Engineering Surveying

Theory and Examination Problems for Students

Volume 2, Second Edition

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Preface

This second edition of Volume 2 of *Engineering Surveying* deals with those more advanced aspects of surveying which generally occur in the final year of degree or diploma courses in engineering. To reinforce a clear understanding of the concepts involved, numerous worked examples, carefully selected for the purpose, are presented after each topic within the chapter. Finally, student exercises complete with answers are supplied for private study purposes.

Chapter 1 has been greatly extended to provide a detailed treatment of the study of errors in surveying observations, the effect of their combination and propagation and the various procedures used to produce a statistically-viable result. The chapter commences with a description of the errors involved, their distribution and the basic statistical techniques used in their treatment. The work then proceeds to the application of least squares in the adjustment and strength analysis of control networks. Theory and application are dealt with, using both classical and matrix algebra, and areas such as unit variance weighting, optimization and pre-survey analysis are covered. In conclusion, a fully-worked example of the adjustment and strength analysis of a control network is supplied to facilitate comprehension of the theory involved.

Chapter 2 dealing with control surveys, covers a wide variety of topics relevant to the basic methodology of position fixing. Included in the treatment are resection/intersection, trigonometric levelling, the theory and application of scale factors, convergence of meridians, and $(t\text{-}T)$ corrections, etc. The importance of electromagnetic distance-measurement is clearly evident by the detailed treatment it receives.

Chapter 3 deals with the broad principles of aerial and terrestrial photogrammetry as befits the engineer, who is more likely to be a user of the end product, rather than a practitioner of the technique. To this end, the elementary theory and methods of obtaining three-dimensional data from aerial photographs is dealt with in detail. This approach not only enables the engineer to utilize the techniques, where necessary, but also provides an introduction to and basic understanding of the subject. It also enables the reader to appreciate the specifications required for vertical air photography, which are provided in full at the end of the chapter.

Chapter 4 deals with the application of field astronomy to position fixing and as such introduces the reader to spherical trigonometry and its application to certain engineering situations. The main difficulty experienced by students is in understanding the concept of time. This topic is, therefore, dealt with at length using the simple concept of clock diagrams.

Throughout the presentation, the accent is on a qualitative and intuitive understanding of the basic concepts of the material.

The book should prove useful to technician and undergraduate students of surveying and civil, mining and municipal engineering, as well as those studying for the various professional examinations which include this subject.

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W. Schofield

1 Errors and adjustments

The basic task in surveying is the establishment of three-dimensional control which is usually achieved by linear and angular measurement. Such measurements must inevitably contain errors; thus statistical techniques are employed not only to distribute these errors but also to assess the reliability of the final accepted value, within specified confidence limits.

1.1 CLASSIFICATION OF ERRORS

(1) *Mistakes* are sometimes called *gross errors*, but should not be classified as errors at all. They are blunders, often resulting from fatigue or the inexperience of the surveyor. Typical examples are: omitting a whole tape length when measuring distance, sighting the wrong target in a round of angles, reading '6' on a levelling staff as '9' and *vice versa*. Mistakes are the largest of the errors likely to arise, therefore great care must be taken to obviate them.

(2) *Systematic errors* can be constant or variable throughout an operation and are generally attributable to known circumstances. The value of these errors can be calculated and applied as a correction to the measured quantity. They can be the result of *natural* conditions, examples of which are: refraction of light rays, variation in the speed of electromagnetic waves through the atmosphere, expansion or contraction of steel tapes due to temperature variations. In all these cases, corrections can be applied to reduce their effect. Such errors may also be produced by instruments, e.g. maladjustment of the theodolite or level; index error in spring balances; ageing of the crystals in electromagnetic distance-measuring (EDM) equipment.

There is the *personal* error of the observer who may have a bias against setting a micrometer or in bisecting a target, etc. Such errors can frequently be self-compensating; for instance, a person setting a micrometer too low when obtaining a bearing will most likely set it too low when obtaining the second bearing, and the resulting angle will be correct.

Systematic errors, in the main, conform to mathematical and physical laws; thus it is argued that appropriate corrections can be computed and applied to *reduce* their effect. It is doubtful, however, whether the effect of systematic errors is ever entirely eliminated, largely due to the inability to obtain an exact measurement of the quantities involved. Typical examples are: the difficulty of obtaining group refractive index

throughout the measuring path of EDM distances; the difficulty of obtaining the temperature of the steel tape, based on air temperature measurements with thermometers. Thus, systematic errors are the most difficult to deal with and therefore they require very careful consideration prior to, during, and after the survey.
(3) *Random errors* are those variates which remain after all other errors have been removed. They are beyond the control of the observer and result from the human inability of the observer to make exact measurements, for reasons already indicated above.

Random variates are assumed to have a continuous frequency distribution called *normal distribution* and obey the law of probability. A random variate x_i, which is normally distributed with a mean μ and standard deviation σ, is written in symbol form as $N(\mu, \sigma^2)$. It should be fully understood that it is random errors *alone* which are treated by statistical processes.

1.1.2 Basic concept of errors

The basic concept of errors in the data captured by the surveyor may be likened to target shooting.

In the first instance, let us assume that a skilled marksman used a rifle with a bent sight, which resulted in his shooting producing a scatter of shots as at *A* in *Figure 1.1*.

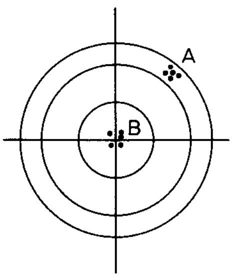

Figure 1.1

That the marksman is skilled (or reliable) is evidenced by the very small scatter which illustrates excellent *precision*. However, as the shots are far from the centre, caused by the bent sight (systematic error), they are completely inaccurate. Such a situation can arise in practice when a piece of EDM equipment produces a set of measurements all agreeing to within a few millimetres (high precision) but, due to an operating fault and lack of calibration, the measurements are all incorrect by several metres (low accuracy).

If the bent sight is now corrected, i.e. systematic errors minimized, the result is a scatter of shots as at *B*. In this case, the shots are clustered near the centre of the target and thus high precision, due to the small scatter, can be related directly to accuracy. The scatter is of course due to the unavoidable random errors.

If the target is now placed face down, the surveyors' task would be to locate the most probable position of the centre based on an analysis of the position of the shots at *B*. From this analogy several important facts emerge, as follows.

(a) Scatter is an 'indicator of precision'. The wider the scatter of a set of results about the mean, the less reliable they will be compared with results having a small scatter.

(b) Precision must not be confused with accuracy; the former is a *relative* grouping without regard to nearness to the truth, whilst the latter denotes absolute nearness to the truth.
(c) Precision may be regarded as an index of accuracy only when all sources of error, other than random errors, have been eliminated.
(d) Accuracy may be defined only by specifying the bounds between which the accidental error of a measured quantity may lie. The reason for defining accuracy thus is that the absolute error of the quantity is generally not known. If it were, it could simply be applied to the measured quantity to give its true value. The *error bound* is usually specified as symmetrical about zero. Thus the accuracy of measured quantity x is $x \pm \varepsilon_x$, where ε_x is greater than or equal to the true but unknown error of x.
(e) Position-fixing by the surveyor, whether it be the co-ordinate position of points in a control network, or the position of topographical detail, is simply an assessment of the most probable position and, as such, requires a statistical evaluation of its reliability.

1.2 FURTHER DEFINITIONS

(1) *True value* of a measurement can never be found, even though such a value exists. This is evident when observing an angle with a one second theodolite; no matter how many times the angle is read, a slightly different value will be obtained.
(2) *True error* (ε_x) similarly can never be found, for it consists of the true value (X) minus the observed value (x), i.e.

$$X - x = \varepsilon_x$$

(3) *Relative error* is a measure of the error in relation to the size of the measurement. For instance, a distance of 10 m may be measured with an error of ± 1 mm, whilst a distance of 100 m may also be measured to an accuracy of ± 1 mm. Although the error is the same in both cases, the second measurement may clearly be regarded as more accurate. To allow for this, the term *relative error* (R_x) may be used, where

$$R_x = \varepsilon_x / x$$

Thus, in the first case $x = 10$ m, $\varepsilon_x = \pm 1$ mm, therefore $R_x = 1/10\,000$; while in the second case $R_x = 1/100\,000$, clearly illustrating the distinction. Multiplying the relative error by 100, gives the *percentage error*. 'Relative error' is an extremely useful definition, and is commonly used in expressing the accuracy of linear measurement. For example, the relative closing error of a traverse is usually expressed in this way. The definition is clearly not applicable to expressing the accuracy to which an angle is measured, however.
(4) *Most probable value* (*MPV*) is the closest approximation to the true value that can be achieved from a set of data. This value is generally taken as the *arithmetic mean* of a set, ignoring at this stage the frequency or weight of the data. For instance, if A is the arithmetic mean, X the true value, and ε_n the errors of a set of n measurements, then

$$A = X - \frac{[\varepsilon_n]}{n}$$

where $[\varepsilon_n]$ is the sum of the errors. As the errors are equally as likely to be + or −, then

for a finite number of observations $[\varepsilon_n]/n$ will be very small and $A \approx X$. For an infinite number of measurements, it could be argued that $A = X$.
(*N.B.* The square bracket is Gaussian notation for 'sum of'.)
(5) *Residual* is the closest approximation to the true error and is the difference between the MPV of a set, i.e. the arithmetic mean, and the observed values. Using the same argument as before, it can be shown that for a finite number of measurements, the residual r is approximately equal to the true error ε.

1.3 PROBABILITY

Consider a length of 29.42 m measured with a tape and correct to ± 0.05 m. The range of these measurements would therefore be from 29.37 m to 29.47 m, giving 11 possibilities to 0.01 m for the answer. If the next bay was measured in the same way, there would again be 11 possibilities. Thus the correct value for the sum of the two bays would lie between $11 \times 11 = 121$ possibilities, and the range of the sum would be $2 \times \pm 0.05$ m, that is, between -0.10 m and $+0.10$ m. Now, the error of -0.10 m can occur only once, that is when both bays have an error of -0.05 m; similarly with $+0.10$. Consider an error of -0.08, this can occur in three ways: (-0.05 and -0.03), (-0.04 and -0.04) and (-0.03 and -0.05). Applying this procedure through the whole range can produce *Table 1.1*, the lower half of which is simply a repeat of the upper half. If the decimal probabilities are added together they equal 1.0000. If the

TABLE 1.1

Error	*Occurrence*	*Probability*
−0.10	1	1/121 = 0.0083
−0.09	2	2/121 = 0.0165
−0.08	3	3/121 = 0.0248
−0.07	4	4/121 = 0.0331
−0.06	5	5/121 = 0.0413
−0.05	6	6/121 = 0.0496
−0.04	7	7/121 = 0.0579
−0.03	8	8/121 = 0.0661
−0.02	9	9/121 = 0.0744
−0.01	10	10/121 = 0.0826
0	11	11/121 = 0.0909
0.01	10	10/121 = 0.0826
etc.	*etc.*	*etc.*

above results are plotted as error against probability the histogram of *Figure 1.2* is obtained, the errors being represented by rectangles. Then, in the limit, as the error interval gets smaller, the histogram approximates to the superimposed curve. This curve is called the *normal probability curve.* The area under it represents the probability that the error must lie between ± 0.10 m, and is thus equal to 1.0000 (certainty) as shown in *Table 1.1.*

More typical bell-shaped probability curves are shown in *Figure 1.3*; the tall thin curve indicates small scatter and thus high precision, whilst the flatter curve represents large scatter and low precision. Inspection of the curve reveals: (i) positive and negative

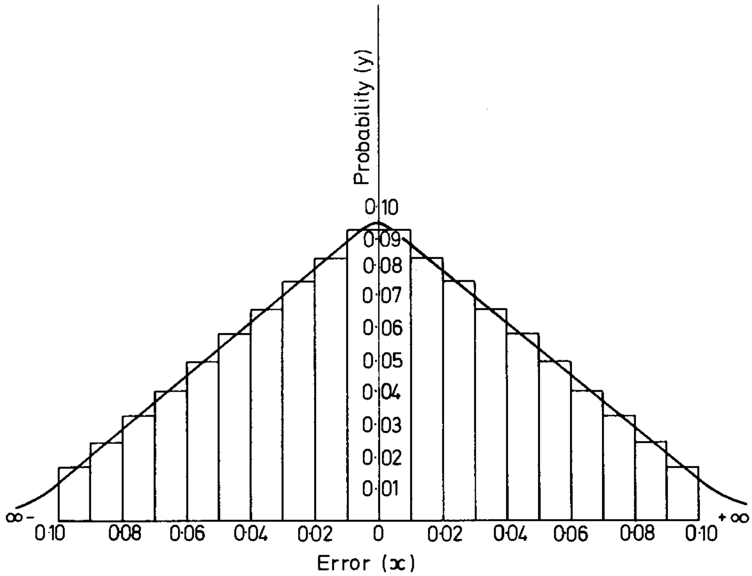

Figure 1.2

errors are equal in size and frequency; (ii) small errors are more frequent than large; (iii) very large errors seldom occur.

The curve can also be used to indicate the probability of an error falling within certain limits. In *Figure 1.3* the shaded portion represents the probability of an error falling within the limits ± 1. The shaded portion represents 68.3% of the total area; thus the error of ± 1 is likely to occur roughly seven times out of a set of ten. This particular area has a special significance, as shown later.

As already illustrated, the area under the curve represents the limit of relative frequency, i.e. probability, and is equal to unity. Thus tables of standard normal curve areas can be used to calculate probabilities provided that the distribution is the standard normal distribution, i.e. $N(0, 1^2)$. If the variable x is $N(\mu, \sigma^2)$ then it must be transformed to the standard normal distribution using $Z = (x - \mu)/\sigma$, where Z has a probability density function equal to $(2\pi)^{-1/2}e^{-Z^2/2}$.

If x is $N(5, 2^2)$ then $Z = (x - 5)/2$, and when $x = 9$ then $Z = 2$.

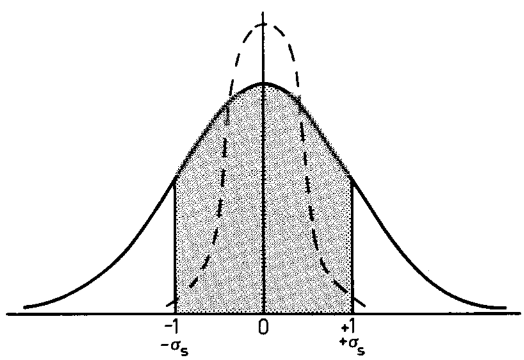

Figure 1.3

The equation of the normal probability distribution curve is

$$y = h\pi^{-1/2}e^{-h^2\varepsilon^2}$$

where y = probability of an occurrence of an error ε, h = index of precision, e = exponential function.

1.4 INDICES OF PRECISION

It is important to be able to assess the precision of a set of observations, and several standards exist for doing this. The most popular is standard deviation (σ), a numerical value indicating the amount of variation about a central value.

In order to appreciate the concept upon which indices of precision devolve, one must consider a measure which takes into account *all* the values in a set of data. Such a measure is the deviation from the mean ($\bar{x}$) of each observed value (x_i), i.e. ($x_i - \bar{x}$), and one obvious consideration would be the mean of these values. However, in a normal distribution the sum of the deviations would be zero; thus the 'mean' of the squares of the deviations may be used, and this is called the *variance* (σ^2).

$$\text{(1)}\quad \sigma^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2/n \qquad (1.1)$$

Theoretically σ is obtained from an infinite number of variates known as the *population.* In practice, however, only a *sample* of variates is available and S is used as an unbiased estimator. Account is taken of the small number of variates in the sample by using ($n - 1$) as the divisor, which is referred to in statistics as the *Bessel correction*; hence, variance is

$$\text{(2)}\quad S^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2/n - 1 \qquad (1.2)$$

As the deviations are squared, the units in which variance is expressed will be the original units squared. To obtain an index of precision in the same units as the original data, therefore, the square root of the variance is used, and this is called *standard deviation* (S), thus

$$\text{(3)}\quad \text{Standard deviation} = S = \pm\left\{\sum_{i=1}^{n} (x_i - \bar{x})^2/n - 1\right\}^{1/2} \qquad (1.3)$$

Standard deviation is represented by the shaded area under the curve in *Figure 1.3*, and so establishes the limits of the error bound within which 68.3% of the values of the set should lie, i.e. seven out of a sample of ten.

Similarly, a measure of the precision of the mean ($\bar{x}$) of the set is obtained using the *standard error* ($S_{\bar{x}}$), thus

$$\text{(4)}\quad \text{Standard error} = S_{\bar{x}} = \pm\left\{\sum_{i=1}^{n} (x_i - \bar{x})^2/n(n - 1)\right\}^{1/2} = S/n^{1/2} \qquad (1.4)$$

Standard error therefore indicates the limits of the error bound within which the 'true' value of the mean lies, with a 68.3% certainty of being correct.

It should be noted that S and $S_{\bar{x}}$ are entirely different parameters. The value of S will *not* alter significantly with an increase in the number (n) of observations, the value of $S_{\bar{x}}$, however, will alter significantly as the number of observations increases. It is important therefore that to describe measured data both values should be used.

Although the weighting of data has not yet been discussed, it is appropriate here to mention several other indices of precision applicable to weighted (w_i) data

(5) Standard deviation (of weighted data)

$$= S_w = \pm\left\{\sum_{i=1}^{n} w_i(x_i - \bar{x})^2/n - 1\right\}^{1/2} \tag{1.5}$$

(6) Standard deviation of a single measure of weight w_i

$$= S_{w_i} = \pm\left\{\sum_{i=1}^{n} w_i(x_i - \bar{x})^2/w_i(n-1)\right\}^{1/2} = S_w/(w_i)^{1/2} \tag{1.6}$$

(7) Standard error (the weighted mean)

$$= S_{\bar{w}} = \pm\left\{\sum_{i=1}^{n} w_i(x_i - \bar{x})^2\Big/\sum_{i=1}^{n}(w_i)(n-1)\right\}^{1/2} = S_w\Big/\left(\sum_{i=1}^{n} w_i\right)^{1/2} \tag{1.7}$$

N.B. The conventional method of expressing *sum of* has been used for the various indices of precision, as this is the format used in texts on statistics, and therefore more easily recognizable. However, for the majority of the expressions the neater Gaussian square bracket format has been used.

1.5 WEIGHT

Weights are expressed numerically and indicate the relative precision of quantities within a set. The greater the weight the greater the precision of the observation to which it relates. Thus an observation with a weight of two may be regarded as twice as reliable as an observation with a weight of one. Consider two mean measures of the same angle: $A = 50° 50' 50''$ of weight one, and $B = 50° 50' 47''$ of weight two. This is equivalent to three observations, 50″, 47″, 47″, all of equal weight, and having a mean value of

$$(50'' + 47'' + 47'')/3 = 48''$$

Therefore the mean value of the angle $= 50° 50' 48''$.

Inspection of this exercise shows it to be identical to multiplying each observation a by its weight w, and dividing by the sum of the weights $[w]$, i.e.

$$\text{weighted mean} = A_m = \frac{a_1w_1 + a_2w_2 + \cdots + a_nw_n}{w_1 + w_2 + \cdots + w_n} = \frac{[aw]}{[w]} \tag{1.8}$$

Weights can be allocated in a variety of ways, such as (i) by personal judgement of the prevailing conditions at the time of measurement; (ii) by direct proportion to the number of measurements of the quantity, i.e. $w \propto n$; (iii) by the use of variance and co-variance factors. This last method is recommended and in the case of the variance factor is easily applied as follows: equation (1.4) shows

$$S_{\bar{x}} = S/n^{1/2}$$

that is, error is inversely proportional to the square root of the number of measures. However, as $w \propto n$, then

$$w \propto 1/S_{\bar{x}}^2 \qquad (1.9)$$

i.e. weight is proportional to the inverse of the variance.

It is important always to consider weights in this way, particularly in, say, the least square adjustment of dissimilar quantities, such as angles and distances. If weights were ignored, then one is assuming that the error in an angle is directly in ratio to the error in length, i.e., say, 1″ to 1 ft, or 1″ to 1 m. Thus if a set of datta was adjusted using its lengths in feet, then simply converting them to metres would result in a different set of adjusted values. (See *Section 1.12.3*).

1.6 REJECTION OF OUTLIERS

It is not unusual, when taking repeated measurements of the same quantity, to find at least one which appears very different from the rest. Such a measurement is called an *outlier*, which the observer intuitively feels should be rejected from the sample. However, intuition is hardly a scientific argument for the rejection of data and a more statistically-viable approach is required.

As already indicated (in *Section 1.4*), standard deviation S represents 68.3% of the area under the normal curve and is therefore representative of 68.3% confidence limits. It follows from this that

$\pm 1.96\ S$ represents 95% confidence limits (0.95 probability)
$\pm 2.57\ S$ represents 99% confidence limits (0.99 probability)
$\pm 3.29\ S$ represents 99.9% confidence limits (0.999 probability)

Thus, any random variate x_i, whose residual error $(x_i - \bar{x})$ is greater than $\pm 3.29\ S$, must lie in the extreme tail ends of the normal curve and should therefore be ignored, i.e. rejected from the sample. In practice, this has not proved a satisfactory rejection criterion due to the limited size of the samples. Logan (*Survey Review*, No. 97, July 1955) has shown that the appropriate rejection criteria are relative to sample size, as follows:

Sample size	*Rejection criteria*
4	1.5 S
6	2.0 S
8	2.3 S
10	2.5 S
20	3.0 S

A similar approach to rejection is credited to Chauvenet. If a random variate x_i, in a sample size n has a deviation from the mean $\bar{x}$ greater than a $1/2n$ probability, it should be rejected. For example, if $n = 8$, then $1/2n = 0.06$ (94% or 0.94) and the probability of the deviate is 1.86 S. Thus, an outlier whose residual error or deviation from the mean was greater than 1.86 S would be rejected. This approach produces the following table:

Sample size	*Rejection criteria*
4	1.53 S
6	1.73 S
8	1.86 S
10	1.96 S
20	2.24 S

It should be noted that *successive* rejection procedures should not be applied to the sample.

1.7 STUDENT'S t-DISTRIBUTION

It was shown in *Section 1.3* that in order to transform a $N(\mu, \sigma^2)$ distribution one uses $Z = (x - \mu)/\sigma$. However, if S is used as an unbiased estimator of σ, then the distribution produced is not the standard normal but is Student's t-distribution with $(n - 1)$ degrees of freedom (DF). As the sample size approaches thirty, then Student's t-distribution approaches the standard normal.

The term *degrees of freedom* refers to the number of measurements in a sample that are free to vary. For instance, in the measurement of a quantity n times, the first measurement defines the quantity, the remaining measures $(n - 1)$ are additional redundant measures taken to confirm the validity of the first. Hence there are $(n - 1)$ DF. Another way to consider it is, if a quantity was measured, say, six times and the mean obtained, then one could vary the first five measurements but the sixth would be fixed relative to the first five measures and the mean, hence there are five, i.e. $(n - 1)$, DF.

If dealing with n equations containing m independent variables, the number of degrees of freedom would be $(n - m)$.

Tables of areas under standard normal curve are available to compute probability, but not under the t-distribution curve due to the variation in the number of DF. Thus t-tables tabulate values of t corresponding to a particular area, and t is the ratio of the difference between the measured mean value and the hypothesized mean, compared with the standard error of the mean, i.e.

$$t = (\bar{x} - \mu)/S_{\bar{x}} \tag{1.10}$$

Thus it can be seen that the t-distribution should be used when the sample size is less than 30 and σ is unknown. Some applications will now be outlined.

1.7.1 Confidence intervals

The t-distribution may be used to calculate the confidence interval of the population mean μ, as follows:

Example 1.1. The angle subtended by a subtense bar was measured 16 times, the mean value obtained was 2° 48′ 34.86″ with a standard deviation of 3.62″. Calculate the 95% and 99% confidence intervals of the mean.

$n = 16 \qquad \bar{x} = 34.86'' \qquad S = 3.62''$

$\therefore \; S_{\bar{x}} = S/n^{\frac{1}{2}} = 0.91''$

From the transformation: $t = (34.86 - \mu)/0.91''$

$\therefore$ Confidence limits for μ are $34.86 \pm 0.91t$

From t-tables (see Appendix, *Table A.1*)

$t_{0.05}$ for 15 DF = 2.13 (95% probability)
$t_{0.01}$ for 15 DF = 2.95 (99% probability)

$\therefore$ Confidence limits for the mean value are

$$2° \, 28' \, 34.86'' \pm 1.93'' \text{ at } 95\% \text{ probability}$$
$$\text{and } 2° \, 48' \, 34.86'' \pm 2.68'' \text{ at } 99\% \text{ probability}$$

An alternative way of expressing the above is that the probability that $\bar{x}$ would lie outside 2° 48′ 32.93″ and 2° 48′ 36.79″ is 0.05 or 5%.

If one assumes a normal distribution then $1.96S_{\bar{x}}$ represents 95% probability and confidence limits for μ are 2° 48′ 34.86 ± 1.78″.

1.7.2 Testing hypotheses

In all tests of significance it is assumed at the outset that there is no significant difference between the distributions under test; this is called the *null hypothesis.*

1.7.2.1 Single-sample problem

As already shown in the previous example, for a t-distribution with 15 DF the probability of a value lying *outside* the limits ±2.13 is 0.95 (95%). These limits are the 5% significance limits and are utilized in hypothesis testing. Consider the following example:

Example 1.2. An angle in a test network has been proven to be 58° 35′ 24.5″ (μ). To check for possible movement of the network the angle was check-measured nine times and this produced a mean value of 58° 35′ 27.3″ ($\bar{x}$) with a standard deviation 2.2″. Is there a significant difference between the population and sample means, thereby indicating possible movement of the stations observed?

(a) Null hypothesis assumes no difference between the sample and assumed population means.
(b) $S_{\bar{x}} = S/n^{\frac{1}{2}} = 2.2/3 = 0.73''$.
(c) $t = (\bar{x} - \mu)/S_{\bar{x}} = (27.3 - 24.5)/0.73 = 3.84$.
(d) From t-tables (see Appendix, *Table A.1*): For a t-distribution with eight DF the 1% significance limits are ±3.36.
(e) The value of $t = 3.84$ lies outside the 1% level, thus there is a significant difference between the two means and the null hypothesis is therefore rejected.

It would appear that movement is taking place in the network; however, it would be unwise to assume this without further measurements and independent investigation. In all such situations one should never discount personal knowledge, experience and judgement.

1.7.2.2 Two-sample problems

When surveying, one is frequently faced with the problem of assessing two sets of observations of varying size, or captured by different observers, to see if they are representative of the same population. However, *prior* to testing this hypothesis one should test the variances for significance using the F-test.

Thus, if there are two samples of size n_1 and n_2, means $\bar{x}_1$ and $\bar{x}_2$ and standard deviations S_1 and S_2, the procedure is as follows:

(a) Set up the null hypothesis that there is no difference between the two means, i.e. $\bar{x}_1 = \bar{x}_2$.
(b) Find the combined standard deviation S of the two samples from

$$S = \left\{\frac{(n_1 - 1)S_1 + (n_2 - 1)S_2}{(n_1 - 1) + (n_2 - 1)}\right\}^{\frac{1}{2}} \tag{1.11}$$

(c) Obtain the standard error using

$$S_{\bar{x}} = S\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{\frac{1}{2}} \tag{1.12}$$

(d) Then $t = |\bar{x}_1 - \bar{x}_2|/S_{\bar{x}}$ gives a distribution with $\{(n_1 - 1) + (n_2 - 1)\}$ DF.
(e) Now obtain t from tables with $\{(n_1 - 1) + (n_2 - 1)\}$ DF at the 5% and/or 1% level of significance.
(f) If computed $t > t$ from tables, then at 1% level there is conclusive evidence of a significant difference between $\bar{x}_1$ and $\bar{x}_2$. At 5% level, there is reasonable evidence. If there is no significant difference the null hypothesis is accepted.

1.8 *F*-DISTRIBUTION

The F-distribution is used to compare the variances of two samples using their ratio. For instance, if the variances were taken from the same population and were equal, then the F ratio equal to S_1^2/S_2^2 with $(S_1 > S_2)$ would be 1. Tables are available that give F values at various levels of significance. Consider the following example:

Example 1.3. Two surveyors, A and B, measure the same angle using the same theodolite. A measures the angle six times with a standard deviation of $\pm 6.8''$. B measures it 14 times with a standard deviation of $\pm 4.9''$. Is there a significant difference in the ability of the two observers?

(a) Null hypothesis: $S_A = S_B$.
(b) $F = S_A^2/S_B^2 = 6.8^2/4.9^2 = 1.93$.
(c) Test significance using F-tables with 5 and 13 DF

1% significance level $F = 4.86$ (see Appendix, *Table A.2*)
5% significance level $F = 3.03$ (see Appendix, *Table A.3*)

(d) $F(1.93)$ is not significant (i.e. greater than 3.03) at the 5% level and so the null hypothesis is accepted that there is no significant difference between the observers.

1.9 CHI-SQUARED DISTRIBUTION

'Chi-squared' is written as χ^2 and is a test of frequency between sets of values to see if the variation is significant. χ^2 represents the sum of the squares of independent, random variates x and must therefore be a random variable itself. It is said to have a *Chi-squared distribution.*

1.9.1 Goodness of fit

In testing a sample for bias one assesses the difference between observed (O) and expected (E) frequencies. Consider the following example:

Example 1.4. Five different types of EDM equipment are tested on a calibration base of known length. Each instrument measured the distance an equal number of times and the frequency with which each instrument obtained the known length was noted as follows:

Instrument	A	B	C	D	E
Observed frequency (O)	8	11	5	13	3
Expected frequency (E)	8	8	8	8	8

Is there any significant difference in the performance of the instruments?

If there was no bias, one would 'expect' each instrument to obtain the same frequency, which from an average of the O values is eight.

(a) Null hypothesis: There is no difference between the O and E values.

(b) $$\chi^2 = \sum_{i=1}^{5} \{(O - E)^2/E\} = \frac{(8-8)^2}{8} + \frac{(11-8)^2}{8} + \frac{(5-8)^2}{8} + \frac{(13-8)^2}{8} + \frac{(3-8)^2}{8} = 8.52$$

(c) From χ^2 distribution tables (see Appendix, *Table A.4*) using (5 − 1) DF

1% significance level $\chi^2 = 13.28$
5% significance level $\chi^2 = 9.49$

(d) At the 5% level there is no significant difference, so the null hypothesis is accepted, and it can be argued that there is no significant difference in the performance of the instruments.

1.9.2 Contingency tables

In the above example the frequencies were classified according to only one criterion, the number of 'true' measurements obtained. If a second criterion was introduced, such as observation under entirely different temperature conditions, then two criteria are present, as shown below.

Contingency table

Instrument	A	B	C	D	E	*Total*
Frequency (25°C)	8	11	5	13	3	40
Frequency (0°C)	4	12	12	18	9	55
Total	12	23	17	31	12	95

The above table having two rows and five columns is known as a *2 × 5 contingency table.* The χ^2 test is used to test the hypothesis that temperature and the frequency of measurements are independent.

(a) Null hypothesis: Frequency is independent of temperature.
(b) Considering the 25°C temperature one would expect the following relationship to be valid

$$\frac{\text{Frequency of A}}{\text{Total frequency}} = \frac{\text{Total frequency at } 25^\circ}{\text{Grand total}}$$

i.e. $8/12 = 40/95$

Thus, if E_A is the *expected* frequency of A at 25°C

We have $E_A/12 = 40/95$
$\therefore\ E_A = (40 \times 12)/95 = 5$

Thus, if R = row total, C = column total, T = grand total then: Expected frequency = $(R \times C)/T$; using this formula the expected frequencies are calculated at 25°C. The expected frequencies at 0°C can then be obtained by subtraction from the totals, as shown below:

Instrument	E_A	E_B	E_C	E_D	E_E	*Total*
Frequency (25°C)	5	10	7	13	5	40
Frequency (0°C)	7	13	10	18	7	55
Total	12	23	17	31	12	95

(c) Using both tables the χ^2 value is obtained as follows

$$\chi^2 = (8-5)^2/5 + (11-10)^2/10 + (5-7)^2/7 + (13-13)^2/13 + (5-3)^2/3 + (4-7)^2/7 + (12-13)^2/13 + (12-10)^2/10 + (18-18)^2/18 + (9-7)^2/7$$
$$= 6.14$$

(d) When using contingency tables the number of DF employed is $(R-1)(C-1) = (2-1)(5-1) = 4$ DF.

Thus, from χ^2 tables (see Appendix, *Table A.4*) using 4 DF

1% level, $\chi^2 = 13.28$ and 5% level, $\chi^2 = 9.49$

Therefore, $\chi^2 = 6.14$ is not significant at the 5% and the null hypothesis is accepted. Thus instrument performance is not relative to temperature.

It should be noted in both the above examples that χ^2 is significant only if it is *greater* than the values given at 5% or 1% levels. Such tests are called *one-tailed.*

1.9.3 Comparison of variances

The χ^2 test may also be used to test the hypothesis that the population variance equals a sample variance, i.e. $\sigma^2 = S^2$. This naturally involves knowing whether σ is significantly *greater* or significantly *less* than S and therefore needs investigation at each tail of the distribution curve. This is called a *two-tailed* test and we cannot therefore use the one-tailed χ^2 tables for 'goodness of fit'.

In order to test the hypothesis that $\sigma = S$, the value of χ^2 is computed using

$$\chi^2 = (n-1)S^2/\sigma^2 \tag{1.13}$$

and tested using χ^2 distribution tables with $(n-1)$ DF.

Example 1.5. A gyro-theodolite was damaged in a tunnelling accident and was subsequently repaired and modified slightly in the process. In the past, repeated calibration measurements on a base line of known azimuth had produced a standard deviation of $\pm 8''$. In order to check the modified instrument, 20 observations were taken on the base and a standard deviation of $12''$ obtained. Had the repairs and modifications significantly altered the performance of the instrument?

(a) Null hypothesis: $\sigma = S$.
(b) $\chi^2 = (n-1)S^2/\sigma^2 = (20-1) \times 12^2/8^2 = 42.75$.
(c) From χ^2 tables (see Appendix, *Table A.4*) for variances with 19 DF

1% significance level, $\chi^2 = 6.844$ and 38.582
5% significance level, $\chi^2 = 8.907$ and 32.852

The value of $\chi^2 = 42.75$ is significant at the 1% level and there is conclusive evidence that $\sigma \neq S$. Thus the null hypothesis is rejected and it would appear that the instrument performance has been affected (reduced) by the repairs and modifications.

If the above question was reworded to ask (i) 'Had the repairs and modifications significantly *improved* performance', or (ii) '. . . significantly *reduced* performance', then although we are still testing the hypothesis $\sigma = S$, it is tested against the 'alternative hypothesis' (i) $\sigma < S$ or (ii) $\sigma > S$. In this case the test is one-tailed, and one-tailed χ^2 tables *for variance* are used.

1.10 COMBINATION OF ERRORS

Much data in surveying is obtained indirectly from various combinations of observed data, for instance the co-ordinates of a point are a function of the length and bearing of a line. As each measurement contains an error, it is necessary to consider the combined effect of these errors on the derived quantity.

The general procedure is to differentiate with respect to each of the observed quantities in turn and sum them to obtain their total effect. Thus if $a = f(x, y, z, \ldots)$, each containing errors $\delta x, \delta y, \delta z, \ldots$ then the total error in a will be

$$\delta a = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z + \cdots \tag{1.14}$$

If it is required to find the standard error in a due to standard errors in x, y and z, etc. the following form is used

$$\sigma_a^2(f) = \left(\frac{\partial f}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial f}{\partial y}\sigma_y\right)^2 + \left(\frac{\partial f}{\partial z}\sigma_z\right)^2 + \cdots \tag{1.15}$$

which is the general equation for the variance of a function. Equation (1.15) is very important and is used extensively in surveying despite its statistical limitations. For instance, in the process of partial differentiation with respect to x, y and z are held temporarily constant, while a is assumed to be a function of x only. This may not be so in practice. In addition, equation (1.15) produces only an approximate answer, which is generally regarded as sufficiently accurate for the purpose.

In the derivations which follow, it is understood that the sample variance S^2 generally replaces the population variance σ^2.

1.10.1 Errors affecting addition or subtraction

Consider a quantity $A(f) = a + b$ where a and b are affected by standard errors σ_a and σ_b, then

$$\sigma_A^2 = \left\{\frac{\partial(a+b)}{\partial a}\sigma_a\right\}^2 + \left\{\frac{\partial(a+b)}{\partial b}\sigma_b\right\}^2 = \sigma_a^2 + \sigma_b^2 \qquad \therefore\ \sigma_A = \pm(\sigma_a^2 + \sigma_b^2)^{\frac{1}{2}} \tag{1.16}$$

As subtraction is simply addition with the signs changed, the above holds for the error in a *difference*.

If $\sigma_a = \sigma_b = \sigma$, then $\qquad \sigma_A = \pm\sigma(n)^{\frac{1}{2}}$ (1.17)

Equation (1.17) should not be confused with equation (1.4) which refers to the *mean*, not the *sum* as above.

Example 1.6. If three angles of a triangle each have a standard error of $\pm 2''$, what is the total error (σ_T) in the triangle?

$$\sigma_T = \pm(2^2 + 2^2 + 2^2)^{\frac{1}{2}} = \pm 2(3)^{\frac{1}{2}} = \pm 3.5''$$

Example 1.7. In measuring a round of angles at a station, the third angle c closing the horizon is obtained by subtracting the two measured angles a and b from 360°. If angle a has a standard error of $\pm 2''$ and angle b a standard error of $\pm 3''$, what is the standard error of angle c?

$$c \pm \sigma_c = 360° - (a \pm \sigma_a) - (b \pm \sigma_b)$$
$$= 360° - (a \pm 2'') - (b \pm 3'')$$

since $\qquad c = 360° - a - b$

then $\qquad \pm\sigma_c = \pm\sigma_a \pm \sigma_b = \pm 2'' \pm 3''$

and $\qquad \sigma_c = \pm(2^2 + 3^2)^{\frac{1}{2}} = \pm 3.6''$

Example 1.8. The standard error of a *mean* angle derived from four measurements is $\pm 3''$; how many measurements would be required, using the same equipment, to halve this error?

From equation (1.4) $\quad \sigma_m = \pm \frac{\sigma_s}{n^{\frac{1}{2}}} \qquad \therefore \; \sigma_s = 3 \times 4^{\frac{1}{2}} = \pm 6''$

i.e. the instrument used had a standard error of $\pm 6''$ for a single observation; thus for $\sigma_m = \pm 1.5''$, when $\sigma_s = \pm 6''$

$$n = \left(\frac{6}{1.5}\right)^2 = 16$$

Example 1.9. If the standard error of a single triangle in a triangulation scheme is $\pm 6.0''$, what is the permissible standard error per angle?

From equation (1.17) $\quad \sigma_T = \sigma_p (n)^{\frac{1}{2}}$

where σ_T is the triangular error, σ_p the error per angle, and n the number of angles.

$$\therefore \; \sigma_p = \frac{\sigma_T}{(n)^{\frac{1}{2}}} = \frac{\pm 6.0''}{(3)^{\frac{1}{2}}} = \pm 3.5''$$

1.10.2 Errors affecting a product

Consider $A(f) = (a \times b \times c)$ where a, b and c are affected by standard errors. The variance

$$\sigma_A^2 = \left\{\frac{\partial(abc)}{\partial a}\sigma_a\right\}^2 + \left\{\frac{\partial(abc)}{\partial b}\sigma_b\right\}^2 + \left\{\frac{\partial(abc)}{\partial c}\sigma_c\right\}^2$$

$$= (bc\sigma_a)^2 + (ac\sigma_b)^2 + (ab\sigma_c)^2$$

$$\therefore \; \sigma_A = \pm abc\left\{\left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2 + \left(\frac{\sigma_c}{c}\right)^2\right\}^{\frac{1}{2}} \qquad (1.18a)$$

The terms in brackets may be regarded as the relative errors R_a, R_b, R_c giving

$$\sigma_A = \pm abc(R_a^2 + R_b^2 + R_c^2)^{\frac{1}{2}} \qquad (1.18b)$$

1.10.3 Errors affecting a quotient

Consider $A(f) = a/b$, then the variance

$$\sigma_A^2 = \left\{\frac{\partial(ab^{-1})}{\partial a}\sigma_a\right\}^2 + \left\{\frac{\partial(ab^{-1})}{\partial b}\sigma_b\right\}^2 = \left(\frac{\sigma_a}{b}\right)^2 + \left(\frac{\sigma_b a}{b^2}\right)^2$$

$$\therefore \; \sigma_A = \pm\frac{a}{b}\left\{\left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2\right\}^{\frac{1}{2}} \qquad (1.19a)$$

$$= \pm\frac{a}{b}(R_a^2 + R_b^2)^{\frac{1}{2}} \qquad (1.19b)$$

1.10.4 Errors affecting powers and roots

The case for the power of a number must not be confused with multiplication, since $a^3 = a \times a \times a$, with each term being exactly the same.

Thus if $A(f) = a^n$, then the variance

$$\sigma_A^2 = \left(\frac{\partial a^n}{\partial a}\sigma_a\right)^2 = (na^{n-1}\sigma_a)^2 \qquad \therefore \; \sigma_A = \pm(na^{n-1}\sigma_a) \tag{1.20a}$$

Alternatively
$$R_A = \frac{\sigma_A}{a^n} = \frac{na^{n-1}\sigma_a}{a^n} = \frac{n\sigma_a}{a} = nR_a \tag{1.20b}$$

Similarly for roots, if the function is $A(f) = a^{1/n}$, then the variance

$$\sigma_A^2 = \left(\frac{\partial a^{1/n}}{\partial a}\sigma_a\right)^2 = \left(\frac{1}{n}a^{1/n-1}\sigma_a\right)^2 = \left(\frac{1}{n}a^{1/n}a^{-1}\sigma_a\right)^2$$

$$= \left(\frac{a^{1/n}}{n}\frac{\sigma_a}{a}\right)^2 \qquad \therefore \; \sigma_A = \pm\left(\frac{a^{1/n}}{n}\frac{\sigma_a}{a}\right) \tag{1.21}$$

The same approach is adopted to general forms which are combinations of the above. Examples can be found throughout both this volume and Volume 1.

1.11 ADJUSTMENT OF OBSERVATIONS BY THE METHOD OF LEAST SQUARES

In the establishment of two- or three-dimensional control networks, the basic measurements are angles and distances. Generally speaking more data is observed than is strictly necessary in order to provide checks on errors and enable a more statistically-viable 'adjustment' and strength analysis to be carried out. The additional data observed are referred to as *redundant measurements.*

Modern instrumentation combined with professional skills enables the capture of the above field data to almost perfect precision, and very high accuracy. Nevertheless, whatever adjustment procedure is adopted, distortion of the shape of the network will occur due to adjustment changes in the observed angles and distances. That is, the angles and distances computed from the final accepted co-ordinates will differ from those originally observed. It follows therefore, that as 'adjustment' is necessary to produce a geometrically correct figure, only those methods of adjustment which produce minimal changes in the observations, should be used. 'Least squares adjustment' is such a method.

It should be noted, however, that least squares is by no means the ideal procedure (Schofield 1979), for the resultant changes to the observed data are frequently at variance with the concept of normally distributed variates and the common sense logic of the observer. Indeed, it could be argued that the words 'distorting procedures' could be used in place of 'adjustment procedures'. However, of all the adjustment procedures available, 'least squares' has the advantages of affecting minimal changes in the data, whilst providing a statistically-viable method that is universally applicable to all types of network. It is also relatively straightforward to apply and provides a strength analysis of the final network.

In order to consider more closely what is meant by a strength analysis of the network, it should be realized that three networks are involved. In the first instance there is the network as set out on the ground, this is the true (but unknown) network. Observation of this network for location, shape and size produces the second network which, due to observational errors, will be different from the true network. Finally, there is the 'adjusted' network which will certainly be different from the observed network and may even be more different from the true network. It is this final network whose strength (or reliability) is analysed, the analysis being based on its differences from the observed network.

Thus, in practice the emphasis should always be on observational procedures to minimize errors rather than sophisticated procedures to 'adjust' and analyse those errors.

The method of least squares has been in use now for over 150 years and is credited to Gauss, although in some instances Laplace and Markov have also been given credit. However, the writings of all three were investigated (Plackett 1969) and Gauss was justified as the definitive creator. Nevertheless, in the realm of statistics the basic theorem in the subject is referred to as the *Gauss–Markov theorem* and this states that, in the case of independent observations of equal weight, the least squares estimates are linear unbiased estimates with minimum variance. In surveying literature, the principle of least squares is shown derived for observations of normal distribution and usually expressed as 'the most probable value or best linear unbiased estimate of an observation is the one for which the sum of the squares of the weighted residuals is a minimum, i.e.

$$[wr^2] = \text{minimum}$$

or in matrix terms, the quadratic form

$$r^T W r = \text{minimum}$$

where r is a vector of residual errors and W a diagonal 'weight' matrix of the inverse of the variances of the observations.

Surveyors tend to use the phrase 'least squares produces the most probable value'; however, least squares is simply a mathematical relationship between the original observations and their adjusted values. Statisticians, however, use minimum variance as a criterion of the performance of an estimator, as it describes more clearly the values produced. A minimum variance estimator is one for which the estimated unknowns have a smaller variance than any other estimator produced. An example of this is the arithmetic mean of a set of observations.

It is worthy of note, that although the principle of least squares is easily derived from the equation for the normal distribution curve, it has been shown (Sunter 1966) that the method gives estimations of minimum variance regardless of the distribution. However, if the data is normally distributed, then a least squares solution will supply the most probable value. From the surveying point of view this is important, for the data used for angles and distances would inevitably be the mean of a set of observations. Thus, whilst the individual variables of the set may not be normally distributed, the 'central limit theorem' shows that the mean value will be. Hence, it could be argued that all data used in the least squares solution of a surveying network would be normally distributed.

1.11.1 Principle of least squares

The equation for the probability curve is given by $\quad y = h\pi^{-\frac{1}{2}} e^{-h^2\varepsilon^2}$

where e is an exponential function, h is the index of precision and y is the probability of the occurrence of an error ε.

Let $\pi^{-\frac{1}{2}} = A$ then $y = Ah\, e^{-h^2\varepsilon^2}$

Differentiating with respect to h

$$\frac{dy}{dh} = A\{e^{-h^2\varepsilon^2} + h(-2h\varepsilon^2\, e^{-h^2\varepsilon^2})\} = A\, e^{-h^2\varepsilon^2}(1 - 2h^2\varepsilon^2)$$

For maximum y $\quad \frac{dy}{dh} = 0 \quad$ i.e. $\quad 1 - 2h^2\varepsilon^2 = 0 \quad \therefore\ \varepsilon^2 = \frac{1}{2h^2}$

Considering errors $\quad \varepsilon_1^2 + \varepsilon_2^2 + \cdots \varepsilon_n^2 = \frac{1}{2h_1^2} + \frac{1}{2h_2^2} + \cdots + \frac{1}{2h_n^2}$

then $\quad [\varepsilon^2] = \left[\frac{1}{2h^2}\right]$

and since h represents precision, the accuracy of the observations will increase as h increases. However, as h increases, $1/2h^2$ decreases, the maximum accuracy will be achieved when

$$\left[\frac{1}{2h^2}\right] = [\varepsilon^2] = \text{a minimum} \tag{1.22}$$

Put into words: *the most probable value of a quantity is the one for which the sum of the squares of the errors (residuals) is a minimum.* This is the principle of least squares as it is generally understood by surveyors; its more rigorous definition has already been stated.

Two basic methods exist for the adjustment of observations by this technique, namely:

(a) The 'indirect method' which uses observation equations, and
(b) the 'direct method' which uses condition equations.

The indirect method has as many observation equations as there are angles and distances in the network; thus, for large networks data handling can become a problem on the computer. The direct method has fewer equations, equal in fact to the number of conditions to be satisfied. Nevertheless, the indirect method of variation of co-ordinates is now universally used because of the ease with which it can be applied to any type of network; thus, a single program suffices for all requirements. Also, in complex networks with many redundancies it is extremely difficult, if not impossible, to formulate satisfactory conditional equations.

1.11.2 Method of observation equations

As the aim of field observations is to produce the true or most probable value (MPV) of that measurement, it follows that provided that the measurements contain only accidental errors, the adjustment should bring about minimal changes in their value. For instance, in the adjustment of an angle, the degrees and minutes would remain unaltered and only the seconds would change. Thus to carry the whole of the quantity, i.e. the degrees, minutes and seconds, into the adjustment, simply results in the added labour of manipulating difficult quantities. The method recommended here is therefore to assume a value for the quantity and by least squares ascertain the correction to that quantity that will produce the MPV. It follows that if the value assumed is as close as possible to the MPV, then the size of the correction will be correspondingly smaller. From the first sentence it can be reasoned that the best value for the assumed value would be the measured value itself. A simple station adjustment will now be solved to illustrate the technique.

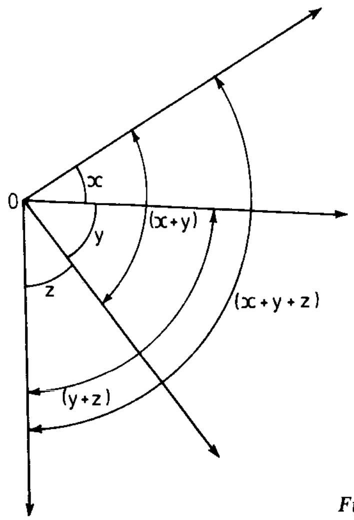

Figure 1.4

Figure 1.4 illustrates the observations taken to surrounding stations from O, the following mean values being recorded

$$\begin{aligned} x &= 25^\circ\ 18'\ 30'' \\ y &= 40^\circ\ 20'\ 25'' \\ z &= 30^\circ\ 30'\ 35'' \\ (x+y) &= 65^\circ\ 38'\ 52'' \\ (y+z) &= 70^\circ\ 51'\ 02'' \\ (x+y+z) &= 96^\circ\ 09'\ 31'' \end{aligned} \left.\vphantom{\begin{aligned} a \\ a \\ a \end{aligned}}\right\} \text{Redundant observations}$$

It is required, by a least squares adjustment, to find the most probable values for x, y and z.

Step 1—assume values (A) for the required quantities. Let them equal the measured values

$$\begin{aligned} A_x &= 25^\circ\ 18'\ 30'' \\ A_y &= 40^\circ\ 20'\ 25'' \\ A_z &= 30^\circ\ 30'\ 35'' \end{aligned}$$

Step 2–formulate the observation equations. By applying a correction v to the assumed values one obtains the MPV

$$A_x + v_1 = \text{MPV of } x$$
$$A_y + v_2 = \text{MPV of } y$$
$$A_z + v_3 = \text{MPV of } z$$

Denoting the observed values as Q, it should be obvious that the difference between the MPV and the observed value is the residual error r, i.e.

$$(A_x + v_1) - Q_x = r_1 \quad \text{(error in angle } x\text{)}$$
$$\text{MPV} - \text{observed value} = \text{residual error}$$

Thus substituting the assumed values from Step 1 and the observed values from the question, one gets

$$25° 18' 30'' + v_1 - 25° 18' 30'' = r_1 \qquad \therefore\ v_1 = r_1$$

similarly $v_2 = r_2$ and $v_3 = r_3$, as indicated by the question.

Also $\quad (A_x + v_1) + (A_y + v_2) - Q_{(x+y)} = r_4$

i.e. $\quad (25° 18' 30'' + v_1) + (40° 20' 25'' + v_2) - 65° 38' 52'' = r_4$

$\therefore\ v_1 + v_2 + 3'' = r_4$

Similarly for $(y + z)$ $\quad (40° 20' 25'' + v_2) + (30° 30' 35'' + v_3) - 70° 51' 02'' = r_5$

$\therefore\ v_2 + v_3 - 2'' = r_5$

and for $(x + y + z)$

$$(25° 18' 30'' + v_1) + (40° 20' 25'' + v_2) + (30° 30' 35'' + v_3) - 96° 09' 31'' = r_6$$
$$\therefore\ v_1 + v_2 + v_3 - 1'' = r_6$$

These then are the *observation equations* summarized below

$$v_1 = r_1$$
$$v_2 = r_2$$
$$v_3 = r_3$$
$$v_1 + v_2 + 3'' = r_4$$
$$v_2 + v_3 - 2'' = r_5$$
$$v_1 + v_2 + v_3 - 1'' = r_6$$

Step 3—formulate the normal equations. The least square condition that

$$[r^2] = [rr] = \text{a minimum}$$

must be expressed in terms of the corrections, i.e.

$$[vv] = \text{a minimum}$$

If $P = [vv]$, the condition for making it a minimum becomes

$$\frac{\partial P}{\partial v_1} = \frac{\partial P}{\partial v_2} = \frac{\partial P}{\partial v_3} = 0$$

From the observation equations

$$P = v_1^2 + v_2^2 + v_3^3 + (v_1 + v_2 + 3'')^2 + (v_2 + v^3 - 2'')^2 + (v_1 + v_2 + v_3 - 1'')^2$$

$$\therefore\ \frac{\partial P}{\partial v_1} = 2v_1 + 2(v_1 + v_2 + 3'') + 2(v_1 + v_2 + v_3 - 1'')$$

Equating to zero, eliminating the unwanted factor of 2 and collecting like terms gives

$$3v_1 + 2v_2 + v_3 + 2'' = 0$$

Similarly
$$\frac{\partial P}{\partial v_2} = v_2 + (v_1 + v_2 + 3'') + (v_2 + v_3 - 2'') + (v_1 + v_2 + v_3 - 1'')$$
$$= 2v_1 + 4v_2 + 2v_3 = 0$$
$$\frac{\partial P}{\partial v_3} = v_3 + (v_2 + v_3 - 2'') + (v_1 + v_2 + v_3 - 1'')$$
$$= v_1 + 2v_2 + 3v_3 - 3'' = 0$$

These then are termed the *normal equations* and are summarized as follows:

$$\begin{aligned} 3v_1 + 2v_2 + v_3 &= -2'' \\ 2v_1 + 4v_2 + 2v_3 &= 0 \\ v_1 + 2v_2 + 3v_3 &= 3'' \end{aligned}$$

These equations are simple enough to be solved by standard methods of substitution giving: $v_1 = -1''$, $v_2 = -0.25''$ and $v_3 = 1.5''$. These values are now applied to the assumed values to give the MPV.

MPV of $x = A_x + v_1 = 25° \, 18' \, 30'' - 1'' = 25° \, 18' \, 29.00''$
MPV of $y = A_y + v_2 = 40° \, 20' \, 25'' - 0.25'' = 40° \, 20' \, 24.75''$
MPV of $z = A_z + v_3 = 30° \, 30' \, 35'' + 1.5'' = 30° \, 30' \, 36.50''$

The MPV may now be rounded off to one second commensurate with the precision of the field data.

Students should carefully note the following:

(1) The *normal equations* possess *symmetry*, in that the coefficients of row 1 are repeated in column 1, and row 2 in column 2, as shown by the arrows.
(2) Although it is advisable to use observed values for the assumed values, it is not mandatory. Assuming a different value would result in a different correction giving ultimately the same MPV.
(3) As the error r has not been utilized in the process it may hereafter be ignored. The observation equations are thus expressed as

$$\begin{aligned} v_1 &= 0 \\ v_2 &= 0 \\ v_3 &= 0 \\ v_1 + v_2 + 3'' = 0 \qquad \text{i.e.} \quad v_1 + v_2 &= -3'' \\ v_2 + v_3 &= 2'' \\ v_1 + v_2 + v_3 &= 1'' \end{aligned}$$

Treated and written in this way makes them very easy to handle when applied in a mechanical solution using the general equations for least squares.

1.11.3 General equations for least squares adjustment

Expressing the observation equations in general terms

$$a_1v_1 + b_1v_2 + c_1v_3 - Q_1 = r_1$$
$$a_2v_1 + b_2v_2 + c_2v_3 - Q_2 = r_2$$
and $$a_nv_1 + b_nv_2 + c_nv_3 - Q_n = r_n$$

i.e. MPV − Observed value = Residual

From least squares $[rr]$ = a minimum. Thus, squaring r_1 gives

$$r_1^2 = a_1^2v_1^2 + 2a_1b_1v_1v_2 + 2a_1c_1v_1v_3 - 2a_1Q_1v_1 + b_1^2v_2^2 + 2b_1c_1v_2v_3 - 2b_1Q_1v_2 + c_1^2v_3^2 - 2c_1Q_1v_3 + Q_1^2$$

Repeating for $r_2 \cdots r_n$ will only change the coefficients to $a_2b_2c_2$ and $a_nb_nc_n$. Thus summing the results and expressing the sum of the squares in the usual manner, i.e. $[r^2]$ as $[rr]$ one gets

$$[rr] = [aa]v_1^2 + 2[ab]v_1v_2 + 2[ac]v_1v_3 - 2[aQ]v_1 + [bb]v_2^2 + 2[bc]v_2v_3 - 2[bQ]v_2 + [cc]v_3^2 - 2[cQ]v_3 + [QQ]$$

As $[rr] = f(v_1, v_2, v_3)$, differentiate and equate to zero for a minimum

$$\frac{\partial f}{\partial v_1} = 2[aa]v_1 + 2[ab]v_2 + 2[ac]v_3 - 2[aQ] = 0$$

$$\frac{\partial f}{\partial v_2} = 2[ab]v_1 + 2[bb]v_2 + 2[bc]v_3 - 2[bQ] = 0$$

$$\frac{\partial f}{\partial v_3} = 2[ac]v_1 + 2[bc]v_2 + 2[cc]v_3 - 2[cQ] = 0$$

These reduce to the general form for normal equations as follows

$$\left.\begin{aligned} [aa]v_1 + [ab]v_2 + [ac]v_3 &= [aQ] \\ [ab]v_1 + [bb]v_2 + [bc]v_3 &= [bQ] \\ [ac]v_1 + [bc]v_2 + [cc]v_3 &= [cQ] \end{aligned}\right\} \qquad (1.23)$$

Note once again the symmetry. If the student now commits this simple expression to memory, it will greatly facilitate the solution of typical least squares problems.

1.11.4 Use of general equations

Considering the problem previously outlined, the observation equations are formed in the usual way and are restated as follows

$$\begin{aligned} v_1 &= 0 \\ v_2 &= 0 \\ v_3 &= 0 \\ v_1 + v_2 &= -3'' \\ v_2 + v_3 &= 2'' \\ v_1 + v_2 + v_3 &= 1'' \end{aligned}$$

where the coefficient of v_1 is a, of v_2 is b and of v_3 is c thus

$$\left.\begin{aligned}\dot{a}v_1 &= 0\\ \ddot{b}v_2 &= 0\\ \bar{c}v_3 &= 0\\ \dot{a}v_1 + \ddot{b}v_2 &= -3''\\ \ddot{b}v_2 + \bar{c}v_3 &= 2''\\ \dot{a}v_1 + \ddot{b}v_2 + \bar{c}v_3 &= 1''\end{aligned}\right\}\ Q \text{ terms} \qquad \begin{aligned}&(1.24a)\\&(1.24b)\\&(1.24c)\\&(1.24d)\\&(1.24e)\\&(1.24f)\end{aligned}$$

and $a = b = c = 1$.

It is simply a matter now of substituting the coefficients into the general equation (1.23) as follows:

(1) $[aa]$ means 'the sum of the squares of the a coefficients' and is obtained from observation equations (1.24a), (1.24d) and (1.24f) thus

$$[aa] = (1 \times 1) + (1 \times 1) + (1 \times 1), \text{(shown with a single dot)} = 3$$

(2) $[ab]$ means 'the sum of the product of the a and b coefficients'—and can only be obtained from equations (1.24d) and (1.24f). For instance, if equation (1.24a) was written in full the coefficients of b and c would be zero, whilst in equation (1.24b) a and c would be zero. Thus it is only necessary to choose those equations in which a and b *together* have a value other than zero

$$[ab] = (1 \times 1) + (1 \times 1) = 2$$

where the single and double dots are together.

(3) $[ac]$ similarly can only be obtained from equation (1.24f), shown by the single dot and bar together

$$[ac] = (1 \times 1) = 1$$

(4) $[aQ]$ is the 'sum of the product of the a and Q coefficients', the Q coefficients being the numerical values shown bracketed and commonly called the *absolute* terms. It can be seen that as the absolute terms are zero in the first three equations, then these equations may be ignored. Thus $[aQ]$ can only be obtained from equations (1.24d) and (1.24f).

$$[aQ] = (1 \times -3'') + (1 \times 1'') = -2''$$

Thus the first line of the block of normal equations is

$$3v_1 + 2v_2 + v_3 = -2''$$

which, due to the symmetry of the equations, can be written as

$$\begin{aligned}&3v_1 + 2v_2 + v_3 = -2''\\&2v_1\\&\ v_1\end{aligned}$$

In exactly the same way, the remainder of the coefficients of the normal equations can be seen at a glance:

(5) $[bb]$ from equations (1.24b, d, e and f) $= (1 \times 1) + (1 \times 1) + (1 \times 1) + (1 \times 1) = 4$.
(6) $[bc]$ from equations (1.24e and f) $= (1 \times 1) + (1 \times 1) = 2$.
(7) $[bQ]$ from equations (1.24d, e and f) $= (1 \times -3'') + (1 \times 2'') + (1 \times 1'') = 0$.

These are also repeated vertically giving

$$\begin{aligned}&3v_1 + 2v_2 + \ \ v_2 = -2''\\&2v_1 + 4v_2 + 2v_3 = 0\\&\ v_1 + 2v_2\end{aligned}$$

(8) Finally, $[cc] = 3$, from equations (1.24c, e and f), and $[cQ] = 3''$, from equations (1.24e and f).

The complete block of normal equations is therefore

$$\begin{aligned} 3v_1 + 2v_2 + v_3 &= -2'' \\ 2v_1 + 4v_2 + 2v_3 &= 0 \\ v_1 + 2v_2 + 3v_3 &= 3'' \end{aligned}$$

Comparison of these equations shows them to be identical to those previously produced from the basic technique. Students should master the mechanical method, which enables them to go direct from observation equations to normal equations. When writing the observation equations, there is no need to letter the coefficients *a*, *b*, *c*, etc., these are easily visualized.

1.11.5 General equations incorporating weights

Up to now the weight of each observation has been ignored in order to facilitate comprehension. However, any least squares adjustment is completely sterile without the inclusion of the relative weights of the observations. Even when all the observations are of equal precision and correlation free, thereby having unit weight, the concept of weighting should still be considered.

Without showing the derivation, the effect of weights (w) is

$$[wvv] = \left[\frac{vv}{\sigma^2}\right] = \text{a minimum}$$

Thus

$$\begin{aligned} [waa]v_1 + [wab]v_2 + [wac]v_3 + \cdots [wan]v_n &= [waQ] \\ [wbb]v_2 + [wbc]v_3 + \cdots [wbn]v_n &= [wbQ] \\ [wcc]v_3 + \cdots [wcn]v_n &= [wcQ] \end{aligned} \tag{1.25}$$

Consider now the original problem with weights incorporated

$$\begin{aligned} x &= 25^\circ\ 18'\ 30'' \text{ wt } 4 \\ y &= 40^\circ\ 20'\ 25'' \text{ wt } 4 \\ z &= 30^\circ\ 30'\ 35'' \text{ wt } 4 \\ (x + y) &= 65^\circ\ 38'\ 52'' \text{ wt } 3 \\ (y + z) &= 70^\circ\ 51'\ 02'' \text{ wt } 2 \\ (x + y + z) &= 96^\circ\ 09'\ 31'' \text{ wt } 1 \end{aligned}$$

Values for x, y and z are assumed and observation equations found in exactly the same manner as before as shown in *Section 1.11.2.* These are now written, with the weights, as follows

$$v_1 = 0 \text{ wt } 4 \tag{1.26a}$$
$$v_2 = 0 \text{ wt } 4 \tag{1.26b}$$
$$v_3 = 0 \text{ wt } 4 \tag{1.26c}$$
$$v_1 + v_2 = -3'' \text{ wt } 3 \tag{1.26d}$$
$$v_2 + v_3 = 2'' \text{ wt } 2 \tag{1.26e}$$
$$v_1 + v_2 + v_3 = 1'' \text{ wt } 1 \tag{1.26f}$$

From equation (1.25)

$$
\begin{aligned}
[waa] &= (4\times 1\times 1) + (3\times 1\times 1) + (1\times 1\times 1) = 8 && \text{(from equations (1.26a,d,f))}\\
[wab] &= (3\times 1\times 1) + (1\times 1\times 1) = 4 && \text{(from equations (1.26d,f))}\\
[wac] &= (1\times 1\times 1) = 1 && \text{(from equation (1.26f))}\\
[waQ] &= (3\times 1\times -3'') + (1\times 1\times 1'') = -8''
\end{aligned}
$$

(from equations (1.26d and f) as $Q = 0$ in (1.26a))

The remainder of the block should be attempted by the student, giving

$$
\begin{aligned}
8v_1 + 4v_2 + v_3 &= -8''\\
4v_1 + 10v_2 + 3v_3 &= -4''\\
v_1 + 3v_2 + 7v_3 &= 5''
\end{aligned}
$$

These equations are solved and the values of $v_1 = -0.97''$, $v_2 = -0.31''$, $v_3 = +0.98''$ applied to the appropriate assumed value in the usual way.

A method favoured in surveying for the solution of large blocks of normal equations is Choleski's decomposition method. However, as the solution would inevitably be carried out on existing computer programs, it will not be dealt with here.

1.11.6 Matrix methods

The format, used to date to express the observation equations in their general form, has been unconventional deliberately in order to facilitate easier understanding and manipulation. A more conventional approach will now be adopted to illustrate the application of matrices.

Observation equations (from *Section 1.11.3*)

$$
\begin{aligned}
a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n - q_1 &= r_1\\
a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n - q_2 &= r_2\\
\vdots \qquad \vdots \qquad\quad \vdots \qquad \vdots \quad &\ \ \vdots\\
a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n - q_m &= r_m
\end{aligned}
$$

The above array of m rows and n columns varies from the initial format of *Section 1.11.3*, in that the coefficients a, b, c, etc. are replaced by a_{mn} and Q by q to conform with vector notation.

Thus a = coefficients of the observation equations
v = corrections
q = absolute terms
r = the residual

In matrix form $\quad r = Av - q \qquad (1.27)$

where r = column vector of m residuals
A = an $m \times n$ matrix of coefficients
v = column vector of n corrections
q = column vector of m absolute terms

A least squares solution is obtained by minimizing the quadratic form r^TWr, i.e. $r^TWr = 0$, where W is an $m \times m$ diagonal matrix of weights

$$
\begin{aligned}
r^TWr &= (Av - q)^TW(Av - q) \\
&= (v^TA^T - q^T)W(Av - q) \\
&= v^T(A^TWA)v - v^T(A^TWq) - (q^TWA)v - q^TWq \\
\partial(r^TWr)/\partial v &= 2(A^TWA)v - (A^TWq) - (q^TWA)^T = 0 \\
2(A^TWA)v &= (A^TWq) + (A^TW^Tq) = 2(A^TWq)
\end{aligned}
$$

Thus, the normal equations are $(A^TWA)v = A^TWq$ (1.28)

and the solution for v is $v = (A^TWA)^{-1}A^TWq$ (1.29)

In equation (1.29), $(A^TWA)^{-1}$ is called the 'variance–covariance' (var–cov) matrix, the application of which is dealt with in *Section 1.13.1.*

The previous example will now be worked by matrix methods.

(1) The observation equations are formulated in the usual way and are as in equations (1.26), i.e.

$$
\begin{aligned}
v_1 &= 0 \text{ wt } 4 \\
v_2 &= 0 \text{ wt } 4 \\
v_3 &= 0 \text{ wt } 4 \\
v_1 + v_2 &= -3'' \text{ wt } 3 \\
v_2 + v_3 &= 2'' \text{ wt } 2 \\
v_1 + v_2 + v_3 &= 1'' \text{ wt } 1
\end{aligned}
$$

(2) From the observation equations the matrices are formed as follows

$$
{}_6A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad {}_6W_6 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
q^T = (0 \quad 0 \quad 0 \quad -3'' \quad 2'' \quad 1'')
$$

where A contains the coefficients (all unity) of v_1, v_2, v_3, q contains the absolute terms and W the weights.

(3)

$$
{}_3(A^TWA)_3 = \begin{bmatrix} 8 & 4 & 1 \\ 4 & 10 & 3 \\ 1 & 3 & 7 \end{bmatrix} \qquad {}_3(A^TWq)_1 = \begin{bmatrix} -8 \\ -4 \\ 5 \end{bmatrix}
$$

The above are the normal equations, as in *Section 1.11.5.*

$$
{}_3(A^TWA)_3^{-1} = \begin{bmatrix} 0.16 & -0.06 & 0.01 \\ -0.06 & 0.14 & -0.05 \\ 0.01 & -0.05 & 0.16 \end{bmatrix}
$$

$$
\begin{aligned}
v &= (A^TWA)^{-1}A^TWq \\
\therefore\ v^T &= (-0.97 \quad -0.31 \quad 0.98)
\end{aligned}
$$

i.e. $v_1 = -0.97''$, $v_2 = -0.31''$, $v_3 = +0.98''$.

The answers, of course, are identical to those obtained by the classical methods.

1.11.7 Method of condition equations (direct method)

In this method condition equations are formed, based on the conditions of adjustment to be satisfied. In order to reduce the number of normal equations, each condition equation is multiplied by an undetermined multiplier called a *correlative* or *Lagrangian multiplier*. The resultant condition equations are then combined in the least squares condition and after differentiation, expressed as a linear function of the correlative. Thereafter back substituting into the condition equations produces a set of correlative normal equations equal in number to the number of conditions. The equations are solved to find the values of the correlatives, which can then be expressed in terms of the corrections.

The method will now be illustrated in detail, using the problem which has been solved already by the previous method. Thus, restating the problem

$$\begin{aligned} x &= 25°\ 18'\ 30'' \\ y &= 40°\ 20'\ 25'' \\ z &= 30°\ 30'\ 35'' \\ (x+y) &= 65°\ 38'\ 52'' \\ (y+z) &= 70°\ 51'\ 02'' \\ (x+y+z) &= 96°\ 09'\ 31'' \end{aligned}$$

Examination of *Figure 1.4* clearly indicates the conditions of adjustment as

$$\begin{aligned} x+y &= (x+y) \\ y+z &= (y+z) \\ x+y+z &= (x+y+z) \end{aligned}$$

However, these conditions are only true of the MPV, thus corrections $v_1, v_2, \ldots, v_6$ are applied to give

$$\begin{aligned} x+v_1+y+v_2 &= (x+y)+v_4 \\ y+v_2+z+v_3 &= (y+z)+v_5 \\ x+v_1+y+v_2+z+v_3 &= (x+y+z)+v_6 \end{aligned}$$

In this method it is important to use just the right number of conditions. Using too few would produce errors, while using too many would produce excessive computation. A rule to decide the correct number is

Number of directly-observed quantities − Number of independent unknowns
= Number of conditions required

In the above case: number of directly observed quantities = 6; number of independent unknowns = 3, i.e. x, y, z; therefore the number of conditions = 3; thus all the above conditions must be used.

When using the method of correlatives, the following should be noted: (i) corrections are applied to all the quantities involved; (ii) these corrections apply to the observed values direct; one does not assume values. Substituting in the conditions of adjustment, we have

$$\underset{\text{MPV}}{x+v_1} + \underset{\text{MPV}}{y+v_2} = \underset{\text{MPV}}{(x+y)+v_4}$$

$$\therefore\ 25°\ 18'\ 30'' + v_1 + 40°\ 20'\ 25'' + v_2 = 65°\ 38'\ 52'' + v_4$$

giving

$$v_1 + v_2 - v_4 + 3'' = 0 \tag{1.30}$$

Similarly for the two remaining conditions

$$40^\circ\,20'\,25'' + v_2 + 30^\circ\,30'\,35'' + v_3 = 70^\circ\,51'\,02'' + v_5$$
$$\therefore\; v_2 + v_3 - v_5 - 2'' = 0 \qquad (1.31)$$

and

$$25^\circ\,18'\,30'' + v_1 + 40^\circ\,20'\,25'' + v_2 + 30^\circ\,30'\,35'' + v_3 = 96^\circ\,09'\,31'' + v_6$$
$$\therefore\; v_1 + v_2 + v_3 - v_6 - 1'' = 0 \qquad (1.32)$$

Each condition equation is now multiplied by a correlative k

$$k_1(v_1 + v_2 - v_4 + 3'') = 0$$
$$k_2(v_2 + v_3 - v_5 - 2'') = 0$$
$$k_3(v_1 + v_2 + v_3 - v_6 - 1'') = 0$$

To facilitate reduction, each of the correlative equations are multiplied by -2; they are then combined in the least squares principle to give the following function

$$v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 + v_6^2 - 2k_1(v_1 + v_2 - v_4 + 3'') - 2k_2(v_2 + v_3 - v_5 - 2'')$$
$$- 2k_3(v_1 + v_2 + v_3 - v_6 - 1'') = \text{a minimum}$$

Differentiating with respect to each variable in turn and equating to zero, we have

$$2v_1 - 2k_1 - 2k_3 = 0 \qquad \therefore\; v_1 = k_1 + k_3$$
$$2v_2 - 2k_1 - 2k_2 - 2k_3 = 0 \qquad \therefore\; v_2 = k_1 + k_2 + k_3$$
$$2v_3 - 2k_2 - 2k_3 = 0 \qquad \therefore\; v_3 = k_2 + k_3$$
$$2v_4 + 2k_1 = 0 \qquad \therefore\; v_4 = -k_1$$
$$2v_5 + 2k_2 = 0 \qquad \therefore\; v_5 = -k_2$$
$$2v_6 + 2k_3 = 0 \qquad \therefore\; v_6 = -k_3$$

The correlative functions are now substituted back into the conditional equations.

Substituting in

$$v_1 + v_2 - v_4 + 3'' = 0$$

gives

$$(k_1 + k_3) + (k_1 + k_2 + k_3) - (-k_1) + 3'' = 0$$
$$\therefore\; 3k_1 + k_2 + 2k_3 + 3'' = 0$$

Similarly

$$v_2 + v_3 - v_5 - 2'' = 0$$
$$\therefore\; (k_1 + k_2 + k_3) + (k_2 + k_3) - (-k_2) - 2'' = 0$$
$$\therefore\; k_1 + 3k_2 + 2k_3 - 2'' = 0$$

and

$$v_1 + v_2 + v_3 - v_6 - 1'' = 0$$
$$\therefore\; (k_1 + k_3) + (k_1 + k_2 + k_3) + (k_2 + k_3) - (-k_3) - 1'' = 0$$
$$\therefore\; 2k_1 + 2k_2 + 4k_3 - 1'' = 0$$

Collecting the correlative normal equations together

$$3k_1 + k_2 + 2k_3 + 3'' = 0$$
$$k_1 + 3k_2 + 2k_3 - 2'' = 0$$
$$2k_1 + 2k_2 + 4k_3 - 1'' = 0$$

note the symmetry.

On solution, $k_1 = -1.75''$, $k_2 = 0.75''$, $k_3 = 0.75''$. These values are now substituted in the above functions, to obtain the values of the corrections

$$v_1 = k_1 + k_3 = -1'' \qquad v_4 = -k_1 = +1.75''$$
$$v_2 = k_1 + k_2 + k_3 = -0.25'' \qquad v_5 = -k_2 = -0.75''$$
$$v_3 = k_2 + k_3 = +1.50'' \qquad v_6 = -k_3 = -0.75''$$

These corrections are now applied to the appropriate observed value to give the MPV

$$\begin{aligned} x &= 25^\circ\ 18'\ 30'' + v_1 = 25^\circ\ 18'\ 29'' \\ y &= 40^\circ\ 20'\ 25'' + v_2 = 40^\circ\ 20'\ 24.75'' \\ z &= 30^\circ\ 30'\ 35'' + v_3 = 30^\circ\ 30'\ 36.50'' \\ (x+y) &= 65^\circ\ 38'\ 52'' + v_4 = 65^\circ\ 38'\ 53.75'' \\ (y+z) &= 70^\circ\ 51'\ 02'' + v_5 = 70^\circ\ 51'\ 1.25'' \\ (x+y+z) &= 96^\circ\ 09'\ 31'' + v_6 = 96^\circ\ 09'\ 30.25'' \end{aligned}$$

The MPV can now be rounded off to the nearest second commensurate with the field data.

Note that: (i) the MPV of x, y and z are identical to the values obtained from the observation equation method; (ii) the MPV above fulfil all the conditions of adjustment specified; (iii) in this particular problem there is no great advantage in using condition equations instead of observation equations; (iv) the order in which the condition equations are set down is immaterial; and (v) students should now study both methods and note the differences in procedure.

1.11.8 General form (correlatives)

Writing the condition equations in a general form

$$\begin{aligned} a_1v_1 + a_2v_2 \cdots + a_nv_n + q_1 &= 0 \\ b_1v_1 + b_2v_2 \cdots + b_nv_n + q_2 &= 0 \\ c_1v_1 + c_2v_2 \cdots + c_nv_n + q_3 &= 0 \end{aligned}$$

Each equation is then multiplied by an unknown correlative and may be written

$$k_1(a_1v_1 + a_2v_2 + \cdots + a_nv_n + q_1) + k_2(b_1v_1 + b_2v_2 + \cdots + b_nv_n + q_2) + k_3(c_1v_1 + c_2v_2 + \cdots + c_nv_n + q_3) = 0$$

According to the principle of least squares, $[vv]$ = a minimum. However, the minimum value is also a function of the conditional equations. Thus, as multiplying the correlatives by -2 will not affect the minimum value and will facilitate reduction, the total function may be written

$$\begin{aligned} F = v_1^2 + v_2^2 + \cdots + v_n^2 &- 2k_1(a_1v_1 + a_2v_2 + \cdots + a_nv_n + q_1) \\ &- 2k_2(b_1v_1 + b_2v_2 + \cdots + b_nv_n + q_2) \\ &- 2k_3(c_1v_1 + c_2v_2 + \cdots + c_nv_n + q_3) \\ &= \text{a minimum} \end{aligned}$$

Differentiating each variable in turn and equating to zero

$$\frac{\partial F}{\partial v_1} = 2v_1 - 2k_1a_1 - 2k_2b_1 - 2k_3c_1 = 0$$

$$\frac{\partial F}{\partial v_2} = 2v_2 - 2k_1a_2 - 2k_2b_2 - 2k_3c_2 = 0$$

$$\frac{\partial F}{\partial v_n} = 2v_n - 2k_1a_n - 2k_2b_n - 2k_3c_n = 0$$

The above equations reduce to

$$v_1 = k_1a_1 + k_2b_1 + k_3c_1$$
$$v_2 = k_1a_2 + k_2b_2 + k_3c_2$$
$$v_n = k_1a_n + k_2b_n + k_3c_n$$

Substituting these values into the original condition equations and substituting K for k simply to emphasize the format, gives the general form for correlative normal equations

$$\left.\begin{aligned} K_1[aa] + K_2[ab] + K_3[ac] + q_1 &= 0 \\ K_1[ab] + K_2[bb] + K_3[bc] + q_2 &= 0 \\ K_1[ac] + K_2[bc] + K_3[cc] + q_3 &= 0 \end{aligned}\right\} \tag{1.33}$$

It is important to note the symmetry of the equations. The student can see that these equations are identical to the previous ones derived for the observation equation method with the correction v replaced by the correlative K.

Once again these equations can be used mechanically to produce the normal equations direct from the condition equations. The condition equations are obtained as has already been shown and are as follows:

$$v_1 + v_2 - v_4 + 3'' = 0$$
$$v_2 + v_3 - v_5 - 2'' = 0$$
$$v_1 + v_2 + v_3 - v_6 - 1'' = 0$$

If the student now multiplies each equation by a correlative they will appear as follows:

$$K_1(v_1 + v_2 - v_4 + 3'') = 0 \tag{1.34a}$$
$$K_2(v_2 + v_3 - v_5 - 2'') = 0 \tag{1.34b}$$
$$K_3(v_1 + v_2 + v_3 - v_6 - 1'') = 0 \tag{1.34c}$$

From the original derivation (see page 30) it can be seen that all the coefficients of equation (1.34a) are a, of equation (1.34b) are b, and of equation (1.34c) are c. The equations will therefore be rewritten purely to facilitate the explanation of the method

$$K_1(a_1v_1 + a_2v_2 - a_4v_4 + 3'') = 0 \tag{1.35a}$$
$$K_2(b_2v_2 + b_3v_3 - b_5v_5 - 2'') = 0 \tag{1.35b}$$
$$K_3(c_1v_1 + c_2v_2 + c_3v_3 - c_6v_6 - 1'') = 0 \tag{1.35c}$$

where $a = b = c = 1$, in this case.

Taking the terms of the general equation in turn

$$K_1[aa] = K_1\{(1 \times 1) + (1 \times 1) + (-1 \times -1)\} = 3K_1$$

obtained from equation (1.35a) only, i.e. the sum of the squares of the a coefficients.

$$K_2[ab] = K_2\{(1 \times 1)\} = 1K_2$$

obtained from equations (1.35a) and (1.35b) by simply multiplying the coefficients of the *like* terms, which in this case is $a_2 \times b_2$.

$$K_3[ac] = K_3\{(1 \times 1) + (1 \times 1)\} = 2K_3$$

obtained from equations (1.35a) and (1.35c), i.e. $a_1 \times c_1 + a_2 \times c_2$.

As the equations are symmetrical the student can immediately write down the first row and column

$$\begin{aligned} &3K_1 + K_2 + 2K_3 + 3'' = 0 \\ &K_1 \\ &2K_1 \end{aligned}$$

The student should now complete the set for himself according to the 'general form' by visual inspection, without the need to write them in the above detailed manner. The complete set should, of course, be

$$\begin{aligned} 3K_1 + K_2 + 2K_3 + 3'' &= 0 \\ K_1 + 3K_2 + 2K_3 - 2'' &= 0 \\ 2K_1 + 2K_2 + 4K_3 - 1'' &= 0 \end{aligned}$$

The condition equations can now be used to give the relationship of v to K. By reference to equations (1.34) it is seen that v_1 is related to K_1 in equation (1.34a) and K_3 in equation (1.34c), therefore $v_1 = K_1 + K_3$. Similarly, v_2 appears in all three equations, therefore $v_2 = K_1 + K_2 + K_3$; v_3 appears in equations (1.34b) and (1.34c) and is thus related to $K_2 + K_3$, and so on. The student should complete the rest and compare his results with those previously obtained in *Section 1.11.7.*

1.11.9 General equations incorporating weights

$$\left.\begin{aligned} K_1\left[\frac{aa}{w}\right] + K_2\left[\frac{ab}{w}\right] + K_3\left[\frac{ac}{w}\right] + q_1 &= 0 \\ K_2\left[\frac{bb}{w}\right] + K_3\left[\frac{bc}{w}\right] + q_2 &= 0 \\ K_3\left[\frac{cc}{w}\right] + q_3 &= 0 \end{aligned}\right\} \quad (1.36)$$

Note that in this case, it is the reciprocals of the weights which are used. However, as weights are an indication of relative accuracy they may be multiplied by an appropriate constant, C, to give a more manageable and similar equation to the ones already outlined. Thus as $(1/w) \times C = W$, the equation may be written

$$K_1[Waa] + K_2[Wab] + K_3[Wac] + q_1 = 0$$
etc.

