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New Theory of the Aether. By *T. J. J. See.*

(Fifth Paper.) (With 2 Plates.)

1. Outlines of a New Theory of Molecular Forces based on Wave-Action, which is also indicated by Laplace's Celebrated Criterion that these Forces become sensible only at Insensible Distances.

Since the renaissance of physical science in the age of *Galileo* natural philosophers have labored patiently for the discovery of the great laws of nature; and thus for about three centuries they have extended their investigations by means of delicate experiments and the most exact methods of mathematical analysis. Yet, notwithstanding this brilliant record of achievement, it remains a somewhat remarkable fact that molecular forces have not yet been assigned to any known physical cause. Accordingly in modern works on physics we still search in vain for an intelligible explanation of the mechanism underlying these forces. The subject therefore has remained very obscure, and continues to challenge the ingenuity of both the geometer and the natural philosopher.

The history of physical science shows that when the solution of a great standing problem at length is attained, it seldom is true that the first attack was wholly successful. Indeed, most of our final solutions of difficult problems result from successive processes of approximation. And thus it may be doubted whether the solution of the problem of molecular forces now in sight is quite complete.

But even if the new effort only opens the way towards the final solution, still it may be of the greatest service to science. For pioneer effort always has to precede the perfect development of science, just as somewhat rude specimens of sculpture and architecture preceded the perfect development of Greek art in the age of *Ictinus, Phidias* and *Praxiteles.*

Accordingly, having arrived at an efficient physical cause of molecular forces which seems to be in general operation throughout nature, we deem it desirable to set forth the results, because the suggestions which this development may convey to others are likely to prove fruitful.

(i) *Laplace*'s criterion, that molecular forces become sensible only at insensible distances, seems to point to wave-action as the underlying physical cause.

In the introduction to his celebrated Theory of Capillarity, (Mécanique Céleste, Tome IV, 1806, with supplement to the theory issued in 1807) *Laplace* examines the theories of his predecessors with characteristic sagacity.

At the very outset of the discussion he alludes to the refractive power exerted by bodies upon light, and says that this force is the result of the attraction of their particles, yet he holds that the law of attraction cannot be determined because »the only condition required is that it must be insensible at sensible distances.« He then proceeds to deal with capillary attraction, in which extensive use is made of this same hypothesis. A part of his reasoning is as follows:

»A long while ago, I endeavored in vain to determine the laws of attraction which would represent these phenomena; but some late researches have rendered it evident that the whole may be represented by the same laws, which satisfy the phenomena of refraction; that is, by laws in which the attraction is sensible only at insensible distances; and from this principle we can deduce a complete theory of capillary attraction.«

»*Clairaut* supposes that the action of a capillary tube may be sensible upon the infinitely thin column, which passes through the axis of the tube. Upon this point I differ wholly from him, and think, with *Hawksbee* and other philosophers, that the capillary attraction is, like the force producing refraction, and all chemical affinities, sensible only at insensible distances. *Hawksbee* observed that in glass tubes, whether the glass is very thick, or very thin, the water rises to the same height, if the interior diameters are the same. Hence it follows that the cylindrical strata of glass, which are at a sensible distance from the interior surface, do not aid in raising the water, though in each one of these strata, taken separately, the fluid ought to rise above the level. It is not the interposition of the strata, which they include between them, which prevents their action upon the water; for it is natural to suppose that the capillary attraction, like the force of gravity, is transmitted through other bodies; this attraction must therefore disappear solely by reason of the distance of the fluid from these strata; whence it follows that the attraction of the glass upon the water is sensible only at insensible distances.«

Laplace justly lays stress upon *Hawksbee*'s observation that in glass tubes, whether very thick or very thin, the water rises to the same height, if the interior diameters are the same. This indicated to *Laplace* that the interior particles of a thick tube of glass exerts no sensible action on the adhering fluid.

Though never suspected heretofore this reasoning of *Laplace* affords the most conclusive evidence that molecular forces really are due to wave-action. It will be shown hereafter that experimental researches by *Rücker* and others on the thickness of soapbubbles, at the critical instant of rupture, make the radius of action of these molecular forces so small that they correspond to the wave-lengths of the ultra-violet region of the spectrum, a fact which may be regarded as an experimental confirmation of the wave-theory of these physical forces.

It appears that *Laplace* himself came near to this line of argument, for in explaining the processes adopted, in the introduction to the theory of capillary attraction, he says that it is evident that »the distance at which the action of the tube ceases to be sensible is imperceptible; so that, if by means of a very powerful microscope, we should be able to make it appear equal to a millimetre, it is probable that the same magnifying power would give to the diameter of the tube an apparent length of several meters. The surface of the tube may therefore be considered as very nearly a plane surface, for an extent which is equal to that of the sphere of its sensible activity; the fluid will therefore be elevated or depressed near that surface, in almost the same manner as if it were a plane. Beyond this point the fluid will be subjected only to the force of gravity and its own action on its particles; its surface will be very nearly that of a spherical segment, of which the extreme tangent planes, being those of the fluid surface at the limits of the sensible sphere of activity of the tube, will be very nearly, in the different tubes, equally inclined to their sides; whence it follows that all segments will be similar. The comparison of these results gives the true cause of the elevation, or depression, of fluids, in capillary tubes, in the inverse ratio of their diameters.«

»Therefore the attraction of a capillary tube has no other influence upon the elevation or depression of the fluid which it contains, than that of determining the inclination of the first tangent planes of the interior fluid surface, situated very near to the sides of the tube; and it is upon this inclination that the concavity or convexity of the surface depends, as well as the magnitude of its radius.«

(ii) The wave-theory underlies the mathematical analysis of *Fourier* and *Poisson*,[1]) based on the solution of partial differential equations.

In the Fourth Paper, near the end of Section 8, I have called attention to the great importance attached to boundary conditions by modern investigators in theoretical physics, and have also pointed out the prominent part played by partial differential equations in the mathematical methods applicable to physical problems. These two independent circumstances seemed to me an overwhelming argument for the wave-theory as representing the true order of nature, which we see exhibited most simply in the refraction of light.

In the New Theory of the Aether we have dwelt on the equation of wave motion developed by *Poisson*:

$$\partial^2\Phi/\partial t^2 = a^2\left(\partial^2\Phi/\partial x^2+\partial^2\Phi/\partial y^2+\partial^2\Phi/\partial z^2\right) \qquad (1)$$

Likewise *Fourier*'s Théorie Analytique de la Chaleur, 1821, leads to the similar expression:

$$\begin{aligned}&\partial^2\Theta/\partial t^2 = a^2\left(\partial^2\Theta/\partial x^2+\partial^2\Theta/\partial y^2+\partial^2\Theta/\partial z^2\right)\\&\Theta = f(x, y, z, t); \quad \Theta = f(x, y, z), \ t = 0\end{aligned} \qquad (2)$$

which holds for the propagation of heat, and other wave motions.

For constant temperature, $\partial\Theta/\partial t = 0$, and therefore

$$\partial^2\Theta/\partial x^2+\partial^2\Theta/\partial y^2+\partial^2\Theta/\partial z^2 = 0\,. \qquad (3)$$

For the disturbances in the theory of sound, *Poisson* usually writes for the velocity-potential φ, thus:

$$\begin{aligned}&\partial^2\varphi/\partial t^2 = a^2\left(\partial^2\varphi/\partial x^2+\partial^2\varphi/\partial y^2+\partial^2\varphi/\partial z^2\right)\\&\partial^2\varphi/\partial t^2 = a^2\nabla^2\varphi;\ \varphi = f(x, y, z), \ t = 0\,.\end{aligned} \qquad (4)$$

In the theory of light, the same differential equation arises (cf. Drude, Theory of Optics, Part I, Chapter III, § 3)

$$\begin{aligned}&\partial^2 s/\partial t^2 = V^2\left(\partial^2 s/\partial x^2+\partial^2 s/\partial y^2+\partial^2 s/\partial z^2\right)\\&\partial^2 s/\partial t^2 = V^2\nabla^2 s;\ s = f(x, y, z), \ t = 0.\end{aligned} \qquad (5)$$

In the theory of waves we have for plane waves along the x-axis:

$$y = A\sin\left[2\pi/\lambda\cdot(Vt-x)+\alpha\right] \qquad (6)$$

But in tri-dimensional space, the disturbance spreads in all directions with the velocity $V = at$ $\quad (at)^2 = x^2+y^2+z^2$ and from any point $P(x, y, z)$, the sphere surface becomes:

$$(at)^2 = (x+at\cos\theta)^2+(y+at\sin\theta\sin\omega)^2+(z+at\sin\theta\cos\omega)^2 \qquad (7)$$

which we have treated in previous papers.

In the treatment of *Poisson*'s equation of wave motion,

$$\partial^2\Phi/\partial t^2 = a^2\left(\partial^2\Phi/\partial x^2+\partial^2\Phi/\partial y^2+\partial^2\Phi/\partial z^2\right);\ \Phi = \Omega\,(x, y, z), \ t = 0 \qquad (8)$$

we have found (AN 5048) that for three variables

$$\Phi = \Omega\,(x, y, z) = (1/8\pi^3)\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty} \Omega\,(\xi, \eta, \zeta)\cos\xi\,(x-\lambda)\cos\eta\,(y-\mu)\cos\zeta\,(z-\nu)\,d\xi\,d\eta\,d\zeta\,d\lambda\,d\mu\,d\nu \qquad (9)$$

in which ξ, η, ζ and λ, μ, ν extend from $-\infty$ to $+\infty$.

This may be transformed into

$$\Phi = \Omega\,(x, y, z) = (1/8\pi^3)\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty} \Omega\,(\xi, \eta, \zeta)\cos\lambda\,(\xi-x)\cos\mu\,(\eta-y)\cos\nu\,(\zeta-z)\,d\xi\,d\eta\,d\zeta\,d\lambda\,d\mu\,d\nu \qquad (10)$$

$$= (1/8\pi^3)\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty} \Omega\,(\xi, \eta, \zeta)\,e^{[\lambda(\xi-x)+\mu(\eta-y)+\nu(\zeta-z)]\sqrt{-1}}\,d\xi\,d\eta\,d\zeta\,dx\,dy\,dz \qquad (11)$$

By including the factor $1/8\pi^3$ in the arbitrary function, this may be written in the well known form of the expression for any time t,

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[1]) In developing the new theory of the aether I found *Poisson*'s Analytical Theory of Wave Motion, (1815–1839) so extraordinarily useful that I was led to apply to M. *Baillaud*, Director of the Observatory of Paris, for an authentic portrait of this unrivaled physical mathematician.

The portrait proved to be somewhat difficult to obtain, but as it reached me on the day this paper is finished, it seems appropriate to acknowledge M. *Baillaud*'s kindness, and at the same time do honor to *Poisson*'s memory and a service to geometers generally by using the portrait as a frontispiece to this Fifth Paper.

In his eulogy of *Poisson*, *Arago* relates that one day the venerable *Lagrange* remarked to the brilliant young geometer: »I am old, and during my intervals of sleeplessness I divert myself by making numerical approximations. Keep this one: it may interest you. *Huyghens* was thirteen years older than *Newton*, I am thirteen years older than *Laplace*; *D'Alembert* was thirty two years older than *Laplace*, *Laplace* is thirty two years older than you.« — which was *Lagrange*'s delicate way of intimating to *Poisson* his destined place in the Pantheon of mathematical fame.

$$\Phi = \Omega(x, y, z, t) = \int\int\int\int\int\int_{-\infty}^{+\infty} e^{(A+Bht)\sqrt{-1}}\, \psi_1(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta\, d\lambda\, d\mu\, d\nu\,. \qquad (12)$$

And finally, in the Fourth Paper, (AN 5085), we have reached *Poisson*'s double integral:

$$\begin{aligned} \Phi &= \Phi' + \Phi'' \\ &= (1/4\pi)\int_0^{\pi}\int_0^{2\pi} F\{x+at\cos\theta,\ y+at\sin\theta\sin\omega,\ z+at\sin\theta\cos\omega\}\, t\sin\theta\, d\theta\, d\omega \\ &\quad + (1/4\pi)(\partial/\partial t)\int_0^{\pi}\int_0^{2\pi} H\{x+at\cos\theta,\ y+at\sin\theta\sin\omega,\ z+at\sin\theta\cos\omega\}\, t\sin\theta\, d\theta\, d\omega\,. \end{aligned} \qquad (13)$$

This expression for the velocity-potential Φ, will hold rigorously for the waves emanating from any mathematical point $P(x, y, z)$ and traversing all space from that centre of disturbance. But in nature the waves proceed from all atoms of a mass, and thus we must extend the integral of *Poisson* by taking the triple integral for the volume and density:

$$\begin{aligned} \Phi = &\int_0^{r}\int_0^{\pi}\int_0^{2\pi} (\sigma/4\pi)\int_0^{\pi}\int_0^{2\pi} F\{x+at\cos\theta,\ y+at\sin\theta\sin\omega,\ z+at\sin\theta\cos\omega\}\, r^2\sin\theta\, dr\, d\theta\, d\omega\cdot t\sin\theta\, d\theta\, d\omega \\ &+ \int_0^{r}\int_0^{\pi}\int_0^{2\pi} (\sigma/4\pi)(\partial/\partial t)\int_0^{\pi}\int_0^{2\pi} H\{x+at\cos\theta,\ y+at\sin\theta\sin\omega,\ z+at\sin\theta\cos\omega\}\, r^2\sin\theta\, dr\, d\theta\, d\omega\cdot t\sin\theta\, d\theta\, d\omega\,. \end{aligned} \qquad (14)$$

This is a double quintuple integral, and by referring to the equations (9) or (12) above we see that (14) corresponds to a single non-nuple integral in the original form of these equations, because the disturbances must be conceived to proceed from each atom of the mass,

$$m = \int_0^{r}\int_0^{\pi}\int_0^{2\pi} \sigma r^2 \sin\theta\, dr\, d\theta\, d\omega\,. \qquad (15)$$

Now in the physical universe, such independent gravitational waves must be imagined to proceed from the several atoms of all bodies whatsoever, just as light waves do from each atom of the self-luminous gases of the stars. Accordingly such integration has to be extended to the waves from all masses severally; and as there is an infinitude of bodies, the result is an integral infinitely repeated, or an infinite integral, though the value of the disturbance remains finite at every point of space.

And not only is there a double infinite or infinite infinite system of interpenetrating waves, but also the resistances — with refraction, dispersion and interference — at the boundaries of all solids and liquid bodies. It is these resistances — refractions, dispersions, diffractions, and other wave transformations — which give rise to molecular forces. They usually are very powerful at the surfaces of bodies, and by their mutual interactions on contiguous atoms and molecules cause cohesion, adhesion, capillarity, and chemical affinity, and other phenomena heretofore utterly bewildering to the natural philosopher.

Now it is our purpose to outline a preliminary theory of these forces, in the hope that the light thus shed on a very obscure problem may induce others to extend these researches. It is obvious that the preliminary theory must necessarily remain very incomplete till the phenomena are carefully studied under a criterion which may operate as an experimentum crucis. But these verifications can only be deduced by investigators of great experience in the several branches of physical science.

2. The Recognized Refraction and Dispersion of Light in a Drop of Rain shows the Cause of the Rainbow, and suggests Similar Molecular Effects when the Source of Light is extended by Double Integration to the Surface of the Entire Celestial Sphere.

(i) Outline of the theory of the rainbow, as an introduction to the wave-theory.

Let the circle in Fig. 1 represent a section of a spherical rain-drop. As water is liquid and yields to the forces acting on the surface this hypothesis of sphericity implies that there are constantly acting forces at work to maintain this figure; and we know from the researches of Lord *Rayleigh* (Proc. Roy. Soc., May 5, 1879, no. 196), on the oscillation-periods of globules of liquid, that the forces at work are quite powerful, otherwise the oscillations of the drops of distorted form would not be so rapid as they are observed to be.

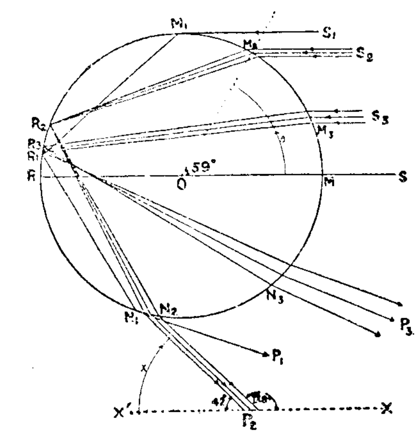

Fig. 1. Path of the sun's rays in the theory of the rainbow. The large circle represents the magnified raindrop, and *OS* the direction of the sun.

As our theory of molecular forces is based upon the action of waves of various lengths, we must be careful not to assume that waves other than those of the visible spectrum are absent, yet in the problem of the rainbow our reasoning of course relates to the visible spectrum. It is believed that waves shorter than the visible spectrum give rise to chemical affinity, capillarity, etc., while the waves of the infra-red region, having enormous wave-lengths, generate heat through breaking up into shorter and shorter wavelengths.

In figure 1 the circle represents a section of a spherical rain-drop, with parallel rays of sunlight $S_1 M_1$, and thus internally refracted along the path $M_1 R_1$, whence it is reflected along the path $R_1 N_1$, and then outwardly refracted along the path $N_1 P_1$. The line from the centre OM_2 makes an angle of 59° with the path of the incident light, which, at this small circle θ about M as a pole, is less deviated by two refractions and reflection, than is the light incident at other small circles about M. It appears that the surface of the rain-drop is divided into different zones about the pole M, and the path within depends on the polar angle θ, and also on the wave-length of the light.

It will be found that for rays of the visible spectrum, the light incident in the narrow zone or surface

$$ds = r\,d\theta\, r\sin\theta \int_0^{2\pi} d\omega = 2\pi r^2 \sin\theta\, d\theta, \quad \theta = 59^\circ \qquad (16)$$

operates to form a parallel pencil $N_2 P_2$ when the rays have undergone their last refraction in leaving the raindrop. In another smaller zone, as M_3, nearer the pole, the incident light forms a divergent pencil $N_3 P_3$, when the originally parallel rays have departed. The direct illumination of the hemisphere of the drop turned towards the sun thus yields successive zones about the pole M:

$$2\pi r^2 \int_0^{1/2\pi} \sin\theta\, d\theta = 2\pi r^2 \left\{ \int_0^{\theta_3} \sin\theta\, d\theta + \int_{\theta_3}^{\theta_2} \sin\theta\, d\theta + \int_{\theta_2}^{\theta_1} \sin\theta\, d\theta \right\} \qquad (17)$$

To understand the illumination of the sky noted in a rainbow, we notice that in the case of an emitted parallel pencil, the only decrease of the light with the distance depends on the absorption in the raindrop as a medium, which is small. But with the divergent pencil the case is very different, because the rays are spread over a greater and greater area as they recede from their point of intersection; and hence the illumination rapidly decreases.

Accordingly, in viewing such a raindrop from a distance, we should receive a considerable amount of refracted light in looking along the conical surface $P_2 N_2$, but very little when we look along any other conical surface about the anti-solar point.

After passing through the falling raindrops the light of the sun thus becomes redistributed in the sky, and a luminous band appears, corresponding to the rays which emerge as parallel pencils; but in the other zones there is relatively increased darkness, owing to the divergence of rays corresponding thereto.

It will be seen from the lower part of the figure that the angle $N_1 P_2 X' = \chi$ is 42°; and hence all raindrops on the surface of the cone 42° from the anti-solar point will be in such position that the light entering them will have undergone minimum deviation, and send to the observer a relatively large amount of light, on a darkened background. This simple theory briefly outlines the foundation of the rainbow, but the dispersion of colors is still to be explained.

We shall now include the effects of refraction and reflection, so as to take account of dispersion. As the sun's rays inlude all the wave-lengths of the spectrum, we must consider the production of color in the rainbow. It is obvious that if the source of light were a point and there were monochromatic light, the luminous band would be reduced to a mere line of one color circling about the anti-solar point. But when light of the whole spectrum is incident upon the drops, the violet rays are deviated more than the average; moreover the width of the source of light lets the waves fall at slightly different angles, and hence the inner side of the cone has an angle χ of about 40°. The rainbow is thus a conical band, about 2° wide, with the red band about 42° from the anti-solar point.

In addition to the primary rainbow thus briefly explained, there is a secondary rainbow due to light which has been twice reflected within the drop, as shown in figure 2.

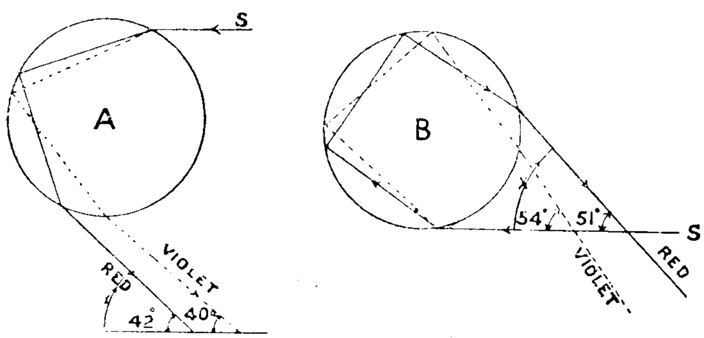

Fig. 2. Explanation of the primary and secondary rainbow, the latter by a reversed double reflection within the raindrop.

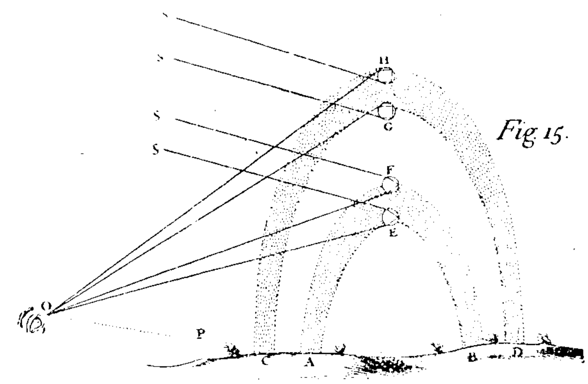

Fig. 3. General outline of the theory of the rainbow given in Newton's Optics, 1704.

Owing to the reversed nature of the reflection, from below upward, we perceive that the colors in the secondary bow should be reversed. Thus whilst the primary bow gives the red above and the violet below, the secondary bow has the violet above and the red below. And the angles χ of the cone are about 54° for the violet, and 51° for the red.

The secondary rainbow is therefore wider than the primary bow, and fainter, while the colors are exactly reversed.

From the reasoning here outlined, it follows that there are two zones for producing the rainbows:

1. The Primary Bow,

$$dS = 2\pi r^2 \int_{\theta_1}^{\theta_2} \sin\theta \, d\theta, \quad \chi_1 = 40^\circ, \; \chi_2 = 42^\circ \qquad (18)$$

2. The Secondary Bow,

$$dS = 2\pi r^2 \int_{\theta_3}^{\theta_4} \sin\theta \, d\theta, \quad \chi_3 = 51^\circ, \; \chi_4 = 54^\circ \qquad (19)$$

where χ_1, χ_2, χ_3, χ_4 are the angles of the cones from the anti-solar point.

Now consider what would be the result for greater changes in wave length than we have considered. Obviously the width of these luminous zones would be increased, and they might attain any width appropriate to the range in wave-length. Thus if the range of wave-length be multiplied say tenfold, the zone of light might become quite wide.

Finally, we should consider the effect of increasing the width of the luminous source, as by putting additional suns to radiating, side by side. Obviously each sun will generate its own rainbow, without regard to that due to the other sun; and thus we should have a superposed, or accumulated integral effect on the background of the sky. If there be suns side by side, from $\omega = 0^\circ$ to $\omega = 360^\circ$, where ω is the azimuth, the circular string of suns near the horizon, would fill the heavens with rainbows superposed three or more layers deep, and the whole lower part of the sky would become very luminous. And if the elevation of the ring of suns be increased, from the horizon to the zenith, $z = 90^\circ$, $z = 0^\circ$, where z is the zenith distance, we should fill the whole heavens several times over with the light of rainbows.

These conceptions, drawn from our theory of the rainbow, as extended by integrating the entire celestial sphere, will perhaps prove of value when we come to deal with the wave theory of molecular forces.

(ii) Sir *John Herschel*'s argument that in refraction the mechanical forces exerted must be termed infinite, may be extended also to dispersion, and the hardness of bodies, as in section 10 below.

In his celebrated article on Light, Encycl. Metrop., 1849, Sect. 561, Sir *John Herschel* has calculated the intensity of the refractive force in comparison with the force of gravity at the earth's surface. Whilst his result is obtained on the *Newton*ian emission theory, and not precisely applicable to the problem before us, yet this reasoning, as *Herschel* remarks, is well calculated to show the greatness of the power of molecular forces. This enormous force we now assign to waves action, and explain by the very high elasticity of the aether, which is $\epsilon = 689321600000$ times more elastic than our air in proportion to its density; and yet this enormously elastic aether not only has the wave surface refracted, and thus suddenly bent into a new position, at the boundary of solids and liquids, but also suffers an unequal refraction or dispersion of the waves according to their length.

Herschel's analysis of the intensity of the forces producing refraction is so worthy of careful study that we quote it as follows:

»Whatever be the forces by which bodies reflect and refract light, one thing is certain, that they must be incomparably more energetic than the force of gravity. The attraction of the earth on a particle near its surface produces a deflexion of only about 16 feet in a second; and, therefore, in a molecule moving with the velocity of light, would cause a curvature, or change of direction, absolutely insensible in that time. In fact, we must consider first, that the time during which the whole action of the medium takes place, is only that within which light traverses the diameter of the sphere of sensible action of its molecules at the surface. To allow so much as a thousandth of an inch for this space is beyond all probability, and this interval is traversed by light in the 1/12672000000000 part of a second. Now, if we suppose the deviation produced by refraction to be 30°, (a case which frequently happens) and to be produced by a uniform force acting during a whole second; since this is equivalent to a linear deflexion of 200000 miles $\times \sin 30^\circ$, or of 100000 miles $= 33000000 \times 16$ feet, such a force must exceed gravity on the earth's surface 33000000 times. But, in fact, the whole effect being produced not in one second, but in the small fraction of it above mentioned, the intensity of the force operating it (see Mechanics) must be greater in the ratio of the square of one second to the square of that fraction; so that the least improbable supposition we can make gives a mean force equal to $4969126272 \times 10^{24}$ times that of terrestrial gravity. But in addition to this estimate already so enormous, we have to consider that gravity on the earth's surface is the resultant attraction of its whole mass, whereas the force deflecting light is that of only those molecules immediately adjoining to it, and within the sphere of the deflecting forces. Now a sphere of 1/1000 of an inch diameter, and of the mean density of the earth, would exert at its surface a gravitating force only

(1/1000) × (1 inch/diameter of the earth)

of ordinary gravity, so that the actual intensity of the force exerted by the molecules concerned cannot be less than

(1000 · earth's diameter)/1 inch (= 46352000000)

times the above enormous number, or upwards of $2 \cdot 10^{44}$ when compared with the ordinary intensity of the gravitating power of matter. Such are the energies concerned in the phenomena of light on the *Newton*ian doctrine. In the undulatory hypothesis, numbers not less immense will occur; nor is there any mode of conceiving the subject which does not call upon us to admit the exertion of mechanical forces which may well be termed infinite.«

3. Outline of New Theory of Surface Tension and of Capillarity based on Wave-Action.

(i) From the small radius of activity of the molecular forces observed by *Quincke* in 1869, — namely 50 micromillimetres, corresponding to a wave-length of only one half that of the shortest wave ever measured — it follows that these forces depend on waves in the invisible chemical spectrum.

