Having cited from authentic sources the early impression formed by *Newton*, as to the nature of gravitation, and how it is transmitted in the superfine medium of the aether, which is treated of at much greater length in my New Theory of the Aether, 1922, we may now resume the argument regarding the interpenetration of the wave fields due to separate stars. Two such wave-fields are illustrated by the following figure; and the number of such wave-fields must be conceived as indefinitely increased (Plate VIII).

4. And as these waves interpenetrate at all points of space, the separate integrations for $\Phi, \Phi', \Phi'', \Phi''', \cdots \Phi^{\nu}$ naturally are not strictly independent; but are mutually connected by the series of approximations incident to integration on integration, to infinite order, for an unlimited number of bodies. The form of the integration thus becomes rigorously:

$$\Phi = \Omega(x,y,z,t) = \int\int\int\int\int\int_{-\infty}^{+\infty} e^{(A+Bht)\sqrt{-1}} \cdot \psi_1(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta\, d\lambda\, d\mu\, d\nu \times$$
$$\times \{1 + \Phi'[1 + \Phi''(1 + \Phi'''(1 + \cdots + \Phi^{\nu-1}(1 + \Phi^{\nu}\cdots))]\} \quad (144)$$

or in full:

$$\Phi = \int\int\int\int\int\int_{-\infty}^{+\infty} e^{(A+Bht)\sqrt{-1}} \cdot \psi_1(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta\, d\lambda\, d\mu\, d\nu \Bigg| 1 + \int\int\int\int\int\int_{-\infty}^{+\infty} e^{(A'+B'h't)\sqrt{-1}} \times$$
$$\times \psi_2(\xi',\eta',\zeta')\, d\xi'\, d\eta'\, d\zeta'\, d\lambda'\, d\mu'\, d\nu' \Big[1 + \int\int\int\int\int\int_{-\infty}^{+\infty} e^{(A''+B''h''t)\sqrt{-1}} \cdot \psi_3(\xi'',\eta'',\zeta'')\, d\xi''\, d\eta''\, d\zeta''\, d\lambda''\, d\mu''\, d\nu'' \cdots$$

. . . .

$$\cdots \Big(1 + \int\int\int\int\int\int_{-\infty}^{+\infty} e^{(A^{\nu-1}+B^{\nu-1}h^{\nu-1}t)\sqrt{-1}} \cdot \psi_{\nu-1}(\xi^{\nu-1},\eta^{\nu-1},\zeta^{\nu-1})\, d\xi^{\nu-1}\, d\eta^{\nu-1}\, d\zeta^{\nu-1}\, d\lambda^{\nu-1}\, d\mu^{\nu-1}\, d\nu^{\nu-1} \times$$
$$\times \Big(1 + \int\int\int\int\int\int_{-\infty}^{+\infty} e^{(A^{\nu}+B^{\nu}h^{\nu}t)\sqrt{-1}} \cdot \psi_{\nu}(\xi^{\nu},\eta^{\nu},\zeta^{\nu})\, d\xi^{\nu}\, d\eta^{\nu}\, d\zeta^{\nu}\, d\lambda^{\nu}\, d\mu^{\nu}\, d\nu^{\nu} \Big) \cdots \Big] \Bigg|_{\nu=1}^{\nu=\infty} . \quad (145)$$

5. This is the Infinite Integral which arises in the Theory of the Aether, owing to the interpenetration of waves from the infinite number of stars constituting the sidereal universe. It results directly from the *Fourier-Poisson* method of integration, for wave-motion,[1]) and affords an impressive illustration of the profound mysteries underlying the observed laws of nature.

Throughout the immensity of space it seems that very perfect order is preserved. This is due to the steady mutual actions of the stars, under universal gravitation, in accordance with the Newtonian law: yet we find difficulty in conceiving the infinite complexity of this action without a study of the above infinite integral.

In spite of the complexity of the integration on integration, the value of the integral is finite, at every point of space, like the force of gravitation itself. In fact the elements of the integral often are nearly insensible, owing to the immense mutual distances of the stars composing the Milky Way or other sidereal systems, to a depth of some two million light-years (cf. my Determination of the Depth of the Milky Way, Proc. Am. Philos. Soc., 1911). This was the extent of the sidereal universe first inferred by Sir *Wm. Herschel*, 1799–1802, and discussed at Paris between *Herschel*, *Laplace*, and *Napoleon*, Aug. 8, 1802 (cf. *Herschel's* Collected Scientific Papers, vol. 1, p. 12), reestablished by me in 1909–1911, by new investigations based on the best modern data, and recently confirmed by *Shapley* at Harvard observatory, *Hubble* at the Mt. Wilson observatory, and Dr. *J. H. Jeans*, Secretary of the Royal Society, and President of the Royal Astronomical Society, London.

Accordingly, it appears that nothing but the infinite integral above described is adequate to define rigorously the

[1]) Since finishing the above discussion of the infinite integral arising in the New Theory of the Aether, it occurs to me that we might derive it also from the point of view of partial differential equations, which apply to wave-motion. As the oscillations of superposed waves always may coexist, the simultaneous waves from the separate stars:

$$\Phi = \Omega(x,y,z,t) \quad \Phi' = \Omega'(x',y',z',t) \quad \Phi'' = \Omega''(x'',y'',z'',t) \quad \Phi''' = \Omega'''(x''',y''',z''',t) \cdots \Phi^{\nu} = \Omega^{\nu}(x^{\nu},y^{\nu},z^{\nu},t)$$

give rise to the corresponding partial differential equations:

$$\nabla^2\Phi, \quad \nabla^2\Phi', \quad \nabla^2\Phi'', \quad \nabla^2\Phi''' \cdots \nabla^2\Phi^{\nu}.$$

Owing to the interpenetration of the waves it is obvious that we should add these differential equations into an expression with an unlimited number of terms:

$$\nabla^2\Phi + \nabla^2\Phi' + \nabla^2\Phi'' + \nabla^2\Phi''' + \cdots + \nabla^2\Phi^{\nu} = \nabla^2\Omega(x,y,z,t) + \nabla^2\Omega'(x',y',z',t) + \nabla^2\Omega''(x'',y'',z'',t) +$$
$$+ \nabla^2\Omega'''(x''',y''',z''',t) + \cdots + \nabla^2\Omega^{\nu}(x^{\nu},y^{\nu},z^{\nu},t)$$

and then integrate successively, with infinite frequency, the arbitrary constants, $c_1, c_2, c_3, c_4, \cdots c_{\nu}$ for the particular integrals, $c_1\Phi, c_2\Phi', c_3\Phi'', c_4\Phi''', \cdots c_{\nu+1}\Phi^{\nu}$, when compounded into the complete integral:

$$c_1\Omega(x,y,z,t) + c_2\Omega'(x',y',z',t) + c_3\Omega''(x'',y'',z'',t) + c_4\Omega'''(x''',y''',z''',t) + \cdots + c_{\nu+1}\Omega^{\nu}(x^{\nu},y^{\nu},z^{\nu},t)$$

being so formed as to unite each particular integral with the whole series, — as by the above brackets, and the particular functions ψ_{ν}, under the separate sextuple integrals, $\psi_1, \psi_2, \psi_3, \cdots \psi_{\nu}$, serving for the effects of the interpenetration of the waves from all the stars of the sidereal universe. Note added Aug. 5, 1925.

actual state of the Aether, as disturbed by the unlimited mass of stars composing the Milky Way. The wave-field is everywhere infinitely complex, yet the value of Φ is finite, because the series of terms, higher than first order, depending on Φ', Φ'' etc., in

$$\{1 + \Phi' \, [1 + \Phi''(1 + \Phi'''(1 + \cdots (1 + \Phi^{\nu} \cdots)]\} \quad (146)$$

rapidly converges, owing to the immense mutual distances $\Delta_{i,j}$, with the wave-action therefore small, like the disturbing gravitational forces to which the wave-fields give rise.

6. Analytical expression for the practical evanescence of the higher terms. We may also show analytically why the smaller terms depending on remote stars tend to become evanescent.