To clarify the situation, consider three quantities x, y, z, with respective weights as shown in *Table 1.2.*

TABLE 1.2

Quantity	w	$1/w$	C	$W = (1/w) \times C$
x	1	1	4	4
y	2	$\frac{1}{2}$	4	2
z	4	$\frac{1}{4}$	4	1

The previous weight example will now be restated and worked using correlatives

$$\begin{aligned} x &= 25°\ 18'\ 30''\text{ wt } 4 \\ y &= 40°\ 20'\ 25''\text{ wt } 4 \\ z &= 30°\ 30'\ 35''\text{ wt } 4 \\ (x+y) &= 65°\ 38'\ 52''\text{ wt } 3 \\ (y+z) &= 70°\ 51'\ 02''\text{ wt } 2 \\ (x+y+z) &= 96°\ 09'\ 31''\text{ wt } 1 \end{aligned}$$

As the ratio of the reciprocal of the weights is to be used, then the multiplying constant C will, in this case, be the common factor 12. However, ignoring the weights at this stage, the condition equations are obtained as before

$$K_1(v_1 + v_2 - v_4 + 3'') = 0$$
$$K_2(v_2 + v_3 - v_5 - 2'') = 0$$
$$K_3(v_1 + v_2 + v_3 - v_6 - 1'') = 0$$

As the inclusion of weights can sometimes cause confusion, the approach illustrated in *Table 1.3* should be used.

TABLE 1.3

v	$W = (1/w) \times C$	a	b	c	Waa	Wab	Wac	Wbb	Wbc	Wcc
v_1	$3 = \frac{1}{4} \times 12$	1		1	3	0	3	0	0	3
v_2	$3 = \frac{1}{4} \times 12$	1	1	1	3	3	3	3	3	3
v_3	$3 = \frac{1}{4} \times 12$		1	1	0	0	0	3	3	3
v_4	$4 = \frac{1}{3} \times 12$	−1			4	0	0	0	0	0
v_5	$6 = \frac{1}{2} \times 12$		−1		0	0	0	6	0	0
v_6	$12 = 1 \times 12$			−1	0	0	0	0	0	12
				[ ] =	10	3	6	12	6	21

Normal equations

$$10K_1 + 3K_2 + 6K_3 + 36'' = 0$$
$$12K_2 + 6K_3 - 24'' = 0$$
$$21K_3 - 12'' = 0$$

N.B. The absolute terms are multiplied by $C = 12$ to balance the equation.

On solution, $K_1 = -5.17$, $K_2 = 2.65$, $K_3 = 1.29$.

Now the relationship of the residuals to the correlatives is obtained in exactly the same way as in *Section 1.11.7*, and then multiplied by $1/w$ as follows

$$v_1 = \frac{1}{w_1}(K_1 + K_3) = \tfrac{1}{4}(-5.17 + 1.29) = -0.97''$$

$$v_2 = \frac{1}{w_2}(K_1 + K_2 + K_3) = \tfrac{1}{4}(-5.17 + 2.65 + 1.29) = -0.31''$$

$$v_3 = \frac{1}{w_3}(K_2 + K_3) = \tfrac{1}{4}(2.65 + 1.29) = 0.99''$$

$$v_4 = \frac{1}{w_4}(-K_1) = \tfrac{1}{3}(+5.17) = 1.72''$$

$$v_5 = \frac{1}{w_5}(-K_2) = \tfrac{1}{2}(-2.65) = -1.33''$$

$$v_6 = \frac{1}{w_6}(-K_3) = (-1.29) = -1.29''$$

These corrections are now applied to the observed values, i.e. $(x + v_1)$, $(y + v_2)$, $(z + v_3)$, $(x + y) + v_4$, $(y + z) + v_4$ and $(x + y + z) + v_6$, to obtain the most probable values. The values will satisfy the least squares condition and the conditions of adjustment.

Method

(1) The data entered in the first two columns are self explanatory.
(2) Enter the coefficients of the *rows* of condition equations in the appropriate *columns.*
(3) Complete the rest of the columns as their headings indicate, i.e.

$$Waa = 3 \times 1 \times 1 = 3 \qquad Wab = 3 \times 1 \times 0 = 0$$
$$Wac = 3 \times 1 \times 1 = 3 \qquad \text{etc.}$$

(4) Complete the above set of normal equations on the basis of its symmetry, i.e. row 1 in column 1, etc.
(5) The relationship of v to K is obtained from the condition equations in exactly the same way as shown in *Section 1.11.7*, then multiplied by $1/w$.

The application of these techniques to station adjustment, levelling circuits and the adjustment of network figures, are dealt with in great detail in the section on *Worked examples*, starting on p. 48.

1.11.10 Matrix methods (direct)

Rewriting the condition equations of *Section 1.11.8* in more conventional terms gives

$$a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n = q_1$$
$$a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n = q_2$$
$$a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n = q_m$$

which in matrix terms is $\qquad Av = q \qquad$ (1.37)

Introducing the weight matrix W and the vector of correlatives k, minimizing the quadratic form v^TWv gives

$$v = W^{-1}A^Tk \tag{1.38}$$

which on substituting in equation (1.37) produces the normal equations

$$(AW^{-1}A^T)k = q \tag{1.39}$$

The normal equations are solved for k which is back-substituted in equation (1.38) to give v. Alternatively, both steps may be combined using

$$v = W^{-1}A^T(AW^{-1}A^T)^{-1}q \tag{1.40}$$

Using matrices the previous example would be as follows:

(1) Condition equation coefficients and absolute terms, as in *Section 1.11.9*, are

$$\begin{matrix} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{matrix}$$
$${}_3A_6 = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & -1 \end{bmatrix} \qquad {}_3q_1 = \begin{bmatrix} -3'' \\ 2'' \\ 1'' \end{bmatrix}$$

(2) Weights are

$$
{}_6W_6 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

(3) Now, the matrices are treated as in equation (1.39)

$$
{}_6W_6^{-1} = \begin{bmatrix} 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.33 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
(AW^{-1}A^T) = \begin{bmatrix} 0.83 & 0.25 & 0.50 \\ 0.25 & 1.00 & 0.50 \\ 0.50 & 0.50 & 1.75 \end{bmatrix}
$$

$$
(AW^{-1}A^T)^{-1}q = \begin{bmatrix} 1.48 & -0.19 & -0.37 \\ -0.19 & 1.19 & -0.29 \\ -0.37 & -0.29 & 0.76 \end{bmatrix} \quad \begin{bmatrix} -3'' \\ 2'' \\ 1'' \end{bmatrix} = k
$$

$$
\therefore\ k^T = [-5.19 \quad 2.66 \quad 1.29] = k_1 \quad k_2 \quad k_3
$$

Now $\qquad v = W^{-1}A^Tk$

$$
\therefore\ v^T = [-0.98'' \quad -0.31'' \quad 0.99'' \quad 1.71'' \quad -1.33'' \quad -1.29'']
$$

The above values for $v_1, v_2 \cdots v_6$ are now applied to the observed values as shown previously.

1.12 VARIATION OF CO-ORDINATES

The 'variation of co-ordinates' method of adjustment, which is basically a least squares method using observation equations, is virtually the standard method of network adjustment. The reasons for this are

(a) It does not rely on the formulation of conditions of adjustment, which may be difficult, if not impossible, to formulate in a complex network containing many redundancies.
(b) The technique can be applied to all types of network, i.e. triangulation, trilateration, triangulateration and traversing, hence a single computer program can be used.
(c) It affords a complete strength analysis of the final 'adjusted' network.

The method is an iterative process, which computes the necessary co-ordinate corrections ($\delta E, \delta N$) to be applied to a set of provisional co-ordinates, in order to render the network geometrically correct.

1.12.1 Observation equations

The above method requires the formation of an observation equation for each and every mean observation comprising the network. These equations take the following form.

(1) Length equations

Consider the length ij in the network with an observed value of O_{ij}. From the provisional co-ordinates of i and j, a computed value of C_{ij} may be obtained.

As the provisional co-ordinates of i and j will be adjusted by amounts δE and δN, so the computed distance will change by an amount δl_{ij}. This final adjusted distance should equal the most probable value, i.e. the observed distance plus its residual correction (v). Thus we have

$$C_{ij} + \delta l_{ij} = O_{ij} + v_{ij}$$

and $$\delta l_{ij} = (O_{ij} - C_{ij}) + v_{ij} \tag{1.41}$$

Now as $l_{ij}^2 = (E_j - E_i)^2 + (N_j - N_i)^2$

$$\delta l_{ij} = (E_j - E_i)(\delta E_j - \delta E_i)/l_{ij} + (N_j - N_i)(\delta N_j - \delta N_i)/l_{ij}$$

but as $(E_j - E_i)/l_{ij} = \sin\alpha_{ij}$ and $(N_j - N_i)/l_{ij} = \cos\alpha_{ij}$ where α_{ij} = the bearing of line ij then equation (1.41) may be expressed as

$$-\delta E_i \sin\alpha_{ij} - \delta N_i \cos\alpha_{ij} + \delta E_j \sin\alpha_{ij} + \delta N_j \cos\alpha_{ij} - (O - C)_{ij} = v_{ij} \tag{1.42}$$

which is the observation equation for length ij.

(2) Bearing equation

If α_{ij} is the bearing of a line ij, then commencing from the same initial argument as above, we have

$$\delta\alpha_{ij} = (O_\alpha - C_\alpha)_{ij} + v_{\alpha ij} \tag{1.43}$$

Now as $\tan\alpha_{ij} = (E_j - E_i)/(N_j - N_i)$

$$\sec^2\alpha_{ij}\,\delta\alpha_{ij} = [(N_j - N_i)(\delta E_j - \delta E_i) - (E_j - E_i)(\delta N_j - \delta N_i)]/(N_j - N_i)^2$$

$$\therefore\ \delta\alpha_{ij} = (N_j - N_i)(\delta E_j - \delta E_i)/l_{ij}^2 - (E_j - E_i)(\delta N_j - \delta N_i)/l_{ij}^2$$

then substituting in equation (1.43) we have the observation equation for the bearing α_{ij} of line ij, as follows

$$-\delta E_i(\cos\alpha_{ij}/l_{ij}) + \delta N_i(\sin\alpha_{ij}/l_{ij}) + \delta E_j(\cos\alpha_{ij}/l_{ij}) - \delta N_j(\sin\alpha_{ij}/l_{ij}) - (O_\alpha - C_\alpha)_{ij} = v_{\alpha ij} \tag{1.44}$$

(3) Angle equation

As an angle (θ) is the difference of two bearings then the observation equation for a clockwise angle jik is

$$-\delta E_j(\cos\alpha_{ij}/l_{ij}) + \delta N_j(\sin\alpha_{ij}/l_{ij}) + \delta E_i[(\cos\alpha_{ij}/l_{ij}) - (\cos\alpha_{ik}/l_{ik})] + \delta N_i[(-\sin\alpha_{ij}/l_{ij}) + (\sin\alpha_{ik}/l_{ik})] + \delta E_k(\cos\alpha_{ik}/l_{ik}) - \delta N_k(\sin\alpha_{ik}/l_{ik}) - (O - C)_{jik} = v_{jik} \tag{1.45}$$

In all the above observation equations

l_{ij} = horizontal length of line ij
α_{ij} = bearing of line ij
$\delta E_i, \delta N_i$ = co-ordinate corrections to E_i and N_i of point i

v_{ij} = residual correction to length ij
$v_{\alpha ij}$ = residual correction to bearing α_{ij} of line ij
v_{jik} = residual correction to horizontal angle jik

The observation equations can be expressed in general form as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1 &= v_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - b_2 &= v_2 \\ \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n - b_m &= v_m \end{aligned}$$

where a = the coefficient of the observation equation
x = the co-ordinate corrections $\delta E, \delta N$
b = the $(O - C)$ term
v = the residual error

Thus in matrix form we have $v = Ax - b$

A is an $m \times n$ matrix, x a column vector of n terms, v and b are column vectors of m terms

m = the number of observed angles and/or lengths
n = twice the number of points *to be adjusted*

As already shown (*Section 1.11.6*), minimizing the quadratic form $v^T W v$ produces the normal equation

$$(A^T W A)x = A^T W b$$

and is solved for x, as follows

$$x = (A^T W A)^{-1} A^T W b$$

1.12.2 Procedure

The procedure involved in the application of the 'variation of co-ordinates' method to network adjustment is as follows:

(1) Obtain provisional co-ordinates for each nodal point of the network. This may be done by scaling them from a plan or computing them from the observed field data.
(2) Using the provisional co-ordinates compute the angles or bearings and/or lengths of the observed data. These are the C values which, with their appropriate observed (O) values, produce the b vector of m $(O - C)$ terms.
(3) Formulate observation equations for each and every observation.
(4) Estimate *a priori* weights for the observations using the inverse of the variances and form a diagonal weight matrix W of size $m \times m$.
(5) Solve the above matrices to obtain the x vector of co-ordinate corrections $(\delta E, \delta N)$. The corrections are applied to the provisional co-ordinates as the first iteration.
(6) The adjusted co-ordinates now replace the provisional co-ordinates and the whole process repeated (only the weights remain fixed), until the x vector of co-ordinate corrections is sensibly zero.

If the provisional co-ordinates were computed from observed data in the first instance, they would be relatively close to their final positions, and one iteration would suffice (assuming the observed data contained only random errors of observation). More than two iterations might be indicative of either poor field data or input errors.

1.12.3 Weights

The coefficients of the observation equations are in radians, thus as all units must be compatible, the $(O - C)$ term for the angles and their weights must also be in radians, with the result that the residual corrections v will be in radians.

For example, if the estimated standard error of the mean measured angles (θ) is $\pm 3''$ then

$$W_\theta = 1/S_\theta^2 = 1/3^2\ \text{sec}^{-2} = 1/2.115 \times 10^{-10}\ \text{rad}^{-2} = 4.727 \times 10^9\ \text{rad}^{-2}$$

If the distance standard error (S_l) is, say, ± 3 mm then

$$W_l = 1/S_l^2 = 1/0.003^2\ \text{m}^{-2} = 1.111 \times 10^5\ \text{m}^{-2}$$

Alternatively, the weight of the angles may be left in sec^{-2} provided that the coefficients of the angle observation equations are changed to seconds, by multiplying throughout by $\rho = (180 \times 3600)/\pi$, thus

$$W_\theta = 1/3^2\ \text{sec}^{-2} = 1.111 \times 10^{-1}\ \text{sec}^{-2}$$

In this case the $(O - C)$ terms and residuals will be in seconds of arc.

Estimating the weight of an observation is perhaps the most difficult aspect of the adjustment process. Whilst it is not too critical to the adjustment process, which in any case only considers the *relative* weights of the data, it is highly critical to the strength analysis.

In the past, attempts have been made to estimate the standard error of angles, by measuring an angle many times and computing its standard error. However, such a process does not consider centring errors nor variable measuring conditions throughout the network, and should be regarded with caution. Similarly, with lengths, one has frequently to rely on the manufacturer's statement of accuracy for the particular piece of EDM equipment used.

An alternative approach to the weighting of the observed data is to use a dimensionless quantity called *unit variance* (Ashkenazi 1970).

1.12.4 Unit variance method of weighting

Weights are related to variance in the following manner

$$W_i = \sigma_0^2/\sigma_i^2 \qquad (1.46)$$

Thus, in a network of inter-related dissimilar quantities σ_0^2 will be unity if the estimates of the standard errors of the data are correct, i.e. $W_i = 1/\sigma_i^2$. This fact can be used to assess the validity of the initial *a priori* estimates of standard errors or to compute *a posteriori* estimates based on the residual corrections to the observed data.

Unit variance (or the variance of an observation of unit weight) is computed from

$$\sigma_0^2 = v^T W v/(m - n) \qquad (1.47)$$

To obtain the value of the vector v one would substitute the final values for the x vector of co-ordinate adjustments into the final equation $v = Ax - b$. However, as the final iteration should result in a value for x of zero, then $v = -b$.

The application of this procedure will now be outlined with respect to a triangulateration. Inevitably in such a network the number of angles exceeds the number of lengths, thus making them statistically stronger:

(a) Estimate the standard error S_θ for the angles and adjust the network using the angles ONLY. Note the value of σ_0^2. A small departure from unity, say $0.5 < \sigma_0^2 < 2.0$ (Ashkenazi *et al* 1978) indicates an incorrect estimate for S_θ.
(b) The correct value for S_θ is now obtained by multiplying S_θ by the computed value for σ_0 i.e. $S'_\theta = S_\theta \times \sigma_0$. Using S'_θ will now produce a value of unity for the recomputed σ_0^2.
(c) Now, enter the lengths (along with the angles) into the computation, using an estimated S_l value. This will result in a further change in the value of σ_0^2. It is restored to unity by altering the standard error of the lengths only as in (b).

Thus, using this approach one produces a statistically-viable estimate of the standard errors (and hence weights) of the observed data for this particular network.

The method has its limitations; for instance, if one enters the lengths first rather than the angles, one would obtain different values for the standard errors. Also, if another network of a different shape, but with the same number of angles and lengths, was measured by the same observer using the same equipment under the same conditions, then different standard errors would result. Thus, wherever possible all sources of information should be analyzed to support the unit variance method.

It should be further pointed out that because of the small number of degrees of freedom (DF) in such a network this method is not satisfactory for traverse networks. For instance, in a basic network with no redundancies there are only three DF, i.e. $(m - n) = 3$.

1.13 STRENGTH ANALYSIS

The strength or reliability of a network is a function of the precision of the observations, expressed through the weight matrix W, and its shape expressed through the A matrix of observation equations.

In order to analyse the strength of the final adjusted network it is necessary to produce what is called the *variance–covariance* matrix.

1.13.1 Variance-covariance matrix (σ_{xx})

The 'var–cov' matrix contains the variances and covariances of the eastings and northings of the nodal points of the adjusted network. The variances are the terms on the main diagonal, whilst the off-diagonal terms are the covariances. Covariance is a measure of correlation and is zero if the random variables are completely independent. The matrix is a square symmetric matrix of size n and is derived from equation (1.29) as follows

$$\sigma_{xx} = \sigma_0^2(A^TWA)^{-1} \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_n} \\ \sigma_{x_2x_1} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2x_n} \\ \vdots & \vdots & & \vdots \\ \sigma_{x_nx_1} & \sigma_{x_nx_2} & \cdots & \sigma_{x_n}^2 \end{bmatrix} \qquad (1.47)$$

The var–cov matrix is fundamental to the strength analysis of the adjusted network, and as shown is a function of the weights (or precision) of the field data and the shape of the network, as defined by the A matrix. As a correct estimate of weights is necessary for a correct estimate of network reliability, it follows that σ_0^2 would be unity and may be dropped from the equation. In the case of a traverse network, σ_0^2 should be taken as unity, for the reasons stipulated in *Section 1.12.4.* If an incorrect value is used for σ_0^2 then the var–cov matrix will be incorrect, resulting in an erroneous strength analysis.

1.13.2 Absolute (point) error-ellipse

The square roots of the diagonal elements of the var–cov matrix are the standard errors of the eastings and northings of the nodal points of the adjusted network, i.e. $\pm\sigma_{En}$, $\pm\sigma_{Nn}$. However, the control point in question may have a standard error greater than σ_E or σ_N, in some other direction (ϕ). This dimension is referred to as the 'semi-major axis of the error-ellipse' ($\pm\sigma_{\max}$), and the semi-minor axis ($\pm\sigma_{\min}$) would be at right-angles to it. The various dimensions are illustrated geometrically in *Figure 1.5*, and are derived from the var–cov matrix as follows

$$\pm\sigma_{\max}^2 = \tfrac{1}{2}(\sigma_{x_1}^2 + \sigma_{x_2}^2) + \{\tfrac{1}{4}(\sigma_{x_1}^2 - \sigma_{x_2}^2)^2 + \sigma_{x_1x_2}^2\}^{\frac{1}{2}} \tag{1.48}$$

$$\pm\sigma_{\min}^2 = \tfrac{1}{2}(\sigma_{x_1}^2 + \sigma_{x_2}^2) - \{\tfrac{1}{4}(\sigma_{x_1}^2 - \sigma_{x_2}^2)^2 + \sigma_{x_1x_2}^2\}^{\frac{1}{2}} \tag{1.49}$$

where $\sigma_{\max}$ and $\sigma_{\min}$ are in effect the eigenvalues of the var–cov matrix.

The bearing ϕ of the semi-major axis is obtained from

$$\tan\phi = \sigma_{x_1x_2}/(\sigma_{\max}^2 - \sigma_{x_1}^2) \tag{1.50}$$

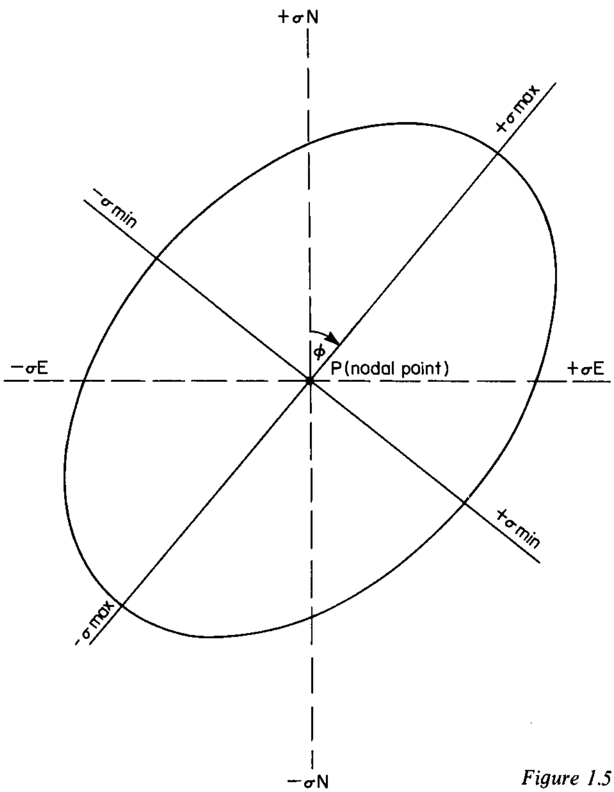

Figure 1.5

The following check on the computation may be obtained from

$$\sigma^2_{\max}\sigma^2_{\min} = \sigma^2_{x_1}\sigma^2_{x_2} - \sigma^2_{x_1x_2} \tag{1.51}$$

In one-dimensional statistics, plus and minus one standard deviation ($\pm\sigma$) represents a probability of 68.3%. However, the probability for the joint event falling within the error-ellipse is only 39.4%. Typical values are

P%	39.4	50.0	90.0	95.0	99.0
σ	1.000	1.177	2.146	2.447	3.035

The error-ellipse may be defined as the 'confidence limits of a point', as it indicates the standard error in the position of the adjusted control point in the network. The bearing (ϕ) of the semi-major axis is also significant for interpretation purposes. If normal to the direction to the fixed origin of the network it implies predominant angular error, if on the same bearing to the fixed origin the predominant error is linear.

The main limitation of the error-ellipse is that it is not an invariant quantity. Simply changing the origin of the network for example, will alter the error-ellipse values. However, for a relatively small engineering network the resultant changes will not be significant. Nevertheless, it is advisable also to compute the *a posteriori* standard errors of the adjusted angles, bearings and distances, which are invariant.

1.13.3 Standard error of the adjusted angle ($\pm\sigma_\theta$)

This quantity can be obtained from the coefficients of the observation equation for the angle and the appropriate elements of the var–cov matrix. For instance, it can be seen that there are six coefficients in the observation equation (1.45) for angle *jik*, i.e. $a_{11}, a_{12} \cdots a_{16}$. The three points *jik* defining the angle, each have an easting and northing, i.e. six values and will be defined by the appropriate 6×6 elements of the var–cov matrix, thus

$$\sigma_\theta^2 = [a_{11} \quad a_{12} \quad a_{13} \quad a_{14} \quad a_{15} \quad a_{16}] \begin{bmatrix} \sigma^2_{x_1} & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_6} \\ \sigma_{x_2x_1} & \sigma^2_{x_2} & \cdots & \sigma_{x_2x_6} \\ \vdots & \vdots & & \vdots \\ \sigma_{x_6x_2} & \sigma_{x_6x_2} & \cdots & \sigma^2_{x_6} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{15} \\ a_{16} \end{bmatrix} \tag{1.52}$$

1.13.4 Standard error of the adjusted bearing ($\pm\sigma_\alpha$)

In a manner similar to the above, the observation equation for the bearing of a line *ij* has only four coefficients (equation (1.44)) and is combined with the appropriate 4×4 elements of the var–cov matrix.

$$\sigma_\alpha^2 = [a_{11} \quad a_{12} \quad a_{13} \quad a_{14}] \begin{bmatrix} \sigma^2_{x_1} & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_4} & a_{11} \\ \sigma_{x_2x_1} & \sigma^2_{x_2} & \cdots & \sigma_{x_2x_4} & a_{12} \\ \vdots & \vdots & & \vdots & a_{13} \\ \sigma_{x_4x_1} & \sigma_{x_4x_2} & \cdots & \sigma^2_{x_4} & a_{14} \end{bmatrix} \tag{1.53}$$

1.13.5 Standard error of the adjusted length ($\pm\sigma_e$)

The observation equation for length ij has four coefficients (equation (1.42)), say $a_{21} \cdots a_{24}$, and will be combined with the same 4 × 4 coefficients of the var–cov matrix defining the eastings and northings of the same two points i and j, as in the case of the bearing ij.

$$\sigma_l^2 = [a_{21} \quad a_{22} \quad a_{23} \quad a_{24}] \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2}^2 & \cdots & \sigma_{x_1x_4} \\ \sigma_{x_2x_1} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2x_4} \\ \vdots & \vdots & & \vdots \\ \sigma_{x_4x_1} & \sigma_{x_4x_2} & \cdots & \sigma_{x_4}^2 \end{bmatrix} \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \\ a_{24} \end{bmatrix} \tag{1.54}$$

The application of the variation of co-ordinates method will now be illustrated fully, through the computation of a traverse network. The traverse used was computer-simulated with unnaturally large standard errors, nevertheless, it clearly illustrates the procedures involved and serves as a model for readers wishing to develop the method (Schofield 1979).

Example 1.10. Figure 1.6 illustrates a link traverse between fixed points W, X, Y and Z. The mean reduced field data is as follows:

Mean horizontal angles				*Mean reduced lengths*	
	°	′	″		(m)
W X A	89	59	13	*X A*	999.769
X A B	180	01	05	*A B*	1000.318
A B C	180	02	26	*B C*	1000.716
B C D	180	00	31	*C D*	1000.151
C D Y	179	57	52	*D Y*	999.372
D Y Z	90	01	24		

Fixed co-ordinates

E_W 1000.000 m N_W 8000.000 m
E_X 1000.000 m N_X 5000.000 m
E_Y 6000.000 m N_Y 5000.000 m
E_Z 6000.000 m N_Z 8000.000 m
Standard error in length = ±0.588 m
Standard error in angle = ±120″

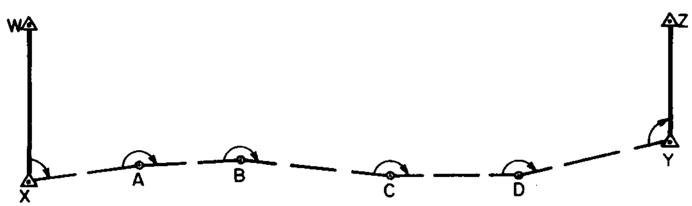

Figure 1.6

Procedure

(1) Using the necessary portions of the above data, the *provisonal* co-ordinates of A, B, C and D were calculated in the usual way:

E_A	1999.769 m	N_A	5000.228 m
E_B	3000.087 m	N_B	5000.141 m
E_C	4000.803 m	N_C	4999.345 m
E_D	5000.953 m	N_D	4998.399 m

(2) The above co-ordinates were computed from X to D using angles X, A, B and C, and lengths XA, AB, BC and CD; thus, as the co-ordinates are now used to compute values for the angles and distances (C-values), they will be identical to the observed values (O-values) except in the case of angles D and Y and length DY. The $(O - C)$ values will therefore be an 11×1 column vector b, i.e.

$$b^T = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.001927 \quad 0.323925 \quad -0.001195)$$

given in the order: angle—length—angle—length—etc.

(3) Using the observation equations for angles and lengths the A matrix of coefficients is formulated. As there are eight stations each having a co-ordinate correction δE and δN and 11 observed values, the initial matrix will be 11×16. However, since points W, X, Y and Z are to be held fixed then their coefficients are set to zero and the final A matrix will be 11×8, as shown overleaf.

(4) Using the standard errors of the angles and lengths the weights are formed as follows

$$\text{Angle weight} = W_\theta = 1/120^2 \text{ sec}^{-2} = 2\,954\,526 \text{ rad}^{-2}$$
$$\text{Length weight} = W_L = 1/0.588^2 \text{ m}^2 = 2.892\,313 \text{ m}^{-2}$$

The square diagonal W matrix of size 11×11 will therefore have the above values as the coefficients of the main diagonal, in the same order as the field data of the A matrix, i.e. angle—length—angle etc.

$$W = \begin{bmatrix} 2\,954\,526 & 0 & 0 & 0 & \cdots & 0_{1.11} \\ 0 & 2.892\,313 & 0 & 0 & \cdots & 0_{2.11} \\ 0 & 0 & 2\,954\,526 & 0 & \cdots & 0_{3.11} \\ 0 & 0 & 0 & 2.892\,313 & \cdots & 0_{4.11} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0_{11.1} & 0_{11.2} & 0_{11.3} & 0_{11.4} & \cdots & 2\,954\,526 \end{bmatrix}$$

(5) Using $x = (A^TWA)^{-1}A^TWb$, the least squares solution produces the column vector x of size 8×1, containing the first corrections to the co-ordinates of points A, B, C and D in the following order δE_A, δN_A, δE_B, δN_B, δE_C, δN_C, δE_D, δN_D

$$x^T = [-0.065, 0.136, -0.131, 0.401, -0.196, 0.792, -0.261, 1.302]$$

Now, using the corrected co-ordinates of the first iteration the whole process is repeated. A new vector of $(O - C)$ terms will this time produce values for all 11 coefficients. A new A matrix is produced but the W matrix remains constant. The second iteration produced the following x vector

$$x^T = [0, 0, 0, 0, 0.001, 0, 0.001, 0]$$

The *A* matrix

			δE_A	δN_A	δE_B	δN_B	δE_C	δN_C	δE_D	δN_D
W	X̂	A	0.00000023	−0.00100023	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
	XA		0.99999997	0.00022786	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
X	Â	B	−0.00000014	0.00199991	−0.00000009	−0.00099968	0.00000000	0.00000000	0.00000000	0.00000000
	AB		−1.00000000	0.00008727	1.00000000	−0.00008727	0.00000000	0.00000000	0.00000000	0.00000000
A	B̂	C	−0.00000009	−0.00099968	0.00000088	0.00199897	−0.00000079	−0.00099928	0.00000000	0.00000000
	BC		0.00000000	0.00000000	−0.99999968	0.00079509	0.99999968	−0.00079509	0.00000000	0.00000000
B	Ĉ	D	0.00000000	0.00000000	−0.00000079	−0.00099928	0.00000174	0.00199913	−0.00000095	−0.00099985
	CD		0.00000000	0.00000000	0.00000000	0.00000000	−0.99999955	0.00094539	0.99999955	−0.00094539
C	D̂	Y	0.00000000	0.00000000	0.00000000	0.00000000	−0.00000095	−0.00099985	−0.00000066	−0.00200080
	DY		0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	−0.99999872	−0.00160220
D	Ŷ	Z	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000160	−0.00100095

and the final adjusted co-ordinates are

Point	*E*	*N*
A	1999.704	5000.363
B	2999.957	5000.542
C	4000.607	5000.137
D	5000.693	4999.701

(6) The residuals are now obtained as shown in *Section 1.12.4*, i.e. $v = -b$, and applied to the observed data to give the final adjusted angles and lengths:

Stations			*Residuals* v	*Final angles and lengths*	v/σ
W	X	A	−27.97″	89° 58′ 45″	0.233
	XA		−0.065 m	999.704 m	0.111
X	A	B	−26.85″	180° 00′ 38″	0.224
	AB		−0.065 m	1000.253 m	0.111
A	B	C	−25.72″	180° 02′ 00″	0.214
	BC		−0.065 m	1000.651 m	0.111
B	C	D	−24.61″	180° 00′ 06″	0.205
	CD		−0.065 m	1000.086 m	0.111
C	D	Y	−23.49″	179° 57′ 29″	0.196
	DY		−0.065 m	999.307 m	0.111
D	Y	Z	−22.36″	90° 01′ 02″	0.186

The final column of the above table is in effect 'rejection criteria'. It is widely acknowledged in surveying literature that if the residual of an observation is greater than *three* times the standard error, that observation should be rejected and re-observed, i.e. $v > 3\,\sigma$. Thus it is obvious that the final column contains dimensionless numbers. Those equal to three or above indicate that the observation should be rejected. If equal to two or thereabouts, the observation is suspect. These values should be used as a guide to the relative precision of the observed data.

In addition, a useful check on the normal equations is obtained from $A^T W b = 0$.

(7) To assess the reliability of the adjusted data the variance–covariance matrix is obtained from the final iteration by using

$$\sigma_{xx} = (A^T W A)^{-1}$$

and assuming $\sigma_0^2 = 1$. This now produces an 8×8 symmetrical matrix as shown overleaf. The variances of the eastings and northings of each point A to D are on the main diagonal. The off-diagonal terms are the covariances; for instance, 4.3992 & −5 relates the E_A to the N_A as shown.

(8) From the var–cov matrix one obtains the standard errors of the co-ordinates and by equations (1.48), (1.49) and (1.50), the error-ellipse values as shown:

The variance–covariance matrix (σ_{xx})

	E_A	N_A	E_B	N_B	E_C	N_C	E_D	N_D
E_A	2.7660 & −1	4.3992 & −5	2.0745 & −1	1.8997 & −6	1.3830 & −1	−1.6787 & −7	6.9149 & −2	1.2024 & −5
N_A	4.3992 & −5	1.6109 & −1	2.4244 & −5	1.9332 & −1	1.9465 & −5	1.4494 & −1	6.9614 & −7	6.4381 & −2
E_B	2.0745 & −1	2.4244 & −5	4.1489 & −1	2.7877 & −6	2.7660 & −1	−7.4050 & −7	1.3830 & −1	2.3948 & −5
N_B	1.8997 & −6	1.9332 & −1	2.7877 & −6	3.6742 & −1	2.4409 & −5	−3.0932 & −1	1.8990 & −6	1.4491 & −1
E_C	1.3830 & −1	1.9465 & −5	2.7660 & −1	2.4409 & −5	4.1489 & −1	−2.3345 & −5	2.0745 & −1	4.1027 & −5
N_C	−1.6787 & −7	1.4494 & −1	−7.4050 & −7	3.0932 & −1	−2.3345 & −5	3.6735 & −1	1.3565 & −6	1.9324 & −1
E_D	6.9149 & −2	6.9614 & −7	1.3830 & −1	1.8990 & −6	2.0745 & −1	1.3565 & −6	2.7660 & −1	2.2973 & −5
N_D	1.2024 & −5	6.4381 & −2	2.3948 & −5	1.4491 & −1	4.1027 & −5	1.9324 & −1	2.2973 & −5	1.6099 & −1

N.B. & −5 signifies 10^{-5}.

	Standard errors of co-ordinates				
Stations	σE (mm)	σN (mm)	σ_{max} (mm)	σ_{min} (mm)	ϕ_{max} (deg)
A	±526	±401	±526	±401	90
B	±644	±606	±644	±606	90
C	±644	±606	±644	±606	90
D	±526	±401	±526	±401	90

In this particular example, the axes of the error-ellipses are identical to the standard errors of the co-ordinates.

(9) Now, using the appropriate terms of the A matrix and var–cov matrix in equations (1.52) and (1.54), the standard errors of the adjusted angles and lengths are obtained

	Standard errors of angles (sec)		*Standard errors of lengths* (mm)
W X A	± 82.8	*X A*	±525.9
X A B	±100.7	*A B*	±525.9
A B C	±108.6	*B C*	±525.9
B C D	±108.6	*C D*	±525.9
C D Y	±100.7	*D Y*	±525.9
D Y Z	± 82.8		

Whilst the above method has been outlined showing the observation equations, the $(O - C)$ values and the weights of the ANGLES in radians, the reader is reminded of the earlier comment regarding this matter. That is, provided that the coefficients of the angle observation equation is changed to seconds, the $(O - C)$ values and the weights may be left in seconds. Further to this, if we expand the normal equations

$$A^TWAx = A^TWb$$
$$A^TW^{\frac{1}{2}}W^{\frac{1}{2}}Ax = A^TW^{\frac{1}{2}}W^{\frac{1}{2}}B$$
$$(W^{\frac{1}{2}}A)^T(W^{\frac{1}{2}}A)x = (W^{\frac{1}{2}}A)^T(W^{\frac{1}{2}}b)$$

the matrices are conveniently formed by multiplying the coefficients of each observation equation by the reciprocal of the corresponding *a priori* standard errors $(1/\sigma)$. Thus, if as above, the units are metres and sexagesimal seconds-of-arc then a standard error of ± 2 sec would result in the angle observation equation being multiplied by $1/2 = 0.5$. Similarly a standard error in length of ± 10 mm, the appropriate equation is multiplied by $1/0.010 = 100$. It must be emphasized that without the inclusion of weighting the resultant least squares adjustment would be completely sterile and meaningless.

1.14 PRE-SURVEY ANALYSIS

It is possible to carry out a strength analysis of a network prior to its observation from the configuration of the network and the proposed precision of the measurements. For instance, if the proposed positions of the survey stations are drawn on a plan and their relative co-ordinates scaled therefrom, the coefficients of the observation equations can be computed and the A matrix produced. Now, knowing the equipment available, an estimate of the precision of the field data can be made and the weight matrix W, produced. Thereafter the variance–covariance matrix can be obtained from

$$\sigma_{xx} = (A^T W A)^{-1}$$

as σ_0^2 will obviously be unity in this situation.

If the strength analysis indicates that the initial estimates will not meet the accuracy specifications, one may alter the configuration of the network and/or the type of equipment proposed and/or the observation technique, until the specifications are met. In this way, the surveyor will know the best positions for his survey stations, the necessary equipment required and the observational technique to adopt, prior to commencing the survey.

This approach would lead to the economic design of networks, to best meet the accuracy specifications and functional requirements.

1.15 NETWORK OPTIMIZATION

Using the same approach as in *Section 1.14*, it should theoretically be possible to optimize surveys, i.e. assess the minimum number of angles and lengths required in a network in order to meet the accuracy specifications. However, in practice, it is doubtful if this is a viable proposition.

In the first instance, traverse networks cannot be optimized, nor can they be analysed with any degree of reliability.

Similarly, trilateration networks cannot be optimized nor therefore can they be analysed. It is a common concept in surveying that the greater the number of redundant measures the stronger the network. However, as one reduces the number of lengths in a trilateration, the network appears to get stronger. The standard errors and error-ellipses are zero when the minimum number of lengths is used. For instance, if only three sides of a triangle are measured, a triangle will always be formed *without apparent error*. Because of this, it is therefore doubtful that triangulateration can be successfully treated. Thus, optimization of engineering networks may not be a practical operation.

WORKED EXAMPLES

Example 1.11. The same angle was measured by two different observers using the same instrument, as follows:

Observer A ° ′ ″	*Observer B* ° ′ ″
86 34 10	86 34 05
33 50	34 00
33 40	33 55
34 00	33 50
33 50	34 00
34 10	33 55
34 00	34 15
34 20	33 44

Calculate: (a) The standard deviation of each set.
(b) The standard error of the arithmetic means.
(c) The most probable value (MPV) of the angle. (KP)

	Observer A			*r*	r^2	*Observer B*			*r*	r^2
	°	′	″	″	″	°	′	″	″	″
	86	34	10	10	100	86	34	05	7	49
		33	50	−10	100		34	00	2	4
		33	40	−20	400		33	55	−3	9
		34	00	0	0		33	50	−8	64
		33	50	−10	100		34	00	2	4
		34	10	10	100		33	55	−3	9
		34	00	0	0		34	15	17	289
		34	20	20	400		33	44	−14	196
Mean =	86	34	00	0	1200 = $[r^2]$	86	33	58	0	624 = $[r^2]$

(a) (i) Standard deviation $([r^2] = \Sigma(x_i - \bar{x})^2)$

$$S_A = \pm\left(\frac{[r^2]}{n-1}\right)^{\frac{1}{2}} = \pm\left(\frac{1200}{7}\right)^{\frac{1}{2}} = \pm 13.1''$$

(b) (i) Standard error $= S_{\bar{x}_A} = \pm\dfrac{S_A}{n^{\frac{1}{2}}} = \pm\dfrac{13.1}{8^{\frac{1}{2}}} = \pm 4.6''$

(a) (ii) $S_B = \pm\left(\dfrac{624''}{7}\right)^{\frac{1}{2}} = \pm 9.4''$

(b) (ii) $S_{\bar{x}_B} = \pm\dfrac{9.4}{8^{\frac{1}{2}}} = \pm 3.3''$

(c) As each arithmetic mean has a different precision exhibited by their $S_{\bar{x}}$ values, they must be weighted accordingly before they can be meaned to give the MPV of the angle

$$\text{Weight of } A \propto \frac{1}{S_{\bar{x}_A}^2} = \frac{1}{21.2} = 0.047$$

$$\text{Weight of } B \propto \frac{1}{10.9} = 0.092$$

The ratio of the weight of A to the weight of B is 1:2

$$\therefore \text{ MPV of the angle} = \frac{(86°\,34'\,00'' + 86°\,33'\,58'' \times 2)}{3}$$

$$= 86°\,33'\,59''$$

As a matter of interest, the following point could be made here: Any observation whose residual is greater than $2.3S$ should be rejected (see *Section 1.6*). As $2.3S_A = 30.2''$ and $2.3S_B = 21.6''$, all the observations should be included in the set. This test should normally be carried out at the start of the problem.

Example 1.12. Discuss the classification of errors in surveying operations, giving appropriate examples.

In a triangulation scheme, the three angles of a triangle were measured and their mean values recorded as 50° 48′ 18″, 64° 20′ 36″ and 64° 51′ 00″. Analysis of each set gave a standard deviation of ±4″ for each of these means. At a later date, the angles were re-measured under better conditions, yielding mean values of 50° 48′ 20″, 64° 20′ 39″ and 64° 50′ 58″. The standard deviation of each value was ±2″. Calculate the most probable values of the angles. (KP)

The angles are first adjusted to 180°. Since the angles within each triangle are of equal weight, then the angular adjustment within each triangle is equal.

50° 48′ 18″ + 2″ =	50° 48′ 20″	50° 48′ 20″ + 1″ =	50° 48′ 21″
64° 20′ 36″ + 2″ =	64° 20′ 38″	64° 20′ 39″ + 1″ =	64° 20′ 40″
64° 51′ 00″ + 2″ =	64° 51′ 02″	64° 50′ 58″ + 1″ =	64° 50′ 59″
179° 59′ 54″	180° 00′ 00″	179° 59′ 57″	180° 00′ 00″

$$\text{Weight of the first set} = w_1 = 1/4^2 = \frac{1}{16}$$

$$\text{Weight of the second set} = w_2 = 1/2^2 = \frac{1}{4}$$

Thus $w_1 = 1$, when $w_2 = 4$.

$$\therefore \text{ MPV} = \frac{(50°\,48'\,20'') + (50°\,48'\,21'' \times 4)}{5} = 50°\,48'\,20.8''$$

Similarly, the MPV of the remaining angles are

64° 20′ 39.6″ 64° 50′ 59.6″

These values may now be rounded off to single seconds.

Example 1.13. A base line of ten bays was measured by a tape resting on measuring heads. One observer read one end while the other observer read the other—the difference in readings giving the observed length of the bay. Bays 1, 2 and 5 were measured six times; bays 3, 6 and 9 were measured five times and the remaining bays were measured four times, the means being calculated in each case. If the standard errors of single readings by the two observers were known to be 1 mm and 1.2 mm, what will be the standard error in the whole line due only to reading errors? (LU)

Standard error in reading a bay $= S_s = (1^2 + 1.2^2)^{\frac{1}{2}} = \pm 1.6$ mm

Consider bay 1, this was measured six times and the mean taken; thus the standard error of the mean is

$$S_{\bar{x}} = \frac{S_s}{n^{\frac{1}{2}}} = \frac{1.6}{6^{\frac{1}{2}}} = \pm 0.6 \text{ mm}$$

This value applies to bays 2 and 5 also. Similarly for bays 3, 6 and 9

$$S_{\bar{x}} = \frac{1.6}{5^{\frac{1}{2}}} = \pm 0.7 \text{ mm}$$

For bays 4, 7, 8 and 10 $S_{\bar{x}} = \dfrac{1.6}{4^{\frac{1}{2}}} = \pm 0.8$ mm

These bays are now summed to obtain the total length. Therefore the standard error of the whole line is

$$(0.6^2+0.6^2+0.6^2+0.7^2+0.7^2+0.7^2+0.8^2+0.8^2+0.8^2+0.8^2)^{\frac{1}{2}} = \pm 2.3 \text{ mm}$$

Example 1.14

(a) A base line was measured using electronic distance-measuring (EDM) equipment and a mean distance of 6835.417 m recorded. The instrument used has a manufacturer's quoted accuracy of 1/400 000 of the length measured ±20 mm. As a check the line was re-measured using a different type of EDM equipment having an accuracy of 1/600 000 ± 30 mm; the mean distance obtained was 6835.398 m. Determine the most probable value of the line.
(b) An angle was measured by three different observers, A, B and C. The mean of each set and its standard error is shown below.

Observer	*Mean angle* ° ′ ″	$S_{\bar{x}}$ ″
A	89 54 36	±0.7
B	89 54 42	±1.2
C	89 54 33	±1.0

Determine the most probable value of the angle. (KP)

(a) Standard error, 1st instrument
$$S_{\bar{x}_1} = \pm\left\{\left(\frac{6835}{400\,000}\right)^2 + (0.020)^2\right\}^{\frac{1}{2}}$$
$$= \pm 0.026 \text{ m}$$

Standard error, 2nd instrument
$$S_{\bar{x}_2} = \pm\left\{\left(\frac{6835}{600\,000}\right)^2 + (0.030)^2\right\}^{\frac{1}{2}}$$
$$= \pm 0.032 \text{ m}$$

These values can now be used to weight the lengths and find their weighted means as shown below.

	Length, L (m)	$S_{\bar{x}}$	*Weight ratio*	*Weight, W*	$L \times W$	
1st instrument	0.417	±0.026	$1/0.026^2 = 1479$	1.5	0.626	
2nd instrument	0.398	±0.032	$1/0.032^2 = 977$	1	0.398	
				$[W] = 2.5$	1.024	$= [LW]$

$$\therefore \text{ MPV} = 6835 + \frac{1.024}{2.5} = 6835.410 \text{ m}$$

(b)

Observer	Mean angle ° ′ ″	$S_{\bar{x}}$ ″	Weight ratio	Weight, W	L × W
A	89 54 36	±0.7	$1/0.7^2 = 2.04$	2.96	6″ × 2.96 = 17.8″
B	89 54 42	±1.2	$1/1.2^2 = 0.69$	1	12″ × 1 = 12″
C	89 54 33	±1.0	$1/1^2 = 1$	1.45	3″ × 1.45 = 4.35″
			[W] = 5.41		34.15 = [LW]

$$\therefore \text{ MPV} = 89° \, 54' \, 30'' + \frac{34.15''}{5.41} = 89° \, 54' \, 36''$$

The student's attention is drawn to the method of finding the weighted mean in both these examples, although since the advent of the pocket calculator there is no need to refine the weights down from the weight ratio, particularly in (b).

Example 1.15. In an underground correlation survey, the sides of a Weisbach triangle were measured as follows:

$W_1W_2 = 5.435$ m $\qquad W_1W = 2.844$ m $\qquad W_2W = 8.274$ m

Using the above measurements in the cosine rule, the calculated angle $WW_1W_2 =$ 175° 48′ 24″. If the standard error of each of the measured sides is 1/20 000, calculate the standard error of the calculated angle in seconds of arc. (KP)

From *Figure 1.7*, by the cosine rule $\qquad c^2 = a^2 + b^2 - 2ab\cos W_1$

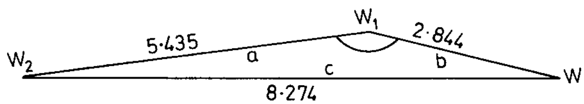

Figure 1.7

Using equation (1.15) from *Section 1.10* and differentiating with respect to each variable in turn

$$2c\delta c = 2ab\sin W_1\delta W_1 \qquad \text{thus} \qquad \delta W_1 = \pm\frac{c\delta c}{ab\sin W_1}$$

Similarly
$$a^2 = c^2 - b^2 + 2ab\cos W_1$$
$$2a\delta a = 2b\cos W_1\delta a - 2ab\sin W_1\delta W_1$$

$$\therefore \delta W_1 = \frac{2a\delta a - 2b\cos W_1\delta a}{2ab\sin W_1} = \frac{\delta a(a - b\cos W_1)}{ab\sin W_1}$$

but, since angle $W_1 \approx 180°$, $\cos W_1 \approx -1$ and $(a + b) \approx c$

$$\therefore \delta W_1 = \pm\frac{\delta ac}{ab\sin W_1}$$

now $b^2 = a^2 - c^2 + 2ab\cos W_1$

and $2b\delta b = 2a\cos W_1 \delta b - 2ab\sin W_1 \delta W_1$

$$\therefore \quad \delta W_1 = \frac{\delta b(b - a\cos W_1)}{ab\sin W_1} = \pm\frac{\delta bc}{ab\sin W_1}$$

Making δW_1, δa, δb and δc equal to the standard deviations gives

$$\sigma_{w_1} = \pm\frac{c}{ab\sin W_1}(\sigma_a^2 + \sigma_b^2 + \sigma_c^2)^{\frac{1}{2}}$$

where $\sigma_a = \frac{5.435}{20\,000} = \pm 2.7\times 10^{-4}$

$$\sigma_b = \frac{2.844}{20\,000} = \pm 1.4\times 10^{-4}$$

$$\sigma_c = \frac{8.274}{20\,000} = \pm 4.1\times 10^{-4}$$

$$\therefore \quad \sigma_{w_1} = \pm\frac{8.274\times 206\,265\times 10^{-4}}{5.435\times 2.844\sin 175^\circ\,48'\,24''}(2.7^2 + 1.4^2 + 4.1^2)^{\frac{1}{2}}$$

$$= \pm 770''$$
$$= \pm 0^\circ\,12'\,50''$$

This is a standard treatment for small errors, and nothing is to be gained by further examples of this type here. The student can find numerous examples of its application in Volume 1 of *Engineering Surveying* and throughout the remainder of this book.

Example 1.16. From a station P, the angles subtended by points Q, R, S and T were measured by two observers A and B. The results are tabulated below.

Observer		*Angle* °	′	″
A	*QPR*	16	02	51
A	*RPS*	40	34	08
A	*SPT*	22	11	04
B	*QPS*	56	37	01
B	*RPT*	62	45	09

In order to apportion weights to their observations, a separate test was carried out, in which both A and B measured a given angle a large number of times. The analysis of the test showed that the standard error of B was twice that of A. Apply appropriate weights to the observations and determine the most probable value of the angles to the nearest 0.1″. (LU)

As the weights are inversely proportional to the square of the standard error, if B has a weight of 1, then A will have a weight of 4, thus

$QPR = x = 16° 02' 51''$ wt 4
$RPS = y = 40° 34' 08''$ wt 4
$SPT = z = 22° 11' 04''$ wt 4
$QPS = (x + y) = 56° 37' 01''$ wt 1
$RPT = (y + z) = 62° 45' 09''$ wt 1

Adopting the above observed values as the assumed values of x, y and z, the observation equations may be formed

$$\underbrace{16° 02' 51'' + v_1}_{\text{MPV}} = \underbrace{16° 02' 51''}_{\text{observed value}}$$

$$\therefore v_1 = 0 \text{ wt } 4$$

i.e. by comparing the MPV with the observed value the difference is the error.

Similarly $v_2 = 0$ wt 4 $\quad v_3 = 0$ wt 4

But $16° 02' 51'' + v_1 + 40° 34' 08'' + v_2 = 56° 37' 01''$ $\quad \therefore v_1 + v_2 = 2''$ wt 1

and $40° 34' 08'' + v_2 + 22° 11' 04'' + v_3 = 62° 45' 09''$ $\quad \therefore v_2 + v_3 = -3''$ wt 1

Assembling the observation equations

$$\begin{aligned} v_1 &= 0 \text{ wt } 4 \\ v_2 &= 0 \text{ wt } 4 \\ v_3 &= 0 \text{ wt } 4 \\ v_1 + v_2 &= 2'' \text{ wt } 1 \\ v_2 + v_3 &= -3'' \text{ wt } 1 \end{aligned}$$

The coefficients of v_1, v_2, v_3, are a, b, c, respectively, and all equal 1. Substituting directly in equation (1.21) gives the normal equations

$$\begin{aligned} 5v_1 + v_2 \qquad &= 2'' \\ v_1 + 6v_2 + v_3 &= -1'' \\ v_2 + 5v_3 &= -3'' \end{aligned}$$

These equations are easily solved by simple algebra

$v_1 = 0.4'' \qquad v_2 = -0.1'' \qquad v_3 = -0.6''$

Thus the MPV are

$$\begin{aligned} QPR &= 16° 02' 51'' + 0.4'' = 16° 02' 51.4'' \\ RPS &= 40° 34' 08'' - 0.1'' = 40° 34' 07.9'' \\ SPT &= 22° 11' 04'' - 0.6'' = 22° 11' 03.4'' \end{aligned}$$

Example 1.17. A straight line *ABCD* was measured as a whole and in sections. Due to variations in accuracy, the measurements have been assigned weights as shown below.

	Measured length (m)	*Weight*
AB	39.231	3
BC	120.716	2
CD	61.256	2
AC	159.935	1
AD	221.218	1

Find by the method of least squares, the most probable lengths of *AB*, *BC* and *CD* to the nearest 0.0001 m. (LU)

$AB = x = 39.231$ wt 3
$BC = y = 120.716$ wt 2
$CD = z = 61.256$ wt 2
$AC = (x + y) = 159.935$ wt 1
$AD = (x + y + z) = 221.218$ wt 1

Taking the observed values of x, y and z as the assumed values, the first three observation equations are

$v_1 = 0$ wt 3 $\quad v_2 = 0$ wt 2 $\quad v_3 = 0$ wt 2

Similarly $\quad v_1 + v_2 = -12$ mm wt 1
$v_1 + v_2 + v_3 = 15$ mm wt 1

By inspection the normal equations are

$5v_1 + 2v_2 + v_3 = 3$ mm $\quad 2v_1 + 4v_2 + v_3 = 3$ mm $\quad v_1 + v_2 + 3v_3 = 15$ mm

On solution $\quad v_1 = -0.0003$ m $\quad v_2 = -0.0004$ m $\quad v_3 = 0.0052$ m

$\therefore$ MPV of $AB = 39.231 - 0.0003 = 39.2307$ m
$BC = 120.716 - 0.0004 = 120.7156$ m
$CD = 61.256 + 0.0052 = 61.2612$ m

Note. Examples 1.16 and 1.17 could equally well have been solved by condition equations with negligible difference in time. The latter method provides a check in the fulfilment of the stipulated conditions, but the student may find it slightly more difficult to form the conditional equations. It is also worth noting that the majority of these problems have the corrections quoted to an accuracy greater than the initial field data; the final values, however, should be rounded to the same accuracy as the field data.

Example 1.18. The measured differences in level in metres between four stations *A*, *B*, *C* and *D* (*Figure 1.8*) are given in the following Table, together with the estimated weights of the values. Determine, by the method of least squares, the most probable values of the differences in level to the nearest 0.0001 m.

From	*To*	*Rise*	*Fall*	*Weight*
A	*B*	5.977		3
B	*C*	8.550		1
C	*D*		2.877	2
D	*A*		11.665	1
D	*B*		5.678	3

(LU)

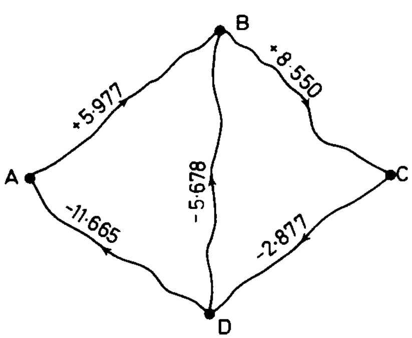

Figure 1.8

To simplify matters, assume an arbitrary value of 100.000 m for station A. In all levelling network adjustments the station from which the levelling commences is assumed to be correct. It is therefore required to find the levels of B, C and D *only*.

Using the first three observed differences in level, the assumed levels for B, C and D are

$B = 105.977$ m $\qquad C = 114.527$ m $\qquad$ and $\qquad D = 111.650$ m

Observation equations are now formed for *each line* of the net as follows:

Most probable level of B − Most probable level of A
= Observed difference in level

$$\text{i.e.}\quad B + v_1 - A = 5.977$$
$$\therefore\ 105.977 + v_1 - 100.000 = 5.977$$
$$\therefore\ v_1 = 0 \text{ wt } 3$$

Similarly $\quad C + v_2 - (B + v_1) = 8.550$

$$\therefore\ 114.527 + v_2 - 105.977 - v_1 = 8.550$$
$$\therefore\ v_2 - v_1 = 0 \text{ wt } 1$$

and $\quad D + v_3 - (C + v_2) = -2.877$

$$\therefore\ 111.650 + v_3 - 114.527 - v_2 = -2.877$$
$$\therefore\ v_3 - v_2 = 0 \text{ wt } 2$$

and $\quad A - (D + v_3) = -11.665$

$$\therefore\ 100.000 - 111.650 - v_3 = 11.665$$
$$\therefore\ v_3 = 15 \text{ mm wt } 1$$

and $\quad B + v_1 - (D + v_3) = -5.678$

$$\therefore\ 105.977 + v_1 - 111.650 - v_3 = -5.678$$
$$\therefore\ v_1 - v_3 = -5 \text{ mm wt } 3$$

Collecting observation equations for easy inspection

$$v_1 = 0 \text{ wt } 3 \qquad v_3 = 15 \text{ mm wt } 1$$
$$v_2 - v_1 = 0 \text{ wt } 1 \qquad v_1 - v_3 = -5 \text{ mm wt } 3$$
$$v_3 - v_2 = 0 \text{ wt } 2$$

Normal equations by inspection of the observation equations and equation (1.25) are

$$7v_1 - v_2 - 3v_3 = -15 \text{ mm}$$
$$-v_1 + 3v_2 - 2v_3 = 0$$
$$-3v_1 - 2v_2 + 6v_3 = 30 \text{ mm}$$

On solution $\qquad v_1 = 2.3$ mm $\qquad v_2 = 6.2$ mm $\qquad v_3 = 8.2$ mm

$\therefore$ Levels of

$$A = 100.000 \text{ m}$$
$$B = 105.977 + 0.0023 = 105.9793 \text{ m}$$
$$C = 114.527 + 0.0062 = 114.5332 \text{ m}$$
$$D = 111.650 + 0.0082 = 111.6582 \text{ m}$$

$\therefore$ Most probable differences in level are

$$A - B = 5.9793 \text{ m}$$
$$B - C = 8.5539 \text{ m}$$
$$C - D = -\ 2.8750 \text{ m}$$
$$D - A = -11.6582 \text{ m}$$

Sum = zero (check)

Since $D - B = -5.6789$ m, this also checks with the sum of the second and third values given above.

Note. The following important points in relation to level networks should be carefully observed:

(1) As errors in levelling are though to follow the laws already outlined they will be proportional to the square root of the length of line levelled, i.e. $\sigma \propto (L)^{\frac{1}{2}}$. However, as weights $\propto (1/\sigma^2)$ then for lines of levelling $w \propto (1/L)$, i.e. the weight is inversely proportional to the length of the line levelled.
(2) The problem could have been just as easily worked out by the method of condition equations, in which case the conditions would have been

   (a) Circuit *ABCDA* should close to zero.
   (b) Circuit *BCDB* should close to zero.
   (c) Circuit *ABDA* should close to zero.

   However, the rule for deciding the number of conditions has already been outlined, i.e.

   Number of directly-observed quantities = 5 (lines *AB*, *BC*, *DC*, *DA*, *DB*)
   Number of independent unknowns = 3 (stations *B*, *C*, *D*)

   ∴ Number of above conditions required = 2

Example 1.19. Angles 'closing the horizon' were measured about a station as follows

$$\begin{aligned} w &= 70^\circ\ 05'\ 31.6'' \\ x &= 164^\circ\ 23'\ 39.8'' \\ y &= 96^\circ\ 50'\ 51.6'' \\ z &= 28^\circ\ 39'\ 50.0'' \\ (w + x) &= 234^\circ\ 29'\ 03.4'' \\ (y + z) &= 125^\circ\ 30'\ 38.2'' \end{aligned}$$

Find, by the method of least squares, the most probable values of angles w, x, y and z. (KP)

This problem is indicated in *Figure 1.9* and shows that there is a condition to be fulfilled, namely that the MPV of $w + x + y + z = 360^\circ$. From this condition it can therefore be seen that if the MPV of w, x and y were known, then the MPV of z must be $360^\circ - (w + x + y)$. Thus z may be regarded as a 'dependent' quantity and the values of

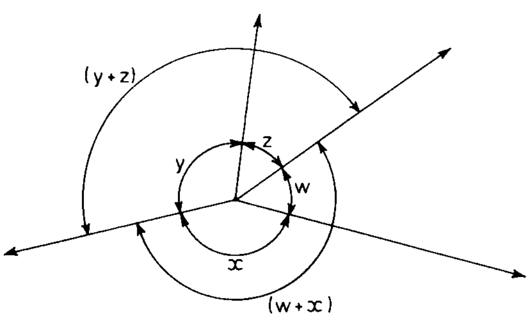

Figure 1.9

w, x and y only need be found. In this way the number of normal equations is reduced from four to three.

The assumed values of w, x and y are taken as equal to the above observed values and therefore the first three observation equations will be

$$v_1 = 0 \qquad v_2 = 0 \qquad v_3 = 0$$

$\therefore$ MPV of $z = 360° - (w + v_1 + x + v_2 + y + v_3) = 28° \; 39' \; 50.0''$

Substituting the assumed values for w, x and y gives $\qquad v_1 + v_2 + v_3 = +7''$

Similarly $\qquad w + v_1 + x + v_2 = 234° \; 29' \; 03.4''$

Substituting assumed values for w and x gives $\qquad v_1 + v_2 = -8''$

$$(y + z) = y + v_3 + 360° - (w + v_1 + x + v_2 + y + v_3) = 125° \; 30' \; 38.2''$$

Substituting assumed values for w, x and y gives $\qquad v_1 + v_2 = 10.4''$

Normal equations direct from observation equations and general equation (1.23) in the usual way are

$$4v_1 + 3v_2 + v_3 = +9.4''$$
$$3v_1 + 4v_2 + v_3 = +9.4''$$
$$v_1 + v_2 + 2v_3 = +7.0''$$

On solution $\qquad v_1 = 0.98'' \qquad v_2 = 0.98'' \qquad v_3 = 2.52''$

$\therefore$ MPV of $w = 70° \; 05' \; 31.6'' + 0.98'' = 70° \; 05' \; 32.6''$
$x = 164° \; 23' \; 39.8'' + 0.98'' = 164° \; 23' \; 40.8''$
$y = 96° \; 50' \; 51.6'' + 2.52'' = 96° \; 50' \; 54.1''$
$z = 360° -$ the MPV of w, x and $y = 28° \; 39' \; 52.5''$

Example 1.20. The measured angles of a geodetic quadrilateral *ABCD* are given below, together with their log sines, differences in log sines for 1″, and their respective weights.

Number	*Measured angle*	°	′	″	Log sin	*Difference in* log sin *for* 1″	*Weight*
1	*CAD*	35	05	09	$\bar{1}$.759 519 0	0.000 031 7	2
2	*BAC*	56	06	57	$\bar{1}$.919 165 2	0.000 014 2	2
3	*DBA*	46	16	00	$\bar{1}$.858 877 0	0.000 020 2	1
4	*CBD*	46	14	08	$\bar{1}$.858 651 2	0.000 020 2	1
5	*ACB*	31	22	49	$\bar{1}$.716 600 8	0.000 034 6	2
6	*DCA*	30	28	41	$\bar{1}$.705 186 3	0.000 035 8	2
7	*BDC*	71	54	02	$\bar{1}$.977 960 7	0.000 006 9	3
8	*ADB*	42	32	02	$\bar{1}$.829 963 4	0.000 023 0	3

Derive the normal equations to determine, by least squares, the most probable values of the angles. The solution of the equations is not required, but the steps that would need to be taken after the solution, to obtain corrections to the angles must be stated.

Note $\qquad \sum$ log sin odds $- \sum$ log sin evens $= -0.000\,008\,6$ $\qquad$ (LU)

For the benefit of the student, this problem will be solved: (1) assuming no weights; (2) in its original form including the weights, by the direct method using correlatives. Modern practice would be to use the method of variation of co-ordinates and would require a computer.

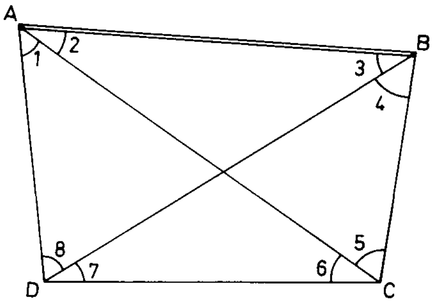

Figure 1.10

(1) It is necessary to obtain the correct number of condition equations required. By reference to *Figure 1.10* the number of directly-observed quantities is eight; the number of independent unknowns is four. This latter statement should now be carefully considered. In any triangulation scheme, one works from a base line *AB* to fix the co-ordinate position of the other points. Thus to fix *C* and *D*, one would require their eastings and northings giving four independent unknowns. The number of condition equations required is then 8 − 4 = 4.

As shown in Chapter 2, there are eight conditions of adjustment for a crossed-quadrilateral, not all of which are independent. The four most independent of the conditions are

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 360°$$
$$2 + 3 - 6 - 7 = 0$$
$$1 + 8 - 4 - 5 = 0$$

$$\log\sin 1 + \log\sin 3 + \log\sin 5 + \log\sin 7 - (\log\sin 2 + \log\sin 4 + \log\sin 6 + \log\sin 8) = 0$$

This latter condition is called the *side condition* and is frequently stated as: $\sum$ log sin of the odd angles = $\sum$ log sin of the even angles (refer to Chapter 2 for its derivation). As the crossed-quadrilateral is a 'unique' figure, the above conditions are the standard conditions always used in its adjustment.

Assuming corrections $v_1 \cdots v_8$, for angles $1 \cdots 8$, the condition equations are derived as

$$1 + v_1 + 2 + v_2 + 3 + v_3 + 4 + v_4 + 5 + v_5 + 6 + v_6 + 7 + v_7 + 8 + v_8 - 360° = 0$$

Substituting the observed values for angles 1, 2, . . . , 8, then

$$v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 + v_8 - 12'' = 0$$

Similarly
$$v_2 + v_3 - v_6 - v_7 + 14'' = 0$$
$$v_1 + v_8 - v_4 - v_5 + 14'' = 0$$

$$317v_1 - 142v_2 + 202v_3 - 202v_4 + 346v_5 - 358v_6 + 69v_7 - 230v_8 - 86 = 0$$

This latter equation is obtained as follows. If angle 1 was corrected by say + 1″, then the correction to its log sin would be +0.000 031 7, as shown in the question. Thus, in general terms a correction of v_1'' to angle 1 would result in a correction of d_1v_1 to its log sin, where d_1 is the *difference in the* log sin for a change of v_1 in the angle. The side condition is therefore written

$$\log\sin 1 + d_1v_1 + \log\sin 3 + d_3v_3 + \log\sin 5 + d_5v_5 + \log\sin 7 + d_7v_7 - (\log\sin 2 + d_2v_2 + \log\sin 4 + d_4v_4 + \log\sin 6 + d_6v_6 + \log\sin 8 + d_8v_8)$$

However, as the $\sum$ log sin 'odds' $-\sum$ log sin 'evens' $= E$, then the above equation may be rewritten

$$d_1v_1 - d_2v_2 + d_3v_3 - d_4v_4 + d_5v_5 - d_6v_6 + d_7v_7 - d_8v_8 \pm E = 0$$

Substituting for d and E from the question, gives the condition equation quoted on p. 59.

For convenience, the condition equations are rewritten here multiplied by their appropriate correlative, etc.

$$K_1(v_1+v_2+v_3+v_4+v_5+v_6+v_7+v_8-12'') = 0 \quad (1.55a)$$
$$K_2(v_2+v_3-v_6-v_7+14'') = 0 \quad (1.55b)$$
$$K_3(v_1+v_8-v_4-v_5+14'') = 0 \quad (1.55c)$$
$$K_4(317v_1-142v_2+202v_3-202v_4+346v_5-358v_6+69v_7-230v_8-86) = 0 \quad (1.55d)$$

Then by direct inspection, as shown in *Section 1.11.8*, the normal equations are

$$8K_1 + 0 + 0 + 2K_4 - 12'' = 0$$
$$0 + 4K_2 + 0 + 349K_4 + 14'' = 0$$
$$0 + 0 + 4K_3 - 57K_4 + 14'' = 0$$
$$2K_1 + 349K_2 - 57K_3 + 507\,802K_4 - 86 = 0$$

The final value of 507 802 is of course the sum of the squares of the coefficients of equation (1.55d). Students should be very careful with negative quantities; for instance $K_2[bd]$ is obtained from the product of the coefficients of equations (1.55b and d) as

$$(1 \times -142) + (1 \times 202) + (-1 \times -358) + (-1 \times 69) = 349$$

The student should study *Section 1.11.8* carefully whilst deducing the normal equations. Also from the condition equations, it can be seen that

$$v_1 = K_1 + K_3 + 317K_4 \qquad v_5 = K_1 - K_3 + 346K_4$$
$$v_2 = K_1 + K_2 - 142K_4 \qquad v_6 = K_1 - K_2 - 358K_4$$
$$v_3 = K_1 + K_2 + 202K_4 \qquad v_7 = K_1 - K_2 + 69K_4$$
$$v_4 = K_1 - K_3 - 202K_4 \qquad v_8 = K_1 + K_3 - 230K_4$$

(2) Weights (tabular form, refer to *Table 1.4*)

(a) The condition equations are obtained in exactly the same way as in method (1) (see equations (1.55a to d)).

(b) Their coefficients are entered vertically in the appropriate column.

(c) The coefficients are then collected according to the appropriate heading, row by row.

(d) The relationship of the correction v to the correlate K is obtained in the usual way from the condition equations and is, of course, identical to that in method (1), but must be multiplied by the inverse of its weight, as shown in *Section 1.11.9*.

Example 1.21. A part of a triangulation scheme consists of a polygon $ABCDE$ within which is a station F. The measured angles are given below, together with the log sins of the outer angles.

TABLE 1.4

V	$1/w \times C = W$	a	b	c	d	Waa	Wab	Wac	Wad	Wbb	Wbc	Wbd	Wcc	Wcd	Wdd
v_1	$\frac{1}{2} \times 6 = 3$	1		1	317	3	0	3	951	0	0	0	3	951	301 467
v_2	3	1	1		−142	3	3	0	−426	3	0	−426	0	0	60 492
v_3	6	1	1		202	6	6	0	1212	6	0	1212	0	0	244 824
v_4	6	1		−1	−202	6	0	−6	−1212	0	0	0	6	1212	244 824
v_5	3	1		−1	346	3	0	−3	1038	0	0	0	3	−1038	359 148
v_6	3	1	−1		−358	3	−3	0	−1074	3	0	1074	0	0	384 492
v_7	2	1	−1		69	2	−2	0	138	2	0	−138	0	0	9 522
v_8	2	1		1	−230	2	0	2	−460	0	0	0	2	−460	105 800
					[ ] =	28	4	−4	167	14	0	1722	14	665	1 710 569

Normal equations

$$28K_1 + 4K_2 - 4K_3 + 167K_4 - 12'' = 0$$
$$4K_1 + 14K_2 + 0 + 1722K_4 + 14'' = 0$$
$$-4K_1 + 0 + 14K_3 + 665K_4 + 14'' = 0$$
$$167K_1 + 1722K_4 + 665K_3 + 1\,710\,569K_4 - 86 = 0 \quad \text{(Note symmetry)}$$

Number	*Measured angle*	°	′	″	Log sin		Log sin *difference for* 1″
1	*BAF*	38	44	54	$\bar{1}.796\ 505\ 5$		0.000 002 6
2	*FBA*	83	48	01		$\bar{1}.997\ 452\ 5$	0.000 000 2
3	*CBF*	42	34	30	$\bar{1}.830\ 303\ 0$		0.000 002 3
4	*FCB*	60	11	18		$\bar{1}.938\ 351\ 7$	0.000 001 2
5	*DCF*	56	02	45	$\bar{1}.918\ 808\ 3$		0.000 001 4
6	*FDC*	37	44	14		$\bar{1}.786\ 780\ 4$	0.000 002 7
7	*EDF*	40	06	22	$\bar{1}.809\ 024\ 3$		0.000 002 5
8	*FED*	86	53	52		$\bar{1}.999\ 363\ 1$	0.000 000 1
9	*AEF*	70	05	48	$\bar{1}.973\ 251\ 8$		0.000 000 8
10	*FAE*	23	48	13		$\bar{1}.605\ 954\ 4$	0.000 004 8
11	*AFB*	57	27	01			
12	*BFC*	77	14	17			
13	*CFD*	86	13	02			
14	*DFE*	52	59	48			
15	*EFA*	86	05	57			
					$\Sigma = 1.327\ 892\ 9$	1.327 902 1	

Derive the normal equations required for a least squares solution, assuming equal weights for all angles. The solution of the equations is *not* required. (LU)

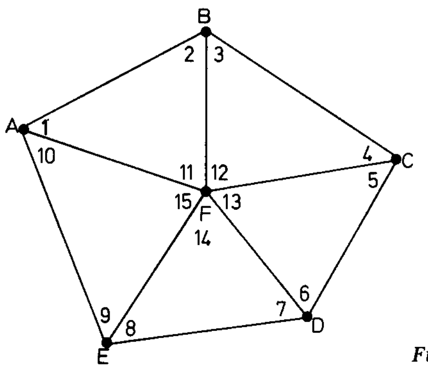

Figure 1.11

From *Figure 1.11*, the conditions of adjustment are

$$\text{Angles}\quad 1 + 2 + 11 = 180^\circ$$
$$3 + 4 + 12 = 180^\circ$$
$$5 + 6 + 13 = 180^\circ$$
$$7 + 8 + 14 = 180^\circ$$
$$9 + 10 + 15 = 180^\circ$$
$$11 + 12 + 13 + 14 + 15 = 360^\circ$$
$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = (2n - 4)90^\circ = 540^\circ$$
$$\sum \text{log sins 'odd'} = \sum \text{log sins 'even'}$$

To find the number of conditions required

Number of directly-observed quantities = 15
Number of independent unknowns = 8

∴ Number of conditions required = 7

Take any line, say AB, as a base; then the number of points to be fixed is four, i.e. C, D, E and F, each with an easting and northing value, giving eight independent unknowns. One may now select any seven of the eight conditions stipulated, but must include the side condition. Thus omitting the first condition, the following observation equations are formed as in the previous problem

$$K_1(v_3+v_4+v_{12}+5'')=0$$
$$K_2(v_5+v_6+v_{13}+1'')=0$$
$$K_3(v_7+v_8+v_{14}+2'')=0$$
$$K_4(v_9+v_{10}+v_{15}-2'')=0$$
$$K_5(v_{11}+v_{12}+v_{13}+v_{14}+v_{15}+5'')=0$$
$$K_6(v_1+v_2+v_3+v_4+v_5+v_6+v_7+v_8+v_9+v_{10}-3'')=0$$
$$K_7(26v_1-2v_2+23v_3-12v_4+14v_5-27v_6+25v_7-v_8+8v_9-48v_{10}-92)=0$$

Normal equations, using general equation (1.33)

$$3K_1+0+0+0+K_5+2K_6+11K_7+5''=0$$
$$0+3K_2+0+0+K_5+2K_6-13K_7+1''=0$$
$$0+0+3K_3+0+K_5+2K_6+24K_7+2''=0$$
$$0+0+0+3K_4+K_5+2K_6-40K_7-2''=0$$
$$K_1+K_2+K_3+K_4+5K_5+0+0+5''=0$$
$$2K_1+2K_2+2K_3+2K_4+0+10K_6+6K_7-3''=0$$
$$11K_1-13K_2+24K_3-40K_4+0+6K_6+5272K_7-92=0$$

Although this completes the problem as stated, the student should also attempt to derive the relationship between v and K, e.g.

$v_1=K_6+26K_7$ $v_2=K_6-2K_7$ $v_{10}=K_4+K_6-48K_7$ $v_4=K_1+K_6-12K_7$ etc.

A point to note in the adjustment of any polygon is that the angles at the centre point are *not* used in the side condition.

EXERCISES

1.1. Explain the meaning of the terms *random error* and *systematic error*, and show by example how each can occur in normal surveying work.

A certain angle was measured ten times by observer A with the following results, all measurements being equally reliable

74° 38′ 18″, 20″, 15″, 21″, 24″, 16″, 22″, 17″, 19″, 13″

(the degrees and minutes remained constant for each observation.)

The same angle was measured under the same conditions by observer B with the following results

74° 36′ 10″, 21″, 25″, 08″, 15″, 20″, 28″, 11″, 18″, 24″

Determine the standard deviation for each observer and the relative weightings. (ICE)

(*Answer:* $\pm 3.4''$; $\pm 6.5''$. A:B is 9:2)

1.2. Derive from first principles an expression for the standard error in the computed angle W_1 of a Weisbach triangle, assuming a standard error of σ_w in the Weisbach angle W, and equal proportional standard errors in the measurement of the sides. What facts, relevant to the technique of correlation using this method, may be deduced from the reduced error equation? (KP)

(*Answer:* see Volume 1)

1.3. What is the difference between an *error* and a *mistake*? How is *weighting* applied to observations?

Four bench marks A, B, C and D were established by precise levelling, the backsights and foresights being kept equal in length. The Table shows the readings obtained, the distances and the number of times the levelling was carried out. If the level of A was 27.091 m AOD, find the probable values of the other points (a) not weighted and (b) weighted. (LU)

Line	*Change* (m)	*Distance* (km)	*Levelled* (number of times)
A to B	rise 6.254	distance 4	once
B to C	rise 5.316	distance 5	twice
C to D	rise 4.639	distance 3	once
D to B	fall 9.970	distance 6	twice
C to A	fall 11.558	distance 6	twice

(*Answer:* (a) B = 33.339 m; C = 38.655 m; D = 43.302 m; (b) 33.337 m; 38.655 m; 43.301 m)

1.4. The differences of level between four triangulation stations A, B, C and D as determined by trigonometric levelling are shown below together with the relative weights of the observations. The reduced level of A = 108.32 m. Calculate the most probable values of the reduced levels of stations B, C and D. (ICE)

Line	*Difference of level* (m)	*Weight of observation*
A to B	18.50	1
B to C	−11.42	2
C to D	5.93	3
D to A	−12.95	1
B to D	−5.64	3

(*Answer:* 126.84 m, 115.36 m and 121.25 m)

1.5. From a station O, the angles to five other stations P, Q, R, S and T were observed as tabulated below; certain groups of angles were read again, the weights to be assigned to the observations being as given in the Table.

Angle	Number	Angle magnitude ° ′ ″	Weight
POQ	1	54 27 34	2
QOR	2	73 21 43	2
ROS	3	86 17 22	2
SOT	4	79 14 35	2
TOP	5	66 38 47	2
POS	6	214 06 38	3
SOP	7	145 53 26	5

Find the most probable values of the angles to the nearest second. (LU)

(*Answer:* hint: 5 is a dependent quantity; examination of normal equations quickly shows $v_1 = v_2 = v_3 = -1''$; $\therefore\ v_4 = 1''$)

1.6. Part of a triangulation scheme consists of a triangle *ABC* with a central point *D*. The measured angles are given below together with the log sins of the outer angles. State the conditions of adjustment of the figure. Thereafter select the exact number of conditions needed, and derive the normal equations required for a least square adjustment of the figure using the method of correlatives. Do not attempt to solve the equations but derive the linear relationship between the correlatives and the residual errors. (KP)

Number	Angle	Angle magnitude ° ′ ″	Log sins		Difference for 1″
1	*DAB*	23 02 45.4	$\bar{1}$.592 697 6		49.5
2	*ABD*	67 43 16.1		$\bar{1}$.966 305 9	8.6
3	*CBD*	37 10 55.4	$\bar{1}$.781 288 2		27.8
4	*BCD*	24 12 01.0		$\bar{1}$.612 707 0	46.8
5	*ACD*	17 12 50.8	$\bar{1}$.471 208 5		68.0
6	*CAD*	10 38 08.3		$\bar{1}$.266 144 3	112.1
7	*ADB*	89 13 57.6			
8	*BDC*	118 37 05.7			
9	*ADC*	152 08 56.7			
			Sum = $\bar{1}$.845 194 3	$\bar{1}$.845 157 2	

(*Answer:* five conditions—answer will vary according to the conditions chosen)

1.7. Five triangulation stations *ABCDE* are in the form of a quadrilateral *ABCD* with an internal station *E*. The measured angles are as follows:

Angle	Measured value ° ′ ″	Log sin	Difference for 1″
EAB	30 59 14	$\bar{1}$.711 678 0	0.000 003 5
ABE	44 41 11	$\bar{1}$.847 094 8	0.000 002 1
EBC	61 21 56	$\bar{1}$.943 343 6	0.000 001 2
BCE	55 19 07	$\bar{1}$.915 045 6	0.000 001 5
ECD	58 01 51	$\bar{1}$.928 566 5	0.000 001 3
CDE	45 54 45	$\bar{1}$.856 292 6	0.000 002 0
EDA	33 16 36	$\bar{1}$.739 321 0	0.000 003 2
DAE	30 25 22	$\bar{1}$.704 473 7	0.000 003 6
BEA	104 19 38		
CEB	63 18 58		
DEC	76 03 41		
AED	116 17 43		

Derive the normal equations required to determine the errors in the observed values by the method of least squares. Assume all readings to be of equal weight and neglect spherical excess. The equations need not be solved. (LU)

(*Answer:* six conditions—answer will vary according to the conditions chosen)

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2

Control surveys

The establishment of three-dimensional control networks for major construction schemes, such as tunnels, bridges, hydroelectric schemes, etc., is generally carried out by:

(1) Triangulation.
(2) Trilateration.
(3) A combination of (1) and (2).
(4) Electromagnetic distance-measurement (EDM) traversing.

In modern engineering surveying, triangulation, once the most popular technique, is rapidly being replaced by EDM traversing. This latter method is proving just as accurate and far more economical. Trilateration, although theoretically sound, is not so widely used, due probably to lack of easy checks and more tedious computations. The third possibility (3) is being used quite extensively to afford greater control of scale error in triangulation.

2.1 TRIANGULATION

Although the areas involved in construction are relatively small compared with national surveys (resulting in the term *microtriangulation*) the accuracy required in establishing the control surveys is frequently of a very high order, e.g. long tunnels or dam deformation measurements.

The principles of the method are illustrated by the typical basic figures shown in *Figure 2.1*. If all the angles are measured, then the scale of the network is obtained by the measurement of one side only, i.e. the base line. Any error therefore, in the measurement of the base line will result in scale error throughout the network. Thus, in order to control this error, check-base lines should be measured at intervals. The scale error is defined as the difference between the measured and computed check base. Using the base line and adjusted angles the remaining sides of the triangles may be found and subsequently the co-ordinates of the control stations.

Triangulation is best suited to open, hilly country, affording long sights, well clear of intervening terrain. In urban areas, roof-top triangulation is used, in which the control stations are situated on the roofs of accessible buildings.