In Poggendorff's Annalen, 137, 1869, *Quincke* gives certain results of his observational researches on capillarity and similar phenomena, and is led to the conclusion that the molecular attraction becomes sensible at a distance of about 50 micro-millimetres, ore 0.000050 mm, one millionth of a millimetre $1 \mu\mu = 0.000001$ mm.

Reinhold and *Rücker* have strikingly confirmed *Quincke*'s conclusions by their researches on soap bubbles. They found that the black film always formed before the stable bubble breaks, and that it has a uniform or nearly uniform thickness of 11 or 12 micro-millimetres, (Proc. Roy. Soc., June 21, 1877; and Phil. Trans. Roy. Soc., Apr. 19, 1883).

In his well known Address on Capillary Attraction Lord *Kelvin* remarks that the abrupt commencement and the permanent stability of the black film bring to light a proposition of fundamental importance in molecular theory: namely the tension of the film, which is sensibly constant when the thickness exceeds $50 \mu\mu$, diminishes to a minimum, and begins to increase again when the thickness is diminished to $10 \mu\mu$. It is not possible, Lord *Kelvin* concludes, to explain this fact by any imaginable law of force between the different portions of the film supposed homogeneous, and we are forced to the conclusion that it depends upon molecular heterogeneousness.

Accordingly, the molecular structure and sustaining forces depend on distances of these dimensions, as if the forces are due to waves in the chemical spectrum. This reasoning is based on well established observational data on the radius of action of molecular forces; and thus it may also throw light on the cause of these forces in such phenomena as capillary attraction. Here is a suggestive summary, in which the micro-millimetre is the unit

1. Wave length of *D*-line of sodium	$590 \mu\mu$
2. Maximum of chemical action in the solar spectrum	400 »
3. Invisible spectrum begins	300 »
4. Shortest wave-length ever measured	100 »
5. *Quincke* observes molecular action effective	50 »
6. *Reinhold* and *Rücker* rupture soap bubbless at thickness of	10 »

It has long been known that chemical action is confined chiefly to the ultra-violet part of the spectrum. And now it appears from this table that the molecular forces, if due to wave action, are chiefly developed in the totally invisible spectrum, the violet *H* and *K* lines of the solar spectrum corresponding to about $400 \mu\mu$. Lord *Kelvin* estimates the radius of action of the molecular forces as less than $250 \mu\mu$, and on the wave theory this result is confirmed.

The question arises: How are we to interpret the development of these short waves? In any new theory there is much which still remains obscure, but the following outline enables us to interpret most if not all of the known phenomena:

1. In *Laplace*'s theory of capillary attraction, based on the theory of molecular forces sensible only at insensible distances, he puts $f(r)$ as the unknown function of the forces, and takes

$$\varphi(r) = \int_{r_1}^{\infty} f(r)\, \mathrm{d}r = 0 \qquad (20)$$

or the action of the forces is insensible beyond a small limiting distance r_1, which is the lower limit of the integral. From the above reasoning we may suppose this value of $r_1 < 250 \mu\mu$.

2. Now *Langley* found by his explorations of the infra-red spectrum, by means of the bolometer, that the heat spectrum was about 20 times the length of the visible spectrum observed by *Newton*, which runs from $A = 759.4 \mu\mu$ to $K = 393.38 \mu\mu$, and terminates quite suddenly at $200.0 \mu\mu$, according to *Cornu*. Thus the heat spectrum, made up of long waves irregularly distributed over a wide space, is of enormous extent, ending in the other direction beyond the red, at say $7340 \mu\mu$.

3. Magnetic and gravitational waves are supposed to be considerably longer than the heat waves, but an instrument to determine their length is not yet available. Thus the planetary forces undoubtedly depend on long waves, while the molecular forces depend on very short waves.

4. It is observed that the longest wave-length of light yet measured is $2500 \mu\mu$, and the shortest electrical oscillation yet measured is some $600000 \mu\mu$. And we know from the phenomena of waves in water of the sea and other fluids that long waves may be broken up into shorter ones by resistance. Accordingly, we conclude that by resistance long electric waves generate heat waves; and an additional breaking up of heat waves gives the still shorter light waves; while a still further disintegration of the light waves, gives the chemical waves of the invisible spectrum beyond the ultra-violet.

5. This transformation by breaking up of the waves appears to be the order of nature. It is exhibited constantly in the surface motions of the sea. And by turning on an electric current, — which was shown in the author's work of 1917, to be aether waves of a certain type — the disturbance is observed to heat a wire till it becomes red, by the resistance opposed to the motion of the longer electric waves. Further operation of the electric current makes the resistance wire glow with the brilliancy of the electric spark or arc, which is filled with violet light, like that of the sun. Still higher action of the current causes the vaporization of the luminous film of the electric light, and thus the generation of chemical waves, as in the light of the sun and stars.

6. The waves producing chemical affinity are thus held to be so short as to be invisible to the human eye. This whole process therefore confirms the following view:

(a) All short waves in nature come from the breaking up of longer waves in the aether.

(b) All molecular forces operative in chemical affinity, capillarity, cohesion, adhesion, surface tension, etc., are due to very short waves in the aether, which lie beyond the ultraviolet, in the region from 10 to $250 \mu\mu$.

7. The maximum of the chemical activity in the solar spectrum, about $400 \mu\mu$, is due to the greater agitation incident to the longer waves, which effect the greatest changes, while the shorter waves exert the greater forces of a steady character.

8. If this conclusion be admissible it confirms *Laplace*'s theory of capillarity, which is mathematically expressed by the formula:

$$\varphi(r) = \int_{r_1}^{\infty} f(r)\, \mathrm{d}r = 0\,. \qquad (20)$$

And it indicates that different substances will exert different forces, according to their resistance or their transmission of the wave-lengths, $\lambda_1, \lambda_2, \lambda_3, \cdots \lambda_i, \lambda_i < r_1$:

$$\varphi(r) = K_1 \int_{r=\lambda_1}^{r=\lambda_2} f(r)\,dr + K_2 \int_{r=\lambda_2}^{r=\lambda_3} f(r)\,dr + K_3 \int_{r=\lambda_3}^{r=\lambda_4} f(r)\,dr + \cdots + K_{i-1} \int_{r=\lambda_{i-1}}^{r=\lambda_i} f(r)\,dr \qquad (21)$$

where $K_1, K_2, K_3, K_4, \cdots K_{i-1}$ are coefficients of conductivities, or resistance for the particular wave-length, the resistance being the reciprocal of the conductivity.

(ii) Direct proof that boundary pressure due to waves is the cause of molecular forces.

1. After the foregoing discussion of the general principles underlying the wave-theory, we now enter upon certain processes of exact calculation. The observational data are incomplete, yet the processes disclosed will prove very instructive. In treatises on physics, (cf. *Daniell*'s Principles of Physics, 3rd edition, 1895, p. 142) we find the conclusion that the Kinetic Energy due to a steady flow of waves is

$$\varepsilon = \tfrac{1}{2} \varrho v^2 \qquad (22)$$

where ϱ is the density of the medium and v the velocity of the waves.

2. Now for simple harmonic motion in a circle of radius a, which corresponds to a wave-amplitude a, we have:

$$v = 2\pi a/t, \quad v^2 = 4\pi^2 a^2/t^2 = 4\pi^2 a^2 \nu^2 \qquad (23)$$

where $\nu = 1/t$ is the wave-frequency.

Using these values in (22) we obtain for the pressure due to the steady flow of waves:

$$\varpi = \tfrac{1}{2} \varrho v^2 = 2\pi^2 a^2 \varrho \nu^2 \qquad (24)$$

dynes per square centimetre, or ergs per cubic cm.

3. When the waves are short, ν is increased, and thus the pressure ϖ is increased, unless the amplitude a is correspondingly decreased. This raises the question as to whether retarded waves have greater or less amplitude than the original unchanged waves. Investigation shows:

(a) The long waves break up into shorter waves, by a process fully outlined for water waves by Sir *George Airy*, Tides and Waves, Encycl. Metr., 1845, (cf. Second Paper on the New Theory of the Aether, AN 5048, pp. 141–142).

It is shown that the wave front becomes steep, owing to resistance, and the crest breaks into two parts, and finally forms two separate waves, the rear wave being shorter and having the smaller amplitude.

(b) The longer of the parts of the broken wave becomes actually of larger amplitude than the original wave. And when subdivision again occurs, the same tendency arises — more waves, and of larger amplitude. This conclusion of *Airy* is verified by the tide heights observed at San Francisco and at Mare Island — the tides at Mare Island being higher by the factor 1.26, which is a noticeable increase of amplitude in traveling 25 miles from the Golden Gate.

(c) In considering waves transformed by resistance we have to sum up the pressure due to all lengths, and the effects of their different amplitudes, which requires an integration of all the variable elements.

4. If n be the index of refraction, the refractive action at the boundary will be (n^2-1), and the wave pressure exerted on the boundary of the fluid will be, in dynes per square cm or ergs per cubic cm:

$$\varpi = 2\pi^2 a^2 \varrho \nu^2 (n^2-1) \qquad (25)$$

But it is well known that ν and n are related, though not in a very simple way. According to the celebrated researches of *Cauchy* on the refraction and dispersion of light,

$$n = A + B\lambda^{-2} + C\lambda^{-4} = A + B\nu^2 + C\nu^4 \qquad (26)$$

where A, B, C, are coefficients, and λ is the wave length, ν the corresponding wave frequency. This formula (26) holds quite accurately for the range of the visible spectrum.

5. Accordingly, for a given wave-amplitude, wave-length, and aether density, we have

$$\begin{aligned}\varpi &= 2\pi^2 a^2 \varrho \nu^2 (n^2-1) \\ &= 2\pi^2 a^2 \varrho \nu^2 [(A+B\nu^2+C\nu^4)^2-1]\,.\end{aligned} \qquad (27)$$

But as the amplitude a, density ϱ, and wave-frequency ν are variable, when waves are resisted by matter and thus transformed, we must take the triple integral for these three independent elements, in order to get a rigorous calculation of the pressure at the boundary of the fluid:

$$\varpi = 2\pi^2 \int_0^a \int_0^\varrho \int_0^\nu 2a\,da\,d\varrho\,[2A\nu^2 + 8AB\nu^3 + 6(B^2+AC)\nu^5 + 16BC\nu^7 + 10\nu^9]\,d\nu\,. \qquad (28)$$

6. It is to be remembered that the elasticity ε and density ϱ are both variable in *Newton*'s formula, for the velocity of a wave in free space:

$$V = K\sqrt{(\varepsilon/\varrho)} = K\sqrt{(\gamma\varepsilon/\gamma\varrho)} \qquad (29)$$

so that the velocity V does not sensibly vary in planetary space (AN 5044). What may occur within transparent bodies is not definitely known, but it is usually assumed that both the density and elasticity varies. If the presence of corpuscular matter did not interfere with the wave propagation, the *Newton*ian formula

$$V^2 = K^2 \varepsilon/\varrho \qquad (30)$$

would give

$$dV = [K^2/(2\varrho^{3/2}\varepsilon^{1/2})] \cdot (\varrho\,d\varepsilon - \varepsilon\,d\varrho)\,. \qquad (31)$$

But it is evident that the resisting forces, which transform waves, would also invalidate the use of this differential equation. In practice we have to rely, for moderately homogeneous waves, on equation (27) or on equation (28) when any process exists by which the triple integral may be evaluated. The difficulty of effecting the integration for the action of waves coming from all directions is increased by the circumstance that they are so short as to be wholly invisible, and the frequency ν thus indeterminate. Hence the amplitude also is indeterminate, and the effects must rest mainly on arguments of probability drawn from a true cause recognized to pervade the physical universe.

4. Physical Theory of the Globular Form of Liquid Drops.

(i) Least action leads to minimum deviation and therefore minimum dispersion in the passing waves, the paths for which are here illustrated for raindrops in the case of the rainbow.

If the direction of an incident beam of light passing through a prism, with section in the form of an equiangular or isosceles triangle, be such that the path within the prism be parallel to the base, it is well known that both the deviation and dispersion will be a minimum, and the external path of the transmitted light will be as nearly as possible identical with that of the incident ray. This result is the outcome of the principle of least action, which may be briefly outlined as follows.

In the case of simple refraction the law of *Snellius*, 1620, is

$$\sin i = n \sin r$$
$$\sin r = 1/n \cdot \sin i\,. \qquad (32)$$

To find the least action along the actual path, we remember that this action is for lengths of path $l_1, l_2, l_3 \cdots l_i$:

$$A = (v_1 l_1 + v_2 l_2 + v_3 l_3 + \cdots + v_i l_i)\,. \qquad (33)$$

And the condition for the minimum of this action is

$$\partial A/\partial s = (\partial/\partial s)(v_1 l_1 + v_2 l_2 + v_3 l_3 + \cdots + v_i l_i) = 0$$
$$= \partial/\partial s\,(l_1 + 1/n_1 \cdot l_2 + 1/n_2 \cdot l_3 + \cdots 1/n_{i-1} \cdot l_i) = 0\,. \qquad (34)$$
$$ds = dt\, \sqrt{[(\partial x/\partial s)^2 + (\partial y/\partial s)^2 + (\partial z/\partial s)^2]}$$
$$= dt\, \sqrt{[(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2]}\,.$$

The action or work is a minimum along the actual path, and there is no change for a small variation in the path: or, in *Hamilton*'s phrase, the action is stationary.

If λ be the wave-length, the velocity $v = f(x, y, z)$, and the time of passage becomes

$$\tau = \int 1/v \cdot ds = \int [1/f(\lambda, x, y, z)] \cdot ds\,. \qquad (35)$$

And for the minimum path our stationary condition is

$$\delta\tau = \delta \int_{x_1 y_1 z_1}^{x_2 y_2 z_2} [1/f(\lambda, x, y, z)] \cdot ds = 0\,. \qquad (36)$$

The solution shows that the time of passage is defined by the function

$$\tau = F(x, y, z, \lambda, \alpha, \beta) \quad (37)$$

where α and β are constants of integration.

We may obtain a better geometrical and physical grasp of these actions by considering the following sketch of the waves of light, in passing through the raindrop for the production of the rainbow.

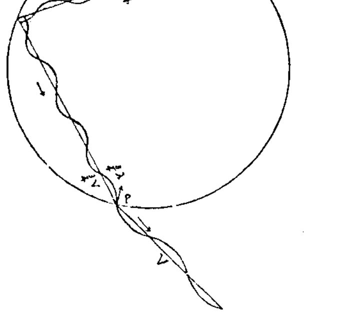

Fig. 4. Illustration of the sudden change of wave-length λ and wave-velocity V at the boundary of a raindrop, by which inward pressure is exerted at the surface of the fluid, as the waves are both coming and going.

1. The waves are of velocity $V = 3 \cdot 10^{10}$ cm in the air, before entering the spherical drop; then at the boundary of the drop, the velocity $V' = V - q$, which for water gives a decrease of speed in the ratio of 4 : 3. Within the drop, therefore, the waves are shorter than without, in the same ratio, because the same number are crowded into a less space $s' = {}^3/_4 s$, as shown in the figure 4.

2. After reflection at the opposite boundary of the drop, the path returns, and the light emerges as shown in the figure. It may be noticed that just as the velocity and wave-length are decreased on entering the raindrop, in the ratio of 4 : 3, so also, on leaving the water, the velocity is increased at the boundary of the raindrop in the same ratio 3 : 4. And just as the retardation of the waves entering the drop gives a pressure of the aether against the surface $\varpi = +2\pi^2 a^2 \varrho\, v^2 (n^2 - 1)$, here indicated by the arrow; so also, on leaving the drop, the sudden acceleration at the boundary, by reaction, gives an equal backward or negative pressure $\varpi = -2\pi^2 a^2 \varrho\, v^2 (n^2 - 1)$. These forces, depending on waves from all directions, applied all over the drop, give rise to surface tension, which is really a central pressure operating through the stress generated in the aether at the boundary of the liquid, by the sudden change in the velocity of the waves.

(ii) The action of passing waves rounds up small masses of liquid into spheres or spheroids of minimum oblateness: Definite geometrical proof based on a theorem of *Archimedes*.

1. The researches of ancient and modern geometers on isoperimetric problems, more especially those of *Euler* and *Lagrange*, *Weierstrass* and *Schwarz*, have shown that a circle has maximum area for a given perimeter; so that for a fixed area, the circle, of all possible geometric figures, has the minimum perimeter.

Many years ago *Weierstrass* placed the Calculus of Variations on a basis of strict rigor; and following his methods, *Schwarz* has dealt extensively with the general problem of minimal surfaces. Of these surfaces the sphere is the simplest, and it is easily shown that it has maximum volume for a fixed surface; or for a fixed volume has the minimal surface.

2. The globular form of liquid drops of water is illustrated by the rainbow, where the smallest deviation from the spherical figure in the water drops would destroy the observed arrangement of colors. Mercury, molten metal, molten glass, suspended globules of oil, and other liquids, have a similar form; and we are naturally led to inquire why nature adopts what mathematicians call minimal surfaces for such masses of liquid.

3. If r be the radius of a sphere, the volume becomes

$$V' = {}^4/_3 \pi r^3\,. \qquad (38)$$

And for the volume of an oblate spheroid, produced by the revolution of an ellipse about its minor axis, we have

$$V = {}^4/_3 \pi a^2 b = {}^4/_3 \pi a^3 \sqrt{(1 - e^2)} \qquad (39)$$

where e is the eccentricity of the sections through the shorter axis b. For equal volumes, $V = V'$, the surfaces $S > S'$, or the surface of the spheroid S is always larger than that of the sphere S', as may be proved by the following analysis. The differential expression for the length of a curve along the x-axis is

$$ds/dx = \sqrt{[1 + (dy/dx)^2 + (dz/dx)^2]} \qquad (40)$$

and the integral:

$$s = \int \sqrt{[1 + (dy/dx)^2 + (dz/dx)^2]}\, dx\,. \qquad (41)$$

T. J. J. See. New Theory of the Aether.

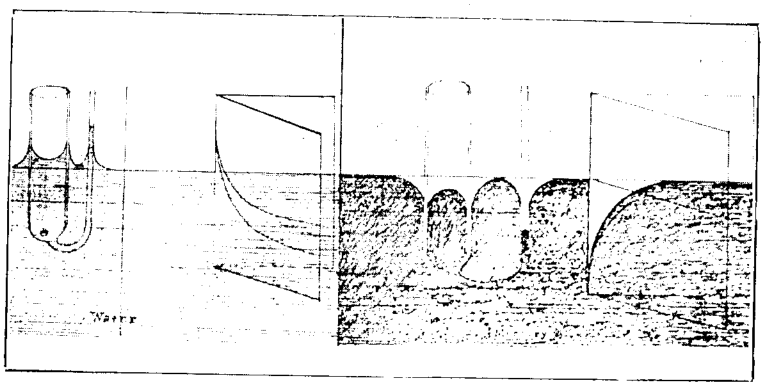

Fig. 9. Illustration of more extraordinary capillary phenomena, for water and mercury.

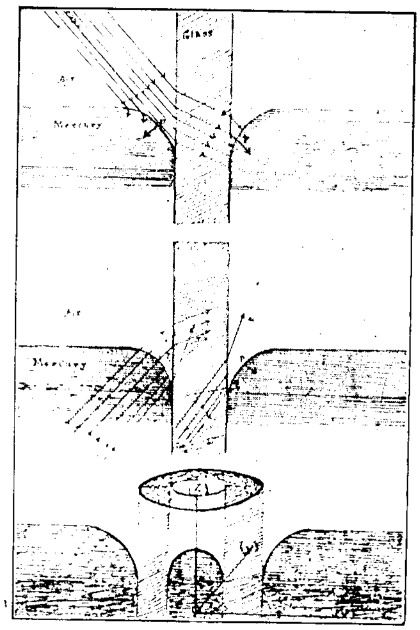

Fig. 13. Illustration of the disturbance of the wave-front when rays pass from air to glass and mercury in contact, giving rise to the observed negative capillary forces.

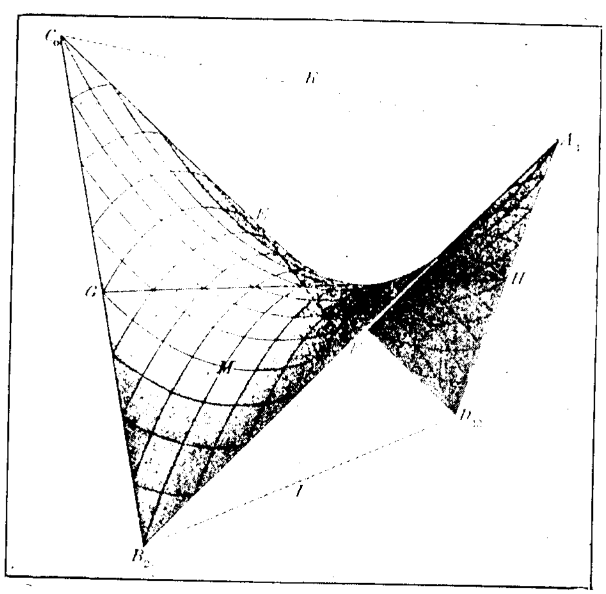

Fig. 14. Prof. *Schwarz*'s illustration of the minimal surface $1/R_1 + 1/R_2 = 0$, with equal but opposite curvature on the two sides.

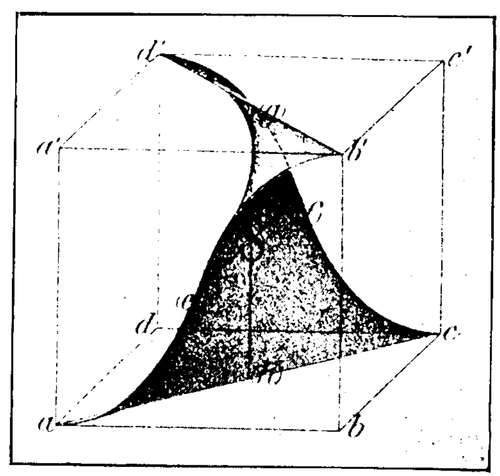

Fig. 15. Prof. *Schwarz*'s illustration of another twisted minimal surface, $1/R_1 + 1/R_2 = 0$.

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4. And for an oblate spheroid we have

$$ds = (1 + b^4 x^2/a^4 y^2)^{1/2}\,dx\,. \qquad (42)$$

If the surface be S we shall have by calculation:

$$S = 2\pi \int x\,ds = 2\pi \cdot ca^2/b^2 \cdot \int [y^2(c^2 - 1)/c^2 + b^4/a^2c^2]^{1/2}\,dx \qquad (43)$$

the solution of which is:

$$\begin{aligned} S &= 2\pi a^2 + \pi a^2 [(1 - c^2)/c] \cdot \log_\varepsilon [(1+c)/(1-c)] \\ &= 2\pi a^2 \{1 + {}^1/_2 (1 - c^2)/c \cdot \log_\varepsilon [(1+c)/(1-c)]\} \end{aligned} \qquad (44)$$

where $\log_\varepsilon$ denotes the natural or Neperian logarithm.

5. For a sphere surface we have the much simpler algebraic expression:

$$S' = 4\pi r^2 = 4\pi a^2 (1 - c^2)^{1/3} \qquad (45)$$

where the radius $r = a(1 - c^2)^{1/6}$, is for the sphere of the equal volume with the spheroid.

6. To apply these formulae to a numerical example, we take the case of the earth with equatorial radius $a = 1$, and the oblateness $1/298.3$; which gives for the eccentricity of the terrestrial meridian

$$\begin{aligned} 1 : 298.3 &= 1 - \sqrt{(1 - c^2)} \\ c &= 0.0818133\,. \end{aligned} \qquad (46)$$

By the second term of the formula for the surface, we find:

$$^1/_2 (1 - c^2)/c = 6.070567\,. \qquad (47)$$

$$\begin{aligned} \log [(1+c)/(1-c)] &= 0.0712213 \\ \log_\varepsilon [(1+c)/(1-c)] &= 0.1639933\,. \end{aligned} \qquad (48)$$

And since $2\pi = 6.2831852$, the second term, with the factors depending on the eccentricity, becomes:

$$2\pi\, {}^1/_2 (1 - c^2)/c \cdot \log_\varepsilon [(1+c)/(1-c)] = 6.2551140\,. \qquad (49)$$

7. On adding the first term, we get for the whole surface of the oblate spheroid S, and of the equal sphere S':

$$\begin{aligned} S &= 12.538299 \\ S' &= 12.538270\,. \end{aligned} \qquad (50)$$

The difference between the surfaces of the spheroid and sphere:

$$S - S' = 0.000029\,. \qquad (51)$$

Accordingly, it thus appears that for small oblateness, there is very little difference between the surface of the spheroid, and the surface of a sphere of equal volume. In case of the earth's oblateness, $1/298.3$, the difference in the surfaces is only 29 parts in 12538270, or one part in 432000.

8. This example proves that the spherical surface is a minimum, because it is the figure to which the oblate spheroid approaches nearer and nearer as the oblateness is made smaller than any assignable quantity.

In the theory of capillarity and similar surfaces, in three dimensions, the surface has the general form:

$$S = \iint \sqrt{\{1 + (dz/dx)^2 + (dz/dy)^2\}}\,dx\,dy\,. \qquad (52)$$

Yet for spheroidal drops of liquid of perfect symmetry the above simpler method of solution is sufficient, and we shall not go into more complex surfaces.

9. For from a physical point of view, we must remember that waves are propagated more rapidly in air than in liquids, such as water, oil, mercury, etc., as shown by the observed refractive indices and electric resistances. Thus, in passing through liquids, the waves encounter sudden resistance at the boundary, and the velocity in the liquid decreases from v to v'; as the waves leave the liquid, the velocity increases from v' to v. Wave energy is thus given up on entering the liquid, owing to internal retardation. On leaving the liquid, the waves are no longer retarded, but actually accelerated, and thus drawing new energy from the unlimited reservoir of the aether they react, or »kick back« correspondingly.

It has long been recognized that a ray of light follows the path of least resistance; electric disturbances follow the same law; and generally throughout nature all physical operations take place according to the principle of Least Action. Therefore if an infinite variety of waves from all directions enter and leave the globule of liquid, the action and reaction of their passage will be such as to make the total resistance a minimum. This can happen only when the figure of the globule is spherical or ellipsoidal, with minimum oblateness.

10. Up to the present time we know but little of these waves, yet they appear to correspond to the forces of surface tension, which are superficial in their character and power. Chemical affinity is known to depend on very short waves, as in [illegible]-violet light, which cannot penetrate solids, through even t[illegible]hinnest layers. Such waves can hardly penetrate solids a[illegible]lli, and pass with difficulty through transparent liquids, and gases. Thus it is natural to attribute the forces of surface tension to waves, chiefly of the ultra-violet spectrum, and they may be of even shorter wave-length.

Owing to lack of penetrating power these short waves could not come directly from the interior of the globe, yet they could come from the stars in the immensity of space, the particles of the air, on all sides, and from the surface of the solid earth in the hemisphere below every drop of liquid. The resistance, on entering the liquid, and the reaction on leaving it, are equal, according to the theory of light, (Sir *Herschel*'s article, Light, Encycl. Metr., 1849, § 561). The total effect of the waves is as if the drop were pressed in on all sides, by central forces. This is our explanation of surface tension, and the globular figures noticed in drops of liquid.

11. Now waves coming and going in all directions, will do least work against the globule when its figure is spherical. For a sphere is a minimal surface, and thus gives least chance of collision with the moving aetherons. And when collision occurs for the waves, the spherical figure yields the shortest average path for the waves which enter the mass of liquid. This spherical figure corresponds therefore to the principle of least action for all the waves of the universe; but the truth of the principle can be made clearest by an illustration.