(a) From the form of the Infinite Integral:

$$\Phi = \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty} e^{(A+Bht)\sqrt{-1}} \cdot \psi_1(\xi, \eta, \zeta) \, d\xi \, d\eta \, d\zeta \, d\lambda \, d\mu \, d\nu \times$$

$$\times \{1 + \Phi' \, [1 + \Phi'' \, (1 + \Phi''' \, (\cdots (1 + \Phi^{\nu} \cdots))]\} \quad (147)$$

we notice that, in another form, the disturbance is the wave-motion:

$$\Phi = \Omega\,(x, y, z, t) \quad \text{or} \quad \Phi = \Omega\,(x, y, z) \qquad t = 0 \qquad (148)$$

which reaches different points at different times; and is strong or feeble according to distances of the stars involved.

(b) For a star, such as our Sun, the coordinates x, y, z, or ξ, η, ζ are everywhere of definite fixed value: but for another star such as Sirius, the coordinates x', y', z', or ξ', η', ζ' (referred to the same origin as $x, y, z, \xi, \eta, \zeta$) have very different values, $x+\alpha$, $y+\beta$, $z+\gamma$, $\xi+\alpha$, $\eta+\beta$, $\zeta+\gamma$, or equivalents, owing to the great distance of Sirius. Thus the ratios of the coordinates x/x', y/y', z/z', or ξ/ξ', η/η', ζ/ζ', would be evanescent, in the whole region about the Sun, where the wave $\Phi = \Omega(x, y, z, t)$ is of importance in defining the value of Φ.

(c) Hence in comparison with Φ, $\Phi' = \Omega'(x', y', z', t)$ would be evanescent, or infinitesimal in value. The higher terms of the infinite integral thus are very minute, so that the series rapidly converges:

$$\Phi = \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty} e^{(A+Bht)\sqrt{-1}} \cdot \psi_1(\xi, \eta, \zeta) \, d\xi \, d\eta \, d\zeta \, d\lambda \, d\mu \, d\nu \times$$

$$\times \{1 + \Phi' \, [1 + \Phi'' \, (1 + \Phi''' \, (\cdots (1 + \Phi^{\nu} \cdots))]\} \quad (149)$$

because at all points where Φ is sensible, or appreciable, Φ' is practically insensible, and Φ'' is still smaller, — of the second order of small quantities!

17. Numerical Example of an Analogous Method of Integration, by Successive Approximations, used in the Theory of the Sun.

The foregoing process of integrating by successive approximations, in order to get the true value of the velocity-potential Φ in superposed wave-fields of the aether, when disturbed by an infinite number of stars, is not essentially different from that long known to geometers for finding the rigorous path of a curve, which may be found roughly by a few terms, but rigorously by many successive approximations.

1. The integration for the law of density in the Sun, treated as a mass of monatomic gas (cf. AN 4053, 4104, 4152).

Let σ denote the density at any point of the Sun's radius, σ_0 the density at the centre; T the temperature at any point, and T' the temperature at a point where the density is σ'. Let x be the distance from the Sun's centre, x' the limiting value for x at the boundary of the photosphere, which proves to be $x' = 3.653962$; and let μ denote the integral

$$\mu = \int_0^x (\sigma/\sigma_0)\, x^2 \, dx = m/4\pi\sigma_0 \qquad (150)$$

where m is the mass included within a sphere of radius x.

Then

$$1 - (\sigma/\sigma_0)^{k-1} = \int_0^x \mu/x^2 \cdot dx \qquad (151)$$

which may be solved by successive approximations.

The law of density thus becomes

$$\sigma = \sigma_0 \left| 1 - \int_0^x 1/x^2 \cdot dx \int_0^x \left| 1 - \int_0^x 1/x^2 \cdot dx \int_0^x \left[1 - \int_0^x 1/x^2 \cdot dx \int_0^x \left(1 - \int_0^x 1/x^2 \cdot dx \int_0^x \cdots \right.\right.\right.\right.$$

$$\left.\left.\left.\left. \cdots \sigma/\sigma_0 \cdot x^2 \, dx, \cdots \right)^{1/(k-1)} x^2 \, dx \right]^{1/(k-1)} x^2 \, dx \right|^{1/(k-1)} x^2 \, dx \right|^{1/(k-1)} . \qquad (152)$$

When the series is thus developed, the value of σ/σ_0 is determined, and the law of density is known; the value of μ may then be found by one additional integration

$$\mu = \int_0^x \sigma/\sigma_0 \cdot x^2 \, dx \, . \qquad (153)$$

The values of μ and $1 - (\sigma/\sigma_0)^{2/3}$ determined by calculation are (cf. AN 4053, p. 327) as follows:

$$\mu = \frac{x^3}{3} - \frac{x^5}{20} + \frac{x^7}{240} - \frac{x^9}{3888} + \frac{19x^{11}}{1425600} - \frac{2719x^{13}}{4447872000} + \frac{20621x^{15}}{800616960000} +$$

$$- \frac{193328x^{17}}{190546836480000} + \frac{39667364x^{19}}{1042672289218560000} - \frac{8078124341x^{21}}{591191587986923520000} + \cdots \qquad (154)$$

$$1 - (\sigma/\sigma_0)^{2/3} = \frac{x^2}{6} - \frac{x^4}{80} + \frac{x^6}{1440} - \frac{x^8}{31104} + \frac{19x^{10}}{14256000} - \frac{2719x^{12}}{53374464000} + \frac{20621x^{14}}{11208637440000} +$$

$$- \frac{193328x^{16}}{3048749383680000} + \frac{39667364x^{18}}{18768101205934080000} - \frac{8078124341x^{20}}{1182390375973847040000000} + \cdots \qquad (155)$$

In these equations x varies between the limits $x=0$, corresponding to the Sun's centre, and $x=x'=3.653962$, which is the value at the boundary of the photosphere.

Such successive approximations give the true law for the density within the Sun's globe, whereas the result of a single integration, under the form:

$$\mu=\int_0^x \sigma/\sigma_0 \cdot x^2 \, dx \qquad 1-(\sigma/\sigma_0)^{k-1}=\int_0^x 1/x^2 \cdot \mu \, dx \qquad (156)$$

would be too rough to give any high degree of rigor for the curve (σ/σ_0) throughout the Sun's mass. In 1905 I determined the curve for μ and $1-(\sigma/\sigma_0)^{2/3}$ to terms in x of the 20^{th} order, by actual calculation involving about fifty successive integrations, and to terms in x of the 50^{th} order, by the resulting differences. Thus I was enabled to compute the law of density with high accuracy down to the very centre of the Sun's mass, which could not have been traced by a less laborious process, owing to the slow convergence of the series.

Accordingly, it appears that the successive integrations, involved in the infinite integral appropriate to the Theory of the Aether, have their analogy in the theory of curves, some of which represent physical laws used in the theory of the universe.

Often a curve may be roughly approximated, by a few terms of the series, yet a very rigorous calculation of the curve may involve many repetitions of the process of integration, for fixing the higher terms. Therefore the process of successive integrations for finding the exact laws of nature is of very great use in physical science; and this general method of approximation deserves increased attention from geometers and natural philosophers.

The investigation described above in section 16 is the first case of a strictly infinite integral which has come under my notice. It appears to be very remarkable for the following reasons:

(a) It arose from the consideration of the Theory of the Aether, under conditions existing in nature, and corresponding closely to the actual state of the Milky Way. It was noticed by Sir *William Herschel*, with characteristic penetration, that the Galaxy is a clustering stream, made up of star clouds and clusters in every degree of condensation. *Barnard*'s magnificent photographs confirm *Herschel*'s conclusion that the Milky Way is gradually breaking up, under the continued action of the clustering power of universal gravitation; yet sensible changes in so stupendous a structure could be expected to become appreciable only with the lapse of millions of ages.

(b) Some 2300 years ago it was suspected by *Democritus*, *Anaxagoras*, and other Greek philosophers that the Milky Way is made up of separate stars too small and too faint to be perceived individually. These impressions of the ancients were confirmed by the invention of the telescope, and now *Barnard*'s splendid photographs illustrate the subject on a stupendous scale. The latest conclusions are somewhat analogous to those of the Greeks in foretelling things not yet seen by mortal eye: namely, owing to the vast extension of the universe in the plane of the Galaxy, to a depth of over two million light-years, the number of stars beyond the reach of telescope or photography many times exceeds those which can be perceived by our most powerful telescopes. Thus the observational indications are that the sidereal universe really is infinite; and thus the velocity-potential of the waves that pervade it could be expressed only by means of an infinite integral, shown to be convergent and everywhere of finite value, like the gravitational forces operating throughout the immensity of space.