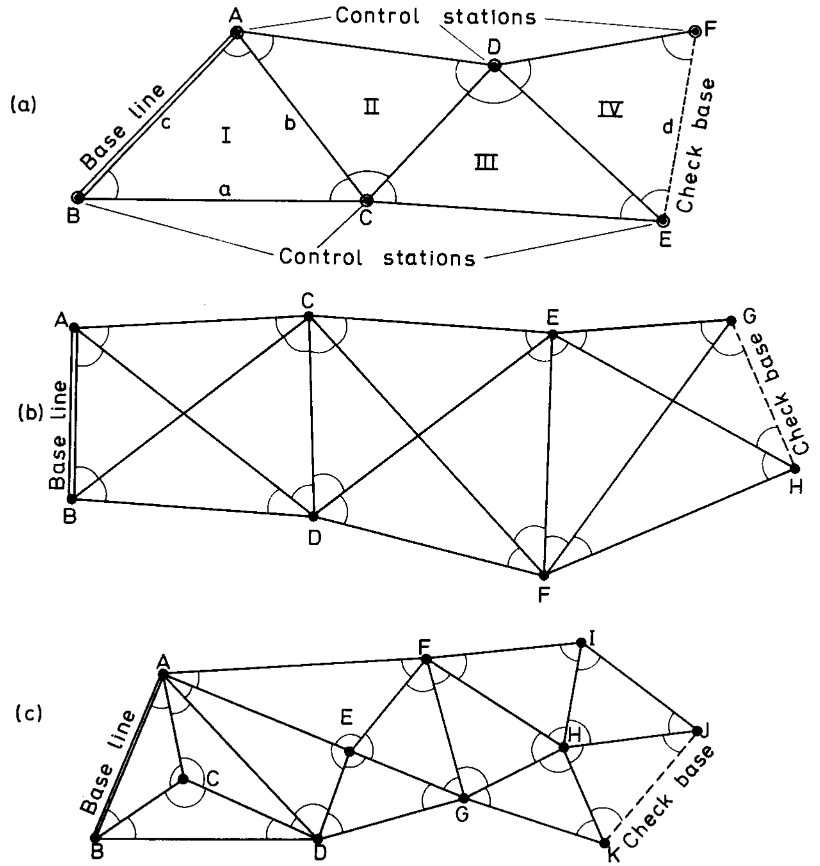

Figure 2.1 (a) Chain of simple triangles, (b) braced quadrilaterals and (c) polygons with central points

2.1.1 Shape of the triangle

The sides of the network are computed by the sine rule. From triangle *ABC* in *Figure 2.1(a)*

$$\log b = \log c + \log \sin B_1 - \log \sin C_1$$

The effect on side *b* of errors in the measurement of angles *B* and *C* is found in the usual way. Consider an error δb in side *b* due to an angular error δB in the measurement of angle *B*, then

$$\frac{\delta b}{b} = \delta B \cot B$$

Similarly for an error δC in angle *C*

$$\frac{\delta b}{b} = -\delta C \cot C$$

If we regard the above errors as standard errors and combine them the result is

$$\frac{\sigma_b}{b} = \{(\sigma_B \cot B)^2 + (\sigma_C \cot C)^2\}^{\frac{1}{2}}$$

Further, assuming equal angular errors, i.e. $\sigma_B = \sigma_C = \sigma$ rad, then

$$\frac{\sigma_b}{b} = \sigma(\cot^2 B + \cot^2 C)^{\frac{1}{2}} \tag{2.1}$$

Equation (2.1) indicates that as angles B and C approach 90°, the effect of angular error on the computed side b would be a minimum. Thus the ideal network for *Figure 2.1(a)* would be to have very small angles opposite the sides which do not enter into the scale error computation, i.e. sides BC, AD, CE and DF. Such a network would not, however, be a practical proposition due to the very limited ground coverage, and the best compromise is the use of equilateral triangles where possible. If small angles are inevitable and cannot be fixed so as not to enter the scale computation, they should be measured with extra precision.

Assuming now that $B = C = 60°$ and $\sigma'' = \pm 1''$, then as $\cot 60° = 3^{-\frac{1}{2}}$ and σ rad $\approx$ 1/200 000

$$\frac{\sigma_b}{b} = \frac{1}{200\,000}\left(\frac{2}{3}\right)^{\frac{1}{2}} = \frac{1}{245\,000}$$

After n triangles the error will be $n^{\frac{1}{2}}$ times the error in each triangle

$$\therefore \frac{\sigma_b}{b} = \frac{n^{\frac{1}{2}}}{245\,000}$$

Thus, after say nine triangles, the scale error would be approximately 1/82 000. This result indicates the need for maximum accuracy in the measurement of the base line and angles, as well as the need for regular check bases and well-conditioned triangles.

It can be shown that when the angles are adjusted, equation (2.1) becomes

$$\frac{\sigma_b}{b} = \sigma\{\tfrac{2}{3}(\cot^2 B + \cot B \cot C + \cot^2 C)\}^{\frac{1}{2}}$$

which theoretically shows no improvement in the scale error if $B = C$.

2.1.2 General procedure

(1) Reconnaissance of the area, to ensure the best possible positions for stations and base lines.
(2) Construction of the stations.
(3) Consideration of the type of target and instrument to be used and also the method of observation. All of these depend on the precision required and the length of sights involved.
(4) Observation of angles and base-line measurements.
(5) Computation—base-line reduction, station and figural adjustment, co-ordinates of stations by direct methods.

A general introduction to triangulation has been presented, aspects of which will now be dealt with in detail.

2.1.3 Catenary base lines

In order to achieve the accuracy required, base lines are measured using either steel or invar types in catenary, or EDM equipment. As EDM equipment becomes more sophisticated and accurate, the use of catenary methods will obviously become less popular. In fact the remarkable accuracies claimed by such short-range equipment as the Kern Mekometer would appear to render catenary methods obsolete. Nevertheless, catenary methods are still in use, and a thorough investigation of their associated systematic errors can only be beneficial to the student.

Figure 2.2 illustrates the technique of measuring in catenary. The tape is suspended clear of the ground over the two measuring heads. The required tension is applied either

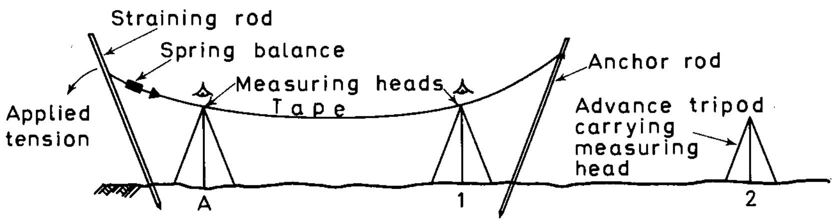

Figure 2.2

by means of a spring balance attached to a ranging rod or by weights suspended over a pulley while the other end is anchored. Both ends of the tape are read off simultaneously against index marks on the measuring heads, at the exact instant the required tension is reached. The procedure is repeated several times until a sufficient sample of reliable results is obtained. To eliminate systematic errors, the temperature, tension, difference in height of the measuring heads, and if necessary the mean level of the line above Ordnance Datum (OD) are taken. Each bay of the line is then reduced to the horizontal as described in *Sections 2.1.3.1* to *2.1.3.7* following.

2.1.3.1 Standardization

During a period of use, a tape will gradually alter in length for a variety of reasons. The amount of change can be found by having the tape standardized at either the National Physical Laboratory (NPL) (invar) or the Department of Trade and Industry (DTI) (steel), or by comparing it with a reference tape kept purely for this purpose. The tape may then be specified as being 30.003 m at 20°C and 10 kg straining mass (tension), or as 30 m exactly at a temperature other than standard.

N.B. The tension applied to a tape should be expressed in newtons (N), the SI unit of force. The spring balances used in the field, however, are graduated in kilograms (kg), a unit of mass; hence the use of the term *straining mass.*

Example 2.1. A distance of 220.450 m was measured with a steel band of nominal length 30 m. On standardization the tape was found to be 30.003 m. Calculate the correct measured distance, assuming the error is evenly distributed throughout the tape.

Error per 30 m = 3 mm

$$\therefore \text{ Correction for total length} = \left(\frac{220.450}{30}\right) \times 3 \text{ mm} = 22 \text{ mm}$$

$\therefore$ Correct length is 220.450 + 0.022 = 220.472 m

Student notes

(1) *Figure 2.3* shows that when the tape is too long, the distance measured appears too short, and the correction is therefore positive. The reverse is the case when the tape is too short.
(2) When setting out a distance with a tape the rules in (1) are reversed.
(3) It is better to compute *Example 2.1* on the basis of the correction (as shown), rather than the total corrected length. In this way fewer significant figures are used.

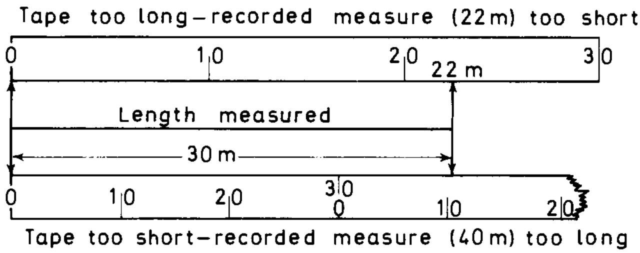

Figure 2.3

Example 2.2. A 30-m band standardized at 20°C was found to be 30.003 m. At what temperature is the tape exactly 30 m? Coefficient of expansion of steel = 0.000 011/°C.

Expansion per 30 m per °C = 0.000 011 × 30 = 0.000 33 m
Expansion per 30 m per 9°C = 0.003 m
$\therefore$ Tape is 30 m at 20°C − 9°C = 11°C

Alternatively, using equation (2.2) where $\Delta t = (t_s - t_a)$, then

$$t_a = -\frac{C_t}{KL} + t_s = -\left(\frac{0.003}{0.000\,011 \times 30}\right) + 20° = 11°\text{C}$$

where t_a = actual temperature and t_s = standard temperature.
This then becomes the standard temperature for future temperature corrections.

2.1.3.2 Temperature

Tapes are usually standardized at 20°C. Any variation above or below this value will cause the tape to expand or contract, giving rise to systematic errors. The difficulty of obtaining the true temperature of the tape resulted in the use of invar tapes. Invar is a nickel–steel alloy with a very low coefficient of expansion.

Coefficient of expansion of steel	$K = 11.2 \times 10^{-6}$ per °C	
Coefficient of expansion of invar	$K = 0.5 \times 10^{-6}$ per °C	
Temperature correction	$C_t = KL\Delta t$	(2.2)

where L = measured length (m) and Δt = difference between the standard and field temperatures (°C).

The sign of the correction is in accordance with the rule specified in (1) of the student notes mentioned earlier.

The effect of an error in the measurement of temperature can be obtained by differentiating equation (2.2)

$$\delta C_t = KL\delta\Delta t$$

thus if $L = 30$ m, $\delta\Delta t = 2$°C and $K = 11 \times 10^{-6}$/°C

$$\delta C_t = 11 \times 10^{-6} \times 30 \times 2 = 0.000\,66 \text{ m}$$

which in 30 m = 1 in 45 000.

2.1.3.3 Tension

Generally the tape is used under standard tension, in which case there is no correction. It may, however, be necessary in certain instances to apply a tension greater than standard. From Hooke's law

stress = strain × a constant

This constant is the same for a given material and is called the *modulus of elasticity* (E). Since strain is a non-dimensional quantity, then E has the same dimensions as stress, i.e. N/mm².

$$\therefore \quad E = \frac{\text{Direct stress}}{\text{Corresponding strain}} = \frac{\Delta T}{A} \div \frac{C_T}{L}$$

$$\therefore \quad C_T = L \times \frac{\Delta T}{AE} \qquad (2.3)$$

ΔT is normally the total stress acting on the cross section but as the tape would be standardized under tension, then ΔT in this case, is the amount of stress *greater* than standard. Therefore ΔT is the difference between field and standard tension. This value is normally measured in the field in kg and should be converted to newtons (N) for compatibility with the other units used in the formula, i.e. 1 kgf = 9.806 65 N.

E is modulus of elasticity in N/mm²; A is cross-sectional area of the tape in mm²; L is measured length in m; and C_T is the extension and thus correction to the tape length in m. As the tape is stretched under the extra tension, the correction is positive.

Errors in tensioning can arise due to (i) index-error of the spring balance; (ii) reading error; (iii) graduation—balances are generally graduated to 0.2 kg only. From equation (2.3)

$$\delta C_T = \frac{L\delta T}{AE}$$

Assuming a tensioning error of $\delta T = 0.5$ kg, $A = 3$ mm², $E = 210 \times 10^3$ N/mm² (210 kN/mm²) and $L = 30$ m

$$\therefore \quad \delta C_T = \frac{30 \times 0.5 \times 9.81}{3 \times 210 \times 10^3} = 1 \text{ in } 128\,000$$

It can be seen from the derived equation that δC_T is directly proportional to δT and inversely proportional to A. Thus any increase in the tensioning error or decrease in the cross-sectional area will result in a directly proportional increase in δC_T. Also as shown in the next Section, errors in the tension affect the sag correction.

2.1.3.4 Sag

When a tape is suspended between two measuring heads, A and B, both at the same level, the shape it takes up is a catenary (*Figure 2.4*). If C is the lowest point on the curve,

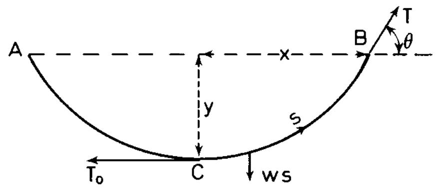

Figure 2.4

then on length CB there are three forces acting, namely the tension T at B, T_0 at C and the mass of portion CB, where w is the mass per unit length and s is the arc length CB. Thus CB must be in equilibrium under the action of these three forces. Hence

Resolving vertically $\quad T \sin\theta = ws$

Resolving horizontally $\quad T\cos\theta = T_0$

$$\therefore \tan\theta = \frac{ws}{T_0}$$

For a small increment of the tape

$$\frac{dx}{ds} = \cos\theta = (1 + \tan^2\theta)^{-\frac{1}{2}} = \left(1 + \frac{w^2s^2}{T_0^2}\right)^{-\frac{1}{2}} = \left(1 - \frac{w^2s^2}{2T_0^2}\cdots\right)$$

$$\therefore x = \int\left(1 - \frac{w^2s^2}{2T_0^2}\right)ds$$

$$= s - \frac{w^2s^3}{6T_0^2} + K$$

When $x = 0$, $s = 0$, $\therefore K = 0 \qquad \therefore x = s - \dfrac{w^2s^3}{6T_0^2}$

The sag correction for the whole span $\qquad ACB = C_s = 2(s - x) = 2\left(\dfrac{w^2s^3}{6T_0^2}\right)$

but $s = L/2 \qquad \therefore C_s = \dfrac{w^2L^3}{24T_0^2} = \dfrac{w^2L^3}{24T^2}$ for small values of θ (2.4)

i.e. $T\cos\theta \approx T \approx T_0$

where w = mass per unit length (kg/m)
T = straining mass (kg)
L = recorded length (m)
C_s = correction (m)

N.B. In equation (2.4) care must be taken to ensure that T and w are in compatible units; thus T may remain in kg.

As $w = W/L$, where W is the total mass of the tape, then by substitution in equation (2.4)

$$C_s = \frac{W^2 L}{24T^2} \tag{2.5}$$

Although this equation is correct, the sag correction is proportional to the cube of the length.

Equations (2.4) and (2.5) apply only to tapes standardized on the flat and are always negative. When a tape is standardized in catenary, i.e. it records the horizontal distance when hanging in sag, no correction is necessary provided the applied tension, say T_A, equals the standard tension T_S. Should the tension T_A exceed the standard, then a sag correction is necessary for the excess tension $(T_A - T_S)$ and

$$C_s = \frac{w^2 L^3}{24}\left(\frac{1}{T_A^2} - \frac{1}{T_S^2}\right) \tag{2.6}$$

In this case the correction will be positive, in accordance with the basic rule. The sag y of the tape may also be found as follows:

$$\frac{dy}{ds} = \sin\theta \approx \tan\theta = \frac{ws}{T_0}, \text{ when } \theta \text{ is small}$$

$$\therefore\; y = \int \frac{ws}{T_0}\, ds = \frac{ws^2}{2T_0}$$

If y is the maximum sag at the centre of the tape, then

$$s = \frac{L}{2} \quad \text{and} \quad y = \frac{wL^2}{8T} \tag{2.7}$$

Equation (2.7) enables w to be found from field measurement of sag, i.e.

$$w = \frac{8Ty}{L^2}$$

which on substitution in equation (2.4) gives

$$C_S = -\frac{8y^2}{3L} \tag{2.8}$$

Equation (2.8) gives the sag correction by measuring sag y and is independent of w and T. The effect of error in tension can be obtained by differentiating equation (2.4) as follows:

$$\delta C_S = -\left(\frac{2L^3 w^2}{24T^3}\right)\delta T$$

Assuming $\delta T = 0.5$ kgf, $w = 0.03$ kgf/m, $T = 10$ kgf and $L = 30$ m, then

$$\delta C_S = -\left(\frac{2 \times 30^3 \times 0.03^2 \times 0.5}{24 \times 10^3}\right) = 1 \text{ in } 30\,000$$

This error will be compounded with the error in tension (p. 72), whilst errors in obtaining the weight and cross-sectional area of the tape should also be considered.

2.1.3.5 Slope

If the difference in height of the two measuring heads is h, the slope distance L and the horizontal equivalent D, then by Pythagoras

$$D = (L^2 - h^2)^{\frac{1}{2}} \qquad (2.9a)$$

Prior to the use of pocket calculators the following alternative approach was generally used, due to the tedium of obtaining square roots

$$\therefore \; D = (L^2 = h^2)^{\frac{1}{2}} = L\left(1 - \frac{h^2}{L^2}\right)^{\frac{1}{2}} = L\left(1 - \frac{h^2}{2L^2} - \frac{h^4}{8L^4}\right)$$

$$\therefore \text{ Slope correction} \qquad C_h = D - L = -\left(\frac{h^2}{2L} + \frac{h^4}{8L^3}\right) \qquad (2.9b)$$

The use of Pythagoras (equation (2.9a)) is advocated due to the small error that can arise when using only two terms of the above expansion on long lines measured by EDM.

Considering errors in the measurement of h

$$\delta C_h = \frac{h\delta h}{L} \quad \text{(error directly proportional to } h\text{)}$$

If $L = 30$ m, $h = 0.500$ m and $\delta h = 0.002$ m, then

$$\delta C_h = \frac{0.500 \times 0.002}{30} = 1 \text{ in } 900\,000$$

2.1.3.6 Altitude

If the surveys are to be connected to the national grid (NG), the distances will need to be reduced to the common datum of that system, namely mean sea level (MSL). Alternatively, if the engineering scheme is of a local nature, distances may be reduced to the mean level of the area. This has the advantage that setting out distances on the ground are, without sensible error, equal to distances computed from co-ordinates in the mean datum plane.

Consider *Figure 2.5*, in which a distance L is measured in a plane situated at a height H above MSL.

$$\text{By similar triangles} \qquad M = \frac{R}{R + H} \times L$$

$$\therefore \text{ Correction} \qquad C_M = L - M = L - \frac{RL}{R + H} = L\left(1 - \frac{R}{R + H}\right) = \frac{LH}{R + H}$$

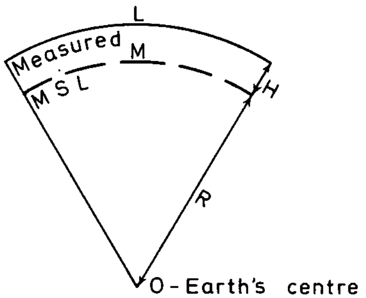

Figure 2.5

As H is negligible compared with R in the denominator

$$C_M = \frac{LH}{R} \tag{2.10}$$

The correction is negative for surface work but may be positive for tunnelling or mining work below MSL. By differentiating equation (2.10) it is seen that the error in C_M is directly proportional to the error in H.

2.1.3.7 Scale factor (SF)

This correction is required only when the survey is connected to the national grid. All distances measured on the ground should be reduced to MSL and then multiplied by the SF for the area, to transform them to grid distances on the *transverse Mercator projection* (see *Section 2.12*). Grid distances are divided by the SF to give distances at MSL. An approximate equation for SF is given in *Section 2.14*.

2.2 ELECTROMAGNETIC DISTANCE-MEASUREMENT (EDM)

Distance-measurement by electromagnetic means has virtually replaced the method of measuring base lines using steel or invar tapes. The advent of EDM equipment has completely revolutionized all surveying procedures and resulted in a change of emphasis and technique, by reason of the fact that distance can now be measured quickly and accurately, regardless of terrain conditions. Examples of resultant procedures are

(a) The easy inclusion of many more base lines in triangulation for the greater control of scale error.
(b) The use of trilateration in which all the sides of the network are measured.
(c) The combination of triangulation with trilateration resulting in a procedure called triangulateration which produces very strong networks.
(d) Traversing on a grandiose scale and with much greater control of swing errors.
(e) Setting-out and photogrammetric control by polar co-ordinates from a single position.
(f) Off-shore position fixing by such techniques as the Tellurometer Hydrodist System.
(g) Three-dimensional trilateration from ground to air by, say, the Tellurometer Aerodist System.

EDM has not only resulted in new techniques, it has also enhanced the accuracy of linear measurement which is now commensurate with the most accurate forms of angular measurement. For instance, the Kern Mekometer quotes an accuracy of $\pm(0.2\text{ mm} + 3.10^{-6}D)$ for distances (D) up to 2.5 km.

The latest developments in instrumentation have integrated electronic digital theodolites with EDM units, thereby providing a single instrument, called a *total station*, capable of performing most surveying and setting-out tasks with economy in manpower, time, speed and reliable accuracy, plus the provision of a data bank. The automatic recording units of these instruments can transfer all data, including point identification, on to punched or magnetic tape or electronic data logger for subsequent interfacing with micro or main frame computers. The computers may in turn be interfaced to a plotter for the automatic production of plans.

2.2.1 Classification of EDM instruments

Equipment in use at the present time falls into three broad categories of operational range

(a) Short-range, electro-optical instruments which use amplitude modulated light, either white light or infra-red for measuring distances up to 5 km.
(b) Medium-range, microwave or electro-optical with ranges up to 25 km.
(c) Long-range radio-wave instruments capable of ranges up to and beyond 100 km.

The engineer is generally concerned only with the short-range equipment which is simple to use and provides a digital readout of the slope distance measured. This is the most basic output supplied and is common to all the wide variety of instruments available. Further refinements are the output of horizontal and vertical distances from some instruments, which sense the angle of inclination and apply appropriate corrections to the slope distance.

2.2.2 Measuring principle

Although there is a wide variety of equipment available the basic principle of operation is the same. Electromagnetic waves are transmitted from the instrument to a retro-reflector, which instantly returns them to the transmitting instrument. The instrument measures the time taken for the waves to travel this double path. Then, from distance (D) = velocity (V) × time (t), the *slant* distance between the instrument and reflector is obtained.

However, as the velocity of light (C) is equal to 299 792.5 ± 0.4 km/s (*in vacuo*), t is extremely small. Indeed for $D = 1$ km, t would be 6×10^{-6} s and to resolve D to 1 mm would require a time interval measurement of 6×10^{-12} s. To facilitate this, recourse is made to a technique called *phase measurement*, which measures the amount by which the reflected wave is out of phase with the transmitted wave, when it is received back at the instrument (*Figure 2.6*).

Any periodic phenomenon which oscillates regularly between maximum and minimum values may be analysed as a simple harmonic motion. With reference to *Figure 2.6*, if P moves in a circle with a constant angular velocity W, the radius vector R makes a phase angle ϕ with the x-axis. A graph of the y values equal to $R \sin \phi$ plotted

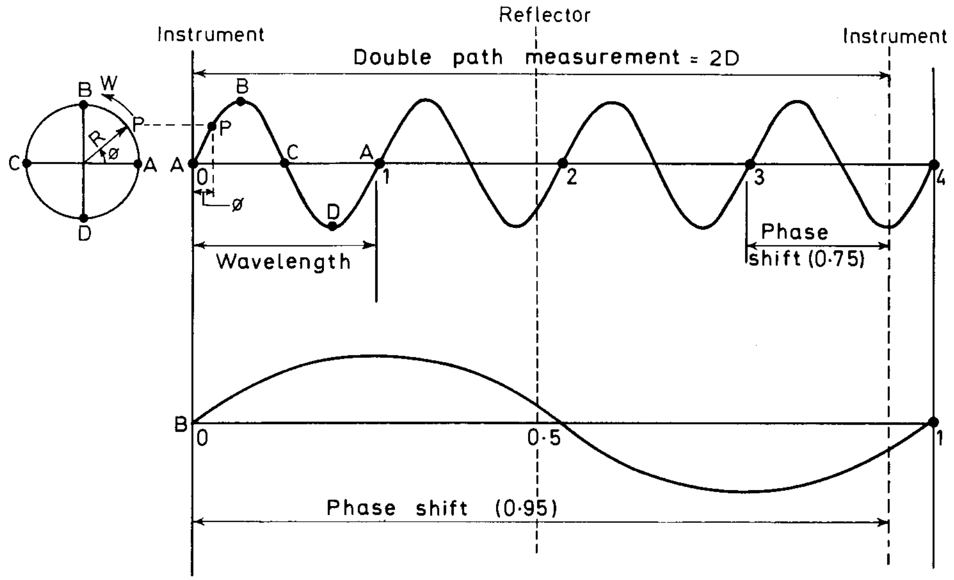

Figure 2.6

against ϕ produces the sine wave illustrated. When $\phi = \pi/2$ the plotted point is at B, for π at C, 1.5π at D and 2π (360°) back to A, completing a single wavelength (λ). The time taken for R to make one complete revolution or cycle is the *period* of the oscillation and is represented by T seconds. Thus since R moves with a constant angular velocity, the phase angle ϕ may be used to measure time or any fraction of time. The number of revolutions per second at which the radius vector rotates is called the frequency f and is related to the wavelength λ by the equation

$$V = f\lambda \tag{2.11}$$

where λ = the wavelength in m, f = the frequency of the energy in hertz and V = the velocity of electromagnetic energy in m/s.

The velocity of electromagnetic energy *in vacuo* is designated by C and $V = C/n$, where n is the atmospheric index of refraction.

With reference to *Figure 2.6*, it can be seen that the distance measured from instrument to reflector and back is

$$2D = N\lambda + \delta\lambda \tag{2.12}$$

where N = an integer number of wavelengths (λ) and $\delta\lambda$ = a fraction of the wavelength = $(\phi/2\pi)\lambda$.

EDM instruments measure only the *fraction* of the wavelength, they do not measure the *number* of wavelengths. This is obtained by transmitting energy of lower frequency and larger wavelength. For example, in *Figure 2.6*, using λ_A the double-path measurement is 3.75 wavelengths. The instrument, however, will simply record the phase angle or phase shift measurement of 0.75, i.e. the difference in phase of the outgoing and reflected incoming waves.

If the distance is now measured using λ_B, which is four times greater than λ_A, the phase is 0.95. As the relationship of A to B is known then $0.95 \times 4 = 3.80$ and gives the whole number of wavelengths = 3. The smaller wavelength will give a more accurate measurement of the remaining portion of the distance, thus total distance = 3 + 0.75 = $3.75\lambda_A$ and knowing the value of λ_A in units of length, the distance is ascertained. This then is the basic principle of EDM measurement called 'phase comparison' and may be expressed from equation (2.12) as

$$D = N(\lambda/2) + \frac{\phi}{2\pi}(\lambda/2) \tag{2.13}$$

from which it can be seen that $\lambda/2$ may be considered as the basic measuring unit.

Implicit in the above explanation is the assumption that λ is constant and known. However, in most EDM instruments (with the exception of the Kern Mekometer 3000) this is not so, only the frequency f is known. However, this is related to λ as follows:

The group refractive index n_g of the atmosphere through which the measuring beam passes is defined as the ratio of the velocity of electromagnetic energy *in vacuo* (C), over the velocity in the atmosphere (V), i.e.

$$n_g = C/V \qquad \text{thus as } V = f\lambda \qquad \therefore\ \lambda = C/fn_g$$

and

$$\lambda/2 = C/2fn_g \tag{2.14}$$

Substituting in equation (2.13) produces the distance-measuring equation

$$D = N\frac{C}{2fn_g} + \frac{\phi}{2\pi}\frac{C}{2fn_g} + k \tag{2.15}$$

where k = the instrument and prism constants.

The instrument constant is the difference between the instrument centre, as set up vertically over the survey station, and its measuring centre which is set at a different position in the instrument. The prism constant is the added distance which the beam travels through the prism on its way back to the transmitter. Both these constants are automatically corrected for in the measuring process. However, for precise work, or when a non-standard prism is used, it will be necessary to determine the constants experimentally.

Equation (2.15) also illustrates the importance of group refractive index in the measuring process and thus the need to take meteorological readings of temperature and atmospheric pressure, to achieve accurate distance measurements.

2.2.3 Reduction of EDM lines

All EDM instruments measure slant length, i.e. the distance from instrument to prism, which must be reduced to the horizontal equivalent, or perhaps its grid equivalent at mean sea level. The corrections are as follows:

(1) *Atmospheric effects*

The velocity of electromagnetic waves is affected by the atmospheric conditions through which they travel during the measuring process. Thus, from equation (2.11), as the frequency f is fixed, the wavelength λ will vary directly as V and the distance recorded by the instrument will require a correction, i.e.

$$V = C/n_g$$

In practice it is impossible to obtain group refractive index (n_g) for the atmosphere throughout the path of the measuring beam. An assessment is therefore made, based on temperature and pressure measurements at the instrument and prism (humidity is insignificant in the case of light wave instruments). The correction, in parts per million (ppm) of the distance measured is then taken from a nomogram supplied with the equipment. In some cases the atmospheric correction may be dialled into the instrument and the measured length automatically corrected.

(2) *Slope correction*
For the lengths generally encountered in engineering, the slope length, corrected for atmospheric conditions, is reduced to the horizontal by Pythagoras or by the cosine of the vertical angle. For maximum accuracy, the vertical angle should be corrected for the effects of curvature and refraction (refer *Section 2.8*).

In some cases the equipment automatically corrects for the vertical angle and displays the horizontal and vertical distances. The vertical angle used, however, is not always corrected for curvature and refraction. Where it *is* corrected, the correction is based on standard atmospheric conditions which may be different from those prevailing at the time of measurement. Where maximum accuracy is required, these facts should be considered.

(3) *Altitude correction*
Where the survey is reduced to a common datum, such as mean sea level or mean site level, the altitude correction (equation (2.10)) is applied as in *Section 2.1.3*.

(4) *Local scale factor* (*LSF*)
If the survey is to be connected into the national grid the horizontal lengths at MSL must be multiplied by the LSF (refer *Section 2.14*).

The above corrections are generally all that is required for the majority of lengths encountered in engineering surveys. However, for lengths in excess of 10 km it may be necessary to adopt the following approach:

(5) *Chord/arc correction*
Because the distance is measured (*Figure 2.7*) in one long length D, reduction to MSL gives the chord distance K, which must then be transformed to its equivalent spheroidal distance S. In triangle ABE using the cosine rule

$$\cos\phi = \frac{(R+H_2)^2 + (R+H_1)^2 - D^2}{2(R+H_2)(R+H_1)}$$

but $\cos\phi = 1 - 2\sin^2(\phi/2)$, and $\sin(\phi/2) = K/2R$

$$\therefore\ 1 - \left(\frac{K^2}{2R^2}\right) = \frac{(R+H_2)^2 + (R+H_1)^2 - D^2}{2(R+H_2)(R+H_1)}$$

from which $$K = R\left[\frac{\{D-(H_2-H_1)\}\{D+(H_2-H_1)\}}{(R+H_1)(R+H_2)}\right]^{\frac{1}{2}}$$

and, if h, equal to the difference in level $(H_2 - H_1)$ of A and B, is substituted

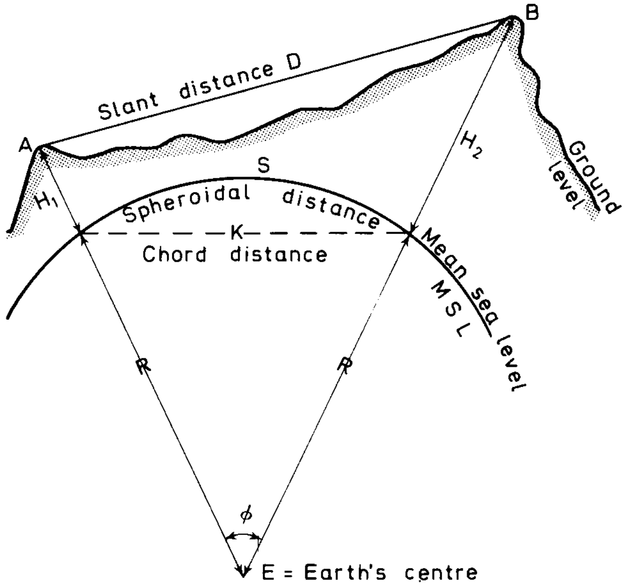

Figure 2.7

$$K = \left[\frac{(D-h)(D+h)}{\left(1+\left[\frac{H_1}{R}\right]\right)\left(1+\left[\frac{H_2}{R}\right]\right)}\right]^{\frac{1}{2}} \tag{2.16}$$

Now $\sin^{-1}\theta = \theta + \frac{\theta^3}{3!} + \frac{9\theta^5}{5!}$

then $\frac{\phi}{2} = \frac{S}{2R} = \sin^{-1}\frac{K}{2R}$

$$\sin^{-1}\left(\frac{K}{2R}\right) = \frac{K}{2R} + \frac{K^3}{8R^3 \times 3!} + \frac{9K^5}{32R^5 \times 5!} \quad \text{(for small values of } \phi\text{)}$$

$$\therefore \frac{S}{2R} = \frac{K}{2R} + \frac{K^3}{48R^3} + \frac{K^5}{3840R^5}$$

$$\therefore S = K + \left(\frac{K^3}{24R^2}\right) \quad \text{(further terms are negligible)} \tag{2.17}$$

Equation (2.17) may be further refined to allow for the fact that the electromagnetic waves travel in a curved path of radius greater than that of the Earth.

For radio waves it has been suggested that $4R/3$ should be used giving

$$S = K + \left(\frac{K^3}{43R^2}\right) \tag{2.18}$$

and for light waves:

$$S = K + \left(\frac{K^3}{38R^2}\right) \tag{2.19}$$

The value of 38 for light waves is dependent upon the coefficient of refraction at the time of measuring (see *Section 2.8.3*). For lines of up to 10 km in length, the chord/arc corrections are less than 1 mm.

The above reduction may also be carried out as follows:

(a) Reduce D to the horizontal D_1, using Pythagoras;
(b) Using D_1, reduce it to K at mean sea level using equation (2.10) where

$$H = \frac{H_1 + H_2}{2}$$

this is now the chord length K;
(c) To K add the chord/are correction, i.e. $K^3/24R^2$, refined as necessary for curvature, giving S.

2.2.4 Sources of error

Sources of error in the measurement of distance by EDM can be split into three basic categories, namely:

(1) *Zero errors*, or errors independent of the distance measured,
(2) *Cyclic errors*, or errors which vary in a periodic fashion with distance measured, and
(3) *Scale error*, or error proportional to the distance measured.

These in turn may be split into non-instrumental and instrumental.

An examination of the basic measuring equation (2.15) shows the main error sources to be:

(a) The modulation frequency (f),
(b) The group refractive index (n_g),
(c) The measurement of the phase angle (θ), and
(d) The additive constant (k),

where (a) and (b) form the main sources of instrumental and non-instrumental scale error, (c) the main instrumental source of cyclic error and (d) the main cause of zero error.

Other error sources which contribute to the final accuracy of the measurement include centring errors, pointing errors and errors in the additional field data necessary to reduce the measured slope length to its final horizontal length on a projection.

The standard error of distance (σ_D) measured by EDM is frequently quoted as

$$\pm\sigma_D = \pm[A^2 + B^2]^{\frac{1}{2}}$$

where A = zero error for EDM and B = proportional error for EDM in ppm of the distance measured.

An average value for most short-range equipment is $\pm$(5 mm + 5 ppm).

However, if the distance is to be reduced to the horizontal on, say, the national grid, a more correct value for standard error would be

$$\pm\sigma_D = \pm[A^2 + B^2 + E^2 + F^2 + G^2]^{\frac{1}{2}}$$

where E = errors due to slope reduction, F = error due to reduction to MSL and G = error due to reduction to the grid by the application of LSF.

It is thus important to examine all the error sources, not only for the better utilization of the equipment but also, in network adjustment, for the correct weighting of the distances in relation to the angles.

2.2.4.1 Modulation frequency (f)

As previously shown, the frequency f is directly related to the wavelength λ; thus, error in the modulation frequency will result in a proportional error in the distance measured.

The modulation frequency is fixed by means of a quartz crystal oscillator which ensures that the frequency remains stable to within ± 5 ppm over an operational temperature range of $-20°C$ to $+50°C$. The modulation frequency can, however, vary from its nominal value due to incorrect factory setting, ageing of the crystal and lack of temperature stabilization. Factory setting of the frequency should be within $\pm 1 \times 10^{-6}$ of the nominal value. Ageing of the crystals will result in frequency errors of about 1×10^{-6} per year, gradually decreasing with time (Hodges 1975). Since most instruments have crystals which operate at ambient temperature they claim no warm-up delay. However, tests have shown (Hodges 1975) that for precise work a warm-up period should be allowed if errors are not to be incurred.

The recommended method of determining the scale factor of an instrument is by measuring the modulation frequency direct, using an electronic frequency counter tuned to one of the standard frequencies, such as Droitwich (call sign MSF) on 200 kHz, which are continuously broadcast.

The alternative is to compare the instrument with another whose modulation frequency is known to be accurate. The procedure is to measure both long and short base lines (say to 1.5 km and 100 m) with both instruments at the same time. The difference in the measured length of the short line determines the difference in the zero constant for the two instruments. Then the measurements of the long lines are corrected accordingly and any residual discrepancy is attributed to the modulation frequency. Care must be taken to ensure equal instrument heights and the procedure repeated several times to ensure a precise mean value. Meteorological effects are common to both instruments and therefore need not be considered.

The correction for frequency error is equal to

$$\frac{\text{Nominal frequency} - \text{Actual frequency}}{\text{Nominal frequency}} \times 10^6 \text{ ppm} \qquad (2.20)$$

If it is suspected that frequency errors exist, the instrument should be returned to its manufacturer for adjustment.

2.2.4.2 Refractive index (n_g)

The ratio between the velocity of light *in vacuo* and the atmospheric velocity is expressed by the atmospheric refractive index n. This value is a constant dependent primarily an atmospheric temperature and pressure and frequently expressed as $(1 + 10^{-6}N)$.

It is unfortunate that various instruments adopt different values for N. For instance the modulation frequency chosen for the Tellurometer CD6 results in a correct direct readout of distance when N is 274 (i.e., $n = 1.000\,274$). This corresponds to the reasonably normal conditions at sea level in temperate countries of 1013 mb of atmospheric pressure and ambient temperature of 20°C. However, for the Wild DI 10 a value of $N = 282$ is used. Such considerations are important when writing computer programs for EDM reduction and when using the nomograms supplied with the equipment. For instance at 760 mm/Hg and 20°C the DI 10 nomogram correction is +8 mm, whilst the CD6 correction is zero. Thus nomograms supplied with various instruments are not interchangeable.

The refractive index of the dry atmosphere at standard temperature and pressure (0°C and 760 mm/Hg) is given by the Barrel and Sears (1939) formula adopted by the International Association of Geodesy (IAG) in 1963. Strictly speaking the formula applied only in the visible range and is not corrected for dispersion effects.

$$(No - 1) \times 10^6 = 287.604 + 1.6288\lambda^{-2} + 0.0136\lambda^{-4} \tag{2.21}$$

where λ = the wavelength of the electromagnetic wave in micrometres (μm) and No = absolute refractive index.

When the light is not homogeneous, as in the case of visible light and infra-red EDM instruments, the value for group refractive index (n_g) should be used and is equal to

$$(n_g - 1) \times 10^6 = 287.604 + (3 \times 1.6288)\lambda^{-2} + (5 \times 0.0136)\lambda^{-4}$$

For $\lambda = 0.90$ μm, $n_g = 1.000\,293\,7$, whilst for $\lambda = 0.93$ μm, $n_g = 1.000\,293\,3$.

The refractive index of the dry atmosphere is proportional to barometric pressure and inversely proportional to absolute temperature, the effect of water vapour generally being regarded as negligible for light-wave instruments. Of the three possible sources of error, temperature has the greatest influence. For instance, a 1 ppm increase in n_g, and hence a 1 ppm change in the measured distance, will result from a change of −1°C in temperature, +3.4 mb in atmospheric pressure and −26 mb in water vapour pressure (Hodges 1980).

In the general situation the atmospheric correction is taken from nomograms, tables or special refractive index slide rules. However, if maximum accuracy is required the atmospheric correction must be computed from first principles. Manufacturers generally quote the formula used in the construction of the nomogram, which is in itself an approximate formula based on an average value for humidity. It has been shown (Curl 1975) that using the formula or nomogram can result in an error of approximately 0.7 ppm compared with computing from first principles. This fact is also significant in those instruments which permit the dialling-in of refractive index from the instrument panel.

The importance of the atmospheric correction is indicated above and is clearly dependent on the accuracy with which temperature and pressure, along the line of measurement, can be obtained. Generally one assumes that the mean of meteorological readings taken at the instrument and reflector, represents average meteorological conditions along the wave path. Hodges (1975) indicates that on a 3-km test line, the error in the above assumption was +2°C and −2 mm Hg. Furthermore, laboratory calibration of the barometers showed a further error of −4 mm Hg, producing a combined error of 12 mm in 3 km. Thus, if this error is to be kept to a minimum it is imperative that all thermometers and barometers used should be carefully calibrated both before and after measuring. At the present time there is no easy solution to this difficult problem of accurately assessing meteorological conditions over the actual measuring path. It is imperative therefore if error from this source is to be minimized that the following procedure be adopted:

(a) Temperature and pressure measurement should be taken at each end of the measured line.
(b) The above measurements should be taken as high as possible (at least three metres above the ground) to avoid ground radiation effects and to reflect properly the mid-line meteorological conditions.
(c) The above measurements should be synchronized with the EDM measurements.
(d) If possible, mid-line meteorological readings should be taken.
(e) Thermometer and barometers should be of the highest quality and carefully calibrated against a reliable standard, before and after use.
(f) Ground grazing lines should be avoided.

A further effect of refraction is that of bending the measuring beam. This is negligible in engineering surveys, about 3 mm in 20 km (Hodges 1980), and will not be considered further.

2.2.4.3 Phase measurement error (θ)

As already shown, the measurement of the phase difference between the transmitted and received waves enables the fractional part of the wavelength to be determined. Thus, errors in the measurement of phase difference will produce errors in the measured distance. Phase errors are cyclic and not proportional to the distance measured, and may be non-instrumental and/or instrumental.

The non-instrumental cause of phase error is by spurious signals from reflective objects illuminated by the beam. Normally the signal returned by the reflector will be sufficiently strong to ensure complete dominance over spurious reflections. However, care should be exercised when using vehicle reflectors or Scotchlite for short-range work.

The main cause of phase error is instrumental and derives from two possible sources. In the first instance if the phase detector were to deviate from linearity around a particular phase value, the resulting error would repeat each time the distance resulted in that phase. Excluding gross malfunctioning, the phase readout is reliably accurate, so maximum errors from this source should not exceed two or three millimetres. The more significant source of phase error arises from electrical cross-talk, or spurious coupling, between the transmit and receive channels. This produces an error which varies sinusoidally with distance and is inversely proportional to the signal strength.

Cyclic errors in phase measurement can be determined by observing to a series of positions distributed over a full wavelength. A bar or rail accurately divided into 10-cm intervals over a distance of 10 m would cover the requirements of most short-range instruments. Details of such an arrangement are given in Hodges (1968). A micrometer on the bar capable of very accurate displacements of the reflector of ± 0.1 mm over 20 cm would enable any part of the error curve to be more closely examined.

The error curve plotted as a function of the distance should be done for strong and weakest signal conditions and may then be used to apply corrections to the measured distance. For the majority of short-range instruments the maximum error will not exceed ± 5 mm.

2.2.4.4 Additive constant (k)

The additive constant is equal to the eccentricity of the optical and physical centres of both instrument and prism. Zero error consists of changes in the additive constant and is not proportional to distance. The claimed accuracy of any EDM measurement cannot be better than that with which the additive constant is known.

Non-instrumental sources of zero error are centring of the instrument and reflector above the survey stations, and incorrect pointing of the instrument. Provided care is exercised when centring, and checked throughout the measuring process to avoid drift due to tripod settlement, this random error should not exceed ± 1 mm. The pointing error results from beam divergence causing the beam diameter to be greater than the reflector. As a result, the reflector samples only a portion of the radiated energy, the particular portion depending on the pointing of the instrument. It is a characteristic of the light-emitting diode that small time delays exist between the radiations from different areas of the collimated beam. The additive constant will therefore depend to some extent on the exact alignment of the instrument relative to the reflector. This error is most likely to occur at distances of less than 50 m. The telescope cannot be used for pointing since the visual aiming point will not coincide with the maximum return signal. However, since the signal must be reduced to prevent overloading, pointing can be done by using the signal monitor meter.

The instrumental errors corresponding to centring and pointing are mechanical misalignment and telescope misalignment. Mechanical misalignment results in the instrument not rotating about a true vertical axis when correctly levelled. Telescope misalignment will of course result in pointing error and subsequent zero error for the reasons already outlined.

In addition to the systematic errors detailed above, normally-distributed random errors due to electrical noise occur in the system. As this error is inversely related to signal strength, it will be small on reasonably strong signals. However, on weak signals it may be very significant. This error may be reduced by using more reflectors to increase the signal where necessary, or by taking the average of a number of readings (n) and thereby reducing the error by a factor of $n^{\frac{1}{2}}$.

Finally, an error of a few millimetres may be caused by a too powerful signal. Most instruments have aperture-reduction facilities to preclude this source of error.

The additive constant (k) can be determined using three points A, B and C in line, and measuring the distances AB, BC and AC. If D is the measured length and L the true length then

$$D_{AB} = L_{AB} + k$$
$$D_{BC} = L_{BC} + k$$
$$D_{AC} = L_{AB} + L_{BC} + k$$

which on solution gives $$D_{AB} + D_{BC} - D_{AC} = k \qquad (2.22)$$

Greater accuracy will be obtained by measuring many more sections $d_1, d_2 \cdots d_n$ and the sum length D, resulting in

$$k = D - \sum_{i=1}^{n} d_i/(n-1) \qquad (2.23)$$

A more rigorous approach is to measure all combinations of the lengths (Schwendener 1972) and adjust by least squares to get the most probable value for k.

The additive constant should always be determined for a definite instrument reflector combination.

2.2.4.5 Measurement of vertical angles

The errors considered above refer to the slant length of the line. However, as the reduced horizontal distance is ultimately required, one must consider the errors resulting from the reduction processes.

The most probable method of reducing to the horizontal is by means of the vertical angle (θ), the appropriate correction (c) being equal to $L - L\cos\theta$, where L is the corrected slant length.

From this equation $\delta c/\delta\theta = L\sin\theta$

and $\delta\theta'' = \delta c \times 206\,265/L\sin\theta$ (2.24)

For $L = 1000$ m, $\theta = 5°\,45'$ and $\delta c = 1$ mm, then $\delta\theta = 2.06''$ which implies that under the specified conditions one would need to measure the vertical angles very carefully indeed. The use of a 1″ theodolite is therefore imperative and, if one accepts a standard error of the vertical angle for one double-face observation of $\pm 4.5''$ to $\pm 6''$, then at least two double-face observations are required. Further examples assuming the same measured distance are for $\theta = 1°$, $\delta\theta = 11.8''$, and for $\theta = 20°$, $\delta\theta = 0.6''$. The measurement of the vertical angles is therefore very critical and all possible error sources must be considered including corrections for curvature and refraction (refer *Section 2.8.3*).

If the reduction is carried out using the difference in height (h) of the two measuring sources, the first term of the correction may be used for a comparable analysis, i.e.

$$c = h^2/2L$$

and $\delta c/\delta h = h/L$ (2.25)

Then for $L = 1000$ m, the equivalent of $\theta = 5°\,45'$ is $h = 100$ m and for $\delta c = 1$ mm as previous, $\delta h = 10$ mm. This order of accuracy could be generally obtained by normal levelling procedures. Further examples are for $h = 1$ m, $\delta h = 1$ m and for $h = 10$ m, $\delta h = 0.1$ m indicating a direct increase in precision with increase in height. Thus it can be seen that obtaining height differences is not as critical as obtaining vertical angles. It should be noted that whilst $h^2/2L$ has been used in the error analysis, Pythagoras should be used to reduce the slant length to the horizontal.

Errors in the measurement of the vertical angle can be classified as either instrumental or non-instrumental. The instrumental causes are well documented and have been covered in Volume 1. The main non-instrumental cause is the difficulty of assessing the effect of refraction on the measured angle. The effect of centring errors is negligible, a combined centring error of instrument and target of ± 50 mm would produce an error of only $\pm 1''$ on a vertical angle of 5° over a distance of 1000 m (Curl 1975). Similarly, in the measurement of reciprocal angles it is necessary to measure instrument and target heights to obtain true reciprocity. An error of ± 5 mm in any one of these heights would produce an error of $\pm 1''$ under the conditions already specified.

Due to the difficulty of obtaining a truly representative model of the atmospheric conditions through which the line of sight passes, it is doubtful if the high accuracies required of the vertical angles is ever achieved. It is therefore reasonable to assume that there will be a fall-off in the accuracy of the measured slope distance when reduced to the horizontal. This fall-off will be substantially reduced if levels rather than vertical angles are used. However, since it is quicker, easier and much more economical to use vertical angles then every precaution should be adopted in their measurement. Wherever possible simultaneous reciprocal angles should be observed. It should always be born in mind that those instruments which automatically reduced the measured distance to the horizontal, do so using vertical angles which are completely uncorrected for refraction, or corrected using a standard value for the coefficient of refraction, which may be very different to the prevailing value at the time of measurement.

2.2.4.6 Reduction to the national grid projection plane

Many engineering networks are connected to the OS national grid; a process which involves reducing the horizontal lengths of the network to mean sea level (MSL) and then to the projection using local scale factors (LSF).

Reduction to MSL is carried out using

$$C_M = \frac{LH}{R} \tag{2.10}$$

where C_M = the altitude correction, H = the mean height of the line above MSL or the height of the measuring station above MSL and R = mean radius of the Earth (6.38×10^6 m).

Differentiating equation (2.10) gives $\delta C = L\delta H/R$ (2.26)

and for $L = 1000$ m, $\delta C = \pm 1$ mm, then $\delta H = \pm 6.38$ m. As Ordnance Survey tertiary bench marks are guaranteed to ± 10 mm, and the levelling process is of more than comparable accuracy, then the errors from this source may be ignored.

Reduction of the horizontal distance to MSL theoretically produces the chord distance, not the arc or spheroidal distance. However, the chord/arc correction is negligible at distances of up to 10 km and will not therefore be considered further.

To convert the spheroidal distance to grid distance it is necessary to calculate the LSF and multiply the spheroidal distance by it. The LSF changes from point to point. In the worst case it changes from one side of a 10-km square to the other by about 6 parts in 100 000 (Ordnance Survey 1950). Thus the value for the middle of the square would be in error by approximately 1 in 30 000.

For details of scale factors, their derivation and application, refer to *Section 2.14*.

The following approximate formula for scale factors will now be used for error analysis, i.e.

$$F = F_0\{1 + (E_m^2/2R^2)\} \tag{2.27}$$

where E_m = the NG easting of the mid-point of the line—4 000 000 m
F_0 = the scale factor at the central meridian = 0.999 601 27
R = the mean radius of the Earth (6.38×10^6 m)

Then the scale factor correction is $C = LF_0\{1 + (E_m^2/2R^2)\} - L$

and $\delta C/\delta E_m = LF_0(E_m/R^2)$ (2.28)

Then for $L = 1000$ m, $\delta C = \pm 1$ mm and $E_m = 120$ km, $\delta E_m = \pm 333$ m; thus the accuracy of assessing one's position on the NG is not critical. Now, differentiating with respect to R

$$\delta C/\delta R = LF_0E_m^2/R^3 \tag{2.29}$$

and for the same parameters as above, $\delta R = \pm 18$ km. The value for $R = 6.38 \times 10^6$ m is a mean value for the whole Earth and is accurate to about 10 km between latitudes 30° and 60°, while below 30° a more representative value is 6362 km.

It can be seen therefore that reduction to MSL and thereafter to NG will have a negligible effect on the reduced horizontal distance.

2.2.4.7 Eccentricity errors

These errors may arise from the manner in which the EDM equipment is mounted on a theodolite and the type of prism used.

(1) Consider telescope-mounted EDM used with a tilting reflector which is offset the same distance, *h*, above the target as the centre of the EDM is above the line of sight of the telescope (*Figure 2.8*).

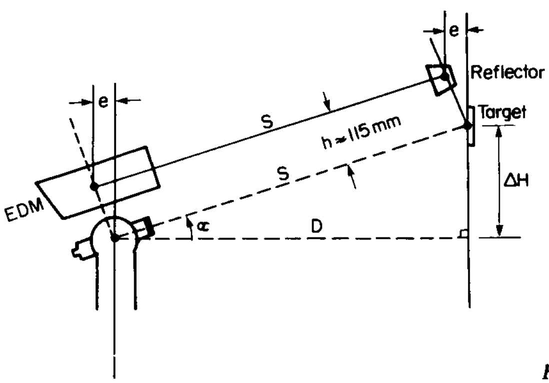

Figure 2.8

In this case the measured distance *S* is equal to the distance from the centre of the theodolite to the target and the eccentricity *e* is self-cancelling at instrument and reflector. Hence *D* and ΔH are obtained in the usual way without further correction.

(2) Consider now telescope-mounted EDM with a non-tilting reflector, as in *Figure 2.9.*

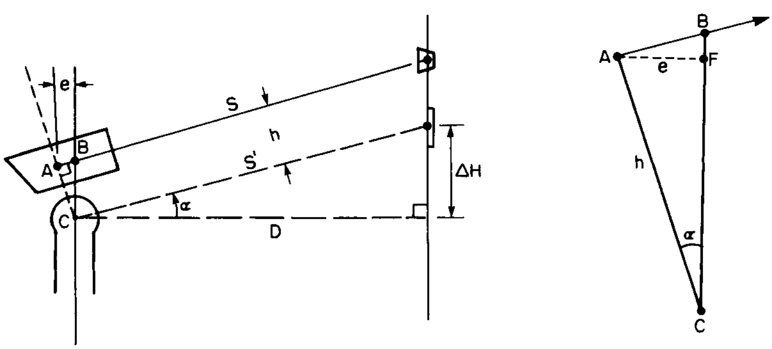

Figure 2.9

The measured slope distance S will be greater than S' by length $AB = h \tan \alpha$. If α is negative, S will be less than S' by $h \tan \alpha$.

Thus if S is used in the reduction to the horizontal D will be too long by $AF = h \sin \alpha$ when α is positive, and too small when α is negative.

If we assume an approximate value of $h = 115$ mm then the error in D when $\alpha = 5°$ is 10 mm, at 10° it is 20 mm and so on to 30° when it is 58 mm. The errors in ΔH for the above vertical angles are 1 mm, 14 mm and 33 mm, respectively.

(3) Instruments mounted on a yoke on the theodolite are generally used with non-tilting reflectors and offset target (*Figure 2.10*). As shown, there is no eccentricity error as the measuring centre of the EDM unit coincides with the axis of tilt.

If used with a tiltable reflector there will be an eccentricity error $e = h \tan \alpha$ on the slope distance as in the previous example. However, as in this case the prism is tilting, the slope distance will be *too small* when α is positive and vice versa.

(4) If yoke-mounted EDM is used with a reflector, the centre of which is also the target

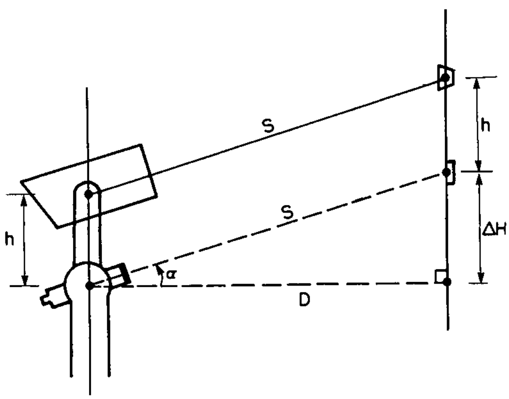

Figure 2.10

(*Figure 2.11*), then eccentricity error results because the measured angle of elevation α is not that of the measured distance S.

In triangle ABC $\qquad h/\sin \theta = S/\sin(90° - \alpha)$

$$\therefore \sin \theta = h \cos \alpha / S$$

Thus, having obtained a value for θ, the horizontal distance D is obtained from

$$D = S \cos(\alpha - \theta) \qquad (2.30a)$$

when α is positive.

For an angle of depression, i.e. when α is negative

$$D = S \cos(\alpha + \theta) \qquad (2.30b)$$

(5) When the EDM unit is co-axial with the telescope line of sight and observations are direct to the centre of the reflector, there are no eccentricity corrections.

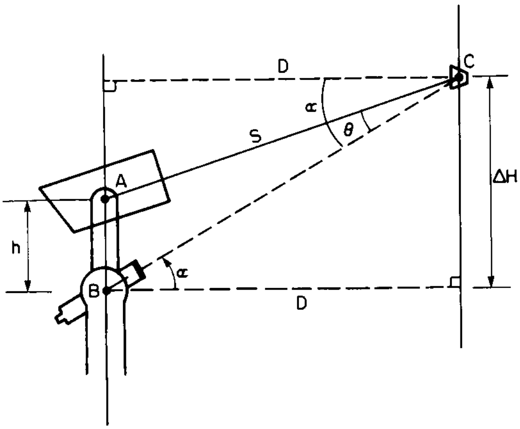

Figure 2.11

WORKED EXAMPLES

Example 2.3. A base line was measured in catenary in four lengths giving 30.126, 29.973, 30.066 and 22.536 m. The differences of level were respectively 0.45, 0.60, 0.30 and 0.45 m. The temperature during observation was 10°C and the straining mass 15 kg. The tape was standardized as 30 m, at 20°C, on the flat with a straining mass of 5 kg. The coefficient of expansion was 0.000 011 per °C, the mass of the tape 1 kg, the cross-sectional area 3 mm². $E = 210 \times 10^3$ N/mm² (210 kN/mm²), gravitational acceleration $g = 9.806\,65$ m/s².

(a) Quote each equation used and calculate the length of the base.
(b) What tension should have been applied to eliminate the sag correction? (LU)

(a) As the field tension and temperature are constant throughout, the first three corrections may be applied to the base as a whole, i.e. $L = 112.701$ m.

Tension

$$C_T = \frac{L\Delta_T}{AE} = \frac{112.701 \times 10 \times 9.806\,65}{3 \times 210 \times 10^2} = \qquad +0.0176$$

Temperature

$$C_t = LK\Delta t = 112.701 \times 0.000\,011 \times 10 = \qquad -0.0124$$

Sag

$$C_s = \frac{LW^2}{24T^2} = \frac{112.701 \times 1^2}{24 \times 15^2} = \qquad -0.0210$$

Slope

$$C_h = \frac{h^2}{2L} = \frac{1}{2 \times 30}(0.45^2 + 0.60^2 + 0.30^2) + \frac{0.45^2}{2 \times 22.536} = \qquad -0.0154$$

+0.0176	−0.0488

Therefore $\quad$ Total correction $= -0.0312$ m

Hence $\quad$ Corrected length $= 112.701 - 0.0312 = 112.6708$ m

N.B. In the slope correction the first three bays have been rounded off to 30 m, the resultant 2nd order error being negligible.

(b) To find the applied tension necessary to eliminate the sag correction, equate the two equations

$$\frac{\Delta T}{AE} = \frac{W^2}{24T_A^2}$$

where ΔT is the difference between the applied and standard tensions, i.e. $(T_A - T_S)$.

$$\therefore \frac{(T_A - T_S)}{AE} = \frac{W^2}{24T_A^2}$$

$$\therefore T_A^3 - T_A^2 T_S - \frac{AEW^2}{24} = 0$$

Substituting for T_S, W, A and E, making sure to convert T_S and W to newtons

gives $\quad T_A^3 - 49T_A^2 - 2\,524\,653 = 0$

Let $\quad T_A = (T + x)$

then $(T + x)^3 - 49(T + x)^2 - 2\,524\,653 = 0$

$$T^3\left(1 + \frac{x}{T}\right)^3 - 49T^2\left(1 + \frac{x}{T}\right)^2 - 2\,524\,653 = 0$$

Expanding the brackets binomially

$$T^3\left(1 + \frac{3x}{T}\right) - 49T^2\left(1 + \frac{2x}{T}\right) - 2\,524\,653 = 0$$

$$\therefore T^3 + 3T^2x - 49T^2 - 98Tx - 2\,524\,653 = 0$$

$$\therefore x = \frac{2\,524\,653 - T^3 + 49T^2}{3T^2 - 98T}$$

assuming $T = 15$ kgf $= 147$ N, then $x = 75$ N

$\therefore$ at the first approximation $T_A = (T + x) = 222$ N

Example 2.4. A base line was measured in catenary with a tape of nominal length 30 m. The tape measured 30.015 m when standardised in catenary at 20°C and 5 kg straining mass. If the mean reduced level of the base was 30.50 m OD, calculate its true length at mean sea level.

Given: mass per unit length of tape $= 0.03$ kg/m (w); density of steel $= 7690$ kg/m^3 (ρ); coefficient of expansion $= 11 \times 10^{-6}$ per °C (K); $E = 210 \times 10^3$ N/mm^2; gravitational acceleration $g = 9.806\,65$ m/s^2; radius of the Earth $= 6.4 \times 10^6$ m (R). (KP)

Bay	*Measured length* (m)	*Temperature* (°C)	*Straining mass* (kg)	*Difference in level* (m)
1	30.050	21.6	5	0.750
2	30.064	21.6	5	0.345
3	30.095	24.0	5	1.420
4	30.047	24.0	5	0.400
5	30.041	24.0	7	—

Standardization

Error/30 m = 0.015 m
Total length of base = 150.297 m

$$\therefore \text{ Correction} = \frac{150.297}{30} \times 0.015 = \qquad +0.0752$$

Temperature

Bays 1 and 2 $\quad C_t = 60 \times 11 \times 10^{-6} \times 1.6 = 0.0010$ m
Bays 3, 4, 5 $\quad C_t = 90 \times 11 \times 10^{-6} \times 4 \quad = 0.0040$ m $\Big\}$ $+0.0050$

(2nd order error negligible in rounding off bays to 30 m)

Tension

Bay 5 only $\quad C_T = \dfrac{L\Delta T}{AE}$, changing ΔT to newtons

where cross-sectional area $\quad A = \dfrac{w}{\rho}$

$$\therefore A = \frac{0.03}{7690} \times 10^6 = 4 \text{ mm}^2$$

$$\therefore C_T = \frac{30 \times 2 \times 9.81}{4 \times 210 \times 10^3} = \qquad +0.0007$$

Slope

$$C_h = \frac{h^2}{2L} - \frac{1}{2 \times 30}(0.750^2 + 0.345^2 + 1.420^2 + 0.400^2) = \qquad -0.0476$$

The 2nd order error in rounding off to 30 m is negligible in this case also. However, care should be taken when many bays are involved, as their accumulative effect may be significant.

Sag

Bay 5 only $\quad C_s = \dfrac{L^3 w^2}{24}\left(\dfrac{1}{T_S^2} - \dfrac{1}{T_A^2}\right)$

$$= \frac{30^3 \times 0.03^2}{24}\left(\frac{1}{5^2} - \frac{1}{7^2}\right) = \qquad +0.0006$$

Altitude

$$C_M = \frac{LH}{R} = \frac{150 \times 30.5}{6.4 \times 10^6} = \qquad -0.0007$$

+0.0815	−0.0483

Therefore Total correction = +0.0332 m
Hence Corrected length = 150.297 + 0.0332 = 150.3302 m

Example 2.5. (a) A standard base was established by accurately measuring with a steel tape the distance between fixed marks on a level bed. The mean distance recorded was 24.984 m at a temperature of 18°C and an applied tension of 155 N. The tape used had recently been standardized in catenary and was 30 m in length at 20°C and 100 N tension. Calculate the true length between the fixed marks given: total mass of the tape = 0.90 kg; coefficient of expansion of steel = 11×10^{-6} per °C; cross-sectional area = 2 mm^2; $E = 210 \times 10^3$ N/mm^2; gravitational acceleration = 9.807 m/s^2.

(b) At a later date the tape was used to measure a 30-m bay in catenary. The difference in level of the measuring heads was 1 m, with an error of 3 mm. Tests carried out on the spring balance indicated that the applied tension of 100 N had an error of 2 N. Ignoring all other sources of error, what is the probable error in the measured bay? (KP)

(a) If the tape was standardized in catenary, then when laid on the flat it would be too long by an amount equal to the sag correction. This amount, in effect, then becomes the standardization correction

$$\text{Error per 30 m} = \frac{LW^2}{24T_s^2} = \frac{30 \times (0.90 \times 9.807)^2}{24 \times 100^2} = 0.0097 \text{ m}$$

$$\therefore \text{ Correction} = \frac{0.0097 \times 24.984}{30} = 0.0081 \text{ m}$$

$$\text{Tension} = \frac{24.984 \times 55}{2 \times 210 \times 10^3} = 0.0033 \text{ m}$$

$$\text{Temperature} = 24.984 \times 11 \times 10^{-6} \times 2 = -0.0006 \text{ m}$$

$$\therefore \text{ Total correction} = 0.0108 \text{ m}$$

$$\therefore \text{ Corrected length} = 24.984 + 0.011 = 24.995 \text{ m}$$

(b) *Effect of levelling error* $\qquad C_h = \frac{h^2}{2L}$

$$\therefore \ \delta C_h = \frac{h \times \delta h}{L} = \frac{1 \times 0.003}{30} = 0.0001 \text{ m}$$

Effect of tensioning error $\qquad$ Sag $\quad C_s = \frac{LW^2}{24T^2}$

$$\therefore \ \delta C_s = -\frac{LW^2}{12T^3}\,\delta T$$

$$\therefore \ \delta C_s = \frac{30 \times (0.9 \times 9.807)^2 \times 2}{12 \times 100^3} = 0.0004 \text{ m}$$

Tension $C_T = \dfrac{L\Delta T}{AE}$

$$\therefore \quad \delta C_T = \frac{L \times \delta(\Delta T)}{A \times E} = \frac{30 \times 2}{2 \times 210 \times 10^3} = 0.0001 \text{ m}$$

$\therefore$ Total error = 0.0006 m

Example 2.6. A 30-m invar reference tape was standardized on the flat and found to be 30.0501 m at 20°C and 88 N tension. It was used to measure the first bay of a base line in catenary, the mean recorded length being 30.4500 m.

Using a field tape, the mean length of the same bay was found to be 30.4588 m. The applied tension was 88 N at a constant temperature of 15°C in both cases.

The remaining bays were now measured in catenary, using the field tape only. The mean length of the second bay was 30.5500 m at 13°C and 100 N tension. Calculate its reduced length given: cross-sectional area = 2 mm^2; coefficient of expansion of invar = 6×10^{-7} per °C; mass of tape per unit length = 0.02 kg/m; difference in height of the measuring heads = 0.5 m; mean altitude of the base = 250 m OD; radius of the Earth = 6.4×10^6 m; gravitational acceleration = 9.807 m/s^2; Young's modulus of elasticity = 210 kN/mm^2. (KP)

To find the corrected length of the 1st bay using the reference tape.

Standardization

Error per 30 m = 0.0501 m

	+	−
$\therefore$ Correction for 30.4500 m =	+0.0508	
Temperature = $30 \times 6 \times 10^{-7} \times 5$ =		−0.0001
Sag = $\dfrac{30^3 \times (0.02 \times 9.807)^2}{24 \times 88^2}$ =		−0.0056
	+0.0508	−0.0057

Therefore Total correction = +0.0451 m

Hence Corrected length = 30.4500 + 0.0451 = 30.4951 m

(using reference tape). Field tape corrected for sag, measures 30.4588 − 0.0056 = 30.4532 m.

Thus the field tape is measuring too short by 0.0419 m (30.4951 − 30.4532) and is therefore too long by this amount. Therefore field tape is 30.0419 m at 15°C and 88 N.

To find length of 2nd bay.

Standardization

Error per 30 m = 0.0419

	+	−
$\therefore$ Correction = $\dfrac{30.5500}{30} \times 0.0419$ =	+0.0427	
Temperature = $30 \times 6 \times 10^{-7} \times 2$ =		−0.000 04

	+	−
Tension $= \dfrac{30 \times 12}{2 \times 210 \times 10^3} =$	$+0.0009$	
Sag $= \dfrac{30^3 \times (0.02 \times 9.807)^2}{24 \times 100^2} =$		-0.0043
Slope $= \dfrac{0.500^2}{2 \times 30.5500} =$		-0.0041
Altitude $= \dfrac{30.5500 \times 250}{6.4 \times 10^6} =$		-0.0093
	$+0.0436$	-0.0177

Therefore Total correction $= +0.0259$ m
Hence Corrected length of 2nd bay $= 30.5500 + 0.0259 = 30.5759$ m

N.B. Rounding off the measured length to 30 m is permissible only when the resulting error has a negligible effect on the final distance.

Example 2.7. A copper transmission line of 12 mm diameter is stretched between two points 300 m apart, at the same level with a tension of 5 kN, when the temperature is 32°C. It is necessary to define its limiting positions when the temperature varies. Making use of the corrections for sag, temperature and elasticity normally applied to base-line measurements by a tape in catenary, find the tension at a temperature of −12°C and the sag in the two cases.

Young's modulus for copper is 70 kN/mm^2, its density 9000 kg/m^3 and its coefficient of linear expansion 17.0×10^{-6}/°C. (LU)

In order first of all to find the amount of sag in the above two cases using equation (2.7), one must find (a) the mass per unit length and (b) the sag length, of the wire.

(a) $w = \text{area} \times \text{density} = \pi r^2 \rho$
$= 3.142 \times 0.006^2 \times 9000 = 1.02$ kg/m

(b) at 32°C, the sag length of wire $= L_H + \left(\dfrac{L^3 w^2}{24T^2}\right)$

where L is itself the sag length. Thus the first approximation for L of 300 m must be used.

$$\therefore \text{ Sag length} = 300 + \left(\frac{300^3 \times (1.02 \times 9.807)^2}{24 \times 5000^2}\right) = 304.5 \text{ m}$$

$$\text{Second approximation} = 300 + \left(\frac{304.5^3 \times (1.02 \times 9.807)^2}{24 \times 5000^2}\right)$$

$$= 304.71 \text{ m} = L_1$$

$$\therefore \text{ Sag} = y_1 = \frac{wL_1^2}{8T} = \frac{(1.02 \times 9.807) \times 304.71^2}{8 \times 5000} = 23.22 \text{ m}$$

At $-12°C$ there will be a reduction in L_1 of

$$(L_1 K \Delta t) = 304.71 \times 17.0 \times 10^{-6} \times 44 = 0.23 \text{ m}$$

$$\therefore \; L_2 = 304.71 - 0.23 = 304.48 \text{ m}$$

From equation (2.7) $y_1 \propto L_1^2$ $\quad \therefore \; y_2 = y_1\left(\dfrac{L_2}{L_1}\right)^2 = 23.22\dfrac{(304.48)^2}{(304.71)^2} = 23.18 \text{ m}$

Similarly, $y_1 \propto 1/T_1$ $\quad \therefore \; T_2 = T_1\left(\dfrac{y_1}{y_2}\right) = 5000\left(\dfrac{23.22}{23.18}\right) = 5009 \text{ N or } 5.009 \text{ kN}$

EXERCISES

2.1. A tape of nominal length 30 m was standardized on the flat at the NPL, and found to be 30.0520 m at 20°C and 44 N of tension. It was then used to measure a reference bay in catenary and gave a mean distance of 30.5500 m at 15°C and 88 N tension. As the mass of the tape was unknown, the sag at the mid-point of the tape was measured and found to be 0.170 m.

Given: cross-sectional area of tape = 2 mm^2; Young's modulus of elasticity = 200×10^3 N/mm^2; coefficient of expansion = 11.25×10^{-6} per °C; and difference in height of measuring heads = 0.320 m. Find the horizontal length of the bay. If the error in the measurement of sag was ± 0.001 m, what is the resultant error in the sag correction? What does this resultant error indicate about the accuracy to which the sag at the mid-point of the tape was measured? (KP)

(*Answer:* 30.5995 m and $\pm 0.000\,03$ m)

2.2. The three bays of a base line were measured by a steel tape in catenary as 30.084, 29.973 and 25.233 m, under respective pulls of 7, 7 and 5 kg, temperatures of 12°, 13° and 17°C and differences of level of supports of 0.3, 0.7 and 0.7 m. If the tape was standardized on the flat at a temperature of 15°C under a pull of 4.5 kg, what are the lengths of the bays? 30 m of tape is exactly 1 kg with steel at 8300 kg/m^3, with a coefficient of expansion of 0.000 011 per °C and $E = 210 \times 10^3$ N/mm^2. (LU)

(*Answer:* 30.057 m, 29.940 m and 25.194 m)

2.3. The details given below refer to the measurement of the first 30-m bay of a base line. Determine the correct length of the bay reduced to mean sea level.

With the tape hanging in a catenary at a straining mass of 10 kg and at a mean temperature of 13°C, the recorded length was 30.0247 m. The difference in height between the ends was 0.456 m and the site was 500 m above MSL. The tape had previously been standardized in catenary at a straining mass of 7 kg and a temperature of 16°C, and the distance between zeros was 30.0126 m.

$R = 6.4 \times 10^6$ m; mass of tape per m = 0.02 kg; sectional area of tape = 3.6 mm^2; $E = 210 \times 10^3$ N/mm^2; temperature coefficient of expansion of tape = 0.000 011 per °C. (ICE)

(*Answer:* 30.0364 m)

2.4. The following data refer to a section of base line measured by a tape hung in catenary.

Bay	*Observed length* (m)	*Mean temperature* (°C)	*Reduced levels of index marks* (m)	
1	30.034	25.2	293.235	293.610
2	30.109	25.4	293.610	294.030
3	30.198	25.1	294.030	294.498
4	30.075	25.0	294.498	294.000
5	30.121	24.8	294.000	293.355

Length of tape between 0 and 30 m graduations when horizontal at 20°C and under 5 kg straining mass is 29.9988 m; cross-sectional area of tape = 2.68 mm^2; straining mass used in the field = 10 kg; temperature coefficient of expansion of tape = 11.16×10^{-6} per °C; elastic modulus for material of tape = 20.4×10^4 N/mm^2; mass of tape per metre length = 0.02 kg; mean radius of the Earth = 6.4×10^6 m. Calculate the corrected length of this section of the line. (LU)

(*Answer:* 150.507 m)

2.3 FIGURAL ADJUSTMENT BY EQUAL SHIFTS

The next step in the computational procedure is that of adjusting the figures in order to make them geometrically correct. The method indicated here is a semi-rigorous approach termed *equal shifts.*

(1) *Simple triangle.* The condition of adjustment of a plane triangle is that all three angles should equal 180°. As the sides increase in length, beyond about 20 km, the triangle becomes spheroidal in shape and the sum of the angles is equal to (180° + spherical excess).

$$\text{Spherical excess} \quad E'' = \frac{\text{Area of triangle}}{R^2} \times 206\,265 \qquad (2.31)$$

(for practical purposes)

Legendre's theorem then stipulates that if one-third of the spherical excess is deducted from each angle, the triangle may be treated as a plane triangle for the computation of side lengths. In calculating the co-ordinates, however, the spheroidal angles are again used.

(2) *Braced quadrilateral.* Conditions of adjustment (*Figure 2.12*)

$$\begin{aligned} \text{Angles } 1+2+3+4+5+6+7+8 &= 360^\circ \\ 1+2+3+4 &= 180^\circ \\ 3+4+5+6 &= 180^\circ \\ 5+6+7+8 &= 180^\circ \\ 7+8+1+2 &= 180^\circ \\ 1+2 &= 5+6 \\ 3+4 &= 7+8 \end{aligned}$$

Side condition: $\sum$ log sins of the *odd* angles = $\sum$ log sins of the *even* angles.

As many of the above conditions are dependent upon each other, only four are used in the actual adjustment. The 'method of adjustment' is: (i) adjust angles 1–8 to equal 360°; (ii) adjust angles (1 + 2) to equal (5 + 6); (iii) adjust angles (3 + 4) to equal (7 + 8); (iv) side condition.

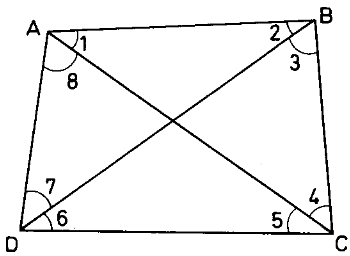

Figure 2.12

Proof of side condition

From *Figure 2.12* it is required to calculate length *CD* from base *AB*.

This may be done via route *BC* or *AD* as follows

$$\frac{AB}{\sin 4} = \frac{BC}{\sin 1} \qquad \therefore\ BC = \frac{AB \sin 1}{\sin 4}$$

Now $$\frac{BC}{\sin 6} = \frac{DC}{\sin 3} \qquad \therefore\ DC = \frac{BC \sin 3}{\sin 6} = \frac{AB \sin 1 \sin 3}{\sin 4 \sin 6}$$

Similarly via *AD* $$DC = \frac{AB \sin 2 \sin 8}{\sin 7 \sin 5}$$

As there can be only one length for *DC*, then cancelling *AB* gives

$$\frac{\sin 1 \sin 3}{\sin 4 \sin 6} = \frac{\sin 2 \sin 8}{\sin 7 \sin 5}$$

Cross-multiplying and takings logs

$$\log \sin 1 + \log \sin 3 + \log \sin 5 + \log \sin 7$$
$$= \log \sin 2 + \log \sin 4 + \log \sin 6 + \log \sin 8$$

The method of adjustment will now be illustrated using the following mean observed angles in *Figure 2.12.*

Number	*Observed angles*				*1st correction*							*2nd correction*		
	°	′	″	″	°	′	″	°	′	″	″	°	′	″
1	50	42	27	−1	50	42	26	117	30	19	1	50	42	27
2	66	47	54	−1	66	47	53				1	66	47	54
3	41	24	32	−1	41	24	31	62	29	36	2	41	24	33
4	21	05	06	−1	21	05	05				1	21	05	06
5	74	13	36	−1	74	13	35	117	30	23	−1	74	13	34
6	43	16	49	−1	43	16	48				−1	43	16	47
7	18	36	14	−1	18	36	13	62	29	42	−1	18	36	12
8	43	53	30	−1	43	53	29				−2	43	53	27
	360	00	08	−8	360	00	00				0	360	00	00

(a) The first step in the method of adjustment is clearly seen.

(b) The second step shows that the difference between angles (1 + 2) and (5 + 6) is 4″, i.e. 1″ per angle which is added to the smaller sum and subtracted from the larger.

(c) The third step is identical to the above, the corrections of 2″ and 1″ have been arbitrarily made to prevent the introduction of decimals of a second (correction per angle = 1.5″).

The three steps have produced corrected angles which satisfy the first seven conditions of adjustment. It is now necessary to find the log sins of these angles and to compare their sums. This can be done very quickly on a pocket calculator.

1	2 *Angles* ° ′ ″	3 Log sin (*odd*)	4 Log sin (*even*)	5 *Difference for 10″ arc*	6	7 *Final values* ° ′ ″
1	50 42 27	$\bar{1}$.888 698		0.000 017	1″	50 42 28
2	66 47 54		$\bar{1}$.963 374	9	−1″	66 47 53
3	41 24 33	$\bar{1}$.820 485		24	1″	41 24 34
4	21 05 06		$\bar{1}$.556 004	55	−1″	21 05 05
5	74 13 34	$\bar{1}$.983 329		6	1″	74 13 35
6	43 16 47		$\bar{1}$.836 046	22	−1″	43 16 46
7	18 36 12	$\bar{1}$.503 810		62	1″	18 36 13
8	43 53 27		$\bar{1}$.840 913	22	−1″	43 53 26
		$\bar{1}$.196 322	$\bar{1}$.196 337	0.000 217		360 00 00
			$\bar{1}$.196 322			
			0.000 015			

$$\therefore \text{ Adjustment} = \frac{15}{217} \times 10'' = 0.7'' \approx 1''$$

(d) Column 5 represents the changes in the log sins of the angles for a change of 10″ in the angle. These values are easily obtained by increasing the value of the angle by 10″ and the finding of its log sin on the pocket calculator. The difference of the two log sin values is the difference for 10″ change in the angle.

Normally the differences for 1″ of arc are used, but in this case 10″ differences are used in order to facilitate understanding of the principles.

(e) Summing columns 3 and 4 shows a difference of 15 (0.000 015) which must be adjusted. The necessary angular correction (0.7″) is obtained by dividing 15 by the sum of column 5, i.e. 217 (0.000 217) as shown. This may be explained as follows: if one alters all the angles by 10″, the total change in the log sins would be 0.000 217. However, the change required is only 0.000 015, which by proportion represents an angular change of $\frac{15}{217} \times 10'' = 0.7''$.

(f) The log sins of the corrected angles are now easily found using columns 5 and 6, to give the corrections to columns 3 and 4.

(g) If any angle is greater than 90°, then a *positive* correction to the angle would require a *negative* correction to its log sin. Thus the difference value in column 5 should have a negative sign which is applied in the summing of this column and throughout.

(h) It is worth noting that the accuracy of a triangulation figure is expressed by the magnitude of the difference in the sum of log sins, i.e. 0.000 015. Compensating errors can occur in angles tending to indicate excellent closure; such errors would, however, substantially unbalance the side equation.

Although the above method can be done quite easily on a pocket calculator, the following approach (Smith 1982) has been produced specifically for a pocket calculator.

The method precludes the use of logarithms and differences for 1″ or 10″, and is as follows: in the side condition assume v is the correction per angle, then

$$\sin(1+v)\sin(3+v)\sin(5+v)\sin(7+v) = \sin(2+v)\sin(4+v)\sin(6+v)\sin(8+v)$$

Now $\sin(1+v) = \sin 1 \cos v + \cos 1 \sin v$ which, as v is very small, $= \sin 1 + \cos 1v$

$$\therefore \frac{(\sin 1+\cos 1v)(\sin 3+\cos 3v)(\sin 5+\cos 5v)(\sin 7+\cos 7v)}{(\sin 2+\cos 2v)(\sin 4+\cos 4v)(\sin 6+\cos 6v)(\sin 8+\cos 8v)} = 1$$

Expanding to first order only

$$\frac{(\sin 1 \sin 3+\sin 1 \cos 3v+\cos 1 \sin 3v)(\sin 5 \sin 7+\sin 5 \cos 7v+\cos 5 \sin 7v)}{(\sin 2 \sin 4+\sin 2 \cos 4v+\cos 2 \cos 4v)(\sin 6 \sin 8+\sin 6 \cos 8v+\cos 6 \sin 8v)}$$

$$= \frac{\sin 1 \sin 3 \sin 5 \sin 7+\sin 1 \sin 3 \sin 5 \sin 7v(\cot 1+\cot 3+\cot 5+\cot 7)}{\sin 2 \sin 4 \sin 6 \sin 8+\sin 2 \sin 4 \sin 6 \sin 8v(\cot 2+\cot 4+\cot 6+\cot 8)}$$

Let $\sin 1 \sin 3 \sin 5 \sin 7 = A$ $\quad \cot 1+\cot 3+\cot 5+\cot 7 = B$
$\sin 2 \sin 4 \sin 6 \sin 8 = C$ $\quad \cot 2+\cot 4+\cot 6+\cot 8 = D$

then the above expansion can be re-arranged and expressed thus

$$v'' = \frac{206\,265(A-C)}{AB+CD}$$

If v'' is positive, then $A > C$ and v'' is subtracted from the 'odd' angles and added to the 'even'. If v'' is negative, then $A < C$ and v'' is added to 'odd' and subtracted from 'even'.