12. *Archimedes* showed, — in a famous theorem which he desired engraved on his tomb, and which was actually found there by *Cicero* when he was consul at Syracuse, 140 years afterwards — that the ratio of the volume of an inscribed sphere to that of the circumscribed cylinder is as 2 : 3. Thus, if waves enter the cylinder at the end they will encounter exactly $^2/_3$ as much resistance from the liquid sphere as from a continuous cylinder of the same liquid.

As the sphere is a minimal surface, and symmetrical in all directions, it is sufficient to consider the waves entering the end of the cylinder from any direction. Let the sphere be imagined to have an expansible but unelastic surface, and after expansion let the surface be punctured, to allow exchange of the fluid. Under these conditions the enclosed and enclosing incompressible fluid may adapt itself to any alteration of the spheroidal volume. Then the altered surface will be greater than the original sphere surface, though the cylinder would still contain all the liquid. The distorted closed surface would thus fill more than $^2/_3$ of the circumscribed Archimedean cylinder; and the total resistance to all the waves within the inner mass of liquid would exceed $^2/_3$ of the total resistance due to the liquid cylinder alone. That is, the surface of the sphere would be increased by dS, so that if the original sphere surface be $S = 4\pi r^2$ the expanded surface would become $S' = S + dS = 4\pi r^2 + dS$; and the original volume $V = {}^4/_3\pi r^3$ would become

$$V' = V + dV = {}^4/_3\pi r^3 + dV. \qquad (53)$$

From this application of the *Weierstrass-Schwarz* mathematical theory of minimal surfaces to a fixed volume of liquid confined within the Archimedean sphere and circumscribed cylinder, it follows therefore that waves passing from all directions through small masses of liquid of any figure whatever, but with greater resistance than air, necessarily will give least action, when the figures of the liquid masses are spherical. If the liquid globes be of appreciable size, the action of gravity on the figures of the liquid will resist the tendency to globular form; for the surface tension is superficial only, while gravity penetrates a mass, and the result is a corresponding spheroidal figure.

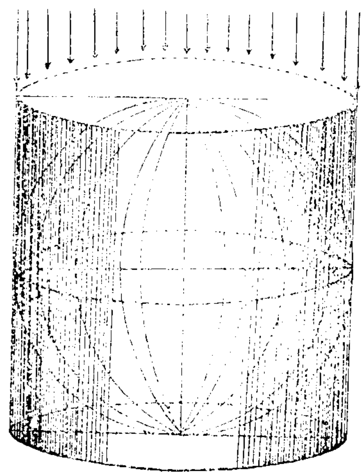

Fig. 5. Illustration of a sphere and circumscribed cylinder, with volumes in the exact ratio of 2 : 3. The illustration is here introduced to prove that if the cylinder be filled with an incompressible liquid, and the sphere surface be dilated into $4\pi r^2 + dS$ above or below, at any parts where the two geometrical figures are not in contact, the continuous fluid within the distorted sphere surface $S' = 4\pi r^2 + dS$ will offer more than 2 : 3 as much resistance to the passing waves as does the whole cylinder of liquid.

The Archimedean Theorem thus yields a rigorous proof that of all possible forms, a distorted drop of liquid may take, the sphere offers minimum resistance to the whole body of waves from all directions: and as nature converts falling globular drops into perfect spheres, this physical fact is a proof that waves incessantly traverse the universe in all directions.

These figures of fluid drops evidently will be of minimum oblateness, or maximum sphericity, but be determined by the balance of forces between gravity on the one hand and surface tension on the other. By equating the observed compression due to gravity to the calculated wave action in the surface tension, we may be able to study the power of the wave action in the case of particular fluids. This method is somewhat analogous to that used by *Quincke* in his researches on surface tension and needs not be further discussed at present.

5. New Theory of Lightning, based on the Accumulating Stress of the Aether at the Boundary of Coalescing Raindrops, and the Oscillatory Discharge.

(i) General remarks on the phenomena of atmospheric electricity and lightning.

In *Ganot*'s Physics, translated by *Atkinson*, 14th ed. 1893, § 995, we find that the treatment of the causes of atmospheric electricity begins with the following suggestive admission that these operations of nature are clouded in impenetrable darkness:

»Although many hypotheses have been propounded to explain the origin of atmospheric electricity, it must be confessed that our knowledge is in an unsatisfactory state.«

Many observations are accordingly detailed, but the physical cause at work is so completely hidden from our view that no intelligible conclusion can be drawn.

In the wave-theory of molecular forces, we hold that all such forces as surface tension are boundary effects of wave-action; and as the boundaries change rapidly, when the small drops are coalescing into larger ones, there is change of aether stress at the surface of the drops. This is called an electric charge on the raindrops, and as the process goes on throughout the cloud, the derangement of the electric equilibrium becomes so pronounced that a discharge occurs, which is called lightning.

For in the condensation of the drops, the capacity for the enlarged drop to hold the collected charge varies only as the radius r, whereas the amount of electricity accumulating under the condensation is proportional to the number of drops collected together or the total volume of water, ${}^4/_3\pi r^3$, and thus varies as the cube of the radius, which is kr^2 times faster than stable electric equilibrium will support. Thus the tendency to discharge increases as kr^2.

It is remarkable that surface tension of a drop does not increase with the size of the drop, which shows that it is a boundary effect, exactly the same whatever be the radius. This is very unusual with the forces of nature, and implies a tendency to a decrease of the central action in proportion as the surface increases, or as $4\pi r^2$. Hence if surface tension be an electric phenomenon, and the drops be condensing to larger size, the tendency to rupture the electric equilibrium at the boundary by oscillatory discharge will increase as kr^2. This corresponds with the known development of lightning when the droplets coalesce into raindrops.

If the electric tension or aether stress at the boundary of a drop attains too high a value, it breaks away in the form of oscillations, as in the discharge of a Leyden jar. Different drops and different parts of the cloud are under unequal electric tension. And as the cloud of moist air (filled with drop-Leyden-jars, so to speak) is a conductor having both capacity and inductance, the discharge necessarily is oscillatory in character. A flash of lightning is thus a series of waves like that shown by photography from an oscillograph in our laboratories.

If electrodynamic forces control the motions of the planets, as shown in the author's work of 1917, and in AN 5044, 5048, it follows that all bodies are centres of waves propagated from their atoms. Thus every star or planet is a great centre of waves; and the waves are in the medium of the aether, under an elastic power $\epsilon = 689321600000$ times greater than that of our air in proportion to its density.

In the First Paper, AN 5044, it is shown that the gravitational potential introduced by *Laplace*, 1782,

$$V = \iiint [(x-x')^2+(y-y')^2+(z-z')^2]^{-1/2} \sigma \, dx \, dy \, dz \quad (54)$$

represents the accumulated stress under the corresponding amplitudes of waves from the mass

$$M = \iiint \sigma \, dx \, dy \, dz \quad (55)$$

thus making the potential have the simple form

$$V = M/r . \quad (56)$$

In the same way all electrical forces depend on wave-action. In the Third Paper on the new theory of the aether, (AN 5079) we have shown that an electric current is nothing but waves of a certain type about a conducting wire; so that aether and electricity are directly connected in a way which can scarcely be denied. Hence as an electric current is simply ordered waves in the aether, arranged in a certain way about a wire, and traveling away from it with the velocity of light, it is natural to inquire into dynamic and static electricity as we see it in the clouds.

In electrical investigations covering a wide field we find that steady waves maintained along a wire may operate as dynamic electricity. Electric current, for example, is generated by a dynamo out of the magnetic field of the earth, which always exists. Hence as the lightning represents dynamic electricity, due to discharge under accumulating aether stress at the surfaces of raindrops, we should study it in connection with the wave-theory as a whole, which includes the earth's magnetic wave-field.

A valid theory of the lightning, conforming to the wave-theory of physical forces, is therefore a most urgent desideratum of science. And until this is outlined, in accordance with the theory of electric waves and oscillations, the mystery of the lightning cannot be intelligently attacked.

Now since the aether has an elastic power $\epsilon = 689321600000$ times greater than that of our air in proportion to its density, we see that lightning is a luminous effect of wave oscillations in this enormously elastic corpuscular medium, which accounts for the violence of the electric shock to material objects of the world. The aether vibrations resulting from lightning as an electric-wave agitation, naturally produce waves in physical bodies, which are then conveyed from the scene of the thunderbolt to other parts of the earth by vibrations in the air, and thus only travel with the speed of about 1100 feet per second.

This view that lightning is oscillatory is proved also by experience in high power telephone lines, which so frequently have their terminals burnt out by the waves induced by lightning. These injuries to the terminals are great sources of loss to telephone companies, and electric engineers thus labor to relieve such inductions in their lines in the hope of saving their terminals as much as possible.

Nothing but a series of electric waves, invisible and generally unsuspected, yet generated in the aether by the successive discharges of parts of the cloud in lightning, could cause these disastrous inductions, which travel with the speed of light, 902000 times faster than sound in the air. The lines so burnt out are interrupted and out of service at the instant we see the flash, but the shock to the earth is felt somewhat later, owing to the slow propagation of the earth wave and the air waves, both of which travel with comparatively low speed.

We conclude therefore that the terrific power shown in the action of lightning has its source in the strain of the enormously elastic aether, and its sudden release, through a series of long waves like those of an oscillograph. This causes the whole aether-field of the earth to oscillate, in a series of waves long enough to penetrate solid bodies. The series of physical oscillations thus set up jar the very earth violently where the lightning strikes.

(ii) The molecular forces operating in raindrops are due to waves traversing the world, and thus lightning depends on such accumulating aether stresses at the boundary of the drops.

In many treatises on the atmosphere it is noted that clouds are in general electrified, usually positively, but sometimes negatively, and only differ electrically from the earth in their higher or lower potential. The formation of a positively electrified cloud is by some authorities attributed to the vapor disengaged from the earth. Our view, however, is that the waves which give rise to molecular forces are always traversing the world, but the state of the cloud, or vapor above the earth, by condensation of droplets, may vary the resistance to the passing waves, and thus give rise to difference of electric potential between the cloud and earth.

It is well known that the electrical capacity of a drop is equal to the radius; which shows that large drops have an increasing capacity, but it augments slowly, as the cube root of the mass. For if m be the mass, we have

$$m = {}^4/_3\pi r^3 \quad \sum_{i=1}^{i=\infty} m_i = {}^4/_3\pi \sum_{i=1}^{i=\infty} r_i^3 \quad (57)$$

and the capacity $\quad r = \sqrt[3]{[{}^3/_4 m/\pi]} . \quad (58)$

After condensation the radius of the large drop becomes

$$R = \sqrt[3]{\sum_{i=1}^{i=\infty} [{}^3/_4 m_i/\pi]} = \sqrt[3]{\sum_{i=1}^{i=\infty} r_i^3} = \sum_{i=1}^{i=\infty} r_i . \quad (59)$$

Now when billions of such droplets coalesce, the capacity of the resulting drop increases as the cube root of the sum of their masses; but the quantity of the electric stress accumulating at the surface of the coalescing drops is merely added. Hence we have:

1. Capacity

$$R = \sum_{i=1}^{i=\infty} r_i . \quad (60)$$

2. Ratio of accumulating total charge to capicity

$$\sum_{i=1}^{i=\infty} m_i : R = {}^4/_3\pi \sum_{i=1}^{i=\infty} r_i^3 : R = kr^2 . \quad (61)$$

where r is the radius of the average droplet.

3. Accordingly, for equal drops, under the same charge, the tendency to rupture the electric equilibrium is equal to kr^2, or increases as the square of the radius.

From this discussion it follows that the electric tension on the surface of the droplets of water increases as the droplets increase in size, in general as the square of the diameter of the drops. The coalescence of the droplets to form raindrops is therefore the one chief condition requisite to the development of lightning.

In the wave-theory of molecular forces, it is held that the retardation of the waves entering the drops, and their corresponding acceleration on leaving the drops, gives rise to aether stress in the boundary of these globules. This surface stress of the aether at the boundary is the cause underlying surface tension. When the aether is so stressed at the boundary, and the droplets are coalescing, there usually is a changing electrical state, and thus the cloud is electrified.

If we consider the infinitely complex aether wave-field about the earth, which we can form some conception of from figure 14 of the Third Paper (AN 5079), illustrating the earth's magnetic field, we shall easily perceive that it is not possible for droplets to coalesce without changing the electrical resistance or total tension in the aether due to the passing waves.

Before condensation this resistance, in modifying the free wave movements of the aether, is proportional to the total space occupied by the droplets of water, or to be cube of their radii. Yet the capacity of a drop to hold a fixed charge is proportional to its electrical capacity, or simply to the radius. In condensation, therefore, whereby many droplets coalesce into a single drop, the wave resistance remains proportional to the space filled with water, $V = {}^4/_3\pi R^3$, or R^3; but the capacity for maintaining electric equilibrium only increases simply as R.

Thus from the relatively inadequate capacity (R) of the growing drop, compared to the relatively rapid growth of mass (R^3) there arises a tendency to rupture the electric equilibrium, proportional to R^2. This occurs on every raindrop, so that the whole cloud becomes electrically charged, with the condensation of the droplets; and as the process proceeds at unequal rates in different parts of the cloud the increasing electric stress (R^2) finally leads to the development of oscillatory discharge or lightning. This happens as soon as the conductivity of the air permits an oscillatory release of the increasing electric stress on the surface of some of the raindrops.

For observation shows that dry air is a non-conductor of electricity, and therefore when the atmosphere is devoid of moisture, a discharge is difficult, except in the form of sheet lightning, so often observed in dry weather. Accordingly, it will not surprise us to note that lightning develops chiefly during rain, especially if there be an atmospheric commotion, or storm, for changing rapidly the coalescence of the droplets, which also may lead to the freezing of some of them into hail. It is well known that hail usually accompanies most violent thunderstorms.

(iii) Under condensation of globules with the electric tension increasing as kr^2 a cloud or part of a cloud becomes charged and forms with another part of the cloud, or with the earth below, a condenser, — the intervening air being the dielectric.

As a flash of lightning may be several kilometres in length, it is obvious that the electric stress accumulates on the cloud as a whole, in respect to the earth below, which is separated by the dielectric of the atmosphere. Friction, condensation of droplets, and similar causes tend to disturb the electric equilibrium of the earth and clouds in the sky. The battery power of a large cloud in respect to the earth may correspond to 3500000 cells, as long ago shown by *De la Rue* and *Müller* for a lightning flash a mile in length.

This enormous electric power accumulating in the condensing droplets makes the electric tension too high for the relatively decreasing capacity of the drops, and tends to rupture the electric equilibrium relative to the earth below. This indicates that some very active physical agency is at work; and in view of the electro-dynamic wave operations of nature as a whole, it is difficult to refer lightning to any cause other than waves. This physical cause alone would make possible this accumulation of aether stress at the boundary of the globules of the clouds, because at this boundary the wave movement changes suddenly, and the result is electric tension released as lightning.

In an address before the Western Society of Engineers at Chicago, 1920, Dr. *Chas. P. Steinmetz*, the eminent electrical engineer, has discussed the older and the newer theories of lightning. He says that experience proves that not over 1 percent of the electrical discharges take place between the clouds and the earth -- the other 99 percent being between parts of the cloud.

He concludes that flashes from one to two miles in length are progressive in their nature. They start with the puncturing of a short space between groups of drops out of electric equilibrium, 20 or 30 feet apart, and spread until the potentials are equalized to a value corresponding to the voltage required to maintain the discharge in the damp air. The period of the discharge is from 0.00001 to 0.25 second, for the slower-acting flashes of more uniform potential distribution. *Steinmetz* concurs in the view above expressed that only a small fraction of the lightning disturbances are due to direct strokes, the vast body of the breaks and burnouts being due to electrical waves with induced voltages of from 500000 to 1000000.

According to the report in the Literary Digest of Nov. 20, »the method of accumulating a charge of 50000000 volts or possibly twice this value on a cloud was explained as involving an initial charge on small particles of condensed moisture, the initial charge being due to the position of the cloud with respect to the earth. It was explained that the earth was surrounded by an electrostatic field with a gradient outward from the surface. Moisture condensing at a distance of one-half mile from the earth would be in a field at a potential of 100000 volts to earth and would assume a charge corresponding to this potential. By collecting into larger particles the charge would be accumulated until values of 50000000 volts or more would be reached when drops of rain were finally formed. Inequalities of 1 or 2 percent of this value, between sections of a cloud quite close together, would suffice to cause a local discharge which would result in a redistribution of potentials and probably in an extended flash. From the effects of direct strokes it has been estimated that the flow of current may be anywhere from 1000 to 100000 ampères, these estimates being based on the size of the conductors that have been melted during the discharge of a stroke to the ground. The illuminating effect of lightning was used to estimate that the light energy of a flash might be equivalent to ten horse-power-hours.«

The discharge is a release of electric tension in the aether at the surface of the drops, but it has to occur through the medium of the atmosphere, in which the cloud floats. As in Geissler-tube experiments, the velocity is great, but less than that of light; and as the electric resistance changes with the discharge, owing to induction in the clouds and other masses, the path may appear zigzag, as shown by actual flashes. Accordingly, although the electric tension is in a fixed direction, the direction alters with partial release, induction and redistribution of electric tension, so as to give the actual zigzag paths presented by lightning.

It must be understood, in viewing these discharge phenomena, that the electric tension is developed between the earth and cloud, or rather between billions of billions of raindrops in different parts of the cloud. Therefore as the discharge, in a group of drops, proceeds from one part of the cloud towards the earth, or towards an adjacent part of the cloud, the local tension is released, and redistributed as the flash advances; this gives rise to a very sudden rearrangement of the electric stress, and as the resistance along the path also changes, by the release, the zigzag path naturally results. In some cases parts of electrified clouds are so situated, that two or more discharges join together and we have forked lightning.

Now in the case of the Leyden jar discharge, we have seen that it is oscillatory, consisting of a series of waves or surges in the medium, coming with such rapid succession as to leave no impression on the eye, yet capable of being photographed by a rapidly rotating mirror called an oscillograph. In like manner, the lightning is an oscillatory discharge, of the very same kind; and if we could see the surging of the medium, we should perceive a very rapid movement to and fro in this agitation of the aether along the path, which thus becomes luminous because of the violent agitations of the particles of the atmosphere, — the length from the cloud to the earth being so great as to make lightning one of the most impressive and terrifying of the phenomena of nature (see fig. 6, plate 2).

6. Wave-Theory of the Adhesion of a Raindrop to a Window Pane: Outline of the Cause of Capillarity and of the Perfect Sphericity of Soap Bubbles.

(i) Wave-theory of the adhesion of a raindrop to a window pane.

The simplest phenomena often give us the most light on the invisible causes underlying the operations of nature; and thus we shall examine somewhat carefully why a raindrop adheres so securely to a windowpane. No phenomenon could be better known than this fact of every day observation. It is everywhere observed, and fortunately we are in a position to attack the problem presented by this phenomenon, because the refractive index of water, and glass, and thus the wave velocities in the two media, are accurately known.

If, therefore, the adhesion of the water to the glass be due to wave action, we shall be able to enter upon the analysis of the forces with some degree of confidence. In figure 7 we show a cross-section of a windowpane, with a drop of rain adhering to it. And we remark that in glass, water, and air, waves of light would have velocities in the ratio of 10, 12, 16: for the refractive indices are inversely as the velocities, and the approximate values of these indices are

$$\text{Air-water, } n = 1.33 = {}^{16}/_{12}$$
$$\text{Air-glass, } n = 1.60 = {}^{16}/_{10}.$$

Accordingly, the following figure is very suggestive; for we see immediately the forces generated, in the propagation of light. It thus becomes clear that in passing from air to water, waves of light are decreased in velocity by ${}^{4}/_{16}$ or ${}^{1}/_{4}$. In passing from water to glass the velocity likewise changes from ${}^{12}/_{16}$ to ${}^{10}/_{16}$, which is ${}^{1}/_{8}$. The wave motion thus changes velocity and generates a strain in the layer of aether and matter containing the surface of the water, and likewise at the surface between the glass and the water.

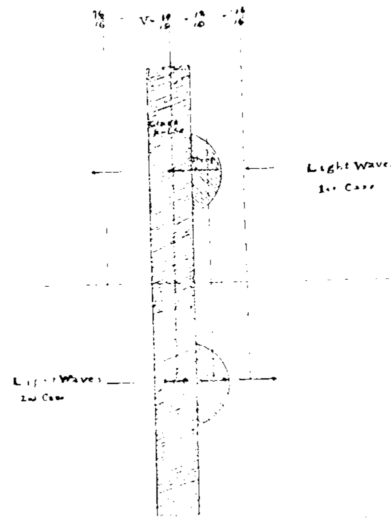

Fig. 7. Illustration of the adhesion of a raindrop to a windowpane, and of the sudden change in the velocity of the waves at the boundary of the three media, air, water, glass, upon which the adhesion depends.

The tension of the molecules due to the aether stress of the surface of the water is called surface tension; that between the water and glass is called adhesion, and makes the water adhere to the windowpane. From the above figure we perceive that light waves passing from the air to the raindrop are delayed at the outer boundary of the drop, and thus the wave front presses in on the water, so as to give the surface tension observed. On passing on into the glass there is a second delay of the wave front at the boundary of the glass; and this aether stress or pressure over the surface between the water and glass causes the adhesion by which the water adheres to the glass.

The raindrop, however, adheres to itself somewhat more strongly than to the glass; because if the glass be inclined and jarred, the water will run down and fall off as a drop, leaving merely a thin layer of moisture on the glass. This recognized and obvious effect will hold for waves passing from the air to the glass, as shown in the upper half of the above figure.

But we must consider waves from all directions, and thus we ask what will happen if the waves move in the opposite direction, and have already traversed the glass, and are leaving it to enter first the water, than the air? In this case the effect will be as shown by the arrows in the lower part of figure 7. As the velocity in glass is small, the waves will speed up on passing from glass to water, and again on passing from water to air. And in both cases they will react or »kick back«, giving an aether stress or adhesion of the water to the glass, and at the outer boundary a surface tension next to the air. Accordingly, whether the waves come from the air or from the glass they will give the aether stresses due to change of velocity, and result in the molecular forces observed.

The theory here briefly traced enables us to understand the adhesion of the rain drop to the windowpane. It is beyond doubt a wave phenomenon, because if the aether be filled with waves moving in all directions, these forces will necessarily result. This will hold true for light waves of the visible spectrum, or for waves of shorter length which are found to correspond to the radius of action for capillarity, as observed by *Rücker*, *Reinhold*, *Kelvin* and others.

(ii) The case of mercury, which gives a depressed column in a tube, and apparently is repelled by the glass.

The above explanation of the adhesion of a raindrop to a windowpane outlines briefly the wave-theory of capillarity, but a liquid like mercury which does not wet the glass must be examined. It will be found that the wave-theory will hold for the case of mercury as well as for that of water, but it is necessary to assume great resistance to the aether waves in the mercury, which is what should hold in the propagation of these waves through this dense medium. For in his experiments at the Physical Laboratory in Turin, 1919, Professor *Q. Majorana* found that even the long waves of gravity are sensibly intercepted by a layer of mercury, (cf. Philosophical Magazine, May, 1920, pp. 488–504).

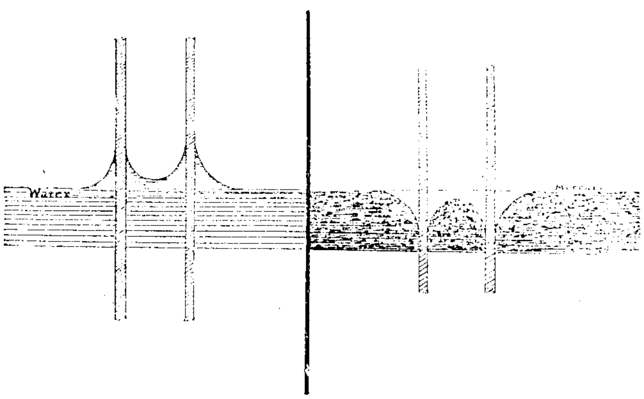

Fig. 8. Illustration of ordinary capillary phenomena, for water and mercury.

All we need to do to explain the negative adhesion of mercury to the glass tube is to take the velocity of propagation of the waves in the several media approximately as follows: air 16, water 12, glass 10, and mercury about half that of glass, or 5.[1] These numbers are approximate only, and in the case of mercury the value is assumed, yet they are sufficiently exact for our present purposes. And thus we see that if mercury offers more resistance to the passage of waves than glass, this fluid will be repelled by the glass tube, and the column of this liquid will be depressed, whereas water and similar fluids are elevated.

This explanation is not so very different from that put forth by *Laplace* in his theory of capillarity a century ago; for this great geometer explained the forces acting on mercury relative to a glass tube as negative, and by his analysis of the forces assumed to be sensible only at insensible distances, obtained a very satisfactory theory of the depression of mercury in capillary tubes. The present theory based on wave action is, however, more logical, and has the advantage of showing why the forces are negative, and can act only at insensible distances.

If mercury will sensibly intercept long gravitational waves, as *Majorana* shows, still more will it resist and quench the shorter waves active in capillarity.

The illustration, fig. 9, pl. 3, of the increasing depression of a mercury column with decreasing diameter of the tube may be regarded as direct proof of the close similarity of the forces which produce elevation of water in capillary tubes with those which depress the corresponding column of mercury. For in both cases the phenomena observed become more extreme with the narrowing of the column — the water rising higher and the mercury sinking lower, relatively to the level of the general free surface.

This effect is well shown in figure 9, plate 3, slightly modified from a work on Practical Physics, by *Black* and *Davis*, the MacMillan Co., New York, 1917, p. 74. Such a contrast in elevations of liquid columns would seem totally inexplicable without a simple and direct theory like that here presented. And if we can prove that with the narrowing of the tubes, wave action may increasingly elevate the height of water in capillarity, it will automatically establish the same cause for the depression of the level of mercury, in similar tubes, which is observed to become more pronounced with the narrowing of the tubes.

About two centuries ago it was observed by *Hawksbee* that if two vertical windowpanes be accurately set at a small angle of mutual inclination in a basin of water, the water line rises in the form of a rectangular hyperbola, showing that in such capillarity the lifting force varies inversely as the diameter or weight of the column to be lifted. I have recently made some observations on the form of the curve of depression for mercury, and confirmed the same law of the rectangular hyperbola referred to its asymptotes. Wherefore it seems impossible to doubt the wave-theory of these capillary phenomena, the cause of which long remained enigmatical and even bewildering to natural philosophers.

In *Atkinson-Ganot*'s Physics, p. 1003, it is shown that the conductivity of mercury for electrical waves is low, 1.6, while for silver it is as high as 100.00, and for copper 99.9. Likewise, (p. 707), we learn that glass offers more electrical resistance than air and other dry gases, while water is a conductor offering much less resistance than either air or glass.

Accordingly, if waves of aether, inclined to the level surface are to pass through water, in contact with glass on one side and air on the other, it will follow that the level

[1] On the scale here used the figure for mercury ought to be not larger than 1, if we judge by the electric resistance of mercury.

of the water should be raised in contact with the glass and be lowered on the side towards the air, in accordance with observation. We shall go into this at greater length in dealing with capillarity, and at present only dwell on it long enough to point out the verification of the wave-theory.

(iii) The perfect sphericity of soap bubbles explained by least action to passing waves, which makes the two concentric sphere surfaces also minimal surfaces.

Just as the *Archimedes-Weierstrass-Schwarz* theory of minimal surfaces, under wave action, will explain the molecular forces which give spherical or spheroidal forms to small masses of liquid; so also will it explain the molecular action of films in such phenomena as soap bubbles.