(c) The splendid mathematical discoveries of *Fourier* and *Poisson*, in the theory of the integration of partial differential equations, have given us a method of analysis as general, elegant and rigorous as the physical problems of nature are complex. Without this method of treatment we could not express the effects of the infinitely complex wave-fields on which the wonders of universal gravitation depend.

(d) Accordingly, the theory of the infinite integral offers several points of special interest to geometers. In his celebrated Researches in the Lunar Theory, 1877, Dr. *G. W. Hill* discovered an Infinite Determinant, which he carefully investigated, proved to be convergent, and always of finite value. The Infinite Determinant has since found applications in the theory of sound and other branches of physical science; so that it has proved to be of deepest interest to geometers and natural philosophers.

May we not hope that the infinite integral here outlined and applied to the waves of the aether incessantly traveling through the sidereal universe will also find new applications in other branches of mathematical and physical science? Such developments in analysis enrich the fertility of our resources in calculation, and alone seem adequate for dealing with the infinitely complex phenomena pervading all nature.

18. Another Example of the Process of Successive Integrations as used in the Planetary and Lunar Theory.

The differential equations of undisturbed planetary motion, in rectangular coordinates, have the form:

$$\begin{aligned} d^2x/dt^2 + k^2(1+m)\,x/r^3 &= 0 \\ d^2y/dt^2 + k^2(1+m)\,y/r^3 &= 0 \qquad (157) \\ d^2z/dt^2 + k^2(1+m)\,z/r^3 &= 0 \end{aligned}$$

k^2 being the Gaussian constant of attraction.

If we multiply the first of these equations by y and the second by x, and subtract the second product from the first, we get by the first integration:

$$(x\,dy - y\,dx)/dt = c$$

and likewise

$$(x\,dz - z\,dx)/dt = c' \qquad (158)$$

$$(y\,dz - z\,dy)/dt = c''$$

whence

$$c\,z - c'\,y + c''\,x = 0, \qquad (159)$$

which shows that the undisturbed motion of the planet lies in a plane passing through the centre of the Sun, the constants of the areas, under the central force, according to *Kepler*'s law of areas, being c, c', c'', in equation (158).

The complete integration of equation (157) yields six constants, which are the elements of the planet's undisturbed motion, usually written:

π = longitude of the perihelion,
Ω = longitude of the ascending node,
i = inclination of the orbit,
$\varphi = \arcsin e$, the function of the eccentricity,
M = mean longitude of the epoch,
n = mean motion.

Accordingly, if θ be any coordinate of the place of the body, $\theta = f(\pi, \Omega, i, \varphi, M, n)$ we have:

$$\Delta\theta = \mathrm{d}\theta/\mathrm{d}\pi\cdot\Delta\pi + \mathrm{d}\theta/\mathrm{d}\Omega\cdot\Delta\Omega + \mathrm{d}\theta/\mathrm{d}i\cdot\Delta i + \mathrm{d}\theta/\mathrm{d}\varphi\cdot\Delta\varphi + \\ + \mathrm{d}\theta/\mathrm{d}M_0\cdot\Delta M_0 + \mathrm{d}\theta/\mathrm{d}n\cdot\Delta n\,. \qquad (160)$$

The place of the perturbed planet thus depends on the variation of these six elements; but as the perturbations usually are small, the solution of the differential equations for the disturbed motion is regarded as similar to that of the undisturbed motion but with varying elements, — the perturbations of the elements to be added to their initial values.

Accordingly, for the disturbed motion of a planet, we have equations of a closely related form, yet with the second members not zero, but very small, and under such constant change that the method of integration is known by the name of variation of parameters, or of the elements, regarded as arbitrary constants.

If r' denote the radius vector of the disturbing planet from the Sun, and ϱ its distance from the planet whose motion is disturbed,

$$r'^2 = x'^2 + y'^2 + z'^2 \quad \varrho^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2 \\ \varrho'^2 = (x''-x)^2 + (y''-y)^2 + (z''-z)^2 \qquad (161)$$

so that $\partial\varrho/\partial x = -(x'-x)/\varrho, \quad \partial\varrho'/\partial x = -(x''-x)/\varrho'$

and the perturbative function be denoted by Ω, we have:

$$\Omega = m'/(1+m)\cdot[1/\varrho - (xx'+yy'+zz')/r'^3] + \\ + m''/(1+m)\cdot[1/\varrho' - (xx''+yy''+zz'')/r''^3] + \cdots \qquad (162)$$

$$\partial\Omega/\partial x = m'/(1+m)\cdot(-1/\varrho^2\cdot\mathrm{d}\varrho/\mathrm{d}x - x'/r'^3) + \\ + m''/(1+m)\cdot(-1/\varrho'^2\cdot\mathrm{d}\varrho'/\mathrm{d}x - x''/r''^3) + \cdots \qquad (163)$$

$$= m'/(1+m)\cdot[(x'-x)/\varrho^3 - x'/r'^3)] + \\ + m''/(1+m)\cdot[(x''-x)/\varrho'^3 - x''/r''^3] + \cdots \qquad (164)$$

Then the differential equations for the disturbed motion become:

$$\begin{aligned} \mathrm{d}^2x/\mathrm{d}t^2 + k^2(1+m)\,x/r^3 &= k^2(1+m)\,\partial\Omega/\partial x \\ \mathrm{d}^2y/\mathrm{d}t^2 + k^2(1+m)\,y/r^3 &= k^2(1+m)\,\partial\Omega/\partial y \\ \mathrm{d}^2z/\mathrm{d}t^2 + k^2(1+m)\,z/r^3 &= k^2(1+m)\,\partial\Omega/\partial z\,. \end{aligned} \qquad (165)$$

As before noted, the right members of these equations differ slightly from zero.

Now since the second members of the equations for the perturbed motion are small, owing to the small masses and great mutual distance of the planets, as we see by the equivalent finite expression:

$$(1+m)\,\partial\Omega/\partial x = \sum m'\,[(x'-x)/\varrho^3 - x'/r'^3] = \\ = \sum_{i=1}^{i=i} m_i\,[(x_i-x)/\varrho^3{}_{i-1} - x_i/r_i^3] \qquad (166)$$

we may in the first approximation integrate the above equations as if the perturbative function were zero.

Accordingly in *Laplace*'s treatment of this question, Mécanique Céleste, Liv. II, Chap. V. § 41, he usually writes the differential equation in the form

$$\mathrm{d}^2y/\mathrm{d}t^2 + a^2y + \alpha\,Q = 0 \qquad (167)$$

where α is a small constant coefficient, and Q a function of the coordinates, and of their differentials.

To integrate the equation of the perturbed motion, therefore, namely,

$$\mathrm{d}^2y/\mathrm{d}t^2 + a^2y + \alpha\,Q = 0$$

we first integrate that for the undisturbed motion,

$$\mathrm{d}^2y/\mathrm{d}t^2 + a^2y = 0 \qquad (168)$$

which gives y in functions of the time,

$$y = c/a\cdot\sin at + c'/a\cdot\cos at \qquad (169)$$

thus corresponding to purely periodic motion.

For $$\mathrm{d}y/\mathrm{d}t = c\cos at - c'\sin at \qquad (170)$$

and $$-\mathrm{d}^2y/\mathrm{d}t^2 = c\,a\sin at + c'a\cos at \qquad (171)$$

which yields, by (168) combined with (169):

$$\mathrm{d}^2y/\mathrm{d}t^2 + a^2y = 0$$

as in (168).