All the digits as displayed on the pocket calculator are significant and should be carried through the computation.

The previous example is now re-worked using this method for the side condition, and it is shown in *Table 2.3* overleaf.

(3) *Polygon with central point.* The basic triangulation figures are shown in *Figure 2.13.*

Conditions of adjustment

(a) Each triangle to equal 180°, i.e. I, II . . . V in *Figure 2.13(c).*
(b) Central angles to equal 360°.
(c) Side condition using the base angles only, i.e. 1, 2, . . . , 10 in *Figure 2.13(c).*

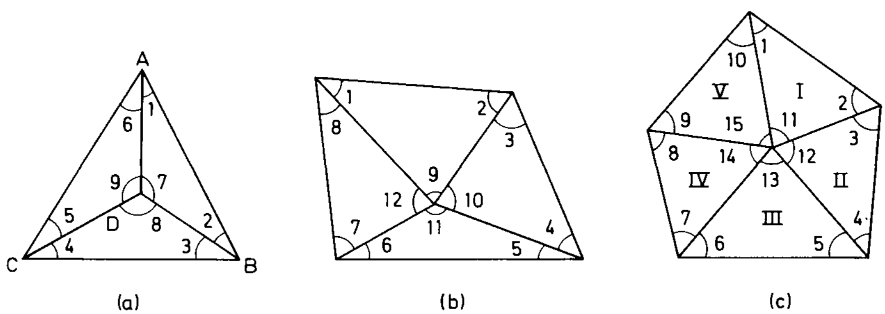

Figure 2.13

TABLE 2.3. Adjustment of braced quadrilateral by equal shifts (using pocket calculator)

Angle no.	*Observed angle* ° ′ ″	*1st corr'n*	*1st corr'd angle* ° ′ ″	*1+2=5+6* *3+4=7+8* ° ′ ″	*2nd corr'n*	*2nd corr'd angle* ° ′ ″	Sin *odd* $\angle_s$ *product*	Cot *odd* $\angle_s$ *sum*	Sin *even* $\angle_s$ *product*	Cot *even* $\angle_s$ *sum*	*3rd corr'n*	*Final corr'd angle* ° ′ ″
1	50 42 27	−1″	50 42 26	} 117 30 19	+1″	50 42 27	0.773 923	0.818 270 3			+1″	50 42 28
2	66 47 54	−1″	66 47 53		+1″	66 47 54	×	+	0.919 124	0.428 634 4	−1″	66 47 53
3	41 24 32	−1″	41 24 31	} 62 29 36	+2″	41 24 33	0.661 432	1.133 911 5	×	+	+1″	41 24 34
4	21 05 06	−1″	21 05 05		+1″	21 05 06	×	+	0.359 753	2.593 582 9	−1″	21 05 05
5	74 13 36	−1″	74 13 35	} 117 30 23	−1″	74 13 34	0.962 342	0.282 479 4	×	+	+1″	74 13 35
6	43 16 49	−1″	43 16 48		−1″	43 16 47	×	+	0.685 561	1.061 927 3	−1″	43 16 46
7	18 36 14	−1″	18 36 13	} 62 29 42	−1″	18 36 12	0.319 014	2.970 867 6	×	+	+1″	18 36 13
8	43 53 30	−1″	43 53 29		−2″	43 53 27			0.693 287	1.039 486 9	−1″	43 53 26
$\sum$	360 00 08	−8″	360 00 00			360 00 00	0.157 152 8	5.205 528 8	0.157 1584	5.123 631 5		360 00 00
							A	B	C	D		

$$\text{3rd correction} = \frac{206\,265(A - C)}{AB + CD}$$

$$= 0.7'' \approx 1''$$

If A > C then add to even $\angle_s$ and subtract from odd $\angle_s$, and *vice versa*

Method of adjustment

(a) Adjust each triangle to 180°.
(b) (i) Adjust the central angles to 360°.
  (ii) Readjust the triangles to 180° using the two base angles in each triangle only.
(c) Side condition adjustment using the base angles only.

Steps (b)(i) and (ii) are in fact only one step, for a correction of say +10″ to each of the central angles would automatically give a correction of −5″ to each base angle of the triangle. The side condition would then be carried out in exactly the same manner already described, in each case excluding the angles at the centre point.

2.4 SATELLITE STATIONS

In *Figure 2.14* it is required to find the angles measured to *A*, *B* and *C* from *D*, or alternatively the bearings *DA*, *DB* and *DC*. If *D* is an 'up-station', e.g. church spire, lightning conductor or tall structure, etc., or the lines of sight are blocked by natural or

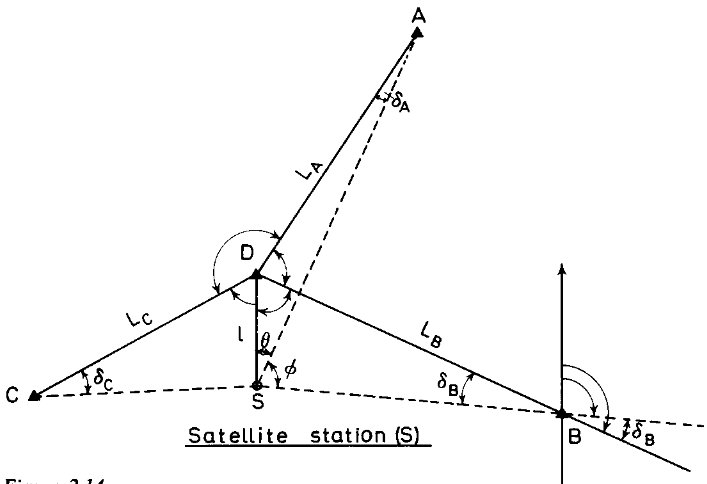

Figure 2.14

man-made obstacles, then it is necessary to establish a satellite station *S* nearby, from which the angles to *A*, *B*, *C* and *D* are measured. These measured angles about *S* are then reduced to their equivalent about *D*. This is best illustrated as follows: if the line *SD* is assumed to be due *N*, then it can be seen that the bearing of *DB* is greater than that of *SB* by the amount δ_B. Thus the measured bearing *SB* is increased by δ_B to give the required bearing *DB*.

If working directly in angles, then regarding *ABSD* as a crossed quadrilateral, it can be seen that

$$A\hat{D}B = A\hat{S}B + \delta_B - \delta_A \quad \text{(with } S \text{ due south of } D\text{)}$$

Students should draw the following and verify for themselves

S due west of D	$A\hat{D}B = A\hat{S}B - \delta_B - \delta_A$
S due east of D	$A\hat{D}B = A\hat{S}B + \delta_B + \delta_A$
S due north of D	$A\hat{D}B = A\hat{S}B - \delta_B + \delta_A$

The method of solving the problem is determined largely by the data supplied. If the angles at A and B to D are given, then one can find an approximate value for $A\hat{D}B$ from $(180° - D\hat{A}B - D\hat{B}A)$, and then use the sine rule with length AB to find L_A and L_B. Then by the sine rule in $\triangle DAS$

$$\delta''_A = \frac{l \sin \theta}{L_A} \times 206\,265 \tag{2.32}$$

To assess the effect of errors in the measured quantities on δ_A, differentiate with respect to each in turn

$$\frac{\delta(\delta_A)}{\delta_A} = \frac{\delta l}{l} = \frac{\delta L}{L} = \cot \theta \, \delta\theta$$

This indicates:

(1) That the fractional error in δ_A is directly proportional to the fractional error in l and L. Thus if $\delta_A = 600'' \pm 1''$, $l = 10$ m and $L = 10$ km, then l need only be measured to the nearest 0.017 m and L to 17 m, i.e. 1 in 600.

(2) That the error in δ_A is proportional to $\cot \theta \, \delta\theta \, \delta_A$, thus the angle θ should be as large as possible and angle δ_A as small as possible, making l as small as possible. The accuracy to which one measures θ, i.e. $\delta\theta$, varies with the value of θ. If it is very large, then $\cot \theta$ is very small and θ need be measured with only normal accuracy.

The sum effect of the standard errors is

$$\frac{\delta(\delta_A)}{\delta_A} = \pm\left\{\left(\frac{\delta l}{l}\right)^2 + \left(\frac{\delta L}{L}\right)^2 + (\cot \theta \, \delta\theta)^2\right\}^{\frac{1}{2}}$$

2.5 RESECTION AND INTERSECTION

Using these techniques, one can establish the co-ordinates of a point P, by observations to at least three known points.

2.5.1 Intersection

This involves sighting in to P from known positions (*Figure 2.15*). If the bearings of the rays are used, then using the rays in combinations of two, the co-ordinates of P are obtained as follows:

In *Figure 2.16* it is required to find the co-ordinates of P, using the *bearings* α and β to P from known points A and B whose co-ordinates are E_A, N_A and E_B, N_B.

$$PL = E_P - E_A \qquad AL = N_P - N_A$$
$$PM = E_P - E_B \qquad MB = N_P - N_B$$

Now as $\quad PL = AL \tan \alpha$

then $\quad E_P - E_A = (N_P - N_A) \tan \alpha \qquad (1)$

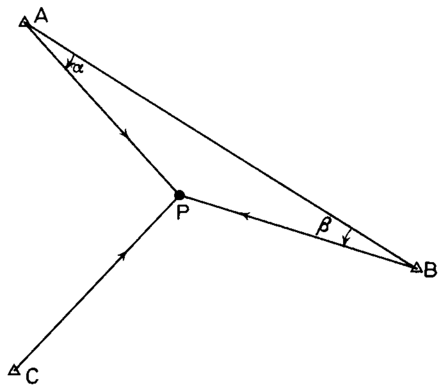

Figure 2.15

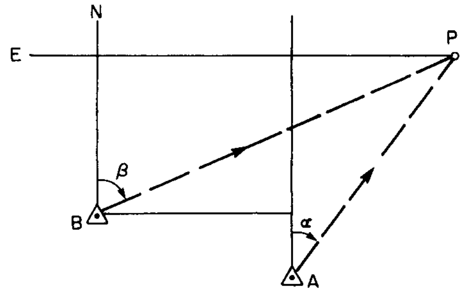

Figure 2.16

Similarly $\quad PM = MB \tan \beta$

then $\quad E_P - E_B = (N_P - N_B) \tan \beta \qquad (2)$

Subtracting (1) from (2) gives

$$E_B - E_A = (N_P - N_A) \tan \alpha - (N_P - N_B) \tan \beta$$
$$= N_P \tan \alpha - N_A \tan \alpha - N_P \tan \beta + N_B \tan \beta$$

$$\therefore \; N_P(\tan \alpha - \tan \beta) = E_B - E_A + N_A \tan \alpha - N_B \tan \beta$$

Thus
$$N_P = \frac{E_B - E_A + N_A \tan \alpha - N_B \tan \beta}{\tan \alpha - \tan \beta} \qquad (2.33a)$$

Similarly
$$N_P - N_A = (E_P - E_A) \cot \alpha$$
$$N_P - N_B = (E_P - E_B) \cot \beta$$

Subtracting
$$N_B - N_A = (E_P - E_A) \cot \alpha - (E_P - E_B) \cot \beta$$

Thus
$$E_P = \frac{N_B - N_A + E_A \cot \alpha - E_B \cot \beta}{\cot \alpha - \cot \beta} \qquad (2.33b)$$

Using equations (2.33a) and (2.33b) the co-ordinates of P are computed. It is assumed that P is always to the right of $A \rightarrow B$, in the equations.

If the observed angles into P are used (*Figure 2.15*) the equations become

$$E_P = \frac{N_B - N_A + E_A \cot \beta + E_B \cot \alpha}{\cot \alpha + \cot \beta} \qquad (2.34a)$$

$$N_P = \frac{E_A - E_B + N_A \cot \beta + N_B \cot \alpha}{\cot \alpha + \cot \beta} \qquad (2.34b)$$

The above equations are also used in the direct solution of triangulation.

2.5.2 Resection

This involves the angular measurement from P out to the known points (*Figure 2.17*).

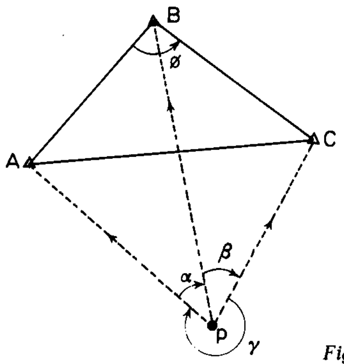

Figure 2.17

Where only three known points are used a variety of analytical methods is available for the solution of P.

(a) The following approach is referred to as the 'analytical method' (from *Figure 2.17*).

Let $BAP = \theta$, then $\quad BCP = (360° - \alpha - \beta - \phi) - \theta = S - \theta$

where ϕ is computed from the co-ordinates of stations A, B and C; thus S is known.

From $\triangle PAB \quad PB = BA \sin\theta/\sin\alpha \qquad (1)$
From $\triangle PBC \quad PB = BC \sin(S - \theta)/\sin\beta \qquad (2)$

Equating (1) and (2) $\quad \dfrac{\sin(S-\theta)}{\sin\theta} = \dfrac{BA \sin\beta}{BC \sin\alpha} = Q \quad$ (known)

then $\quad (\sin S \cos\theta - \cos S \sin\theta)/\sin\theta = Q$

$\sin S \cot\theta - \cos S = Q$

$\therefore \cot\theta = (Q + \cos S)/\sin S \qquad (3)$

Thus, knowing θ and $(S - \theta)$, the triangles can be solved for lengths and bearings AP, BP and CP, and three values for the co-ordinates of P obtained if necessary.

The method fails, as do all three-point resections, if P lies on the circumference of a circle passing through A, B and C and thereby has an infinite number of positions.

(b) This second approach is presented to illustrate the diversity of methods available. A, B and C (in *Figure 2.18*) are fixed points whose co-ordinates are known, and the co-ordinates of the circle centres O_1 and O_2 are

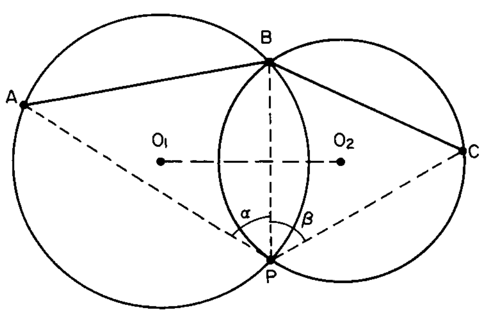

Figure 2.18

$$E_1 = \tfrac{1}{2}\{E_A + E_B + (N_A - N_B)\cot\alpha\}$$
$$N_1 = \tfrac{1}{2}\{N_A + N_B - (E_A - E_B)\cot\alpha\}$$
$$E_2 = \tfrac{1}{2}\{E_B + E_C + (N_B - N_C)\cot\beta\}$$
$$N_2 = \tfrac{1}{2}\{N_B + N_C - (E_B - E_C)\cot\beta\}$$

Thus the bearing α of $O_1 \rightarrow O_2$ is obtained in the usual way, i.e.

$$\alpha = \tan^{-1}\{(E_2 - E_1)/(N_2 - N_1)\}$$

then
$$E_P = E_B + 2\{(E_B - E_1)\sin\alpha - (N_B - N_1)\cos\alpha\}\sin\alpha \qquad (2.35a)$$
$$N_P = N_B + 2\{(E_B - E_1)\sin\alpha - (N_B - N_1)\cos\alpha\}\cos\alpha \qquad (2.35b)$$

(c) Dr T. L. Thomas of Imperial College offers the following solution for a three-point resection; from *Figure 2.17*

$$E_P = E_A + \frac{ZV}{(V^2 + W^2)} \qquad N_P = N_A + \frac{ZW}{(V^2 + W^2)} \qquad \tan(\text{bg}\ \overrightarrow{PA}) = \frac{V}{W}$$

where
$$V = \Delta E_1 \cot\alpha - \Delta E_2 \cot(\alpha + \beta) + (N_C - N_B)$$
$$W = \Delta N_1 \cot\alpha - \Delta N_2 \cot(\alpha + \beta) + (E_B - E_C)$$
$$X = \Delta E_1 \Delta E_2 + \Delta N_1 \Delta N_2$$
$$Y = \Delta E_1 \Delta N_2 - \Delta N_1 \Delta E_2$$
$$Z = X\cot\alpha - X\cot(\alpha + \beta) + Y + Y\cot\alpha\cot(\alpha + \beta)$$

$$\Delta E_1 = E_B - E_A \qquad \Delta E_2 = E_C - E_A \qquad \Delta N_1 = N_B - N_A \qquad \Delta N_2 = N_C - N_A$$

2.5.3 Semigraphic solution of resection/intersection

Where more than three points are used, a semigraphic solution may be used. This approach is fast becoming obsolete due to the almost universal use of machine computation.

The principle of an intersection is shown in *Figure 2.19*. As one cannot plot the whole of the ray at a large scale, only the area about the point of intersection is plotted. In order to fix the direction of the rays, the *double cutting points* E_1, E_2, and N_1, N_2, are required.

Procedure

(1) Obtain provisional co-ordinates for the point of intersection by scaling from a plan or using only two rays in equations (2.33a) and (2.33b). This value P' is plotted in the centre of the graph paper.

(2) From the scale of the plot and the size of the paper, it is now possible to fix the northing value of the top and bottom of the paper and the easting value of the left and right hand sides, i.e. N_t, N_b, E_l, E_r.

(3) When the quadrant or reduced bearing of the ray is less than 45°, the best cut is obtained on the east-west axis, i.e. E_1, E_2. If greater than 45° it is obtained on the north-south axis, i.e. N_1, N_2.

(4) Assuming α is less than 45° then

$$E_1 = E_A + AA_1 = E_A + A_1E_1 \tan\alpha$$

$$\therefore\ E_1 = E_A + (N_A - N_t)\tan\alpha \qquad (2.36a)$$

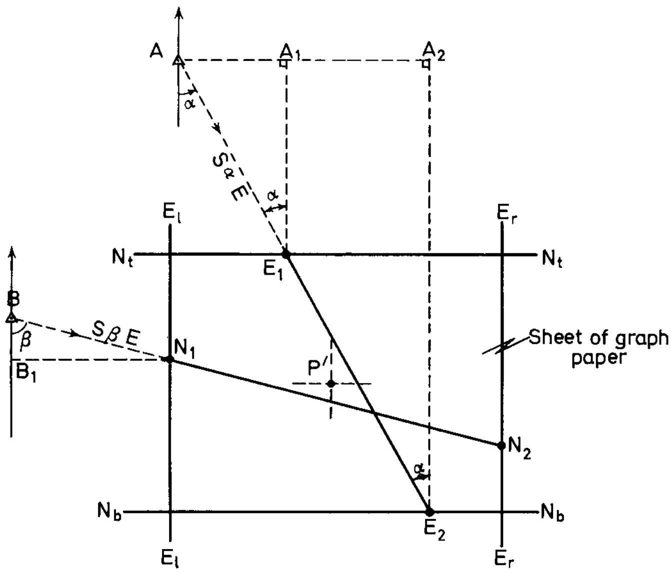

Figure 2.19

Similarly $\qquad E_2 = E_A + (N_A - N_b)\tan\alpha$ (2.36b)

If the bearing of the ray is west the sign before the brackets is negative, i.e. $E_A - (N_A - N_t)\tan\alpha$, etc. Assuming β is greater than 45° then

$$N_1 = N_B - BB_1 = N_B - B_1N_1\cot\beta$$

$$\therefore\ N_1 = N_B + (E_l - E_B)\cot\beta. \qquad (2.37a)$$

Similarly $\qquad N_2 = N_B - (E_r - E_B)\cot\beta$ (2.37b)

If the bearing of the ray in this case is north, the sign before the brackets is positive, i.e. $N_B + (E_l - E_B)\cot\beta$.

(5) Double cuts are calculated in this way for all the rays, which are then plotted and their mean point of intersection scaled off to give the co-ordinates of P.

(6) In the case of a resection, the bearing of, say, PA is obtained using the co-ordinates of A and the provisional co-ordinates of P. By adding the observed angles to the bearing PA, the remaining bearings are obtained. These bearings are then reversed and the computation treated as for an intersection.

2.6 TRILATERATION

Trilateration, based exclusively on measured horizontal distances, has gained acceptance because of the advent of EDM instrumentation. The geometric figures used are similar to those employed in triangulation, although not as standardized due to greater control of scale error. It was originally considered that trilateration would supersede triangulation as a method of control due to the scale error factor. However, subsequent results have shown that the system is liable to a rapid accumulation of azimuth error, thereby requiring a dense system of azimuth control points.

The fact that there is no horizontal angle measurement required in trilateration would appear to make it more rapid and thus, at first glance, more economical than triangulation. However, much depends on the length of line involved and the accuracy requirements.

All EDM equipment measures slope distance, which therefore needs to be reduced to the horizontal at some datum level. This requires then not only the measurement of slope length, but also the relative levels of the control points and instrument heights, or the measurement of vertical angles.

EDM instruments are calibrated for the velocity of electromagnetic waves under certain standard meteorological conditions. Thus actual meteorological conditions along the measuring path need to be known in order to correct the measured distance. At the present time this is not a practical proposition and one has to be content with the measurement of temperature and pressure at each end of the line being measured. For the best possible results under these conditions one requires carefully calibrated thermometers and barometers hung as high as possible by the instruments and read at the same instant of measurement. In order to comply with this latter requirement some form of inter-communication is necessary.

Similar precautions are also required when measuring the vertical angles. In order to achieve the accuracy required one needs to use highly precise theodolites preferably with automatic vertical circle indexing. Ideally, simultaneous reciprocal observations are necessary. If vertical angles are possible at only one end of the line, then corrections for curvature and refraction must be applied. Also depending on the terrain and accuracy requirement it may be necessary to consider the effect of 'deviation of the vertical' on the angles measured.

It would appear therefore that not only is trilateration possibly less economical than triangulation but on consideration of the above error sources (Chrzanowski and Wilson 1967) it may also prove less accurate. There appears to be conflicting evidence on this point (Burke 1971), although (Hodges 1967) has shown conclusively that angles computed through a trilateration are as accurate as those measured with a 1″ theodolite on the same control net.

A further reason why trilateration has not superseded triangulation must be in the superior internal checks given by triangulation. For instance, a triangle with three angles measured has an angle check whereas with three sides measured there is none; a braced quadrilateral with angles observed has four conditions (three angles, one side) to be satisfied, whereas with the sides there is only the single condition that the computed total angle at one corner equals the sum of the two computed component angles.

Network design is therefore especially critical in trilateration. In order to obtain sufficient redundancy for checks on the accuracy, the geometric figures become quite complicated. For instance, to obtain the same redundancy as a triangulation braced quadrilateral, a pentagon with all ten sides measured would have to be used. Indeed, experts in trilateration analysis have proposed the use of the hexagon, with all sides measured (20 giving *10* checks) as the basic network figure. However, from the practical viewpoint, pentagons and hexagons with all stations intervisible are difficult to establish in the field. Thus, from the logistic viewpoint, trilateration would require as much organization as triangulation.

The network may be computed by the method of variation of co-ordinates as already indicated (Chapter 1), or the following less rigorous approaches used:

(1) The simplest approach is to derive the angles of the figures from the lengths using the half-angle equation

$$\tan\frac{A}{2} = \left(\frac{s(s-a)}{(s-b)(s-c)}\right)^{\frac{1}{2}}, \qquad \text{where} \quad 2s = (a+b+c) \tag{2.38}$$

These angles are then used to calculate bearings around the network. In this way a closed traverse is produced and adjusted to give the final co-ordinates. Alternatively, the co-ordinates may be found directly using equations (2.34). If the survey is to be tied into the national grid, the lengths would need to be reduced to the spheroid and used in the above manner to produce 'provisional' co-ordinates. The provisional co-ordinates would be used to compute $(t - T)$ and scale factor (SF) corrections (refer to *Sections 2.11, 2.14* and *2.15*) which would be applied to the angles and lengths respectively to produce their grid equivalents. These values used in the direct equations (2.34) would give the grid co-ordinates of the points.

(2) Direct co-ordination of the control points can be made, without the use of angles. To find the co-ordinates of C, given the co-ordinates of A and B, and the length of the sides a, b, c of the triangle

$$E_c = \tfrac{1}{2}(E_A + E_B) + \frac{a^2 - b^2}{2c^2}(E_A - E_B) - \frac{2\Delta}{c^2}(N_A - N_B) \tag{2.39a}$$

and

$$N_c = \tfrac{1}{2}(N_A + N_B) + \frac{a^2 - b^2}{2c^2}(N_A - N_B) + \frac{2\Delta}{c^2}(E_A - E_B) \tag{2.39b}$$

where A, B and C are in clockwise order, and

$$\Delta = \{s(s-a)(s-b)(s-c)\}^{\frac{1}{2}}$$

If the survey is to be tied into the national grid, the SF would need to be found from 'provisional co-ordinates' and applied to the spheroidal lengths to give the grid lengths. These latter lengths are then used in the formula to give the grid co-ordinates.

Dr T. L. Thomas (1971) offers the following alternative equations for trilateration computation

$$\Delta E = E_B - E_A \qquad \Delta N = N_B - N_A \qquad c^2 = \Delta E^2 + \Delta N^2$$

$$p = \frac{\Delta E}{c} \qquad q = \frac{\Delta N}{c} \qquad k = \frac{(b^2 + c^2 - a^2)}{2c} \qquad h = (b^2 - k^2)^{\frac{1}{2}}$$

Then $\quad E_C = E_A + pk - qh \qquad N_C = N_A + qk + ph$

Checks

$$a^2 = (E_C - E_B)^2 + (N_C - N_B)^2$$
$$b^2 = (E_C - E_A)^2 + (N_C - N_A)^2$$

It is assumed in the above that C is to the left of $\overrightarrow{AB}$.

2.6.1 Triangulateration

As its name implies, triangulateration is simply the combining of triangulation and trilateration to produce a control system in which all the angles and sides are measured.

From the accuracy point of view, the system should be very strong, possessing all the advantages of both systems from which it is derived. The improvement in the redundancy checks for a braced quadrilateral and a central point pentagon are shown below:

Quadrilateral	*Triangulation*	*Trilateration*	*Triangulateration*
No. of directions	12	0	12
No. of sides	1	6	6
No. of checks	4	1	9
Pentagon			
No. of directions	20	0	20
No. of sides	1	10	10
No. of checks	6	4	15

Whilst it is generally acknowledged that triangulateration is more accurate than the previously-mentioned systems, one must consider whether or not it is economically justified. The logistics of the system will certainly not be equal to sum of the previous two methods, for once one has set up at the observation station and established targets/reflectors on the stations to be observed, a skilled surveyor could acquire all the necessary field data with little extra time and effort. The use of electronic 'total stations' makes the prospect even more viable and may justify the initial high capital expenditure involved. Further, as there would be little or no accumulation of scale and azimuth error, ill-conditioned figures could be utilized, thereby reducing the reconnaissance time.

It should be possible through pre-survey analysis to optimize the system so that every station in the network need not be occupied thus further improving the viability. The adjustment of such a network containing dissimilar quantities presents no difficulty if computer facilities are available. Using the variation of co-ordinates method, all the data can be adjusted *en masse* to produce the corrected co-ordinates of the network plus a complete error analysis and an *a posteriori* weighting of the field data.

It is thus evident that triangulateration is to be preferred over the use of triangulation or trilateration and thus seems to be modern practice. However, it is unlikely to supersede traversing because of the basic difference between the two systems and the accuracy/economy factor.

2.7 TRAVERSING

Since the advent of EDM equipment, traversing has emerged as the most popular method of establishing control networks not only in engineering surveying but also in geodetic work. In underground mining it is the only method of control applicable, whilst in civil engineering it lends itself ideally to surveys and dimensional control of route type projects such as highway and pipeline construction.

Traverse networks are, to a large extent, free of the limitations imposed on the other systems and compared with them, have the following advantages:

(a) Much less reconnaissance and organization required in establishing a single line of easily accessible stations compared with the laying-out of well-conditioned geometric figures.

(b) In conjunction with (a), the limitations imposed on the other systems by topographic conditions do not apply to traversing.
(c) The extent of observations to only two stations at a time is relatively small and flexible as to fluctuating atmospheric conditions compared with the extensive angular and/or linear observations at stations in the other systems. It is thus much easier to organize.
(d) Traverse networks are free of the strength of figure considerations so characteristic of triangular structures. Thus once again the organizational requirements are further reduced.
(e) Scale error does not accrue as in triangulation, whilst the use of longer sides, easily measured with EDM equipment, reduces azimuth swing errors.
(f) Traverse stations can usually be chosen so as to be easily accessible, as well as convenient for the subsequent densification of lower order control.
(g) Traversing permits the control to closely follow the route of a highway, pipeline or tunnel etc., with the minimum number of stations.

From the accuracy point of view it has been shown (Chrzanowski and Konecny 1965, and Adler and Schmutter 1971) that traversing is superior to triangulation and trilateration and, in some instances, even to triangulateration. However, it must be said that these findings are disputed by Phillips (1967). Nevertheless, it can be argued that, from the accuracy point of view, traversing compares more than favourably with the other methods.

Thus, from a consideration of all the above statements it is obvious that from the logistical point of view, traversing is far superior to all the other methods and offers at least equivalent accuracy. Refer to Volume 1 for further details on traversing.

2.8 TRIGONOMETRICAL LEVELLING

Trigonometrical levelling is used where difficult terrain precludes the use of conventional spirit levelling. The method is generally less accurate than spirit levelling, although in stable atmospheric conditions results comparable with precise levelling have been obtained.

2.8.1 Single observations

The principles of the method are shown in *Figure 2.20*. If the spheroïdal distance D between stations A and B is known, the difference in height may be computed using the observed vertical angle. Refraction of the line of sight through the atmosphere to B results in the telescope pointing to E. Thus the observed vertical angle of elevation (measured from the horizontal) is α, the angle of refraction is $\hat{r}$ and the angle due to the curvature of the Earth's surface is $\hat{c}$. Treating ABA' as a plane right-angled triangle (the chords to the arcs have been omitted from the *Figure*)

Difference in height of A and $B = A'B = H = D \tan \phi$

where $$\phi = \alpha + (\hat{c} - \hat{r}) \qquad (2.40a)$$

$$\therefore \; H = D \tan[\alpha + (\hat{c} - \hat{r})] \qquad (2.40b)$$

The term $(\hat{c} - \hat{r})$ is the combined correction for curvature and refraction, and is

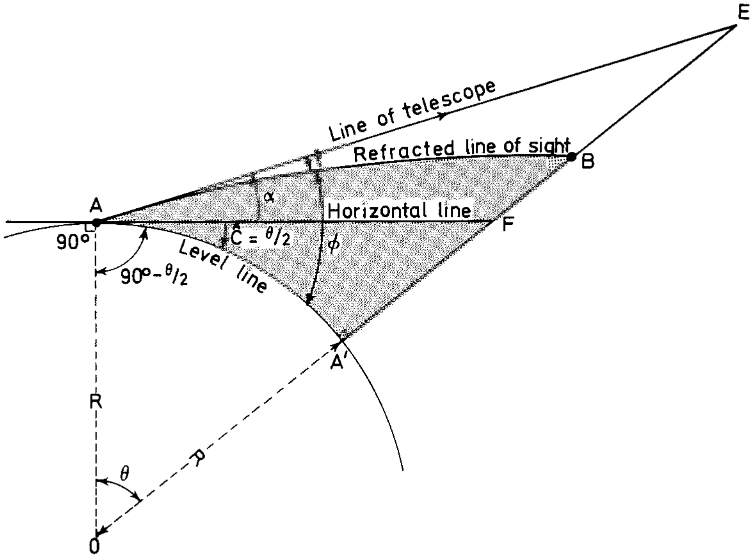

Figure 2.20

expressed in angular terms above. When expressed linearly (see Volume 1) the equation becomes

$$H = D \tan \alpha + (c - r) \tag{2.40c}$$

Similarly, considering the angle of depression β, measured from B to A, as in *Figure 2.21*, and treating $AB'B$ as the plane right-angled triangle

Difference in height of B and $A = B'A = H = D \tan \phi$

where $\phi = \beta - \hat{c} + \hat{r} = \beta - (\hat{c} - \hat{r})$ (2.41a)

$\therefore$ expressed angularly $H = D \tan[\beta - (\hat{c} - \hat{r})]$ (2.41b)

or expressed linearly $H = D \tan \beta - (c - r)$ (2.41c)

The difference in height of the instrument and signal has not been considered in the above analysis. Correction for this variation is most easily made by means of a simple sketch as in *Figure 2.22*

h_T = height of theodolite and h_s = height of signal

$\therefore$ Difference in height $AB = H - h_s + h_T = H - (h_s - h_T)$

$$\therefore\ H = D \tan \alpha + (c - r) - (h_s - h_T) \tag{2.42a}$$

Similarly in the observation from B to A (*Figure 2.23*)

Difference in height $BA = H + h'_s - h'_T = H + (h'_s - h'_T)$

$$\therefore\ H = D \tan \beta - (c - r) + (h'_s - h'_T) \tag{2.42b}$$

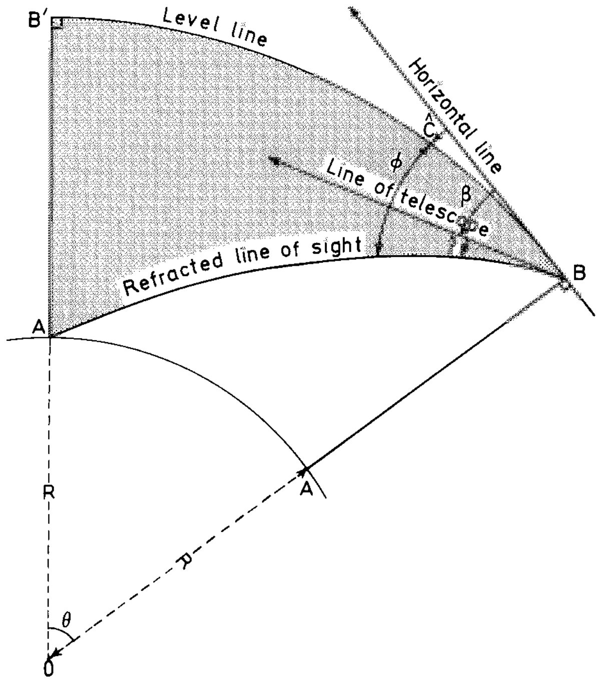

Figure 2.21

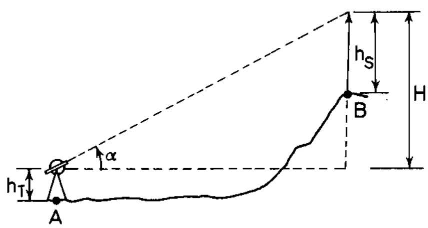

Figure 2.22

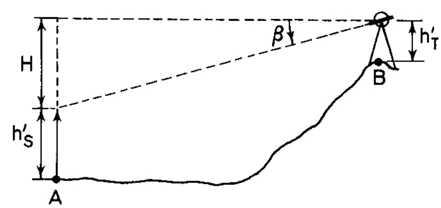

Figure 2.23

2.8.2 Reciprocal trigonometrical levelling

If the above observations from A and B are carried out simultaneously, the technique is called *reciprocal trigonometric levelling*, and the effect of curvature and refraction is considered to be eliminated. Thus summing equations (2.40a) and (2.41a) gives $2\phi = (\alpha + \beta)$ and $H = D \tan \phi$.

$$\therefore \quad H = D \tan\left(\frac{\alpha + \beta}{2}\right) \tag{2.43a}$$

The correction for the variations in the theodolite and signal heights is made by taking the mean of the values involved, noting that one of the corrections is $-(h_s - h_T)$, thus

$$H = D\tan\left(\frac{\alpha+\beta}{2}\right) + \frac{(h_s' - h_T') - (h_s - h_T)}{2} \tag{2.43b}$$

An alternative approach to the problem of variation in instrument and signal heights, is to adjust the observed angles to the values that would have been obtained, had the instrument and signal been the same height. From *Figure 2.24*, assuming the general case of the signal being taller than the theodolite, the correction e must be subtracted from the observed angle α, to give the adjusted angle α_0. This value is used in equation (2.43a) in place of α.

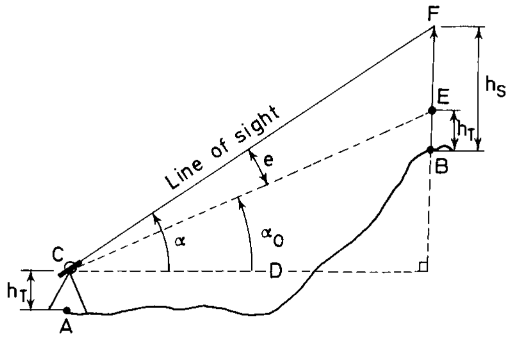

Figure 2.24

Assuming $CF \approx CE \approx D$, then

$$e'' = \left(\frac{h_s - h_T}{D}\right) \times 206\,265 \tag{2.44}$$

A similar sketch would show that for angles of depression β the correction is added to give β_0. Finally, in the event of both angles α and β being angles of depression, the basic equation becomes

$$H = D\tan\left(\frac{\alpha-\beta}{2}\right) \tag{2.45}$$

2.8.3 Curvature and refraction

In the case of single observations, corrections must be made for curvature and refraction. Whilst curvature is directly related to distance D and radius of the Earth R, refraction varies chiefly with temperature. Therefore for most accurate results several sets of reciprocal observations are carried out over the working area in order to arrive at a mean value for the coefficient of refraction K, which can then be used in the adjustment of single observations.

K is a measure of the curvature of the line of sight and is the ratio of the radius of curvature of the Earth R, to the radius of curvature of the line of sight R_s, therefore $K = R/R_s$. For instance, if $K = 1$, then $R = R_s$, and from the observational point of view, the Earth appears flat. The average value for K is 0.14, although substantially different values are obtained in unusual conditions, such as over icecaps. It should be noted that an alternative method for defining K exists, resulting in an average value of 0.07.

Considering the correction in angular terms, then from *Figure 2.20*

$$\hat{c} = \theta/2 \tag{2.46}$$

and $$\hat{r} = K\hat{c} = K(\theta/2) \tag{2.47}$$

From equations (2.40b) and (2.41b), as H and D are common, then

$$\alpha + (\hat{c} - \hat{r}) = \beta - (\hat{c} - \hat{r}) \qquad \therefore\ 2\hat{r} = \alpha - \beta + 2\hat{c}$$

Substituting for $\hat{c}$ from equation (2.46) gives $$\hat{r} = \tfrac{1}{2}(\theta + \alpha - \beta) \tag{2.48}$$

When α and β are both angles of depression $\hat{r} = \frac{1}{2}(\theta - \alpha - \beta)$. Substituting $\hat{r} = K(\theta/2)$ into equation (2.48) gives

$$K = \frac{\theta + \alpha - \beta}{\theta} \tag{2.49}$$

N.B. The angles α and β in equations (2.48) and (2.49) are first corrected for variation in instrument and signal heights—see *Worked example 2.13*, p. 121.

Considering the correction in linear terms (*Figure 2.20*)

$$c = FA' = D^2/2R \tag{2.50a}$$

where all the units are the same. Taking $R = 6372$ km and quoting D in km, then c in m is

$$c = \frac{1000D^2}{2 \times 6372} = 0.0785D^2 \text{ m} \tag{2.50b}$$

The linear value of $r = BE \approx D\hat{r}$ rad but $\hat{r} = K(\theta/2)$ and $\theta = D/R$, thus $\hat{r} = KD/2R$.

$$\therefore\ \text{Linear } r = BE = D\hat{r} = \frac{KD^2}{2R} \tag{2.51}$$

Combining the two $$(c - r) = \frac{D^2}{2R} - \frac{D^2K}{2R} = \frac{D^2}{2R}(1 - K) \tag{2.52}$$

where K is obtained from equation (2.49). If $K = 0.14$ is used, the equation approximates to $0.0674D^2$ m, where D is in km.

Thus these angular and/or linear values for c and r are used in the appropriate equations for single observations.

WORKED EXAMPLES

Example 2.8. The mean values of the angles A, B and C of a triangle as measured in a major triangulation were as follows, with the weights shown: A 50° 22′ 32.5″, 5; B 65° 40′ 47.5″, 3; C 63° 56′ 46.5″, 6. The length of the side BC was 37.5 km and the radius of the Earth 6267 km.

Calculate: (a) the spherical excess; (b) the probable values of the spherical angles. (LU)

(a) Spherical excess $$E'' = \frac{\frac{1}{2}ab \sin \hat{C} \times 206\,265}{R^2}$$

From the sine rule $b = a \sin B/\sin A$

$$\therefore \; E'' = \frac{a^2 \sin B \sin C}{2R^2 \sin A} \times 206\,265 = 3.9''$$

(b) Sum of adjusted spheroidal angles should equal $180° + E''$, i.e. $180°\,00'\,03.9''$.

Angle	*Mean value* ° ′ ″	*Weight*	*Reciprocal weight*	*Correction*	*Corrected angles* ° ′ ″
A	50 22 32.5	5	$\frac{1}{5} \times 30 =$ 6	$\frac{-2.6 \times 6}{21} = -0.7''$	50 22 31.8
B	65 40 47.5	3	$\frac{1}{3} \times 30 =$ 10	$\frac{-2.6 \times 10}{21} = -1.3''$	65 40 46.2
C	63 56 46.5	6	$\frac{1}{6} \times 30 =$	$\frac{-2.6 \times 5}{21} = -0.6''$	63 56 45.9
Sum	180 00 06.5		*Sum* = 21	*Sum* = $-2.6''$	180 00 03.9
	03.9				

$\therefore$ Correction $= -2.6''$

Example 2.9. Four triangulation stations are in the form of a triangle *ABC*, within which lies the fourth station *D*. The measured angles with the log sins of the outer angles are given below. Adjust the angles to the nearest second by the method of equal shifts.

Number		*Measured angle* ° ′ ″	Log sin	*Difference in LS for* 1″
1	*BAD*	26 31 32	$\bar{1}$.649 915 6	0.000 004 2
2	*ABD*	20 57 35	$\bar{1}$.553 532 9	5 5
3	*DBC*	35 05 09	$\bar{1}$.759 519 0	3 2
4	*BCD*	30 28 41	$\bar{1}$.705 186 3	3 6
5	*ACD*	26 59 46	$\bar{1}$.656 989 0	4 1
6	*CAD*	39 57 26	$\bar{1}$.807 680 7	2 5
7	*ADB*	132 30 50		
8	*BDC*	114 26 04		
9	*CDA*	113 03 06		

(LU)

Refer to *Figure 2.13(a)* and use the method outlined in *Section 2.3(3)*, p. 101.

Δ	Number	Angles °	′	″	First corr'n ″	Corrected angles ″	Central angles °	′	″	Second corr'n ″	Corrected angles ″
	1	26	31	32	1	33				−0.5	32.5
ABD	2	20	57	35	1	36				−0.5	35.5
	7	132	30	50	1	51	132	30	51	1	52
Sum		179	59	57							
	3	35	05	09	2	11				−0.5	10.5
BCD	4	30	28	41	2	43				−0.5	42.5
	8	114	26	04	2	06	114	26	06	1	7
Sum		179	59	54							
	5	26	59	46	−6	40				−0.5	39.5
CAD	6	39	57	26	−6	20				−0.5	19.5
	9	113	03	06	−6	00	113	03	00	1	1
Sum		180	00	18			359	59	57		

As the correction to the central angles is 1″, this automatically gives a correction of −0.5″ to each of the base angles of the triangles to restore them to 180°.

Number	Side condition Log sin (odd)	Number	Side condition Log sin (even)	Difference 1″	Correction ″	Final angles °	′	″
1	$\bar{1}$.649 915 6			42	−1	26	31	31.5
		2	$\bar{1}$.553 532 9	55	1	20	57	36.5
3	$\bar{1}$.759 519 0			32	−1	35	05	09.5
		4	$\bar{1}$.705 186 3	36	1	30	28	43.5
5	$\bar{1}$.656 989 0			41	−1	26	59	38.5
		6	$\bar{1}$.807 680 7	25	1	39	57	20.5
Sum	$\bar{1}$.066 423 6		$\bar{1}$.066 399 9	231				
	399 9							
Difference = 237								

But $\frac{237}{231}$ of 1″ ≈ 1″

The central angles are as shown at the end of the second correction. The final angles shown may now be rounded off to the nearest second.

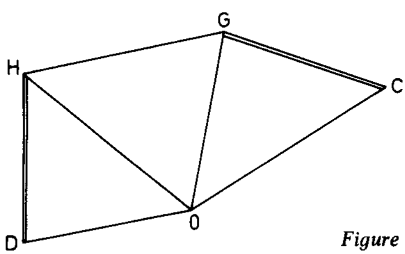

Figure 2.25

Example 2.10. In the triangulation network shown in *Figure 2.25* all the angles have been observed and the sides *DH* and *GC* measured as base and check base respectively, with the following results:

ΔDHO ° ′ ″	ΔHGO ° ′ ″	ΔGCO ° ′ ″
$\hat{D} = 79\ 47\ 05$	$\hat{H} = 77\ 28\ 58$	$\hat{G} = 82\ 22\ 17$
$\hat{H} = 58\ 32\ 35$	$\hat{G} = 36\ 02\ 38$	$\hat{C} = 71\ 29\ 47$
$\hat{O} = 41\ 40\ 05$	$\hat{O} = 66\ 28\ 48$	$\hat{O} = 26\ 08\ 17$
$DH = 426.58$ m		$GC = 486.83$ m

Adjust the observed angles by 'equal shifts' to give a consistent figure. (ICE)

The requirement in this question is that the figure should be adjusted so that the 'computed' value of the check base equals the 'measured' value.

First, adjust each triangle. Summing the angles of each triangle gives: $DHO = 179°\ 59'\ 45''$, $HGO = 180°\ 00'\ 24''$ and $GCO = 180°\ 00'\ 21''$. There is thus a correction per angle of 5″, = 8″ and −7″ per triangle, respectively. The corrected angles are now as follows:

ΔDHO ° ′ ″	ΔHGO ° ′ ″	ΔGCO ° ′ ″
$\hat{D} = 79\ 47\ 10$	$\hat{H} = 77\ 28\ 50$	$\hat{G} = 82\ 22\ 10$
$\hat{H} = 58\ 32\ 40$	$\hat{G} = 36\ 02\ 30$	$\hat{C} = 71\ 29\ 40$
$\hat{O} = 41\ 40\ 10$	$\hat{O} = 66\ 28\ 40$	$\hat{O} = 26\ 08\ 10$

By the sine rule through *Figure 2.25*, the computed value for

$$GC = \frac{HD \sin H\hat{D}O \sin G\hat{H}O \sin G\hat{O}C}{\sin H\hat{O}D \sin O\hat{G}H \sin O\hat{C}G}$$

Taking logs	*Difference for 10″*	° ′ ″	*Difference for 10″*
log 426.58 = 2.630 001			
log sin 79° 47′ 10″ = $\bar{1}$.993 063	3.7	log sin 41 40 10 = $\bar{1}$.822 712	23.7
log sin 77° 28′ 50″ = $\bar{1}$.989 548	4.7	log sin 36 02 30 = $\bar{1}$.769 653	29.0
log sin 26° 08′ 10″ = $\bar{1}$.643 951	42.8	log sin 71 29 40 = $\bar{1}$.976 943	7.2
$\sum$ = 2.256 563		$\sum$ = $\bar{1}$.569 308	

∴ Log *GC* = 2.687 255 = 486.69 m (computed)
Log *GC* = 2.687 378 = 486.83 m (measured)

Difference = 0.000 123

This difference must be adjusted among the six angles used in the computation so that the final log value of *GC* (computed) would equal that of *GC* (measured).

Sum of differences for 10″ = 111.1

$$\therefore \text{ Correction per angle} = \left(\frac{123}{111}\right) \times 10'' = 11''$$

As the final log value of *GC* (computed) needs to be increased, then inspection of the log computation shows that angles *HDO*, *GHO* and *GOC* would be adjusted by +11″ each, whilst *HOD*, *OGH* and *OCG* are adjusted by −11″ each. The three angles not used in the computation remain as shown in the first correction.

Example 2.11. In a triangle *ABC*, *AB* = 5205.0 m, *AC* = 5113.8 m and the angles *B* and *C* were 55° 01′ 05″ and 62° 04′ 20″, respectively. Station *A* could not be occupied and observations were taken from satellite station *P*, 11.1 m from *A* and inside the triangle. Instrument readings at *P* were: on *A*, 0° 00′ 00″; on *C*, 148° 28′ 40″; on *B*, 211° 31′ 10″. Calculate the angular error in the triangle. (LU)

As the theodolite is a clockwise-measuring instrument, the instrument readings at *P* serve to fix the relative positions of *A*, *B* and *C* (*Figure 2.26*), as well as the following angular values: $A\hat{P}C = 148° 28' 40''$, $C\hat{P}B = 63° 02' 30''$, $B\hat{P}A = 148° 28' 50''$.

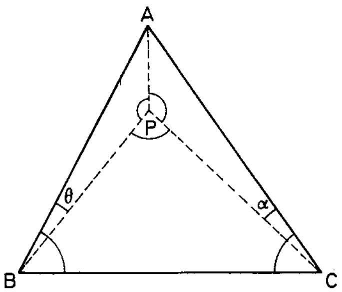

Figure 2.26

By the sine rule in ΔAPC

$$\alpha'' = \frac{AP \sin A\hat{P}C}{AC} \times 206\,265 = \frac{11.1 \sin 148° 28' 40''}{5113.8} \times 206\,265$$

$$= 234'' = 0° 03' 54''$$

Similarly in ΔAPB

$$\theta'' = \frac{AP \sin B\hat{P}A}{AB} \times 206\,265 = \frac{11.1 \sin 148° 28' 50''}{5205.0} \times 206\,265$$

$$= 230'' = 0° 03' 50''$$

$$\therefore \quad C\hat{A}B = C\hat{P}B - \alpha'' - \theta''$$
$$= 63° 02' 30'' - 03' 54'' - 03' 50'' = 62° 54' 46''$$

$$\therefore \text{ Angular error} = 180° - (\hat{A} + \hat{B} + \hat{C})$$
$$= 180° - (62° 54' 46'' + 55° 01' 05'' + 62° 04' 20'')$$
$$= +11''$$

Example 2.12. (a) Define the Coefficient of Refraction K, and show how its value may be obtained from simultaneous reciprocal trigonometric levelling observations.

(b) Two triangulation stations A and B are 2856.85 m apart. Observations from A to B gave a mean vertical angle of $+01°\ 35'\ 38''$, the instrument height being 1.41 m and the target height 2.32 m. If the level of station A is 156.86 m OD and the value of K for the area is 0.16, calculate the reduced level of B (radius of Earth = 6372 km). (KP)

(a) Refer to *Section 2.8.3.*

(b) This part will be answered using both the angular and the linear approaches.

Angular method

Difference in height of $AB = H = D \tan[\alpha + (\hat{c} - \hat{r})]$ where $\hat{c} = \theta/2$ and

$$\theta = \frac{D}{R} = \frac{2856.85}{6\,372\,000} = 0.000\,448 \text{ rad}$$

$$\therefore\ \hat{c} = 0.000\,224 \text{ rad}$$

$$\hat{r} = K(\theta/2) = 0.16 \times 0.000\,224 = 0.000\,036 \text{ rad}$$

$$\therefore\ (\hat{c} - \hat{r}) = 0.000\,188 \text{ rad} = 0°\ 00'\ 38.8''$$

$$\therefore\ H = 2856.85 \tan(01°\ 35'\ 38'' + 0°\ 00'\ 38.8'') = 80.03 \text{ m}$$

From *Figure 2.22*

$$\text{RL of } B = \text{RL of } A + h_T + H - h_s$$

$$= 156.86 + 1.41 + 80.03 - 2.32 = 235.98 \text{ m}$$

Linear method

$$H = D \tan \alpha + (c - r)$$

where $$(c - r) = \left(\frac{D^2}{2R}\right)(1 - K) = \frac{2856.85^2}{2 \times 6\,372\,000} \times 0.84 = 0.54 \text{ m}$$

$$D \tan \alpha = 2856.85 \tan(01°\ 35'\ 38'') = 79.49 \text{ m}$$

$$\therefore\ H = 79.49 + 0.54 = 80.03 \text{ m}$$

Example 2.13. Two stations A and B are 1713 m apart. The following observations were recorded: height of instrument at A 1.392 m, and at B 1.464 m; height of signal at A 2.199 m, and at B 2 m. Elevation to signal at B $1°\ 08'\ 08''$, depression angle to signal at A $1°\ 06'\ 15''$. If 1″ at the Earth's centre subtends 30.393 m at the Earth's surface, calculate the difference of level between A and B and the refraction correction.(LU)

From equation (2.43b) $$H = D \tan\left(\frac{\alpha + \beta}{2}\right) + \frac{(h'_s - h'_T) - (h_s - h_T)}{2}$$

where h_T = height of instrument at A; h_s = height of signal at B; h'_T = height of instrument at B; h'_s = height of signal at A.

$$\therefore\ H = 1713 \tan\left(\frac{(1°\ 08'\ 08'') + (1°\ 06'\ 15'')}{2}\right) + \frac{(2.199 - 1.464) - (2.000 - 1.392)}{2}$$

$$= 33.490 + 0.064 = 33.55 \text{ m}$$

Using the alternative approach (equation (2.44))

Correction to angle of elevation

$$e'' = \frac{1.392 - 2.000}{1713.0} \times 206\,265 = -73.2''$$

$$\therefore \quad \alpha = (1^\circ\ 08'\ 08'') - (01'\ 13.2'') = 1^\circ\ 06'\ 54.8''$$

Correction to angle of depression

$$e'' = \frac{(2.199 - 1.464)}{1713.0} \times 206\,265 = 88.5''$$

$$\therefore \quad \beta = (1^\circ\ 06'\ 15'') + (01'\ 28.5'') = 1^\circ\ 07'\ 43.5''$$

$$\therefore \quad H = 1713 \tan\left(\frac{1^\circ\ 06'\ 54.8'' + 1^\circ\ 07'\ 43.5''}{2}\right) = 33.55\text{ m}$$

Refraction correction $\hat{r} = \frac{1}{2}(\theta + \alpha - \beta)$

where $\theta'' = 1713.0/30.393 = 56.4''$

$$\therefore \quad \hat{r} = \tfrac{1}{2}(56.4'' + (1^\circ\ 06'\ 54.8'') - (1^\circ\ 07'\ 43.5'')) = 3.8''$$

and also $K = \dfrac{\hat{r}}{\theta/2} = \dfrac{3.8''}{28.2''} = 0.14$

Example 2.14. Two points A and B are 8 km apart and at levels of 102.50 m and 286.50 m OD, respectively. The height of the target at A is 1.50 m and at B 3.00 m, while the height of the instrument in both cases is 1.50 m. If 31 m on the Earth's surface subtends 1″ of arc at the Earth's centre and the effect of refraction is one seventh that of curvature, predict the observed angles from A to B and B to A. (KP)

With reference to *Figure 2.20*, it is required to find α, the observed angle, given the value for ϕ.

Difference in level A and $B = H = 286.50 - 102.50 = 184.00$ m

$$\therefore \text{ by radians } \phi'' = \frac{184}{8000} \times 206\,265 = 4744'' = 1^\circ\ 19'\ 04''$$

Angle subtended at the centre of the Earth $\theta'' = \dfrac{8000}{31} = 258''$

$\therefore$ Curvature correction $\hat{c} = \theta/2 = 129''$ and $\hat{r} = \hat{c}/7 = 18''$

Now $H = D \tan \phi$

where $\phi = \alpha + (\hat{c} - \hat{r})$

$$\therefore \quad \alpha = \phi - (\hat{c} - \hat{r}) = 4744'' - (129'' - 18'') = 4633'' = 1^\circ\ 17'\ 13''$$

Similarly from equation (2.41) $\phi = \beta - (\hat{c} - \hat{r})$

$$\therefore \quad \beta = \phi + (\hat{c} - \hat{r}) = 4855'' = 1^\circ\ 20'\ 55''$$

The observed angle α must be corrected for variation in instrument and signal heights. Normally the correction is subtracted from the observed angle to give the truly reciprocal angle. In this example, α is the truly reciprocal angle, thus the correction must be added in this reverse situation

$e'' = [(h_s - h_T)/D] \times 206\,265 = [(3.00 - 1.50)/8000] \times 206\,265 = 39''$

$\therefore\ \alpha = 4633'' + 39'' = 4672'' = 1°\,17'\,52''$

Example 2.15. A gas drilling-rig is set up on the sea bed 48 km from each of two survey stations which are on the coast and several kilometres apart. In order that the exact position of the rig may be obtained, it is necessary to erect a beacon on the rig so that it may be clearly visible from theodolites situated at the survey stations, each at a height of 36 m above the high water mark.

Neglecting the effects of refraction, and assuming that the minimum distance between the line of sight and calm water is to be 3 m at high water, calculate the least height of the beacon above the high water mark, at the rig. Prove any equations used.

Calculate the angle of elevation that would be measured by the theodolite when sighted on to this beacon, taking refraction into account and assuming that the error due to refraction is one seventh of the error due to curvature of the Earth. Mean radius of Earth = 6273 km. (ICE)

From *Figure 2.27*

$D_1 = (2h_1R)^{\frac{1}{2}}$ (equation (2.50a), see Volume 1 for proof)

$\therefore\ D_1 = (2 \times 33 \times 6\,273\,000)^{\frac{1}{2}} = 20.35$ km

$\therefore\ D_2 = 48 - D_1 = 27.65$ km

$\therefore$ since $D_2 = (2h_2R)^{\frac{1}{2}}$

$h_2 = 61$ m, and to avoid grazing by 3 m, height of beacon = 64 m

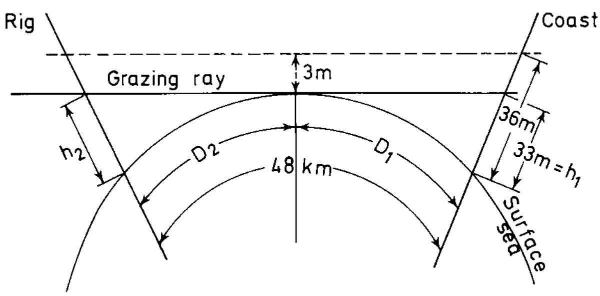

Figure 2.27

From *Figure 2.20*

Difference in height of beacon and theodolite = 64 − 36 = 28 m; observed vertical angle $\alpha = \phi - (\hat{c} - \hat{r})$ for angles of elevation, where

$$\phi'' = \frac{28 \times 206\,265}{48\,000} = 120.3''$$

$$\hat{c} = \theta/2$$

where $$\theta'' = \left(\frac{48}{6273}\right) \times 206\,265 = 1578.3''$$

$$\therefore\ \hat{c} = 789.2'' \qquad \text{and} \qquad \hat{r} = \hat{c}/7 = 112.7''$$

$$\therefore\ \alpha = 120.3'' - 789.2'' + 112.7'' = -556.2'' = -0^\circ\, 09'\, 16''$$

The negative value indicates α to be an angle of depression, not elevation, as quoted in the question.

Example 2.16. The co-ordinates of station P are to be found from a semi-graphic solution of an intersection. Considering only two of the rays, calculate the values of the double cutting points, given: bearing AP = S 05° 20′ 20″ E; bearing BP = N 84° 10′ 30″ E; co-ordinates of A = E 3500.05 m, N 5085.38 m; co-ordinates of B = E 1054.66 m, N 2980.08 m; provisional co-ordinates of P = E 3640 m, N 3720. Co-ordinates of graph paper: top edge = N 3800 m and bottom edge = N 3600 m; left-hand edge = E 3600 m and right-hand edge = E 3800 m. (KP)

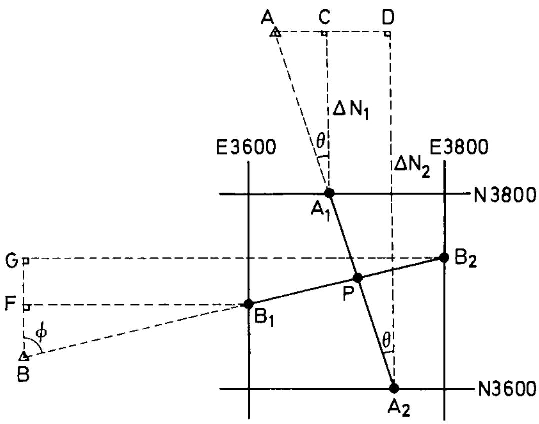

Figure 2.28

From *Figure 2.28*

$$CA_1 = \Delta N_1 = 5085.38 - 3800 = 1285.38\ \text{m}$$
$$DA_2 = \Delta N_2 = 1485.38\ \text{m}$$
$$AC = \Delta N_1 \tan\theta = 1285.38 \tan 05^\circ\, 20'\, 20'' = 120\,121\ \text{m} = \Delta E_{A_1}$$
$$AD = \Delta N_2 \tan\theta = 138.812\ \text{m} = \Delta E_{A_2}$$

Therefore, double cutting points A_1 and A_2

$$A_1 = E_A + \Delta E_{A_1} = 3500.05 + 120.12 = \text{E}\ 3620.17\ \text{m}$$
$$A_2 = E_A + \Delta E_{A_2} = 3500.05 + 138.81 = \text{E}\ 3638.86\ \text{m}$$

Similarly for B

$$FB_1 = \Delta E_{B_1} = 3600 - 1054.60 = 2545.40\ \text{m}$$
$$GB_2 = \Delta E_{B_2} = 2745.34\ \text{m}$$
$$\therefore\ BF = \Delta E_{B_1} \cot\phi = 2545.34 \cot 84^\circ\, 10'\, 30'' = 259.668\ \text{m} = \Delta N_{B_1}$$
$$BG = \Delta E_{B_2} \cot\phi = 280.071\ \text{m} = \Delta N_{B_2}$$

Double cuts B_1 and B_2

$$B_1 = N_B + \Delta N_{B_1} = 2980.08 + 259.668 = \text{N } 3239.75 \text{ m}$$
$$B_2 = N_B + \Delta N_{B_2} = \text{N } 3260.15 \text{ m}$$

EXERCISES

2.5. A polygon *ABCDEA* with a central station *O* forms part of a triangulation scheme. The angles in each of the figures which form the complete network are being adjusted, and in this case the angles in each of the triangles *DOE* and *EOA* have already been adjusted and need no further correction.

Making use of the information given in the Table below, use the method of equal shifts to determine the correction that must be applied to each of the remaining angles. (ICE)

Triangle	*Angle*	*Observed value* ° ′ ″	Log sin	Log sin *difference for* 1″
AOB	*OAB*	40 17 57	$\bar{1}$.810 755 7	25
	OBA	64 11 20	$\bar{1}$.954 355 6	10
	AOB	75 30 52		
BOC	*OBC*	37 22 27	$\bar{1}$.783 201 4	28
	OCB	71 10 50	$\bar{1}$.976 139 0	7
	BOC	71 26 22		
COB	*OCD*	24 51 25	$\bar{1}$.623 615 4	46
	ODC	51 48 47	$\bar{1}$.895 421 4	17
	COD	103 19 33		
		Adjusted values ° ′ ″		
DOE	*ODE*	67 18 59	$\bar{1}$.965 036 2	
	OED	51 02 00	$\bar{1}$.890 707 1	
	DOE	61 39 01		
EOA	*OEA*	116 47 40	$\bar{1}$.950 671 4	
	OAE	15 08 02	$\bar{1}$.416 766 2	
	EOA	48 04 18		

(*Answer: OAB* 4.8″; *OBA* −5.8″; *AOB* −8.0″; *OBC* 14.8″; *OCB* 4.2″; *BOC* 2.0″; *OCD* 12.8″; *ODC* 2.2″; *COD* 0″)

2.6. A bridge is to be built across a river where it is approximately 1.5 km wide and a survey station has been established on each bank to mark the centre line.

Excluding the use of electronic devices, describe how the distance between these two stations can be determined to a high degree of accuracy. Outline the calculations involved and quote the relevant equations at each stage. (ICE)

(*Answer:* Triangulation; braced quadrilateral; base line; figural adjustment)

2.7. In order to demonstrate how a triangulation is adjusted by the method of equal shifts, consider a figure which consists of a triangle *ABC* with a central (internal) point *D* and in which the following fictitious angles are given as 'observed angles': *BAD* = *ABD* = *CBD* = *BCD* = *ACD* = 30° 00′; *ADB* = *BDC* = *CDA* = 120° 00′; *CAD* = 33° 00′.

Although the error in ΔADC is so large that a gross mistake appears to have been made, adjust the angles of triangulation (to the nearest minute) to give a consistent figure. What are the five equations of condition to which the adjusted angles must conform? (ICE)

(*Answer:* $BAD = CBD = 30° 21'$; $ACD = 29° 21'$; $ABD = BCD = 29° 19'$; $CAD = 31° 19'$; $ADB = BDC = 120° 20'$; $CDA = 119° 20'$)

2.8. The details given below refer to observations made at a satellite station O, in order to determine the angle at an inaccessible station A in a triangle ABC. Compute the angle BAC.

Length $OA = 9.435$ m; bearing of side $OA = 0° 00' 00''$; length AB 2925 m; bearing of side $OB = 78° 46' 00''$; length AC 3426 m; bearing of side $OC = 100° 12' 00''$; $\log \sin 1'' = \bar{6}.685\,575$. (ICE)

(*Answer:* $B\hat{A}C = 21° 24' 26''$)

N.B. $\dfrac{1}{\sin 1''} = 206\,265''$. Throughout the author has avoided the use of $\sin 1''$, and converted radians to seconds using 206 265, the number of seconds in one radian.

2.9. Describe the difference between the techniques of reciprocal levelling and reciprocal trigonometrical levelling, and discuss the conditions under which each is most effectively used.

The horizontal distance between two stations P and Q is 5951.30 m. A theodolite at P is sighted onto a beacon adjacent to station Q at the same time as a theodolite at Q sights onto a beacon adjacent to station P. The following measurements are obtained: angle of elevation recorded at $P = 01° 19' 38''$; angle of depression recorded at $Q = 01° 21' 01''$; height of beacon at $P = 2.85$ m; height of beacon at $Q = 2.36$ m; height of instrument at $P = 1.36$ m; height of instrument at $Q = 1.47$ m.

Determine the difference of level between the two stations and the coefficient of atmospheric refraction. Assume the radius of the Earth is 6.37×10^6 m. (ICE)

(*Answer:* 139.18 m, 0.14)

2.10. The distance between two points A and B was 6336 m. B was 150 m above A. Calculate the angles observed from A and B with a theodolite assuming the instrument and signal heights to be equal and the effect of refraction to be one seventh that of curvature. Take the radius of the Earth as 6336 km. (LU)

(*Answer:* $A = 1° 19' 54''$, $B = 1° 22' 50''$)

2.9 THE SPHEROID

As engineering schemes grow in size and complexity so too must the surveying operations associated with the control of such schemes. This is already apparent in the motorway surveys extending for many kilometres. However, commensurate with the increase in scale of these surveys is the effect of the Earth's curvature. As the Earth may be regarded as flat only for surveys of limited extent, very large schemes require the use of spheroidal co-ordinates and projections.

2.9.1 The spheroid of reference

In order to compute surveys in spheroidal co-ordinates, i.e. latitude (ϕ) and longitude (λ), one must reference the co-ordinates to an arbitrarily defined geometrical figure, commonly called the *spheroid of reference.*

In this context the Earth may be considered as comprising three surfaces:

(a) The *physical surface*: this is the actual ground surface of the Earth, which, although a physical reality, is mathematically non-definable. Because of this, it cannot be used as a datum on which to compute position.
(b) The *geoid*: consider a series of interconnecting channels cut through the continents, which permit the entry of the sea into them. These waters, flowing freely under gravity and neglecting tidal effects, would form a *near* spheroidal-shaped equipotential surface called the *geoid* (meaning 'Earth-shaped'). This surface is everywhere normal to the direction of gravity, but because of variations of mass within the Earth, it is an irregular surface, requiring an infinite number of parameters to define it mathematically. It follows that this mean sea level surface also cannot be used as a basis for the computation of position. It is worth noting that levelling and astronomical observations are related to the direction of gravity by the plate bubbles of the instruments used, and so are referred to the geoid.
(c) The *spheroid of reference*: had the Earth been more spherical in shape, the ideal mathematical figure on which to compute the position of widely separated points would have been the sphere. However, geodetic observations for the figure of the Earth have shown that the polar radius is smaller than the equatorial radius, by about 20 km. Thus the simplest mathematically-definable figure which best fits the shape of the Earth, is the one produced by rotating an ellipse about its minor axis, and is called an *oblate spheroid.* Such a figure when used as a basis for the computation of position, is termed a *spheroid of reference.*

Figure 2.29 shows the relationship of the three surfaces to each other over a small section of the Earth's surface. At point A, normals to the spheroid and geoid are shown. The angle δ between these two directions is termed the *deviation of the vertical*, and is a measure of how much the two surfaces are out of coincidence. For instance, had $\delta = 0$, then the two surfaces in question would be parallel, and assuming no vertical

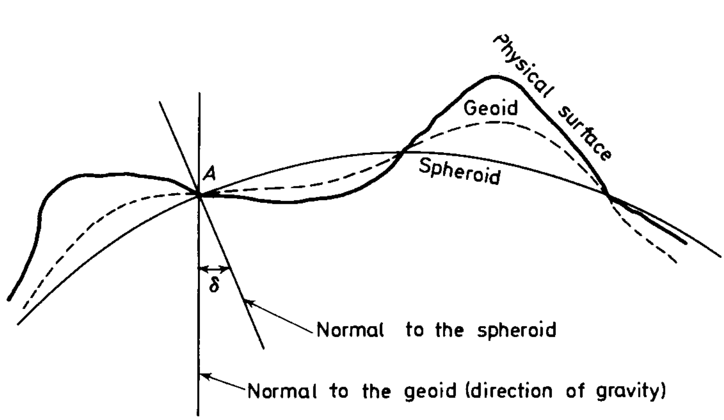

Figure 2.29

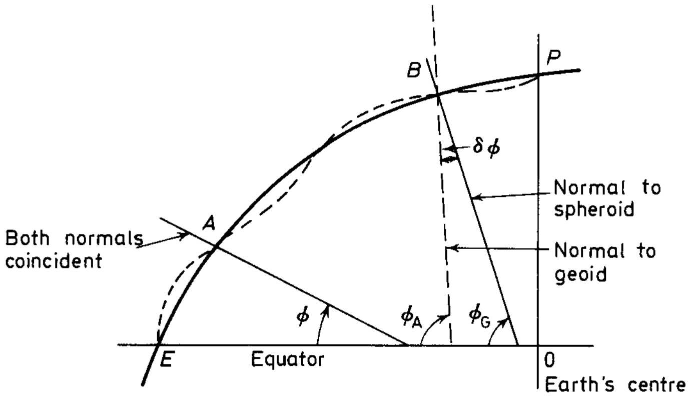

Figure 2.30

separation between the two, the surfaces would coincide or fit together. Thus the best fitting spheroid for use in any country is the one giving minimum values for the deviation of the vertical, throughout the country in question.

It is the work of the geodetic surveyor to find the best fitting spheroid of reference for any country. The procedure, reduced to its simplest elements, may be outlined as follows. *Figure 2.30* shows a meridional section of the geoid and spheroid (the geoid is grossly exaggerated). The angle which the normal to the geoid makes with the Equator is the astronomical latitude ϕ_A, in the meridional plane *POE*. Similarly, the geodetic or spheroidal latitude ϕ_G is formed by the normal to the spheroid. In the same way a distinction exists between the astronomical and geodetic longitudes λ_A and λ_G. Thus the astronomical co-ordinates of a point differ from the geodetic or spheroidal co-ordinates by the components, in their respective planes, of the deviation of the vertical. For example, $\phi_A - \phi_G = \delta\phi$, whilst in the case of the respective longitudes the deviation is $\delta\lambda \cos\phi$.

It is required to find the best fitting spheroid between *A* and *B*. To do this, let us assume the astronomical and geodetic co-ordinates are equal at the origin *A*, that is the deviations of the vertical are zero. A ground survey from *A* to *B* will enable the geodetic co-ordinates of *B* to be calculated using an assumed spheroid of reference. These values may then be compared with the astronomical co-ordinates observed at *B*, including astronomical and geodetic azimuth, and values for the deviation obtained. If the origin and spheroid assumptions are in error, then systematic deviations of the vertical will be noted as the survey proceeds. As previously stated, the best fitting spheroid of reference will be the one giving minimum deviation values throughout. The survey stations at which these comparisons are made are termed *Laplace* stations.

Computations on the spheroid for very long lines are extremely complicated and beyond the scope of this book. However, for lines less than about 40 km, semi-rigorous methods may be used, as described in the following sections.

2.10 COMPUTATION ON THE SPHEROID

2.10.1 Convergence of meridians

On the spheroid, directions are referred to the meridian (line of longitude) through the point concerned. Considering *Figure 2.31*, *A* and *B* are two points on the Earth's

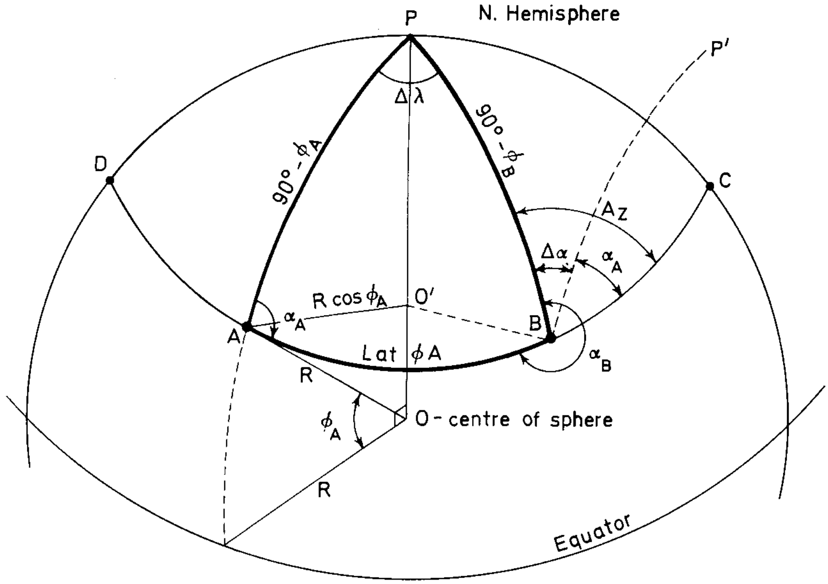

Figure 2.31

surface, P is the pole and BC an extension of line AB: the azimuth of AB is α_A and the azimuth of BA is α_B where $\alpha_B = \alpha_A + 180° + \Delta\alpha$ (BP' parallel to AP).

Thus, because of the convergence of meridians AP and BP, the reverse and forward azimuths of AB differ not by 180°, as in plane surveying, but by 180° + Δα. For most practical work

$$\Delta\alpha = \Delta\lambda \sin \phi_m \qquad (2.53)$$

where $\Delta\lambda$ = difference in longitude from A to B,
and ϕ_m = mean latitude of A and $B = \frac{1}{2}(\phi_A + \phi_B)$

A correction for convergence is essential when checking the bearing of a line by astronomy or gyro-theodolite. Its effect is zero at the Equator and increases with increase in latitude, and is a maximum if the line runs due east-west.

2.10.2 Azimuth and bearing

The *azimuth* of a line is its direction relative to true north, i.e. relative to the meridian circle passing through it. The *bearing* of a line is its direction relative to the meridian circle passing through the origin of the survey.

Thus from *Figure 2.31*, the bearing of BC is α_A, assuming A is the origin of the survey and ABC a straight line, while its azimuth A_Z is $\alpha_A + \Delta\alpha$. This fact is very important in the following computations on the spheroid.

For points in the southern hemisphere (*Figure 2.32*) it can be seen that the azimuth $A_Z = \alpha_A - \Delta\alpha$.

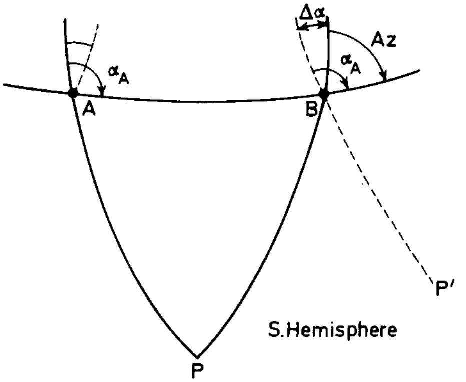

Figure 2.32

2.10.3 Latitude and longitude by the method of mean latitudes

Many methods exist for finding the latitude and longitude of points on the Earth's surface, all of which vary with the length of line involved and the accuracy required. If working on the sphere, spherical trigonometry may be used, but large errors can result due to the difficulty of finding the sines and tangents of the very small angles subtended at the centre of the sphere by the relatively short lines on the Earth's surface. The method of mean latitudes for short lines under 40 km is recommended for all such questions set in engineering examinations.

Consider line AB on the Earth's surface (*Figure 2.33*). As AB is relatively short, the meridians and parallels may be represented as straight lines forming a rectangular grid. The dotted line represents the mean position, with a mean azimuth of $\alpha_m = (\alpha_A + \Delta\alpha/2)$. Thereafter, the right-angle triangle ABC may be solved by plane trigonometry for the sides AC and BC, representing the difference in latitude ($\Delta\phi$) and longitude ($\Delta\lambda$).

On the spheroid it can be shown that

$$\Delta\lambda = \frac{L \sin \alpha_m}{\nu_m \cos \phi_m} \tag{2.54}$$

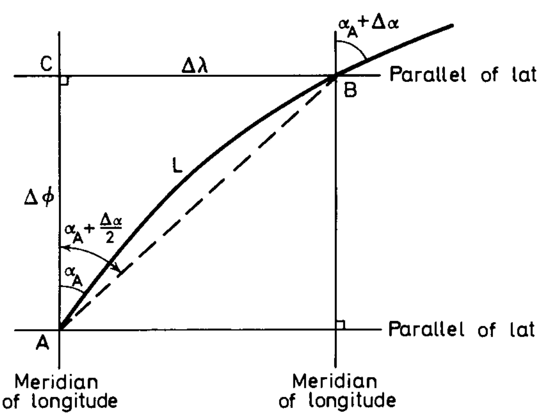

Figure 2.33

$$\text{and} \quad \Delta\phi = \frac{L \cos \alpha_m}{\rho_m} \tag{2.55}$$

where α_m is the mean of the forward and reverse azimuths, ignoring 180°, and ν_m and ρ_m are the values of the principle radii of curvature of the spheroid at the mean latitude. For work of limited extent, the surface of the sphere that best fits the spheroid at that point may be used, i.e. $R = (\nu\rho)^{\frac{1}{2}}$.

Thus, from *Figure 2.34(a)* Length $AB = \Delta\lambda R \cos \phi_m$

which is equivalent to side CB in *Figure 2.33*.

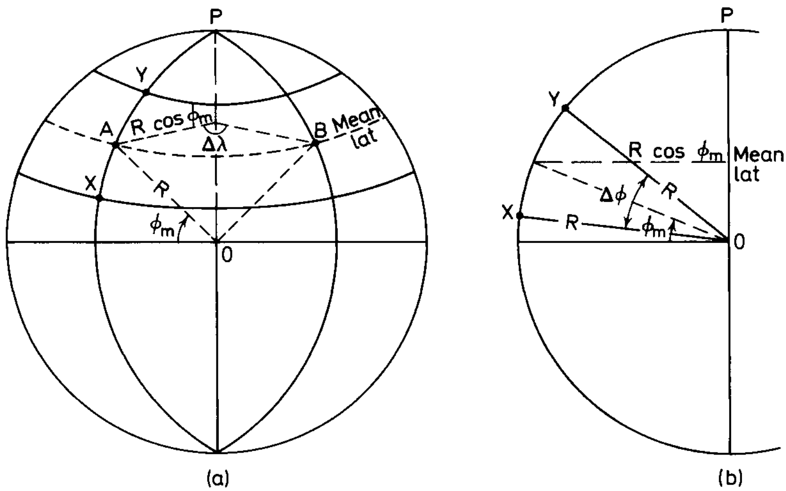

Figure 2.34

$$CB = L \sin\left(\alpha_A + \frac{\Delta\alpha}{2}\right) = L \sin \alpha_m \qquad \therefore \; \Delta\lambda R \cos \phi_m = L \sin \alpha_m$$

$$\Delta\lambda = \frac{L \sin \alpha_m}{R \cos \phi_m} \quad \text{(rad)} \tag{2.56a}$$

Similarly (*Figure 2.34(b)*) Length $XY = R\Delta\phi$

which is equivalent to AC in *Figure 2.33*.

$$AC = L \cos \alpha_m \qquad \therefore \; L \cos \alpha_m = R\Delta\phi$$

$$\Delta\phi = \frac{L \cos \alpha_m}{R} \quad \text{(rad)} \tag{2.56b}$$

The following examples will illustrate the various applications of the above data.

WORKED EXAMPLES

Example 2.17. The latitudes and longitudes of two stations A and B are given below, together with the distances on the Earth's surface corresponding to 1″ of latitude and longitude. Determine the azimuths of the lines AB and BA, and the distance AB. (LU)

Station	*Latitude* ° ′ ″	*Longitude* ° ′ ″
A	N 54 52 30	W 2 08 05
B	N 54 51 42	W 2 02 33

Latitude ° ′	*Distance on Earth's surface* 1″ *latitude*	1″ *longitude*
54 50	30.44 m	17.57 m
54 55	30.44 m	17.53 m

Student Note. As the meridians and parallels form a rectangular grid defining the north-south and east-west directions, the student may find it easier to use the quadrantal bearing system in the solution of these problems.

From *Figure 2.35*

Average latitude $= \phi_m =$ N 54° 52′ 06″ $\quad \Delta\phi = 48''\quad \Delta\lambda = 05'\,32''$

Therefore $\Delta\phi$ in terms of linear distance on the Earth's surface using the table provided in the question

$$\Delta\phi = 48'' \times 30.44\text{ m} = 1461.12\text{ m}$$

Similarly for $\Delta\lambda$, but one must first interpolate in the Table to find the value of 1″ longitude at latitude ϕ_m.

$$\therefore \text{ at } 54°\,52'\,06'' \qquad 1'' \text{ longitude} = 17.57\text{ m} - \frac{0.04}{300''} \times 126'' = 17.55\text{ m}$$

$$\therefore \Delta\lambda = 332'' \times 17.55 = 5826.60\text{ m}$$

In plane triangle ABC

$$\tan^{-1}\left(\alpha_B + \frac{\Delta\alpha}{2}\right) = \frac{\Delta\lambda}{\Delta\phi} = \frac{5826.60}{1461.12}$$

$$\therefore (\alpha_B + \Delta\alpha/2) = \text{N } 76°\,20'\,22'' \text{ W} = \alpha_m$$

now $\quad \Delta\alpha = \Delta\lambda \sin\phi_m = 332'' \sin 54°\,52'\,06'' = 04'\,32''$

$\therefore\ \alpha_B = 76°\,20'\,22'' - 02'\,16'' =$ N 76° 18′ 06″ W

$\therefore$ Azimuth of BA at B (measured clockwise from due north) = 283° 41′ 54″

Azimuth BA at A = N $(\alpha_B + \Delta\alpha)$ W = N 76° 22′ 38″ W

$\therefore$ Azimuth AB at A = S 76° 22′ 38″ E = 103° 37′ 22″

Length $AB = \Delta\phi \sec\alpha_m = 1461.12 \sec 76°\,20'\,22'' = 6186.75$ m

These problems frequently take the form of calculating the latitude and longitude of station B, given the latitude and longitude of A and the azimuth and distance AB. It is obvious then that the mean latitude is unknown and so the value for the convergence of meridians $\Delta\alpha$ cannot be computed. The solution therefore takes the form of successive approximations.