For a soap bubble is made up of two concentric sphere surfaces — the outer surface and the inner surface. The pressure of the cushion of air within the bubble prevents it from collapsing; and the waves traversing the outer surface act in the same way as in the case of a solid drop of liquid, and thus round up the mass from without.

On the inner surface there is an analogous wave pressure directed towards the liquid and thus acting in an outward direction. This is not from the confined air, which is a discontinuous cushion, but from the infinitely fine network of passing waves. The resistance to the waves through the entire bubble, with the double liquid wall, is least when the path in the water is the shortest; that is, when the waves go as near the centre of the hollow sphere as possible, as may be shown by mathematical analysis. But it may be seen immediately from the geometrical indications of the accompanying figure.

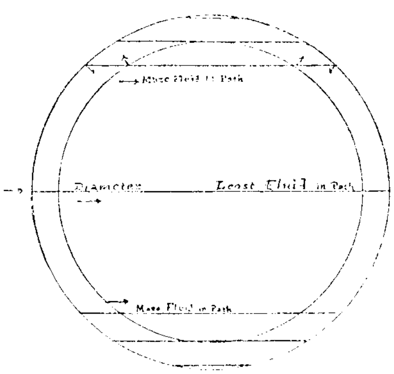

Fig. 10. Magnified cross section of a soap bubble, showing minimum thickness of liquid film at the centre, and least resistance to waves passing in that direction, which explains the central contraction and symmetrical form of a soap bubble.

Just as the film of water is pressed together into a thin layer, by the inward passage of the waves from the outside, so also will the thickness of the bubble as a whole be compressed by inside wave pressure everywhere directed towards the outside. For as the waves near the tangent to the inner boundary of the fluid they react against the adjacent liquid, owing to the greater resistance along these adjacent paths.

By this reaction on the inner walls the liquid is pressed to itself from both sides, and the layer between the outer and inner surfaces made as thin as possible. As the waves keep the confined layer of liquid symmetrically compressed on both sides, rupture of the soap bubble is not very easy. In time it comes about, however, owing to the water trickling down under its own gravity, and thus rounding up into a liquid sphere or spheroid formed right on the lower side of the soap bubble, as it becomes unsymmetrical.

(iv) Direct proof of wave pressure at the boundary of a drop.

The celebrated argument made by *Maxwell*, Treatise on Electricity and Magnetism, 1870, §§ 792–793, and the well known experiments of *Nichols* and *Hull* and *Lebedew*, 1901, shows that aether waves do exert pressure against any surface on which they impinge. Yet in order to have an objective proof of this important theorem drawn directly from nature, by observations which we can easily verify, it is advisable to go into this reasoning somewhat more carefully. We therefore consider the form and action of a series of steady waves in the sea.

1. It is well known that when the waves of the sea approach the shore, where the water is shallow, the motion of the base of the wave is retarded, while the top of it tends to move on as before. The result is the formation of breakers: the base of the wave is so held back that the top becomes steep, and finally curls over till the wave breaks in a whirling rush of foam.

2. Now this delay of the movement of the base causes the wave to exert a pressure against the shore which resists its advance. Accordingly we may thus verify *Maxwell's* conclusion that waves exert pressure against resisting objects. We see also the effects of such resistance in the wearing away of the sea shore when exposed to the dashing inrush of the waves. Sand and soft earth are carried bodily along with the waves, and even solid rock is slowly worn away by the incessant beating of the waves.

3. In order to make an experimentum crucis directly applicable to the problem now in hand, we shall imagine an island table-land in the open sea covered by the water to a depth comparable to the length of the waves which pass over it. Under these circumstances the waves will be retarded as they enter upon the submerged table-land; and in advancing across it they will be shorter and steeper as shown in figure 11. This is similar to the vertical surging of the surface, in a stream, which thus shows where rocks are in the bed; for the resistance of underlying movement manifests itself in alterations of the surface, so that the fluid is thrown into surface irregularities.

4. Accordingly, we perceive that as the resisted waves advance over the submerged table-land, which is not taken to be near enough to the surface to form breakers, they are shorter and steeper than the original waves as they come in from the deeper sea.

Now what will happen when the resisted waves at length depart from the table-land, and again enter the deep sea on the opposite side?

First, it is evident that the waves will take on greater speed in the deeper water; they will therefore become longer in the freely yielding deep sea, just as they become shortened by resistance as they ran over the shallow water.

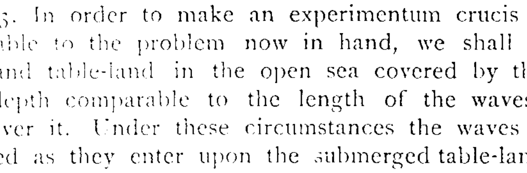

Fig. 11. Illustration of the pressure exerted by sea-waves against a submerged island plateau.

Second, as the longer waves, moving more freely into the deep sea, have the elements of their rotations accelerated, this more rapid whirl of the water will exert a backward pressure against the submerged table-land. Accordingly there is not only a pressure against the resisting land on approaching it, but also on leaving it.

5. We may easily satisfy ourselves of the correctness of this conclusion by the following independent experiment. Suppose an athlete standing on a spring board, and imagine the whole foundation carried along at a uniform rate, as if moving at a fixed speed on a railroad track. If the athlete wishes to accelerate his speed temporarily he will have to jump forward, by the exertion of his muscles, which will be sustained by the elastic rebound of the springboard. In other words, to give the athlete a greater velocity, forward, he must kick back against the foundation on which he is transported along. This is analogous to sea waves speeding up on leaving the submerged table-land: action and reaction are equal and opposite, and this general law is applicable to all nature.

6. Now consider the waves of light entering the raindrop and leaving it by the paths shown in the foregoing figure 4. Then, we know from *Maxwell's* reasoning, and these practical experiments, that there is an inward pressure against the surface of the water at the point of entrance, and a corresponding reaction against the surface at the point of emergence, because there is a sudden change of wave velocity at both points. This is the physical basis of our theory of surface tension.

7. If waves fill the world having all directions and wave lengths, it will follow that at the boundary of liquid drops, there is a sudden transition: the waves enter from all directions, but they also leave, in all directions, along various paths. And in every direction within the drop the speed is less than the original speed. There is thus a surface reaction towards the centre, owing to the decrease of action at the boundary, but coming and going.

7. The Fundamental Facts of Observation appear to furnish Criteria for a Wave-Theory of Capillarity.

(i) Detailed examination of the distortion of the wave-front and reaction of the waves in air, water, glass.

1. Consider waves traversing the universe, in all possible directions, and of any required length. What will happen when the waves pass from air to water and glass respectively? Take the refractive index of water at $n = 1.33$, and of glass at about 1.60; then it is evident that the velocity of waves of light or chemical activity will be swiftest in air, next swiftest in water, and slowest in glass. The relative velocities in the media air, water, glass are as 16, 12, 10 respectively.

2. Case 1, waves passing from air to water and glass.

Let figure 12 represent a section of plate glass partly surrounded by water: the ray will traverse the successive media, air-water-glass, and the wave surface will suffer distortion as shown in the figure. As the ray spreads out, under the effects of refraction, and the velocity is decreased both in the water and in the glass, the wave front will take the convex form shown by the heavy line.

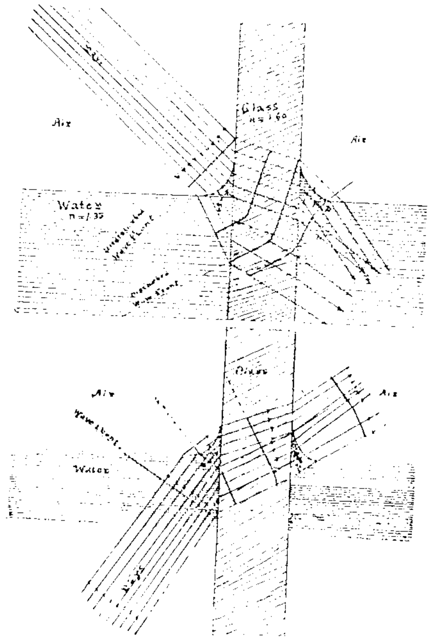

Fig. 12. Illustration of the disturbance of the wave-front when rays pass from air to water and glass in contact, giving rise to the observed capillary forces.

3. This change of the wave front from a plane surface into a convex surface will occur right at the surface of the water and glass. The ray r will spread in two directions, and its progress is most resisted by the glass and less resisted by the water at its contact with the glass. The water is fluid, while the glass is solid; and thus if the wave front is to remain continuous, the fluid must so adopt itself to the glass as to secure continuity — that is the level of the water must rise around the glass.

4. *Maxwell* showed (Treatise on Electricity and Magnetism, 1870, § 793) that all waves exert pressure against a surface on which they fall. Hence if there be pressure against the liquid surface, it will thereby be carried up around the glass which is solid. Thus it is easy to see why waves make the liquid film rise around the glass, as observed in the phenomenon of capillarity. The above argument and illustration will apply to all cases where the waves descend. For if they emerge on the opposite side of the glass, the reaction of the waves will carry the water up on that side also, as shown by the heavy line on the right.

5. Let us now consider the case in which the waves ascend, as shown by the arrows in the lower part of figure 12, of a cross-section of the wave front.

This figure shows what will happen in all cases of ascending waves, propagated more rapidly in air, less rapidly in water, and least rapidly in glass. On the right the reaction of the emerging wave front will force the level of the water up about the glass, by the reversed wave pressure p.

(ii) Detailed study of the distortion of the wave front when the waves are propagated in air, glass, mercury.

1. In the first instance suppose as before that the

waves descend from above, then the cross-section of the wave front is shown by figure 13, plate 3.

(a) If the resistance to the waves in mercury be greater than in air and glass, then the refractions and reactionary pressures to the wave surface as it advances will be of the type here pictured: the waves escaping from the mercury and speeding on more rapidly in the glass than in the mercury will by the rebound, $-p$, press the fluid back from the glass, on the left. On the right, the increased resistance to the waves due to the mercury, as the waves leave the glass and travel more slowly in the liquid mercury, will push that liquid away with a positive pressure $+p$.

(b) The result is a forcing of the fluid downward, by rebound, on the left; and forcing of it forward by equal direct pressure on the right. In both cases therefore the mercury is pressed away from the glass. The mercury therefore seems to attract itself more than it does the glass: which is the usual explanation of the negative capillarity of mercury. But we must also consider why the tube of mercury is below the normal level of the liquid, and hence we proceed to view the action of ascending waves.

2. Waves from below would act as shown in the central part of figure 13, plate 3. In all cases the level of the mercury is depressed.

(a) On the left, the speeding up of waves leaving the mercury for the air and glass, gives a reaction with negative pressure, $-p$, and the mercury is forced back, or lowered around the glass, as if the fluid were repelled by the glass.

(b) On the right, the increased resistance due to the mercury, when the waves emerge from the glass into the air or mercury gives a direct action or positive pressure, forcing the mercury away, the reaction at the corner giving the most decided downward pressure. Thus the level of the fluid is lowered, whether the waves descend or ascend; and, as the waves come from every direction, the apparent repulsion of the mercury from the glass is symmetrical, as found by observation.

3. The change of level in the case of a column of mercury depressed in a tube is due to the above causes also. For when the mercury is acted upon powerfully on all sides, the action conforms to *Laplace*'s integral (Mécanique Céleste, Liv. X. supplement 2)

$$I_\lambda = {}^1/_2 H \iint \mathrm{d}x\,\mathrm{d}y \left\{\partial\left[p\,(1+p^2+q^2)^{-1/2}\right]/\partial x + \partial\left[q\,(1+p^2+q^2)^{-1/2}\right]/\partial y\right\} = {}^1/_2 H\,c\cos\varpi \qquad (62)$$

where c denotes the circumference of the tube, and the angle ϖ is a constant found by observation.

To explain this formula, we remark that if $u = f(x, y, z)$ be the equation of the spherical surface of radius of curvature r touching the capillary surface at any point,

$$\mathrm{d}r = \left[(\partial u/\partial x)^2 + (\partial u/\partial y)^2 + (\partial u/\partial z)^2\right]^{-1/2} \mathrm{d}u\,. \qquad (63)$$

The z-axis is vertical, and $\quad \partial u/\partial z = 1;\ \partial u/\partial x = -p;\ \partial u/\partial y = -q \qquad (64)$

$$\partial r/\partial z = \left[1 + (\partial u/\partial x)^2 + (\partial u/\partial y)^2\right]^{-1/2} = (1+p^2+q^2)^{-1/2} \qquad (65)$$

$$\partial r/\partial x = \left[1 + (\partial u/\partial x)^2 + (\partial u/\partial y)^2\right]^{-1/2} \partial u/\partial x = -(1+p^2+q^2)^{-1/2}\,p \qquad (66)$$

$$\partial r/\partial y = \left(1 + (\partial u/\partial x)^2 + (\partial u/\partial y)^2\right]^{-1/2} \partial u/\partial y = -(1+p^2+q^2)^{-1/2}\,q\,. \qquad (67)$$

Accordingly, if g be the acceleration due to gravity, σ the density of the fluid, and V the volume elevated by the force of capillarity, we get *Laplace*'s equation, founded on the above integral:

$$g\,\sigma\,V = {}^1/_2 H\,c\cos\varpi\,. \qquad (68)$$

After deriving the above double integral and this formula *Laplace* remarks: »Thus the mass of the fluid elevated above the level by the capillary action is proportional to the circumference of the section of the inner surface of the tube.« That is, the lifting force is proportional to the extent of the glass surface acting on the fluid, — which again very strongly points to wave-action, exerting sensible influence only at insensible distances. The tube of glass is solid and cannot be raised, and the reaction simply sinks the central column of mercury as if repelled by the glass. Hence the marginal depression of the fluid is also accompanied by a lowering of the central column below that in the exterior basin of mercury.

4. From these sketches of the wave fronts taken by liquids of various power of resistance, under wave action from all directions, we perceive that the fundamental facts of capillarity established by observation agree qualitatively with the wave-theory. A better concordance probably could not be expected, and it is difficult to imagine such conformity in theory without a true physical cause underlying the observed laws of nature.

Let us now recur to the theory of the rainbow, and note the shortening of the waves within the drop shown in the foregoing figure 4. As the velocity of the aether waves is changed suddenly at the boundary, both on entering and emerging from the drop, the pressure exerted at the boundary is obvious.

(iii) Method of calculating the kinetic pressure when the waves are resisted.

A mass of water m, in which waves are advancing with the velocity v, has the corresponding kinetic energy ${}^1/_2 m v^2 = E$. After a certain amount of resistance, suppose the velocity of the waves becomes v_1, then the kinetic energy becomes less, as $v_1 < v$, and we have:

$$E_1 = {}^1/_2 m\,v_1^{\,2}\,. \qquad (69)$$

Therefore the loss of energy due to retardation of velocity of wave motion becomes:

$$E - E_1 = {}^1/_2 m\,(v^2 - v_1^{\,2})\,. \qquad (70)$$

Accordingly, since a decrease of depth delays the propagation of sea waves, and constantly reduces the velocity, we infer that so long as the waves of the sea beat upon the shallow shore, there is decrease of energy in the waves. Part of the energy is lost by the dashing of the water against the shore. But any action which delays the speed of the water is equivalent to holding it back; and when the rush of the water is hold back it exerts a steady pressure against the resisting shore.

If the shore were made by an artificial platform of boards, under water, the inrush of the waves would tend to sweep the platform away; and observation shows that on the sea shore vast banks of sand, and loose gravel are hurled inland by the whirling of the inrushing waves against the resisting sea beach. These phenomena are well known, and are familiar to all observers of nature.

Let us now consider what will happen when aether waves in the form of light, heat, electric current, etc., fall upon a medium in which the velocity is less than in air. As the aether waves travel less rapidly in the fluid than in air, there must be an arrest or stopping of the velocity of the wave motion at the surface of the denser medium. Here we have a definite physical boundary, where the velocity changes suddenly. In any two media, the velocities are directly as the refractive indices: thus in air-water $n = 4/3$, and we know by *Foucault's* experiment of 1853, that the velocity in air is to that in water about as 4 to 3.

But the energy of the wave motion is as the squares of the velocities; and hence for air-water $n = 4/3$, we have $n^2 = 16/9$, and $n^2 - 1 = 7/9$. Accordingly, when the aether waves pass from air to water, they are so retarded at the boundary as to suddenly surrender $7/9$th of their kinetic energy to the molecules constituting the boundary of the liquid. This loss of energy at the boundary

$$n^2 - 1 = E - E_1 = \tfrac{1}{2} m (v^2 - v_1^2) = 7/9 \qquad (71)$$

is incessant.

Along with the loss of energy as the ray enters the drop, there is refraction, dispersion, etc., such as we see in the rainbow. In his celebrated article Light, Encycl. Metr., § 561, Sir *John Herschel* dwells on the fact that the forces producing refraction or dispersion are of practically infinite intensity. For the light not only is retarded in its forward motion, but also turned out of its rectilinear course, and the waves have increasing dispersion with decreasing wave-length.

Moreover, since on leaving the liquid drop for the air again, the velocity of the waves increase from about 3 to 4, this increases the energy in the aether waves in the ratio of $(4/3)^2$, so that the waves outside, in virtue of speeding away with an energy measured by n^2, have $n^2 - 1$ more energy than those within the drop. Hence the receding waves react on the boundary of the liquid drop with an energy amounting to $7/9$ of that they have in free space.

Taking the refractive index as the most certain of physical data, we have: glass, $n = 1.608$; water, 1.336; air, 1.000.

Thus the wave disturbances travel 1.608 times faster in air than in glass; and 1.336 faster in water than in air. The progress in water, however, is also faster than in glass by the differences:

$$\begin{array}{ll} \text{Air-Glass} & 0.608 = v - v_1 \\ \text{Air-Water} & 0.336 = \quad » \\ \text{Water-Glass} & 0.272 = \quad » \end{array} \qquad (72)$$

Accordingly, from these data on the refractive indices, and the easily verified phenomena presented by sea waves, we see clearly that the inward pressure of the aether waves at the boundaries of liquids and solids cannot be denied. This pressure is easily shown to be in dynes per sq. cm.:

$$\tfrac{1}{2} \varrho v^2 = 2\pi^2 a^2 \varrho \nu^2 \qquad (7$$

where ϱ is the density of the medium, ν the frequenc a the amplitude of the waves.

When the waves are resisted at the boundary of a soli or liquid body, the refractive action $n^2 - 1$ enters as a facto and the loss of energy must be introduced. Thus on enterin and on emerging from such boundaries the energy of th wave motion becomes

$$E = 2\pi^2 a^2 \varrho \nu^2 (n^2 - 1) \qquad (74$$

as already found in section 2.

8. Geometrical Criteria for the Types of Minima Surfaces possible in Nature, and their Physica Significance under the Wave-Theory.

(i) The geometrical criteria for minimal surfaces recog nized by *Schwarz*.

In his Gesammelte Mathematische Abhandlungen, vol. 1 1890, Berlin, pp. 325–329, Prof. *H. A. Schwarz*, the eminen geometer of the University of Berlin, has condensed the results of his extensive researches on minimal surfaces. It is not practicable to develop his results in any detail here, but we remark that the final equations resemble those of *Laplace* for the surface of the fluid elevated in capillary tubes; and that the criteria were begun by *Euler* and have been improved by later geometers — *Laplace, Gauss, Roberts, Riemann, Jellett, Weierstrass, Schwarz*.

Let x, y, z denote the rectangular coordinates of any point of the minimal surface. The coordinate z may be considered as a function of the two other coordinates x, y, as in the capillary formula of *Laplace* above explained. Moreover for symmetry and simplicity we put:

$$p = \partial z/\partial x, \; q = \partial z/\partial y,$$

$$r = \partial^2 z/\partial x^2, \; s = \partial^2 z/\partial x\,\partial y, \; t = \partial^2 z/\partial y^2 . \qquad (75)$$

$$P = (1 + p^2 + q^2)^{-1/2} (z - p\,x - q\,y) . \qquad (76)$$

$$1/R_1 + 1/R_2 = -(1 + p^2 + q^2)^{-3/2} [(1 + q^2)\, r + - 2p\,q\,s + (1 + p^2)\, t] \qquad (77)$$

$$\xi = -p\,P - x\,(1 + p^2 + q^2)^{1/2}$$

$$\eta = -q\,P - y\,(1 + p^2 + q^2)^{1/2} \qquad (78)$$

$$\mathrm{d}S = (1 + p^2 + q^2)^{1/2} \cdot \mathrm{d}x\,\mathrm{d}y \qquad (79)$$

$$S = \iint (1 + p^2 + q^2)^{1/2} \cdot \mathrm{d}x\,\mathrm{d}y . \qquad (80)$$

In this double integral the integration for the surface is to be extended to all its elements. Under these suppositions we have the differential equation for the minimal surface:

$$(\partial \xi/\partial x + \partial \eta/\partial y)\, \mathrm{d}x\,\mathrm{d}y = P\,(1/R_1 + 1/R_2)\, \mathrm{d}S - 2\mathrm{d}S \qquad (81)$$

of which the integral becomes:

$$2S = \iint P\,(1/R_1 + 1/R_2)\, \mathrm{d}S + \int (\eta\, \mathrm{d}x + \xi\, \mathrm{d}y) . \qquad (82)$$

The double integral is to be extended over the entire surface considered, and the single integral over all elements of the boundary taken in the sense indicated by the derivation of the formula.

Schwarz remarks that if we apply the formula thus given by *Jellett* in 1853, (Sur la surface dont la courbure est constante, Liouville's Journal de Math. pur. et appl., Tome 18.163–167) to minimal surfaces, the theorem indicated in equation (82) will hold true. Yet another proof may be derived by the following process.

We take the normals as drawn from every point of the surface, and lay off thereon on the same side of the surface the length h. Let the volume thus arising be denoted by V, and we have

$$V = hS + {}^1/_2 h^2 \iint (1/R_1 + 1/R_2)\, dS + {}^1/_3 h^3 \iint (1/R_1 R_2)\, dS\,. \tag{83}$$

But there is another expression for this volume as follows:

$$V = {}^1/_3 h S + {}^1/_3 h \iint P(1/R_1 + 1/R_2)\, dS + {}^1/_3 h^2 \iint (P/R_1 R_2)\, dS + {}^1/_3 h^2 \iint (1/R_1 + 1/R_2)\, dS + {}^1/_3 h^3 \iint (1/R_1 R_2)\, dS +$$
$$+ {}^1/_3 h \int \begin{vmatrix} X & Y & Z \\ x & y & z \\ dx & dy & dz \end{vmatrix} + {}^1/_6 h^2 \int \begin{vmatrix} X & Y & Z \\ x & y & z \\ dX & dY & dZ \end{vmatrix} \tag{84}$$

In this equation X, Y, Z denote the cosines of the angles which the normals to the surface make with the directions of the coordinate axes, and the quantity

$$P = Xx + Yy + Zz\,. \tag{85}$$

The double integrals are to be extended over all elements of the surface, while the single integrals are to be extended over all elements of the boundary line.

By comparison of the coefficients of the terms in (83) and (84) multiplied by h and h^2 respectively, we find the equations:

$$2S = \iint P(1/R_1 + 1/R_2)\, dS + \int \begin{vmatrix} X & Y & Z \\ x & y & z \\ dx & dy & dz \end{vmatrix} \tag{86}$$

$$\iint \left(\frac{1}{R_1} + \frac{1}{R_2}\right) dS = 2 \iint \frac{P\, dS}{R_1 R_2} + \int \begin{vmatrix} X & Y & Z \\ x & y & z \\ dX & dY & dZ \end{vmatrix} \tag{87}$$

If $1/R_1 + 1/R_2 = 0$, which is the condition that the curvatures on the opposite sides are equal and opposite, equation (86) gives *Schwarz*'s criterion for the area of a minimal surface (p. 178). In less rigorous form this criterion was first indicated by *Euler*, but *Weierstrass* and *Schwarz* have greatly improved the demonstration. Moreover *Schwarz* has discovered from theory a new surface afterwards verified by experiment, as shown below.

Minimal Surfaces of special physical interest.

1. The sphere,

$$x^2 + y^2 + z^2 = r^2\,. \tag{88}$$

2. The spheroid,

$$x^2/a^2 + (y^2 + z^2)/[a^2(1 - e^2)] = 1\,. \tag{89}$$

3. The spherical soap bubble, a film of liquid bounded by two concentric spheres:

$$\begin{aligned} x^2 + y^2 + z^2 &= r^2, \text{ outer surface,} \\ x_1^2 + y_1^2 + z_1^2 &= r_1^2, \text{ inner surface.} \end{aligned} \tag{90}$$

4. Surfaces with equal but opposite curvature at every point,

$$1/R_1 + 1/R_2 = 0\,. \tag{91}$$

(see figure 14, 15 on plate 3.)

5. Surfaces stretched from fixed frames, and bending under gravity (see figure 16).

6. *Schwarz*'s theoretically predicted surface afterwards experimentally verified.

The accompanying figure 17, from *Poincaré*'s Capillarité, Paris, 1895, p. 66, exhibits to the eye the form of one of Prof. *Schwarz*'s helicoidal surfaces, which he was able to realize experimentally.

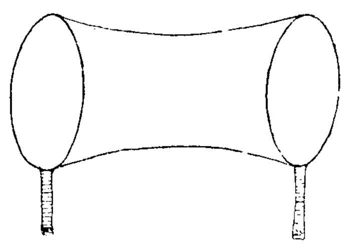

Fig. 16. Illustration of a cylindrical soap film drawn out by separating two wire frames to which the film is attached.

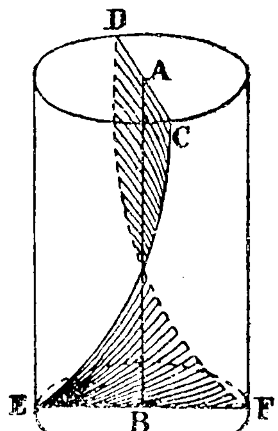

Fig. 17. *Poincaré*'s figure of the helicoidal minimal surface, first theoretically predicted by Prof. *Schwarz*, and afterwards verified by experiment.

»M. *Schwarz* a pu réaliser expérimentalement cette surface d'équilibre.«

»Pour cela, il tend un fil suivant l'axe AB d'un cylindre de verre au moyen de deux fils metalliques CD et EF s'appuyant sur les bases du cylindre. Ces deux fils étant parallèles, il forme une lame plane s'appuyant sur les trois fils AB, CD, EF. En tournant CD, la lame se déforme et engendre une surface $ECDF$ passant par AB et coupant normalement la surface du cylindre. Cette surface est un hélicoide.«

»Avec quelques précautions on peut obtenir ainsi une surface à plusieurs spires. Si on supprime le fil vertical AB, l'hélicoide n'en reste pas moins une surface d'équilibre, mais alors l'équilibre n'est plus stable et il est impossible d'obtenir expérimentalement une surface à plusieurs spires. M. *Schwarz* a même constaté que si, après avoir formé une telle surface à l'aide du fil central, on coupe ce fil, la lame hélicoidale disparaît et se transforme en deux lames planes fermant les bases du cylindre de verre.«

(ii) The types of minimal surfaces which would result from waves coming from all directions.

These minimal surfaces obviously would be classed as follows:

1. Sphere surfaces, around liquid drops of any and every description, in accordance with observation, for all natural liquids known in nature, or produced artificially from

solids by the application of high temperatures. The perfect sphericity of figure for the globules implies that the waves come and go in every direction, that the pressure of the entering and the reaction of the departing waves on the average is exactly central.

