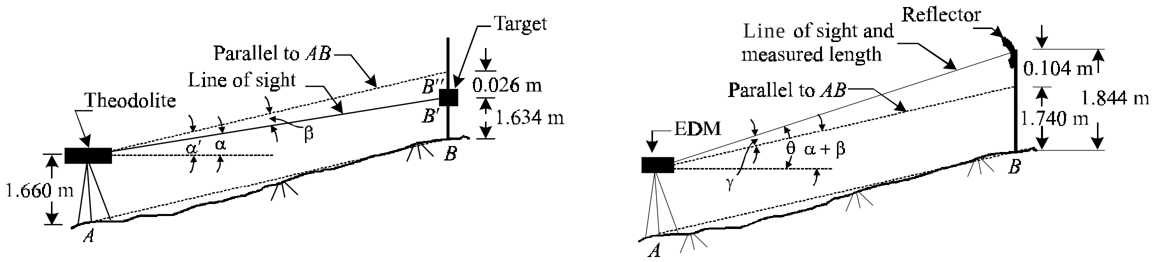
(d) if the reduced horizontal distance is to have a standard error of ± 1.8 mm.

Solution:

If the heights of the instrument and the target are not same, a correction known as *eye and object correction* is required to be determined to get the correct vertical angle. In Fig. 2.20a, the observed vertical angle is α. If the correct vertical angle α' and the correction to α is β then

$$\alpha' = \alpha + \beta$$

While measuring the distance as shown Fig. 2.20b, the height of the EDM and the reflector should also be same and the inclination of the line of sight due to height difference between the EDM and the reflector, a correction of β is to be applied to the slope angle θ, to make the line of sight inclined at α + β to the horizontal.



(a) (b)

Fig. 2.20

From Fig. 2.19a, we have

$$\beta = \frac{1.660 - 1.634}{113.893} = \frac{0.026}{113.893} \text{ radians}$$

$$= \frac{0.026}{113.893} \times 206265'' = 47.1''$$

From Fig. 2.19b, we have

$$\gamma = \frac{1.844 - 1.740}{113.893} = \frac{0.104}{113.893} \text{ radians}$$

$$= \frac{0.104}{113.893} \times 206265'' = 3'08.4''$$

The correct slope angle

$$\theta = \alpha + \beta + \gamma$$
$$= 4°23'18'' + 47.1'' + 3'08.4''$$
$$= 4°27'13.5''$$

Therefore the horizontal distance

$$L = L' \cos\theta$$
$$= 113.893 \times \cos 4°27'13.5''$$
$$= \mathbf{113.549 \text{ m}}.$$

We have $L = L' \cos\theta$

$$dL = -L' \sin\theta \, d\theta \qquad \ldots(a)$$

The negative sign in the above expression shows decrease in L with increase in $d\theta$. Thus the relative accuracy

$$\frac{dL}{L} = \frac{L' \sin\theta}{L' \cos\theta} . d\theta$$
$$= \tan\theta \, . \, d\theta$$

It is given that $\dfrac{dL}{L} = \dfrac{1}{100000}$,

therefore,

$$\frac{1}{100000} = \tan 4°27'13.5'' \times d\theta$$

or $d\theta$ = 0.000128 radians

$$= 0.000128 \times 206265''$$
$$= 26''$$

It is given that

$$dL = \pm 1.5 \text{ mm},$$

therefore, from Eq. (a), we get

$$0.0015 = 113.893 \times \sin 4°27'13.5'' \times d\theta$$
$$d\theta = 0.000169 \text{ radians}$$
$$= 0.000169 \times 206265''$$
$$= \pm \mathbf{35''}.$$

OBJECTIVE TYPE QUESTIONS

1. A metallic tape is
(*a*) a tape made of any metal.
(*b*) another name of a steel tape.
(*c*) another name of an invar tape.
(*d*) is a tape of water proof fabric into which metal wires are woven.

2. Spring balance in linear measurements is used
(*a*) to know the weight of the tape
(*b*) to apply the desired pull.
(*c*) to know the standard pull at the time of measurement.
(*d*) none of the above.

3. Ranging in distance mesurements is
(*a*) another name of taping.
(*b*) a process of establishing intermediate points on a line.
(*c*) putting the ranging rod on the hill top for reciprocal ranging.
(*d*) a process of determining the intersectoion of two straight lines.

4. Reciprocal ranging is employed when
(*a*) the two ends of a line are not intervisible.
(*b*) one end of a line is inaccessible.
(*c*) both the ends are inaccessible.
(*d*) the ends of the line are not visible even from intermediate points.

5. The following expression gives the relative accuracy in linear measurements when the slope angle is α.
(*a*) $\frac{dD}{D} = \tan 2\alpha . d\alpha.$ (*b*) $\frac{dD}{D} = \tan^2 \alpha . d\alpha.$
(*c*) $\frac{dD}{D} = 2 \tan \alpha . d\alpha.$ (*d*) $\frac{dD}{D} = \tan \alpha . d\alpha.$

6. If the slope angle 64°08′07″ is measured to an accuracy of 10″ the expected relative accuracy in the linear measurements is
(*a*) 1/10. (*b*) 1/100.
(*c*) 1/1000. (*d*) 1/10000.

7. The temperature correction and pull correction
(*a*) may have same sign. (*b*) always have same sign.
(*c*) always have opposite signs. (*d*) always have positive sign.

8. The sag corrections on hills
(*a*) is positive. (*b*) is negative.
(*c*) may be either positive or negative. (*d*) is zero

9. The correction for reduced length on the mean sea level is proportional to

(*a*) *H*. (*b*) H^2.

(*c*) 1/*H*. (*d*) 1/2*H*.

where *H* is the mean elevetion of the line.

10. If the difference in the levels of the two ends of a 50 m long line is 1 m and its ends are out of alignment by 5 m then the corrections for slope (c_s) and alignment (c_m) are related to each other as

(*a*) $c_s = 4c_m$. (*b*) $c_s = 0.4c_m$.

(*c*) $c_s = 0.04c_m$. (*d*) $c_s = 0.004c_m$.

11. Stadia is a form of tacheometric mesurements that relies on

(*a*) fixed intercept. (*b*) fixed angle intercept

(*c*) varying angle intercept (*d*) none of the above.

12. The tacheometric method of surveying is generally preferred for

(*a*) providing primary control. (*b*) large scale survey.

(*c*) fixing points with highest precision. (*d*) difficult terrain.

13. If two points *A* and *B* 125 m apart, have difference in elevation of 0.5 m, the slope correction to the measured length is

(*a*) + 0.001 m. (*b*) 0.001 m.

(*c*) + 0.0125 m. (*d*) 0.001 m.

14. The branch of surveying in which an optical instrument is used too determine both horizontal and vertical positions, is known as

(*a*) Tachemetry. (*b*) Tachometry.

(*c*) Tacheometry. (*d*) Telemetry.

15. If the vertical angle from one station to another 100 m apart, is 60°, the staff intercept for a tacheometer with $k = 100$ and $c = 0$, would be

(*a*) 1. (*b*) 4.

(*c*) 5. (*d*) 0.1.

16. Electronic distance measurement instruments use

(*a*) X-rays. (*b*) Sound waves.

(*c*) Light waves. (*d*) Magnetic flux.

17. Modern EDM instruments work on the principle of measuring

(*a*) the reflected energy generated by electromagnetic waves.

(*b*) total time taken by electromagnetic wave in travelling the distance.

(*c*) the change in frequency of the electromagnetic waves.

(*d*) the phase difference between the transmitted and the reflected electromagnetic waves.

18. The range of infrared EDM instrument is generally limited to measuring the distances

(*a*) 2 to 3 km. (*b*) 20 to 30 km.

(*c*) 200 to 300 km. (*d*) more than 300 km.

19. Electromagnetic waves are unaffected by

(*a*) air temperature.

(*b*) atmospheric pressure.

(*c*) vapour pressure.

(*d*) wind speed.

ANSWERS

1. (*d*)	**2.** (*b*)	**3.** (*b*)	**4.** (*a*)	**5.** (*d*)	**6.** (*d*)
7. (*a*)	**8.** (*b*)	**9.** (*a*)	**10.** (*c*)	**11.** (*b*)	**12.** (*d*)
13. (*b*)	**14.** (*c*)	**15.** (*b*)	**16.** (*c*)	**17.** (*d*)	**18.** (*a*)
19. (*d*).					

3 LEVELLING

3.1 LEVELLING

Levelling is an operation in surveying performed to determine the difference in levels of two points. By this operation the height of a point from a *datum*, known as *elevation*, is determined.

3.2 LEVEL SURFACE

A *level surface* is the equipotential surface of the earth's gravity field. It is a curved surface and every element of which is normal to the plumb line.

3.3 DATUM

A *datum* is a reference surface of constant potential, called as a level surface of the earth's gravity field, for measuring the elevations of the points. One of such surfaces is the mean sea level surface and is considered as a standard datum. Also an arbitrary surface may be adopted as a datum.

3.4 LEVEL LINE

A line lying in a level surface is a *level line*. It is thus a curved line.

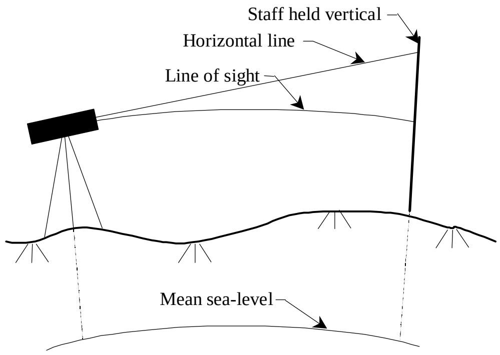


Fig. 3.1

A level in proper adjustment, and correctly set up, produces a horizontal line of sight which is at right angles to the direction of gravity and tangential to the level line at the instrument height. It follows a constant height above mean sea level and hence is a curved line, as shown in Fig. 3.1.

Over short distances, such as those met in civil engineering works, the two lines can be taken to coincide. Over long distances a correction is required to reduce the staff readings given by the horizontal line of sight to the level line equivalent. Refraction of the line of sight is also to be taken into account. The corrections for the curvature of the level line C_c and refraction C_r are shown in

Fig. 3.2. The combined correction is given by

$$C_{cr} = -\frac{3}{7}\frac{d^2}{R} \qquad ...(3.1)$$

where

C_{cr} = the correction for the curvature and refraction,

d = the distance of the staff from the point of tangency, and

R = the mean earth's radius.

For the value of R = 6370 km and d in kilometre, the value of C_{cr} in metre is given as

$$C_{cr} = -0.067d^2 \qquad ...(3.2)$$

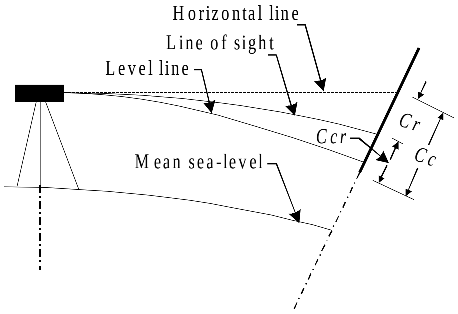


Fig. 3.2

3.5 DIRECT DIFFERENTIAL OR SPIRIT LEVELLING

Differential levelling or spirit levelling is the most accurate simple direct method of determining the difference of level between two points using an instrument known as *level* with a *levelling staff*. A level establishes a horizontal line of sight and the difference in the level of the line of sight and the point over which the levelling staff is held, is measured through the levelling staff.

Fig. 3.3 shows the principle of determining the difference in level Δh between two points A and B, and thus the elevation of one of them can be determined if the elevation of the other one is known. S_A and S_B are the staff readings at A and B, respectively, and h_A and h_B are their respective elevations.

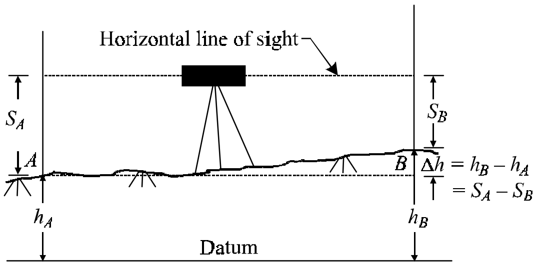


Fig. 3.3

From the figure, we find that

(*i*) if $S_B < S_A$, the point *B* is higher than point *A*.

(*ii*) if $S_B > S_A$, the point *B* is lower than point *A*.

(*iii*) to determine the difference of level, the elevation of ground point at which the level is set up, is not required.

Booking and Reducing the Levels

Before discussing the booking and methods of reducing levels, the following terms associated with differential levelling must be understood.

Station: A station is the point where the levelling staff is held. (Points *A, a, b, B, c,* and *C* in Fig. 3.4).

Height of instrument (H.I.) or height of collimation: For any set up of the level, the elevation of the line of sight is the height of instrument. (H.I. = $h_A + S_A$ in Fig. 3.3).

Back sight (B.S.): It is the first reading taken on the staff after setting up the level usually to determine the height of instrument. It is usually made to some form of a bench mark (B.M.) or to the points whose elevations have already been determined. When the instrument position has to be changed, the first sight taken in the next section is also a back sight. (Staff readings S_1 and S_5 in Fig. 3.4).

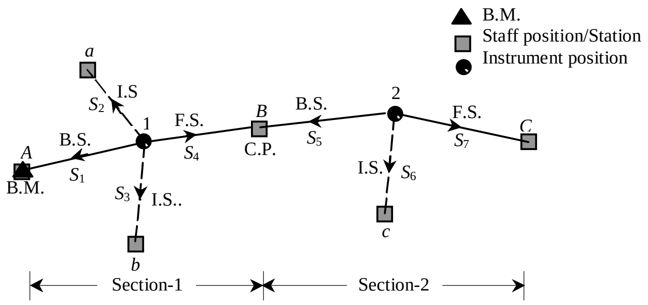


Fig. 3.4

Fore sight (F.S.): It is the last reading from an instrument position on to a staff held at a point. It is thus the last reading taken within a section of levels before shifting the instrument to the next section, and also the last reading taken over the whole series of levels. (Staff readings S_4 and S_7 in Fig. 3.4).

Change point (C.P.) or turning point: A change point or turning point is the point where both the fore sight and back sight are made on a staff held at that point. A change point is required before moving the level from one section to another section. By taking the fore sight the elevation of the change point is determined and by taking the back sight the height of instrument is determined. The change points relate the various sections by making fore sight and back sight at the same point. (Point *B* in Fig. 3.4).

Intermediate sight (I.S.): The term 'intermediate sight' covers all sightings and consequent staff readings made between back sight and fore sight within each section. Thus, intermediate sight station is neither the change point nor the last point. (Points *a*, *b*, and *c* in Fig. 3.4).

Balancing of sights: When the distances of the stations where back sight and fore sight are taken from the instrument station, are kept approximately equal, it is known as balancing of sights. Balancing of sights minimizes the effect of instrumental and other errors.

Reduced level (R.L.): Reduced level of a point is its height or depth above or below the assumed datum. It is the elevation of the point.

Rise and fall: The difference of level between two consecutive points indicates a rise or a fall between the two points. In Fig. 3.3, if (S_A – S_B) is positive, it is a rise and if negative, it is a fall. Rise and fall are determined for the points lying within a section.

Section: A section comprises of one back sight, one fore sight and all the intermediate sights taken from one instrument set up within that section. Thus the number of sections is equal to the number of set ups of the instrument. (From *A* to *B* for instrument position 1 is section-1 and from *B* to *C* for instrument position 2 is section-2 in Fig. 3.4).

For booking and reducing the levels of points, there are two systems, namely the *height of instrument* or *height of collimation method* and *rise and fall method*. The columns for booking the readings in a level book are same for both the methods but for reducing the levels, the number of additional columns depends upon the method of reducing the levels. Note that except for the change point, each staff reading is written on a separate line so that each staff position has its unique reduced level. This remains true at the change point since the staff does not move and the back sight from a forward instrument station is taken at the same staff position where the fore sight has been taken from the backward instrument station. To explain the booking and reducing levels, the levelling operation from stations *A* to *C* shown in Fig. 3.4, has been presented in Tables 3.1 and 3.2 for both the methods. These tables may have additional columns for showing chainage, embankment, cutting, etc., if required.

Table 3.1 Height of instrument method

Station	B.S.	I.S.	F.S.	H.I.	R.L.	Remarks	
A	S_1			H.I.$_A$ = h_A +S_1	h_A	B.M. = *ha*	Section-1
a		S_2			h_a = H.I.$_A$ – S_2		
b		S_3			h_b = H.I.$_A$ – S_3		
B	S_5		S_4	H.I.$_B$ = h_B + S_5	h_B = H.I.$_A$ – S_4	C.P.	–
c		S_6			h_c = H.I.$_B$ – S_6		Section-2
C			S_7		H_C = H.I.$_B$ – S_7		
	Σ B.S.		Σ F.S.				
Check: Σ B.S. – Σ F.S. = Last R.L. – First R.L.							

In reducing the levels for various points by the height of instrument method, the height of instrument (H.I.) for the each section highlighted by different shades, is determined by adding the elevation of the point to the back sight reading taken at that point. The H.I. remains unchanged for all the staff readings taken within that section and therefore, the levels of all the points lying in that section are reduced by subtracting the corresponding staff readings, i.e., I.S. or F.S., from the H.I. of that section.

In the rise and fall method, the rises and the falls are found out for the points lying within each section. Adding or subtracting the rise or fall to or from the reduced level of the backward station obtains the level for a forward station. In Table 3.2, r and f indicate the rise and the fall, respectively, assumed between the consecutive points.

Table 3.2 Rise and fall method

Station	B.S.	I.S.	F.S.	Rise	Fall	R.L.	Remarks	
A	S_1					h_A	B.M. = h_a	Section-1
a		S_2		$r_1 = S_1 - S_2$		$h_a = h_A + r_1$		
b		S_3			$f_1 = S_2 - S_3$	$h_b = h_a - f_1$		
B	S_5		S_4		$f_2 = S_3 - S_4$	$h_B = h_b - f_2$	C.P.	
c		S_6			$f_3 = S_5 - S_6$	$h_c = h_B - f_3$		
C			S_7	$r_2 = S_6 - S_7$		$H_C = h_c + r_2$		Section-2
	Σ B.S.		Σ F.S.	Σ Rise	Σ Fall			
Check:	Σ B.S. – Σ F.S. = Σ Rise – Σ Fall = Last R.L. – First R.L.							

The arithmetic involved in reduction of the levels is used as check on the computations. The following rules are used in the two methods of reduction of levels.

(*a*) For the height of instrument method

(*i*) Σ B.S. – Σ F.S. = Last R.L. – First R.L.

(*ii*) Σ [H.I. × (No. of I.S.'s + 1)] – Σ I.S. – Σ F.S. = Σ R.L. – First R.L.

(*b*) For the rise and fall method

Σ B.S. – Σ F.S. = Σ Rise – Σ Fall = Last R.L. – First R.L.

3.6 COMPARISON OF METHODS AND THEIR USES

Less arithmetic is involved in the reduction of levels with the height of instrument method than with the rise and fall method, in particular when large numbers of intermediate sights is involved. Moreover, the rise and fall method gives an arithmetic check on all the levels reduced, i.e., including the points where the intermediate sights have been taken, whereas in the height of instrument method, the check is on the levels reduced at the change points only. In the height of instrument method the check on all the sights is available only using the second formula that is not as simple as the first one.

The height of instrument method involves less computation in reducing the levels when there are large numbers of intermediate sights and thus it is faster than the rise and fall method. The rise and fall method, therefore, should be employed only when a very few or no intermediate sights are taken in the whole levelling operation. In such case, frequent change of instrument position requires determination of the height of instrument for the each setting of the instrument and, therefore, computations involved in the height of instrument method may be more or less equal to that required in the rise and fall method. On the other hand, it has a disadvantage of not having check on the intermediate sights, if any, unless the second check is applied.

3.7 LOOP CLOSURE AND ITS APPORTIONING

A *loop closure* or *misclosure* is the amount by which a level circuit fails to close. It is the difference of elevation of the measured or computed elevation and known or established elevation of the same point. Thus loop closure is given by

$$e = \text{computed value of R.L.} - \text{known value of R.L.}$$

If the length of the loop or circuit is L and the distance of a station to which the correction c is computed, is l, then

$$c = -e\frac{l}{L} \qquad ...(3.3)$$

Alternatively, the correction is applied to the elevations of each change point and the closing point of known elevation. If there are n_1 change points then the total number points at which the corrections are to be applied is

$$n = n_1 + 1$$

and the correction at each point is

$$= -\frac{e}{n} \qquad ...(3.4)$$

The corrections at the intermediate points are taken as same as that for the change points to which they are related.

Another approach could be to apply total of $-e/2$ correction equally to all the back sights and total of $+e/2$ correction equally to all the fore sights. Thus if there are n_B back sights and n_F fore sights then

$$\text{correction to each back sight} = -\frac{e}{2n_B}$$

$$\text{correction to each fore sight} = +\frac{e}{2n_F} \qquad ...(3.5)$$

3.8 RECIPROCAL LEVELLING

Reciprocal levelling is employed to determine the correct difference of level between two points which are quite apart and where it is not possible to set up the instrument between the two points for balancing the sights. It eliminates the errors due to the curvature of the earth, atmospheric refraction and collimation.

If the two points between which the difference of level is required to be determined are A and B then in reciprocal levelling, the first set of staff readings (a_1 and b_1) is taken by placing the staff on A and B, and instrument close to A. The second set of readings (a_2 and b_2) is taken again on A and B by placing the instrument close to B. The difference of level between A and B is given by

$$\Delta h = \frac{(a_1 - b_1) + (a_2 - b_2)}{2} \qquad ...(3.6)$$

and the combined error is given by

$$e = \frac{(b_1 - a_1) - (b_2 - a_2)}{2} \qquad \text{...(3.7)}$$

where

$e = e_l + e_c - e_r$ ...(3.8)

e_l = the collimation error assumed positive for the line of sight inclined upward,

e_c = the error due to the earth's curvature, and

e_r = the error due to the atmospheric refraction.

We have

$$e_c - e_r = \text{the combined curvature and refraction error}$$
$$= 0.067d^2.$$

The collimation error is thus given by

$$e_l = e - 0.067d^2 \text{ in metre} \qquad \text{...(3.9)}$$

where d is the distance between A and B in kilometre.

3.9 TRIGONOMETRIC LEVELLING

Trigonometric levelling involves measurement of vertical angle and either the horizontal or slope distance between the two points between which the difference of level is to be determined. Fig. 3.5 shows station A and station B whose height is to be established by reciprocal observations from A on to signal at B and from B on to signal at A. Vertical angles α (angle of elevation) and β (angle of depression), are measured at A and B, respectively. The refracted line of sight will be inclined to the direct line AB and therefore, the tangent to the refracted line of sight makes an angle υ with AB. The vertical angle α is measured with respect to this tangent and to the horizontal at A.

Similarly, from B the angle of depression β is measured from the horizontal to the tangent to the line of sight. The point C lies on the arc through A, which is parallel to the mean sea level surface. d is the geodetic or spheroidal distance between A and B, and could be deduced from the geodetic coordinates of the two points. Angle BAC between AB and chord AC is related to angle θ between the two verticals at A and B which meet at the earth's centre, since AC makes an angle of $\theta/2$ with the horizontal at A.

If the elevations of A and B are h_A and h_B, respectively, then the difference in elevation Δh between A and B, is found out by solving the triangle ABC for BC. Thus in triangle ABC

$$BC = AC \frac{\sin\left(\alpha + \frac{\theta}{2} - \upsilon\right)}{\sin\left[180^\circ - \left(90^\circ + \frac{\theta}{2}\right) - \left(\alpha + \frac{\theta}{2} - \upsilon\right)\right]}$$

$$= AC \frac{\sin(\alpha + \frac{\theta}{2} - \upsilon)}{\cos(\alpha + \theta - \upsilon)} \qquad \text{...(3.10)}$$

The correction for curvature and refraction at A and B is ($\theta/2 - \upsilon$) and this refers angles a and β to chords AC and BD, respectively.

Thus angle of elevation $BAC = \alpha + \theta/2 - \upsilon$ and angle of depression $DBA = \beta - \theta/2 + \upsilon$.

Since AC is parallel to BD, $\angle BAC = \angle DBA$,

Therefore, $\angle BAC + \angle DBA = (\alpha + \theta/2 - \upsilon) + (\beta - \theta/2 + \upsilon)$

or $$2\ \angle BAC = \alpha + \beta$$

or $$\angle BAC = \frac{\alpha + \beta}{2} = \angle DBA.$$

This is the correct angle of elevation at A or angle of depression at B and is mean of the two angles α and β. In addition, we can write Eq. (3.10) as

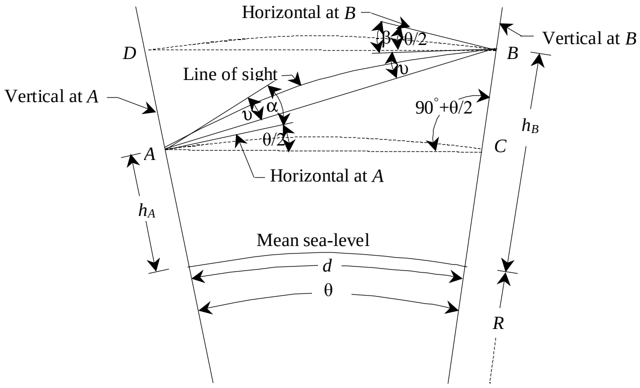


Fig. 3.5

$$BC = AC \tan (\alpha + \theta/2 - \upsilon)$$

or $$\Delta h = AC \tan\left(\frac{\alpha + \beta}{2}\right).$$

Therefore, $$h_B = h_A + \Delta h$$

$$= h_A + AC \tan\left(\frac{\alpha + \beta}{2}\right) \qquad \ldots (3.11)$$

Note that in Eq. (3.11) it is assumed that α is the angle of elevation and β is angle of depression. In practice, only magnitudes need to be considered, not signs provided one angle is elevation and the other is depression.

The coefficient of refraction K in terms of the angle of refraction υ and the angle θ subtended at the centre of the spheroid by the arc joining the stations, is given by

$$K = \frac{\upsilon}{\theta} \qquad \ldots (3.12)$$

3.10 SENSITIVITY OF A LEVEL TUBE

The *sensitivity of a level tube* is expressed in terms of angle in seconds subtended at the centre by the arc of one division length of the level tube. The radius of curvature of the inner surface of the upper portion of the level tube is also a measure of the sensitivity. The sensitivity of a level tube depends upon radius of curvature of the inner surface of the level tube and its diameter. It also depends upon the length of the vapour bubble, viscosity and surface tension of the liquid and smoothness of the inner surface of the tube.

If α' is the sensitivity of the level tube it is given by

$$\alpha' = \frac{s}{nD\sin 1''} \text{ seconds} \qquad \ldots(3.13)$$

where

s = the change in the staff reading for movement of the bubble by n divisions (Fig.3.6), and

D = the distance of the staff from the instrument.

The radius of curvature of the level tube is expressed as

$$R = \frac{nlD}{s} \qquad \ldots(3.14)$$

where l is the length of one division of the level tube.

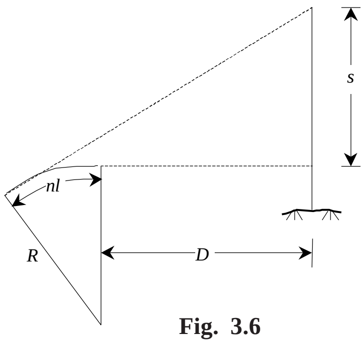


Fig. 3.6

3.11 TWO-PEG TEST

Two-peg test is conducted for checking the adjustment of a level. Fig. 3.7 shows the method of conducting the test. Two rigid points A and B are marked on the ground with two pegs and the instrument is set up exactly between them at point C. Readings are taken on the staff held at A and B, and the difference between them gives the correct difference in level of the pegs. The equality in length of back sight and fore sight ensures that any instrumental error, e, is equal on both sights and is cancelled out in the difference of the two readings. The instrument is then moved to D so that it is outside the line AB and it is near to one of the pegs. Readings are again taken on the staff held at A and B. The difference in the second set of the staff readings is equal to the difference in level of the points A and B, and it will be equal to that determined with the first set of readings if the instrument is in adjustment. If the two values of the difference in level differ from each other, the instrument is out of adjustment.

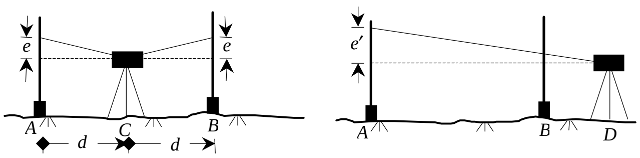


Fig. 3.7

The adjustment of the instrument can also be tested by determining the difference in level of the points A and B by placing the instrument at C and D as shown in Fig. 3.8.

Fig. 3.8

3.12 EYE AND OBJECT CORRECTION

The heights of the instrument and signal at which the observation is made, are generally not same and thus the observed angle of elevation θ' as shown in Fig. 3.9, does not refer to the ground levels at *A* and *B*. This difference in height causes the observed vertical angle θ' to be larger than that θ which would have been observed directly from those points. A correction (ε) termed as *eye and object correction,* is applied to the observed vertical angle to reduce it to the required value.

The value of the eye and object correction is given by

$$\varepsilon = \frac{h_s - h_i}{d} \text{ radians} \qquad \ldots(3.15)$$

$$= 206265 \frac{h_s - h_i}{d} \text{ seconds} \qquad \ldots(3.16)$$

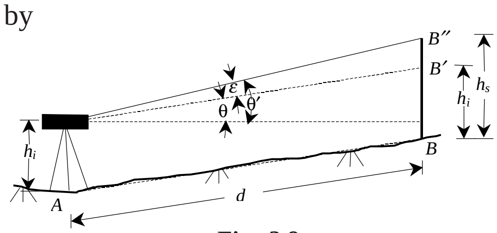


Fig. 3.8

Example 3.1. The following readings were taken with a level and 4 m staff. Draw up a level book page and reduce the levels by the height of instrument method.

0.578 B.M.(= 58.250 m), 0.933, 1.768, 2.450, (2.005 and 0.567) C.P., 1.888, 1.181, (3.679 and 0.612) C.P., 0.705, 1.810.

Solution:

The first reading being on a B.M., is a back sight. As the fifth station is a change point, 2.005 is fore sight reading and 0.567 is back sight reading. All the readings between the first and fifth readings are intermediate sight-readings. Similarly, the eighth station being a change point, 3.679 is fore sight reading, 0.612 is back sight reading, and 1.888, 1.181 are intermediate sight readings. The last reading 1.810 is fore sight and 0.705 is intermediate sight-readings. All the readings have been entered in their respective columns in the following table and the levels have been reduced by height of instrument method. In the following computations, the values of B.S., I.S., H.I., etc., for a particular station have been indicated by its number or name.

Section-1:

$$\text{H.I.}_1 = h_1 + \text{B.S.}_1 = 58.250 + 0.578 = 58.828 \text{ m}$$

$$h_2 = \text{H.I.}_1 - \text{I.S.}_2 = 58.828 - 0.933 = 57.895 \text{ m}$$

$$h_3 = \text{H.I.}_1 - \text{I.S.}_3 = 58.828 - 1.768 = 57.060 \text{ m}$$

$$h_4 = \text{H.I.}_1 - \text{I.S.}_4 = 58.828 - 2.450 = 56.378 \text{ m}$$

$$h_5 = \text{H.I.}_1 - \text{F.S.}_5 = 58.828 - 2.005 = 56.823 \text{ m}$$

Section-2:

$$\text{H.I.}_5 = h_5 + \text{B.S.}_5 = 56.823 + 0.567 = 57.390 \text{ m}$$
$$h_6 = \text{H.I.}_2 - \text{I.S.}_6 = 57.390 - 1.888 = 55.502 \text{ m}$$
$$h_7 = \text{H.I.}_2 - \text{I.S.}_7 = 57.390 - 1.181 = 56.209 \text{ m}$$
$$h_8 = \text{H.I.}_2 - \text{F.S.}_8 = 57.390 - 3.679 = 53.711 \text{ m}$$

Section-3:

$$\text{H.I.}_8 = h_8 + \text{B.S.}_8 = 53.711 + 0.612 = 54.323 \text{ m}$$
$$h_9 = \text{H.I.}_8 - \text{I.S.}_9 = 54.323 - 0.705 = 53.618 \text{ m}$$
$$h_{10} = \text{H.I.}_8 - \text{F.S.}_{10} = 54.323 - 1.810 = 52.513 \text{ m}$$

Additional check for H.I. method: Σ [H.I. $\times$ (No. of I.S.s + 1)] – Σ I.S. – Σ F.S. = Σ R.L. – First R.L.

[58.828 × 4 + 57.390 × 3 + 54.323 × 2] – 8.925 – 7.494 = 557.959 – 58.250 = 499.709 (*O.K.*)

Table 3.3

Station	B.S.	I.S.	F.S.	H.I.	R.L.	Remarks
1	0.578			58.828	58.250	B.M.=58.250 m
2		0.933			57.895	
3		1.768			57.060	
4		2.450			56.378	
5	0.567		2.005	57.390	56.823	C.P.
6		1.888			55.502	
7		1.181			56.209	
8	0.612		3.679	54.323	53.711	C.P.
9		0.705			53.618	
10			1.810		52.513	
Σ	1.757	8.925	7.494		557.956	
Check: 1.757 – 7.494 = 52.513 – 58.250 = – 5.737 (*O.K.*)						

Example 3.2. Reduce the levels of the stations from the readings given in the Example 3.1 by the rise and fall method.

Solution:

Booking of the readings for reducing the levels by rise and fall method is same as explained in Example 3.1. The computations of the reduced levels by rise and fall method is given below and the results are tabulated in the table. In the following computations, the values of B.S., I.S., Rise (r), Fall (f), etc., for a particular station have been indicated by its number or name.

(i) Calculation of rise and fall

Section-1:

$$f_2 = \text{B.S.}_1 - \text{I.S.}_2 = 0.578 - 0.933 = 0.355$$
$$f_3 = \text{I.S.}_2 - \text{I.S.}_3 = 0.933 - 1.768 = 0.835$$
$$f_4 = \text{I.S.}_3 - \text{I.S.}_4 = 1.768 - 2.450 = 0.682$$
$$r_5 = \text{I.S.}_4 - \text{F.S.}_5 = 2.450 - 2.005 = 0.445$$

Section-2: $f_6 = \text{B.S.}_5 - \text{I.S.}_6 = 0.567 - 1.888 = 1.321$

$$f_7 = \text{I.S.}_6 - \text{I.S.}_7 = 1.888 - 1.181 = 0.707$$

$$f_8 = \text{I.S.}_7 - \text{F.S.}_8 = 1.181 - 3.679 = 2.498$$

Section-3: $f_9 = \text{B.S.}_8 - \text{I.S.}_9 = 0.612 - 0.705 = 0.093$

$$f_{10} = \text{I.S.}_9 - \text{F.S.}_{10} = 0.705 - 1.810 = 1.105$$

(*ii*) Calculation of reduced levels

$$h_2 = h_1 - f_2 = 58.250 - 0.355 = 57.895 \text{ m}$$

$$h_3 = h_2 - f_3 = 57.895 - 0.835 = 57.060 \text{ m}$$

$$h_4 = h_3 - f_4 = 57.060 - 0.682 = 56.378 \text{ m}$$

$$h_5 = h_4 + r_5 = 56.378 + 0.445 = 56.823 \text{ m}$$

$$h_6 = h_5 - f_6 = 56.823 - 1.321 = 55.502 \text{ m}$$

$$h_7 = h_6 + r_7 = 55.502 + 0.707 = 56.209 \text{ m}$$

$$h_8 = h_7 - f_8 = 56.209 - 2.498 = 53.711 \text{ m}$$

$$h_9 = h_8 - f_9 = 53.711 - 0.093 = 53.618 \text{ m}$$

$$h_{10} = h_9 - f_{10} = 53.618 - 1.105 = 52.513 \text{ m}$$

Table 3.4

Station	B.S.	I.S.	F.S.	Rise	Fall	R.L.	Remarks
1	0.578					58.250	B.M.=58.250 m
2		0.933			0.355	57.895	
3		1.768			0.835	57.060	
4		2.450			0.682	56.378	
5	0.567		2.005	0.445		56.823	C.P.
6		1.888			1.321	55.502	
7		1.181		0.707		56.209	
8	0.612		3.679		2.498	53.711	C.P.
9		0.705			0.093	53.618	
10			1.810		1.105	52.513	
Σ	1.757		7.494	1.152	6.889		
Check: 1.757 – 7.494 = 1.152 – 6.889 = 52.513 – 58.250 = – 5.737 (*O.K.*)							

Example 3.3. The following consecutive readings were taken with a level on continuously sloping ground at a common interval of 20 m. The last station has an elevation of 155.272 m. Rule out a page of level book and enter the readings. Calculate

(*i*) the reduced levels of the points by rise and fall method, and (*ii*) the gradient of the line joining the first and last points.

0.420, 1.115, 2.265, 2.900, 3.615, 0.535, 1.470, 2.815, 3.505, 4.445, 0.605, 1.925, 2.885.

Solution:

Since the readings have been taken along a line on a continuously sloping ground, any sudden large change in the reading such as in the sixth reading compared to the fifth reading and in the eleventh reading compared to the tenth reading, indicates the change in the instrument position. Therefore, the sixth and eleventh readings are the back sights and fifth and tenth readings are the fore sights. The first and the last readings are the back sight and fore sight, respectively, and all remaining readings are intermediate sights.

The last point being of known elevation, the computation of the levels is to be done from last point to the first point. The falls are added to and the rises are subtracted from the known elevations. The computation of levels is explained below and the results have been presented in the following table.

(i) Calculation of rise and fall

Section-1: $f_2 = \text{B.S.}_1 - \text{I.S.}_2 = 0.420 - 1.115 = 0.695$

$f_3 = \text{I.S.}_2 - \text{I.S.}_3 = 1.115 - 2.265 = 1.150$

$f_4 = \text{I.S.}_3 - \text{I.S.}_4 = 2.265 - 2.900 = 0.635$

$f_5 = \text{I.S.}_4 - \text{F.S.}_5 = 2.900 - 3.615 = 0.715$

Section-2: $f_6 = \text{B.S.}_5 - \text{I.S.}_6 = 0.535 - 1.470 = 0.935$

$f_7 = \text{I.S.}_6 - \text{I.S.}_7 = 1.470 - 2.815 = 1.345$

$f_8 = \text{I.S.}_7 - \text{I.S.}_8 = 2.815 - 3.505 = 0.690$

$f_9 = \text{I.S.}_8 - \text{F.S.}_9 = 3.505 - 4.445 = 0.940$

Section-3: $f_{10} = \text{B.S.}_9 - \text{I.S.}_{10} = 0.605 - 1.925 = 1.320$

$f_{11} = \text{I.S.}_{10} - \text{F.S.}_{11} = 1.925 - 2.885 = 0.960$

(ii) Calculation of reduced levels

$h_{10} = h_{11} + f_{11} = 155.272 + 0.960 = 156.232$ m

$h_9 = h_{10} + f_{10} = 156.232 + 1.320 = 157.552$ m

$h_8 = h_9 + f_9 = 157.552 + 0.940 = 158.492$ m

$h_7 = h_8 + f_8 = 158.492 + 0.690 = 159.182$ m

$h_6 = h_7 + f_7 = 159.182 + 1.345 = 160.527$ m

$h_5 = h_6 + f_6 = 160.527 + 0.935 = 161.462$ m

$h_4 = h_5 + f_5 = 161.462 + 0.715 = 162.177$ m

$h_3 = h_4 + f_4 = 162.177 + 0.635 = 162.812$ m

$h_2 = h_3 + f_3 = 162.812 + 1.150 = 163.962$ m

$h_1 = h_2 + f_2 = 163.962 + 0.695 = 164.657$ m

Table 3.5

Station	Chainage (m)	B.S.	I.S.	F.S.	Rise	Fall	R.L.	Remarks
1	0	0.420					164.657	
2	20		1.115			0.695	163.962	
3	40		2.265			1.150	162.812	
4	60		2.900			0.635	162.177	
5	80	0.535		3.615		0.715	161.462	C.P.
6	100		1.470			0.935	160.527	
7	120		2.815			1.345	159.182	
8	140		3.505			0.690	158.492	
9	160	0.605		4.445		0.940	157.552	C.P.
10	180		1.925			1.320	156.232	
11	200			2.885		0.960	155.272	Elevation = 155.272 m
Σ		1.560		10.945	0.000	9.385		
Check: 1.560 – 10.945 = 0.000 – 9.385 = 155.272 – 164.657 = – 9.385 (*O.K.*)								

(*iii*) Calculation of gradient

The gradient of the line 1-11 is

$$= \frac{\text{difference of level between points } 1\text{-}11}{\text{distance between points } 1\text{-}11}$$

$$= \frac{155.272 - 164.657}{200} = \frac{-9.385}{200}$$

$$= 1 \text{ in } 21.3 \ (\textit{falling})$$

Example 3.4. A page of level book is reproduced below in which some readings marked as (×), are missing. Complete the page with all arithmetic checks.

Solution:

The computations of the missing values are explained below.

$$B.S._4 - I.S._5 = f_5, \quad B.S._4 = f_5 + I.S._5 = -0.010 + 2.440 = \mathbf{2.430}$$

$$B.S._9 - F.S._{10} = f_{10}, \quad B.S._9 = f_{10} + F.S._{10} = -0.805 + 1.525 = \mathbf{0.720}$$

$$B.S._1 + B.S._2 + B.S._4 + B.S._6 + B.S._7 + B.S._9 = \Sigma B.S.$$

$$3.150 + 1.770 + 2.430 + B.S._6 + 1.185 + 0.720 = 12.055$$

$$B.S._6 = 12.055 - 9.255 = \mathbf{2.800}$$

Table 3.6

Station	B.S.	I.S.	F.S.	Rise	Fall	R.L.	Remarks
1	3.150					×	
2	1.770		×		0.700	×	C.P.
3		2.200			×	×	
4	×		1.850	×		×	C.P.
5		2.440			0.010	×	
6	×		×	1.100		×	C.P.
7	1.185		2.010	×		222.200	C.P.
8		–2.735		×		×	Staff held inverted
9	×		1.685		4.420	×	C.P.
10			1.525		0.805	×	
Σ	12.055		×	×	×		

$\text{B.S.}_1 - \text{F.S.}_2 = f_2, \quad \text{F.S.}_2 = \text{B.S.}_1 - f_2 = 3.150 - (-0.700) = \mathbf{3.850}$

$\text{I.S.}_5 - \text{F.S.}_6 = r_6, \quad \text{F.S.}_6 = \text{I.S.}_5 - r_6 = 2.440 - 1.100 = \mathbf{1.340}$

$\text{B.S.}_2 - \text{I.S.}_3 = 1.770 - 2.200 = -0.430 = \mathbf{0.430}\ (fall) = f_3$

$\text{I.S.}_3 - \text{F.S.}_4 = 2.200 - 1.850 = \mathbf{0.350} = r_4$

$\text{B.S.}_6 - \text{F.S.}_7 = 2.800 - 2.010 = \mathbf{0.790} = r_7$

$\text{B.S.}_7 - \text{I.S.}_8 = 1.185 - (-2.735) = \mathbf{3.920} = r_8$

For the computation of reduced levels the given reduced level of point 7 is to be used. For the points 1 to 6, the computations are done from points 6 to 1, upwards in the table and for points 8 to 10, downwards in the table.

$$h_6 = h_7 - r_7 = 222.200 - 0.790 = 221.410 \text{ m}$$
$$h_5 = h_6 - r_6 = 221.410 - 1.100 = 220.310 \text{ m}$$
$$h_4 = h_5 + f_5 = 220.310 + 0.010 = 220.320 \text{ m}$$
$$h_3 = h_4 - r_4 = 220.320 - 0.350 = 219.970 \text{ m}$$
$$h_2 = h_3 + f_3 = 219.970 + 0.430 = 220.400 \text{ m}$$
$$h_1 = h_2 + f_2 = 220.400 + 0.700 = 221.100 \text{ m}$$
$$h_8 = h_7 + r_8 = 222.200 + 3.920 = 226.120 \text{ m}$$
$$h_9 = h_8 - f_9 = 226.120 - 4.420 = 221.700 \text{ m}$$
$$h_{10} = h_9 - f_{10} = 221.700 - 0.805 = 220.895 \text{ m}$$

The computed missing values and the arithmetic check are given Table 3.7.

Table 3.7.

Station	B.S.	I.S.	F.S.	Rise	Fall	R.L.	Remarks
1	3.150					221.100	
2	1.770		3.850		0.700	220.400	C.P.
3		2.200			0.430	219.970	
4	2.430		1.850	0.350		220.320	C.P.
5		2.440			0.010	220.310	
6	2.800		1.340	1.100		221.410	C.P.
7	1.185		2.010	0.790		222.200	C.P.
8		-2.735		3.920		226.120	Staff held inverted
9	0.720		1.685		4.420	221.700	C.P.
10			1.525		0.805	220.895	
Σ	12.055		12.266	6.610	6.365		
Check: 12.055 – 12.266 = 6.610 – 6.365 = 220.895 – 221.100 = – 0.205 (*O.K.*)							

Example 3.5. Given the following data in Table 3.8, determine the R.L.s of the points 1 to 6. If an uniform upward gradient of 1 in 20 starts at point 1, having elevation of 150 m, calculate the height of embankment and depth of cutting at all the points from 1 to 6.

Table 3.8

Station	Chainage (m)	B.S.	I.S.	F.S.	Remarks
B.M.	–	10.11			153.46 m
1	0		3.25		
2	100		1.10		
3	200	6.89		0.35	
4	300		3.14		
5	400	11.87		3.65	
6	500			5.98	

Solution:

Reduced levels of the points by height of instrument method

$$\text{H.I.}_{\text{B.M.}} = \text{R.L.}_{\text{B.M.}} + \text{B.S.}_{\text{B.M.}} = 153.46 + 10.11 = 163.57 \text{ m}$$

$$h_1 = \text{H.I.}_{\text{B.M.}} - \text{I.S.}_1 = 163.57 - 3.25 = 160.32 \text{ m}$$

$$h_2 = \text{H.I.}_{\text{B.M.}} - \text{I.S.}_2 = 163.57 - 1.10 = 162.47 \text{ m}$$

$$h_3 = \text{H.I.}_{\text{B.M.}} - \text{F.S.}_3 = 163.57 - 0.35 = 163.22 \text{ m}$$

$$\text{H.I.}_3 = h_3 + \text{B.S.}_3 = 163.22 + 6.89 = 170.11 \text{ m}$$

$$h_4 = \text{H.I.}_3 - \text{I.S.}_4 = 170.11 - 3.14 = 166.97 \text{ m}$$
$$h_5 = \text{H.I.}_3 - \text{F.S.}_5 = 170.11 - 3.65 = 166.46 \text{ m}$$
$$\text{H.I.}_5 = h_5 + \text{B.S.}_5 = 166.46 + 11.87 = 178.33 \text{ m}$$
$$h_6 = \text{H.I.}_5 - \text{F.S.}_6 = 178.33 - 5.98 = 172.35 \text{ m}$$

Levels of the points from gradient

Since the gradient is 1 in 20, for every 100 m the rise is 5 m.

Level of point 1, $h'_1 = 150$ m (given)

Level of point 2, $h'_2 = 150 + 5 = 155$ m

Level of point 3, $h'_3 = 155 + 5 = 160$ m

Level of point 4, $h'_4 = 160 + 5 = 165$ m

Level of point 5, $h'_5 = 165 + 5 = 170$ m

Level of point 6, $h'_6 = 170 + 5 = 175$ m

Height of embankment and depth of cutting

At point 1 $h_1 - h'_1 = 160.32 - 150.00 = +\ 10.32$ m (embankment)

At point 2 $h_2 - h'_2 = 162.47 - 155.00 = +\ 7.47$ m (embankment)

At point 3 $h_3 - h'_3 = 163.22 - 160.00 = +\ 3.22$ m (embankment)

At point 4 $h_4 - h'_4 = 166.97 - 165.00 = +\ 1.97$ m (embankment)

At point 5 $h_5 - h'_5 = 166.46 - 170.00 = -\ 3.54$ m (cutting)

At point 6 $h_6 - h'_6 = 172.35 - 175.00 = -\ 2.65$ m (cutting)

The computed values of the height of embankment and depth of cutting are tabulated below.

Table 3.9

Station	Chainage (m)	B.S.	I.S.	F.S.	H.I.	R.L.	Gradient level	Embankment/cutting	
								Height (m)	Depth (m)
B.M.	–	10.11			163.57	153.46	–		
1	0		3.25			160.32	150	10.32	
2	100		1.10			162.47	155	7.47	
3	200	6.89		0.35	170.11	163.22	160	3.22	
4	300		3.14			166.97	165	1.97	
5	400	11.87		3.65	178.33	166.46	170		3.54
6	500			5.98		172.35	175		2.65

Example 3.6. The readings given in Table 3.10, were recorded in a levelling operation from points 1 to 10. Reduce the levels by the height of instrument method and apply appropriate checks. The point 10 is a bench mark having elevation of 66.374 m. Determine the loop closure and adjust the calculated values of the levels by applying necessary corrections. Also determine the mean gradient between the points 1 to 10.

Table 3.10

Station	Chainage (m)	B.S.	I.S.	F.S.	Remarks
1	0	0.597			B.M.= 68.233 m
2	20	2.587		3.132	C.P
3	40		1.565		
4	60		1.911		
5	80		0.376		
6	100	2.244		1.522	C.P
7	120		3.771		
8	140	1.334		1.985	C.P
9	160		0.601		
10	180			2.002	

Solution:

Reduced levels of the points

$$\text{H.I.}_1 = h_1 + \text{B.S.}_1 = 68.233 + 0.597 = 68.830 \text{ m}$$
$$h_2 = \text{H.I.}_1 - \text{F.S.}_2 = 68.233 + 3.132 = 65.698 \text{ m}$$
$$\text{H.I.}_2 = h_2 + \text{B.S.}_2 = 65.698 + 2.587 = 68.285 \text{ m}$$
$$h_3 = \text{H.I.}_2 - \text{I.S.}_3 = 68.285 - 1.565 = 66.720 \text{ m}$$
$$h_4 = \text{H.I.}_2 - \text{I.S.}_4 = 68.285 - 1.911 = 66.374 \text{ m}$$
$$h_5 = \text{H.I.}_2 - \text{I.S.}_5 = 68.285 - 0.376 = 67.909 \text{ m}$$
$$h_6 = \text{H.I.}_2 - \text{F.S.}_6 = 68.285 - 1.522 = 66.763 \text{ m}$$
$$\text{H.I.}_6 = h_6 + \text{B.S.}_6 = 66.763 + 2.244 = 69.007 \text{ m}$$
$$h_7 = \text{H.I.}_6 - \text{I.S.}_7 = 69.007 - 3.771 = 65.236 \text{ m}$$
$$h_8 = \text{H.I.}_6 - \text{F.S.}_8 = 69.007 - 1.985 = 67.022 \text{ m}$$
$$\text{H.I.}_8 = h_8 + \text{B.S.}_8 = 67.022 + 1.334 = 68.356 \text{ m}$$
$$h_9 = \text{H.I.}_8 - \text{I.S.}_9 = 68.356 - 0.601 = 67.755 \text{ m}$$
$$h_{10} = \text{H.I.}_8 - \text{F.S.}_{10} = 68.356 - 2.002 = 66.354 \text{ m}$$

Loop closure and loop adjustment

The error at point 10 = computed R.L. – known R.L.

= 66.354 – 66.374 = –0.020 m

Therefore correction = +0.020 m

Since there are three change points, there will be four instrument positions. Thus the total number of points at which the corrections are to be applied is four, i.e., three C.P.s and one last F.S. It is reasonable to assume that similar errors have occurred at each station. Therefore, the correction for each instrument setting which has to be applied progressively, is

$$= +\frac{0.020}{4} = 0.005 \text{ m}$$

i.e., the correction at station 1 0.0 m

the correction at station 2 + 0.005 m

the correction at station 6 + 0.010 m

the correction at station 8 + 0.015 m

the correction at station 10 + 0.020 m

The corrections for the intermediate sights will be same as the corrections for that instrument stations to which they are related. Therefore,

$$\text{correction for I.S.}_3\text{, I.S.}_4\text{, and I.S.}_5 = +\ 0.010 \text{ m}$$

$$\text{correction for I.S.}_7 = +\ 0.015 \text{ m}$$

$$\text{correction for I.S.}_9 = +\ 0.020 \text{ m}$$

Applying the above corrections to the respective reduced levels, the corrected reduced levels are obtained. The results have been presented in Table 3.11.

Table 3.11

Station	Chainage (m)	B.S.	I.S.	F.S.	H.I.	R.L.	Correction	Corrected R.L.
1	0	0.597			68.830	68.233	–	68.233
2	20	2.587		3.132	68.285	65.698	+ 0.005	65.703
3	40		1.565			66.720	+ 0.010	66.730
4	60		1.911			66.374	+ 0.010	66.384
5	80		0.376			67.909	+ 0.010	67.919
6	100	2.244		1.522	69.007	66.763	+ 0.010	66.773
7	120		3.771			65.236	+ 0.015	65.251
8	140	1.334		1.985	68.356	67.022	+ 0.015	67.037
9	160		0.601			67.755	+ 0.020	67.775
10	180			2.002		66.354	+ 0.020	66.374
Σ		6.762		8.641				
Check: 6.762 – 8.641 = 66.354 – 68.233 = – 1.879 (*O.K.*)								

Gradient of the line 1-10

The difference in the level between points 1 and 10, Δh = 66.324 – 68.233 = –1.909 m

The distance between points 1-10, D = 180 m

$$\text{Gradient} = -\frac{1.909}{180} = -\ 0.0106$$

$$= \mathbf{1 \text{ in } 94.3} \quad (\textit{falling})$$

Example 3.7. Determine the corrected reduced levels of the points given in Example 3.6 by two alternative methods.

Solution: Method-1

From Eq. (3.3), the correction $c = -e\frac{l}{L}$

The total correction at point 10 (from Example 3.6) = + 0.020 m

The distance between the points 1 and 10 = 180 m

$$\text{Correction at point 2} = +\frac{0.020}{180}\times 20 = +0.002 \text{ m}$$

$$\text{Correction at point 6} = +\frac{0.020}{180}\times 100 = +0.011 \text{ m}$$

$$\text{Correction at point 8} = +\frac{0.020}{180}\times 140 = +0.016 \text{ m}$$

$$\text{Correction at point 10} = +\frac{0.020}{180}\times 180 = +0.020 \text{ m}$$

Corrections at points 3, 4, and 5 = + 0.011 m

Correction at point 7 = + 0.016 m

Correction at point 9 = + 0.020 m

The corrections and the corrected reduced levels of the points are given in Table 3.12.

Table 3.12

Station	R.L.	Correction	Corrected R.L.
1	68.233	–	68.233
2	65.698	+ 0.002	65.700
3	66.720	+ 0.011	66.731
4	66.374	+ 0.011	66.385
5	67.909	+ 0.011	67.920
6	66.763	+ 0.011	66.774
7	65.236	+ 0.016	65.252
8	67.022	+ 0.016	67.038
9	67.755	+ 0.020	67.775
10	66.354	+ 0.020	66.374

Method-2

In this method half of the total correction is applied negatively to all the back sights and half of the total correction is applied positively to all the fore sights.

Total number of back sights = 4

Total number of fore sights = 4

$$\text{Correction to each back sight} = -\left(\frac{-0.020}{2\times 4}\right) = +0.0025 \text{ m}$$

$$\text{Correction to each fore sight} = +\left(\frac{-0.020}{2\times 4}\right) = -0.0025 \text{ m}$$

The correction to each intermediate sight is also the same as for the fore sights, i.e., - 0.0025 m. The correction and the corrected values of the reduced levels are tabulated in Table 3.13.

Table 3.13

Station	Observed		Correction	Corrected			F.S.	H.I.	Corrected
	B.S.	I.S.		F.S.	B.S.	I.S.			R.L.
1	0.597	+ 0.0025	0.5995			68.8325	68.233		
2	2.587	3.132	+ 0.0025	– 0.0025	2.5895		3.1295	68.2925	65.703
3		1.565		– 0.0025		1.5625			66.730
4		1.911		– 0.0025		1.9085			66.384
5		0.376		– 0.0025		0.3735			67.919
6	2.244	1.522	+ 0.0025	– 0.0025	0.2465		1.5195	69.0195	66.773
7		3.771		– 0.0025		3.7685			65.251
8	1.334	1.985	+ 0.0025	– 0.0025	1.3365		1.9825	68.3735	67.037
9		0.601		– 0.0025		0.5985			67.775
10			2.002	– 0.0025			1.9995		66.374

Example 3.8. Reciprocal levelling was conducted across a wide river to determine the difference in level of points *A* and *B*, *A* situated on one bank of the river and *B* situated on the other. The following results on the staff held vertically at *A* and *B* from level stations 1 and 2, respectively, were obtained. The level station 1 was near to *A* and station 2 was near to *B*.

Instrument at	Staff reading on	
	A	*B*
1	1.485	1.725
2	1.190	1.415

(a) If the reduced level of *B* is 55.18 m above the datum, what is the reduced level of *A*?

(b) Assuming that the atmospheric conditions remain unchanged during the two sets of the observations, calculate (*i*) the combined curvature and refraction correction if the distance *AB* is 315 m, and (*ii*) the collimation error.

Solution:

To eliminate the errors due to collimation, curvature of the earth and atmospheric refraction over long sights, the reciprocal levelling is performed.

From the given data, we have

$$a_1 = 1.485 \text{ m}, \quad a_2 = 1.725 \text{ m}$$

$$b_1 = 1.190 \text{ m}, \quad b_2 = 1.415 \text{ m}$$

The difference in level between *A* and *B* is given by

$$\Delta h = \frac{(a_1 - b_1) + (a_2 - b_2)}{2}$$

$$= \frac{(1.485 - 1.190) + (1.725 - 1.415)}{2} = 0.303 \text{ m}$$

R.L. of B = R.L. of A + Δh

R.L. of A = R.L. of B – Δh

= 55.18 – 0.303 = **54.88 m**.

The total error $e = e_l + e_c - e_r$

where

$$e = \frac{(b_1 - a_1) - (b_2 - a_2)}{2}$$

$$= \frac{(1.190 - 1.485) - (1.415 - 1.725)}{2} = 0.008 \text{ m}$$

and $e_c - e_r = 0.067\ d^2$

$= 0.067 \times 0.315^2 = 0.007$ m.

Therefore collimation error $e_l = e - (e_c - e_r)$

= 0.008 – 0.007 = **0.001 m**.

Example 3.9. To determine difference in level between two stations A and B, reciprocal vertical angles have been observed as + 6°32′58.3″ from A to B and – 6°33′36.7″ from B to A, the horizontal distance AB being 1411.402 m.

Compute

(*i*) the corrected vertical angle,

(*ii*) the coefficient of refraction,

(*iii*) the correction for the earth's curvature and atmospheric refraction, and

(*iv*) the elevation of B if the elevation of A is 116.73 m.

Take the mean radius of the earth equal to 6383.393 km.

Solution: (Fig. 3.10)

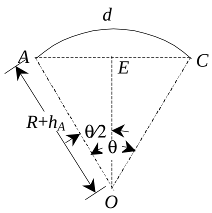


Fig. 3.10

In Fig. 3.10, from Δ AEO, we have

$$\text{Chord } AC = 2\ (R + h_A) \sin \frac{\theta}{2}$$

But for all practical purposes we can take

$$\text{Chord } AC = \text{arc } AC = d\,\frac{(R + h_A)}{R}$$

Unless h_A is appreciable, chord $AC = d$ since h_A becomes negligible compared to R. From Fig. 3.10, we get

$$\sin \frac{\theta}{2} = \frac{AE}{AO}\ \frac{\text{chord } AC/2}{R} = \frac{d}{2R}$$

$$= \frac{1411.402}{2 \times 6383.393 \times 1000}$$

$$\frac{\theta}{2} = 22.8''$$

The observed angle of elevation α = 6°32′58.3″
and the observed angle of depression β = 6°33′36.7″

(*i*) Correct vertical angle $= \dfrac{\alpha + \beta}{2}$

$$= \frac{6^\circ 32' 58.3'' + 6^\circ 33' 36.7''}{2} = 6^\circ 33' 17.5''$$

We know that

$$\alpha + \theta/2 - \upsilon = \frac{\alpha + \beta}{2}$$

$$\upsilon = \alpha + \theta/2 - \frac{\alpha + \beta}{2}$$

$$= 6^\circ 32' 58.3'' + 22.8'' - 6^\circ 33' 17.5''$$

$$= \mathbf{3.6''}.$$

(*ii*) Coefficient of refraction $K = \dfrac{\upsilon}{\theta}$

$$= \frac{3.6}{2 \times 22.8} = \mathbf{0.079}.$$

Combined correction for curvature and refraction

$$C_{cr} = -\frac{3}{7}\frac{d^2}{R}$$

$$= -\frac{3}{7} \times \frac{1.411402^2 \times 1000}{6383.393} = \mathbf{-\ 0.134\ m}.$$

(*iii*) The difference of level in *A* and *B* is given by

$$\Delta h = AC \tan\left(\frac{\alpha + \beta}{2}\right)$$

$$= d \tan\left(\frac{\alpha + \beta}{2}\right)$$

$$= 1411.402 \times \tan 6^\circ 33' 17.5'' = 162.178 \text{ m}$$

Elevation of *B* $h_B = h_A + \Delta_h$

$$= 116.73 + 162.178 = \mathbf{278.91\ m.}$$

Example 3.10. The following observations were made to determine the sensitivity of two bubble tubes. Determine which bubble tube is more sensitive.The distance of the staff from the instrument was 80 m and the length of one division of both the bubble tubes is 2 mm.

Bubble tube	Bubble reading			Staff reading
		L.H.S.	R.H.S.	
A	(*i*)	13	5	1.618
	(*ii*)	18	12	1.767
B	(*i*)	15	3	1.635
	(*ii*)	6	14	1.788

Solution:

Bubble tube *A*

The distance of the bubble from the centre of its run

(*i*) $n_1 = \frac{1}{2} \times (13 - 5) = 4$ divisions

(*ii*) $n_2 = \frac{1}{2} \times (12 - 8) = 2$ divisions

The total number of divisions *n* through which bubble has moved $= n_1 + n_2 = 6$

The staff intercept $s = 1.767 - 1.618 = 0.149$ m

The sensitivity of the bubble tube

$$\alpha'_A = 206265 \times \frac{s}{nD} \text{ seconds}$$

$$= 206265 \times \frac{0.149}{6 \times 80} \text{ seconds} = 1'4''$$

Bubble tube *B*

The distance of the bubble from the centre of its run

(*i*) $n_1 = \frac{1}{2} \times (15 - 3) = 6$ divisions

(*ii*) $n_2 = \frac{1}{2} \times (14 - 6) = 4$ divisions

The total number of divisions *n* through which bubble has moved $= n_1 + n_2 = 10$

The staff intercept $s = 1.788 - 1.635 = 0.153$ m

The sensitivity of the bubble tube

$$\alpha'_B = 206265 \times \frac{0.153}{10 \times 80} \text{ seconds} = 40''$$

Since $\alpha'_A > \alpha'_B$, the bubble *A* is more sensitive than *B*.

Example 3.11. If sensitivity of a bubble tube is 30″ per 2 mm division what would be the error in staff reading on a vertically held staff at a distance of 200 m when the bubble is out of centre by 2.5 divisions?

Solution:

The sensitivity of a bubble tube is given by

$$\alpha' = 206265 \frac{s}{nD} \text{ seconds}$$

where s can be taken as the error in staff reading for the error in the bubble tube.

Therefore
$$s = \frac{nD\alpha'}{206265}$$
$$= \frac{2.5 \times 200 \times 30}{206265} = \mathbf{0.073\ m}.$$

Example 3.12. Four stations C, A, B, and D were set out in a straight line such that $CA = AB = BD = 30$ m. A level was set up at C and readings of 2.135 and 1.823 were observed on vertically held staff at A and B, respectively, when bubble was at the centre of its run. The level was then set up at D and readings of 2.026 and 1.768 were again observed at A and B, respectively. Determine the collimation error of the level and correct difference in level of A and B.

Solution: (Fig. 3.8)

Apparent difference in level of A and B when instrument at C

$$\Delta h_1 = 2.135 - 1.823 = 0.312 \text{ m}$$

Apparent difference in level of A and B when instrument at D

$$\Delta h_2 = 2.026 - 1.768 = 0.258 \text{ m}$$

Since the two differences in level do not agree, the line of collimation is inclined to the horizontal and not parallel to the axis of the bubble tube. Let the inclination of the line of collimation with the horizontal be θ, directed upwards. The distance d between consecutive stations is 30 m. If the errors in the staff readings at A and B for the instrument position at C are e_{A1} and e_{B1} and that for the instrument position D are e_{A2} and e_{B2}, respectively, then

$$e_{A1} = d\theta$$
$$e_{B1} = 2\,d\theta$$
$$e_{A2} = 2\,d\theta$$
$$e_{B2} = d\theta$$

The correct staff readings for the instrument position at C are $(2.135 - d\theta)$ and $(1.823 - 2\,d\theta)$ and that for the instrument position at D are $(2.026 - 2\,d\theta)$ and $(1.768 - d\theta)$

Substituting $d = 30$ m, the correct difference in level are

$$\Delta h_1 = (2.135 - 30\theta) - (1.823 - 60\theta) = 0.312 + 30\theta$$
$$\Delta h_2 = (2.026 - 60\theta) - (1.768 - 30\theta) = 0.258 - 30\theta$$

Since both $\Delta h_1$and Δh_2 are the correct differences in level, they must be equal.

Therefore
$$\Delta h_1 = \Delta h_2$$

$$0.312 + 30\theta = 0.258 - 30\theta$$
$$\theta = -0.0009 \text{ radians}$$
$$= -0.0009 \times 206265 \text{ seconds}$$
$$= -3'5.64''$$

The negative sign shows the line of collimation is inclined downwards rather upwards as assumed.

The correct difference of level between *A* and *B*

$$\Delta h = 0.312 + 30 \times (-0.0009)$$
$$= \mathbf{0.285\ m}.$$

The correct difference in level can also be obtained from Eq. (3.6) by reciprocal levelling

$$\Delta h = \frac{(a_1 - b_1) + (a_2 - b_2)}{2}$$

= half the sum of apparent difference in level

$$= \frac{0.312 + 0.258}{2} = 0.285 \text{ m}.$$

Example 3.13. Fig. 3.11 shows a rectangle *ABCD*, in which *A*, *B*, and *C* are the stations where staff readings were obtained with a level set up at *E* and *D*. The observed readings are given in Table 3.14.

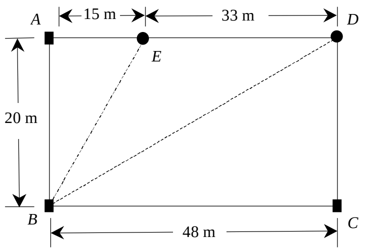


Fig. 3.11

Table 3.14

Level at	Staff reading at		
	A	*B*	*C*
E	1.856	0.809	–
D	2.428	1.369	1.667

If *A* is a bench mark having elevation of 150 m, calculate the correct elevations of *B* and *C*.

Solution: Since $\angle DAB = 90°$

$$EB = \sqrt{(15^2 + 20^2)} = 25 \text{ m}$$
$$DB = \sqrt{(48^2 + 20^2)} = 52 \text{ m}$$

Assuming the line of sight is inclined upwards by angle θ,

the correct staff reading on *A* when level at $E = 1.856 - 15\theta$

the correct staff reading on *B* when level at $E = 0.809 - 25\theta$

the correct staff reading on *A* when level at $D = 2.428 - 48\theta$

the correct staff reading on *B* when level at $D = 1.369 - 52\theta$

The correct differences in level of A and B from the two instrument positions must be equal. Therefore

$$(1.856 - 15\theta) - (0.809 - 25\theta) = (2.428 - 48\theta) - (1.369 - 52\theta)$$

$$= \frac{0.012}{6} = 0.002 \text{ radians}$$

The correct level difference of A and B = $(1.856 - 15\theta) - (0.809 - 25\theta)$

$$= 1.047 + 10\theta$$

$$= 1.047 + 10 \times 0.002 = 1.067 \text{ m}$$

Reduced level of B = $150 + 1.067$

= **151.067 m**.

The correct staff reading at C = $1.667 - 20\theta$

The correct level difference of A and C = $(2.428 - 48\theta) - (1.667 - 20\theta)$

$$= 0.761 - 28 \times 0.002 = 0.705 \text{ m}$$

Reduced level of C = $150 + 0.705$ = **150.705 m**.

Example 3.14. To determine the difference in level between two stations A and B, 4996.8 m apart, the reciprocal trigonometric levelling was performed and the readings in Table 3.15,were obtained. Assuming the mean earth's radius as 6366.20 km and the coefficient of refraction as 0.071 for both sets of observations, compute the observed value of the vertical angle of A from B and the difference in level between A and B.

Table 3.15

Instrument at	Height of Instrument (m)	Target at	Height of Target (m)	Mean vertical angle
A	1.6	*B*	5.5	+ 1°15′32″
B	1.5	*A*	2.5	–

Solution (Fig. 3.9):

Height of instrument at A, $h_i = 1.6$ m

Height of target at B, $h_S = 5.5$ m

Correction for eye and object to the angle α' observed from A to B

$$\varepsilon_A = \frac{h_S - h_i}{d}.206265 \text{ seconds}$$

$$= \frac{5.5 - 1.6}{4996.8} \times 206265 \text{ seconds}$$

$$= 2'41''$$

Similarly, the correction for eye and object to the angle β' observed from B to A

$$\varepsilon_B = \frac{2.5 - 1.5}{4996.8} \times 206265 \text{ seconds} = 41.3''$$

Length of arc at mean sea level subtending an angle of 1″ at the centre of earth

$$= \frac{R \times 1''}{206265} \times 1000$$

$$= \frac{6366.2 \times 1''}{206265} \times 1000 = 30.86 \text{ m}$$

Therefore angle θ subtended at the centre of earth by AB

$$= \frac{4996.8}{30.86}$$

$$\theta = 2'41.9''$$

Refraction $\upsilon = K\theta$

$$= 0.071 \times 2'41.9'' = 11.5''$$

Therefore correction for curvature and refraction

$$\theta/2 - \upsilon = \frac{2'41.5''}{2} - 11.5'' = 1'9.5''$$

Corrected angle of elevation for eye and object

$$\alpha = \alpha' - \varepsilon_A$$

$$= 1°15'32'' - 2'41'' = 1°12'51''$$

Corrected angle of elevation for curvature and refraction

$$\alpha + \theta/2 - \upsilon = 1°12'51'' + 1'9.5''$$

$$= 1°14'0.5''$$

If b is the angle of depression at B corrected for eye and object then

$$\alpha + \theta/2 - \upsilon = \beta - (\theta/2 - \upsilon)$$

or $\beta = 1°14'0.5'' + 1'9.5'' = 1°15'10''$

If the observed angle of depression is β' then

$$\beta' = \beta' - \varepsilon_B$$

or $\beta' = \beta + \varepsilon_B$

$$= 1°15'10'' + 41.3'' = \mathbf{1°15'51.3''}$$

Now the difference in level

$$\Delta h = AC \tan\left(\frac{\alpha' + \beta'}{2}\right)$$

$$= 4996.8 \times \tan\left(\frac{1°12'51'' + 1°15'51.3''}{2}\right)$$

$$= \mathbf{108.1 \text{ m}}$$

OBJECTIVE TYPE QUESTIONS

1. A datum surface in levelling is a
(*a*) horizontal surface.
(*b*) vertical surface.
(*c*) level surface.
(*d*) non of the above.

2. Reduced level of a point is its height or depth above or below
(*a*) the ground surface.
(*b*) the assumed datum.
(*c*) assumed horizontal surface.
(*d*) the line of collimation.

3. The correction for the atmospheric refraction is equal to
(*a*) + 1/7 of the correction for curvature of the earth.
(*b*) 1/7 of the correction for curvature of the earth.
(*c*) + 6/7 of the correction for curvature of the earth.
(*d*) 6/7 of the correction for curvature of the earth.

4. If the back sight reading at point *A* is greater than the fore sight reading at point *B* then
(a) *A* is higher than *B*.
(b) *B* is higher than *A*.
(*c*) height of the instrument is required to know which point is higher.
(*d*) instrument position is required to know which point is higher.

5. Change points in levelling are
(*a*) the instrument stations that are changed from one position to another.
(*b*) the staff stations that are changed from point to point to obtain the reduced levels of the points.
(*c*) the staff stations of known elevations.
(*d*) the staff stations where back sight and fore sight readings are taken.

6. Balancing of sights mean
(*a*) making fore sight reading equal to back sight reading.
(*b*) making the line of collimation horizontal.
(*c*) making the distance of fore sight station equal to that of the back sight station from the instrument station.
(*d*) taking fore sight and back sight readings at the same station.

7. The height of instrument method of reducing levels is preferred when
(*a*) there are large numbers of intermediate sights.
(*b*) there are no intermediate sights.
(*c*) there are large numbers of fore sights.
(*d*) there are no fore sights.

8. Sensitivity of a bubble tube depends on

(*a*) the radius of curvature.

(*b*) the length of the vapour bubble.

(*c*) the smoothness of the inner surface of the buble tube.

(*d*) all the above.

9. Reciprocal levelling is employed to determine the accurate difference in level of two points which

(*a*) are quite apart and where it is not possible to set up the instrument midway between the points.

(*b*) are quite close and where it is not possible to set up the instrument midway between the points.

(*c*) have very large difference in level and two instrument settings are required to determine the difference in level.

(*d*) are at almost same elevation.

10. When a level is in adjustment, the line of sight of the instrument is

(*a*) perpendicular to the vertical axis of the instrument and parallel to the bubble tube axis.

(*b*) perpendicular to the vertical axis of the instrument and bubble level axis.

(*c*) perpendicular to the bubble tube axis and parallel to the vertical axis.

(*d*) none of the above.

11. A Dumpy level is preferred to determine the elevations of points

(*a*) lying on hills.

(*b*) lying on a line.

(*c*) lying in moderately flat terrain.

(*d*) on a contour gradient.

ANSWERS

1. (*c*)	**2.** (*b*)	**3.** (*a*)	**4.** (*b*)	**5.** (*d*)	**6.** (*c*)
7. (*a*)	**8.** (*d*)	**9.** (a)	**10.** (*a*)	**11.** (*c*)	

4 THEODOLITE AND TRAVERSE SURVEYING

4.1 THE THEODOLITE

A theodolite is a versatile instrument basically designed to measure horizontal and vertical angles. It is also used to give horizontal and vertical distances using stadia hairs. Magnetic bearing of lines can be measured by attaching a trough compass to the theodolite. It is used for horizontal and vertical alignments and for many other purposes.