Therefore the original equation (168) is integrated, since we find directly:

$$-\int\mathrm{d}^2y/\mathrm{d}t^2\cdot\mathrm{d}t = \int c\,a\sin at\,\mathrm{d}t + \int c'\,a\cos at\,\mathrm{d}t = \\ = c\cos at - c'\sin at = \mathrm{d}y/\mathrm{d}t\,. \qquad (172)$$

By repeating the integration we have:

$$y = \int\mathrm{d}y/\mathrm{d}t\cdot\mathrm{d}t = \int c\cos at\,\mathrm{d}t - \int c'\sin at\,\mathrm{d}t = \\ = c/a\cdot\sin at + c'/a\cdot\cos at\,. \qquad (173)$$

To determine the constants of the integration, c and c', we proceed as follows. First, to find c we multiply (169) by $a\sin at$, and (170) by $\cos at$; and add:

$$\begin{aligned} a\,y\sin at &= c\sin^2 at + c'\sin at\cos at \\ \mathrm{d}y/\mathrm{d}t\cdot\cos at &= c\cos^2 at - c'\sin at\cos at \end{aligned} \qquad (174)$$

wherefore $$c = a\,y\sin at + \mathrm{d}y/\mathrm{d}t\cdot\cos at\,. \qquad (175)$$

Second, to find c', we multiply (169) by $a\cos at$, and (170) by $-\sin at$, and add:

$$\begin{aligned} a\,y\cos at &= c\sin at\cos at + c'\cos^2 at \\ -\mathrm{d}y/\mathrm{d}t\cdot\sin at &= -c\cos at\sin at + c'\sin^2 at \end{aligned} \qquad (176)$$

wherefore $$c' = a\,y\cos at - \mathrm{d}y/\mathrm{d}t\cdot\sin at\,. \qquad (177)$$

Now we have found that in the equation of undisturbed motion, the purely periodic function y depends on a double integration of the equation (168), and thus we may write the process in the form:

$$y = -\int\!\!\int\mathrm{d}^2y/\mathrm{d}t^2\cdot\mathrm{d}t\,\mathrm{d}t = a^2\int\!\!\int y\,\mathrm{d}t\,\mathrm{d}t = \\ = \int\!\!\int a^2\{(c/a\cdot\sin at + c'/a\cdot\cos at)\}\,\mathrm{d}t\,\mathrm{d}t \qquad (178)$$

which is easily verified.

In the theory of perturbations, involving equations of the form (167) above, it is shown, (Mécanique Céleste, Liv. II, Ch. V. § 41), that an additional integration is required for the terms involving αQ.

Thus we get for the perturbed motion

$$y = c/a\cdot\sin at + c'/a\cdot\cos at - \alpha/a\cdot\sin at\int Q\,\mathrm{d}t\cos at + \\ + \alpha/a\cdot\cos at\int Q\,\mathrm{d}t\cdot\sin at\,. \qquad (179)$$

The double integration for y, in the theory of the undisturbed motion, with the additional integration for the terms involving the perturbations, — in all therefore the three integrations here explained — give a fair view of the processes employed in the planetary theory: yet it may be noticed that in fixing the perturbations in longitude, there is a double integral involved, in addition to the double integrals for the perturbation of the radius vector:

$$\delta r = a\cos v\int n\,\mathrm{d}t\cdot r\sin v\{2\int \mathrm{d}R + r(\partial R/\partial r)\} + \\ - a\sin v\int n\,\mathrm{d}t\cdot r\cos v\{2\int \mathrm{d}R + r(\partial R/\partial r)\} \quad (180)$$

the integral for the longitude having the form:

$$\sqrt{(1-e^2)}\cdot\delta v = (2r\,\mathrm{d}\delta r + \mathrm{d}r\,\delta r)/(a^2\,n\,\mathrm{d}t) + \\ + 3a/\mu\cdot\int\int n\,\mathrm{d}t\,\mathrm{d}R + 2a/\mu\cdot\int n\,\mathrm{d}t\cdot r\,(\partial R/\partial r)\,. \quad (181)$$

These equations of *Laplace* give the perturbations in the radius vector and longitude, R being the disturbing function, and $\mu = M + m$ the combined masses of the Sun and disturbed planet, and n the mean motion.

For our present purposes it is not necessary to go into greater detail, as to the mathematical processes involved, in these calculations. But we may describe the physical significance of such approximations, which are accurate to the first powers of the attracting masses, or disturbing forces. It is that when two planets mutually disturb each other, the effect upon one body is calculated just as if the motion of the other body were undisturbed, and vice versa. Thus for either body the undisturbed coordinates of the other are used, as if its motion had not been perturbed. When these first order perturbations are obtained, for correcting the elements, the resulting new osculating elements may be employed to calculate the perturbations involving the second powers of the masses or disturbing forces. And if these second order corrections are applied, to give a second set of osculating elements, the latter may then be used to calculate the perturbations to the third order of the masses or perturbing forces.

The process is therefore one of successive approximations. It presents no difficulty in numerical calculation, by the method of special perturbations, which is so much used for the minor planets; and may be employed in the above formulae of *Laplace*, if in the repetition of the integrations, we successively correct the elements, to take account of the perturbations of the first, second, or higher orders.

The formulae of *Laplace*, however, are intended primarily for general perturbations: that is for perturbations incident to the expansion of the perturbative function in a series, thus involving the time as an independent variable, which may serve for indefinite time. The usual mode of expansion involves the powers of the masses, and the various powers of the eccentricities and the mutual inclinations of the orbits.

Objection is sometimes made to this method, even as refined by *Hansen*, to ensure increased rigor in *Laplace*'s processes. It is held that the method of expansion in series is inexact, because the number of terms is indefinite, and all of them cannot be fully taken into account, without enormous labor; and even then an uncertainty will remain as to the inclusion of all the terms, which could effect the result.

In 1895 Professor *Newcomb* expressed to me this fundamental objection to the theory of general perturbations, and said that if he had his planetary researches to do again, he would adopt the method of numerical integrations employed in the theory of special perturbations, as applied to the asteroids.

Newcomb remarked that it took Dr. *G. W. Hill* 11 years to complete his New Theory of Jupiter and Saturn; while the method of special perturbations, to a degree of accuracy adequate for several centuries, could be applied in very much less time. The problem is one which Dr. *Hill* used to describe as depending on »pushing the approximations to sufficiently high order as respects the masses, eccentricities, mutual inclinations«, etc.

The criticism of *Newcomb* is of weight, not so much in theory as in practice — where the real difficulty of picking out all sensible terms arises. The trouble in practice is so serious that purely numerical processes are increasingly sought, yet the methods of *Laplace*, *Hansen* and *Hill* continue to be used in the planetary and lunar theory.

Part IV. Theorems of *Laplace* and *Poisson* on the Invariability of the Major Axes and Mean Motions of the Planets generalized for all the Higher Powers of the Masses: Criticism of Traditional Theories from the Age of *Newton*, Criteria which overthrow the Theory of Ultra-Mundane Corpuscles: The Calculus of Probability establishes the Wave-Theory as the Order of Nature.

19. *Laplace*'s Theorem of 1773, $\delta_1 a_i = 0$, $\delta_1 n_i = 0$, and the Mean Planetary Potential Theorem of 1799.

1. In addition to the theorem of 1773,

$$\delta_1 a_i = 0 \quad \delta_1 n_i = 0 \qquad (182)$$

we find in the Mécanique Céleste, Liv. II, Chap. VII, § 61, 1799, that *Laplace* reaches for the planetary system a formula connecting the masses and major axes very similar to that of the potential, 1782, expressed as follows:

$$\text{constant} = m/a + m'/a' + m''/a'' + \text{etc.} = \sum_{i=0}^{i=i} m_i/a_i = H\,. \qquad (183)$$

»All these equations take place in relation to the inequalities of very long period which might affect the elements of the orbits of m, m', etc.«.

This really represents the sum of the mutual potential energies of the bodies of the solar system. The theorem reached in 1799 thus corresponds to a modern extension of the early conclusion of *Laplace*, in 1773, $\delta_1 a_i = 0$, $\delta_1 n_i = 0$, that to the first powers of the disturbing masses, the major axes of the planetary orbits are invariable. This bears upon the celebrated problem of the gravitational stability of the planetary system.