2. Hollow spherical globules, such as soap bubbles, which have the liquid enclosed between two concentric sphere surfaces. These fluid films are perfectly symmetrical about the centre and tend to become thinner by the external and internal wave pressure at the boundaries. The compression of the liquid layer on the two surfaces of the bubble increases the elastic power of the aether and matter enclosed between the concentric surfaces of fluid, and thereby gives the film a certain tensile strength, as observed in soap bubbles.

3. The next most natural class of minimal surfaces would be those which fulfill the condition $1/R_1 + 1/R_2 = 0$, as first given by *Michael Roberts* (Liouville's Journal de Math. pur. et app., II. 300–312). *Roberts'* paper bore the title: Sur les surfaces dont les rayons de courbure sont égaux, mais dirigés en sens opposés. At the time of his investigation no one had the slightest idea why the curvature had to be equal but opposite at every point.

4. Now, under the wave-theory, we see that if waves come from every direction, and therefore also depart in every direction, after traversing the layer of fluid, in the minimal surface, it would be necessary for the curvature to be exactly equal and opposite at every point, — otherwise an unsymmetrical tendency in the liquid film would result, from the direct pressure of the entering and the reaction of the departing waves.

5. This type of minimal surfaces is actually observed, and as the mathematical criteria are rigorously fulfilled, the question arises whether any other cause except wave action could fulfill these geometrical laws of minimal surfaces. We may hold that no cause other than wave action could conform to these rigorous criteria, because of the infinite order of accuracy involved in the theory and found by observation to be fulfilled by liquid films in actual practice.

6. Thus we conclude that under wave action the only two chief types of surfaces which could result are:

(a) Solid spheres or drops of liquid, with the modification (b).

(b) Bubbles, symmetrical about a centre, or other double sheeted films symmetrical about an axis, on which the surface is extended. Symmetry is a fundamental condition of stability, as when a sheet of soap suds is stretched on a plane ring, symmetrical on the two sides.

(c) Minimal surfaces fulfilling the geometrical criterion of equal and opposite curvature at every point, $1/R_1 + 1/R_2 = 0$.

7. A profound argument could be drawn from the theory of probability to the following effect:

(a) The rigorous conformity of the complicated surfaces which would theoretically result from wave action with the surfaces fulfilled by liquids in actual nature, cannot possibly be due to mere coincidence.

(b) In view of the extreme rigor of these geometrical criteria, as applied to actual liquids, the chances against any theory of mere accidental coincidence is more than infinity to one.

8. For the disposition of the molecules in the required films involves the arrangement of an infinite number of these molecules in perfect order, in three dimensions, and thus the observed coincidence of the liquid films with the geometrical minimal surfaces, becomes at least an infinity of the third order (∞^3) to one, that the observed coincidence rests on a true physical cause, which therefore can be nothing but wave-action.

9. The fact mentioned by *Poincaré* (Capillarité, p. 66, 1895) that *Schwarz* concluded from geometrical considerations what form a certain helicoidal type of surface should take, and on the basis of this geometrical prediction it was shown by experiment to really exist, is a very remarkable example of the laws of geometry being used to fulfill the process of physical discovery. It is only established laws, founded on the true order of nature, which may thus be used to guide the explorer of the physical universe.

10. There are many examples of theoretical discovery handed down in the history of science. In all the celebrated cases they rest on the mathematical application of true laws of nature. *Laplace*, who used this method to discover the cause of the great inequality in the mean motion of Jupiter and Saturn, 1785, regarded the confirmation of mathematical theory by observation as the sublimest of triumphs. Similar views have been held by the successors of the illustrious author of the Mécanique Céleste, as in the theoretical discovery of Neptune by *Adams* and *Leverrier*, 1846, and of external conical refraction by Sir *W. R. Hamilton*, 1833.

(iii) The concluded cause of the minimal surfaces observed in nature.

From the above discussion of the minimal surfaces found in nature we conclude that the observed surfaces all fulfill rigorous and very remarkable mathematical criteria. They present either the minimal closed surface for a given volume, as in the globules of rain, dew, quicksilver, and other fluids which confront us on all sides; or an unclosed surface of double but opposite curvature fulfilling the geometrical condition $1/R_1 + 1/R_2 = 0$.

It is easy to show that if the minimal surface be closed, — like that of a globule of dew, with a single spherical surface, or the soap bubble, with two concentric spherical surfaces, — the action of waves from all directions will generate actions and reactions at the boundaries which will physically round up the figure of the fluid, and render the surfaces true minimal surfaces.

Of all the possible forms which mass of fluid may take, the sphere has minimal surface for given volume, or maximum volume for given surface. It is not by accident that in all liquid drops nature presents us with a never-failing recurrence of this beautiful and wonderful symmetry of figure. Accordingly we naturally conclude that the observed law can rest on no cause other than wave-action.

For the chances are infinity to one that an infinite multitude of drops of one fluid would not attain this figure except by the steady action of a true physical cause. And as the same law holds for an unlimited series of natural liquids the chances that a true cause is at work are again indefinitely increased. Finally, as the globular form for liquid drops is observed to hold for every solid rock, metal, and other solid compound of the crust of the globe, when rendered molten, by an infinite series of changing temperatures, and pressures, we see another independent infinite probability that the assumption of the globular figure depends on a true physical cause, which can be nothing but wave-action.

Accordingly the compound probability of all these several events, everywhere recurring constantly, is not less than the maximum infinity of the third order, equivalent to all the points in space to one, namely:

$$P = \int_0^\infty \int_0^\infty \int_0^\infty dx\, dy\, dz = \infty^3 . \qquad (92)$$

We therefore dismiss the subject, and consider the cause of the globular figure proved incontestably by the most varied phenomena of nature.

When we come to the minimal surfaces of equal but opposite curvature, $1/R_1 + 1/R_2 = 0$, we see that the problem is physically less simple, yet the cause involved is of the same general type, because the surface is kept taught by stretching, just like the rubber layer of a toy balloon. As the surface tends incessantly to contract, it follows that it must be acted upon by forces tending to make its extent a minimum, as in the case of globules of liquid under the surface tension due to wave-action.

Thus as the tension, in the surfaces fulfilling the geometrical condition $1/R_1 + 1/R_2 = 0$, is similar to that in liquid globules, it follows that in this case also the effect is due to wave-action.

Moreover, the waves come equally from all directions, and thus the opposite curvatures of the surfaces necessarily are equal. Any lack of perfect symmetry, in the distribution of the wave-action from the opposite directions would render the surfaces unsymmetrical; so that they would not fulfill the required *Euler-Weierstrass-Schwarz* geometrical condition. The fact that *Schwarz* could predict a theoretical form of minimal surface, which was subsequently verified by experiment renders these laws capable of use in discovery.

The wave-theory of physical forces thus approaches the degree of accuracy characteristic of the theory of gravitation, in the celebrated case of the planet Neptune, discovered by *Adams* and *Leverrier*, 1846; or the dependability of the wave-theory of light, which enabled *Sir W. R. Hamilton* to predict the external conical refraction, observed by *Lloyd* for crystals of arragonite, 1833.

After careful consideration of these recondite researches of *Schwarz* on minimal surfaces, which we have now applied to the wave-actions incessantly going on in nature, we are of the opinion that few more useful results have been obtained since the foundation of modern mathematico-physical science; and thus in view of their usefulness these researches on minimal surfaces deserve to rank with the most celebrated discoveries in astronomy and natural philosophy.

9. Historical Survey of the Researches of Geometers on the Cause of Capillarity.

(i) References to the older scientific literature of capillarity, 1500–1804, A. D.

The literature of this subject is so extensive that a brief descriptive summary alone will enable the reader to appreciate the successive steps in our progress.

In an extensive examination of the history of the subject, quoted by *Maxwell*, *Poggendorff* held that *Leonardo da Vinci* (1452–1519) must be considered the discoverer of capillary phenomena. But the scientific observations on capillarity practically have all been made since the age of *Newton*, and begin with *Hawksbee*.

1. *Hawksbee*, Physico-mechanical Experiments, London, 1709, pp. 139–169: also Phil. Trans., R. Soc., 1709–13.

(a) *Hawksbee* ascribed capillarity to an attraction between water and the glass tubes or plate; and observed that the effect is the same whether the tubes be thick or thin, and thus held that only those particles of glass which are very near the surface have any sensible influence on the phenomenon. This early observation of *Hawksbee* thus laid the foundation of the celebrated hypothesis afterwards developed by *Laplace* that molecular forces are sensible only at insensible distances.

(b) Dr. *Jurin*, in the Phil. Trans., 1718, no. 355, p. 739, and 1719, no. 363, p. 1083, extended the observations on capillarity, and discovered the law that the height to which the same liquid, such as water, rises in tubes is inversely proportional to their radii. This is easily verified by the curve taken by the surface of water between two vertical panes of glass, set mutually inclined at a small angle, which is a rectangular hyperbola referred to its asymptotes, as shown on the left of the foregoing figure 9.

(c) We may calculate the capillary elevation by *Jurin*'s method as follows. Let r denote the radius of the tube, ϱ the density of the liquid, α the angle of contact reckoned from the downward vertical, T the tension of the surface film, and h the mean height to which the fluid is elevated.

Then the vertical component of the whole tension round the edge of the film obviously balances the weight of the liquid column, and we have the equation

$$\pi r^2 h \varrho g = 2\pi r T \cos\alpha . \qquad (93)$$

This gives for the mean height h and surface tension T

$$h = 2T\cos\alpha / r\varrho g . \qquad (94)$$

$$T = hr\varrho g / 2\cos\alpha . \qquad (95)$$

In the case of water the density $\varrho = 1$, while the angle $\alpha = 0°$, $\cos\alpha = 1$, and by observing h, and r we may calculate T directly.

In the 5th edition of the Properties of Matter, 1907, p. 265, Professors *Tait* and *Peddie* give the following table [1])

[1]) It will be noticed that in *Quincke*'s table the surface tension T is quite large for mercury: 540 in air, 418 in water. Thus in air mercury has a surface tension which is nearly 7 times that of water. Considering the high reflecting power of mercury and its great resistance to an electric current, which is simply ordered wave motion, such a result was to be expected, and in fact strikingly confirms the wave-theory.

This result also conforms to *Majorana*'s celebrated experiments at Turin, as communicated to the Acc. dei Lincei, Rome, 1919, in which it is shown that gravitational waves are sensibly intercepted by a layer of mercury placed symmetrically about a weight in one arm of a delicate balance. The present results for molecular forces, as well as *Majorana*'s experiments of 1919, thus confirm the writer's discovery of 1916, that the fluctuations of the moon's mean motion are due to the interposition of the solid body of the earth in the path of the sun's gravitative action, near the time of lunar eclipses. (cf. Elect. Wave-Theory of Phys. Forc., vol. 1, 1917). (Continuation see p. 323.)

of T from observations by Professor *Quincke*:

	Air	Water	Mercury
Water	81	—	418
Mercury	540	418	—
Alcohol	25.5	—	399

The unit here employed is one dyne per (linear) centimetre. More elaborate tables of *Quincke*'s data will be found in *Maxwell*'s celebrated article Capillary Action, Scient. Pap., vol. 2. p. 589.

2. *Clairault*, Théorie de la Figure de la Terre, Paris, 1743.

(a) *Clairault* was the first to attempt to reduce capillarity to the laws of the equilibrium of fluids, by an exact analysis of the forces concurring to elevate the liquid in a glass tube. He explained the elevation as the result of two forces, one due to the attraction of the meniscus of the liquid, and the other due to the direct attraction of the tube itself on the molecules of the liquid.

(b) But *Clairault* erred in regarding the attraction of the tube as the principle force — imagining its power to extend as far as the central axis — which is contrary to the careful researches of *Laplace*, who subsequently showed, (1806–1807), that molecular forces are to be sharply distinguished from long range forces, such as gravity and magnetism, because observations prove that these molecular forces are sensible only at insensible distances.

Laplace's theory thus becomes a strong argument for the wave-theory, because of the minute range of action of the molecular forces, but as the undulatory theory of light was rejected by *Laplace*, he naturally did not suspect that waves could underlie the action of molecular forces.

(c) Owing to these defective hypotheses, *Clairault* failed to demonstrate from his theory that the ascent of the liquid should be inversely proportional to the diameter of the tube — as noted by observers since the time of *Jurin*, 1718. And although *Clairault* arrived at a number of hypotheses, which would account for the observed elevation of the fluid, none were based on molecular forces sensible only at insensible distances, and thus it subsequently required all the mathematical ingenuity of *Laplace* to deduce from *Clairault*'s theory an explanation of the elevation in tubes.

(d) *Clairault*, however, showed that if the attraction of the matter of the tube differs only by its intensity or coefficient, from that of the fluid on itself, then the fluid will rise above the surrounding level when the action of the tube exceeds half that of the fluid on itself — a remarkable theorem afterwards more fully developed by *Laplace* in his theory of capillary action.

3. *Segner*, Comment. Soc. Reg. Götting. I, 1751, p. 301.

(a) Eight years after the publication of *Clairault*'s Theory of the Figure of the Earth, 1743, containing the outline of his theory of capillarity above described, there appeared in the commentaries of the Royal Society of Göttingen an important work by *Segner*, who introduced the useful and suggestive conception of the surface tension of liquids. He ascribed this surface tension to attractive forces, the sphere of whose action is taken to be so small as heretofore not to be perceived by the senses.

(b) *Segner* thus laid the foundation of the theory of surface tension, produced by attractive forces sensible only at insensible distances, — as in the theory of *Laplace*; and he used this new theory of surface tension to calculate the curvature of a meridional section of a drop of liquid, but did not investigate the curvature in a plane at right angles to the meridian. It is probable that *Segner* regarded the tense membrane of the surface of the liquid as stretched equally in all directions, unless the contrary was shown by the curvature of the surface observed.

(c) *Segner*'s theory found confirmation in observations of *Leidenfrost* of Duisburg, 1756, on the contraction of soap bubbles, in which it was shown that the air in the bubble is expelled by the contraction of the membrane of the soap bubble. In the Mem. de l'Acad. d. Sc., 1787, p. 506, *Monge* adapts the view that the adherence of the particles of fluid in capillarity have influence only at the surface itself and in the direction of the surface which thus follows *Segner*'s theory of surface tension. *Monge* applied the theory of surface tension to explain the apparent attractions and repulsions between bodies floating on a liquid.

4. *Young*'s theory of capillarity, 1804, founded on the action of surface tension like that of *Segner*, 1751.

In his essay on the Cohesion of Fluids, (Phil. Trans. 1805, p. 65) Dr. *Thomas Young* developed a theory of capillarity as *Segner* had done half a century before, founded on the principle of surface tension. He observed the constancy of the angle of contact of the liquid surface with the solid, and showed how the constancy of the angle and the tension of the surface enable the fluid to exhibit capillary phenomena. Whilst the theory of *Young* involves both cohesion and surface tension for explaining capillarity, it avoids as far as possible the use of mathematical symbols, yet it is held by *Maxwell* to be essentially correct.

(ii) The more recent researches on capillarity by *Laplace*, *Poisson*, *Gauss*, *Quincke*, *Maxwell*, *Kelvin*.

5. *Laplace*, Memoir on capillary attraction, Supplement to the tenth book of the Mécanique Céleste, 1806, and Supplement to the theory of capillary attraction, 1807.

The theory of *Laplace* is so well known, and so much used by all students of the subject that we shall not here go into it in detail, except to note certain difficulties to which attention should be called. In his Capillarité, Paris, 1895, p. 1–3, *Poincaré* condenses *Laplace*'s theory into a few leading formulae, by taking the attraction to have the general form

$$A = m\,m' f(r) = \partial V/\partial r \qquad (96)$$

where m and m' are the masses of the molecules, r their distance apart, and $f(r)$ an unknown function of the distance.

The wave-theory therefore is strikingly confirmed both by a large body of well established terrestrial phenomena, and by the discovery of the cause of the fluctuations of the moon's mean motion.

After mathematical researches extending over more than 40 years, *Newcomb*, in 1909, pronounced these lunar fluctuations to be the most enigmatical phenomena presented by the celestial motions; so that the discovery of the cause underlying these perplexing perturbations in the moon's mean longitude is a notable triumph in celestial mechanics.

Laplace's celebrated hypothesis that the capillary forces are sensible only at insensible distances, leads to the formula for zero forces at all distances greater than r:

$$\varphi(r) = \int_r^\infty f(r)\,\mathrm{d}r = 0 \qquad (97)$$

where r is the radius of molecular activity, shown by experiment of *Quincke*, 1869, to be $r < 50\mu\mu$, 50 millionths of a millimetre.

In consequence of the first hypothesis in equation (96) the molecular forces admit of a potential

$$V = \sum_{r=0}^{r=r} m\,\varphi(r) \qquad (98)$$

with the components for unit mass at x, y, z under the action of the mass m:

$$X = \partial V/\partial x,\quad Y = \partial V/\partial y,\quad Z = \partial V/\partial z\,. \qquad (99)$$

If the whole of the attracting molecules form a volume, the expression of the potential becomes, for the density ϱ:

$$V = \iiint \varrho\,\varphi(r)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \qquad (100)$$

Owing to the high incompressibility of liquids, *Laplace* adopts in effect the hypothesis that the density ϱ is constant. In his Nouvelle Théorie de l'Action Capillaire, 1831, *Poisson* rejects *Laplace*'s hypothesis; likewise *Poincaré* remarks that *Laplace*'s assumption is illegitimate, because it is probable that the density is not the same at a distance less than the radius of molecular activity from the surface as at a distance greater than this radius.

But whatever be the exactitude of *Laplace*'s hypothesis, it leads to the expression for the potential

$$V = \varrho \iiint \varphi(r)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \qquad (101)$$

And if $\varrho = 1$ in the liquid considered

$$V = \iiint \varphi(r)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \qquad (102)$$

The only other point in this theory to which we shall call special attention relates to *Laplace*'s K, (Méc. Cél., Liv. X, Suppl. à la Théorie de l'Action Capillaire), which implies that in every liquid there is a great internal pressure. Near the end of this supplement *Laplace* derives the formulae for this internal pressure,

$$n^2 - 1 = 4K/V^2 \qquad (103)$$

$$s = K/g = (n^2 - 1)\,V^2/4g \qquad (104)$$

where V = velocity of light, $n^2 - 1 = {}^7/_9$, the refractive action, and g = acceleration of gravity, s = length of the column of water of equal pressure, in units of the sun's distance. This gives $s = 12000$, approximately, or a column of water over 10000 times longer than the distance from the earth to the sun. *Laplace* himself adds that »une aussi prodigieuse valeur ne peut pas être admise avec vraisemblance«, so that he apparently did not regard the value of $K = gs$ as probable. Accordingly it will not surprise us to find modern physicists rejecting it. In his Properties of Matter, 1899, p. 244, *Tait* says:

»In some statical theories of molecular action, especially that of *Laplace*, one of the most striking deductions is that there must be a very great internal pressure in every liquid mass: — a pressure wholly independent of the form and size of the bounding surface. This is usually known as ,*Laplace*'s K'. *Laplace*'s own estimate of its value in water is given (with the caution ,Une aussi prodigieuse valeur ne peut pas être admise avec vraisemblance') as the weight of the water which would fill a tube of unit section whose length is 10000 times the distance of the earth from the sun: i. e. something like 10^{12} tons weight per square inch. This was based on the corpuscular theory of light, the numerical data being the refractive index of water and the speed of light.«

It would be easy to show from practical experience, — as in human diving, and from the survival of fresh water and marine life, in such delicate animals as fish, which have bladders filled with air, etc., — that the view of an enormous internal pressure for water is not valid. If these views were true we could not dive without having our lungs crushed, and the bladders of the fish could not operate as they actually do; for the fish not merely survive, but are not injured when taken from the water a short time.

Laplace's final expression for the pressure in the interior of a fluid has the form

$$p = K + {}^1/_2 H\,(1/R_1 + 1/R_2)\,. \qquad (105)$$

Here K is the assumed constant pressure, in that theory very large, which however does not influence observed capillary phenomena, H is another constant on which all capillarity phenomena depend, and R_1 and R_2 are the radii of curvature of any two normal sections of the surface at right angles to each other.

If in the above formula we put $1/R_1 + 1/R_2 = 0$ in the second term of the right member, as in minimal surfaces, we see that within such films the pressure p would be equal to K only, which shows the connection between such films as soap bubbles, with double surface tension, and capillarity.

6. *Gauss*, Principia generalia theoriae figurae fluidorum statu aequilibrii, 1830, (Werke, V. p. 29).

Gauss forms the force-function for the potentials of all the pairs of particles in their mutual action. With the sign reversed he thus obtains the potential energy of the system. *Gauss* treats the problem of the forces urging the fluid with his usual lucidity, in three parts: the first depending on gravity; the second, on the mutual attraction between the particles of the fluid; and the third, on the action between the particles of the fluid and the particles of the solid or fluid in contact with it.

Gauss makes this aggregate expression a minimum:

$$\Omega = -gc\int z\,\mathrm{d}s + {}^1/_2 cc \iint \mathrm{d}s\,\mathrm{d}s'\,\varphi(\mathrm{d}s\cdot\mathrm{d}s') + cC\iint \mathrm{d}s\,\mathrm{d}S\,\Phi(\mathrm{d}s\,\mathrm{d}S)\,. \qquad (106)$$

In this formula g = the force of gravity, z = elevation above the base plane H, c = density, taken as uniform in the spaces s and s', C = density of the solid, or fluid of different kind; and the spaces s and s' are filled by the mobile fluid, and S by the solid or fluid of different kind. With this explanation of *Gauss*' fundamental equation, made up of three terms, it only remains to add that the potential so constituted is a minimum, and therefore for such a level surface, the sum of the space differentials vanishes:

$$\delta\Omega = (\partial\Omega/\partial x)\,\mathrm{d}x + (\partial\Omega/\partial y)\,\mathrm{d}y + (\partial\Omega/\partial z)\,\mathrm{d}z = 0\,. \qquad (107)$$

In deriving more general conditions for the free surface than *Laplace* had done, *Gauss* thus improved the theory. At the close of his paper he recommends the method of *Segner* and *Gay-Lussac*, which *Quincke* has since extensively applied, by measuring the dimensions of large drops of mercury on a horizontal plane, and those of large bubbles of air or other gases, in transparent liquids resting against the under side of a horizontal plate of a substance wetted by the liquid.

7. *Poisson*'s Nouvelle théorie de l'action capillaire, Paris, 1831, pp. 1–300.

Poisson's new theory of capillary action is developed with such geometrical elegance, that it must always occupy a prominent place in any survey of the subject. But it is justly remarked that although *Poisson* adopts processes different from those employed by *Laplace*, yet in general the conclusions are identical, except in respect to uniformity of the internal density of the liquid, explained in equations (100) and (101) above. At the close of paragraph 5 above we have indicated reasons for adopting *Laplace*'s view that ϱ may be taken from under the integral signs in equation (100). *Gauss*' procedure accords with this view also, as we see by his principal equation for Ω above.

Poisson's criticism that *Laplace*'s theory makes the constant pressure K very large, whereas it must be in fact very small, is undoubtedly valid, from considerations already pointed out in treating of *Laplace*'s theory. Thus *Poisson* reached results in general accord with those of *Laplace*, but did not confirm the great constant pressure K in liquids, and added the claim of a rapid variation of density near the surface, which does not admit of experimental determination.

The theory developed in the present paper, that the wave stress undergoes sudden change at the surface of liquids appears to reconcile these several difficulties; for whilst it assigns to this surface tension the globular form of drops, and the elasticity of the film of soap bubbles, it does not give a great internal pressure for liquids, but only the somewhat feeble surface stress noticed in oscillating drops and elastic films.

In dealing with the *Segner-Young* contribution to the theory of capillarity, 1751–1804, we noted the fact that they successfully explained capillarity by surface tension, and thus it is appropriate for us to draw attention to the rather feeble intensity of the surface tension of various liquids, as determined by *Quincke*, and given briefly in the above table.

For water in air, the surface tension, $T = 81$, by the formula

$$T = hr\varrho g/2\cos\alpha\,. \qquad (95)$$

Here α is the angle of contact reckoned from the downward vertical. For water in contact with glass, $\cos\alpha = 1$, and T is found from the radius of the tube r, and observed height of column h. For mercury in air the value of $T = 540$, but in this case $\alpha = 128°\,52'$, so that the cosine is negative and the column depressed.

The dyne is the force producing an acceleration of one centimetre per second in a gram mass, and in view of the feebleness of these forces of surface tension, we see why we cannot explain capillarity by such a feeble force, and at the same time admit the enormous constant fluid pressure found by *Laplace*'s theory.

8. *Maxwell*'s article on capillary action, Encycl. Brit. 9th ed., 1875.

In concluding this section it only remains to point out the last great contribution to the whole theory of capillarity, in recent times, the article on capillary action by *Maxwell*, Encyc. Brit., 9th ed., reprinted in *Maxwell*'s Scientific Papers, vol. II, pp. 541–591. Though written about half a century ago, it is still the most extensive and accurate survey of the subject yet available. On page 589 he gives a table of *Quincke*'s experimental results much more elaborate than that quoted above.

The more recent contributions by Lord *Kelvin*, Lord *Rayleigh*, and other investigators, have added to the extensive literature already cited; but *Quincke*'s researches will long remain the chief source of experimental data.

10. New Theory of Cohesion and Adhesion.

(i) Refraction and dispersion of waves at the surface of solids, may produce hardness.

Up to this point we have treated with special attention the cause underlying surface tension and capillarity, because we have felt that if these causes could be definitely assigned, it would not be very difficult to pass over to the related cause of cohesion and adhesion.

In fact adhesion is directly related to capillarity, for when liquids rise in tubes the fluid always wets the tube, so as to adhere to it, and lift the column upward by the force of surface tension. And when the liquid does not rise, but is depressed in the tubes, as in the case of mercury, there is no adhesion, but rather repulsion, or as is commonly said, a greater attraction of the liquid for itself than for the solid. Thus the molecular forces in adhesion are the same as in capillarity; and cohesion is similar to the attraction of liquid particles for one another, except that the cohesive force depends greatly on temperature, and thus becomes much more powerful in solids.

Two centuries ago, in the 3rd edition of the Optics, 1721, p. 365, *Newton* discussed the mystery of the molecular forces as follows:

»And how such very hard particles (in solids) which are only laid together and touch only in a few points, can stick together, and that so firmly as they do, without the assistance of something which causes them to be attracted or pressed towards one another, is very difficult to conceive.«

Accordingly, it appears that *Newton* was very much baffled in his efforts to conceive of the cause which underlies cohesion and adhesion, more especially in solids. The difficulty in liquids was no doubt almost equally great, but our treatment of it is already outlined, and we shall therefore deal chiefly with cohesion and adhesion as exhibited by solids.

In the wave-theory we hold that no refraction of the wave front can occur without the expenditure of energy, drawn from the general reservoir of the aether; therefore as waves move more slowly in solids than in free space, there necessarily is wave energy exerted against the solid owing to derangement of the wave front at the boundaries of such masses.

Moreover, the refraction of waves usually is associated with dispersion, or separation of waves, owing to the unequal refraction. Both of these changes in the wave field involve work done at the boundaries of solid bodies, and the result is wave stresses and reactions which give rise to cohesion [1]) and adhesion.

The full development of this theory of cohesion and adhesion involves the complicated problem of wave transformation and separation in a medium 689321600000 times more elastic than air in proportion to its density. This problem is new in science, and as it has not yet been treated exhaustively, we first outline the physical considerations which must be borne in mind.