A theodolite has three important lines or axes, namely the *horizontal axis* or *trunion axis*, the *vertical axis*, and the *line of collimation* or the *line of sight*. It has one *horizontal circle* perpendicular to the vertical axis of the instrument for measuring horizontal angles and one *vertical circle* perpendicular to the trunion axis for measuring vertical angles. For leveling the instrument there is one *plate level* having its axis perpendicular to the vertical axis. The instrument also has one *telescope level* having its axis parallel to the line of sight for measuring vertical angles.

The three axes of a perfectly constructed and adjusted theodolite have certain geometrical requirements of relationship between them as shown in Fig. 4.1. The line of collimation has to be perpendicular to the trunion axis and their point of intersection has to lie on the vertical axis. The intersection of the horizontal axis, the vertical axis and the line of collimation, is known as the *instrumental centre*. The line of sight coinciding with the line of sight describes a vertical plane when the telescope is rotated about the trunion axis. The vertical axis defined by plumb bob or optical plummet, has to be centered as accurately as possible over the station at which angles are going to be measured.

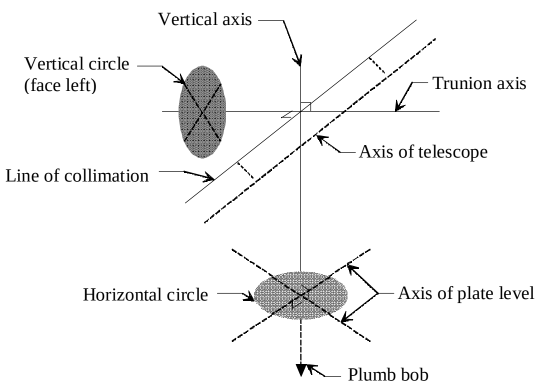


Fig. 4.1

4.2 ERRORS DUE TO MALADJUSTMENTS OF THE THEODOLITE

Errors in horizontal circle and vertical circle readings arise due to certain maladjustments of the theodolite.

Error in the Horizontal Circle Reading

Error in the horizontal circle readings are due to the following maladjustments of the instrument:

(*i*) The line of collimation not perpendicular to the trunion axis by a small amount *c*.

(*ii*) The trunion axis not perpendicular to the vertical axis by a small amount *i*.

The Line of Collimation Not Perpendicular to the Trunion Axis by a Small Amount *c*

Let us consider a sphere the centre *O* of which is the *instrumental centre* of the theodolite as shown in Fig. 4.2. The line of sight of the theodolite is out of adjustment from the line perpendicular to the trunion axis by a small angle *c*. When the theodolite is rotated about the trunion axis for the pointing on *P* the line of sight sweeps along circle Z_1PT_1. The reading on the horizontal circle, however, as if *P* were in vertical circle ZS_1 (to which Z_1PT_1 is parallel) whereas it is actually in vertical circle ZP_1. Consequently, the error in the horizontal circle reading is S_1P_1 for this sighting, and is positive on a clockwise reading circle.

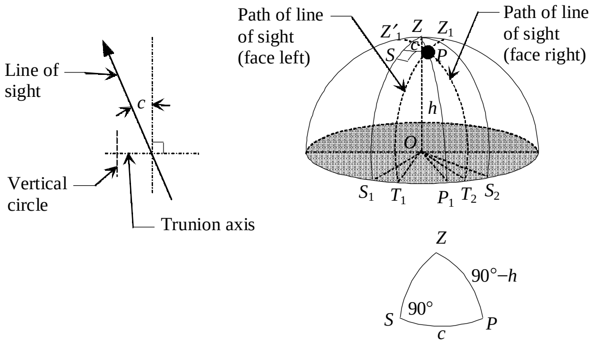


Fig 4.2

Let *SP* be at right angles to ZS_1 in the spherical triangle *ZSP* and the altitude of *P* be *h* (= PP_1). Then

$$\frac{\sin Z}{\sin SP} = \frac{\sin S}{\sin ZP}$$

or

$$\sin Z = \frac{\sin SP \sin S}{\sin ZP}.$$

For small angle, we can write the error in horizontal circle reading

$$Z = \frac{c \sin 90°}{\sin (90° - h)} = \frac{c}{\cos h}$$

$$= c \sec h. \tag{4.1}$$

The Trunion Axis not Perpendicular to the Vertical Axis by a Small Amount *i*

In Fig. 4.3, it has been assumed that the left-hand support of the trunion axis is higher than the right-hand support and, consequently, the line of sight sweeps along circle Z_2PS_3, making angle i with the vertical circle ZS_3. P appears to be on that circle but is in fact on vertical circle ZP_2, and therefore, the error in the horizontal circle reading is S_3P_2 and it is negative.

Considering the right-angled spherical triangle PP_2S_3 in which $\angle P_2 = 90°$, from Napier's rule we have

$$\sin P_2S_3 = \tan i . \tan PP_2 .$$

For small angles we can write the error in horizontal circle reading

$$P_2S_3 = i \tan PP_2 = i \tan h \qquad \ldots(4.2)$$

where $$PP_2 = PP_1 = h.$$

In the case of depression angles the errors in horizontal circle reading due to collimation error c are the same since path Z_1PT_1 is parallel to ZS_1 throughout (Fig. 4.2). However, for the error in trunion axis, path Z_2PS_3 in Fig. 4.3, is inclined to ZS_3, the two paths effectively crossing at S_3 when moving from elevation to depression. Thus there is a change in the direction of error for depression angles.

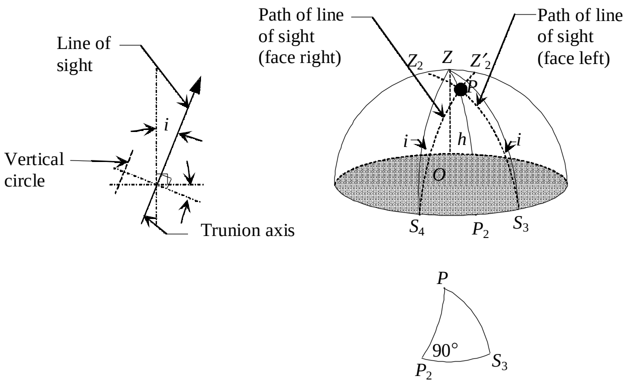


Fig. 4.3

For face right observations the path of the lines of sight change from Z_1 to Z_1' in Fig. 4.2, thus giving an error of P_1S_2 in the horizontal circle reading. This is of similar magnitude but of opposite sign to S_1P_1. Similarly, in Fig. 4.3, Z_2 moves to Z_2' and S_3 to S_4, giving error S_4P_2 which is of same magnitude but of opposite sign to error P_2S_3.

It may be noted that in each case taking means of face left and face right observations cancels out the errors, and the means will be the true values of the horizontal circle readings.

Errors in Vertical Circle Reading

Errors in circle readings due to the line of collimation not being perpendicular to the trunion axis and the trunion axis not being perpendicular to the vertical axis may be taken as negligible. When the vertical axis is not truly vertical the angle i by which the trunion axis is not perpendicular to the vertical axis, varies with the pointing direction of the telescope of the instrument. Its value is maximum when the trunion axis lies in that plane which contains the vertical axis of the instrument and the true vertical. However, in this case the horizontal circle, reading will be in error as the trunion axis is inclined to the vertical axis of the instrument. The error in this case is of the form (*ii*) above but *i* is now variable.

4.3 TRAVERSE

A traverse consists a series of straight lines of known length related one another by known angles between the lines. The points defining the ends of the traverse lines are called the *traverse stations*.

Traverse survey is a method of establishing control points, their positions being determined by measuring the distances between the traverse stations which serve as control points and the angles subtended at the various stations by their adjacent stations. The anglers are measured with a theodolite and the distances are measured by the methods discussed in Chapter 2 depending on the accuracy required in the survey work. Chain and compass traverse may be run for ordinary surveys.

Types of Traverse

There are two types of traverse, namely the *open traverse* and the *closed traverse*. An open traverse originates at a point of known position and terminates at a point of unknown position (Fig. 4.4a), whereas a closed traverse originates and terminates at points of known positions (Fig. 4.4b). When closed traverse originates and terminates at the same point, it is called the *closed-loop traverse* (Fig. 4.4 c). For establishing control points, a closed traverse is preferred since it provides different checks for included angles, deflection angles and bearings for adjusting the traverse. When an open traverse is used the work should be checked by providing cut off lines and by making observations on some prominent points visible form as many stations as possible.

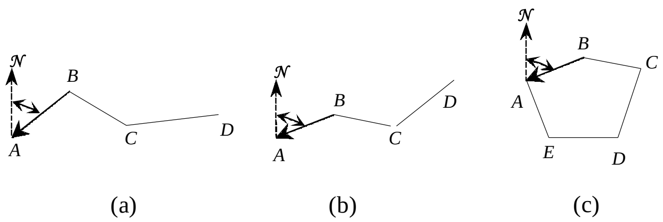


Fig. 4.4

4.4 COORDINATES

Normally, plane rectangular coordinate system having x-axis in east-west direction and y-axis in north-south direction, is used to define the location of the traverse stations. The y-axis is taken as the reference axis and it can be (a) true north, (b) magnetic north, (c) National Grid north, or (d) a chosen arbitrary direction.

Usually, the origin of the coordinate system is so placed that the entire traverse falls in the first quadrant of the coordinate system and all the traverse stations have positive coordinates as shown in Fig. 4.5.

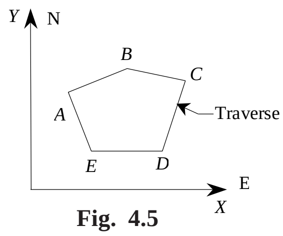


Fig. 4.5

4.5 BEARING

Bearing is defined as the direction of any line with respect to a given meridian as shown in Fig. 4.6. If the bearing θ or θ' is measured clockwise from the north side of the meridian, it is known as the *whole-circle bearing* (W.C.B.).The angle θ is known as the *fore bearing* (F.B.) of the line *AB* and the angle θ' as the *back bearing* (B.B.). If θ and θ' are free from errors, $(\theta - \theta')$ is always equal to 180°.

The acute angle between the reference meridian and the line is known as the *reduced bearing* (R.B.) or *quadrantal bearing*. In Fig. 4.7, the reduced bearings of the lines *OA*, *OB*, *OC*, and *OD* are Nθ_AE, Sθ_BE, Sθ_CW, and Nθ_DW, respectively.

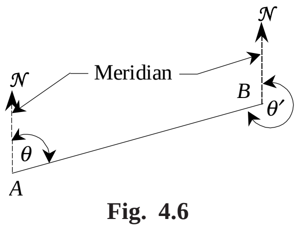


Fig. 4.6

4.6 DEPARTURE AND LATITUDE

The coordinates of points are defined as departure and latitude. The latitude is always measured parallel to the reference meridian and the departure perpendicular to the reference meridian. In Fig. 4.8, the departure and latitude of point *B* with respect to the preceding point *A*, are

$$\text{Departure} = BC = l \sin \theta$$

$$\text{Latitude} = AC = l \cos \theta. \qquad \ldots(4.3)$$

where l is the length of the line *AB* and θ its bearing. The departure and latitude take the sign depending upon the quadrant in which the line lies. Table 4.1 gives the signs of departure and latitude.

Table 4.1

	Quadrant			
	N-E	S-E	S-W	N-W
Departure	+	+	–	–
Latitude	+	–	–	+

Departure and latitude of a forward point with respect to the preceding point is known as the *consecutive coordinates*.

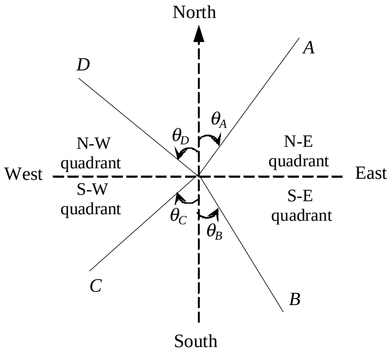


Fig. 4.7

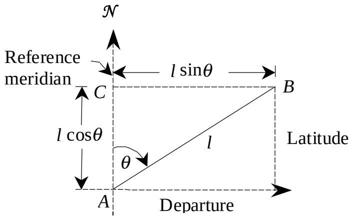


Fig. 4.8

4.7 EASTING AND NORTHING

The coordinates (X,Y) given by the perpendicular distances from the two main axes are the eastings and northings, respectively, as shown in Fig. 4.9. The easting and northing for the points P and Q are (E_P, N_P,) and (E_P, N_P,), respectively. Thus the relative positions of the points are given by

$$\Delta E = E_Q - E_P$$
$$\Delta N = N_Q - N_P. \qquad \ldots(4.4)$$

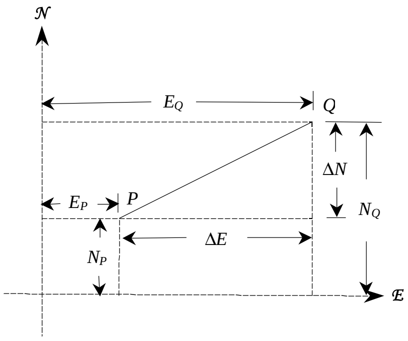


Fig. 4.9

4.8 BALANCING THE TRAVERSE

In a closed traverse the following conditions must be satisfied:

$$\Sigma \text{ Departure} = \Sigma D = 0$$
$$\Sigma \text{ Latitude} = \Sigma L = 0 \qquad \ldots(4.5)$$

If the above conditions are not satisfied, the position A of the originating stations and its computed position A' will not be the same as shown in Fig. 4.10, due to the observational errors. The distance AA' between them is known as the *closing error*. The closing error is given by

$$e = \sqrt{(\Sigma D)^2 + (\Sigma L)^2} \qquad \ldots(4.6)$$

and its direction or reduced bearing is given by

$$\tan\theta = \frac{(\Sigma D)}{(\Sigma L)}. \qquad \ldots(4.7)$$

The term *balancing* is generally applied to the operation of adjusting the closing error in a closed traverse by applying corrections to departures and latitudes to satisfy the conditions given by the Eq. (4.5).

The following methods are generally used for balancing a traverse:

(a) *Bowditch's method* when the linear errors are proportional to $\sqrt{l}$ and angular errors are proportional to $1/\sqrt{l}$, where l is the length of the line. This rule can also be applied graphically when the angular measurements are of inferior accuracy such as in compass surveying. In this method the total error in departure and latitude is distributed in proportion to the length of the traverse line. Therefore,

$$c_D = \Sigma D \frac{l}{\Sigma l}$$

$$c_L = \Sigma L \frac{l}{\Sigma l} \qquad \ldots(4.8)$$

where

c_D and c_L = the corrections to the departure and latitude of the line to which the correction is applied,

l = the length of the line, and

Σl = the sum of the lengths of all the lines of the traverse, i.e., perimeter p.

(b) *Transit rule* when the angular measurements are more precise than the linear measurements. By transit rule, we have

$$c_D = \Sigma D \frac{D}{D_T} \qquad \ldots(4.9)$$

$$c_L = \Sigma L \frac{L}{L_T}$$

where

D and L = the departure and latitude of the line to which the correction is applied, and

D_T and L_T = the arithmetic sum of departures and latitudes all the lines of the traverse, (i.e., ignoring the algebraic signs).

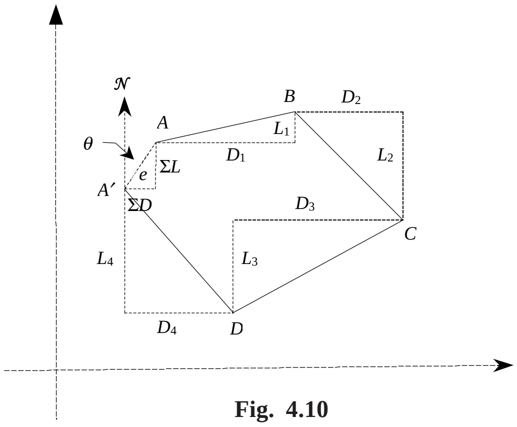


Fig. 4.10

4.9 OMITTED OBSERVATIONS

In a closed traverse if lengths and bearings of all the lines could not be measured due to certain reasons, the omitted or the missing measurements can be computed provided the number of such omissions is not more than two. In such cases, there can be no check on the accuracy of the field work nor can the traverse be balanced. It is because of the fact that all the errors are thrown into the computed values of the omitted observations.

The omitted quantities are computed using Eq. (4.5), i.e.

$$\Sigma D = l_1 \sin\theta_1 + l_2 \sin\theta_2 + \ldots\ldots. + l_n \sin\theta_n = 0$$

$$\Sigma L = l_1 \cos\theta_1 + l_2 \cos\theta_2 + \ldots\ldots. + l_n \sin\theta_n = 0 \qquad \ldots(4.10)$$

It may be noted that

length of the traverse lines $l = \sqrt{D^2 + L^2}$

departure of the line $D = l \sin\theta$

latitude of the line $L = l \cos\theta$

bearing of the line $q = \tan^{-1}(D/L)$.

4.10 CENTERING ERROR OF THEODOLITE

If the theodolite is not correctly centered over the ground station mark at which the horizontal angles are to be measured, its vertical angle will not pass through the station mark and the measured horizontal angle will be in error.

In Fig. 4.11, let the true centering position of the theodolite be *S*. When the theodolite is not correctly centered over station *S* the vertical axis of the theodolite may lie any where within a circle of radius *x* from *S*, *x* being the centering error. However, there are two points S_1 and S_2 on the perimeter of the circle at which the true horizontal angle *PSQ* will be subtended. S_1 and S_2 lie on the circumference of the circle passing through *P*, *S* and *Q*. Accordingly $\angle PS_1Q = \angle PSQ = \angle PS_2Q$, because all the three angles stand on chord *PQ*.

To determine the maximum angular error due to a centering error, let us assume that the theodolite is centered at S' which is *x* distance away from the correct position *S*.

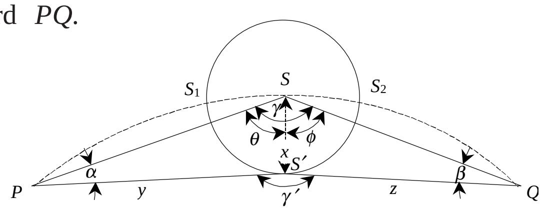


Fig. 4.11

$$\text{Correct horizontal angle } \angle PSQ = \gamma = \theta + \phi$$

$$\text{Measured horizontal angle } \angle PS'Q = \gamma'$$

$$= \alpha + \theta + \phi + \beta$$

$$\text{Therefore the error in measurement } E = \gamma' - \gamma$$

$$= \alpha + \beta$$

From $\Delta SS'P$, we have

$$\frac{\sin \alpha}{SS'} = \frac{\sin \theta}{S'P} \quad \ldots(4.11)$$

From $\Delta SS'Q$, we have

$$\frac{\sin \beta}{SS'} = \frac{\sin \phi}{S'Q}\,. \quad \ldots(4.12)$$

Taking $S'P = y$ and $S'Q = z$, for small angles Eqs. (4.11) and (4.12) become

$$\alpha = \frac{x \sin \theta}{y}$$

$$\beta = \frac{x \sin \phi}{z}\,.$$

Therfore

$$E = x\left(\frac{\sin \theta}{y} + \frac{\sin \phi}{z}\right)$$

$$= \frac{x}{\sin 1''}\left(\frac{\sin \theta}{y} + \frac{\sin \phi}{z}\right) \quad \text{seconds} \quad \ldots(4.13)$$

where $$\frac{x}{\sin 1''} = 206265.$$

The maximum absolute error E occurs when

(a) sin θ and cos ϕ are maximum, i.e., $\theta = \phi = 90°$.

(b) y and z are minimum.

For the given values if γ, y and z, the maximum angular error can be determined as under.

For a given value of γ,

$$\theta = \gamma - \phi$$

$$E = \frac{x}{\sin 1''}\left(\frac{\sin(\gamma-\phi)}{y}+\frac{\sin\phi}{z}\right) \qquad \ldots(4.14)$$

$$= \frac{x}{\sin 1''}\left(\frac{\sin\gamma\cos\phi-\sin\phi\cos\gamma}{y}+\frac{\sin\phi}{z}\right)$$

$$\frac{dE}{d\phi}=\frac{x}{\sin 1''}\left(\frac{-\sin\gamma\sin\phi-\cos\phi\cos\gamma}{y}+\frac{\cos\phi}{z}\right) = 0$$

$$\frac{\cos\phi}{z}=\frac{\sin\gamma\sin\phi+\cos\phi\cos\gamma}{y}$$

$$y = z(\sin\gamma\tan\phi+\cos\gamma)$$

$$\tan\phi = \frac{y-z\cos\gamma}{z\sin\gamma}. \qquad \ldots(4.15)$$

For the given values of γ, y and z, the maximum value of α can be determined from Eq. (4.15) which on substitution in Eq. (4.14) will give the maximum absolute error E_{Max}.

4.11 COMPATIBILITY OF LINEAR AND ANGULAR MEASUREMENTS

The precision achieved in a measurement depends upon the instruments used and the methods employed. Therefore, the instruments and methods to be employed for a particular survey should be so chosen that the precision in linear and angular measurements are consistent with each other. This can be achieved by making the error in angular measurement equal to the error in the linear measurement.

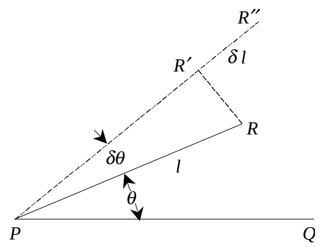


Fig. 4.12

In Fig. 4.12, a point R is located with reference to the traverse line PQ by making angular measurement θ and linear measurement l. But due to the errors $\delta\theta$ in the angular measurement R is located at R' and due to error δl in the linear measurement the location of R' is further displaced to R'' from R'.

The displacement due to the angular error

$$RR' = l \tan \delta\theta.$$

The displacement due to the linear error

$$R'R'' = \delta l \ .$$

For consistency in linear and angular errors

$$RR' = R'R''$$

or

$$l \tan \delta\theta = \delta l$$

$$\tan \delta\theta = \frac{\delta l}{l} \ .$$

For small values of $\delta\theta$, $\tan \delta\theta = \delta\theta$, thus

$$\delta\theta = \frac{\delta l}{l} \ . \qquad ...(4.16)$$

Example 4.1. The fore bearings and back bearings of the lines of a closed traverse *ABCDA* were recorded as below:

Line	Fore bearing	Back bearing
AB	77°30′	259°10′
BC	110°30′	289°30′
CD	228°00′	48°00′
DA	309°50′	129°10′

Determine which of the stations are affected by local attraction and compute the values of the corrected bearings.

Solution (Fig. 4.13):

Method-I

In this method the errors in the bearings of the lines are determined and the bearings are corrected for the respective errors.

By observing the values of the fore bearings and back bearing of the lines, it is found that the fore bearing and back bearing of the line *CD* differ exactly by 180°, i.e., 228° – 48° = 180°. Therefore both the stations *C* and *D* must be free from local attraction. Since for other lines the difference is not 180°, the stations *A* and *B* are affected by local attraction.

Since station *D* is free from local attraction, the fore bearing of *DA* must be correct.

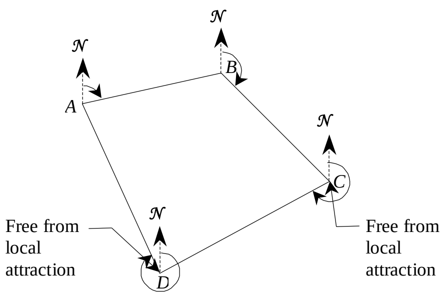


Fig. 4.13

Calculation of corrected bearings

$$\text{Correct fore bearing of } DA = 309°50' \text{ (given)}$$

$$\text{Correct back bearing of } DA = 180° + 309°50' = 129°50'$$

Observed back bearing of DA = 129°10′
Error at A = 129°10′ – 129°50′ = – 40′
Correction at A = + 40′
Observed fore bearing of AB = 77°30′
Correct fore bearing of AB = 77°30′ + 40′ = 78°10′
Correct back bearing of AB = 180° + 78°10′ = 258°10′
Observed back bearing of AB = 259°10′
Error at B = 259°10′ – 258°10′ = 1°
Correction at B = – 1°
Observed fore bearing of BC = 110°30′
Correct fore bearing of BC = 110°30′ – 1° = 109°30′
Correct back bearing of BC = 180° + 109°30′ = 289°30′. (*Check*)

Since the computed back bearing of BC is equal to its observed back bearing, the last computation provides a check over the entire computation.

Method-II

Since all the bearings observed at a station are equally affected, the difference of bearings of two lines originating from that station will be the correct included or interior angle between the lines at that station. In this method, from the observed bearings included angles are computed and starting from a correct bearing, all other bearings are computed using the included angles.

In Method-I we have found that the stations C and D are free from local attraction and therefore, the bearings observed at these stations are the correct bearings.

Calculation of included angles

Included angle A = back bearing of DA – fore bearing of AB
= 129°10′ – 77°30′ = 51°40′.
Included angle B = back bearing of AB – fore bearing of BC
= 259°10′ – 110°30′ = 148°40′.
Included angle C = back bearing of BC – fore bearing of CD
= 289°30′ – 228°00′ = 61°30′.
Included angle D = back bearing of CD – fore bearing of DA
= (48°00′ – 309°50′) + 360° = 98°10′.
Σ Included angles = 51°40′ + 148°40′ + 61°30′ + 98°10′ = 360°
Theoretical sum = $(2n - 4)$. 90° = (2 × 4 – 4) × 90° = 360°. (*Check*)

Since the sum of the included angles is equal to their theoretical sum, all the angles are assumed to be free from errors.

Calculation of corrected bearings

Correct back bearing of DA = 180° + 309°50′ – 360° = 129°50′
Correct fore bearing of AB = 129°50′ – ∠A = 129°50′ – 51°40′ = 78°10′

Correct back bearing of AB = 180° + 78°10′ = 258°10′

Correct fore bearing of BC = 258°10′ – ∠B = 258°10′ – 148°40′ = 109°30′

Correct back bearing of BC = 180° + 109°30′ = 289°30′. (*Check*)

The corrected bearings of the lines are tabulated in Table 4.2.

Table 4.2

Line	Corrected	
	Fore bearing	Back bearing
AB	78°10′	258°10′
BC	109°30′	289°30′
CD	228°00′	48°00′
DA	309°50′	129°50′

Example 4.2. The angles at the stations of a closed traverse *ABCDEFA* were observed as given below:

Traverse station	Included angle
A	120°35′00″
B	89°23′40″
C	131°01′00″
D	128°02′20″
E	94°54′40″
F	155°59′20″

Adjust the angular error in the observations, if any, and calculate the bearings of the traverse lines in the following systems if whole circle bearing of the line AB is 42°:

(a) Whole circle bearing in sexagesimal system.

(b) Quadrantal bearing in sexagesimal system.

(c) Corresponding values in centesimal system for (a).

Solution:

Adjustment of angular error

The sum of the internal angles of a polygon having n sides is (2n - 4). 90°, therefore for six sides polygon

$$\Sigma \text{ Internal angles} = (2 \times 6 - 4) \times 90° = 720°$$

$$\Sigma \text{ Observed internal angles} = 719°56'00''$$

$$\text{Total error} = 719°56'00'' - 720° = -4'$$

$$\text{Total correction} = 4' \text{ or } 240''.$$

Since the error is of some magnitude, it implies that the work is of relatively low order; therefore, the correction may be applied equally to each angle assuming that the conditions were constant at the time of observation and the angles were measured with the same precision.

Hence the correction to each angle $= \frac{240'}{6} = 40''$.

The corrected included angles are given in the following table:

Traverse station	Included angle	Correction	Adjusted value
A	120°35′00″	+ 40″	120°35′40″
B	89°23′40″	+ 40″	89°24′20″
C	131°01′00″	+ 40″	131°01′40″
D	128°02′20″	+ 40″	128°03′00″
E	94°54′40″	+ 40″	94°55′20″
F	155°59′20″	+ 40″	156°00′00″
Σ	719°56′00″	+ 240″	720°00′00″

(a) Calculation of W.C.B.

W.C.B. of *AB* = 42° (given)
W.C.B. of *BA* = 180° + 42° = 222°00′00″
W.C.B. of *BC* = W.C.B. of *BA* – ∠*B*
= 222°00′00″ – 89°24′20″ = 132°35′40″
W.C.B. of C*B* = 180° + 132°35′40″ = 312°35′40″
W.C.B. of *CD* = W.C.B. of C*B* – ∠*C*
= 312°25′40″ – 131°01’40'’ = 181°34′00″
W.C.B. of *DC* = 180° + 181°34′00″ = 361°34′00″ – 360° = 1°34′00″
W.C.B. of *DE* = W.C.B. of *DC* – ∠*D*
= 361°34′00″ – 128°03′00″ = 233°31′00″
W.C.B. of *ED* = 180° + 233°31′00″ = 413°31′00″ – 360° = 53°31′00″
W.C.B. of *EF* = W.C.B. of ED – ∠*E*
= 413°31′00″ – 94°55′20″ = 318°35′40″
W.C.B. of *FE* = 180° + 318°35′40″ = 498°35′40″ – 360° = 138°35′40″
W.C.B. of *FA* = W.C.B. of *FE* – ∠*F*
= 498°35′40″ – 156°00′00″ = 342°35′40″
W.C.B. of *AF* = 180° + 342°35′40″ = 522°35′40″ – 360° = 162°35′40″
W.C.B. of A*B* = W.C.B. of AF – ∠*A*
= 162°35′40″ – 120°35′40″ = 42°00′00″. (*Check*)

(b) Computation of Quadrantal bearings (R.B.)

W.C.B. of *AB* = 42°
AB being N-E quadrant, R.B. = N42°E
W.C.B. of *BC* = 132°35′40″
BC being S-E quadrant, R.B. = S (180° – 132°35′40″) E = S47°24′20″E
W.C.B. of *CD* = 181°34′00″

CD being S-W quadrant, R.B. = S(181°34′00″ – 180°)W = S1°34′00″W

W.C.B. of *DE* = 233°31′00″

DE being S-W quadrant, R.B. = S(233°31′00″ – 180°)W = S53°31′00″W

W.C.B. of *EF* = 318°35’40″

EF being N-W quadrant, R.B. = N(360° – 318°35’40″)W = N41°24′20″W

W.C.B. of *FA* = 342°35′40″

FA being N-W quadrant, R.B. = N(360° – 342°35′40″)W = N17°24′20″W.

W.C.B. in centesimal system

In the centesimal system a circle is divided equally in 400 gon, as against 360° in the sexagesimal system as shown in Fig. 4.14.

Thus $1 \text{ degree} = \frac{100}{90} = \frac{10}{9} \text{ gon}$

$1 \text{ minute} = \frac{10}{60 \times 9} = \frac{1}{54} \text{ gon}$

$1 \text{ second} = \frac{1}{60 \times 54} = \frac{1}{3240} \text{ gon}$

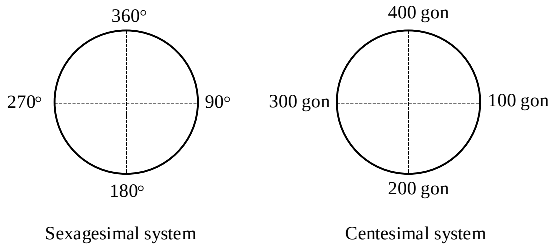


Fig. 4.14

Hence for the line *BC* its bearing 132°35′40″ in centesimal system is

$$132° = 132 \times \frac{10}{9} = 146.6667 \text{ gon}$$

$$35' = 35 \times \frac{1}{54} = 0.6481 \text{ gon}$$

$$40^2 = 40 \times \frac{1}{3240} = 0.0123 \text{ gon}$$

$$\text{Total} = 147.327 \text{ gon}$$

Making similar calculations other bearings can be converted into centesimal system. All the results are given in Table 4.3.

Table 4.3

Line	Bearing		
	Sexagesimal system		Centesimal system
	W.C.B.	R.B.	W.C.B.
AB	42°00′00″	N42°00′00″E	46.6667 gon
BC	132°35′40″	S47°24′20″E	147.3271 gon
CD	181°34′00″	S1°34′00″W	201.7407 gon
DE	233°31′00″	S53°31′00″W	259.4630 gon
EF	318°35′40″	S41°24′20″W	353.9938 gon
FA	342°35′40″	N17°24′20″W	380.6605 gon

Example 4.3. A closed-loop traverse *ABCDA* was run around an area and the following observations were made:

Station at	Station to	Length (m)	Included angle	W.C.B.
A	*B*	187.4	86°30′02″	140°11′40″
B	*C*	382.7	80°59′34″	
C	*D*	106.1	91°31′29″	
D	*A*	364.8	100°59′15″	

Adjust the angular error, if any, and calculate the coordinates of other stations if the coordinates of the station *A* are E1000 m and N1000 m.

Solution:

For systematic computations, the observations are recorded in a tabular form suggested by Gale. The Gale's traverse table is given in Table 4.4 and all results of various computations have been entered in the appropriate columns of the table.

Adjustment of angular error

Σ Included angles = 86°30′02″ + 80°59′34″ + 91°31′29″ + 100°59′15″ = 360°00′2040″

Theoretical sum of the included angles = $(2n - 4). 90° = (2 \times 4 - 4) \times 90° = 360°$

Total error = 360°00′2040″ – 360° = + 20″

Total correction = – 20″.

Assuming the conditions of observation at different stations constant, the total correction can be distributed equally to each angle.

$$\text{Thus the correction to individual angle} = \frac{20}{4} = -5''.$$

Therefore the corrected included angles are

$\angle A$ = 86°30′02″ – 5″ = 86°29′57″

$\angle B$ = 80°59′34″ – 5″ = 80°59′29″

$\angle C$ = 91°31′29″ – 5″ = 91°31′24″

$\angle D$ = 100°59′15″ – 5″ = 100°59′10″

Total = 360°00′00''. (*Check*)

Table 4.4 Gale's traverse table

Station		Length (m)	Included angle	Correction	Corrected angle	W.C.B.	Consecutive coordinates		Independent	
at	to						Departure	Latitude	Easting (E)	Northing (N)
A	*B*	187.4	86°30′02″	–5″	86°29′57″	140°11′40″	120.0	–144.0	1000	1000
B	*C*	382.7	80°59′34″	–5″	80°59′29″	41°11′09″	252.0	288.0	1120.0	856.0
C	*D*	106.1	91°31′29″	–5″	91°31′24″	312°42′33″	– 78.0	72.0	1370.0	1144.0
D	*A*	364.8	100°59′15″	–5″	100°59′10″	233°41′43″	– 294.0	– 216.0	1294.0	1216.0
Σ			360°00′20″	–20″	360°00′00″		0.0	0.0		

Computation of W.C.B.

W.C.B. of AB = 140°11′40″ (given)

W.C.B. of BA = 180° + 140°11′40″ = 320°11′40″

W.C.B. of BC = W.C.B. of BA + $\angle B$

= 320°11′40″ + 80°59′29″ – 360° = 41°11′09″

W.C.B. of BA = 180° + 40°11′09″ = 221°11′09″

W.C.B. of CD = W.C.B. of CB + ∠C

= 221°11′09″ + 91°31′24″ = 312°42′33″

W.C.B. of DC = 180° + 312°42′33″ – 360° = 132°42′33″

W.C.B. of DA = W.C.B. of DC + $\angle D$

= 132°42′33″ + 100°59′10″ = 233°41′43″

W.C.B. of AD = 180° + 233°41′43″ – 360° = 53°41′43″

W.C.B. of AB = W.C.B. of AD + $\angle A$

= 53°41′43″ + 86°29′57″ = 140°11′40″. (*Check*)

Computation of consecutive coordinates

Departure of a line $D = l \sin \theta$

Latitude of a line $L = l \cos \theta$.

Line *AB*

$$D_{AB} = 187.4 \times \sin 140°11'40'' = +\ 120.0 \text{ m}$$

$$L_{AB} = 187.4 \times \cos 140°11'40'' = -\ 144.0 \text{ m.}$$

Line *BC*

$$D_{BC} = 382.7 \times \sin 41°11'09'' = +252.0 \text{ m}$$

$$L_{BC} = 382.7 \times \cos 41°11'09'' = -\ 288.0 \text{ m.}$$

Line *CD*

$$D_{CD} = 106.1 \times \sin 312°42'33'' = -78.0 \text{ m}$$
$$L_{CD} = 106.1 \times \cos 312°42'33'' = +72.0 \text{ m.}$$

Line *DA*

$$D_{DA} = 364.8 \times \sin 233°41'43'' = -294.0 \text{ m}$$
$$L_{DA} = 364.8 \times \cos 233°41'43'' = -216.0 \text{ m.}$$
$$\Sigma D = 120.0 + 252.0 - 78.0 - 294.0 = 0.0 \qquad (Check)$$
$$\Sigma L = -144.0 + 288.0 + 72.0 - 216.0 = 0.0 \qquad (Check)$$

Computation of independent coordinates (Easting and Northing)

Coordinates of *A*

$$E_A = 1000.0 \text{ m (given)}$$
$$N_A = 1000.0 \text{ m (given)}$$

Coordinates of *B*

$$E_B = E_A + D_{AB} = 1000.0 + 120.0 = \mathbf{1120.0\ m}$$
$$N_B = N_A + L_{AB} = 1000.0 - 144.0 = \mathbf{856.0\ m}.$$

Coordinates of *C*

$$E_C = E_B + D_{BC} = 1120.0 + 252.0 = \mathbf{1370.0\ m}$$
$$N_C = N_B + L_{BC} = 856.0 + 288.0 = \mathbf{1144.0\ m}$$

Coordinates of *D*

$$E_D = E_C + D_{CD} = 1370.0 - 78.0 = \mathbf{1294.0\ m}$$
$$N_D = N_C + L_{CD} = 1144.0 + 72.0 = \mathbf{1216.0\ m}$$

Coordinates of *A*

$$E_A = E_D + D_{DA} = 1294.0 - 294.0 = \mathbf{1000.0\ m} \qquad (Check)$$
$$N_A = N_A + L_{DA} = 1216.0 - 216.0 = \mathbf{1000.0\ m}. \qquad (Check)$$

Example 4.4. The data given in Table 4.5, were obtained for an anti-clockwise closed-loop traverse. The coordinates of the station A are E1500 m and N1500 m. Determine the correct coordinates of all the traverse stations after adjusting the traverse by

(*i*) Bowditch's method

(*ii*) Transit rule.

Table 4.5

Internal angles	Length (m)	Bearing
$\angle A = 130°18'45''$	$AB = 17.098$	$AF = 136°25'12''$
$\angle B = 110°18'23''$	$BC = 102.925$	
$\angle C = 99°32'35''$	$CD = 92.782$	
$\angle D = 116°18'02''$	$DE = 33.866$	
$\angle E = 119°46'07''$	$EF = 63.719$	
$\angle F = 143°46'20''$	$FA = 79.097$	

Solution (Fig. 4.15):

(*i*) Adjusting the traverse by Bowditch's method

Adjustment of angular errors

$$\sum \text{Internal angles} = 720°00'12''$$

$$\text{Expected } \sum \text{Internal angles} = (2n - 4).\ 90° = (2 \times 6 - 4) \times 90° = 720°$$

$$\text{Total error} = +\ 12''$$

$$\text{Total correction} = -\ 12''.$$

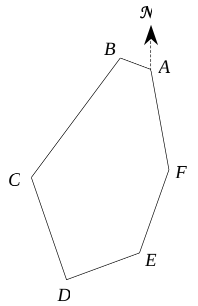


Fig. 4.15

Thus, assuming that all the angles have been measured with same precision, the correction to the individual angles

$$= -\frac{12}{6} = -\ 2''.$$

The corrected internal angles are given in Table 4.5.

Computation of bearings

Since the traverse is an anti-clockwise traverse, the measured bearing of the line *AF* is the back bearing (B.B.) of *FA*. The computation of bearings of other lines can be done in anti-clockwise direction by using the B.B. of *AF* or in clockwise direction using the fore bearing (F.B.) of *FA*, and the adjusted internal angles. Here the first approach has been used.

$$\text{F.B.}_{FA} = \text{B.B.}_{AF} = 136°25'12'' \text{ (given)}$$

$$\text{F.B.}_{AB} = \text{B.B.}_{AF} + \angle A$$

$$= 136°25'12'' + 130°18'43'' = 266°43'55''$$

$$\text{B.B.}_{AB} = 180° + 266°43'55'' = 86°43'55''$$

$$\text{F.B.}_{BC} = \text{B.B.}_{AB} + \angle B$$

$$= 86°43'55'' + 110°18'21'' = 197°02'16''$$

$$\text{B.B.}_{BC} = 180° + 197°02'16^2 = 17°02'16''$$

$$\text{F.B.}_{CD} = \text{B.B.}_{BC} + \angle C$$

$$= 17°02'16'' + 99°32'33'' = 116°34'49''$$

$$\text{B.B.}_{CD} = 180° + 116°34'49'' = 296°34'49''$$

$$\text{F.B.}_{DE} = \text{B.B.}_{CD} + \angle D$$

$$= 296°34'49'' + 116°18'00'' = 52°52'49''$$

$$\text{B.B.}_{DE} = 180° + 52°52'49'' = 232°52'49''$$

$$\text{F.B.}_{EF} = \text{B.B.}_{DE} + \angle E$$

$$= 232°52'49'' + 119°46'05'' = 352°38'54''$$

$$\text{B.B.}_{EF} = 180° + 352°38'54'' = 172°38'54''$$

$$\text{F.B.}_{FA} = \text{B.B.}EF + \angle F$$

$$= 172°38'54'' + 143°46'18'' = 316°25'12''$$

$$\text{B.B.}_{FA} = 180° + 316°25'12'' = 136°25'12''. \qquad (\textit{Check})$$

Computation of consecutive coordinates

Line *AB*

$$D_{AB} = l_{AB} \sin \theta_{AB} = 17.098 \times \sin 266°43'55'' = -17.070 \text{ m}$$

$$L_{AB} = l_{AB} \cos \theta_{AB} = 17.098 \times \cos 266°43'55'' = -0.975 \text{ m.}$$

Line *BC*

$$D_{BC} = l_{BC} \sin \theta_{BC} = 102.925 \times \sin 197°02'16'' = -30.157 \text{ m}$$

$$L_{BC} = l_{BC} \cos \theta_{BC} = 102.925 \times \cos 197°02'16'' = -98.408 \text{ m.}$$

Line *CD*

$$D_{CD} = l_{CD} \sin \theta_{CD} = 92.782 \times \sin 116°34'49'' = +82.976 \text{ m}$$

$$L_{CD} = l_{CD} \cos \theta_{CD} = 92.782 \times \cos 116°34'49'' = -41.515 \text{ m.}$$

Line *DE*

$$D_{DE} = l_{DE} \sin \theta_{DE} = 33.866 \times \sin 52°52'49'' = +27.004 \text{ m}$$

$$L_{DE} = l_{DE} \cos \theta_{DE} = 33.866 \times \cos 52°52'49'' = +20.438 \text{ m.}$$

Line *EF*

$$D_{EF} = l_{EF} \sin \theta_{EF} = 63.719 \times \sin 352°38'54'' = -8.153 \text{ m}$$

$$L_{EF} = l_{EF} \cos \theta_{EF} = 63.719 \times \cos 352°38'54'' = +63.195 \text{ m.}$$

Line *FA*

$$D_{FA} = l_{FA} \sin \theta_{FA} = 79.097 \times \sin 316°25'12'' = -54.527 \text{ m}$$

$$L_{FA} = l_{FA} \cos \theta_{FA} = 79.097 \times \cos 316°25'12'' = +57.299 \text{ m.}$$

Algebraic sum of departures = total error in departure = $\Sigma D = +0.073$ m

Algebraic sum of latitudes = total error in latitude = $\Sigma L = +0.034$ m

Arithmetic sum of departures $D_T = 219.887$ m

Arithmetic sum of latitudes $L_T = 281.830$ m

Balancing the traverse

(*i*) By Bowditch's method

Correction to (departure/latitude) of a line

$$= -\text{ Algebraic sum of (departure/latitude)} \frac{\text{length of that line}}{\text{perimeter of the traverse}}$$

$$c_D = -\Sigma D \frac{l}{\Sigma l}$$

$$c_L = -\Sigma L \frac{l}{\Sigma l}$$

$$c_{D,AB} = -0.073 \times \frac{17.098}{389.487} = -0.003 \text{ m}$$

$$c_{D,BC} = -0.073 \times \frac{102.925}{389.487} = -\ 0.019 \text{ m}$$

$$c_{D,CD} = -0.073 \times \frac{92.782}{389.487} = -\ 0.017 \text{ m}$$

$$c_{D,DE} = -0.073 \times \frac{33.866}{389.487} = -\ 0.006 \text{ m}$$

$$c_{D,EF} = -0.073 \times \frac{63.719}{389.487} = -\ 0.013 \text{ m}$$

$$c_{D,FA} = -0.073 \times \frac{79.097}{389.487} = -\ 0.015 \text{ m}$$

Total $= -\ 0.073$ m (*Check*)

$$c_{L,AB} = -0.034 \times \frac{17.098}{389.487} = -\ 0.001 \text{ m}$$

$$c_{L,BC} = -0.034 \times \frac{102.925}{389.487} = -\ 0.009 \text{ m}$$

$$c_{L,CD} = -0.034 \times \frac{92.782}{389.487} = -\ 0.008 \text{ m}$$

$$c_{L,DE} = -0.034 \times \frac{33.866}{389.487} = -\ 0.003 \text{ m}$$

$$c_{L,EF} = -0.034 \times \frac{63.719}{389.487} = -\ 0.006 \text{ m}$$

$$c_{L,FA} = -0.034 \times \frac{79.097}{389.487} = -\ 0.007 \text{ m}$$

Total $= -\ 0.034$ m (*Check*)

Corrected consecutive coordinates

$D'_{AB} = -\ 17.070 - 0.003 = -\ 17.073$ m, $L'_{AB} = -\ 0.975 - 0.001 = -\ 0.976$ m

$D'_{BC} = -\ 30.157 - 0.019 = -\ 30.176$ m, $L'_{BC} = -\ 98.408 - 0.009 = -\ 98.417$ m

$D'_{CD} = +\ 82.976 - 0.017 = +\ 82.959$ m, $L'_{CD} = -\ 41.515 - 0.008 = -\ 41.523$ m

$D'_{DE} = +\ 27.004 - 0.006 = +\ 26.998$ m, $L'_{DE} = +\ 20.438 - 0.003 = +\ 20.435$ m

$D'_{EF} = -\ 8.153 - 0.013 = -\ 8.166$ m, $L'_{EF} = +\ 63.195 - 0.006 = +\ 63.189$ m

D'_{FA} = – 54.527 – 0.015 = – 54.542 m, L'_{FA} = + 57.299 – 0.007 = + 57.292 m

$\Sigma D'$ = 0.000 m (*Check*) $\Sigma L'$ = 0.000 m (*Check*)

Independent coordinates

E_A = 1500 m (given), N_A = 1500 m (given)

$E_B = E_A + D'_{AB}$ = 1500 – 17.073 = **1482.927 m**

$N_B = N_A + L'_{AB}$ = 1500 – 0.976 = **1499.024 m**

$E_C = E_B + D'_{BC}$ = 1482.927 – 30.176 = **1452.751 m**

$N_C = N_B + L'_{BC}$ = 1499.024 – 98.417 = **1400.607 m**

$E_D = E_C + D'_{CD}$ = 1452.751 + 82.959 = **1535.710 m**

$N_D = N_C + L'_{CD}$ = 1400.607 – 41.523 = **1379.519 m**

$E_E = E_D + D'_{DE}$ = 1535.710 + 26.998 = **1562.708 m**

$N_D = N_D + L'_{DE}$ = 1359.084 + 20.435 = **1379.519 m**

$E_F = E_E + D'_{EF}$ = 1562.708 – 8.166 = **1554.542 m**

$N_F = N_F + L'_{EF}$ = 1379.519 + 63.189 = **1442.708 m**

$E_A = E_F + D'_{FA}$ = 1554.542 – 54.542 = 1500 m (*Check*)

$N_A = N_F + L'_{FA}$ = 1442.708 + 57.292 = 1500m. (*Check*)

The results of all the computations are given in Table 4.6.

(*ii*) By Transit rule

Correction to (departure/latitude) of a line

$$= -\text{ Algebraic sum of departure/latitude)} \frac{\text{(departure/latitude) of that line}}{\text{arithmetic sum of (departure/latitude)}}$$

$$c_D = -\Sigma D \frac{D}{D_T}$$

$$c_L = -\Sigma L \frac{L}{L_T}$$

Table 4.6

Station	Length (m)	Internal angle	Correction	Corrected Int. angle	Bearing	Consecutive coordinates	
						Departure	Latitude
A	17.098	130°18′45″	– 2″	130°18′43″	266°43′55″	– 17.070	– 0.975
B	102.925	110°18′23″	– 2″	110°18′21″	197°02′16″	– 30.157	– 98.408
C	92.782	99°32′35″	– 2″	99°32′33″	116°34′49″	+ 82.976	– 41.515
D	33.866	116°18′02″	– 2″	116°18′00″	52°52′49″	+ 27.004	+ 20.438
E	63.719	119°46′07″	– 2″	119°46′05″	352°38′54″	– 8.153	+ 63.195
F	79.097	143°46′20″	– 2″	143°46′18″	316°25′12″	– 54.527	+ 57.299
Σ	389.487	720°00′12″	– 12″	720°00′00″		+ 0.073	– 0.034

Table 4.6 (*continued*)

Correction by Bowditch's method		Corrected consecutive coordinates		Independent coordinates	
Departure	Latitude	Departure	Latitude	Easting (m)	Northing (m)
– 0.003	– 0.001	– 17.073	– 0.976	1500	1500
– 0.019	– 0.009	– 30.176	– 98.417	1482.927	1499.024
– 0.017	– 0.008	+ 82.959	– 41.523	1452.751	1400.607
– 0.006	– 0.003	+ 26.998	+ 20.435	1535.710	1359.084
– 0.013	– 0.006	– 8.166	+ 63.189	1562.708	1379.519
– 0.015	– 0.007	– 54.542	+ 57.292	1554.542	1442.708
– 0.073	– 0.034	0.000	0.000		

$$c_{D,AB} = -0.073 \times \frac{17.070}{219.887} = -\ 0.006 \text{ m}$$

$$c_{D,BC} = -0.073 \times \frac{30.157}{219.887} = -\ 0.010 \text{ m}$$

$$c_{D,CD} = -0.073 \times \frac{82.976}{219.887} = -\ 0.027 \text{ m}$$

$$c_{D,DE} = -0.073 \times \frac{27.004}{219.887} = -\ 0.009 \text{ m}$$

$$c_{D,EF} = -0.073 \times \frac{8.153}{219.887} = -\ 0.003 \text{ m}$$

$$c_{D,FA} = -0.073 \times \frac{54.527}{219.887} = -\ 0.018 \text{ m}$$

$$\text{Total} = -\ 0.073 \text{ m} \quad (\textit{Check})$$

$$c_{L,AB} = -0.034 \times \frac{0.957}{281.830} = -\ 0.000 \text{ m}$$

$$c_{L,BC} = -0.034 \times \frac{98.408}{281.830} = -\ 0.012 \text{ m}$$

$$c_{L,CD} = -0.034 \times \frac{41.515}{281.830} = -\ 0.005 \text{ m}$$

$$c_{L,DE} = -0.034 \times \frac{20.438}{281.830} = -\ 0.002 \text{ m}$$

$$c_{L,EF} = -0.034 \times \frac{63.195}{281.830} = -\ 0.008 \text{ m}$$

$$c_{L,FA} = -0.034 \times \frac{57.299}{281.830} = -\ 0.007 \text{ m}$$

$$\text{Total} = -\ 0.034 \text{ m} \quad (Check)$$

Corrected consecutive coordinates

$D'_{AB} = -\ 17.070 - 0.006 = -\ 17.076$ m, $L'_{AB} = -\ 0.975 - 0.000 = -\ 0.975$ m

$D'_{BC} = -\ 30.157 - 0.010 = -\ 30.167$ m, $L'_{BC} = -\ 98.408 - 0.012 = -\ 98.420$ m

$D'_{CD} = +\ 82.976 - 0.027 = +\ 82.949$ m, $L'_{CD} = -\ 41.515 - 0.005 = -\ 41.520$ m

$D'_{DE} = +\ 27.004 - 0.009 = +\ 26.995$ m, $L'_{DE} = +\ 20.438 - 0.002 = +\ 20.436$ m

$D'_{EF} = -\ 8.153 - 0.003 = -\ 8.156$ m, $L'_{EF} = +\ 63.195 - 0.008 = +\ 63.187$ m

$D'_{FA} = -\ 54.527 - 0.018 = -\ 54.545$ m, $L'_{FA} = +\ 57.299 - 0.007 = +\ 57.292$ m

$\Sigma D' = 0.000$ m (*Check*) $\Sigma L' = 0.000$ m (*Check*)

Independent coordinates

$E_A = 1500$ m (given), $N_A = 1500$ m (given)

$E_B = E_A + D'_{AB} = 1500 - 17.076 =$ **1482.924 m**

$N_B = N_A + L'_{AB} = 1500 - 0.975 =$ **1499.025 m**

$E_C = E_B + D'_{BC} = 1482.924 - 30.167 =$ **1452.757 m**

$N_C = N_B + L'_{BC} = 1499.025 - 98.420 =$ **1400.605 m**

$E_D = E_C + D'_{CD} = 1452.757 + 82.949 =$ **1535.706 m**

$N_D = N_C + L'_{CD} = 1400.605 - 41.520 =$ **1359.085 m**

$E_E = E_D + D'_{DE} = 1535.706 + 26.995 =$ **1562.701 m**

$N_D = N_D + L'_{DE} = 1359.085 + 20.436 =$ **1379.521 m**

$E_F = E_E + D'_{EF} = 1562.701 - 8.156 =$ **1554.545 m**

$N_F = N_F + L'_{EF} = 1379.521 + 63.187 =$ **1442.708 m**

$E_A = E_F + D'_{FA} = 1554.545 - 54.545 = 1500$ m (*Check*)

$N_A = N_F + L'_{FA} = 1442.708 + 57.292 = 1500$ m. (*Check*)

The results of all the computations for Transit rule are given in Table 4.7.

Table 4.7

Correction by Transit rule		Corrected consecutive coordinated		Independent coordinates	
Departure	Latitude	Departure	Latitude	Easting (m)	Northing (m)
– 0.006	– 0.000	– 17.076	– 0.975	1500	1500
– 0.010	– 0.012	– 30.167	– 98.420	1482.924	1499.025
– 0.027	– 0.005	+ 82.949	– 41.520	1452.757	1400.605
– 0.009	– 0.002	+ 26.995	+ 20.436	1535.706	1359.085
– 0.003	– 0.008	– 8.156	+ 63.187	1562.701	1379.521
– 0.018	– 0.007	– 54.545	+ 57.292	1554.545	1442.708
– 0.073	– 0.034	0.000	0.000		

Example 4.5. The following data were collected while running a closed traverse *ABCDA*. Calculate the missing data.

Line	Length (m)	Bearing
AB	330	181°25′
BC	?	89°50′
CD	411	355°00′
DA	827	?

Solution (Fig. 4.16):

Let the length of *BC* be l and the bearing of *DA* be θ then the consecutive coordinates of the lines are

$$D_{AB} = l_{AB} \sin \theta_{AB} = 330 \times \sin 181°25' = -8.159 \text{ m}$$

$$L_{AB} = l_{AB} \cos \theta_{AB} = 330 \times \cos 181°25' = -329.899 \text{ m}$$

$$D_{BC} = l_{BC} \sin \theta_{BC} = l \times \sin 89°50'$$

$$L_{BC} = l_{BC} \cos \theta_{BC} = l \times \cos 89°50'$$

$$D_{CD} = l_{CD} \sin \theta_{CD} = 411 \times \sin 355°00' = -35.821 \text{ m}$$

$$L_{CD} = l_{CD} \cos \theta_{CD} = 411 \times \cos 355°00' = +409.436 \text{ m}$$

$$D_{DA} = l_{DA} \sin \theta_{DA} = 827 \times \sin \theta$$

$$L_{DA} = l_{DA} \cos \theta_{DA} = 827 \times \cos \theta$$

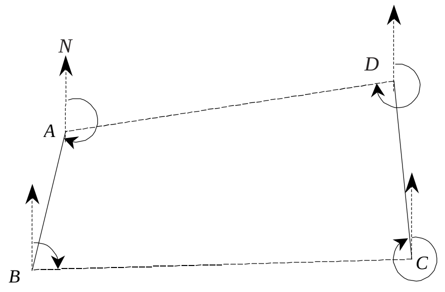


Fig. 4.16

In the closed traverse *ABCDA*

$$\Sigma D = 0.0$$

$$\Sigma L = 0.0$$

$$-8.159 + l \times \sin 89°50 - 35.821 = 827 \times \sin \theta$$

$$-329.899 + l \times \cos (89°50') + 409.436 = 827 \times \cos \theta$$

$$827 \times \sin \theta = -43.980 + 0.999\, l \qquad \ldots(a)$$

$$827 \times \cos \theta = +79.537 + 0.003\, l \qquad \ldots(b)$$

Taking 0.003 l = 0 in Eq. (a), we get

$$\cos \theta = \frac{79.537}{827}$$

$$\theta = 84°29'.$$

Substituting the value of θ in Eq. (a), we get

$$l = \frac{43.980 + 872 \times \sin 84^\circ 29'}{0.999}$$

$$= 867.15 \text{ m.}$$

Now substituting the value of l = 867.15 m in Eq. (b), we get the value of θ in the second iteration as

$$\cos \theta = \frac{79.537 + 0.003 \times 867.15}{827}$$

$$= 84°18'$$

and the value of l as

$$l = \frac{43.980 + 827 \times \sin 84^\circ 18'}{0.999}$$

$$= 867.78 \text{ m.}$$

Taking the value of θ and l for the third iteration, we get

$$\cos \theta = \frac{79.537 + 0.003 \times 867.78}{827}$$

$$\theta = 84°18¢.$$

Since the value of θ has not changed, the value of l will be same as in the second iteration, and therefore, the length of BC = **867.78 m** and the bearing of DA = **84°18′**.

Alternative solution Fig. (4.17):

In this method, the two sides *BC* and *DA* with omitted measurements, are made adjacent lines by drawing parallel lines. In this way the lines *DA* and *DA′* are the adjacent lines in the traverse *ADA′BA*. To achieve this, draw *DA′* from *D* parallel to and equal to *BC* and *BA′* from *B* parallel to and equal to *CD*.

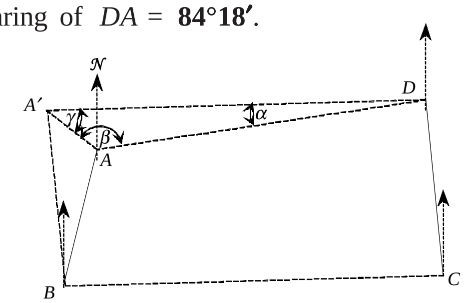


Fig. 4.17

Considering the length and bearing of the line *AA′* in the closed traverse *ABA′A* as l and θ, respectively, we get

$$D_{AB} + D_{BA'} + D_{A'A} = 0$$

$$L_{AB} + L_{BA'} + L_{A'A} = 0$$

$$D_{A'A} = l \sin \theta = -330 \times \sin 181°25' - 411 \times \sin 355° = +43.980$$

$$L_{A'A} = l \cos \theta = -33 \times \cos 181°25' - 411 \times \cos 355° = -79.537$$

$$\tan \theta = \frac{l \sin \theta}{l \cos \theta} = \frac{43.980}{79.537}$$

$$\theta = 28°56'26''$$

(in S-E quadrant, since departure is positive and latitude is negative)

or W.C.B. of $AA' = 180° - 28°56'26'' = 151°03'34''$

and $AA' = l = \sqrt{D^2{}_{A'A} + L^2{}_{A'A}} = \sqrt{43.980^2 + 79.537^2} = 90.887$ m.

In $\Delta AA'D$, we have

$$\frac{AD}{\sin \beta} = \frac{AA'}{\sin \alpha} = \frac{AD}{\sin \gamma}$$

$$\gamma = \text{bearing of } A'A - \text{bearing of } A'D$$

$$= \text{bearing of } A'A - \text{bearing of } BC$$

$$= 51°03'34'' - 89°50'$$

$$= 61°13'34''.$$

$$\sin \alpha = \frac{AA' \sin \gamma}{AD}$$

$$= \frac{90.887 \times \sin 61°13'34''}{827} = 0.09633$$

$$\alpha = 5°31'40''.$$

In $\Delta AA'D$, we have

$$\beta = 180° - (\alpha + \gamma)$$

$$= 180° - (5°31'40'' + 61°13'34'')$$

$$= 113°14'46''.$$

$$A'D = \frac{AD \sin \beta}{\sin \gamma}$$

or

$$BC = \frac{827 \times \sin 113°14'46''}{\sin 61°13'34''}$$

= **866.90 m**.

Bearing of DA = bearing of $DA' - \alpha$

$= (180° + 89°50') - 5°31'40''$

$= \mathbf{264°18'20''}$.

Example 4.6. P, Q, R, and S are four stations whose coordinates are as below:

Station	Easting (m)	Northing (m)
P	1000.00	1000.00
Q	1180.94	1075.18
R	1021.98	1215.62
S	939.70	1102.36

Another station X is to be fixed at the intersection of the lines PR and QS. What are the coordinates of X ?

Solution Fig. (4.18):

$$PQ = \sqrt{QB^2 + PB^2} = \sqrt{(1075.18 - 1000.00)^2 + (1180.94 - 1000.00)^2} = 195.937 \text{ m}$$

$$QR = \sqrt{QC^2 + CR^2} = \sqrt{(1215.62 - 1075.18)^2 + (1180.94 - 1021.98)^2} = 212.112 \text{ m}$$

$$PR = \sqrt{PD^2 + DR^2} = \sqrt{(1215.62 - 1000.00)^2 + (1021.98 - 1000.00)^2} = 216.737 \text{ m}$$

$$SQ = \sqrt{SE^2 + EQ^2} = \sqrt{(1102.36 - 1075.18)^2 + (1180.94 - 939.70)^2} = 242.766 \text{ m}$$

$$SP = \sqrt{SA^2 + AP^2} = \sqrt{(1102.36 - 1000.00)^2 + (1000.00 - 939.70)^2} = 118.801 \text{ m}$$

From ΔPRQ, we have

$$\cos\alpha = \frac{PR^2 + PQ^2 - RQ^2}{2PR.PQ}$$

$$= \frac{216.373^2 + 195.937^2 - 212.112^2}{2 \times 216.373 \times 195.937}$$

$$\alpha = 61°37'00''.$$

From ΔSPQ, we have

$$\cos\beta = \frac{SQ^2 + PQ^2 - SP^2}{2SQ.PQ}$$

$$= \frac{242.766^2 + 195.937^2 - 118.801^2}{2 \times 242.766 \times 195.937}$$

$$= 28°59'28''.$$

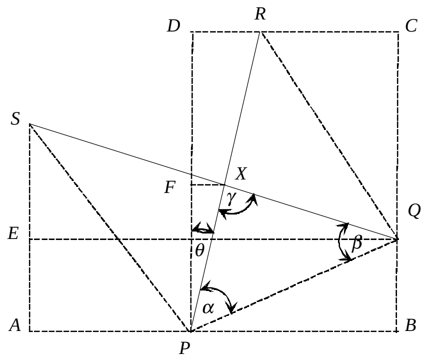


Fig. 4.18

From ΔPQB, we have

$$\tan\,\omega = \frac{QB}{PB}$$

$$= \frac{(1075.180 - 1000.00)}{(1180.940 - 1000.00)}$$

$$\omega = 22°23'46''.$$

From ΔPQX, we have

$$= 180° - (\alpha + \beta) = 180° - (61°37'00'' + 28°59'28'')$$

$$= 89°23'32''.$$

From sin law in ΔPQX, we get

$$\frac{PX}{\sin\beta} = \frac{PQ}{\sin\gamma}$$

$$PX = \frac{PQ.\sin\beta}{\sin\gamma}$$

$$= \frac{195.937 \times \sin\,(28°59'28'')}{\sin\,(89°23'32'')} = 94.971 \text{ m.}$$

Let bearing of PX be θ then

$$\theta = 90° - (\alpha + \omega)$$

$$\theta = 90° - (61°37'00'' + 22°23'46'') = 5°59'14''.$$

$$\text{Departure of } PX = PX \sin\,\theta = 94.971 \times \sin\,(5°59'14'') = +\,9.91 \text{ m}$$

$$\text{Latitude of } PX = PX \cos\,\theta = 94.971 \times \cos\,(5°59'14'') = 94.45 \text{ m}$$

Coordinates of X

Easting of X = easting of P + departure of PX = 1000.00 + 9.91 = 1009.91 m

Northing of X = northing of P + latitude of PX = 1000.00 + 94.45 = 1094.45 m

Therefore the coordinates of X are **E1009.91 m** and **N1094.45 m**.

Alternative solution-I (Fig. 4.19):

Let the coordinates of X be (X, Y) and

$$XD = m, \quad DQ = n$$

$$XB = x, \quad BR = y$$

From Δs RBX, and RAP, we have

$$\frac{y}{x} = \frac{RA}{PA}$$

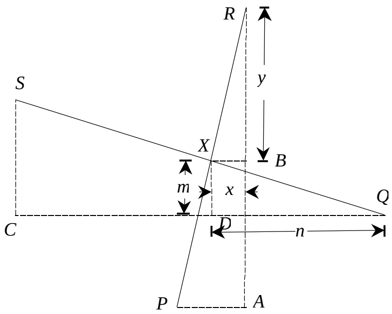


Fig. 4.19

or
$$\frac{1215.62 - Y}{1021.98 - X} = \frac{1215.62 - 1000.00}{1021.98 - 1000.00} = \frac{215.62}{21.98}$$

$$Y = +\,9.810\ X - 8809.827. \qquad \ldots\text{(a)}$$

From Δs XDQ, and QSC, we have

$$\frac{m}{n} = \frac{SC}{CQ}$$

or
$$\frac{Y - 1075.18}{1180.94 - X} = \frac{1102.36 - 1075.18}{1180.94 - 939.70} = \frac{27.18}{241.24}$$

$$Y = -\,0.113\ X + 1208.234. \qquad \ldots\text{(b)}$$

Equating Eqs. (a) and (b), we get

$$9.810\ X - 8809.827 = -\,0.113\ X + 1208.234$$

$$90.923\ X = 10018.061$$

$$X = 1009.58 \text{ m}.$$

Substituting the value of X in (a), we get

$$Y = 9.810 \times 1009.58 - 8809.827$$

$$= 1094.15 \text{ m}$$

Thus the coordinates of X are **E 1009.58 m**, **N 1094.15 m**.

Alternative solution-II (Fig. 4.19):

Since PR and QS are two straight lines, their intersection can be determined if their equations are known.

Equation of a straight line is

$$y = ax + b.$$

Equation of the line PR

$$y_1 = a_1x_1 + b_1$$

$$a_1 = \frac{RA}{PA} = \frac{215.62}{21.98} = 9.801$$

At the point P, $x_1 = 1000.00$ and $y_1 = 1000.00$

therefore, $1000.0 = 9.801 \times 1000.00 + b_1$

$$b_1 = -8801$$

Similarly, equation of the line SQ

$$y_2 = a_2 x_2 + b_2$$

$$a_2 = \frac{SC}{CQ} = \frac{-27.18}{241.24} = -0.113$$

At the point S, $x_2 = 939.70$ and $y_2 = 1102.36$

therefore, $1102.36 = -0.113 \times 939.70 + b_2$

$$b_2 = 1208.55.$$

Thus the equations of the lines PR and SQ are

$$y_1 = 9.801\ x_1 - 8801$$

$$y_2 = -0.113\ x_2 + 1208.55.$$

At the intersection X of the two lines $y_1 = y_2 = y$ and $x_1 = x_2 = x$, we have

$$9.801\ X - 8801 = -0.113\ X + 1208.55$$

$$X = 1009.64 \text{ m}$$

$$Y = 9.801 \times 1009.64 - 8801 = 1094.48 \text{ m}.$$

Thus the coordinates of X are **E 1009.64 m**, **N 1094.48 m**.

Example 4.7. A theodolite was set up at station PO and horizontal and vertical angles were observed as given in the following table:

Station	Face	Horizontal circle reading	Vertical circle reading
P	L	26°36′22″	– 40°17′18″
Q	L	113°25′50″	+ 26°14′32″

Calculate the true value of angle POQ when the line of collimation is inclined to the trunion axis by (90° – c) and the trunion axis is not perpendicular to the vertical axis by (90° – i) where c = 22″ and i = 16″ down at right.

Solution (Figs. 4.2 and 4.3):

The error e_c in horizontal circle reading for face left for the line of collimation to the trunion axis, is given by

$$e_c = z = c \sec h$$

For the sighting OP

$$e_{cOP} = +22'' \sec 40°17'18'' = +28.9''$$

$$\text{Correction} = -\ 28.9''$$

and for the sighting *OQ*

$$e_{cOQ} = +\ 22'' \sec 26°14'32'' = +\ 24.5''$$

$$\text{Correction} = -\ 24.5''$$

It may be noted that the error in the horizontal circle readings remains same for the angles of elevation and depression. It changes sign only when the face is changed.

The error in the horizontal circle reading for the trunion axis not perpendicular to the vertical axis, is given by

$$e_i = i \tan h$$

and for the sighting *OP*

$$e_{iOP} = +\ 16'' \tan 40°17'18'' = +\ 13.6'' \quad \text{(for depression angle)}$$

$$\text{Correction} = -\ 13.6''.$$

For the sighting *OQ*

$$e_{iOQ} = -\ 16'' \tan 26°14'32'' = -\ 7.9'' \quad \text{(for elevation angle)}$$

$$\text{Correction} = +\ 7.9''.$$

It may be noted that for this error in horizontal circle readings the signs are different for the angles of depression and elevation.

Total correction for the sighting *OP* = – 28.9″ – 13.6″ = – 42.5″

Total correction for the sighting *OQ* = – 24.5″ + 7.9″ = – 16.6″

Therefore,

the correct horizontal circle reading for *OP* = 26°36′22″ – 42.5″
= 26°35′39.5″

the correct horizontal circle reading for *OQ* = 113°25′50″ – 16.6″
= 113°25′33.4″

Therefore

the correct horizontal angle *POQ* = 113°25′3.4″ – 26°35′39.5″
= **86°49′53.9″**.

OBJECTIVE TYPE QUESTIONS

1. A theodolite can measure

(*a*) difference in level.

(*b*) bearing of a line.

(*c*) zenith angle.

(*d*) all the above.

2. The error in the horizontal circle readings, is due to
- (*a*) the late axis bubble not being parallel to the line of collimation.
- (*b*) the line of sight not being parallel to the telescope axis.
- (*c*) the line of collimation not being perpendicular to the trunion axis.
- (*d*) none of the above.

3. The error in the horizontal circle readings due the line of collimation not being perpendicular to the trunion axis is eliminated by
- (*a*) taking readings on the different parts of the horizontal circle.
- (*b*) taking readings on both the faces.
- (*c*) removing the parallax.
- (*d*) transiting the telescope.

4. Quadrantal bearing is always measured from
- (*a*) the north end of the magnetic meridian only.
- (*b*) the south end of the magnetic meridian only.
- (*c*) the north end or the south end of the magnetic meridian.
- (*d*) either the north end or the south end of the magnetic meridian as the case may be.

5. If the departure and latitude of a line are + 78.0 m and – 135.1 m, respectively, the whole circle bearing of the line is
- (*a*) 150°.
- (*b*) 30°.
- (*c*) 60°.
- (*d*) 120°.

6. If the departure and latitude of a line are + 78.0 m and – 135.1 m, respectively, the length of the line is
- (*a*) 213.1 m.
- (*b*) 57.1 m.
- (*c*) 156.0 m.
- (*d*) non of the above.

7. Transit rule of balancing a traverse is applied when
- (*a*) the linear and angular measurements are of same precision.
- (*b*) the linear measurements are more precise than the angular measurements.
- (*c*) the angular measurements are more precise than the linear measurements.
- (*d*) the linear measurements are proportional to l and the angular measurements are proportional to $(1/l)$ where l is the length of the line.

8. The error due to the non-verticality of the vertical axis of a theodolite
- (*a*) is eliminated in the method of repetition only.
- (*b*) is eliminated in the method of reiteration only.
- (*c*) is eliminated in the method of repetition as well as in reiteration.
- (*d*) cannot be eliminated by any method.

9. Random method of running a line between two points *A* and *B* is employed when

(*a*) *A* and *B* are not intervisible even from an intermediate point.

(*b*) *A* and *B* are only intervisible from an intermediate point.

(*c*) the difference of level between the points is large.

(*d*) it is not a method at all for running a line.

10. The error in the horizontal circle reading of 41°59′13.96″ and vertical circle reading of + 36°52′11.63″ for any pointing due to the trunion axis not being perpendicular to the vertical axis by (90° – *i*) where *i* is 20″, is

(*a*) + 15″.

(*b*) + 18″.

(*c*) – 15″.

(*d*) – 18″.

ANSWERS

1. (*d*)	**2.** (*c*)	**3.** (*b*)	**4.** (*d*)	**5.** (*a*)	**6.** (*c*)
7. (*c*)	**8.** (*d*)	**9.** (*a*)	**10.** (*a*)		

5 ADJUSTMENT OF SURVEY OBSERVATIONS

5.1 ADJUSTMENT OF OBSERVATIONS

While making a measurement, certain amount of error is bound to creep into the measurements and hence, no observation is free from error. Gross errors are checked by designing the field procedures of observations. Systematic errors are expressed by functional relationships and therefore, they can be completely eliminated from the observations. The remaining error in the observations is the accidental or random error. The random errors use probability model and can only be minimized or adjusted, and the adjusted value of a quantity is known as the *most probable value* of the measured quantity. It is the most probable value of a measured quantity which is used for computing other quantities related to it by mathematical relationships.

5.2 METHOD OF LEAST SQUARES

It is a general practice in surveying to always have redundant observations as they help in detection of mistakes or blunders. Redundant observations require a method which can yield a unique solution of the model for which the observations have been made. The least squares method provides a general and systematic procedure which yields a unique solution in all situations.

Assuming that all the observations are uncorrelated then the least squares method of adjustment is based upon the following criterion:

"*The sum of the weighted squares of the residuals must be a minimum*".

If υ_1, υ_2, υ_3, etc., are the residuals and ω_1, ω_2, ω_3, etc., are the weights then

$$\phi = \omega_1\upsilon_1^2 + \omega_2\upsilon_2^2 + \omega_3\upsilon_3^2 + \ldots\ldots + \omega_n\upsilon_n^2 = \text{a minimum} \quad \ldots(5.1)$$

$$= \sum_{i=1}^{n} \Sigma\omega_i \nu_i = \text{a minimum}.$$

The above condition which the residuals have to satisfy is in addition to the conditions which the adjusted values have to satisfy for a given model.