2. Extension of *Laplace*'s theorem by *Poisson*, 1808. We are reminded that *Laplace*'s early result of 1773, and the above theorem of 1799, was first given a notable extension in 1808, when the brilliant young geometer *Poisson* extended such reasoning also to the second powers of the disturbing masses, and rigorously demonstrated that even under this second approximation the major axes of the planetary orbits remain constant and the mean motions invariable.

In his Mécanique Céleste, Tome I, Chap. XXV, pp. 391–403, 1889, *Tisserand* has a careful discussion of this celebrated problem. Designating by $\delta_1 a_i$ the perturbations of the semi-axes depending on the first powers of the masses, and by $\delta_2 a_i$ the corresponding perturbations depending on the second powers of the masses, he comes to the following conclusions (p. 400): »It is therefore demonstrated that $d\delta_2 a_i/dt$ contains only periodic terms, and consequently that $\delta_2 a_i$ does not include any secular term. Accordingly, a_i has no secular inequality when we take account of the first and second powers of the masses.« Hence *Poisson*'s conclusion: »The major axes of the orbits described by the planets about the Sun have no secular inequalities, neither to the first nor to the second approximation.«

In a careful Historical Critique (p. 402-3) *Tisserand* reviews the efforts of later geometers to extend these researches to higher powers of the masses, yet in pointing out the uncertainties in their conclusions, finds nothing definitely established.

3. The author's generalization of the theorems of *Laplace* and *Poisson* for the higher powers of the masses. During 1924 I was able to devise a mathematical demonstration, based on the recognized properties of infinite series, that if the terms in $\delta_1 a_i$ and $\delta_2 a_i$ disappear, as purely periodic, not secular, then all higher orders as $\delta_3 a_i$, $\delta_4 a_i$, etc. will also disappear, because the higher order terms are very much smaller than $\delta_1 a_i$ and $\delta_2 a_i$, and necessarily similar in character, since *Lagrange*'s force function has no other terms than combinations of the masses in pairs:

$$U = m_0 \sum_{j=1}^{j=j} m_j/\Delta_{0,j} + \sum_{j=1}^{j=j} \sum_{k=1}^{k=k} m_j m_k/\Delta_{j,k} \qquad (184)$$

in which expression m_0 represents the mass of the Sun.

In respect to the mutual perturbative action of the planets, in their orbital revolutions about the Sun, this expression for U, reduced to the simplest form, may be still further simplified, and put in the usual form:

$$U = \sum_{i=0}^{i=i} \sum_{j=1}^{j=j} m_i m_j/\Delta_{ij}\,. \qquad (185)$$

Viewing the major axes and mean motions of the planets as functions conforming analytically to *Taylor*'s theorem:

$$u_1 = u + du/dx \cdot h/1 + d^2u/dx^2 \cdot h^2/(1 \cdot 2) + d^3u/dx^3 \cdot h^3/(1 \cdot 2 \cdot 3) + d^4u/dx^4 \cdot h^4/(1 \cdot 2 \cdot 3 \cdot 4) + \cdots \qquad (186)$$

we may expand these assumed variables, with *Tisserand*, Mécanique Céleste, Tome I, p. 394, in a series proceeding according to the powers of the disturbing masses:

$$a_i + \delta a_i = a_i + \delta_1 a_i + \delta_2 a_i + \delta_3 a_i + \delta_4 a_i + \cdots \qquad (187)$$

Likewise the mean motions may be expanded in a similar series:

$$n_i + \delta n_i = n_i + \delta_1 n_i + \delta_2 n_i + \delta_3 n_i + \delta_4 n_i + \cdots \qquad (188)$$

In accordance with experience it is assumed that the major axes and mean motions a_i and n_i are constants connected by the Keplerian relation, for unchanging masses:

$$n_i^2\, a_i^3 = k(m_0 + m_i)\,. \qquad (189)$$

The assumed changes, therefore, would give, by the use of the above series:

$$(n_i + \delta_1 n_i + \delta_2 n_i + \cdots)^2 (a_i + \delta_1 a_i + \delta_2 a_i + \cdots)^3 = = k(m_0 + m_i) = n_i^2\, a_i^3 \qquad (190)\,(A)$$

whence we find the terms depending on the first and second powers of the masses to be:

$$\delta_1 n_i = -\tfrac{3}{2}\, n_i/a_i \cdot \delta_1 a_i$$
$$\delta_2 n_i = -\tfrac{3}{2}\, n_i/a_i \cdot \delta_2 a_i + \tfrac{15}{8}(\delta_1 a_i)^2\,. \qquad (191)\,(B)$$

But these terms are shown by the researches of *Laplace*, 1773, and *Poisson*, 1808, to vanish identically, term by term; and therefore our equation (190) (A), to terms of the third order in respect to the masses, becomes merely:

$$(n_i + \delta_3 n_i + \cdots)^2 (a_i + \delta_3 a_i + \cdots)^3 = n_i^2\, a_i^3 \qquad (192)\,(C)$$

by which it appears that the expansion yields:

$$\{n_i^2 + 2n_i\, \delta_3 n_i + (\delta_3 n_i)^2\} \{a_i^3 + + 3a_i^2\, \delta_3 a_i + 3a_i(\delta_3 a_i)^2 + (\delta_3 a_i)^3\}\,. \qquad (193)\,(D)$$

Now *Lagrange*'s force function U is made up of terms of the type $m_i m_j/\Delta_{ij}$, and therefore any term depending on higher powers of the masses than the second do not represent forces in nature, but only analytical terms in a series, where an expression is expanded, in approximations, depending on the powers of the masses, which is usual in the planetary theory.

4. Criticism on this method of series, as not according with the laws of nature. *Newcomb* and many other mathematicians have complained of the unsatisfactory character of these methods which depend on infinite series. Without rejecting the result on any ground except the hopeless character of definitely evaluating the series, owing to the unlimited number of terms, we may remark:

a) That variations of the type $\delta_3 n_i$, $\delta_3 a_i$, $\delta_4 n_i$, $\delta_4 a_i$, ⋯ depending on powers of the disturbing masses higher than the second, seem to be excluded from consideration under the actual law of nature, by the conditions defined in *Lagrange*'s force function U.

b) And this inference is confirmed by the identical vanishing of $\delta_1 n_i$, $\delta_2 n_i$, $\delta_1 a_i$, $\delta_2 a_i$, under the careful analysis of *Laplace* and *Poisson*. The chance of terms depending on $\delta_3 n_i$, $\delta_3 a_i$, $\delta_4 n_i$, $\delta_4 a_i$, ⋯ being different from zero is easily seen to be infinitely small. For if the terms depending on the first two terms of the series $\delta_1 n_i$, $\delta_1 a_i$, $\delta_2 n_i$, $\delta_2 a_i$, vanish, then the probability of the higher terms becoming sensible is practically zero.

c) The conclusion that $\delta_3 a_i$ vanishes like $\delta_1 a_i$, $\delta_2 a_i$ must be held to be infinitely probable, in view of the character of the forces operating in nature. It is inconceivable that $\delta_1 a_i$, $\delta_2 a_i$ should introduce no secular terms, and yet $\delta_3 a_i$ or $\delta_4 a_i$ do so!

For under the laws of nature, we may deny the existence of terms of the type considered by *Tisserand*, Méc. Cél. I, p. 393:

$$F = m_i^{\lambda}\, m_j^{\mu}\, m_k^{\nu}\, A \qquad (194)$$

where A is a function of the coordinates only: since, by analogy with *Lagrange*'s force function U, we are restricted in the planetary theory to terms of the order $m_i\, m_j / \Delta_{i,j}$ which results from the grouping of the bodies in pairs under the Newtonian law.

This criticism is equivalent to the claim that in nature all the actions of the bodies of the solar system are of the type included in *Lagrange*'s force function U. Accordingly, judging from the law of nature, *Tisserand* had no right to consider a term of the form F, equation (194) above. At most such terms would arise from the imperfections of our methods of approximation, by expansion in series, not from the forces actually operating in nature!