1. When a ray of light enters a drop of water, with refractive index $n = {}^4/_3$, the so-called refractive action is $n^2 - 1 = {}^7/_9$. The wave velocity is diminished or accelerated at the boundary by ${}^1/_3$; and ${}^7/_9$ of the energy is exerted against the surface layer of the drop. This slowing down of the wave speed or its acceleration thus exerts a pressure against the boundary of the drop: for long ago *Maxwell* recognized, (Treatise on Electricity and Magnetism, §§ 781–793) that partial stopping of wave motion leads to the exertion of pressure on the surface obstructing the progress of the waves. If waves leave the drop for free space, there is corresponding reaction of the free aether at the boundary, and thus a similar development of central pressure.

2. Moreover, it is evident that in proportion as these wave actions and reactions are sudden and violent at the boundary of a body, so that the refractive action $n^2 - 1$ is large, in the same proportion the related dispersive action also is large. Accordingly, as diamond has the greatest of known refractive indices, $n = 2.49$, and is so powerful in the dispersion of colors as to yield the unapproached lustre which gives the great value to this crystal, it ought theoretically to be the hardest of bodies, and is so found by experiment.

In his »Six Lectures on Light«, second edition, New York, 1886, p. 20, *Tyndall* lucidly explains total reflection, the limiting angle for which in water is 48° 30′; for flint-glass 38° 41′; for diamond 23° 41′; thus rapidly diminishing with increase of the refractive index.

»Thus all the light incident from two complete quadrants, or 180°, in the case of diamond, is condensed into an angular space of 47° 22′ (twice 23° 41′) by refraction. Coupled with its great refraction are the great dispersive and great reflective powers of diamond; hence the extraordinary radiance of the gem, both as regards white light and prismatic light.«

Tyndall's remark that all the light incident from two complete quadrants, or 180°, in the case of the diamond, is condensed by refraction to an angular space of 47° 22′, $47^\circ\!.37/180^\circ = 1/3.8$, contains the germ of the secret of the most powerful molecular forces, such as those which produce hardness. For just as the rays in a plane angle are thus condensed, so the rays from the solid angle of a whole hemisphere are condensed into 1/3.8 of their original distribution; so that on any area the concentration of energy increases as the square of 3.8 and becomes 14.44 times greater. As the dispersion is in about the same proportion, the combined effect of refraction and dispersion is magnified some 200 times. [2]) Considering the tendency to rupture the aether by this sudden discontinuity at the boundary of the diamond, it is not remarkable that the actions and reactions of the more powerful invisible waves give the cohesion underlying the hardness of diamond. It is evident that the hardness of diamond and other crystals, the great tenacity of steel and other wires, depend on wave-action and -reaction at the surface; and therefore the strength of such a solid depends on some such combination as the following:

1. Refractive action, $(n^2 - 1)$, which depends on the density of the solid, σ, and the changing wave-length λ and thus on some unknown function, $\pi(\sigma/\lambda)$;

2. The violence of the incessant bending of the wave-front, for waves coming from all directions, $\varphi(\beta)$;

3. The violence of the incessant dispersion of these incident waves, $\psi(\delta)$;

4. The combination of systematic stresses due to the crystalline arrangement of the atomic planes with the effects of the two latter violent tendencies, thus leading almost to the disruption of the medium, $\chi(\varkappa)$;

5. The enormous power of reflection with very slight absorption of energy, at the surface, $\theta(\varrho \cdot c^{-\alpha})$;

6. The great central pressure due to the integration of the steady action of the sheath of partially disrupted waves always enveloping the solid, $\varpi(\omega)$.

Accordingly if the condition be imposed that the normal elastic power of the aether is not greater than unity, which is $\varepsilon = 689321600000$ times that of our air in proportion to its density, then we shall have (cf. *Todhunter*'s Integral Calculus, edition 1910, § 277, p. 262):

$$\Omega = \int_0^{\sigma/\lambda}\int_0^{\beta}\int_0^{\delta}\int_0^{\varkappa}\int_0^{\varrho c^{-\alpha}}\int_0^{\omega} \varepsilon(n^2 - 1)\,\pi(\sigma/\lambda)\,\varphi(\beta)\,\psi(\delta)\,\chi(\varkappa)\,\theta(\varrho\, c^{-\alpha})\,\varpi(\omega)\cdot d(\sigma/\lambda)\,d\beta\,d\delta\,d\varkappa\,d(\varrho\, c^{-\alpha})\,d\omega\,. \qquad (108)$$

[1]) In his »Aether of Space, 1909«, p. 109, Sir *Oliver Lodge* treats of cohesion in a very simil r way to that here adopted:

»Why the whole of a rod should follow, when one end is pulled, is a matter requiring explanation; and the only explanation that can be given involves, in some form or other, a continuous medium connecting the discrete and separated particles or atoms of matter.«

»When a steel spring is bent or distorted, what is it that is really strained? Not the atoms — the atoms are only displaced; it is the connecting links that are strained — the connecting medium — the aether. Distortion of a spring is really distortion of the aether. All stresses exist in the aether. Matter can only be moved. Contact does not exist between the atoms of matter as we know them; it is doubtful if a piece of matter ever touches another piece, any more than a comet touches the sun when it appears to rebound from it; but the atoms are connected, as the comet and the sun are connected, by a continuous plenum without break or discontinuity of any kind. Matter acts on matter only through the aether.«

[2]) Should the other variable elements indicated below increase in about the same proportion as the two well known elements here calculated, the result would be an increase of stress of the order of 8000000 times the value otherwise effective. And as there are sudden discontinuities in the physical state of bodies, as in passing from fluid to solid, owing no doubt to closeness of contact of the molecules of the solid, — assumed to be less than the wave-lengths to which the forces are due — the whole wave-action in the aether seems ample to account for the hardness of the diamond.

But although the nature of the wave function producing solidity and rigidity is thus recognized, yet we cannot at present evaluate the resulting sextuple integral, because the part contributed by each variable is ill defined.

(ii) The assigned cause of the hardness in diamond suggests a similar origin of tenacity.

The theory of the hardness of diamond here outlined will also explain tenacity, or the great breaking strength of such substances as steel[1]), which attains maximum power in pianoforte wire.

1. It is a remarkable fact of observation, drawn from experience from the early ages of history, that tenacity is increased through drawing and rolling, by which the metal is given a smoother and more compressed surface. For example, we make strong wire ropes by first drawing the metal into fine wire, each strand being given a compact and compressed surface large compared to the cylindrical content of the solid wire, and then twisting the wire into a rope, which thus becomes not only strong, but also flexible.

2. The fact that in fine wire there is rapid increase of surface compared to the cylindrical content, when the wire is small, shows that the large amount of smooth surface is the essential element of strength, and points to waves in the aether as giving the force of cohesion. The relation of surface to volume in a circular cylinder of length h and radius r is easily found, thus:

Volume of cylinder $V = \pi r^2 h$ (109)
Curved surface of cylinder $S = 2\pi r h$

Ratio of $S/V = 2\pi r / \pi r^2 = 2/r = \eta$. (110)

Accordingly, as r diminishes the ratio η rapidly increases, according to the curve for a rectangular hyperbola referred to its asymptotes.

3. On account of the finite dimensions of the molecules of the wire, and the finite but greater length of the light waves, it is of course not possible to decrease the radius of the wire below a certain limit, without the metal losing the power of cohesion and breaking. Along with this property, by which a finite radius is required for strength in a metal, goes also the closely related problems of malleability and ductility.

(a) Gold is the most malleable of metals, gold leaf having been reduced to a thickness of $1/300000^{\text{th}}$ of an inch, or 1/11800 of a millimetre.

(b) Platinum is the most ductile of metals. By coating a platinum wire 0.01 inch in diameter with silver till the thickness of the whole was 0.2 inch, Dr. *Wallaston* drew the cylinder out into a wire as fine as possible, and by boiling with dilute nitric acid, he removed the silver coating and obtained the platinum wire alone with a diameter of approximately 1/10000 mm nearly the same thickness as the thin gold leaf described in (a) above.

4. The metallic coating used to draw the platinum wire into such fineness was of silver, which is the most perfect of all electrical conductors, and thus the wave-action was no doubt very perfectly equilibrated at all times in Dr. *Wallaston's* ductility experiment. As the gold leaf was in similar sheets while being hammered the conditions also were favorable for malleability, without disruption of the molecular forces holding the gold leaf together.

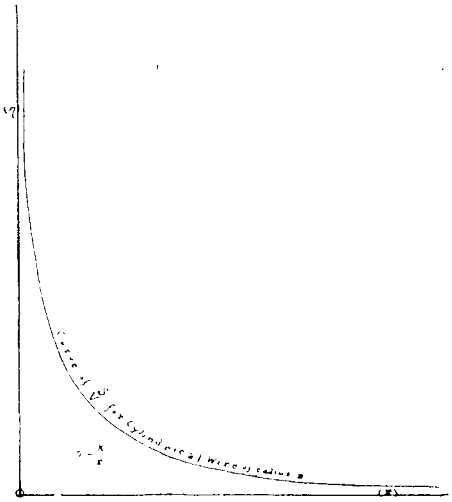

Fig. 18. Curve of the ratio of the surface to volume, in a drawn cylindrical wire of radius r. As it is an observed fact that the strength of a given mass of metal is increased by drawing it into wire, this curve shows that tenacity depends on the relative increase of surface compared to volume, and therefore on wave-action at the surface of the wire. Increase of tenacity, with $\eta = S/V$, equ. 110, begins to fail as short wave-lengths and molecular dimensions are approached.

Notwithstanding these favorable conditions it is a little remarkable that gold and platinum, with their very high atomic weights, should prove among the most continuously yielding of metals. This gradual yielding of metals is directly opposed to hardness, which leads to rupture.

Glass threads drawn by Dr. *Boys* have been reduced to a diameter of about 1 : 100000 of an inch, or three times that of the diameter of the minimum platinum wire above mentioned, with strength however approaching that of steel wire, which shows that surface wave-action probably has increased the strength greatly.

5. It is noticed in modern metallurgy that the pure metals generally are softer than their alloys. Both hardening and increase of strength may be effected, however, by the admixture of a small percentage of certain other metals. Nickel and vanadium are used in the manufacture of hard steel: and such compounds as phosphor bronze, and aluminium bronze, have greatly increased tenacity. We may explain all these effects by the wave-theory, the molecular forces being augmented not merely at the boundary, but throughout the mass. This same reasoning applies also to the internal strength and structure of crystals, such as diamond, quartz and other substances.

[1]) Steel is a mechanical mixture of a very fine matrix of carbon in iron, and as diamond is cristallized pure carbon, it would seem that the great strength of steel, over iron, must depend in some way on such wave-transformations as refraction, dispersion, etc., to which the non-conducting carbon contributes at the boundary of the wire. It surely is not accidental that the strength of steel depends on the same element, carbon, which in crystallized form gives diamond its unparalleled hardness.

6. The metals increase in hardness somewhat in the following order: lead, tin, aluminium, gold, silver, platinum, zinc, copper, iron, steel. These results may be explained by the assumption of molecular properties varying from metal to metal, but on the whole depending on the grip of the waves about the atoms and molecules, under their state of condensation, and electrical conductivity or non-conductivity. Lead, for example, is loosely held together, and yields easily to powerful forces. It somewhat resembles pitch, which is a viscous fluid, or solid for quick acting forces, while the lead is an easily yielding solid.

In the same way hardness is measured by the following scale.

1.	Talc	6.	Feldspar
2.	Rock Salt	7.	Quartz
3.	Calc Spar	8.	Topaz
4.	Fluorspar	9.	Corundum
5.	Apatite	10.	Diamond.

(iii) *Newton*'s problem of cohesion may be solved by noticing that waves which have difficulty in passing through between two compact masses will naturally take the path of Least Action around them, and thus force them together.

1. So long as two bodies are not near together the waves from the atoms of each mass, as well as from the rest of the universe, will easily pass between and around the two masses. Each mass will have its own wave field, and they will not sensibly interfere. But when the two bodies are brought very near together, each obstructs the waves from the other, and the wave fields become entangled. When they are brought very near, so as to form a smooth contact, the whole wave action is so much less, when they are pressed tightly together, that nature adopts this method for Least Action, and forces them as solidly together as possible. This gives us a general idea of the cause of cohesion, which so much puzzled *Newton* that he had extreme difficulty in conceiving the cause at work.

2. In order to make the contact effective and powerful, it must be very close indeed, so that the molecules are not whole wave-lengths apart. The fact that observation shows that the contact must be close, appears to point directly to the wave-theory. What explanation other than the wave-theory is possible? The problem is like the hypothesis underlying *Laplace*'s theory of capillarity, that the molecular forces become sensible only at insensible distances, which as we have shown, can point to nothing but wave-action.

3. The measurements of *Rücker* show that the ultra-violet waves are of the required order of magnitude, and we know that their working at small distances, in a medium 689321600000 times more elastic than air in proportion to its density, should produce very great power of attraction, since the path of Least Action usually is around the outside of solid masses, and thus they are forced together by reaction of the whole wave field.

This gives us a very tangible conception of the practical working of the wave-theory when applied to molecular forces. We may verify the conclusion here drawn by observations on the dashing of water waves upon two floats anchored so close together that the waves do not pass freely between them. The result is a full pressure of the waves without, not perfectly balanced by the diminished pressure within, so that the floats are drawn together as if by an attractive force.

4. Now it is very remarkable that nature should be filled with such a multitude of minimal surfaces: raindrops, drops of dew, globules of mercury, iodine, or any chemical liquid which does not adhere to the supporting surface. All melted metals, such as leaden shot, take the same figure, and so also of molten rock of any and every description. Accordingly, globules of liquid, with minimal surfaces, actually are universal in nature. What is the cause of this universal tendency to minimal surfaces? It must be related to the coalescence of contiguous small drops into larger ones, as in the phenomena of rain, accompanied by lightning.

5. It has long been held, first by *Maupertuis*, *Euler* and *Lagrange*, and subsequently by *Hamilton*, *Jacobi*, *Kelvin*, *Helmholtz*, *Tait*, *Poincaré*, *Larmor*, and many other eminent mathematicians, that nature always follows the principle of Least Action. *Fermat*'s principle was of earliest date, and gave the first indication of the more general theorem of Least Action devised by *Maupertuis* and confirmed by *Euler* and *Lagrange*. It is known that the forces governing the mechanical operations of material systems obey the principle of Least Action, and correspond to the wave-theory of physical forces. Can it be possible that the figures of globules of fluid masses and elastic surfaces would exhibit minimal surfaces, without also depending on waves which are resisted in their progress at the border, and thus transform the liquid into minimal surfaces?

6. In this wave-theory, we find a direct and simple explanation of adhesion, cohesion, capillarity, surface tension, chemical affinity and even of explosive forces. The waves cannot but offer different resistance in their penetrating power when different substances are in contact; and moreover they are refracted unequally in passing through liquid, whence we may explain at once adhesion and capillarity. Cohesion is somewhat different: the particles of solid bodies offer least resistance when the particles are closest together.

7. If the particles are separated appreciably some of the waves pass between them, and on the whole the two separate bodies offer greater resistance to passing waves than would be offered by one mass made by a solid union of the two particles firmly together. This offers a simple theory of the great difficulty discussed by *Newton* in 1721. Surface tension has already been explained in describing the ray of light entering a drop of water; but we may have to include waves both longer and shorter than those of light, to complete the general theory. Experiment shows that chemical affinity is greatly promoted by ultra-violet light, and this confirms the wave-theory of physical forces. Thus, it only remains to say a word about explosive forces, which are related to chemical affinity, and of which no suitable theory has been put forth heretofore.

8. It appears to me that in the molecules of explosive bodies there is a certain resistance to the passing waves, as the atoms are then arranged; but if the atoms mutually are so readjusted as to come closer together suddenly, and re-arranged into a molecule of maximum symmetry and conden-

sation of its parts, there will suddenly be much less resistance to the passing waves. The great energy of the waves always passing through the aether, is thus released or set free by the readjustment of the atoms in the molecule; and this suddenly available energy is so powerful, in view of the aether's enormous elastic power, — which is 689321600000 times that of our air in proportion to its density, and thus much more powerful than our air in any readjustment of the wave field, — that when the release occurs by a sudden readjustment, a violent oscillation of the molecular structure results, — in which disruption and new combination of the oscillating atoms takes place.

9. This new theory derives the store of explosive energy from the elastic power of the aether. This power is shown to exist by the enormous observed speed of the propagation of light and electricity, 300000 km per second. We cannot deny the observed fact of such a velocity for waves in the aether. Accordingly, the enormous elastic power, 689321600000 times greater than that of air in proportion to its density, necessarily follows. And if the universe be filled with waves, of various lengths, from the short waves, most effective in chemical affinity, to the longer waves of light, heat, and radio-telegraphy, it naturally will follow that sudden change in the power of resistance of bodies incident to the rearrangement of the atoms into a new and more compact, less resisting molecular structure, would generate vast stores of energy hitherto latent.

10. This is best illustrated by the new theory of the cause of lightning, a phenomenon which has been equally mysterious and bewildering to natural philosophers. Here is what occurs in lightning:

(a) First, water exists in the atmosphere in the form of invisible vapor. Lowering temperature, usually with currents of colder air, produces a cloud, which is visible, because the light does not pass through it. At first the cloud is made up of very minute particles of water — microscopic in size — but if the cooling and tendency to precipitation continues, the particles of water grow in size, and decrease in number.

(b) When the separate water globules coalesce, into fewer but larger globules, their resistance to passing waves is decreased. And if the region of the earth and atmosphere previously was in electrodynamic equilibrium — the aether waves of this region departing at the same rate that they arrive, so as to give rise to no accumulating aether strain — it will experience with the condensation of drops an accumulating stress on the surface of the globules. The waves will flow from the earth and celestial spaces at the old rate, but the resistance to their passage at the surfaces of the enlarged drops is decreased with the condensation of the drops. A positive state of the rain cloud results, and augments rapidly as the rain drops grow.

(c) The result is accumulation of such a strain in the electric medium, or the aether, that lightning develops for restoring the electric equilibrium.

If so terrific a power as lightning can result from the changing electric stress or resistance of the enlarging drops to the waves traversing the universe, it naturally will be easy to imagine that explosive forces and similar atomic powers of incredible magnitude may have their seat in the elastic power of the aether, and the changes in the equilibrium of this medium.

11. Geometrical Conditions fulfilled by an Infinite System of Waves coming from all Directions and passing through a Liquid Sphere under Least Action.

(i) Geometrical conditions of minimum action.

1. We consider a sphere of fluid, whose surface is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2. \qquad (111)$$

The part of a right line intercepted between any two points $p'(x', y', z')$, $p''(x'', y'', z'')$ in the sphere surface, is equal to the length of the chord:

$$\chi = \sqrt{[(x'-x'')^2+(y'-y'')^2+(z'-z'')^2]}. \qquad (112)$$

Now waves passing through the fluid sphere, after refraction and dispersion at the boundary, follow some of the systems of chords from every point of the surface in every possible direction; so that the paths of minimum action within the surface are along the infinite system of chords drawn from every point of the surface, and therefore doubly infinite in number.

2. For if we suppose waves to originate within the sphere, it is clear that they will be propagated spherically, along these chords, and no deviation from rectilinear motion will occur till the wave front reaches the boundary of the liquid sphere. Refraction and dispersion will take place at the boundary when the wave is going outward, exactly the reverse of what occurs in coming inward; so that from one of these motions the other can be calculated.

3. In AN 5048, p. 165-6, it is shown that the wave function $\Omega(x, y, z, t)$ has the velocity potential Φ:

$$\Phi = \Omega(x, y, z, t) = (1/8\pi^3) \int\int\int\int\int\int e^{\Lambda\sqrt{(-1)}}\Omega(\xi, \eta, \zeta, t)\, d\xi\, d\eta\, d\zeta\, d\lambda\, d\mu\, d\nu$$
$$\Lambda = (\xi-x)\lambda+(\eta-y)\mu+(\zeta-z)\nu. \qquad (113)$$

4. And *Poisson* has reduced this sextuple integral to the double integral:

$$\Phi = (1/4\pi)\int_0^\pi\int_0^{2\pi} F(x+at\cos\theta, y+at\sin\theta\sin\omega, z+at\sin\theta\cos\omega)\, t\sin\theta\, d\theta\, d\omega$$
$$+(1/4\pi)(\partial/\partial t)\int_0^\pi\int_0^{2\pi} H(x+at\cos\theta, y+at\sin\theta\sin\omega, z+at\sin\theta\cos\omega)\, t\sin\theta\, d\theta\, d\omega. \qquad (114)$$

5. These integrals are rigorous for the wave disturbances from any point, so long as the movement remains within the liquid sphere, and they will hold true right up to the boundary. In the fourth paper, section 6, (AN 5085) we have extended the integration so as to include the waves from every atom within the boundary of the sphere (r, θ, ω).

$$\Phi = \int_0^r\int_0^\pi\int_0^{2\pi} (\sigma/4\pi) \int_0^\pi\int_0^{2\pi} F[l(x+at\cos\theta)+m(y+at\sin\theta\sin\omega)+n(z+at\sin\theta\cos\omega)-(at+s)]\, r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\omega\; t\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\omega$$

$$+\int_0^r\int_0^\pi\int_0^{2\pi} (\sigma/4\pi)(\partial/\partial t) \int_0^\pi\int_0^{2\pi} \Pi[l(x+at\cos\theta)+m(y+at\sin\theta\sin\omega)+n(z+at\sin\theta\cos\omega)-(at+s)]\times$$
$$\times r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\omega\; t\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\omega\,. \qquad (115)$$

6. As the wave disturbances emerging from all atoms will yield a perfect reverse image of those coming in from all directions, it suffices to find the geometrical condition under which the velocity potential yields minimum action. This condition obviously is attained when the mass of fluid is perfectly spherical; for it may be shown that any departure from perfect sphericity yields a resulting action by all the waves greater than the minimum. If the total wave action be given by

$$\Omega = \iiint \Phi\,\sigma\, r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\omega \qquad (116)$$

then it will follow that for a sphere only is the action a minimum:

$$\delta\Omega = \partial\Omega/\partial x\cdot \mathrm{d}x + \partial\Omega/\partial y\cdot \mathrm{d}y + \partial\Omega/\partial z\cdot \mathrm{d}z = 0\,. \qquad (117)$$

We may reach a similar conclusion also from the wave-theory of gravitation, by noting that the force of gravity is due to waves receding from the centre of mass. The effect of the accumulating aether stress is the central force, which gives a body like our sun a sensibly spherical figure. This conclusion from the wave-theory is confirmed by observation, which shows that the heavenly bodies would be perfectly spherical except for rotations about their axes. The oblateness of the sun is found to be wholly insensible, and the oblatenesses of the different planets correspond severally to their respective rotatory motions.

Accordingly, in the case of immense masses the receding gravitational waves generate the central aether stresses which produce globular figures of the sun and planets; whilst in the case of small liquid drops the globular figures are maintained by the minimum action of the passing waves.

(ii) Geometrical criteria for the theory of minimal surfaces as applied to liquid masses and films.

In our previous discussion we found that in general a minimal surface is a surface of double curvature, such that the fundamental condition fulfilled is that

$$1/R_1 + 1/R_2 = \varrho_1 + \varrho_2 = 0 \qquad (91)$$

where ϱ_1 and ϱ_2 are the radii of the curvature of the two principal sections at any point of the surface. The radii of curvature are equal but of opposite sign, as shown in such figures as those of a saddle, a mountain pass, the surface of a glove between thumb and forefinger, etc.

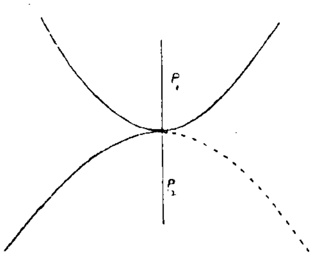

Fig. 19. Sketch of the radii of curvature for a minimal surface, $\rho_1+\rho_2 = 1/R_1+1/R_2 = 0$.

The two principal sections lie in different planes, but may be projected as shown in figure 19.

The theory of minimal surfaces involves the treatment of functions of three quantities $F(x, y, z)$ which may be determined as functions of two independent variables u and v, of the general type:

$$u = \int_{t_0}^{t_1} F(x, y, x_1, y_1, \mathrm{d}x_1/\mathrm{d}t, \mathrm{d}y_1/\mathrm{d}t)\,\mathrm{d}t \qquad (118)$$

where $\mathrm{d}x/\mathrm{d}t = x_1$, $\mathrm{d}y/\mathrm{d}t = y_1$, are equations of condition, each involving two variables, as x and y.

For the minimal surfaces, then, we have not a single but a double integral of the form:

$$\Omega = \iint F(x, y, z, \partial x/\partial u, \partial y/\partial u, \partial z/\partial u, \partial x/\partial v, \partial y/\partial v, \partial z/\partial v)\times$$
$$\times\,\mathrm{d}u\,\mathrm{d}v\,. \qquad (119)$$

(cf. Dr. *Hancock*'s Lectures on the Calculus of Variations, Cincinnati, 1904, p. 209.)

In the problem of molecular forces now before us we are concerned chiefly with the sphere, which for a given volume has the minimal surface. The problem of the sphere is therefore one in maxima and minima, corresponding to that of the circle, originally due to *Zenodorus*, (150 B. C.), who sought the plane figure with minimum perimeter. A treatment of it will be found in Dr. *Hancock*'s Lectures on Maxima and Minima, p. 92. In the same work, p. 75, there is a solution of the problem: To determine the greatest and smallest curvature at a regular point of a surface $F(x, y, z) = 0$.

That the sphere is a minimal surface is fairly obvious without any elaborate mathematical treatment. In the more general surface of double curvature, the fundamental condition $\varrho_1+\varrho_2 = 0$ will always hold true. For in the case of twisted surfaces it is obvious that the curvature must be opposite on the two sides; and every point of the surface must be about a centre of curvature lying in the principal planes.

Now imagine a physical surface, like a film of liquid, to depart from the minimal surface; then obviously our condition would be $\varrho_1+\varrho_2 = \alpha$, so that $\varrho_1 = -\varrho_2+\alpha$. And therefore the curvature in different planes would not be the same on opposite sides at the same point. The result of this condition would be that the film could not be of equal thickness or equal tension in different directions at any point. This obviously would not be a minimal surface; for it could be stretched and somewhat thinned out at the point, without altering the curvature on one side.

In fact the mathematical condition

$$\varrho_1 = -\varrho_2+\alpha \qquad (120)$$

would imply a swelling in the physical surface, or a sheet of unequal thickness. If this inequality existed, it would gradually augment, under wave action, and the lump of liquid would tend to increase to a drop. But this would disrupt the liquid surface.

Hence the condition of physical stability is

$$\varrho_1+\varrho_2 = 0$$

and a liquid sheet fulfilling this condition is stable so long as the tendency to a drop does not develop under gravitational action on the fluid.

A drop is a load, and may be slightly unsymmetrical, so that it leads to instability; the more it is augmented the more unstable it becomes till the liquid film is disrupted. For in passing the waves tend to make the drop round by everywhere decreasing its surface, and thus they operate to disrupt the film, by drawing in the liquid on all sides. These inferences are easily verified by actual experiments with soap bubbles and other films of soap water containing enough glycerine to make the surfaces elastic.

Now in view of the above reasoning we see why a liquid surface of soap water may stretch and hold taut, as a minimal surface, even when it is a surface of double curvature. It may take the form of a saddle, and yet be perfectly stable, because on the two sides of the film $\varrho_1 + \varrho_2 = 0$, enables the passing waves from all directions to traverse the liquid film with minimum resistance. If, however, $\varrho_1 + \varrho_2 = \alpha$, the integral for the action of the passing waves from all directions is not a minimum; and the principle of least action is violated.