5.3 OBSERVATION EQUATIONS AND CONDITION EQUATIONS

The relation between the observed quantities is known as *observation equation.* For example, if α and β are the angles observed at a station then $\alpha + \beta = d$ is the observation equation.

A *condition equation* expresses the relation existing between several dependent quantities. For example, the three angles α, β and γ of a plane triangle are related to each other through the condition equation $\alpha + \beta + \gamma = 180°$.

5.4 NORMAL EQUATION

Normal equations are the equations which are formed from the observation equations using the criterion of least squares. The solution of normal equations yields the most probable values or the adjusted values of the unknowns in the equations.

A normal equation of an unknown is formed by multiplying each observation equation by the coefficient of that unknown in the observation equation and the weight of the equation, and by adding the equations thus formed. The number of normal equations is same as the number of unknowns.

For example, let there be three unknowns x, y, and z having n observation equations as below:

$$\begin{aligned} &a_1x + b_1y + c_1z - d_1 = 0 \\ &a_2x + b_2y + c_2z - d_2 = 0 \\ &. \quad . \quad . \quad . \quad . \\ &. \quad . \quad . \quad . \quad . \\ &. \quad . \quad . \quad . \quad . \\ &a_nx + b_ny + c_nz - d_n = 0 \end{aligned}$$

The normal equations of x, y, and z are respectively

$$\begin{aligned} &(\Sigma a^2)x + (\Sigma ab)y + (\Sigma ac)z - (\Sigma ad) = 0 \\ &(\Sigma ab)x + (\Sigma b^2)y + (\Sigma bc)z - (\Sigma bd) = 0 \\ &(\Sigma ac)x + (\Sigma bc)y + (\Sigma c^2)z - (\Sigma cd) = 0 \end{aligned}$$

In matrix form, the above equations are written as

$$\begin{vmatrix} \Sigma a^2 & \Sigma ab & \Sigma ac \\ \Sigma ab & \Sigma b^2 & \Sigma bc \\ \Sigma ac & \Sigma bc & \Sigma c^2 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} - \begin{vmatrix} \Sigma ad \\ \Sigma bd \\ \Sigma cd \end{vmatrix} = 0 \qquad \ldots(5.2)$$

or $$\boldsymbol{CX} = \boldsymbol{D} \qquad \ldots(5.3)$$

where

C = the coefficient matrix of normal equations

X = the column vector of unknowns, and

D = the column vector of constants.

The normal equations given by Eq. (5.2), have the following characteristics:

(*i*) The number of normal equations is equal to the number of unknowns.

(*ii*) The matrix coefficients of the unknowns is a symmetric matrix, i.e., the elements of i^{th} row are the same as the elements of i^{th} column.

5.5 LEAST SQUARES METHOD OF CORRELATES

Correlates are the unknown multipliers used to determine the most probable values of unknown parameters which are the errors (or the corrections) considered directly. The number of correlates is equal to the number of condition equations, excluding the one imposed by the least squares principle. The method of determining the most probable values has been explained in Example 5.6.

5.6 METHOD OF DIFFERENCES

If the normal equations involve large numbers, the solution of simultaneous equations becomes very laborious. The method of differences simplifies the computations. In this method, the observation equations are written in terms of the quantity whose most probable values is to be determined by least squares method. The solution of Example 5.8 is based on this method.

5.7 METHOD OF VARIATION OF COORDINATES

In the method of variation of coordinates, provisional coordinates are allocated to points requiring adjustment. The amount of displacement for the adjustment is determined by the method of least squares.

In Fig 5.1, there are two points A and B having coordinates (x_A, y_A) and (x_B, y_B), respectively, from where the observations were made on the point C, whose coordinates are to be determined. Let the provisional coordinates assigned to C be (x_C', y_C') and

the bearing of AC = θ_{AC}

the bearing of BC = θ_{BC}

the length of AC = l_{AC}

the length of BC = l_{BC}

the angle ACB = α

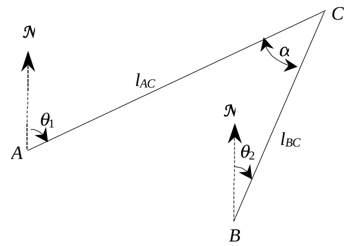


Fig. 5.1

The length of AC is given by

$$l^2_{AC} = (x_C' - x_A)^2 + (y_C' - y_A)^2$$

By differentiating the above equation, we get

$$dl_{AC} = \frac{1}{l_{AC}}\left[(x_C' - x_A)dx_C + (y_C' - y_A)dy_C - (x_C' - x_A)dx_A - (y_C' - y_A)dy_A\right] \quad \ldots(5.4)$$

where dl_{AC} is the displacement in l_{AC} due to small displacements dx_C, dx_A, dy_C, and dy_A in C and A, respectively.

The bearing of AC is given by

$$\tan\theta_{AC} = \frac{x_C' - x_A}{y_C' - y_A} \quad \ldots(5.5)$$

The change $d\theta_{AC}$ in the bearing due to the displacements dx_C, dx_A, dy_C, and dy_A, is

$$d\theta_{AC} = \frac{1}{l^2_{AC}}\left[(y_C' - y_A)dx_C - (x_C' - x_A)dy_C - (y_C' - y_A)dx_A + (x_C' - x_A)dy_A\right] \quad \ldots(5.6)$$

Similar expressions can be obtained for *BC* and then the change in the angle *ACB* can be related to $d\theta_{AC}$ and $d\theta_{BC}$. Using the method of least squares, the displacements *dx* and *dy* in the points can be determined. Residuals υ can be derived in the form

$$\upsilon = O - C - d\gamma$$

where

O = the observed value of quantity, i.e., length, bearing, or angle,

C = the calculated value of that quantity from the coordinates, and

$d\gamma$ = the change in that quantity due the displacements of the respective points.

The best value of the quantity will be $C + d\gamma$.

5.8 GENERAL METHOD OF ADJUSTING A POLYGON WITH A CENTRAL STATION (*By The Author*)

Let a polygon with a central station *O* has *n* sides and the observed angles be $\theta_1, \theta_2, \ldots, \theta_{3n}$ as shown in Fig. 5.2. The total number of angles observed in this polygon will be 3*n* with *n* angles around the central station.

The computations are done step-wise as explained below.

Step-1: Determine the total corrections for (*i*) each triangle, (*ii*) the central station and (*iii*) the side conditions

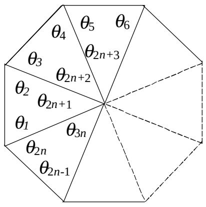


Fig. 5.2

(*i*) $C_1 = 180° - (\theta_1 + \theta_2 + \theta_{2n+1})$

$C_2 = 180° - (\theta_3 + \theta_4 + \theta_{2n+2})$

$C_3 = 180° - (\theta_5 + \theta_6 + \theta_{2n+3})$

. . .

. . .

$Cn = 180° - (\theta_{2n-1} + \theta_{2n} + \theta_{3n})$

(*ii*) $C_{n+1} = 360° - (\theta_{2n+1} + \theta_{2n+2} + \ldots + \theta_{3n})$

(*iii*) $C_{n+2} = -$ [log sin(odd angles) – log sin(even angles)] $\times 10^6$ for angles $(\theta_1, \theta_2 \ldots \theta_{2n})$.

Calculate log sin of the angles using pocket calculator and ignore the negative sign of the values.

Step-2: Framing the normal equations of the correlates

There will be $(n + 2)$ correlates and the normal equations for the correlates will be written in matrix form. The size of the coefficient matrix will be $(n + 2) \times (n + 2)$. The column vectors of correlates and constants will have $(n + 2)$ elements.

Let f_1, f_2, f_3, etc., = the differences for 1″ of log sin of the angles $\times 10^6$

$$F_{12} = f_1 - f_2$$

$$F_{34} = f_3 - f_4$$

$$F_{56} = f_5 - f_6$$

. .

. .

$$F_{(2n-1)2n} = f_{2n-1} - f_{2n}$$
$$F^2 = f_1^{\,2} + f_2^{\,2} + \ldots\ldots.. + f_{2n}^{\,2}$$

The coefficient matrix of normal equations is formed as below.

(*i*) Write the diagonal elements from (1,1) to (n + 2, n + 2) as 3, 3, 3,…..,3, n, and F^2. The (n + 2, n + 2)th element is F^2 and (n + 1, n + 1)th element is n.

	1	2	3	4	.	.	.	n+1	n+2
1	3								
2		3							
3			3						
4				3					
.									
.									
.							3		
n+1								n	
n+2									F^2

(*ii*) Write F_{12}, F_{34}, F_{56},….., $F_{(2n-1)2n}$, and 0 in the (n + 2)th column from 1st row to (n + 1)th row and in the (n + 2)th row from 1st column to (n + 1)th column. It may be noted that the values of the (n + 2, n + 1)th element and (n + 1, n + 2)th element are zero.

	1	*2*	*3*	*4*	.	.	.	n+1	n+2
1	3								F_{12}
2		3							F_{34}
3			3						F_{56}
4				3					F_{78}
.									.
.									.
.							3		$F_{(2n-1)\,2n}$
n+1								n	0
n+2	F_{12}	F_{34}	F_{56}	F_{78}	.	.	$F_{(2n-1)\,2n}$	0	F^2

(*iii*) Write 1 as the value of the remaining elements of the (n+1)th column and the (n+1)th row, and zero for all the remaining elements of the matrix.

	1	*2*	*3*	*4*	.	.	.	n+1	n+2
1	3	0	0	0			0	1	F_{12}
2	0	3	0	0			0	1	F_{34}
3	0	0	3	0			0	1	F_{56}
4	0	0	0	3			0	1	F_{78}
.							.	.	.
.							.	.	.
.	0	0	0	0	.	.	3	1	$F_{(2n-1)\,2n}$
n+1	1	1	1	1	.	.	1	n	0
n+2	F_{12}	F_{34}	F_{56}	F_{78}	.	.	$F_{(2n-1)\,2n}$	0	F^2

(*iv*) Write the elements of column vectors of the unknown correlates and the constants as shown below and this is the final form of the matrix of the normal equations.

	1	2	3	4	.	.	.	$n+1$	$n+2$				
1	3	0	0	0			0	1	F_{12}	λ_1		C_1	
2	0	3	0	0			0	1	F_{34}	λ_2		C_2	
3	0	0	3	0			0	1	F_{56}	λ_3		C_3	
4	0	0	0	3			0	1	F_{78}	λ_4	−	C_4	= 0
.							.	.	.	.		.	
.							.	.	.	.		.	
.	0	0	0	0	.	.	3	1	$F_{(2n-1)\,2n}$	λ_n		C_n	
$n+1$	1	1	1	1	.	.	1	n	0	λ_{n+1}		C_{n+1}	
$n+2$	F_{12}	F_{34}	F_{56}	F_{78}	.	.	$F_{(2n-1)\,2n}$	0	F^2	λ_{n+2}		C_{n+2}	

Now the normal equations can be solved directly using the matrix for the values of the correlates. For small values of n, the equations can be written as under and can be solved for λ_1, λ_2, λ_3, etc.

$$3\lambda_1 + \lambda_{n+1} + F_{12}\lambda_{n+2} - C_1 = 0$$

$$3\lambda_2 + \lambda_{n+1} + F_{34}\lambda_{n+2} - C_2 = 0$$

$$3\lambda_3 + \lambda_{n+1} + F_{56}\lambda_{n+2} - C_3 = 0$$

$$3\lambda_n + \lambda_{n+1} + F_{(2n-1)\,2n}\lambda_{n+2} - C_n = 0$$

$$\lambda_1 + \lambda_2 + ... + \lambda_n + n\lambda_{n+1} - C_{n+1} = 0$$

$$F_{12}\lambda_1 + F_{34}\lambda_2 + ... + F_{(2n-1)\,2n}\lambda_n + F^2\lambda_{n+2} - C_{n+2} = 0$$

Step-3: Calculate the values of the corrections e_1, e_2, e_3,… e_{3n} to the angles using the following expressions which can be framed up to the desired numbers by noticing their pattern and similarity in Type-1 and Type-2 equations.

Type-1	Type-2
$e_1 = \lambda_1 + f_1\lambda_{n+2}$	$e_{2n+1} = \lambda_1 + \lambda_{n+1}$
$e_2 = \lambda_1 - f_2\lambda_{n+2}$	$e_{2n+2} = \lambda_2 + \lambda_{n+1}$
	$e_{2n+3} = \lambda_3 + \lambda_{n+1}$
$e_3 = \lambda_2 + f_3\lambda_{n+2}$	. .
$e_4 = \lambda_2 - f_4\lambda_{n+2}$	. .
	$e_{3n-2} = \lambda_{n-2} + \lambda_{n+1}$
$e_5 = \lambda_3 + f_5\lambda_{n+2}$	$e_{3n-1} = \lambda_{n-1} + \lambda_{n+1}$
$e_6 = \lambda_3 - f_6\lambda_{n+2}$	$e_{3n} = \lambda_n + \lambda_{n+1}$
. .	
. .	

$$e_{2n-3} = \lambda_{n-1} + f_{2n-3}\lambda_{n+2}$$
$$e_{2n-2} = \lambda_{n-1} - f_{2n-2}\lambda_{n+2}$$

$$e_{2n-1} = \lambda_n + f_{2n-1}\lambda_{n+2}$$
$$e_{2n} = \lambda_n - f_{2n}\lambda_{n+2}$$

Step-4: The most probable values of the angles are

$$\theta_1 + e_1$$
$$\theta_2 + e_2$$
$$\theta_3 + e_3$$
$$\cdot \quad \cdot$$
$$\cdot \quad \cdot$$
$$\theta_{3n-1} + e_{3n-1}$$
$$\theta_{3n} + e_{3n}$$

Example 5.15 has been solved by this method to help the students understand the method more clearly.

Example 5.1. A distance is measured six times with following results:

74.31 m, 74.28 m, 74.32 m, 74.33 m, 74.30 m, 74.31 m.

Determine the most probable value of the distance by least square method.

Solution:

Let the most probable value of the distance be $\hat{l}$. If the six observed values of the distance are l_1, l_2, l_3, l_4, l_5, and l_6 and the respective residuals are υ_1, υ_2, υ_3, υ_4, υ_5, and υ_6 then

$$\upsilon_1 = \hat{l} - l_1; \qquad \upsilon_2 = \hat{l} - l_2; \qquad \upsilon_3 = \hat{l} - l_3$$

$$\upsilon_4 = \hat{l} - l_4; \qquad \upsilon_5 = \hat{l} - l_5; \qquad \upsilon_6 = \hat{l} - l_6.$$

From the least squares principle, we have

$$\phi = \upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2 + \upsilon_4^2 + \upsilon_5^2 + \upsilon_6^2 = \text{a minimum}$$

$$= (\hat{l} - l_1)^2 + (\hat{l} - l_2)^2 + (\hat{l} - l_3)^2 + (\hat{l} - l_4)^2 + (\hat{l} - l_5)^2 + (\hat{l} - l_6)^2 = \text{a minimum.}$$

For ϕ to be a minimum

$$\frac{d\phi}{d\hat{l}} = 2(\hat{l} - l_1) + 2(\hat{l} - l_2) + 2(\hat{l} - l_3) + 2(\hat{l} - l_4) + 2(\hat{l} - l_5) + 2(\hat{l} - l_6) = 0$$

$$6\hat{l} = l_1 + l_2 + l_3 + l_4 + l_5 + l_6$$

$$\hat{l} = \frac{l_1 + l_2 + l_3 + l_4 + l_5 + l_6}{6}$$

$$= \frac{74.31 + 74.28 + 74.32 + 74.33 + 74.30 + 74.31}{6}$$

$$= \mathbf{74.31\ m}.$$

Example 5.2. An angle was measured six times by different observers and the following values were obtained:

42°25′10″ (2), 42°25′08″ (1), 42°25′09″ (3), 42°25′07″ (2), 42°25′11″ (3), 42°25′09″ (2).

The values given in the parentheses are the weights of the observations. Determine the most probable value of the angle using least squares.

Solution:

Let the observations be α_1, α_2, α_3, α_4, α_5, and α_6 and their respective weights are ω_1, ω_2, ω_3, ω_4, ω_5, and ω_6. If the most probable value of the angle is $\hat{\alpha}$ and the respective residuals are υ_1, υ_2, υ_3, υ_4, υ_5, and υ_6 then

$$\upsilon_1 = \hat{\alpha} - \alpha_1; \qquad \upsilon_2 = \hat{\alpha} - \alpha_2; \qquad \upsilon_3 = \hat{\alpha} - \alpha_3$$

$$\upsilon_4 = \hat{\alpha} - \alpha_4; \qquad \upsilon_5 = \hat{\alpha} - \alpha_5; \qquad \upsilon_6 = \hat{\alpha} - \alpha_6.$$

From the least squares principle, we get

$$\phi = \omega_1\upsilon_1^2 + \omega_2\upsilon_2^2 + \omega_3\upsilon_3^2 + \omega_4\upsilon_4^2 + \omega_5\upsilon_5^2 + \omega_6\upsilon_6^2 = \text{a minimum}$$

$$= (\hat{\alpha} - \alpha_1)^2 + (\hat{\alpha} - \alpha_2)^2 + (\hat{\alpha} - \alpha_3)^2 + (\hat{\alpha} - \alpha_4)^2 + (\hat{\alpha} - \alpha_5)^2 + (\hat{\alpha} - \alpha_6)^2$$

$$= \text{a minimum.}$$

For ϕ to be a minimum

$$\frac{d\phi}{d\hat{\alpha}} = \begin{bmatrix} 2\omega_1(\hat{\alpha} - \alpha_1) + 2\omega_2(\hat{\alpha} - \alpha_2) + 2\omega_3(\hat{\alpha} - \alpha_3) + 2\omega_4(\hat{\alpha} - \alpha_4) \\ + 2\omega_5(\hat{\alpha} - \alpha_5) + 2\omega_6(\hat{\alpha} - \alpha_6) \end{bmatrix} = 0$$

$$\hat{\alpha} = \frac{\omega_1\alpha_1 + \omega_2\alpha_2 + \omega_3\alpha_3 + \omega_4\alpha_4 + \omega_5\alpha_5 + \omega_6\alpha_6}{(\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6)}$$

$$= 42°25' + \frac{2\times10'' + 1\times8'' + 3\times9'' + 2\times7'' + 3\times11'' + 2\times9''}{(2+1+3+2+3+2)}$$

$$= \mathbf{42°25'9.2''}.$$

Example 5.3. Three angles of a plane triangle were measured and the following values were obtained:

$$\theta_1 = 52°33'; \qquad \theta_2 = 64°45'; \qquad \theta_3 = 62°39'.$$

Determine the least squares estimates of the angles.

Solution:

Let the most probable values of the angle = $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$

the respective residuals = υ_1, υ_2, υ_3.

We can write now that

$$\hat{\theta}_1 = \theta_1 + \upsilon_1$$

$$\hat{\theta}_2 = \theta_2 + \upsilon_2$$

$$\hat{\theta}_3 = \theta_3 + \upsilon_3.$$

In a plane triangle, we have

$$\hat{\theta}_1 + \hat{\theta}_2 + \hat{\theta}_3 = 180°$$

or $$(\theta_1 + \upsilon_1) + (\theta_2 + \upsilon_2) + (\theta_3 + \upsilon_3) = 180°$$

$$\upsilon_1 + \upsilon_2 + \upsilon_3 = 180° - (\theta_1 + \theta_2 + \theta_3)$$

$$= 180° - (52°33' + 64°45' + 62°39')$$

$$= 180° - 179°57' = +3'$$

or $$\upsilon_3 = 3' - (\upsilon_1 + \upsilon_2).$$

From the least squares principle, we have

$$= \upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2 = \text{a minimum}$$

$$= \upsilon_1^2 + \upsilon_2^2 + (3' - \upsilon_1 - \upsilon_2)^2 = \text{a minimum.}$$

Therefore, $$\frac{\partial \phi}{\partial \upsilon_1} = 2\upsilon_1 - 2(3' - \upsilon_1 - \upsilon_2) = 0$$

$$\frac{\partial \phi}{\partial \upsilon_2} = 2\upsilon_2 - 2(3' - \upsilon_1 - \upsilon_2) = 0$$

or $$2\upsilon_1 + \upsilon_2 = 3'$$

$$\upsilon_1 + 2\upsilon_2 = 3'.$$

The above equations are the normal equations for υ_1 and υ_2. The solution of these equations yields

$$\upsilon_1 = 1'$$

$$\upsilon_2 = 1'$$

$$\upsilon_3 = 3' - 1' - 1' = 1'.$$

Thus the most probable values of the angles are

$$\hat{\theta}_1 = 52°33' + 1' = \mathbf{52°34'}$$

$$\hat{\theta}_2 = 64°45' + 1' = \mathbf{64°46'}$$

$$\hat{\theta}_3 = 62°39' + 1' = \mathbf{62°40'}$$

$$\text{Total} = 180° \qquad (\textit{Check}).$$

Example 5.4. The angles for the figure of a triangulation scheme shown in Fig. 5.3 were measured as under.

$$\theta_1 = 44°42'00''; \quad \theta_3 = 43°48'00''; \quad \theta_5 = 42°06'00''$$
$$\theta_2 = 46°00'00''; \quad \theta_4 = 44°31'12''; \quad \theta_6 = 48°52'48''$$

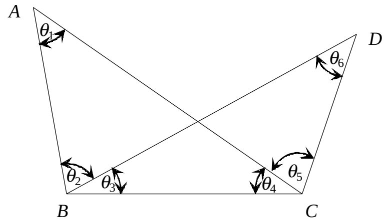


Fig. 5.3

Solution (Fig. 5.3):

For the most probable values of the angles from Δ′s *ABC* and *BCD*, we get

$$\hat{\theta}_1 + \hat{\theta}_2 + \hat{\theta}_3 + \hat{\theta}_4 = 180°$$
$$\hat{\theta}_3 + \hat{\theta}_4 + \hat{\theta}_5 + \hat{\theta}_6 = 180°.$$

Therefore for the respective residuals υ_1, υ_2, υ_3. etc., we have

$$\begin{aligned} \upsilon_1 + \upsilon_2 + \upsilon_3 + \upsilon_4 &= 180° - (\theta_1 + \theta_2 + \theta_3 + \theta_4) \\ &= 180° - 179°01'12'' \\ &= 58'48'' \\ &= 0.98° \\ \upsilon_3 + \upsilon_4 + \upsilon_5 + \upsilon_6 &= 180° - (\theta_3 + \theta_4 + \theta_5 + \theta_6) \\ &= 180° - 179°18'00'' \\ &= 42'00'' \\ &= 0.70°. \end{aligned}$$

The above two condition equations can be used to have four independent unknowns. Thus

$$\upsilon_1 = 0.98° - (\upsilon_2 + \upsilon_3 + \upsilon_4) \quad \text{...(a)}$$
$$\upsilon_6 = 0.70° - (\upsilon_3 + \upsilon_4 + \upsilon_5). \quad \text{...(b)}$$

From the least squares theory, we have

$$\phi = \upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2 + \upsilon_4^2 + \upsilon_5^2 + \upsilon_6^2 = \text{a minimum} \quad \text{...(c)}$$

Now using Eqs. (a) and (b), Eq. (c) becomes

$$\phi = (0.98 - \upsilon_2 - \upsilon_3 - \upsilon_3 - \upsilon_4)^2 + \upsilon_2^2 + \upsilon_3^2 + \upsilon_4^2 + \upsilon_5^2 + (0.70 - \upsilon_3 - \upsilon_4 - \upsilon_5)^2$$
$$= \text{a minimum.}$$

Differentiating ϕ partially, we get

$$\frac{\partial\phi}{\partial\upsilon_2} = 2(0.98-\upsilon_2-\upsilon_3-\upsilon_4)\times(-1)+2\upsilon_2 = 0$$

$$\frac{\partial\phi}{\partial\upsilon_3} = 2(0.98-\upsilon_2-\upsilon_3-\upsilon_4)\times(-1)+2\upsilon_3+2(0.70-\upsilon_3-\upsilon_4-\upsilon_5)\times(-1) = 0$$

$$\frac{\partial\phi}{\partial\upsilon_4} = 2(0.98-\upsilon_2-\upsilon_3-\upsilon_4)\times(-1)+2\upsilon_4+2(0.70-\upsilon_3-\upsilon_4-\upsilon_5)\times(-1) = 0$$

$$\frac{\partial\phi}{\partial\upsilon_5} = 2\upsilon_5+2(0.70-\upsilon_3-\upsilon_4-\upsilon_5)\times(-1) = 0.$$

By clearing and rearranging, we get

$$2\upsilon_2 + \upsilon_3 + \upsilon_4 = 0.98$$
$$\upsilon_2 + 3\upsilon_3 + 2\upsilon_4 + \upsilon_5 = 1.68$$
$$\upsilon_2 + 2\upsilon_3 + 3\upsilon_4 + \upsilon_5 = 1.68$$
$$\upsilon_3 + \upsilon_4 + 2\upsilon_5 = 0.70$$

The above equations solve for

$$\upsilon_2 = 12'36''$$
$$\upsilon_3 = 16'48''$$
$$\upsilon_4 = 16'48''$$
$$\upsilon_5 = 4'12''$$

and from Eqs. (a) and (b), we have

$$\upsilon_1 = 12'36''$$
$$\upsilon_6 = 4'12''.$$

Therefore the most probable values of the angles are

$$\hat{\theta}_1 = 44°42'00'' + 12'36'' = 44°54'36''$$
$$\hat{\theta}_2 = 46°00'00'' + 12'36'' = 46°12'36''$$
$$\hat{\theta}_3 = 43°48'00'' + 16'48'' = 44°04'48''$$
$$\hat{\theta}_4 = 44°31'12'' + 16'48'' = 44°48'00''$$

Total = 180° (*Check*).

$$\hat{\theta}_3 = 43°04'48''$$
$$\hat{\theta}_4 = 44°48'00''$$

$$\hat{\theta}_5 = 42°06'00'' + 04'12'' = 42°10'12''$$

$$\hat{\theta}_6 = 48°52'48'' + 04'12'' = 48°57'00''$$

$$\text{Total} = 180° \quad (\textit{Check}).$$

Example 5.5. In Fig. 5.4, the observed values of the distances *AB*, *BC*, *CD*, *AC*, and *BD* are as 50.000 m, 50.070 m, 50.050 m, 100.090 m, and 100.010 m, respectively. Determine the adjusted values of *AD* assuming that all the observations are of equal reliability and uncorrelated.

Fig. 5.4

Solution (Fig. 5.4):

Let the distances *AB*, *BC*, *CD*, *AC*, and *BD* be l_1, l_2, l_3, l_4, and l_5, respectively. Since to determine *AD* minimum of three distances are l_1, l_2, and l_3 are required, let their most probable values be $\hat{l}_1$, $\hat{l}_2$ and $\hat{l}_3$, respectively. Assuming the residuals of the five observations as υ_1, υ_2, υ_3, υ_4, and υ_5, we have

$$\upsilon_1 = \hat{l}_1 - l_1$$

$$\upsilon_2 = \hat{l}_2 - l_2$$

$$\upsilon_3 = \hat{l}_3 - l_3$$

$$\upsilon_4 = \hat{l}_1 + \hat{l}_2 - l_4$$

$$\upsilon_5 = \hat{l}_2 + \hat{l}_3 - l_5.$$

From the theory of least squares, we have

$$= \upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2 + \upsilon_4^2 + \upsilon_5^2 = \text{a minimum}$$

$$= (\hat{l}_1 - l_1)^2 + (\hat{l}_2 - l_2)^2 + (\hat{l}_3 - l_3)^2 + (\hat{l}_1 + \hat{l}_2 - l_4)^2 + (\hat{l}_2 + \hat{l}_3 - l_5)^2 = \text{a minimum.}$$

Differentiating the above equation partially, we get

$$\frac{\partial \phi}{\partial \hat{l}_1} = 2(\hat{l}_1 - l_1) + 2(\hat{l}_1 + \hat{l}_2 - l_4) = 0$$

$$\frac{\partial \phi}{\partial \hat{l}_2} = 2(\hat{l}_2 - l_2) + 2(\hat{l}_1 + \hat{l}_2 - l_4) + 2(\hat{l}_2 + \hat{l}_3 - l_5) = 0$$

$$\frac{\partial \phi}{\partial \hat{l}_3} = 2(\hat{l}_3 - l_3) + 2(\hat{l}_2 + \hat{l}_3 - l_5) = 0$$

or $$2\hat{l}_1 + \hat{l}_2 = l_1 + l_4$$

$$\hat{l}_1 + 3\hat{l}_2 + \hat{l}_3 = l_2 + l_4 + l_5$$

$$2\hat{l}_2 + \hat{l}_3 = l_3 + l_5$$

or $$2\hat{l}_1 + \hat{l}_2 = 50.000 + 100.090 = 150.090 \quad \text{...(a)}$$

$$\hat{l}_1 + 3\hat{l}_2 + \hat{l}_3 = 50.070 + 100.090 + 10.010 = 250.170 \quad \text{...(b)}$$

$$2\hat{l}_2 + \hat{l}_3 = 50.050 + 100.010 = 150.060 \quad \text{...(c)}$$

From Eq. (c), we get $$\hat{l}_3 = \frac{150.060 - \hat{l}_2}{2} \quad \text{...(d)}$$

From Eq. (a), we get $$\hat{l}_2 = 150.090 - 2\hat{l}_1 \quad \text{...(e)}$$

Now substituting the values of $\hat{l}_2$ and $\hat{l}_3$ in Eq. (b), we get

$$\hat{l}_1 + 3 \times (150.090 - 2\hat{l}_1) + \frac{1}{2} \times [150.060 - (150.090 - 2\hat{l}_1)] = 250.170$$

$$4\hat{l}_1 = 200.085$$

$$\hat{l}_1 = 50.022 \text{ m}$$

From Eq. (e), we get $$\hat{l}_2 = 150.090 - 2 \times 50.022$$

$$= 50.046 \text{ m}$$

From Eq. (d), we get $$\hat{l}_3 = \frac{150.060 - 50.046}{2}$$

$$= 50.007 \text{ m}$$

Thus the adjusted distance $$AD = 50.022 + 50.046 + 50.007$$

$$= \mathbf{150.075 \text{ m}}.$$

Example 5.6. Find the least square estimate of the quantity x from the following data:

x (m)	Weight
2x = 292.500	$\omega_1 = 1$
3x = 438.690	$\omega_2 = 2$
4x = 585.140	$\omega_3 = 3$

Solution:

Let $\hat{x}$ = the least square estimate of x and

$$\upsilon_1,\ \upsilon_2,\ \upsilon_3 = \text{the residuals of the three observations.}$$

Therefore

$$\upsilon_1 = 2\hat{x} - 292.500$$

$$\upsilon_2 = 3\hat{x} - 438.690$$

$$\upsilon_3 = 4\hat{x} - 585.140.$$

From the theory of least squares, we have

$$\phi = \omega_1 \upsilon^2{}_1 + \omega_2 \upsilon^2{}_2 + \omega_3 \upsilon^2{}_3 = \text{a minimum}$$

$$= 1\times(2\hat{x} - 292.500)^2 + 2\times(3\hat{x} - 438.690)^2 + 3´(4\hat{x} - 585.140)^2 = \text{a minimum.}$$

Therefore

$$\frac{d\phi}{d\hat{x}} = 2\times1\times(2\hat{x} - 292.500)\times2 + 2\times2\times(3\hat{x} - 438.690)\times3 + 2\times3\times(4\hat{x} - 585.140)\times4 = 0$$

$$4\hat{x} - 2 \times 292.500 + 18\hat{x} - 6 \times 438.690 + 48\hat{x} - 12 \times 585.140 = 0$$

$$70\hat{x} = 10238.830$$

$$\hat{x} = \mathbf{146.269\ m}.$$

Example 5.7. Adjust the following angles of a triangle *ABC* by the method of correlates.

$$\angle A = 86°35'11.1'' \qquad \omega_1 = 2$$

$$\angle B = 42°15'17.0'' \qquad \omega_2 = 1$$

$$\angle C = 51°09'34.0'' \qquad \omega_3 = 3$$

Solution:

For a triangle, we have

$$A + B + C = 180°$$

$$(A + B + C - 180°) = \text{Error} = E.$$

Therefore

$$E = (86°35'11.1'' + 42°15'17.0'' + 51°09'34.0'') - 180°$$

$$= 180°00'2.1'' - 180°$$

$$= +\ 2.1''$$

or Correction = – 2.1″.

Let the corrections to the angles *A*, *B* and *C* be e_1, e_2 and e_3, respectively, then

$$e_1 + e_2 + e_3 = -\ 2.1'' \qquad \ldots\text{(a)}$$

From the least squares criterion, we have

$$\phi = \omega_1 e^2{}_1 + \omega_2 e^2{}_2 + \omega_3 e^2{}_3 = \text{a minimum.} \qquad \ldots\text{(b)}$$

Differentiating Eqs. (a) and (b), we get

$$\partial e = \partial e_1 + \partial e_2 + \partial e_3 = 0 \qquad \ldots\text{(c)}$$

$$\partial\phi = \omega_1 \partial e_1 + \omega_2\, \partial e_2 + \omega_3 \partial e_3 = 0 \qquad \ldots\text{(d)}$$

Multiplying Eq. (c) by $-\lambda$ and adding the result to Eq. (d), we get

$$-\lambda \times (\partial e_1 + \partial e_2 + \partial e_3) + \omega_1 \partial e_1 + \omega_2 \partial e_2 + \omega_3 \partial e_3 = 0$$

or

$$(\omega_1 e_1 - \lambda)\partial e_1 + (\omega_2 e_2 - \lambda)\partial e_2 + (\omega_3 e_3 - \lambda)\partial e_3 = 0$$

Therefore

$$(\omega_1 e_1 - \lambda) = 0 \quad \text{or} \quad e_1 = \frac{\lambda}{\omega_1}$$

$$(\omega_2 e_2 - \lambda) = 0 \quad \text{or} \quad e_2 = \frac{\lambda}{\omega_2} \qquad \ldots\text{(e)}$$

$$(\omega_3 e_3 - \lambda) = 0 \quad \text{or} \quad e_3 = \frac{\lambda}{\omega_3}$$

Substituting the values of e_1, e_2 and e_3 in Eq. (a), we get

$$\frac{\lambda}{\omega_1} + \frac{\lambda}{\omega_2} + \frac{\lambda}{\omega_3} = -2.1$$

$$\lambda = \frac{-2.1}{\left(\frac{1}{2} + \frac{1}{1} + \frac{1}{3}\right)} = -1.15''.$$

Now from Eqs. (e), we get

$$e_1 = \frac{-1.15}{2} = -0.57''$$

$$e_2 = \frac{-1.15}{1} = -1.15'' \quad e_3 = \frac{-1.15}{3} = -0.38''.$$

Therefore, the most probable values of the angles are

$\angle A = 86°35'11.1'' - 0.57'' =$ **86°35′10.53″**

$\angle B = 42°15'17.0'' - 1.15'' =$ **42°15′15.85″**

$\angle C = 51°09'34.0'' - 0.38'' =$ **51°09′33.62″**

Total = 180° (*Check*).

Example 5.8. Determine the adjusted values of the angles of the angles *A*, *B* and *C* from the following observed values by the method of differences.

A = 39°14′15.3″

B = 31°15′26.4″

C = 42°18′18.4″

$$A + B = 70°29'45.2''$$
$$B + C = 73°33'48.3''$$

Solution:

If k_1, k_2 and k_3 are the corrections to the angles A, B and C, respectively, then

$$A = 39°14'15.3'' + k_1$$
$$B = 31°15'26.4'' + k_2$$
$$C = 42°18'18.4'' + k_3 \quad \ldots\text{(a)}$$
$$A + B = (39°14'15.3'' + 31°15'26.4'') + k_1 + k_2$$
$$= 70°29'41.7'' + k_1 + k_2$$
$$B + C = (31°15'26.4'' + 42°18'18.4'') + k_2 + k_3$$
$$= 73°33'44.8'' + k_2 + k_3$$

Equating Eq. (a) to the respective observed values, i.e.,

$$39°14'15.3'' + k_1 = 39°14'15.3''$$
$$31°15'26.4'' + k_2 = 31°15'26.4''$$
$$42°18'18.4'' + k_3 = 42°18'18.4'' \quad \ldots\text{(b)}$$
$$70°29'41.7'' + k_1 + k_2 = 70°29'45.2''$$
$$73°33'44.8'' + k_2 + k_3 = 73°33'48.3''$$

Eq. (b) reduce to

$$k_1 = 0$$
$$k_2 = 0$$
$$k_3 = 0 \quad \ldots\text{(b)}$$
$$k_1 + k_2 = 3.5$$
$$k_2 + k_3 = 3.5$$

Forming the normal equations for k_1, k_2 and k_3, we get

$$2k_1 + k_2 = 3.5$$
$$k_1 + 3k_2 + k_3 = 7.0$$
$$k_2 + 2k_3 = 3.5$$

The solution of the above normal equations gives

$$k_1 = 0.88''$$
$$k_2 = 1.75''$$
$$k_3 = 0.88''.$$

Therefore, the most probable values or the adjusted values of the angles are

$$A = 39°14'15.3'' + 0.88'' = \mathbf{39°14'16.18''}$$
$$B = 31°15'26.4'' + 1.75'' = \mathbf{31°15'28.15''}$$
$$C = 42°18'18.4'' + 0.88'' = \mathbf{42°18'19.28''}.$$

Example 5.9. The observed differences in level for the points in a level net shown in Fig. 5.5 are given below:

From (Lower point)	To (Higher point)	Level difference (m)
Q	P	h_1 = 6.226
S	Q	h_2 = 5.133
S	P	h_3 = 11.368
Q	R	h_4 = 23.521
S	R	h_5 = 28.639
P	R	h_6 = 17.275

Determine the most probable values of the elevations of Q, R and S if the observations are uncorrelated and of equal reliability.

Solution (Fig. 5.5):

Let $\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_6$ = the most probable values of the differences in level, and

$\upsilon_1, \upsilon_2, \ldots, \upsilon_6$ = the respective residuals.

Designating the elevation of the point by its own name, we can write

$$Q - P + \hat{h}_1 = Q - 150.020 + h_1 + \upsilon_1 = 0$$

$$S - Q + \hat{h}_2 = S - Q + h_2 + \upsilon_2 = 0$$

$$S - P + \hat{h}_3 = S - 150.020 + h_3 + \upsilon_3 = 0 \quad \ldots(a)$$

$$Q - R + \hat{h}_4 = Q - R + h_4 + \upsilon_4 = 0$$

$$S - R + \hat{h}_5 = S - R + h_5 + \upsilon_5 = 0$$

$$P - R + \hat{h}_6 = 150.020 - R + h_6 + \upsilon_6 = 0$$

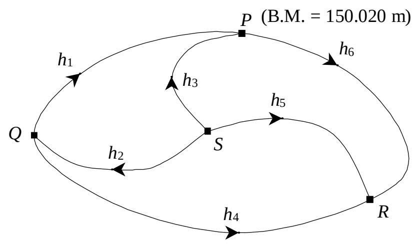


Fig. 5.5

Substituting the values of h_1, h_2, h_3, etc., in Eq. (a), we get

$$\upsilon_1 = 143.794 - Q$$
$$\upsilon_2 = Q - S - 5.133$$
$$\upsilon_3 = 138.652 - S \quad \ldots(b)$$
$$\upsilon_4 = R - Q - 23.521$$
$$\upsilon_5 = R - S + 28.639$$
$$\upsilon_6 = R - 167.295$$

Applying the least squares criterion, we get

$$\phi = \upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2 + \upsilon_4^2 + \upsilon_5^2 + \upsilon_6^2 = \text{a minimum} \quad \ldots(c)$$

$$= (143.794 - Q)^2 + (Q - S - 5.133)^2 + (138.652 - S)^2 + (R - Q - 23.521)^2 + (R - S + 28.639)^2 + (R - 167.295)^2 = \text{a minimum.}$$

To minimize ϕ

$$\frac{\partial\phi}{\partial Q} = -2(143.794 - Q) + 2(Q - S - 5.133) - 2(R - Q - 23.521) = 0$$

$$\frac{\partial\phi}{\partial R} = 2(R - Q - 23.521) + 2(R - S + 28.639) + 2(R - 167.295) = 0$$

$$\frac{\partial\phi}{\partial S} = -2(Q - S - 5.133) - 2(138.652 - S) - 2(R - S + 28.639) = 0$$

By clearing and collecting terms, we get

$$3Q - R - S = 125.406$$
$$-Q + 3R - S = 219.455 \quad \ldots(d)$$
$$-Q - R + 3S = 104.880$$

The solution of the above equations yields

Q = **143.786 m**

R = **167.298 m**

S = **138.654 m**.

Alternative solution:

The normal equations given by Eqs. (d) can directly be formed as given in Sec. 5.4. Let us write the coefficients of the unknowns Q, R and S and the constants of Eqs. (b) in the tabular form as given in below:

Coefficients			Constant
Q	R	S	
– 1	0	0	143.794
+ 1	0	– 1	– 5.133
0	0	– 1	138.652
– 1	+ 1	0	– 23.521
0	+ 1	– 1	– 28.639
0	+ 1	0	– 167.295

To obtain the normal equation for Q

The coefficients of Q appear in first, second and fourth lines. Multiply the first line by (–1), the second line by (+ 1) and the fourth line by (– 1), and add them. The result is

$[(-1) \times (-1)Q + (-1) \times (0)R + (-1) \times (0)S + (-1) \times 143.794] + [(+1) \times (+1)Q + (+1) \times (0)R + (+1) \times (-1)S + (+1) \times (-5.133)] + [(-1) \times (-1)Q + (-1) \times (+1)R + (-1) \times (0)S + (-1) \times (-23.521)] = 0$

or $$3Q - R - S = 125.406 \quad \ldots(e)$$

To obtain the normal equation for R

The coefficients of R appear in fourth, fifth and sixth lines. Multiply the fourth line by (+ 1), the fifth line by (+ 1) and the sixth line by (+ 1), and add them. The result is

$[(+1) \times (-1)Q + (+1) \times (+1)R + (+1) \times (0)S + (+1) \times (-23.521)] + [(+1) \times (0)Q + (+1) \times (+1)R + (+1) \times (-1)S + (+1) \times (-28.639)] + [(+1) \times (0)Q + (+1) \times (+1)R + (+1) \times (0)S + (+1) \times (-167.295)] = 0$

or $$-Q + 3R - S = 219.455 \quad \ldots(f)$$

To obtain the normal equation for S

The coefficients of S appear in second, third and fifth lines. Multiply the second line by (– 1), the third line by (– 1) and the fifth line by (– 1), and add them. The result is

$[(-1) \times (+1)Q + (-1) \times (0)R + (-1) \times (-1)S + (-1) \times (-5.133)] + [(-1) \times (0)Q + (-1) \times (0)R + (-1) \times (-1)S + (-1) \times 138.652] + [(-1) \times (0)Q + (-1) \times (+1)R + (-1) \times (-1)S + (-1) \times (-28.639)] = 0$

or $$-Q - R + 3S = 104.880. \quad \ldots(g)$$

Comparing the Eqs. (e), (f) and (g) with Eqs. (d), we find that they are same. It will be realized that we have automatically carried out the partial differentiation of ϕ demanded by the principle of least squares.

Example 5.10. Determine the least square estimates of the levels of *B*, *C*, and *D* from the following data for the level net shown in Fig. 5.6. The level difference for the line *A* to *B* is the mean of the two runs, all other lines being observed once only.

Line	Length	Level difference (m)	
	(km)	Rise (+)	Fall (–)
A to *B*	22	42.919	–
B to *C*	12	–	12.196
B to *D*	25	–	20.544
C to *D*	22	–	8.236
D to *A*	32	–	22.557

Solution (Fig. 5.6):

Assuming the distance being equal for the back sights and fore sights, the accidental errors may be taken as proportional to $\sqrt{(\text{number of instrument stations})}$ and hence proportional to $\sqrt{(\text{length of line})}$. Accordingly, the weights of the observations can be taken as inversely proportional to the square of the errors and, therefore, can be taken as the reciprocal of the length of the line, i.e.,

$$\propto \frac{1}{e^2}$$

$$\propto \frac{1}{\left(\sqrt{l}\right)^2}$$

$$\propto \frac{1}{l}$$

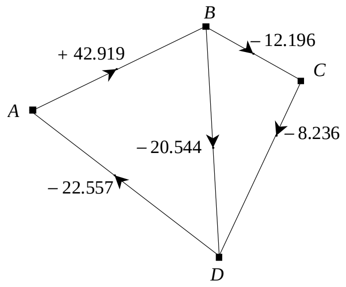


Fig. 5.6

From Fig. 5.6, we find that there are three closed circuits of level runs, namely *ABCDA*, *ABDA*, and *BCDB*. Only two out of three are needed to determine the most probable values of differences in level. In choosing the two out of three, the compatibility of the level nets must be ensured through the directions of levelling. The level nets shown in Fig. 5.7a are compatible but the level nets shown in Fig. 5.7b are not compatible as to maintain conformity along *BD* in the two nets *ABDA* and *BCDB*, the second net has to be considered as *BDCB*. In doing so the falls from *B* to *C* and *C* to *D* are to be taken as rises from *C* to *B* and *D* to *C*, respectively. Therefore, the level nets in Fig. 5.7 a will be used to avoid any confusion.

In each of the closed circuit the following condition must be satisfied:

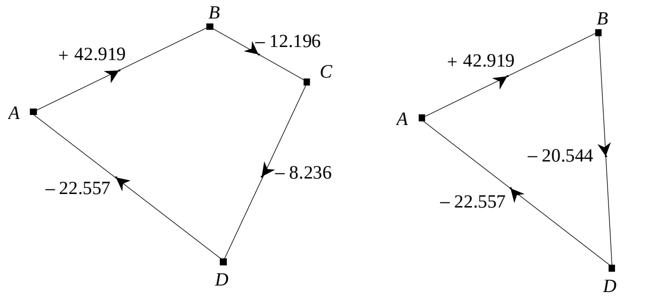
$$\Sigma \text{ Rise} = \Sigma \text{ Fall.}$$

Let the corrections to be applied to the differences in level be e_1, e_2, e_3, e_4, and e_5.

In circuit *ABCDA*

$$\text{Error} = +\ 42.919 - 12.196 - 8.236 - 22.557 = -\ 0.070 \text{ m}$$

$$\text{Correction} = +\ 0.070 \text{ m.}$$



(a)

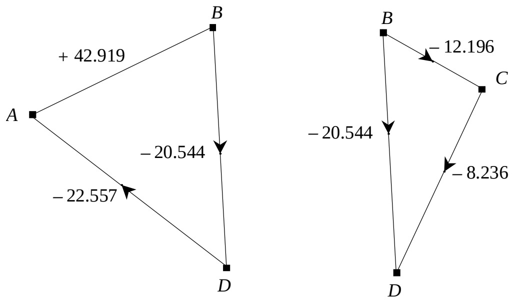
Fig. 5.7

In circuit *ABDA*

$$\text{Error} = +\ 42.919 - 20.544 - 22.557$$

$$= -\ 0.182 \text{ m}$$

$$\text{Correction} = +\ 0.182 \text{ m.}$$



(b)

Fig. 5.7

Thus we have the following condition equations.

$$e_1 + e_2 + e_3 + e_4 = 0.070 \quad \ldots(a)$$

$$e_1 + e_4 + e_5 = 0.182.$$

From the least squares principle, we have

$$\phi = \omega_1 e^2_1 + \omega_2 e^2_2 + \omega_3 e^2_3 + \omega_4 e^2_4 + w_5 e^2_5 = \text{a minimum} \quad \ldots(b)$$

Differentiating partially Eqs. (a) and (b), we get

$$\partial e_1 + \partial e_2 + \partial e_3 + \partial e_4 = 0 \quad \ldots(c)$$

$$\partial e_1 + \partial e_4 + \partial e_5 = 0 \quad \ldots(d)$$

$$\omega_1 \partial e_1 + \omega_2 \partial e_2 + \omega_3 \partial e_3 + \omega_4 \partial e_4 + \omega_5 \partial e_5 = 0 \quad \ldots(e)$$

Multiplying Eq. (c) by $-\lambda_1$, Eq. (d) by $-\lambda_2$, and adding the results to Eq. (e), we get

$$-\lambda_1(\partial e_1 + \partial e_2 + \partial e_3 + \partial e_4) - \lambda_2(\partial e_1 + \partial e_4 + \partial e_5) + \omega_1 \partial e_1 + \omega_2 \partial e_2 + \omega_3 \partial e_3 + \omega_4 \partial e_4 + \omega_5 \partial e_5 = 0$$

or

$$(\omega_1 e_1 - \lambda_1 - \lambda_2)\partial e_1 + (\omega_2 e_2 - \lambda_1)\partial e_2 + (\omega_3 e_3 - \lambda_1)\partial e_3$$
$$+ (\omega_4 e_4 - \lambda_1 - \lambda_2)\partial e_4 + (\omega_5 e_5 - \lambda_2)\partial e_5 = 0$$

Since ∂e_1, ∂e_2, etc., are independent quantities, we have

$$(\omega_1 e_1 - \lambda_1 - \lambda_2) = 0 \quad \text{or} \quad e_1 = \frac{\lambda_1 + \lambda_2}{\omega_1}$$

$$(\omega_2 e_2 - \lambda_1) = 0 \quad \text{or} \quad e_2 = \frac{\lambda_1}{\omega_2}$$

$$(\omega_3 e_3 - \lambda_1) = 0 \quad \text{or} \quad e_3 = \frac{\lambda_1}{\omega_3}$$

$$(\omega_4 e_4 - \lambda_1 - \lambda_2) = 0 \quad \text{or} \quad e_4 = \frac{\lambda_1 + \lambda_2}{\omega_4}$$

$$(\omega_5 e_5 - \lambda_2) = 0 \qquad \text{or} \qquad e_5 = \frac{\lambda_2}{\omega_5}$$

Now taking weights inversely proportional to the length of run, we have

$$\omega_2 = \frac{1}{12}, \quad \omega_3 = \frac{1}{25}, \quad \omega_4 = \frac{1}{22}, \quad \omega_5 = \frac{1}{32}.$$

Two runs were made on the line *AB* and the mean value of level difference is given. We know that

$$\sigma_m = \frac{\sigma}{\sqrt{n}}$$

and

$$\omega_m \sigma_m^2 = \omega \sigma^2$$

therefore,

$$\omega_m = \frac{\omega \sigma^2}{\sigma_m^2} = n\omega. \qquad \ldots(f)$$

Eq. (f) relates the weight of the mean of n observations to that of the single observation. Hence, for the observation made for the line *AB*, which is mean of two observations

$$n = 2$$

$$\omega = \frac{1}{l} = \frac{1}{20}$$

$$\omega_1 = \omega_m = n\omega = \frac{2}{22} = \frac{1}{11}.$$

Therefore

$$e_1 = 11(\lambda_1 + \lambda_2), \quad e_2 = 12\lambda_1, \; e_3 = 22\lambda_1, \; e_4 = 32(\lambda_1 + \lambda_2), \; e_5 = 25\lambda_2.$$

Substituting the values of e_1, e_2, e_3, e_4, and e_5 in Eq. (a), we get

$$77\lambda_1 + 43\lambda_2 = 0.070$$

$$43\lambda_1 + 68\lambda_2 = 0.182$$

$$\lambda_1 = -0.0009$$

$$\lambda_2 = +0.0032.$$

Therefore

$$e_1 = 11 \times (-0.0009 + 0.00032) = +0.025 \text{ m}$$

$$e_2 = 12 \times (-0.0009) = -0.012 \text{ m}$$

$$e_3 = 22 \times (-0.0009) = -0.020 \text{ m}$$

$$e_4 = 32 \times (-0.0009 + 0.0032) = +0.074 \text{ m}$$

$$e_5 = 25 \times 0.0032 = +0.080 \text{ m}.$$

Therefore the most probable values of the level differences are

$$A \text{ to } B = 42.919 + 0.025 = \mathbf{+\ 42.944\ m}$$
$$B \text{ to } C = -\ 12.196 - 0.012 = -\ \mathbf{12.208\ m}$$
$$B \text{ to } D = -\ 20.544 + 0.080 = -\ \mathbf{20.464\ m}$$
$$C \text{ to } D = -\ 8.236 - 0.020 = -\ \mathbf{8.256\ m}$$
$$D \text{ to } A = -\ 22.557 + 0.074 = -\ \mathbf{22.483\ m}.$$

If the level nets shown in Fig. 5.7b are considered, the falls from *B* to *C* and from *C* to *D* in the net *BCDA* are to be taken as rises from *C* to *B* and from *D* to *C*, respectively.

$$\text{Error in } ABDA = +\ 42.919 - 20.544 - 22.557 = -\ 0.182 \text{ m}$$
$$\text{Error in } BCDB = +\ 12.196 - 20.544 + 8.236 = -\ 0.112 \text{ m}$$

Therefore,

$$e_1 + e_5 + e_4 = -\ (-\ 0.179) = 0.182 \text{ m}$$
$$e_2 + e_5 + e_3 = -\ (-\ 0.112) = 0.110 \text{ m}.$$

Following the steps as given above for Fig. 5.7a, the values of the corrections would be

$$e_1 = +\ 0.025 \text{ m}$$
$$e_2 = +\ 0.012 \text{ m}$$
$$e_3 = +\ 0.020 \text{ m}$$
$$e_4 = +\ 0.074 \text{ m}$$
$$e_5 = +\ 0.080 \text{ m}$$

and the most probable values would be

$$A \text{ to } B = 42.919 + 0.025 = +\ 42.309 \text{ m}$$
$$C \text{ to } B = +\ 12.196 + 0.012 = +\ 12.025 \text{ m}$$
$$B \text{ to } D = -\ 20.544 + 0.080 = -\ 20.157 \text{ m}$$
$$D \text{ to } C = +\ 8.236 + 0.020 = +\ 8.132 \text{ m}$$
$$D \text{ to } A = -\ 22.557 + 0.074 = -\ 22.152 \text{ m}.$$

Now taking rises from *C* to *B* and from *D* to *C* as falls from *B* to *C* and from *C* to *D*, respectively, we get the same values of the most probable values of the level differences as obtained considering the nets shown in Fig. 5.7a, i.e.,

$$A \text{ to } B = +\ 42.309 \text{ m} \qquad B \text{ to } C = -\ 12.025 \text{ m}$$
$$B \text{ to } D = -\ 20.157 \text{ m} \qquad C \text{ to } D = -\ 8.132 \text{ m}$$
$$D \text{ to } A = -\ 22.152 \text{ m}.$$

Example 5.11. The mean observed values of a spherical triangle *ABC* are as follows:

$$\alpha = 55°18'24.45'' \qquad \omega_1 = 1$$
$$\beta = 62°23'34.24'' \qquad \omega_2 = 2$$
$$\gamma = 62°18'10.34'' \qquad \omega_3 = 3$$

The length of the side BC was also measured as 59035.6 m. If the mean earth's radius is 6370 km, determine the most probable values of the spherical angles.

Solution (Fig. 5.8):

In a Spherical triangle *ABC*, we have

$$A + B + C - 180° = \text{Spherical excess} = \varepsilon.$$

Thus for the given angles

$$\varepsilon = \alpha + \beta + \gamma - 180°$$
$$= 55°18'24.45'' + 62°23'34.24'' + 62°18'10.34'' - 180°$$
$$= 180°00'9.03'' - 180°$$
$$= 9.03''.$$

The spherical excess is given by

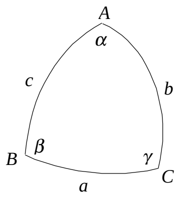


Fig. 5.8

$$\varepsilon = \frac{A_0}{R^2 \sin 1''} \quad \text{seconds}$$

where

$$A_0 = \frac{1}{2} a^2 \frac{\sin\beta \sin\gamma}{\sin\alpha},$$

R = the mean radius of earth, and

a = the measured length of BC.

In determination of A_0, spherical excess being a small quantity, taking the observed angles directly would not cause appreciable error in ε. Therefore,

$$A_0 = \frac{1}{2} \times 59035.6^2 \frac{\sin(62°23'34.24'')\sin(62°18'10.34'')}{\sin(55°18'24.45'')}$$
$$= 1665.5940 \text{ sq km}$$

and

$$\varepsilon = \frac{1665.5940}{6370^2} \times 206265 = 8.47''.$$

Theoretical sum of the angles = 180° + ε = 180° + 8.47″ = 180°00′8.47″

Thus the total error in the angles = 180°00′9.03″ – 180°00′8.47″

= 0.56″

Correction = – 0.56″.

If the corrections to the angles α, β, and γ are e_1, e_2, and e_3, respectively, then

$$e_1 + e_2 + e_3 = -0.56'' \qquad \text{...(a)}$$

From theory of least squares, we have

$$\phi = \omega_1 e_1^2 + \omega_2 e_2^2 + \omega_3 e_3^2 = \text{a minimum.} \qquad \text{...(b)}$$

Differentiating Eqs. (a) and (b), we get

$$\partial e_1 + \partial e_2 + \partial e_3 = 0 \qquad \text{...(c)}$$

$$\omega_1 \partial e_1 + \omega_2 \partial e_2 + \omega_3 \partial e_3 = 0 \qquad \text{...(d)}$$

Multiplying Eq (c) by $-\lambda$ and adding the result to Eq. (d), we get

$$(\omega_1 e_1 - \lambda)\partial e_1 + (\omega_2 e_2 - \lambda)\partial e_2 + (\omega_3 e_3 - \lambda)\partial e_3 = 0$$

or $$(\omega_1 e_1 - \lambda) = 0 \quad \textbf{or} \quad e_1 = \frac{\lambda}{\omega_1} = \lambda$$

$$(\omega_2 e_2 - \lambda) = 0 \quad \textbf{or} \quad e_2 = \frac{\lambda}{\omega_2} = \frac{\lambda}{2}$$

$$(\omega_3 e_3 - \lambda) = 0 \quad \textbf{or} \quad e_3 = \frac{\lambda}{\omega_3} = \frac{\lambda}{3}$$

Now substituting the values of e_1, e_2, and e_3 in Eq. (a), we get

$$\lambda + \frac{\lambda}{2} + \frac{\lambda}{3} = -0.56''$$

$$\lambda = -\frac{6 \times 0.56}{11}$$

$$\lambda = -0.305''.$$

Thus
$$e_1 = -0.305''$$
$$e_2 = -0.153''$$
$$e_3 = -0.102''.$$

Therefore, the most probable values of the spherical angles are

α = 55°18′24.45″ – 0.305″ = **55°18′24.14″**

β = 62°23′34.24″ – 0.153″ = **62°23′34.09″**

γ = 62°18′10.34″ – 0.102″ = **62°18′10.24″**

Total = 80°00′08.47″ (*Check*).

Alternative solution:

The corrections to the individual angles can be taken as inversely proportion to their weights, i.e.,

$$e_1 : e_2 : e_3 = \frac{1}{1} : \frac{1}{2} : \frac{1}{3}$$

Therefore
$$e_2 = \frac{1}{2} e_1$$

$$e_3 = \frac{1}{3} e_1 .$$

Substituting the values of e_2, and e_3 in Eq. (a), we get

$$e_1 = -\frac{6 \times 0.56}{11}$$

$$e_1 = -0.305''$$

and $$e_2 = -0.153''$$

$$e_3 = -0.102''.$$

These corrections are the same as obtained from the least squares principle and, therefore, the most probable values by applying these corrections will be the same as obtained above.

Example 5.12. A braced quadrilateral *ABCD* as shown in Fig. 5.8, was set out to determine the distance between *A* and *B*. The mean observed angles are given below:

$$\theta_1 = 43°48'22''; \qquad \theta_5 = 49°20'43'';$$

$$\theta_2 = 38°36'57''; \qquad \theta_6 = 33°04'56'';$$

$$\theta_3 = 33°52'55''; \qquad \theta_7 = 50°10'43'';$$

$$\theta_4 = 63°41'24''; \qquad \theta_8 = 47°23'28'';$$

Adjust the quadrilateral by

(a) Approximate method

(b) Rigorous method.

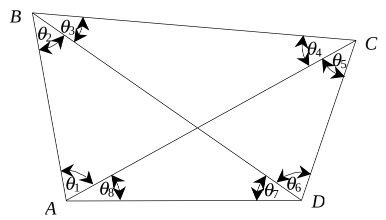


Fig. 5.8

Solution (Fig. 5.8):

A braced quadrilateral has the following four conditions to be satisfied:

(*i*) $\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7 + \theta_8 = 360°$

or $$\Sigma\theta = 360°$$

(*ii*) $\theta_1 + \theta_2 = \theta_5 + \theta_6$

(*iii*) $\theta_3 + \theta_4 = \theta_7 + \theta_8$

(*iv*) $\log \sin \theta_1 + \log \sin \theta_3 + \log \sin \theta_5 + \log \sin \theta_7$

$$= \log \sin \theta_2 + \log \sin \theta_4 + \log \sin \theta_6 + \log \sin \theta_8$$

or Σ log sin (odd angle) = Σ log sin (even angle).

(a) Approximate method

Satisfying the condition (*i*)

$$\Sigma\theta = 359°59'28''$$

$$\text{Total error} = 359°59'28'' - 360° = -32''$$

$$\text{Correction } C_1 = +32''.$$

If the corrections to the angles be e_1, e_2, e_3, etc., then

$$e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 = +32''$$

Distributing the total correction to each angle equally, we get

$$e_1 = e_2 = e_3 = e_4 = e_5 = e_6 = e_7 = e_8 = +\frac{32}{8} = 4''$$

Therefore the corrected angles are

$$\theta_1 = 43°48'22'' + 4'' = 43°48'26''$$
$$\theta_2 = 38°36'57'' + 4'' = 38°37'01''$$
$$\theta_3 = 33°52'55'' + 4'' = 33°52'59''$$
$$\theta_4 = 63°41'24'' + 4'' = 63°41'28''$$
$$\theta_5 = 49°20'43'' + 4'' = 49°20'47''$$
$$\theta_6 = 33°04'56'' + 4'' = 33°05'00''$$
$$\theta_7 = 50°10'43'' + 4'' = 50°10'47''$$
$$\theta_8 = 47°23'28'' + 4'' = 47°23'32''$$
$$\text{Total} = 360°00'00'' \quad (\textit{Check}).$$

Satisfying the condition (*ii*)

$$\theta_1 + \theta_2 = \theta_5 + \theta_6$$
$$\theta_1 + \theta_2 = 43°48'26'' + 38°37'01'' = 82°25'27''$$
$$\theta_5 + \theta_6 = 49°20'47'' + 33°05'00'' = 82°25'47''$$
$$\text{Difference} = 20''$$

$$\text{Correction to each angle} = \frac{20}{4} = 5''$$

The signs of the corrections C_2 to angles θ_1, θ_2, θ_5, and θ_6 are determined as below.

Since the sum of θ_1 and θ_2 is less than the sum of θ_5 and θ_6, the correction to each θ_1 and θ_{2+}, is + 5″ and the correction to each θ_5 and θ_{6+}, is – 5″.

Therefore the corrected angles are

$$\theta_1 = 43°48'26'' + 5'' = 43°48'31''$$
$$\theta_2 = 38°37'01'' + 5'' = 38°37'06''$$
$$\theta_5 = 49°20'47'' - 5'' = 49°20'42''$$
$$\theta_6 = 33°05'00'' - 5'' = 33°04'55''$$
$$\theta_1 + \theta_2 = 82°25'37'' = \theta_5 + \theta_6 \quad (\textit{Check}).$$

Satisfying the condition (*iii*)

$$\theta_3 + \theta_4 = \theta_7 + \theta_8$$
$$\theta_3 + \theta_4 = 33°52'59'' + 63°41'28'' = 97°34'27''$$
$$\theta_7 + \theta_8 = 50°10'47'' + 47°23'32'' = 97°34'19''$$
$$\text{Difference} = 8''$$

$$\text{Correction to each angle} = \frac{8}{4} = 2''$$

The signs of the corrections C_3 to angles θ_3, θ_4, θ_7, and θ_8 are determined as below:

Since the sum of θ_3 and θ_4 is more than the sum of θ_7 and θ_8, the correction to each θ_3 and θ_4, is – 2″ and the correction to each θ_7 and θ_8, is +2″.

Therefore the corrected angles are

$$\theta_3 = 33°52'59'' - 2'' = 33°52'57''$$

$$\theta_4 = 63°41'28'' - 2'' = 63°41'26''$$

$$\theta_7 = 50°10'47'' + 2'' = 50°10'49''$$

$$\theta_8 = 47°23'32'' + 2'' = 47°23'34''$$

$$\theta_3 + \theta_4 = 97°34'23'' = \theta_7 + \theta_8 \qquad (Check).$$

Satisfying the condition (*iv*)

The following computations involve the values of (log sin) of the angles which were determined using a log table before the inception of digital calculators. As the students now, will be using the calculators, the method of determining the values of (log sin θ) and other quantities using a calculator has been used here.

The corrections to the individual angles for satisfying the condition (*iv*), is given as

$$c_n = \frac{f_n\delta}{\Sigma f^2} \text{ seconds} \qquad \ldots(a)$$

where

f_n = the difference 1″ for log sinθ_n multiplied by 10^6, i.e., [log sin(θ_n + 1″) – log sin θ_n] × 10^6,

d = [Σ log sin(odd angle) – Σ log sin(even angle)] × 10^6 ignoring the signs of Σ log sin(odd angle) and Σ log sin (even angle), and

Σf^2 = the sum of squares of f_1, f_2, f_3, etc., i.e.,

$$f_1^2 + f_2^2 + f_3^2 + \ldots\ldots + f_n^2 .$$

The sign of the corrections C_n is decided as below.

If Σ log sin(odd angle) > Σ log sin(even angle), the corrections for odd angles will be positive and for the even angles negative and *vice-versa*.

Calculating the values of log sin(odd angle)'s, ignoring the signs

log sin θ_1 = log sin (43°48′31″) = 0.1597360, f_1 = 22

log sin θ_3 = log sin (33°52′57″) = 0.2537617, f_3 = 31

log sin θ_5 = log sin (49°20′42″) = 0.1199607, f_5 = 18

log sin θ_7 = log sin (50°10′49″) = 0.1146031, f_7 = 18

Σ log sin(odd angle) = 0.6480615

Calculating the values of log sin(even angle)'s, ignoring the signs

$$\log \sin \theta_2 = \log \sin (38°37'06'') = 0.2047252, \; f_2 = 26$$
$$\log \sin \theta_4 = \log \sin (63°41'26'') = 0.0474917, \; f_4 = 10$$
$$\log \sin \theta_6 = \log \sin (33°04'55'') = 0.2629363, \; f_6 = 32$$
$$\log \sin \theta_8 = \log \sin (47°23'34'') = 0.1331152, \; f_8 = 19$$
$$\Sigma \log \sin \text{(even angle)} = 0.6482684$$

Therefore,
$$\delta = [\Sigma \log \sin \text{(odd angle)} - \Sigma \log \sin \text{(even angle)}] \times 10^6$$
$$= (0.6480615 - 0.6482684) \times 10^6$$
$$= 2069 \text{ (ignoring the sign)}$$

and
$$\Sigma f^2 = 22^2 + 26^2 + 31^2 + 10^2 + 18^2 + 32^2 + 18^2 + 19^2 = 4254.$$

Since Σ log sin (odd angle) < Σ log sin (even angle) the corrections c_1, c_3, c_5, and c_7 will be negative, and c_2, c_4, c_6, and c_8 will be positive.

Thus
$$c_1 = 22 \times \frac{2069}{4254} = -\,10.7''; \; c_5 = 18 \times \frac{2069}{4254} = -\,8.8''$$
$$c_2 = 26 \times \frac{2069}{4254} = +\,12.6''; \; c_6 = 32 \times \frac{2069}{4254} = +\,15.6''$$
$$c_3 = 31 \times \frac{2069}{4254} = -\,15.1''; \; c_7 = 18 \times \frac{2069}{4254} = -\,8.8''$$
$$c_4 = 10 \times \frac{2069}{4254} = +\,4.9''; \; c_8 = 19 \times \frac{2069}{4254} = +\,9.2''.$$

Therefore the adjusted values of the angles are

$$\theta_1 = 43°48'31'' - 10.7'' = \mathbf{43°48'20.3''}$$
$$\theta_2 = 38°37'06'' + 12.6'' = \mathbf{38°37'18.4''}$$
$$\theta_3 = 33°52'57'' - 15.1'' = \mathbf{33°52'41.9''}$$
$$\theta_4 = 63°41'26'' + 4.9'' = \mathbf{63°41'30.9''}$$
$$\theta_5 = 49°20'42'' - 8.8'' = \mathbf{49°20'33.2''}$$
$$\theta_6 = 33°04'55'' + 15.6'' = \mathbf{33°05'10.6''}$$
$$\theta_7 = 50°10'49'' - 8.8'' = \mathbf{50°10'40.2''}$$
$$\theta_8 = 47°23'34'' + 9.2'' = \mathbf{47°23'43.2''}$$
$$\text{Total} = 359°59'58.7'' \qquad (\textit{Check}).$$

Since there is still an error of 1.3″, if need be one more iteration of all the steps can be done to get better most probable values of the angles. Alternatively, since the method is approximate, one can add $\frac{1.3''}{8} = 0.163''$ to each angle and take the resulting values as the most probable values.

Thus,

$$\theta_1 = 43°48'20.3'' + 0.163'' = 43°48'20.463''$$
$$\theta_2 = 38°37'18.6'' + 0.163'' = 38°37'18.763''$$
$$\theta_3 = 33°52'41.9'' + 0.163'' = 33°52'42.063''$$
$$\theta_4 = 63°41'30.9'' + 0.163'' = 63°41'31.063''$$
$$\theta_5 = 49°20'33.2'' + 0.163'' = 49°20'33.363''$$
$$\theta_6 = 33°05'10.6'' + 0.163'' = 33°05'10.763''$$
$$\theta_7 = 50°10'40.2'' + 0.163'' = 50°10'40.363''$$
$$\theta_8 = 47°23'43.2'' + 0.163'' = 47°23'43.363''$$

Total = 360°00′00.204″ (*Check*).

For systematic computations, the computations may be done in tabular form as given in Table 5.1.

(*b*) Rigorous method

Let the corrections to the angles be e_1, e_2,......, e_8 then from the conditions to be satisfied, we get

$$e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 = +32'' \quad \ldots(a)$$

$$e_1 + e_2 - e_5 - e_6 = 82°25'27'' - 82°25'47'' = +20'' \quad \ldots(b)$$

$$e_3 + e_4 - e_7 - e_8 = 97°34'27'' - 97°34'19'' = -8'' \quad \ldots(c)$$

$$f_1e_1 - f_2e_2 + f_3e_3 - f_4e_4 + f_5e_5 - f_6e_6 + f_7e_7 - f_8e_8 = -\delta = -2069 \quad \ldots(d)$$

where $\delta = [\Sigma \log \sin (\text{odd angle}) - \Sigma \log \sin (\text{even angle})] \times 10^6$.