Accordingly, the general theorem is that in the mutual actions of the planets, we may always take $\delta a_i = 0$, in equation (187) above, since we have:

$$a_i + \delta a_i = a_i + \delta_1 a_i + \delta_2 a_i + \delta_3 a_i + \delta_4 a_i + \cdots + \delta_\nu a_i = a_i\,. \qquad (195)$$

And, in like manner, since by (188)

$$n_i + \delta n_i = n_i + \delta_1 n_i + \delta_2 n_i + \delta_3 n_i + \delta_4 n_i + \cdots + \delta_\nu n_i = n_i \qquad (196)$$

we may put:

$$\delta n_i = 0\,, \quad \delta a_i = 0 \qquad (197)$$

which is a comprehensive generalization resulting from the theorems of *Laplace* 1773, 1799, and of *Poisson*, 1808.

In works on Celestial Mechanics, such as *Tisserand*'s Mécanique Céleste, I, p. 300, it is shown that, to the second approximation, any element may be put in the form:

$$\eta = P + P' t + P'' t^2 + \sum A \frac{\cos}{\sin}(\alpha t + \beta) + \\ + t \sum A' \frac{\cos}{\sin}(\alpha' t + \beta') \qquad (\alpha)\ (197a)$$

yet owing to the theorems of *Laplace*, 1773, and *Poisson*, 1808, that $\delta_1 a_i = 0$, $\delta_1 n_i = 0$, $\delta_2 a_i = 0$, $\delta_2 n_i = 0$, we have $P' = 0$, $P'' = 0$, for the major axes and mean motions. Under double integration the terms in the mean longitude take the form (cf. *Laplace*, Mécanique Céleste, Liv. II, chap. VII, § 54, chap. VIII, § 65):

$$\delta L = k \int\!\!\int \mathrm{d}R/\mathrm{d}\epsilon \cdot \mathrm{d}t\, \mathrm{d}t = \\ = k\, i\, P \sin[(i n - i' n') t + n - n' + Q] / (i n - i' n')^2 \qquad (\beta)\ (197b)$$

and with the small divisors incident to near approach to commensurability in the mean motions, this gives rise to a great inequality of Jupiter and Saturn, in 918 years, and of Uranus and Neptune, in about 4000 years.

The above diagram represents the theory of the great inequality of Jupiter and Saturn, under successive conjunctions in the lines $O J_1 S_1$, $O J_2 S_2$, $O J_3 S_3$, $O J_4 S_4$, etc., separated each time by about 240°. From the nature of the potential of the planetary forces:

$$U^{(2)} = m m' / \Delta_{i,j} = \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int [(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{-1/2} \times \\ \times \sigma\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \sigma'\, \mathrm{d}x'\, \mathrm{d}y'\, \mathrm{d}z' \qquad (\gamma)\ (197c)$$

with the resulting forces themselves, acting mutually on the planets, in right lines, we see that these forces are given in direction by sines and cosines of angles depending on differences in the mean motions, and thus are chiefly periodic, as in the two latter terms of equation (α) above. The secular terms for the changes of the orbits themselves are given by the terms P' and P'', while the long-period inequalities are given by (β).

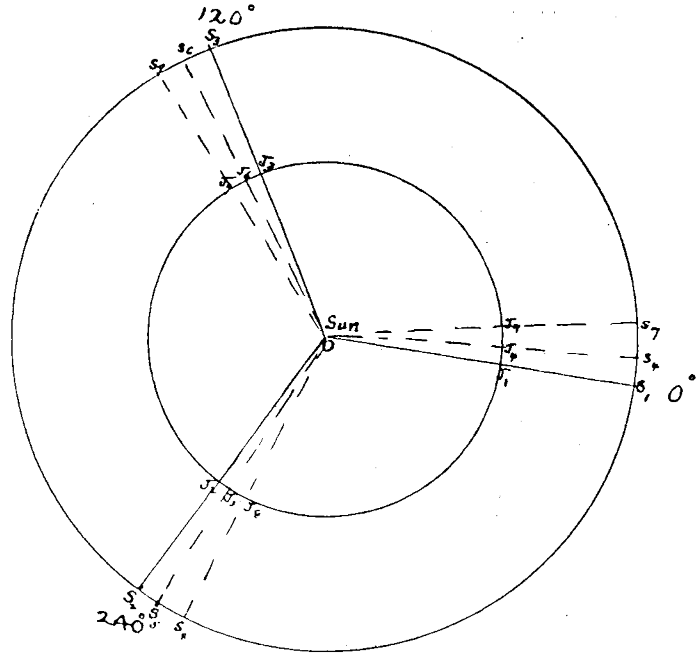

Fig. 9. Diagram representing the theory of the great inequality of Jupiter and Saturn, with line of conjunction through the points $O J_1 S_1$, $O J_2 S_2$, $O J_3 S_3$, $O J_4 S_4$ etc., slowly revolving so as to complete their cycle in 918 years. The accelerating or retarding forces, after acting for 459 years in one direction, finally reverse themselves and begin over again, as explained by *Laplace*, 1785.

20. Confirmation of the above Theorem by the Method used in *Laplace*'s Supplement to Vol. III of Mécanique Céleste, 1808, and in *Poisson*'s Researches, 1816.

After the development of the celebrated paper showing that $\delta_2 a_i = 0$, $\delta_2 n_i = 0$, which *Poisson* presented to the Institute of France, June 20, 1808, and published in Tome VIII of the Journal de l'École Polytechnique, Cahier XV, pp. 1–56, *Laplace* resumed the subject, and on Aug. 17, 1808, presented to the Board of Longitude of France a very simple demonstration, which he offered as an appendix to the Mécanique Céleste, vol. III, pp. 325–350, édition of the Paris Academy, 1878.

1. *Laplace* takes the perturbative function R as a function of the mean motion, $\int n\, \mathrm{d}t$, and of the elements $a, e, \varpi, \epsilon, p, q$, whence under conditions of invariability, and then variability, the differential expression results:

$$0 = (\partial R/\partial a)\, \mathrm{d}a + (\partial R/\partial e) \mathrm{d}e + (\partial R/\partial \varpi)\, \mathrm{d}\varpi + (\partial R/\partial \epsilon)\, \mathrm{d}\epsilon + \\ + (\partial R/\partial p)\, \mathrm{d}p + (\partial R/\partial q)\, \mathrm{d}q\,. \quad (A)\ (198)$$

And for the finite variation, retaining only the first powers of the increments $\delta \int n\, \mathrm{d}t$, δa, δe, $\delta \varpi$, $\delta \epsilon$, δp, δq, the equation for δR becomes:

$$\delta R = \mathrm{d}R/n\,\mathrm{d}t \cdot [\delta \textstyle\int n\,\mathrm{d}t + \delta\epsilon] + (\partial R/\partial a)\,\delta a + (\partial R/\partial e)\,\delta e +$$
$$+ (\partial R/\partial \varpi)\,\delta\varpi + (\partial R/\partial p)\,\delta p + (\partial R/\partial q)\,\delta q\,. \qquad \text{(B)}\ (199)$$

2. It is observed from the form of the function R,

$$R = m'\,(x\,x' + y\,y' + z\,z')/r'^3 - m'/\varrho$$

where
$$r' = \sqrt{(x'^2 + y'^2 + z'^2)} \qquad (200)$$
$$\varrho = \sqrt{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]}$$

that R is of the order of the disturbing mass m'; moreover, the variations of the elements involve $\mathrm{d}R$, or partial differentials of R in respect to one of the elements, and thus the variations $\delta\epsilon$, δe, etc., which depend on R, are also of the order m': therefore in equation (B) the terms of the second member are of the order m'^2.

Accordingly, if terms of the type $R\,\delta\epsilon^2$, $R\,\delta e^2$, $(\partial R/\partial\epsilon)\,\delta\epsilon^2$, $(\partial R/\partial e)\,\delta e^2$, etc., were considered they would be of the order m'^3, but they usually are neglected by *Laplace* and *Poisson*, yet we shall consider the effects of such terms in our final estimate.

3. *Laplace* then considers the variations in δR depending on the planet m, and finds results similar to those just cited, namely, that to terms of the order m^2, only periodic changes will take place in the coordinates of the two planets m and m', and thus $\delta_2 a_i = 0$, $\delta_2 n_i = 0$, as before cited.