Accordingly we conclude:

1. That minimal surfaces correspond to the principle of least action for all passing waves.

2. Any departure from minimal surfaces renders the wave action greater than the least possible, and therefore is not mathematically admissible, nor will it occur in physical nature.

3. Therefore drops of liquid always take a form as nearly globular as possible; and liquid films follow the mathematical law of minimal surfaces so as to make the physical action of the passing waves a minimum.

4. The instant a liquid film departs mathematically from the minimal form $\varrho_1 + \varrho_2 = 0$, as by the partial development of a drop, the inequality rapidly augments, and the surface is disrupted.

(iii) Examination of the wave-lengths appropriate to the several forces.

From the theory of physical forces resulting from the new theory of the aether it follows that waves of different lengths give rise to different physical effects. In a general way we know that the chemical forces correspond to the ultra-violet region of the spectrum; there also probably will be found the waves producing surface tension, capillarity, cohesion, adhesion, etc. Next in order of increasing wave length comes light, then heat, with the infra-red rays investigated by *Langley*, twenty times longer than the space covered by the visual rays known to *Newton*.

In an earlier section above, we have found the general expression for the potential of the molecular forces:

$$V = \sum_{i=1}^{i=i} m_i \varphi(r) = \sum_{i=1}^{i=i} m_i \int_0^r f(r)\,\mathrm{d}r + \sum_{i=1}^{i=i} m_i \int_r^\infty f(r)\,\mathrm{d}r \quad (121)$$

the second integral of which becomes zero when the distance r exceeds the radius of activity of the molecular forces at work.

Accordingly, we could make a table of wave lengths with their corresponding forces somewhat as follows:

$$V = \sum_{i=1}^{i=i} m_i \int_{\lambda_0}^{\lambda_1} f(r)\,\mathrm{d}r \quad \lambda < r \quad \partial V/\partial r = \text{Molecular forces: Chemical affinity, Surface tension, Capillarity, Cohesion, Tenacity, Adhesion.}$$

$$+ \sum_{i=1}^{i=i} m_i \int_{\lambda_1}^{\lambda_2} f(r)\,\mathrm{d}r \quad \begin{pmatrix} \lambda > \lambda_1 \\ < \lambda_2 \end{pmatrix} \quad \partial V/\partial r = \text{Light and heat.}$$

$$+ \sum_{i=1}^{i=i} m_i \int_{\lambda_2}^{\lambda_3} f(r)\,\mathrm{d}r \quad \begin{pmatrix} \lambda > \lambda_2 \\ < \lambda_3 \end{pmatrix} \quad \partial V/\partial r = \begin{cases} \text{Magnetism, Gravi-} \\ \text{tation, etc.} \end{cases}$$

$$+ \sum_{i=1}^{i=i} m_i \int_{\lambda_3}^{\lambda_4} f(r)\,\mathrm{d}r \quad \begin{pmatrix} \lambda > \lambda_3 \\ < \lambda_4 \end{pmatrix} \quad \partial V/\partial r = \begin{cases} \text{Electrodynamic} \\ \text{action.} \end{cases} \quad (122)$$

This table gives us an inspiring view of an immense subject, and may enable us to understand the types of waves effective in the various operations of nature. The first region of wave length here outlined, λ_0 to λ_1, is undoubtedly the region of very short range molecular forces. It could be further subdivided, in the order indicated on the right:

$$V = \sum_{i=1}^{i=i} m_i \int_{\lambda_1}^{\lambda_\alpha} f(r)\,\mathrm{d}r \quad \begin{pmatrix} \lambda > \lambda_0 \\ < \lambda_\alpha \end{pmatrix} \quad \partial V/\partial r = \begin{cases} \text{Chemical affinity,} \\ \text{Explosive forces.} \end{cases}$$

$$+ \sum_{i=1}^{i=i} m_i \int_{\lambda_\alpha}^{\lambda_\beta} f(r)\,\mathrm{d}r \quad \begin{pmatrix} \lambda > \lambda_\alpha \\ < \lambda_\beta \end{pmatrix} \quad \partial V/\partial r = \text{Cohesion, Tenacity.}$$

$$+ \sum_{i=1}^{i=i} m_i \int_{\lambda_\beta}^{\lambda_\gamma} f(r)\,\mathrm{d}r \quad \begin{pmatrix} \lambda > \lambda_\beta \\ < \lambda_\gamma \end{pmatrix} \quad \partial V/\partial r = \text{Adhesion}$$

$$+ \sum_{i=1}^{i=i} m_i \int_{\lambda_\gamma}^{\lambda_\delta} f(r)\,\mathrm{d}r \quad \begin{pmatrix} \lambda > \lambda_\gamma \\ < \lambda_\delta \end{pmatrix} \quad \partial V/\partial r = \begin{cases} \text{Capillarity, Surface} \\ \text{tension} \end{cases} \quad (123)$$

In a general scheme of this kind, it is obvious that if the forces pointed out be due to waves similar in type but of different lengths, the corresponding actions in many phenomena will somewhat overlap, and be more or less merged together. Thus chemical affinity is a maximum in the ultraviolet spectrum, which is very slightly visible as light. And in the same way the infra-red spectrum investigated by *Langley* is of such immense extent that in all probability the magnetic and gravitational waves will overlap at least part of this region. But these questions must be left to the future, in the hope that greater experience will enable us to illuminate problems which still remain quite obscure. For the present, suffice it to say that magnetic and gravitational waves must be long, otherwise they would be lacking in power of penetration; so that the sun's action on the moon would be almost wholly cut off at the time of lunar eclipses, which is contrary to observation in the lunar fluctuations.

(iv) New theory of acoustic attraction and repulsion: Confirmation of the wave-theory of gravitation.

In the New Theory of the Aether, AN 5044, 5048, 5079, 5085, we have treated of the waves between two bodies and shown that in the process of mutual interpenetration by the independent waves from each centre the medium is thinned out most in the straight line joining the bodies. As the kinetic exchange tends to keep the aether of uniform density, the tension is thus a maximum in this line, while the increase of stress or pressure is a maximum beyond the two masses. This could be otherwise expressed by saying that under the wave-action some of the aetherons are worked out from between the bodies, and transferred beyond them, as will be readily understood from the double wave field shown in Fig. 8, AN 5048, p. 163.

In order to illuminate this subject still further by well established physical data from known gases, we now [1]) treat very briefly of acoustic attraction and repulsion, which has been experimentally investigated, but not correctly explained by the following authorities.

1. The Philosophical Magazine, for April, 1871, p. 283, where Prof. *Challis* cites the experiment of *Clement* and considers hydrodynamical conditions.

2. The Philosophical Magazine for June, 1871, with experiments on acoustic attraction and repulsion by *Guyot*, *Schellbach*, *Guthrie*, and Sir *W. Thomson* (Lord *Kelvin*).

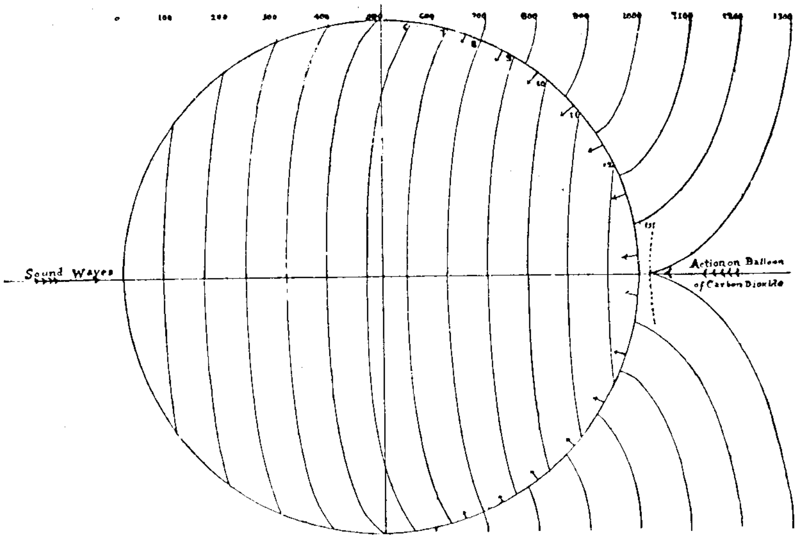

Fig. 20. Illustration of the progress of the wave-front when sound waves advance through the air with velocity 1, and through a balloon containing carbon dioxide, with velocity 0.78. Any phase of the sound wave thus reaches the rear of the balloon by going around through the air quicker than directly through the CO_2 of the balloon, and the reaction on the rear elastic membrane of the balloon impels it towards the source of the sound, which explains the observed acoustic attraction.

These experiments, as understood by physicists, have led to the conclusion that the vibrations of an elastic medium attract bodies which are specifically heavier than itself and repel those which are specifically lighter. (cf. *Ganot*'s Physics. 14th English edition, by *E. Atkinson*, 1893, p. 274). In proof of this view it is pointed out that a balloon of goldbeater's skin filled with carbonic acid gas is attracted towards the opening of a resonance box, bearing a vibrating tuning fork; while a similar balloon filled with hydrogen gas and tied down by a string is repelled. Experimenters have found that this result always follows, even when the hydrogen balloon is made heavier than air by loading it with wax, or other substances.

This last remark leads me to see in these experiments, not a law based on the relative specific densities of the bodies, but one based on their rate of conductivity of the sound vibrations.

In studying the phenomena of attraction and repulsion, due to electrodynamic action, we are placed at great disadvantage by the enormous speed of such action, which conceals from our view the nature of the process involved. It is therefore well to consider the slower processes which may be more accessible to investigation by laboratory experiments, chiefly in sound.

It is well known that as hydrogen has the greatest molecular velocity of any of the gases, it conducts sound vibrations more rapidly than any other gas. The following data are taken from the table in *Wüllner*'s Experimental-Physik, Leipzig, 1882, Vol. I, p. 804.

Gas	Density	Velocity of Sound in
Air	1	1
Oxygen	1.1056	0.9524
Hydrogen	0.06926	3.8123
Carbonic oxide	0.9678	1.0158
Carbon dioxide	1.5290	0.7812
Ammonia	0.59767	1.2534

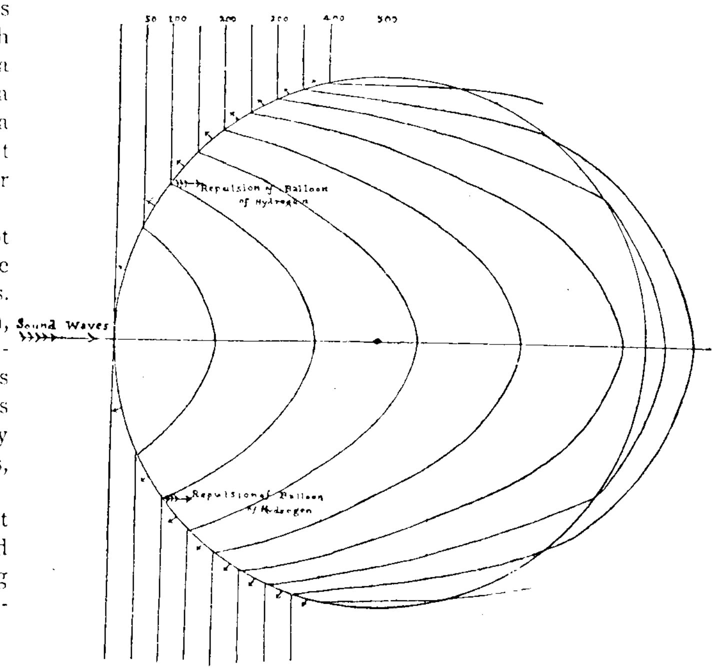

Fig. 21. Illustration of the enormously rapid advance of the sound wave-front in a balloon filled with hydrogen, $V = 3.81$. The internal advance of the sound wave is so rapid that the wave front reverses itself before the centre of the balloon is reached, and the elastic reaction against the surrounding air thus repels the balloon from the source of the sound. This accounts for the observed phenomenon of acoustic repulsion.

[1]) This explanation, based on the wave-theory, with the following plates for balloons of carbon dioxide and hydrogen, was developed in the year 1916, but publication has been deferred till the present time.

It appears from the numbers here given that in the very light gas hydrogen sound has 3.81 times the velocity that it has in air; while ammonia, a gas with relative density of 0.60, has a velocity of 1.25 times that in atmosphere. The facts thus support the view that a balloon filled with ammonia would also be noticeably repelled by the sound waves emitted from a resonance box bearing a vibrating tuning fork.

If this phenomenon of repulsion were due to the smaller average density, it would not persist after the balloon was loaded with wax, or other material, as has been found by observation in the case of hydrogen. It therefore must be due not to the relative lightness of the body floating in the air, but to the great velocity of the sound vibrations in the hydrogen, the waves of which are conducted through the body of the balloon more rapidly than through the air about it.

It is obvious that the rapid advance of the sound waves through the better conducting hydrogen gives a reaction against the surrounding air before this enveloping gas is agitated by the waves coming through the atmosphere alone; the effect of this advance agitation through the hydrogen is an elastic reaction of the hydrogen balloon against the greater part of its as yet undisturbed envelope. It thus rebounds against the inert air, and is repelled from the resonance box, as found by observation.

In the light of this explanation, which is the only one admissible, we readily see also by the above table why a balloon filled with carbon dioxide should be attracted to the resonance box. For the density of carbon dioxide is 1.529, and its conductivity of sound only 0.78 that of air. The sound waves on entering such a balloon will be appreciably outrun by those in the surrounding atmosphere; thus the outside air will give an advance elastic reaction against the enclosed sluggish balloon of carbon dioxide.

Viewed kinetically it is obvious that some of the lighter and more rapidly moving molecules of the air, under sound agitation, are thus transferred beyond the heavy mass of CO_2; and as the air between the balloon and the sounding box is thus somewhat thinned out, the increase of external pressure and internal tension incident to this kinetic transfer of some of the air particles to the space beyond the balloon, causes it to be attracted to the sounding box.

I find on examination that all the other phenomena of acoustic attraction and repulsion, which are reported by the eminent experimenters above named, can be satisfactorily explained in the same way; so that it is natural to infer that we have here a remarkable general law of nature. As the experiments are definite and decisive, it would seem that there is no escape from this conclusion, and the resulting law must therefore be taken as fundamental.

These well established experiments on acoustic attraction and repulsion, in the air, which we can experiment with in our laboratories, confirms our theory of gravitational attraction through the aether, with particles moving 1.57 times faster than light. For in the fourth paper, (AN 5085), we have shown from the confirmation of *Poisson*'s researches, how intimate is the connection between the theories of light and sound, as correctly held by that illustrious geometer a century ago.

The aether is of decreasing density in the direction of a central mass such as the sun, and when a body like the earth is also introduced, with decreasing density towards its centre, — thus giving two independent decreases of density incident to the waves from each centre — it follows that the actual density between the bodies is less than if the other body were absent. There is also increase of stress or pressure beyond the bodies. The result is the incessant pulling in the right line between the masses — which we call planetary forces. We could view either body as operating by its wave-action to agitate and expel some of the aetherons from the region between the masses, and increase them beyond so that the density is a minimum along the line sun-earth.

12. General Considerations on the Wave-Theory in Relation to *Planck*'s Quantum Theory, with an extension of *Planck*'s views, 1913.

(i) Tendency to geometrical forms explained by the wave-theory.

The plausibility of the wave-theory appears from the fact that if we take a solid and heat it, we get a glowing mass with predominant waves of heat, — leaving the action of the shorter waves of the molecular forces weakened, but still largely intact. If we still further increase the heat, the solid fuses into a liquid — the increasing agitation of the long heat waves have so far overcome the shorter waves underlying molecular forces that the molecules become released and the fluid is thus free to flow about. Still higher temperature will vaporize the liquid and convert it into a gas, with particles flying about with high velocities.

Now when we consider such transitions of the state of matter through various temperatures and crucial states, with predominant aether waves of various lengths, what explanation of the discontinuity in physical conditions is so plausible as that afforded by the wave-theory? Rising temperature liquifies and vaporizes all bodies; decreasing temperature and increasing pressure has enabled the experimenter to liquify and solidify all bodies, including the most permanent gases, such as oxygen, hydrogen, helium. The wave-theory is directly involved in all temperature problems, and we have shown how molecular forces depending on short waves develop and become effective when the long heat waves are withdrawn. Is not such a general indication in nature significant of the underlying cause?

It has long been recognized that the raindrops are spherical, but far too little attention has been directed to the question of this exact sphericity of figure, — how it arises and how it is maintained. Molecular forces indeed are the assigned cause of the sphericity, but as nothing is known as to the laws of these forces, or how they act, the current assumptions are admissible only in default of a better explanation. Under the circumstances it becomes advisable to inquire into the degree of sphericity of figure actually maintained, with a view of throwing light on the cause of molecular forces.

1. It is generally agreed that the colors of the rainbow are well separated, except for the overlap of images, due to the finite dimensions of the sun, which renders the spectrum impure. So far as one can see, therefore, the drops

of rain are exceedingly spherical; and no departure from perfect sphericity can be inferred from the observed colors of the rainbow, or from the conical form of that splendid arch of light.

2. Nor is there any evidence indicating noticeable oscillation of figure in the drops which produce the rainbow. Oscillations of figure would render the refraction and reflection irregular and variable; so that the angle of the cone of the rainbow from the anti-solar point would be variable. It is true that the rapidity of the oscillation would render the phenomenon difficult of detection; yet if a great number of drops, enough of them to constitute a considerable fraction of the whole, were incessantly in oscillation, it seems certain that the separation of colors along the conical outline would be blurred, and the rainbow appear as an overlapping hazy arch of light, devoid of distinct colors.

3. Now this hazy arch is not observed in the sky when natural rain is falling. We cannot say that this absence of a noticeably hazy border to the rainbow proves that no drops depart from the true spherical figure; but only that such departure from very perfect sphericity, if they exist, are exceedingly few or of excessively short duration for any individual drop, and thus the corresponding oscillations of figure exert no sensible diffusion of coloration, in comparison with the integral effect of the light from all the spherical drops.

In the Proc. Roy. Soc., May 5, 1870, no. 106, the late Lord *Rayleigh* has devised a means of determining the time of vibration of a dew drop. Lord *Kelvin* found the formula for the period of vibration to be

$$\tau = {}^1/_4 a^{3/2} \text{ second} \qquad (124)$$

where a is the radius of the sphere of water measured in centimetres. For a radius $^1/_4$ cm the period is $^1/_{32}$ second; and hence the table:

r	τ
$^1/_4$ cm	$^1/_{32}$ second.
1	$^1/_4$ »
2.54	1 »
4	2 »
16	16 »
36	36 »
1407	13200 »

Accordingly, when the drop is one inch in diameter, 2.54 cm, a whole second is required for the vibration. It is only for small drops that the vibration is rapid, and the forces powerful.

Before the time of *Plato* the Greek geometers had noticed the spherical figures of the sun and moon, and inferred a like spherical figure for the earth, from the circular section of the earth's shadow at the time of lunar eclipses. As the orbits of the planets also appeared to be essentially circular, and the Greek natural philosophers noted the tendency of drops of dew, oil, and other liquids to assume the spherical form, which was then held to be a perfect figure, it is now possible to understand *Plato*'s doctrine that the deity always geometrizes, — ὁ θεὸς ἀεὶ γεωμετρεῖ.

Apparently this conclusion was not an idle remark, but represented a genuine philosophical induction from the observed order of nature, which we are only beginning to interpret after the lapse of some two thousand three hundred years. To the modern natural philosophers it will appear as wonderful as it did to the Greeks that nature approximates these very beautiful geometrical figures. Thus the cause of such observed phenomena should engage the attention of the leading geometers of our own age.

In view of the foregoing discussion, it appears that the physical cause of the rainbow is a two-fold one.

1. The exact sphericity of the raindrops, the spherical figures of which are maintained by passing waves shorter than those of light. The cause of these minimal spherical surfaces is now assigned for the first time and shown to accord with the *Weierstrass-Schwarz* mathematical theory of minimal surfaces.

2. The dispersion of the light is due to the spherical figures of the drops with the refraction of the incident light following the law of *Snellius*, as *Descartes* found by actual calculation, 1637. The true theory was originally discovered by *Theodorich*, about 1311 A.D., but his explanation was not published till 1814. Meanwhile it was independently discovered by *Antonius de Dominis*, Archbishop of Spalato, about 1591. *Newton* first developed the complete theory of spectral colors through the decomposition and recomposition of white light, in a series of experiments begun in 1666, and fully published in his Treatise on Optics, 1704.

In view of this development, it is well to dwell on the real physical significance of the rainbow. We should remember that the very existence of this great natural phenomenon implies an infinite variety of waves. Otherwise this splendid bow of color would never span the heavens. Just as the colors in the sky are a perpetual reminder that some 500 trillion waves enter the eye every second, so also do they tacitly imply not merely waves from the region of the visible spectrum, but also invisible waves from the region of the ultra-violet.

It would be in the highest degree improbable that waves come only from the visible spectrum; for the longer heat waves always accompany the sun's light, and are known to come from the red and infra-red regions of the spectrum; and as chemical processes always are going on in nature, and are known to depend on the shorter violet and ultra-violet waves, it follows from the chemical processes of the world alone, that ultra-violet rays also fill the sky, though quite invisible on the blue border of the rainbow.

Accordingly, when we behold the glorious arch of the rainbow, we are at the same time reminded of quadrillions of waves too short to be visible, yet entering the eye every second. They too fill every part of the sky and traverse every drop of rain just as the waves of the visible spectrum enter the correspondingly small pupil of the eye, to the number of about half a quadrillion. Under these circumstances, it is strange that we have not sooner recognized how the waves give the raindrops such mathematically perfect sphericity, and by the resulting dispersion add to the beauty of the world.

(ii) General outline of *Planck*'s Quantum Theory, with inference suggested by the wave-theory.

From the wave-theory thus briefly outlined it follows that all the phenomena of the physical universe should depend on the mutual interaction of waves and the corresponding forces of nature. This relationship is shown in the theory of the correlation of forces, and the doctrine of the conservation of energy, which have become fundamental in modern science. But there are some difficulties to be overcome, and heretofore a method of attacking them has not been developed, even by the most eminent authorities. It seems likely that most of the supposed difficulties of the wave-theory will disappear the moment we attribute the forces of nature to wave action; for then we may use the forces of nature to study the waves by which the forces are produced, and also investigate the forces observed with a view of inferring the type of waves from which they might arise.

Accordingly, after this sketch of the wave-theory, we have now to consider the views announced by Professor *Planck*. In his address as Rector of the University of Berlin, Oct. 15, 1913, reported in the Revue Scientifique, Paris, Feb. 14, 1914, *Planck* gives a summary of his chief conclusions, to the effect that neither motion nor physical force is strictly continuous in character, but each of them made up of small jumps or sudden alterations in value. This quantum theory probably is not identical with the wave-theory, yet it has enough elements in common to be worthy of careful examination, on the probability that the two theories may be reconciled by future developments.

Planck's theory is described very briefly in the following account:

»Suppose a mass of water in which violent winds have produced a train of very high waves. After the wind has ceased, the waves still maintain themselves and go from one shore to another. Then takes place a characteristic change. The energy of motion of the longer and larger waves gradually changes, especially when they meet the shore or other solid objects, into that of shorter and smaller waves, until finally the waves become so small as to be quite invisible. This is the well-known change of visible motion into heat, of mass movement into molecular movement.«

»But this process does not go on indefinitely; it finds a natural limit in the size of the atoms. The larger the atoms are, the sooner comes the end of this subdivision of the total energy of movement.«

»Now suppose a similar process with undulations of light and heat; suppose that the rays emitted by a powerfully incandescent body are concentrated into a closed cavity by mirrors and there continually reflected to and fro. Here also will take place a progressive transformation of the radiant energy into shorter and shorter waves. According to classic theory we should expect that the whole energy of the radiation should finally be confined to the ultra-violet part of the spectrum.«

»Now, not the slightest trace of any such phenomenon can be discovered in nature. The transformation reaches, sooner or later, a perfectly clear and well-determined limit, and after this the state of the radiation is stable in all respects.«

»To make this fact agree with the classic theory the most diverse attempts have been made; but it has been shown that the contradiction penetrates too deeply into the roots of the theory to leave them intact. So the only thing to do is to overhaul the foundations of the theory.«

»In the case of the water-waves the subdivision of their energy of motion came to an end because the atoms retained the energy in a certain way, because each atom represents a determinate quantity of matter, which can move only as a whole. Also in the light and heat radiation, although it is quite immaterial in its nature, there must be certain processes in action that retain the energy in determinate quantities and retain them the more powerfully as the waves are shorter and the vibrations more rapid.«

This outline of *Planck*'s theory assures us that the transformation of energy waves into shorter and shorter wavelength would lead one to expect »that the whole energy of the radiation would finally be confined to the ultra-violet part of the spectrum. Now, not the slightest trace of any such phenomenon can be discovered in nature. The transformation reaches, sooner or later, a perfectly clear and well defined limit, and after this the state of the radiation is stable in all respects.«

It should be pointed out that molecular forces furnish evidence of such shorter and shorter waves, at least up to a certain limit hitherto quite unknown. And it is found from the observed thickness of soap bubbles, just before their rupture, that this length corresponds to the wave-length of the ultra-violet spectrum and beyond. Accordingly, it seems to me that *Planck* has not drawn all the admissible conclusions. For if we concede that molecular forces be due to waves, the evidence is that shorter and shorter waves really exist, at least to atomic and perhaps electronic dimensions.

Nature therefore presents to us a book of mysteries which is not yet opened, but securely sealed, as with seven seals. As we have to explain cohesion, adhesion, hardness, tenacity, etc., we cannot yet truthfully say what is the limit of the shortness of the waves, unless this is finally set by the dimensions of the atoms and electrons.

In the last paragraph of the above quotation *Planck* describes the smallness of the masses as fixing limits to the shortness of waves, because such small masses can only move as a whole. He does not show how these elementary quanta of matter vibrating as a whole are represented, but the inference is that no source is able to give out energy till the energy has reached a certain value, by natural sympathy of the vibrating system or otherwise, as in *Helmholtz*'s resonators, with which the atoms have many properties in common.

(iii) Discontinuities in the quantum-theory naturally accounted for by the wave-theory.

It is chiefly by the differences of wave-lengths in the integrals for the molecular forces that we explain the different forces of nature. The bolometer-researches of *Langley* on the solar spectrum, showed that the wave-lengths are quite irregularly distributed over the infra-red region. If therefore the operations of heating, at different temperatures t_1 and t_2 should bring into prominence the part played by waves of lengths between λ_1 and λ_2, it might be possible to account for the discontinuities noted by *Planck*.

For as changing resistance breaks up electrical waves from longer to shorter wave-length, and at the same time heat waves appear from this disintegration, it is very probable that in summing up the effects of waves over a great range of wave-lengths many special phenomena would appear suddenly at certain temperatures. This probable connection between the quantum-theory and the wave-theory seems to make intelligible a great body of phenomena involving sudden transition, which heretofore have been quite obscure to the natural philosopher. It is necessary to have some mental picture of the cause of the apparent discontinuity, and at present this can only be supplied by the wave-theory.