Example 2.18. A triangulation station P has a latitude of 45° 05′ 00″ N and longitude 90° 10′ 11″ W, and the distance from P to the next station Q is 7600 m. The azimuth of the line PQ at P is 140° 30′ 04″ and the Table below gives the lengths a and b in metres on

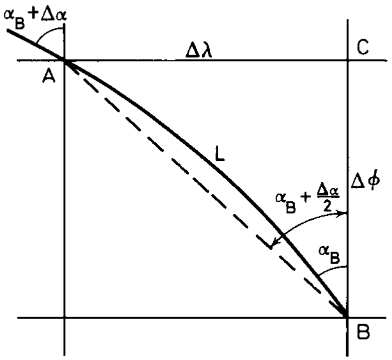

Figure 2.35

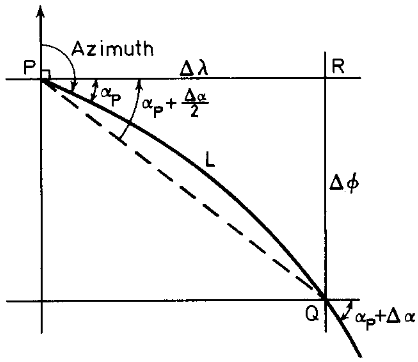

Figure 2.36

the Earth's surface corresponding to changes of 1″ in latitude and longitude, respectively. Determine to the nearest second the latitude and longitude of Q and the azimuth of QP at Q. (LU)

Latitude ° ′ ″	*Length a* (m)	*Length b* (m)
45 00 00	30.384 12	21.558 21
45 05 00	30.384 57	21.526 98

Refer to *Figure 2.36*; *first approximation*

$$\text{Azimuth of } PQ = A_Z = 140° \, 30' \, 04''$$

$$\alpha_p = 50° \, 30' \, 04''$$

$$\Delta\phi'' = \frac{L \cos \alpha_p}{a}$$

where value for a is taken at ϕ_p.

$$\therefore \; \Delta\phi = \frac{7600 \cos 50° \, 30' \, 04''}{30.384\,57} = 159.14''$$

$$\phi_m = \phi_p - \frac{\Delta\phi}{2} = (45° \, 05' \, 00'') - 80'' = 45° \, 03' \, 40'' \text{ north}$$

but $$\Delta\lambda'' = \frac{L \sin \alpha_p}{b}$$

where value for b is at ϕ_m

and $$b = 21.558\,21 - \frac{(0.031\,23 \times 220'')}{300''} = 21.535\,31 \text{ m}$$

$$\therefore \; \Delta\lambda = \frac{7600 \sin(50° \, 30' \, 04'')}{21.535\,31} = 272.4''$$

$$\therefore \; \text{Convergence} = \Delta\alpha = \Delta\lambda \sin \phi_m = 272.4 \sin 45° \, 03' \, 40'' = 192.8''$$

Final approximation $\qquad \alpha_p + \dfrac{\Delta\alpha}{2} = 50° 30' 04'' = 50° 31' 40''$

$$\Delta\phi = \frac{L \cos(\alpha_p + \Delta\alpha/2)}{a}$$

where a is abstracted for ϕ_m.

$$a = 30.384\,12 + \frac{0.000\,45 \times 220''}{300''} = 30.384\,45 \text{ m}$$

$$\therefore \quad \Delta\phi = \frac{7600 \cos 50° 31' 40''}{30.384\,45} = 159.05''$$

It can now be seen that the change in $\Delta\phi$ is negligible

$$\therefore \quad \Delta\lambda'' = \frac{L \sin(\alpha_p + \Delta\alpha/2)}{b}$$

where the value for b is as in the first approximation.

$$\therefore \quad \Delta\lambda = \frac{7600 \sin 50° 31' 40''}{21.535\,31} = 272.5''$$

Normally $\Delta\alpha$ would be recomputed using the final values for $\Delta\lambda$ and ϕ_m

$\therefore$ Latitude $Q = 45° 05' 00''$ N $- 02' 39'' = 45° 02' 21''$ N

and Longitude $Q = 90° 10' 11''$ W $- 04' 32'' = 90° 05' 39''$ W

Azimuth at $Q = 90° + (\alpha_p + \Delta\alpha) = 140° 30' 04'' + 192.8'' = 140° 33' 17''$

$\therefore$ Azimuth QP at $Q = 140° 33' 17'' + 180° = 320° 33' 17''$

Example 2.19. A straight line AB which forms part of a land boundary is to be set out. The geographical co-ordinates of A and B are as follows:

Point	*Latitude* ° ′ ″	*Longitude* ° ′ ″
A	34 40 28 S	148 12 02 E
B	34 44 05 S	148 04 20 E

The line is to be set out from A and B simultaneously so that the two parts may join somewhere between the points. Calculate the azimuth of each part of the line and describe briefly the method which should be used in setting out these azimuth angles. Assume that the mean radius of the Earth is 6.37×10^6 m. (ICE)

In *Figure 2.37* as the boundary is in the southern hemisphere the convergence is negative.

$$\therefore \quad \Delta\alpha = \Delta\lambda'' \sin\phi_m = 462'' \sin 34° 42' 16'' = 263'' = 4' 23''$$

From *Figure 2.34a* it is seen that length $\Delta\lambda = \Delta\lambda'' R \cos\phi_m$

$\therefore$ in *Figure 2.37* $\qquad AC = \dfrac{462'' \times 6.37 \times 10^6 \times \cos 34° 42' 16''}{206\,265} = 11\,730$ m

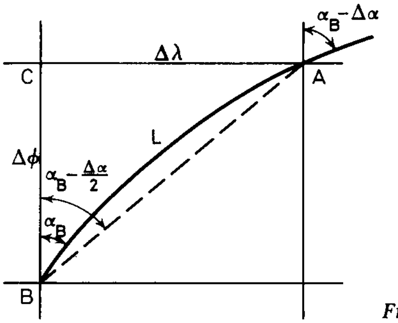

Figure 2.37

Similarly, in *Figure 2.34(b)* Length $\Delta\phi = \Delta\phi'' \times R$

$$\therefore \text{ in } \textit{Figure 2.37} \qquad BC = \frac{6.37 \times 10^6 \times 217''}{206\,265} = 6700 \text{ m}$$

N.B. That in both cases $\Delta\lambda$ and $\Delta\phi$ are brought to radians.

$$\therefore \text{ in } \Delta ABC \qquad \tan(\alpha_B - \Delta\alpha/2) = \frac{AC}{BC} = \frac{11\,730}{6700} = 60° \, 15' \, 32''$$

$\therefore \ \alpha_B = 60° \, 15' \, 32'' + 2' \, 11'' = 60° \, 17' \, 43''$ = Azimuth BA at B

$\therefore$ Azimuth BA at $A = \alpha_B - \Delta\alpha$ = N 60° 13′ 20″ E

$\therefore$ Azimuth AB at A = S 60° 13′ 20″ W = 240° 13′ 20″

Example 2.20. A ten-leg traverse from A to K was run alongside a motorway, running almost due east. The traverse was oriented by a gyro-theodolite at A, and the bearing of the final bay JK was computed as 87° 43′ 02″. A similar observation gave the azimuth of JK at K as 87° 50′ 48″. The total departure AK, from co-ordinates, was 12 545 m and the mean latitude N 52° 20′ 20″. Determine the angular adjustments to the traverse leg bearings. Radius of the Earth = 6.37×10^6 m. (KP)

Convergence of meridians = $\Delta\alpha'' = \Delta\lambda'' \sin\phi_m$

$$\text{From } \textit{Figure 2.34(a)} \qquad \Delta\lambda'' = \frac{L}{R\cos\phi_m} \times 206\,265$$

$$\therefore \ \Delta\alpha'' = \frac{L\tan\phi_m}{R} \times 206\,265 = \frac{12\,545 \tan 52° \, 20' \, 20''}{6.37 \times 10^6} \times 206\,265 = 526'' = 08' \, 46''$$

Azimuth of JK by gyro-theodolite = 87° 50′ 48″ = bearing $JK + \Delta\alpha$

$\therefore$ True bearing of JK = 87° 50′ 48″ − 08′ 46″ = 87° 42′ 02″

Computed bearing of JK = 87° 43′ 02″

$\therefore$ Error = +01′ 00″

Hence correction = −6″ per angle, which is distributed accumulatively on the bearings as follows: first bearing −6″, second bearing −12″, third bearing −18″, . . . , 10th bearing −60″.

2.11 SETTING OUT PARALLELS OF LATITUDE

The location of land boundaries along parallels of latitude has frequently formed examination questions in the past. Two methods are available.

2.11.1 Tangent method

In *Figure 2.38* tangent AF is established from A by turning off an angle of 90° from true north. Offsets such as ED and FB are set off from the tangent in the direction of the

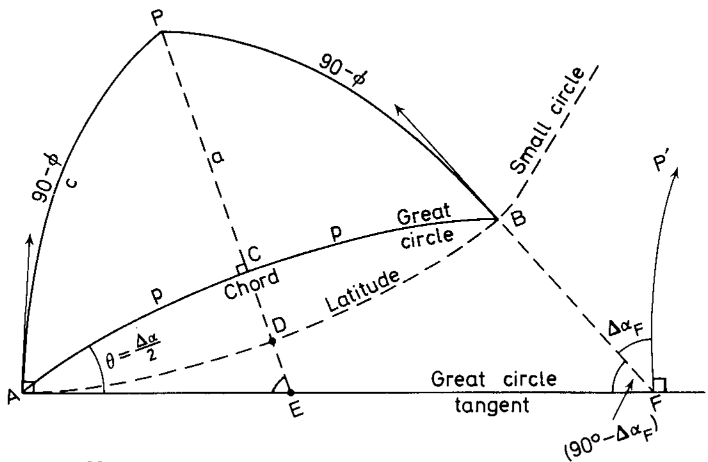

Figure 2.38

meridian, i.e. at angles of $(90° - \Delta\alpha)$, to locate the parallel at D, B, etc. The equation for the calculation of the offsets may be deduced using Napier's rules in the right-angled spherical triangle PCA

$$\sin(90° - P\hat{A}C) = \tan p \tan(90° - c)$$

If $AB = L$, then the angular value for $p \doteq L/2R$, c = co-latitude $(90 - \phi)$ and $P\hat{A}C = (90° - \theta)$

$$\therefore \sin\theta = \tan(L/2R)\tan[90° - (90° - \phi)]$$

which, as angles θ and p are small, may be written $\quad \theta = \dfrac{L\tan\phi}{2R}$

As length $AB \approx AF$, then $\quad BF = AB\theta = L\theta = \dfrac{L^2\tan\phi}{2R} \quad (2.57)$

For offsets on the spheroid, $R = \nu$.

It can be seen that offsets are proportional to the distance L, along the tangent, squared.

2.11.2 Chord method

In *Figure 2.38* a chord ACB to the parallel of latitude, is established from A by turning off an angle of $(90° - \theta)$ from true north. From *Worked example 2.20* (p. 135)

$$\Delta\alpha = \frac{L \tan \phi}{R}$$

but $$\theta = \frac{L \tan \phi}{2R} \qquad \therefore \; \theta = \frac{\Delta\alpha}{2} \tag{2.58}$$

Thus as $\Delta\alpha$ is calculated using a specific length L, the measurement of this distance from A along the line of sight will fix B on the parallel of latitude.

Intermediate points defining the parallel may be fixed by offsets from the chord. In *Figure 2.38*, CD is the maximum offset. Corresponding offsets equidistant each side of CD will be equal in length. The offset equation is as follows:

$$\begin{aligned} CD = CE - ED &= \left(\frac{\Delta\alpha}{2}\right)l_1 - \left(\frac{l_1^2 \tan \phi}{2R}\right) \\ &= l_1\left(\frac{L \tan \phi}{2R}\right) - \left(\frac{l_1^2 \tan \phi}{2R}\right) \\ &= \frac{l_1(L - l_1) \tan \phi}{2R} \end{aligned} \tag{2.59a}$$

Equation (2.59a) is the general equation for offsets, but as CD is the maximum offset, then $l_1 = L/2$, and

$$\text{Maximum offset} = \frac{\frac{1}{2}L(L - L/2) \tan \phi}{2R} = \frac{L^2 \tan \phi}{8R} \tag{2.59b}$$

Further chords may be established in a manner similar to setting out a simple curve, as shown in *Figure 2.39*.

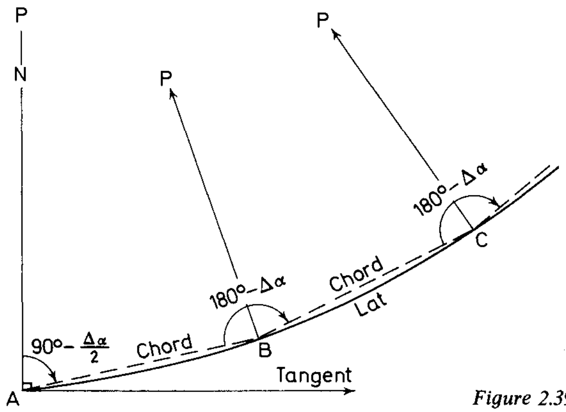

Figure 2.39

WORKED EXAMPLES

Example 2.21. Prove that a great circle arc of length D (small compared with the Earth's radius) joining two points of equal latitude ϕ, has azimuths at its ends differing from 90° or 270° by α given by: $\sin \alpha = \left(\frac{D}{2R}\right) \tan \phi$, where R is the radius of the Earth. Show also that the maximum offset Δ from the great circle to the line of latitude is given by

$$\Delta = \frac{D^2}{8R} \tan \phi$$

From a position of latitude 42° N a great circle is to be set out to a position 12 km eastward and of the same latitude as the starting point. What should be the azimuth of the line at the start, and what would be the maximum distance from the great circle to the line of latitude? Take the radius of the Earth as 6390 km. (LU)

Refer to *Section 2.10* for the answer to the first part.

$$\text{Azimuth of the line at the start} = \left(90° - \frac{\Delta\alpha}{2}\right)$$

where $$\frac{\Delta\alpha}{2} = \frac{L \tan \lambda}{2R} = \frac{12 \tan 42° \times 206\,265}{2 \times 6390} = 174''$$

$\therefore$ Azimuth $= 89° \, 57' \, 06''$

and maximum offset $$\Delta = \frac{12^2 \tan 42°}{8 \times 6390} = 2.536 \text{ m}$$

Example 2.22. A 3-km length of the line of latitude 51° 30′ N is to be set out with boundary stones at 1-km intervals.

Given that 1″ of longitude subtends 18.986 m at the Earth's surface in latitude 51° 30′, calculate the data required and state how you would set out the boundary knowing true north at the start of the line. (LU)

Chord method

Difference in longitude $= \Delta\lambda = 3000/18.986 = 158''$

Convergence $\Delta\alpha = \Delta\lambda \sin \phi_m = 158'' \sin 51° \, 30' = 124''$

From true north at the start, turn off an angle of $(90° - \Delta\alpha/2) = 89° \, 58' \, 58''$ and set out markers at 1-km intervals for 3 km. The offsets, to fix boundary stones at 1 and 2 km on the parallel, will be equal, as they are equidistant each side of the maximum offset.

From *Figure 2.34(a)*, if $\Delta\lambda = 1''$, then $AB = 18.986$ m, thus by radians

$$R \cos \phi \Delta\lambda = 18.986 \text{ m}$$

$$\therefore \; R = \frac{18.986 \times 206\,265}{\cos 51° \, 30'} = 6\,290\,848 \text{ m}$$

$\therefore$ Offset at 1 and 2 km from equation (2.59a)

$$= \frac{1000(3000 - 1000) \tan 51° \, 30'}{2 \times 6\,290\,848} = 0.200 \text{ m}$$

Tangent method
From true north, turn off an angle of 90° to establish the line of the tangent.

$$\text{Offset at 1 km} = \frac{l_1^2 \tan \phi}{2R}$$

$$= \frac{1000^2 \tan 51^\circ 30'}{2 \times 6\,290\,848} = 0.100 \text{ m}$$

Offset at 2 km = 0.400 m
Offset at 3 km = 0.900 m

These offsets would be set off in the direction of the meridian, i.e. at angles of $(90 - \Delta\alpha)$ from the tangent.

$$\therefore \quad \Delta\alpha \text{ at 1 km} = \frac{L \tan \phi}{R} = \frac{1 \times \tan 51^\circ 30'}{6\,290\,848} \times 206\,265 = 0.04''$$

at 2 km $\Delta\alpha = 0.08''$, and at 3 km $\Delta\alpha = 0.12''$.

2.12 TRANSVERSE MERCATOR PROJECTION

Having defined the position of points by their spheroidal co-ordinates ϕ and λ on a spheroid of reference, a map must be compiled on a plane surface. To this end, a projection must be adopted which best suits the requirements of the country to be mapped.

A map projection is a means of representing the lines of latitude and longitude of the spheroidal Earth on a flat sheet of paper. The lines so produced form what is called a *graticule*. Since it is impossible to represent a curved surface on a plane, there is no such thing as the perfect projection. However, certain projections fulfil certain requirements.

In the case of the British Isles, the Ordnance Survey (OS) adopted as their spheroid of reference Airy's spheroid, which has the following dimensions:

Equatorial semi-axis = 6 377 563.4 m
Polar semi-axis = 6 356 256.9 m

In the case of the projection, the prime requirement of the OS was that at any point on the projection the scale was the same in all directions. The result of this requirement is that small areas on the ground retain their true shape on the map, and angles calculated from rectangular co-ordinates correspond, almost exactly, with angles observed on the ground. Such projections are termed *orthomorphic*. With this in mind, plus the fact that it is eminently suited to a country having its greatest extent in a north-south direction, the transverse Mercator projection (TMP) was adopted.

The TMP is a cylindrical projection as illustrated in *Figure 2.40*. The cylinder is in contact with the Earth along a meridian of longitude, and lines of latitude and longitude are projected onto the cylinder from a point source at the Earth's centre. Orthomorphism is achieved by stretching the scale along the meridians to keep pace with the increasing scale along the parallels. By opening up the cylinder and spreading it out flat the lines of latitude and longitude form a graticule of complex curves intersecting at right angles, the central meridian being straight.

It is obvious from *Figure 2.40* that scale, i.e. the ratio of distance on the ground to that on the projection, would be correct only along the meridian where the cylinder and

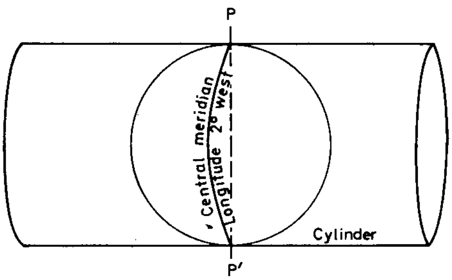

Figure 2.40

Earth are in contact. Thus the projection scale factor along the meridian of contact would be unity. As scale factor varies as the square of the distance from the central meridian, it can be shown that at about 270 km east or west the scale factor is 1.000 90.

The central meridian adopted for the British Isles is longitude 2° W. The intersection of this meridian with the parallel of latitude 49° N is the true point of origin of the projection. However, in order to reduce the scale error at the extreme east and west edges of the country, the scale at the central meridian was arbitrarily reduced by the factor of 2499/2500. This has the effect of making the scale 0.04% too small at the central meridian, and 0.04% too large near the east and west coasts. The apparent result of reducing the scale at the central meridian is to reduce the diameter of the projection cylinder, as shown in *Figure 2.41*. Thus, at the central meridian the scale factor is 0.999 601 27, and 180 km east and west of the central meridian it is unity.

2.13 THE NATIONAL GRID

It was recommended by the Davidson Committee, set up in 1935 to study OS maps and plans, that a national grid (NG) should be superimposed on all OS maps. This had the effect of providing a single reference system for the whole country, on which the position of a point may be defined by 'plane rectangular co-ordinates'. The origin of the national grid was established 400 000 m west and 100 000 m north of the true origin, with the result that the co-ordinates of a point are always positive, and so defined by its easting and northing.

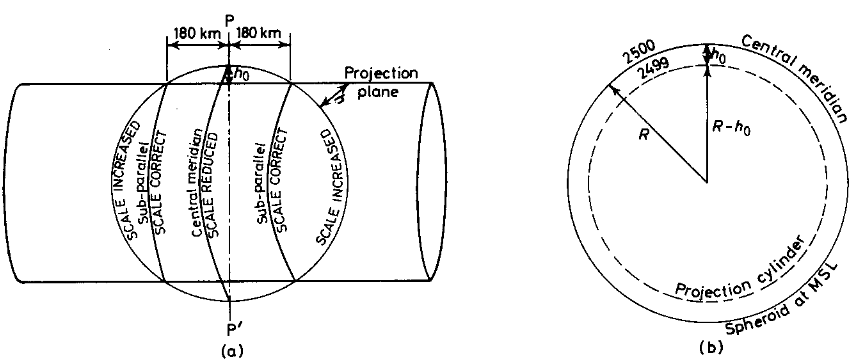

Figure 2.41

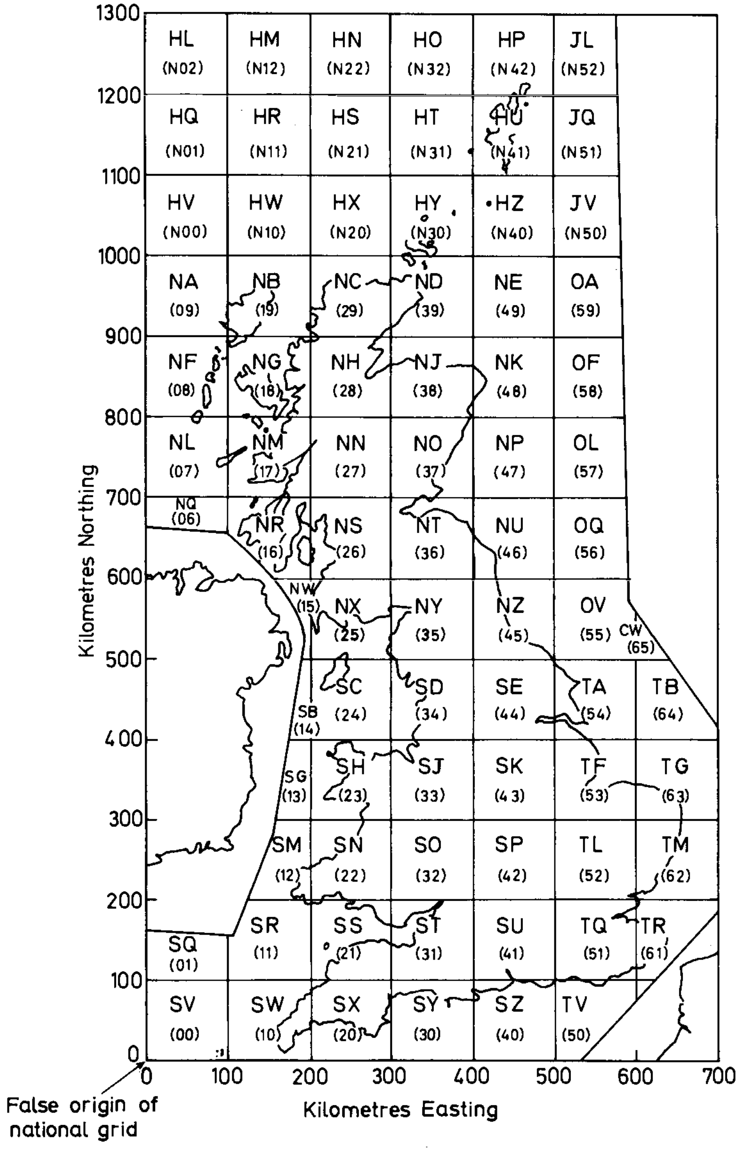

Figure 2.42. National reference system of Great Britain showing 100-km squares, the figures used to designate them in the former system, and the letters which have replaced the figures. (Courtesy Ordnance Survey, Crown Copyright Reserved)

As shown in *Figure 2.42*, the grid is simply a series of lines parallel and at right angles to the central meridian of the TMP. Thus at the central meridian, grid north and true or geographical north, are the same. To the east and west of the central meridian, however, grid and true north will differ by a variable amount due to the convergence of the meridians to the central meridian.

2.14 SCALE FACTORS

The origins of scale factors having been fully expounded, it simply remains to define them more specifically, thus

$$F = G/S \tag{2.60}$$

where F = local scale factor (LSF)
G = grid distance (as computed from NG co-ordinates) and
S = distance on the spheroid at MSL

Figure 2.43 shows the relationship of these various distances and serves to explain the application of LSF.

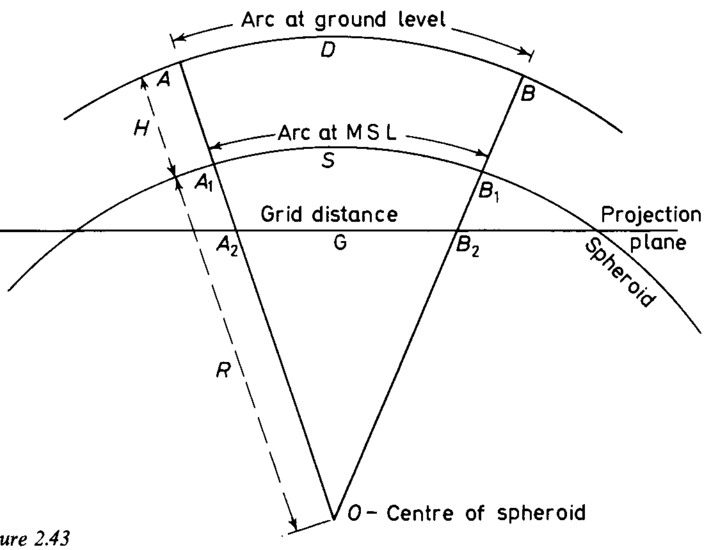

Figure 2.43

A semi-rigorous formula for F may be deduced as follows. Scale error (SE) is the difference between the scale factor at any point (F) and that at the central meridian (F_0), and varies as the square of the distance from the central meridian. Thus, SE $= K(\Delta E)^2$ where ΔE is the difference in eastings of the point in question and the central meridian, i.e. $\Delta E = (E - 400\,000)$ m.

$$\therefore\ F = F_0 + \text{SE} = 0.999\,601\,27 + K(\Delta E)^2$$

(F_0 is the LSF at the central meridian and equals 0.999 601 27). Considering a point 180 000 m east or west of the central meridian, its value for F is unity, thus

$$1.000\,000\,00 = 0.999\,601\,27 + K(180\,000)^2 \qquad \therefore\ K = 1.228 \times 10^{-14}$$

and hence
$$F = 0.999\,601\,27 + [1.228 \times 10^{-14} \times (E - 400\,000)^2] \tag{2.61}$$

Thus the value of F for a point whose NG co-ordinates are E 638 824 m, N 309 912 m is

$$F = 0.999\,601\,27 + (1.228 \times 10^{-14} \times 238\,824^2) = 1.000\,301\,6$$

The OS recommend that for very accurate work the LSF should be computed at each end of the line and in the middle, and the mean value obtained from Simpson's rule. However, for all practical purposes it is sufficient to compute the LSF at the midpoint of the line.

It is very important to realize that the scale factor relates only to distances on the spheroid at MSL and grid distances on the projection. Thus horizontal distances on the ground must first be reduced to their equivalent at MSL before the application of LSF to convert them to grid distances. The application of the altitude correction formula has already been illustrated in *Section 2.1.3.6.*

2.14.1 Application of scale factors

Grid to ground distance

Any distance calculated from NG co-ordinates will be grid distance. If this distance is to be set out on the ground it must:

(a) Be *divided* by the LSF to give the spheroidal distance at MSL, i.e. $S = G/F$.
(b) Have the altitude correction applied to give the horizontal ground distance.

Consider two points, A and B, whose co-ordinates are

A: E 638 824.076	N 307 911.843
B: E 644 601.011	N 313 000.421
$\therefore\ \Delta E = 5\,776.935$	$\therefore\ \Delta N = 5\,088.578$

Grid distance $= (\Delta E^2 + \Delta N^2)^{\frac{1}{2}} = 7698.481\text{ m} = G$

Mid-easting of AB = E 641 712 m

$\therefore\ F = 1.000\,318\,8$ (from equation (2.61))

$\therefore$ Spheroidal distance at MSL $= S = G/F = 7696.027$ m

Now assuming AB at a mean height (H) of 250 m above MSL, the altitude correction C_m is

$$C_m = \frac{SH}{R} = \frac{7696 \times 250}{6\,384\,100} = +0.301\text{ m}$$

$\therefore$ Horizontal distance at ground level = 7696.328 m

Ground to grid distance

When connecting surveys to the national grid, horizontal distances measured on the ground must be:

(a) Reduced to their equivalent on the spheroid at MSL.
(b) *Multiplied* by the LSF to produce the equivalent grid distance, i.e. $G = S \times F$.

Consider now the previous problem worked in reverse

Horizontal ground distance	= 7696.328 m
Altitude correction C_m	= −0.301 m
$\therefore$ Spheroidal distance S at MSL	= 7696.027 m
F	= *1.000 318 8*
$\therefore$ Grid distance $G = S \times F$	= 7698.481 m

The above computations are greatly simplified if the LSF and altitude corrections are combined to give an adjusted LSF at mean elevation. This may be carried out as follows (*Figure 2.41(b)*)

LSF at the central meridian $F_0 = \dfrac{\text{Grid distance}}{\text{Spheroidal distance}} = \dfrac{R - h_0}{R}$

$\therefore\ h_0 = R(1 - F_0)$

and at any position on the spheroid

$$h = R(1 - F) \qquad (2.62)$$

Referring to *Figure 2.41(a)*, it can be seen that when $F < 1$, the spheroid is *above* the projection plane and h is *positive*; when $F > 1$, the spheroid is *below* the projection plane and h is *negative*. Consider once again line AB whose value for F is 1.000 318 7 (*Figure 2.44*). Height of projection plane above or below MSL at the mid-point of $AB = h = R(1 - F) = 6\,384\,100(1 - 1.000\,318\,7) = -2034.6$ m.

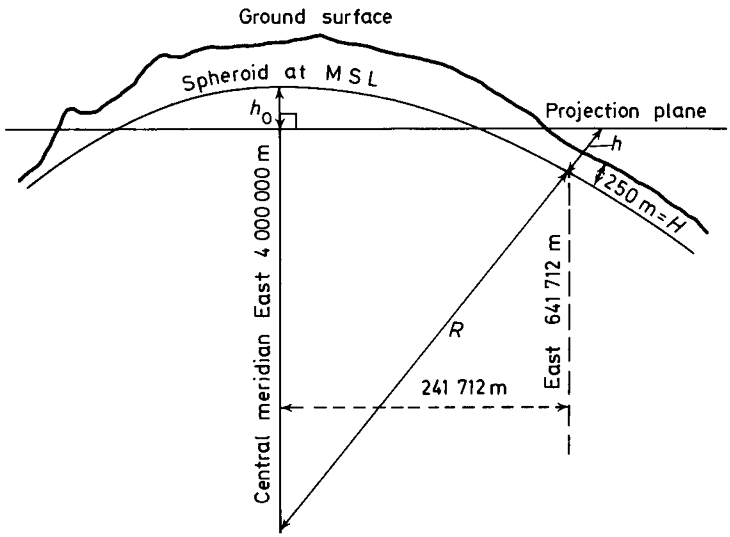

Figure 2.44

Hence F at MSL $= 1.000\,318\,7$
F at the projection plane $= 1.000\,000\,0$

$\therefore$ Difference $\Delta F = 0.000\,318\,7$

Then by proportion, F at 250 m above MSL, i.e. F_a is

$$F_a = F \pm \frac{\Delta F \times H}{h} \qquad (2.63)$$

$$= 1.000\,318\,7 - \frac{0.000\,318\,7 \times 250}{2034.6}$$

$\therefore\ F_a = 1.000\,279\,5$

Doubts about the sign of this correction to F may be resolved by considering *Figure 2.44*. The point in question has an LSF of 1.000 318 7 at MSL, thus the projection plane must lie *above* MSL. Now as the ground is 250 m above MSL, it approaches closer to the projection plane and so closer to unity, hence the correction to 1.000 318 7 is negative, giving 1.000 279 5. Consider now the central meridian where the LSF at MSL is 0.999 6. If the ground is above MSL it is further from the projection plane, and so the LSF at ground level is further removed from unity; thus again the correction to F is negative. If the area in question was below MSL, as occurs in mining and tunnelling, the correction would be positive.

Whilst it is important to understand the principles, these considerations can be eliminated by combining equations (2.62) and (2.63) to give

$$F_a = F(1 - H/R) \tag{2.64}$$

where H is the ground height relative to MSL, and is positive when above and negative when below MSL.

$$\therefore \quad F_a = 1.000\,318\,7\left(1 - \frac{250}{6\,384\,100}\right) = 1.000\,279\,5$$

It is only necessary to apply LSFs to surveys that are to be connected to the national grid, and in the reverse case, i.e. when setting out distances on the ground are required from NG co-ordinate values. Also it is sufficient to use an LSF for an area, the extent of the area depending on the accuracy required.

An alternative approach to the use of scale factors in engineering surveys can be found in Schofield (1973).

2.15 CONVERGENCE OF MERIDIANS

This topic has already been discussed; however, with particular reference to the national grid it may be further outlined.

The central meridian on the transverse Mercator projection (longitude 2° W), and the grid line 400 000 m east of the false origin, are coincident. East and west of the central meridian, all other meridian lines converge on the central meridian, i.e. they converge towards and meet at the North Pole. However, the grid lines are all parallel to the central meridian, hence the clockwise angle at any point between the direction of the grid lines (grid north) and the meridian lines (true north) is defined as the *convergence of meridians* ($\Delta\alpha$).

$$\Delta\alpha = \Delta\lambda \sin \phi_m$$

From *Figure 2.34(a)*

$$\Delta\lambda = L/R \cos \phi_m$$

$$\therefore \quad \Delta\alpha = \frac{L \tan \phi_m}{R} \tag{2.65}$$

where L is the distance from the central meridian.

These approximate equations for $\Delta\alpha$ will give values correct to 25″, with ϕ_m taken from a map to the nearest 05′. For example, the NG co-ordinates of a station are E 626 238, N 302 646. If the latitude is N 52° 34′, calculate the convergence of meridians, taking $R = 6\,384\,100$ m.

$$L = 626\,238 - 400\,000 = 226\,238 \text{ m}$$

$$\Delta\alpha'' = \frac{226\,238 \tan 52° \, 34'}{6\,384\,100} \times 206\,265 = 9549''$$

$$\therefore \; \Delta\alpha = 2° \, 39' \, 09''$$

2.16 THE $(t - T)$ CORRECTION

The transverse Mercator projection is orthomorphic, thus angles measured on the ground need not be altered when used in the plane of projection. However, when long lines of sight are involved, a small correction, called the $(t - T)$ correction, is required.

Lines observed on the ground are curved due to the spheroidal shape of the Earth, and will always be *concave* to the central meridian as shown in *Figure 2.45*. Thus, the

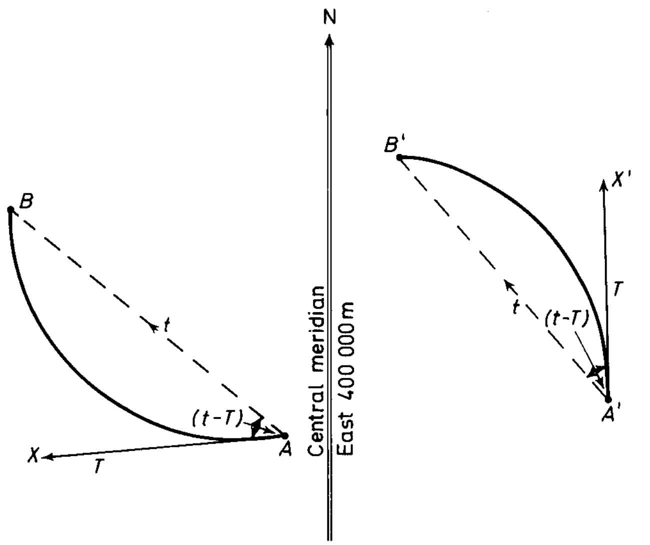

Figure 2.45

line AB observed on the ground will have the direction AX, whilst on the projection it is the straight line AB shown dotted. The difference between these two directions is called the $(t - T)$ correction. From this it can be seen that where long lines are involved, the angle between OS stations observed on the ground will differ slightly from the equivalent angle computed from NG co-ordinates.

The equation for correcting the bearing of line AB is

$$(t_A - T_A)'' = (2\Delta E_A + \Delta E_B)(N_A - N_B)K \tag{2.66}$$

where ΔE = NG easting − 400 000, expressed in km
N = NG northing expressed in km
A = station at which the correction is required
B = station observed
$K = 845 \times 10^{-6}$

A maximum value for the $(t - T)$ correction would be about $\pm 7''$; it is zero for points having equal northings and increases with distance from the central meridian.

The application of the $(t - T)$ correction and convergence of meridians is clearly illustrated in the following worked examples.

WORKED EXAMPLES

Example 2.23. The national grid co-ordinates of two points, A and B, are A: E_A 238 824.076, N_A 307 911.843; and B: E_B 244 601.011, N_B 313 000.421

Calculate (1) The grid bearing and length of $\overrightarrow{AB}$.
(2) The azimuth of $\overrightarrow{AB}$ and $\overrightarrow{BA}$.
(3) The ground length AB.

Given (a) Mean latitude of the line = N 54° 00′.
(b) Mean altitude of the line = 250 m AOD.
(c) Local radius of the Earth = 6 384 100 m. (KP)

(1) $E_A = 238\,824.076$ $N_A = 307\,911.843$
$E_B = 244\,601.011$ $N_B = 313\,000.421$

$\Delta E = 5776.935$ $\Delta N = 5088.578$

$$\text{Grid distance} = (\Delta E^2 + \Delta N^2)^{\frac{1}{2}} = 7698.481 \text{ m}$$

$$\text{Grid bearing } \overrightarrow{AB} = \tan^{-1}\frac{\Delta E}{\Delta N} = 48°\ 37'\ 30''$$

(2) In order to calculate the azimuth, i.e. the direction relative to true north, one must compute (a) the convergence of meridians at A and $B(\Delta\alpha)$ and (b) the $(t - T)$ correction at A and B (*Figure 2.46*).

(a) Convergence of meridians at $A = \Delta\alpha_A = \dfrac{L_A \tan\phi}{R}$

where L_A = Distance from the central meridian
$= 400\,000 - E_A = 161\,175.924$ m

$$\therefore\ \Delta\alpha''_A = \frac{161\,176 \tan 54°}{6\,384\,100} \times 206\,265 = 7167'' = 1°\ 59'\ 27''$$

Similarly $$\Delta\alpha''_B = \frac{155\,399 \tan 54°}{6\,384\,100} \times 206\,265 = 6911'' = 1°\ 55'\ 11''$$

(b) $$(t_A - T_A) = (2\Delta E_A + \Delta E_B)(N_A - N_B)K$$
$$= 477.751 \times -5.089 \times 845 \times 10^{-6} = -2.05''$$

N.B. The eastings and northings are in km.

$$(t_B - T_B) = (2\Delta E_B + \Delta E_A)(N_B - N_A)K$$
$$= 471.974 \times 5.089 \times 845 \times 10^{-6} = +2.03''$$

Although the signs of the $(t - T)$ correction are obtained from the equation the student is advised always to draw a sketch of the situation.

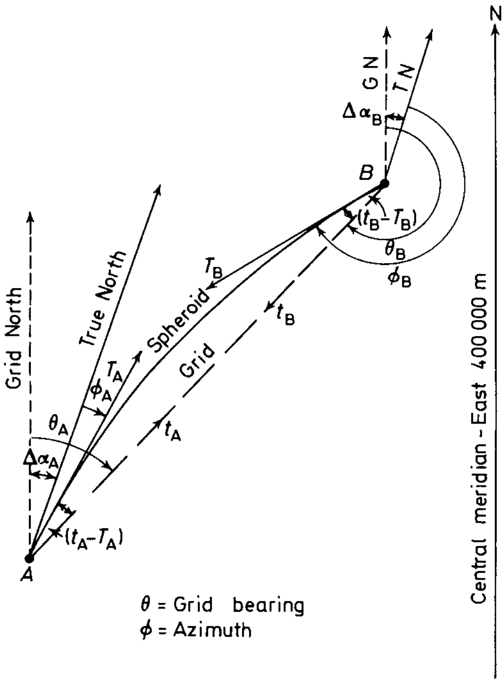

Figure 2.46

Referring to *Figure 2.46*

$$\text{Azimuth } \overrightarrow{AB} = \phi_A = \theta_A - \Delta\alpha_A - (t_A - T_A)$$
$$= 48^\circ\ 37'\ 30'' - 1^\circ\ 59'\ 27'' - 02'' = 46^\circ\ 38'\ 01''$$

$$\text{Azimuth } \overrightarrow{BA} = \phi_B = \theta_B - \Delta\alpha_B + (t_B - T_B)$$
$$= (48^\circ\ 37'\ 30'' + 180^\circ) - 1^\circ\ 55'\ 11'' + 2''$$
$$= 226^\circ\ 42'\ 21''$$

(3) To obtain ground length from grid length one must obtain the LSF adjusted for altitude.

Mid-easting of $AB = 241\,712.544$ m $= E$

$$\text{LSF} = 0.999\,601 + [1.228 \times 10^{-14} \times (E - 400\,000)^2] = F$$

$\therefore\ F = 0.999\,908$

The altitude is 250 m OD, i.e. $H = +250$. LSF F_a adjusted for altitude is

$$F_a = F\left(1 - \frac{H}{R}\right) = 0.999\,908\left(1 - \frac{250}{6\,384\,100}\right) = 0.999\,869$$

$\therefore$ Ground length AB = Grid length $\div F_a$

$\therefore\ AB = 7698.481/0.999\,869 = 7699.483$ m

Example 2.24. As part of the surveys required for the extension of a large underground transport system, a base line was established in an existing tunnel and connected to the national grid via a wire correlation in the shaft and precise traversing therefrom.

Thereafter the azimuth of the base was checked by gyro-theodolite using the reversal point method of observation as follows:

Reversal points	*Horizontal circle readings* ° ′ ″	*Remarks*
r_1	330 20 40	Left reversal
r_2	338 42 50	Right reversal
r_3	330 27 18	Left reversal
r_4	338 22 20	Right reversal

Horizontal circle reading of the baseline = 28° 32′ 46″
Convergence of meridians = 0° 20′ 18″
$(t - T)$ correction = 0° 00′ 04″
NG easting of baseline = 500 000 m

Prior to the above observations, the gyro-theodolite was checked on a surface base line of known azimuth. The following mean data were obtained

Known azimuth of surface base = 140° 25′ 54″
Gyro azimuth of surface base = 141° 30′ 58″

Determine the national grid bearing of the underground base line. (KP)

Refer to Volume 1 for information on the gyro-theodolite.

Using Schuler's mean

$$N_1 = \tfrac{1}{4}(r_1 + 2r_2 + r_3) = 334° 33' 24''$$
$$N_2 = \tfrac{1}{4}(r_2 + 2r_3 + r_4) = 334° 29' 54''$$
$$\therefore\ N = (N_1 + N_2)/2 = 334° 31' 39''$$

Horizontal circle reading of the base = 28° 32′ 46″
∴ Gyro azimuth of the base line = 28° 32′ 46″ − 334° 31′ 39″
= 54° 01′ 07″

However, observations on the surface base show the gyro-theodolite to be 'over-reading' by (141° 30′ 58″ − 140° 25′ 54″) = 1° 05′ 04″.

∴ True azimuth of base line ϕ = Gyro azimuth − Instrument constant
= 54° 01′ 07″ − 1° 05′ 04″
= 52° 56′ 03″

Now by reference to *Figure 2.47*, the sign of the corrections to give the NG bearing can be seen, i.e.

Azimuth ϕ = 52° 56′ 03″
Convergence of meridians $\Delta\alpha$ = −0° 20′ 18″
$(t - T)$ = −0° 00′ 04″

∴ NG bearing θ = 52° 35′ 41″

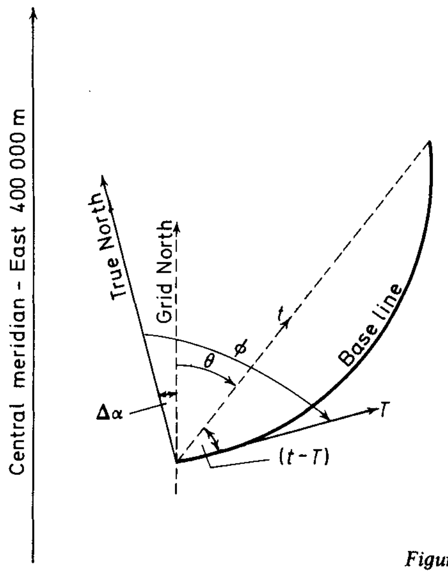

Figure 2.47

EXERCISES

2.11. Explain the meaning of the term *convergence of meridians.* Show how this factor has to be taken into account when running long survey lines by theodolite.

From a point *A* in latitude 53° N, longitude 2° W, a line is run at right angles to the initial meridian for a distance of 31 680 m in a westerly direction to point *B*.

Calculate the true bearing of the line at *B*, and the longitude of that point. Calculate also the bearing and distance from *B* of a point on the meridian of *B* at the same latitude as the starting point *A*. The radius of the Earth may be taken as 6273 km. (LU)

(*Answer:* 269° 37′ 00″; 2° 28′ 51″ W; 106.5 m)

2.12. Two points, *A* and *B*, have the following co-ordinates:

	Latitude ° ′ ″				*Longitude* ° ′ ″			
A	52	21	14	N	93	48	50	E
B	52	24	18	N	93	42	30	E

Given the following values:

Latitude	1″ *of latitude*	1″ *of longitude*
52° 20′	30.423 45 m	18.638 16 m
52° 25′	30.423 87 m	18.603 12 m

find the azimuths of *B* from *A* and of *A* from *B*, also the distance *AB*. (LU)

(*Answer:* 308° 23′ 36″, 128° 18′ 35″, 9021.9 m)

2.13. At a terminal station A in latitude N 47° 22′ 40″, longitude E 0° 41′ 10″, the azimuth of a line AB of length 29 623 m was 23° 44′ 00″.

Calculate the latitude and longitude of station B and the reverse azimuth of the line from station B to the nearest second. (LU)

Latitude	*1″ of longitude*	*1″ of latitude*
47° 30′	20.601 m	30.399
47° 35′	20.568 m	30.399

(*Answer:* N 47° 37′ 32″; E 0° 50′ 50″; 203° 51′ 08″)

2.14. A boundary is to be a line 60 km in length along the 45° parallel of latitude. Calculate the data for setting out boundary markers at 30-km intervals and describe the method of setting out.

Assume that 30.45 m on a great circle subtends 1″ at the Earth's centre, that the mean radius of the Earth is 6367 km and log R (in metres) = 6.803 935. (LU)

(*Answer:* Chord method; setting out angle = 89° 43′ 48″, maximum offset 282.7 m)

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3

Aerial photogrammetry

As the word 'photogrammetry' implies, it means measurements from photographs, and in the case of aerial photogrammetry it is measurements from aerial photographs.

The major use of aerial photogrammetry is in the preparation of contoured plans from the aerial photographs. With the aerial camera in the body of the aircraft, photographs are taken along prearranged flight paths, with the optical axis of the camera pointing *vertically* down (*Figure 3.4*). Such photographs are termed *vertical photographs* and are the only type that will be considered here.

The essential processes involved in the production of a contoured plan or digital ground model from aerial photographs are:

(a) Photography.
(b) Ground control.
(c) Restitution system.

3.1 PHOTOGRAPHY

Basically the photographs are taken using dimensionally-stable film in precision-built cameras (*Figure 3.1*). It is important that all topographic detail must be clearly reproduced and therefore recognizable on the photograph, and that the geometric relationships between the ground objects and the photo images are rigorously maintained. These conditions are governed largely by the atmospheric conditions prevailing at the time of photography, aircraft movement, the characteristics of the camera and film, the scale used, and the eventual processing of the film.

The choice of photographic materials and the subsequent processing of the film are the province of highly-experienced photographers who utilize their professional skills and knowledge to reduce distortions to a minimum. In general, the most popular film used is panchromatic which produces the well-known black-and-white photographs used in the map-making process. This film is in 120-m lengths and has speed ratings in the region of 200 to 400 ASA.

Colour film may replace panchromatic where interpretation of detail is also of importance. For instance, in black-and-white photography the variation and distinguishability of shades between the extremes of black and white is in the region of 200, whereas in colour photography this number increases to 5000. Also, infra-red

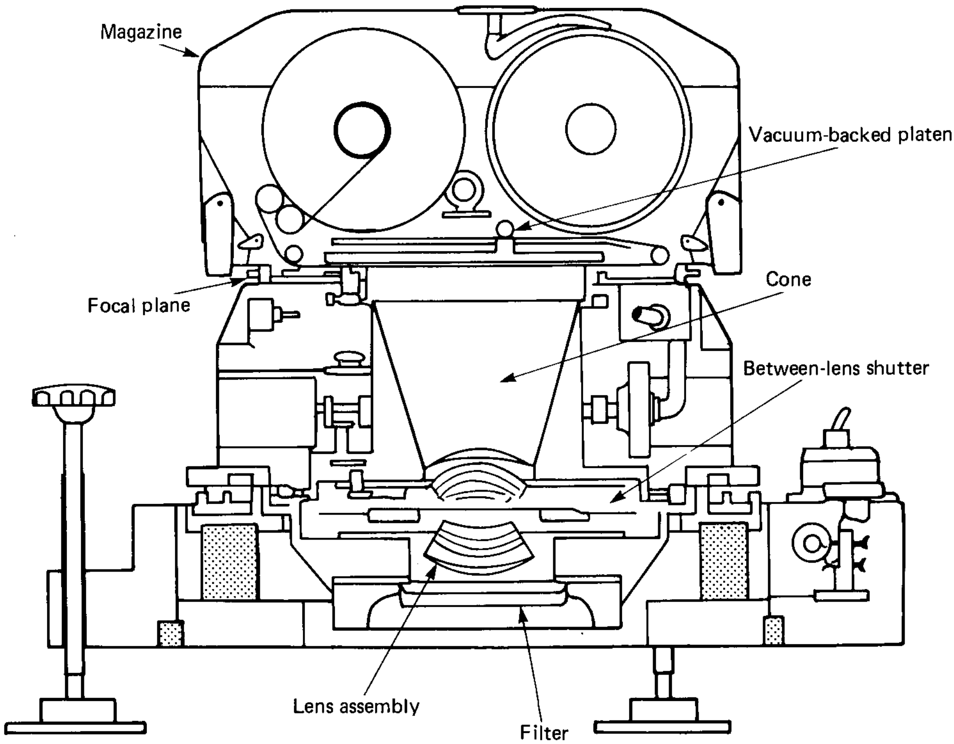


Figure 3.1

photography is widely used for the clearer detection of coastlines, rivers, drainage patterns, changes in the vitality of vegetation due to growth patterns or disease, and other phenomena such as the amount of pollutants in bodies of water.

Cameras used for air survey, as with all other survey equipment, are precision-built, and their lenses are of such high quality that aberrations are practically negligible. As is shown diagrammatically in *Figure 3.2*, lenses may be classified broadly as

(a) Normal-angle (60°) – Principal distance (f) = 82.5 mm
(b) Wide-angle (90°) – Principal distance (f) = 152.4 mm
(c) Super-wide-angle (125°) – Principal distance (f) = 305.0 mm

Normal-angle lenses are not now in common use, and the super-wide-angle is limited to small-scale mapping. From the engineering point of view the most popular lens is the wide-angle combined with a format size of 230 mm × 230 mm. It is the camera which produces the fundamental geometry of the air photo; that of a central perspective projection with the lens as origin.

After the lens system, the next major consideration is the shutter which, ideally, should be capable of exposing the total film format for the required interval at the same instant of time. In addition, image movement, caused largely by the apparent movement of the ground relative to the aircraft, must also be reduced to negligible proportions. To attain such efficiency, a shutter of the rotating-disc type situated between the lens elements (*Figure 3.1*) is used, and the complete camera is mounted in an anti-vibration holder.

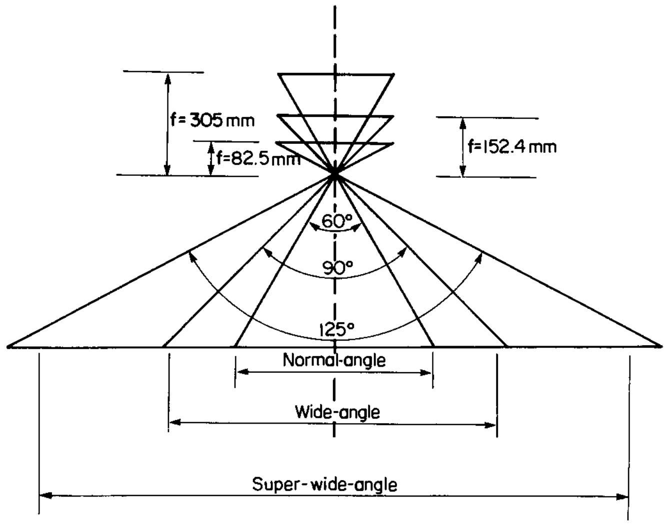


Figure 3.2

The aerial camera is capable of shutter speeds ranging from 1/50th to 1/2000th sec, the speeds most commonly used in practice being from 1/100th to 1/1000th sec.

Film-flattening devices available are low-pressure vacuum systems, and physical flattening by pressure pads acting against the film. The complete cycling time of the camera is in the region of 1.5 to 2 seconds. Maximum distortions, after careful calibration of the camera, are in the region of 3 to 10 μm.

The so-called 'camera constants' obtained from the calibration process are:

(a) The position of the principal point.
(b) The focal length of the lens.
(c) The pattern and magnitude of distortion over the effective photographic field.

3.2 GEOMETRY OF THE AERIAL PHOTOGRAPH

Before one can appreciate the need for ground control and a restitution system, one requires a knowledge of the errors present in the air photograph. These errors are largely caused by *tilt* in the plane of the film at the instant of exposure, and *displacement* of object position due to *ground relief.* It should be understood that a photograph is not a plan, except where the terrain is absolutely flat and level and the photograph axis is truly vertical.

3.2.1 Definitions

Because of the pitch and roll of aircraft in flight it is rare for a truly vertical photograph to be taken. *Figure 3.3* illustrates a near-vertical photograph with the optical axis tilted at θ to the vertical. In practice, θ is usually less than 3°. The definitions of commonly-used terms are as follows:

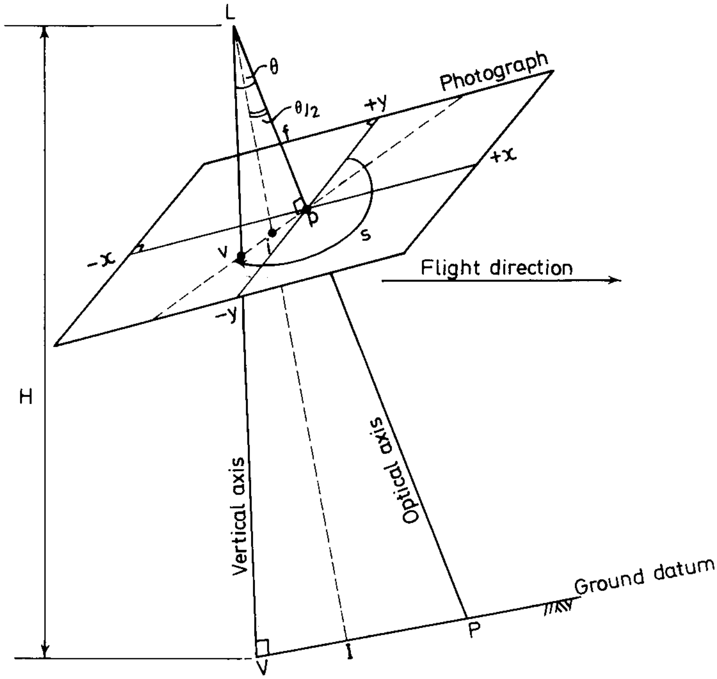


Figure 3.3

Photo axis: the right-angled $x - y$ axis formed by joining the opposite fiducial marks of the photograph. This is the axis from which photo co-ordinates are measured. The x-axis approximates to the direction of flight.

Optical axis: the line LpP from the lens centre and at 90° to the plane of the photograph.

Principal distance: the distance $Lp = f$, from the lens to the plane of the photograph. Alternatively the principal distance may be referred to as the *focal length.*

Vertical axis: the line LvV following the direction of gravity and thus at 90° to a level datum plane.

Tilt: the angle θ formed by the vertical and optical axes (see also *principal line*, below).

Principal point (*PP*)*:* the point p where the optical axis cuts the photograph, and coincides with the origin of the photo axes.

Plumb point: the point v where the vertical axis cuts the photograph.

Isocentre: the point i, where the bisector of the angle of tilt cuts the photograph.

Principal line: the line vip in the plane of the photograph giving the direction of maximum tilt of the photograph. It is therefore at the angle θ to the horizontal.

Plate parallels: the lines at 90° to the principal line which are similar to strike lines in geology, i.e. they are level lines.

Isometric parallel: the plate parallel passing through the isocentre and forming the axis of tilt of the photograph.

Flying height: the vertical height of the *lens* above ground at the instant of exposure of the film, and is equal to $(H - h)$, where H is the height of the lens above datum (usually MSL) and h is the mean height of the terrain.

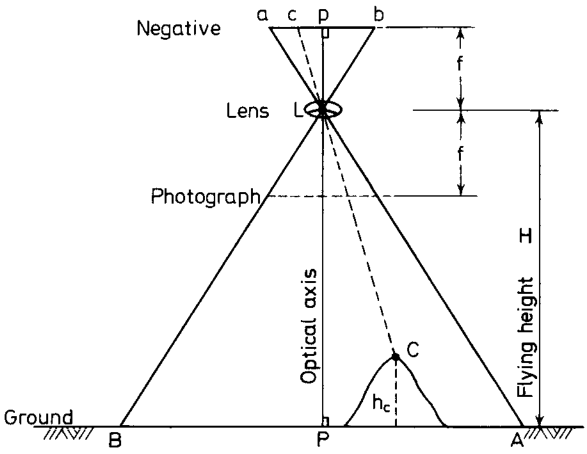


Figure 3.4

Swing: the angle s measured in the plane of the photograph, clockwise from the $+y$ axis to the plumb point. It defines the direction of tilt relative to the photo axes.

The main sources of error in the air photo will now be outlined.

3.2.2 Scale and its variation due to ground relief

In *Figure 3.4* the scale of a photograph is the ratio of the distance on the ground to its imaged distance on the photograph. Hence, by similar triangles

$$\text{Scale} = \frac{ab}{AB} = \frac{f}{H}$$

At point C, it is obvious that the scale $S_c = f/(H - h_c)$. Thus scale S varies with relief throughout the photograph and for any elevation (h) is given by

$$S = \frac{f}{H - h} \tag{3.1}$$

3.2.3 Scale and its variation due to tilt

Figure 3.5 assumes flat terrain and indicates i as the axis of tilt; thus the scale at the isocentre is common to both a tilted and a truly vertical photograph

(a) *Scale at isocentre* $\quad S_I = \dfrac{Li}{LI} = \dfrac{Lv_1}{LV} = \dfrac{f}{H}$

(b) *Scale at principal point* $\quad S_p = \dfrac{Lp}{LP} = \dfrac{f}{H \sec \theta}$

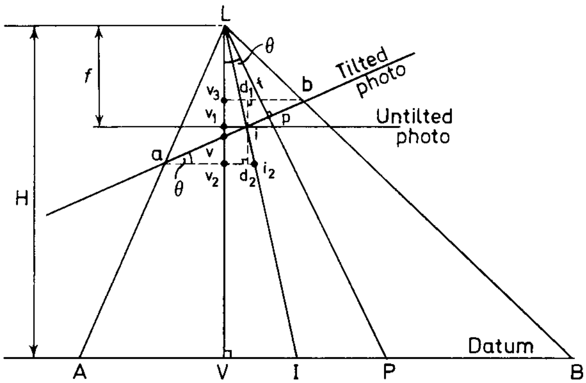


Figure 3.5

(c) *Scale at plumb point* $S_v = \dfrac{Lv}{LV} = \dfrac{f \sec \theta}{H}$

(d) *Scale at random point* a $S_a = \dfrac{La}{LA} = \dfrac{Lv_2}{LV} = \dfrac{Lv_1 + id_2}{LV}$

$$= \frac{f + ai \sin \theta}{H}$$

Let $ai = y_a$, the distance from the isocentre, then $S_a = \dfrac{f + y_a \sin \theta}{H}$

(e) *Scale at random point* b $S_b = \dfrac{Lb}{LB} = \dfrac{Lv_3}{LV} = \dfrac{Lv_1 - id_1}{Lv} = \dfrac{f - ib \sin \theta}{H}$

Let $ib = y_b$, then $S_b = \dfrac{f - y_b \sin \theta}{H}$

Thus it can be seen that the scale continually varies along the principal line with distance from the isocentre. By definition, however, the scale along a plate parallel at a particular point will be constant providing the ground is level. The basic equation considering ground relief h is therefore

$$S = \frac{f \pm y \sin \theta}{H - h} \qquad (3.2)$$

By substituting the appropriate distance from the isocentre for points v, i and p, the equations shown in (a) to (c) are obtained. For example, consider p

$$S_p = \frac{f - ip \sin \theta}{H}$$

where $ip = y_p = f \tan(\theta/2)$

Hence $$S_p = \frac{f - f\tan(\theta/2)\sin\theta}{H} = \frac{f - f\left(\dfrac{\sin\theta/2}{\cos\theta/2}\right)[2\sin(\theta/2)\cos(\theta/2)]}{H}$$

$$= \frac{f - f[2\sin^2(\theta/2)]}{H} = \frac{f[1 - 2\sin^2(\theta/2)]}{H} = \frac{f\cos\theta}{H}$$

$$\therefore\ S_p = \frac{f}{H\sec\theta}$$

Similarly for v; $y_v = vp - ip = f[\tan\theta - \tan(\theta/2)]$, which on substitution reduces to the equation already given in (c). Note that on the lower side of the tilted photograph the formula is +ve, and *vice versa*. Scale change along the principal line can be found by differentiating the basic equation with respect to y, i.e.

$$\frac{dS}{dy} = \pm\frac{\sin\theta}{H}$$

3.2.4 Image displacement due to ground relief

Figure 3.6 shows an untilted photograph of undulating terrain. Point A, if projected orthogonally onto a plan, would appear at B. Its true position on the photograph is therefore at b, and distance ab is the displacement resulting from the height of A above datum. By similar triangles

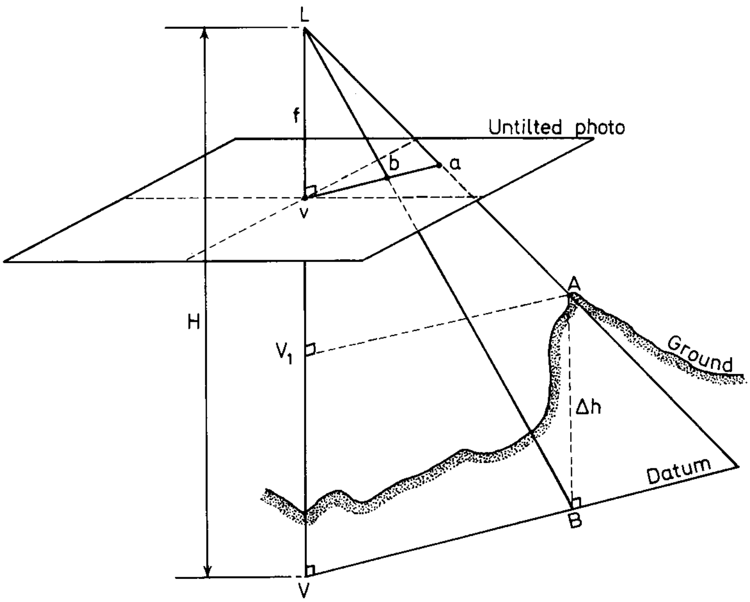


Figure 3.6

$$\frac{va}{V_1A} = \frac{f}{H - \Delta h} = \frac{va}{VB} \qquad \therefore\ va(H - \Delta h) = fVB$$

but $$\frac{vb}{VB} = \frac{f}{H} \qquad \therefore\ VB = \frac{vbH}{f}$$

thus $$va(H - \Delta h) = vbH = (va - ab)H \qquad \therefore\ \frac{va}{va - ab} = \frac{H}{H - \Delta h}$$

$$\therefore\ vaH - va\Delta h = vaH - abH$$

$$\therefore\ ab = va\Delta h/H \tag{3.3}$$

From equation (3.3) it can be seen that any increase in flying height would reduce the amount of displacement *ab*, which in turn is directly proportional to the height Δh of the object. It can also be seen that if Δh and H remain constant, displacement will increase with distance *va* from the plumb point. This latter point is important in the construction of mosaics, and may result in the central portions only of the photograph being used.

From *Figure 3.6* it can be clearly seen that LV is parallel to AB and both are vertical. Thus $LABV$ forms a plane containing *v*, *b* and *a*, showing the displacement *ba* as being radial from the vertical LV at *v*, which in a near-vertical photograph is the plumb point.

3.2.5 Image displacement due to tilt

Considering point *a* in *Figure 3.7*, whose distance from the isocentre on the tilted photograph is *ia*, and on the untilted ia_1, then the displacement Δt due to tilt is given by

$$\Delta t = ia - ia_1$$

$$\therefore\ \frac{\Delta t}{ia} = \frac{ia - ia_1}{ia} = 1 - \left(\frac{ia_1}{ia}\right) = 1 - \left(\frac{CL}{Ca}\right) = 1 - \left(\frac{f \operatorname{cosec}\theta}{Ci + ia}\right)$$

but, as $C\hat{L}i = C\hat{i}L = (90^\circ - \theta/2)$, then $Ci = CL = f \operatorname{cosec}\theta$

$$\therefore\ \frac{\Delta t}{ia} = \frac{f \operatorname{cosec}\theta}{f \operatorname{cosec}\theta + ia} = \frac{f \operatorname{cosec}\theta + ia - f \operatorname{cosec}\theta}{f \operatorname{cosec}\theta + ia} = \frac{ia}{f \operatorname{cosec}\theta + ia}$$

$$\therefore\ \Delta t = \frac{(ia)^2}{f \operatorname{cosec}\theta + ia} = \frac{(ia)^2 \sin\theta}{f + ia\sin\theta}$$

Similarly for point *b* on the upper side of the photograph, it can be shown that

$$\Delta t = \frac{(ib)^2 \sin\theta}{f - ib\sin\theta}$$

$\therefore$ the general equation is $$\Delta t = \frac{y^2 \sin\theta}{f \pm y\sin\theta} \tag{3.4}$$

where y is the distance from the isocentre measured along the principal line. For any point *off* the principal line, the situation is as shown in *Figure 3.8*.

It can be seen that $ia = id\cos\phi$ and $ia_1 = id_1\cos\phi$. Therefore the displacement $dd_1 = \Delta t_1$, projected on to the principal line gives

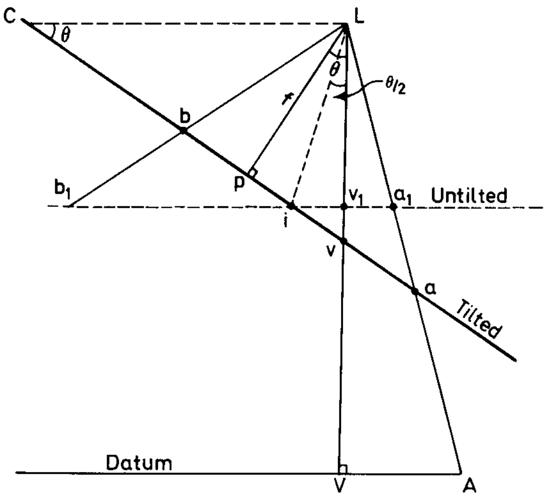


Figure 3.7

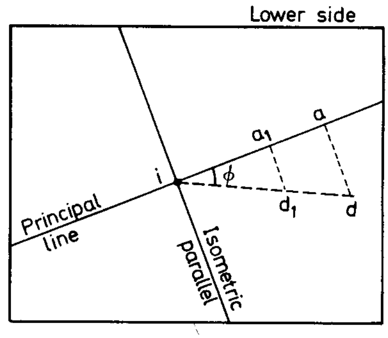


Figure 3.8

$$\Delta t = \Delta t_1 \cos \phi = \frac{(y \cos \phi)^2 \sin \theta}{f \pm y \cos \phi \sin \theta}$$

$\therefore$ the general equation becomes $$\Delta t_1 = \frac{y^2 \sin \theta \cos \phi}{f \pm y \sin \theta \cos \phi} \qquad (3.5)$$

and, if θ is small then $y \sin \theta \cos \phi$ is negligible, compared with f

$$\therefore \ \Delta t_1 = \frac{y^2 \sin \theta \cos \phi}{f} \qquad (3.6)$$

The equation shows that displacement is proportional to the distance from the isocentre squared, and will therefore be greatest at the edges of the photograph. It shows also that increasing the focal length of the camera will help to reduce the displacement.

As shown in *Section 3.2.6* following, angles measured about the isocentre on a tilted photograph are equal to their corresponding angles on the ground. It follows, therefore, that image displacement due to tilt must be radial from the isocentre.

3.2.6 Angular ratios on a tilted photograph

From the geometry of plane surfaces it can be shown that any line on the tilted photograph in *Figure 3.9* would intersect its corresponding line on the ground, along the perspective axis.

Considering the angle $PIA = \beta$ on the ground, and its equivalent $pia = \alpha$ on the photograph, about the isocentre, then in right-angled triangles BiC and BIC

$\tan \alpha = BC/Bi \qquad \tan \beta = BC/BI \qquad \therefore \ \tan \alpha / \tan \beta = BI/Bi$

but, in $\triangle BiI \qquad B\hat{i}I = B\hat{I}i = 90° - \theta/2 \qquad \therefore \ BI = Bi$

and $\qquad \tan \alpha = \tan \beta \qquad (3.7)$

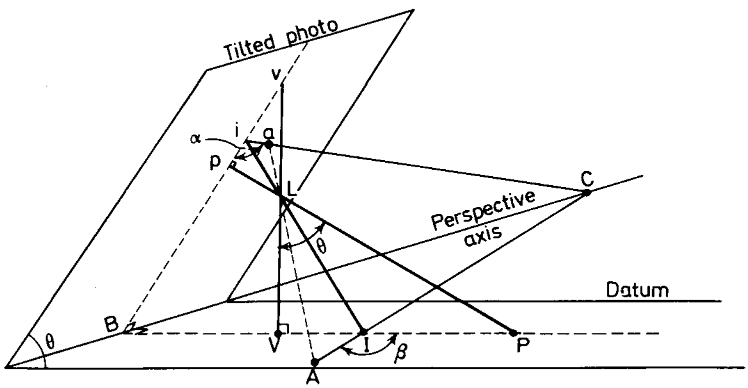


Figure 3.9

Consider now a similar construction through the plumb points v and V and any other point on the ground similar to A with the two lines meeting on the perspective axis. In right-angled triangles

BvC and BVC $\quad \tan\alpha = BC/Bv \quad \tan\beta = BC/BV$

$$\therefore \frac{\tan\alpha}{\tan\beta} = \frac{BV}{Bv} = \cos\theta \tag{3.8}$$

Similarly at p $\quad \tan\alpha = BC/Bp \quad \tan\beta = BC/BP$

$$\therefore \frac{\tan\alpha}{\tan\beta} = \frac{BP}{Bp} = \sec\theta \tag{3.9}$$

This latter ratio is particularly important when considering *radial line plotting*, which assumes the angles about the principal point on the photograph are equal to their equivalents on the ground. This is, of course, only true for the isocentre.

In addition to the complications already outlined, further displacements may result due to variation in flying height, refraction of the rays of light (particularly near the body of the aircraft), camera and photographic errors, etc. It can now be clearly seen that a photograph is not a plan, except where the axis of the photograph is truly vertical and the ground is flat and level.

3.2.7 Combined effect of tilt and relief

As already shown, displacement due to tilt and relief are not radial from any one point on the photograph. *Figure 3.10* shows ab as the top and bottom of a tall structure. The height displacement ab is radial from the plumb point v, whilst the tilt displacements aa_1 and bb_1 are radial from the isocentre. Note the reverse direction of displacement on the upper side of the photograph.

The treatment of such effects is to: (i) eliminate tilt displacement by a mathematical or optical rectification of the photograph, i.e. reduce the tilted photograph to its horizontal equivalent; (ii) consider height displacement on the rectified photograph—for instance, after rectification the equivalent position of the plumb point v is at v_1 (see also *Figure 3.7*) from which the height displacement a_1b_1 is radial.

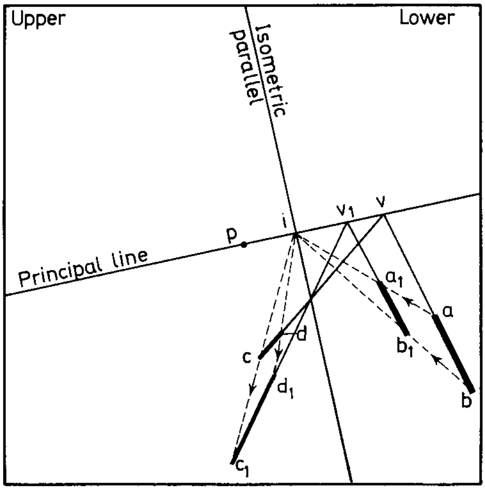


Figure 3.10

3.2.8 To find the *x* and *y* tilts of a photograph

Given the focal length of the camera and the co-ordinates of the plumb point, the *x* and *y* tilts may be found as follows (from *Figure 3.11(b)*):

$$\sin\theta_1 = \frac{ab}{ap} = \frac{vc}{ap} = \frac{vc}{pv} \times \frac{pv}{ap} = \sin\theta\cos S$$

Thus for tilts in the *y* direction (θ_y) $\qquad \sin\theta_y = \sin\theta\cos S \qquad$ (3.10)

Now from *Figure 3.11(a)*

$$\cos S = y_v/pv \qquad \text{and} \qquad pv = f\tan\theta \quad (\text{see } \textit{Figure 3.3})$$

$$\therefore\ \sin\theta_y = \sin\theta \times \frac{y_v}{f\tan\theta} = \sin\theta \times \frac{y_v}{f} \times \frac{\cos\theta}{\sin\theta}$$

$$\therefore\ \sin\theta_y = \frac{y_v\cos\theta}{f} \qquad (3.11)$$

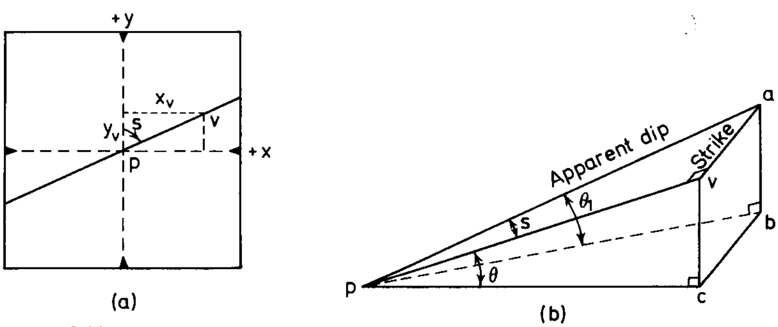


Figure 3.11

Similarly $\sin\theta_x = \sin\theta\cos(90° - S) = \sin\theta\sin S$

But $\sin S = x_v/pv$

Thus for tilts in the x direction (θ_x) $$\sin\theta_x = \frac{x_v\cos\theta}{f} \quad (3.12)$$

Putting $S = 0°$ in equation (3.10) gives $\theta y = \theta$; therefore, all lines parallel to the principal line have the same maximum tilt. Putting $S = 90°$, gives $\theta y = 0$; therefore, all lines at 90° to the principal line (i.e. plate parallels) are horizontal.

3.2.9 Ground co-ordinates from a tilted photograph of flat terrain

In *Figure 3.12* consider point a whose photo co-ordinates x_a and y_a are measured about the fiducial axes; also known are the flying height H, the focal length f, the tilt θ and the swing S. It is first necessary to obtain the co-ordinates of a relative to the principal line

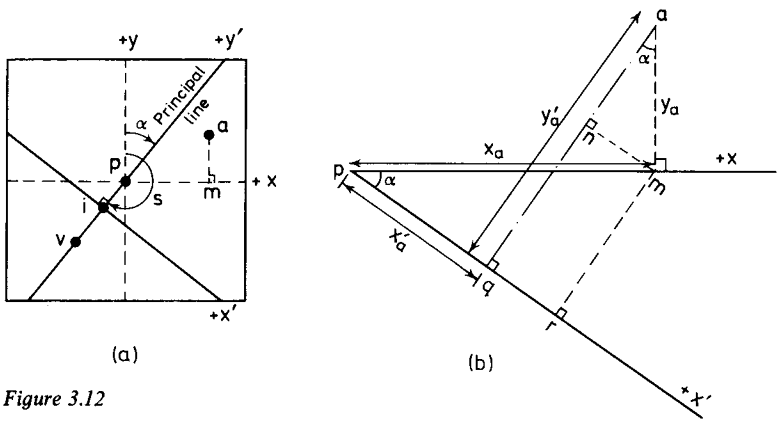


Figure 3.12

axes, with the isocentre as origin (tilt displacement radial from isocentre), i.e. x'_a, y'_a. *Figure 3.12*(*b*) illustrates the rotational effects where the angle between the respective axes is $\alpha = 180° + S$. The amount of translation necessary is $pi = f\tan(\theta/2)$. Assuming a parallel through p, an examination of *Figure 3.12*(*a*) shows that the angle between the x axes is α. Then from *Figure 3.12*(*b*)

$$x'_a = pr - qr = x_a\cos\alpha - y_a\sin\alpha \quad \text{and} \quad y'_a = an + mr = y_a\cos\alpha + x_a\sin\alpha$$

To obtain the general form for these equations, substitute $(180° + S)$ for α and add the translation amount $pi = f\tan(\theta/2)$ to obtain the new origin at i, then

$$x' = -x\cos S + y\sin S \quad (3.13)$$

$$y' = -x\sin S - y\cos S + f\tan(\theta/2) \quad (3.14)$$

where x and y are the photo co-ordinates measured about the fiducial axes.

Equation (3.2), for scale on a tilted photograph, can now be applied to the reduced co-ordinates to give the ground co-ordinates X and Y as follows:

$$X = Kx' \qquad \text{and} \qquad Y = Ky' \qquad \text{where} \quad K = H/(f - y \sin \theta)$$

3.2.10 Ground co-ordinates from a tilted photograph of rugged terrain

The data in this case are exactly the same as in *Section 3.3.1*, plus the elevations h of the points in question.

As the effect of ground relief is radial from the plumb point, then the rotation and translation is this time relative to v, where $pv = f \tan \theta$, then

$$x' = -x \cos S + y \sin S \tag{3.15}$$

$$y' = -x \sin S - y \cos S + f \tan \theta \tag{3.16}$$

Thus, from *Figure 3.13*, the new co-ordinates of a are

$$y'_a = vr \qquad \text{and} \qquad x'_a = ra$$

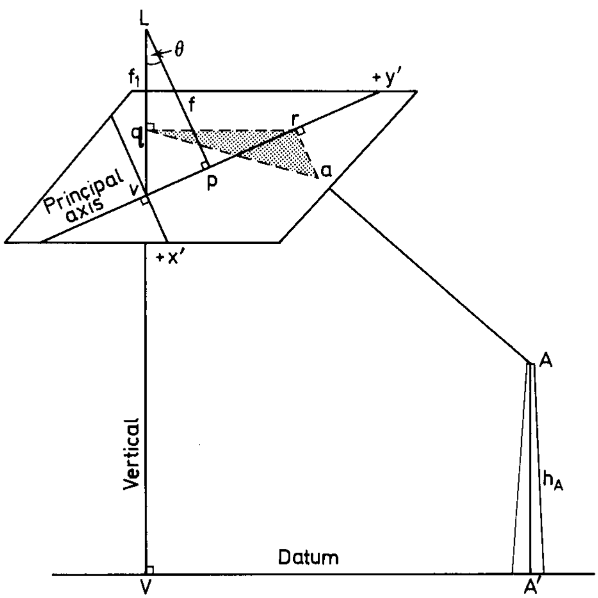


Figure 3.13

Mathematical rectification for each point is now carried out by considering horizontal planes passing through the plate parallels of the points in question. For example, in the case of point a, the horizontal plane through it is qra, and the rectified co-ordinates are therefore

$$y''_a = qr = y'_a \cos \theta \qquad \text{and} \qquad x''_a = ra = x'a$$

The new focal length appropriate to the plane of rectification is now

$$f_1 = Lq = Lv - qv = f \sec\theta - y'_a \sin\theta$$

We now have an untilted photograph of ground point A with a new focal length f_1. It now only remains to multiply the photo co-ordinates by their appropriate scale as in *Section 3.2.2*, giving

$$X_A = x''_a \frac{(H - h_a)}{f} \quad \text{and} \quad Y_A = y''_a \frac{(H - h_a)}{f}$$

Example 2.25. Two points A and B situated 10 and 40 m, respectively, above datum, are imaged on a near-vertical aerial photograph, taken from an altitude of 2000 m with a camera of focal length 152 mm. The photo co-ordinates of the points about the fiducial axes are measured by a comparator as follows:

	x (mm)	y (mm)
a	+50.00	+100.00
b	−100.00	+80.00

If the tilt and swing of the photograph are 2° and 20°, respectively, calculate the horizontal ground distance *AB*.

$$x'_a = -50\cos 20° + 100\sin 20° = -12.78 \text{ mm}$$
$$y'_a = -50\sin 20° - 100\cos 20° + 152\tan 2° = -105.76 \text{ mm}$$
$$x'_b = 100\cos 20° + 80\sin 20° = +121.33 \text{ mm}$$
$$y'_b = 100\sin 20° - 80\cos 20° + 152\tan 2° = -35.66 \text{ mm}$$

Rectified co-ordinates

$$x''_a = x'_a = -12.78 \text{ mm}$$
$$y''_a = y'_a\cos 2° = -105.70 \text{ mm}$$
$$x''_b = x'_b = +121.33 \text{ mm}$$
$$y''_b = y'_b\cos 2° = -35.64 \text{ mm}$$

New focal length per point

$$f_{1a} = f\sec\theta - y'_a\sin\theta = 152\sec 2° + 105.76\sin 2° = 155.78 \text{ mm}$$
$$f_{1b} = f\sec\theta - y'_b\sin\theta = 152\sec 2° + 35.66\sin 2° = 153.33 \text{ mm}$$

Ground co-ordinates

$$X_A = \frac{x''_a(H - h_a)}{f_{ia}} = \frac{-12.78(2000 - 10.00)}{155.78} = -163.26 \text{ m}$$

$$Y_A = \frac{y''_a(H - h_a)}{f_{ia}} = \frac{-105.70 \times 1990}{155.78} = -1350.26 \text{ m}$$

$$X_B = \frac{x''_b(H - h_b)}{f_{ib}} = \frac{121.33(2000 - 40.00)}{153.33} = +1550.95 \text{ m}$$

$$Y_B = \frac{y''_b(H - h_b)}{f_{ib}} = \frac{-35.64 \times 1960}{153.33} = -455.58 \text{ m}$$

Ground distance

$$D = (\Delta X^2 + \Delta Y^2)^{\frac{1}{2}} = (1714.21^2 + 849.68^2)^{\frac{1}{2}} = 1933.64 \text{ m}$$

The above example serves to illustrate the need for a restitution system to correct photo measurements for the effects of tilt and relief displacement.