Another condition to be satisfied from least squares theory is

$$\phi = e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + e_7^2 + e_8^2 = \text{a minimum.} \quad \ldots(e)$$

Differentiating the Eq. (a) to (e), we get

$$\partial e_1 + \partial e_2 + \partial e_3 + \partial e_4 + \partial e_5 + \partial e_6 + \partial e_7 + \partial e_8 = 0 \quad \ldots(f)$$

$$\partial e_1 + \partial e_2 - \partial e_5 - \partial e_6 = 0 \quad \ldots(g)$$

$$\partial e_3 + \partial e_4 - \partial e_7 - \partial e_8 = 0 \quad \ldots(h)$$

$$f_1\partial e_1 - f_2\partial e_2 + f_3\partial e_3 - f_4\partial e_4 + f_5\partial e_5 - f_6\partial e_6 + f_7\partial e_7 - f_8\partial e_8 = 0 \quad \ldots(i)$$

$$e_1\partial e_1 + e_2\partial e_2 + e_3\partial e_3 + e_4\partial e_4 + e_5\partial e_5 + e_6\partial e_6 + e_7\partial e_7 + e_8\partial e_8 = 0 \quad \ldots(j)$$

Multiplying Eqs. (f), (g), (h) and (i) by $-\lambda_1$, $-\lambda_2$, $-\lambda_3$, and $-\lambda_4$, respectively, and adding to Eq. (j), and equating the coefficients of ∂e_1, ∂e_2, ∂e_3, etc., in the resulting equation, to zero, we get

$$(e_1 - \lambda_1 - \lambda_2 - f_1\lambda_4) = 0 \quad \text{or} \quad e_1 = \lambda_1 + \lambda_2 + f_1\lambda_4$$
$$(e_2 - \lambda_1 - \lambda_2 + f_2\lambda_4) = 0 \quad \text{or} \quad e_2 = \lambda_1 + \lambda_2 - f_2\lambda_4$$
$$(e_3 - \lambda_1 - \lambda_3 - f_3\lambda_4) = 0 \quad \text{or} \quad e_3 = \lambda_1 + \lambda_3 + f_3\lambda_4$$
$$(e_4 - \lambda_1 - \lambda_3 + f_4\lambda_4) = 0 \quad \text{or} \quad e_4 = \lambda_1 + \lambda_3 - f_4\lambda_4$$

Table 5.1

Angle	C_1	Corrtd. angle	$C_{2,3}$	Corrtd. angle	Log sin (odd)	Log sin (even)	f	f^2	C_4	Corrtd. angle
θ_1 = 43°48′22″	+4″	43°48′26″	+5″	43°48′26″	0.1597360		22	484	– 10.7″	43°48′20.3″
θ_2 = 38°36′57″	+4″	38°37′01″	+5″	38°37′01″		0.2047252	26	676	+ 12.6″	38°37′18.6″
θ_3 = 33°52′55″	+4″	33°52′59″	–2″	33°52′59″	0.2537617		31	961	– 15.1″	33°52′41.9″
θ_4 = 63°41′24″	+4″	63°41′28″	–2″	63°41′28″		0.0474917	10	100	+ 4.9″	63°41′30.9″
θ_5 = 49°20′43″	+4″	49°20′47″	–5″	49°20′47″	0.1199607		18	324	– 8.8″	49°20′33.2″
θ_6 = 33°04′56″	+4″	33°05′00″	–5″	33°05′00″		0.2629363	32	1024	+ 15.6″	33°05′10.6″
θ_7 = 50°10′43″	+4″	50°10′47″	+2″	50°10′47″	0.1146031		18	324	– 8.8″	50°10′40.2″
θ_8 = 47°23′28″	+4″	47°23′32″	+2″	47°23′32″		0.1331152	19	361	+ 9.2″	47°23′43.2″
Σ = 359°59′28″	+32″	360°00′00″		360°00′00″	0.6480615	0.6482684		4254		359°59′58.9″

$$(e_5 - \lambda_1 + \lambda_2 - f_5\lambda_4) = 0 \quad \text{or} \quad e_5 = \lambda_1 - \lambda_2 + f_5\lambda_4$$
$$(e_6 - \lambda_1 + \lambda_2 + f_6\lambda_4) = 0 \quad \text{or} \quad e_6 = \lambda_1 - \lambda_2 - f_6\lambda_4$$
$$(e_7 - \lambda_1 + \lambda_3 - f_7\lambda_4) = 0 \quad \text{or} \quad e_7 = \lambda_1 - \lambda_3 + f_7\lambda_4$$
$$(e_8 - l_1 + \lambda_3 + f_8\lambda_4) = 0 \quad \text{or} \quad e_8 = \lambda_1 - \lambda_3 - f_8\lambda_4.$$

Substituting the values of the corrections $e_1, e_2, \ldots\ldots, e_8$ in Eq. (a) to (e), we get

$$8\lambda_1 + F\lambda_4 = 32''$$
$$4\lambda_2 + (F_{12} - F_{56})\lambda_4 = 20''$$
$$4\lambda_3 + (F_{34} - F_{78})\lambda_4 = -8''$$
$$(F_{12} + F_{34} + F_{56} + F_{78})\lambda_1 + (F_{12} - F_{56})\lambda_2 + (F_{34} - F_{78})\lambda_3 + F^2\ \lambda_4 = -2069$$

where $F = \Sigma f_{odd} - \Sigma f_{even} = (22 + 31 + 18 + 18) - (26 + 10 + 32 + 19) = 2$

$$F_{12} = f_1 - f_2 = 22 - 26 = -4$$
$$F_{34} = f_3 - f_4 = 31 - 10 = 21$$
$$F_{56} = f_5 - f_6 = 18 - 32 = -14$$
$$F_{78} = f_7 - f_8 = 18 - 19 = -1$$
$$F^2 = 4254.$$

Therefore

$$8\lambda_1 + 2\lambda_4 = 32''$$
$$4\lambda_2 + 10\lambda_4 = 20''$$
$$4\lambda_3 + 22\lambda_4 = -8'' \qquad \ldots(k)$$
$$2\lambda_1 + 10\lambda_2 + 22\lambda 3 + 4254\lambda 4 = -2069$$

or

$$\lambda_1 = \frac{1}{8}(32 - 24\lambda_4)$$
$$\lambda_2 = \frac{1}{4}(20 - 10\lambda_4)$$
$$\lambda_3 = \frac{1}{4}(-8 - 22\lambda_4)..$$

Substituting the values of λ_1, λ_2 and λ_3 in Eq. (k), we get

$$\lambda_4 = -0.507$$

and then

$$\lambda_1 = +4.127$$
$$\lambda_2 = +6.268$$
$$\lambda_3 = +0.789.$$

Thus,

$$e_1 = +4.127 + 6.268 - 22 \times (-0.507) = -0.759''$$
$$e_2 = +4.127 + 6.268 + 26 \times (-0.507) = +23.577''$$
$$e_3 = +4.127 + 0.789 - 31 \times (-0.507) = -10.801''$$

$$e_4 = +4.127 + 0.789 + 10 \times (-0.507) = +9.986''$$
$$e_5 = +4.127 - 6.268 - 18 \times (-0.507) = -11.267''$$
$$e_6 = +4.127 - 6.268 + 32 \times (-0.507) = +14.083''$$
$$e_7 = +4.127 - 0.789 - 18 \times (-0.507) = -5.788''$$
$$e_8 = +4.127 - 0.789 + 19 \times (-0.507) = +12.971''$$
$$\text{Total} = 32.002'' \approx 32'' \qquad (Check).$$

In view of the Eqs. (a) to (d), the following checks may be applied on the calculated values of the corrections:

$$e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 = +32.002'' \approx +32''$$
$$e_1 + e_2 - e_5 - e_6 = +20.002'' \approx +20''$$
$$e_3 + e_4 - e_7 - e_8 = -7.998'' \approx -8''$$
$$f_1e_1 - f_2e_2 + f_3e_3 - f_4e_4 + f_5e_5 - f_6e_6 + f_7e_7 - f_8e_8 = -\delta = -2068.486 \approx -2069$$

Applying the corrections to the observed values, the adjusted values of the angles are

$$\theta_1 = 43°48'22'' - 0.759'' = \mathbf{43°48'21.241''}$$
$$\theta_2 = 38°36'57'' + 23.577'' = \mathbf{38°37'20.577''}$$
$$\theta_3 = 33°52'55'' - 10.801'' = \mathbf{33°52'44.199''}$$
$$\theta_4 = 63°41'24'' + 9.986'' = \mathbf{63°41'33.986''}$$
$$\theta_5 = 49°20'43'' - 11.267'' = \mathbf{49°20'31.733''}$$
$$\theta_6 = 33°04'56'' + 14.083'' = \mathbf{33°05'10.083''}$$
$$\theta_7 = 50°10'43'' - 5.788'' = \mathbf{50°10'37.212''}$$
$$\theta_8 = 47°23'28'' + 12.971'' = \mathbf{47°23'40.971''}$$
$$\text{Total} = 360°00'00.002'' \qquad (Check).$$

Example 5.13. Fig. 5.9 shows a quadrilateral *ABCD* with a central station *O*. The angles measured are as below:

$\theta_1 = 29°17'00''$; $\theta_2 = 28°42'00''$

$\theta_3 = 62°59'49''$; $\theta_4 = 56°28'01''$

$\theta_5 = 29°32'06''$; $\theta_6 = 32°03'54''$

$\theta_7 = 59°56'06''$; $\theta_8 = 61°00'54''$

$\theta_9 = 122°00'55''$; $\theta_{10} = 60°32'05''$

$\theta_{11} = 118°23'50''$; $\theta_{12} = 59°03'10''$

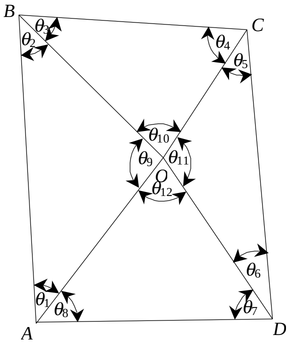


Fig. 5.9

Determine the most probable values of the angles assuming that the angles are of same reliability and have been adjusted for station adjustment and spherical excess.

Solution (Fig. 5.9):

There are four triangles in which the following conditions are to be satisfied:

$$\theta_1 + \theta_2 + \theta_9 = 180°$$
$$\theta_3 + \theta_4 + \theta_{10} = 180°$$
$$\theta_5 + \theta_6 + \theta_{11} = 180°$$
$$\theta_7 + \theta_8 + \theta_{12} = 180°$$

At station *O*, the condition to be satisfied is

$$\theta_9 + \theta_{10} + \theta_{11} + \theta_{12} = 360°$$

and the side condition to be satisfied is

$$\sum \log \sin \text{(odd angle)} = \sum \log \sin \text{(even angle)}$$

or $$\log \sin \theta_1 + \log \sin \theta_3 + \log \sin \theta_5 + \log \sin \theta_7 = \log \sin \theta_2 + \log \sin \theta_4 + \log \sin \theta_6 + \log \sin \theta_8.$$

For the various notations used here and in other problems, Example 5.12 may be referred.

$$29°17'00'' + 28°42'00'' + 122°00'55'' - 180° = E_1 = -5''$$
$$e_1 + e_2 + e_9 = C_1 = +5''$$
$$62°59'49'' + 56°28'01'' + 60°32'05'' - 180° = E_2 = -5''$$
$$e_3 + e_4 + e_{10} = C_2 = +5''$$
$$29°32'06'' + 32°03'54'' + 118°23'50'' - 180° = E_3 = -10''$$
$$e_5 + e_6 + e_{11} = C_3 = +10''$$
$$59°56'06'' + 61°00'54'' + 59°03'10'' - 180° = E_4 = +10''$$
$$e_7 + e_8 + e_{12} = C_4 = -10''$$
$$122°00'55'' + 60°32'05'' + 118°23'50'' + 59°03'10'' - 360° = E_5 = 0''$$
$$e_9 + e_{10} + e_{11} + e_{12} = C_5 = 0''$$

Calculating the values of log sin (odd angle)'s, ignoring the signs

$$\log \sin \theta_1 = \log \sin (29°17'00'') = 0.3105768, \quad f_1 = 38$$
$$\log \sin \theta_3 = \log \sin (62°59'49'') = 0.0501309, \quad f_3 = 11$$
$$\log \sin \theta_5 = \log \sin (29°32'06'') = 0.3071926, \quad f_5 = 37$$
$$\log \sin \theta_7 = \log \sin (59°56'06'') = 0.0627542, \quad f_7 = 12$$
$$\Sigma \log \sin \text{(odd angle)} = 0.7306545$$

Calculating the values of log sin(even angle)'s, ignoring the signs

$$\log \sin \theta_2 = \log \sin (28°42'00'') = 0.3185566, \quad f_2 = 38$$
$$\log \sin \theta_4 = \log \sin (56°28'01'') = 0.0790594, \quad f_4 = 14$$
$$\log \sin \theta_6 = \log \sin (32°03'54'') = 0.2750028, \quad f_6 = 34$$
$$\log \sin \theta_8 = \log \sin (61°00'54'') = 0.0581177, \quad f_8 = 12$$
$$\Sigma \log \sin \text{(even angle)} = 0.7307365$$

Therefore $$\delta = [\Sigma \log \sin \text{(odd angle)} - \Sigma \log \sin \text{(even angle)}] \times 10^6$$
$$= (0.7306545 - 0.7307365) \times 10^6 = E_6$$
$$= -820$$

or $$f_1e_1 - f_2e_2 + f_3e_3 - f_4e_4 + f_5e_5 - f_6e_6 + f_7e_7 - f_8e_8 = C_6 = +820.$$

Since there are six condition equations, there will be six correlates and the equations to determine them using the theory of least squares, will be

$$3\lambda_1 + \lambda_5 + F_{12}\lambda_6 = 5$$
$$3\lambda_2 + \lambda_5 + F_{34}\lambda_6 = 5$$
$$3\lambda_3 + \lambda_5 + F_{56}\lambda_6 = 10$$
$$3\lambda_4 + \lambda_5 + F_{78}\lambda_6 = -10 \qquad \ldots(a)$$
$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 4\lambda_5 = 0$$
$$F_{12}\lambda_1 + F_{34}\lambda_2 + F_{56}\lambda_3 + F_{78}\lambda_4 + F^2\lambda_6 = 820$$

From the values of f_1, f_2, f_3, etc., we have

$$F_{12} = 0, \quad F_{34} = -3, \quad F_{56} = 3, \quad F_{78} = 0, \quad F^2 = 6018.$$

Substituting the above values in Eqs. (a) and solving them, we get

$$\lambda_1 = +2.083, \; \lambda_4 = -2.917$$
$$\lambda_2 = +2.219, \; \lambda_5 = -1.250$$
$$\lambda_3 = +3.614, \; \lambda_6 = +0.136.$$

Now the corrections to the angles are

$$e_1 = \lambda_1 + f_1\lambda_6 = 2.083 + 38 \times 0.136 = +7.251''$$
$$e_2 = \lambda_1 - f_2\lambda_6 = 2.083 - 38 \times 0.136 = -3.085''$$
$$e_3 = \lambda_2 + f_3\lambda_6 = 2.219 + 11 \times 0.136 = +3.715''$$
$$e_4 = \lambda_2 - f_4\lambda_6 = 2.219 - 14 \times 0.136 = +0.315''$$
$$e_5 = \lambda_3 + f_5\lambda_6 = 3.614 + 37 \times 0.136 = +8.646''$$
$$e_6 = \lambda_3 - f_6\lambda_6 = 3.614 - 34 \times 0.136 = -1.010''$$
$$e_7 = \lambda_4 + f_7\lambda_6 = -2.917 + 12 \times 0.136 = -1.285''$$
$$e_8 = \lambda_4 - f_8\lambda_6 = -2.917 - 12 \times 0.136 = -4.549''$$
$$\text{Total} = 9.998'' \approx 10''$$

Check:

$$\text{Total correction} = (2n - 4)\,90° - (\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 + \theta_7 + \theta_8)$$
$$= 360° - 359°59'50''$$
$$= 10'' \quad (\textit{Okay}).$$

$$e_9 = \lambda_1 + \lambda_5 = 2.083 - 1.250 = 0.833''$$
$$e_{10} = \lambda_2 + \lambda_5 = 2.219 - 1.250 = 0.969''$$
$$e_{11} = \lambda_3 + \lambda_5 = 3.614 - 1.250 = 2.364''$$
$$e_{12} = \lambda_4 + \lambda_5 = 2.917 - 1.250 = -4.167''$$
$$\text{Total} = -0.001'' \approx 0$$

Check:

$$\text{Total correction} = 360° - (\theta_9 + \theta_{10} + \theta_{11} + \theta_{12})$$
$$= 360° - 360°$$
$$= 0 \quad (\textit{Okay}).$$

Applying the above corrections to the observed angles, we get the most probable values as under:

$$\theta_1 + e_1 = 29°17'00'' + 7.251'' = \mathbf{29°17'07.25''}$$
$$\theta_2 + e_2 = 28°42'00'' - 3.085'' = \mathbf{28°41'56.92''}$$
$$\theta_9 + e_9 = 122°00'55'' + 0.833'' = \mathbf{122°01'55.83''}$$
$$\text{Total} = 180°00'00'' \quad (\textit{Check}).$$
$$\theta_3 + e_3 = 62°59'49'' + 3.715'' = \mathbf{62°59'52.71''}$$
$$\theta_4 + e_4 = 56°28'01'' + 0.315'' = \mathbf{56°28'01.31''}$$
$$\theta_{10} + e_{10} = 60°32'05'' + 0.969'' = \mathbf{60°32'05.97''}$$
$$\text{Total} = 180°00'00'' \quad (\textit{Check}).$$
$$\theta_5 + e_5 = 29°32'06'' + 8.646'' = \mathbf{29°32'14.65''}$$
$$\theta_6 + e_6 = 32°03'54'' - 1.010'' = \mathbf{32°03'52.99''}$$
$$\theta_{11} + e_{11} = 118°23'50'' + 2.364'' = \mathbf{118°23'52.36''}$$
$$\text{Total} = 180°00'00'' \quad (\textit{Check}).$$
$$\theta_7 + e_7 = 59°56'06'' - 1.285'' = \mathbf{59°56'04.72''}$$
$$\theta_8 + e_8 = 61°00'54'' - 4.549'' = \mathbf{61°00'49.45''}$$
$$\theta_{12} + e_{12} = 59°03'10'' - 4.167'' = \mathbf{59°03'05.83''}$$
$$\text{Total} = 180°00'00'' \quad (\textit{Check}).$$

Example 5.14. For two stations *A* (E 4006.99 m, N 11064.76 m) and *B* (E 7582.46 m, N 8483.29 m), the following observations were made on station *C*.

Line	Length (m)	Bearing	Angle
AC	4663.08 ± 0.05	76°06′29″ ± 4.0″	*ACB* = 61°41′57″ ± 5.6″
BC	3821.21 ± 0.05	14°24′27″ ± 4.0″	

If the provisional coordinates of *C* have been taken as E 8533.38 m, N 12184.52 m, determine the coordinates of *C* by the method of variation of coordinates.

Solution (Fig. 5.1):

(*i*) Calculation of (*O* – *C*) values

For θ_{AC}

$$x_C - x_A = 8533.38 - 4006.99 = 4526.39 \text{ m}$$
$$y_C - y_A = 12184.52 - 11064.76 = 1119.76 \text{ m}$$
$$\tan \theta_{AC} = \frac{4526.39}{1119.76} = 4.0422858$$
$$\text{Computed value of } \theta_{AC} = 76°06'17.5'' = C$$
$$\text{Observed value of } \theta_{AC} = 76°06'29'' = O$$
$$(O - C) = 76°06'29'' - 76°06'17.5'' = +11.5''$$
$$= +\frac{11.5}{206265} = +5.57535210 \times 10^{-5} \text{ radians.}$$

For θ_{BC}

$$x_C - x_B = 8533.38 - 7582.46 = 950.92 \text{ m}$$

$$y_C - y_B = 12184.52 - 8483.29 = 3701.23 \text{ m}$$

$$\tan\theta_{BC} = \frac{950.92}{3701.23} = 0.2569200$$

$$\text{Computed value of } \theta_{BC} = 14°24'31.7'' = C$$

$$\text{Observed value of } \theta_{BC} = 14°24'27'' = O$$

$$(O - C) = 14°24'27'' - 14°24'31.7'' = -4.7''$$

$$= -\frac{4.7}{206265} = -2.27862216 \times 10^{-5} \text{ radians.}$$

For l_{AC}

$$l_{AC} = [(x_C - x_A)^2 - (y_C - y_A)^2]$$

$$= [(4526.39^2 - 1119.76^2)]$$

$$\text{Computed value of } l_{AC} = 4662.84 \text{ m} = C$$

$$\text{Observed value of } l_{AC} = 4663.08 \text{ m} = O$$

$$(O - C) = 4663.08 - 4662.84 = +0.24 \text{ m.}$$

For l_{BC}

$$l_{BC} = [(x_C - x_B)^2 - (y_C - y_B)^2]$$

$$= [(950.92^2 - 3701.23^2)]$$

$$\text{Computed value of } l_{BC} = 3821.43 \text{ m} = C$$

$$\text{Observed value of } l_{BC} = 3821.21 \text{ m} = O$$

$$(O - C) = 3821.21 - 3821.43 = -0.22 \text{ m.}$$

For angle ACB

$$\angle ACB = \text{Back bearing of } AC - \text{back bearing of } BC$$

$$= (180° + 76°06'17.5'') - (180° + 14°24'31.7'')$$

$$\text{Computed value of } \angle ACB = 61°41'45.8'' = C$$

$$\text{Observed value of } \angle ACB = 61°41'57'' = O$$

$$(O - C) = 61°41'57'' - 61°41'45.8'' = +11.2''.$$

$$= +\frac{11.2}{206265} = +5.42990813 \times 10^{-5} \text{ radians.}$$

(*ii*) Calculation of residuals υ

Since A and B are fixed points

$$dx_A = dx_B = dy_A = dy_B = 0.$$

Therefore, from Eq. (5.4), we have

$$dl_{AC} = \frac{1}{l_{AC}}\left[(x_C - x_A)\,dx_c + (y_c - y_A)\,dy_c\right]$$

$$dl_{AC} = \frac{1}{4662.84} \times \left(4526.39\,dx_c + 1119.76\,dy_c\right)$$

$$= 0.70736718 \times 10^{-1} dx_c + 2.40145491 \times 10^{-1}\, dy_c.$$

$$dl_{BC} = \frac{1}{3821.43} \times (950.92\; dx_c + 3701.23\; dy_c)$$

$$= 2.48838785 \times 10^{-1} dx_c + 9.68545806 \times 10^{-1} dy_c..$$

From Eq. (5.6), we have

$$d\theta_{AC} = \frac{1}{l^2 AC}\left[(y_c - y_A)\,dx_c - (x_c - x_A)\,dy_c\right]$$

$$= \frac{1}{4662.84^2} \times (1119.76\; dx_c - 4526.39\; dy_c)$$

$$= 5.15019796 \times 10^{-5} dx_c - 2.08185723 \times 10^{-4} dy_c.$$

$$d\theta_{BC} = \frac{1}{3821.43^2} \times (3701.23 dx_c - 950.92 dy_c)$$

$$= 2.53451144 \times 10^{-4} dx_c - 6.51166672 \times 10^{-5}\; dy_c.$$

$$d\alpha = d\theta_{AC} - d\theta_{BC}$$

$$= (5.15019796 \times 10^{-5} - 2.53451144 \times 10^{-4})\; dx_c$$

$$-(2.08185723 \times 10^{-4} - 6.51166672 \times 10^{-5})\, dy_c$$

$$= -2.01949164 \times 10^{-4}\; dx_c - 1.43069056 \times 10^{-4}\, dy_c..$$

Now the residuals can be computed as below:

$$\upsilon_{\theta AC} = (O - C) - d\theta_{AC}$$

$$= +5.57535210 \times 10^{-5} - 5.15019796 \times 10^{-5} dx_c + 2.08185723 \times 10^{-4} dy_c$$

$$\upsilon_1 = d_1 + a_1 X + b_1 Y$$

$$\upsilon_{\theta BC} = \upsilon_2 = (O - C) - d\theta_{BC}$$

$$= -2.27862216 \times 10^{-5} - 2.53451144 \times 10^{-4} dx_c + 6.51166672 \times 10^{-5} dy_c$$

$$\upsilon_2 = d_2 + a_2 X + b_2 Y$$

$$\upsilon_{lAC} = \upsilon_3 = (O - C) - dl_{AC}$$

$$= +\,0.24 - 9.70736718 \times 10^{-1}\; dx_c - 2.4014591 \times 10^{-1}\; dy_c$$

$$\upsilon_3 = d_3 + a_3X + b_3Y$$

$$\upsilon_{lAC} = \upsilon_4 = (O - C) - dl_{BC}$$

$$= -0.22 - 2.48838785 \times 10^{-1}\, dx_c - 9.68545806 \times 10^{-1}\, dy_c$$

$$\upsilon_4 = d_4 + a_4X + b_4Y$$

$$\upsilon_\alpha = \upsilon_5 = (O - C) - d\alpha$$

$$= +5.42990813 \times 10^{-5} + 2.01949164 \times 10^{-4}\, dx_c + 1.43069056 \times 10^{-4}\, dy_c$$

$$\upsilon_5 = d_5 + a_5X + b_5Y$$

where in the above equations

$$X = dx_c,$$

$$Y = dy_c,$$

a, b = the coefficients of terms, and

d = the constant term.

Since the standard errors of the observations are given, the weights of the observations can be computed taking $\omega \propto \frac{1}{\sigma^2}$.

For lengths $\omega_l = \frac{1}{0.05^2}$ or $\sqrt{\omega_l} = 200$

For bearings $\omega_\theta = \frac{1}{(4/206265)^2}$ or $\sqrt{\omega_\theta} = 51566.25$

For angle $\omega_\alpha = \frac{1}{(5.6/206265)^2}$ or $\sqrt{\omega_\alpha} = 36833.04.$

From the least squares principle, we have

$$\phi = (\omega_\theta\ \upsilon_1)^2 + (\omega_\theta\ \upsilon_2)^2 + (\omega_l\, \upsilon_3)^2 + (\omega_l\, \upsilon_4)^2 + (\omega_\alpha\, \upsilon_5)^2 = \text{a minimum}$$

$$= \omega_\theta\, (d_1 + a_1X + b_1Y)^2 + \omega_\theta\, (d_2 + a_2X + b_2Y)^2 + \omega_l\, (d_3 + a_3X + b_3Y)^2$$

$$+ \omega_l\, (d_4 + a_4X + b_4Y)^2 + \omega_\alpha\, (d_5 + a_5X + b_5Y)^2 = \text{a minimum.}$$

Differentiating the above equation, we get

$$\frac{\partial \phi}{\partial X} = 2\omega_\theta\, (d_1 + a_1X + b_1Y)a_1 + 2\omega_\theta\, (d_2 + a_2X + b_2Y)\, a_2$$

$$+ 2\omega_l\, (d_3 + a_3X + b_3Y)\, a_3 + 2\omega_l\, (d_4 + a_4X + b_4Y)\, a_4$$

$$+ 2\omega_\alpha\, (d_5 + a_5X + b_5Y)\, a_5 = 0$$

$$\frac{\partial \phi}{\partial Y} = 2\omega_\theta\,(d_1 + a_1X + b_1Y)b_1 + 2\omega_\theta\,(d_2 + a_2X + b_2Y)b_2 + 2\omega_l$$

$$(d_3 + a_3X + b_3Y)b_3 + 2\omega_l\,(d_4 + a_4X + b_4Y)b_4 + 2\omega_\alpha$$

$$(d_5 + a_5X + b_5Y)b_5 = 0$$

By rearranging the terms, we get

$$[\omega_\theta(a^2{}_1 + a^2{}_2) + \omega_l(a^2{}_3 + a^2{}_4) + \omega_\alpha a^2{}_5]X + [\omega_\theta(b_1a_1 + b_2a_2) + \omega_l(b_3a_3 + b_4a_4) + \omega_\alpha b_5a_5]Y$$
$$= -[\omega_\theta(d_1a_1 + d_2a_2) + \omega_l(d_3a_3 + d_4a_4) + \omega_\alpha d_5a_5]$$

$$[\omega_\theta(b_1a_1 + b_2a_2) + \omega_l(b_3a_3 + b_4a_4) + \omega_a b_5a_5]X + [\omega_\theta(b^2{}_1 + b^2{}_2) + \omega_l(b^2{}_3 + b^2{}_4) + \omega_\alpha b^2{}_5]Y$$
$$= -[\omega_\theta(d_1b_1 + d_2b_2) + \omega_l(d_3b_3 + d_4b_4) + \omega_\alpha d_5b_5].$$

The above two equations are the normal equations in X (i.e., dx_C) and Y (i.e., dy_c). Now Substituting the values of a, b, d, ω_θ, ω_l; and ω_α, we get

$$634.895456\ dx_c + 156.454114\ dy_c - 48.6944390 = 0$$

$$156.454114\ dx_c + 552.592599\ dy_c + 99.6361111 = 0$$

The above equations solve for

$$dx_c = +\ 1.30213821 \times 10^{-1} = +\ 0.13 \text{ m}$$

$$dy_c = -\ 2.17173736 \times 10^{-1} = -\ 0.22 \text{ m}.$$

Hence the most probable values of the coordinates of C are

Easting of C = E 8533.38 + 0.13 = **E 8533.51 m**

Northing of C = N 12184.52 – 0.22 = **N 12184.30 m**.

Example 5.15. A quadrilateral *ABCD* with a central station O is a part of a triangulation survey. The following angles were measured, all have equal weight.

$\theta_1 = 57°55'08''$; $\theta_2 = 38°37'27''$

$\theta_3 = 62°36'16''$; $\theta_4 = 34°15'39''$

$\theta_5 = 36°50'25''$; $\theta_6 = 51°54'24''$

$\theta_7 = 27°57'23''$; $\theta_8 = 49°52'50''$

$\theta_9 = 83°27'17''$; $\theta_{10} = 83°08'06''$

$\theta_{11} = 91°15'09''$; $\theta_{12} = 102°09'32''$.

Adjust the quadrilateral by the method of least squares.

Solution (Fig. 5.9):

This example has been solved by the general method of adjusting a polygon with a central station discussed in Sec. 5.8.

For the given polygon

$$n = 4$$

$$(n + 2) = 6.$$

Therefore, the matrix of coefficients of normal equations will be a 6 × 6 matrix.

Step-1: Total corrections

(*i*) $C_1 = 180° - (\theta_1 + \theta_2 + \theta_9) = +8''$

$C_2 = 180° - (\theta_3 + \theta_4 + \theta_{10}) = -1''$

$C_3 = 180° - (\theta_5 + \theta_6 + \theta_{11}) = +2''$

$C_4 = 180° - (\theta_7 + \theta_8 + \theta_{12}) = +15''$

(*ii*) $C_5 = 360° - (\theta_9 + \theta_{10} + \theta_{11} + \theta_{12}) = -4''$

(*iii*) $C_6 = -$ [log sin(odd angles) – log sin(even angles)] $\times 10^6$

(for angles $(\theta_1, \theta_2 \ldots \theta_{2n})$.

$= -[(0.071964298 + 0.051659872 + 0.222148263 + 0.329012985) - (0.20466985 + 0.24952153 + 0.104021508 + 0.116507341)] \times 10^6$

$= -651.9''$.

Step-2: Normal equations

$f_1 = 13, \quad f_2 = 26, \quad f_3 = 11, \quad f_4 = 31, \quad f_5 = 28, \quad f_6 = 17, \quad f_7 = 40, \quad f_8 = 18$

$F_{12} = 13 - 26 = -13, \quad F_{34} = 11 - 31 = -20$

$F_{56} = 28 - 17 = 11, \quad F_{78} = 40 - 18 = 22$

$F^2 = 4924.$

Coefficient matrix of normal equations

	1	*2*	*3*	*4*	5	6
1	3	0	0	0	1	–13
2	0	3	0	0	1	–20
3	0	0	3	0	1	11
4	0	0	0	3	1	22
5	1	1	1	1	4	0
6	–13	–20	11	22	0	4924

Matrix of normal equations

	1	*2*	*3*	*4*	5	6					
1	3	0	0	0	1	–13	λ_1		8	
2	0	3	0	0	1	–20	λ_2		–1	
3	0	0	3	0	1	11	λ_3	–	2	= 0
4	0	0	0	3	1	22	λ_4		+15	
5	1	1	1	1	4	0	λ_5		–4	
6	–13	–20	11	22	0	4924	λ_6		–651.9	

Normal equations

$$3\lambda_1 + \lambda_5 - 13\lambda_6 - 8 = 0$$

$$3\lambda_2 + \lambda_5 - 20\lambda_6 + 1 = 0$$

$$3\lambda_3 + \lambda_5 + 11\lambda_6 - 2 = 0$$
$$3\lambda_4 + \lambda_5 + 22\lambda_6 - 15 = 0$$
$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 4\lambda_5 + 4 = 0$$
$$-13\lambda_1 - 20\lambda_2 + 11\lambda_3 + 22\lambda_4 + 4924\lambda_6 + 651.9 = 0$$

The above equations solve for

$$\lambda_1 = +3.456, \qquad \lambda_4 = +7.703$$
$$\lambda_2 = +0.073, \qquad \lambda_5 = -4.500$$
$$\lambda_3 = +2.768, \qquad \lambda_6 = -0.164.$$

Now the corrections to the angles are

$$e_1 = \lambda_1 + f_1\lambda_6 = 3.456 + 13 \times (-0.164) = +1.324''$$
$$e_2 = \lambda_1 - f_2\lambda_6 = 3.456 - 26 \times (-0.164) = +7.720''$$
$$e_3 = \lambda_2 + f_3\lambda_6 = 0.073 + 11 \times (-0.164) = -1.731''$$
$$e_4 = \lambda_2 - f_4\lambda_6 = 0.073 - 31 \times (-0.164) = +5.157''$$
$$e_5 = \lambda_3 + f_5\lambda_6 = 2.768 + 28 \times (-0.164) = -1.824''$$
$$e_6 = \lambda_3 - f_6\lambda_6 = 2.768 - 17 \times (-0.164) = +5.556''$$
$$e_7 = \lambda_4 + f_7\lambda_6 = 7.703 + 40 \times (-0.164) = +1.143''$$
$$e_8 = \lambda_4 - f_8\lambda_6 = 7.703 - 18 \times (-0.164) = +10.655''$$

$$e_9 = \lambda_1 + \lambda_5 = 3.456 - 4.500 = -1.044''$$
$$e_{10} = \lambda_2 + \lambda_5 = 0.073 - 4.500 = -4.427''$$
$$e_{11} = \lambda_3 + \lambda_5 = 2.768 - 4.500 = -1.732''$$
$$e_{12} = \lambda_4 + \lambda_5 = 7.703 - 4.500 = +3.203''.$$

Checks:

$$C_1 = e_1 + e_2 + e_9 = +8''$$
$$C_2 = e_3 + e_4 + e_{10} = -1''$$
$$C_3 = e_5 + e_6 + e_{11} = +2''$$
$$C_4 = e_7 + e_8 + e_{12} = +15''$$
$$C_5 = e_9 + e_{10} + e_{11} + e_{12} = -4''$$

Applying the above corrections to the observed angles, we get the most probable values as under:

$$\theta_1 + e_1 = 57°55'08'' + 1.324'' = \mathbf{57°55'09.3''}$$
$$\theta_2 + e_2 = 38°37'27'' + 7.720'' = \mathbf{38°37'34.7''}$$
$$\theta_9 + e_9 = 83°27'17'' - 1.044'' = \mathbf{83°27'16.0''}$$
$$\text{Total} = 180°00'00'' \ (\textit{Check}).$$
$$\theta_3 + e_3 = 62°36'16'' - 1.731'' = \mathbf{62°36'14.0''}$$
$$\theta_4 + e_4 = 34°15'39'' + 5.157'' = \mathbf{34°15'44.0''}$$

$$\theta_{10} + e_{10} = 83°08'06'' - 4.427'' = \mathbf{83°08'02.0''}$$
$$\text{Total} = 180°00'00''\ (\textit{Check}).$$
$$\theta_5 + e_5 = 36°50'25'' - 1.824'' = \mathbf{36°50'23.0''}$$
$$\theta_6 + e_6 = 51°54'24'' + 5.556'' = \mathbf{51°54'30.0''}$$
$$\theta_{11} + e_{11} = 91°15'09'' - 1.732'' = \mathbf{91°15'07.0''}$$
$$\text{Total} = 180°00'00''\ (\textit{Check}).$$
$$\theta_7 + e_7 = 27°57'23'' + 1.143'' = \mathbf{27°57'24.0''}$$
$$\theta_8 + e_8 = 49°52'50'' + 10.655'' = \mathbf{49°53'01.0''}$$
$$\theta_{12} + e_{12} = 102°09'32'' + 3.203'' = \mathbf{102°09'35.0''}$$
$$\text{Total} = 180°00'00''\ (\textit{Check}).$$

OBJECTIVE TYPE QUESTIONS

1. Theory of errors is applied to minimize
(*a*) the gross errors.
(*b*) the systematic errors.
(*c*) the random errors.
(*d*) all the above.

2. Most probable value of a quantity is equal to
(*a*) observed value + correction.
(*b*) the observed value – correction.
(*c*) the true value + correction.
(*d*) the true value – correction.

3. The method of least squares of determining the most probable value of a quantity is based upon the criterion that
(*a*) Σ Correction^2 = a minimum.
(*b*) Σ Error^2 = a minimum.
(*c*) Σ (Weight × correction)^2 = a minimum.
(*d*) Σ Residual^2 = a minimum.

4. If the observations of a quantity contains systematic and random errors, the most probable value of the quantity is obtained by
(*a*) removing the systematic and random errors from the observations.
(*b*) removing the systematic errors and minimizing the residuals from the observations.
(*c*) removing the random errors and minimizing the systematic errors from the observations.
(*d*) minimizing the systematic and random errors from the observations.

5. The most probable value of a quantity is the quantity which is nearest to
(*a*) the true value of the quantity.
(*b*) the true value of the quantity ± standard deviation.
(*c*) the true value of the quantity ± probable error.
(*d*) the observed value of the quantity ± weight of the observation.

6. The theory of least squares is used in

(*a*) the method of differences.

(*b*) in the normal equation method.

(*c*) the method of correlates.

(*d*) all the above.

7. In a braced quadrilateral the number of conditions required to be satisfied for adjustment excluding the condition imposed by least squares theory, is

(*a*) 2.

(*b*) 3.

(*c*) 4.

(*d*) 5.

8. The spherical excess for a triangle of area 200 sq km is approximately

(*a*) 0.5″.

(*b*) 1.0″.

(*c*) 1.5″.

(*d*) 2.0″.

9. Correlate is the unknown multiplier used to determine the most probable values by multiplying it with

(*a*) normal equation.

(*b*) observation equation.

(*c*) condition equation.

(*d*) condition imposed by the least squares theory .

10. Station adjustment of observation means

(*a*) making sum of the angles observed around a station equal to 360°.

(*b*) checking the permanent adjustment of the instrument at every station.

(*c*) adjusting the instrument so that it is exactly over the station.

(*d*) shifting the station location to make it intervisible from other stations.

ANSWERS

1. (*c*)	**2.** (*a*)	**3.** (*d*)	**4.** (*b*)	**5.** (*a*)	**6.** (*d*)
7. (*c*)	**8.** (*b*)	**9.** (*c*)	**10.** (*a*)		

6 TRIANGULATION AND TRILATERATION

6.1 TRIANGULATION SURVEYS

Triangulation is one of the methods of fixing accurate controls. It is based on the trigonometric proposition that if one side and two angles of a triangle are known, the remaining sides can be computed. A triangulation system consists of a series of joined or overlapping triangles in which an occasional side called as *base line,* is measured and remaining sides are calculated from the angles measured at the vertices of the triangles, vertices being the control points are called as *triangulation stations*.

Triangulation surveys are carried out

1. to establish accurate control for plane and geodetic surveys covering large areas,
2. to establish accurate control for photogrammetric surveys for large areas,
3. to assist in the determination of the size and shape of the earth,
4. to determine accurate locations for setting out of civil engineering works such as piers and abutments of long span bridges, fixing centre line, terminal points and shafts for long tunnels, measurement of the deformation of dams, etc.

When all the sides of a triangulation system are measured it is known as the *trilateration system*. However, the angular measurements define the shape of the triangulation system better than wholly linear measurements and so it is preferred that a number of angles are included in trilateration system.

A combined triangulation and trilateration system in which all the angles and all the sides are measured, represents the strongest network for creating horizontal control.

6.2 STRENGTH OF FIGURE

The *strength of figure* is a factor considered in establishing a triangulation system to maintain the computations within a desired degree of precision. It plays an important role in deciding the layout of a triangulation system.The expression given by the U.S. Coast and Geodetic Survey for evaluation of strength of figure is

$$L^2 = \frac{4}{3} d^2 R \qquad \ldots(6.1)$$

where L^2 = the square of the probable error that would occur in the sixth place of the logarithm of any side,

d = the probable error of an observed direction in seconds of arc,

R = a term which represents the shape of a figure

$$= \frac{D-C}{D} \sum \left(\delta_A^2 + \delta_A \delta_B + \delta_B^2 \right) \qquad \ldots(6.2)$$

D = the number of directions observed excluding the known side of the figure,

$\delta_A, \delta_B, \delta_C$ = the difference in the sixth place of logarithm of the sine of the distance angles A, B, C, etc., respectively,

$$C = (n' - S' + 1) + (n - 2S + 3) \qquad \ldots(6.3)$$

n' = the total number of sides including the known side of the figure,

n = the total number of sides observed in both directions including the known side,

S' = the number of stations occupied, and

S = the total number of stations.

6.3 DISTANCE OF VISIBLE HORIZON

If there is no obstruction due to intervening ground between two stations A and B, the distance D of *visible horizon* as shown in Fig. 6.1 from a station A of known elevation h above mean sea level, is calculated from the following expression:

$$h = \frac{D^2}{2R}(1 - 2m) \qquad \ldots(6.4)$$

where h = the elevation of the station above mean sea level,

D = the distance of visible horizon,

R = the mean earth's radius ($\simeq$6373 km), and

m = the mean coefficient of refraction (taken as 0.07 for sights over land, 0.08 for sights over sea).

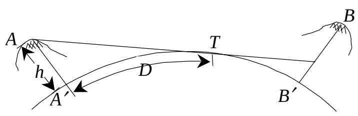


Fig. 6.1

For the sights over land

$$h = 0.6735\, D^2 \text{ metres} \qquad \ldots(6.5)$$

where D is in kilometers.

The expression given by Eqs. (6.4) or (6.5), is used to determine the intervisibility between two triangulation stations.

6.4 PHASE OF A SIGNAL

When cylindrical opaque signals are used they require a correction in the observed horizontal angles due to an error known as the *phase*. When sunlight falls on a cylindrical opaque signal it is partly illuminated, and the remaining part being in shadow as shown in Fig. 6.2 becomes invisible to the observer. While making the observations the observations may be made on the bright portion (Fig. 6.2a) or the bright line (Fig. 6.2b). Since the observations are not being made on the centre of the signal, an error due to incorrect bisection is introduced in the measured horizontal angles at *O*.

Observations Made on Bright Portion (Fig. 6.2a)

When the observations are made on the two extremities *A* and *B* of the bright portion *AB* then the phase correction β is given by the following expression:

$$\beta = \frac{206265}{D} r \cos^2(\theta / 2) \text{ seconds} \qquad \ldots(6.6)$$

where θ = the angle between the sun and the line *OP*,

D = the distance *OP*, and

r = the radius of the cylindrical signal.

Observations Made on Bright Line (Fig. 6.2b)

In the case of the observations made on the bright line at *C*, the phase correction is computed from the following expression:

$$\beta = \frac{20625}{D} r \cos(\theta / 2) \text{ seconds} \qquad \ldots(6.7)$$

While applying the correction, the directions of the phase correction and the observed stations with respect to the line *OP* must be noted carefully.

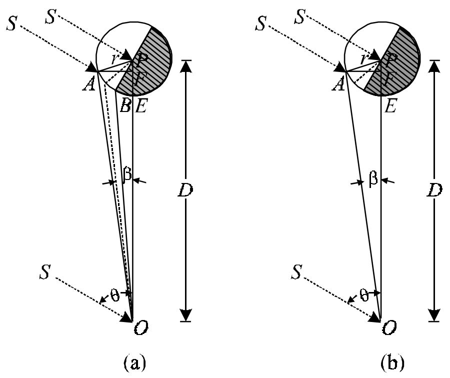


Fig. 6.2

6.5 SATELLITE STATION, REDUCTION TO CENTRE, AND ECCENTRICITY OF SIGNAL

In triangulation surveys there can be two types of problems as under:

(a) It is not possible to set up the instrument over the triangulation station.

(b) The target or signal is out of centre.

The first type of problem arises when chimneys, church spires, flag poles, towers, light houses, etc., are selected as triangulation stations because of their good visibility and forming well-conditioned triangles. Such stations can be sighted from other stations but it is not possible to occupy them directly below such excellent stations for making observations on other stations. The problem is solved as shown in Fig. 6.3, by taking another station S in the vicinity of the main triangulation station C from where the other stations A and B are visible. Such stations are called the *satellite stations*, and determining the unobserved angle ACB from the observations made is known as the *reduction to centre*.

In the second type of problem, signals are blown out of position. Since the signal S is out of centre as shown in Fig. 6.3, i.e., not over the true position of the station, the observations made from other stations A and B will be in error, but the angle ϕ observed at C will be correct. Therefore the observed angles at A and B are to be corrected. Such type of problem is known as the *eccentricity of signal*.

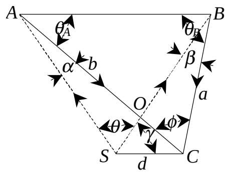


Fig. 6.3

The problems reduction to centre and eccentricity of signal are solved by determining the corrections α and β. It may be noted that if the instrument is not centered over the true position of the station it will also introduce error in the angles measured at that station but it cannot be corrected since the observer has unknowingly made the centering error and therefore, the displacement of the instrument from the correct position of the station is not known.

The observations made on the station C from the stations A and B, are θ_A, θ_B, respectively. At the satellite station S the measured angles are θ and γ.If the line AS is moved to AC by an angle α, and the line BS is moved to BC by an angle β, the satellite station S moves to the main station C by amount d, and the angle θ becomes ϕ by getting corrections α and β. The values of the corrections α and β are given by

$$\alpha = 206265 \frac{d}{b} \sin (\theta + \gamma) \text{ seconds} \quad \text{...(6.8)}$$

$$\beta = 206265 \frac{d}{a} \sin \gamma \text{ seconds} \quad \text{...(6.9)}$$

From Δ's AOS and BOC, we have

$$\angle AOS = \angle BOC$$

$$\theta + \alpha = \phi + \beta$$

$$\phi = \theta + \alpha - \beta \quad \text{...(6.10)}$$

Eq. (6.10) gives the value of ϕ when the satellite station S is in S_1 position shown in Fig. 6.4a. In general, depending upon the field conditions the following four cases may occur as shown in Fig. 6.4.

Case-1: S_1 position of the satellite station (Fig. 6.4a)

$$\phi = \theta + \alpha - \beta$$

Case-2: S_2 position of the satellite station (Fig. 6.4b)

$$\phi = \theta - \alpha + \beta \quad \text{...(6.11)}$$

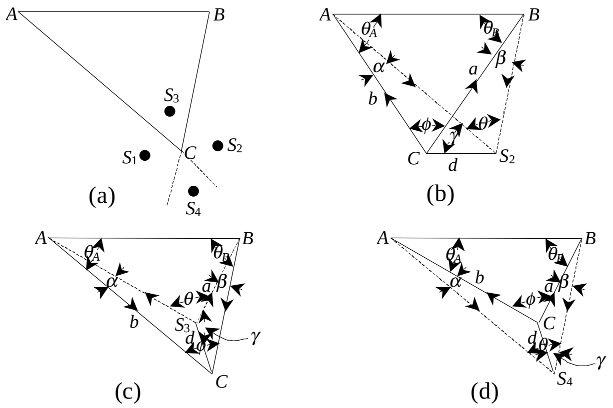


Fig. 6.4

Case-3: S_3 position of the satellite station (Fig. 6.4c)

$$\phi = \theta - \alpha - \beta \qquad \text{...(6.12)}$$

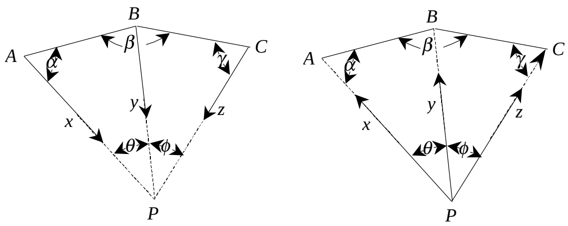
Case-4: S_4 position of the satellite station (Fig. 6.4d)

$$\phi = \theta + \alpha + \beta. \qquad \text{...(6.13)}$$

6.6 LOCATION OF POINTS BY INTERSECTION AND RESECTION

The points located by observing directions from the points of known locations, are known as the *intersected points* (Fig. 6.5a). When a point is established by taking observations from the point to the points of known locations, such points are known as the *resected points* (Fig. 6.5b). Generally the intersected points are located for subsequent use at the time of plane tabling to determine the plane-table station by solving three-point problem. Therefore, their locations are such that they are visible from most of the places in the survey area. The resected points are the additional stations which are established when the main triangulation stations have been completed and it is found necessary to locate some additional stations for subsequent use as instrument stations as in topographic surveys. This problem also arises in hydrographic surveys where it is required to locate on plan the position of observer in boat.

In Fig. 6.6a, let A, B, and C be the main triangulation stations whose locations are known. P is the point whose location is to be determined.



(a) Intersected point (b) Resected point

Fig. 6.5

Since the coordinates of A, B, and C are known, the lengths a, b, and the angle β are known. The angle α and γ are observed at A and C.

Let

$$\angle APB = \theta$$

$$\angle BPC = \phi$$

$$AP = x$$

$$BP = y$$

$$CP = z$$

From Δ's ABP and BCP, we get

$$\frac{y}{\sin\alpha} = \frac{a}{\sin\theta}$$

and

$$\frac{y}{\sin\gamma} = \frac{b}{\sin\phi}$$

or

$$y = a\frac{\sin\alpha}{\sin\theta} = b\frac{\sin\gamma}{\sin\phi}\,. \qquad \text{...(6.14)}$$

Also for the quadrilateral $ABCP$, we have

$$\alpha + \beta + \gamma + \phi + \theta = 360°. \qquad \text{...(6.15)}$$

There are two unknowns θ and ϕ in two equations (6.14) and (6.15). The solution of these equations gives the values of θ and ϕ, and then the values of x, y, and z can be calculated by sine law. Knowing the distances x, y, and z, the point P can be plotted by intersection.

If the coordinates of P, A and B are (X_P, Y_P), (X_A, Y_A), and (X_B, Y_B), the reduced bearings θ_1 and θ_2 of AP and BP are respectively given by

$$\tan\theta_1 = \frac{X_P - X_A}{Y_P - Y_A} \qquad \text{...(6.16)}$$

$$\tan\theta_2 = \frac{X_P - X_B}{Y_P - Y_B}\,. \qquad \text{...(6.17)}$$

Since θ_1 and θ_2 are now known, by solving the simultaneous equations (6.16) and (6.17) the coordinates X_P are Y_P are determined, and the point P is located. The check on the computations is provided by computing the distance CP.

In the case of resected points shown in Fig. 6.6b, the angles α and γ are unknowns, and the angles θ and ϕ are measured by occupying the station P. The Eqs. (6.14) and (6.15) solve for α and γ, and then x, y, and z are computed by sine law. The point P can be located now by intersection. Since in this case the angles α and γ are not measured, the solution of Eqs. (6.14) and (6.15) is obtained as explained below.

From Eq. (6.15), we get

$$\alpha + \gamma = 360^\circ - (\beta + \theta + \phi)$$

or
$$\frac{1}{2}(\alpha + \gamma) = 180^\circ - \frac{1}{2}(\beta + \theta + \phi) \qquad \ldots(6.18)$$

From Eq. (6.14), we have

$$\frac{\sin\gamma}{\sin\alpha} = \frac{a\sin\phi}{b\sin\theta}.$$

Let
$$\frac{\sin\gamma}{\sin\alpha} = \tan\lambda.$$

Therefore
$$\frac{a\sin\phi}{b\sin\theta} = \tan\lambda. \qquad \ldots(6.19)$$

We can also write

$$\frac{\sin\gamma}{\sin\alpha} + 1 = \tan\lambda + 1$$

$$1 - \frac{\sin\gamma}{\sin\alpha} = 1 - \tan\lambda$$

or
$$\frac{\sin\alpha + \sin\gamma}{\sin\alpha} = \tan\lambda + \tan 45^\circ$$

and
$$\frac{\sin\alpha - \sin\gamma}{\sin\alpha} = 1 - \tan\lambda\tan 45^\circ$$

or
$$\frac{\sin\alpha - \sin\gamma}{\sin\alpha + \sin\gamma} = \frac{1 - \tan\lambda\tan 45^\circ}{\tan\lambda + \tan 45^\circ}$$

$$\frac{2\cos\frac{1}{2}(\alpha + \gamma)\sin\frac{1}{2}(\alpha - \gamma)}{2\sin\frac{1}{2}(\alpha + \gamma)\cos\frac{1}{2}(\alpha - \gamma)} = \frac{1}{\tan(45^\circ + \lambda)}$$

$$\cot\left(\frac{\alpha + \gamma}{2}\right)\tan\left(\frac{\alpha - \gamma}{2}\right) = \cot(45^\circ + \lambda)$$

$$\tan\left(\frac{\alpha + \gamma}{2}\right) = \cot(45^\circ + \lambda)\tan\left(\frac{\alpha + \gamma}{2}\right). \qquad \ldots(6.20)$$

Substituting the values of $\left(\frac{\alpha+\gamma}{2}\right)$ from Eq. (6.18) and the value of λ from Eq. (6.19) in Eq. (6.20), we get the value of $\left(\frac{\alpha-\gamma}{2}\right)$. Now the values of α and γ can be obtained from the values of $\left(\frac{\alpha+\gamma}{2}\right)$ and $\left(\frac{\alpha-\gamma}{2}\right)$.

Obviously,

$$\alpha = \left(\frac{\alpha+\gamma}{2}\right) + \left(\frac{\alpha-\gamma}{2}\right)$$

$$\gamma = \left(\frac{\alpha+\gamma}{2}\right) - \left(\frac{\alpha-\gamma}{2}\right).$$

Now the lengths x, y, and z are obtained by applying sine law in the triangles *APB* and *BPC*. The length y of the common side *BP* gives a check on the computations. The azimuths of the three sides *AP*, *BP*, and *CP* are computed, and the coordinates of *P* are determined from the azimuths and lengths of the lines.

The following points should be noted:

(a) Due regard must be given to the algebraic sign of $\tan\left(\frac{\alpha+\gamma}{2}\right)$ and cot (45° + λ) to get the correct sign of $\left(\frac{\alpha-\gamma}{2}\right)$.

(b) If $\left(\frac{\alpha+\gamma}{2}\right)$ is equal to 90°, the point P lies on the circle passing through *A*, *B*, and *C*, and the problem is indeterminate.

(c) Mathematical checks do not provide any check on the field work.

(d) The angles θ and ϕ measured in the field can be checked by measuring the exterior angle at *P*. The sum of all the angles should be 360°.

(e) A fourth control station may be sighted to check the accuracy of a three-point resection. The two independent solutions should give the same position of the point *P*. It checks both the field work and computation.

(f) The least squares method can be used to adjust the overdetermined resection, and for the determination of the most probable values.

6.7 REDUCTION OF SLOPE DISTANCES

In trilateration system, the lengths of the sides of the figures are obtained by measuring the slope distances between the stations using EDM equipment, and reducing them to equivalent mean sea level distances. There are the two following methods of reducing the slope distances:

1. Reduction by vertical angles
2. Reduction by station elevations.

Reduction by Vertical Angles

Fig. 6.6 shows two stations A and B established by trilateration. α is the angle of elevation at A, and β is the angle of depression at B. The angle υ is the refraction angle assumed to be the same at both stations. For simplicity in computations it is further assumed that the height of instrument above the ground and the height of the signal above the ground at both stations are same. Thus the axis-signal correction is zero.

The angle subtended at the centre of the earth O, between the vertical lines through A and B, is θ. The angle OVB is $(90° – \theta)$ as the angle VBO in triangle VBO, is 90°. The distance L between A and B, is measured along the refracted line (curved) from A to B by the EDM. The distance L within the range of the instrument can be taken as the straight-line slope distance AB. The difference between the two distances in 100 km is less than 20 cm.

In ΔVAB,

$$\angle VAB + \angle ABV + \angle BVA = 180°$$

$$(90° - \alpha + \upsilon) + (\beta + \upsilon) + (90° - \theta) = 180°$$

$$2\upsilon = \alpha + \theta - \beta$$

$$\upsilon = \frac{1}{2}(\alpha + \theta - \beta) \qquad \ldots(6.21)$$

In $\Delta BAB'$,

$$\angle BAB' = \alpha - \upsilon + \theta/2 \qquad \ldots(6.22)$$

Substituting the value of υ from Eq. (6.21) in Eq. (6.22), we get

$$\angle BAB' = \alpha - \frac{1}{2}(\alpha + \theta - \beta) - \frac{\theta}{2}$$

$$= \frac{1}{2}(\alpha + \beta) \qquad \ldots(6.23)$$

Taking $\angle AB'B$ approximately equal to 90° in the triangle ABB', the distance AB' is very nearly given by

$$AB' \simeq AB \cos BAB' \qquad \ldots(6.24)$$

With approximate value of AB', the value of θ can be computed with sufficient accuracy from the following relationship.

In ΔAOE,

$$\sin\frac{\theta}{2} = \frac{AE}{OE} = \frac{AB'}{2R} \qquad \ldots(6.25)$$

taking $OE = R$ which the radius of earth for the latitude of the area in the direction of the line.

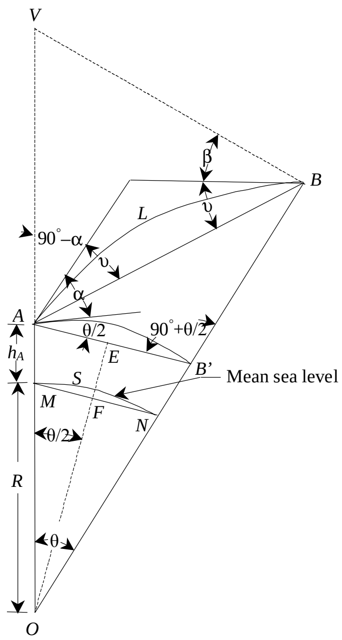


Fig. 6.6

In $\Delta ABB'$,

$$\angle ABB' = 180° - (\angle BAB' + \angle AB'B)$$

$$= 180° - \frac{1}{2}(\alpha + \beta) - \left(90° + \frac{\theta}{2}\right)$$

$$= 90° - \frac{1}{2}(\alpha + \beta + \theta)$$

The triangle ABB' can now be solved by the sine law to give a more exact value of AB'. Thus

$$AB' = AB \frac{AB \sin ABB'}{\sin\left(90° + \frac{\theta}{2}\right)} \qquad \ldots(6.26)$$

The points M and N are the sea level positions of the stations A and B, respectively. The chord length MN which is less than the chord length AB', is given by

$$MN = AB' - AB'\frac{h_A}{R}$$

$$= AB'\left(1 - \frac{h_A}{R}\right) \qquad \ldots(6.27)$$

where h_A is the elevation of the station A.

The sea level distance S which is greater than the chord length MN, is given by

$$S = MN + \frac{MN^3}{24R^2} \qquad \ldots(6.28)$$

or

$$\delta l = S - MN$$

$$= \frac{MN^3}{24R^2} \qquad \ldots(6.29)$$

The difference δl is added to the chord distance MN to obtain the sea level distance S of the line. The value of δl is about 250 cm in a distance of 100 km, and decreases to 1 ppm of the measured length at about 30 km. Thus, for distances less than 30 km, the chord-to-arc distance correction is negligible.

Reduction by Station Elevation

If the difference in elevation Δh of the two stations A and B as shown in Fig. 6.7 is known, the measured slope distance L between the stations can be reduced to its equivalent sea level distance without making any further observations. Assuming the distance AB as the straight-line distance, the difference between the slope distance and the horizontal distance AC can be written as

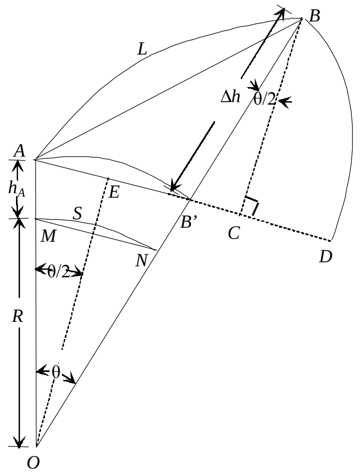


Fig. 6.7

$$CD = AB - AC = \frac{\Delta h^2}{2AB},$$

since the slope of the line *AB* is small.

From $\Delta CBB'$, we have

$$B'C = \Delta h \sin\frac{\theta}{2}.$$

The value of $\theta/2$ is determined from Eq. (6.25) by calculating the approximate value of AB' as below.

$$AB \simeq AB - \frac{\Delta h^2}{2AB} \qquad \ldots(6.30)$$

We have

$$AB' = AD - B'D$$

$$= AD - (B'C + CD)$$

$$= AB - \left(\Delta h \sin\frac{\theta}{2} + \frac{\Delta h^2}{2AB}\right) \qquad \ldots(6.31)$$

Now the value of *S* is determined from Eq. (6.28) by substituting the value of *MN* from Eq. (6.27).

6.8 SPHERICAL TRIANGLE

The theodolite measures horizontal angles in the horizontal plane, but when the area becomes large, such as in the case of primary triangulation, the curvature of the earthmeans that such planes in large triangles called as *spherical triangles* or *geodetic triangles* are not parallel at the apices as shown in Fig. 6.8. Accordingly, the three angles of a large triangle do not total 180°, as in the case of plane triangles, but to 180° + ε, where ε is known as *spherical excess*. The spherical excess depends upon the area of the triangle, and it is given by

$$\varepsilon = \frac{A_0}{R^2 \sin 1''} \text{ seconds} \quad \ldots(6.32)$$

where A_0 = the area of the triangle in sq km, and

R = the mean radius of the earth in km ($\simeq$ 6373 km).

The triangular error is given by

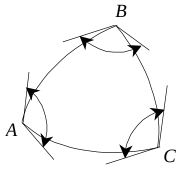


Fig. 6.8

$$\varepsilon = \Sigma \text{ Observed angles} - (180° + \varepsilon)$$

$$= A + B + C - (180° - \varepsilon) \quad \ldots(6.33)$$

6.9 EFFECT OF EARTH'S CURVATURE

Due to the curvature of the earth causes the azimuth or bearing of a straight line to constantly changes resulting into the bearing of X from Y being not equal to the bearing of Y from $X \pm 180°$. As shown in Fig. 6.9, let XY be the line in question, and ϕ_1 and ϕ_2 be the latitudes of its two ends X and Y, respectively. N and S are the terrestrial north and south poles, and the meridians of X and Y are SXN and SYN, respectively. At the equator these meridians are parallel, but this changes with progress towards the poles. Let the angles between the meridians at the poles be θ, which is equal to the difference in longitude of X and Y. The bearings of XY at X and Y are α and $\alpha + \delta\alpha$, respectively. Here $\delta\alpha$ indicates the convergence of the meridians, i.e., the angle between the meridians at X and Y.

We have

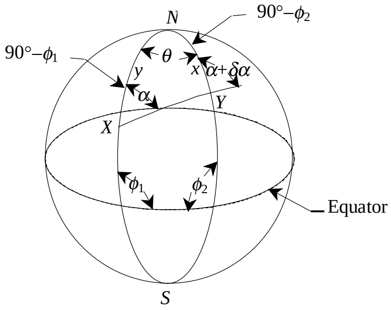


Fig. 6.9

$$XN = 90° - \phi_1$$

$$YN = 90° - \phi_2$$

$$\tan \frac{X+Y}{2} = \frac{\cos \frac{x-y}{2}}{\cos \frac{x+y}{2}} \cot \frac{\theta}{2}$$

$$\tan \frac{\alpha + \left(180° - \alpha - \delta\alpha\right)}{2} = \frac{\cos \frac{(90° - \phi_2) - (90° - \phi_1)}{2}}{\cos \frac{(90° - \phi_2) + (90° - \phi_1)}{2}} \cot \frac{\theta}{2}$$

$$\tan \left(90° - \frac{\delta\alpha}{2}\right) = \frac{\cos \frac{\phi_1 - \phi_2}{2}}{\sin \frac{\phi_1 + \phi_2}{2}} \cot \frac{\theta}{2}$$

$$\tan \frac{\delta\alpha}{2} = \frac{\sin \frac{\phi_1 + \phi_2}{2}}{\cos \frac{\phi_1 - \phi_2}{2}} \tan \frac{\theta}{2}$$

For survey purposes, taking $(\phi_1 - \phi_2)$ and θ small, we have

$$\delta\alpha = \theta \sin \frac{\phi_1 - \phi_2}{2}$$
$$= \theta \sin \bar{\phi}$$

where $\bar{\phi}$ is the mean latitude of X and Y.

Within the limits of calculations it is quite adequate to assume that the meridians at X and Y are parallel when the two points are relatively close together (say < 40 km apart). With this assumption a rectangular grid can be established as in Fig. 6.10, in which the lines are spaced apart at mid-latitude distances. Thus λ represents the length of 1″ of latitude, and μ represents the length of 1″ of longitude at the mean latitude $\bar{\phi}$ of X and Y in each case. l is the length of XY which is actually part of a great circle, and $\left(\alpha + \frac{\delta\alpha}{2}\right)$ is the average bearing of XY.

Thus

$$\lambda\delta\phi = l \cos\left(\alpha + \frac{\delta\alpha}{2}\right) \quad \text{...(6.34)}$$

$$\mu\delta\phi = l \sin\left(\alpha + \frac{\delta\alpha}{2}\right) \quad \text{...(6.35)}$$

and

$$\delta\alpha = \delta\theta \sin \bar{\phi} \quad \text{...(6.36)}$$

If X and Y are widely separated, the triangle XNY is solved for angle X and Y using standard expressions for $\tan \frac{X+Y}{2}$ and $\tan \frac{X-Y}{2}$.

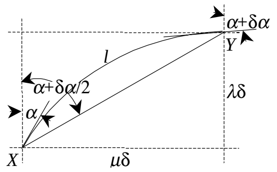


Fig. 6.10

6.10 CONVERGENCE

The Grid North has its direction of the central meridian but elsewhere a meridian does not align with Grid North. Thus, in general, the grid bearing of a line will not equal the true bearing of that line if measured at a station by an astronomical method or by gyro-theodolite. To convert the former to the latter a convergence factor has to be applied.

Example 6.1. In a triangulation survey, four triangulation stations A, B, C, and D were tied using a braced quadrilateral $ABCD$ shown in Fig. 6.12. The length of the diagonal AC was measured and found to be 1116.40 m long. The measured angles are as below:

α = 44°40′59″ $\quad$ γ = 63°19′28″

β = 67°43′55″ $\quad$ δ = 29°38′50″.

Calculate the length of BD.

Solution (Fig. 6.11):

In ΔABD, we have

$$\angle BAD = \theta = 180° - (\alpha + \gamma)$$
$$= 180° - (44°40'59'' + 63°19'28'')$$
$$= 71°59'33''.$$

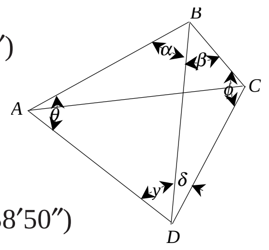


Fig. 6.11

In ΔBCD, we have

$$\angle BCD = \phi = 180° - (\beta + \delta)$$
$$= 180° - (67°43'55'' + 29°38'50'')$$
$$= 82°37'15''.$$

By sine rule in ΔABD, we have

$$\frac{AB}{\sin\gamma} = \frac{BD}{\sin\theta}$$

$$AB = BD\frac{\sin\gamma}{\sin\theta} \qquad \text{...(a)}$$

By sine rule in ΔBCD, we have

$$\frac{BC}{\sin\delta} = \frac{BD}{\sin\phi}$$

$$BC = BD\frac{\sin\delta}{\sin\phi} \qquad \text{...(b)}$$

By cosine rule in ΔABC, we have

$$AC^2 = AB^2 + BC^2 - 2\,AB\;BC\cos ABC \qquad \text{...(c)}$$

Substituting the values of AB and BC from (a) and (b), respectively, in (c), we get

$$AC^2 = BD^2\frac{\sin^2\gamma}{\sin^2\theta} + BD^2\frac{\sin^2\delta}{\sin^2\phi} + 2BD\frac{\sin\gamma}{\sin\theta}BD\frac{\sin\delta}{\sin\phi}\cos ABC$$

$$= BD^2\left(\frac{\sin^2\gamma}{\sin^2\theta} + \frac{\sin^2\delta}{\sin^2\phi} + 2\frac{\sin\gamma}{\sin\theta}\frac{\sin\delta}{\sin\phi}\cos ABC\right)$$

$$1116.40^2 = BD^2\left(\begin{array}{l}\dfrac{\sin^2 63°19'28''}{\sin^2 71°59'33''} + \dfrac{\sin^2 29°38'50''}{\sin^2 82°37'15''} + 2\dfrac{\sin 63°19'28''}{\sin 71°59'33''}\\ \times\dfrac{\sin 29°38'50''}{\sin 82°37'15''}\cos(44°40'59'' + 67°43'55'')\end{array}\right)$$

$$= BD^2 \times 1.489025$$

$$BD = \sqrt{\frac{1116.40^2}{1.489025}} = 914.89 \text{ m.}$$

Example 6.2. Compute the value of *R* for the desired maximum probable error of 1 in 25000 if the probable error of direction measurement is 1.20″.

Solution:

L being the probable error that would occur in the sixth place of logarithm of any side, we have

$$\log\left(1 \pm \frac{1}{25000}\right) = 17.37 \times 10^{-6}.$$

The sixth place in log of $\left(1 \pm \frac{1}{25000}\right) = 17.$

Thus $L = 17.$

It is given that

$$d = 1.20''.$$

From Eq. (6.1), we have $L^2 = \frac{4}{3} d^2 R$.

Therefore $R = \frac{3}{4}\left(\frac{L}{d}\right)^2 = \frac{3}{4}\left(\frac{17}{1.20}\right)^2 = 151.$

Example 6.3. Compute the value of $\frac{D - C}{D}$ for the triangulation figures shown in Fig. 6.13. The broken lines indicate that the observation has been taken in only one direction.

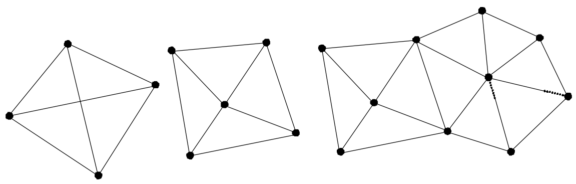


Fig. 6.12

Solution (Fig. 6.12):

Figure (a)

$$n = 6$$
$$n' = 6$$
$$S = 4$$
$$S' = 4$$
$$D = 2(n - 1) = 2 \times (6 - 1) = 10$$

$$C = (n' - S' + 1) + (n - 2S + 3)$$

$$= (6 - 4 + 1) + (6 - 2 \times 4 + 3)$$

$$= 3 + 1 = 4.$$

Therefore $$\frac{D - C}{D} = \frac{10 - 4}{10} = 0.6$$

Figure (b) $$n = 8$$

$$n' = 8$$

$$S = 5$$

$$S' = 5$$

$$D = 2(n - 1)$$

$$= 2 \times (8 - 1) = 14$$

$$C = (n' - S' + 1) + (n - 2S + 3)$$

$$= (8 - 5 + 1) + (8 - 2 \times 5 + 3)$$

$$= 4 + 1 = 5.$$

Therefore $$\frac{D - C}{D} = \frac{14 - 5}{14} = 0.64$$

Figure (c)

$$n = 20$$

$$n' = 18$$

$$S = 10$$

$$S' = 10$$

$$D = 2(n - 1) - 2$$

$$= 2 \times (20 - 1) - 2 = 36$$

$$C = (n' - S' + 1) + (n - 2S + 3)$$

$$= (18 - 10 + 1) + (20 - 2 \times 10 + 3)$$

$$= 9 + 3 = 12.$$

Therefore $$\frac{D - C}{D} = \frac{36 - 12}{36}$$

$$= 0.67.$$

Example 6.4. Compute the strength of figure *ABCD* (Fig. 6.13) for all the routes by which the length *CD* can be determined from the known side *AB* assuming that all the stations have been occupied, and find the strongest route.

Solution (Fig. 6:13):

There are four routes by which the length of *CD* can be computed. These are

Route-1: ΔABD and ΔBDC having common side *BD*

Route-2: ΔABD and ΔADC having common side *AD*

Route-3: ΔABC and ΔADC having common side *AC*

Route-4: ΔABC and ΔBCD having common side *BC*.

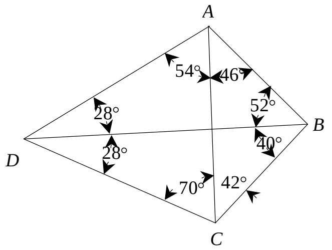


Fig. 6.13

The relative strength of figures can be computed quantitatively in terms of the factor *R* given by Eq. (6.2). By means of computed values of *R* alternative routes of computations can be compared, and hence, the best or strongest route can be selected. As strength of a figure is approximately equal to the strength of the strongest chain, the lowest value of *R* is a measure of the strongest route.

For a braced quadrilateral, the value of $\frac{D-C}{D}$ has been calculated in Example 6.3, and therefore

$$\frac{D-C}{D} = 0.60.$$

For each route the value of $\sum\left(\delta_A^2 + \delta_A\delta_B + \delta_B^2\right)$ is calculated as below.

Let $$\Delta = \sum\left(\delta_A^2 + \delta_A\delta_B + \delta_B^2\right).$$

Route-1:

In ΔABD, the distance angles for the sides *AB* and BD are 28° and (54° + 46° = 100°), respectively.

$$\delta_{28} = \text{sixth place of decimal of } \left[\log\sin\left(28° + 1''\right) - \log\sin\left(28°\right)\right] = 3.96$$

$$\delta_{100} = \text{sixth place of decimal of } \left[\log\sin\left(100° + 1''\right) - \log\sin\left(100°\right)\right] = -0.37$$

$$\delta_{28}^2 + \delta_{28}\delta_{100} + \delta_{100}^2 = 3.96^2 + 3.96\times(-0.37) + 0.37^2 = 14.34 \simeq 14.$$

In ΔBDC the distance angles for the sides *BD* and *CD* are (70° + 42° = 112°) and 40°, respectively.

$$\delta_{112} = \text{Sixth place of decimal of } \left[\log\sin\left(112° + 1''\right) - \log\sin\left(112°\right)\right] = -0.85$$

$$\delta_{40} = \text{Sixth place of decimal of } \left[\log\sin\left(40° + 1''\right) - \log\sin\left(40°\right)\right] = -2.51$$

$$\delta_{112}^2 + \delta_{112}\delta_{40} + \delta_{40}^2 = 0.85^2 + (-0.85)\times 2.51 + 2.51^2 = 5.$$

Therefore $$\Delta_1 = 14 + 5 = 19.$$

In the similar manner Δ_2, Δ_3, and Δ_4 for the remaining routes are calculated.

Route-2:

In ΔABD the distance angles for the sides AB and AD are 28° and 52°, respectively.

$$\delta_{28} = 3.96$$

$$\delta_{52} = 1.65$$

In ΔADC the distance angles for the sides AD and CD are 70°and 54°, respectively.

$$\delta_{70} = 0.77$$

$$\delta_{54} = 1.53$$

$$\Delta_2 = 29.$$

Route-3:

In ΔABC the distance angles for the sides AB and AC are 42° and (52° + 40° = 92°), respectively.

$$\delta_{42} = 2.34$$

$$\delta_{92} = -0.07$$

In ΔADC the distance angles for the sides AC and CD are (28° + 28° = 56°) and 54°, respectively.

$$\delta_{56} = 1.42$$

$$\delta_{54} = 1.53$$

$$\Delta_3 = 12.$$

Route-4:

In ΔABC the distance angles for the sides AB and BC are 42° and 46°, respectively.

$$\delta_{42} = 2.34$$

$$\delta_{46} = 2.03$$

In ΔBCD the distance angles for the sides BC and CD are 28° and 40°, respectively.

$$\delta_{28} = 3.96$$

$$\delta_{40} = 2.51$$

$$\Delta_4 = 46.$$

Thus

$$R_1 = 0.6 \times \Delta_1 = 0.6 \times 19 = 11$$

$$R_2 = 0.6 \times \Delta_2 = 0.6 \times 29 = 17$$

$$R_3 = 0.6 \times \Delta_3 = 0.6 \times 12 = 7$$

$$R_4 = 0.6 \times \Delta_4 = 0.6 \times 46 = 28.$$

The route-3 has the minimum value of R = 7, therefore the strongest route.

Example 6.5. In a triangulation survey, the altitudes of two stations *A* and *B*, 110 km apart, are respectively 440 m and 725 m. The elevation of a peak P situated at 65 km from *A* has an elevation of 410 m. Ascertain if *A* and *B* are intervisible, and if necessary, find by how much *B* should be raised so that the line of sight nowhere be less than 3 m above the surface of ground. Take earth's mean radius as 6400 km and the mean coefficient of refraction as 0.07.