Professor *Planck* describes the apparently discontinuous and explosive character of certain natural phenomena as follows:

»In any case the hypothesis of quanta has led to the idea that there are in nature changes that are not continuous, but explosive. I need only remind you that this representation is made acceptable by the discovery and close study of radioactive phenomena. The hypothesis of quanta has so far enabled us to obtain results in better accord with existing measurements of radiation than those of all preceding theories.«

»But there is something further. If it is a point in favor of a new hypothesis that it is verified even in regions to which it was not expected to apply, at the outset, the hypothesis of quanta may surely claim an advantage. I desire to call attention here only to a single striking circumstance. Since we have succeeded in liquifying air, hydrogen and helium, an abundant and new field of experimentation has opened to research in the domain of the lower temperatures, and already a whole series of new and extremely surprising results have come to light.«

»To heat a piece of copper from -250° to -249°, that is, by one degree centigrade, not the same quantity of heat is required as to heat it from 0° to 1°, but about thirty times less. If we started with an initial temperature still lower, we should find that the corresponding quantity of heat was still smaller, without assignable limit. This is directly contrary not only to all customary statements, but also to the requirements of the classic theory, for although we learned more than a century ago to distinguish strictly between temperature and quantity of heat, we have nevertheless been led to the conclusion that even if these magnitudes are not exactly proportional, they vary at least in some parallel way.«

»The hypothesis of quanta has completely cleared up this difficulty, and moreover has furnished another result of high importance, namely, that the forces which provoke heat-vibrations in a solid are precisely the same as those that produce elastic vibrations. We may thus now calculate from the elastic properties of a monatomic body its heat energy at different temperatures a service that the classic theory has never been able to perform.«

The researches heretofore made are too incomplete for us to affirm that these phenomena of quanta can all be explained by the wave-theory; yet the indications of a hitherto unsuspected connection are so plain that the cause underlying the observed phenomena will necessarily become an object of attention in future investigations. Heretofore the phenomena of quanta have appeared as deep mysteries.

(iv) Conclusion to the fifth paper on the new theory of the aether.

From the foregoing comprehensive but necessarily incomplete survey of an extensive subject, it appears that the wave-theory of molecular forces is overwhelmingly indicated by the minimal surfaces pervading nature. The tendency to perfect sphericity of figure is so remarkable a phenomenon that it can not fail to become an object of research among philosophers, as to why these physical laws exist.

It appears that *Plato* saw in the nearly circular orbits of the planets, and in the spherical figures of the sun and moon and all fluid globules the geometrizing of the Deity — *ὁ θεὸς ἀεὶ γεωμέτρει.*

But *Newton*, *Clairaut*, and *Laplace* showed that the theory of universal gravitation fully accounts for the figures of the heavenly bodies. And recently it has been recognized, from the writer's Researches in Cosmogony 1908–10, that the observed roundness of the orbits of the planets and satellites, which had so profoundly impressed both *Newton* and *Laplace*, is due to the secular action of the nebular resisting medium formerly pervading the solar system.

Thus, to complete the solution of the problem of the Greek philosophers, it remained to account for the perfect sphericity of figure of liquid drops. This production of perfect liquid spheres in nature we have now explained by the wave-theory, which yields minimal surfaces with very remarkable geometrical properties. The proof deduced from the *Archimedean* theorem, Fig. 5, section 4, that spherical drops of liquid are true minimal surfaces, for the whole of the waves traversing the universe in every direction, doubtless will be of more than ordinary interest to geometers and natural philosophers. I am not aware of any previous use of this beautiful theorem, in physical investigations, since the days of *Archimedes*.

In this paper no considerable outline of the wave-theory of chemical affinity, and of explosive forces has been attempted. That is reserved for a sixth paper, in which I hope to deal also with the living forces. These vital processes long have been considered electrical in character, and yet beyond the reach of research so long as molecular forces could not be definitely referred to wave-action. The problems of crystallography likewise are many and promising, and I have left the wave-theory of the hardness of diamond incomplete, yet sufficiently outlined to be suggestive to others.

It only remains to add that *Maxwell*, *Boltzmann*, and other eminent natural philosophers, have taken the molecular forces to vary inversely as the fourth or fifth power of the distance $f(r) = K/r^4$, or $f(r) = K'/r^5$, which will be found to accord well with the wave-theory. According to *Laplace*'s hypothesis these forces are sensible only at insensible distances, and thus manifest themselves chiefly in the immediate proximity of physical matter, where the refractions, dispersions, diffractions, interferences, etc. appear separately or conjointly and in unknown intensity.

If, on the average, about two or three of these influences be at work near physical bodies, — the intensity of each being as the inverse square of the distance r, — the compound effect of their joint action would be approximately

$$\varphi(r) = K/r^4, \; \psi(r) = K'/r^5, \text{ or } \chi(r) = K''/r^6 . \quad (125)$$

This conclusion accords well with observations, but as the distances at which phenomena are noted are nearly insensible, we must not expect great observational accuracy, nor attach much importance to the theoretical agreement with the wave-theory.

After outlining this new theory of molecular forces, it only remains to call attention to certain definite steps in the theory of the rainbow, the study of which, under the undulatory theory of light, has now enabled us to assign the cause of molecular forces.

1. About the year 1311 A.D. the first analysis of the colors of the rainbow, with correct explanation of the refractions producing the primary and secondary bows, was made by *Theodorich* (cf. *Venturi*, Commentarii sopra la storia e le teorie dell'ottica, Bologna, 1814), who was a contemporary of *Dante*, and thus flourished in the darkest period of the Middle Ages. But *Theodorich*'s researches were not published until 1814, — after a delay of 503 years! — so that they first became known early in the 19th century.

2. Meanwhile about the year 1591, the celebrated *Antonius de Dominis*, Archbishop of Spalato, independently discovered and experimentally demonstrated the origin of the colors of the rainbow. In his Treatise on Optics, 1704, p. 126, *Newton* says:

»This bow never appears but where it rains in the sunshine, and may be made artificially by spouting up water which may break aloft, and scatter into drops, and fall down like rain. For the sun shining upon these drops certainly causes the bow to appear to a spectator standing in a due position to the rain and sun. Hence it is now agreed upon that this bow is made by refraction of the sun's light in drops of falling rain. This was understood by some of the ancients, and of late more fully discovered and explained by the famous *Antonius de Dominis*, Archbishop of Spalato, in his book De Radiis Visus et Lucis, published by his friend *Bartolus* at Venice, in the year 1611, and written about twenty years before. For he teaches there how the interior bow is made in round drops of rain by two refractions of the sun's light, and one reflexion between them, and the exterior by two refractions and two sorts of reflexions between them in each drop of water, and proves his explications by experiments made with a phial full of water, and with globes of glass filled with water, and placed in the sun to make the colours of the two bows appear in them.«

3. It is well known that *Newton*'s experiments on colors, with the decomposition and recomposition of white light by means of prisms, were begun in 1666, but not published in full until 1704, when the celebrated Treatise on Optics appeared. Since *Newton*'s day there has been no material change in the theory of the colors of the rainbow.

4. Having had occasion to examine the theory of the rainbow with much attention in the year 1916, I was led to conceive that the waves entering and leaving the drops would exert a pressure towards the centre, and thus to form a new theory of molecular forces depending on wave-action. At length, after several years of research, I have been able to outline a proof that heretofore we have recognized only a small part of the wave secrets of nature.

5. Accordingly, it appears that the study of the rainbow has finally led to the cause of molecular forces, including the phenomenon of lightning, which so long proved terrifying to mankind, and utterly bewildering even to the most learned natural philosophers. The rainbow itself is beautiful, but its wave origin was suggestive of deeper secrets of nature. In fact, if our new theory of molecular forces be admissible, this great arch of light so splendidly spanning the heavens during rains and thunderstorms now becomes nothing less than a triumphal arch of discovery. By the study of the illumination of this glorious arch we are enabled to penetrate the much deeper mystery of atmospheric electricity and of the lightning, which in all ages has spread fear and terror in those who rejoiced to behold the splendor of the rainbow.

6. Accordingly, if mankind should hereafter be able to view the rainbow, and the lightning so frequently associated with it, in calm reassurance that both phenomena depend on the all-pervading aether and represent the same wave-order in nature, it ought to afford some consolation to philosophers to realize that their researches, beginning with *Theodorich*'s pioneer effort in the age of *Dante*, subsequently extended by *Antonius de Dominis*, *Descartes* and *Newton* in the 16th, 17th and 18th centuries, have finally brought to light an even greater secret of the universe. Under the circumstances a torch-bearer of the Greeks, who had brought down the lightning, as the most dazzling flash of the aether of the skies, doubtless would have exclaimed with *Aeschylus*:

Ναρθηκοπλήρωτον δὲ θηρῶμαι πυρὸς
Πηγὴν κλοπαίαν, ἣ διδάσκαλος τέχνης
Πάσης βροτοῖς πέφηνε καὶ μέγας πόρος.
Prom. Vinct. 109.

»I brought to earth the spark of heavenly fire,
Concealed at first, and small, but spreading soon
Among the sons of men, and burning on,
Teacher of art and use, and fount of power.«

Very grateful acknowledgements are due to Mr. *W. S. Trankle* for facilitating the completion of this paper.

Starlight on Loutre, Montgomery City, Missouri, 1920 Dec. 10.
T. J. J. See.

I. Postscript. Since finishing the above paper it has occurred to me that the nature of the wave-action in maintaining the oscillations of a globule of liquid might be examined somewhat more critically. When a drop is disturbed from the spherical form its figure oscillates from a prolate to an oblate ellipsoid, or vice versa. Thus it may be worth while to consider these extreme spheroidal forms of the globule.

1. The Prolate Spheroid. The equation of the meridional section is

$$a^2 y^2 + b^2 x^2 = a^2 b^2 \qquad (\alpha)$$

which gives

$$y^2 = (b^2/a^2)(a^2 - x^2)\,. \qquad (\beta)$$

The differential element of the volume is

$$dv = \pi\,(b^2/a^2)(a^2 - x^2)\,dx \qquad (\gamma)$$

which by integration gives:

$$v = \pi\,(b^2/a^2) \int (a^2 - x^2)\,dx \qquad (\delta)$$

$$= \pi\,(b^2/a^2)(a^2 x - {}^1/_3 x^3) + c \qquad (\varepsilon)$$

If we calculate the volume from the plane passing through the centre, we have for $x = 0$, $v = 0$, and therefore also $c = 0$. Hence between the limits $x = 0$, $x = a$, we have

$$\begin{aligned} {}^1/_2 v &= {}^2/_3 \pi\, b^2 a \\ v &= {}^4/_3 \pi\, b^2 a\,. \end{aligned} \qquad (\zeta)$$

As πb^2 is the area of the circle described on the conjugate axis, and $2a$ is the transverse axis, and the volume

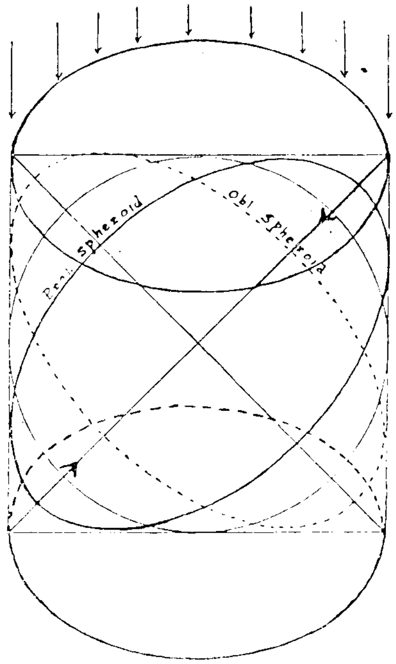

Fig. 22. Theory of wave-action on oscillating drop of liquid, alternately prolate and oblate, the circumscribing cylinder having axis oblique, and being somewhat variable in form and dimensions, which however for the sake of simplicity is not represented in the figure.

of the circumscribed cylinder therefore $2\pi b^2 a$, it follows that the volume of the prolate ellipsoid is to that of the circumscribed cylinder as $2:3$, which is a remarkable extension of the celebrated theorem of *Archimedes* illustrated above in Fig. 5.

2. The Obiate Spheroid. In this case we have obviously

$$dv = \pi x^2 \, dy \,. \qquad (\eta)$$

And on substituting for x^2 its value from the equation of the ellipse, namely:

$$x^2 = (a^2/b^2)\cdot(b^2 - y^2) \qquad (\theta)$$

we get in like manner:

$$v = \pi \, (a^2/b^2) \int (b^2 - y^2) \, dy = \pi \, (a^2/b^2) \, (b^2 y - {}^1/_3 y^3) + c \,. \qquad (\iota)$$

And between the proper limits, this expression for v becomes:

$$v = {}^1/_3 \pi \, a^2 \, b \qquad (\varkappa)$$

which is another remarkable extension of the celebrated theorem of *Archimedes* illustrated in Fig. 5 above.

If we compare the volumes of the two spheroids here considered, we find:

$$\text{Obl. Spheroid} : \text{Prolate Spheroid} = a : b \,. \qquad (\lambda)$$

It thus appears that the volumes of the two spheroids are as their greater and lesser axes respectively. Accordingly, the volumes of the cylinders orthogonally circumscribed about them would also be in the ratio of $a : b$. And the *Archimed*ean theorem on the ratio of the volumes of the spheroids to the orthogonally circumscribed cylinders in each case is $^2/_3$.

3. Now when we consider a drop of water or other liquid oscillating about its mean figure, which is spherical, we perceive that there is not only an alteration of figure, from a prolate spheroid to a sphere, and from a sphere to an oblate spheroid, or vice versa, but also that an alteration of volume would be expected to occur except for the incompressibility of the fluid under the slight force of surface tension. The incompressibility of liquids, however, imposes the condition:

$$\text{Obl. Spher.} = \text{Prol. Spher.} = \text{Sphere}$$
$${}^4/_3 \pi \, a^2 \, b = {}^4/_3 \pi \, b'^2 \, a' = {}^4/_3 \pi \, r^3 \qquad (\mu)$$

or

$$r = \sqrt[3]{(a^2 \, b)} = \sqrt[3]{(a' \, b'^2)} \qquad (\nu)$$

This requires that for an oscillating globule the axes a and b in the two spheroidal forms must take successively appropriate values, yet when the form of the spheroid has alternated, the axes are not identical in the two cases, and should be written as in equation (ν).

4. If we consider the resistance to the waves, due to the fluids in the prolate and oblate spheroids, when the axis of the circumscribing cylinders coincides with the major axis of the prolate and the minor axis of the oblate spheroid, it is evident from the above equations that the *Archimed*ean theorem will hold rigorously true for these two orthogonally coincident axial positions, just as in the case of the sphere treated in Fig. 5 above. In these cases the resistance to the waves due to the fluid spheroids is exactly $^2/_3$ of that due to the whole cylinder of fluid.

But when the axes of the fluid spheroids are oblique or inclined at any angle to the axis of the circumscribed cylinder, this theorem of the ratio $^2/_3$ for the resistance of the passing waves will not hold. In the oblique position of the axes the section of the cylinder is not circular, but really elliptical. And even if the circumscribing cylinder be elliptical, the wave resistance due to the enclosed obliquely tilted spheroids will be less than $^2/_3$ of that due to the whole cylinder.

(a) The wave pressure at the two ends of the spheroids, parallel to the polar axes, is relatively greater than from the various oblique directions.

(b) Whilst the axes of the spheroids remain fixed in position the sides of the figure are thus forced in or out, as the case may be, till the motion is checked by inertia balancing momentum, as the globule maintains its vibration; and this oscillation, heretofore attributed to unknown molecular forces, is really due to the unequal wave pressure accumulating at the boundary of the fluid in the different directions.

(c) The above figure will convey some impressions of this oscillation in a typical case, but the enclosing cylinder must be conceived as somewhat variable in figure and dimensions. These additional considerations show that the wave-theory may be adapted to the behavior of drops in oscillation as well as to those which have settled down to the figure of equilibrium, which when free from external forces, is that of a sphere.

1921 Feb. 19. *T. J. J. See.*

2. Postscript.

Theory of the Flow of Waves in Right Lines through any Conical Space ω, and of the Change of the Double Integral of the Waves over any Closed Surface S, when Refraction occurs within the Enclosed Space [1]).

[1]) Written about 3 years, but not heretofore published.

In the theory of the brightness of the stellar universe, under an equal distribution of the stars as conceived by *Herschel*, *W. Struve* shows that for a small solid angle ω, the number of stars $d\nu$ included in the cone thus defined between the distances r and $r+dr$ is given by the expression

$$d\nu = k\,\omega\,r^2\,dr \qquad (1)$$

in which k is a constant.

As the total light is determined by the accumulated effect of the stars at their several distances, the whole amount of light received from such a cone will be found by integrating this expression between the limits 0 and ∞:

$$\sum_{i=1}^{i=i}(1/r_i^2)\,d\nu_i =$$

$$= A = k\,\omega\int(1/r^2)\,r^2\,dr = k\,\omega\int dr = k\,\omega\cdot\infty \qquad (2)$$

In practice this expression is finite and less than the brightness of the sun's disc, and thus either the universe is finite, or an absorption of light by cosmical dust in space is considered probable.

Let λ be the flow of light in straight lines, from a luminous point, under constant wave velocity; then if r be the distance of the luminous point, the intensity, or quantity of light which passes through unit of surface perpendicular to the ray in unit of time, will be proportional to the illumination of steady intensity defined by the equation:

$$I = \lambda/r^2 \qquad (3)$$

If the surface illuminated be inclined at an angle ϑ to the ray, we have for the intensity of the illumination of such a tilted area: $I' = (1/r^2)\,\lambda\cos\vartheta\,.$ (4)

In the article Light, Encycl. Brit., 9[th] ed., *Tait* points out that these two intensities are exactly similar to the following expressions: Equation (3) is the expression for the gravitational force exerted by a particle of mass λ on a unit of matter at distance r; and (4) for the resolved component of this force on a given direction. This is an additional indication that gravitation and light are both due to waves.

Accordingly, if there be any number of separate sources of light, we may employ, for calculation of the effect, an expression exactly analogous to that of the gravitational or electric potential, namely:

$$\Omega = \sum_{i=1}^{i=i}(1/r_i)\,\lambda_i =$$

$$= \iiint[(x-x')^2+(y-y')^2+(z-z')^2]^{-1/2}\,\sigma\,dx\,dy\,dz\,. \qquad (5)$$

And the differential of this expression with respect to r

$$\partial\Omega/\partial r_i = \sum_{i=1}^{i=i}(1/r_i^2)\,\lambda_i =$$

$$= \iiint[(x-x')^2+(y-y')^2+(z-z')^2]^{-1}\,\sigma\,dx\,dy\,dz \qquad (6)$$

will give the total radiation due to any number of separate sources of light, when the waves are propagated in right lines, in cones composing spheres, separately homogeneous in wave distribution, about the several centres of radiation.

Moreover, if n be the external normal at any point of a closed surface, we may, as in the fundamental proposition of potentials, take the double integral over the surface:

$$\iint d\Omega/dn\cdot dS = -4\pi\,\lambda_0 \qquad (7)$$

where λ_0 is the sum of the values of the light λ_i from the several sources lying within the surface.

As every source external to the closed surface sends in light which goes out again, and thus leaves the wave-distribution in the cones of space unchanged, while the light from the internal source goes wholly out, we perceive that the amount of light lost through the surface per second for each unit source is 4π, the total area of the unit sphere surrounding the source. Hence we verify the above formula (7), that for all the internal sources the integral is the sum of the several sources of radiation λ_i, and thus equal to $-4\pi\,\lambda_0$.

Now consider light waves flowing in conical streams from the objects of the material universe in every direction. It is easily seen that the light received from a uniformly illuminated plane surface, when the normal is inclined at the angle ϑ, is represented by the double integral:

$$A = \iint(1/r^2)\cos\vartheta\,dS\,. \qquad (8)$$

It may be shown that for a closed surface, which has no inside source of light, this integral vanishes, because the original wave distribution in the cones of space is unchanged. And for all shells of equal uniform brightness whose edges lie on the same cone its value is constant.

This theorem, that when propagated in right lines, the expression for the light passing through the closed surface vanishes for all external sources, is of the highest significance: it affords an experimentum crucis as to the flow of light from all sources in spherical cones in which the light distribution remains homogeneous — free from refracting or dispersing disturbances — and any other kind of flow gives this integral a finite value different from zero. Hence in general we have

$$\iint d\Omega/dn\cdot dS = \iint(1/r^2)\cos\vartheta\,dS = -4\pi\,\lambda_0 + A\,. \qquad (9)$$

To apply the above theorems to capillarity and other molecular forces, it suffices to enclose the fluid at the point under investigation (x, y, z), with a spherical surface of convenient radius, so that the waves from external sources are redistributed by refraction, dispersion, etc., within the sphere surface.

(a) We may neglect the collective actions of the waves originating from the particles within the enclosing spherical surface: such aggregate action yields the expression

$$\iint d\Omega/dn\cdot dS = -4\pi\,\lambda_0\,. \qquad (10)$$

As the enclosed matter, by hypothesis, is not a chief source of radiation, we know that λ_0 is small, and of the order of intensity seen in gravitative forces, which are always very feeble.

(b) Thus we are left to consider the effects of the waves passing through the sphere surface enclosed about the fluid and solid at the point (x, y, z). These waves are refracted, dispersed, and unequally resisted by the matter in the paths of the cones which make up the surface 4π of the unit sphere. If no refraction, dispersion or resistance occurred, the integral of these passing waves would be zero:

$$\iint (1/r^2) \cos\theta \, dS = 0 \,. \qquad (11)$$

But under the refractions, dispersions, retardations, etc., actually occurring along the paths of certain cones, the integral does not vanish, but always reduces to a finite quantity:

$$\iint (1/r^2) \cos\theta \, dS = A \qquad (12)$$

This failure of the integral over the closed surface to vanish, implies that the aethereal medium is stressed by the refractions, dispersions, etc., along the paths of certain cones, thus developing forces, which may become quite large in certain cases.

It is upon this integral (12) that the molecular forces depend: and as the integral for the effects of the redistributed waves over the closed surface is not zero, the wave-principle of Least Action always makes the integral for the sum total of the action of the waves along all their actual paths a minimum. Thus the residue A in (12) is made as small as possible.

Accordingly there are physical limitations imposed by nature upon the geometrical conditions underlying *Gauss*'s theorems that in the theory of the potential:

1. For an internal point

$$\iint d\Omega/dn \cdot dS = -4\pi \lambda_0$$

2. For an external point

$$\iint (1/r^2) \cos\theta \, dS = 0 \,.$$

1. These celebrated theorems (Allgemeine Lehrsätze, § 22, *Gauss*'s Werke, Bd. V, p. 224) are based upon rectilinear actions in nature which follow the law of the inverse squares, as specifically pointed out by *Gauss* in his introductory remarks, §§ 1, 2.

2. If, therefore, there be in nature forces due to waves, — which suffer refraction or dispersion when the wave path is through heterogeneous matter, as when a fluid is in contact with a solid or of such shape as to cause refraction or dispersion, — these theorems of *Gauss* cease to hold rigorously true.

3. It is upon such principles that the fluctuations of the moon depend. And in a different way, the stress arising from wave action gives rise to molecular and atomic forces (cf. section 7 above).

1921 July 4. *T. J. J. See.*

Photographische Oppositionshelligkeit des Neptun 1920. Von *K. Schütte.*

In den Jahren 1920–21 machte ich auf Anregung von Herrn Prof. *Wilkens* und in Fortsetzung von dessen eigenen, bisher nicht veröffentlichten Beobachtungen extrafokale Aufnahmen der 4 helleren Jupitertrabanten, zwecks Ermittelung ihrer photographischen Helligkeitsschwankungen.

Da im Februar und März 1920 Jupiter und Neptun zur Zeit der Opposition sehr nahe aneinander vorbeigingen, so befindet sich auf mehreren Platten auch der Neptun, womit hier unbeabsichtigt die Gelegenheit gegeben war, seine photographische Helligkeit zu messen. Da die Untersuchungen über die Jupitermonde noch nicht abgeschlossen sind, sei hier — eine spätere ausführlichere Veröffentlichung vorbehaltend — das Ergebnis der Ausmessung, soweit es den Neptun betrifft, mitgeteilt.

Der 8-zöllige Refraktor diente als Richtfernrohr, während die Aufnahmen selbst mit einem der Sternwarte von den Zeiß-Werken zur Verfügung gestellten 4-zölligen photographischen Objektiv von etwa 1.90 m Brennweite ausgeführt wurden. Die Platten (9×12 cm) befanden sich 12–13 mm extrafokal und zeigen kleine Sternscheibchen von ca. 0.6 mm Durchmesser; dieselben sind in der Mitte homogen geschwärzt und haben nur am Rande einen mehr oder weniger dunklen schmalen Ring.

Als Vergleichsterne wurde die in der Nähe des Jupiter stehende Praesepe im Krebs gewählt, für welche die *Schwarzschild*schen photographischen Größen angenommen wurden [1]). Die photographische Extinktion ist gleich der 2.5-fachen visuellen gesetzt. Die Schwärzung jedes Scheibchens wurde 2–4 mal unabhängig voneinander geschätzt und dann gemittelt. Nach dem *Schwarzschild*schen Schwärzungsgesetz für gleiche Expositionszeiten folgt [2]):

$$\delta m = \sigma \cdot (y - s) + e_\alpha - e_\beta$$

wobei σ = Skalenwert der benutzten Skala, nach der Methode der kleinsten Quadrate aus den Praesepesternen bestimmt; y = Schwärzungsmittel der Vergleichsterne; e_α = photographische Extinktion der Vergleichsterne; e_β = photographische Extinktion des Jupiter und Neptun; s = Schwärzungsziffer des Neptun, korrigiert wegen Lage auf der Platte und Skala, und δm die an das Mittel M_0 der photographischen Helligkeit der Vergleichsterne (von Extinktion befreit) anzubringende Größendifferenz des Neptun ist. In unserem Falle, wo 10 der hellsten Praesepesterne benutzt wurden, ist $M_0 = 7^m 14$. Die Entwickelung der Platten geschah mit Rodinal 1:15 acht Minuten lang bei Zimmertemperatur.

Die Daten 6 brauchbarer Platten (Agfa-Isolar-Platten der Emulsion 2672) enthält folgende Tabelle:

Datum m. Z. Gr.	Pl.	t_α	t_β	e_α	e_β	Sk.	s	d	y	σ	δm	Δm	m_p
1920 März 10.468	46	24^m	24^m	0^m18	0^m22	52	2.2	3.1	8.1	0^m244	$+1^m40$	$+0^m06$	8^m60
» 22.390	49	24	24	0.12	0.15	52	1.7	3.6	7.3	0.251	+1.38	+0.05	8.57
» 24.467	50	10, 24	24	0.30	0.40	52	0.8	3.7	6.3	0.265	+1.36	+0.05	8.55
» 25.331	51	10, 16	16	0.09	0.09	52	2.2	3.7	7.1	0.261	+1.28	+0.04	8.46
» 25.474	54	10, 24	24	0.35	0.46	52	2.4	3.8	8.4	0.258	+1.44	+0.04	8.62
April 17.336	59	10, 16	16	0.28	0.18	58	0.8	3.3	5.3	0.266	+1.30	+0.02	8.46

Spalte 1 gibt die m. Z. Gr. der Beobachtung, d. h. die Mitte der Expositionszeit; Spalte 2 gibt die Nr. der Platte; Spalte 3, 4 t_α und t_β die zur Ausmessung benutzten Expositionszeiten von Praesepe und Neptun; Spalte 5, 6 e_α und e_β die zugehörigen photographischen Extinktionen; Spalte 7 die Nr. der zur Ausmessung benutzten Skalenplatte; Spalte 8 die Schwärzungsziffer s des Neptun (korrigiert); Spalte 9 Abstand d des Neptun von der Plattenmitte in cm; Spalte 10, 11

[1]) Publikation der Kuffnerschen Sternwarte, Bd. V p. C 99.

[2]) *A. Wilkens*, Photographisch-photometrische Untersuchungen. AN 4124-5 p. 335.