3.3 GROUND CONTROL

The establishment of ground control points, which are clearly distinguishable on the air photographs, is very important to the photogrammetric process.

The minimum number of points required per photograph comprises two plan points to control scale and three height points to control level in the spatial model. It is important to realize that ground control, fixed by normal survey methods, should be more accurate than that attainable by the photogrammetric restitution system used.

An indication of the distribution, location and accuracy of the control points would be provided by the photogrammetrist after stereoscopic examination of the photographs and annotated on the photographs. It is obvious from this that the control points must consist of detail already clearly visible on the photographs.

The type of detail chosen must be consistent with the photogrammetric measuring process of placing a floating dot on the stereo-model. For instance, for 1/2500 scale mapping, the photo-scale would be in the region of 1/10 000. If the diameter of the floating dot within the plotter was 40 μm, then the target would need to be 400 mm in diameter in order to accommodate the floating dot without being obscured by it. Thus an appropriate control point might be the centre of a large manhole cover and not the fine points normally associated with control stations. Similarly, for height control the points chosen should lie in flat, horizontal ground free from vegetation. Steep slopes or peaks should be avoided to reduce the large height errors that would result from bad positioning of the floating dot.

The amount of control required depends largely on the scale and accuracy of the finished plan. For engineering plans at 1/500 scale, it is usual to supply at least two points per photo to control scale and orientation and three to control level.

3.3.1 Pre-marked control

In the production of large scale engineering plans the control points are generally pre-marked targets. Their locations are indicated from initial photography and then, when established, the area is re-flown. A popular type of target used consists of a large white cross of durable material with arms in the region of 2 m long and 0.25 m wide. The size, however, is very much a function of photo scale, as already shown, and must be large enough to be clearly visible on the photographs and small enough to form a suitable area for the floating dot. Although pre-marking is more expensive than the use of existing detail, they may be so constructed as to be used for the control of setting-out, at a later stage.

3.3.2 Accuracy requirements

General rules quoted for the accuracy of fixing ground control are

(a) for large-scale engineering plans $\pm 0.02\% H$
(b) for medium-scale engineering plans $\pm 0.03\% H$
(c) for small-scale topographic plans $\pm 0.05\% H$

where H is the flying height and related to the accuracy of the photogrammetric plotter. The specifications apply to both planimetric position and height control. Thus for 1/10 000 photography using a wide-angle camera (f = 150 mm), H would be 1500 m and ground control fixed to an accuracy of ± 0.3 m in case (a) above. Based on this information, an appropriate survey procedure could be instituted.

Normal survey procedures will also be used to complete detail on the plan which may have been obscured on the photographs by cloud, glare, shadow, trees or other factors.

3.3.3 Aerial triangulation

For mapping at smaller scales and lower accuracies, the process of aerial triangulation may be used. This procedure provides control direct from the photographs, thereby reducing the amount, and thus cost, of ground control fixed by normal survey techniques. Aerial triangulation may be used to establish two- or three-dimensional control points, either by analogue methods in precision plotters, or by purely analytical processes. Radial-line plotting and slotted template assemblies are graphical and mechanical methods, respectively, of two-dimensional aerial triangulation, in which the minor control points are fixed in position, relative to only two ground control points, one at each end of the strip (refer to *Section 3.5.3*).

In the analogue process, each stereo model is connected to the next, thus forming a strip of model co-ordinates. Each strip is then connected to the next, ultimately forming a set of block co-ordinates. Due to error propagation in the process, strip and block adjustment of the co-ordinates are necessary before they are transformed to fit the ground control.

Aerial triangulation forms a very important aspect of photogrammetry but is mentioned here only very briefly as it is beyond the scope of this book.

3.4 FLIGHT PLANNING

The flight specifications for a particular project will vary with the type of project. For instance, photography required for interpretation purposes will not require the same detailed planning as that required for large-scale mapping.

The main factors to consider are the directions of the flight lines, the overlaps, scale and flying height. Some of the factors cannot be obtained until the flight has commenced. For instance, the heading direction and the time interval between exposures can only be calculated when the wind velocity at the time of flight is known. One also needs some idea of the number of photographs required in order to decide on the number of magazines of films to take. A plot of the flight lines would also serve to indicate the appropriate turns to utilize for changing magazines. The flying height of the aircraft is dependent on many factors ranging from aircraft capabilities, terrain conditions, and survey requirements, to the type of restitution (or plotting) system to be adopted. Flight planning is thus a skilled procedure requiring careful planning at all its various stages.

3.4.1 Direction of flight lines

Generally the area is flown parallel to the longest side to give the minimum number of strips. In this way the number of turns and run-ins, which are non-productive, are reduced to a minimum.

If large areas having different levels exist, such as mountain ranges or plateaux, the area may be flown parallel to these in order to avoid rapid variation in scale.

Each photograph in a strip overlaps the previous one by 60%, thus the new ground covered on each photograph is 40%. The purpose of the overlap is to permit stereoscopic viewing of the area. Each strip overlaps the previous one by 20 to 30% (*Figure 3.14*), thus complete coverage of the area is obtained.

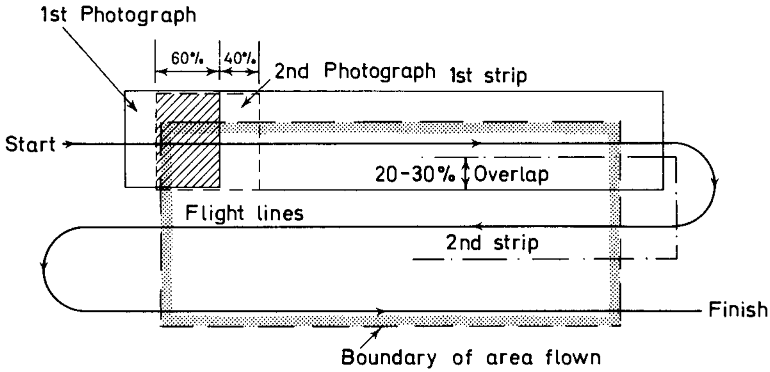


Figure 3.14

The overlapping, which is automatically controlled by an intervalometer on the air camera, is illustrated more clearly in *Figure 3.15*. The distance *B* between each photograph in the air is called the *air base*, while its equivalent on the photograph, *b*, is called the *photo base*. Due to the overlap, both of the principal points of the adjacent photographs will appear on the central photograph. The photo bases are in fact the direction of flight of the aircraft.

Care must be exercised when flying over steadily-rising ground, as failure to do so may result in the complete loss of the overlap required in both forward and lateral directions (*Figure 3.16*). The loss of forward overlap may be overcome by decreasing the exposure interval, whilst to ensure lateral overlap the flight lines must be based on the minimum lateral overlap over the highest ground.

The flight may also be affected by cross-winds (*Figure 3.17*) causing the aircraft to drift off the planned course. The triangle *ABC* may be solved to give the value of θ, and the craft is corrected on to course *AD* at a specific *air speed*, the wind velocity causing it to 'crab' the planned course *AC* at a different speed called the *ground speed*. Thus if the camera is not squared to the direction of flight the photographs will be crabbed, as shown, with resultant gaps in the coverage. This adjustment is carried out by the 'drift-ring setting', which rotates the camera through θ about the vertical axis of the camera mount. Modern viewers and air cameras have largely eliminated this problem.

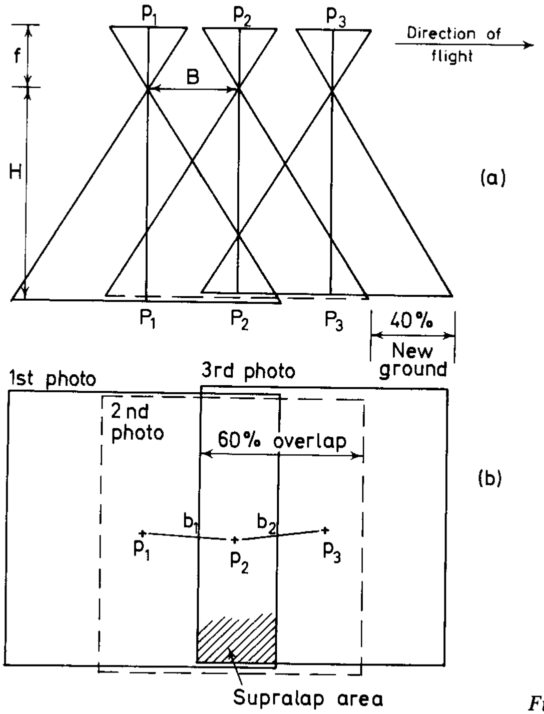


Figure 3.15

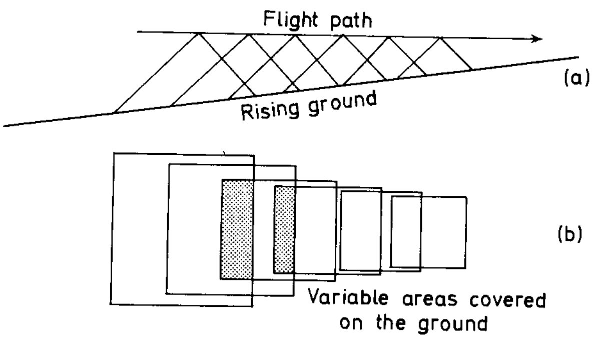


Figure 3.16

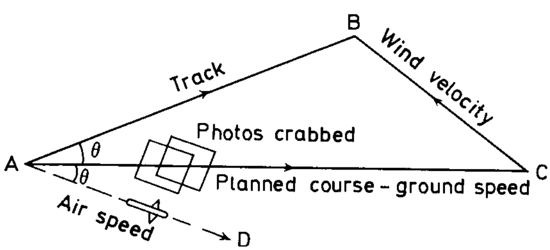


Figure 3.17

3.4.2 Scale and flying height

The scale of the photography will depend upon the map compilation techniques used. For simple graphical or mechanical radial line methods, the photograph scale is usually greater than the map scale, in order to reduce the effect of compilation errors. If the map is to be produced by stereo-plotters, the photo scale is usually smaller. For example, 1/12 500 photography is frequently used to produce 1/2500 plans. As already shown, the flying height H is a function of the scale, thus using a normal wide-angle camera ($f = 152$ mm) for 1/12 500 photography gives

$$f/H = 1/12\,500 = 0.152/H \qquad \therefore\ H = 1900 \text{ m}$$

Where there is great variation in ground relief or the area contains many tall buildings, the flying height may need to be increased. One of the limitations of radial-line plotting is that variation in ground relief must not exceed $H/10$. Thus, if this method of plotting is to be used, H will be related to variation in ground levels.

The method of heighting may also control the flying height. Many stereo-plotters are given a C-factor which relates flying height to the minimum contour interval. Thus a machine with a C-factor of 2000 would use photography taken at a height of 2000 times the contour interval adopted.

Image movement, caused by the camera being in motion at the instant of exposure, can greatly affect the quality of the photograph. It can be reduced by flying higher, at slower speeds and using faster shutter speeds. As there is an acceptable limit to image movement, it will have an effect on the value of the flying height.

Where very hilly ground is encountered (*Figure 3.18*), or high urban construction with narrow streets, the use of a wide-angle lens at flying height H may result in much ground detail being obscured, i.e. dead ground at B and C. It may also be difficult to handle this photography stereoscopically. However, the use of a narrow-angle lens ($2f$) at twice the flying height ($2H$) would produce equivalent photography at the same scale. Also, all three points A, B and C are imaged on the normal-angle photography, whereas B and C are not imaged on the wide-angle. This situation may therefore be a contributory factor in deciding flying height.

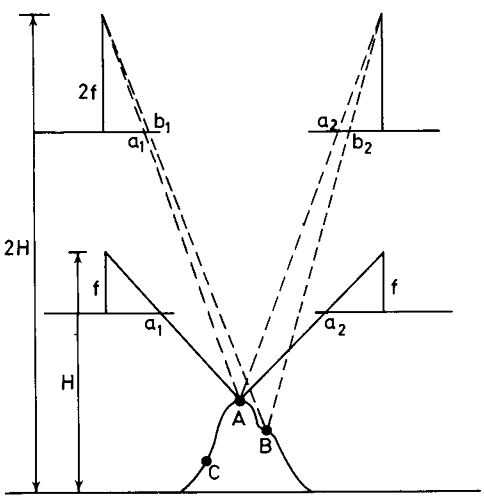


Figure 3.18

The above examples serve to illustrate the complications involved in deciding appropriate flight data.

3.4.3 Costing the project

The number of photographs needed to cover an area will be required not only to cost the work, but also to estimate the amount of film required, and the points at which the film magazines should be changed. If possible, the magazines should be changed in the turns.

Consider an area 200 km by 100 km to be flown at an average scale of 1/10 000. At this scale the area is 20 m by 10 m. The photography has a format size of 230 mm × 230 mm, of which 60% is overlapped; thus the new ground covered at this scale is 40% of 230 mm = 92 mm.

Therefore the number of photographs per strip = 20 000 ÷ 92 mm = 218, plus say two each end to ensure complete coverage, i.e. a total of 222.

In the same way the number of strips, assuming a 30% lateral overlap = 10 000 ÷ (70% of 230 mm) = 63 strips.

$\therefore$ Total number of photographs = 222 × 63 = 13 986

In all, it can be seen that careful planning is needed to ensure a satisfactory and economic completion of a project. The necessary flight maps must be carefully prepared for use by the navigator, who, with the aid of modern viewers and cameras, can concentrate on the execution of the project.

3.5 RADIAL-LINE PLOTTING

Due to the errors and distortions inherent in the air photo, some form of restitution system must be used to produce a plan. In general terms the accuracy of the restitution system used is directly proportional to its cost. Enlargement from photo scale to plan scale would also result in a proportional enlargement of existing errors, unless these had been minimized by an appropriate restitution system. Thus large-scale plans, most frequently used by the engineer, may require the most precise restitution, and therefore entail higher costs. A broad classification of available systems is:

(a) Precise —reads to 0.01 mm—enlarges × 8
(b) Topographic—reads to 0.01 mm—enlarges × 4
(c) Approximate—reads to 0.4 mm—enlarges × 2
(d) Direct —reads to 0.4 mm—enlarges × 0.1

Radial-line plotting falls into the latter category and is a simple direct method of mapping from single photographs.

As already shown, the effects of tilt and relief cause displacement of the image points from their true plan positions:

(a) Tilt displacement is radial from the isocentre.
(b) Relief displacement is radial from the plumb point.

However, provided that tilt and ground relief are limited, image displacement may be assumed radial from the easily-located principal point of the photograph.

Provided that the above limitations prevail, the method produces quite reasonable results and could usefully be employed by the engineer for updating existing plans and producing reconnaissance plans for initial project investigation.

3.5.1 Principles

The method assumes that angles to points of detail on the photograph measured about the principal point (PP) are equal to their equivalent angles on the ground. However, as already shown in the analysis of the photograph, this assumption is incorrect due to height and tilt displacements.

Consider now three truly vertical photographs (*Figure 3.19*). As there is no tilt, the plumb point will coincide with the principal point, and the height displacement of the chimney imaged at a_1, a_2 and a_3, will be radial from the principal point. Thus, trisection

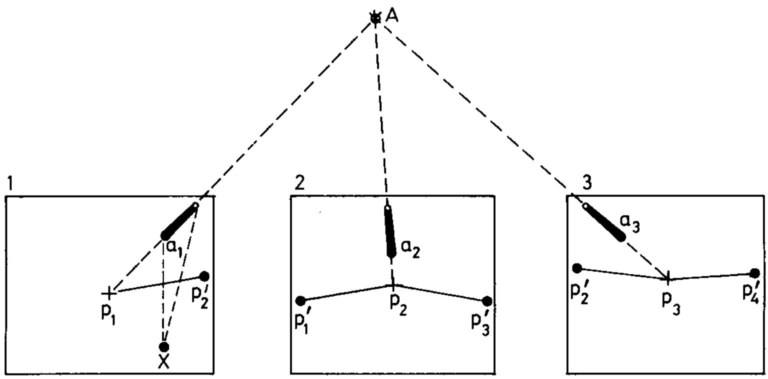


Figure 3.19

of the chimney from each of the photographs gives the position of the chimney as on the orthogonal projection of a plan. If one had chosen any other point on the photograph, such as X, then two positions for the chimney would be obtained, which is of course unacceptable. The method then is very similar to plane table resection.

In practice, however, practically every photograph would contain tilt, which renders no position on the photograph 'angle true'. It is obvious then that the principal point may be used only when the effects of height and tilt displacement are kept within acceptable limits. It can be shown that these limits are achieved when 'tilt is limited to no more than 2° and the variation in ground relief is never more than 10% of the flying height'. This statement is called the *Arundel assumption*, Arundel being the place where the method was thoroughly tried and tested.

3.5.2 Proof of the Arundel assumption

Consider first the effects of tilt, and assume that a plotting error of less than 0.5 mm is negligible.

Figure 3.20 shows point a on a tilted photograph. If it was plotted by radial-line

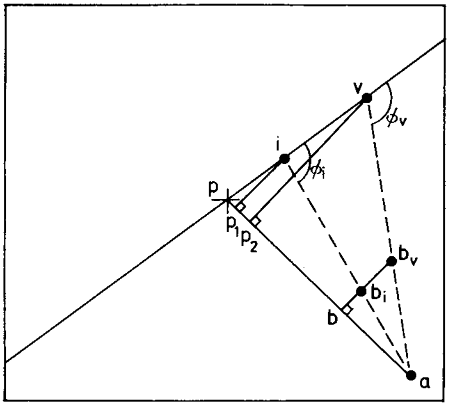


Figure 3.20

methods using the isocentre i, it would be fixed in its true position at b_i, i.e. corrected for tilt. In *Section 3.2.5* equation (3.6) shows that the tilt displacement ab_i is

$$ab_i = \frac{ia^2 \sin\theta \cos\phi_i}{f}$$

However, as p is to be used for radial line plotting then b_i would be fixed at b, with a resultant plotting error of b_ib. As the amount of tilt will generally be less than 3°, then $i\hat{p}a \approx \phi_i$

$$\therefore \; ip_1 = ip \sin\phi_i = f \tan(\theta/2) \sin\phi_i$$

From similar triangles

$$b_ib = \frac{ip_1 \times ab_i}{ia}$$

$$= \frac{f \tan(\theta/2) \sin\phi_i \times ia^2 \sin\theta \cos\phi_i}{ia \times f}$$

$$= ia \sin\theta \tan(\theta/2) \sin\phi_i \cos\phi_i$$

But

$$\tan(\theta/2) = \frac{\sin(\theta/2)}{\cos(\theta/2)}$$

and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ and $\sin\phi_i \cos\phi_i = \frac{1}{2}\sin 2\phi_i$

which on substitution gives $b_ib = ia\frac{1}{2}\sin(2\phi_i)2\sin^2(\theta/2)$

and, as θ is small, $2\sin^2(\theta/2) \approx \theta^2/2$ $\quad \therefore \; b_ib = \dfrac{ia\theta^2 \sin 2\phi_i}{4}$

From this equation, the error is a maximum when $\phi = 45°$, which with a 230 mm × 230 mm format would give a maximum value for ia of 162 mm. If a value of 2° for tilt is assumed then substituting in the equation gives $b_ib = 0.05$ mm. Thus, *provided that tilt is limited to less than 2°*, the error resulting from using the PP is negligible.

Considering now the effect of ground relief. In this case if a was plotted by radial line methods using the plumb point v, it would be fixed at b_v, i.e. corrected for height displacement. Equation (3.3) in *Section 3.2.4* shows that

$$ab_v = av(\Delta h/H)$$

However, if plotted from p, it would be fixed at b, with a resultant error of bb_v. From similar triangles

$$bb_v = \frac{vp_2 \times ab_v}{av}$$

but $vp_2 = pv \sin \phi_v$ (assuming $vpa \approx \phi_v$), thus $vp_2 = f \tan \theta \sin \phi_v$

$$\therefore \ bb_v = \frac{f \tan \theta \sin \phi_v \Delta h}{H}$$

which shows that the maximum value for bb_v occurs when $\phi_v = 90°$. Assuming then that $\theta = 2°$ (fixed by previous analysis), $f = 152$ mm and $bb_v \not> 0.5$ mm

$$\frac{\Delta h}{H} = \frac{bb_v}{f \tan \theta} = \frac{0.5}{152 \times 0.035} \approx \frac{1}{10}$$

Thus, for negligible plotting error when using the PP, the ground relief should not exceed 10% of the flying height. These two factors (termed the *Arundel assumption*) are the limitations of the radial-line method.

3.5.3 Preparation of photographs

The following steps are necessary in the preparation of the photographs.

(1) *Base lining*

The PP of each photograph is located in the usual way, pricked through and identified using a suitable symbol. The PP in the overlap areas are now located, i.e. the position of P_2 on photograph 1 for instance. If these points fall on points of detail easily located on common photographs, then they are simply pricked through. If they fall in featureless terrain or on water, they are best transferred under a stereoscope, using some form of floating mark. Two small pieces of transparent plastic sheet with identical crosses on, are ideal. Assuming one wants to transfer P_2 to photograph 1: both photographs are assembled under the stereoscope to form a stereo model. One mark is placed precisely over P_2 on photograph 2, whilst the other mark is moved about the required area on photograph 1 until both marks appear to fuse into one. The mark on photograph 1 is now moved in the x-direction until the fused mark appears to rest on the ground; this then is the position of P_2 on photograph 1. The procedure is reversed when transferring P_1 to photograph 2. When all the PP are located they are joined up to form base lines, and will appear as the PP in *Figure 3.21.*

(2) *Minor control points (MCP)*

MCP are easily-identifiable points of detail on the photographs, such as road intersections, fence corners, etc. and are selected in the positions shown in *Figure 3.21.* The distance of the MCP from the PP should be roughly equal to the mean length of the base lines appearing on that print, to ensure good intersection. If possible they should be at the mean height of the terrain and clearly visible on three consecutive photographs.

(3) *Ground control points (GCP) and tie points (TP)*

Horizontal ground control points are now located, pricked through and identified by drawing a triangle round them.

Where a block of strips is concerned, further points known as *tie* points (TP) will be required in the lateral overlap in order to connect adjacent strips. The first two and last two photographs in the strip have specially selected TP which are sometimes called *major* points. The TP should occur at every fourth PP and may be the already-fixed MCP.

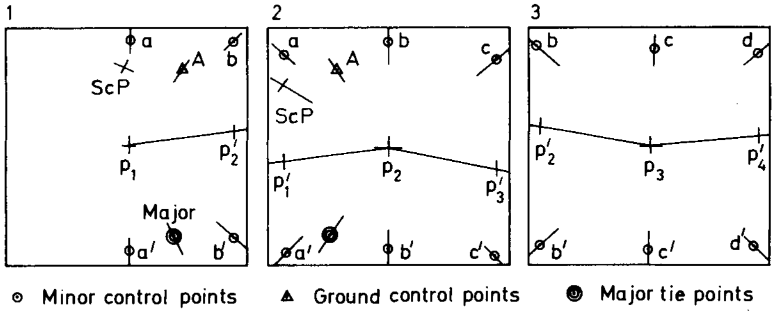


Figure 3.21

(4) *Scale point (ScP)*

The ScP is chosen to fix the scale of the whole strip and should be a point situated at the mean height of the strip. It may be an MCP, or even a GCP, if they meet the requirement.

When all these points have been fixed, short radials from the central PP of each photograph are drawn through them. This completes the preparation of the photographs, which will appear as in *Figure 3.21.*

3.5.4 Construction of the radial-line plot

Due to the many factors already discussed, it should be realized that all of the photographs in the strip are at slightly different scales. The first step, therefore, is to produce a strip of photographs at a 'common' scale, as follows.

A strip of stable plastic drawing material is placed over the first photograph and the following data traced: the exact position of p_1 and ScP only, plus radials through the remaining detail and the base line. The plot will appear as shown in *Figure 3.22(a)*. The first photograph is removed and the sheet oriented over the second photograph by overlying the drawn base over the base p'_1p_2 on the photograph. In this position it is moved back and forth until the radial through ScP intersects the plotted position of ScP. The sheet is held, and p_2 and the remaining radials are marked off. The sheet now appears as in *Figure 3.22(b)*. It is now placed over the third photograph correlated to the appropriate base p'_2p_3, and moved back and forth until the radials b, b' trisect the intersections b, b'. The sheet is held and the remaining detail taken off as shown in *Figure 3.22(c)*. The procedure is continued to the end of the strip thereby forming a 'principal point traverse' to a common but unknown scale. The scale will be the

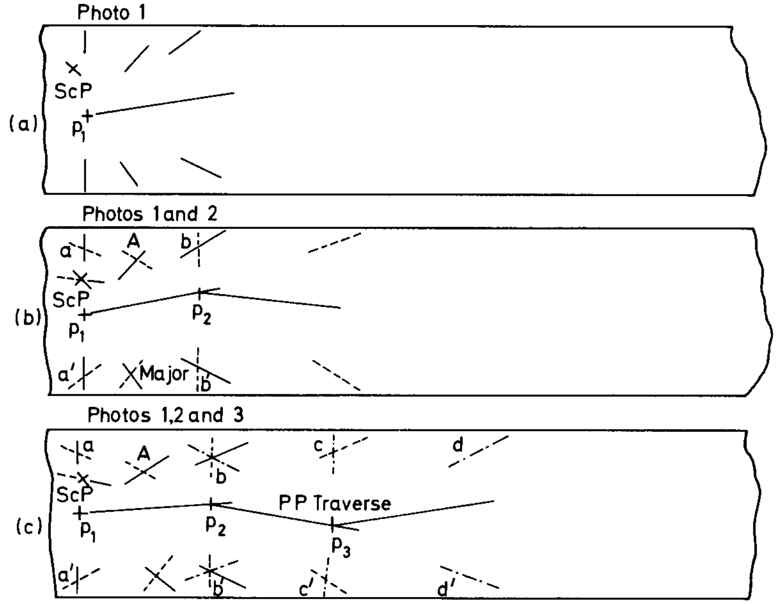


Figure 3.22

relationship of distance p_1–ScP on the photograph to its equivalent on the ground, and as ScP is at the mean height of the terrain it will be roughly the mean scale of all the photographs in the strip.

The next step is to bring the strip to a known scale, namely the compilation scale of the plan being prepared. This is easily carried out using the three-point method. Assume the strip when completed appears as in *Figure 3.23* (*a*). GC points *A* and *B* have been joined by a straight line and a further point *C* selected, to form a well-conditioned triangle. A second strip is now accurately gridded to the scale required and the GC plotted thereon at *Ap* and *Bp*. This strip is overlain on the first strip with *Ap* over *A*, line *ApBp* aligned with *B* and the radial is drawn through *C*. The strip is now moved to put *Bp* over *B*, aligned with *A* and a second radial drawn through *C*; this gives the correct scaled position *Cp*. The procedure is repeated using all three points in turn and radials

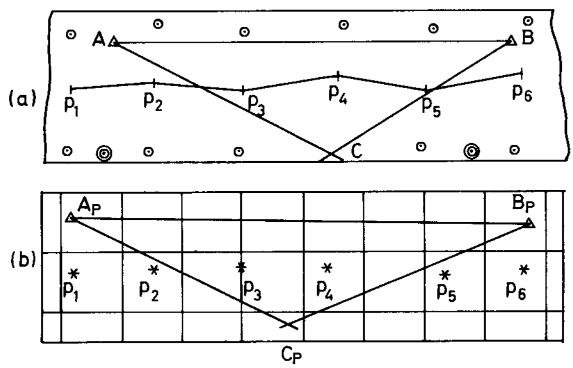


Figure 3.23

drawn through the PP. These trisections give the PP in their correct scaled positions (*Figure 3.23*(*b*)), the procedure being repeated to give the positions of the MCP, etc.

The plotting of detail may now be carried out by placing photograph 1 under the gridded sheet with its PP under p_1 and the base aligned. Radials are then drawn from p_1 through all the important points of detail. Similarly, photograph 2 is placed under p_2 and intersecting radials drawn to locate the points in question. The remaining detail, between these accurately-located points of detail, may now be drawn in freehand, by interpolating from the appropriate photograph.

3.5.5 Block adjustment

If there is more than one strip involved they will need to be fixed relative to each other at the correct scale.

The main grid is constructed and all GCP plotted thereon. The first strip with GCP at *Ap* and *Bp* is adjusted to the main grid. The second strip is then adjusted to the first using the major TP in the lateral overlap and so on with subsequent strips. Generally, discrepancies between the remaining TP will be found which, if small (< 1 mm), may be adjusted by a simple equal shifts movement of the strips. If the discrepancies are excessive they are measured in terms of easting and northing differences. These values are algebraically summed and meaned. The strips are then uniformly shifted through this mean value in order to give agreement.

The description of block adjustment has been kept to a minimum, as it is inevitably carried out by the mechanical method of slotted templates.

3.6 SLOTTED-TEMPLATE ASSEMBLY

Slotted templates is a semi-mechanical method of producing a radial-line plot, the principles being identical to those already outlined.

3.6.1 Preparation of template

The photographs are prepared in exactly the same way as for radial-line plotting, with the exception of TP and ScP. The MCP in the lateral overlap are sufficient to tie in adjacent strips.

The photograph is then covered with a sheet of cellulose acetate, the PP marked and the radials scribed with a fine needle. A hole is then punched out precisely on the PP, which enables the template to be placed over a stud in a slotted template cutter, and slots cut precisely on the scribed radials. The corners of the template are then rounded off to facilitate easy movement. Considering photograph 2 of *Figure 3.21*, its template would appear as in *Figure 3.24*(*a*).

3.6.2 Template assembly

A plotting board is now accurately gridded, the ground control plotted on it and fine pins fixed vertically in the GCP. Studs are now placed over these pins as shown in *Figure 3.24*(*b*). The studs are of different colours to denote GCP, MCP and PP.

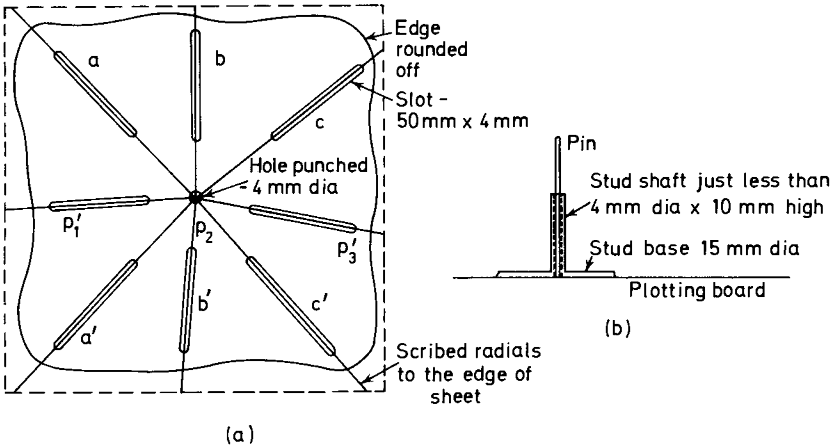


Figure 3.24

When laying the templates to the plotting board it is usual to start with the strip having the most GCP. This enables the scale and azimuth of the assembly to be fixed to the greatest degree of accuracy. Considering the photographs in *Figure 3.21*, the slot representing GCP *A* would be placed over the appropriate stud fixed to the board. A stud is now placed in each remaining slot and PP hole, the slot studs being free to move along the slot. The slots of the second template are placed over the appropriate studs of the first, the remaining slots being again filled with studs. The procedure is continued until a second GCP is reached. It is unlikely that the appropriate slot will fit over the GC stud pinned to the board. Thus the whole assembly may have to be squeezed together or gently stretched apart, the studs moving accordingly in their slots. Thus all the studs take up their correct scaled positions in the strip, by reason of the fact that they are free to move along the trisecting slots of the templates (*Figure 3.25*).

The scale having now been fixed, the remaining strips are similarly laid down to the

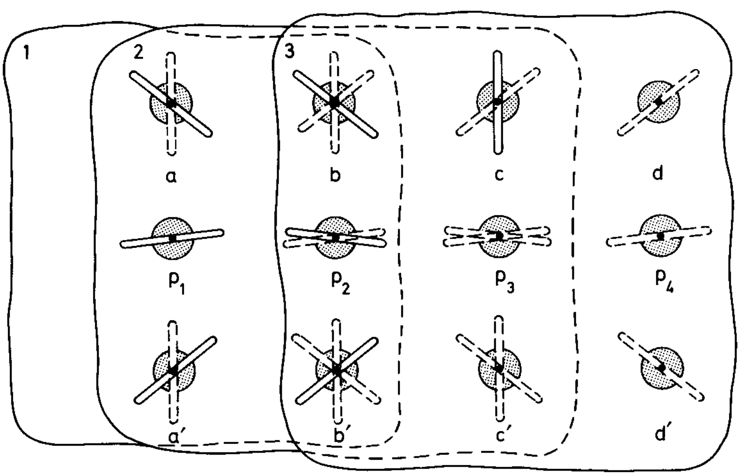


Figure 3.25

GC and MCP in the lateral overlap. The whole assembly is then allowed to settle, pins implanted in the empty studs and then the templates removed.

The main advantage of this method is that it eliminates the tedious scaling and block adjustments required in graphical radial-line methods.

3.6.3 Errors

Errors may result in the buckling of templates when forcing them into position, the main reasons being: (i) bad photograph preparation; (ii) inaccurate template cutting; (iii) mis-plotting of ground control; (iv) mis-computation of ground control; (v) mis-identification of points; (vi) excessive tilt in the photograph, in which case it may have to be rectified and a new template prepared; (vii) excessive ground relief, which may result in having to use the plumb point in place of the principal point.

3.6.4 Radial-line plotter

The radial-line plotter (*Figure 3.26*) is an extremely simple instrument, affording approximate restitution of the air photo. It works on exactly the same principle of graphical intersection already described.

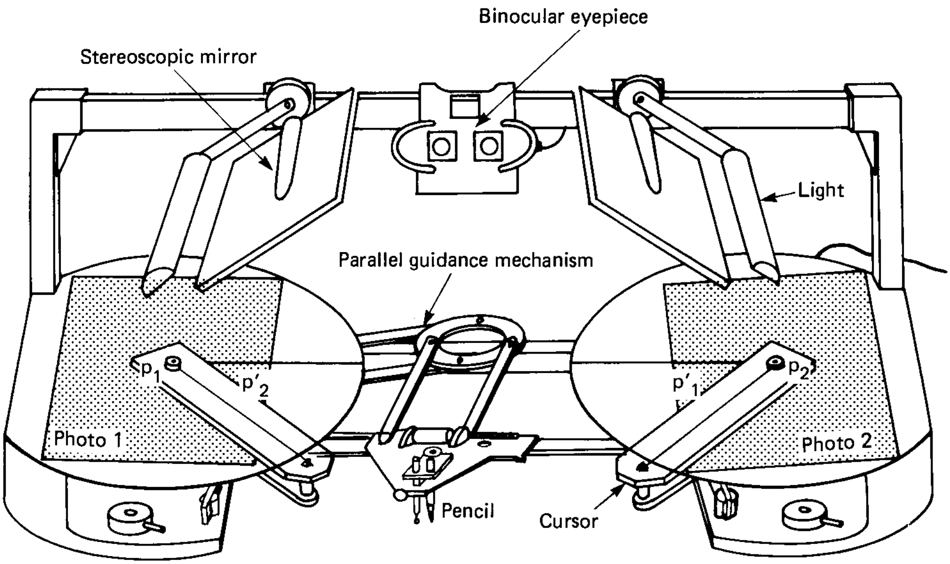


Figure 3.26 Radial-line plotter

The photographs are baselined and mounted about their respective principal points on the horizontal plates of the instrument. The photographs are then rotated until the baselines are co-linear and are then fixed to the plates ready for stereoscopic viewing (refer to *Section 3.7*). As shown, each photograph is overlapped by a cursor which rotates about the principal point. When viewed through the stereoscope, the radial lines intersect to form a cross on the three-dimensional stereo model of the overlap area of the photographs. Movement of the plotting pencil will cause the cross to traverse a selected piece of detail, which is thus plotted on to the plan which is being updated.

The machine has limited enlargement facilities (× 0.5 to × 2) and does not cater for tilt. Nevertheless, as an approximate restitution system it requires no prior knowledge or training, is relatively cheap, and affords an answer much simpler and speedier than do graphical methods.

3.6.5 Sketchmaster

It is convenient at this stage to mention another simple instrument which could easily be utilized by the engineer for the updating of existing plans from air photographs. Such an instrument is the Sketchmaster (*Figure 3.27*), which is a simple reflecting plotter affording an approximate rectification of a single air photo.

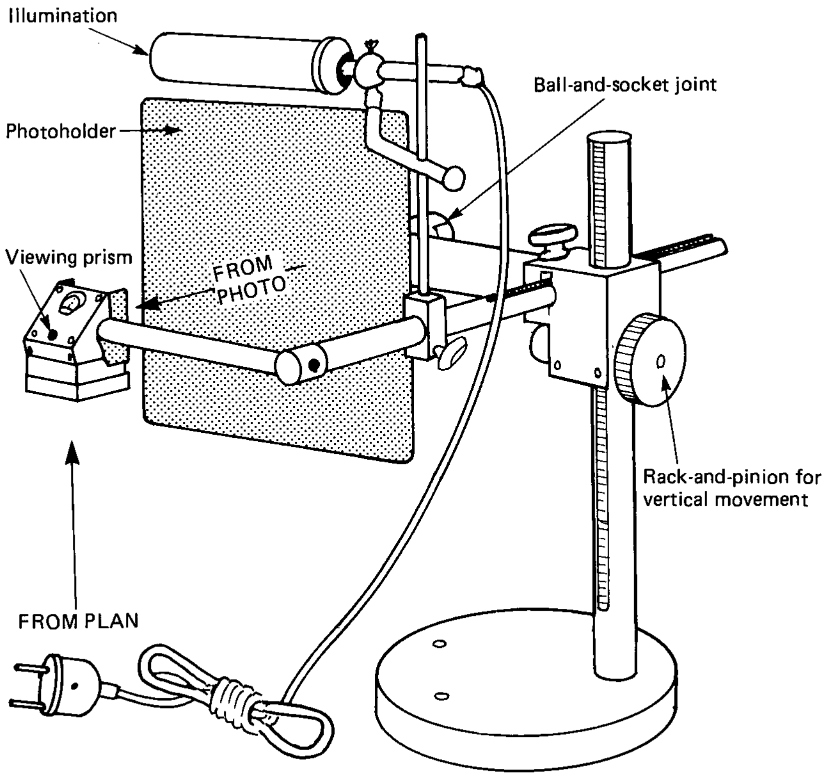


Figure 3.27 Zeiss Sketchmaster

It consists essentially of a photo holder and viewing prism, which can be moved in the vertical and horizontal planes by rack-and-pinion mechanism. The photo holder is on a ball-and-socket arrangement, thereby affording universal movement. When viewed through the prism eyepiece, the photograph can be seen overlapping the plan. The above-mentioned degrees of movement enable the overlapping image of the photograph to be scaled to that of the plan. However, as the scale of the photograph will not be uniform throughout, small areas will need to be dealt with in turn. In this way the photographic detail can be transferred by hand directly on to the existing plan.

3.7 STEREOSCOPY

To date, only the production of planimetric detail has been dealt with; *stereoscopy*, the process of seeing in three dimensions, enables the vertical dimension to be obtained. The application of stereoscopy to air survey will now be illustrated by relating the human sight processes to that of the air camera producing overlapping pairs of photographs.

3.7.1 Stereoscopy in air survey

Consider first a simplified explanation of the seeing processes when looking down at a survey arrow sticking in the ground (*Figure 3.28*). The arrow is viewed simultaneously

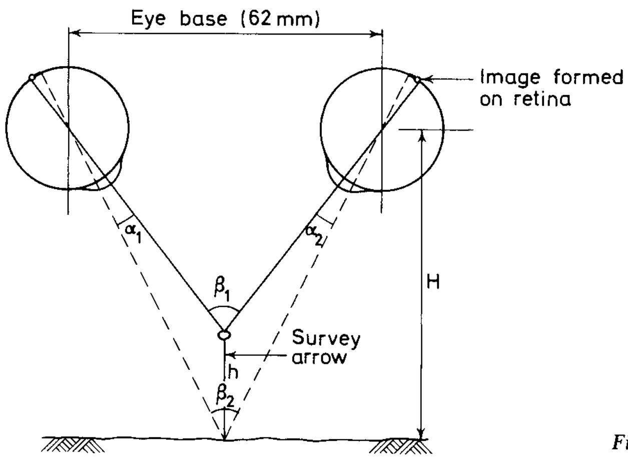


Figure 3.28

from two different positions, the two images fusing together to form a three-dimensional image in the mind. The angles α_1 and α_2 are termed the *angles of convergence*, and may be defined as the ability of the eyes to rotate simultaneously in their sockets. The ability to focus for varying distance is called *accommodation*, while the aperture control variation of the pupil of the eye is called *adaptation*. The angles β_1 and β_2 are called the *parallactic angles* and are a function of the stereoscopic perception of height, i.e.

$$h = f(\beta_1 - \beta_2) = f(\alpha_1 - \alpha_2)$$

Commonsense tells us that if a person was taken to a height of, say, 2000 m, the parallactic angles would be so small as to render height perception impossible. The 'horizontal parallaxes' of the survey arrow are shown on the retina of the eye. That these are a function of the parallactic angles is obvious from the diagram.

As commonsense has shown, in the determination of height there is a definite relationship of the eye or air base to the flying height. Thus when flying, the eye separation must be greatly increased as shown in *Figure 3.29*. The identical situation of human viewing to air survey can now be clearly seen.

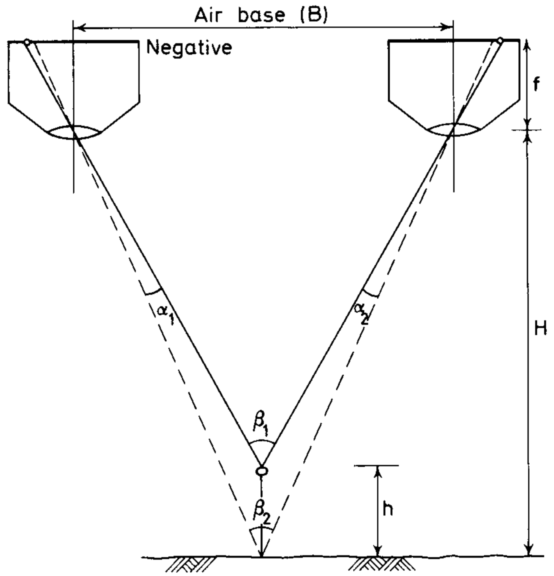


Figure 3.29

If the negatives are now printed as photographs (positives), and viewed simultaneously, so that the left eye sees only the left photograph and the right eye only the right photograph, then a three-dimensional image will form in the mind. The above condition can be most easily obtained by viewing the photographs under a stereoscope, as in *Figure 3.30*. The 3-D image formed is termed a *stereo model*, and the two photographs used are termed *stereo pairs*. Generally speaking the stereo model is exaggerated and this can be useful in the heighting process, particularly where the terrain is relatively flat. Photography can, however, be planned to increase or reduce this effect. If the value of f is fixed, then from the base/height ratio, it can be seen that to halve the flying height would double the impression of height. It can also be shown that increasing the viewing distance of the stereoscope produces a proportionate increase in the impression of height.

3.7.2 Parallax

As already shown, stereoscopic height is a function of the parallactic angles, which are in turn a function of the horizontal parallaxes. As the angles occur in space, they cannot be measured on an aerial photograph. However, the horizontal parallaxes can be used to ascertain vertical heights.

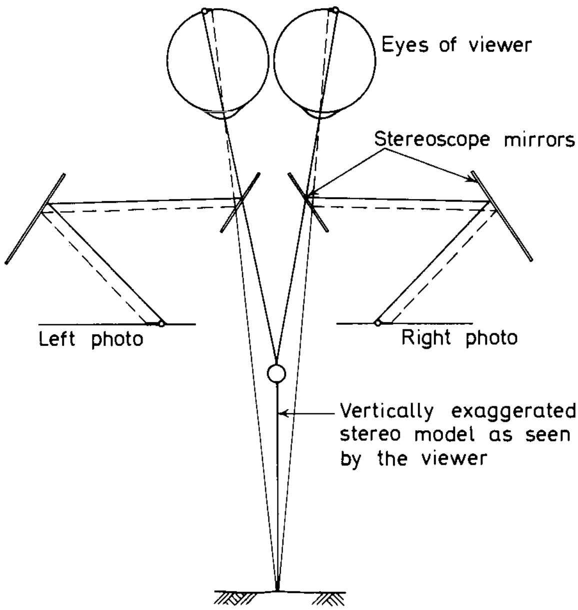


Figure 3.30

Figure 3.31 illustrates a stereo pair of photographs in plan and elevation, on which it is intended to measure the parallax of A (P_A). By definition the parallax of a point is its apparent movement, parallel to the eye base, when viewed from two different positions. Thus A appears at a_1 when viewed from L_1, and at a_2 when viewed from L_2. By overlapping the two photographs, the apparent movement of A is shown as a_1a_2', i.e. L_1a_2' is parallel to L_2a_2. It is thus shown that the parallax of A is the 'algebraic difference of the x-ordinates'.

$$\therefore \; P_A = a_1a_2' = [x_1 - (-x_2)] = (x_1 + x_2)$$

N.B. The x-ordinates are always measured parallel to the photo base and not the fiducial axes. Whilst this indicates that the parallax of a point could easily be measured from the photograph using a simple ruler, in fact it is the difference in parallax between points which is measured, as will be shown later.

3.7.3 Basic parallax equation

This is easily deduced from *Figure 3.31* in which triangles L_1L_2A and $a_2'L_1a_1$ are similar

$$\frac{a_2'a_1}{L_1p_1} = \frac{L_1L_2}{H - h_A}$$

but $\quad a_2'a_1 = P_A \qquad L_1p_1 = f \qquad$ and $\qquad L_1L_2 = B$

$$\therefore \; P_A = \frac{fB}{(H - h_A)} \tag{3.17}$$

As shown in *Figure 3.31*, equation (3.17) assumes absolutely vertical photographs taken at exactly the same flying height. This state of affairs rarely exists; thus, heights obtained using this formula are frequently termed *crude heights*.

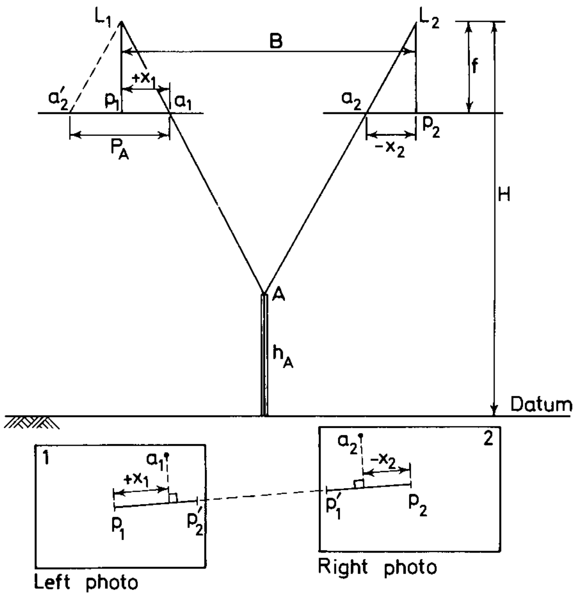


Figure 3.31

From the scale of the photograph it is known that

Scale $= b/B = f/(H - h)$

where b is the mean photo base $(b_1 + b_2)/2$, and h is the mean height of the terrain.

$$\therefore \quad B = \frac{b(H - h)}{f}$$

which on substitution into equation (3.17) gives

$$P_A = \frac{b(H - h)}{H - h_A}$$

as $(H - h)$ is the mean flying height, it is frequently written

$$P_A = \frac{bH_o}{H - h_A} \tag{3.18}$$

However, as previously stated, it is normal practice to measure the difference in parallax (ΔP) between two points, using an instrument called a *parallax bar*. Thus, considering two points A and C

$$P_A = \frac{fB}{H - h_A} \quad \text{and} \quad P_C = \frac{fB}{H - h_C}$$

$$\therefore \quad (H - h_A) = fB/P_A \quad \text{and} \quad (H - h_C) = fB/P_C$$

$$\therefore \quad (H - h_C) - (H - h_A) = h_A - h_C = \Delta h_{AC}$$

$$= fB(1/P_C - 1/P_A) = fB\left(\frac{P_A - P_C}{P_A P_C}\right)$$

but $P_A - P_C = \Delta P_{AC}$ = difference in parallax between A and C.

$$\therefore \quad \Delta h_{AC} = \frac{fB}{P_A} \times \frac{\Delta P_{AC}}{P_C}$$

but since

$$P_C = P_A + \Delta P_{AC}$$

$$\Delta h_{AC} = \frac{fB}{P_A} \times \frac{\Delta P_{AC}}{P_{AC} + \Delta P_{AC}}$$

Hence

$$\Delta h_{AC} = \frac{(H - h_A)\Delta P_{AC}}{P_A + \Delta P_{AC}} \tag{3.19}$$

In relatively flat terrain ΔP_{AC} in the denominator is negligible.

Hence

$$\Delta h_{AC} = \frac{(H - h_A)\Delta P_{AC}}{P_A} \tag{3.20}$$

An inspection of the basic equation (3.17) shows that as h_A increases, then P_A must also increase; thus an important rule of parallax heighting is: the higher the point, the greater its parallax.

3.7.4 Measurement of parallax

Parallax heighting is usually carried out with the aid of a parallax bar (*Figure 3.32*). This instrument is essentially a rod, carrying two glass plates with fine dots etched on them. The smaller pair of dots is used when the stereo model is viewed under

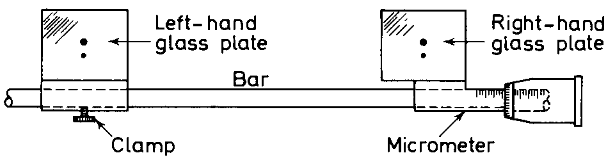


Figure 3.32

magnification. The left plate can be moved anywhere along the bar and clamped in position; the right plate can be moved only by manipulation of the micrometer. Parallax measurements are made to an accuracy of 0.01 mm.

The heighting procedure is as follows. First the photographs are set with their bases co-planar for viewing under the stereoscope (*Figure 3.31*). It is important therefore that the bar is kept parallel to the base ($p_1 p_2$) when measuring. Consider now the measurement of height AC in *Figure 3.33*. The bar is set to mid-run on the micrometer and the right-hand dot (RHD) is placed over the image a_2, the left-hand plate is unclamped and the left-hand dot (LHD) placed over a_1. When viewed through the stereoscope the two dots will have fused into one, and appear to be resting on the point A in the stereo model. The parallax bar is read equal to M_A. This is *not* a measure of the distance $a_1 a_2$, for the micrometer could have been set to any reading prior to the operation. The LHD now remains clamped in this position on the bar for all future heighting operations on this pair of photographs. It is now moved to c_1, and as the separation has not yet been altered, the RHD will be at d, causing the fused image to

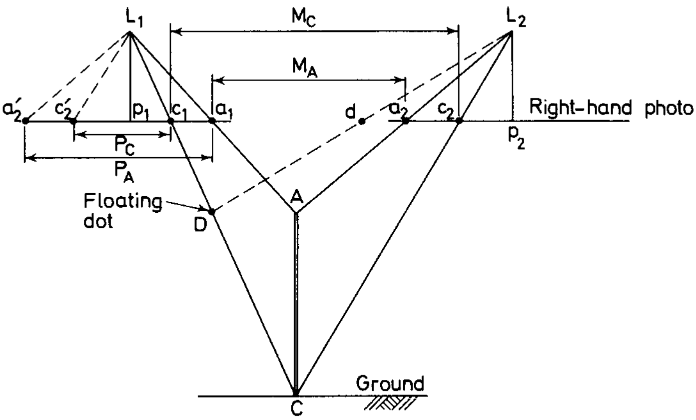


Figure 3.33

appear floating in space at D. While looking through the stereoscope the RHD is moved by manipulating the micrometer until the floating dot appears resting on the ground at C, in which case the RHD will, as *Figure 3.33* shows, be over c_2 on the photograph; the reading M_C is noted. As parallax has already been defined it should be obvious that the individual readings are meaningless. However, as $L_1a'_2$ and $L_1c'_2$ are parallel to L_2a_2 and L_2c_2, respectively, it can be seen that the 'difference' in the bar readings ($M_C - M_A$) is equal to the 'difference' in parallax ($P_A - P_C = \Delta P_{AC}$), which in turn is a function of the 'difference' in height of A and C (Δh_{AC}) and can be computed using equation (3.19).

3.7.5 Basic procedure

Assuming that it is required to find the levels of a grid of points in the stereoscopic overlap of a pair of photographs, one must commence from a GCP of known level, as follows:

(a) Using the basic parallax formula ($P_A = fB/H - h_A$) calculate the parallax P_A of ground control point A whose level h_A is known. fB and H will also be known (refer to *Section 3.7.6*).
(b) Obtain a parallax bar reading on the image points a_1 and a_2 of ground control point A.
(c) Now obtain a bar reading on point C. The difference between the readings on A and C will be equal to ΔP_{AC}.
(d) As P_A is known, then $P_C = P_A \pm \Delta P_{AC}$ (+ve if C is higher than A and −ve if lower). Whether or not C is higher than A may be detected from an examination of the stereo model and/or the bar readings.
(e) Now calculate the level of point C, i.e. h_C, from the basic formula $P_C = fB/H - h_C$. Alternatively, one may calculate $\Delta h_{AC} = (H - h_A)\Delta P_{AC}/P_A + \Delta P_{AC}$ and knowing the level of A thereby obtain the level of C.
(f) This process is now continued. For example, a bar reading on point D will give ΔP_{AD}, from which h_D or Δh_{AD} can be obtained as shown in (d) and (e) above.

3.7.6 Error in the parallax equation

The following analysis is *not* intended to give an indication of the accuracy to which parallax heights can be obtained, but rather to give an indication of the accuracy to which the various components of the parallax equations should be measured.

(1) *Parallax measurement* (ΔP)

The precision required of the bar readings may be indicated as follows: with the aid of a simple sketch, it can be seen that the parallax of principal point p_1 is equal to the photo base length b_2, while the parallax of p_2 is b_1 (see *Figure 3.34*). Thus the parallax of the principal points is b which, substituted into equation (3.20), gives

$$\frac{\Delta h}{\Delta P} \approx \frac{H_o}{b} \tag{3.21}$$

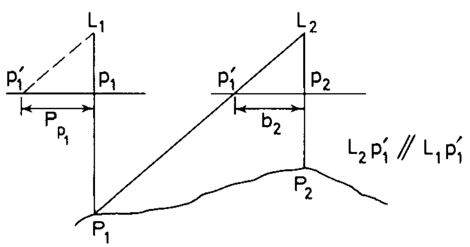


Figure 3.34

Consider 230 mm × 230 mm photography from a height of 3000 m with a 60% overlap, in which case $b \approx 92$ mm.

$$\therefore \quad \frac{\Delta h}{\Delta P} \approx \frac{3000}{92} \approx 33 \text{ m/mm}$$

As 3000 m is the ceiling for most air surveys and ±0.03 mm an average for parallax bar error, then the heighting error would be less than 1 m, i.e. 33 × 0.03 = 0.99 m.

(2) *Flying height* (H)

Error in the value of H has greater critical effect on the equations than any of the other components. It can be shown that proportional error in the flying height, i.e.

$$\frac{\delta(\Delta h)}{\Delta h} = \frac{2\delta H_o}{H_o} \tag{3.22}$$

where H_o is the flying height above the ground.

Thus, if the difference in parallax heighting requires an accuracy of 1 in 100, then the flying height would need to be accurate to 1 in 200. For $\Delta h = 100$ m, and so accurate to 1 m, and $H_o = 2000$ m, it would need to be accurate to 10 m. Such an accuracy is not possible from altimeter readings and the following approach is usually adopted.

Consider two points, A and C, whose levels are known. They may be points of detail on a plan or GCP whose photo images are a and c. Let the mean level of the points equal h_o and the scaled distance between the photo image points equal l, then if the actual distance between the ground points is L we have l/L = photo scale = $f/H - h_o$, from which

$$H = fL/l + h_o \tag{3.23}$$

If a and c are positioned symmetrically about the PP, then the effects of tilt and height differences between them are small.

This procedure should be carried out for as many pairs of points as possible in each photograph and the mean value adopted.

(3) *Focal length* (f)

In this case the proportional error in the parallax height differences is directly proportional to the proportional error in the focal length, i.e.

$$\frac{\delta(\Delta h)}{\Delta h} = \frac{\delta f}{f} \tag{3.24}$$

Thus, if as in the previous case height differences of 100 m were required to an accuracy of 1 m, then $f = 152.4$ mm would need to be accurate to 1.5 mm, which is well within the limits of a calibrated camera.

(4) *Air base* (B)

As above $$\frac{\delta(\Delta h)}{\Delta h} = \frac{\delta B}{B} \tag{3.25}$$

If $f = 150$ mm, and $H = 3000$ m, then $B = 1840$ m for a 60% overlap. Thus, a proportional error of 1 in 100 for $\Delta h = 100$ m would be equivalent to an error of 18.4 m in B.

A value for the air base may be obtained by measuring the distance between consecutive principal points from a radial line plot, and then applying the scale of the plot to this distance.

Alternatively, if two GCP (A and C) whose distance apart is known, appear in the overlap area of a pair of photographs, their co-ordinates may be carefully measured using the photo base as the x-axis and line at right-angles as the y-axis, i.e. x_a, y_a and x_c, y_c. One also requires the parallax of the two points then

$$X_A = Bx_a/p_a \qquad Y_A = By_a/p_a$$
$$X_B = Bx_b/p_b \qquad Y_B = By_b/p_b$$

where X and Y are the ground co-ordinates of the two points.

Horizontal distance = $D_{AB} = [(X_A - X_B)^2 + (Y_A - Y_B)^2]^{\frac{1}{2}}$

Then, substituting for X and Y, and re-arranging gives

$$B = \frac{D_{AB}}{\left[\left\{\frac{x_a}{p_a} - \frac{x_b}{p_b}\right\}^2 + \left\{\frac{y_a}{p_a} - \frac{y_b}{p_b}\right\}^2\right]^{\frac{1}{2}}} \tag{3.26}$$

As cautioned in the first sentence, the preceding analysis should not be regarded as an indication of the accuracy of parallax heighting. The reason for this is the fundamental one, of the theory being at variance to the practical situation. That is, the basic parallax formula is derived on the assumption that the photographs are perfectly vertical and possess no tilt, and that there is no variation in flying height, a situation which rarely exists in practice. For this reason, the uncorrected heights obtained from parallax measurements are termed *crude heights.*

3.7.7 Parallax height corrections

The import of the problem of uncorrected parallax heights is clearly illustrated by the following statement taken from 'Heights from Parallax Measurements' by Prof. E. H. Thompson (*Photogrammetric Record*, Vol. I, No. 4, Oct. 1954), 'The necessity for the adjustment of crude heights is forcibly brought home when one considers that, with wide-angle photographs taken from an altitude of 18 000 ft (4900 m) over country with height differences of a few hundred feet, it is as accurate to assume the ground to be flat as to trust parallax readings uncorrected for tilt'.

The convention adopted for tilts is to denote rotation of each camera about three mutually-perpendicular axes, X, Y and Z, as ω, ϕ and K, respectively, as shown in *Figure 3.35*. Translations along each axis are bx, by and bz, respectively.

(1) *Tip* ($\delta\phi$)

Pitching of the aircraft from nose to tail would cause rotation about the y-axis ($\delta\phi$) as shown in *Figure 3.36*. The result of this tilt is to image points 3 and 4 at 3′ and 4′ with

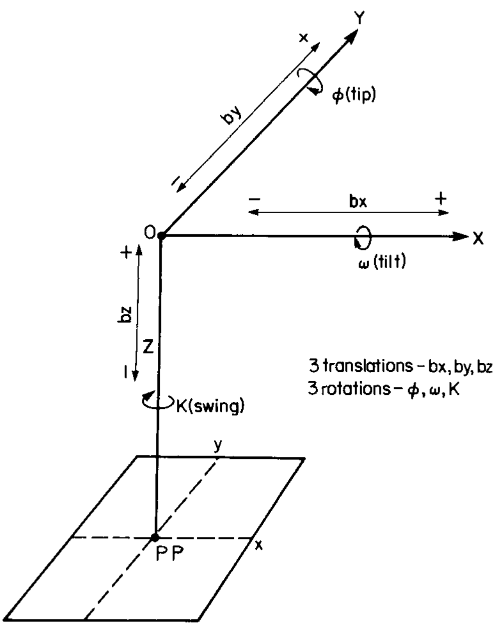


Figure 3.35 Co-ordinate axes of the camera, showing six degrees of freedom

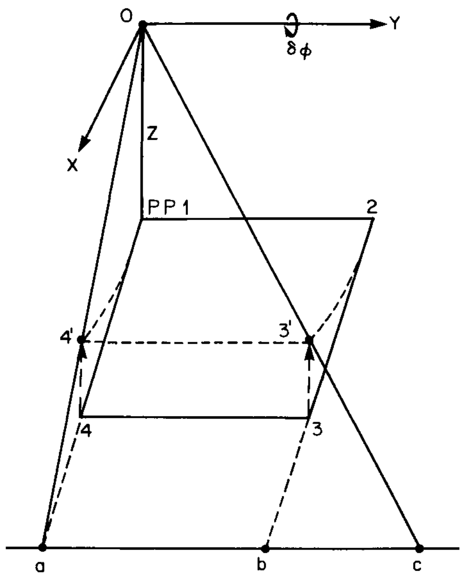


Figure 3.36

resultant error in the parallax measurement of $3b$ and $4a$. As shown, parabolic deformation of the stereo model takes place and the error in ground heights of any point can be shown to be equal to $(X^2 + Z^2)\delta\phi/B$.

(2) *Tilt* ($\delta\omega$)

Tilting of the aircraft from wing-tip to wing-tip would cause rotation about the x-axis ($\delta\omega$), as shown in *Figure 3.37*. The result of this tilt is to displace points 2 and 3 radially from PP1. The displacement of 2 is in the y-direction, hence its parallax measurement is unaffected. The parallax measurement of point 3 will be in error by distance $3b$ which will distort the height by an amount equal to $XY\delta\omega/B$. This expression represents a rectangular hyperboloidal deformation of the stereo model.

(3) *Swing* (δk)

Swing error is eliminated by careful baselining of the photographs. Error in this process will result in a raising and lowering of each half of the stereo model, as shown in *Figure 3.38*.

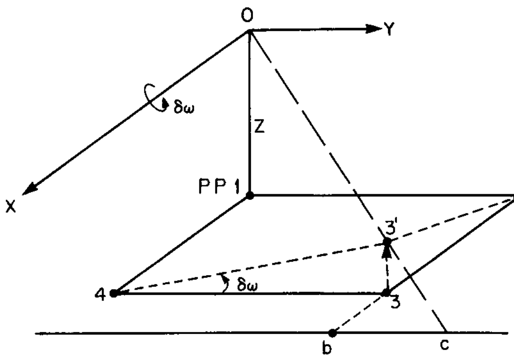


Figure 3.37

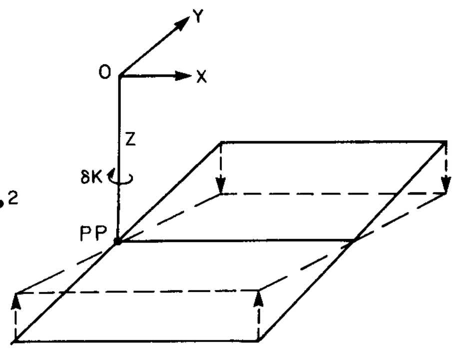


Figure 3.38

(4) *Variation in flying height* (Φ)

Variation in the heights of the camera at adjacent exposures, tilts the air base (Φ) and so tilts the stereo model about the y-axis, as shown in *Figure 3.39*.

The combined effect of all the above errors in each photograph comprising a stereo pair is to transform the stereo model into a hyperbolic paraboloid. It may be shown to first order, that if the tilts are small and ground relief not excessive, the error in parallax heights may be expressed in terms of the photo co-ordinates (x, y) of the image point of the left-hand photograph as

$$\delta h = \frac{Z^2}{fB}\left[(K_1 - K_2)y - \phi_1(f + x^2/f) + \phi_2(f + (x - b)^2/f) + w_1\frac{xy}{f} - w_2(x - b)y/f\right] - \frac{\Phi xz}{f} \tag{3.27}$$

which is more easily expressed as

$$\delta h = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 \quad (3.28)$$

The first four terms represent a hyperbola and the last a parabola.

The method therefore of correcting crude heights requires five GCP, whose levels (h_i) are known, distributed throughout the overlap area, as shown in *Figure 3.40*. The

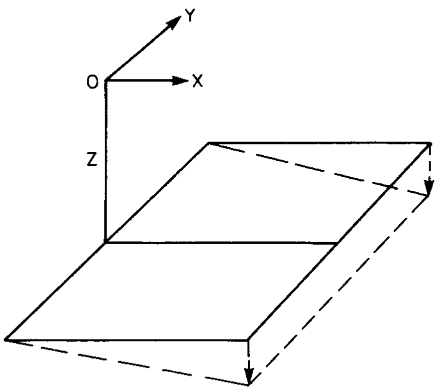


Figure 3.39

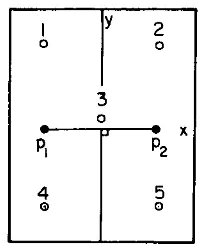


Figure 3.40

parallax heights (h_i') of the five points are found taking, say, point 1 as datum and $h_i - h_i' = \delta h_i$. The centre of the photo base $p_1 p_2$ is taken as the origin for the x, y co-ordinate system and the co-ordinates of all five points scaled from the left-hand photograph and inserted in the five equations, i.e.

$$\begin{aligned} h_1 - h_1' &= \delta h_1 = a_0 + a_1 x_1 + a_2 y_1 + a_3 x_1 y_1 + a_4 x_1^2 \\ h_2 - h_2' &= \delta h_2 = a_0 + a_1 x_2 + a_2 y_2 + a_3 x_2 y_2 + a_4 x_2^2 \\ \vdots \quad & \quad \vdots \quad \quad \vdots \\ h_5 - h_5' &= \delta h_5 = a_0 + a_1 x_5 + a_2 y_5 + a_3 x_5 y_5 + a_4 x_5^2 \end{aligned}$$

The equations are then solved for the coefficients $a_0, a_1 \ldots a_4$. Thereafter, the crude height (h') of any point in the overlap may be corrected (δh), using the above coefficients and its photo co-ordinates in equation (3.28).

The whole process can be quickly and easily carried out using an appropriate computer program.

3.8 RESTITUTION SYSTEMS

Restitution is the fundamental problem in photogrammetry and involves establishing the photographs (or diapositives) in exactly the same positions as they had at the time of flight, and thereafter relating the correctly-formed stereo model to ground control.

Approximate methods of restitution have already been dealt with. The remaining methods may be either analogue, using universal or precision plotters, or analytical, using precise photo co-ordinates measured in a comparator.

The problem may be illustrated as follows. *Figure 3.41* shows two photographs in a

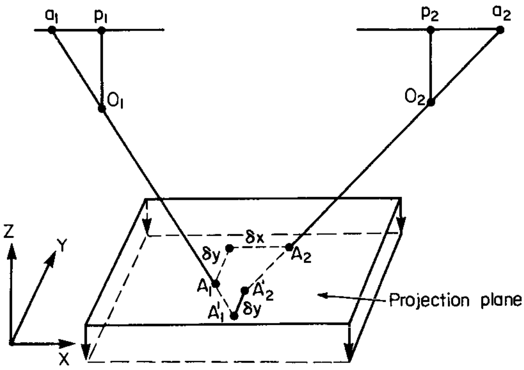


Figure 3.41

horizontal plane projecting images of a single point A on to a parallel plane. Because the projection system has not been properly oriented a_1 and a_2 intersect at A_1 and A_2 instead of coinciding at A. The discrepancy between A_1 and A_2 may be expressed in co-ordinate terms as δx and δy, and are called the x and y parallax.

The δx value is eliminated by lowering the projection plane in the Z direction (bz) until the two points are imaged at A_1' and A_2', separated only by δy.

y-parallax is eliminated by moving the projectors through the five degrees of freedom, i.e. bz, by, ϕ, ω, K as illustrated in *Figure 3.35*. This procedure, known as *relative orientation*, is carried out over six standard points distributed throughout the stereo model, and when complete it establishes the projectors in their correct relative positions. Failure to achieve correct relative orientation will result in a distorted model and will have a particular effect on the accuracy of height measurements.

The above model must now be correctly scaled and oriented to the ground co-ordinate system. This process is called *absolute orientation* and can only be achieved by the use of ground control.

The first step is the scaling of the model based, in its simplest form, on the known distance between two ground control points. This is achieved by altering the separation of the projectors in the x-direction, i.e. bx translation.

The second step is the levelling of the spatial model by rotating it about its X and Y axis, i.e. ϕ and ω, until it conforms to the height data of at least three GCP.

When this process is completed the plotting of detail, spot heighting and contouring can commence. Alternatively, the three-dimensional co-ordinates of points can be measured to produce a digital ground model (DGM).

It should be noted that the accuracy of restitution and thus the final plan is dependent on the accuracy of the ground control and its correct identification.

The method of restitution of the light rays comprising the model may be by optical projection or by the use of space rods forming a mechanical analogue of the light rays. Measurements to an accuracy of 10 μm can be made on 230 mm × 230 mm photography, whilst enlargement from photograph to stereo model is a function of the ratio Z/f. To illustrate this, consider some technical details of the Wild A8 plotter

Principal distance (f)	98 mm to 215 mm
Dimensions of floating dot	0.07 mm for 6 × magnification
	0.045 mm for 8.5 × magnification
Z	300 mm

Thus, for photography at 1/10 000 scale using $f = 152$ mm, we have

Model scale $1/10\,000 \times Z/f$	$\simeq 1/5000$
Enlargement of model to plot $\times 4$	$= 1/1250$
Measurement accuracy on photo	$= 10$ µm
Accuracy of point on plot	$= 0.1$ mm

The above comments relate entirely to plan and DGM production by analogue methods, which are the most economical for such purposes. However, analytical photogrammetry is also worthy of mention. In this process photo co-ordinates are measured precisely to an accuracy of 1 or 2 µm using instruments called *comparators* or *stereo comparators*. The relative and absolute orientation of the photo co-ordinates is done mathematically by computer. The co-ordinates produced are more accurate than those obtained by analogue methods and are particularly useful in aerial triangulation.

3.9 MOSAICS AND ORTHOPHOTOMAPS

Although the principal end product of photogrammetry, as far as the engineer is concerned, is a plan or DGM, other types of plan are available in the form of mosaics or orthophotomaps.

(1) *Uncontrolled mosaic*

An uncontrolled mosaic is formed by matching up the photographs, usually at contact scale, to get the best possible fit. No account is taken of displacements due to tilt and ground relief, nor is the assembly fitted to any form of ground control.

(2) *Controlled mosaic*

In this case the photographs are first corrected for tilt in an optical rectifier and all brought to the same scale. The prints are then carefully assembled and fitted to the ground control. A more cohesive picture is produced in this case, but the effects of displacement due to ground relief are still present.

(3) *Orthophotograph*

An orthophotograph is one that has been mechanically corrected for tilt and relief displacements using an orthoprojector or orthophotoscope linked directly to a stereo plotter. In this way the correctly oriented stereo model is exposed and photographed in small slits (4 mm × 2 mm) at true map position and correct scale. The end product is a contoured photograph to correct scale containing very small errors of position and height. Although not quite as accurate as the line drawn plan, it can be produced much more quickly and will contain all the land form detail not usually shown on a plan. This latter point is the reason why the mosaic or orthophoto may be preferred, in some instances, to a plan. For example, in flood control, geological investigation or irrigation works, the ability actually to see the areas involved could be extremely useful.

3.10 SPECIFICATION FOR VERTICAL AIR PHOTOGRAPHY

Although engineers will not generally be involved in the practice of photogrammetry, they will most certainly be involved with the end products. To this end they will be required to draw up appropriate contracts for the work to be done and will therefore need to stipulate the necessary specifications.

The following specification was originally prepared by the British Air Survey Association, examined and adopted by the Royal Institution of Chartered Surveyors, and is reproduced here verbatim.

SECTION ONE: Summary of requirements and materials to be delivered

1.1 Area

1.1.1 The area or route to be photographed stereoscopically measures approximately .... square kilometres or line kilometres and is defined as follows.

EITHER On the contract map or photomosaic attached as Annexure ..... of the specification document,

OR By the geographical, grid, of other reference co-ordinates as listed below and/or indicated in the sketch in Annexure .....

....

....

....

related to .... datum/projection.

1.2 Scale of photography and focal length of lens (*see* Section Three)

1.2.1

EITHER The nominal scale of the photography shall be 1/ ....

OR The photography shall be flown from a computed altitude or altitude for each block as follows:

.... metres above mean sea level

.... metres above mean sea level

.... metres above mean sea level

1.2.2 The camera shall have a lens of nominal focal length .... mm and a nominal negative format of 230 mm × 230 mm or of .... mm × .... mm.

1.3 Photography (*see* Section Four)

1.3.1 The film used shall be

EITHER Black-and-white aerial, panchromatic,

OR Black-and-white aerial, infra-red.

1.3.2 The photography shall be of an image quality and geometric quality suitable for photogrammetric mapping or photomosaicing or general interpretation.

1.4 Film negatives

1.4.1

EITHER All films exposed on the contract shall be retained by the producer for a period of not less than .... years and then they will be .....
....
....

OR All films exposed on the contract shall be delivered to the client or user.

1.4.2 Each processed film shall be kept in roll form on a spool and in a metal or plastics container as supplied by the film manufacturer. Rejected negatives shall not be removed from the roll.

1.5 Other material to be delivered (*see* Section Five) Delete items not required

1.5.1 An index plot *and/or* a photoindex supplied in the form of
...... sets of transparencies
...... sets of negatives
...... sets of paper prints
....

1.5.2 ...... sets of paper contact prints.

1.5.3 One copy of all film reports (*see* Clause 6.4).

1.5.4 Other products
....
....

SECTION TWO: Cameras and associated equipment

This section refers to 230 mm × 230 mm-format metric cameras only. Other cameras used shall be specified separately.

2.1 Camera

2.1.1 A metric survey camera shall be used, fitted with a lens that is designed to give a residual radial distortion not exceeding 15 micrometers within 100 millimetres of the principal point. The film shall be held in the intended image plane during exposure to maintain sharp focus and hold image distortion within the limits specified in Clause 4.5.

2.1.2 The format of the negative and the focal length of the lens unit(s) shall be as specified in Clause 1.2.2.

2.1.3 The lens shall be corrected for the spectral range of the film used.

2.2 Calibration

2.2.1 Each camera lens unit to be used on the contract shall have been calibrated, tested and certified by the camera manufacturer or by a calibration centre recognised internationally or approved by the camera manufacturer. The certificate will show that the camera has been calibrated within twelve months at commencement of the photography.

2.2.2 The producer shall hold a valid calibration certificate and shall supply a copy to the client or user on request.

2.2.3 The calibration certificate shall contain the following information:

- name and address of the calibration centre
- date of calibration
- camera manufacturer's serial number of the lens unit
- calibrated focal length (principal distance) of the lens unit
- radial distortion in micrometres at intervals not exceeding 10 mm along each of the four semi-diagonals referred to the axis of best symmetry
- distances between fiducial marks – sides and diagonals, or their co-ordinates in a rectangular reference system
- position of the principal point of autocollimation or of best symmetry with respect to the fiducial centre
- radial and tangential resolution figures for the lens unit issued by the manufacturer, at the time of manufacture or after optical readjustments of the lens unit
- measured reseau co-ordinates (if any) in a rectangular reference system.

2.2.4 The measured distortion shall fall within the limit defined by the manufacturer for the lens type.

2.2.5 If, during the course of the contract, any damage to the camera is suspected that is liable to affect the calibration, the camera shall be recalibrated.

2.2.6 If, up to six months after completion of this contract, significant changes in calibration are found, the producer shall inform the client or user.

2.3 Camera mounting

The camera shall be installed in a mounting which attenuates the effects of aircraft vibration.

2.4 Filters

2.4.1

EITHER Only optical filters provided by the lens manufacturer or meeting the same optical specification shall be used,

OR Filters as specified below shall be used:

....

2.4.2 The light fall-off in cameras having an angle of view larger than 60 degrees shall be compensated by a graded filter.

2.5 Camera windows

2.5.1 Any camera window used shall be checked by the calibration centre to ensure that it will not adversely affect lens resolution and distortion and that it is substantially free from veins, striations and other inhomogeneities.

2.5.2 The camera window shall be mounted in material eliminating mechanical stress to the window.

SECTION THREE: Flying and photographic coverage

3.1 Photographic coverage

3.1.1 The area shall be covered by approximately straight runs (strips) of near-vertical photographs at the approximate altitude required in Clause 1.2.1.

3.1.2 The direction of flight lines shall

EITHER Be selected by the contractor and a copy of the flight plan shall be supplied to the client on request,

OR Conform to the flight plan attached as Annexure ..... of the specification document.

3.1.3 The forward overlap (forelap) between successive exposures in each run shall be between 55 and 65 per cent, except where specified otherwise.

3.1.4 The lateral overlap (sidelap) between adjacent strips should normally be:

- between 20 and 40 per cent for flying heights of less than 1500 m above mean ground level
- between 15 and 35 per cent for flying heights of 1500 m and above.

Where ground heights within the area of overlap vary by more than 10 per cent of the flying height, a reasonable variation in the stated overlaps shall be permitted, provided that the forward overlap does not fall below 55 per cent and the lateral overlap does not fall below 10 per cent or exceed 45 per cent.

In extreme terrai.. relief, where the lateral overlap specified above is impossible to maintain in straight and parallel flight lines, the 'gaps' created by excessive relief shall be filled by short runs flown between the main runs and parallel to them.

3.1.5 Where a run crosses a shoreline, the forward overlap shall be increased to a nominal 90 per cent subject to the constraints imposed by the camera cycle time. The increase in overlap shall include at least three photo-centres on land.

3.1.6 Runs that would fall along a shoreline may be repositioned to reduce the proportion of water covered, provided that the coverage extends beyond the limit of any land feature by at least 10 per cent of the run width, despite the increased lateral overlap.

3.1.7 Where the ends of runs of photography join the ends of other runs flown in the same general direction, there shall be an overlap of at least two stereoscopic models, which if the scales of photography are different shall be at the smaller photo-scale.

3.1.8 Crab shall not exceed 5 degrees when measured between the base line and a line parallel to the frame of the negative, nor create stereoscopic gaps in the photography.

3.1.9 Tilt should not normally exceed 2 degrees. Isolated exposures with up to 4 degrees may be permitted.

3.1.10 Where a few exposures in a long run are rejected because of cloud, quality or inadequate overlap they may be replaced by a short run, provided that an overlap of at least two stereoscopic models is supplied at both ends.

3.2 Flying conditions

3.2.1 Photography may be taken at any suitable solar altitude above 15 degrees, except where specified otherwise below:

Minimum solar altitude .... degrees
Maximum solar altitude .... degrees.

3.2.2 Photography shall be flown only in conditions when the visibility does not significantly impair the tone reproduction in the negative. Relevant detail shall not be lost as a result of atmospheric haze or dust.

3.2.3

EITHER Photography shall be substantially free from cloud, dense shadow or smoke. Isolated areas of cloud, dense shadow or smoke shall not be cause for rejection of the photography provided that the intended use is not impaired,

OR Photography shall be completely free of cloud, dense shadow or smoke, OR ....

3.2.4 Special conditions related to timing or season for photography as specified below:

....

SECTION FOUR: Aerial film and image quality of negative

4.1 Aerial film

4.1.1 The type of aerial film to be used shall be as specified in Clause 1.3.1.

4.1.2 The emulsion shall be coated on a stable-base film.

4.1.3 The conditions of the film stock to be used shall be such that when exposed film is processed

- it shall be free of stains, discolouration, or brittleness that can be attributed to ageing or improper storage; and
- the fog density (emulsion only) shall not exceed a value of 0.2 using the same developer, time and temperature as will be used on the contract, except for film nominally rated at a speed in excess of 250 EAFS (Effective Aerial Film Speed) which shall not exceed a value of 0.4.

4.2 Exposure

4.2.1 A shutter speed shall be chosen that meets the requirements of minimal image movement, at an adequate lens aperture for the prevailing illumination conditions.

4.2.2 The calculated forward image movement shall not normally exceed 30 micrometres.

Up to 60 micrometres shall be acceptable in cases of very low subject luminance and/or photography at scales of 1/5000 and larger.

Up to 90 micrometres shall be acceptable in cases of extremely low subject luminance and/or photography at scales of 1/2000 and larger.

4.3 Filter

4.3.1 The contractor shall select filters to provide suitable tone reproduction, except where the filters to be used are specified below:

....

4.4 Processing and drying

4.4.1 Equipment used for processing and drying of the film shall be capable of achieving consistent negative quality specified under Clauses 4.5 and 4.6 below without causing deformation of the film.

4.4.2 Processing and drying of the film shall be carried out without affecting its dimensional stability. In any negative, the differential lengths between any pairs of fiducial marks shall not exceed 0.03 per cent and the overall scale change shall not exceed 0.08 per cent.

4.4.3 The residual thiosulphate content of processed film shall not exceed 20 milligrammes per square metre.

4.4.4 All processed negatives should be substantially free of blisters, bubbles, inclusions, coating lines, stress or static marks, bar marks, pin-holes, abrasions, streaks, stains, chemical marks, drying marks or scratches, on both the emulsion side and the base side, apparent either in diffuse or specular light.

Some tolerances shall be allowed where processing has to be carried out in sub-standard conditions, provided that the intended purpose of the negatives is not impaired.

4.5 Metric quality of negatives

The original negatives or contact diapositives produced from them shall not contain residual y-parallaxes after relative orientation in excess of 20 micrometres anywhere in the model.

4.6 Image quality of negatives

4.6.1 The density and contrast of all negatives shall be such that commercially available grades of paper, covering log exposure ranges of 0.6 up to 1.6 can be used to produce prints with detail in dark and bright areas of interest. Suitable dodging methods shall be permitted.

4.6.2 The fog density (emulsion only) of the negatives shall not normally exceed a value of 0.2 measured in an area clear of any exposure to light, except for film nominally rated at a speed in excess of 250 EAFS which shall not exceed 0.4.

4.6.3 Useful minimum shadow detail should not normally have a density of less than 0.2 above base-plus-fog, except in the corners of super-wide-angle photographs where a minimum density of 0.1 above base-plus-fog shall be acceptable.

4.6.4 The maximum density in useful areas of the negatives shall not exceed 1.5 above base-plus-fog, other than in small areas of high reflectance where a maximum density of 2.0 shall be permissible. In exceptional cases, where very dense spots caused by specular reflection of the sun from highly-reflective objects occur, they shall be accepted.

4.6.5 All fiducial marks shall be clearly visible and sharp on every negative.

4.6.6 The camera panel of instruments recorded on the film should be clearly legible on all processed negatives. Failure of instrument illumination during a sortie shall not be cause for rejection of the photography except where specified below:

....