Solution (Fig.6:15):

The distance of visible horizon given by Eq. (6.4), is

$$h = \frac{D^2}{2R}(1-2m)$$

$$D^2 = \frac{2Rh}{(1-2m)}$$

$$D = \sqrt{\frac{2 \times 6400h}{(1-2\times 0.07)\times 1000}}$$

$$= 3.85794\sqrt{h} \text{ kilometre}$$

$$= 3.85794\sqrt{440} = 80.92 \text{ km.}$$

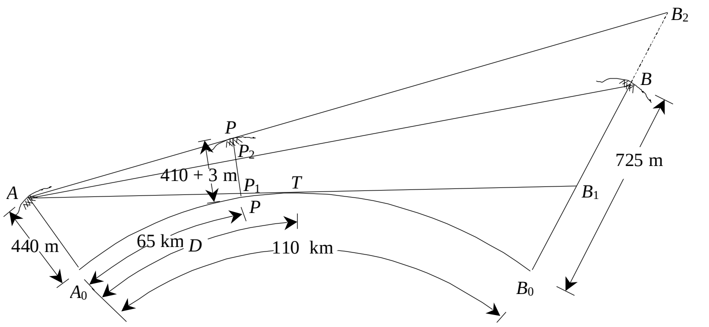


Fig. 6.15

Therefore $\quad P_0T = D - A_0P_0$

$$= 80.92 - 65 = 15.92 \text{ km.}$$

$$P_0P_1 = \left(\frac{P_0T}{3.85794}\right)^2$$

$$= \left(\frac{15.92}{3.85794}\right)^2 = 17.03 \text{ m.}$$

$$TB_0 = A_0B_0 - A_0T$$

$$= 110 - 80.92 = 29.08 \text{ km.}$$

$$B_0B_1 = \left(\frac{TB_0}{3.85794}\right)^2$$

$$= \left(\frac{29.08}{3.85794}\right)^2 = 56.82 \text{ m.}$$

$$BB_1 = B_0B - B_0B_1$$

$$= 725 - 56.82 = 668.18 \text{ m.}$$

From similar Δ's AP_2P_1 and ABB_1, we get

$$\frac{P_2P_1}{A_0P_0} = \frac{BB_1}{A_0B_0}$$

$$P_2P_1 = \frac{A_0P_0\ BB_1}{A_0B_0}$$

$$= \frac{65 \times 668.18}{110} = 394.83 \text{ m.}$$

Therefore $P_2P_0 = P_2P_1 - P_1P_0$

$$= 394.83 + 17.03 = 411.86 \text{ m.}$$

Since the line of sight has to be 3 m above the ground surface at P, the elevation of P may be taken as 410 + 3 = 413 m, or PP_0 = 413 m. The line of sight fails to clear P by PP_2 = 413 – 411.86 = 1.14 m. Thus the amount of raising required at B is BB_2.

From similar Δ's APP_2 and AB_2B, we get

$$\frac{BB_2}{A_0B_0} = \frac{P_2P}{A_0P_0}$$

$$BB_2 = \frac{A_0B_0\ P_2P}{A_0P_0}$$

$$= \frac{110 \times 1.14}{65} = 1.93 \text{ m} \simeq 2 \text{ m.}$$

Example 6.6. Solve the Example 6.5 by Capt. McCaw's method.

Solution (Fig.6:16):

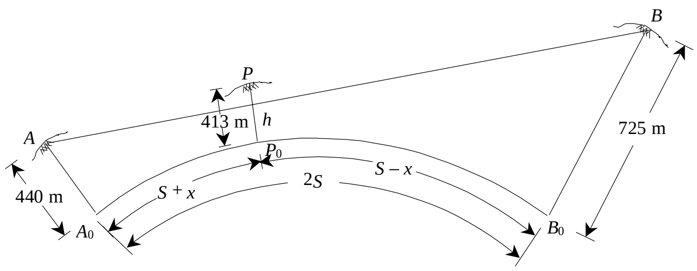


Fig. 6.16

It is given that

$$2S = 110 \text{ km}$$

$$S = \frac{110}{2} = 55 \text{ km}$$

and $S + x = 65$ km

$x = 65 - S$

$= 65 - 55 = 10$ km.

From Capt. McCaw's formula, we have

$$h = \frac{1}{2}(h_B + h_A) + \frac{1}{2}(h_B - h_A)\frac{x}{s} - (S^2 - x^2)\operatorname{cosec}^2 \xi \frac{(1-2m)}{2R}$$

$$= \frac{1}{2} \times (725 + 440) + \frac{1}{2} \times (725 - 440) \times \frac{10}{55} - (55^2 - 10^2) \times 1 \times \frac{(1-2\times 0.07)}{2\times 6400} \times 1000$$

$= 582.5 + 25.909 - 196.523$

$= 411.89$ m.

Here h is same as P_2P_0 obtained in Example 6.5. The line of sight fails to clear the line of sight by 413 – 411.89 = 1.11 m. The amount of raising BB_2 required at B is calculated now in a similar manner as in Example 6.5. Thus

$$= \frac{110\times 1.11}{65} = 1.88 \text{ m} \approx 2 \text{ m}.$$

Example 6.7: Two triangulation stations *A* and *B*, *B* being on the right of *A*, have cylindrical signals of diameter 4 m and 3 m, respectively. The observations were made on *A* on the bright portion and on *B* on the bright line, and the measured angle *AOB* at a station *O* was 44°15′32″.The distances *OA* and *OB* were measured as 4015 m and 5635 m, respectively. The angle which the sun makes with the lines joining the station *O* with *A* and *B* was 42°. Calculate the correct angle *AOB*.

Solution (Fig.6:17):

Given that

$D_1 = 4015$ m; $D_2 = 5635$ m

$r_1 = 2$ m; $r_2 = 1.5$ m

$\theta_1 = \theta_2 = 42°$ m

$\phi' = 44°15'32''$.

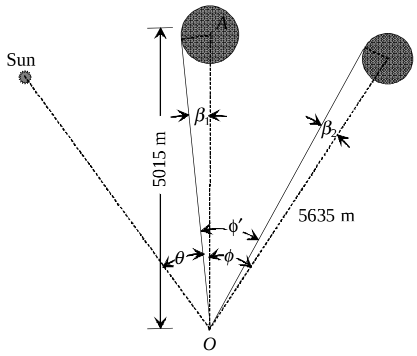


Fig. 6.17

If the phase corrections for the signals *A* and *B* are β_1 and β_2, respectively, the correct angle *AOB*

$$\phi = \phi' - \beta_1 + \beta_2$$

The observation for *A* was made on the bright portion, therefore the phase correction

$$\beta_1 = \frac{206265 r_1}{D_1}\cos^2\frac{\theta}{2} \text{ seconds}$$

$$= \frac{206265\times 2}{4015}\cos^2\frac{42°}{2}$$

$= 89.55''$.

The observation for B was made on the bright line, therefore, the phase correction

$$\beta_2 = \frac{206265 r_2}{D_2} \cos\frac{\theta}{2} \text{ seconds}$$

$$= \frac{206265 \times 1.5}{5635} \cos\frac{42°}{2}$$

$$= 51.26''.$$

Thus the correct angle AOB

$$= 44°15'32'' - 89.55'' + 51.26''$$

$$= 44°14'53.71''.$$

Example 6.8. The directions observed from a satellite station S, 70 m from a triangulation station C, to the triangulation station A, B, and C are 0°00′00″, 71°32′54″ and 301°16′15″, respectively. The lengths of AB, and AC are 16.5 km and 25.0 km, respectively. Deduce the angle ACB.

Solution (Fig.6.18):

Given that $\theta = 71°32'54''$

$\xi = 301°16'15''$

$a = AC = 16.5$ km

$b = BC = 25.0$ km

$d = SC = 70$ m

$\gamma = 360° - 301°16'15'' = 58°43'45''$.

From Eq. (6.11), we have

$$\phi = \theta - \alpha + \beta$$

From Eq. (6.9), we have

$$\alpha = 206265 \frac{d}{a} \sin\gamma \quad \text{seconds}$$

$$= 206265 \times \frac{70}{16.5 \times 1000} \sin 58^\circ 43'45''$$

$$= 747.94'' = 12'27.94''.$$

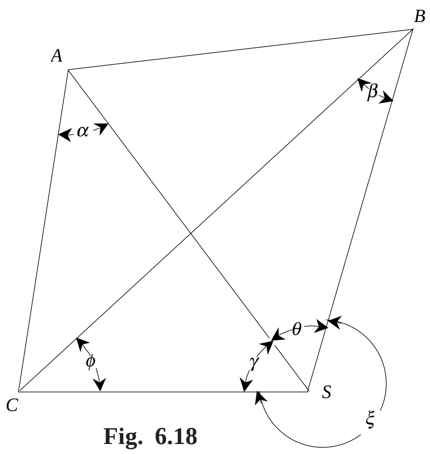


Fig. 6.18

From Eq. (6.8), we have

$$\beta = 206265 \frac{d}{b} \sin(\theta + \gamma) \quad \text{seconds}$$

$$\beta = 206265 \times \frac{70}{25.0 \times 1000} \sin(71^\circ 32'56'' + 58^\circ 43'45'') \quad \text{seconds}$$

$$= 440.62'' = 7'20.62''.$$

Thus $$\phi = 71^\circ 32'56'' - 12'27.94'' + 7'20.62''$$

$$= \mathbf{71^\circ 27'48.68''}.$$

Example 6.9. *S* is a satellite station to a triangulation station *A* at a distance of 12 m from *A*. From *S* the following bearings were observed:

$$A = 0^\circ 00'00''$$

$$B = 143^\circ 36'20''$$

$$C = 238^\circ 24'48''$$

$$D = 307^\circ 18'54''$$

The lengths of lines *AB*, *AC*, and *AD* were measured and found to be 3190.32 m, 4085.15, and 3108.60 m, respectively. Determine the directions of *B*, *C*, and *D* from *A*.

Solution (Fig.6.19):

If α_1, α_2, and α_3 are the corrections to the observed directions, the required directions from the stations *A* will be $(\theta_1 + \alpha_1)$ for *B*, $(\theta_2 + \alpha_2)$ for *C*, and $(\theta_3 + \alpha_3)$ for *D*. Since the distances *SB*, *SC*, and *SD* are quite large compared to the distance *SA*, the distances *AB*, *AC*, and *AD*, respectively can be taken equal to them. The corrections can be computed from the following relationship that is same as Eq. (6.9), i.e.,

$$\alpha = 206265 \frac{d}{a} \sin\theta \text{ seconds}$$

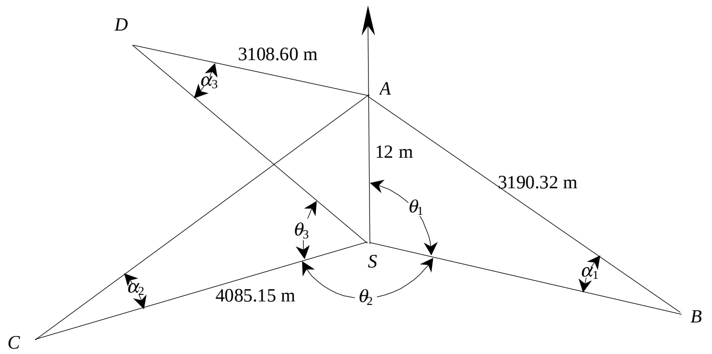


Fig. 6.19

Thus $$\alpha_1 = 206265 \times \frac{12}{3190.32} \sin 143^\circ 36'20''$$

$$= 460.34'' = 7'40.34''$$

$$\alpha_2 = 206265 \times \frac{12}{4085.15} \sin 238^\circ 24'48''$$

$$= -516.13'' = -8'36.13''$$

$$\alpha_3 = 206265 \times \frac{12}{3108.60} \sin 307^\circ 18'54''$$

$$= -633.24'' = -10'33.24''.$$

Therefore, the directions from *A* to

B = 143°36′20″ + 7′40.34″ = **143°44′00.34″**

C = 238°24′48″ – 8′36.13″ = **238°16′11.87″**

D = 307°18′54″ – 10′33.24″ = **307°08′20.76″**.

Example 6.10. In a triangulation survey, the station *C* could not be occupied in a triangle *ABC*, and a satellite station *S* was established north of *C*. The angles as given in Table 6.1 were measured at *S* using a theodolite.

Table 6.1

Pointing on	Horizontal circle reading
A	14°43′27″
B	74°30′35″
C	227°18′12″

Approximate lengths of *AC* and *BC* were found by estimation as 17495 m and 13672 m, respectively, and the angle *ACB* was deduced to be 59°44′53″. Calculate the distance of *S* from *C*.

Solution (Fig.6.20):

$\angle ASB$ = Angle to B – angle to A

= 74°30′35″ – 14°43′27″ = 59°47′08″

$\angle BSC$ = Angle to C – angle to B

= 227°18′12″ – 74°30′35″ = 152°47′37″

$\angle ASC$ = 360° – ($\angle ASB$ + $\angle BSC$)

= 360° – (59°47′08″ + 152°47′37″) = 147°25′15″.

By sine rule in Δ's *ASC* and *BCS*, we get

$$\frac{AC}{\sin ASC} = \frac{d}{\sin \alpha} \quad \text{or} \quad \sin \alpha = \frac{d \sin ASC}{AC}$$

and

$$\frac{BC}{\sin BSC} = \frac{d}{\sin \beta} \quad \text{or} \quad \sin \beta = \frac{d \sin BSC}{BC}.$$

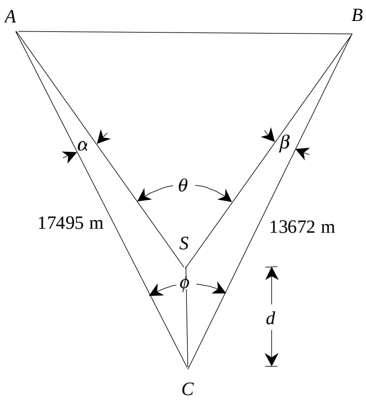


Fig. 6.20

For small angles, we can take

$$\sin \alpha = \alpha'' \sin 1'' = \frac{\alpha''}{206265}$$

$$\sin \beta = \beta'' \sin 1'' = \frac{\beta''}{206265}.$$

Therefore $$\alpha'' = \frac{206265 \times d \times \sin 147^\circ 25' 15''}{17495} = 6.348d$$

$$\beta'' = \frac{206265 \times d \times \sin 152°47'37''}{13672} = 6.898d.$$

The angle ϕ is given by Eq. (6.12)

$$\phi = \angle ASB - \alpha - \beta$$

$$= 59°47'08'' - (6.348 + 6.898)\ d$$

$$= 59°47'08'' - 13.246\ d.$$

But the value of ϕ is given as 59°44′53″. Therefore

$$59°44'53'' = 59°47'08'' - 13.246d$$

$$d = \frac{59°47'08'' - 59°44'53''}{13.246''}$$

$$= \frac{59.7855556 - 59.7480556}{(13.246/3600)} = \textbf{10.192 m}.$$

Example 6.11. Determine the coordinates of a point R from the following data:

Coordinates of P = E1200 m, N1200 m

Coordinates of Q = E400 m, N1000 m

Bearing of PR = 62°13′40″

Bearing of QR = 38°46′25″.

Solution (Fig. 6.21):

Let the coordinates of R be (X, Y), and the bearings of PR and QR be α and β, respectively.

$$\alpha = 38°46'25'',\quad \beta = 62°13'40''$$

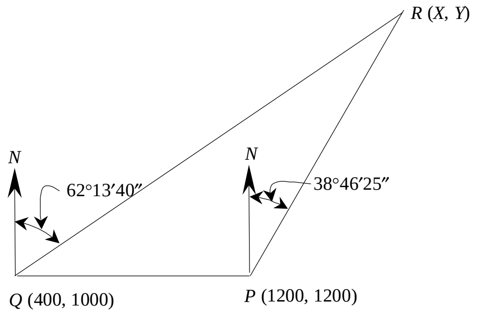


Fig. 6.21

$$\tan\alpha = \frac{X - 1200}{Y - 1200}$$

$$\tan\beta = \frac{X - 400}{Y - 1000}$$

or $$Y\tan\alpha - 1200\tan\alpha = X - 1200$$

$$Y\tan\beta - 1000\tan\alpha = X - 400 \quad \ldots(a)$$

or $$X = Y\tan\alpha - 1200\tan\alpha + 1200 = Y\tan\beta - 1000\tan\beta + 400$$

$$Y = \frac{1200\tan\alpha - 1000\tan\beta - 800}{\tan\alpha - \tan\beta}. \quad \ldots(b)$$

Substituting the values of α and β in (b), we get

$$Y = \textbf{1583.54 m}$$

and from (a), we get

$$X = \mathbf{1508.08\ m}.$$

Example 6.12. In a triangulation survey it was required to establish a point *P* by taking observations from three triangulation stations *A*, *B*, and *C*. The observations made are as under:

$$\angle PAB = 65°27'48''$$
$$\angle PBA = 72°45'12''$$
$$\angle PBC = 67°33'24''$$
$$\angle PCB = 78°14'55''$$
$$AB = 2600 \text{ m}$$
$$BC = 2288 \text{ m}.$$

Determine three distances *AP*, *BP*, and *CP* to fix *P*.

Solution (Fig. 6.6a):

In the quadrilateral *ABCP*, we have

$$\theta + \phi + \angle PAB + \angle PBA + \angle PBC + \angle PCB = 360°$$

or
$$\theta = 360° - (65°27'48'' + 72°45'12'' + 67°33'24'' + 78°14'55'') - \phi$$
$$= 75°58'41'' - \phi$$
$$= \varepsilon - \phi \qquad \ldots(a)$$

where ε is = 75°58′41″.

From Eq. (6.14), we have

$$\sin\phi = \frac{BC \sin PCB \sin\theta}{AB \sin PAB}$$

$$= \frac{2288 \times \sin 78°14'55''}{2600 \times \sin 65°27'48''} \sin\theta$$

$$= 0.94708 \sin \theta. \qquad \ldots(b)$$

Substituting the value of θ from (a) in (b), we get

$$\sin \phi = 0.94708 \sin(\varepsilon - \phi)$$
$$= 0.94708 (\sin\varepsilon \cos\phi - \cos\varepsilon \sin\phi)$$
$$1 = 0.94708 (\sin\varepsilon \cot\phi - \cos\varepsilon)$$
$$\cot\phi = \cot\varepsilon + \frac{1}{0.94708 \sin\varepsilon}$$

Substituting the value of ε, we get

$$\cot \phi = 1.33804$$
$$\phi = 36°46'23''.$$

Therefore $\theta = 75°58'41'' - 36°46'23''$

$= 39°12'18''.$

From ΔAPB, we get

$$\frac{AP}{\sin PBA} = \frac{AB}{\sin\theta} = \frac{BP}{\sin PBA}$$

$$AP = \frac{2600 \times \sin 72°45'12''}{\sin 39°12'18''} = 3928.35 \text{ m}$$

$$BP = \frac{2600 \times \sin 65°27'48''}{\sin 39°12'18''} = 3741.85 \text{ m.}$$

From ΔBPC, we get

$$\frac{CP}{\sin PBC} = \frac{BC}{\sin\phi} = \frac{BP}{\sin PCB}$$

$$CP = \frac{2288 \times \sin 67°33'24''}{\sin 36°46'23''} = 3532.47 \text{ m}$$

$$BP = \frac{2288 \times \sin 78°14'55''}{\sin 39°12'18''} = 3741.85 \text{ m.}$$

The two values of BP being same, give a check on the computations.

Example 6.13. To establish a point P, three triangulation stations A, B, and C were observed from P with the following results:

$$\angle APB = 39°12'18''$$
$$\angle BPC = 36°46'23''$$
$$\angle ABC = 140°18'36''$$
$$AB = 2600 \text{ m}$$
$$BC = 2288 \text{ m.}$$

Compute the lengths of the lines PA, PB, and PC to establish P by the method of resection.

Solution (Fig. 6.6b):

It is given that

$$\theta = 39°12'18''$$
$$\beta = 140°18'36''$$
$$\phi = 36°46'23''$$
$$a = 2600 \text{ m}$$
$$b = 2288 \text{ m.}$$

In the quadrilateral *ABCP*, we have

$$\alpha + \beta + \gamma + \theta + \phi = 360°$$

$$\alpha + \gamma = 360° - (\beta + \theta + \phi)$$

$$= 360° - (140°18'36'' + 39°12'18'' + 36°46'23'')$$

$$= 143°42'43''$$

$$\frac{1}{2}(\alpha + \gamma) = 71°51'21.5''. \quad \ldots\text{(a)}$$

From Eq. (6.14), we get

$$\frac{\sin\gamma}{\sin\alpha} = \frac{a\sin\phi}{b\sin\theta} \quad \ldots\text{(b)}$$

$$\tan\Delta = \frac{\sin\gamma}{\sin\alpha} \quad \ldots\text{(c)}$$

Adding unity to both sides, we have

$$\frac{\sin\gamma}{\sin\alpha} + 1 = \tan\Delta + 1$$

$$\frac{\sin\alpha + \sin\gamma}{\sin\alpha} = \tan\Delta + \tan 45° \quad \ldots\text{(d)}$$

Subtracting both sides of (c) from unity, we get

$$1 - \frac{\sin\gamma}{\sin\alpha} = 1 - \tan\Delta$$

$$\frac{\sin\alpha - \sin\gamma}{\sin\alpha} = 1 - \tan 45° \tan\Delta \quad \ldots\text{(e)}$$

Dividing (e) by (d), we have

$$\frac{\sin\alpha - \sin\gamma}{\sin\alpha + \sin\gamma} = \frac{1 - \tan 45° \tan\Delta}{\tan\Delta + \tan 45°}$$

$$\frac{2\cos\frac{1}{2}(\alpha + \gamma)\sin\frac{1}{2}(\alpha - \gamma)}{2\sin\frac{1}{2}(\alpha + \gamma)\cos\frac{1}{2}(\alpha - \gamma)} = \frac{1}{\tan\left(45° + \Delta\right)}$$

$$\cot\frac{1}{2}(\alpha + \gamma)\tan\frac{1}{2}(\alpha - \gamma) = \cot\left(45° + \Delta\right)$$

$$\tan\frac{1}{2}(\alpha-\gamma) = \cot\left(45^\circ + \Delta\right)\tan\frac{1}{2}(\alpha+\gamma) \qquad \text{...(f)}$$

From (b) and (c), we get

$$\tan\Delta = \frac{2600\times\sin 36^\circ 46'23''}{2288\times\sin 39^\circ 12'18''} = 1.076228$$

$$\Delta = 47^\circ 06' 09.54''.$$

Substituting the value of Δ and ½(α + β) from (a) in (f), we get

$$\tan\frac{1}{2}(\alpha-\gamma) = \cot\left(45^\circ + 47^\circ 06' 9.54''\right)\tan 71^\circ 51' 21.5''$$

$$= -\ 0.112037$$

$$\frac{1}{2}(\alpha-\gamma) = -\ 6^\circ 23' 33.33''.$$

Thus
$$\frac{1}{2}(\alpha+\gamma) + \frac{1}{2}(\alpha-\gamma) = \alpha = 71^\circ 51' 21.5'' - 6^\circ 23' 33.33''$$

$$= 65^\circ 27' 48.17''$$

and
$$\frac{1}{2}(\alpha+\gamma) - \frac{1}{2}(\alpha-\gamma) = \gamma = 71^\circ 51' 21.5'' + 6^\circ 23' 33.33''$$

$$= 78^\circ 14' 54.83''.$$

In ΔABP, we have

$$\angle ABP + \alpha + \theta = 180^\circ$$

$$\angle ABP = 180^\circ - (\alpha + \theta)$$

$$= 180^\circ - (65^\circ 27' 48.17'' + 39^\circ 12' 18'')$$

$$= 75^\circ 19' 53.83''.$$

In ΔCBP, we have

$$\angle CBP + \gamma + \phi = 180^\circ$$

$$\angle CBP = 180^\circ - (\gamma + \phi)$$

$$= 180^\circ - (78^\circ 14' 54.83'' + 36^\circ 46' 23'')$$

$$= 64^\circ 58' 42.17''.$$

Check:
$$\angle ABC = \angle ABP + \angle CBP = 75^\circ 19' 53.83'' + 64^\circ 58' 42.17''$$

$$= 140^\circ 18' 36'' \qquad (\textit{Okay}).$$

Now from Δ's ABP and CBP, we have

$$\frac{x}{\sin ABP} = \frac{a}{\sin\theta} = \frac{y}{\sin\alpha}$$

$$x = \frac{a \sin ABP}{\sin \theta}$$

$$= \frac{2600 \times \sin 75°19'53.83''}{\sin 39°12'18''} = \mathbf{3979.23\ m}.$$

$$y = \frac{a \sin \alpha}{\sin \theta}$$

$$= \frac{2600 \times \sin 65°27'48.17''}{\sin 39°12'18''} = \mathbf{3741.85\ m}.$$

$$\frac{y}{\sin \gamma} = \frac{b}{\sin \phi} = \frac{z}{\sin CBP}$$

$$y = \frac{b \sin \gamma}{\sin \phi}$$

$$= \frac{2288 \times \sin 78°14'54.83''}{\sin 36°46'23''} = \mathbf{3741.85\ m}.$$

$$z = \frac{b \sin CBP}{\sin \phi}$$

$$= \frac{2288 \times \sin 64°58'42.17''}{\sin 36°46'23''} = \mathbf{3463.26\ m}.$$

Example 6.14. To determine the sea- level distance between two stations *A* and *B*, *A* being lower than B, the measured slope distance between *A* and *B*, and corrected for meteorological conditions, is 32015.65 m. The measured vertical angles at *A* and *B* are + 3°40′15″ and –3°51′32″, respectively. The elevation of point *A* is 410.22 m. Determine the sea level distance between the points. Take mean earth's radius as 6370 m.

Solution (Fig. 6.7):

From Eq. (6.23), we get

$$\angle BAB' = \frac{1}{2}(\alpha + \beta)$$

$$= \frac{1}{2} \times \left(3°14'15'' + 3°51'32'' \right)$$

$$= 3°45'53.5''.$$

From Eqs. (6.24) and (6.25), we get

$$\sin\frac{\theta}{2} \approx \frac{\theta}{2} \approx \frac{AB\cos BAB'}{2R} \quad \text{radians}$$

$$= \frac{32015.65 \times \cos 30°45'53.5''}{2 \times 6370 \times 10^3} \times 206265 \text{ seconds}$$

$$= 517.23'' = 8'37.23''.$$

and

$$\angle AB'B = 90° + \frac{\theta}{2}$$

$$= 90 + 8'37.23'' = 90°08'37.23''$$

$$\angle ABB' = 90° - \frac{1}{2}(\alpha + \beta) - \frac{\theta}{2}$$

$$= 90° - 3°45'53.5'' - 8'37.23'' = 86°05'29.27''.$$

From Eq. (6.26), we have

$$AB' = \frac{AB\sin ABB'}{\sin\left(90° + \theta/2\right)}$$

$$= \frac{32015.65 \times \sin 86°05'29.27''}{\sin 90°08'37.23''}$$

$$= 31941.286 \text{ m.}$$

From Eq. (6.27), we have

$$MN = AB'\left(1 - \frac{h_A}{R}\right)$$

$$= 31941.286 \times \left(1 - \frac{410.22}{6370 \times 1000}\right)$$

$$= 31939.229 \text{ m.}$$

From Eq. (6.29), the sea level distance

$$S = MN + \frac{MN^3}{24R^2}$$

$$= 31939.229 + \frac{31939.229^3}{24 \times 6370^2 \times 10^6}$$

$$= \mathbf{31939.262 \text{ m}}.$$

Example 6.15. Solve the Example 6.14 when the elevations of stations *A* and *B* are respectively 1799.19 m and 4782.74 m, and the vertical angles not observed.

Solution (Fig. 6.8):

The difference in elevations

$$\Delta h = 4782.74 - 1799.19$$

$$= 2983.55 \text{ m.}$$

From Eq. (6.30), we have

$$AB' \approx AB - + \frac{\Delta h^2}{2AB}$$

$$= 32015.65 - \frac{2983.55^2}{2 \times 32015.65} = 31876.631 \text{ m.}$$

From Eq. (6.26), we have

$$\frac{\theta}{2} \approx \frac{AB'}{2R} \quad \text{radians}$$

$$= \frac{31876.631 \times 206265}{2 \times 6370 \times 1000} \quad \text{radians}$$

$$= 516.096'' = 8'36.09''$$

From Eq. (6.31), the more accurate value of *AB′*

$$= AB - \left(\Delta h \sin \frac{\theta}{2} + \frac{\Delta h^2}{2AB} \right)$$

$$= 32015.65 - \left(2983.55 \times \sin 8'36.09'' + \frac{2983.55^2}{2 \times 32015.65} \right)$$

$$= 31869.166 \text{ m.}$$

From Eq. (6.27), we have

$$MN = AB' \left(1 - \frac{h_A}{R} \right)$$

$$= 31869.166 \times \left(1 - \frac{1799.19}{6370 \times 10^3} \right)$$

$$= 31854.239 \text{ m.}$$

From Eq. (6.29), the sea level distance

$$S = MN + \frac{MN^3}{24R^2}$$

$$= 31854.239 + \frac{31854.239^3}{24 \times 6370^2 \times 10^6}$$

$$= \mathbf{31854.272\ m}.$$

Example 6.16. In a geodetic survey, the mean angles of a triangle *ABC* having equal weights, are as below:

$$\angle A = 62°24'18.4''$$

$$\angle B = 64°56'09.9''$$

$$\angle C = 52°39'34.4''.$$

Side *AB* has length of 34606.394 m. Estimate the corrected values of the three angles. Take the radius of the earth to be 6383.393 km.

Solution (Fig. 6.9):

In order to estimate the spherical excess using Eq. (6.32), it is necessary to estimate the area of the triangle *ABC*. For this purpose it is sufficiently accurate to assume the triangle to be as a plane triangle. Thus the sum of the three angles should be equal to 180°. To satisfy this condition the value (ΣObserved angles – 180°)/3 must be deducted from each angle.

$$\frac{1}{3}(\Sigma\text{Observed angles} - 180°)$$

$$= \frac{1}{3} \times (62°24'18.4'' + 64°56'09.9'' + 52°39'34.4'' - 180°)$$

$$= 0.9''.$$

Thus the plane triangles are

$$A = 62°24'18.4'' - 0.9'' = 62°24'17.5''$$

$$B = 64°56'09.9'' - 0.9'' = 64°56'09.0''$$

$$C = 52°39'34.4'' - 0.9'' = 52°39'33.5''$$

$$\text{Sum} = 180°00'00''.$$

By sine rule in ΔABC, we get

$$\frac{AB}{\sin C} = \frac{BC}{\sin A}$$

$$BC = \frac{AB \sin A}{\sin C}$$

$$= \frac{34606.394 \times \sin 62°24'17.5''}{\sin 52°39'33.5''} = 38576.121 \text{ m}.$$

Area of the ΔABC

$$A_0 = \frac{1}{2} AB\, BC \sin B$$

$$= \frac{1}{2} \times 34606.394 \times 38576.121 \times \sin 64^\circ 56' 09''$$

$$= 604.64 \text{ km}^2.$$

From Eq. (6.32), we have

$$\varepsilon = \frac{A_0}{R^2 \sin 1''}$$

$$= \frac{604.64}{6383.393^2} \times 206265 = 3.06''.$$

Thus the theoretical sum of the spherical angles

$$= 180^\circ + \varepsilon$$

$$= 180^\circ + 3.06''$$

$$= 180^\circ 00' 3.06''.$$

The sum of the observed angles

$$= 180^\circ + 3 \times 0.9''$$

$$= 180^\circ 00' 02.7''.$$

Therefore, the triangular error

$$= 180^\circ 00' 02.7'' - 180^\circ 00' 3.06'' = -0.36''.$$

Since the angles were measured with equal reliability, the correction to each angle will be $+ 0.36''/3 = + 0.12''$.

Therefore the corrected angles are

$$A = 62^\circ 24' 18.4'' + 0.12'' = 62^\circ 24' 18.52''$$

$$B = 64^\circ 56' 09.9'' + 0.12'' = 64^\circ 56' 10.02''$$

$$C = 52^\circ 39' 34.4'' + 0.12'' = 52^\circ 39' 34.52''$$

$$\text{Sum} = 180^\circ 00' 00''.$$

OBJECTIVE TYPE QUESTIONS

1. Control for survey can be provided by

(*a*) Triangulation.

(*b*) Trilateration.

(*c*) Traversing.

(*d*) All of the above.

2. The distance of visible horizon for a point having an elevation of 637.5 m is

(*a*) 6.735 km.

(*b*) 67.35 km.

(*c*) 10 km.

(*d*) 100 km.

3. A strongest route in a triangulation net has

(a) minimum value of *R*.

(b) maximum value of *R*.

(c) minimum value of $\sqrt{R}$.

(d) maximum value of $\sqrt{R}$.

where $R = \frac{D-C}{D} \Sigma \left(\delta_A^2 + \delta_A\delta_B + \delta_B^2\right)$

4. In a braced quadrilateral, the position of unknown corner points can be determined by

(*a*) a single route only.

(*b*) two alternative routes only.

(*c*) three alternative routes only.

(*d*) four alternative routes only.

5. Phase correction is required when the observations are made on

(*a*) Pole signals.

(*b*) Cylindrical signals.

(*c*) Pole and brush signals.

(*d*) Beacons.

6. The errors in horizontal angle measurements due to eccentricity of signal, is eliminated completely by

(*a*) the method of repetition.

(*b*) the method of reiteration.

(*c*) both the above method.

(*d*) none of the above.

7. The problem of reduction to center is solved by

(*a*) taking a long base line.

(*b*) removing the error due to phase.

(*c*) taking a satellite station.

(*d*) taking well-conditioned triangles.

8. A satellite station is a station

(*a*) close to the main triangulation station that cannot be occupied for making observations.

(*b*) also known as an intersected point.

(*c*) also known as a resected point.

(*d*) which falls on the circumference of the circle passing through three main triangulation stations.

9. The horizontal refraction is minimum between

(*a*) 6 AM to 9 AM.

(*b*) 10 AM to 2 PM.

(*c*) 8 AM to 12 Noon.

(*d*) 2 PM to 4 PM.

10. The vertical refraction is minimum between

(*a*) 6 AM to 9 AM.

(*b*) 10 AM to 2 PM.

(*c*) 8 AM to 12 Noon.

(*d*) 2 PM to 4 PM.

11. A grazing line of sight is that line which

(*a*) joins two stations which are not intervisible.

(*b*) is at least 3 m above the intervening ground between two stations.

(*c*) touches the intervening ground between two stations.

(*d*) joins the signals at two stations kept on towers.

12. Sum of the three angles of spherical triangle

(*a*) is always less than 180°.

(*b*) is always more than 180°.

(*c*) is less or more than 180° depending the location of the triangle on spheroid.

(*d*) is equal to 180°.

ANSWERS

1. (*d*)	**2.** (*d*)	**3.** (*a*)	**4.** (*d*)	**5.** (*b*)	**6.** (*d*)
7. (*c*)	**8.** (*a*)	**9.** (*a*)	**10.** (*b*)	**11.** (*c*)	**12.** (*b*)

7 CURVE RANGING

7.1 CURVES

In highways, railways, or canals the curve are provided for smooth or gradual change in direction due the nature of terrain, cultural features, or other unavoidable reasons. In highway practice, it is recommended to provide curves deliberately on straight route to break the monotony in driving on long straight route to avoid accidents.

The horizontal curve may be a simple circular curve or a compound curve. For a smooth transition between straight and a curve, a transition or easement curve is provided. The vertical curves are used to provide a smooth change in direction taking place in the vertical plane due to change of grade.

7.2 CIRCULAR CURVES

A simple circular curve shown in Fig. 7.1, consists of simple arc of a circle of radius R connecting two straights AI and IB at tangent points T_1 called the *point of commencement* (P.C.) and T_2 called the *point of tangency* (P.T.), intersecting at I, called the *point of intersection* (P.I.), having a deflection angle Δ or angle of intersection ϕ. The distance E of the midpoint of the curve from I is called the *external distance*. The arc length from T_1 to T_2 is the *length of curve*, and the chord T_1T_2 is called the *long chord*. The distance M between the midpoints of the curve and the long chord, is called the *mid-ordinate*. The distance T_1I which is equal to the distance IT_2, is called the *tangent length*. The tangent AI is called the *back tangent* and the tangent IB is the *forward tangent*.

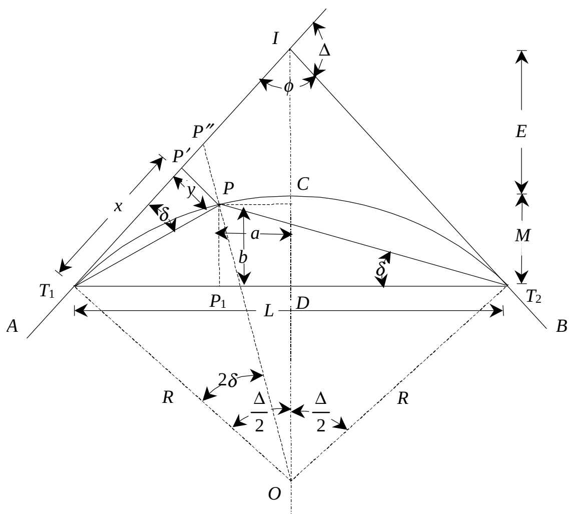


Fig. 7.1

Formula to calculate the various elements of a circular curve for use in design and setting out, are as under.

$$\text{Tangent length } (T) = R \tan\frac{\Delta}{2} \qquad ...(7.1)$$

$$\text{Length of curve } (l) = \frac{\pi R\Delta}{180} \qquad ...(7.2)$$

$$\text{Long chord } (L) = 2R\sin\frac{\Delta}{2} \qquad ...(7.3)$$

$$\text{External distance } (E) = R\left(\sec\frac{\Delta}{2} - 1\right) \qquad ...(7.4)$$

$$\text{Mid-ordinate } (M) = R\left(1 - \cos\frac{\Delta}{2}\right) \qquad ...(7.5)$$

$$\text{Chainage of } T_1 = \text{Chainage of P.I.} - T \qquad ...(7.6)$$

$$\text{Chainage of } T_2 = \text{Chainage of } T_1 + l. \qquad ...(7.7)$$

Setting out of Circular Curve

There are various methods for setting out circular curves. Some of them are:

Perpendicular Offsets from Tangent (Fig. 7.1)

$$y = R - \sqrt{(R^2 - x^2)} \quad (exact) \qquad ...(7.8)$$

$$= \frac{x^2}{2R} \quad (approximate) \qquad ...(7.9)$$

where x = the measured distance from T_1 along the tangent.

Radial Offsets (Fig. 7.1)

$$r = \sqrt{(R^2 + x^2) - R^2} \quad (exact) \qquad ...(7.10)$$

$$= \frac{x^2}{2R} \quad (approximate) \qquad ...(7.11)$$

where x = the measured distance from T_1 along the tangent.

Offsets from Long Chord (Fig. 7.1)

$$b = \sqrt{(R^2 - a^2)} - \sqrt{\left(R^2 - \frac{L^2}{4}\right)} \qquad ...(7.12)$$

where a = the measured distance from D along the long chord.

Offsets from Chords Produced (Fig. 7.2)

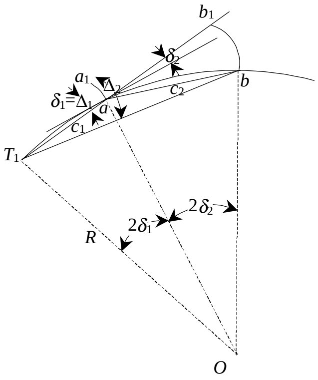


Fig. 7.2

Offset from tangent

$$a_1a = \frac{T_1a^2}{2R} \quad ...(7.13)$$

Offset from chords produced

$$b_1b = \frac{ab(T_1a + ab)}{2R} \quad ...(7.14)$$

Rankin's Method or Deflection Angle Method

$$\text{Tangential angle } \delta_n = 1718.9\frac{c_n}{R} \text{ minutes} \quad ...(7.15)$$

$$\text{Deflection angles } \Delta_n = \Delta_{n-1} + \delta_n \quad ...(7.16)$$

7.3 COMPOUND CURVES

A compound curve (Fig. 7.3) has two or more circular curves contained between the two main straights or tangents. The individual curves meet tangentially at their junction point. Smooth driving characteristics require that the larger radius be more than 1 times larger than the smaller radius.

The elements of a compound curve shown in Fig. 7.3 are as below:

Tangent lengths

$$t_s = R_S \tan\frac{\Delta_S}{2} \quad ...(7.17)$$

$$t_L = R_L \tan\frac{\Delta_L}{2} \quad ...(7.18)$$

$$T_s = (t_S + t_L)\frac{\sin\Delta_L}{\sin\Delta} + t_S \quad ...(7.19)$$

$$T_L = (t_S + t_L)\frac{\sin\Delta_S}{\sin\Delta} + t_L \quad ...(7.20)$$

where $\Delta = \Delta_S + \Delta_L$.

Lengths of curves

$$l_s = \frac{\pi R_S \Delta_S}{180} \quad ...(7.21)$$

$$l_L = \frac{\pi R_L \Delta_L}{180} \quad ...(7.22)$$

$$l = l_S + l_L \quad ...(7.23)$$

$$= \frac{\pi}{180}\left(R_S\Delta_S + R_L\Delta_L\right) \qquad ...(7.24)$$

(*iii*) Chainages

$$\text{Chainage of } T_1 = \text{Chainage of P.I.} - T_S \qquad ...(7.25)$$

$$\text{Chainage of } T_3 = \text{Chainage of } T_1 + l_S \qquad ...(7.26)$$

$$\text{Chainage of } T_2 = \text{Chainage of } T_3 + l_L \qquad ...(7.28)$$

7.4 REVERSE CURVES

A reverse curve is one in which two circular curves of same or different radii have their centre of curvature on the opposite sides of the common tangent (Fig. 7.4).Two straights to which a reverse curve connects may be parallel or non-parallel.

We have the following cases:

Case-1: Non-parallel straights when $R_1 = R_2 = R$, and $\Delta_2 > \Delta_1$ ($\Delta = \Delta_2 - \Delta_1$) given Δ_1, Δ_2, d, and chainage of I

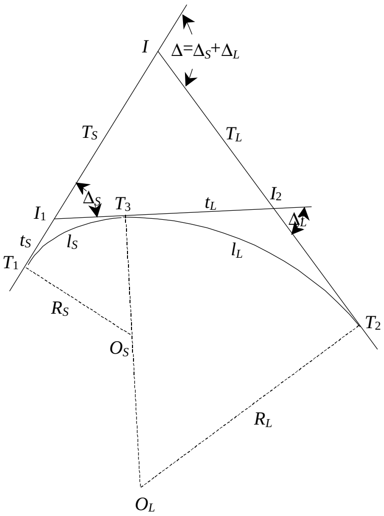


Fig.7.3

$$R = \frac{d}{\tan\frac{\Delta_1}{2} + \tan\frac{\Delta_2}{2}} \qquad ...(7.29)$$

where d = the length of the common tangent.

$$\text{Chainage of } T_1 = \text{Chainage of P.I.} - \left(R\tan\frac{\Delta}{2} + d\,\frac{\sin\Delta_2}{\sin\Delta}\right) \qquad ..(7.30)$$

$$\text{Chainage of } T_3 = \text{Chainage of } T_1 + \frac{\pi R\Delta_1}{180} \qquad ...(7.31)$$

$$\text{Chainage of } T_2 = \text{Chainage of } T_3 + \frac{\pi R \Delta_2}{180} \quad \text{...(7.32)}$$

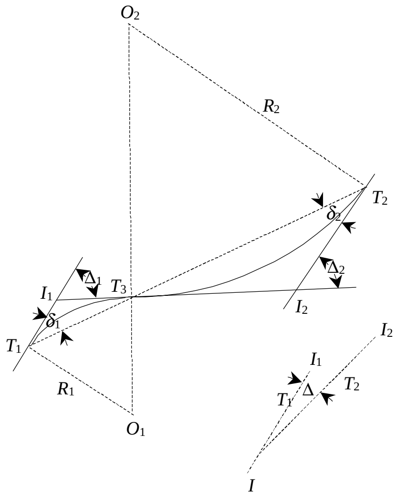


Fig. 7.4

Case-2: Non-parallel straights when $R_1 = R_2 = R$, given δ_1, δ_2, and L

$$R = \frac{L}{\sin\delta_1 + 2\cos\theta + \sin\delta_2} \quad \text{...(7.33)}$$

where
$$\theta = \sin^{-1}\left(\frac{\cos\delta_1 + \cos\delta_2}{2}\right)$$

$$\Delta_1 = \delta_1 + (90° - \theta)$$

$$\Delta_2 = \delta_2 + (90° - \theta).$$

Case-3: Non-parallel straights when $R_1 = R_2$, given δ_1, δ_2, L and R_1 (or R_2)

$$R_2 = \frac{L^2 - 2LR_1\sin\delta_1}{2L\sin\delta_2 + 4R_1\sin^2\left(\frac{\delta_1 - \delta_2}{2}\right)} \quad \text{...(7.35)}$$

Case-4: Parallel straights when $\Delta_1 = \Delta_2$, given R_1, R_2, and Δ_1 $(=\Delta_2)$ (Fig. 7.5)

$$D = 2(R_1 + R_2)\sin^2\frac{\Delta_1}{2} \quad \text{...(7.36)}$$

$$L = \sqrt{[2D(R_1 + R_2)]} \quad \text{...(7.37)}$$

$$H = (R_1 + R_2)\sin\Delta_1 \quad \text{...(7.38)}$$

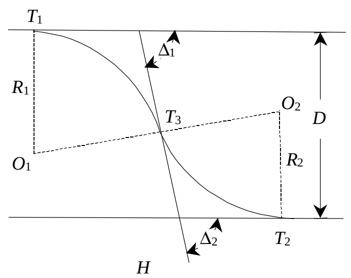


Fig. 7.5

7.5 TRANSITION CURVES

Transition curves permit gradual change of direction from straight to curve and *vice-versa*, and at the same time gradual introduction of cant or superelevation (raising of the outer edge over the inner). Transition curve is also required to be introduces between two circular curves of different radii. The radius of transition curve at its junction with the straight is infinity, i.e., that of the straight, and at the junction with the circular curve that of the circular curve (Fig. 7.6).

A clothoid is a curve whose radius decreases linearly from infinity to zero. It fulfills the condition of an ideal transition curve, i.e.,

$$rl = \text{constant} = RL = K \quad ...(7.39)$$

For a transition curve, we have

$$\text{Deflection angle of curve} = \phi_1 = \frac{L}{2R} \text{ radians} \quad ...(7.40)$$

$$\text{Deflection angle of a specific chord} = \frac{1800 l^2}{\pi RL} \text{ minutes} \quad ...(7.41)$$

$$\delta_n = \frac{1800L}{\pi R} \text{ minutes} \quad ...(7.42)$$

$$= \frac{\phi_1}{3} \text{ radians (when } \phi_1 \text{ is small)} \quad ...(7.43)$$

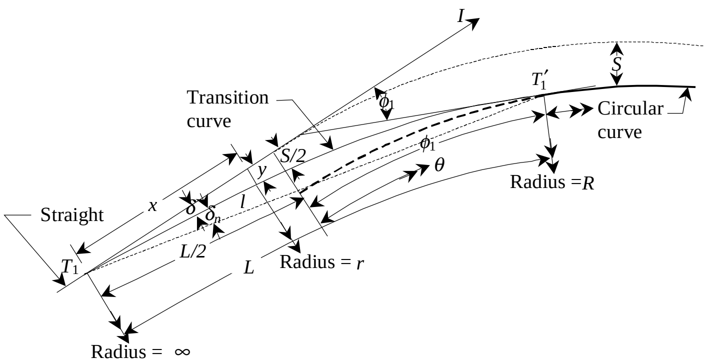


Fig. 7.6

Offsets from tangent

$$y \approx l,\; x = \frac{l^3}{6RL} \text{ (Cubic spiral)} \quad ...(7.44)$$

$$= \frac{y^3}{6RL} \text{ (Cubic parabola)} \quad ...(7.45)$$

Shift of circular curve
$$S = \frac{L^2}{24R} \quad ...(7.46)$$

Tangent length
$$IT_1 = (R+S)\tan\frac{\theta}{2} + \frac{L}{2} \quad ...(7.47)$$

Rate of Change of Radial Acceleration

The centrifugal force P acting on a vehicle moving at a velocity of V on a curve shown in Fig. 7.7, having weight W is given by

$$P = \frac{WV^2}{gR} \quad ...(7.48)$$

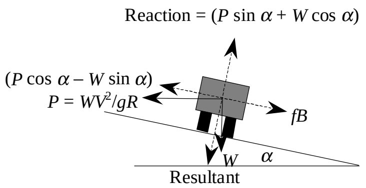


Fig. 7.7

The ratio $\frac{P}{W} = \frac{V^2}{gR}$ is known as the *centrifugal ratio*. By lifting the outer edge of the road or rail, the resultant can be made to act perpendicular to the running surface. In practice to avoid large superelevations, for the amount by which the outer edge is raised, an allowance (fB) for friction is made. In the case of transition curve, radial acceleration given by expression $\frac{V^2}{R}$, changes as the vehicle moves along the curve due to change in radius. For a constant velocity v the rate of change of radial acceleration (assumed uniform) is

$$\alpha = \frac{v^2/R}{L/v}$$

$$= \frac{v^3}{RL} \quad ...(7.49)$$

where v is in m/s.

7.6 VERTICAL CURVES

Vertical curves are introduced at the intersection of two gradients either as summit curves (Fig. 7.8a), or sag curves (Fig. 7.8b).

The requirement of a vertical curve is that it should provide a constant rate of change of grade, and a parabola fulfills this requirement. As shown in Fig. 7.9, for flat gradients it is normal to assume the length of curve ($2L$) equal to the length along the tangents the length of the long chord AB its horizontal projection.

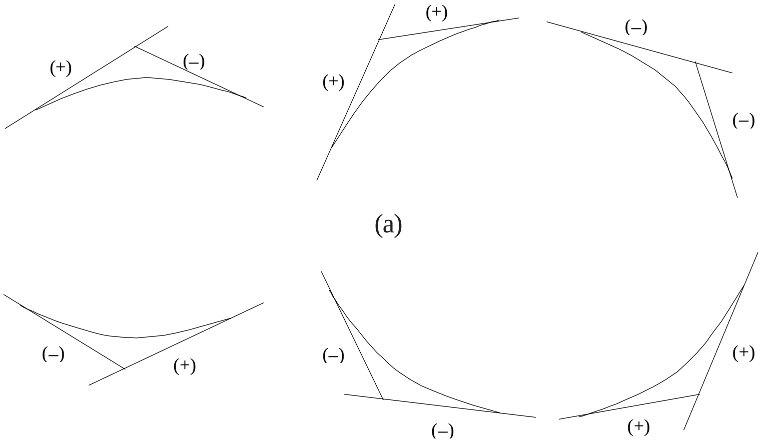


Fig. 7.8

Tangent correction $h = kN^2$ ...(7.50)

where

$$k = \frac{e_1 - e_2}{4n} \qquad \text{...(7.51)}$$

N = the number of chords counted from A (total length $l = 2n$),

n = the number of chords of length l on each side of apex of the curve, and

e_1, e_2 = the rise and fall per chord length of l corresponding to $+g_1$ and $-g_2$, respectively.

Chainage of A = Chainage of apex $C - nl$

Chainage of B = Chainage of apex $C + nl$

Elevation of A = Elevation of apex $C \mp ne_1$ (take – ive when e_1 + ive and *vice-versa*)

Elevation of B = Elevation of apex $C \pm ne_2$ (take + ive when e_2 –ive and *vice-versa*)

Elevation of tangent at any point (n') = Elevation of $A + n'e_1$

Elevation of corresponding point on curve

= Elevation of tangent + tangent correction (algebraically)

For the positive value of K, the tangent correction is negative, and *vice-versa*.

n^{th} chord gradient $= e_1 - (2n - 1)\ k$ ...(7.52)

Elevation of n^{th} point on curve = Elevation of $(n - 1)^{th}$ point + n^{th} chord gradient

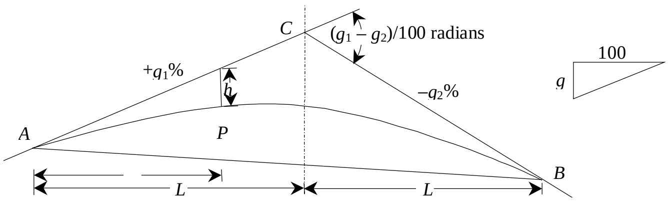


Fig. 7.9

Example 7.1. The chainage of the intersection point of two straights is 1060 m, and the angle of intersection is 120°. If radius of a circular curve to be set out is 570 m, and peg interval is 30 m, determine the tangent length, the length of the curve, the chainage at the beginning and end of the curve, the length of the long chord, the lengths of the sub-chords, and the total number of chords.

Solution (Fig. 7.10):

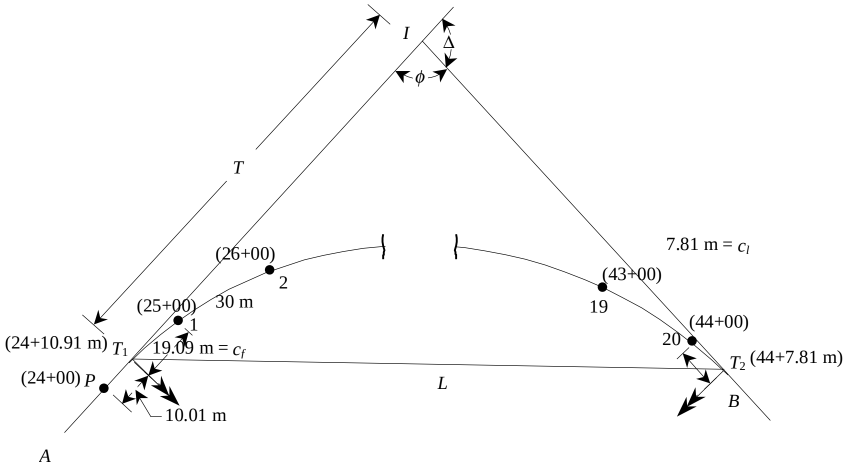


Fig. 7.10

$$\text{Deflection angle } \Delta = 180° - \phi$$
$$= 180° - 120° = 60°$$
$$\frac{\Delta}{2} = 30°$$

(i) Tangent length
$$T = R \tan \frac{\Delta}{2}$$
$$= 570 \times \tan 30° = \mathbf{329.09\ m}$$

(ii) Length of curve
$$l = \frac{\pi R \Delta}{180}$$
$$= \frac{\pi \times 570 \times 60}{180} = \mathbf{596.90\ m}$$

(iii) Chainage of P.I. = 1060 m
= (35 × 30 + 10) m
= 35 Full chain + 10 m
= 35 + 10

$$T = 329.09 \text{ m} = 10 + 29.09$$

$$l = 596.90 \text{ m} = 19 + 26.90$$

$$\text{Chainage of } T_1 = \text{Chainage of P.I.} - T$$

$$= (35 + 10) - (10 + 29.09)$$

$$= \mathbf{24 + 10.91}$$

$$\text{Chainage of } T_2 = \text{Chainage of } T_1 + l$$

$$= (24 + 10.91) + (19 + 26.90) = \mathbf{44 + 7.81}$$

$$\text{Long chord } L = 2R \sin \frac{\Delta}{2}$$

$$= 2 \times 570 \times \sin 30° = \mathbf{570 \text{ m}}.$$

(iv) On the straight *AI*, the chainage of T_1 is (24 + 10.91). Therefore a point *P* having chainage (24 + 00) will be 10.91 m before T_1 on *AI*. Since the peg interval is 30 m, the length of the normal chord is 30 m. The first point 1 on the curve will be at a distance of 30 m from *P* having chainage (25 + 00) and 30 – 10.91 = 19.09 m from T_1. Thus the length of the first sub-chord

= **19.09 m**.

Similarly, the chainage of T_2 being (44 + 7.81) a point *Q* on the curve having chainage of (44 + 00) will be at a distance of 7.81 m from T_2. Thus the length of the last sub-chord

= **7.81 m**.

To calculate the length of the sub-chords directly, the following procedure may be adopted. If chainage of T_1 is $(F_1 + m_1)$ and the length of the normal chord *C* is *m* then

the length of the first sub-chord $c_f = m - m_1$

$= 30 - 10.91 = 19.09$ m.

If the chainage of T_2 is $(F_2 + m_2)$

the chainage of the last sub-chord $c_l = m_2 = 7.81$ m.

(v) The total number of chords $N = n + 2$

where n = Chainage of the last peg – chainage of the first peg

= (44 + 00) – (25 + 00) = 19

Thus $N = 19 + 2 =$ **21**.

Example 7.2. Two straight roads meet at an angle of 130°. Calculate the necessary data for setting out a circular curve of 15 chains radius between the roads by the perpendicular offsets method. The length of one chain is 20 m.

Making the use of the following data, determine the coordinates of P.C., P.T., and apex of the curve.

(a) Coordinates of a control point *X* = *E* 1200 m, *N* 1500 m

(b) Distance of *X* from P.I. = 100 m

(c) Bearing of line joining *IX* = 320°

(d) Angle between *IX* and back tangent = 90°.

Solution (Fig. 7.11):

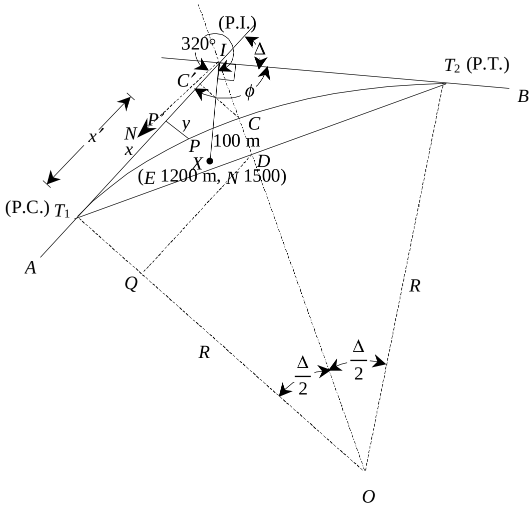


Fig. 7.11

The given data are

$$R = 15 \text{ chains} = 15 \times 20 = 300 \text{ m}$$

$$\Delta = 180° - 130° = 50°$$

$$\frac{\Delta}{2} = 25°$$

(a) The value of x' to fix the apex C of the curve, is determined from ΔOQC

$$x_C = QC = R \sin\frac{\Delta}{2}$$

$$= 300 \times \sin 25°$$

$$= 126.79 \text{ m.}$$

The maximum value of x being 126.79 m, the offsets are to be calculated for

$$x = 20, 40, 60, 80, 100, 120, \text{ and } 126.79 \text{ m.}$$

The perpendicular offsets are calculated from

$$y = R - \sqrt{(R^2 - x^2)}$$

Thus

$$y_{20} = 300 - \sqrt{(300^2 - 20^2)} = 0.67 \text{ m}$$

$$y_{40} = 300 - \sqrt{(300^2 - 40^2)} = 2.68 \text{ m}$$

$$y_{60} = 300 - \sqrt{(300^2 - 60^2)} = 6.06 \text{ m}$$

$$y_{80} = 300 - \sqrt{(300^2 - 80^2)} = 10.86 \text{ m}$$

$$y_{100} = 300 - \sqrt{(300^2 - 100^2)} = 17.16 \text{ m}$$

$$y_{120} = 300 - \sqrt{(300^2 - 120^2)} = 25.05 \text{ m}$$

$$y_{126.79} = 300 - \sqrt{(300^2 - 126.79^2)} = 28.11 \text{ m}$$

(*b*) Tangent length $T = T_1C = R \tan \dfrac{\Delta}{2}$

$= 300 \times \tan 25° = 139.892$ m

External distance $E = IC = R\left(\sec \dfrac{\Delta}{2} - 1\right)$

$= 300 \times (\sec 25° - 1) = 31.013$ m

Bearing of IT_1 = Bearing $IX - \angle XIT_1$

$= 320° - 90° = 230°$

Bearing of IC = Bearing of $IT_1 - \dfrac{\phi}{2}$

$= 230° - \dfrac{130°}{2} = 165°$

Bearing of $XI = (180° + 320°) - 360° = 140°$

Bearing of IT_2 = Bearing of $IT_1 - \phi$

$= 230° - 130° = 100°$

Coordinates of I

Departure of $XI = D_{XI} = 100 \times \sin 140° = +\ 64.279$ m

Latitude of $XI = L_{XI} = 100 \times \cos 140° = -\ 76.604$ m

Easting of $I = E_I$ = Easting of $X + D_{XI} = 1200 + 64.279$

$= E$ 1264.279 m ≈ ***E* 1264.28 m**

Northing of $I = N_I$ = Northing of $X + L_{XI} = 1500 - 76.604$

$= N$ 1423.396 m ≈ ***N* 1423.40 m**

Coordinates of T_1

Departure of $IT_1 = D_{IT1} = 139.892 \times \sin 230° = -\ 107.163$ m

Latitude of $IT_1 = L_{IT1} = 139.892 \times \cos 230° = -\ 89.921$ m

Easting of $T_1 = E_{T1}$ = Easting of $I + D_{IT1} = 1264.279 - 107.163$

$= E$ 1157.116 m ≈ ***E* 1157.12 m**

$$\text{Northing of } T_1 = N_{T1} = \text{Northing of } I + L_{IT1} = 1423.396 - 89.921$$
$$= N\ 1333.475 \text{ m} \approx \mathbf{N\ 1333.48\ m}$$

Coordinates of C

$$\text{Departure of } IC = D_{IC} = 31.013 \times \sin 165° = +\ 8.027 \text{ m}$$
$$\text{Latitude of } IC = L_{IC} = 31.013 \times \cos 165° = -\ 29.956 \text{ m}$$
$$\text{Easting of } C = E_C = \text{Easting of } I + D_{IC} = 1264.279 + 8.027$$
$$= E\ 1272.306 \text{ m} \approx \mathbf{E\ 1272.31\ m}$$
$$\text{Northing of } C = N_C = \text{Northing of } I + L_{IC} = 1423.396 - 29.956$$
$$= N\ 1393.440 \text{ m} \approx \mathbf{N\ 1393.44\ m}$$

Coordinates of T_2

$$\text{Departure of } IT_2 = D_{IT2} = 139.892 \times \sin 100° = +\ 137.767 \text{ m}$$
$$\text{Latitude of } IT_2 = L_{IT2} = 139.892 \times \cos 100° = -\ 24.292 \text{ m}$$
$$\text{Easting of } T_2 = E_{T2} = \text{Easting of } I + D_{IT2} = 1264.279 + 137.767$$
$$= E\ 1402.046 \text{ m} \approx \mathbf{E\ 1402.05\ m}$$
$$\text{Northing of } T_2 = N_{T2} = \text{Northing of } I + L_{IT2} = 1423.396 - 24.292$$
$$= N\ 1399.104 \text{ m} \approx \mathbf{N\ 1399.10\ m}$$

Checks:

$$\text{Long chord } T_1T_2 = 2R \sin \frac{\Delta}{2}$$
$$= 2 \times 300 \times \sin 25° = 253.57 \text{ m}$$
$$= \sqrt{(E_{T2} - E_{T1})^2 + (N_{T2} - N_{T1})^2}$$
$$= \sqrt{(1402.046 - 1157.116)^2 + (1399.104 - 1333.475)^2}$$
$$= \sqrt{244.930^2 + 65.629^2} = 253.57 \text{ m}$$

$$\text{Mid-ordinate } EC = R\left(1 - \cos\frac{\Delta}{2}\right)$$
$$= 300 \times (1 - \cos 25°) = 28.11 \text{ m}$$

$$\text{Easting of } E = E_E = E_{T1} + \frac{1}{2}(E_{T2} - E_{T1})$$
$$= 1157.116 + \frac{1}{2} \times 244.930 = 1279.581 \text{ m}$$

$$\text{Northing of } N = N_E = N_{T1} + \frac{1}{2}(N_{T2} - N_{T1})$$
$$= 1333.475 + \frac{1}{2} \times 65.629 = 1366.290 \text{ m}$$

$$EC = \sqrt{(E_E - E_C)^2 + (N_E - N_C)^2}$$
$$= \sqrt{(1279.581 - 1272.36)^2 + (1366.290 - 1393.44)^2}$$
$$= \sqrt{7.275^2 + (-27.151)^2}$$
$$= 28.11 \text{ m.} \quad (\textit{Okay})$$

Example 7.3. A circular curve of 250 m radius is to be set out between two straights having deflection angle of 45°20′ right, and chainage of the point of intersection as 112 + 10. Calculate the necessary data for setting out the curve by the method of offsets from the chords produced taking the length of one chain as 20 m.

Solution (Fig. 7.2):

$$\Delta = 45°20' = 45.333°$$
$$\frac{\Delta}{2} = 22°40'$$
$$\text{Tangent length } T = R \tan\frac{\Delta}{2}$$
$$= 250 \times \tan 22°40' = 104.41 \text{ m}$$
$$\text{Length of curve } l = \frac{\pi R \Delta}{180}$$
$$= \frac{\pi \times 250 \times 45.333}{180} = 197.80 \text{ m}$$
$$\text{Chainage of } T_1 = \text{Chainage of P.I.} - T$$
$$= (112 + 10) - 104.41$$
$$= (112 \times 20 + 10) - 104.41$$
$$= 2145.59 \text{ m} = 107 + 5.59$$
$$\text{Chainage of } T_2 = \text{Chainage of } T_1 + l$$
$$= 2145.59 + 197.80$$
$$= 2343.39 \text{ m} = 117 + 3.39.$$
$$\text{Length of first sub-chord } c_f = (107 + 20) - (107 + 5.59) = 14.41 \text{ m}$$
$$\text{Length of last sub-chord } c_l = (117 + 3.39) - (117 + 0) = 3.39 \text{ m}$$
$$\text{Number of normal chords } N = 117 - 108 = 9$$
$$\text{Total number of chords } n = 9 + 2 = 11.$$

Offsets from chords produced

$$O_1 = \frac{c_f^2}{2R} = \frac{14.41^2}{2 \times 250} = \mathbf{0.42\ m}$$

$$O_2 = \frac{C(c_f + C)}{2R}$$

$$= \frac{20 \times (14.41 + 20)}{2 \times 250} = \mathbf{1.38\ m}$$

$$O_3 \text{ to } O_{10} = \frac{C^2}{R}$$

$$= \frac{20^2}{250} = \mathbf{1.60\ m}$$

$$O_{11} = \frac{c_l(c_l + C)}{2R}$$

$$= \frac{3.39 \times (3.39 + 20)}{2 \times 250} = \mathbf{0.16\ m}.$$

The chainages of the points on the curve, chord length, and offsets are given in Table 7.1.

Table 7.1

Point	Chainage	Chord length (m)	Offset (m)	Remarks
0	107+5.59	14.41	–	T_1 (P.C.)
1	108+00	20	0.42	
2	109+00	20	1.38	
3	110+00	20	1.60	
4	111+00	20	1.60	
5	112+00	20	1.60	
6	113+00	20	1.60	
7	114+00	20	1.60	
8	115+00	20	1.60	
9	116+00	20	1.60	
10	117+00	20	1.60	
11	117+3.39	3.39	0.16	T_2 (P.T.)

Example 7.4. A circular curve of radius of 17.5 chains deflecting right through 32°40′, is to be set out between two straights having chainage of the point of intersection as (51 + 9.35). Calculate the necessary data to set out the curve by the method of deflection angles. Also calculate the necessary data indicating the use of the control points shown in Fig. 7.12. The length of one chain is 20 m. Present the values of the deflection angles to be set out using a theodolite of least count (a) 20″, and (b) 1″ in tabular form.

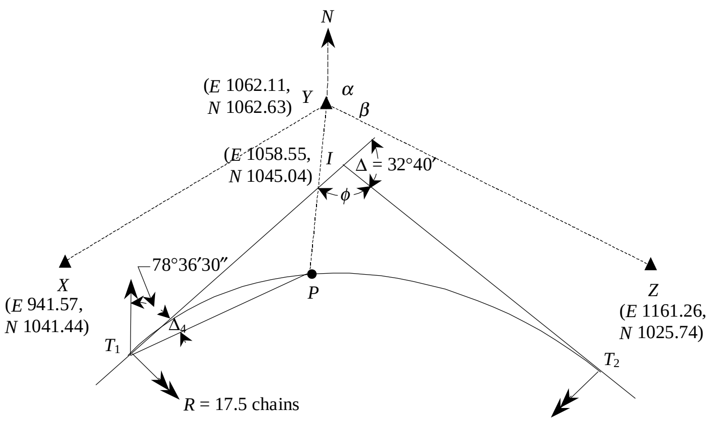


Fig. 7.12

Solution (Fig. 7.12):

$$R = 17.5 \times 20 = 350 \text{ m}$$

$$\Delta = 32°40' = 32.667°$$

$$\frac{\Delta}{2} = 16°20'$$

$$\text{Tangent length } T = R \tan \frac{\Delta}{2}$$

$$= 350 \times \tan 16°20' = 102.57 \text{ m}$$

$$\text{Length of curve } l = \frac{\pi R \Delta}{180}$$

$$= \frac{\pi \times 250 \times 32.667}{180} = 199.55 \text{ m}$$

$$\text{Chainage of } T_1 = \text{Chainage of P.I.} - T$$

$$= (51 + 9.35) - 102.57$$

$$= (51 \times 20 + 9.35) - 102.57$$

$$= 926.78 \text{ m} = 46 + 6.78$$

$$\text{Chainage of } T_2 = \text{Chainage of } T_1 + l$$

$$= 926.78 + 199.55 = 1126.33 \text{ m}$$

$$= 56 + 6.33.$$

$$\text{Length of first sub-chord } c_f = (46 + 20) - (46 + 6.78) = 13.22 \text{ m}$$

$$\text{Length of last sub-chord } c_l = (56 + 6.33) - (56 + 0) = 6.33 \text{ m}$$

$$\text{Number of normal chords } N = 56 - 47 = 9$$

$$\text{Total number of chords } n = 9 + 2 = 11.$$

Coordinates of T_1 and T_2

$$\text{Bearing of } IT_1 = \alpha = 180° + \text{bearing of } T_1I$$
$$= 180° + 78°36'30''$$
$$= 258°36'30''$$
$$\text{Bearing of } IT_2 = \beta = \text{Bearing of } IT_1 - \phi$$
$$= \text{Bearing of } IT_1 - (180° - \Delta)$$
$$= 258°36'30'' - (180° - 32°40')$$
$$= 111°16'30''$$

Coordinates of T_1

$$\text{Easting of } T_1 = E_{T1} = \text{Easting of } I + T \sin \alpha$$
$$= 1058.55 + 102.57 \times \sin 258°36'30''$$
$$= E\ 958.00 \text{ m}$$
$$\text{Northing of } T_1 = N_{T1} = \text{Northing of } I + T \cos \alpha$$
$$= 1045.04 + 102.57 \times \cos 258°36'30''$$
$$= N\ 1024.78 \text{ m}$$

Coordinates of T_2

$$\text{Easting of } T_2 = E_{T2} = \text{Easting of } I + T \sin \beta$$
$$= 1058.55 + 102.57 \times \sin 111°16'30''$$
$$= E\ 1154.13 \text{ m}$$
$$\text{Northing of } T_2 = N_{T2} = \text{Northing of } I + T \cos \beta$$
$$= 1045.04 + 102.57 \times \cos 111°16'30''$$
$$= N\ 1007.812 \text{ m.}$$

Tangential angles

$$\delta = 1718.9 \frac{c}{R} \text{ minutes}$$

$$\delta_1 = 1718.9 \frac{13.22}{350} = 64.925'$$

$$\delta_2 \text{ to } \delta_{10} = 1718.9 \frac{20}{350} = 98.223'$$

$$\delta_{11} = 1718.9 \frac{6.33}{350} = 31.088'$$

Deflection angles

$$\Delta_1 = \delta_1 = 64.925' = 1°04'55''$$
$$\Delta_2 = \Delta_1 + \delta_2 = 64.925' + 98.223' = 163.148' = 2°43'09''$$

$$\Delta_3 = \Delta_2 + \delta_3 = 163.148' + 98.223' = 261.371' = 4°21'22''$$
$$\Delta_4 = \Delta_3 + \delta_4 = 261.371' + 98.223' = 359.594' = 5°59'36''$$
$$\Delta_5 = \Delta_4 + \delta_5 = 359.594' + 98.223' = 457.817' = 7°37'39''$$
$$\Delta_6 = \Delta_5 + \delta_6 = 457.817' + 98.223' = 556.040' = 9°16'02''$$
$$\Delta_7 = \Delta_6 + \delta_7 = 556.040' + 98.223' = 654.263' = 10°54'16''$$
$$\Delta_8 = \Delta_7 + \delta_8 = 654.263' + 98.223' = 752.486' = 12°32'29''$$
$$\Delta_9 = \Delta_8 + \delta_9 = 752.486' + 98.223' = 850.709' = 14°10'43''$$
$$\Delta_{10} = \Delta_9 + \delta_{10} = 850.709' + 98.223' = 948.932' = 15°48'56''$$
$$\Delta_{11} = \Delta_{10} + \delta_{11} = 948.932' + 31.088' = 980.020' = 16°20'00''$$

Check: $\Delta_{11} = \dfrac{\Delta}{2} = 16°20'$ (*Okay*)

The deflection angles for theodolites having least counts of 20″ and 1″, respectively, are given in Table 7.2.

Table 7.2

Point	Chainage	Chord length (m)	Tangential angle (′)	Deflection angle (′)	Angle set on 20″ theodolite	Angle set on 1″ theodolite
0 (T_1)	46+6.78	–	0.0	0.0	0.0	0.0
1	47+00	13.22	64.925	64.925	1°05′00″	1°04′55″
2	48+00	20	98.233	163.148	1°43′00″	1°43′09″
3	49+00	20	98.233	261.371	4°21′20″	4°21′22″
4	50+00	20	98.233	359.594	5°59′40″	5°59′36″
5	51+00	20	98.233	457.817	7°37′40″	7°37′49″
6	52+00	20	98.233	556.040	9°16′00″	9°16′02″
7	53+00	20	98.233	654.263	10°54′20″	10°54′16″
8	54+00	20	98.233	752.486	12°32′20″	12°32′29″
9	55+00	20	98.233	850.709	14°10′40″	14°10′43″
10	56+00	20	98.233	948.932	15°49′00″	15°48′56″
11(T_2)	56+6.33	6.33	31.088	980.0230	16°20′00″	16°20′00″

Use of Control Points

To make the use of control points to set out the curve, the points on the curve can be located by setting up the theodolite at any control point, and setting of bearing of a particular point on the theodolite, and measuring the length of the point from the control point. From the figure we find that to set out the complete curve, use of control point *Y* would be most appropriate.