He then considers the action of m'' on m, hence adding to R the part

$$R = m''(xx'' + yy'' + zz'')/r''^3 - m''/\varrho'\,. \qquad (201)$$

And in the action of m' on m, as before, he takes

$$R = m'(xx' + yy' + zz')/r'^3 - m'/\varrho\,. \qquad (202)$$

Noticing in the development of R the terms which are independent of $n't$ and $n''t$, he finds the terms to be of the form

$$m'\,m'' \int \mathrm{d}X = m'\,m''\;X = \int \mathrm{d}R \qquad (203)$$

where X is a function of the coordinates of m only.

Accordingly, the variation of the expression $\int \mathrm{d}R = m'\,m''\,X$ produces in $\int \mathrm{d}R$ non-periodic quantities of the third order of the masses m, m', etc., which usually are neglected.

4. But we may consider the effect of such quantities as follows:

a) The elements of the orbit of a planet may be represented by systems of terms of the forms

$$N \sin(gt + \beta), \quad N \cos(gt + \beta) \qquad (204)$$

in which g is of the same order as that of the disturbing masses m', m'', etc., and hence the elements will vary slowly with the time, but will be brought into the divisor by the integrations for the mean motions of the planets.

b) The double integration of a quantity, depending on an angle of this kind, yields in the mean motion ζ, and its variation $\delta\zeta$, the following integral expressions:

$$\zeta = \int n\,\mathrm{d}t = 3\iint a\,n\,\mathrm{d}t\,\mathrm{d}R$$
$$\delta\zeta = 3\,a\,n \iint \mathrm{d}t\,\mathrm{d}\delta R + 3\,a^2 \iint n\,\mathrm{d}t\,\mathrm{d}R \int \mathrm{d}R \qquad (205)$$

and therefore upon actual integration, divisors are introduced of the form g^2, which is of the order m'^2, m''^2, etc.

But since $\mathrm{d}\delta R$ and $\mathrm{d}R\int\mathrm{d}R$ both vanish, these divisors can operate only upon terms of the third or higher orders,

$$H\,m'^3/g^2 = \lambda\,m' \qquad (206)$$

yielding a term of the first order only, which is of the same order of importance as the periodical inequalities. We shall first examine a little more closely the nature of these integrals, and then resume the consideration of the terms depending on the third and higher powers of the masses.

5. This latter integration in greater detail is as follows. From the Keplerian equation

$$a^3\,n^2 = 1 \quad \text{we get} \quad a\,n = a^{-1/2} \qquad (207)$$

wherefore
$$a\,n\,\mathrm{d}R = a^{-1/2}\,\mathrm{d}R\,. \qquad (208)$$

And since
$$\delta a = -2 \int a^2\,\mathrm{d}R \qquad (209)$$

we find, under variation, as in differentiation,

$$\delta(a\,n\,\mathrm{d}R) = \delta(a^{-1/2}\,\mathrm{d}R) = a^{-1/2}\,\mathrm{d}\delta R - \tfrac{1}{2}a^{-3/2}\,\mathrm{d}R\,\delta a =$$
$$= a\,n\,\mathrm{d}\delta R + a^2\,n\,\mathrm{d}R \int \mathrm{d}R\,. \qquad (210)$$

Multiplying by $3\mathrm{d}t$, and integrating twice, we obtain:

$$3\delta \iint (a\,n\,\mathrm{d}t\,\mathrm{d}R) = 3a\,n \iint \mathrm{d}t\,\delta R + 3a^2 \iint n\,\mathrm{d}t\,\mathrm{d}R \int \mathrm{d}R \qquad (211)$$

the terms $a\,n$, a^2, being placed without the double integration signs, which may be done by neglecting terms of the order m^3. Therefore, by integrating relative to δ, we get

$$\zeta = 3 \iint a\,n\,\mathrm{d}t\,\mathrm{d}R$$
$$\delta\zeta = 3a\,n \iint \mathrm{d}t\,\mathrm{d}\delta R + 3a^2 \iint n\,\mathrm{d}t\,\mathrm{d}R \int \mathrm{d}R\,. \qquad (212)$$

6. In approximations to the square of the masses, m'^2, both $\mathrm{d}\delta R = 0$ and $\mathrm{d}R\int\mathrm{d}R = 0$, yet terms of the form

$$H\,m'^3/g^2 = \lambda\,m' \qquad (213)$$

appear, owing to the double integrations of such trigonometric expressions for the elements as:

$$N \sin(gt + \beta), \quad N \cos(gt + \beta)\,.$$

It remains to add, however, that the above result, $\lambda\,m'$, is of the first order only; and as the differential equations for the elements contain R, or $(\partial R/\partial e)$, for example, therefore of the form $\lambda'\,m'$, the combined result yields a term of the type:

$$\lambda\,m' \cdot (\partial R/\partial e) = L\,m'^2 \qquad (214)$$

which is still only of the second power of the masses, for which *Poisson* and *Laplace* proved their theorems of invariability.

Accordingly, since

$$\delta_1 a_i = 0 \quad \delta_2 a_i = 0 \quad \delta_1 n_i = 0 \quad \delta_2 n_i = 0 \qquad (215)$$

even when we include terms of the third order as respects the masses, owing to the above double integrations for the mean longitude, we may write:

$$\delta_3 a_i = 0 \quad \delta_3 n_i = 0\,. \qquad (216)$$

By similar reasoning it may be shown that $\delta_4 a_i = 0$, $\delta_4 n_i = 0$, and so on, for all higher powers of the masses. Accordingly, it appears that under the double integrations required for terms in the mean longitude, the order of the terms as respects the powers of masses is reduced by two: and therefore the theorems of *Laplace* and *Poisson* are again valid for terms in ζ and $\delta\zeta$ of the order m'^3. And, owing to the Keplerian relation $a^3\,n^2 = 1$, also for the major axes.

Finally, this fundamental theorem,

$$\delta_3 a_i = 0 \quad \delta_3 n_i = 0 \quad \delta_4 a_i = 0 \quad \delta_4 n_i = 0 \qquad (217)$$

etc., is confirmed by an inference of *Poisson*, who resumed the subject in Tome VIII of the Journal de l'École Polytechnique, Cahier XV, pp. 1–56, 1808, and published in the Mémoires de l'Académie des Sciences, 1816, pp. 55–67, a further investigation in which he sought to show that the major axes have no secular inequalities of the third order in respect to the masses, when we have regard wholly to the variation of the elements of the disturbed planet. He included terms of the third order of the disturbing masses (m'^3) arising from those of the second order of the disturbed planet (m^2); and then, by induction, infers that this will hold for all powers of the masses, so far as they depend on the elements of the disturbed planet (la planète troublée).

The fundamental theorems for the invariability of the major axes and mean motions of the planets,

$$\delta_j a_i = 0 \quad \delta_j n_i = 0 \qquad \text{(s. Footnote)} \qquad (218)$$

where $j = 1, 2, 3, 4, \cdots$ are powers of the masses, thus appears to be confirmed by the careful use of the methods of *Laplace* and *Poisson*.

No satisfactory proof has ever been brought forward to invalidate this generalized theorem. In his Mécanique Céleste, Tome I, pp. 402–3, *Tisserand*'s discussion shows that he is skeptical as to the claims of *Heretu* that the major axes have secular inequalities of the third order with respect to the masses. He points out that *Leverrier*'s small third order term in t^2, in the development of the part of $\int n\,dt$ in the mean longitude of Saturn disturbed by Jupiter, (Ann. de l'Obs., Tome XI, Chap. XXI, p. 126) was got by a purely numerical calculation, by interpolation, which therefore is not satisfactory from the point of view of planetary theory.

In conclusion, therefore, the constancy of the major axes and mean motions of the planets for all powers of the masses, being deduced from a critical review of the planetary theory, and confirmed by the New Theory of the Aether, must be held to be permanently established. Any denial of the theorem

$$\delta_j a_i = 0 \quad \delta_j n_i = 0 \qquad (218)$$

so long as the masses are unchanged, is in violation of the conservation of energy, which acquires a new physical justification from the functions U and V under the Kinetic Theory of the Aether.

The physical cause of gravitation, — namely the steady stress due to the aether waves from all the particles — shows that so long as the mass is constant, no change is possible in a_i or n_i, under the mutual action of the planets.