SECTION FIVE: Photographic products

5.1 Index plots and/or photoindices

An index plot and/or photoindex shall be supplied as specified in Clause 1.4.1 to show the relative positions of all accepted photography.

The index plots and/or photoindices shall contain the following information:

- base map references
- area designation
- period of photography
- scale of index
- scale of photography
- indication of North
- camera type and focal length of lens unit
- contractor's name
- approximate geographical or grid co-ordinates.
- film numbers and run (strip) numbers at both edges of each sheet and where changes occur within a sheet
- photo numbers.

5.1.1 Index plots, where required, shall indicate the position and number of sufficient exposures to facilitate the approximate positioning of intervening exposures.

5.1.2 Photo indices, where required, shall be prepared using the first and last and every alternate print. The prints shall be trimmed to the edge of the photographic image, and the photo number shall be visible on the first and last prints and on every fifth print used.

5.2 Paper prints

Contact prints shall be made on an automatic dodging printer on medium-weight resin-coated paper or double-weight fibre-based paper on which ink and pencil can be used on both sides unless otherwise specified below:

....

Sets of prints to be produced away from the contractor's laboratory may be made on a manual dodging printer.

5.3 Diapositives

Where required diapositives shall be produced on a stable-base film using an automatic dodging printer unless otherwise specified below:

....

5.4 Duplicate negatives

5.4.1 Where required, duplicate negatives shall be supplied

EITHER Produced via an intermediate positive to produce a conventional wrong-reading image when viewed emulsion up,

OR Produced on direct duplicating film to produce a right-reading image when viewed emulsion up.

5.4.2 The duplicate negatives shall be produced on a stable-base film with tone reproduction (density distribution) as close to that of the original negatives as is reasonably feasible.

5.4.3 The negatives shall be produced using an automatic dodging printer unless otherwise specified below.

....

SECTION SIX: Documentation and annotation

6.1 Film annotation

The following information shall be supplied as leaders at the start and the end of each film:

– *start* or *end* (as appropriate)
– project number *and/or* area designation
– where parts of more than one project or area are recorded on the film, all areas shall be mentioned
– film number
– year(s), month(s) and day(s) of photography
– nominal scale(s) of photography
– type of camera
– the principal distance or calibrated focal length of the lens unit.

6.2 Negative numbering (photo numbering) and print annotation

Numbering of negatives shall be carried out using heat foil or indelible ink, or other methods. The numbers shall be printed in a neat and clearly legible type. The height of type shall be approximately 3 millimetres.

Each negative to be used shall be provided with the following annotation, using heat foil or indelible ink or titling strips or other methods, which shall appear on all the contact prints:

- producer's identification
- project number *and/or* area designation
- film number and photo number
- year(s), month(s) and day(s) of photography
- altitude above mean sea level or height above ground level
- nominal scale of photography
- principal distance or calibrated focal length of the lens unit.

6.3 Film container annotation (label)

The outside of each film container shall clearly show:

- project number *and/or* area designation
- where parts of more than one project or area are recorded on the film, all areas shall be mentioned
- year(s), month(s) and day(s) of photography
- run numbers and photo numbers
- nominal scale(s) of photography
- camera type
- focal length of the lens unit.

6.4 Film report

A report shall be included in the film container with each film giving the following information:

- contractor's identification
- film number
- camera type and number, lens type, number and focal length
- filter type and number
- magazine number(s) or cassette and cassette-holder unit number(s)
- film type and manufacturer's emulsion number
- lens aperture and shutter speed (exposure time)
- run number and flight direction
- year(s), month(s) and day(s) of photography
- aircraft type and identification

- names of pilot(s), navigator and photographer
- start and end times for each run in local time
- photo numbers of all offered photography
- computed altitude above mean sea level (true altitude)
- nominal scale of photography
- weather conditions – cloud type, degree of haze
- degree of turbulence
- date of processing
- method of development
- developer used and dilution
- time and temperature of development or film transport speed
- length of film processed
- general comment on quality.

6.5 Other documentation as specified below:

....

....

3.11 APPLICATIONS OF PHOTOGRAMMETRY IN ENGINEERING

Apart from its many applications in civil engineering, photogrammetry is widely used in forestry, town planning, architecture and even dentistry and medicine. Only the more prominent applications in civil engineering will be considered here.

(1) *Highway optimization*

In the well-mapped UK it is possible to reduce the area required for a new route into a relatively small band. Aerial photographic interpretation will serve as a very useful aid in this initial decision. The examination of stereo pairs can supply an enormous amount of information to the trained eye, such as—geology of the area, main soil types, faults, land-slip areas, areas presenting drainage problems, location of borrow and quarry sites, major obstacles, expensive land, best grades, etc.

The photographs in glass plate form (diapositives) are then oriented in stereo plotters, which have rotary digitizers attached to their lead screws. The operator scans the stereo model with the floating dot obtaining the x, y and z machine co-ordinates of the area. These values appear as a permanent record on an electric typewriter, in addition to being punched on card or tape. The computer converts these machine co-ordinates to full-scale ground co-ordinates by a comparison with certain ground control points supplied. The computer now has in its memory store a mathematical model of the ground commonly called a *digital terrain model.* The engineer now supplies certain parameters to the computer, such as the co-ordinates of the route, limiting grades, lengths of vertical and horizontal transition curves and typical cross-section templates. From this data the computer selects the best route, supplies earth-work quantities, plots cross-sections and longitudinal sections along the centre line of the proposed route and produces mass-haul diagrams. Further optimization may now be carried out from this data enabling the computer to supply final quantities (compensated for bulking and compaction) including a differentiation of the various types of material and quantities for top-soil stripping, respreading and seeding. Final drawings of plans, long sections and cross-sections are supplied, along with all the necessary setting-out data. Even perspective views of the proposed route at regular intervals are supplied, which, if flashed rapidly on a screen, give the impression of travelling the proposed route.

(2) *Traffic engineering*

Photographs may be used in land use studies to enable travel patterns to be estimated and predicted.

Time-lapse photography of a traffic route can provide information such as traffic speeds and density, concentration of traffic for selected time periods, *en route* travel time and the relative productivity of the various route segments. This technique is called the *sky-count technique*, although the tedium of actually counting the cars on the photographs has been eliminated by using electronic scanners capable of sensing vehicles on infra-red photography.

Traffic management, broadly aimed at improving traffic flow, can be aided in many ways by air photographs. The photographs provide a visual inspection of a large area at a glance and can be taken to show on- and off-peak flows, normal and congested routes, non-utilization of streets, parking characteristics, junction studies, effect of public transport on traffic flows, etc. They may also be used to produce traffic density contour maps, and to provide a permanent inventory of roads, streets and car parks.

(3) *Remote sensing*

Remote sensing is the most recent development in the field of air survey. However, all aerial photographs are in effect examples of remote sensing, in that they can detect the nature of an object without actually touching it.

All photographs portray detail by a comparison of the visible light reflected from various objects. This light comprises electromagnetic energy with wavelengths from 0.4 μm to 0.7 μm. Energy whose wavelength is less than 0.4 μm is called *ultra-violet*, and above 0.7 μm *infra-red*. The camera is capable of recording energy within the 0.3 μm to 1.2 μm range, but above this upper value special equipment is required.

In colour photography, each distinctive colour is a function of the light reflected by the objects, which is in turn a function of the energy absorption and reflection characteristics. Thus, as blue has different reflection characteristics to red, it is possible to distinguish between them. However, by sensing in the infra-red spectrum it is possible to distinguish different objects having the same colour, due to the variable energy reflection characteristics. This is particularly impressive in the field of ecology, where healthy and diseased vegetation will appear as different colours on infra-red colour film (false colour), even though to the human eye apparently identical.

To record energy in the 1 μm to 20 μm band, thermal infra-red line-scanning devices are used. These devices record variation in energy due to variation in temperature. The terrain is sensed from the air, a strip at a time, and a thermal image built up. A typical example of its use is in river pollution where pollutants having different temperatures are recorded as shades varying from white through to black. Correlation with the various origins of these pollutants enables a more detailed analysis of the river to be made. This form of sensing can operate day or night, but cannot penetrate atmospheric cloud conditions. Other prospective uses of this technique are the detection of various rock and soil types, and assessment of the moisture content of various soils.

Using side-scanning airborne radar it is possible to build up terrain images over a wide area at a single scan. As radar functions in the spectral range of 0.5 mm to 1 m, it is operable day or night under any sort of weather conditions. Image detail is built up on the basis of the time differences between the reflected electromagnetic waves. For instance, the travel time from a point directly below the aircraft would be less than that from the edge of the line scan. These time differences are converted to amplitude video signals and imaged on one line of a cathode ray tube. In this way a terrain picture is built up in a manner similar to existing TV pictures.

Thus, using these devices, it is possible to operate under any weather conditions, at day or night, and still differentiate between any sort of detail having different reflection characteristics. The possible applications of these techniques in engineering are therefore multifarious.

WORKED EXAMPLES

Example 3.1. A rectangular area 100 km × 50 km is to be mapped from aerial photographs. The camera has a film format of 230 mm × 230 mm and a focal length of 152 mm. If the area is flown at a mean altitude of 3040 m, the forward overlap is 60% and the lateral overlap 30%, calculate:

(a) the number of photographs required to cover the area, adding two to each end of the strip to ensure coverage;
(b) the interval between successive exposures if the ground speed was 130 km/h;
(c) the amount of image blur for a shutter speed of 1/300 s. (KP)

(a) Scale of photography = f/H = 152/3 040 000 = 1/20 000
Length of area at scale = 100 000/20 000 = 5 m = 5000 mm
Effective cover per photograph = 40% of 230 mm = 92 mm
∴ Number of photographs per strip = 5000/92 = 54.3 ≈ 55
∴ Number of photographs required = 55 + 4 = 59
Width of area at scale = Half length = 2500 mm
Effective cover per strip = 70% of 230 mm = 161 mm
∴ Number of strips = 2500/161 = 15.5 ≈ 16
∴ Total number of photographs = 16 × 59 = 944

(b) A new photograph is taken every 40% of the ground cover per photograph. Thus 92 mm is equivalent to 92 × 20 000 mm = 1840 m on the ground. This length is, in effect, the air base B.

Ground speed = 130 km/h = 36 m/s
∴ Interval between exposures = 1840/36 = 51.1 s

(c) Ground covered in 1/300 s = 36 000/300 = 120 mm
∴ Image blur on the photograph = 120/20 000 = 0.006 mm

Example 3.2. Define the terms *plumb point* and *isocentre* in connection with photogrammetry. Show that the angle subtended by two points at the isocentre of a photograph is equal to the corresponding ground angle.

A photograph taken with a 254-mm focal length camera has a tilt of 7°. Find the distances in mm from the photograph principal point to the plumb point and to the isocentre. (LU)

For the answer to the first part of the question refer to *Sections 3.2.1* and *3.2.6.*

Refer to *Figure 3.3* (p. 155).

Distance from principal to plumb point $= pv = f \tan\theta = 254 \tan 7° = 31.19$ mm
Distance from principal to isocentre $= pi = f \tan(\theta/2) = 15.54$ mm

Example 3.3. Prove that the angle α on a photograph, between any line through the principal point and the line of greatest tilt, is related to the corresponding horizontal angle β on the ground by

$$\tan\beta = \tan\alpha\cos\theta$$

where θ is the angle of tilt of the camera.

On a photograph taken with a 250-mm focal length camera, the following are the co-ordinates of the photograph plumb point v, and the images a and b of ground points A and B.

Point	x (mm)	y (mm)
v	−27.8	13.0
a	−8.0	57.2
b	96.1	20.4

What would be the horizontal angle subtended by A and B at the ground principal point? (LU)

For the answer to the first part refer to *Section 3.2.6*, equation (3.9).

It must be remembered that photo co-ordinates are measured from the fiducial axes with p as origin (*Figure 3.42*).

$$\therefore \text{ Distance } pv = f\tan\theta = (27.8^2 + 13.0^2)^{\frac{1}{2}} = 30.7 \text{ mm}$$
$$\therefore\ 250\tan\theta = 30.7$$
$$\therefore\ \theta = 7°\,00'$$

From *Figure 3.42*
$$\phi_1 = \tan^{-1} 13/27.8 = 25°\,04'$$
$$\phi_2 = \tan^{-1} 57.2/8.0 = 82°\,02'$$
$$\therefore\ \alpha_1 = \phi_2 - \phi_1 = 56°\,58'$$

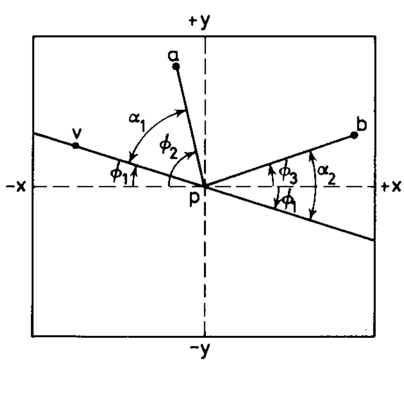


Figure 3.42

The equivalent angle on the ground is now obtained from

$$\tan\beta_1 = \tan\alpha_1 \cos\theta = \tan 56^\circ 58' \cos 7^\circ \qquad \therefore\ \beta_1 = 56^\circ 46'$$

Similarly $\phi_3 = \tan^{-1} 20.4/96.1 = 12^\circ 00' \therefore\ \alpha_2 = \phi_3 + \phi_1 = 37^\circ 04'$

but $\tan\beta_2 = \tan\alpha_2 \cos\theta \qquad \therefore\ \beta_2 = 36^\circ 52'$

$\therefore$ Angle $A\hat{P}B$ on the ground $= 180^\circ - (\beta_1 + \beta_2) = 86^\circ 22'$

Example 3.4. From a radial line plot, the distance between the ground principal points of two consecutive air photographs is found to be 654 m. The 254-mm focal length camera was vertical for each photograph and the aircraft maintained level flight. If the parallax of a point A is 93.60 mm, find the height of the aircraft above A. Find also the reduced levels of points B and C, whose differences in parallax from that at A are $+0.36$ mm and -0.20 mm, respectively. The reduced level at A is known to be 29.96 m AOD. (LU)

The problem is illustrated in *Figure 3.31*. From similar triangles L_1AL_2 and $L_1a_1a_2'$

$$(H - h_A) = \frac{Bf}{P_A} = \frac{654 \times 254}{93.6} = 1775 \text{ m}$$

which is the height of the aircraft above A.

Equation (3.19) lends itself ideally to this particular problem

$$\Delta h_{AB} = \frac{(H - h_A)\Delta P_{AB}}{P_A + \Delta P_{AB}} = \frac{1775 \times 0.36}{93.6 + 0.36} = 6.80 \text{ m}$$

$\therefore$ RL of $B = 29.96 + 6.80 = 36.76$ m AOD

Similarly

$$\Delta h_{AC} = \frac{(H - h_A)\Delta P_{AC}}{P_A + \Delta P_{AC}} = \frac{1775 \times (-0.20)}{93.6 - 0.20} = -3.80 \text{ m}$$

$\therefore$ RL of $C = 29.96 - 3.80 = 26.16$ m AOD

Example 3.5. Explain what is meant by parallax in relation to aerial photographs. How would you measure the parallax of a point appearing on two overlapping photographs if your only equipment was a millimetre rule?

On the overlap of a pair of vertical aerial photographs taken at a height of 2500 m above sea level with a 152-mm focal length camera, are shown two points A and B. Point A is the centre of a bridge in the valley while B is a point on a pass through a range of hills. In order to estimate the amount of rise between these two points, parallax bar measurements were taken as follows:

Point A—mean reading 11.43 mm
Point B—mean reading 5.90 mm

The mean level of the valley containing the principal points of the photographs is 82 m AOD, whilst a BM on a bridge near point A was 74.55 m AOD.

If the respective photograph bases are 89.1 mm and 91.4 mm, calculate the height of B above A. (KP)

Refer to *Section 3.7.2* for the answer to the first part of the question.

As the level of point A is known sufficiently accurately, then its parallax may be calculated from equation (3.18)

$$P_A = \frac{bH_o}{H - h_A}$$

where $\quad b = \frac{b_1 + b_2}{2} = \frac{89.1 + 91.4}{2} = 90.25\ \text{mm}$

$H_o = 2500 - 82 = 2418\ \text{m} \qquad H = 2500\ \text{m} \qquad h_A = 74.55\ \text{m}$

$$\therefore\ P_A = \frac{90.25 \times 2418}{2500 - 74.55} = 89.97\ \text{mm}$$

$P_B = P_A + \Delta P_{AB}$, where ΔP_{AB} is the difference in parallax $= 11.43 - 5.90$, and is *added* as B is obviously higher than A.

$$\therefore\ P_B = 89.97 + 5.53 = 95.50\ \text{mm}$$

But since $\quad P_B = \frac{bH_o}{H - h_B} \quad$ then $\quad h_B = 214.9\ \text{m AOD}$

$\therefore$ Height of B above $A = 214.9 - 74.55 = 140.4\ \text{m}$

Example 3.6. When an aircraft in level flight takes two successive vertical photographs, its height above a ground station A is H_A. The parallax of A on the photographs is P_A, and the parallax of a ground point B, a height h above A, is $(P_A + \Delta P)$. Show that the height of B above A may be obtained approximately from the equation

$$h = H_A \frac{\Delta P}{P_A}\left(1 - \frac{\Delta P}{P_A}\right)$$

In a pair of vertical photographs the parallax of a point A of known height 112.82 m above MSL is 91.4 mm. The changes of parallax to two points P and Q are -1.25 mm and $+0.87$ mm, respectively. Find the heights of P and Q above MSL if the camera focal length is 254 mm and the air base is 722 m.

If all of the ground was at the same level as A, what would be the percentage overlap for 230 mm × 230 mm photographs? (LU)

By reference to *Figure 3.31* and taking the height of the aircraft above A as H_A

then $\quad P_A = \frac{fB}{H_A}$

Similarly, as B is height h above A, then

$$P_B = \frac{fB}{H_A - h} = (P_A + \Delta P)$$

But $\quad H_A - (H_A - h) = \frac{fB}{P_A} - \frac{fB}{P_A + \Delta P}$

$$\therefore\ h = \frac{fB}{P_A}\left(1 - \frac{P_A}{P_A + \Delta P}\right) = H_A\left(\frac{P_A + \Delta P - P_A}{P_A + \Delta P}\right)$$

$$= H_A\left(\frac{\Delta P}{P_A + \Delta P}\right) = H_A \frac{\Delta P}{P_A}\left(\frac{P_A}{P_A + \Delta P}\right)$$

$$= H_A \frac{\Delta P}{P_A}\left(\frac{1}{1 + \frac{\Delta P}{P_A}}\right) = H_A \frac{\Delta P}{P_A}\left(1 + \frac{\Delta P}{P_A}\right)^{-1}$$

which, expanded binomially to the first term only, gives

$$h = H_A \frac{\Delta P}{P_A}\left(1 - \frac{\Delta P}{P_A}\right)$$

From $P_A = \frac{fB}{H_A}$ $\qquad H_A = \frac{fB}{P_A} = \frac{254 \times 722}{91.4} = 2006 \text{ m}$

$$\therefore \text{ The height of } P \text{ below } A = h_p = H_A \frac{\Delta P}{P_A}\left(1 - \frac{\Delta P}{P_A}\right)$$

$$= 2006\left(\frac{-1.25}{91.4}\right)\left(1 + \frac{1.25}{91.4}\right)$$

$$= -28.48 \text{ m}$$

$\therefore$ Height of P above MSL $= 112.82 - 28.48 = 84.34$ m

Similarly $\qquad h_Q = 2006 \times \left(\frac{0.87}{91.4}\right)\left(1 - \frac{0.87}{91.4}\right) = 19.86 \text{ m}$

$\therefore$ Height of Q above MSL $= 112.82 + 19.86 = 132.68$ m

It is required to find first the length of ground covered on a 230 mm × 230 mm format

$$\text{Scale} = \frac{f}{H_A} = \frac{\text{Format size}}{\text{Equivalent ground}} \quad \text{(see } \textit{Figure 3.4}\text{)}$$

$$= \frac{254}{2006} = \frac{230}{G}$$

$\therefore \; G = 1817$ m

The air base = 722 m which is also the *effective cover* (see *Worked example 3.1*).

$\therefore$ Effective cover $= \frac{100\%}{1817} \times 722 = 40\%$ $\qquad \therefore$ Overlap = 60%

Example 3.7. Clearly define what is meant by *parallax* on a pair of overlapping vertical aerial photographs.

The following table gives parallax bar readings on several points in the stereoscopic overlap of a pair of photographs:

Photograph points	*Mean bar readings* (mm)	*Remarks*
a	6.85	
b	11.31	
p_1	5.98	PP of photograph 1
c	2.62	

If the length of the photograph bases on photographs 1 and 2 are 88.30 mm and 84.28 mm, respectively, then without using any further data, calculate the parallax of the above points.

Thereafter, find the reduced levels of the points *A*, *B* and *C*, given the following information: focal length of lens, $f = 150$ mm; flying height above datum, $H = 500$ m; mean ground level, $h = 224.68$ m. (The parallax bar used gave increased readings as the distance between the dots increased.) (KP)

This final statement in the question may require some explanation. It should be obvious from an examination of *Figure 3.33* that as the distance between the dots increases so the level of the point decreases. Some parallax bars are graduated to give increased readings as the distance between the dots increases (Glauser bars), thus on this type of bar an increased reading indicates a decrease in level and so a decrease in parallax. The American or direct bar gives reduced readings as the distance between the dots increases; thus on this type of bar a decrease in reading indicates a decrease in level.

Figure 3.34 shows that the parallax of $p_1 = b_2$, similarly the parallax of p_2 would be b_1, where b_1 and b_2 are the respective photograph bases.

$\therefore$ Parallax of $p_1 = 84.28$ mm

$$\Delta P_{p_1a} = (5.98 - 6.85) = -0.87 \text{ m} \qquad \therefore\ P_A = 84.28 - 0.87 = 83.41$$
$$\Delta P_{p_1b} = (5.98 - 11.31) = -5.33 \text{ mm} \qquad P_B = 84.28 - 5.33 = 78.95$$
$$\Delta P_{p_1c} = (5.98 - 2.62) = +3.36 \text{ mm} \qquad P_C = 84.28 + 3.36 = 87.64$$

From the type of bar used it is obvious that points *A* and *B* are lower than P_1, thus their parallax is less and the difference must be subtracted.

$$b = \frac{b_1 + b_2}{2} = 86.29 \text{ mm}$$

and from
$$P_A = \frac{b(H-h)}{H-h_A} = \frac{86.29(500-224.68)}{500-h_A} = 83.41 \text{ mm}$$

$$h_A = 215.17 \text{ m}$$

Similarly $h_B = 199.08$ m and $h_c = 228.92$ m

Example 3.8. In a pair of overlapping air photographs, the following co-ordinate measurements to the top of an electric pylon were obtained:

Top of pylon (*t*)	*x* (mm)	*y* (mm)
Photograph 1	82.45	52.00
Photograph 2	−74.88	48.84

The origin of the above co-ordinates is the PP, the axis being formed by the fiducial marks of the photographs. The relevant photograph base on each photograph is at an angle of 80° and 258°, respectively, measured clockwise from the $+y$ axis.

Thereafter parallax bar measurements were taken as follows: top of pylon 35.63 mm and bottom of pylon 41.89 mm.

If the focal length of the camera lens was 152 mm, the flying height 2000 m, the photograph bases 92.84 and 90.16 mm, respectively, and the mean level of the terrain 112 m OD, calculate the approximate height of the pylon. (KP)

From *Figure 3.43*

Photograph 1 $\quad p_1b_1 = p_1c_1/\cos 10° = 82.45/\cos 10° = 83.72$ mm

$b_1c_1 = p_1c_1 \tan 10° = 82.45 \tan 10° = 14.54$ mm

$\therefore\ t_1b_1 = t_1c_1 - b_1c_1 = 52.00 - 14.54 = 37.46$ mm

and $\quad b_1a_1 = t_1b_1 \sin 10° = 37.46 \sin 10° = 6.50$ mm

$\therefore\ p_1a_1 = p_1b_1 + b_1a_1 = 83.72 + 6.50 = 90.22$ mm

Photograph 2 $\quad b_2c_2 = t_2c_2 \tan 12° = 48.84 \tan 12° = 10.39$ mm

$b_2p_2 = p_2c_2 - b_2c_2 = 74.88 - 10.39 = 64.49$ mm

$a_2p_2 = b_2p_2 \cos 12° = 64.49 \cos 12° = 63.08$ mm

$\therefore$ Parallax of top of pylon $P_t = 90.22 + 63.08 = 153.30$ mm

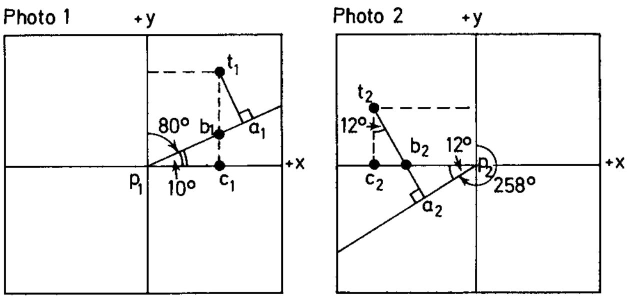


Figure 3.43

From bar measurements

$\Delta P_{tb} = 41.89 - 35.63 = 6.26$ mm

$\therefore$ Parallax of bottom of pylon $P_b = 153.30 - 6.26 = 147.04$ mm

From the basic parallax formula $P_t = b(H - h)/(H - h_t)$

$$\therefore\ h_t = 2000 - \left[\frac{91.5(2000 - 112)}{153.3}\right] = 873 \text{ m OD}$$

Similarly

$$h_b = 2000 - \left[\frac{91.5(2000 - 112)}{147.04}\right] = 825 \text{ m OD}$$

$\therefore$ Height of pylon = 48 m

3.12 TERRESTRIAL PHOTOGRAMMETRY

This form of photogrammetry utilizes photographs taken from a ground station. The instrument used is called a *photo theodolite* and comprises a precision camera integrated with a theodolite. The theodolite enables the direction of the principal axis of the camera to be found, relative to a base line.

3.12.1 Principle

At each station the camera is carefully centred and levelled such that the principal axis of the camera is horizontal and the plane of the photograph vertical. The plan position of a ground point can then be fixed analytically, graphically or instrumentally from the terrestrial photograph. *Figure 3.44* indicates the position of a point A relative to the fiducial axes of the photograph. The horizontal axis x is called the *horizon line*, while the vertical axis y is called the *principal line*.

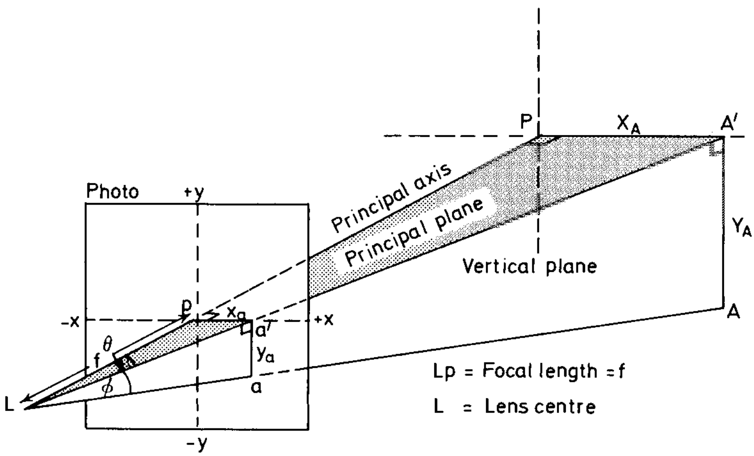


Figure 3.44

The horizontal and vertical angles, θ and ϕ, respectively, may be defined as follows

$$\tan\theta = \frac{x_a}{f} \qquad (3.29a)$$

$$\tan\phi = \frac{-y_a}{La'} \quad \text{but} \quad La' = \frac{x_a}{\sin\theta} \quad \text{or} \quad \frac{f}{\cos\theta}$$

$$\therefore\ \tan\phi = \frac{-y_a\sin\theta}{x_a} \qquad (3.29b)$$

or

$$\tan\phi = \frac{-y_a\cos\theta}{f} \qquad (3.29c)$$

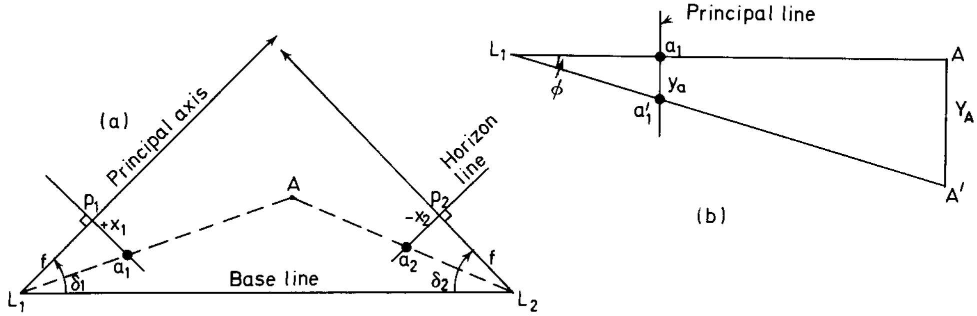


Figure 3.45. (a) Plan view, and (b) elevation.

3.12.2 Method of intersection

In this method (*Figure 3.45*) the camera axis is oriented at any angle to the base line L_1L_2, photographs being taken from both ends of the base line. The position of a point A may be fixed graphically by plotting the base line to the scale required. The direction of the principal axis of each camera station is drawn with the horizon line plotted at right angles to the axis. The x co-ordinates of the points are plotted and drawn through to intersect at A. The level of the point relative to the principal plane can be found by similar triangles (*Figure 3.45(b)*)

$$\frac{Y_A}{L_1A} = \frac{y_a}{L_1a_1}$$

but from *Figure 3.45(a)*

$$L_1a_1 = (x_1^2 + f^2)^{\frac{1}{2}}$$

$$\therefore \quad Y_A = \frac{L_1A \times y_a}{(x_1^2 + f^2)^{\frac{1}{2}}} \tag{3.30}$$

These simple techniques have largely been replaced by stereoscopic methods.

3.12.3 Stereoscopic methods

To facilitate stereoscopic viewing the photographs are taken from each end of a base line with the principal axis at 90° to the base (*Figure 3.46*). The base should be of such

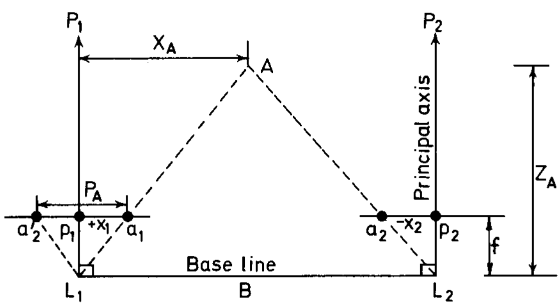


Figure 3.46. Plan view.

length as to give a well-conditioned intersection of rays, and accurately measured to reduce the propagation of errors from this source.

From *Figure 3.46*, triangles L_1AL_2 and $a_2'L_1a_1$ are similar

$$\therefore\ Z_A/L_1L_2 = L_1p_1/a_1a_2'$$

$$\therefore\ Z_A = \frac{fB}{P_A} \tag{3.31}$$

where P_A is the parallax of A.

Similarly $\quad X_A/Z_A = x_1/f \qquad \therefore\ X_A = \dfrac{Z_A x_1}{f} \quad (3.32)$

From *Figure 3.45(b)* $\quad Y_A/y_a = \dfrac{L_1A}{L_1a_1} = \dfrac{Z_A}{f}$ (from *Figure 3.46*)

$$\therefore\ Y_A = \frac{Z_A y_a}{f} \tag{3.33}$$

Note that in equation (3.31), $P_A = [x_1 - (-x_2)] = (x_1 + x_2)$; if A was to the right of P_2 then $P_A = (x_1 - x_2)$, whilst if it was to the left of P_1, then $P_A = (x_2 - x_1)$.

Apart from this analytical solution, the photographs can be oriented in some of the larger universal plotters (Wild A7) and plans produced in this way.

3.12.4 Application

The method was originally devised for topographic surveys of very rugged terrain, and, as such, was widely utilized in Switzerland. The following instances of its use will serve to indicate present-day applications:

(a) Survey of sheer rugged faces in quarries, dam sites, etc. It was used to survey the face of Edinburgh Castle.
(b) Short-base methods are used to make road-accident plans.
(c) These latter methods have been used for wriggle surveys in tunnels (Snowy Mountains, Mersey Tunnel).
(d) Recording architectural details for the restoration of ancient buildings.
(e) It has been used in many scientific projects, such as stereoscopic photographs of objects in an intensely hot state which require measuring.
(f) It has even been used to produce contoured plans of animals for husbandry purposes.

WORKED EXAMPLES

Example 3.9. In order to determine the focal length of a camera a ground photograph was taken at station O of two markers P and Q whose images appear on a print and have the co-ordinates shown in the Table below. The horizontal angle subtended by P and Q at the camera was found to be 32° 50′.

The camera was then set up at each end of a base line AB of length 350.52 m and photographs were taken of a signal X. The angles between the optical axis of the camera and the base line were 64° 20′ and 48° 30′ at A and B, respectively. The co-ordinates of the images of X on the prints are shown below:

Station	*Object on print*	*Distance from principal line* (mm)	*Distance from horizon line* (mm)
O	*P*	+57.15	
O	*Q*	−27.94	
A	*X*	−23.37	+47.75
B	*X*	−70.10	−19.05

The optical axis of the camera was horizontal throughout and the RL of A was 35.78 m. The height of the camera at A and B was 1.3 m.

Calculate the reduced levels of B and X, and the distance of X from the base line. (ICE)

From *Figure 3.47(a)*

$$(\alpha + \beta) = 32^\circ\ 50'$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{(27.94/f) + (57.15/f)}{1 - (27.94/f)(57.15/f)} = \tan 32^\circ\ 50'$$

$$\therefore\ \frac{(85.09/f)}{1 - (1596.77/f^2)} = 0.645\ 28$$

$$\therefore\ 0.645\ 28f^2 - 85.09f - 1030.36 = 0$$

$$\therefore\ f = 143 \text{ mm}$$

From *Figure 3.47(b)*

$\theta_A = \tan^{-1}(23.37/143) = 9^\circ\ 17''$ $\qquad \theta_B = \tan^{-1}(70.1/143) = 26^\circ\ 07'$

$\therefore$ In $\triangle AXB$ $\qquad \hat{A} = 73^\circ\ 37'$ $\qquad \hat{B} = 22^\circ\ 23'$ $\qquad \hat{X} = 84^\circ\ 00'$

$$AX = \frac{AB \sin\hat{B}}{\sin\hat{X}} = 134.21 \text{ m} \quad \text{and} \quad BX = \frac{AB\sin\hat{A}}{\sin\hat{X}} = 338.14 \text{ m}$$

$\therefore$ Perpendicular distance of X from $AB = AX \sin\hat{A} = 128.76$ m

From *Figure 3.47(c)*
$$\tan\phi_A = \frac{y_A \sin\theta_A}{x_A} \qquad (3.34)$$

$$\therefore\ \phi_A = \tan^{-1}(47.75 \sin 9^\circ\ 17')/23.37 = 18^\circ\ 15'$$
$$\therefore\ XX' = AX\tan\phi_A = 44.24 \text{ m}$$
$$\text{Level of principal plane} = 35.78 + 1.30 = 37.08$$
$$\therefore\ \text{Level of point } X = 37.08 + 44.24 = 81.32 \text{ m}$$

Similarly from B
$$\phi_B = \tan^{-1}(-19.05 \sin 26^\circ\ 07')/70.10 = -6^\circ\ 49'$$
$$\therefore\ XX' = BX\tan\phi_B = -40.45$$

As the level of X from A is 81.32 m,

then $\qquad$ The level of $B = 81.32 + 40.45 = 121.77$ m

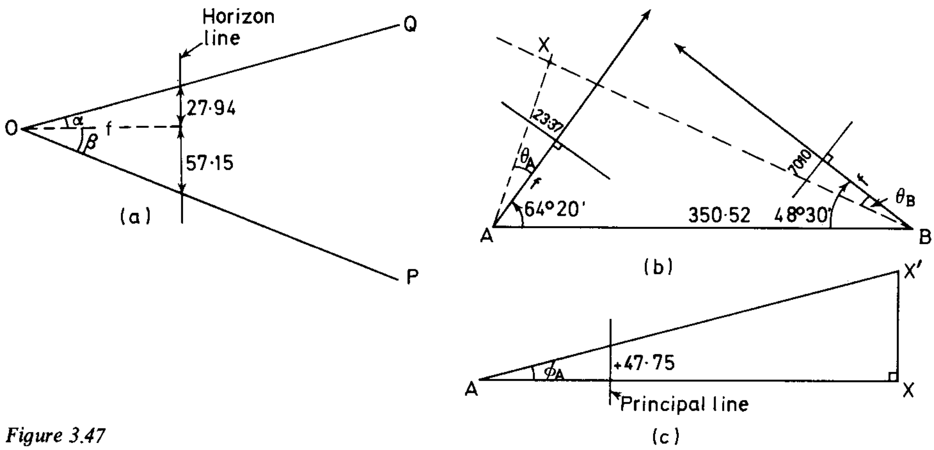


Figure 3.47

Example 3.10. In order to determine the area of a plot of ground *PQRS*, a distinctive pole was placed at each of the four corners of the plot. The poles are clearly visible on 230 mm × 230 mm ground photographs taken of the plot from each end of a base line 213.36 m long. For each photograph the optical axis of the camera was horizontal and was set at right-angles to the base line. The base line ran in an east-west direction and the plot was to the north of it. The focal length of the camera lens was 152 mm.

The information given below shows the *x* co-ordinates of the images of the four poles measured from the two photographs:

Pole	*Photograph 1 (west)* x *co-ordinate measured from principal point* (mm)	*Photograph 2 (east)* x *co-ordinate measured from principal point* (mm)
P	8.3	−56.0
Q	71.6	14.0
R	106.1	20.7
S	11.0	−74.4

Calculate the area of the plot in square metres. (ICE)

From *Section 3.12.3*, equations (3.31) and (3.32), respectively

$Z = fB/P$ and $X = Zx/f$

To find the ground co-ordinates of the poles

$Z_P = 153 \times 213.36/(8.3 + 56.0) = 507.68$ m (north of base station 1)

and $X_P = 507.68 \times 8.3/153 = 27.51$ m (east of base station 1)

Similarly Q = north 566.74 m, east 265.15 m
R = north 382.25 m, east 264.15 m
S = north 382.25 m, east 27.51 m

The co-ordinates give a figure as shown in *Figure 3.48*.

$$\therefore \text{ Area} = \frac{125.43 + 184.49}{2} \times 237.64 = 36\,825 \text{ m}^2$$

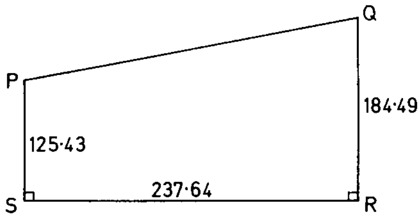


Figure 3.48

EXERCISES

3.1. Describe the method of producing a minor control plot from a set of overlapping aerial photographs. Assuming that two ground control points appear towards each end of the run, how are these used to adjust the scale of the plot?

Describe also how slotted templates are prepared and used, and how scale adjustment is effected in this case. (LU)

3.2. Explain how a stereoscopic pair of vertical aerial photographs having negligible tilt distortion are base lined, and how they are positioned for viewing under a stereoscope.

Explain how a particular contour could be located using a parallax bar and establish from first principles the parallax equation on which this work is based. (ICE)

3.3. It is proposed to map an area of 30 km × 12 km to a scale of 1/20 000 by strip photography from the air, using a camera fitted with a lens of focal length 152 mm and giving prints 230 mm square. The operating speed of the plane is to be 200 km/h and provision is to be made for 60% longitudinal overlap of prints and 25% lateral overlap.

Find (a) the average height above ground at which the plane must operate; (b) the time interval between exposures in any one run; (c) the minimum number of photographs required. (LU)

(*Answer:* (a) 3040 m, (b) 33.1 s and (c) 5705)

3.4. Explain why the heights of buildings, trees, etc. appear to be exaggerated when a pair of vertical air photographs is viewed stereoscopically.

A factory chimney 122 m high appears at the principal point of a truly vertical photograph. On the next photograph, taken shortly after, and also truly vertical, the base of the chimney is on the x axis and 83.82 mm to the left of the principal point. Each of the photographs is 203 mm × 203 mm.

Given that the flying height of the aircraft above the ground was 792 m, and that the focal length of the camera lens was 127 mm, determine: (a) the distance of the top of the chimney from the y axis on the second photograph, and (b) the percentage overlap between the two photographs. (ICE)

(*Answer:* (a) 91.2 mm and (b) 58.7%)

3.5. The following table illustrates a schedule of parallax bar readings on three points, *A*, *B* and *C*.

Reading	*A*	*B*	*C*
1st	6.98	5.56	7.82
2nd	6.99	5.58	7.84
3rd	6.96	5.55	7.79
4th	6.99	5.55	7.83

Point A is a ground control station whose level is 184.00 m OD. The photographs were taken at a height of 3500 m above MSL, using a 150-mm focal length lens. On the photographs, the photo base lines measured 84.20 mm and 86.28 mm, respectively, while the mean height of the terrain in the overlap was 120 m OD.

Calculate the approximate reduce levels of B and C. (The parallax bar used gave reduced readings as the distance between the measuring plates increased.) (KP)

(*Answer:* 122 m and 216 m OD)

3.6. Derive the expression for the parallax of a point which appears on a pair of overlapping air photographs.

The operating speed of an aircraft engaged on an aerial survey was 200 km/h. If the flying height was 2000 m above datum, and exposures were taken at intervals of 20 s using a camera of focal length 254 mm, find the length of the air base.

Hence, determine the height of a tower which appears on consecutive photographs if the difference in parallax measurements for the base and top of the tower was 1.51 mm. The base of the tower was known (from ground-control survey) to be 15.00 m above datum.

What was the scale of the photographs, assuming that the average level of the terrain was at datum? (KP)

(*Answer:* 1111.2 m; 21 m; 1 in 7874)

3.7. A photo-theodolite having a focal length of 150 mm is used to take a photograph at each end of a base line AB 250 m long. In each case the optical axis of the camera is horizontal and the height of instrument is constant. The horizontal angles between the optical axis and the base line as measured at A and B are 60° and 48°, respectively.

The co-ordinates (related to the principal point of each print) of two points P and Q whose images appear on the prints are shown below:

	Co-ordinates of P (mm)		*Co-ordinates of Q* (mm)	
	x	*y*	*x*	*y*
Print at *A*	−10.8	0	17.6	8.4
Print at *B*	−24.8	−2.0	36.0	Not measured

Calculate the horizontal distance and the difference of level between P and Q and also the difference of level between A and B. (ICE)

(*Answer:* $PQ = 90.50$ m; 13.46 m; 3.02 m)

3.8. What are the advantages and disadvantages of the use of a photo-theodolite in surveying?

Two stereo photographs have been taken from the ends of a base line *PQ* 60.00 m long with a camera having a lens of 165 mm focal length. A well-defined point *R* appears in both photographs and in each case it is to the right of the vertical cross-hair. In the photograph from *P* the horizontal and vertical measurements from the cross-hairs are 45.72 mm and 6.10 mm, respectively, while the corresponding measurements on the other photograph are 4.32 mm and 3.05 mm.

Establish the plan position of *R* relative to *P* and *Q*, and determine the difference in level between the camera axis at *P* and at *Q*. (ICE)

(*Answer:* $Z_R = 239.13$ m; X_R from $P = 66.26$ m; X_R from $Q = 6.26$ m; difference in level = 4.42 m)

4

Field astronomy

Astronomical observations are generally used to find the azimuth of a line relative to the true meridian and to find the latitude and longitude of a point. In this way surveys may be located and oriented on the Earth's surface, the position of control points for small-scale mapping established and azimuth controlled in large-scale traversing.

4.1 SPHERICAL TRIGONOMETRY

The computation of astronomical observations requires the use of spherical trigonometry.

A spherical triangle is more clearly defined by assuming the Earth to be a perfect sphere and considering lines on its surface. It is important to realize that all three sides of a spherical triangle must be arcs of great circles. A *great circle* is one which has as its centre the centre of the sphere, and a radius equal to the radius of the sphere. Thus, on the Earth's surface the Equator and all meridians of longitude are great circles, while all parallels of latitude are *small circles*. In *Figure 4.1* therefore, *PAB* is a spherical triangle but *PCD* is not a spherical triangle. As great circles are circles of maximum radii, it follows that the shortest distance between two points on the Earth's surface is the arc of a great circle joining the two points.

As in plane triangles the spherical triangle has three sides and three angles, and the method of defining these quantities is as follows (see *Figure 4.2*):

(a) The angles X, Y and Z are measured normal to the planes subtending them; they do not necessarily sum to 180°.
(b) The sides x, y and z are defined by the angles which they subtend at the centre of the sphere. Thus the 'length' of side y is angle XOZ.

The following spherical trigonometry equations are all the student requires for the solution of astronomy problems:

(1) Given three sides x, y and z (*Figure 4.2*), and required to find the angles; or, given two sides and the included angle and required to find the remaining side; use the cosine rule

$$\cos x = \cos y \cos z + \sin y \sin z \cos X \tag{4.1}$$

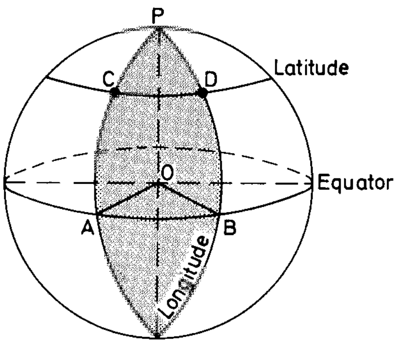


Figure 4.1

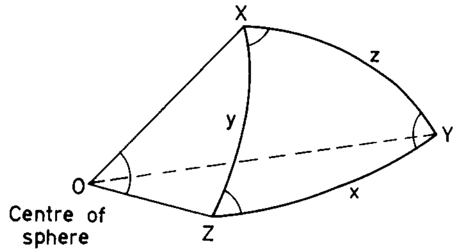


Figure 4.2

(2) Given two angles and a side opposite and required to find the remaining opposite side; or, given two sides and an angle opposite and required to find the remaining opposite angle; use the sine rule

$$\frac{\sin X}{\sin x} = \frac{\sin Y}{\sin y} = \frac{\sin Z}{\sin z} \tag{4.2}$$

(3) Given two angles and the included side and required to find a side; or, given two sides and the included angle and required to find an angle; use the four-parts rule

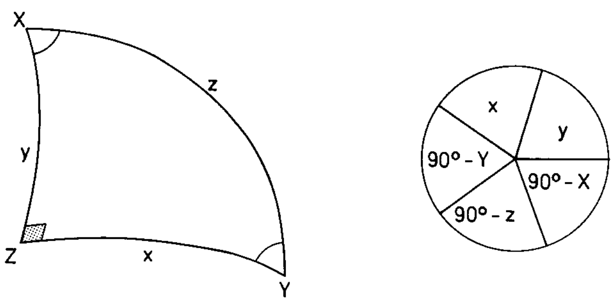
$$\sin X \cot Y = \sin z \cot y - \cos z \cos X \tag{4.3}$$

(4) Napier's rules are used in the solution of right-angled spherical triangles (*Figure 4.3(a)*). Excluding the right-angle, the five remaining parts are defined as the two sides x and y forming the arms of the right-angle, and the complements of the remaining three parts.

The five parts are entered in consecutive clockwise order in the circle (*Figure 4.3(b)*). Any part may now be defined as the middle part, the parts on either side are then the adjacent parts; the remaining two parts the opposite parts. Then Napier's rule may be written

sine of mid-part = product of cosines of opposite parts (4.4)
or = product of tangents of adjacent parts (4.5)

e.g. $\sin(90° - X) = \cos x \cos(90° - Y) = \cos x \sin Y$
or $\sin(90° - X) = \tan y \tan(90° - z) = \tan y \cot z$



(a) (b) *Figure 4.3*

The first three equations, i.e. (4.1) to (4.3), can also be used to solve a right-angled spherical triangle, but the application of Napier's rule is simpler. Similarly, any triangle can be split into two right-angled triangles and solved using Napier's rule.

The application of these equations to the solution of spherical triangles can result in ambiguities; generally, however, sufficient information is given defining the position of the celestial body observed to enable these ambiguities to be resolved.

The application of spherical trigonometry to engineering problems, other than astronomy, will now be illustrated in the following *Worked examples.*

WORKED EXAMPLES

Example 4.1. A submarine cable is to be laid by the shortest route from a station M (lat 34° 55′ S, long 56° 10′ W) to another station T (lat 33° 56′ S, long 18° 28′ E). Taking the Earth to be a sphere such that 31 m on the surface of the Earth subtends 1″ of arc at the centre, determine: (a) the length of cable in kilometres (ignoring differences in level); (b) the direction in which the cable-laying vessel should set out from M; (c) the most southerly latitude reached. (LU)

Referring to *Figure 4.4*

Length of side $MP = t = (90° - 34° 55') = 55° 05'$ (co-latitude of M)
Length of side $TP = m = (90° - 33° 56') = 56° 04'$ (co-latitude of T)
Angle $M\hat{P}T = P = (56° 10' + 18° 28') = 74° 38'$ (difference in longitude)

(a) By the cosine rule

$$\cos p = \cos t \cos m + \sin t \sin m \cos P$$
$$= \cos 55° 05' \times \cos 56° 04' + \sin 55° 05' \times \sin 56° 05' \times \cos 74° 38'$$
$$\therefore\ p = \cos^{-1} 0.499\,82 = 60° 00' \quad \text{or} \quad 300° 00'$$
$$\therefore\ \text{Length of cable } p = 60° \times 3600 \times 31 \text{ m} = 6696 \text{ km}$$

N.B. It is obvious from the difference in longitude that 300° 00′ cannot be the required value.

(b) To find angle M using the sine rule

$$\frac{\sin M}{\sin m} = \frac{\sin P}{\sin p} \qquad \therefore\ \sin M = \frac{\sin 56° 04' \times \sin 74° 38'}{\sin 60° 00'}$$

$$\therefore\ M = \sin^{-1} 0.923\,79 = 67° 29' \quad \text{or} \quad 112° 31'$$
$$\therefore\ \text{Direction in which to set out} = \text{S } 67° 29' \text{ E}$$

The above ambiguity is resolved from the fact that the three angles of the triangle should add to 180° + the spherical excess, and that the angle at T will be somewhat similar to that at M. As the spherical excess will not be great the angle cannot be 112° 31′.

(c) In right-angled triangle MDP, it is necessary to find side DP; i.e. at the most southerly point D, the angle formed is a right-angle. Entering the components in *Figure 4.5*

$$\sin DP = \cos(90° - P\hat{M}D)\cos(90° - t)$$
$$= \sin P\hat{M}D \sin t = \sin 67° 29' \times \sin 55° 05'$$
$$\therefore\ DP = \sin^{-1} 0.757\,48 = 49° 14' \quad \text{or} \quad 130° 46'$$

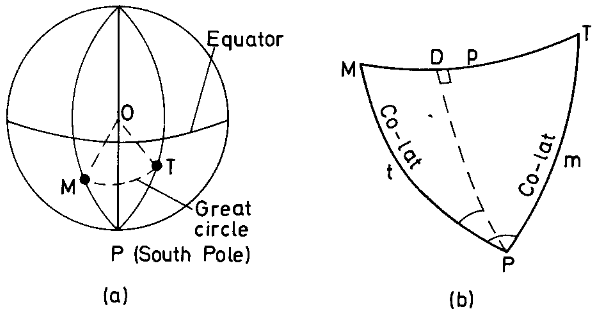


Figure 4.4

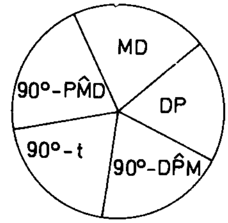


Figure 4.5

i.e. DP is the co-latitude (90° − lat) of D, and is obviously 49° 14′

$$\therefore \text{ Most southerly latitude} = (90° - 49°\,14')\text{ S}$$
$$= 40°\,46'\text{ S}$$

Example 4.2. Three stations A, O and C have reduced levels of 646.2, 457.2 and 364.2 m, respectively, and a sextant records the angle AOC as 55° 20′ 20″. If the distance from A to O is 405.4 m and from O to C 731.5 m, both measured horizontally, determine the difference of azimuth between the lines OA and OC. (LU)

It may be remembered from Volume 1 that the sextant measures the angle in the plane AOC, i.e. θ = length of side AC in spherical triangle APC (*Figure 4.6*).

Taking O as datum, angles δA and δC can be calculated thus supplying the lengths of the two remaining sides AP and PC. The angle required from a solution of the spherical triangle is ϕ, the horizontal angle between the planes AOP and COP; this is the angle which would have been obtained by a theodolite. Thus

$$AA' = (646.2 - 457.2) = 189.0\text{ m} \qquad CC' = (457.2 - 364.2) = 93.0\text{ m}$$

$$\therefore\ \delta A = \tan^{-1}\left(\frac{189.0}{405.4}\right) = 24°\,59'\,43'' \qquad \delta C = \tan^{-1}\left(\frac{93.0}{731.5}\right) = 7°\,14'\,44''$$

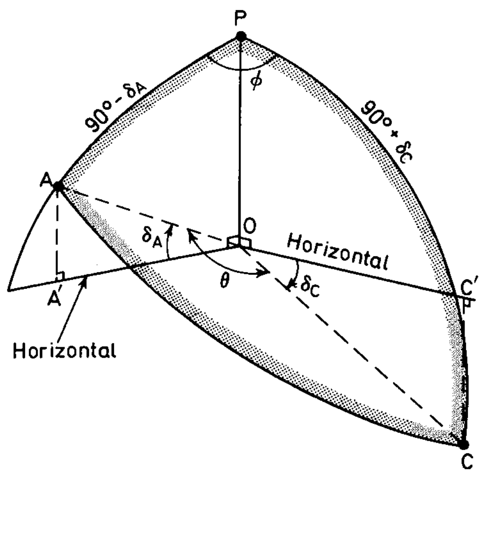


Figure 4.6

By the cosine rule

$$\cos P = \frac{\cos AC - \cos(90^\circ - \delta A)\cos(90^\circ + \delta C)}{\sin(90^\circ - \delta A)\sin(90^\circ + \delta C)}$$

$$= \frac{\cos(55^\circ\ 20'\ 20'') - \cos(65^\circ\ 00'\ 17'')\cos(97^\circ\ 14'\ 44'')}{\sin(65^\circ\ 00'\ 17'')\sin(97^\circ\ 14'\ 44'')}$$

$\therefore\ \cos P = 0.691\,813$

N.B. $\cos 97^\circ\ 14'\ 44'' = -\cos 82^\circ\ 45'\ 16''$, thus changing the sign in the above equation.

$\therefore\ P = 46^\circ\ 13'\ 34''$

the other possible value for P lies in the fourth quadrant and is of course not acceptable.

Example 4.3. A pipeline is to be set out (see *Figure 4.7*) between three pegs A, B and C on the ground. The pipe is to be laid to rising grades of 1 in 20 from A to B and 1 in 50 from B to C. The horizontal angle ABC was measured by theodolite as 45° 30′ 30″.
Calculate the angle to which the pipe must be bent. (KP)

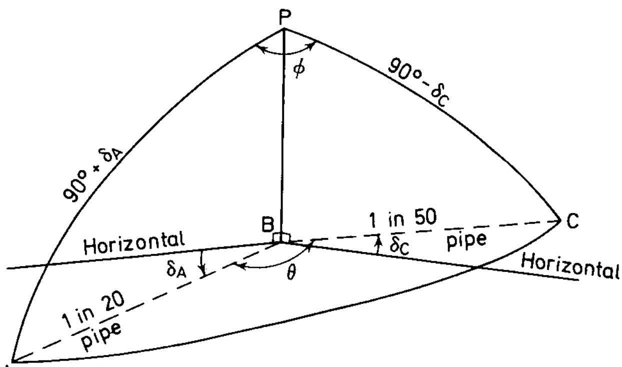


Figure 4.7

This problem is practically the reverse of the previous one. In this case the horizontal angle ϕ is known, and θ is the one required

$\delta A = \cot^{-1} 20 = 2^\circ\ 51'\ 45''$ $\qquad \delta C = \cot^{-1} 50 = 1^\circ\ 08'\ 45''$

By the cosine rule

$$\begin{aligned}\cos AC &= \cos AP \cos PC + \sin AP \sin PC \cos P\\ &= \cos(90^\circ + \delta A)\cos(90^\circ - \delta c) + \sin(90^\circ + \delta A)\sin(90^\circ - \delta c)\cos\phi\\ &= \cos 92^\circ\ 51'\ 45'' \times \cos 88^\circ\ 51'\ 15'' + \sin 92^\circ\ 51'\ 45'' \times \sin 88^\circ\ 51'\ 15''\\ &\qquad \times \cos 45^\circ\ 30'\ 30''\\ &= -0.000\,999 + 0.699\,792 = 0.698\,793\end{aligned}$$

$\therefore$ Side $AC = \theta = 44^\circ\ 40'\ 12''$ (the angle of bend of the pipe)

Example 4.4. Explain why a ship travelling in open sea between two ports usually navigates along the arc of the great circle on which poth ports are situated.

If a ship sails along the great circle joining two places each of latitude 45° north, show that the highest latitude L reached during the voyage is given by $\cot L = \cos(D/2)$, where D is the difference between the longitudes of the two places.

Calculate the shortest distance measured along the Earth's surface between New York (lat 40° 35′ N, long 74° 00′ W) and Cape Town (lat 33° 56′ S, long 18° 26′ E). The radius of the Earth may be taken as 6370 km. (LU)

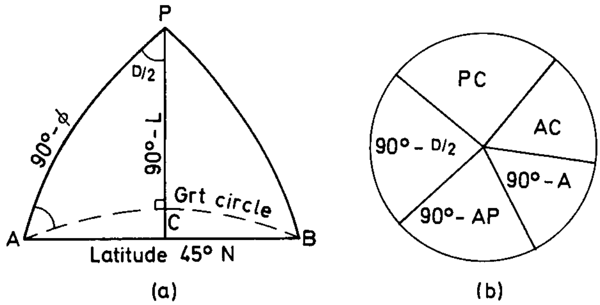


Figure 4.8

The reason why a ship on the open sea usually follows the arc of the great circle on which two ports are situated is that a great circle, being a circle of maximum radius, is the shortest distance between two points. By Napier's rule in *Figure 4.8*

$$\sin(90° - D/2) = \tan PC \times \tan(90° - AP)$$

But $PC = (90° - L)$ and $AP = 45°$

$$\therefore \cos(D/2) = \tan(90° - L) \tan 45° = \cot L$$

From the latitude and longitude of the two places given it is possible to obtain two sides and the included angle of spherical triangle APB in *Figure 4.9*. It is required to calculate side AB.

$$\theta = (74° 00' + 18° 26') = 92° 26'$$
$$AP = (90° - 40° 35') = 49° 25'$$
$$BP = (90° + 33° 56') = 123° 56'$$

By the cosine rule

$$\cos AB = \cos 49° 25' \times \cos 123° 56' + \sin 49° 25' \times \sin 123° 56' \times \cos 92° 26'$$
$$\therefore \cos AB = -0.389\,908$$
$$\therefore AB = -67° 03' 04'' = 112° 56' 56'' = \Delta$$

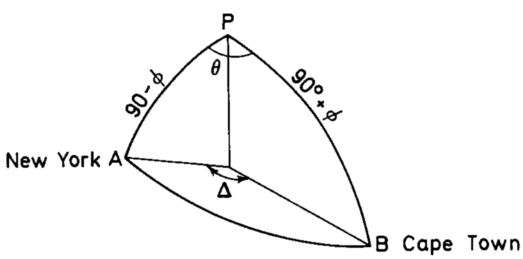


Figure 4.9

∴ Shortest distance between New York and Cape Town = $R\Delta$ rad

$$\therefore\ R\Delta = \frac{6370 \times 406\,616''}{206\,265} = 12\,557 \text{ km}$$

N.B. 1 radian = 206 265″.

4.2 DEFINITIONS OF TERMS IN ASTRONOMY

It is very important to familiarize oneself with the following definitions and to understand them thoroughly.

(1) *Celestial sphere*

This is the first and basic conception in astronomy (*Figure 4.10*). The Earth is assumed to be stationary and situated at the centre of a sphere of infinite radius. This is termed the *celestial sphere*, and all the heavenly bodies are imagined as fixed to its surface as it apparently rotates from east to west (clockwise looking south in the northern hemisphere).

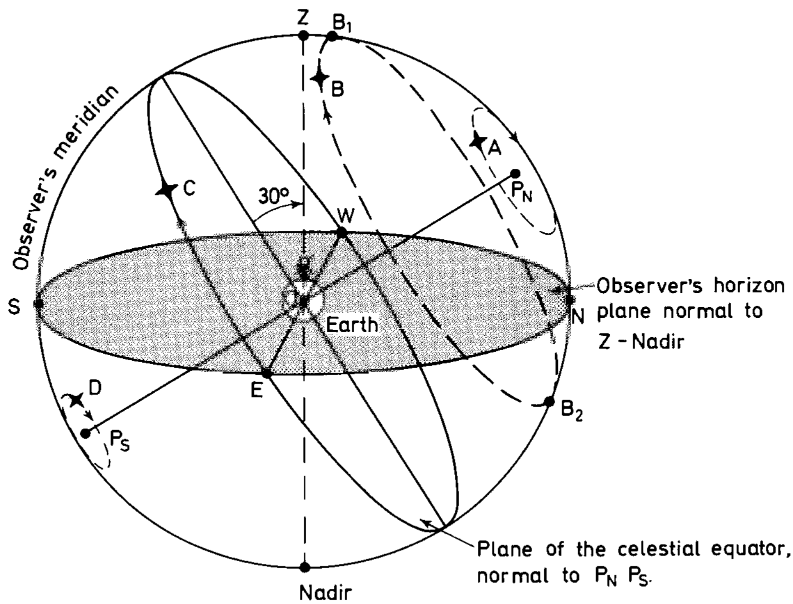


Figure 4.10

(2) *Celestial poles and equator*

In *Figure 4.10* the celestial poles P_N and P_S are simply extensions of the Earth's north and south poles, while the celestial equator is an extension of the Earth's Equator. It follows that the centre of the Earth, *O*, is also the centre of the celestial sphere.

(3) *Zenith* (*Z*)

Figure 4.10 shows an observer on the Earth's surface in latitude 30° N. If a plumb line to the centre of the Earth (direction of gravity) is extended upwards it will cut the celestial sphere at *Z*. This is the observer's position on the celestial sphere.

If the line is extended downwards it will cut the sphere at the observer's *nadir*. This term may hereafter be ignored.

(4) *Horizon plane* (*NESW*)

This plane is normal to the observer's zenith (*Figure 4.10*), passing through the centre of the Earth. As the Earth's size is insignificant relative to the celestial sphere, it may be regarded as a point *O*. Thus the observer's horizon may be regarded as the plane from which the vertical angles to the heavenly bodies are measured by theodolite.

(5) *Celestial meridians*

Celestial meridians are more easily understood if one regards them as extensions of the meridians of longitude on the Earth's surface. *SZN* in *Figure 4.10* represents the observer's celestial meridian. The plane of the celestial meridian is always normal to the observer's horizon. Where a celestial meridian passes through a star or other heavenly body, it is generally termed the *declination circle* of the star.

(6) *Prime vertical*

The celestial equator cuts the horizon plane at the east (*E*) and west (*W*) cardinal points. The celestial meridian passing through *EZW* is the *prime vertical.*

(7) *Transit or culmination*

Figure 4.10 shows four stars *A*, *B*, *C* and *D* revolving from east to west about the polar axis. If these stars were viewed from outside the north celestial pole P_N, the apparent motion of the stars is clockwise. It is anticlockwise when viewed from outside the south pole P_S. Star *A*, only a few degrees from the pole, describes a small circle entirely above the observer's meridian and is thus visible for 24 hours a day. Stars *B* and *C* do, for some period of the day, set below the observer's horizon and are therefore not visible during that period, while star *D* is never visible to an observer in the northern hemisphere. Stars such as *A*, which never set, are called *circum-polar* stars.

Considering star *B*, when it arrives on the observer's meridian at B_1, it is said to *culminate* or *transit* (there is a fine distinction between the two terms which may reasonably be ignored; hereafter the term *transit* is preferred). When the star crosses the meridian from east to west, as at B_1, it is called *upper transit*, and *lower transit* when it crosses from west to east, as at B_2.

It is worth noting here that the interval of time elapsing between two successive upper or lower transits of a star is called a *sidereal day*. Similarly for the Sun it would be a *solar day*.

(8) *Elongation*

If the celestial sphere of *Figure 4.10* is viewed from the outside through the pole, it will appear as in *Figure 4.11*. The star *S* is shown moving clockwise around the pole *P*. At S_W it reaches its greatest angular distance from the observer's meridian *PZ*. In this position it is said to be at *elongation* and an observation from *Z* at this instant will form the right-

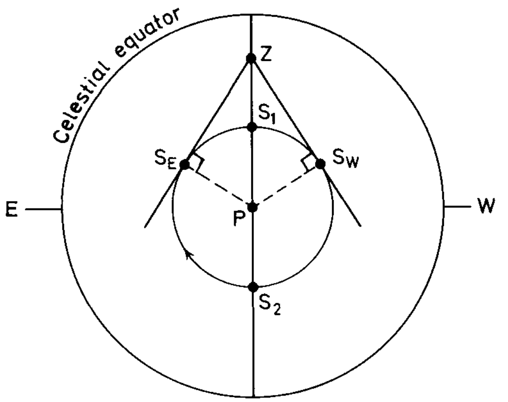


Figure 4.11

angled spherical triangle PZS_W. As the star is in the western hemisphere it is said to be at *western elongation.* Its continued motion around the pole will eventually result in its *eastern elongation* at S_E.

At S_1, when the star is on the same side of the pole as the observer, it is at upper transit; at S_2 it is at lower transit (see B_1 and B_2, *Figure 4.10*). Referring to *Figure 4.14*, it can be seen that if $\phi > \delta$ the star would revolve around P and Z, thus making it impossible to obtain a tangential sight from within the circle of the star's motion.

Summary

To summarize, it can be seen that the celestial sphere contains a system of parallels and meridians identical to the lines of latitude and longitude on the Earth. The declination circles through the stars and zenith of the observer are similar to the lines of longitude. The small circles described by the stars are similar to the parallels of latitude. These facts are very important when considering the co-ordinate position of a star.

4.2.1 Celestial co-ordinate systems

(1) Declination and right-ascension system

In the same way that a position on the Earth's surface may be fixed by its latitude and longitude, so the position of a star on the celestial sphere may be fixed by its declination (δ) and right ascension (RA).

From *Figure 4.12* it can be seen that the declination of a star is measured from the plane of the celestial equator along the meridian arc or declination circle through the star in question; it is analogous to latitude on the Earth. Hence the declinations are measured north and south of the equator. Arc $P_N S_N$ is thus the co-declination of the star and is $(90° - \delta_N)$. If the declination had been south (δ_S) then the co-dec would be $P_N S_S$ equal to $(90° + \delta_S)$. It is easy to see that the reverse is the case in the southern hemisphere.

While the position of the star on the declination circle is now fixed, the position of the circle itself needs locating. This is done by means of the RA which is the horizontal angle between a reference circle and the declination circle in question (*Figure 4.12*), and is similar to the longitude on Earth. The point on the celestial sphere chosen for reference is called the *First Point of Aries* (♈) which is a purely imaginary point. At the vernal (or spring) and autumn equinoxes the Sun is directly over the Equator. A line from the centre of the Earth through the centre of the Sun, at these times, would cut the Equator at ♈. The RA is measured in the direction opposite to the motion of the stars from 0° to 360° or 0 to 24 h.

It should be noted that neither the declination nor the RA can be measured by an observer, but are obtained from a current edition of the *Star Almanac for Land Surveyors* (HMSO, published annually).

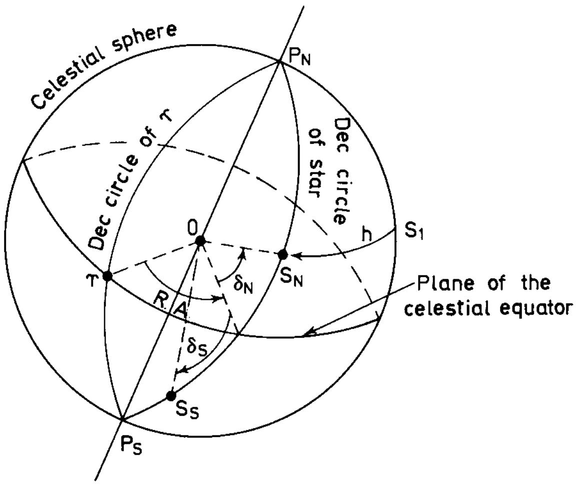


Figure 4.12

(2) Hour-angle and declination system

An alternative system of co-ordinates is obtained by substituting the hour angle (h) of a star for the RA in (1).

The hour angle of a star is the angle from the declination circle of the observer to the declination circle of the star. In *Figure 4.12*, if $P_N S_1 P_S$ is considered as the observer's declination circle, the star will transit at S_1 and the hour angle h will be 0° or 0 h. When the star reaches S_N, then h will have a value, say 45° or 3 h west of the observer's meridian. It is generally measured in units of time, east or west of the observer's meridian. For example, h equal to 3 h east, would be equivalent to 21 h measured in the direction of the star's motion. It should be carefully noted that measurement of the hour angle commences from upper transit and that upper transit occurs when the star is on the same side of the pole as the zenith of the observer.

(3) Altitude and azimuth system

The final system of fixing a star is direct measurement of its altitude (H) and azimuth (A) (*Figure 4.13*).

The altitude of the star S is the vertical angle H measured from the observer's horizon plane along the declination circle of the star. The arc ZS is therefore the co-altitude of the star equal to $(90° - H)$.

The azimuth of the star is the angle PZS. The student may prefer to visualize this as the observer at Z looking towards the true north at P. The azimuth will then be the bearing ZS equal to $A°$ as azimuths are measured clockwise from north, i.e. its quadrant bearing is north-east. If the star had been the other side of the observer's meridian, its quadrant bearing would be north-west and its azimuth $(360° - A)$.

An observer in the southern hemisphere (Z_S) looking towards the south pole (P_S) would be looking due south, i.e. 180°. Thus the star at S in the east would have an azimuth of $(180° - A)$, whilst a star in the west would be $(180° + A)$.

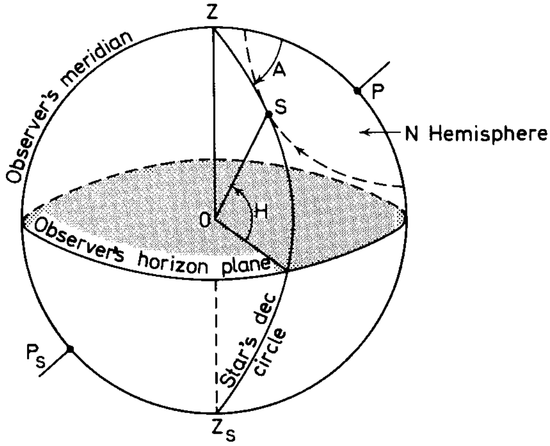


Figure 4.13

4.2.2 The astronomical triangle

The reader is now in a position to visualize the elements of the spherical triangle as shown in *Figure 4.14*.

To facilitate understanding of the triangle, the reader should remember that the zenith Z and the observer's horizon form a pair, while the pole P and the Equator also form a pair. Thus considering line ZS, as Z is present the observer's horizon must be present, and from the horizon to S is the measured altitude, therefore ZS is the *co-altitude* $(90° - H)$. Similarly with PS, as P is present—the Equator must also be. The element measured from the Equator to S is the declination, therefore PS is the *co-declination* $(90° - \delta)$. Finally with ZP, the element from the Equator to the observer at Z is of course latitude, therefore ZP is the *co-latitude* $(90° - \phi)$. The remaining two elements have already been defined; the azimuth of the star, i.e. angle A, is always at Z, and the hour angle h always at P. It can be seen that S is to the east of ZP, thus its azimuth is east of ZP, and its hour angle h east of PZ.

Basically, field astronomy now consists of obtaining sufficient elements of the astronomical triangle to enable the remainder to be calculated.

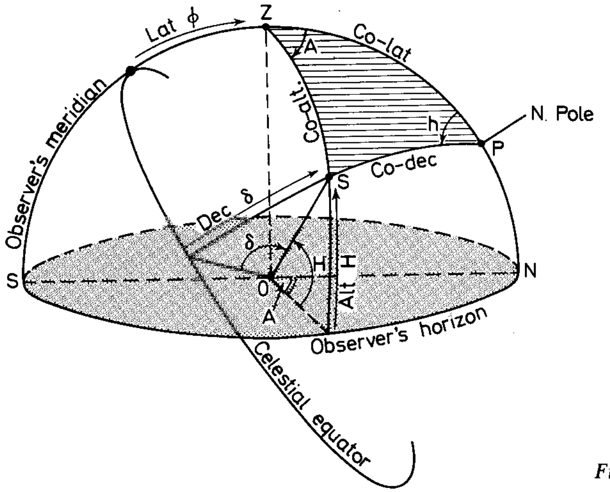


Figure 4.14

4.3 TIME IN ASTRONOMY

If the altitude of a star is measured, this altitude is fixed only for the instant of time at which it was measured. It follows then that this time must be recorded. In the United Kingdom the time would be measured in terms of *Greenwich mean time* (GMT) or, as it is now called, *Universal time* (UT). However, the star's motion is measured in *sidereal time* (ST) while the actual Sun is measured in apparent time (AT). These time systems will be dealt with in detail to attempt to clarify this topic.

A good-class radio for the reception of Greenwich time signals is a very important piece of equipment when carrying out astronomical observations. Whilst the Greenwich pips are well recognized in the UK, the radio should be capable of clearly receiving the radio time signals transmitted from stations throughout the world.

A quartz and atomic clock records time to an accuracy greater than 1/1000th sec/day and is therefore used to provide international time and frequency standards. However, such is the accuracy of atomic time, that it varies from UT which is based on the slightly irregular rotation of the Earth, but which nevertheless forms the basis of astronomical observations. The time signals transmitted by radio are at the rate of atomic time approximately corrected to UT by step adjustments of exactly one leap second. The correction applied is the last second on 30 June or 31 December, as appropriate, and the resulting emitted time signal is called *co-ordinated Universal time* (UTC). The maximum discrepancy between UTC and UT will never be greater than 0.7 sec.

The above discrepancy between UTC and UT, and referred to as DUT1, is transmitted in simple code form along with the time signals. Utilizing this code enables the surveyor to obtain UT to 0.1 sec. For greater accuracy, reference should be made to the *Time Service Circulars* (published weekly by the Royal Greenwich Observatory) where the difference is given in milliseconds. Similar information can be obtained from the Bureau Internationale de l'Heure (BIH).

As indicated above, UT = UTC ± DUT1 where DUT1 is indicated by emphasizing seven consecutive seconds markers, following the minute marker.

For a POSITIVE correction	DUT1 = ($n \times 0.1$) sec	where ($1 < n < 7$)
A NEGATIVE correction is	DUT1 = ($m \times 0.1$) sec	where ($9 < m < 15$)

i.e., the negative value is indicated by emphasizing the 9th to the $(8 + m)^{\text{th}}$ second. The form of emphasizing may be by lengthening, doubling, splitting, or altering the tone of the signal. If DUT1 is zero, then naturally there will be no emphasized seconds markers. The UK emissions from Rugby (call sign MSF on frequency 2.5 MHz, 120 m wavelength) use double pulses, limited to the first 15 seconds markers of each minute, with a maximum of 7 at any one time. It should be realized that the above coding system applies only to primary time signals and not to secondary systems such as telephones.

Thus, when observing a star the observer requires the UT of the instant it crosses the cross-hair of the theodolite. To obtain this he requires a chronometer, probably of the quartz crystal type. The instant the star is bisected, the observer starts a stopwatch. He then moves to the chronometer and at the instant of taking the reading, he stops the stopwatch. Thus, the chronometer time of the instant he bisected the star is

(Chronometer time − Stopwatch interval)

Then from his radio, he obtains UT = UTC ± DUT1 which is, at the instant of transmission, compared with the chronometer. Any difference is the *chronometer error*, which is applied to the chrono time of the observation to get the equivalent UT

i.e. UT = (Chrono time − Stopwatch interval ± Chrono error)

Time problems are much simplified with the aid of a time diagram. An examination of *Figure 4.11* shows that moving Z to the outside of the circle in no way alters the line of the meridian PZ. Similarly, moving the star S_W to the periphery in the line PS_W would not alter the direction of the star from P. Thus a time diagram can be constructed with P at the centre and the heavenly bodies, including ♈, on the circumference. The observer's meridian and Greenwich meridian can be denoted by Z and G, respectively, on the circumference.

4.3.1 Sidereal day

A sidereal day commences on any particular meridian when the First Point of Aries (♈) is at upper transit there, i.e. it is zero hours (0 h) local sidereal time (LST). If the particular meridian in question had been Greenwich, the time would be 0 h GST. The ♈ now moves clockwise round the Equator (northern hemisphere) returning to upper transit 24 sidereal hours later. Thus a sidereal day is the interval between two successive transits of the ♈.

Figure 4.15 shows ♈ defining LST in relation to the observer's meridian PZ. The star's declination circle PS is now added defining the position of the star, the angle ZPS is the star's western hour angle (HA). The angle from ♈ to S, measured anticlockwise, is by definition of the Right Ascension (RA). It can thus be seen that

$$\text{Sidereal time} = \text{HA} + \text{RA} \tag{4.6}$$

When the star is at upper transit it will be on PZ

$$\therefore \text{ Sidereal time of transit} = \text{Star's RA} \tag{4.7}$$

Figure 4.16 clearly shows that when the star's HA is easterly, then

$$\text{LST} = \text{RA} - \text{HA} \tag{4.8}$$

There is no need to commit these equations to memory if the definitions already outlined are clearly understood and a time diagram is used.

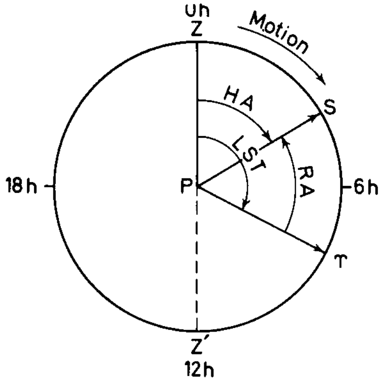


Figure 4.15

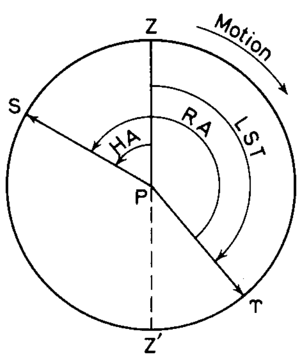


Figure 4.16

4.3.2 Solar day

An *apparent solar day* may similarly be defined as the interval between successive upper transits of the *actual Sun* (*A*). However, when the Sun is at upper transit we call it noon or 12 h; thus the commencement of a solar day at 0 h will need to be measured from the antipodes of the observer's meridian, i.e. *Z′*. This statement is very important and should be carefully noted. (*Figure 4.15* with 0 h sidereal time at *Z*, should not be confused with *Figure 4.17* having 0 h solar time at *Z′*.)

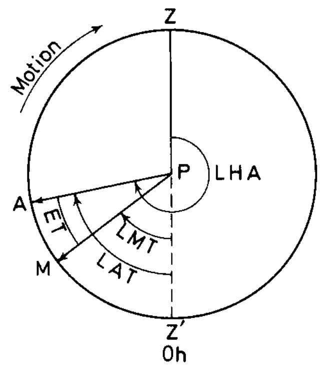


Figure 4.17

The movement of the actual Sun is irregular for a variety of reasons, and using it as a timekeeper would result in days of varying length. Consequently, there has been evolved the conception of a *mean Sun* (*M*) which travels round the Equator at a uniform rate and therefore transits on any meridian at equal intervals of time. The interval between two successive transits of the mean Sun on the antipodal meridian is called a *mean solar day*.

Figure 4.17 indicates *local mean time* (LMT) and *local apparent time* (LAT), both measured from the antipodes of the observer in the direction of the Sun's motion. It is important to note, however, that the *local hour angle* of the Sun (LHA) is still measured from the observer's meridian as shown. In order to fix the position of *M* relative to *A*, the interval between mean and apparent time, called the *equation of time* (ET), is used. By definition then

$$ET = LMT - LAT \tag{4.9}$$

If, for instance, LMT is 4 h and LAT 4 h 10 m, then ET = −10 min, indicating that M is behind A by 10 m. The position of A relative to M varies throughout the year with the result that the ET ranges from maximum values of +14 m 20 s to −16 m 20 s. In order to avoid the use of positive and negative signs, the *Star Almanac* replaces ET with a quantity called E, such that

$$E = 12\text{ h} - ET \tag{4.10}$$

Therefore, in this case $E = 12\text{ h} - (-10\text{ m}) = 12\text{ h }10\text{ m}$

It can clearly be seen from the diagram that

$$LHA = LMT + 12\text{ h} - ET = 4\text{ h} + 12\text{ h} - (-10\text{ m}) = 16\text{ h }10\text{ m}$$

$$\therefore\ LHA = LMT + E \tag{4.11}$$

It is appropriate at this point to mention a quantity R analogous to E, which is also tabulated in the *Star Almanac* and may be defined as

$$R = RAMS \pm 12\text{ h} \tag{4.12}$$

where RAMS is the *right ascension of the mean Sun.* This quantity is used to relate instants of mean time to sidereal time, or *vice versa.* This is illustrated by *Figure 4.18*; assume for instance that the LMT = 4 h, and $R = 18$ h, then from equation (4.12) $RAMS = 6$ h. Then $Z'M = 4$ h fixes the position of M, and as RA is measured anticlockwise from ♈, then ♈ is fixed relative to M at that instant. Thus LST, which is in effect the LHA of ♈, can be easily seen to be 22 h. This should more properly be expressed as

LST at 4 h LMT = 22 h

This example, then, indicates that

$$LST \text{ or } (LHA\ ♈) = LMT + RAMS + 12\text{ h} = LMT + R \tag{4.13}$$

In finding RAMS from R, it is generally best to add 12 h when $R < 12$ h, and subtract when $R > 12$ h. In fact, it is immaterial; for instance, if $R = 11$ h, then RAMS = 11 + 12 = 23 h, or 11 − 12 = −1 h = 24 − 1 = 23 h. As negative time is unknown then simply adding 24 h will give the correct result.