Let us take a point *P* having chainage 50 + 00 = 50 × 20 + 00 = 1000.00 m, and calculate the necessary data to locate it from *Y*.

$$\text{Bearing of } T_1P = \text{Bearing of } T_1I + \angle IT_1P$$
$$= \text{Bearing of } T_1I + \Delta_4$$
$$= 78°36'30'' + 5°59'36'' = 84°36'06''$$
$$\text{Length } T_1P = 2R \sin \Delta_4$$
$$= 2 \times 350 \times \sin 5°59'36'' = 73.09 \text{ m}$$
$$\text{Departure of } T_1P = D_{T1P} = 73.09 \times \sin 84°36'06'' = 72.77 \text{ m}$$
$$\text{Latitude of } T_1P = L_{T1P} = 73.09 \times \cos 84°36'06'' = 6.88 \text{ m}$$
$$\text{Easting of } P = E_P = E_{T1} + D_{T1P}$$
$$= 958.00 + 72.77 = 1030.77 \text{ m}$$
$$\text{Northing of } P = N_P = N_{T1} + L_{T1P}$$
$$= 1024.78 + 6.88 = 1031.66 \text{ m}$$
$$\text{Length } YP = \sqrt{(E_P - E_Y)^2 + (N_E - N_Y)}$$
$$= \sqrt{(1030.77 - 1062.11)^2 + (1031.66 - 1062.63)^2}$$
$$= \sqrt{(-31.34)^2 + (-30.97)^2}$$
$$= 44.06 \text{ m}$$
$$\text{Bearing of } YP = \tan^{-1}\left(\frac{-31.34}{-30.97}\right) = 45°30'25''$$
$$\text{WCB of } YP = 180° + 45°30'25'' = 225°20'25''$$
$$\text{Bearing of } YZ = \tan^{-1}\left(\frac{E_Z - E_Y}{N_Z - N_Y}\right)$$
$$= \tan^{-1}\left(\frac{1161.26 - 1062.11}{1025.74 - 1062.63}\right)$$
$$= \tan^{-1}\left(\frac{99.15}{-36.89}\right) = 69°35'30''$$
$$\text{WCB of } YZ = 90° + 69°35'30'' = 110°24'30''$$
$$\angle PYZ = \text{WCB of } YP - \text{WCB of } YZ$$
$$= 225°20'25'' - 110°24'30'' = 114°55'55''.$$

The surveyor can sight Z, and P can be located by setting off an angle 114°55′55″ on the horizontal circle of the theodolite, and measuring distance YP equal to 44.06 m. For other points similar calculations can be done to locate them from Y.

Example 7.5. Points *A*, *B*, *C*, and *D* lie on two straights as shown in Fig. 7.13, having coordinates given in Table 7.3. The chainage of *A* is 1216.165 m. Calculate necessary data to set out a circular curve of radius 220 m from a point *P* (*E* 1114.626 m, *N* 710.012 m) at through chainage interval of 20 m.

Table 7.3

Point	Easting (m)	Northing (m)
A	935.922	657.993
B	1051.505	767.007
C	1212.840	778.996
D	1331.112	712.870

Solution (Fig. 7.13):

Coordinates of *I*

Let the coordinates of *I* be E_I and N_I. The equation of a straight line is

$$Y = mx + c$$

At the intersection *I*, we have

$$m_1E_1 + c_1 = m_2E_1 + c_2$$

$$E_1 = \frac{c_2 - c_1}{m_1 - m_2}$$

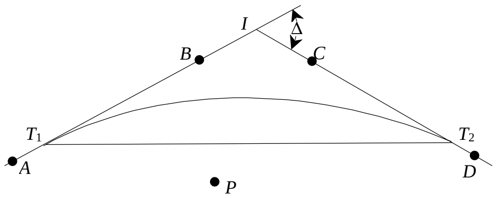


Fig. 7.13

But $$m_1 = \frac{N_B - N_A}{E_B - E_A}$$

$$= \frac{767.007 - 657.993}{1051.505 - 935.922}$$

$$= \frac{109.014}{115.583} = 0.9432$$

$$m_2 = \frac{N_D - N_C}{E_D - E_C}$$

$$= \frac{712.870 - 778.996}{1331.112 - 1212.840}$$

$$= \frac{-66.126}{118.272} = -\ 0.5591$$

At Point *A* $$657.993 = 0.9432 \times 935.922 + C_1$$

$$C_1 = -\ 224.769$$

At Point D $\quad 712.870 = -0.5591 \times 1331.112 + C_2$

$C_2 = 1457.095$

Thus the coordinates of I are

$$E_I = \frac{1457.095 + 224.769}{0.9432 + 0.5591} = 1119.526 \text{ m}$$

$$N_I = 0.9432 \times 1119.526 - 224.769$$

$$= 831.168 \text{ m}$$

The deflection angle

$$\Delta = \tan^{-1}(m_1) - \tan^{-1}(m_2)$$

$$= \tan^{-1}(0.9432) - \tan^{-1}(-0.5591)$$

$$= 43°19'33'' - (-29°12'34'')$$

$$= 72°32'07''$$

$$\frac{\Delta}{2} = 36°16'03.5''.$$

$$\text{Tangent length } (T) = R \tan \frac{\Delta}{2}$$

$$= 220 \times \tan 36°16'03.5'' = 161.415 \text{ m}$$

$$\text{Length of curve } (l) = \frac{\pi R \Delta}{180}$$

$$= \frac{\pi \times 200 \times 72°32'07''}{180} = 278.515 \text{ m.}$$

$$\text{Length } IA = \sqrt{\left[(E_A - E_I)^2 + (N_A - N_I)^2\right]}$$

$$= \sqrt{\left[(935.922 - 1119.526)^2 + (657.993 - 831.168)^2\right]}$$

$$= \sqrt{\left[(-183.604)^2 + (-173.175)^2\right]}$$

$$= 44.06 \text{ m.}$$

Thus chainage of I = Chainage of A + IA

= 1216.165 + 252.389 = 1468.554 m

Thus chainage of T_1 = Chainage of I + T

= 1468.554 – 161.415

= 1307.139 m = 65 + 7.139.

$$\text{Thus chainage of } T_2 = \text{Chainage of } T_2 + l$$
$$= 1307.139 + 278.515$$
$$= 1585.654 \text{ m} = 79 + 5.654.$$

$$\text{Bearing of } AI\ \theta = \tan^{-1}\left(\frac{E_I - E_A}{N_I - N_A}\right)$$
$$= \tan^{-1}\left(\frac{183.604}{173.175}\right) = 46°40'28''.$$

$$\text{Length of } AT_1 = AI - T$$
$$= 252.389 - 161.415 = 90.974 \text{ m}.$$

$$\text{Easting of } T_1 = E_{T1} = \text{Easting of } A + AT_1 \sin\theta$$
$$= 935.922 + 90.974 \times \sin 46°40'28''$$
$$= 1002.103 \text{ m}.$$

$$\text{Northing of } T_1 = N_{T1} = \text{Northing of } A + AT_1 \cos\theta$$
$$= 657.993 + 90.974 \times \cos 46°40'28''$$
$$= 720.414 \text{ m}.$$

Let the centre of the curve be O.

$$\text{Bearing of } T_1O = \alpha = \text{Bearing of } T_1I + 90°$$
$$= \text{Bearing of } AI + 90°$$
$$= 46°40'28'' + 90° = 136°40'28''.$$

Coordinates of O

$$\text{Easting of } O = E_O = E_{T1} + R \sin\alpha$$
$$= 1002.103 + 220 \times \sin 136°40'28''$$
$$= 1153.054 \text{ m}.$$

$$\text{Northing of } O = N_O = N_{T1} + R \cos\alpha$$
$$= 720.414 + 220 \times \cos 136°40'28''$$
$$= 560.371 \text{ m}.$$

$$\text{Bearing of } OT_1 = \text{Bearing of } T_1O + 180°$$
$$= 136°40'28'' + 180°$$
$$= 316°40'28''.$$

$$\text{Length of first sub-chord } c_f = (65 + 20) - (65 + 7.139)$$
$$= 12.861 \text{ m}.$$

$$\text{Length of last sub-chord } c_l = (79 + 5.654) - (79 + 0)$$
$$= 5.654 \text{ m}.$$

$$\text{Total number of chords } n = (79 - 66) + 2$$
$$= 13 + 2 = 15.$$

Point 1 on the curve

Angle subtended at O by the first sub-chord for point 1

$$= \frac{c_f}{R} \text{ radians}$$

$$= \frac{12.861}{220} \times \frac{180}{\pi} \text{ degree} = 3°20'58''$$

$$\text{Bearing of } O1 = \text{Bearing of } OT_1 + 3°20'58''$$

$$= 316°40'28'' + 3°20'58''$$

$$= 320°01'26''.$$

$$\text{Easting of point 1} = E_1 = E_O + R \sin 320°01'26''$$

$$= 1153.054 + 220 \times \sin 320°01'26''$$

$$= 1011.711 \text{ m.}$$

$$\text{Northing of point 1} = N_1 = N_O + R \cos 320°01'26''$$

$$= 560.371 + 220 \times \cos 320°01'26''$$

$$= 728.960 \text{ m.}$$

$$\text{Length } P1 = \sqrt{(E_1 - E_P)^2 + (N_1 - N_P)^2}$$

$$= \sqrt{(1011.711 - 1114.626)^2 + (728.960 - 710.012)^2}$$

$$= \sqrt{(-102.915)^2 + 18.948^2}$$

$$= 104.64 \text{ m.}$$

$$\text{Bearing of } P1 = \tan^{-1}\left(\frac{183.604}{173.175}\right) = -79°34'05''.$$

The line being in fourth quadrant, the bearing will be 360° – 79°34′05″ = 280°25′55″.

Point 2 on the curve

$$\text{Chord length} = c_f + C = 12.861 + 20 = 32.861 \text{ m.}$$

$$\text{Angle subtended at } O = \frac{32.861}{220} \times \frac{180}{\pi} \text{ degrees}$$

$$= 8°33'29''.$$

$$\text{Bearing of } O2 = 316°40'28'' + 8°33'29''$$

$$= 325°13'57''.$$

$$\text{Easting of point 2} = E_2 = E_O + R \sin 325°13'57''$$

$$= 1153.054 + 220 \times \sin 325°13'57''$$

$$= 1027.600 \text{ m.}$$

$$\text{Northing of point 2} = N_2 = N_O + R \cos 320°01'26''$$
$$= 560.371 + 220 \times \cos 320°01'26''$$
$$= 728.960 \text{ m.}$$

$$\text{Length } P2 = \sqrt{(E_2 - E_P)^2 + (N_2 - N_P)^2}$$
$$= \sqrt{(1027.600 - 1114.626)^2 + (741.095 - 710.012)^2}$$
$$= \sqrt{(-87.026)^2 + 31.083^2} = 92.41 \text{ m.}$$

$$\text{Bearing of } P2 = \tan^{-1}\left(\frac{-87.026}{31.083}\right)$$
$$= -70°20'41'' = 289°39'19''.$$

Point 3 on the curve

$$\text{Chord length} = c_f + C + C = 12.861 + 20 + 20 = 52.861 \text{ m.}$$

$$\text{Angle subtended at } O = \frac{52.861}{220} \times \frac{180}{\pi} \text{ degrees}$$
$$= 13°46'01''.$$

$$\text{Bearing of } O3 = 316°40'28'' + 13°46'01''$$
$$= 330°26'29''.$$

$$\text{Easting of point 3} = E_3 = E_O + R \sin 330°26'29''$$
$$= 1153.054 + 220 \times \sin 330°26'29''$$
$$= 1044.525 \text{ m.}$$

$$\text{Northing of point 2} = N_3 = N_O + R \cos 330°26'29''$$
$$= 560.371 + 220 \times \cos 330°26'29''$$
$$= 751.738 \text{ m.}$$

$$\text{Length } P3 = \sqrt{(E_3 - E_P)^2 + (N_3 - N_P)^2}$$
$$= \sqrt{(1044.525 - 1114.626)^2 + (7851.738 - 710.012)^2}$$
$$= \sqrt{(-70.101)^2 + 41.726^2} = 81.579 \text{ m.}$$

$$\text{Bearing of } P3 = \tan^{-1}\left(\frac{-70.101}{41.726}\right)$$
$$= -59°14'16'' = 300°45'44''.$$

In similar manner as above, the necessary data for the remaining points have been calculated, and are presented in Table 7.3.

Table 7.3

Point	Chainage	Chord length (m)	Coordinates		Bearing of *Pp* (*p*=1, 2, 3,..)	Distance of *Pp* (*p*=1, 2, 3,..)	Bearing of *Op* (*p*=1, 2, 3,..)	Angle subtended by T_1p (*p*=1, 2, 3...)
			E (m)	*N* (m)				
0–T_1	65+7.139	–	1002.103	720.414	275°16′54″	113.003	316°40′28″	–
1	66+00	12.861	1011.711	728.960	280°25′55″	106.640	320°01′26″	3°20′58″
2	67+00	32.861	1027.600	741.095	289°39′19″	92.410	325°13′57″	8°33′29″
3	68+00	52.861	1044.525	751.738	300°45′44″	81.579	330°26′29″	13°46′01″
4	69+00	72.861	1062.346	760.801	314°10′16″	72.888	335°39′00″	18°58′32″
5	70+00	92.861	1080.916	768.208	329°55′06″	67.254	340°51′31″	24°11′03″
6	71+00	112.861	1100.083	773.899	347°10′33″	65.521	346°04′03″	29°23′35″
7	72+00	132.861	1119.686	777.826	4°16′02″	68.003	351°16′34″	34°36′06″
8	73+00	152.861	1139.565	779.957	19°37′26″	74.258	356°29′05″	39°48′37″
9	74+00	172.861	1159.556	780.275	32°35′49″	83.400	1°41′47″	45°01′09″
10	75+00	192.861	1179.493	778.777	43°19′45″	94.532	6°54′08″	50°13′40″
11	76+00	212.861	1199.212	775.474	52°15′54″	106.958	12°06′40″	55°26′12″
12	77+00	232.861	1218.549	770.396	59°50′29″	120.192	17°09′11″	60°38′43″
13	78+00	252.861	1237.345	763.583	66°25′02″	133.902	22°31′42″	65°51′14″
14	79+00	272.861	1255.446	755.091	72°14′57″	147.859	27°44′14″	71°03′46″
15–T_2	79+5.654	278.515	1260.416	752.396	73°47′23″	151.826	29°12′35″	72°32′07″ = Δ (*Okay*)

As a check, the bearing of T_1 and T_2, and their distances from P have also been calculated, and presented in the above Table.

A further check on the computations can be applied by calculating the coordinates of T_2 directly as below.

$$\angle IT_1T_2 = \frac{\Delta}{2} = 36°16'03.5''$$

$$\text{Bearing of } T_1T_2 = \theta' = \text{Bearing of } AI + \frac{\Delta}{2}$$

$$= 46°40'28'' + 36°16'03.5'' = 82°56'31.5''.$$

$$\text{Length of } T_1T_2 = 2R \sin \frac{\Delta}{2}$$

$$= 2 \times 220 \times \sin 36°16'03.5'' = 260.285 \text{ m.}$$

Thus

$$E_{T2} = E_{T1} + T_1T_2 \sin \theta'$$

$$= 1002.103 + 260.285 \sin 82°56'31.5''$$

$$= 1260.416 \quad (Okay)$$

$$N_{T2} = N_{T1} + T_1T_2 \cos \theta'$$

$$= 720.414 + 260.285 \cos 82°56'31.5''$$

$$= 752.396 \quad (Okay).$$

Example 7.6. Two straights *AP* and *QC* meet at an inaccessible point *I*. A circular curve of 500 m radius is to be set out joining the two straights. The following data were collected:

$$\angle APQ = 157°22', \angle CQP = 164°38', PQ = 200 \text{ m.}$$

Calculate the necessary data for setting out the curve by the method of offsets from long chord. The chain to be used is of 30 m length, and the chainage of *P* is (57 + 17.30) chains.

Solution (Fig. 7.14):

In the figure

$$\alpha = 157°22'$$

$$\beta = 164°38'$$

$$\theta_1 = 180° - \alpha = 22°38'$$

$$\theta_2 = 180° - \beta = 15°22'$$

$$\phi = 180° - (\theta_1 + \theta_2)$$

$$= 180° - (22°38' + 15°22') = 142°00'$$

$$\Delta = 180° - \phi$$

$$= 180° - 142°00' = 38°00'$$

$$\frac{\Delta}{2} = 19°00'.$$

From sine rule in ΔPIQ, we have

$$\frac{PI}{\sin\theta_2} = \frac{PQ}{\sin\phi}$$

$$PI = \frac{PQ \sin\theta_2}{\sin\phi}$$

$$= \frac{200 \times \sin 15° 22'}{\sin 142°} = 86.085 \text{ m.}$$

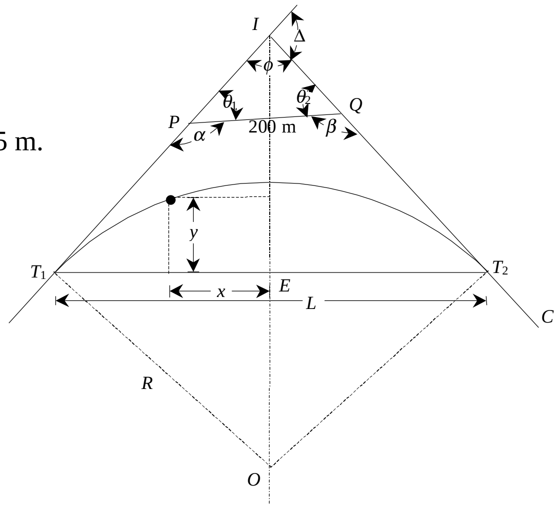


Fig. 7.14

Tangent length $T = R \tan\frac{\Delta}{2}$

$= 500 \times \tan 19°$

$= 172.164$ m.

Length of curve $l = \frac{\pi R \Delta}{180}$

$= \frac{\pi \times 500 \times 38}{180} = 331.613$ m.

Chainage of T_1 = Chainage of $P + PI - T$

$= 57 \times 30 + 17.30 + 86.085 - 172.164$

$= 1641.221$ m

$= (54 + 21.22)$ chains.

Chainage of T_2 = Chainage of $T_1 + l$

$= 1641.221 + 331.613$

$= 1972.834$ m

$= (65 + 22.83)$ chains.

Long chord $= 2R \sin\frac{\Delta}{2}$

$= 2 \times 500 \times \sin 19° = 325.57$ m.

Mid-ordinate $= R\left(1 - \cos\frac{\Delta}{2}\right)$

$= 500 \times (1 - \cos 19°) = 27.24$ m.

Locate E at $\frac{1}{2} T_1 T_2 = \frac{1}{2} L = \frac{1}{2} \times 325.57 = 162.79$ m, and C at 27.24 m on the perpendicular to T_1T_2 at E. To locate other points, the distances to be measured along T_1T_2 from E on either sides of E are

$$x = 0, 30, 60, 90, 120, 150, 162.785 \text{ m.}$$

The ordinates are given by

$$y = \sqrt{\left(R^2 - x^2\right)} - \sqrt{\left(R^2 - \frac{L}{4}\right)^2}$$

$$= \sqrt{\left(500^2 - x^2\right)} - 472.759$$

The calculated values of y are given in Table 7.5.

Table 7.5

Distance x (m)	0	30	60	90	120	150	162.79
Offset y (m)	27.24	26.34	23.63	19.07	12.63	4.21	0.00

Check:

y at x = 0.00 m is equal to M = 27.24 m

y at x = 162.79 m is equal to = 0.00 m.

Example 7.7. Two straights AB and BC are to be connected by a right-hand circular curve. The bearings of AB and BC are 70° and 140°, respectively. The curve is to pass through a point P at a distance of 120 m from B, and the angle ABP is 40°. Determine

(*i*) Radius of the curve,

(*ii*) Chainage of the tangent points,

(*iii*) Total deflection angles for the first two pegs.

Take the peg interval and the length of a normal chord as 30 m. The chainage of the *P.I.* is 3000 m.

Solution (Fig. 7.15):

From the given data, we have

Bearing of AB = 70°

Bearing of BC = 140°

$\angle ABP = \alpha = 40°$

BP = 120 m.

Thus the deflection angle Δ = 140° – 70° = 70°

$$\frac{\Delta}{2} = 35°.$$

If $\angle POT_1 = \theta$, the following expression can be derived for θ.

$$\theta = \cos^{-1} \frac{\cos\left(\frac{\Delta}{2} + \alpha\right)}{\cos\frac{\Delta}{2}} - \alpha$$

$$= \cos^{-1}\frac{\cos(35° + 40°)}{\cos 35°} - 40°$$

$$= 31°34'52.46''.$$

The radius of the curve passing through P which is at a distance of x from B along the tangent BT_1, is given by

$$R = \frac{x \tan\alpha}{1 - \cos\theta}$$

$$= \frac{BP\cos\alpha \dfrac{\sin\alpha}{\cos\alpha}}{1-\cos\theta}$$

$$= \frac{BP \sin\alpha}{1-\cos\theta}$$

$$= \frac{120 \times \sin 40°}{1 - \cos 31°34'52.46''}$$

$$= 520.82 \text{ m.}$$

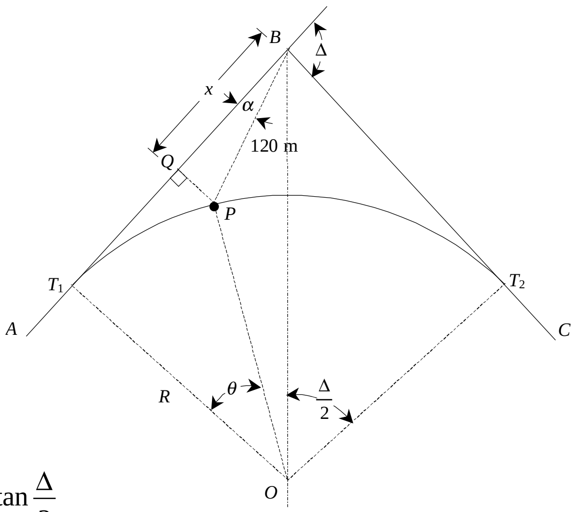


Fig. 7.15

Tangent length $T = R\tan\frac{\Delta}{2}$

$$= 520.82 \times \tan 35° = 364.68 \text{ m.}$$

Length of curve $l = \frac{\pi R\Delta}{180}$

$$= \frac{\pi \times 520.82 \times 70}{180} = 636.30 \text{ m.}$$

Chainage of T_1 = Chainage of $B - T$
= 3000 – 364.68
= 2635.32 m = (87 + 25.32) chains.

Chainage of T_2 = Chainage of $T_1 + l$
= 2635.32 + 636.30
= 3271.62 m = (109 + 1.62) chains.

Length of first sub-chord c_f = (87 + 30) – (87 + 25.32)
= 4.68 m.

Length of last sub-chord c_l = (109 + 1.62) – (109 + 0)
= 1.62 m.

$$\text{Tangential angle} = \frac{1718.9c}{R} \text{ minutes}$$

For the first peg $\delta_1 = \dfrac{1718.9 \times 4.68}{520.82} = 0°15'27''$

For the second peg $\delta_1 = \dfrac{1718.9 \times 30}{520.82} = 1°39'01''$

Total deflection angles

For the first peg $\Delta_1 = \mathbf{0°15'27''}$

For the second peg $\Delta_2 = 0°15'27'' + 1°39'01'' = \mathbf{1°54'28''}$.

Example 7.8. Two straights *AB* and *BC* intersect at chainage 1530.685 m, the total deflection angle being 33°08′. It is proposed to insert a circular curve of 1000 m radius and the transition curves for a rate of change of radial acceleration of 0.3 m/s^3, and a velocity of 108 km/h. Determine setting out data using theodolite and tape for the transition curve at 20 m intervals and the circular curve at 50 m intervals.

Solution (Fig. 7.16):

Given that

Rate of change of radial acceleration $\alpha = 0.3$ m/s^3

$$\text{Velocity } V = 110 \text{ km/h, or } v = \frac{108 \times 1000}{60 \times 60} = 30.000 \text{ m/s}$$

$$\text{Deflection angle } \Delta = 33°08', \text{ or } \frac{\Delta}{2} = 16°34'$$

Radius of circular curve = 1000 m

Chainage of *I* = 1530.685 m

Peg interval for the transition curve = 20 m

Peg interval for the circular curve = 50 m.

$$\alpha = \frac{v^3}{LR}$$

$$L = \frac{v^3}{\alpha R} = \frac{30.000^3}{0.3 \times 1000} = 90.000 \text{ m.}$$

$$\text{Shift } S = \frac{L^2}{24R} = \frac{90.000^2}{24 \times 1000} = 0.338 \text{ m.}$$

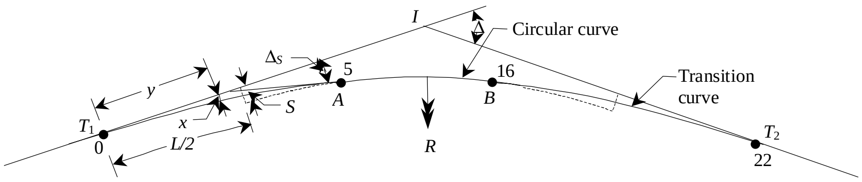


Fig. 7.16

$$\text{Tangent length } IT_1 = (R + S)\tan\frac{\Delta}{2} + \frac{L}{2}$$

$$= (1000 + 0.338) \times \tan 16°34' + \frac{90.000}{2}$$

$$= 342.580 \text{ m.}$$

$$\Delta S = \frac{L}{2R}$$

$$= \frac{90.000}{2 \times 1000} \text{ radians}$$

$$= \frac{90.000}{2 \times 1000} \times \frac{180}{\pi} \text{ degrees} = 2°34'42''.$$

Angle subtended by the circular curve at its centre

$$\theta = \Delta - 2\Delta S$$

$$= 32°24' - 2 \times 2°43'28'' = 27°58'36''.$$

$$\text{Length of circular curve } l = \frac{\pi RA}{180}$$

$$= \frac{\pi \times 1000 \times 27°58'36''}{180}$$

$$= 488.285 \text{ m.}$$

Chainage of T_1 = Chainage of $I - IT_1$

= 1530.685 – 342.580 = 1188.105 m

= (59 + 8.105) chains for peg interval 20 m.

Chainage of A = Chainage of $T_1 + L$

= 1188.105 + 90.000 = 1278.105 m

= (25 + 28.105) chains for peg interval 50 m.

Chainage of B = Chainage of $A + l$

= 1278.105 + 488.285

= 1766.390 m

= (35 + 16.390) chains for peg interval 50 m.

= (88 + 6.390) chains for peg interval 20 m.

Chainage of T_2 = Chainage of $B + L$

= 1766.390 + 90.000 = 1856.390 m

= (92 + 16.390) chains for peg interval 20 m.

Length of first sub-chord for the transition curve

$$c'_f = (59 + 20) - (59 + 8.105) = 11.895 \text{ m.}$$

Length of first sub-chord for the circular curve

$$c_f = (25 + 50) - (25 + 28.105) = 21.895 \text{ m.}$$

Length of last sub-chord for the circular curve

$$c_l = (35 + 16.390) - (35 + 0) = 16.390 \text{ m.}$$

Deflection angle for the transition curve

$$\delta = \frac{1800 l^2}{\pi R L} \text{ minutes}$$

$$= \frac{1800}{\pi \times 1000 \times 90.000} l^2 \text{ minutes}$$

$$= 0.006366 \; l^2 \text{ minutes.}$$

For different values of l, the deflection angles are given in Table 7.5.

Deflection angle for the circular curve

$$\delta = 1718.9 \frac{c}{R} \text{ minutes}$$

$$= \frac{1718.9}{1000} c \text{ minutes}$$

$$= 1.7189 \; c \text{ minutes}$$

Table 7.5

Point	Chainage (m)	Chord length (m)	l (m)	Deflection angle (δ)
0 (T_1)	1188.105	0.0	0.0	0.0
1	1200.000	11.895	11.895	0°00′54″
2	1220.000	20	31.895	0°06′29″
3	1240.000	20	51.895	0°17′09″
4	1260.000	20	71.895	0°32′54″
5 (A)	1278.105	18.105	90.000	0°51′34″

The values of the tangential angles and total deflection angles to be set out at A are given in Table 7.6.

Table 7.6

Point	Chainage (m)	Chord length (m)	Tangential angle (δ)	Deflection angle (Δ)
5 (*A*)	1278.105	0.0	0.0	0.0
6	1300.000	21.895	0°37′38.1″	0°37′38″
7	1350.000	50	1°25′56.7″	2°03′35″
8	1400.000	50	1°25′56.7″	3°29′32″
9	1450.000	50	1°25′56.7″	4°55′28″
10	1500.000	50	1°25′56.7″	6°21′25″
11	1550.000	50	1°25′56.7″	7°47′22″
12	1600.000	50	1°25′56.7″	9°13′18″
13	1650.000	50	1°25′56.7″	10°39′15″
14	1700.000	50	1°25′56.7″	12°05′12″
15	1750.766	50	1°25′56.7″	13°31′08″
16 (*B*)	1766.390	16.390	0°28′10.4″	13°59′19″

The second transition curve is set out from tangent point T_2 by measuring distances in the direction from B to T_2, and deflection angles for the corresponding points from T_2 measured in counterclockwise from T_2I.

Length of first sub-chord for the transition curve measured from B

$$c_f'' = (88 + 20) - (88 + 6.390) = 13.610 \text{ m.}$$

Length of last sub-chord

$$c_l'' = (92 + 16.390) - (92 + 0) = 16.390 \text{ m.}$$

Table 7.7

Point	Chainage (m)	Chord length (m)	l (m)	Deflection angle (δ)
16 (*B*)	1766.390	13.610	90.000	0°51′34″ (from Table 7.5)
17	1780.000	20.000	76.390	0°37′09″
18	1800.000	20.000	56.390	0°20′15″
19	1820.000	20.000	36.390	0°08′26″
20	1840.000	16.390	16.390	0°01′43″
21 (T_2)	1856.390	0.0	0.0	0.0

Number of normal chords of 20 m = 92 – 89 = 3

Deflection angles

For point 17, the chord length from T_2 = 95.097 – 2.234 = 92.863 m

$$\delta_{17} = 0.006025 \times 92.863^2 = 51'57''.$$

For point 18, the chord length from T_2 = 92.863 – 20 = 72.863 m

$$\delta_{18} = 0.006025 \times 72.863^2 = 31'59''.$$

For other points δ can be computed in the similar manner. The data are tabulated inTable-7.7.

Example 7.9. For Example 7.8, determine the offsets required to set out the first transition curve.

Solution (Fig. 7.16):

In the previous example, the length of the transition curve has been determined as

$$L = 90.000 \text{ m}$$

The offsets from the straight T_1I for cubic spiral

$$x = \frac{l^3}{6RL}$$

$$= \frac{l^3}{6 \times 1000 \times 90.000} = \frac{l^3}{540000}$$

Chord length for setting out a transition curve is taken as 1/2 or 1/3 of the corresponding chord length taken for a circular curve which is $R/20$.Therefore, for the circular curve chord length = 1000/20 = 50 m, and for the transition curve 50/2 or 50/3 ≈ 20 m. Thus the given chord lengths in the problem, are justified. The offsets calculated using the above formula, are given in Table 7.8.

Table 7.8

Point	Chainage (m)	Chord length (m)	l (m)	x (m)
0 (T_1)	1188.105	0.0	0.0	0.0
1	1200.000	11.895	11.895	0.003
2	1220.000	20	31.895	0.060
3	1240.000	20	51.895	0.259
4	1260.000	20	71.895	0.688
5 (T_2)	1278.105	18.105	90.000	1.350

Example 7.11. It is proposed to connect two straights, their point of intersection being inaccessible, by a curve wholly transitional. The points *A* and *B* lie on the first straight, and *C* and *D* lie on the second straight. These points were connected by running a traverse *BPQC* between *B* and *C*. The data given in Table 7.9 were obtained.

Table 7.9

Line	Length (m)	Bearing
AB	0.0	3°05′10″
BP	51.50	19°16′50″
PQ	70.86	29°02′10″
QC	66.25	53°25′30″
CD	0.0	66°45′10″

Determine the locations of the tangent points for the design of the rate of change of radial acceleration as 1/3 m/s^3, and the design velocity as 52 km/h.

Solution (Fig. 7.17, 7.19 and 7.20):

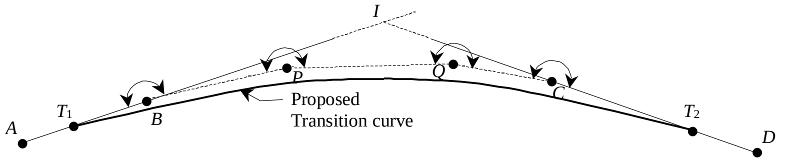


Fig. 7.17

Latitudes and departures of the lines

$$\text{Latitude } L = l \cos \theta$$

$$\text{Departure } D = l \sin \theta$$

$$L_{BP} = 51.50 \times \cos 19°16'50'' = 48.61 \text{ m}$$

$$D_{BP} = 51.50 \times \sin 19°16'50'' = 17.00 \text{ m}$$

$$L_{PQ} = 70.86 \times \cos 29°02'10'' = 61.95 \text{ m}$$

$$D_{PQ} = 70.86 \times \sin 29°02'10'' = 34.39 \text{ m}$$

$$L_{QC} = 66.25 \times \cos 53°25'30'' = 39.48 \text{ m}$$

$$D_{QC} = 66.25 \times \sin 53°25'30'' = 53.20 \text{ m}$$

$$\Sigma L = 150.04 \text{ m}$$

$$\Sigma D = 104.59 \text{ m.}$$

$$\text{Length } BC = \sqrt{150.04^2 + 104.59^2} = 182.90 \text{ m}$$

$$\text{Bearing of } BC\ \theta = \tan^{-1}\left(\frac{150.04}{104.59}\right) = 55°07'13''.$$

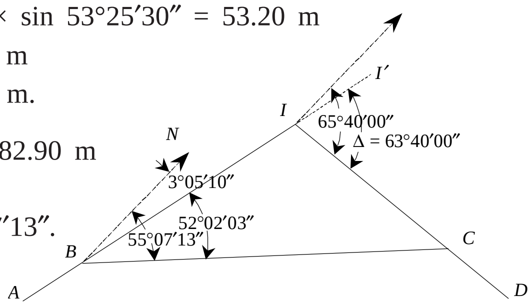


Fig. 7.18

In Fig. 7.18, we have

$$\Delta = \angle I'IC = \text{Bearing of } CD - \text{bearing of } AB$$

$$= \text{Bearing of } IC - \text{bearing of } BI$$

$$= 66°45'10'' - 3°05'10''$$

$$= 63°40'00''.$$

In ΔBIC, we have

$$\angle BIC = 180° - \angle I'IC$$
$$= 180° - 63°40'00'' = 116°20'00''$$
$$\angle IBC = \text{Bearing of } BC - \text{Bearing of } BI$$
$$= 55°07'13'' - 3°05'10'' = 52°02'03''$$
$$\angle BCI = 180° - (\angle BIC + \angle IBC)$$
$$= 180° - (116°20'00'' + 52°02'03'') = 11°37'57''.$$

By sine rule, we get

$$\frac{IB}{\sin BCI} = \frac{BC}{\sin BIC} = \frac{IC}{\sin IBC}$$

$$IB = \frac{BC \sin BCI}{\sin BIC}$$

$$= \frac{182.90 \times \sin 11°37'57''}{\sin 116°20'00''} = 41.15 \text{ m}$$

$$IC = \frac{BC \sin IBC}{\sin BIC}$$

$$= \frac{182.90 \times \sin 52°02'03''}{\sin 116°20'00''} = 160.89 \text{ m}.$$

In $\Delta BI'C$, we have

$$BI' = BC \cos IBC$$
$$= 182.90 \times \cos 52°02'03'' = 112.52 \text{ m}$$
$$CI' = BC \sin IBC$$
$$= 182.90 \times \sin 52°02'03'' = 144.19 \text{ m}.$$

The transitional curve is to be wholly transitional and, therefore, there will be two transition curves meeting at common point T as shown in Fig. 7.19, having the common tangent EF at T. The tangent ET to the first transition curve at T makes angle ϕ_1 with that tangent T_1I or the tangent TF to the second transition curve makes angle ϕ_1 with the tangent IT_2.

$$\phi_1 = \frac{\Delta}{2} = \frac{63°40'00''}{2} = 31°50'00'' = 0.5555965 \text{ radians}.$$

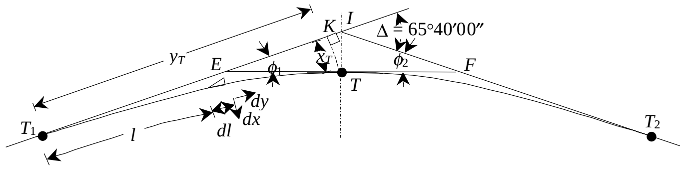


Fig. 7.19

The angle ϕ_1 being of large magnitude, the first order equations used between ϕ_1 and $\frac{\Delta}{2}$ in this case, will not be valid.

Taking $$dx = dl \sin \phi$$

$$dy = dl \cos \phi$$

we have $$dx = \left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!}\right) dl = \left(\frac{l^2}{2k} - \frac{l^6}{48k^3} + \frac{l^{10}}{3840k^5}\right) dl$$

$$dy = \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{6!}\right) dl = \left(1 - \frac{l^4}{8k^2} + \frac{l^8}{384k^4}\right) dl.$$

By integrating the above equations, we get

$$x = \frac{l^3}{6k} - \frac{l^7}{336k^3} + \frac{l^{11}}{42240k^5}$$

$$y = l - \frac{l^5}{40k^2} + \frac{l^9}{3456k^4}$$

In the above equations, the constant of integration are zero since at $l = 0$, $\phi = 0$. If L is the length of each transition curve with the minimum radius R at the junction point T, then

$$\phi_1 = 0.5555965 = \frac{L}{2R}$$

$$L = 1.111193 \quad R.$$

Also $$\alpha = \frac{v^3}{LR}$$

$$\frac{1}{3} = \frac{(52 \times 1000 / 3600)^3}{LR}$$

$$LR = k = \left(52 \times \frac{1000}{3600}\right)^3 \times 3 = 9041.15.$$

$$(1.1460996 \quad R)\ R = 9041.15$$

$$R_2 = \frac{9041.15}{1.111193}$$

$$R = \sqrt{\frac{9041.15}{1.111193}} = 90.20 \text{ m}$$

and $$L = 10023 \text{ m}.$$

Thus, when $l = L$ at T, the values of x and y are

$$x_T = \frac{100.23^3}{6 \times 9041.15} - \frac{100.23^7}{336 \times 9041.15^3} = 18.15 \text{ m}$$

$$y_T = 100.23 - \frac{100.23^5}{40 \times 9041.15^2} = 97.14 \text{ m.}$$

Now

Tangent length $\quad T_1I = T_2I = y_T + x_T \tan \phi_1$

$= 97.14 + 18.15 \times \tan 31°50'00'' = 108.41$ m.

To locate T_1 from BI, and T_2 from C, we require

$$BT_1 = IT_1 - IB$$

$= 108.41 - 41.15 =$ **67.26 m**

$$CT_2 = IT_2 - IC$$

$= 108.41 - 160.89 =$ **– 52.48 m**.

The negative sign of CT_2 shows that T_2 is between I and C.

Example 7.11. Two straights having a total deflection angle of 65°45′ are connected with a circular curve of radius 1550 m. It is required to introduce a curve of length 120 m at the beginning and end of the circular curve without altering the total length of the route. The transition curve to be inserted is a cubic spiral, and the chainage of the point of intersection is 5302.10 m. Calculate

(*i*) the distance between the new and the previous tangent points,

(*ii*) the setting out data for transition curve taking peg intervals 20 m, and

(*iii*) the data for locating the midpoint of the new circular curve from the point of intersection.

Solution (Fig. 7.20 and 7.22):

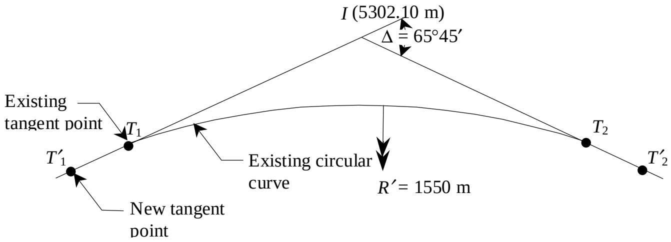


Fig. 7.20

$$R' = 1550 \text{ m}$$

$$\Delta = 65°45' = 65.75°$$

$$\frac{\Delta}{2} = 32°52'30''$$

Existing circular curve

$$\text{Curve length } l = \frac{\pi R' \Delta}{180}$$

$$= \frac{\pi \times 1550 \times 65.75}{180} = 1778.71 \text{ m.}$$

$$\text{Tangent length } (IT_1)\ T = R' \tan \frac{\Delta}{2}$$

$$= 1550 \times \tan 32°52'30'' = 1001.78 \text{ m.}$$

Transition curves

$$\phi_1 = \frac{L}{2R} \text{ radians}$$

where L is the length of the transition curve, and R is the radius of the new circular curve.

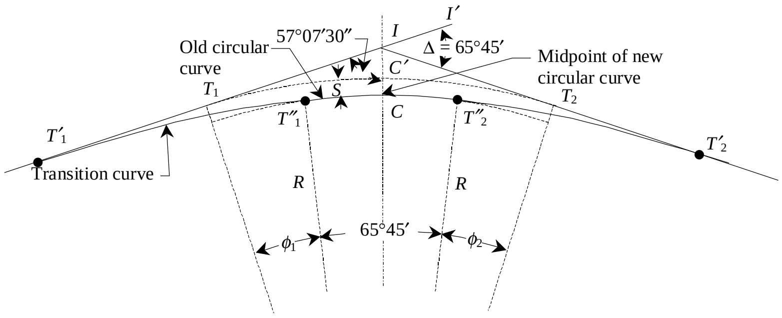


Fig. 7.21

Total length of the curve

$$T_1' T_2' = 2L + \left[\frac{\pi R \Delta}{180} - 2(\phi_1 R)\right]$$

$$= 2L + \left(\frac{\pi \times 65.75}{180} - \frac{L}{R}\right)R$$

$$= 2L + \left(1.1475540 - \frac{L}{R}\right)R$$

$$\text{Shift of circular curve} \quad S = \frac{L^2}{24R}$$

The new tangent length

$$IT_1' = \frac{L}{2} + (R+S)\tan\frac{\Delta}{2}$$

$$= \frac{L}{2} + \left(R + \frac{L^2}{24R}\right)\tan 32°52'30''$$

$$T_1'\,T_1 = IT_1' - T = IT_1' - 1001.78$$

Since the total length of the route must remain unchanged, the total curve length

$$T_1'\,T_2' = \text{Length of the existing circular curve} + 2\;T_1'\,T_1'$$

or

$$2L + \left(1.1475540 - \frac{L}{R}\right)R = 1778.71 + 2\,T_1'\,T_1'$$

$$= 1778.71 + 2\left[\frac{L}{2} + \left(R + \frac{L^2}{24R}\right)\tan 32°52'30''\right] - 2 \times 1001.78$$

By clearing and rearranging the terms, and substituting the value of L = 120 m, we get

$$R^2 - 1549.98\;R + 5346.32 = 0$$

Solving the above equation, we get

$$R = 1546.52 \text{ m.}$$

Thus

$$S = \frac{L^2}{24R} = \frac{120^2}{24 \times 1546.52} = 0.39 \text{ m}$$

and

$$IT_1' = \frac{120}{2} + (1546.52 + 0.39)\tan 32°52'30'' = 1059.78 \text{ m.}$$

(a) Therefore the distance between the new tangent point T_1' and the previous tangent point T_1

$$T_1'\,T_1 = IT_1' - IT_1$$

$$= 1059.78 - 1001.78 = \mathbf{58.00\ m}.$$

(b) Setting out transition curve $T_1'\,T_1''$

$$\text{Chainage of } T_1' = \text{Chainage of } I - T_1'I$$

$$= 5302.10 - 1059.78 = 42425.32 \text{ m}$$

$$= 212 + 2.32 \text{ for } 20 \text{ m chain.}$$

The deflection angle for chord length l

$$\delta = \frac{1800 l^2}{\pi RL} \text{ minutes}$$

$$= \frac{1800}{\pi \times 1546.52 \times 120} l^2 \text{ minutes}$$

$$= 0.0030873 \; l^2 \text{ minutes.}$$

The length of the first sub-chord

$$c_f = (212 + 20) - (212 + 2.32) = 17.68 \text{ m}$$

$$\delta_{17.68} = 0.0030873 \times 17.68^2 \text{ minutes}$$

$$= 00'58''$$

$$\delta_{37.68} = 0.0030873 \times 37.682 \text{ minutes}$$

$$= 4'23''.$$

Other values of δ are given in Table 7.10.

Table 7.10

Point	Chainage (m)	Chord lenght (m)	Deflection angle (δ)
0 (T_1')	4242.32	0.0	0.0
1	4260.00	17.68	0′58″
2	4280.00	37.68	4′23″
3	4300.00	57.68	10′16″
4	4320.00	77.68	18′38″
5	4340.00	97.68	29′27″
6	4360.00	117.68	42′45″
7 (T_1'')	4362.32	120.00	44′27″

(c)

$$IO = \frac{OT_1}{\cos\frac{\Delta}{2}} = \frac{R+S}{\cos\frac{\Delta}{2}}$$

$$= \frac{1546.52 + 0.39}{\cos\frac{65^\circ 45'}{2}} = 1841.87 \text{ m}$$

$$IC' = IO - OC'$$

$$= 1841.87 - (1546.52 + 0.39) = 294.96 \text{ m}$$

$$IC = IC' + C'C$$

$$= 294.96 + 0.39 = \mathbf{295.35 \text{ m}}$$

$$\angle II'O = 65°45' + 57°07'30'' = \mathbf{122°52'30''}$$

Now C can be located by setting out an angle of 122°52′30″ from $T_1'I$ produced, and measuring a distance of 295.35 m from I.

Example 7.12. Two parallel railway tracks, centre lines being 60 m apart, are to be connected by a reverse curve, each section having the same radius. If the maximum distance between the tangent points is 220 m calculate the maximum allowable radius of the reverse curve that can be used.

Solution (Fig. 7.22):

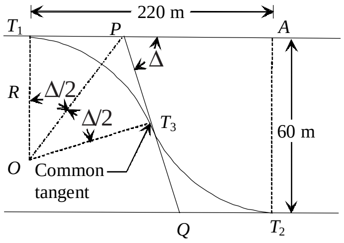


Fig. 7.22

We have

$$T_1P = PT_3 = T_3Q = QT_2 = R \tan\frac{\Delta}{2}$$

$$T_1T_3 = T_3T_2 = \sin\frac{\Delta}{2} = \sqrt{\left(AT_1^2 + AT_2^2\right)}$$

$$= \sqrt{\left(200^2 + 60^2\right)}$$

$$= 228.035 \text{ m} \qquad \ldots\text{(a)}$$

$$PQ = (PT_3 + T_3Q) = 2PT_3 = 2R \tan\frac{\Delta}{2} = \frac{60}{\sin\Delta}$$

or

$$2R\frac{\sin\frac{\Delta}{2}}{\sin\frac{\Delta}{2}} = \frac{60}{2\sin\frac{\Delta}{2}\cos\frac{\Delta}{2}}$$

$$2R\sin\frac{\Delta}{2} = \frac{30}{\sin\frac{\Delta}{2}}$$

From (a), we have

$$228.035 = \frac{30}{\sin\frac{\Delta}{2}}$$

$$\sin\frac{\Delta}{2} = \frac{30}{228.035}$$

Substituting the value of $\sin\frac{\Delta}{2}$ in (a) we get

$$2R \times \frac{30}{228.035} = 228.035$$

$$R = \frac{228.035^2}{60} = \mathbf{866.7 \ m}.$$

Example 7.13. The first branch of a reverse curve has a radius of 200 m. If the distance between the tangent points is 110 m, what is the radius of the second branch so that the curve can connect two parallel straights, 18 m apart ? Also calculate the length of the two branches of the curve.

Solution (Fig. 7.23):

We have

$$R_1 = 200 \text{ m}$$
$$D = 18 \text{ m}$$
$$L = 110 \text{ m.}$$

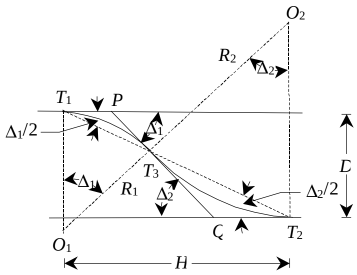


Fig. 7.23

From Eq. (7.37), we have

$$L = \sqrt{\left[2D\left(R_1 + R_2\right)\right]}$$

$$R_2 = \frac{L^2}{2D} - R_1$$

$$= \frac{100^2}{2 \times 18} - 200 = 136.11 \text{ m.}$$

From Eq. (7.36), we have

$$D = 2\left(R_1 + R_2\right)\sin^2\frac{\Delta_1}{2}$$

$$\Delta_1 = 2\sin^{-1}\left[\sqrt{\frac{D}{2\left(R_1 + R_2\right)}}\right]$$

$$= 2 \times \sin^{-1}\left[\sqrt{\frac{18}{2 \times \left(200 + 136.11\right)}}\right]$$

$$= 18°50'10'' = 18.836° = \Delta_2$$

The lengths of the first and second branches of the curve

$$l_1 = \frac{\pi R_1 \Delta_1}{180}$$

$$= \frac{\pi \times 200 \times 18.836}{180} = \mathbf{65.75 \text{ m}}$$

$$l_1 = \frac{\pi \times 136.11 \times 18.836}{180} = \mathbf{44.75 \text{ m}}.$$

Example 7.14. It is proposed to introduce a reverse curve between two straights *AB* and *CD* intersecting at a point *I* with $\angle CBI = 30°$ and $\angle BCI = 120°$. The reverse curve consists of two circular arcs *AX* and *XD, X* lying on the common tangent *BC.* If *BC* = 791.71, the radius

R_{AX} = 750 m, and chainage of B is 1250 m, calculate

(*i*) the radius R_{XD},

(*ii*) the lengths of the reverse curve, and

(*iii*) The chainage of D.

Solution (Fig. 7.24):

From the given data, we have

$$\Delta_1 = 30°, \frac{\Delta_1}{2} = 15°$$

$$\Delta_2 = 180° - 120° = 60°, \frac{\Delta_2}{2} = 30°$$

$$R_{AX} = 750 \text{ m}$$

$$BC = 791.71 \text{ m}$$

$$\text{Chainage of } B = 1250 \text{ m.}$$

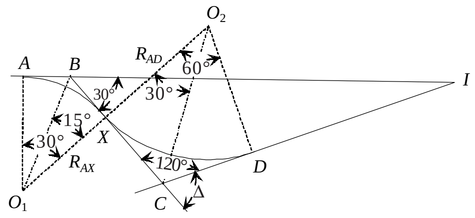


Fig. 7.24

(a) Tangent lengths

$$BX = R_{AX} \tan \frac{\Delta_1}{2}$$

$$XC = R_{XD} \tan \frac{\Delta_2}{2}$$

$$BC = BX + XD$$

$$BC = R_{AX} \tan \frac{\Delta_1}{2} + R_{XD} \tan \frac{\Delta_2}{2}$$

$$R_{XD} = \frac{BC - R_{AX} \tan \frac{\Delta_1}{2}}{\tan \frac{\Delta_2}{2}}$$

$$= \frac{791.71 - 750 \times \tan 15°}{\tan 30°} = \mathbf{1023.21 \ m}.$$

(b) Length of the reverse curve l

$$l_{AX} = \frac{\pi R_{AX} \Delta_1}{180}$$

$$l_{XD} = \frac{\pi R_{XD} \Delta_2}{180}$$

$$l = l_{AX} + l_{XD}$$

$$= \frac{\pi}{180}\left(R_{AX}\Delta_1 + R_{XD}\Delta_2\right)$$

$$= \frac{\pi}{180} \times (750 \times 30 + 1023.21 \times 60) = \mathbf{1464.20\ m}.$$

(*c*) Chainage of *D*

$$AB = BX = R_{AX} \tan \frac{\Delta_1}{2}$$

$$= 750 \times \tan 15° = 200.96 \text{ m}$$

$$\text{Chainage of } D = \text{Chainage of } B - AB + l$$

$$= 1250 - 200.96 + 146.20 = \mathbf{2513.24\ m}.$$

Example 7.15. Two straights *AB* and *BC* falling to the right at gradients 10% and 5%, respectively, are to be connected by a parabolic curve 200 m long. Design the vertical curve for chainage and reduce level of *B* as 2527.00 m and 56.46 m, respectively. Take peg interval as 20 m.

Also calculate the sight distance for a car having headlights 0.60 m above the road level, and the headlight beams inclined upwards at an angle of 1.2°.

Solution (Fig. 7.25):

The total number of stations at 20 m interval

$$= 2n = \frac{L}{20} = \frac{200}{20} = 10 \text{ m}$$

or

$$n = \frac{10}{2} = 5.$$

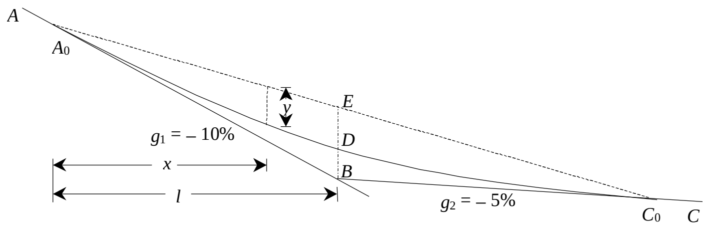


Fig. 7.25

Fall per chord length

$$e_1 = \frac{g_1}{100} \times 20 = \frac{-10}{100} \times 20 = -2 \text{ m}$$

$$e_2 = \frac{g_2}{100} \times 20 = \frac{-5}{100} \times 20 = -1 \text{ m}.$$

Elevation of the beginning of the curve at A_0

$$= \text{Elevation of } B - ne_1$$

$$= 56.46 - 5 \times (-2) = 66.46 \text{ m.}$$

Elevation of the end of the curve at C_0

$$= \text{Elevation of } B + ne_2$$

$$= 56.46 - 5 \times (-1) = 51.46 \text{ m.}$$

Tangent correction with respect to the first tangent

$$h = kN^2$$

where $$k = \frac{e_1 - e_2}{4n}$$

$$= \frac{-2-(-1)}{4 \times 5} = -0.05$$

Reduced levels (R.L.) of the points on the curve

$$= \text{Tangential elevation} - \text{tangent correction}$$

$$= H - h$$

where H is the tangential elevation of a point.

R.L. on the n'^{th} point on the curve = R.L. of $A_0 + n'e_1 - kn'^2$

R.L. of point 1 (A_0) $= 66.46 + 1 \times (-2) - (-0.05) \times 1^2 = 64.51$ m

R.L. of point 2 $= 6.46 + 2 \times (-2) - (-0.05) \times 2^2 = 62.66$ m

R.L. of point 3 $= 6.46 + 3 \times (-2) - (-0.05) \times 3^2 = 60.91$ m

R.L. of point 4 $= 66.46 + 4 \times (-2) - (-0.05) \times 4^2 = 59.26$ m

R.L. of point 5 $= 66.46 + 5 \times (-2) - (-0.05) \times 5^2 = 57.71$ m

R.L. of point 6 $= 66.46 + 6 \times (-2) - (-0.05) \times 6^2 = 56.26$ m

R.L. of point 7 $= 66.46 + 7 \times (-2) - (-0.05) \times 7^2 = 54.91$ m

R.L. of point 8 $= 66.46 + 8 \times (-2) - (-0.05) \times 8^2 = 53.66$ m

R.L. of point 9 $= 66.46 + 9 \times (-2) - (-0.05) \times 9^2 = 52.51$ m

R.L. of point 10 (C_0) $= 66.46 + 10 \times (-2) - (-0.05) \times 10^2 = 51.46$ m (*Okay*).

Chainage of the intersection point $B = 2527.00$ m

Chainage of A_0 = Chainage of $B - 20n$

$= 2527.00 - 20 \times 5 = 2427.00$ m.

Chainage of C_0 = Chainage of $B + 20n$

$= 2527.00 + 20 \times 5 = 2627.00$ m.

The chainage of the points and the reduced levels of the corresponding points on the curve are tabulated in Table 7.11.

Table 7.11

Point	Chainage (m)	R.L. of points on curve (m)	Remarks
0	2427.00	66.46	A_0 (P.C.)
1	2447.00	64.51	
2	2467.00	62.66	
3	2487.00	60.91	
4	2507.00	59.26	
5	2527.00	57.71	Apex
6	2547.00	56.26	
7	2567.00	54.91	
8	2587.00	53.66	
9	2607.00	52.51	
10	2627.00	51.46	C_0 (P.T.)

With the car at tangent point A_0, the headlight beams will strike the curved road surface at a point where the offset y from the tangent at A_0 is (0.60 + x tan 1.2°), x being the distance from A_0. The offset y at a distance x from A_0 is given by

$$y = \frac{g_1 - g_2}{400l} x^2$$

where l is half of the total length of the curve = 200/2 = 100 m. Thus

$$y = \frac{-2+1}{400 \times 100} x^2 = \frac{x^2}{40000} \text{ ignoring the sign}$$

or $$0.60 + x \tan 1.2° = \frac{x^2}{40000}$$

$$x^2 - 837.88x - 24000 = 0$$

$$\text{Sight distance } x = \frac{+837.88 + \sqrt{937.88^2 + 4 \times 1 \times 2400}}{2 \times 1} = \mathbf{865.61 \ m}.$$

Example 7.16. Two straights *AB* having gradient rising to the right at 1 in 60 and *BC* having gradient falling to the right at 1 in 50, are to be connected at a summit by a parabolic curve. The point *A*, reduced level 121.45 m, lies on *AB* at chainage 1964.00 m, and *C*, reduced level 120.05 m, lies on *BC* at chainage 2276.00 m. The vertical curve must pass through a point *M*, reduced level 122.88 at chainage 2088.00 m.

Design the curve, and determine the sight distance between two points 1.06 m above road level.

Solution (Fig. 7.26):

Given that

$$g_1 = +\frac{100}{60} = \frac{10}{6}\,\%$$

$$g_2 = +\frac{100}{50} = \frac{10}{5} = 2\%.$$

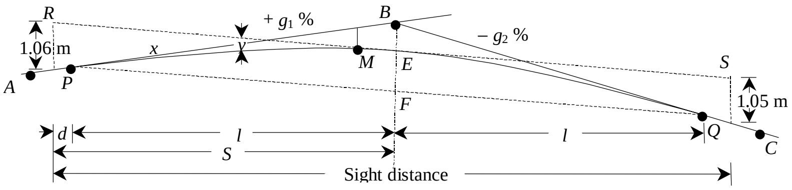


Fig. 7.26

Let the horizontal distance $AB = x_1$.

$$\text{Level of } B = \text{Level of } A + \frac{x_1}{60}$$

$$= 121.45 + \frac{x_1}{60}$$

$$= \text{Level of } C + \frac{2276 - 1964 - x_1}{50}$$

$$= 120.05 + \frac{312 - x_1}{50}$$

Therefore

$$121.45 + \frac{x_1}{60} = 120.05 + \frac{312 - x_1}{50}$$

$$x_1 = \frac{50 \times 60}{110}\left(120.05 + \frac{312}{50} - 121.45\right) - 132.00 \text{ m.}$$

$$\text{Chainage of } B = \text{Chainage of } A + x_1$$

$$= 1964.00 + 132.00 = 2096.00 \text{ m.}$$

$$\text{Distance of } M \text{ from } B = x' = \text{Chainage of } B - \text{chainage of } M$$

$$= 2096.00 - 2088.00 = 8.00 \text{ m.}$$

$$\text{Distance of } M \text{ from } A = \text{Chainage of } M - \text{chainage of } A$$

$$= 2088 - 1964.00 = 124.00 \text{ m .}$$

$$\text{Grade level at the chainage of } M = 121.45 + \frac{124.00}{60} = 123.52 \text{ m.}$$

Curve level at M = 122.88 m

Offset at M = 123.52 – 122.88 = 0.64 m.

If the tangent length of the curve is l then the offset at M

$$= \frac{g_1 - g_2}{400l}(l - x')^2$$

or
$$0.64 = \frac{g_1 - g_2}{400l}(l - 8)^2$$

$$= \frac{\frac{10}{6} + 2}{400l}(l - 8)^2$$

$$l^2 - 85.818182\ l + 64 = 0$$

$$l = 85.066 \text{ m.}$$

The value of l can be taken as 85 m for the design purposes. Therefore

chainage of P = 2096.00 – 85 = 2011.00 m

chainage of Q = 2011.00 + 2 × 85 = 2181.00 m .

Taking peg interval as 20 m, the values of x and chainage of the points are

x = 0.0 m	chainage of 0 (P)	= 2011.00 m
= 9.0 m	1	= 2020.00 m
= 29.0 m	2	= 2040.00 m
= 49.0 m	3	= 2060.00 m
= 69.0 m	4	= 2080.00 m
= 89.0 m	5	= 2100.00 m
= 109.0 m	6	= 2120.00 m
= 129.0 m	7	= 2140.00 m
= 149.0 m	8	= 2160.00 m
= 169.0 m	9	= 2180.00 m
= 170.0 m	10	= 2181.00 m

Grade levels

Distance between A and P = 2011.00 – 1964.00 = 47.00 m

Grade level at P = $\frac{47}{60} + 121.45$ = 122.23 m.

The grade levels of other points can be obtained from 122.23 + $\frac{x}{60}$, and the offsets y from $\frac{g_1 - g_2}{400l}x^2$ by substituting the values of x. The curve levels can be calculated by subtracting the offsets from the corresponding grade levels. The calculated values of the design data are presented in Table 7.12.

The level of a point on the curve above P is obtained by the expression

$$h = \frac{g_1}{100}x - \frac{g_1 - g_2}{400l}x^2$$

where x is the distance of the point P. The highest point on the curve is that point for which h is maximum. By differentiating the above expression and equating to zero, we get the value x_{max} at which $h = h_{max}$.

$$h = \frac{g_1 x}{100} - \frac{g_1 - g_2}{400l}x^2$$

$$\frac{dh}{dx} = \frac{g_1}{100} - 2\frac{(g_1 - g_2)}{400l}x = 0$$

$$x = \frac{400l\, g_1}{200(g_1 - g_2)}$$

$$= \frac{400 \times 85 \times \frac{10}{6}}{200 \times \left(\frac{10}{6} + 2\right)} = 77.27 \text{ m.}$$

$$= x_{max} \text{ for } h_{max}.$$

Table 7.11

Point	Chainage (m)	Grade level (m)	x (m)	y (m)	Curve level (m)
0 (P)	2011.00	122.23	0.0	0.0	122.23
1	2020.00	122.38	9.00	0.01	122.37
2	2040.00	122.71	29.00	0.09	122.62
3	2060.00	123.05	49.00	0.26	122.79
4	2080.00	123.38	69.00	0.51	122.87
5	2100.00	123.71	89.00	0.85	122.86
6	2120.00	124.05	109.00	1.28	122.77
7	2140.00	124.38	129.00	1.79	122.59
8	2160.00	124.71	149.00	2.39	122.32
9	2180.00	125.05	169.00	3.08	121.97
10 (Q)	2181.00	125.06	170.00	3.11	121.95

$$\text{The grade level at } x_{max} = 122.23 + \frac{77.27}{60} = 123.52 \text{ m}$$

and the offset
$$= \frac{\left(\frac{10}{6} + 2\right)}{400 \times 85} \times 77.27^2 = 0.64 \text{ m.}$$

Thus the reduced level of the highest point

$$= 123.52 - 0.64 = 122.88 \text{ m.}$$

$$\text{The offset } BE \text{ at } B = \frac{\left(\frac{10}{6} + 2\right)}{400 \times 85} \times 85^2 = 0.78 \text{ m} = EF.$$

The sight line can be taken as the tangent *RS* to the curve at *E*. *RS* is parallel to *PQ* which has a slope of $\dfrac{\frac{g_1 l}{100} + \frac{g_2 l}{100}}{2l} = \dfrac{g_1 + g_2}{200}$ radians, and the slope of *PB* is $\dfrac{g_1}{100}$ radians. Thus the angle θ between *PB* and *PQ* as shown in Fig. 7.27,

$$= \frac{g_1}{100} - \frac{g_1 + g_2}{200}$$

$$= \frac{g_1 - g_2}{200}.$$

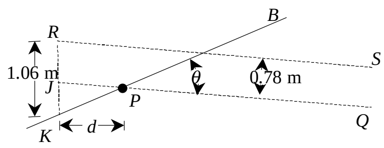


Fig. 7.27

$$\text{Distance } JK = 1.06 - 0.78 = 0.28 \text{ m.}$$

Thus in ΔJPK, we have

$$\frac{JK}{d} = \theta$$

$$d = \frac{JK}{\theta}$$

$$= \frac{JK}{\frac{g_1 - g_2}{200}} = \frac{0.28 \times 200}{\left(\frac{10}{6} + 2\right)} = 15.27 \text{ m.}$$

Thus the total sight distance

$$RS = 2\,(d + l)$$

$$= 2 \times (15.27 + 85) = 200.54 \text{ m}$$

$$= \mathbf{200\ m} \text{ (say)}$$

OBJECTIVE TYPE QUESTIONS

1. A circular curve is most suited for connecting

(*a*) two straights in horizontal plane only.

(*b*) two straights in vertical plane only.

(*c*) two straights, one in horizontal plane and the second in vertical plane.

(*d*) two straights in horizontal plane or vertical plane.

2. A compound curve consists of

(*a*) two circular arcs of same radius only.

(*b*) two circular arcs of different radii only.

(*c*) two circular arcs of different radii with their centers of curvature on the same side of the common tangent only.

(*d*) two or more circular arcs of different radii with their centers of curvature on the same side of the common tangent.

3. A reverse curve consists of

(*a*) two circular arcs of different radii with their centers of curvature on the same side of the common tangent only.

(*b*) two circular arcs of same radius with their centers of curvature on the same side of the common tangent only.

(*c*) two circular arcs of different radii with their centers of curvature on the opposite side of the common tangent only.

(*d*) two circular arcs of same or different radii with their centers of curvature on the opposite side of the common tangent.

4. A transition curve is a special type of curve which satisfies the condition that

(*a*) at the junction with the circular curve, the angle between the tangents to the transition curve and circular curve should be 90°.

(*b*) at the junction with the circular curve, the angle between the tangents to the transition curve and circular curve should be zero.

(*c*) its curvature at its end should be infinity .

(*d*) its curvature at its end should be infinity.

5. The most widely used transition curve for small deviation angles for simplicity in setting out is

(*a*) cubic parabola.

(*b*) cubic spiral.

(*c*) lemniscate curve.

(*d*) hyperbola.

6. The following curve has the property that the rate of change of curvature is same as the rate of change of increase of superelevation:

(*a*) Reverse curve.

(*b*) Compound curve.

(*c*) Transition curve.

(*d*) Vertical curve.

7. A parabola is used for

(*a*) summit curves alone. (*b*) sag curves alone.

(*c*) both summit and sag curves. (*d*) none of the above.

8. A parabola is preferred for vertical curves because it has the following property:

(*a*) The slope is constant throughout.

(*b*) The rate of change of slope is constant throughout.

(*c*) The rate of change of radial acceleration is constant throughout.

(*d*) None of the above.

9. The shortest distance between the point of commencement and the o\point of tangency of a circular curve is known as

(*a*) Long chord. (*b*) Normal chord.

(*c*) Sub-chord. (*d*) Half-chord.

10. The long chord of a circular curve of radius R with deflection angle Δ is given by

(*a*) $2R\cos(\Delta/2)$. (*b*) $2R\sin(\Delta/2)$.

(*c*) $2R\tan(\Delta/2)$. (*d*) $2R\sec(\Delta/2)$.

11. The lengths of long chord and tangent of a circular curve are equal for the deflection angle of

(*a*) 30°. (*b*) 60°.

(*c*) 90°. (*d*) 120°.

12. The degree of a circular curve of radius 1719 m is approximately equal to

(*a*) 1°. (*b*) 10°.

(*c*) 100°. (*d*) None of the above.

13. If the chainage of point of commencement of a circular curve for a normal chord of 20 m is 2002.48 m, the length of the first sub-chord will be

(*a*) 2.48 m. (*b*) 17.52 m.

(*c*) 20 m. (*d*) 22.48 m.

14. If the chainage of point of tangency of a circular curve for a normal chord of 20 m is 2303.39 m, the length of the last sub-chord will be

(*a*) 3.39 m. (*b*) 16.61 m.

(*c*) 23.39 m. (*d*) none of the above.

15. For an ideal transition curve, the relation between the radius r and the distance l measured from the beginning of the transition curve, is expressed as

(*a*) $l \propto r$. (*b*) $l \propto r^2$.

(*c*) $l \propto 1/r$. (*d*) $l \propto 1/r^2$.

16. For a transition curve, the shift S of a circular curve is given by

(*a*) $\frac{L}{24R^2}$. (*b*) $\frac{L^2}{24R^2}$.

(*c*) $\frac{L^3}{24R^2}$. (*d*) $\frac{L^3}{24R}$.

17. For a transition curve, the polar deflection angle α_s and the spiral angle Δ_s are related to each other by the expression

(*a*) $\alpha_S = \Delta_S/2$. (*b*) $\alpha_S = \Delta_S/3$.

(*c*) $\alpha_S = \Delta_S/4$. (*d*) $\alpha_S = \Delta^2{}_S/3$.

18. To avoid inconvenience to passengers on highways, the recommended value of the centrifugal ratio is

(*a*) 1. (*b*) 1/2.

(*c*) 1/4. (*d*) 1/8.

19. The following value of the change in radial acceleration passes unnoticed by the passengers:

(*a*) 0.003 m/s^2/sec. (*b*) 0.03 m/s^2/sec.

(*c*) 0.3 m/s^2/sec. (*d*) 3.0 m/s^2/sec.

20. The curve preferred for vertical curves is a

(*a*) circular arc. (*b*) spiral.

(*c*) parabola. (*d*) hyperbola.

21. If an upgrade of 2% is followed by a downgrade of 2%, and the rate of change of grade is 0.4% per 100 m, the length of the vertical curve will be

(*a*) 200 m. (*b*) 400 m.

(*c*) 600 m. (*d*) 1000 m.

22. For a vertical curve if *x* is the distance from the point of tangency, the tangent correction is given by

(*a*) Cx. (*b*) Cx^2.

(*c*) Cx^3. (*d*) Cx^4.

ANSWERS

1. (*a*)	**2.** (*d*)	**3.** (*d*)	**4.** (*b*)	**5.** (*a*)	**6.** (*c*)
7. (*c*)	**8.** (*b*)	**9.** (*a*)	**10.** (*b*)	**11.** (*d*)	**12.** (*a*)
13. (*b*)	**14.** (*a*)	**15.** (*c*)	**16.** (*d*)	**17.** (*b*)	**18**. (*c*)
19. (*c*)	**20.** (*c*)	**21.** (*d*)	**22.** (*b*)		

8 AREAS AND VOLUMES

8.1 AREAS AND VOLUMES

In civil engineering works such as designing of long bridges, dams, reservoirs, etc., the area of catchments of rivers is required. The areas of fields are also required for planning and management of projects. The area is required for the title documents of land.

In many civil engineering projects, earthwork involves excavation and removal and dumping of earth, therefore it is required to make good estimates of volumes of earthwork. Volume computations are also needed to determine the capacity of bins, tanks, and reservoirs, and to check the stockpiles of coal, gravel, and other material.

Computing areas and volumes is an important part of the office work involved in surveying.

8.2 AREAS

The method of computation of area depends upon the shape of the boundary of the tract and accuracy required. The area of the tract of the land is computed from its plan which may be enclosed by straight, irregular or combination of straight and irregular boundaries. When the boundaries are straight the area is determined by subdividing the plan into simple geometrical figures such as triangles, rectangles, trapezoids, etc. For irregular boundaries, they are replaced by short straight boundaries, and the area is computed using approximate methods or Planimeter when the boundaries are very irregular. Standard expressions as given below are available for the areas of straight figures.

$$\text{Area of triangle } = \frac{1}{2}ab \sin C$$

in which C is the included angle between the sides a and b.

$$\text{Area of trapezium } = \frac{a+b}{2}h$$

in which a and b are the parallel sides separated by perpendicular distance h.

Various methods of determining area are discussed below.

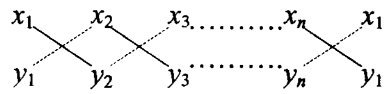
Area Enclosed by Regular Straight Boundaries

If the coordinates of the points A, B, C, etc., for a closed traverse of n sides shown in Fig. 8.1, are known, the area enclosed by the traverse can be calculated from the following expression.

$$A = \frac{1}{2}\left[x_1(y_2 - y_n) + x_2(y_3 - y_1) + \ldots\ldots\ldots + x_{n-1}(y_n - y_{n-2}) + x_n(y_1 - y_{n-1})\right] \qquad \ldots(8.1)$$

$$= \frac{1}{2}\left[y_1\left(x_n - x_2\right) + y_2\left(x_1 - x_3\right) + \dots\dots\dots + y_{n-1}\left(x_{n-2} - x_n\right) + y_n\left(x_{n-1} - x_1\right)\right] \quad \dots(8.2)$$

The area can also be computed by arranging the coordinates in the determinant form given below.



The products of the coordinates along full lines is taken positive and along dashed lines negative.

Thus the area

$$A = \frac{1}{2}\left(x_1 y_2 - y_1 x_2 + x_2 y_3 - y_2 x_3 \dots\dots\dots + x_n y_1 - y_n x_1\right) \quad .(8.3)$$

Area Enclosed by Irregular Boundaries

Two fundamental rules exist for the determination of areas of irregular figures as shown in Fig 8.2. These rules are (i) Trapezoidal rule and (ii) Simpson's rule.

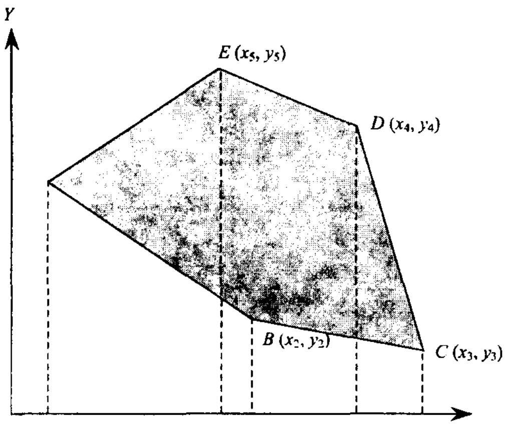


Fig. 8.1

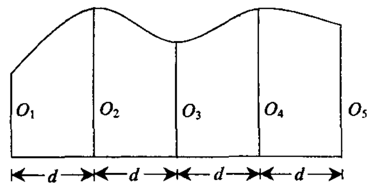


Fig. 8.2

Trapezoidal Rule

In trapezoidal rule, the area is divided into a number of trapezoids, boundaries being assumed to be straight between pairs of offsets. The area of each trapezoid is determined and added together to derive the whole area. If there are n offsets at equal interval of d then the total area is

$$A = d\left(\frac{O_1 + O_n}{2} + O_2 + O_3 + \dots\dots\dots + O_{n-1}\right) \quad \dots(8.4)$$

While using the trapezoidal rule, the end ordinates must be considered even they happen to be zero.

Simpson's Rule

In Simpson's rule it is assumed that the irregular boundary is made up of parabolic arcs. The areas of the successive pairs of intercepts are added together to get the total area.

$$A=\frac{d}{3}\left[(O_1+O_n)+4(O_2+O_4+.....+O_{n-2})+2(O_3+O_5+.....+O_{n-1})\right] \qquad ...(8.5)$$

Since pairs of intercepts are taken, it will be evident that the number of intercepts n is even. If n is odd then the first or last intercept is treated as a trapezoid.

Planimeter

An integrating device, the planimeter, is used for the direct measurement of area of all shapes, regular or irregular, with a high degree of accuracy.

The area of plan is calculated from the following formula when using Amsler polar planimeter.

$$A=M(R_F-R_I\pm 10N+C) \qquad ...(8.6)$$

where M = the multiplying constant of the instrument. Its value is marked on the tracing arm of the instrument,

R_F and R_I = the final and initial readings,

N = the number of complete revolutions of the dial taken positive when the zero mark passes the index mark in a clockwise direction and negative when in counterclockwise direction, and

C = the constant of the instrument marked on the instrument arm just above the scale divisions. Its value is taken as zero when the needle point is kept outside the area. For the needle point inside, the value of MC is equal to the area of zero circle.

8.3 VOLUMES

Earthwork operations involve the determination of volumes of material that is to be excavated or embanked in engineering project to bring the ground surface to a predetermined grade. Volumes can be determined via cross-sections, spot levels or contours. It is convenient to determine the volume from 'standard-type' cross-sections shown in Fig. 8.3, provided that the original ground surface is reasonably uniform in respect of the cross-fall, or gradient transverse to the longitudinal centre line. The areas of various standard-type of cross-sections have been discussed in various examples.

Having computed the cross-sections at given intervals of chainage along the centre line by standard expressions for various cross-sections, or by planimeter, etc., volumes of cut in the case of excavation or volumes of fill in the case of embankment, can be determined using end-area rule or prismoidal rule which are analogous to the trapezoidal rule and Simpson's rule, respectively.

If the cross-sections are considered to be apart by distance d then by end-areas rule and prismoidal rule the volume is given by the following formulae:

End-areas rule

$$V=d\left(\frac{A_1+A_n}{2}+A_2+A_3+.........+A_{n-1}\right) \qquad ...(8.7)$$

Prismoidal rule

$$V=\frac{2d}{6}\left[A_1+A_n+4(A_2+A_4+.....+A_{n-2})+2(A_3+A_5+.....+A_{n-1})\right] \qquad ...(8.8)$$

The prismoidal rule assumes that the earth forms a prismoid between two cross-sections $2d$ apart, and for this to apply the linear dimensions of the mid-section between them should be the mean of the corresponding dimensions at the outer sections. The prismoidal formula gives very nearly correct volume of earthwork even for irregular end sections and sides that are warped surfaces.

The prismoidal formula though being more accurate than end-areas rule, in practice the end-areas rule is more frequently adopted because of the ease of its application. End-areas rule gives the computed volumes generally too great which is in favour of contractor.

Since the end-areas rule gives volume larger than the prismoidal rule, accurate volume by the former can be obtained by applying a correction known as *prismoidal correction* given below.

$$C_{pc} = \text{Volume by end-areas rule} - \text{volume by prismoidal rule}$$

$$= d(A_1 + A_2) - \frac{d}{3}(A_1 + A_2 + 4A_m) \qquad \ldots(8.9)$$

where A_m = the area of the middle section.

If h_1 and h_2 are the depths or heights at the midpoints of the sections A_1 and A_2, respectively, h_m for the middle section is $= \frac{h_1 + h_2}{2}$.

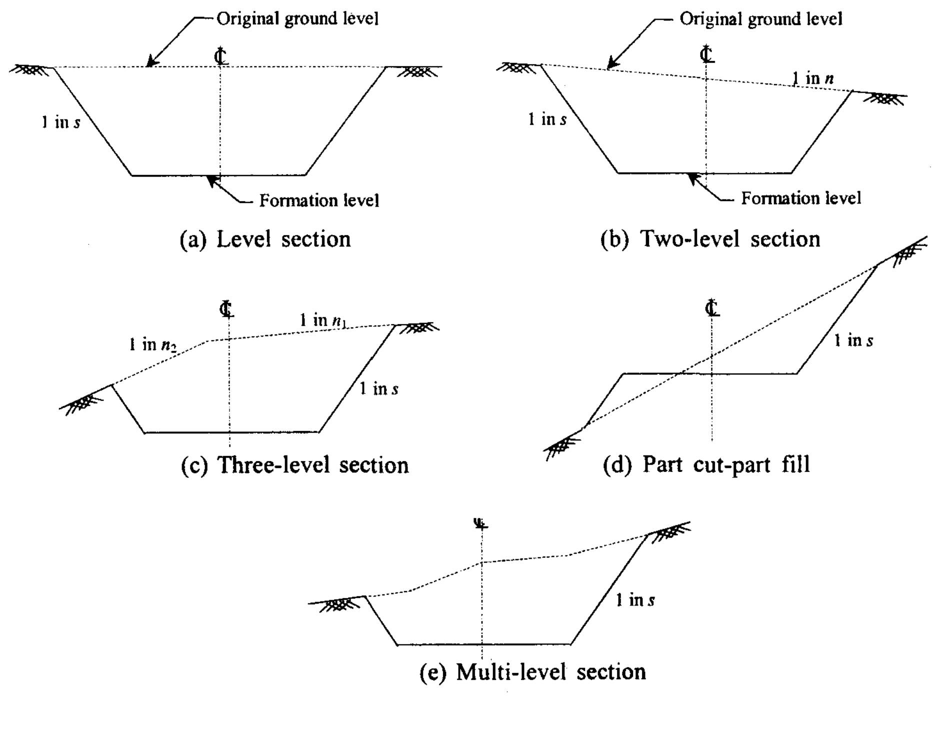


Fig. 8.3

For a two-level section, we have

$$A_1 = \frac{n^2 s}{n^2 - s^2}\left(h_1 + \frac{b}{2s}\right)^2 - \frac{b^2}{2s}$$

$$A_2 = \frac{n^2 s}{n^2 - s^2}\left(h_2 + \frac{b}{2s}\right)^2 - \frac{b^2}{2s}$$

$$A_m = \frac{n^2 s}{n^2 - s^2}\left(h_m + \frac{b}{2s}\right)^2 - \frac{b^2}{2s}$$

where n = the cross-fall (1 in n) of the original ground,

s = the side slope (1 in s), and

b = the width of formation.

Substituting the values of A_1, A_2, and in Eq. (8.9), the prismoidal correction for a two-level section is

$$= \frac{ds}{6}\left(\frac{n^2}{n^2 - s^2}\right)(h_1 - h_2)^2 \qquad \text{...(8.10)}$$

In a similar manner, the prismoidal corrections can be obtained for other types of sections.

The end-areas rule for calculating the volume is based on parallel end-areas, i.e., the two cross-sections are normal to the centre line of the route. When the centre line is curved, the cross-sections are set out in radial direction and, therefore they are not parallel to one another. Hence the volume computed by end-areas rule will have discrepancies and need correction for curvature.

According to Pappus's theorem, the volume swept by an area revolving about an axis is given by the product of the area and the length of travel of centroid of the area, provided that the area is in the plane of the axis and to one side thereof.

If an area A is revolved along a circular path of radius R through a distance L, the volume of the solid generated is

$$V = \frac{AL(R \pm e)}{R} \qquad \text{...(8.11)}$$

where e is the distance of the centroid of the cross-sections from the centre line AB as shown in Fig. 8.4, and it is called as *eccentricity* of cross-sections. The eccentricity e is taken negative when it is inside the centre O and the centre line AB, and positive when outside.

Volume by Spot Levels

If the spot levels have been observed at the corners of squares or rectangles forming grid, the volume is calculated by determining the volume of individual vertical prisms of square or rectangular base A and depth h, and adding them together. The following expression gives the total volume.

$$V = \frac{A(\sum h_1 + 2\sum h_2 + 3\sum h_3 + 4\sum h_4)}{4} \qquad \ldots(8.12)$$

where

A = the area of the square or rectangle,

$\sum h_1$ = the sum of the vertical depths common to one prism,

$\sum h_2$ = the sum of the vertical depths common to two prism,

$\sum h_3$ = the sum of the vertical depths common to three prism, and

$\sum h_4$ = the sum of the vertical depths common to four prism.

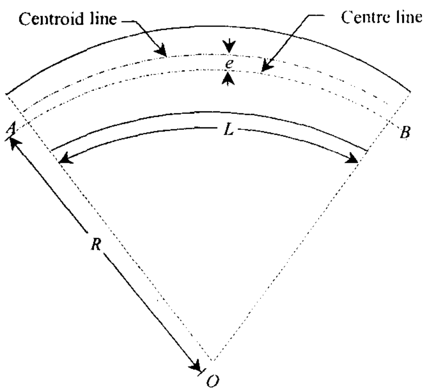


Fig. 8.4

Volume from Contours

Contours are used in a manner similar to cross-sections and the distance between the cross-sections is taken equal to the contour interval. The area enclosed by a contour is determined using a planimeter. This method of determining volume is approximate since the contour interval is generally not sufficiently small to fully depict the irregularities of the ground.

8.4 HEIGHTS OF POINTS FROM A DIGITAL TERRAIN MODEL

A digital terrain model (DTM) also is a digital representation of terrain surface by densely distributed points of known (X, Y, Z) coordinates. It is also known as digital elevation model (DEM) when the Z coordinate represent only elevation of the points. Data for a DTM may be gathered by land survey, photogrammetry or from an existing map, of these the photogrammetric methods are most widely used. By photogrammetric method the DTM is produced in a photogrammetric plotter by supplementing it with special processing components which make model digitization possible. The stored digital data in the form of (X, Y, Z) coordinates of characteristic points in a computer, allow the interpolation of Z coordinates of other points of given coordinates (X, Y).

When the characteristic points are located in the form of square, rectangular or triangular grid layouts, linear interpolation is used to determine the Z coordinates of other points. Fig. 8.5a shows the spot levels Z_A, Z_B, Z_C, and Z_D at the nodes A, B, C, and D, respectively, of the square of side L.

Interpolating linearly between A and B, in Fig. 8.5b,

$$Z_a = Z_A + (Z_B - Z_A)\frac{x}{L}$$

$$= \frac{(L-x)}{L}Z_A + \frac{x}{L}Z_B$$

Similarly, interpolating between C and D, in Fig. 8.5c,

$$Z_b = \frac{(L-x)}{L} Z_C + \frac{x}{L} Z_D$$

Now interpolating between a and b, in Fig. 8.5d, we get the Z coordinate of P.

$$Z_P = \frac{(L-x)}{L} Z_a + \frac{y}{L} Z_b$$

$$= \frac{(L-y)}{L}\left[\frac{(L-x)}{L} Z_A + \frac{x}{L} Z_B\right] + \frac{y}{L}\left[\frac{(L-x)}{L} Z_C + \frac{x}{L} Z_D\right]$$

$$= \frac{1}{L^2}\left[(L-x)(L-y)Z_A + x(L-y)Z_B + y(L-x)Z_C + xyZ_D\right] \qquad ..(8.13)$$

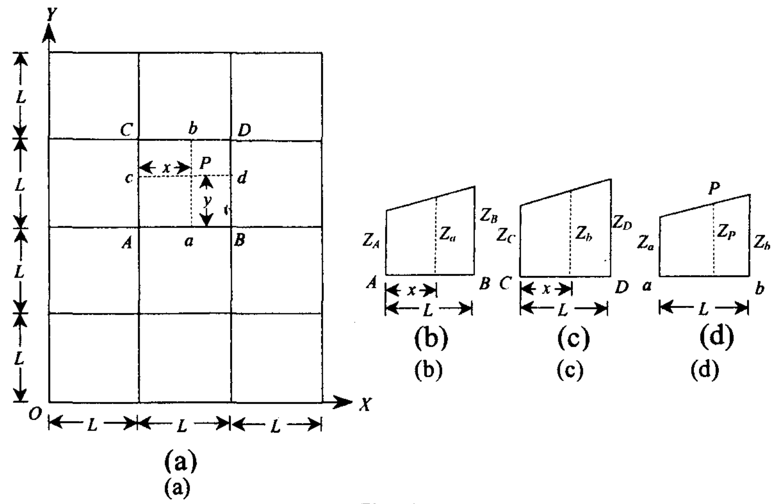


Fig. 8.5

8.5 MASS-HAUL DIAGRAM

A *mass-haul diagram* or *curve* is drawn subsequent to the calculation of earthwork volumes, its ordinates showing cumulative volumes at specific points along the centre line. Volumes of cut and fill are treated as positive and negative, respectively. Compensation can be made as necessary, for shrinkage or bulking of the excavated material when placed finally in an embankment.

Fig. 8.6 shows a typical mass-haul diagram in which the following characteristics of a mass-haul diagram may be noted:

(a) A to G and S to M indicate decreasing aggregate volume which imply the formation of embankment.

(b) Rising curve from R to B indicates a cut.

(c) R having a minimum ordinate is a point which occurs in the curve at the end of an embankment.

(d) S having the maximum ordinate is a point which indicates the end of a cut.

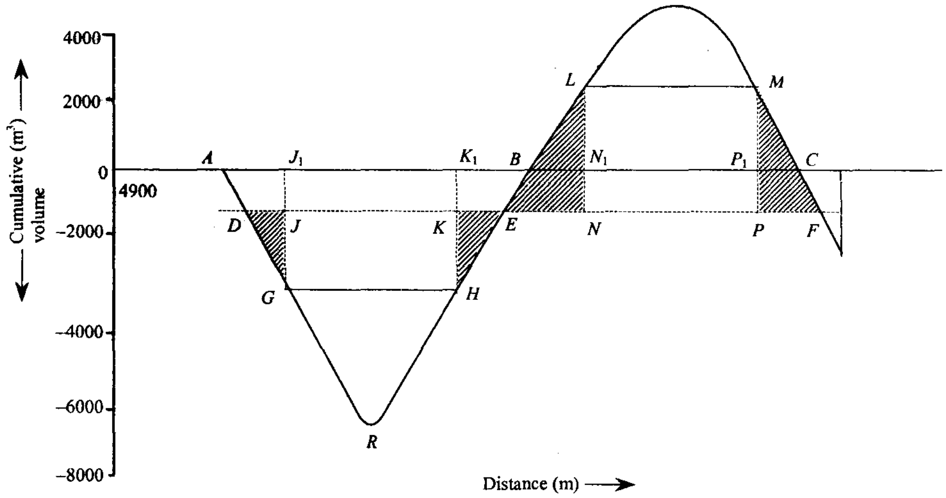


Fig. 8.6

In a mass-haul diagram for any horizontal line, e.g., *GH*, the ordinates at *G* and *H* will be equal, and therefore, over the length *GH*, the volume of cut and fill are equal, i.e., they are balanced out. When the curve lies between the trace line *ABC*, earth is moved to left, *B-R-A*, and, similarly, when the curve lies above, earth is moved to right, i.e., *B-S-C*. The length of the balancing line indicates the maximum distance that the earth will be transported within the particular loop of the diagram formed by the line. Though the base line *ABC* gives continuous balancing lines *AB* and *BC*, but for ensuring the most economical solution, the balancing lines should be taken at appropriate place without caring for continuity.

Haul: It is defined as the total volume of excavation multiplied by average haul distance. It is the area between the curve and balancing line, i.e., area *GRH* is the haul in length *GH*.

Free-haul: It is the distance up to which the hauling is done by the contractor free of charge. For this distance the cost of transportation of the excavated material is included in the excavation cost.

Overhaul: It is the excavated material from a cutting moved to a greater distance than the free-haul, the extra distance is overhaul.

Example 8.1. The length of a line originally 100 mm long on a map plotted to a scale of 1/1000, was found to be 96 mm due to shrinkage of the map. The map prepared using a tape of length 20 m was later found to be actually 20.03 m. If a certain area on the map, measured using a planimeter, is 282 mm^2, determine the correct area on the ground.

Solution:

Due to shrinkage in the map, the scale of the map will change from 1/1000 to 1/*S* where

$$S = 1000 \times \frac{100}{96}.$$

Further, since the 20 m tape was actually 20.03 m, a correction factor c of $\frac{20.03}{20}$ has to be applied to all the linear measurements.

The correct area on the ground due to change in scale

$$A' = \text{Measured area} \times (\text{new scale})^2$$
$$= \text{Measured area} \times S^2$$

Now the corrected area due to incorrect length of the tape

$$A = A'c^2$$

$$= 282 \times \left(1000 \times \frac{100}{96}\right)^2 \times \left(\frac{20.03}{20}\right)^2 \text{ mm}^2$$

$$= 282 \times \left(1000 \times \frac{100}{96}\right)^2 \times \left(\frac{20.03}{20}\right)^2 \times \frac{1}{1000^2} \text{ m}^2 = \mathbf{307\ m^2.}$$

Example 8.2. The coordinates of traverse stations of a closed traverse *ABCDE* are given in Table 8.1.

Table 8.1

Station	X (m)	Y (m)
A	0	0
B	+ 170	+ 320
C	+ 470	+ 90
D	+ 340	– 110
E	– 40	– 220

Calculate the area enclosed by the traverse.

Solution (Fig. 8.7):

Writing the coordinates in determinant form, we get

$$\begin{array}{cccccc} A & B & C & D & E & A \\ 0 & +170 & +470 & +340 & -40 & 0 \\ 0 & +320 & +90 & -110 & -220 & 0 \end{array}$$

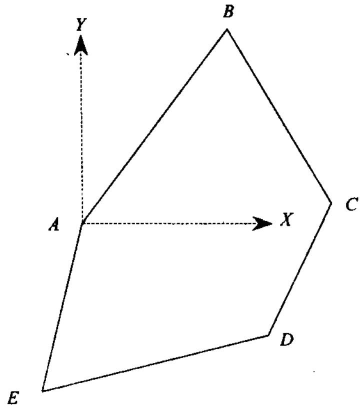


Fig. 8.7

Thus the area

$$A = \frac{1}{2} \times [0 \times 320 - 0 \times 170 + 170 \times 90 - 320 \times 470 + 470 \times (-110) - 90 \times 340$$

$$+ 340 \times (-220) - (-110) \times (-40) + (-40 +) \times 0 - (-220) \times 0]$$

$$= \frac{1}{2} \times (-296600) = -148300 \text{ m}^2$$

$= 148300 \text{ m}^2$ (neglecting the sign) = **14.83 hectares.**

It may be noted that the computed area has negative sign since the traverse has been considered clockwise.

Example 8.3. A tract of land has three straight boundaries *AB, BC,* and *CD*. The fourth boundary *DA* is irregular. The measured lengths are as under:

AB = 135 m, *BC* = 191 m, *CD* = 126 m, *BD* = 255 m.

The offsets measured outside the boundary *DA* to the irregular boundary at a regular interval of 30 m from *D*, are as below:

Distance from D (m)	0.0	30	60	90	120	150	180
Offsets (m)	0.0	3.7	4.9	4.2	2.8	3.6	0.0

Determine the area of the tract.

Solution (Fig. 8.8):

Let us first calculate the areas of triangles *ABD* and *BCD*.

The area of a triangle is given by

$$A = \sqrt{S(S-a)(S-b)(S-c)}$$

in which a, b, and c are the lengths of the sides, and $S = \frac{a+b+c}{2}$

For ΔABD

$$S = \frac{135 + 255 + 180}{2} = 285 \text{ m}$$

$$A_1 = \sqrt{285 \times (285 - 135) \times (285 - 255) \times (285 - 180)}$$

$= 11604.42 \text{ m}^2$.

For ΔBCD

$$S = \frac{191 + 126 + 255}{2} = 286 \text{m}$$

$$A_2 = \sqrt{286 \times (286 - 191) \times (286 - 126) \times (286 - 255)}$$

$= 11608.76 \text{ m}^2$.

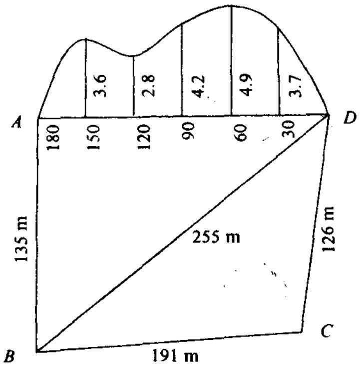


Fig. 8.8

Now to calculate the area of the irregular figure, use of trapezoidal rule or Simpson's rule can be made. The Simpson's rule require even number of increments, whereas the trapezoidal rule can be used for odd as well as even number of increments. In the present case since the number of increments is even, the area can be determined with either trapezoidal rule or Simpson's rule.

Area by trapezoidal rule

$$A = d\left(\frac{O_1 + O_7}{2} + O_2 + O_3 + O_4 + O_5 + O_6\right)$$

In this case O_1 and O_7 are the end offsets, and therefore $O_1 = O_7 = 0$ m.

Thus
$$A_3 = 30 \times \left(\frac{0+0}{2} + 3.6 + 2.8 + 4.2 + 4.9 + 3.7\right)$$
$$= 30 \times 19.2 = 576.00 \text{ m}^2.$$

Hence the total area of the tract

$$= A_1 + A_2 + A_3$$
$$= 11604.42 + 11608.76 + 576.00$$
$$= 23789.18 \text{ m}^2 = \mathbf{2.4\ hectares}.$$

Example 8.4. The area of an irregular boundary was measured using a planimeter. The initial and final readings were 9.036 and 1.645, respectively. The zero mark on the dial passed the index mark twice. The tracing point was moved in clockwise direction and needlepoint was outside the plan. Calculate the area of the plan if the multiplying constant of the planimeter is 100 cm^2.

Solution:

$$A = M\left(R_F - R_I \pm 10N + C\right)$$

Since the needlepoint was kept outside the plan, $C = 0$, and the tracing point was moved in clockwise direction, $N = +2$. Thus

$$A = 100 \times (1.645 - 9.036 + 10 \times 2) = \mathbf{1260.9\ m^2}.$$

Example 8.5. An area defined by the lines of a traverse *ABCDEA* is to be partitioned by a line *XY*, *X* being on *AB*, and *Y* on *CD*, having bearing 196°58′. The area of *XCDYX* has to be 30100 m^2. The coordinates of the traverse stations are given in Table 8.2.

Table 8.2

Station	*A*	*B*	*C*	*D*	*E*
Easting (*E*)	610	1010	760	580	460
Northing (*N*)	760	760	260	380	510

Solution (Fig. 8.9):

Let *X* be at a distance of *x* from *B* on *AB*, and *Y* be at a distance of *y* from *C* on *CD*. Since *A* and *B* have equal northings, i.e., N 760, the coordinates of *X* can be written as

$$\text{Easting of } X = \text{Easting of } B - x$$
$$= (1010 - x) = \gamma \text{ (say)}$$

$$\text{Northing of } X = \text{Northing of } A \text{ or } B$$
$$= 760 = \lambda \text{ (say).}$$

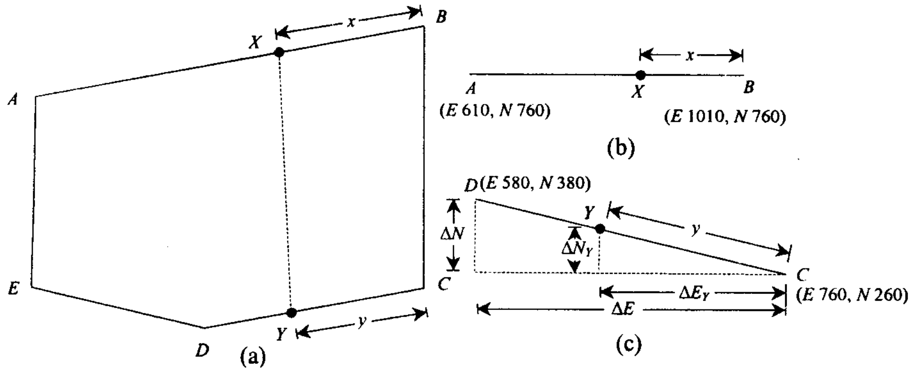


Fig. 8.9

From Fig. 8.9b, we find that

$$\frac{\Delta E_Y}{\Delta E} = \frac{Y}{CD} = \frac{\Delta N_Y}{\Delta N}$$

We have

$$\Delta E = 760 - 580 = 180 \text{ m}$$
$$\Delta N = 380 - 260 = 120 \text{ m}$$
$$CD = \sqrt{(\Delta E^2 + \Delta N^2)}$$
$$= \sqrt{(180^2 + 120^2)} = 216.33 \text{ m.}$$

Therefore

$$\Delta E_Y = \frac{\Delta E}{CD} y = \frac{180}{216.33} y = 0.832\ y$$
$$\Delta N_Y = \frac{\Delta N}{CD} y = \frac{120}{216.33} y = 0.555\ y$$

and

$$\text{Easting of } Y = \text{Easting of } C - \Delta E_Y$$
$$= 760 - 0.832\ y = \alpha \text{ (say)}$$
$$\text{Northing of } Y = \text{Northing of } C + \Delta N_Y$$
$$= 260 - 0.555\ y = \beta \text{ (say).}$$

Now the bearing of XY being 196°58′, the bearing of YX will be 196°58′ − 180° = 16°58′.

Thus

$$\tan 16°58' = \frac{E_X - E_Y}{N_X - N_Y} = \frac{\gamma - \alpha}{\lambda - \beta}$$
$$0.30509 = \frac{\gamma - \alpha}{\lambda - \beta}$$

$$\gamma - \alpha = 0.30509\ (\lambda - \beta)$$

$$1010 - x - 760 + 0.832\ y = 0.30509 \times (760 - 260 - 0.555\ y)$$

$$x = 97.455 + 1.001\ y.$$

Therefore

$$\text{Easing of } X = 1010 - (97.455 + 1.001\ y)$$

$$= 912.545 - 1.001\ y = x' \quad \text{(say)}$$

$$\text{Northing of } X = 760 \text{ m.}$$

Now writing the coordinates of *X*, *B*, *C*, *Y*, and *X* in determinant form

X	*B*	*C*	*Y*	*X*
x'	1010	760	α	x'
760	760	260	β	760

The area of the traverse *XBCYX* is

$$A = \frac{1}{2} \times [x' \times 760 - 760 \times 1010 + 1010 \times 260 - 760 \times 760 + 760 \times \beta - 260 \times \alpha$$

$$+ \alpha \times 760 - \beta \times x']$$

$$= \frac{1}{2} \times [(760 - \beta)x' + 760\beta + 500\alpha - 1082600]$$

$$= \frac{1}{2} \times [(760 - 260 - 0.555y) \times (912.545 - 1.001y) + 760 \times (260 + 0.555y)$$

$$+ 500 \times (760 - 0.832y) - 1082600]$$

$$= \frac{1}{2} \times [0.556y^2 - 1001.312y - 48727.5]$$

Since the traverse *XBCYX* has been considered in clockwise direction, the sign of the computed area will be negative. Therefore

$$\frac{1}{2} \times [0.556y^2 - 1001.312y - 48727.5] = -30100$$

$$0.556y^2 - 1001.312y - 48727.5 = -60200$$

$$0.556y^2 - 1001.312y + 11472.5 = 0$$

The solution of the above equation gives

$$y = 23.06 \text{ m}$$

and $$x = 97.455 + 1.001 \times 23.06 = 120.54 \text{ m}.$$

Thus the coordinates of X and Y are

$$E_X = 912.545 - 1.001 \times 23.06$$
$$= E\ 889.46 \text{ m} \approx \mathbf{E\ 889\ m}$$
$$N_X = \mathbf{N\ 760\ m}$$
$$E_Y = 760 - 0.832 \times 23.06$$
$$= E\ 740.81 \text{ m} \approx \mathbf{E\ 741\ m}$$
$$N_Y = 260 + 0.555 \times 23.06$$
$$= N\ 272.80 \text{ m} \approx \mathbf{N\ 273\ m}.$$

Example 8.6. Calculate the area of cross-section that has breadth of formation as 10 m, center height as 3.2 m and side slopes as 1 vertical to 2 horizontal.

Solution (Fig. 8.10):

A cross-section having no cross-fall, i.e., the ground transverse to the center line of the road is level, is called as a *level- section*. The area of a level-section is given by

$$A = h\ (b + sh)$$

where

h = the depth at the center line in case of cutting , and the height in case of embankment,

b = the formation width, and

1 in s = the side slope.

The widths w are given by

$$w = \frac{b}{2} + sh$$

It is given that

$$b = 10 \text{ m}$$
$$h = 3.2 \text{ m}$$
$$s = 2.$$

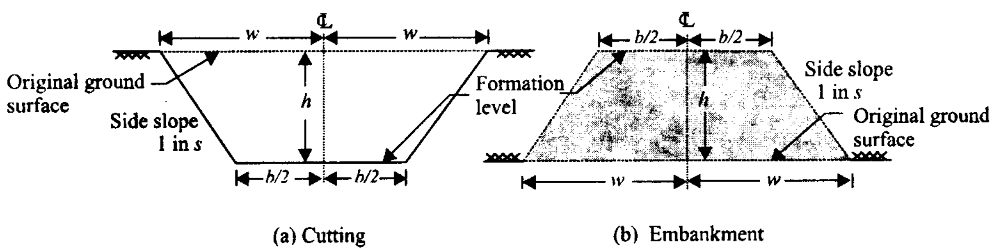


Fig. 8.10

Hence the area

$$A = 3.2 \times (10 + 2 \times 3.2) = \mathbf{52.48\ m^2}.$$

Example 8.7. Compute the area of cross-section if the formation width is 12 m, side slopes are 1 to 1, average height along the center line is 5 m, and the transverse slope of the ground is 10 to 1.

Solution (Fig. 8.11):

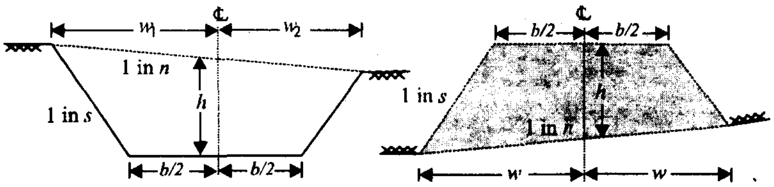
A cross-section that has a cross-fall is known as *two-level section*. For such sections the area is given by

$$A = \frac{1}{2s}\left[\left(\frac{b}{2} + sh\right)(w_1 + w_2) - \frac{b^2}{2}\right]$$

where
$$w_1 = \frac{n}{n-s}\left(\frac{b}{2} + sh\right),$$

$$w_2 = \frac{n}{n+s}\left(\frac{b}{2} + sh\right), \text{ and}$$

1 in n = the cross-fall of the ground.



(a) Cutting (b) Embankment

Fig. 8.11

From the given data, we have

$$b = 12 \text{ m}$$
$$h = 5 \text{ m}$$
$$s = 1$$
$$n = 10.$$

To calculate the area, let us first calculate w_1 and w_2.

$$w_1 = \frac{10}{10-1} \times \left(\frac{12}{2} + 1 \times 5\right) = 12.22 \text{ m}$$

$$w_2 = \frac{10}{10+1} \times \left(\frac{12}{2} + 1 \times 5\right) = 10.00 \text{ m}.$$

Therefore
$$A = \frac{1}{2 \times 1} \times \left[\left(\frac{12}{2} + 1 \times 5\right)(12.22 + 10.00) - \frac{12^2}{2}\right] = \mathbf{157.00\ m^2}.$$

Example 8.8. The following data pertains to a cross-section:

Formation width = 18 m

Depth of cut at the midpoint of formation = 3 m

Transverse slope on the right side of midpoint = 10 to 1

Transverse slope on the left side of midpoint = 7 to 1

Side slope = 2.5 to 1.

Compute the area of the cross-section.

Solution (Fig. 8.12):

In this problem the original ground surface has three different levels at A, B, and C, and such sections are known as *three-level section*. The area of a three level-section is given by the following expression.

$$A = \frac{1}{2}(w_1 + w_2)\left(h + \frac{b}{2s}\right) - \frac{b^2}{4s}$$

where

$$w_1 = \frac{n_1}{n_1 - s}\left(\frac{b}{2} + sh\right),$$

$$w_2 = \frac{n_2}{n_2 + s}\left(\frac{b}{2} + sh\right).$$

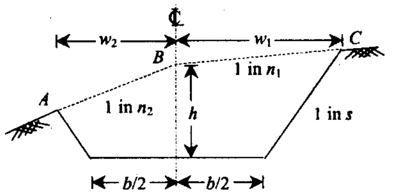


Fig. 8.12

It is given that

$$b = 18 \text{ m}$$
$$h = 3 \text{ m}$$
$$s = 2.5$$
$$n_1 = 10$$
$$n_2 = 7.$$

Now

$$w_1 = \frac{10}{10-2} \times \left(\frac{18}{2} + 2 \times 3\right) = 18.75 \text{ m}$$

$$w_2 = \frac{10}{7+2} \times \left(\frac{18}{2} + 2 \times 3\right) = 11.67 \text{ m}.$$

Therefore

$$A = \frac{1}{2} \times (18.75 + 11.67) \times \left(3 + \frac{18}{2 \times 2}\right) - \frac{18^2}{4 \times 2}$$

$$= \mathbf{73.58 \ m^2}.$$

Example 8.9. Calculate the area of a cut section shown in Fig. 8.13.

Solution (Fig. 8.12):

Since the point B at which the level is changing from 1 in 18 to 1 in 10 is not on the center line, there is no standard expression for such a three-level section. Let us derive the expression for determination of the area.

$$GE = b_1 = \frac{b}{2} + FE$$

$$FD = b_2 = \frac{b}{2} - FE$$

$$w_1 = \frac{n_1}{n_1 - s}(b_1 + sh)$$

$$w_2 = \frac{n_2}{n_2 + s}(b_2 + sh).$$

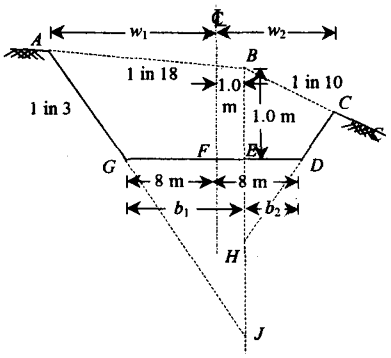


Fig. 8.13

The area of the cross-section *ABCDGA* is given by

A = Area of ΔABJ – area of ΔGFJ + area of ΔBCH – area of ΔEDH

$$= \frac{1}{2}w_1 BJ - \frac{1}{2}GE\, EJ + \frac{1}{2}w_2 BH - \frac{1}{2}ED\, EH$$

$$= \frac{1}{2}(w_1 BJ - b_1 EJ + w_2 BH - b_2 EH)$$

Now

$$BJ = BE + EJ = BE + \frac{b_1}{s}$$

$$EJ = \frac{b_1}{s}$$

$$BH = BE + EH = BE + \frac{b_2}{s}$$

$$EH = \frac{b_2}{s}.$$

From the figure, we have

$$b_1 = \frac{16}{2} + 1.0 = 9.0 \text{ m}$$

$$b_2 = \frac{16}{2} - 1.0 = 7.0 \text{ m}$$

$$s = 3,\ n_1 = 18,\ n_2 = 10$$

$$h = 1.0 \text{ m} = BE,\ b = 16 \text{ m}.$$

Therefore

$$w_1 = \frac{18}{18-2} \times (9.0 + 3 \times 1.0) = 13.5 \text{ m}$$

$$w_2 = \frac{10}{10+2} \times (7.0 + 3 \times 1.0) = 8.33 \text{ m}$$

$$BJ = 1.0 + \frac{9.0}{2} = 5.5 \text{ m.}$$

$$EJ = \frac{9.0}{2} = 4.5$$

$$BH = 1.0 + \frac{7.0}{2} = 4.5$$

$$EH = \frac{7.0}{2} = 3.5 \text{ m.}$$

Therefore the area of cross-section

$$A = \frac{1}{2} \times (13.5 \times 5.5 - 9.0 \times 4.5 + 8.33 \times 4.5 - 7.0 \times 3.5)$$

$$= \mathbf{23.37 \ m^2}.$$

Example 8.10. The width of a certain road at formation level is 9.50 m with side slopes 1 in 1 for cut and 1 in 2 for filling. The original ground has a cross-fall of 1 in 5. If the depth of excavation at the center line of the section is 0.4 m, calculate the areas of the cross-section in cut and fill.

Solution (Fig. 8.14):

The transverse slope of the ground as shown in Fig. 6.13, may intersect the formation level such that one portion of the section is in cutting and the other in filling. In such cases the section is partly in cut and partly in fill, and they are known as *side hill two-level section.*

Since the side slopes in fill and cut are different, the areas of fill and cut will different as below.

$$\text{Area of fill} = \frac{1}{2} \frac{\left(\frac{b}{2} - nh\right)^2}{n - s}$$

$$\text{Area of cut} = \frac{1}{2} \frac{\left(\frac{b}{2} + nh\right)^2}{n - r}.$$

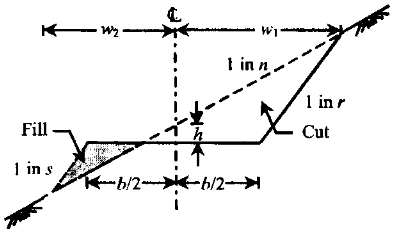


Fig. 8.14

The side widths can be calculated from the following expressions.

$$w_1 = \frac{n}{n - r}\left(\frac{b}{2} + rh\right)$$

$$w_1 = \frac{n}{n - s}\left(\frac{b}{2} - sh\right).$$

In Fig. 8.13, the center of the formation is lying in cut. If the center of the cross-section lies in the fill, in the expressions the $+h$ is replaced by $-h$ and *vice-versa* to get the areas, i.e.,

$$\text{Area of fill} = \frac{1}{2}\frac{\left(\frac{b}{2}+nh\right)^2}{n-s}$$

$$\text{Area of cut} = \frac{1}{2}\frac{\left(\frac{b}{2}-nh\right)^2}{n-r}.$$

The given data are

$$b = 9.5 \text{ m}$$
$$h = 0.4 \text{ m}$$
$$s = 2$$
$$r = 1$$
$$n = 5.$$

Since the center of formation is lying in cut, we have

$$\text{Area of cut} = \frac{1}{2}\times\frac{\left(\frac{9.5}{2}+5\times 0.4\right)^2}{5-1} = \mathbf{5.70\ m^2}$$

$$\text{Area of fill} = \frac{1}{2}\times\frac{\left(\frac{9.5}{2}-5\times 0.4\right)^2}{5-2} = \mathbf{1.26\ m^2}.$$

Example 8.11. A new road is to be constructed with formation width of 20 m, side slopes of 1 vertical to 2.5 horizontal. The heights of fill at the center line of three successive cross-sections, 50 m apart, are 3.3 m, 4.1 m, and 4.9 m, respectively. The existing ground has a cross-fall of 1 in 10. Calculate the volume of the fill.

Solution (Fig. 8.11b):

It is the case of two- level section for which the area is given by

$$A = \frac{1}{2s}\left[\left(\frac{b}{2}+sh\right)(w_1+w_2)-\frac{b^2}{2}\right]$$

where

$$w_1 = \frac{n}{n-s}\left(\frac{b}{2}+sh\right),$$

$$w_2 = \frac{n}{n+s}\left(\frac{b}{2}+sh\right).$$

The volume of fill is given by the end-areas rule, i.e.,

$$V = d\left(\frac{A_1 + A_3}{2} + A_2\right)$$

and by prismoidal rule

$$V = \frac{d}{3}(A_1 + A_3 + 4A_2)$$

where A_1, A_2, and A_3 are the areas of the three cross-sections.

The following are given

$$b = 20 \text{ m}$$
$$s = 2.5$$
$$n = 10$$
$$d = 50 \text{ m}$$
$$h_1, h_2, h_3 = 3.3 \text{ m}, 4.1 \text{ m}, 4.9 \text{ m}.$$

For section-1

$$w_1 = \frac{10}{10-2.5} \times \left(\frac{20}{2} + 2.5 \times 3.3\right) = 24.27 \text{ m}$$

$$w_2 = \frac{10}{10+2.5} \times \left(\frac{20}{2} + 2.5 \times 3.3\right) = 14.60 \text{ m}$$

$$A_1 = \frac{1}{2 \times 2.5} \times \left[\left(\frac{20}{2} + 2.5 \times 3.3\right) \times (24.27 + 14.60) - \frac{20^2}{2}\right]$$

$$= 101.88 \text{ m}^2.$$

For section-2

$$w_1 = \frac{10}{10-2.5} \times \left(\frac{20}{2} + 2.5 \times 4.1\right) = 26.93 \text{ m}$$

$$w_2 = \frac{10}{10+2.5} \times \left(\frac{20}{2} + 2.5 \times 4.1\right) = 16.20 \text{ m}$$

$$A_1 = \frac{1}{2 \times 2.5} \times \left[\left(\frac{20}{2} + 2.5 \times 4.1\right) \times (26.93 + 16.20) - \frac{20^2}{2}\right]$$

$$= 134.68 \text{ m}^2.$$

For section-3

$$w_1 = \frac{10}{10-2.5} \times \left(\frac{20}{2} + 2.5 \times 4.9\right) = 29.59 \text{ m}$$

$$w_2 = \frac{10}{10+2.5} \times \left(\frac{20}{2} + 2.5 \times 4.9\right) = 17.80 \text{ m}$$

$$A_1 = \frac{1}{2 \times 2.5} \times \left[\left(\frac{20}{2} + 2.5 \times 4.9\right) \times (29.59 + 17.80) - \frac{20^2}{2}\right]$$

$$= 170.89 \text{ m}^2.$$

Therefore

by end-areas rule

$$V = 50 \times \left(\frac{101.88 + 170.89}{2} + 134.68\right)$$

$$= 13553.3 \approx \mathbf{13553\ m^3}.$$

and by prismoidal rule

$$V = \frac{50}{3} \times (101.88 + 170.89 + 4 \times 134.68)$$

$$= 13524.8 \approx \mathbf{13525\ m^3}.$$

Since the end-areas rule gives the higher value of volume than the prismoidal rule, the volume by the former can be corrected by applying prismoidal correction given by the following formula for a two-level section.

$$C_{pc} = \frac{d}{6} s \left(\frac{n^2}{n^2 - s^2}\right)(h_1 - h_2)^2$$

$$\frac{d}{6} s \left(\frac{n^2}{n^2 - s^2}\right) = \frac{50}{6} \times 2.5 \times \left(\frac{10^2}{10^2 - 2.5^2}\right) = 22.22$$

Now

$$C_{pc1} = 22.22 \times (3.3 - 4.1)^2 = 14.22 \text{ m}^3$$

$$C_{pc2} = 22.22 \times (4.1 - 4.9)^2 = 14.22 \text{ m}^3.$$

Total correction

$$C_{pc} = C_{pc1} + C_{pc2} = 2 \times 14.22 = 28.44 \text{ m}^3.$$

Thus the corrected end-areas volume

$$= 13553 - 28.44 = 13524.56 \approx 13525 \text{ m}^3.$$

Example 8.12. Fig. 8.15 shows the distribution of 12 spot heights with a regular 20 m spacing covering a rectangular area which is to be graded to form a horizontal plane having an elevation of 10.00 m. Calculate the volume of the earth.

Solution (Fig. 8.15):

Since the finished horizontal surface has the elevation of 10.00 m, the heights of the corners above the finished surface will be (h – 10.00) where h is the spot heights of the points.

Now

$\Sigma h_1 = 17.18 + 17.76 + 18.38 + 17.76 = 71.08$ m

$\Sigma h_2 = 17.52 + 18.00 + 18.29 + 18.24 + 17.63 + 17.32 = 107.00$ m

$\Sigma h_3 = 0$

$\Sigma h_4 = 17.69 + 18.11 = 35.80$ m

$A = 20 \times 20 = 400$ m^2.

The volume is given by

$$V = \frac{A(\Sigma h_1 + 2\Sigma h_2 + 3\Sigma h_3 + 4\Sigma h_4)}{4}$$

$$= 400 \times \frac{(17.08 + 2 \times 107.00 + 3 \times 0 + 4 \times 35.80)}{4}$$

$$= \mathbf{37428.00\ m^3}.$$

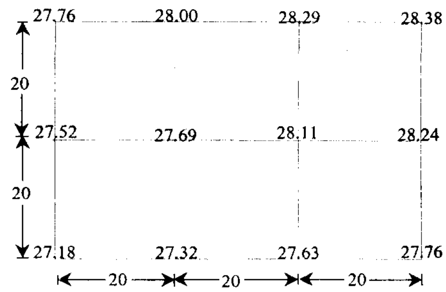


Fig. 8.15

Example 8.13. The area having spot heights given in Example 8.12 to be graded to form a horizontal plane at a level where cut and fill are balanced. Assuming no bulking or shrinking of the excavated earth and neglecting any effects of side slopes, determine the design level.

Solution (Fig. 8.15):

Let the design level be h. Thus

$\Sigma h_1 = (27.18 - h) + (27.76 - h) + (28.38 - h) + (27.76 - h) = 111.08 - 4h$

$\Sigma h_2 = (27.52 - h) + (28.00 - h) + (28.29 - h) + (28.24 - h) + (27.63 - h) + (27.32 - h)$

$= 167.00 - 6h$

$\Sigma h_3 = 0$

$\Sigma h_4 = (27.69 - h) + (28.11 - h) = 55.80 - 2h.$

Since the cut and fill are balanced, there will be no residual volume of excavated earth, therefore

$$V = \frac{400}{4} \times [(111.08 - 4h) + 2(167.00 - 6h) + 3 \times 0 + 4(55.80 - 2h)]$$

$$0 = 668.28 - 24h$$

$$h = \frac{668.28}{24} = \mathbf{27.85\ m}.$$

Example 8.14. The stations P and Q were established on the top of a spoil heap as shown in Fig. 8.16. Seven points were established at the base of the heap and one at the top on the line PQ. The observations given in Table 8.3 were recorded using a total station instrument from the stations P and Q keeping the heights of instrument and prism equal at both the stations.

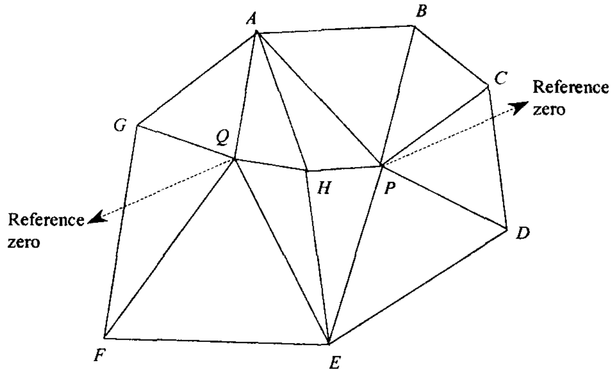


Fig. 8.16

Table-8.3

Station observed	Observations taken from station					
	P			*Q*		
	Horizontal angle	Horizontal distance	Difference in level	Horizontal angle	Horizontal distance	Difference in level
A	230°15′24″	50.23	− 5.65	128°23′25″	42.89	− 5.37
B	281°26′18″	41.69	− 5.34	–	–	–
C	322°47′32″	37.28	− 5.13	–	–	–
D	24°41′56″	46.17	− 5.44	–	–	–
E	104°17′46″	48.33	− 5.76	246°43′35″	58.96	− 5.48
F	–	–	–	323°46′32″	45.41	− 5.67
G	–	–	–	62°30′51″	34.35	− 5.71
H	183°37′10″	20.13	+ 0.25	191°14′00″	28.72	+ 0.53

Determine the volume of heap above a horizontal plane 6.0 m below the station *P*.

Solution (Fig. 8.17):

For such problems, the volume is calculated from a ground surface model consisting of a network of triangles. Further, it may be noted that if the point *H* does not lie on the ridge joining the stations *P* and *Q*, the top of the ridge would be truncated.

As shown in Fig. 8.17, the volume bounded between the ground surface and the horizontal surface is the sum of the volumes of the triangular prisms with base in horizontal plane and having heights equal to mean of the three heights of the corners. For example, the volume of the prism 1-2-3-1′-2′-3′ is

$$\frac{h_1 + h_2 + h_3}{3} \times \text{area of triangle } 1'\text{-}2'\text{-}3'$$

$$= h \times \text{area of triangle } 1'\text{-}2'\text{-}3'$$

where $$h = \frac{h_1 + h_2 + h_3}{3}.$$

In the present case, since the lengths of two sides of a triangle and the included angle are known, the area of the triangle can be computed from the following formula for a plane triangle.

$$A = \frac{1}{2} ab \sin C.$$

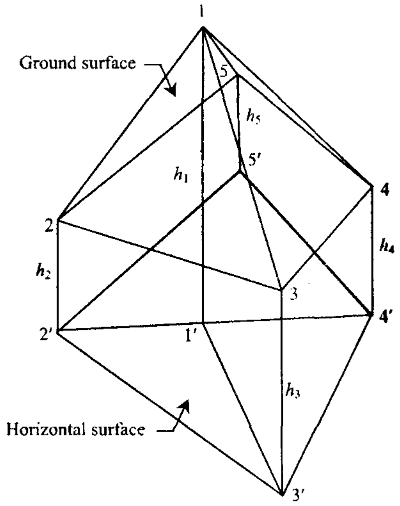


Fig. 8.17

Let us calculate the volume under the triangle *PAB*.

$\angle APB$ = Direction to *B* from *P* − direction to *A* from *P*

$= 281°26'18'' - 230°15'24'' = 51°10'54''$

$PA = L_1 = 50.23$ m

$PB = L_2 = 41.69$ m

Height of $P = h_1 = 6.00$ m

Height of $A = h_2 = 6.00 - 5.65 = 0.35$ m

Height of $B = h_3 = 6.00 - 5.34 = 0.65$ m

$$\text{Mean height } h = \frac{1}{3} \times (6 + 0.35 + 0.65) = 2.33 \text{ m}$$

$$\text{Volume} = 2.33 \times \frac{1}{2} \times 50.23 \times 41.69 \times \sin 51°10'54'' = 1900.79 \text{ m}^3.$$

Table 8.4

Triangle	Angle	L_1 (m)	L_2 (m)	h_1 (m)	h_2 (m)	h_3 (m)	h (m)	Volume (m^3)
PAB	51°10'54"	50.23	41.69	6.00	0.35	0.65	2.33	1900.79
PBC	41°21'14"	41.69	37.28	6.00	0.65	0.87	2.51	1288.73
PCD	61°54'24"	37.28	46.17	6.00	0.87	0.56	2.48	1882.85
PDE	79°35'50"	46.17	48.33	6.00	0.56	0.24	2.27	2491.00
PEH	79°19'24"	48.33	20.13	6.00	0.24	6.25	4.16	1988.56
PHA	46°38'14"	20.13	50.23	5.72	6.25	0.35	4 20	1543.74
QAH	62°50'35"	42.89	28.72	5.72	0.35	6.25	4.10	2246.81
QHE	55°29'35"	28.72	58.96	5.72	6.25	0.24	4.07	2839.64
QEF	77°02'57"	58.96	45.41	5.72	0.24	0.05	2.00	2609.27
QFG	98°44'19"	45.41	34.35	5.72	0.05	0.01	1.93	1487.77
QGA	65°52'34"	34.35	42.89	5.72	0.01	0.35	2.03	1364.77
					Total =			21643.93
								≈ **21644**

The horizontal plane is 6 m below P, or (6 + 0.25) = 6.25 m below H, or (6.25 – 0.53) = 5.72 m below Q. Therefore, to calculate the heights of the corners of the triangles, observed from P, the height 6.00 m of P, and observed from Q, the height 5.72 m of Q, above the horizontal plane have to be considered.

Table 8.4 gives the necessary data to calculate the volume of triangular prisms.

Example 8.15. Fig. 8.18 shows the spot heights at the nodes of the squares of 20 m side in certain area. With origin at O, three points P, Q, and R were located having coordinates as (23.0 m, 44.0 m), (86.0 m, 48.0 m), and (65.0 m, 2.0 m), respectively. It is proposed to raise the area PQR to a level of 50.00 m above datum. Determine the volume of earthwork required to fill the area.

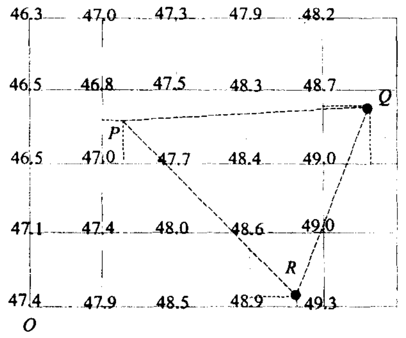


Fig. 8.18

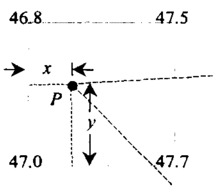


Fig. 8.19

Solution (Fig. 8.18):

To determine the heights of the points P, Q, and R above datum, let us define the squares as shown in Fig. 8.19.

For the point P

$$L = 20$$

$$x = 23.0 - 20 = 3.0 \text{ m}$$

$$L - x = 20 - 3.0 = 17.0 \text{ m}$$

$$y = 44.0 - 40 = 4.0 \text{ m}$$

$$L - y = 20 - 4.0 = 16.0 \text{ m}$$

$$Z_A = 47.0 \text{ m}, \quad Z_B = 47.7 \text{ m}, \quad Z_C = 46.8 \text{ m}, \quad Z_D = 47.5 \text{ m}.$$

$$Z_P = \frac{1}{L^2}[(L-x)(L-y)Z_A + x(L-y)Z_B + y(L-x)Z_C + xyZ_D]$$

$$= \frac{1}{20^2} \times [17.0 \times 16.0 \times 47.0 + 3.0 \times 16.0 \times 47.7 + 4.0 \times 17.0 \times 46.8 + 3.0 \times 4.0 \times 47.5]$$

$$= 47.1 \text{ m}.$$

For the point Q

$$x = 86.0 - 80 = 6.0 \text{ m}$$
$$L - x = 20 - 6.0 = 14.0 \text{ m}$$
$$y = 48.0 - 40 = 8.0 \text{ m}$$
$$L - y = 20 - 8.0 = 12.0 \text{ m}$$
$$Z_A = 49.0 \text{ m}, \quad Z_B = 49.3 \text{ m}, \quad Z_C = 48.7 \text{ m}, \quad Z_D = 48.9 \text{ m}.$$
$$Z_Q = \frac{1}{20^2} \times [14.0 \times 12.0 \times 49.0 + 6.0 \times 12.0 \times 49.3 + 8.0 \times 14.0 \times 48.7 + 6.0 \times 8.0 \times 48.9]$$
$$= 49.0 \text{ m.}$$

For the point R

$$x = 65.0 - 60 = 5.0 \text{ m}$$
$$L - x = 20 - 5.0 = 15.0 \text{ m}$$
$$y = 2.0 - 0 = 2.0 \text{ m}$$
$$L - y = 20 - 2.0 = 18.0 \text{ m}$$
$$Z_A = 48.9 \text{ m}, \quad Z_B = 49.3 \text{ m}, \quad Z_C = 48.6 \text{ m}, \quad Z_D = 49.0 \text{ m}.$$
$$Z_R = \frac{1}{20^2} \times [15.0 \times 18.0 \times 48.9 + 5.0 \times 18.0 \times 49.3 + 2.0 \times 15.0 \times 48.6 + 5.0 \times 2.0 \times 49.0]$$
$$= 49.0 \text{ m.}$$

The area of ΔPQR

$$PQ = \sqrt{(23.0 - 86.0)^2 + (44.0 - 48.0)^2} = 63.13 \text{ m}$$
$$QR = \sqrt{(86.0 - 65.0)^2 + (48.0 - 2.0)^2} = 50.57 \text{ m}$$
$$RP = \sqrt{(65.0 - 23.0)^2 + (2.0 - 44.0)^2} = 59.40 \text{ m}$$
$$S = \frac{1}{2}(PQ + QR + RP)$$
$$= \frac{1}{2} \times (63.13 + 50.57 + 59.40) = 86.55 \text{ m.}$$
$$\text{Area } A = \sqrt{[S(S - PQ)(S - QR)(S - RP)]}$$
$$= \sqrt{86.55 \times (86.55 - 63.13) \times (86.55 - 50.57) \times (86.55 - 59.40)}$$
$$= 1407.2 \text{ m}^2.$$

Depth of fills at P, Q, and R

$$h_P = 50.0 - 47.1 = 2.9 \text{ m}$$
$$h_Q = 50.0 - 49.0 = 1.0 \text{ m}$$
$$h_R = 50.0 - 49.0 = 1.0 \text{ m.}$$

Therefore volume of fill

$$= A\frac{(h_P + h_Q + h_R)}{3}$$

$$= 1407.2 \times \frac{(2.9 + 1.0 + 1.0)}{3} = 2298.4 \approx \mathbf{2298\ m^3}.$$

Example 8.16. Along a proposed road, the volumes of earthwork between successive cross-sections 50 m apart are given in Table 8.5. Plot a mass-haul diagram for chainage 5000 to 5600, assuming that the earthworks were balanced at chainage 5000. The positive volumes denote cut and the negative volumes denote fill. Draw the balancing lines for the following cases:

(a) Balance of earthworks at chainage 5000 and barrow at chainage 5600.

(b) Equal barrow at chainages 5000 and 5600.

Determine the costs of earthworks in the above two cases using the rates as under.

(i)	Excavate, cart, and fill within a free-haul distance of 200 m	Rs. 10.00 /m^3
(ii)	Excavate, cart, and fill for overhaul	Rs. 15.00 /m^3
(iii)	Barrow and fill at chainage 5000	Rs. 20.00 /m^3
(iv)	Barrow and fill at chainage 5600	Rs. 25.00 /m^3.

Table 8.5

Chainage (m)	Volume (m^3)
5000	–
5050	– 2100
5100	– 2400
5150	– 1500
5200	+ 1800
5250	+ 2200
5300	+ 2100
5350	+ 1700
5400	+ 1300
5450	+ 300
5500	– 600
5550	– 2300
5600	– 2500

Solution (Fig. 8.20):

Taking cut as positive and fill as negative the cumulative volumes are determined at each chainage with zero volume at chainage 5000 since at this chainage the earthworks were balanced, and the same have been tabulated in Table 8.6. Plot the mass-haul diagram Fig. 8.19, taking the chainage on x-axis and the cumulative volumes on y-axis.

Case (a): Since the earthworks have been balanced at the chainage 5000, there is no surplus or barrow at this chainage, therefore the balancing must commence from A which is the origin of the mass-haul diagram. This balancing line will intersect the mass-haul diagram at B and C leaving a barrow of 2000 m^3 at chainage 5600.

Case (b): For equal barrow at chainages 5000 and 5600, the balancing line must bisect the ordinate at chainage 5600, i.e., the balancing line EJ must pass through J such that DJ = JQ = $DQ/2$ = – 2000/2 = – 1000 m^3 or 1000 m^3 barrow at chainages 5000 and 5600.

Table 8.6

Chainage (m)	Volume (m^3)		Cumulative Volume (m^3)
	Cut (+)	Fill (–)	
5000	–	–	0
5050	–	– 2100	– 2100
5100	–	– 2400	– 4500
5150	–	– 1500	– 6000
5200	+ 1800	–	– 4200
5250	+ 2200	–	– 2000
5300	+ 2100	–	+ 100
5350	+ 1700	–	+ 1800
5400	+ 1300	–	+ 3100
5450	+ 300	–	+ 3400
5500	–	– 600	+ 2800
5550	–	– 2300	+ 500
5600	–	– 2500	– 2000

The free-haul distance is the distance up to which carting of the excavated material is done without any extra payment to the contractor, the cost of transportation being included in excavation cost. Beyond the free-haul distance if the excavation is to cart it is overhaul and a different unit rate 55 is applied. To show free-haul distance of 200 m, draw the balancing lines LM and NP 200 m long. The volumes of excavation involved are given by the intercepts from LM to H, and NP to K. Since the balancing line indicates that cut balances fill over the length of the balancing line the earth will be cart a maximum distance of 200 m from M to L, and N to P, respectively.

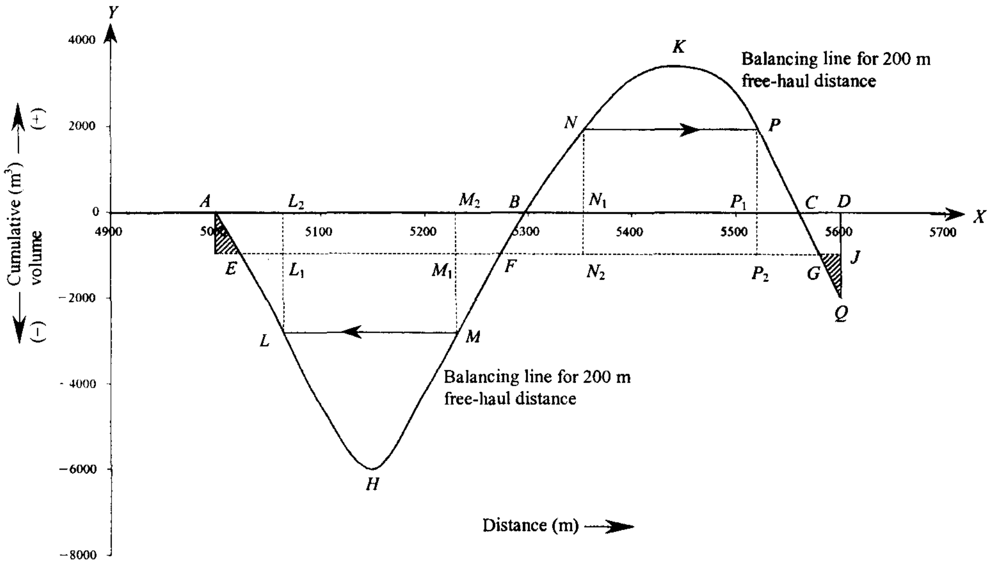


Fig. 8.20

Cost of excavation

Case (a): The balancing line AC shows balance of earthworks at chainage 5000 with barrow at chainage 5600. The balancing lines LM and NP show free-haul distance of 200 m. In this case the remaining volumes to be overhauled are given by the ordinates LL_2 and NN_1.

From the mass-haul diagram, we get

$$LL_2 = 2100 \text{ m}^3$$

$$NN_1 = 1400 \text{ m}^3$$

And let r_1 = Rs 10.00 /m^3, r_2 = Rs 15.00 /m^3, r_3 = Rs 20.00 /m^3, r_4 = Rs 25.00 /m^3.

Free-haul volume V_1 = Intercept between LM and H + intercept between NP and K

= (Ordinate of H – ordinate of L) + (ordinate of K – ordinate of N)

= (6000 – 2100) + (3400 – 1400)

= 5900 m^3

Overhaul volume V_2 = Intercept between AB and LM + intercept between BC and NP

= (Ordinate of L – ordinate of A) + (ordinate of N – ordinate of B)

= (2100 – 0) + (1400 – 0)

= 3500 m^3

Barrow at chainage 5600 V_3 = 2000 m^3

Therefore cost = $V_1 r_1 + V_2 r_2 + V_3 r_4$

= 5900 × 10.00 + 3500 × 15.00 +2000 × 25.00

= **Rs. 161500.00**

Case (b): The balancing line EJ shows equal barrow at chainages 5000 and 5600. The balancing lines LM and NP show free-haul distance of 200 m. In this case the remaining volumes to be overhauled are given by the ordinates LL_1 and NN_2.

From the mass-haul diagram, we get

$$L_1L_2 = N_1N_2 = \text{DJ} = \frac{1}{2}\text{DQ} = \frac{2000}{2} = 1000 \text{ m}^3$$

$$NN_2 = NN_1 + N_1N_2 = 1400 + 1000 = 2400 \text{ m}^3$$

$$LL_1 = LL_2 - L_1L_2 = 2100 - 1000 = 1100 \text{ m}^3$$

Free-haul volume V_1 = Same as in the case (a)

= 5900 m^3

Overhaul volume V_2 = Intercept between LM and EF + intercept between FG and NP

$= LL_1 + NN_2 = 1100 + 2400 = 3500 \text{ m}^3$

Barrow at chainage 5000 $V_3 = 1000 \text{ m}^3$

Barrow at chainage 5600 $V_4 = 1000 \text{ m}^3$

Therefore cost $= V_1r_1 + V_2r_2 + V_3r_3 + V_4r_4$

$= 5900 \times 10.00 + 3500 \times 15.00 + 1000 \times 20.00 + 1000 \times 25.00$

= **Rs. 156500.00**.

OBJECTIVE TYPE QUESTIONS

1. If area calculated by end - areas rule and prismoidal rule are A_e and A_p, respectively, then $(A_e - A_p)$
   (*a*) is always positive.
   (*b*) is always negative.
   (*c*) may be positive or negative.
   (*d*) is equal to zero.
2. Prismoidal correction is required to correct the volume calculated
   (*a*) using contours.
   (*b*) using spot heights.
   (*c*) for a curved section.
   (*d*) by end-areas rule.
3. Curvature correction to the computed volume is applied when
   (*a*) the formation levels at the cross-sections are at different levels.
   (*b*) the successive cross-sections are not parallel to each other.
   (*c*) the distance between the successive cross-sections is quite large.
   (*d*) none of the above.

4. Free-haul distance is

(*a*) the length of a balancing line.

(*b*) the distance between two balancing lines.

(*c*) the distance between two successive points where the mass-haul diagram intersects the line of zero ordinate.

(*d*) the distance up to which carting of excavated material is done without extra payment.

5. A mass-haul diagram indicates cutting if the curve

(*a*) rises.

(*b*) falls.

(*c*) becomes horizontal.

(*d*) none of the above.

6. A mass-haul diagram indicates fill if the curve

(*a*) rises.

(*b*) falls.

(*c*) becomes horizontal.

(*d*) none of the above.

7. Maximum ordinate on a mass-haul diagram occurs

(*a*) at the end of a cut.

(*b*) at the end of an embankment.

(*c*) where cut and fill are balanced.

(*d*) none of the above.

8. A minimum ordinate on a mass-haul diagram occurs

(*a*) at the end of a cut.

(*b*) at the end of an embankment.

(*c*) where cut and fill are balanced.

(*d*) none of the above.

ANSWERS

1. (*a*)	**2.** (*d*)	**3.** (*b*)	**4.** (*d*)	**5.** (*a*)	**6.** (*b*)
7. (*a*)	**8.** (*b*)				

9 Point Location and Setting Out

9.1 SETTING OUT

Setting out is a procedure adopted to correctly position a specific design feature such as a building, a road, a bridge, a dam, etc., on the ground at the construction site. It requires location of the control fixed during the original survey. These may be subsidiary stations which are located by the method of intersection or resection (see Sec. 6.6) from controls already fixed. Setting out is thus the reverse process of detail surveying in that the control stations are used to fix the points on the ground in their correct relative positions.

9.2 POINT LOCATION

If two control points *A* and *B* are known, a third point *C* as shown in Fig. 9.1, can be located in a number of ways.

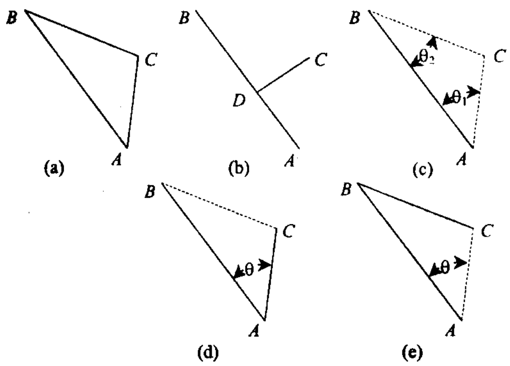


Fig. 9.1

(*a*) Set out distances *AC* and *BC*.

(*b*) Set out distance *AD* and then perpendicular distance *DC*.

(*c*) Set off angles θ_1 and θ_2.

(*d*) Set off angle θ and distance *AC*.

(*e*) Set off angle θ and distance *BC*.

9.3 INTERSECTION AND RESECTION

Subsidiary stations close to the work site can be fixed using intersection or resection. The method of locating points by intersection and resection is discussed in Sec. 6.6.

In Fig. 9.1c, the point C can either be coordinated by observing the angles θ_1 and θ_2 or be located if its coordinates are known, since the angles θ_1 and θ_2 can be calculated.

Location of points by resection requires pointings made on at least three known stations. This technique is very useful in setting out works since it allows the instrument to be sited close to the proposed works, and its coordinates to be obtained from two angle observations.

The coordinates of appoint C, shown in Fig. 9.2, can be obtained by usual method from the given data. However, if many points are to be fixed it is useful to employ the following general formulae. The points of known coordinates A and B, and the point C to be located, are considered clockwise.

$$E_C = \frac{E_A \cot\theta_B + E_B \cot\theta_A + (N_B - N_A)}{\cot\theta_A + \cot\theta_B} \qquad \ldots(9.1)$$

$$N_C = \frac{N_A \cot\theta_B + N_B \cot\theta_A - (N_B - N_A)}{\cot\theta_A + \cot\theta_B} \qquad \ldots(9.2)$$

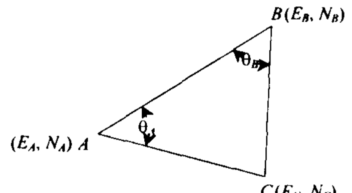


Fig. 9.2

9.4 TRANSFER OF SURFACE ALIGNMENT-TUNNELLING

Transferring the surface alignment through a vertical shaft is difficult operation in view of the small size of the shaft. Generally, plumbwires are used to transfer directions underground. Essentially, the plumbwires produce a vertical reference plane, and on the surface the plane can be placed in the line of sight; below ground, the line of sight can be sighted into that plane. This is known as *co-planing*, and the line of sight when established can be used to set up floor or roof stations within the tunnel.

Accurate transfer of surface alignment down a vertical shaft using two plumbwires can be achieved by *Weisbach triangle* method.

In Fig. 9.3, p and q are plan positions of the plumbwires P and Q on the ground surface alignment above the tunnel, respectively. A theodolite, reading directly one second, is set up at A', approximately in line with p and q. In triangle $pA'q$, the angle $pA'q$ is measured by the method of repetition, and the lengths of sides are also measured correct up to millimeter. The angle $pq\,A'$ is also calculated by applying sine rule.

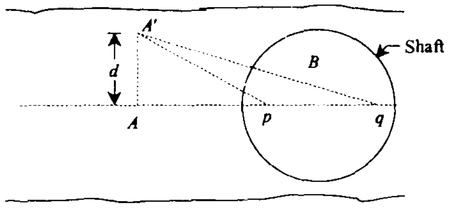


Fig. 9.3

Now, the perpendicular distance d of A' from the line qp produced, is calculated from the following expression.

$$d = \frac{pA'\,qA'}{pq} \sin pA'q \qquad \ldots(9.3)$$

The point p and q are joined by a fine thread, and a perpendicular AA' equal to d in length is dropped from A' on the thread. The foot of perpendicular A is the required point on the line qp produced which may be occupied by the theodolite for fixing the points on the floor or roof of the tunnel.

9.5 SETTING OUT BY BEARING AND DISTANCE

Setting out by bearing and distance, or by *polar rays* as it is often referred to, is a common task using modern instrument. The computed values of the angles should always be whole circle bearings (WCB) and not angles reduced to the north-south axis, i.e., the reduced bearings. If the WCB is always calculated by subtracting the northing and easting values of the point where the line starts from point being aimed at, then positive and negative signs of the angle obtained will be correct.

9.6 MONITORING MOVEMENTS

One of the important tasks of surveying is measurement of small movements due to deformation in civil engineering structures or in industrial measurement system. When structure, such as a dam, is loaded, it will move, and accurate theodolite observations on to targets attached to the structure can be used to measure this movement. The horizontal position of the target can be calculated by the process of intersection whilst the vertical movement is calculated by tangent trigonometry.

9.7 CONSTRUCTION LASERS

A rotating construction laser produces a horizontal plane of laser light. This may be manifest as a line on a staff or, in the case of infrared instruments, located by a photo-electric cell mounted on the staff. The equipment allows the task of setting out levels to be carried out by one person, and the calculations involved are essentially the same as those for a levelling exercise.

9.8 SIGHT RAILS FOR A TRENCH SEWER

The sight rails are positioned so that the line connecting their upper edges reflects the gradient of the trench bottom or the pipe invert, as applicable. A boning rod or traveller of correct length is held with the upper edge of its horizontal sight bar just in the line of sight given by the sight rails; in this position its lower end stands at the required level (Fig. 9.4). The horizontal sight rails are nailed to stout uprights, firmly installed on alternate sides of the trench. These uprights must be well clear of the sides of the trenches. Frequent checking of their integrity is essential.

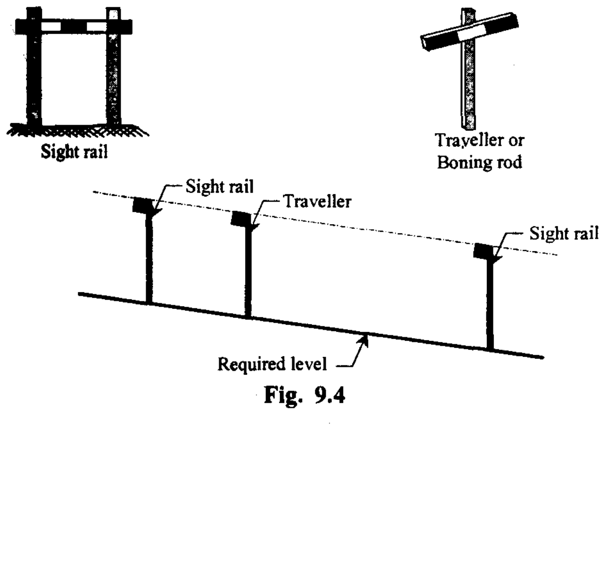


Fig. 9.4

9.9 EMBANKMENT PROFILE BOARDS

The profile boards are nailed to two uprights which are firmly driven into the ground near the toes of the embankment (Fig. 9.5). The inner uprights must have clearances from the toes of the order of 1.0 m to prevent disturbance. The inner and outer uprights can be spaced up to 1.0 m apart, since the sloping boards reflecting the side gradients, need to be of reasonable length for sighting purposes. A traveller is used in conjunction with the upper surface of the boards to achieve the gradients.

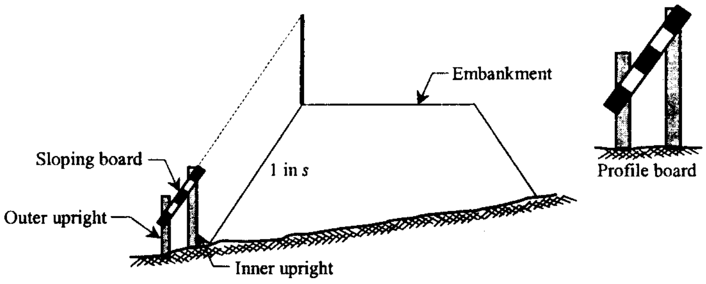


Fig. 9.5

Example 9.1. The four corners *A, B, C,* and *D* of a rectangular building having the coordinates given in Table 9.1, are to be set out from control station *P* by a total station instrument. Calculate the WCB and distance to establish each corner of the building.

Table 9.1

Corner	Easting (m)	Northing (m)
A	117.984	92.849
B	82.629	128.204
C	33.132	78.707
D	68.487	43. 352

The coordinates of *P* are *E* 110.383 m, *N* 81.334 m.