The student should note that the inclusion of the Greenwich meridian in no way alters the basic approach. Had the local meridian ZZ' been Greenwich CG' in *Figure 4.18*, then equation (4.13) would be written

$$GST = GMT + R$$

which is perhaps more appropriate, as R is tabulated against GMT (UT) in the *Star Almanac.*

4.3.3 Effect of longitude on time

In order to eliminate the need for equations the student is required to use his imagination in the following manner. Imagine you are situated at the centre of the

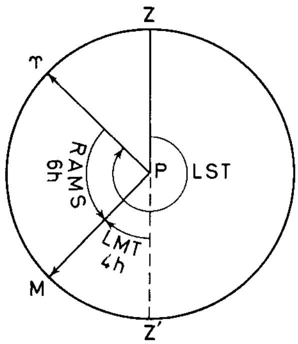


Figure 4.18

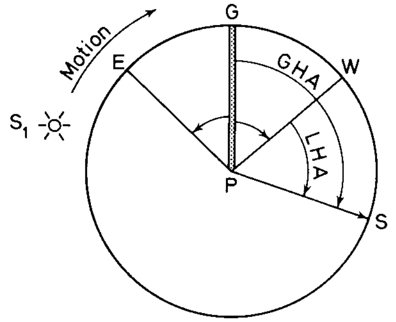


Figure 4.19

celestial sphere *P* looking out towards the Greenwich meridian *G* (*Figure 4.19*). The Sun *S* is approaching the meridian *E*, east of Greenwich. When it is overhead at *E* the time there is noon; it has not yet reached *G* of *W* and so the time there is still early morning. In fact, if the longitude of *E* was 45° E and that of *W* 30° W, then as 360° = 24 h, it would be 9 h (or 9 a.m.) at *G* and 7 h (or 7 a.m.) at *W*. As the Sun reaches *G*, 3 h later, it is now noon at *G*, 15 h (or 3 p.m.) at *E* and 10 h (or 10 a.m.) at *W*. Finally when the Sun reaches *W*, it will be 2 p.m. at *G* and 5 p.m. at *E*. This commonsense approach may be used regardless of the form the heavenly body takes. Also from the diagram, when the Sun reaches *S*, the LHA, for an observer on the meridian *PW*, is *WPS*, and the GHA is *GPS*, thus

West longitude = Greenwich HA − Local HA

As this holds for any body, it follows that

West longitude = GMT − LMT
= GAT − LAT
= GST − LST

It is obvious from the diagram that for east longitude the reverse is the case. The above equations are not meant to be memorized—a time diagram will soon indicate the situation.

Standard time is the official time of a country or zone of a country. If, for instance, *E* in *Figure 4.19* was the east coast of the United Kingdom, *G* Greenwich and *W* the west coast, then, as above, when it is 12 h at *G*, it would be 15 h at *E* and 10 h at *W*; thus all the clocks throughout the country would be showing different times. Britain therefore adopted the time at Greenwich as the standard time for the country. Finding the standard time from GMT (UT) is a straightforward application of longitude. For instance, the time at, say, 30° E, would be 2 h ahead of GMT.

4.3.4 Time intervals

All heavenly bodies have an apparent rotation around the Earth of 360° in 24 h, thus

15° ≡ 1 h
15′ ≡ 1 m
15″ ≡ 1 s (permitting easy conversion)

However, if the body in question was ϒ or a star, the hours in question would be sidereal hours. If the body was M or A they would be mean or apparent solar hours, respectively.

If ϒ and M started at the same instant, from the same point, on their journey around the Earth, ϒ would arrive back 3 m 55.91 s earlier than M. Each day ϒ would get further ahead until by the end of one year it would have completed one more rotation than M.

$$\therefore \text{ 366 sidereal days} = \text{365 mean solar days}$$

$$\therefore \text{ 24 sidereal hours} = \frac{365 \times 24}{366} \text{ mean solar hours}$$

$$= \text{23 h 56 m 04.09 s} \quad \text{(mean solar time)}$$

This may be written as 1 sidereal hour = 1 solar hour − 9.83 s
or 1 solar hour = 1 sidereal hour + 9.86 s

It is this difference in time, gradually accumulating during the year, which is tabulated as R in the *Star Almanac*.

Various time problems will now be considered in detail.

WORKED EXAMPLES

Example 4.5. Find the GST at 20 h 00 m 00 s GMT on 17 November 1965, if the value for R is 3 h 20 m 30 s. What would be the equivalent LST for a place on longitude 60° W?

(a) Construct a time diagram showing the Greenwich meridian GG' (G' is the antipodes) (*Figure 4.20*).
(b) From the antipodes G' turn off 20 h GMT to fix position M (mean Sun).
(c) As R = 3 h 20 m 30 s, then RAMS = R + 12 h = 15 h 20 m 30 s. This is the angular distance, *measured anticlockwise*, from ϒ to M thus fixing the relative position of ϒ at this instant.
(d) From the diagram, GST is the angular distance, measured clockwise, from upper transit G to ϒ = 23 h 20 m 30 s. This is obtained as follows

as $G'GM$ = 20 h then $G'M$ = 4 h
ϒ$G'M$ 15 h 20 m 30 s
thus ϒG' measured anticlockwise = 11 h 20 m 30 s
∴ GMG'ϒ = ϒG' + 12 h = 23 h 20 m 30 s

This problem can be done much quicker by simply using $GST = GMT + R$, but time diagrams, even for relatively easy problems, lead to a better understanding of time in astronomy.

If the student now adds the local meridian of 4 h west and measures clockwise to ϒ from this point, then LST is obviously 19 h 20 m 30 s.

Example 4.6. Find the GHA of the Sun at GMT 11 h 00 m 00 s on 17 October 1965, if the appropriate value for E was 12 h 14 m 36.6 s. What is the LHA at 45° E?

If E = 12 h 14 m 36.6 s, then the 'equation of time' ET = −14 m 36.6 s. The minus sign indicates that M is behind A. Thus the time diagram (*Figure 4.21*) can be constructed as follows:

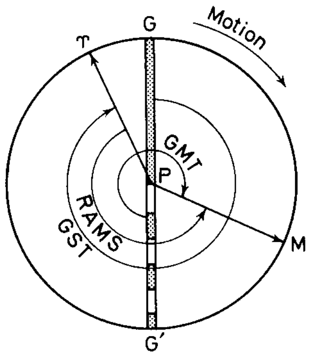


Figure 4.20

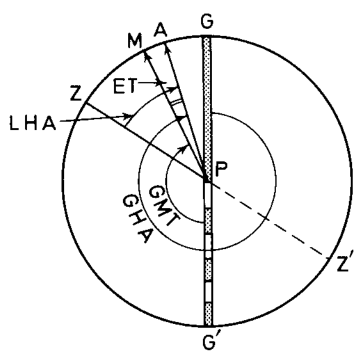


Figure 4.21

First construct the Greenwich meridian GG'. Then

GMT defines the position of M (measured clockwise from antipodes) = 11 h.
ET defines the position of A relative to M, and the GHA from G clockwise to the actual Sun A is easily deduced as 23 h 14 m 36.6 s.

From *Figure 4.21*

ZG = 3 h (45° east)
AG = 12 h − 11 h 14 m 36.6 s $(G'A)$ = 0 h 45 m 23.4 s
∴ ZA = LHA = 2 h 14 m 36.6 s (i.e. $ZG - AG$)

Using formula   GHA = GMT + E = 23 h 14 m 36.6 s
and   LHA = LMT + E = 14 h + 12 h 14 m 36.6 s
= 2 h 14 m 36.6 s

N.B. In this last equation LMT is measured from Z'.

The reverse operation of finding GMT given GHA is not quite as simple as the equation indicates, for E is tabulated against GMT in the *Star Almanac*. It is necessary therefore to carry out successive approximation for E. As in most engineering examinations the correct value for E will be quoted; the operation is just the reverse of the above. In any event, taking E from the *Star Almanac* for 12 h on the date in question produces only a small error.

Example 4.7. An observer in longitude 105° E observed the Sun at upper transit. The instant of transit was recorded by chronometer as LMT 12 h 08 m 25 s on 17 March 1965, and the value of E from the *Star Almanac* was 11 h 51 m 25 s. Find the error of the chronometer.

Construct GG', then using longitude 7 h E fix Z relative to G, and subsequently the antipodes Z' (*Figure 4.22*). From E, the value of ET is +08 m 35 s, thus M is ahead of A, and A is on Z as it is at upper transit.

∴ LMT is 12 h 08 m 35 s (and the chronometer is slow by 10 s)

Example 4.8. Find the LST at LMT 8 h 00 m 00 s on 17 March 1965, in longitude 30° W if the appropriate R value is 11 h 37 m 26 s.

Draw a diagram showing GG', and locate ZZ' from longitude (2 h W) (*Figure 4.23*). LMT is measured from Z', thereby fixing the position of M. R = 11 h 37 m 26 s,

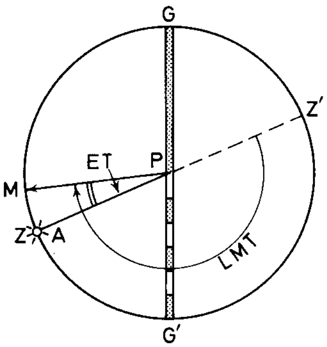


Figure 4.22

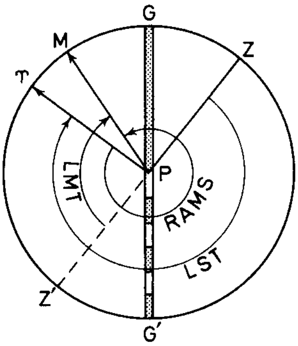


Figure 4.23

therefore RAMS = 23 h 37 m 26 s, which is measured anticlockwise from ♈, thus fixing ♈ by measuring clockwise from *M*. Now LST is measured clockwise from the observer to ♈, i.e. $Z♈$

$\therefore\ Z♈ = 19\text{ h } 37\text{ m } 26\text{ s}$ i.e. $ZZ' + Z'M - ♈M = 12\text{ h} + 8\text{ h} - 22\text{ m } 34\text{ s}$

Using equation (4.13) $\text{LST} = \text{LMT} + R = 19\text{ h } 37\text{ m } 26\text{ s}$

Example 4.9. The GST at GMT (UT) 0 h is 14 h 00 m 00 s. Find the LMT of the transit of ♈ in longitude (a) 120° E and (b) 120° W.

(a) Draw a diagram and indicate *ZZ′* relative to *GG′* (*Figure 4.24*(*a*)). As GMT is 0 h, then *M* will be over the antipodes at *G′*. As GST is 14 h, measured clockwise from *G*, the position of ♈ is now fixed relative to *M*. *P*♈ and *PM* can be imagined to be the hands of a watch fixed relative to each other. Then as ♈ moves to *Z* (i.e. transits on the observer's meridian) a distance of 2 sidereal hours, *M* will move an equivalent amount of solar hours, creating *Figure 4.24*(*b*).

$\therefore$ As 2 sidereal hours = 2 − (2 × 9.8 s) = 1 h 59 m 40.4 s MT

$\text{LMT} = Z'M = Z'G' + G'M = 8\text{ h} + 1\text{ h } 59\text{ m } 40.4\text{ s}$

$\therefore\ \text{LMT} = 9\text{ h } 59\text{ m } 40.4\text{ s}$

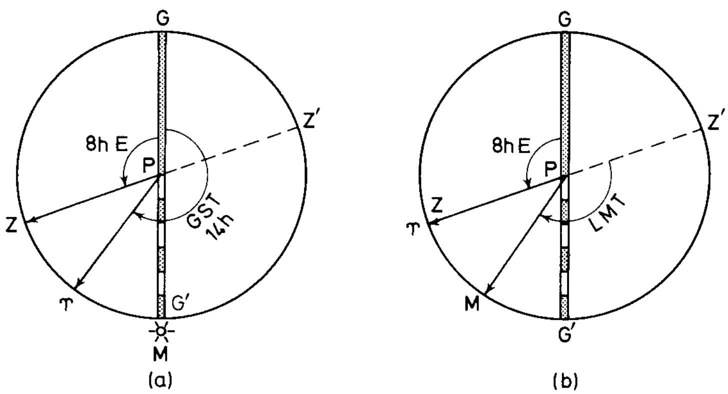


Figure 4.24

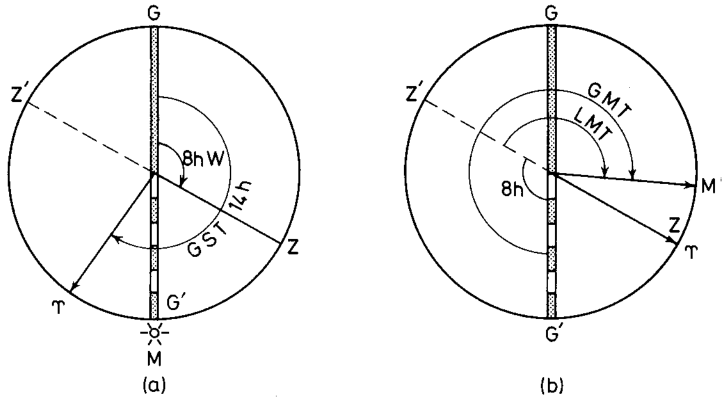


Figure 4.25

(b) The diagram is drawn exactly as before, only the longitude changes (*Figure 4.25(a)*). To transit ϒ at Z it may be moved clockwise through 18 sidereal hours or anticlockwise through 6 sidereal hours. Regardless of the direction, ϒ and M will appear in the same position on the diagram but two different answers will be obtained in the computation. This is because travelling through 18 h gives a change of 18 × 9.8 s to MT, but only 6 × 9.8 s if travelling in the opposite direction. The direction of travel must be such that the transit of ϒ will occur on the same day for which the information is required.

Moving ϒ clockwise through 18 sidereal hours moves M clockwise through 17 h 57 m 3.6 s solar time (*Figure 4.25(b)*).

Therefore, as GMT is obviously 17 h 57 m 3.6 s, then from *Figure 4.25(b)*

LMT = 9 h 57 m 3.6 s

Explanation. In *Figure 4.25(a)*, as the GMT is 0 h, the LMT is 0 h − 8 h = 16 h on the *previous* day. Thus 16 h + 17 h 57 m 3.6 s = 9 h 57 m 3.6 s on the *same* day.

Thus, when in doubt the student should find the LMT of the meridian, as above. Then using the sidereal interval (as an approximation) through which ϒ moves, clockwise or anticlockwise, decide which direction gives the transit on the required day.

Example 4.10. If the GST is 14 h at 0 h GMT, find the LST in longitude 105° W at LMT 10 h.

The student may at first glance find this problem rather confusing, but even with no set idea on how to commence it, by simply adding the data to the time diagram the answer will soon become clear.

Commence with the Greenwich line GG' as usual and then add the observer's meridian ZZ', at 7 h W. As the GST is 14 h at 0 h GMT, this fixes ϒ and M relative to each other as shown (*Figure 4.26(a)*). What is now required is LST at LMT 10 h. Thus moving M anticlockwise through 7 h solar time will fix it so that $Z'GM$ = 10 h = LMT (*Figure 4.26(b)*). It follows that ϒ will move back through 7 h solar time = 7 h + 1 m 9 s = 7 h 1 m 9 s sidereal time.

∴ LST = 23 h 58 m 51 s (measured clockwise from Z to ϒ)

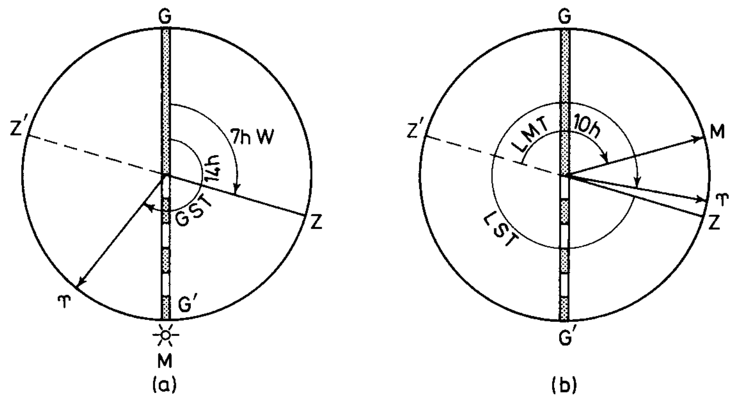


Figure 4.26

N.B. If *M* was moved clockwise the distance would be 17 h and the sidereal interval a little more. From *Figure 4.26(a)* at the instant that GST = 14 h, LST = 7 h. Thus adding a sidereal interval of more than 17 h on to 7 h would give a value greater than 24 h, thus moving into the next day. *M* must therefore move anticlockwise, as above.

Example 4.11. If the GST is 9 h at GMT 0 h, find the LMT of transit of a star at longitude 120° east. The RA of the star is 14 h.

Construct Greenwich and fix *Z* at 8 h each of *G* (*Figure 4.27*). 9 h GST measured clockwise from *G* fixes Υ. 14 h RA measured anticlockwise from Υ fixes *S*. 0 h GMT fixes *M* on the antipodes of *G*. *S* and *M* are now fixed relative to each other. By simple deduction from the diagram it can be seen that *S* needs to move 3 h anticlockwise or 21 h clockwise to transit on *Z*. From the diagram, LMT (*Z'M*) is 8 h at that instant. Moving *S* through 21 h would give an LMT on 29 h, i.e. 5 h on the next day, thus *S* is moved back through 3 h to *Z*, and *M* will move back the equivalent amount in solar time, i.e. 2 h 59 m 31 s to *M'*.

$\therefore$ LMT = *Z'M'* = 8 h − 2 h 59 m 31 s = 5 h 00 m 29 s

Example 4.12. If the GST is 15 h at GMT 0 h, find the LMT of the western elongation of a star in longitude 90° W, whose RA = 9 h and whose LHA = 4 h W.

Fix *GG'* and *ZZ'* in the usual way (*Figure 4.28*). 15 h GST fixes Υ_1 and 0 h GMT fixes *M*. At the instant of elongation the HA of the star is 4 h W, thus 4 h clockwise from *Z*

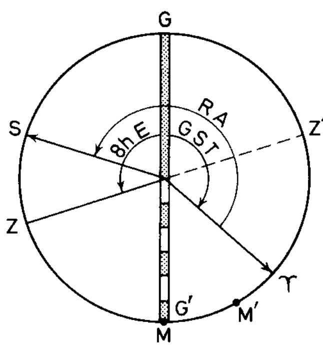


Figure 4.27

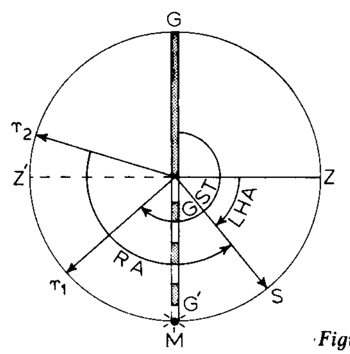


Figure 4.28

fixes the star at S. Also at this instant the RA is 9 h, measured clockwise from S; this fixes this position of the First Point of Aries at the instant of elongation at Υ_2. The position of M at this instant is found by moving Υ_1 forward to coincide with Υ_2, a distance of 4 h sidereal time; thus M moves forward 3 h 59 m 20.8 s solar time. However, when GMT was 0 h, LMT (measured clockwise from Z') was 18 h, but as Z is west of Greenwich it must be 18 h on the *preceding day*. Thus the LMT calculated for elongation will be

18 h + 3 h 59 m 20.8 s = 21 h 59 m 20.8 s

still on the preceding day. It follows that for the star to elongate on the day required it must go round again another 24 h sidereal time. M will then go round another 23 h 56 m 04.1 s solar time and the required LMT of elongation is

21 h 59 m 20.8 s + 23 h 56 m 04.1 s = 21 h 55 m 24.9 s

EXERCISES

The student is advised to re-work the preceding examples without reference to the methods or diagrams given. In this way familiarity with the elements of the time diagrams will be attained.

4.1. At a certain place the LMT was 17 h 10 m 20 s when the LST was 9 h 40 m 30 s. The Greenwich time signal revealed that the mean time was 1 h 20 m fast of GMT. (a) What is the longitude of the place? (b) What is the LST of LMN? (c) What is the GST of GMN? (LU)

(*Answer:* (a) 20° E, (b) 4 h 29 m 19 s and (c) 4 h 29 m 32 s)

N.B. LMN is *local mean noon*, or LMT 12 h.

4.2. Define the *First Point of Aries* and show how its movement is related to that of the mean Sun.

In order to determine the latitude of a survey station, a meridian observation at upper transit is to be made on a star of RA 2 h 06 m 30 s. The longitude of the station is 93° 37′ 03″ W and the HA of the Υ with respect to the Greenwich meridian at GMN on the day on which the observation is to be made is 11 h 41 m 19 s. Determine the GMT of transit. (ICE)

(*Answer:* 8 h 39 m 32 s)

4.3. Define the terms *sidereal time, mean time, apparent time* and *equation of time.*

The Sun is observed to be at upper transit from a certain longitude when the GMT is known to be 4 h 10 m 04 s. Given that the equation of time at the instant of transit is known to be 11 m 41 s, the clock being ahead of the Sun, determine the longitude of the observing station. (ICE)

(*Answer:* 120° 24′ 15″ E)

Further problems on time occur in the following sections.

4.4 OBSERVATIONAL AND INSTRUMENTAL CORRECTIONS

The observed quantities necessary for the solution of the astronomical triangle are: (i) the vertical angle to the star or the Sun; or (ii) the precise time of the instant of observation.

In the case of the azimuth of a line, the horizontal angle between the line and the star (or the Sun) will also be required at the instant of observation.

4.4.1 Observational corrections to the vertical angle

(a) *Refraction*

This phenomenon has already been discussed in *Section 2.8.3*. In astronomy the correction is usually taken as

$$r = -58'' \cot H \tag{4.14}$$

where H = observed altitude.

However, as r varies with temperature and pressure a more accurate value can be obtained from *Star Almanac* tables using

$$r = f \times r_o \tag{4.15}$$

where r_o is a correction under standard conditions of 100.5 kN/m^2 and 7.2°C and f = a factor varying with measured temperature and pressure. The refraction correction is applied to all heavenly bodies.

(b) *Parallax*

This correction is applicable only to the Sun and corrects the vertical angle to that value which would be obtained had it been measured from the centre of the Earth. From *Figure 4.29*, maximum parallax $\delta p = \tan(R/D) = 9''$ of arc; the parallax correction used is then

$$\delta p = +9'' \cot H \tag{4.16}$$

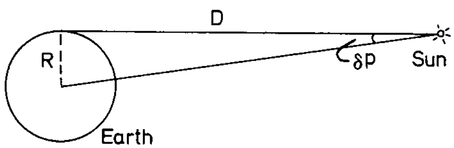


Figure 4.29

(c) *Semi-diameter*

Accurate bisection of the Sun's centre is not possible due to its relatively large viewing size, unless the theodolite is fitted with a Roelof solar prism. The solar prism presents four images of the Sun, overlapping to give a cross-shaped image, the centre of which is the Sun's centre.

Normally, the observation is to the Sun's lower limb A as in *Figure 4.30*, in which case the altitude will be too low by the semi-diameter. Observations to the upper limb B will similarly be too high. The value for semi-diameter varies from 16′ 18″ to 15′ 45″ and can be obtained from the *Star Almanac*.

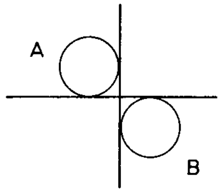


Figure 4.30

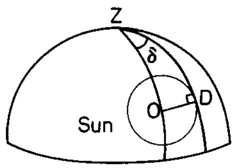


Figure 4.31

It can also be seen that as the vertical cross-hair touches the left or right limb of the Sun, the horizontal angle will also require a semi-diameter correction. From *Figure 4.31*, as the limb is observed tangentially, spherical triangle ZDO is right-angled at D.

By Napier's rules $\sin\delta = \sin OD \operatorname{cosec} ZO$

as δ is small $ZO \approx ZD$ and $\delta = OD \operatorname{cosec} ZD$

Hence, as ZD is the co-altitude, then

$$\delta = \text{Semi-diameter} \times \text{Secant altitude} \tag{4.17}$$

In practice these corrections are eliminated by observing both limbs (A and B) on alternate faces of the instrument and taking the mean.

(d) *Vertical-axis error*

The error in the horizontal bearing of a line due to the axis of the instrument being inclined at e to the vertical is $e \tan H$ (see Volume 1). This effect is not eliminated by changing face and must be applied as a correction.

The value of e is determined by reading the left (L) and right (R) ends of the plate bubble as viewed by the observer.

Then $e = (\sum L - \sum R)d/n$

where d is the value per division of the bubble, and n is the number of readings on the ends of the bubble. For instance

	(L)	(R)	
Face left (FL)	3.5	1.5	bubble readings
Face right (FR)	2.0	3.0	
$\sum$	5.5	4.5	

If $d = 20''$ then $e = \dfrac{(5.5 - 4.5)20''}{4} = 5''$

Thus, if the altitude $H = 50°$ then the correction is $+5'' \tan 50°$.

4.5 METHODS OF DETERMINING LATITUDE

4.5.1 Meridian altitudes of Sun or star

This method requires the altitude of a body at the instant of transit on the observer's meridian.

If the Sun was used, the instant of transit would be attained by following the Sun through the theodolite until maximum elevation was obtained. This would give a single-face observation to the Sun's extreme limbs. To avoid this a rapid double-face observation could be made and the mean taken, ignoring the slight movement off transit.

Alternatively, knowing the longitude of the station and the appropriate value for E, the UT (GMT) of transit can be computed, and observation carried out at this time.

Now by constructing a section through the observer's meridian, the latitude of the observer is easily deduced. *Figure 4.32* is a reproduction of *Figure 4.10* in two parts, (*a*) represents the northern hemisphere and (*b*) the southern. The student should study

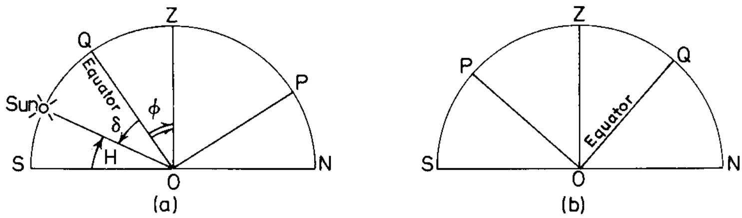


Figure 4.32

Figure 4.10 in conjunction with *Figure 4.32*. If the altitude H and declination δ are now shown on the diagram, it is easy to deduce the latitude $\phi = 90° - (H + \delta)$. Frequently the altitude is specified as north or south; this infers that it is measured from the north or south direction or limb in the diagram. For instance, in *Figure 4.32* (*a*) the altitude is south, the declination south and the latitude north of the Equator. Refer to *Worked example 4.13*.

4.5.2 Zenith pairs of stars

The technique outlined in *Section 4.5.1* can be used with a single star to obvious advantage. The centre of the star is bisected directly and the corrections for semi-diameter and parallax eliminated. However, as theoretically only a single face pointing is possible, refraction and vertical collimation error of the instrument must be considered. Errors in these two corrections will produce subsequent error in the latitude.

By using two stars at roughly the same altitude (zenith pair), these errors are cancelled. Consider two such stars S_1 and S_2 in *Figure 4.33*

Using S_1 $\qquad \phi = 90° - H_1 + \delta_1$

Using S_2 $\qquad \phi = H_2 + \delta_2 - 90°$

Adding $\qquad 2\phi = (\delta_1 + \delta_2) + (H_2 - H_1)$

Now H_2 and H_1 are the observed altitudes corrected for vertical collimation error e and refraction r. As ϕ is a function of the difference of the two altitudes and the altitudes are similar, then the errors e_1 and e_2 will be equal and thus cancel out. Similarly, as the observations will be close together, the errors e_r in the refraction corrections r, will be

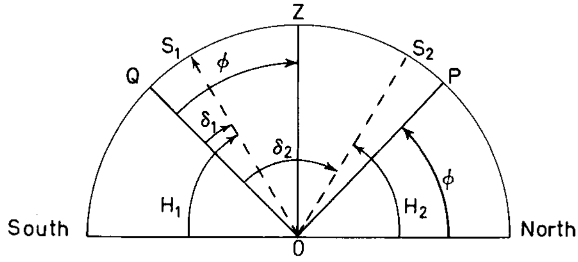


Figure 4.33

virtually equal and so cancel out. If the refraction corrections themselves are equal, i.e. $r_1 = r_2$, they will also be cancelled.

The advantages of zenith pair observations to stars which transit in close succession are clearly seen.

4.5.3 Close circumpolar stars

This method involves observations of stars that are close to the poles. In the northern hemisphere this would be Polaris and in the southern hemisphere Octantis.

The mean altitude H of several double face pointings is taken and the mean chronometer time deduced from the chronometer times of the instants of observation. From the latter it is possible to calculate the LHA of Polaris h, and the declination is taken from the *Star Almanac*. Then, using

$$\phi = H - p\cos h + \tfrac{1}{2}p^2 \sin 1'' \sin^2 h \tan H$$

where p is the co-declination in seconds, the latitude may be found. This equation is derived basically from the cosine rule.

A rapid solution can be obtained using Pole Star Tables in the *Star Almanac* where the latter part of equation (4.18) is tabulated as a_0, with corrections a_1 and a_2, then $\phi = H + a_0 + a_1 + a_2$. The table for a_0 is entered with the argument LST. (Refer to *Worked example 4.15*.)

WORKED EXAMPLES

Example 4.13. A certain star is observed at upper transit to be at an altitude of 51° 17′ 47″ in the southern sky. The RA of the star is 6 h 30 m 17 s and the declination 11° 38′ 55″ S. If the GMT of the observation is 20 h 6 m 19 s and the value of R at this instant is 8 h 10 m 33 s, determine the latitude and longitude of the observer and the LMT of observation. (ICE)

Construct a semicircle with the diameter as the north-south plane of observation (see *Figure 4.34*) and thus Z at 90° to this. Where one places north and south is immaterial. As the altitude H is south, the position of the star is fixed at S in the southern quadrant. As the declination δ is south, the position of the equator OQ is fixed. Then

$$\phi = 90° - \delta - H$$

where H is the observed altitude corrected for refraction r.

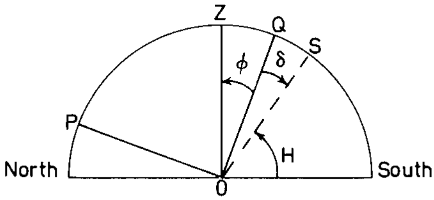


Figure 4.34

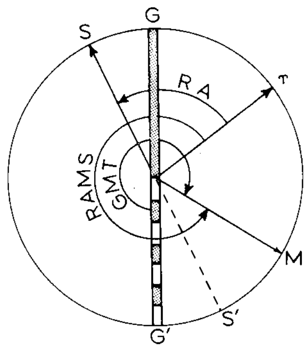


Figure 4.35

$\therefore$ Altitude $(H_o) = 51° 17' 47''$
$r = -58'' \cot H_o = \quad -46.5''$

$\therefore H = 51° 17' 00.5''$

$\therefore$ Latitude $\phi = 90° - 11° 38' 55'' - 51° 17' 00.5''$
$= 27° 04' 04.5''$ N

N.B. From *Figure 4.34*, *Z* is measured towards the north from *OQ*, hence it is a northern latitude.

Construct a time diagram with the Greenwich meridian *GG′* (*Figure 4.35*). GMT 20 h 6 m 19 s measured clockwise from *G′* will fix the position of *M* (mean Sun). *R* = 8 h 10 m 33 s, thus RAMS = *R* + 12 h = 20 h 10 m 33 s measured anticlockwise from ♈ to *M*, thus the relative position of ♈ is fixed.

RA of the stars is 6 h 30 m 17 s; measured anticlockwise from ♈ fixes *S*. As *S* is at upper transit, then this position is the observer's meridian. From *Figure 4.35*

MG′ = 24 h − GMT = 3 h 53 m 41 s
G♈ = RAMS − 12 h − *MG′* = 4 h 16 m 52 s
∴ *GS* = RA − *G*♈ = 2 h 13 m 25 s E = 33° 21′ 15″ E

LMT is measured clockwise from *S′* to *M* = GMT + *G′S′*
= 22 h 19 m 44 s

Example 4.14. Zenith pair observations are used to find the latitude of an observer in the southern hemisphere. The necessary data are given below.

Star	*Declination*	°	′	″	*Observed altitude* °	′	″	*RA*
1	North	19	58	20	50	01	20	13 h 12 m 02.0 s
2	South	61	07	00	48	55	30	13 h 20 m 02.0 s

If the value of *R* at UT (GMT) 12 h is 16 h 35 m 20 s, determine the UT of transit of the stars on the observer's meridian east 60° 15′ 15″. What is the local standard time (LStT) if the standard meridian of the place is 15° 30′ west of the observer? (KP)

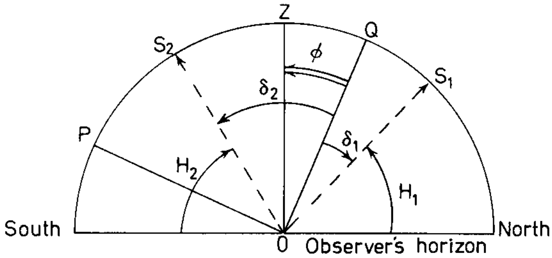


Figure 4.36

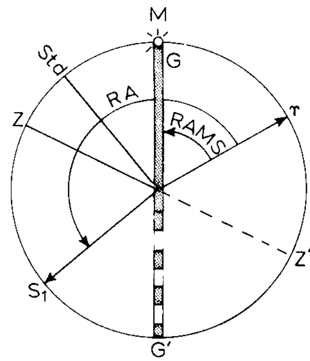


Figure 4.37

Figure 4.36 is constructed starting with the observer's horizon and Z, P and Q are constructed so that Z is in the southern hemisphere. The values for δ and H will fix the position of the stars as shown

Refraction correction for $S_1 = -58'' \cot 50° \, 01' \, 20'' = -49''$
Refraction correction for $S_2 = -58'' \cot 48° \, 55' \, 30'' = -51''$

$\therefore \; H_1 = 50° \, 01' \, 20'' - 49'' = 50° \, 00' \, 31''$
and $\; H_2 = 48° \, 55' \, 30'' - 51'' = 48° \, 54' \, 49''$

Then from *Figure 4.36*

$$\phi_1 = 90° - \delta_1 - H_1 = 20° \, 01' \, 09'' \text{ S}$$
$$\phi_2 = \delta_2 + H_2 - 90° = 20° \, 01' \, 49'' \text{ S}$$
$$\therefore \; \phi = 20° \, 01' \, 29'' \text{ S} \quad \text{(mean)}$$

Construct a time diagram as in *Figure 4.37*. First construct GG' then from the value of R the RAMS = 4 h 35 m 20 s, which fixes ϒ relative to M at UT 12 h. The RA measured anticlockwise from ϒ fixes S_1. S_1 can transit at Z by moving clockwise for roughly 4 h, or anticlockwise for roughly 20 h. As the UT is 12 h, moving anticlockwise will give an approximate UT of 8 h on the next day, thus S_1 is moved clockwise.

From the diagram $\qquad S_1Z = \text{RA} - \text{RAMS} - \text{Longitude} = 4 \text{ h } 35 \text{ m } 41 \text{ s}$

If S_1 moves forward this amount, M will move the equivalent amount in solar time = 4 h 34 m 56 s.

$\therefore$ UT of transit of S_1 = 16 h 34 m 56 s

From the RA it can be seen that S_2 will transit 8 min later than S_1

$\therefore$ UT of transit of S_2 = 16 h 42 m 56 s

If the UT is as above, then the LMT measured from Z' will obviously be greater by the longitude (4 h 01 m 01 s)

$\therefore$ LMT of transit of S_1 = 20 h 35 m 57 s

As the standard meridian (Std) is 1 h 02 m west of Z, the time there will be earlier by this amount.

$\therefore$ LStT of transit of S_1 = 19 h 33 m 57 s

Example 4.15. An observer, at longitude 5° 15′ 00″ west, obtained the following mean field data to Polaris: mean observed altitude = 52° 46′ 18″; mean chronometer time of observation = 11 h 56 m 04 s; chronometer fast on UT by 7 h 36 m 04 s; mean barometer reading = 760 mm; mean temperature = 12.0°C.

The remaining data required were taken from the *Star Almanac* for the date in question

Declination = 89° 08′ 08″ north R = 18 h 47 m 25.0 s
RA = 2 h 04 m 17 s r_0 = 44″ f = 1.04

Determine the latitude of the observer, given

$$\phi = H - p\cos h + \tfrac{1}{2}p^2 \sin 1'' \sin^2 h \tan H \qquad \text{(KP)}$$

(1) From a time diagram find the LHA of Polaris (*Figure 4.38*).

From R RAMS = 6 h 47 m 25.0 s

ZZ' is the observer's longitude

UT = 11 h 56 m 04 s − 7 h 36 m 04 s = 4 h 20 m 00 s = $G'M$

RAMS measured from M fixes the position of ♈

∴ ♈G = 12 h − UT − RAMS = 0 h 52 m 35 s
GZ = Longitude = 0 h 21 m 00 s

then LHA = 24 h − RA − ♈Z = 20 h 42 m 08 s
or = 3 h 17 m 52 s E = 49° 28′ 00″ E = h

(2) Observed altitude = 52° 46′ 18″
$r_0 \times f$ = −46″

Corrected altitude = 52° 45′ 32″ = H

(3) δ = 89° 08′ 08″, therefore co-declination = p = 0° 51′ 52″

Now, substituting in the given equation ϕ = 52° 12′ 08″ N

Example 4.16. A star of declination 47° 20′ 17″ S is observed at upper transit at an altitude of 56° 48′ 41″ N. What will be the observed altitude at lower transit and what is the latitude of the place of observation?

From this latitude what will be the observed altitude of the top of the Sun at transit, if the declination of the Sun is 22° 22′ 25″ S and the Sun's semi-diameter is 16′ 18″?

Explain why you could not use any of the observations alone to determine the azimuth of a survey line with any accuracy. What method would you use? (ICE)

Figure 4.39 illustrates the situation with S at upper transit S_1. The observed altitude corrected for refraction = 56° 48′ 06″ N.

∴ $\phi = 90° - H + \delta$ = 80° 32′ 11″ S

S at lower transit is indicated by S_2, thus

$S_1OP = S_2OP = (90° - \delta)$ = 42° 39′ 43″
$ZOP = (90° - \phi)$ = 9° 27′ 49″

∴ ZOS_2 = 52° 07′ 32″

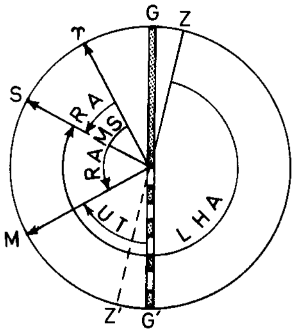


Figure 4.38

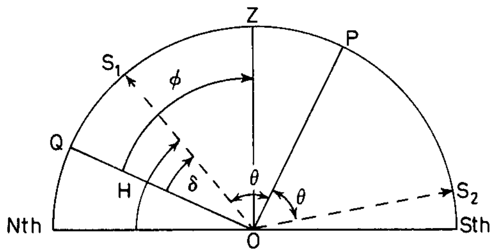


Figure 4.39

and the altitude at lower transit $= 90° - ZOS_2 = 37° 52' 28''$

$\therefore$ Observed altitude $= 37° 52' 28'' +$ Refraction $= 37° 53' 44''$ S

Similarly, altitude to Sun's centre at upper transit is

$90° - \phi + \delta_s = 31° 50' 14''$ N

This value must now be brought to the observed value by applying the following corrections: semi-diameter $= +16' 18''$; refraction $= +1' 33''$; parallax $= -08''$.

$\therefore$ Observed altitude at upper transit $= 32° 07' 57''$ N, and
Observed altitude at lower transit $= 13° 15' 00''$ S

EXERCISES

4.4. The following Table gives data relating to a zenith-pair meridian observation for latitude:

Star	Declination ° ′ ″	Observed altitude ° ′ ″	Level: Object	Level: Eye
X	18 42 38 S	58 32 47 N	6.2	3.8
Y	83 50 22 S	56 21 18 S	3.6	6.4

Calculate the latitude of the observing station, taking the refraction correction as $-57''$ cot altitude, and one division of the bubble to be equal to 15″. What is the advantage of observing stars which culminate on opposite sides of the zenith? (LU)

(*Answer:* 50° 10′ 24″ S)

N.B. The level readings refer to the altitude bubble giving vertical circle index corrections of $+18''$ (*X*) and $-21''$ (*Y*) from $\left(\frac{o - e}{n} \times d''\right)$, where *d* is the value of one division of the bubble, and *n* the *N*° of readings.

4.5. The right ascensions and declinations of two stars to be used to determine the latitude of a place whose longitude is 30° 50′ 39″ E and whose approximate latitude is 20° S are given below:

Star	RA	Declination ° ′ ″
1	10 h 17 m 56.5 s	20 01 45 N
2	10 h 30 m 43.8 s	61 30 00 S

If the value of R at the previous Greenwich mean noon is 10 h 33 m 48.2 s determine the GMT of local transit of the stars.

The maximum observed altitudes of stars 1 and 2 at transit are 49° 55′ 23″ and 48° 34′ 32″, respectively. What is the latitude of the place? (LU)

(*Answer:* 2 h 16 m 53.5 s; 2 h 29 m 38.7 s; 20° 03′ 41″ S)

4.6. The following are the recorded meridian altitudes of six stars and their declinations:

Star	Meridian altitude ° ″ ″	Declination ° ′ ″
X_1	30 12 02 N	69 49 55 N
X_2	30 41 57 S	8 59 10 S
X_3	47 26 32 N	87 05 10 N
X_4	47 25 42 S	7 45 20 N
X_5	57 29 17 N	82 51 50 N
X_6	57 00 27 S	17 20 20 N

It may be assumed that the altitudes are free from instrumental errors. From these observations deduce the effects of refraction and compare your results with the usual values. (LU)

(*Answer:* ±97″, ±52″, ±37″)

Hint: compare the mean latitude per pair with the individual latitude.

4.6 DETERMINATION OF AZIMUTH

Apart from the basic idea of orienting a base line of a survey relative to the meridian, the determination of azimuth as an aid to controlling azimuth error has become even more important with the increase in popularity of EDM traversing (see Chapter 2).

All the methods used require the horizontal angle, between the base line and the observed body, combined with either the altitude or the time of observation. This enables the necessary elements of the astro-triangle to be found and used in a solution for the azimuth angle at Z (*Figure 4.14*).

4.6.1 Ex-meridian observations to Sun or star, measuring 'altitude'

Considering first a Sun observation. The observer sets up the instrument at one end of the base line and bisects the reference object (RO) at the other end. The instrument is then swung to the Sun and the mean altitude H and horizontal angle from base line to Sun recorded. The mean value is recorded from at least three double-face readings to alternate limbs of the Sun.

The time of the instant of observation is recorded, to the nearest minute, for the purpose of abstracting declination δ from the *Star Almanac*. Temperature and pressure are recorded for the computation of refraction. The observer's latitude ϕ is obtained from a map or by observation. Thus the three sides of the astro-triangle are known, i.e. co-altitude (co-alt), co-declination (co-dec) and co-latitude (co-lat), enabling a solution for angle Z by the cosine rule (see *Worked example 4.17*).

The obvious advantages of using a star are that: (i) direct bisection is possible; (ii) no corrections for parallax or semi-diameter are necessary; (iii) time of observation is not required since declination is constant for that day; (iv) it is possible to observe at or near elongation, which can be shown to be the best condition for reducing the effect of errors in δ, H or ϕ.

4.6.2 Ex-meridian observations to Sun or star, measuring 'time'

This technique is similar to the previous one; the exact chronometer time of the observation, however, replacing the measured altitude. A rough value for latitude may be necessary (to the nearest degree) for the computation of instrument corrections.

From the UT of the observation and the longitude of the observer, the LHA h is computed, and the declination δ taken from the *Star Almanac*. The latitude ϕ is found as in the previous method and the astro-triangle solved by the four-parts formula using h, δ and ϕ.

The advantages of using a star instead of the Sun have already been outlined. Error analysis shows that the effect of error in time is a minimum when angles S and δ are 90°. S will be 90° when the star is at elongation and δ approximates to 90° in the case of Polaris or Octantis. Thus observations to circumpolar stars near elongation are used when high precision is required.

If Polaris is used and accuracy to 0.2′ is acceptable, a rapid solution is possible from Pole Star Tables in the *Star Almanac*, where

$$\text{Azimuth of Polaris} = (b_0 + b_1 + b_2) \sec \phi$$

It is necessary to know the LST to take out b_0, the latitude to take out b_1, and the month of the year, to take out b_2.

A further advantage of using Polaris is its relatively slow movement in space due to its small circle of motion around the pole in 24 sidereal hours.

4.6.3 Star at or near elongation

If it is intended to observe the star at the exact instant of elongation (*Figure 4.11*) then a right-angled spherical triangle is formed. In such a case, as δ and ϕ will be known prior to the observation, it is possible to pre-calculate the time, azimuth and altitude of the star, from

$$\sin H = \frac{\sin \phi}{\sin \delta} \qquad \sin Z = \frac{\cos \delta}{\cos \phi} \qquad \sin h = \frac{\cos H}{\cos \phi}$$

Using this data the instrument can be pre-set some minutes before elongation. However, since in theory elongation occurs for only an instant, and time is required for several double-face readings, it is generally preferred to observe stars near elongation using either of the methods already outlined.

4.7 POSITION LINES

The method of position lines is a semi-graphical technique for the determination of latitude and longitude.

It requires the measurement of both time and altitude to at least two stars plus the approximate position of the observer.

4.7.1 Principle

Consider an observer measuring the altitude to a star whose position in space is fixed. If he now moves the instrument to an infinite number of positions such that the altitude remains the same, he will trace out a circle on the Earth's surface. This circle would have as its centre the position of the star projected vertically on to the Earth, and its angular radius equal to the co-altitude (*Figure 4.40*). If two stars are observed, two circles are

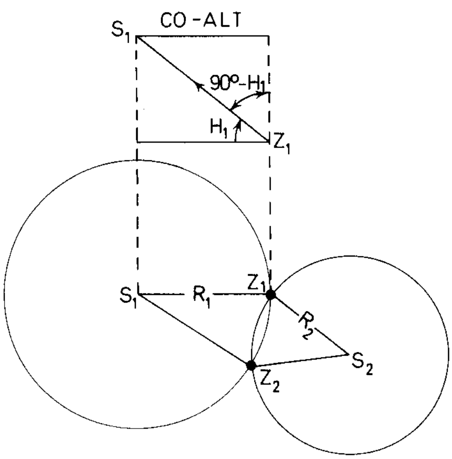


Figure 4.40

formed cutting at two places Z_1 and Z_2, one of which is the observer's position, i.e. there are only two places on the Earth's surface where the respective co-altitudes will be the same at a given instant. As the approximate position of the observer is known, the correct position is easily defined.

4.7.2 Method

Considering position Z_1 only. The intersecting arcs in the immediate vicinity are plotted to a large scale thus appearing as straight lines called *position* lines. The lines

Z_1S_1 and Z_1S_2 are the directions or azimuths of the stars relative to the observer at Z_1 and are normal to the position lines.

Thus the altitude and times of the stars are measured, and using the approximate value for ϕ, LHA (h) and δ the astro-triangle is solved for azimuth and altitude. Each star now has a calculated and measured altitude, plus an azimuth from the approximate position.

Next a meridian is drawn and a point Z_c selected on it to define the approximate position of the observer. Using a protractor, the azimuths of the stars are turned off (Z_cS_1, Z_cS_2) (*Figure 4.41*). The position lines are drawn at 90° to the azimuths, at points fixed using *Figure 4.42*. This shows that if the observed altitude H_o is greater than the

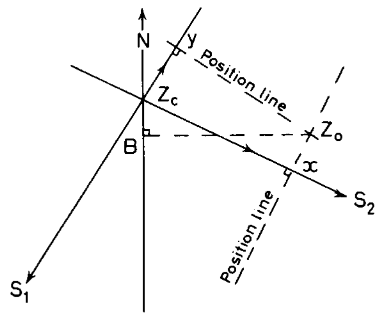


Figure 4.41

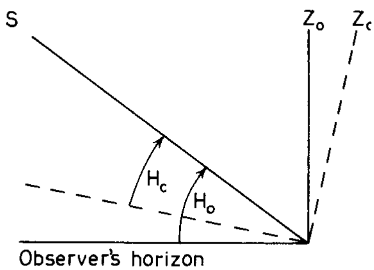


Figure 4.42

calculated altitude H_c, the true position of the observer Z_o is towards the star, measured from the approximate position Z_c. This distance Z_oZ_c, which is a linear function of $H_o - H_c$, is then plotted to scale towards the star fixing point x in *Figure 4.41*. Assuming $H_o < H_c$, the intercept would be measured from Z_c away from S_1, fixing point y. Position lines are drawn through x and y to intersect at Z_o, the observer's true position. This simple sketch as in *Figure 4.42* will indicate quickly whether the intercept is measured towards or away from Z_c.

Distance Z_cB is the difference in latitude ($\Delta\phi$) between the approximate and true positions of the observer and is scaled directly from the construction. Distance BZ_o is the difference in longitude $\Delta\lambda$. As $\Delta\lambda$ is a variable distance on the Earth's surface from Equator to the pole, it must be corrected as follows

$$\text{True } \Delta\lambda = BZ_o \sec \phi$$

Where three or four stars are used, the intersection of the position lines may give a triangle or quadrilateral of error, the required position lying at the centre of a circle inscribed within the figure. The student is now advised to consider the principles outlined, in conjunction with a careful study of the *Worked examples*.

WORKED EXAMPLES

Example 4.17. The mean values of a Sun observation for azimuth are: mean observed altitude = 38° 21′ 55″; mean circle reading to Sun = 13° 01′ 35″; mean circle reading to reference target (RT) = 64° 44′ 27″; the difference of plate level readings from a double face observation = +0.3 (1 division = 20″); the latitude of the observer = 51° 25′ 18″

N. From the *Star Almanac* the following additional data were taken: declination = 3° 01′ 36″ N; refraction = 01′ 10″; parallax = 06″. Find the azimuth of the line to the RT if the Sun was in the south-east at the instant of observation.

What would be the effect on the azimuth of an error in the measured altitude? What does the resulting error equation indicate? (KP)

Observed altitude	=	38° 21′ 55″
Refraction	=	−1′ 10″
Parallax	=	+6″

∴ Corrected altitude = H = 38° 20′ 51″

The remaining elements of the astro-triangle are given, i.e. ϕ (latitude) and δ (declination), permitting its solution by the cosine rule. The student should draw the astro-triangle PZS and indicate its various elements.

$$\cos(\text{co-dec}) = \cos(\text{co-lat})\cos(\text{co-alt}) + \sin(\text{co-lat})\sin(\text{co-alt})\cos\hat{Z}$$

$$\therefore \cos\hat{Z} = \frac{\sin\delta - \sin\phi\sin H}{\cos\phi\cos H} = \frac{\sin 3^\circ\,01'\,36'' - \sin 51^\circ\,28'\,18''\sin 38^\circ\,20'\,51''}{\cos 51^\circ\,28'\,18''\cos 38^\circ\,20'\,51''}$$

$$\therefore \hat{Z} = 152^\circ\,09'\,10''$$

This is the angle PZS, but as the Sun is in the south-east it is also the azimuth of the Sun.

Clockwise angle between Sun and RT = 51° 42′ 52″

∴ Azimuth to RT = 203° 52′ 02″

The effect of an error in altitude is obtained in the usual way by differentiating the basic equation with respect to H

$$\cos\hat{Z} = \frac{\sin\delta - \sin\phi\sin H}{\cos\phi\cos H}$$

$$\therefore -\sin Z\,\delta Z = \frac{\cos\phi\cos H(-\sin\phi\cos H) - (\sin\delta - \sin\phi\sin H)(-\cos\phi\sin H)\,\delta H}{\cos^2\phi\cos^2 H}$$

$$= \frac{-\sin\phi\cos^2 H + \sin\delta\sin H - \sin\phi\sin^2 H\,\delta H}{\sin Z\cos\phi\cos^2 H}$$

From the cosine rule $\sin\phi - \sin\delta\sin H = \cos\delta\cos H\cos S$

$$\therefore \delta Z = \frac{\cos\delta\cos H\cos S\,\delta H}{\sin A\cos\phi\cos^2 H}$$

From the sine rule $\sin A\cos\phi = \sin S\cos\delta$

$$\therefore \delta Z = \frac{\cos\delta\cos H\cos S\,\delta H}{\sin A\cos\phi\cos^2 H} = S\sec H\,\delta H$$

This indicates that δZ will be a minimum as S approaches 90°, i.e. Sun or star at elongation, and H is a low altitude.

Example 4.18. From a place of longitude 80° 20′ 15″ W the star Polaris was observed in order to find the latitude of the place and the azimuth of a line from the instrument to a reference target (RT). The mean times and angles are given below:

Sight	*Horizontal angle* ° ′ ″	*Mean altitude* ° ′ ″	*Mean time GMT*
RT	246 13 22	—	—
Polaris	63 41 16	42 31 40	05 h 01 m 43.6 s

Also: declination of Polaris = 89° 04′ 24″ N; RA of Polaris = 1 h 55 m 49.2 s; R = (GST − GMT) = 18 h 17 m 18.0 s; refraction correction = −1′ 03″. Calculate the latitude of the place and the azimuth of the line to the reference target. (LU)

This problem gives the values of H and δ, and it is required to solve the astro-triangle for Z and ϕ. It is obvious then that a further element of the triangle is required, which must be LHA (h).

Construct a time diagram (*Figure 4.43*) showing Z at 5 h 21 m 21 s west. GMT fixes the position of M. R = 18 h 17 m 18 s, thus RAMS = 6 h 17 m 18 s measured in effect anticlockwise from ϒ to M, fixes the relative position of ϒ. RA of Polaris fixes the position of S; then LHA is ZS.

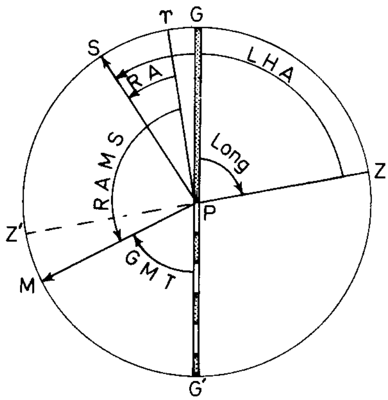


Figure 4.43

From *Figure 4.43* ϒG = 12 h − (GMT + RAMS) = 0 h 40 m 58.4 s

∴ LHA (h) = Longitude + ϒG + RA = 7 h 58 m 08.6 s E

∴ h = 119° 32′ 01″ E

Corrected altitude = H = 42° 30′ 37″

Declination = δ = 89° 04′ 24″

Now by the sine rule

$$\sin Z = \frac{\sin h \sin(90° - \delta)}{\sin(90° - H)} = \frac{\sin h \cos \delta}{\cos H} = \frac{\sin 119° 32' 09'' \cos 89° 04' 24''}{\sin 42° 30' 37''}$$

∴ $\hat{Z}$ = 1° 11′ 35.6″ (east, as indicated by the LHA)

Clockwise angle from Polaris to RT = 182° 32′ 06″

∴ Azimuth of RT = 183° 43′ 41.6″

The latitude could now be found by using equation (4.18) for ‘close circumpolar stars’

(*Section 4.5.3*). However, as the azimuth angle is available, the astro-triangle may be solved using

$$\tan \tfrac{1}{2}(x + y) = \frac{\cos \frac{1}{2}(X - Y)}{\cos \frac{1}{2}(X + Y)} \times \tan\left(\frac{Z}{2}\right)$$

This equation is generally used in conjunction with

$$\tan \tfrac{1}{2}(x - y) = \frac{\sin \frac{1}{2}(X - Y)}{\sin \frac{1}{2}(X + Y)} \times \tan\left(\frac{Z}{2}\right)$$

when both sides x and y are required.

Using the former gives

$$\tan \tfrac{1}{2}(\text{co-alt} + \text{co-dec}) = \frac{\cos \frac{1}{2}(\hat{h} - \hat{Z})}{\cos \frac{1}{2}(\hat{h} + \hat{Z})} \times \tan \tfrac{1}{2}(\text{co-lat})$$

$$\therefore \quad \frac{\tan 24^\circ\, 12'\, 30'' \cos 60^\circ\, 21'\, 52''}{\cos 59^\circ\, 10'\, 16''} = \tan \tfrac{1}{2}(90^\circ - \phi)$$

$$\therefore \quad (90^\circ - \phi) = 46^\circ\, 54'\, 12''$$

$$\therefore \quad \text{Latitude } \phi = 43^\circ\, 05'\, 48'' \text{ N}$$

Example 4.19. From a theodolite station observations were made on each limb of the rising Sun and also on a reference target (RT). The latitude of the station had previously been determined as 50° 29′ 25″ N. The following Table gives the mean values of the observed readings:

Sight	*Horizontal circle* ° ′ ″	*Observed altitude* ° ′ ″	*GMT*
Sun	215 44 55	30 43 45	15 h 32 m 18.6 s
RT	043 31 15	—	—

The appropriate value of E was 11 h 59 m 20 s, the corrections for refraction and parallax were 1′ 38″ and 8″, respectively, and the Sun's declination was 9° 00′ 40″ N.

Calculate the longitude of the station and the azimuth of the line joining the station and the RT. (LU)

$$\begin{aligned} \text{Corrected altitude } H &= 30^\circ\, 43'\, 45'' - 1'\, 38'' + 8'' = 30^\circ\, 42'\, 15'' \\ \text{Latitude } \phi &= 50^\circ\, 29'\, 25'' \\ \text{Declination } \delta &= 9^\circ\, 00'\, 40'' \end{aligned}$$

Thus knowing the three sides of the astro-triangle, the azimuth angle A at Z and the LHA (h) may be found. By the cosine rule

$$\cos(90^\circ - \delta) = \cos(90^\circ - \phi)\cos(90^\circ - H) + \sin(90^\circ - \phi)\sin(90^\circ - H)\cos A$$

$$\therefore \quad \cos A = \frac{\sin \delta - \sin \phi \sin H}{\cos \phi \cos H} = \frac{\sin 9^\circ\, 00'\, 40'' - \sin 50^\circ\, 29'\, 25'' \sin 30^\circ\, 42'\, 15''}{\cos 50^\circ\, 29'\, 25'' \cos 30^\circ\, 42'\, 15''}$$

$$\therefore \quad \cos A = -0.433\,829 \qquad \text{and hence} \qquad A = 115^\circ\, 42'\, 39''$$

This angle could obviously be measured east or west from north. However, as the

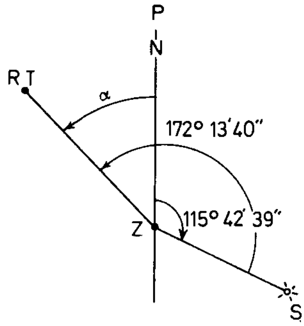


Figure 4.44

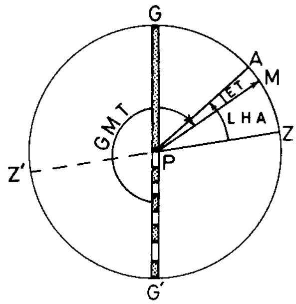


Figure 4.45

question states a rising Sun observed from a northern latitude, then the Sun must be in the south-east (*Figure 4.44*).

$\therefore$ Azimuth of RT = N α W = N 56° 31′ 01″ W = 307° 28′ 59″

A second application of the cosine rule will now give h

$\cos h = \sin H - \sin\phi \sin\delta/\cos\phi \cos\delta = 0.620\,292$
$\therefore\ h = 38°\,20'\,15'' = 2\text{ h } 33\text{ m } 21\text{ s E}$

Now construct the time diagram (*Figure 4.45*) with GG' as the Greenwich meridian. GMT fixes the position of M (mean Sun). E = 11 h 59 m 20 s, therefore from equation (4.10) E = 12 h − ET. The ET = +40″, thus M is ahead of A (actual Sun) by this amount. As the LHA (h) is east of the observer's meridian, this fixes the position of Z in the west, as shown

$\therefore$ Longitude $GZ = GM - AM + AZ$
= 3 h 32 m 18.6 s − 40 s + 2 h 33 m 21 s
= 6 h 04 m 59.6 s W
= 91° 14′ 46.3″ W

Example 4.20. The mean values of time, horizontal and vertical circle readings of an observation on Polaris are given in the table. The vertical angle has already been corrected for refraction. At the time of observation, R was 10 h 47 m 06.8 s and RA and declination of Polaris were 1 h 55 m 26.3 s and 89° 04′ 52″ N.

Sight	*Mean corrected vertical angle* ° ′ ″	*Mean horizontal angle* ° ′ ″	*Mean time (GMT)*
Target		213 04 35	
Star	43 43 00	146 38 20	0 h 33 m 30.7 s

If the longitude of the station was assumed to be 21° 52′ 40″ W, find the azimuth of the line joining the theodolite station and the RT. (LU)

δ and H are the only elements of the astro-triangle supplied directly, a further element is thus required which is obviously the LHA (h). Then from *Figure 4.46*: longitude fixes the position of Z and Z'; GMT fixes the position of M (mean Sun); R =

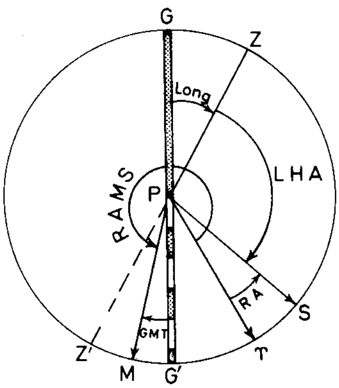


Figure 4.46

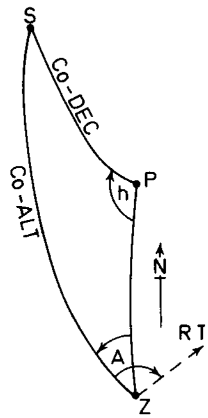


Figure 4.47

10 h 47 m 06.8 s, therefore RAMS = 22 h 47 m 06.8 s, fixing ♈; RA fixes the position of S (Polaris).

$\therefore\ \Upsilon G' = 24\text{ h} - (\text{RAMS} + \text{GMT}) = 0\text{ h } 39\text{ m } 23.3\text{ s}$

and $\quad \text{LHA }(h) = ZS = 12\text{ h} - \Upsilon G' - \text{RA} - \text{Longitude}$
$\quad = 8\text{ h } 01\text{ m } 40\text{ s W} = 120° 25' 00''\text{ W}$

The resultant astro-triangle (*Figure 4.47*) can now be solved by the sine rule

$$\sin A = \frac{\sin h \sin(90° - \delta)}{\sin(90° - H)} = \frac{\sin h \cos \delta}{\cos H}$$

$$\therefore\ \sin A = \frac{\sin 120° 25' 00'' \times \cos 89° 04' 52''}{\cos 43° 43' 00''} = 0.019\,135$$

$\therefore\ A = 01° 05' 47''$ (west of north as defined by LHA)

From the horizontal angles supplied, the reference target (RT) is 66° 26′ 15″ clockwise of the star (*Figure 4.47*).

$\therefore$ Azimuth of RT = 66° 26′ 15″ − 01° 05′ 47″ = 65° 20′ 28″

Example 4.21. Two stars are observed at elongation from a station A in a northern latitude as follows:

Star	*Declination* °	′	″		*Clockwise angle from AB* °	′	″
S_1	+56	40	50	in the west	20	10	19
S_2	+76	07	48	in the east	104	17	10

Determine the azimuth of AB and then the latitude. (LU)

As the latitude ϕ is not supplied, the azimuth per star cannot be found from $\sin Z = \cos \delta/\cos \phi$, and meaned. The following method is adopted which, as it precludes ϕ, is not adversely affected by error in ϕ and is thus more accurate.

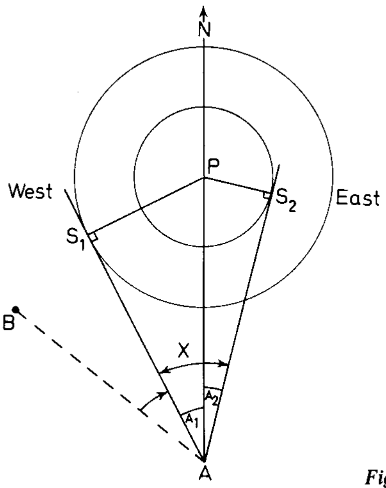


Figure 4.48

The situation is shown in *Figure 4.48*

$$\sin A_1 = \cos\delta_1/\cos\phi \qquad \text{and} \qquad \sin A_2 = \cos\delta_2/\cos\phi$$

$$\therefore \sin A_1/\sin A_2 = \cos\delta_1/\cos\delta_2 = K \quad \text{(a known constant)}$$

Now $\quad A_1 + A_2 = X$

and $\quad \sin A_1 = \sin(X - A_2) = \sin X \cos A_2 - \cos X \sin A_2$

but $\quad \sin A_1 = K \sin A_2$

$$\therefore K = \sin X \cot A_2 - \cos X$$

$$\therefore \cot A_2 = \frac{K + \cos X}{\sin X}$$

But $\quad K = \dfrac{\cos 56° \, 40' \, 50''}{\cos 76° \, 07' \, 48''} = 2.291\,453$

and from *Figure 4.48*

$$X = 104° \, 17' \, 10'' - 20° \, 10' \, 19'' = 84° \, 06' \, 51''$$

$$\therefore \cot A_2 = \frac{2.291\,453 + \cos 84° \, 06' \, 51''}{\sin 84° \, 06' \, 51''}$$

$$\therefore A_2 = 22° \, 33' \, 48'' \quad \text{(the azimuth of } S_2\text{)}$$

$$\therefore \text{Azimuth } AB = 22° \, 33' \, 48'' - 104° \, 17' \, 10'' = 278° \, 16' \, 38''$$

Example 4.22. From a place of approximate latitude and longitude 52° N and 1° W, it is intended to observe four stars to obtain a position line fix. The stars are each to have an altitude of about 50° and their azimuths are to be about 45°, 135°, 225° and 315°. If the time of observation is to be about 01 hours GMT when the approximate value of R (GST − GMT) will be about 8 h 34 m, calculate the declinations (δ) and right ascensions (RA) of the stars to the nearest degree and five minutes, respectively.

The nearest star suitable for the south-eastern observation has a right ascension of 11 h 12 m and a declination of 20° 44′ N. At what approximate GMT will its altitude be 50° and what will then be its approximate azimuth?

Three-figure accuracy is considered to be adequate in this question. (LU)

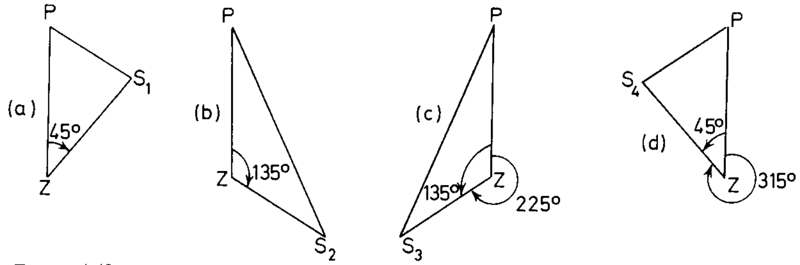


Figure 4.49

Figure 4.49 illustrates the azimuths of the four stars and shows that the solution of the astro-triangle will be identical for stars S_1, S_4 and S_2, S_3. In addition the latitude and altitude are given, then

By the cosine rule $\quad \sin \delta = \sin H \sin \phi + \cos H \cos \phi \cos A$

Stars S_1 and S_4 $\quad \sin \delta = \sin 50° \sin 52° + \cos 50° \cos 52° \cos 45°$

$\therefore \ \delta = 19°$ N

Stars S_2 and S_3 $\quad \sin \delta = \sin 50° \sin 52° + \cos 50° \cos 52° \cos 135°$

$\therefore \ \delta = 62°$ N

To find the RA the HA of the stars must first be computed.

Stars S_1 and S_4 (using the four parts equation)

$$\sin A \cot h = \sin(90° - \phi) \cot(90° - H) - \cos(90° - \phi) \cos A$$

$$\therefore \ \cot h = \frac{\cos 52° \tan 50° - \sin 52° \cos 45°}{\sin 45°}$$

$\therefore \ h = 76° = 5$ h 5 m E for S_1; W for S_4

Stars S_2 and S_3 $\quad \cot h = \dfrac{\cos 52° \tan 50° - \sin 52° \cos 135°}{\sin 135°}$

$\therefore \ h = 1$ h 55 m E for S_2; W for S_3

To find the RA construct a time diagram in the usual way (*Figure 4.50*). Longitude 4 m W fixes Z relative to G. GMT measured clockwise from the antipodes fixes M. $R = 8$ h 34 m, thus RAMS = 20 h 34 m which serves to fix the position of ♈. For star S_1, the LHA of 5 h 5 m east measured from Z fixes S_1. Then RA of S_1 is the angle measured anticlockwise from ♈

$MZ = 11$ h 4 m

$\therefore \ Z♈ = \text{RAMS} - MZ = 9$ h 30 m

$\therefore$ RA of $S_1 = Z♈ + \text{LHA} = 14$ h 35 m

As the LHA of S_4 is 5 h 5 m W

then $\quad$ RA of $S_4 = 14$ h 35 m $-$ 10 h 10 m $= 4$ h 25 m

Stars S_2 and S_3 can be fixed on the diagram using their respective HA

$\therefore$ RA of $S_2 = 11$ h 25 m $\qquad$ RA of $S_3 = 7$ h 35 m

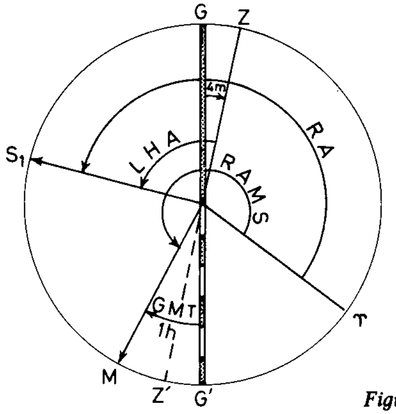


Figure 4.50

In the second part of the question the value of δ is changed for S_2

$$\cos A = \frac{\sin \delta - \sin H \sin \phi}{\cos H \cos \phi} = \frac{\sin 20° 44' - \sin 50° \sin 52°}{\cos 50° \cos 52°}$$

$\therefore \ A = 129°$

Similarly $\quad \cos h = \dfrac{\sin 50° - \sin 20° 44' \sin 52°}{\cos 20° 44' \cos 52°} \qquad \therefore \ h = 2\text{ h } 05\text{ m E}$

The student should attempt a time diagram for himself using h = 2 h 05 m; RA = 11 h 12 m; R = 8 h 34 m, giving

GMT = 0 h 37 m

Example 4.23. Observations were taken on four stars to determine the latitude and longitude of a station, whose assumed position was 50° 30′ N, 104° 37′ W. The azimuths and altitudes for the times of observation were calculated on the basis of the assumed position. These values are shown below together with the measured altitude. Correct the measured altitude using the refraction table. Plot on squared paper the position lines and hence estimate the latitude and longitude of the station.

Star	*Calculated azimuth* ° ′	*Calculated altitude* ° ′ ″	*Observed altitude* ° ′ ″
1	042 15	53 27 41	53 28 30
2	148 10	48 10 51	48 12 16
3	245 45	49 21 24	49 22 03
4	319 30	54 59 35	54 59 52

Altitude	47° 55′		48° 28′		49° 01′		49° 35′		50° 10′		50° 45′		51° 21′
Refraction, r″		52		51		50		49		48		47	
Altitude	51° 21′		51° 58′		52° 35′		53° 12′		53° 50′		54° 29′		55° 09′
Refraction, r″		46		45		44		43		42		41	

(LU)

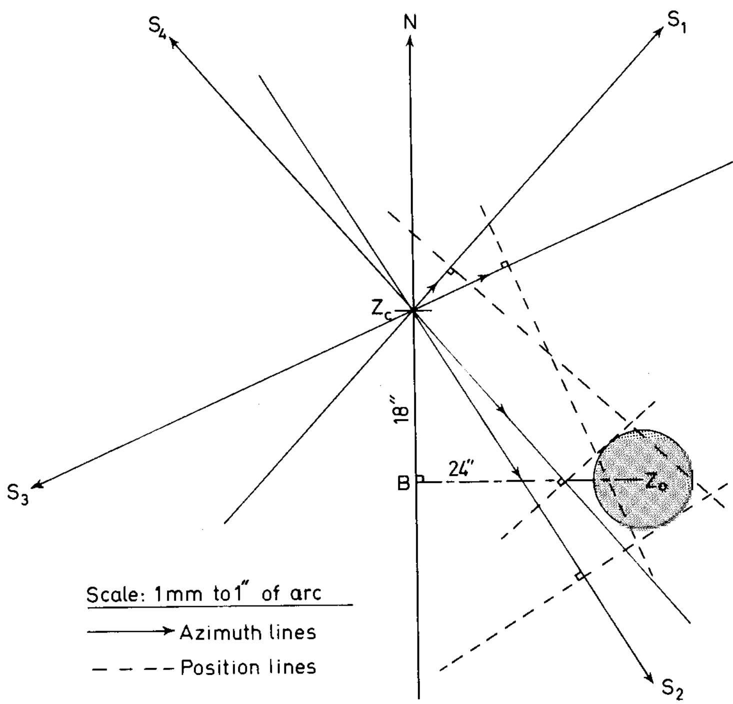


Figure 4.51

Star	*Observed altitude* ° ′ ″	*Refraction, r″*	*Corrected observed altitude* ° ′ ″	*Calculated altitude* ° ′ ″	*Difference* ″	*Remarks*
1	53 28 30	−43	53 27 47	53 27 41	+6	Towards star
2	48 12 16	−52	48 11 24	48 10 51	+33	Towards star
3	49 22 03	−50	49 21 13	49 21 24	−11	Away from star
4	54 59 52	−41	54 59 11	54 59 35	−24	Away from star

These above differences are now plotted as outlined in *Section 4.7.2* to give *Figure 4.51* from which

Difference in latitude $= Z_cB = 18''$
∴ Latitude of observer $= 50° 30' - 18'' = \text{N } 50° 29' 42''$ (ϕ)
Difference in longitude $= BZ_0 \sec \phi = 24'' \sec 50° 29' 42'' = 37''$
∴ Longitude of observer $= 104° 37' - 37'' = 104° 36' 23''$ W

Students should note the method of solving the quadrilateral of error using a circle, the centre of which is Z_o.

EXERCISES

4.7. In order to determine the azimuth of a survey line *XY*, a theodolite was set up at *X*, west of Greenwich, and an afternoon extra-meridian observation made on the Sun.

Using the following data, determine the azimuth of the line. Corrected altitude of Sun's centre = 17° 38′ 11″; GMT of observation 15 h 18 m 14 s; Sun's declination at previous Greenwich mean midnight (GMM), 9° 46′ 47.8″ S, decreasing by 1327″ per day; latitude of station X = 51° 32′ 25″ N; horizontal angle from Sun to station Y = 55° 26′ 28″ (measured clockwise). (LU)

(*Answer:* 282° 35′ 48″)

4.8. Calculate the approximate GMT at which a star (declination 29° 50′ N, RA 8 h 38 m 36 s) will attain a corrected altitude of 30° on the east side of the meridian in latitude 52° 10′ N and longitude 1 h 30 m E, when GST of GMM is 1 h 25 m 15 s. (LU)

(*Answer:* 0 h 29 m 44 s)

4.9. Using the equation (or any other one you may know) and information given below, compute the azimuth of RO from the observer's position:

Mean observed altitude H of the Sun's lower limb in the eastern sky = 30° 51′ 43″; latitude, ϕ = 51° 24′ 00″ N; declination, δ = 5° 30′ 00″ N and $P = (90° - \delta)$; correction for refraction = 1′ 14″; correction for semi-diameter = 15′ 24″; correction for parallax = 07″; horizontal angle measured clockwise from RO to Sun = 86° 40′ 30″. The equation is

$$\tan \tfrac{1}{2}A = [\sec s \sin(s - H) \sin(s - \phi) \sec(s - P)]^{\frac{1}{2}}$$

where $s = \tfrac{1}{2}(H + \phi + P)$ (ICE)

(*Answer:* 77° 10′ 24″)

4.10. With the aid of a sketch explain how to determine whether a particular star will elongate when viewed from a particular latitude.

A star of declination 61° 57′ 24″ N is observed at eastern elongation from a point A, which is at latitude 43° 58′ 12″ N. At elongation the horizontal clockwise angle from the star to a point B is 83° 12′ 12″. Compute the azimuth of B from A. (ICE)

(*Answer:* 123° 59′ 23″)

4.11. The following observations were taken from a station of longitude 6° 30′ 00″ west.

Sight	*Mean horizontal angle* ° ′ ″	*Mean altitude* ° ′ ″	*GMT*
RO	246 18 32	—	—
Polaris	13 37 40	55 08 20	23 h 44 m 52.1 s

At the time of observation the value of R (taken from the *Star Almanac*) was 8 h 31 m 21.4 s and the RA and declination of Polaris were 2 h 00 m 16.1 s and 89° 07′ 03″ N.

Apply a refraction correction of 41″ to the altitude and find the latitude of the station and the azimuth of the line from the station to the RO. (LU)

(*Answer:* 52° 42′ 04″ N; 227° 08′ 20″)

4.12. From a station O in latitude 51° 38′ 24″ N the following observations were taken on the Sun, seen in the western sky. All corrections have been made and the readings apply to the Sun's centre at the instant of observation: mean corrected altitude = 30° 19′ 20″; mean horizontal circle reading on A = 00° 00′ 00″; mean horizontal circle reading on Sun = 225° 02′ 39″; declination of Sun = 23° 18′ 24″ N; GMT of observation = 16 h 39 m 29.4 s; E = 11 h 59 m 41.9 s.

Find the longitude of O and the azimuth of OA. (LU)

(*Answer:* 0° 00′ 55.7″ E; 14° 55′ 51″)

4.13. The following data refer to observations on Arcturus, seen in the western sky, from a station O in latitude 51° 38′ 24″ N; mean corrected altitude = 47° 11′ 05″; mean horizontal circle reading on A = 220° 36′ 04″; mean horizontal circle reading on star = 80° 10′ 10″; declination of star = 19° 23′ 57″ N; RA of star = 14 h 13 m 47.2 s; UT of observation = 23 h 02 m 15.4 s; R = 17 h 34 m 51.6 s.

Find the longitude of O and the azimuth of OA. (LU)

(*Answer:* 0° 00′ 58.2″ E; 339° 46′ 10″)

4.14. The following are the relevant data for determining the position of a station from observations on four stars:

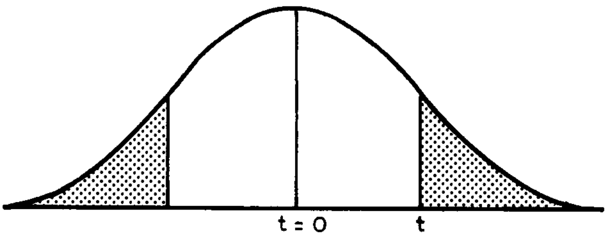
Star	*Observed altitude* ° ′ ″	*Refraction* ″	*Calculated azimuth* ° ′	*Calculated altitude* ° ′ ″
1	55 12 11	40	061 30	55 11 48
2	66 48 20	25	129 30	66 48 23
3	47 35 56	53	215 00	47 35 03
4	49 08 43	50	326 40	49 07 39

What is the correct latitude and longitude of the station if the assumed position is 29° 20′ N, 30° 00′ E? (LU)

(*Answer:* 29° 20′ 10″ N; 29° 59′ 37″ E)

Appendix

TABLE A.1. Student's *t*-distribution



v or DF	*A = 0.05*	*A = 0.01*
1	12.706	63.657
2	4.303	9.925
3	3.182	5.841
4	2.776	4.604
5	2.571	4.032
6	2.447	3.707
7	2.365	3.499
8	2.306	3.355
9	2.262	3.250
10	2.228	3.169
11	2.201	3.106
12	2.179	3.055
13	2.160	3.012
14	2.145	2.977
15	2.131	2.947
16	2.120	2.921
17	2.110	2.898
18	2.101	2.878
19	2.093	2.861
20	2.086	2.845
21	2.080	2.831
22	2.074	2.819
23	2.069	2.807
24	2.064	2.797
25	2.060	2.787
26	2.056	2.779
27	2.052	2.771
28	2.048	2.763
29	2.045	2.756
30	2.042	2.750
40	2.021	2.704
60	2.000	2.660
120	1.980	2.617
∞	1.960	2.576

TABLE A.2. ***F*-distribution (1% significance level)**

Degrees of freedom for denominator	*Degrees of freedom for numerator* 1	2	3	4	5	6	7	8	9
1	4052	5000	5403	5625	5764	5859	5928	5982	6023
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
17	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
19	8.19	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
25	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41

TABLE A.3. ***F*-distribution (5% significance level)**

Degrees of freedom for denominator	Degrees of freedom for numerator: *1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
1	161	200	216	225	230	234	237	239	241	242
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83

TABLE A.4. Chi-square distribution (one-tailed tests)

Null hypothesis $\sigma = \sigma_0$
Alternative hypothesis $\sigma > \sigma_0$

DF	*Levels of significance* 5 per cent	1 per cent
1	3.841	6.635
2	5.991	9.210
3	7.815	11.345
4	9.488	13.277
5	11.070	15.086
6	12.592	16.812
7	14.067	18.475
8	15.507	20.090
9	16.919	21.666
10	18.307	23.209
11	19.675	24.725
12	21.026	26.217
13	22.362	27.688
14	23.685	29.141
15	24.996	30.578
16	26.296	32.000
17	27.587	33.409
18	28.869	34.805
19	30.144	36.191
20	31.410	37.566
21	32.671	38.932
22	33.924	40.289
23	35.172	41.638
24	36.415	42.980
25	37.652	44.314
26	38.885	45.642
27	40.113	46.963
28	41.337	48.278
29	42.557	49.588
30	43.773	50.892

TABLE A.5. Chi-square distribution (one-tailed tests)

Null hypothesis $\sigma = \sigma_0$
Alternative hypothesis $\sigma < \sigma_0$

DF	*Levels of significance* 5 per cent	1 per cent
1	$0.0^2\,393$	$0.0^3\,157$
2	0.103	0.0201
3	0.352	0.115
4	0.711	0.297
5	1.145	0.554
6	1.635	0.872
7	2.167	1.239
8	2.733	1.646
9	3.325	2.088
10	3.940	2.558
11	4.575	3.053
12	5.226	3.571
13	5.892	4.107
14	6.571	4.660
15	7.261	5.229
16	7.962	5.812
17	8.672	6.408
18	9.390	7.015
19	10.117	7.633
20	10.851	8.260
21	11.591	8.897
22	12.338	9.542
23	13.091	10.196
24	13.848	10.856
25	14.611	11.524
26	15.379	12.198
27	16.151	12.879
28	16.928	13.565
29	17.708	14.256
30	18.493	14.953

TABLE A.6. Chi-square distribution (two-tailed test)

Null hypothesis $\sigma = \sigma_0$
Alternative hypothesis $\sigma \neq \sigma_0$

	Levels of significance			
DF	*5 per cent*		*1 per cent*	
1	$0.0^3 982$	5.024	$0.0^4 393$	7.879
2	0.0506	7.378	0.0100	10.597
3	0.216	9.348	0.0717	12.838
4	0.484	11.143	0.207	14.860
5	0.831	12.832	0.412	16.750
6	1.237	14.449	0.676	18.548
7	1.690	16.013	0.989	20.278
8	2.180	17.535	1.344	21.955
9	2.700	19.023	1.735	23.589
10	3.247	20.483	2.156	25.188
11	3.816	21.920	2.603	26.757
12	4.404	23.337	3.074	28.300
13	5.009	24.736	3.565	29.819
14	5.629	26.119	4.075	31.319
15	6.262	27.488	4.601	32.801
16	6.908	28.845	5.142	34.267
17	7.564	30.191	5.697	35.718
18	8.231	31.526	6.265	37.156
19	8.907	32.852	6.844	38.582
20	9.591	34.170	7.434	39.997
21	10.283	35.479	8.034	41.401
22	10.982	36.781	8.643	42.796
23	11.689	38.076	9.260	44.181
24	12.401	39.364	9.886	45.558
25	13.120	40.646	10.520	46.928
26	13.844	41.923	11.160	48.290
27	14.573	43.194	11.808	49.645
28	15.308	44.461	12.461	50.993
29	16.047	45.722	13.121	52.336
30	16.791	46.979	13.787	53.672

Index