Solution (Fig. 9.6):

For corner *A*

$$\text{WCB of } PA = \theta_{PA} = \tan^{-1}\left[\frac{E_A - E_P}{N_A - N_P}\right]$$

$$= \tan^{-1}\left[\frac{117.984 - 110.383}{92.849 - 81.334}\right] = \tan^{-1}\left[\frac{7.601}{11.515}\right]$$

$$= \mathbf{33°25'43''}$$

$$PA = \sqrt{(E_A - E_P)^2 + (N_A - N_P)^2}$$

$$= \sqrt{7.601^2 + 11.515^2}$$

$$= \mathbf{13.797\ m.}$$

For corner B

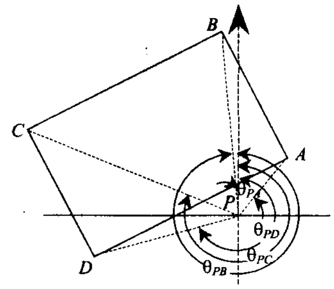


Fig. 9.6

$$\text{WCB of } PB = \theta_{PB} = \tan^{-1}\left[\frac{E_B - E_P}{N_B - N_P}\right]$$

$$= \tan^{-1}\left[\frac{82.629 - 110.383}{128.204 - 81.334}\right]$$

$$= \tan^{-1}\left[\frac{-27.754}{46.870}\right]$$

$= 30°37'55''$ (from a calculator)

The whole circle bearings are never negative. The computed value using a calculator is from trigonometric functions relative to north-south axis with positive and negative signs depending upon the quadrant containing the angle. The tangent value of an angle is positive or negative as shown in Fig. 9.7.

In this case northing of B is greater than that of P so B must lie in the fourth quadrant. Therefore

$$\theta_{PB} = 360° - 30°37'55'' = \mathbf{329°22'05''}$$

$$PB = \sqrt{(-27.754)^2 + 46.870^2} = \mathbf{54.471\ m.}$$

For corner C

$$\text{WCB of } PC = \theta_{PC} = \tan^{-1}\left[\frac{E_C - E_P}{N_C - N_P}\right]$$

$$= \tan^{-1}\left[\frac{33.132 - 110.383}{78.707 - 81.334}\right]$$

$$= \tan^{-1}\left[\frac{-77.251}{-2.627}\right]$$

$= 88°03'08''$ (from a calculator)

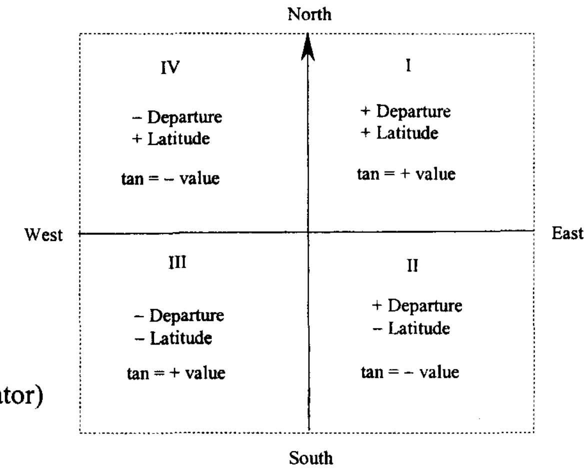


Fig. 9.7

Since both easting and northing of C are less that those of P as indicated by the negative signs in numerator and denominator of the calculation of $\tan^{-1}$, the point C must be in the third quadrant to get the positive value of the tangent of WCB. Therefore

$$\theta_{PC} = 180° + 88°03'08'' = \mathbf{268°03'08''}$$

$$PC = \sqrt{(-77.251)^2 + (-2.627)^2} = \mathbf{77.296\ m.}$$

For corner D

$$\text{WCB of } PD = \theta_{PD} = \tan^{-1}\left[\frac{E_D - E_P}{N_D - N_P}\right]$$

$$= \tan^{-1}\left[\frac{68.487 - 110.383}{43.352 - 81.334}\right]$$

$$= \tan^{-1}\left[\frac{-41.896}{-37.982}\right] = 47°48'19'' \text{ (from a calculator)}$$

The point D is also in the third quadrant due to the same reasons as for C above. Therefore

$$\theta_{PD} = 180° + 47°48'19'' = \mathbf{227°48'19^2}$$

$$PD = \sqrt{(-41.896)^2 + (-37.982)^2} = \mathbf{56.55\ m.}$$

Example 9.2. From two triangulation stations A and B the clockwise horizontal angles to a station C were measured as $\angle BAC = 50°05'26^2$ and $\angle ABC = 321°55'44^2$. Determine the coordinates of C given those of A and B are

A	E 1000.00 m	N 1350.00 m
B	E 1133.50 m	N 1450.00 m.

Solution (Fig. 9.8):

$$\alpha = 50°05'26''$$

$$\beta = 360° - 321°55'44^2 = 38°04'16''.$$

If the coordinates of A, B, and C are (E_A, N_A), (E_B, N_B), and (E_C, N_C), respectively, then

$$BD = E_B - E_A = 1133.50 - 1000.00 = 133.50 \text{ m}$$

$$AD = N_B - N_A = 1450.00 - 1350.00 = 100.00 \text{ m.}$$

Therefore bearing of AB

$$\theta = \tan^{-1}\frac{BD}{AD}.$$

$$= \tan^{-1}\frac{133.50}{100.0} = 53°09'52''.$$

$$\text{Bearing of } AC = \theta_{AC} = \text{Bearing of } AB + \alpha$$

$$= 53°09'52'' + 50°05'26''$$

$$= 103°15'18''.$$

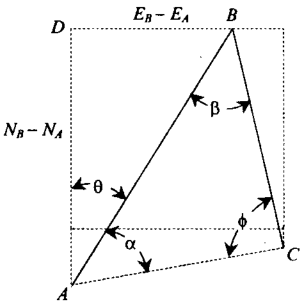


Fig. 9.8

Bearing of $BA = \theta_{BC}$ = Bearing of AB + 180°
= 53°09′52″ + 180° = 233°09′52″.

Bearing of $BC = \theta_{BC}$ = Bearing of $BA - b$
= 233°09′52″ − 38°04′16″ = 195°05′36″.

In ΔABC, we have

$$\angle ACB = 180° - (\alpha + \beta)$$
$$= 180° - (50°05'26'' + 38°04'16'') = 91°50'18''.$$

$$AB = \sqrt{BD^2 + AD^2}$$
$$= \sqrt{133.50^2 + 100.00^2} = 166.80 \text{ m.}$$

$$AC = \frac{AB \sin\beta}{\sin\phi}$$
$$= \frac{166.80 \times \sin 38°04'16''}{\sin 91°50'18''} = 102.90 \text{ m}$$

$$BC = \frac{AB \sin\alpha}{\sin\phi}$$
$$= \frac{166.80 \times \sin 50°05'26''}{\sin 91°50'18''} = 128.01 \text{ m.}$$

Let the latitude and departure of C, considering the line AC and BC, be respectively L_{AC}, D_{AC}, and L_{BC}, D_{BC}.

$$L_{AC} = AC \cos\theta_{AC} = 102.90 \times \cos 103°15'18'' = -23.59 \text{ m}$$
$$D_{AC} = AC \sin\theta_{AC} = 102.90 \times \sin 103°15'18'' = 100.16 \text{ m}$$
$$L_{BC} = BC \cos\theta_{BC} = 128.01 \times \cos 195°05'36'' = -123.59 \text{ m}$$
$$D_{BC} = BC \sin\theta_{BC} = 128.01 \times \sin 195°05'36'' = -33.33 \text{ m.}$$

Coordinates of C

$$E_C = E_A + D_{AC} = 1000.00 + 100.16 = 1100.16 \text{ m}$$
$$= E_B + D_{BC} = 1133.50 - 33.33 = 1100.17 \text{ m} \quad (Okay)$$
$$N_C = N_A + L_{AC} = 1350.00 - 23.59 = 1326.41 \text{ m}$$
$$= N_B + L_{BC} = 1450.00 - 123.59 = 1326.41 \text{ m} \quad (Okay).$$

Thus, the coordinates of C are ***E* 1100.17 m** and ***N* 1326.41 m**.

Example 9.3. The coordinates of two control points A and B are

A	E 3756.81 m	N 1576.06 m
B	E 3614.09 m	N 1691.63 m.

A subsidiary station P having coordinates E 3644.74 m, N 1619.74 m, is required to be established. Determine the angles at A and B to be set out, and distances AP and BP to check the fixing of P.

Solution (Fig. 9.9):

Let the bearings of the lines AB, AP, and BP be θ_1, θ_2, and θ_3, respectively. If the coordinates of A, B, and P are (E_A, N_A), (E_B, N_B), and (E_P, N_P), respectively, then the bearings and lengths of the lines are as below.

$$\Delta E_{AB} = E_B - E_A = 3614.09 - 3756.81 = -142.72 \text{ m}$$

$$\Delta N_{AB} = N_B - N_A = 1691.63 - 1576.06 = +115.57 \text{ m}$$

$$\Delta E_{AP} = E_P - E_A = 3644.74 - 3756.81 = -112.07 \text{ m}$$

$$\Delta N_{AP} = N_P - N_A = 1619.74 - 1576.06 = +43.68 \text{ m}$$

$$\Delta E_{BP} = E_P - E_B = 3644.74 - 3614.09 = +30.65 \text{ m}$$

$$\Delta N_{BP} = N_P - N_B = 1619.74 - 1691.63 = -71.89 \text{ m.}$$

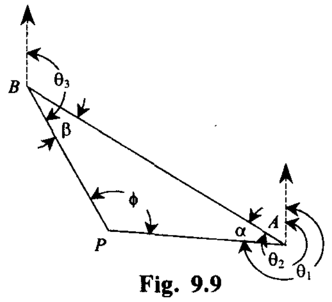


Fig. 9.9

(i) Bearing of the lines

$$\theta_1 = \tan^{-1}\frac{\Delta E_{AB}}{\Delta N_{AB}}$$

$$= \tan^{-1}\frac{-142.72}{115.57} = -51°00'02''.$$

Since ΔE_{AB} is negative is positive and ΔN_{AB} is positive, the line AB is fourth quadrant, therefore

$$\theta_1 = 360° - 51°00'02'' = 308°59'58''.$$

$$\theta_2 = \tan^{-1}\frac{\Delta E_{AP}}{\Delta N_{AP}}$$

$$= \tan^{-1}\frac{-112.07}{43.68} = -68°42'23''.$$

The line AP is fourth quadrant, therefore

$$\theta_2 = 360° - 68°42'23'' = 291°17'37''.$$

$$\theta_3 = \tan^{-1}\frac{\Delta E_{BP}}{\Delta N_{BP}} = \tan^{-1}\frac{30.65}{-71.89}$$

$$= -23°05'27''.$$

The line BP is second quadrant, therefore

$$\theta_3 = 180° - 23°05'27'' = 156°54'33''.$$

Now the angles α and β can be calculated as below.

$$\alpha = \text{Bearing of } AB - \text{bearing of } AP$$

$$= \theta_1 - \theta_2 = 308°59'58'' - 291°17'37'' = 17°42'21''$$

β = Bearing of BP – back bearing of AB

$= \theta_1 - \theta_2 = 156°54'33'' - (180° + 308°59'58'' - 360°) = 27°54'35''$.

To check the computations

ϕ = (360° – back bearing of BP) + back bearing of AP

$= [360° - (180° - 156°54'33'')] + (180° + 291°17'37'') = 134°23'04''$

Now, in ΔAPB we should have

$$\alpha + \beta + \phi = 180°$$

$$17°42'21'' + 27°54'35'' + 134°23'04'' = 180° \quad (Okay).$$

Lengths of the lines

$$AP = \sqrt{\Delta E_{AP}^2 + \Delta N_{AP}^2} = \sqrt{(-112.07)^2 + 43.68^2} = 120.28 \text{ m.}$$

$$BP = \sqrt{\Delta E_{BP}^2 + \Delta N_{BP}^2} = \sqrt{30.65^2 + (-71.89)^2} = 78.15 \text{ m.}$$

Thus

$$\angle BAP = \mathbf{17°42'21''}$$

$$\angle ABP = \mathbf{27°54'35''}$$

$$AP = \mathbf{120.28 \text{ m}}$$

$$BP = \mathbf{78.15 \text{ m.}}$$

Example 9.4. To monitor the movement of dam, the observations were made on a target C attached to the wall of the dam from two fixed concrete pillars A and B, situated to the north-west of the dam. The coordinates and elevations of the pillar tops on which a theodolite can be mounted for making observations, are:

A	E 1322.281 m,	N 961.713 m,	241.831 m
B	E 1473.712 m	N 1063.522 m,	242.262 m

The observations given in Table-9.2, were made with a theodolite having the height of collimation 486 m above the pillar top. If the height of C is given by the mean observations from A and B, determine its movement after the reservoir is filled.

Table 9.2

Station	Angle	Before filling	After filling
A	Horizontal $\angle BAC$	55°11′23″	55°11′12″
	Vertical angle $\angle C$	+5°33′12″	+5°33′06″
B	Horizontal $\angle ABC$	48°31′18″	48°31′05″
	Vertical angle $\angle C$	+4°54′42″	+4°54′38″

Solution (Fig. 9.10):

The given data are

$$E_A, N_A = E \text{ } 1322.281 \text{ m}, N \text{ } 961.713 \text{ m}$$

$$E_B, N_B = E \text{ } 1473.712 \text{ m}, N \text{ } 1063.522 \text{ m}$$

$$h_A = 241.831 \text{ m}$$
$$h_B = 242.262 \text{ m}$$
$$h_i = 486 \text{ mm}$$
$$\theta_A = \angle BAC$$
$$\theta_B = \angle ABC$$
α = Vertical angle to C at A

β = Vertical angle to C at B.

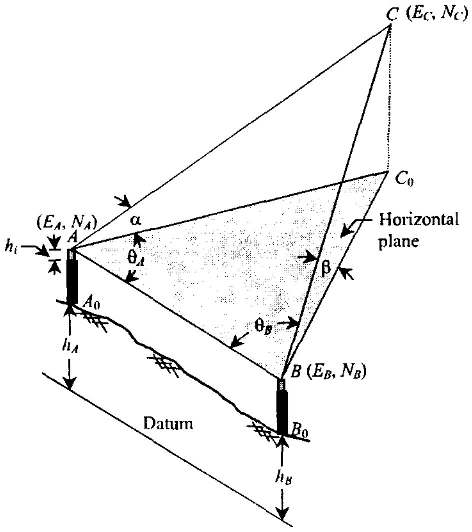


Fig. 9.10

$$\text{Bearing } \phi \text{ of } AB = \tan^{-1}\left[\frac{E_B - E_A}{N_B - N_A}\right]$$
$$= \tan^{-1}\left[\frac{1473.712 - 1322.281}{1063.522 - 961.713}\right]$$
$$= \tan^{-1}\left[\frac{151.431}{101.809}\right] = 56°05'11.6''.$$

The line AB is in first quadrant, therefore

$$\phi = 56°05'11.6''.$$
$$AB = \sqrt{(E_B - E_A)^2 + (N_B - N_A)^2}$$
$$= \sqrt{151.431^2 + 101.809^2} = 182.473 \text{ m}.$$

(i) Before filling the reservoir

$$\text{Bearing of } AC = \theta_{AC} = \text{Bearing of } AB + \theta_A$$
$$= 56°05'11.6'' + 55°11'23'' = 111°16'34.6''.$$
$$\text{Bearing of } BC = \theta_{BC} = \text{Bearing of } BA - \theta_B$$
$$= (180° + 56°05'11.6^2) - 48°31'18''$$
$$= 187°33'53.6''.$$

In ΔABC, we have

$$\gamma = 180° - (\theta_A + \theta_B)$$
$$= 180° - (55°11'23'' + 48°31'18'') = 76°17'19''.$$
$$AC_0 = \frac{AB\sin\theta_B}{\sin\gamma}$$
$$= \frac{182.473 \times \sin 48°31'18''}{\sin 76°17'19''} = 140.720 \text{ m}$$
$$AC_0 = \frac{AB \sin \theta\ A}{\sin\gamma}$$
$$= \frac{182.473 \times \sin 55°11'23''}{\sin 76°17'19''} = 154.214 \text{ m}.$$

Latitude and departure of C from A

$$L_{AC} = AC_0 \cos\theta_{AC} = 140.720 \times \cos 111°16'34.6'' = -51.062 \text{ m}$$
$$D_{AC} = AC_0 \sin\theta_{AC} = 140.720 \times \sin 111°16'34.6'' = +131.129 \text{ m}.$$

Therefore
$$E_C = E_A + D_{AC}$$
$$= 1322.281 + 131.129 = E\ 1453.410 \text{ m}$$
$$N_C = N_A + L_{AC}$$
$$961.713 - 51.062 = N\ 910.651 \text{ m}.$$

Latitude and departure of C from B

$$L_{BC} = BC_0 \cos\theta_{BC} = 154.214 \times \cos 187°33'53.6'' = -152.872 \text{ m}$$
$$D_{BC} = BC_0 \sin\theta_{BC} = 154.214 \times \sin 187°33'53.6'' = -20.302 \text{ m}.$$

Therefore
$$E_C = E_B + D_{BC}$$
$$= 1473.712 - 20.302 = E\ 1453.410 \text{ m}$$
$$N_C = N_B + L_{BC}$$
$$= 1063.522 - 152.872 = N\ 910.650 \text{ m}.$$

Thus
$$\text{mean } E_C = 1453.4100 \text{ m}$$
$$\text{mean } N_C = 910.6505 \text{ m}.$$

Height of point C

$$\text{Height of instrument at } A = h_A + h_i$$

$$\text{H.I.} = 241.831 + 0.486 = 242.317 \text{ m}$$

Height of C above height of instrument

$$V = AC_0 \tan \alpha$$
$$= 140.720 \times \tan 5°33'12'' = 13.682 \text{ m.}$$

$$\text{Elevation of } C = \text{H.I.} + V$$
$$= 242.317 + 13.682 = 255.999 \text{ m.}$$

$$\text{Height of instrument at } B = h_B + h_i$$

$$\text{H.I.} = 242.262 + 0.486 = 242.748 \text{ m}$$

Height of C above height of instrument

$$V = BC_0 \tan \beta$$
$$= 154.214 \times \tan 4°54'42'' = 13.252 \text{ m.}$$

$$\text{Elevation of } C = \text{H.I.} + V$$
$$= 242.748 + 13.252 = 256.000 \text{ m.}$$

$$\text{Mean elevation of } C = \frac{255.999 + 256.000}{2} = 255.9995 \text{ m}$$

(ii) After filling the reservoir

$$\text{Bearing of } AC = \theta_{AC} = 56°05'11.6'' + 55°11'12'' = 111°16'23.6''.$$

$$\text{Bearing of } BC = \theta_{BC} = (180° + 56°05'11.6'') - 48°31'05^2 = 187°34'06.6''.$$

In ΔABC, we have

$$\gamma = 180° - (55°11'12'' + 48°31'05'') = 76°17'43''.$$

$$AC_0 = \frac{182.473 \times \sin 48°31'05''}{\sin 76°17'43''} = 140.708 \text{ m}$$

$$BC_0 = \frac{182.473 \times \sin 55°11'12''}{\sin 76°17'43''} = 154.204 \text{ m.}$$

Latitude and departure of C from A

$$L_{AC} = AC_0 \cos \theta_{AC} = 140.708 \times \cos 111°16'23.6'' = -51.051 \text{ m}$$

$$D_{AC} = AC_0 \sin \theta_{AC} = 140.708 \times \sin 111°16'23.6'' = +131.120 \text{ m.}$$

Therefore

$$E_C = E_A + D_{AC}$$
$$= 1322.281 + 131.120 = E\ 1453.401 \text{ m}$$

$$N_C = N_A + L_{AC}$$
$$= 961.713 - 51.051 = N\ 910.662 \text{ m.}$$

Latitude and departure of C from B

$$L_{BC} = BC_0 \cos \theta_{BC} = 154.204 \times \cos 187°34'06.6'' = -152.861 \text{ m}$$

$$D_{BC} = BC_0 \sin \theta_{BC} = 154.204 \times \sin 187°34'063.6'' = -20.310 \text{ m.}$$

Therefore

$$E_C = E_B + D_{BC}$$

$$= 1473.712 - 20.310 = E\ 1453.402 \text{ m}$$

$$N_C = N_B + L_{BC}$$

$$= 1063.522 \quad 152.861 = N\ 910.661 \text{ m.}$$

Thus

$$\text{mean } E_C = 1453.4015 \text{ m}$$

$$\text{mean } N_C = 910.6615 \text{ m.}$$

Height of point C

$$\text{Elevation of } C = \text{H.I. at } A + AC_0 \tan \beta$$

$$= 242.317 + 140.708 \times \tan 5°33'06'' = 255.994 \text{ m.}$$

$$\text{Elevation of } C = \text{H.I. at } B + BC_0 \tan \beta$$

$$= 242.748 + 154.204 \times \tan 4°54'38'' = 255.997 \text{ m.}$$

$$\text{Mean elevation of } C = \frac{255.994 + 255.997}{2} = 255.9955 \text{ m.}$$

Movement of dam

$$\delta_E = 1453.4100 - 1453.4015$$

$$= +0.0085 \text{ m} = 8.5 \text{ mm west}$$

$$\delta_N = 910.6505 - 910.6615 = -0.011 \text{ m} = 11.0 \text{ mm north.}$$

The horizontal movement

$$= \sqrt{8.5^2 + 11.0^2} = 13.9 \text{ mm north-west}$$

$$\approx \textbf{14 mm north-west.}$$

Example 9.5. From two stations A and B a third station C, not intervisible from A and B, is to be fixed by making linear measurements along and perpendicular to the line AB. The coordinates of the main stations are:

A	E 908.50 m,	N 1158.50 m
B	E 942.00 m,	N 1298.50 m
C	E 933.50 m,	N 1224.50 m

Determine the required data to fix C.

Solution (Fig. 9.11):

Let the coordinates of the points A, B, and C be (E_A, N_A), (E_B, N_B), and (E_C, N_C), respectively. Also let

$$\angle BAC = \phi_A$$

$$\angle ABC = \phi_B$$

$$AB = L$$

$$AD = x$$
$$DB = y = (L - x)$$
$$CD = d$$

From Eqs. (9.1) and (9.2), we have

$$E_C = \frac{E_A \cot \theta_B + E_B \cot \theta_A + (N_B - N_A)}{\cot \theta_A + \cot \theta_B} \quad \text{...(a)}$$

$$N_C = \frac{N_A \cot \theta_B + N_B \cot \theta_A - (E_B - E_A)}{\cot \theta_A + \cot \theta_B} \quad \text{...(b)}$$

But $\cot \theta_A = \dfrac{x}{d}$

$$\cot \theta_B = \frac{y}{d}$$

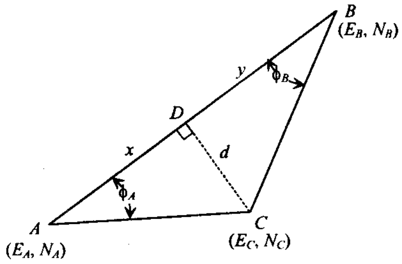


Fig. 9.11

Substituting the values of $\cot \theta_A$ and $\cot \theta_B$ in (a) and (b), we get

$$E_C \frac{x}{d} + E_C \frac{y}{d} = E_A \frac{y}{d} + E_B \frac{x}{d} + (N_B - N_A)$$

$$(E_C - E_B)x + (E_C - E_A)y = (N_B - N_A)d$$

and $(N_C - N_B)x + (N_C - N_A)y = -(E_B - E_A)d$

Let $E_B - E_A = E_{BA}$

$$N_B - N_A = N_{BA}$$
$$E_C - E_A = E_{CA}$$
$$N_C - N_A = N_{CA}$$
$$E_C - E_B = E_{CB}$$
$$N_C - N_B = N_{CB}$$

Thus $E_{CB}\, x + E_{CA}\, (L - x) = N_{BA}\, d$

$$N_{CB}\, x + N_{CA}\, (L - x) = -\, E_{BA}\, d$$

$$(E_{CB} - E_{CA})\, x - N_{BA}\, d = -\, E_{CA}\, L \quad \text{...(c)}$$

$$(N_{CB} - N_{CA})\, x + E_{BA}\, d = -\, N_{CA}\, L \quad \text{...(d)}$$

In (c) and (d), there are two unknowns x and d, and solution of these two equations will give their values.

$$E_{BA} = 942.00 - 908.50 = 33.50 \text{ m}$$
$$N_{BA} = 1298.50 - 1158.50 = 140.00 \text{ m}$$

$$E_{CA} = 933.50 - 908.50 = 25.00 \text{ m}$$
$$N_{CA} = 1224.50 - 1158.50 = 66.00 \text{ m}$$
$$E_{CB} = 933.50 - 942.00 = -8.50 \text{ m}$$
$$N_{CB} = 1224.50 - 1298.50 = -74.00 \text{ m}$$
$$L = \sqrt{33.50^2 + 140.00^2}$$
$$= 143.95 \text{ m.}$$

Thus (c) and (d) are

$$(-8.50 - 25.00)\, x - 140.00\, d = -25.00 \times 143.95$$
$$(-74.00 - 66.00)\, x + 33.50\, d = -66.00 \times 143.95$$
$$-33.50\, x - 140.00\, d = -3598.75$$
$$-140.00\, x + 33.50\, d = -9500.70$$
$$d = \frac{185551.55}{20722.25} = \mathbf{8.95\ m}$$
$$x = \mathbf{70.02\ m.}$$

Alternative solution

$$\text{Bearing of } AB = \tan^{-1}\frac{33.50}{140.00} = 13°27'25.3''$$
$$\text{Bearing of } AC = \tan^{-1}\frac{25.00}{66.00} = 20°44'45.9''$$
$$\theta_A = 20°44'45.9'' - 13°27'25.3'' = 7°17'20.6''$$
$$AC = \sqrt{25.00^2 + 66.00^2} = 70.58 \text{ m.}$$

Therefore $AD = 70.58 \times \cos 7°17'20.6'' = 70.01$ m

$$CD = 70.58 \times \sin 7°17'20.6'' = 8.95 \text{ m.}$$

Example 9.6. During the installation of plumbwires in a shaft, two surface stations A and B were observed from a surface station P near to a line XY. The observations are given in Table-9.4. If PB = 79.056 m, PX = 8.575 m, and XY = 6.146 m, determine the bearing of XP given that X was the nearer of the wires to P.

Table 9.4

Pointing on	Horizontal circle reading
Plumb wire Y	0°00'00"
Plumb wire X	0°02'40"
A (E 1550.00 m, N 1600.00 m)	85°45'44"
B (E 1500.00 m, N 1450.00 m)	265°43'58"

Solution (Fig. 9.12):

From the given data, we get

$$AB = \sqrt{(E_A - E_B)^2 + (N_A - N_B)^2}$$

$$= \sqrt{(1550-1500)^2 + (1600-1450)^2}$$

$$= \sqrt{50^2 + 150^2} = 158.114 \text{ m.}$$

$$\text{Bearing of } AB = \theta_{AB} = \tan^{-1}\left[\frac{50}{150}\right] = 18°26'05.8''$$

$$\angle APB = \text{Pointing to } B - \text{pointing to } A$$

$$= 265°43'58'' - 85°45'44'' = 179°58'14''.$$

Since P is very close to the line AB, it may be assumed that

$$PA + PB = AB$$

Thus $$PA = AB - PB = 158.114 - 79.056 = 79.058 \text{ m.}$$

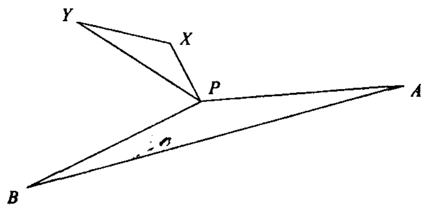


Fig. 9.12

Now in ΔAPB, we have

$$\frac{\sin PAB}{PB} = \frac{\sin PAB}{PA} = \frac{\sin APB}{AB}$$

$$\frac{\sin PAB}{78.855} = \frac{\sin PBA}{79.259} = \frac{\sin 179°58'14''}{158.114}$$

$$\sin 179°58'14'' = \sin(180° - 1'46'') = \sin 1'46'' = \sin 106''$$

$$= 106'' \times \sin 1'' \text{ (angle being small)}$$

Similarly $$\sin PAB = PAB'' \times \sin 1''$$

$$\sin PBA = PBA'' \times \sin 1''$$

Therefore $$\angle PAB = \frac{79.056 \times 106}{158.114} = 52.999''$$

$$\angle PBA = \frac{79.058 \times 106}{158.114} = 53.001''$$

$$\text{Bearing } BP = \text{Bearing } AB - \angle PBA$$

$$= 18°26'05.8'' - 53.001'' = 18°25'12.8''.$$

Now $\angle XPY$ = Pointing on X – pointing on Y

$= 2°02'40'' - 0°00'00''$

$= 2'40'' = 160''$.

In ΔXPY, we have

$$\sin XYP = XYP'' \times \sin 1''$$

$$\sin XPY = XPY'' \times \sin 1''$$

$$\frac{XYP}{PX} = \frac{XPY}{XP}$$

$$XYP = \frac{PX\,XPY}{XP} = \frac{8.575 \times 160}{6.146}\text{ seconds} = 3'43.2''.$$

Therefore $\angle PXY = 180° - (3'43.2'' + 2'40'') = 179°53'36.8''$

$\angle BPX = 360°$ – (pointing on B – pointing on X)

$= 360° - 265°43'58'' + 0°02'40'' = 94°18'42''$

Bearing of PB = Bearing of $BP + 180°$

$= 18°25'12.8'' + 180° = 198°25'12.8''$

Bearing of PX = Bearing of $PB + \angle BPX$

$= 198°25'12.8'' + 94°18'42'' = 292°43'54.8''$

Bearing of XP = Bearing of $PX + 180°$

$= 292°43'54.8'' + 180° = 112°43'54.8''$

Bearing of XY = Bearing of $XP + \angle PXY$

$= 112°43'54.8'' + 179°53'36.8''$

$= 292°37'31.6''$ = **292°37'32''**.

Example 9.7. For setting out a rectangular platform *ABCD*, a rotating construction laser was used. It gave a reading of 0.878 m on a temporary B.M., having a level 45.110 m. The platform has a cross fall of 1 in 1000 longitudinally and 1 in 250 transversely. If the platform is 8 m longitudinally, i.e., along *AD* or *BD*, and 40 m transversely, i.e., along *AB* or *DC*, determine the offsets from the laser beam to the corners of the platform. The lowest corner *A* has a level 45.30 m.

Solution (Fig. 9.13):

A construction laser produces a horizontal plane of laser light. In this case the horizontal plane produced by the laser beam has a level

$= 45.110 + 0.878 = 45.988$ m.

The levels of all the corners should be found out and difference from the level of the horizontal plane produced would be the reading for the particular corner.

Level of A = 45.30m

$$\text{Level of } B = 45.30 + \frac{1}{250} \times 40 = 45.460 \text{ m}$$

$$\text{Level of } C = 45.460 + \frac{1}{1000} \times 8 = 45.468 \text{ m}$$

$$\text{Level of } D = 45.468 - \frac{1}{250} \times 40 = 45.308 \text{ m}$$

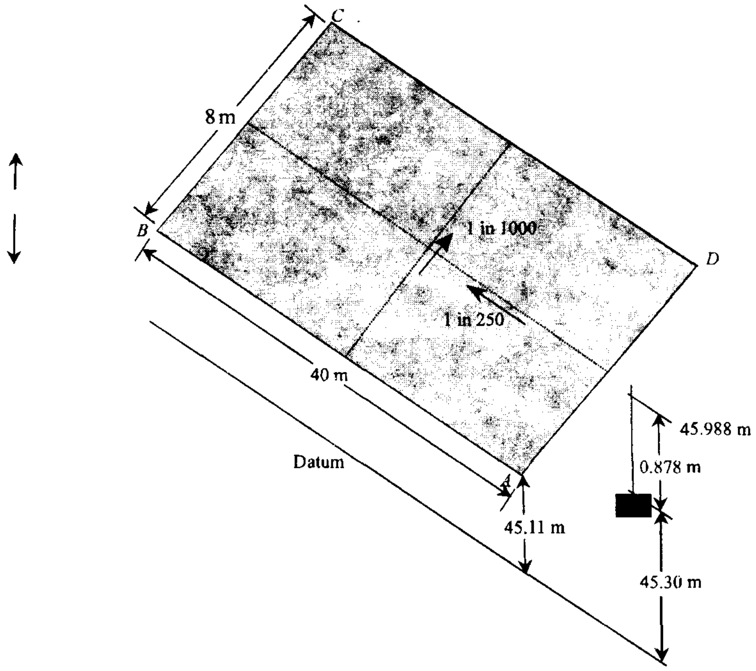


Fig. 9.13

Check: Level of $A = 45.468 - \frac{1}{1000} \times 8 = 45.300$ m (*Okay*).

Offsets

to corner $A = 45.988 - 45.300 = \mathbf{0.688\ m}$

to corner $B = 45.988 - 45.460 = \mathbf{0.528\ m}$

to corner $C = 45.988 - 45.468 = \mathbf{0.520\ m}$

to corner $D = 45.988 - 45.308 = \mathbf{0.680\ m}$.

Example 9.8. To lay a sewer is to be laid between two points *A* and *B*, 120 m apart, the data of profile levelling are given in Table 9.4. The invert level at *A* is to be 112.250 m, and the gradient of *AB* is to be 1 in 130, *B* being at lower level than *A*.

Table-9.4

Point	B.S.	I.S.	F.S.	H.I.	Distance (m)	Remarks
1	0.744			117.064	–	B.M. = 116.320 m
2		3.036			0	*A*
3		2.808			30	
4		2.671			60	
5		3.026			90	
6		3.131			120	*B*
7			0.744			B.M.

At the setting-out stage, the level was set up close to its previous position, and a back sight of 0.698 was recorded on the staff held at the B.M. Determine

(a) the length of the traveler,

(b) the height of rails above ground level at *A* and *B*, and

(c) the staff reading required for fixing of sight rails at *A* and *B*.

Solution (Fig. 9.14):

Let us first reduce the existing ground levels from the given levelling record.

Table 9.5

Point	B.S.	I.S.	F.S.	H.I.	Distance (m)	R.L.	Remarks
1	0.744			117.064	–	116.320	B.M.= 116.320 m
2		3.036			0	114.028	*A*
3		2.808			30	114.256	
4		2.671			60	114.393	
5		3.026			90	114.038	
6		3.131			120	113.933	*B*
7			0.744			116.320	B.M.

Now the invert levels for the intermediate points and point *B* are as below:

at 0 m (*A*) = 112.250 m

$$30 \text{ m} = 112.250 - \frac{1}{130} \times 30 = 112.019 \text{ m}$$

$$60 \text{ m} = 112.250 - \frac{1}{130} \times 60 = 111.788 \text{ m}$$

$$90 \text{ m} = 112.250 - \frac{1}{130} \times 90 = 111.558 \text{ m}$$

$$120 \text{ m } (B) = 112.250 - \frac{1}{130} \times 120 = 111.327 \text{ m.}$$

The level differences at the intermediate points have been given in Table-9.6.

Table 9.6

Point	Distance (m)	Ground level	Invert level	Difference (m)
A	0	114.028	112.250	1.778
	30	114.256	112.019	2.237
	60	114.393	111.788	2.605
	90	114.038	111.558	2.480
B	120	113.933	111.327	2. 606

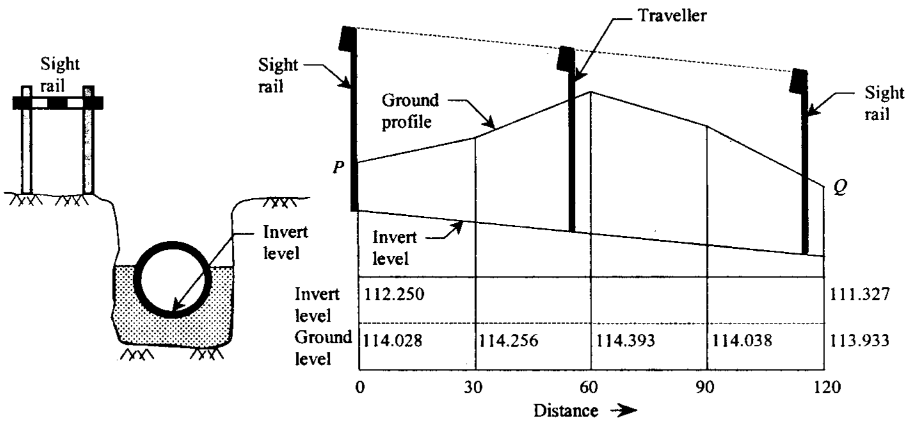


Fig. 9.14

Seeing the level difference between the ground level and invert level, a traveller or boning rod of 3 m length should be sufficient. Further, the line of sight given by the sight rails should have gradient of 1 in 130, and it must have clearance of 1 m above ground and also one of the sight rails should be about 1 m or so above the ground for convenient sighting.

Hence the levels of the top sight rails

at A = 112.250 + 3 = 115.250 m

at B = 111.327 + 3 = 114.327 m.

The height of the top of sight rails above ground

at A = 115.250 − 114.028 = **1.222 m** *(Okay)*

at B = 114.327 − 113.933 = **0.394 m.**

To achieve the levels of top of the sight rails at A and B as 115.250 m and 114.327 m, respectively, the staff readings required are calculated below.

Level of the B.M. = 116.320 m

B.S. reading on B.M. = 0.698 m

Height of instrument = 116.320 + 0.698 = 117.018 m

Staff reading at A = 117.018 − 115.250 = **1.768** m

Staff reading at B = 117.018 − 114.327 = **2.691 m.**

To fix the sight rails, the staff is moved up and down the uprights to give readings of 1.768 m and 2.691 m, respectively, at A and B, marks are made thereon corresponding to the base of the staff. The sight rail is then nailed in position and checked. Alternatively, the tops of the uprights could be leveled, and measurements made down the uprights to locate the finished levels of 115.250 m and 114.327 m.

Example 9.9. An embankment is to be constructed on ground having a transverse cross fall of 1 in 10. At a cross-section the formation level is 296.63 m, ground level at the centre line being 291. 11 m. Side slope of 1 in 2.5 have been specified together with a formation width of 20 m. Determine the necessary data to establish the profile boards to control the construction.

Solution (Fig. 9.15):

The inner upright of the profile board should have a clearance of about 1 m from the toes to avoid disturbance. The inner and outer uprights can be spaced to 1 m apart. A traveller is required in conjunction with the upper surface of the boards to achieve the gradient.

As shown in the figure

$$w_1 = \frac{b}{2} + \frac{ns}{n-s}\left(h + \frac{b}{2n}\right)$$

$$w_2 = \frac{b}{2} + \frac{ns}{n+s}\left(h - \frac{b}{2n}\right)$$

where

b = the formation width (=20 m),

h = the height of the embankment at centre line (= 296.63 − 291.11 = 5.52),

n = the transverse cross fall (= 10), and

s = the side slope of the embankment (= 2.5).

Thus

$$w_1 = \frac{20}{2} + \frac{10 \times 2.5}{10 - 2.5}\left(5.52 + \frac{20}{2 \times 10}\right) = 31.73 \text{ m}$$

$$w_2 = \frac{20}{2} + \frac{10 \times 2.5}{10 + 2.5}\left(5.52 - \frac{20}{2 \times 10}\right) = 19.04 \text{ m.}$$

The reduced level of C = 291.11 m

The reduced level of A = $291.11 + \frac{1}{10} \times 19.04 = 293.01$ m

The reduced level of A = $291.11 - \frac{1}{10} \times 31.73 = 287.94$ m

The difference of level between C and A = $\frac{19.04}{10} = 1.904$ m

The difference of level between C and B = $\frac{31.73}{10} = 3.173$ m.

In the first instant let us assume the length of a traveller as 1.25 m with centre lines of the uprights at 1 m and 2 m, respectively, from toes A and B.

The level of the bottom of uprights

inner near A = $293.01 + \frac{1}{10} \times 1 = 293.11$ m

$$\text{outer near } A = 293.01 + \frac{1}{10} \times 2 = 293.21 \text{ m}$$

$$\text{inner near } B = 287.94 - \frac{1}{10} \times 1 = 287.84 \text{ m}$$

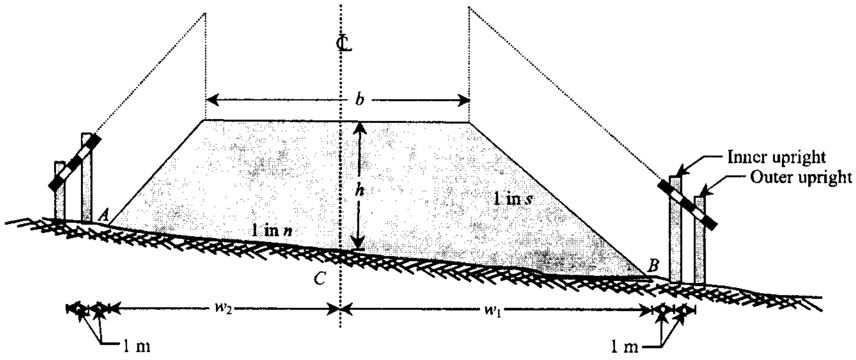


Fig. 9.15

$$\text{outer near } A = 287.94 - \frac{1}{10} \times 2 = 287.74 \text{ m.}$$

The level of sight line at A = 293.01 + 1.25 = 294.26 m

The level of sight line at B = 287.94 + 1.25 = 289.19 m.

The level of sight line at

$$\text{inner upright near } A = 294.26 - \frac{1}{2.5} \times 1 = 293.86 \text{ m}$$

$$\text{outer upright near } A = 294.26 - \frac{1}{2.5} \times 2 = 293.46 \text{ m}$$

$$\text{inner upright near } B = 289.19 - \frac{1}{2.5} \times 1 = 288.79 \text{ m}$$

$$\text{outer upright near } B = 289.19 - \frac{1}{2.5} \times 2 = 288.39 \text{ m.}$$

The height of the uprights, i.e., altitude of sight line above ground near A

inner upright = 293.86 – 293.11 = 0.75 m

outer upright = 293.46 – 293.21 = 0.25 m

near B

inner upright = 288.79 – 287.84 = 0.95 m

outer upright = 288.39 – 287.74 = 0.65 m.

Taking traveller length as 1.5 m for convenience, the altitude of the sight line above ground line can be computed in a similar manner as above.

OBJECTIVE TYPE QUESTIONS

1. A third point *C* cannot be located using two points *A* and *B* of known locations by measuring
   (*a*) all the sides of the triangle *ABC*.
   (*b*) two angles *A* and *B* and the length *AB*.
   (*c*) all the angles of the triangle *ABC*.
   (*d*) the angle *A*, and the lengths *AB* and *BC*.
2. Location of a point *P* by resection is done by observing
   (*a*) one control point from *P*.
   (*b*) two control points from *P*.
   (*c*) three control points from *P*.
   (*d*) *P* from three control points.
3. Co-planing is a process of
   (*a*) bringing points in same horizontal plane.
   (*b*) establishing points in a vertical plane at different levels.
   (*c*) centering the instrument over the ground station mark.
   (*d*) transferring the surface alignment underground through a narrow shaft.
4. Accurate surface alignment down a vertical shaft using two plumb wires is achieved by
   (*a*) Weisbach triangle method.
   (*b*) reducing the size of triangle of error to zero.
   (*c*) by adjusting the closing error.
   (*d*) none of the above.
5. Sight rails are used for setting out
   (*a*) large buildings.
   (*b*) bridges.
   (*c*) the gradient of canal bed.
   (*d*) the gradient of trench of bottom or pipe invert.
6. Weisbach triangle method is a method
   (*a*) of locating the plane table position on paper by minimizing the size of triangle of error.
   (*b*) used in transferring the ground surface alignment down the shaft using plumb wires.
   (*c*) of determining spherical excess in spherical triangles.
   (*d*) none of the above.

ANSWERS

1. (*c*) **2.** (*c*) **3.** (*d*) **4.** (*a*) **5.** (*d*) **6.** (*b*)

SELECTED PROBLEMS

1. The difference in elevation Δh between two points was measured repeatedly for 16 times giving the following results:

7.8621, 7.8632, 7.8630, 7.8646, 7.8642, 7.8652, 7.8620, 7.8638, 7.8631, 7.8641, 7.8630, 7.8640, 7.8630, 7.8637, 7.8633, 7.8630.

Compute the mean and the standard error of the mean of the observations.

2. Calculate the standard error of the volume of a cuboid whose sides x, y, and z have the values and standard errors as under:

$x = 60 \pm 0.03$ cm, $y = 50 \pm 0.02$ cm, $z = 40 \pm 0.01$ cm.

3. Compute the area, error in the area, and the standard error of the area for Fig. 1 using the following data:

$a = 50.30 \pm 0.01$ m

$b = 82.65 \pm 0.03$ m

$r = 9.50 \pm 0.02$ m.

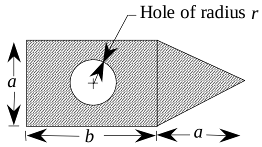


Fig. 1

The measurements were made with 30 m tape standardized at 30° C and the field temperature during the measurements was 50° C. Take coefficient of linear expansion = 1.15×10^{-5} per ° C.

4. A single measurement of an angle has the standard deviation of ± 2.75″. To get the standard error of the mean of a set of angles as 1″ how many measurements should be made in similar conditions?

5. In Fig. 2, the sides PQ and QR of a Weisbach triangle used in an underground traverse measure (2.965 ± 3) m and (2.097 ± 2) m, respectively. The Approximate value of the angle QRP is 6′50″ and the standard error of the angle PRS is ± 4″. Calculate how many times this angle QRP should be measured in order that the standard error of the bearing of RS is not to exceed 5″. The standard error of a single measurement of the angle QRP is estimated to be ±10² and assume that the bearing of PQ is known without error.

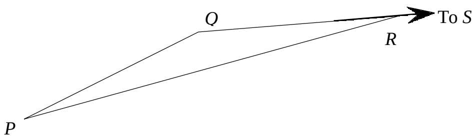


Fig. 2

6. A line AB was measured in segments along sloping ground with a 30 m tape, and the following measurements were recorded:

Slope distance (m)	Difference in elevation (m)
30.00	1.58
30.00	0.95
18.92	0.90
11.46	0.56
30.00	3.01
8.23	0.69

What is horizontal length of the line *AB* ?

7. A line was measured by a 30 m tape in five bays, and the following results were obtained:

Bay	Span length (m)	Rise/fall between the ends of span (m)
1	29.60	0.0
2	29.80	– 0.21
3	29.40	0.0
4	29.60	+ 0.31
5	26.00	– 0.04

The field temperature and pull as measured were 10° C and 175 N.

The tape was standardized on the flat under a pull of 125 N at temperature 20° C. If the top of the first peg is at 250.00 m from the mean sea level, calculate the correct length of the line reduced to mean sea level. Take mean radius of the earth as 6372 km.

Tape details:

Tape density ρ = 77 kg/m^3

Cross-sectional area A = 6 mm^2

Coefficient of linear expansion α = 0.000011/°C

Modulus of elasticity of tape material E = 207 kN/mm^2

8. The length of a steel tape found to be exactly 30 m at a temperature of 30° C under pull of 5 kg when lying on the flat platform. The tape is stretched over two supports between which the measured distance is 300.000 m. There are two additional supports in between equally spaced. All the supports are at same level; the tape is allowed to sag freely between the supports. Determine the actual horizontal distance between the outer supports, and its equivalent reduced mean sea level distance if the mean temperature during the measurements was 37° C and the pull applied was 9kg. The average elevation of the terrain is 1500 m. Take tape details as below:

Weight = 1.50 kg

Area of cross-section = 6.5 mm^2

Coefficient of linear expansion = 1.2×10^{-5}/°C

Modulus of elasticity of tape material E = 2.1×10^6 kg/cm^2

Mean earth, radius = 6372 km.

9. A 30 m tape weighing 0.900 kg has a cross-sectional area 0.0485 cm^2. The tape measures 30.000 m when supported throughout under a tension of 5 kg. The modulus of elasticity is 2.1×10^6 kg/cm^2. What tension is required to make the tape measure 30 m when supported only at the two ends ?

10. The length of an embankment was measured along its surface as 25.214 m using a steel tape under a pull of 25 N at a temperature of 10° C. If the top and bottom of the embankment are at levels of 75.220 m and 60.004 m, respectively, and the tape was standardized on the flat at 20° C under a pull of 49 N, what is the embankment gradient ?

Tape details

Cross-sectional area = 6 mm^2

Coefficient of linear expansion = 0.000011/°C

Modulus of elasticity of tape material = 207000 MN/mm^2.

11. A steel tape of nominal length 200 m with a plumb bob of mass 16 kg attached to it was used to measure a length down a shaft as 160.852 m. The mean temperature during the measurement was 4° C. If the tape was standardized to be 200.0014m under a tension of 115 N at 20° C temperature, determine the correct measured length.

The following data may be used:

Mass of tape = 0.07 kg/m

Cross-sectional area of tape = 10 mm^2

Coefficient of linear expansion α = 11×10^{-6}/°C

Modulus of elasticity = 2×10^5 N/mm^2

Acceleration due to gravity = 9.807 m/s^2.

12. A distance is measured along a slope with an EDM which when corrected for meteorological conditions and instrument constants, is 714.652 m. The EDM is 1.750 m above the ground, and the reflector is located 1.922 m above ground. A theodolite is used to measure a vertical angle of + 4°25′15″ to a target placed 1.646 m above ground. Determine the horizontal length of the line.

13. With a theodolite set 1.58 m above station *A*, a sight is taken on a staff held at station *B*. The staff intercept 1.420 m with middle cross hair reading 3.54 m, and vertical angle – 5°13′. With the instrument set 1.55 m above station *B*, a sight is taken on the staff held at station *A*. The staff intercept is 1.430 m with middle cross hair reading as 2.35 m, and the vertical angle + 6°00′. The instrument is internal focussing with constants $k = 101$ and $c = 0$. What is the average length of *AB* and the average difference in elevation between the two points ?

14. A tacheometer was set up at station *A* with trunion axis 1.18 m above ground, and due to some obstruction in line of sight only reading of upper stadia wire could be recorded as 2.022 with vertical angle as + 3°05′, on the staff held vertically at station *B*. The line joining *A* and *B* has a gradient of 1 in 20. If the tacheometric constants are as $k = 100$ and $c = 0$, calculate the other staff readings and the horizontal distance *AB*.

15. To determine the gradient of a line *AB*, the following data were collected from a station *T* with staff held vertical using a tacheometer having the constants as $k = 100$ and $c = 0$ m.

Tacheometer at	Staff at	Bearing observed at *T*	Vertical angle	Staff readings (m)
T	*A*	120°15′	+ 7°35′	1.410, 1.965, 2.520
	B	206°15′	+ 4°10′	1.655, 2.475, 3.295

Determine the gradient of the line *AB*.

16. A base line *AB* was measured in two parts *AC* and *CB* of lengths 1540 m and a9a9 m, respectively, with a steel tape, which was exactly 30 m at 20° C at pull of 10 kg. The applied pull during the measurements for both parts was 25 kg, whereas the respective temperatures were 40° C and 45° C. The ground slope for *AC* and *CB* were + 2°40′ and + 3°10′, respectively, and the deflection angle for *CB* was 11° R. Determine the correct length of the base line. The cross-section of the tape is 0.025 cm^2, the coefficient of linear expansion is 2.511×10^{-6}/°C, and the modulus of elasticity is $2.1´10^5$ kg/cm^2.

17. The following observations were made using a tacheometer (k = 100 and c = 0 m). A point *P* is on the line *AB* and between the stations *A* and *B*, and another point *Q* is such that the angle *ABQ* is 120°.

Tacheometer at	Staff at	Vertical angle	Staff readings (m)
P (h_I = 1.45 m)	*A* (h_A = 256.305 m)	– 7°15′	2.225, 2.605, 2.985
	B	– 3°30′	1.640, 1.920, 2.200
Q (h_I = 1.51 m)	*B*	+ 9°34′	0.360, 0.900, 1.440

h_I = Height of instrument above ground

h_A = Elevation of *A* above m.s.l.

Determine

(*i*) the distance *AQ*,

(*ii*) the elevation of *B*and *Q*, and

(*iii*) the gradient of line *AQ*.

18. A back sight of 3.0545 m is taken on a point 50 m from the level. A for sight 2.1604 m is taken on a point 200 m from the level. Compute the correct difference in level between the two points, taking into effect of

(*i*) curvature, and

(*ii*) curvature and refraction.

19. Sighting across a lake 40 km wide through a pair of binoculars, what is the height of a shortest tree on the opposite shore whose tip the observer can see if the eyes are 1.70 m above the shoreline on which he stands ?

20. The line of sight rises at the rate of 0.143 m in 100 m when the level bubble is centered. A back sight of 1.713 m is taken on a point *P* at a distance of 25 m from the level, and a fore sight of 1.267 m is taken from a point *Q* at a distance of 60 m from the level. If the elevation of *P* is 111.000 m, what is the elevation of *Q* ?

21. A line of levels is run from B.M.-1 (elevation = 100.00 m) to B.M.-2 (elevation = 104.00 m). The field observations were recorded as given below:

Station	B.S.	I.S.	F.S.	Remarks
1	4.95			B.M.-1
2		2.65		
3	5.60		3.45	
4	– 3.90		– 2.60	
5		2.50		
6			1.50	B.M.-2

Reduce the levels of points 2, 3, 4, and 5. Determine the total error of closure, and adjust the values.

22. The following figures were extracted from a level filed book; some of the entries eligible because of exposure to rain. Insert the missing figures, and check your results.

Station	B.S.	I.S.	F.S.	H.I.	R.L.	Remarks
1	?			279.08	277.65	B.M.-1
2		2.01			?	
3		?			278.07	
4	3.37		0.40		278.68	
5		2.98			?	
6		1.41			280.64	
7			?		281.37	B.M.-2

23. A staff is held at a distance of 200 m from a level, and a reading of 2.587 m is obtained. Calculate the correct reading for curvature and refraction.

24. The following results were obtained in reciprocal leveling across a river for staff held vertically at stations at *X* and *Y* from level stations *A* and *B* on each bank of a river, respectively.

Staff reading at *X* from *A* = 1.753 m

Staff reading at *Y* from *A* = 2.550 m

Staff reading at *X* from *B* = 2.080 m

Staff reading at *Y* from *B* = 2.895 m

Calculate the elevation of *Y* if the elevation of *X* is 101.30 m above mean sea level.

25. In order to check the adjustment of a tilting level, the following procedure was followed. Pegs *A*, *B*, *C*, and *D* were set out on a straight line such that *AB* = *BC* = *CD* = 30 m. The level was set up at *B*, and readings were taken to a staff at *A* then at *C*. The level was then moved to *D*, and readings were again taken to a staff held first at *A* and then at *C*. The readings are given below:

Instrument position	Staff reading (m)	
	A	*C*
B	1.926	1.462
D	2.445	1.945

Determine whether or not the instrument is in adjustment and if not, explain how the instrument can be corrected.

26. A tilting level is set up with the eyepiece vertically over a peg *A*. The height of the center of the eyepiece above *A* was measured to be 1.516 m. The reading on a vertically held staff at a peg *B*, was found to be 0.0696 m. The positions of the level and staff were interchanged, and the measured height of the center of the eyepiece above *B* was 1.466 m, and the staff reading at *A* was 2.162 m. Determine the difference in level between *A* and *B*. Also ascertain from the readings the adjustment of line of collimation, and calculate the correct reading on the staff at *A*.

27. During levelling it was found that the bubble was displaced by two divisions off the centre when the length of the sight was 100 m. If the angular value of one division of the bubble tube is 20², find the consequent error in the staff reading. What is the radius of the bubble tube if one graduation is 2 mm long?

28. Levelling was done to determine the levels of two pegs *A* and *B*, and to determine the soffit level of an over bridge. Using the values of staff readings given the following table, and given that the first back sight is taken on a bench mark at a temple (R.L. = 75.630 m), and the final fore sight is on a bench mark at P.W.D. guest house (R.L. = 75.320 m) determine the closing error.

Could a lorry 5.5 m high pass under the bridge?

Staff reading (m)	Remarks
1.275	B.S. on B.M. at temple (R.L. = 75.630 m)
2.812	F.S. on C.P.1
0.655	B.S. on C.P.1
– 3.958	Inverted staff to soffit of bridge
1.515	Ground level beneath center of bridge
1.138	F.S. on C.P.2
2.954	B.S. on C.P.2
2.706	Peg *A*
2.172	Peg *B*
1.240	F.S. on B.M. at P.W.D. (R.L. = 75.320 m)

29. To check the rail levels of an existing railway, seven points were marked on the rails at regular intervals of 20 m, and the following levels were taken:

Point	B.S.	I.S.	F.S.	Remarks
	2.80			B.M. *A* = 38.40 m
1		0.94		
2		0.76		
3		0.57		
4	1.17		0.37	
5		0.96		
6		0.75		
7			0.54	

Reduce the levels of the points by rise and fall method, and carry out appropriate checks. If the levels of the points 1 and 7 were correct, calculate the amount by which the rails are required to be lifted at the intermediate points to give a uniform gradient throughout.

30. To determine the collimation adjustment of a level, readings were taken on two bench marks *X* and *Y* 53.8 m apart having elevations of 187.89 m and 186.42 m, respectively, the readings being 0.429 and 1.884, respectively. The distance of the level at *P* from *Y* was 33.8 m. What is the collimation error per 100 m ? If further reading of 2.331 is taken from *P* on a point *Z* 71.6 m from *P*, what is the elevation of *Z* ?

31. To determine the reduced level of a point *B*, the vertical angle to *B* was measured as + 1°48′15″ from a point *A* having reduced level of 185.40 m. The vertical angle from *B* to *A* was also measured as – 1°48′02″. The signal heights and instrument heights at *A* and *B* were 3.10 mand 4.50 m, and 1.35 m and 1.36 m, respectively. The geodetic distance *AB* is 5800 m. If the mean radius of the earth is 6370 km determine (a) the reduced level of *B* and (b) the refraction correction.

32. The following observations were obtained for a closed-link traverse *ABCDE*.

Station	Clockwise angle	Length (m)
A	260°31′18″	–
B	123°50′42″	129.352
C	233°00′06″	81.700
D	158°22′48″	101.112
E	283°00′18″	94.273

The observations were made keeping the bearings of lines *XA* and *EY*, and the coordinates of *A* and *E*, fixed as below:

W.C.B. of *XA* = 123°16′06″
W.C.B. of *EY* = 282°03′00″
Coordinates of *A* = E 782.820 m N 460.901 m
Coordinates of *E* = E 740.270 m N 84.679 m

Obtain the adjusted values of the coordinates of stations *B*, *C*, and *D* by Bowditch's method.

33. Determine the coordinates of the intersection of the line joining the traverse stations *A* and *B* with the line joining the stations *C* and *D* if the coordinates of the traverse stations are as below.

Station	Easting (m)	Nothing (m)
A	4020.94	5915.06
B	4104.93	6452.93
C	3615.12	5714.61
D	4166.20′	6154.22

34. A theodolite traverse was run between two points *A* and *B* with the following observations:

Line	Bearing	Length (m)
A-1	86°37′	128.88
1-2	165°18′	208.56
2-3	223°15′	96.54
3-*B*	159°53′	145.05

Calculate the bearing and distance of point *B* from point *A*.

35. Calculate the lengths of the lines *BC* and *CD* from the following observations made for a closed traverse *ABCDE*.

Line	Length (m)	Bearing	ΔE (m)	ΔN (m)
AB	104.85	14°31′	+ 26.29	+ 101.50
BC	–	319°42′	–	–
CD	–	347°15′	–	–
DE	91.44	5°16′	+ 8.39	+ 91.04
EA	596.80	168°12′	+ 122.05	– 584.21

36. The following table giving lengths and bearings of a closed-loop traverse contains an error in transcription of one of the values of length. Determine the error.

Line	*AB*	*BC*	*CD*	*DE*	*EA*
Length (m)	210.67	433.67	126.00	294.33	223.00
Bearing	20°31′30″	357°16′00″	120°04′00″	188°28′30″	213°31′00″

37. A traverse was run between two points *P* and *Q* having the coordinates as E1268.49 m, N1836.88 m, and E1375.64 m, N1947.05 m, respectively. The field observations yielded the following values of eastings and northings of the traverse lines.

Line	Length (m)	ΔE (m)		ΔN (m)	
		+	–	+	–
AB	104.65	26.44	–	101.26	–
BC	208.96	136.41	–	158.29	–
CD	212.45	203.88	–	–	59.74
DE	215.98	–	146.62	–	158.59
EA	131.18	–	112.04	68.23	–

Calculate the adjusted coordinates of *A*, *B*, *C*, and *D* using Bowditch's method.

38. The two legs *AB* and *BC* of a traverse and the angle *ABC* as measured are 35.50 m, 26.26 m, and 135°, respectively. Calculate the resulting maximum error in the measurement of the angle due to the centering error of ± 2 mm.

39. In a certain theodolite it was found that the left-hand end of the trunion axis is higher than the right-hand end making it to incline by 30″ to the horizontal. Determine the correct horizontal angle between the targets *A* and *B* at the theodolite station from the following observations.

Pointings	Horizontal circle reading (face right)	Vertical circle reading
A	246°18′53″	+ 63°22′00″
B	338°41′28″	+ 12°16′20″

40. The bearing and length of a traverse line are 38°45′20″ and 169.08 m, respectively. If the standard deviations of the two observations are ±20″ and ±50 mm, respectively, calculate the standard deviations of the coordinate differences of the line.

41. In some levelling operation, rise (+) and fall (–) between the points with their weights given in parentheses, are shown in Fig. 3.

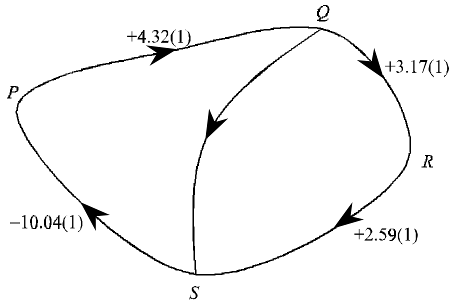


Fig. 3

Given that the reduced level of *P* as 134.31 m above datum, determine the levels of *Q*, *R*, and *S*.

42. In a triangulation network shown in Fig. 4, the measured angles are as follows:

θ_1 = 67°43′04″, θ_4 = 29°38′52″

θ_2 = 45°24′10″, θ_5 = 63°19′35″

θ_3 = 37°14′12″, θ_6 = 49°47′08″

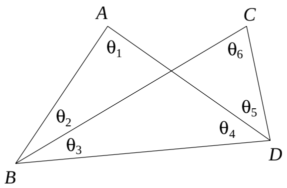


Fig. 4

Adjust the angles to the nearest seconds assuming that θ_1 and θ_6 are of twice the weight of the other four.

43. Three stations *A*, *B*, and *C* have the coordinates E 2000.00 m, N 2000.00 m, E 1000.00 m, N 267.95 m, and E 0.00 m, N 2000.00 m, respectively. The following readings have been recorded by a theodolite set up at a station *P* very near to the centre of the circle circumscribing stations *A*, *B*, and *C* :

Pointings on	*A*	*B*	*C*
Horizontal circle reading	00°00′00″	119°59′51.0″	240°00′23.5″

Determine the coordinates of *P*.

44. By application of the principle of least squares Determine the most probable values of *x* and *y* from the following observations using the least squares method. Assume that the observations are of equal weights.

$$2x + y = +1.0$$
$$x + 2y = -1.0$$
$$x + y = +0.1$$
$$x - y = +2.2$$
$$2x = +1.9.$$

45. In a braced quadrilateral shown in Fig. 5, the angles were observed as plane angles with no spherical excess:

θ_1 = 40°08′17.9″, θ_2 = 44°49′14.7″

θ_3 = 53°11′23.7″, θ_4 = 41°51′09.9″

$\theta_5 = 61°29'34.3''$, $\theta_6 = 23°27'51.2''$

$\theta_7 = 23°06'37.3''$, $\theta_8 = 71°55'49.0''$

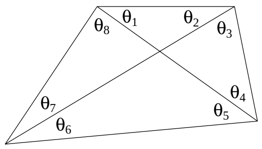


Fig. 5

Determine the most probable values of the angles by rigorous method.

46. Directions ware observed from a satellite station *S*, 200 m from a station *C*, with the following results:

$$\angle A = 00°00'00''$$
$$\angle B = 62°15'24''$$
$$\angle C = 280°20'12''.$$

The approximate lengths of *AC* and *BC* are 25.2 km and 35.5 km, respectively. Calculate the $\angle ACB$.

47. The altitudes of the proposed triangulation stations *A* and *B*, 130 km apart are respectively 220 m and 1160 m. The altitudes of two peaks *C* and *D* on the profile between *A* and *B*, are respectively 308 m and 632 m, the distances *AC* and *AD* being 50 km and 90 km. Determine whether *A* and *B* are intervisible, and if necessary, find the minimum height of scaffolding at *B*, assuming *A* as the ground station.

48. Calculate the data for setting out the kerb line shown in Fig. 6 if R = 12 m and Δ = 90°. Calculate the offsets at 2 m interval.

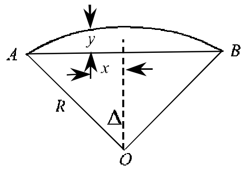


Fig. 6

49. Derive data needed to set out a circular curve of radius 600 m by theodolite and tape (deflection angle method) to connect two straights having a deflection angle 18°24′, the chainage of the intersection point being 2140.00 m. The least count of the is 20″.

50. A circular curve of radius 300 m is to be set out using two control stations *A* and *B*, their coordinates being E 2134.091 m, N 1769.173 m and E 2725.172 m, N 1696.142 m, respectively. The chginage of the first tangent point having coordinates E 2014.257 m, N 1542.168 m is 1109.27 m. If the coordinates of the point of intersection are E 2115.372 m and N 1593.188 m, calculate the bearing and distance from *A* required to set out a point *X* on the curve at chainage 1180 m.

51. Two straights *AB* and *CD* having bearings as 30° and 45°, respectively, are to be connected *CD* by a continuous reverse curve consisting of two circular curves of equal radius and four transition curves. The straight *BC*, 800 m long, having bearing of 90°, is to be the common tangent to the two inner transition curves. What is the radius of the circular curves if the maximum speed is to be restricted to 80 km/h and a rate of change of radial acceleration is 0.3 m/s^3 ? Give (a) the offset, and (b) the deflection angle with respect to *BC* to locate the intersection of the third transition curve with its circular curve.

52. It is required to connect two straights having a total deflection angle of 18°36′ right by a circular curve of 450 m radius and two cubic spiral transition curves at the ends. The design velocity is 70 km/h, and the rate of change of radial acceleration along the transition curve is not to exceed 0.3 m/s^3. Chainage of the point of intersection is 2524.20 m. Determine (a) length of the transition curve, (b) shift of the circular curve, (c) deflection angles for the transition curve to locate the points at 10 m interval, and (d) deflection angles for the circular curve at 20 m intervals.

53. If the sight distance equals half the total length of the curve, g_1 = +4% and g_2 = –4%, and the observer's eye level h = 1.08 m, calculate the length of the vertical curve.

54. A gradient of +2% is to be joined to a gradient –4/3% by a vertical curve with a sight distance of 200 m at 1.05 m above ground level. Determine the chainage and level of the two tangent points for the highest point on the curve at chainage 2532.00 m and level 100.23 m.

55. In a modification, a straight sloping at –1 in 150 is to be changed to a gradient of +1 in 100, the intervening curve being 400 m in length. Assuming the curve to start at a point *A* on the negative gradient, calculate the reduced levels of pegs at 50 intervals which you set out in order to construct the curve, assuming the reduced level of *A* to be 100 m.

56. Determine the area in hectares enclosed by a closed traverse *ABCDE* from the following data.

Station	Easting (m)	Nothing (m)
A	181.23	184.35
B	272.97	70.05
C	374.16	133.15
D	349.78	166.37
E	288.21	270.00

57. Using the data given below to determine the volume of earth involved a length of the cutting to be made in ground:

Transverse slope	= 1 in 5
Formation width	= 8.00 m
Side slopes	= 1 in 2
Depths at the centre lines of three sections, 20 m apart	= 2.50, 3.10, 4.30 m.

Get the results considering

(a) The whole cutting as one prismoid.

(b) End-areas formula with prismoidal correction.

58. A 10 m wide road is to be constructed between two point *A* and *B*, 20 m apart, the transverse slope of the original ground being 1 in 5. The cross-sections at *A* and *B* are partly in cut with 0.40 m cut depth at the centre line and partly in fill with 0.26 m fill at the centre line. The respective side slopes of cut and fill are 1 vertical in 2 horizontal, and the centre line of the road *AB* is a curved line of radius 160 m in plan. Determine the net volume of earthworks between the two sections.

59. A dam is to be constructed across a valley to form a reservoir, and the areas in the following table enclosed by contour loops were obtained from a plan of the area involved.

(a) If the 660 m level represents the level floor of the reservoir, use the prismoidal formula to calculate the volume of water impounded when the water level reaches 700 m.

(b) Determine the level of water at which one-third of the total capacity is stored in the reservoir.

(c) On checking the calculations it was found that the original plan from which the areas of contour loops had been measured had shrunk evenly by approximately 1.2 % of linear measurement. What is the corrected volume of water?

Contour (m)	660	665	670	675	680	685	690	695	670
Area	5200	9400	16300	22400	40700	61500	112200	198100	272400

60. A point *X* having the cordinates as E1075.25 m, N 2000.25 m, is to be set out using two control points *P* (E 1050.25 m, N 2050.25 m) and *Q* (E 1036.97 m, N 1947.71 m) by two linear mesurements alone. Calculate the necessary data to fix *X*.

61. A length of sewer *PRQ* is to be constructed in which the bearings of *RP* and *RQ* are 205°02′ and 22°30′, respectively. The coordinates of the manhole at *R* are E 134.17 m, N 455.74 m. A station *A* on the nearby traverse line *AB* has coordinates E 67.12 m, N 307.12 m, the bearing of *AB* being 20°55′. Considering a point *M* on *AB* such that *MR* is at right angles to *AB*, derive the data to set out the two lengths of sewer.

62. The coordinates of three stations *A*, *B*, and *C* are respectively E 11264.69 m, N 21422.30 m, E 12142.38 m, N 21714.98 m, and E 12907.49 m, N 21538.66 m. Two unknown points *P* and *Q* lie to the southerly side of *AC* with *Q* on the easterly side of point *P*. The angles measured are ∠*APB* = 38°15′47″, ∠*BPC* = 30°38′41″, ∠*CPQ* = 33°52′06″, and ∠*CQP* = 106°22′20″. Calculate the length and bearing of the line *PQ*.

63. During a shaft plumbing exercis, two surface reference stations *A* (E 1000.00 m, N 1000.00 m) and *B* (E 1300.00 m, N 1500.00 m) were observed with a theodolite placed at a surface station *X* near to the line *AB*. The observations were also made on two plumb wires *P* and *Q*, the distances *XA*, *XP*, and *PQ* being 269.120 m, 8.374 m, and 5.945 m, respectively, when *P* is nearer to *X* than *Q*. The measured horizontal angles at *X* from a reference mark *M* are as ∠*MXA* = 273°42′24″, ∠*MXB* = 93°42′08″, ∠*MXP* = 98°00′50″, and ∠*MXQ* = 98°00′40″. Calculate the bearing of *PQ*.

ANSWERS TO SELECTED PROBLEMS

1. 17.8635 m, ± 2.128×10^{-4} m.
2. ± 82 cm^3.
3. 5138.82 m^2, 2.43 m^2, ± 2.34 m^2
4. 8.
5. 6.
6. 128.34 m.
7. 144.38 m.
8. 299.92 m.
9. 16.95 kg.
10. 1 in 1.32.
11. 160.835 m.
12. 712.512 m.
13. 142.54 m, 14.58m.
14. 1.122, 1.072, 94.73 m.
15. 1/60.33.
16. 3420.65 m.
17. 204.43 m, 263.092 m, 244.783 m, 1 in 18.
18. (*i*) 0.8969 m, (*ii*) 0.8966 m.
19. 14.62 m.
20. 111.496 m.
21. R.L.'s: 102.30 m, 101.50 m, 109.70 m, 103.30 m, Closing error: + 0.30 m, Adjusted values: 102.30 m, 101.40 m, 109.50 m, 103.10 m.
22. B.S.: 1.43 m, I.S.: 1.01 m, F.S.: 0.68 m, H.I.: 282.05 m, R.L.'s: 277.07 m, 279.07 m.
23. 2.584 m.
24. 100.49 m.
25. Line of collimation up by + 2′4″.
26. 0.758 m, 2.224 m.
27. 19 mm, 20.627 m.
28. 73.858 m, 74.392 m, 4 mm, No, Clearance under bridge: 5.473 m.
29. 0.02, 0.03, 0.03, 0.02, 0.01.
30. 27.9 mm, 185.984 m.
31. (a) 367.21 m, (b) 13.4″.
32. E 730.630 m and N 342.553 m, E 774.351 m and N 273.541 m, E 738.688 m and N 178.933 m.
33. E 4042.93 m and N 6055.88 m.
34. 157°35′, 433.41 m.
35. 143.73 m, 289.15 m.
36. BC = 343.67 m.
37. Coordinates of C: E 1634.67 m, N 2037.13 m.
38. 25.3″.
39. 92°23′28″.
40. $\sigma_{\Delta E} = \pm 34$ mm, $\sigma_{\Delta N} = \pm 40$ mm.
41. Q 138.64 m, R 141.78 m, S 144.34 m.
42. 45°23′59″, 29°38′51″, 67°42′59″, 37°14′11″, 63°19′45″, 49°47′13″.
43. E 999.92 m, N 1422.57 m.
44. $x = +1.02$, $y = -1.05$.
45. θ_1 = 48°08′15.23″, θ_8 = 71°55′52.62″.
46. 62°00′31″.
47. 5.50 m.
48. At $x = 2$ m, $y = 3.34$ m.
49. At chainage 2100.00 m, 02°44′00″.
50. 194°47′01″, 209.791 m.
51. 758.5 m, 0.51 m, 00°36′20″.
52. (a) 54.44 m, (b) 0.27 m, (c) At chainage 2450.00 m, 00°16′40″, (d) At chainage 2560.00 m, 05°14′20″.
53. 216 m.
54. 2443.20 m, 2591.20 m, 99.34 m, 99.83 m.
55. 99.719, 99.541, 99.469, 99.500, 99.635, 99.875, 100.219, 100.667.
56. 1.944 hectares.
57. (a) 2283.3 m^3, (b) 2331.9 m^3.
58. 2.23 m^3 (cut).
59. (a) 29.693 × 106 m^3, (b) 669.8 m, (c) 30.406×10^6 m^3.
60. AS = 55.90 m, BS = 65.01 m.
61. AR = 163.04 m, AM = 162.76 m, PM = 9.59 m, $\angle MRP$ = 85°53′, $\angle MRQ$ = 91°35′.
62. 1013.27 m, 68°10′10″.
63. 35°16′00″.

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