In order to confirm the foregoing generalization, on undeniable physical grounds, we need only point out that the planetary forces are due to the derivatives of *Lagrange*'s force function U, equation (72a), or for unit mass, of the potential function as defined by *Laplace*, 1782:

$$U = \sum_{i=0}^{i=i} \sum_{j=1}^{j=j} \iiiiiint [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{-1/2} \times \sigma_i\, dx_i\, dy_i\, dz_i\, \sigma_j\, dx_j\, dy_j\, dz_j \qquad (219)$$

$$V = \iiint [(a-x)^2 + (b-y)^2 + (c-z)^2]^{-1/2}\, \sigma\, dx\, dy\, dz\,.$$

This latter expression is an integral for the superposed amplitudes of the waves coming from each atom of the mass.

And since the forces are as the squares of the integrated amplitudes of the waves, or

$$X = -\partial V/\partial a \quad Y = -\partial V/\partial b \quad Z = -\partial V/\partial c \quad R = \sqrt{(X^2 + Y^2 + Z^2)}$$

$$-\partial V/\partial a = \iiint [(a-x)^2 + (b-y)^2 + (c-z)^2]^{-3/2} \times (a-x)\, \sigma\, dx\, dy\, dz \qquad (220)$$

etc.

we see why this force of gravitation obeys the law of inverse squares, observed in nature.

The curve of the varying amplitude for the waves, and the square of the amplitude at double distance, yielding surfaces S and s, of area 4 and 1, are shown to scale in Plate I. A study of this illustration shows exactly why the law of gravitation follows the law of *Newton*, 1686:

$$f = m\,m'/r^2 = \iiiiiint [(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{-1} \times \sigma\, dx\, dy\, dz \cdot \sigma'\, dx'\, dy'\, dz'\,. \qquad (221)$$

The geometrical law of the inverse squares follows from the physical law of the conservation of energy in the aether, as the waves expand spherically in free space. This alone proves that energy cannot be lost in the aether. Thus the conservation of energy rests on the kinetic character of the aether as the ultimate medium underlying the operations of nature.

And the accuracy of the Newtonian law becomes in fact a superb test of the validity of the principle of the conservation of energy. If there is no dissipation of energy, by unknown transformation of the waves, as in free planetary space, where no resistance is encountered, the conservation is rigorous. And this in turn assures us that no other medium underlies the aether. It is the ultimate medium for the wave stresses producing physical forces. Hence under the mutual actions of the planets a_i and n_i are invariable, or $\delta_j a_i = 0$, $\delta_j n_i = 0$.

Brief Criticism of Certain Traditional Theories.

Before attempting the preparation of a succinct summary of the chief conclusions at which we have arrived, it remains to review briefly the theories of a few of the earlier investigators, yet without attempting exhaustive historical critiques. Indeed a detailed examination of the work of the earlier philosophers scarcely seems necessary, in view of the bewildering difficulties of the problem of the nature and cause of gravitation, which are admitted by all writers since the initial epoch of *Huyghens*, *Hooke*, *Wren*, *Halley* and *Newton*, 250 years ago.

Accordingly we are called upon to notice chiefly the progress of two theories, both of which have come down to us in an obscure and unsatisfactory state: namely, the theory of ultramundane corpuscles, and the associated problem of the velocity of the propagation of gravitational action across the celestial spaces.

21. *Arago*'s historical review of the theory of Ultramundane Corpuscles, 1842: the theory now definitely rejected as disproved.

The chief considerations in the progress of the theory of ultramundane corpuscles, from the age of *Newton* to 1842, are

Note. These theorems may be established with entire rigor and somewhat more directly, from the differential equations of the disturbed motion, by showing that $m \cdot \partial\Omega/\partial x$, etc., involve only products of the masses in pairs. On Jan. 6, 1926, I communicated this briefer analysis, to M. *E. Picard*, Perpetual Secretary of the Paris Academy of Sciences, on account of the interest of French geometers in the generalizations of the theorems of *Laplace* and *Poisson*. (Jan. 16, 1926.)

incorporated in the report by *Arago*, of a committee of the French Chamber of Deputies, relative to the publication of a new edition of the works of *Laplace*, and developed somewhat more fully in the two volumes of *Arago*'s Eloges or Biographies of Distinguished Scientific Men, English translation by *Smyth, Powell* and *Grant*, 2 vols., Ticknor and Fields, Boston, 1859.

The only other suitable discussion of this question of the cause of gravitation known to me is that by Professor *J. Clerke Maxwell*, articles Atom and Attraction, Encyc. Brit., 9th ed., reprinted in *Maxwell*'s Scientific Papers, vol. II, pp. 473–476, 487–490. *Maxwell*'s discussion is as clear and lucid as is usual with this distinguished physicist; yet it is historically not quite so complete as that given by the Perpetual Secretary of the Paris Academy. For *Arago* speaks of the ideas of the Abbe *Varignon* (1654–1722), and of *Fatio de Duillier* (1664–1753), both intimate friends of *Newton*, and says these ideas were subsequently reinvented and perfected by *Le Sage* (1724–1803).

The work entitled: Traité Physique Mécanique par *George Louis Le Sage*, was published at Paris in 1818, by *Pierre Prevost*, who was encouraged in the publication by Dr. *Wollaston* of London, and by *Laplace, Biot, Arago, Fourier* and other members of the Paris Academy of Sciences.

In view of the two centuries which have elapsed since the establishment of the law of gravitation, *Le Sage*'s work therefore has had a certain currency, and was carefully considered by the most eminent philosophers of the age of *Laplace*, 100 years after the age of *Newton*, and more recently by *Clerke Maxwell* and Lord *Kelvin*, towards the end of the 19th century. The defects in *Le Sage*'s theory of ultramundane corpuscles have always been recognized, yet the theory has had a certain favor in default of a better one.

In his carefully written biography of *Laplace*, *Arago* says, in the lucid style characteristic of that eminent astronomer:

»Nature engenders the gravity of bodies by a process so recondite, so completely beyond the reach of our senses and the ordinary resources of human intelligence, that the philosophers of antiquity, who supposed that they could explain everything mechanically according to the simple evolutions of atoms, excepted gravity from their speculations.«

»*Descartes* attempted what *Leucippus*, *Democritus*, *Epicurus*, and their followers thought to be impossible.«

»He made the fall of terrestrial bodies depend upon the action of a vortex of very subtle matter circulating around the earth. The real improvements which the illustrious *Huyghens* applied to the ingenious conception of our countryman were far, however, from imparting to it clearness and precision, those characteristic attributes of truth.«

»Those persons form a very imperfect estimate of the meaning of one of the greatest questions which has occupied the attention of modern inquirers, who regard *Newton* as having issued victorious from a struggle in which his two immortal predecessors had failed. *Newton* did not discover the cause of gravity any more than *Galileo* did. Two bodies placed in juxtaposition approach each other. *Newton* does not inquire into the nature of the force which produces this effect. The force exists, he designates it by the term attraction; but, at the same time, he warns the reader that the term as thus used by him does not imply any definite idea of the physical process by which gravity is brought into existence and operates.«

»The force of attraction being once admitted as a fact, *Newton* studies it in all terrestrial phenomena, in the revolutions of the moon, the planets, satellites, and comets; and, as we have already stated, he deduced from this incomparable study the simple, universal, mathematical characteristics of the forces which preside over the movements of all the bodies of which our solar system is composed.«

»The applause of the scientific world did not prevent the immortal author of the Principia from hearing some persons refer the principle of gravitation to the class of occult qualities. This circumstance induced *Newton* and his most devoted followers to abandon the reserve which they had hitherto considered it their duty to maintain. Those persons were then charged with ignorance, who regarded attraction as an essential property of matter, as the mysterious indication of a sort of charm; who supposed that two bodies may act upon each other without the intervention of a third body. This force was then either the result of the tendency of an aethereal fluid to move from the free regions of space, where its density is a maximum, towards the planetary bodies around which there exists a greater degree of rarefaction, or the consequence of the impulsive force of some fluid medium.«

»*Newton* never expressed a definitive opinion respecting the origin of the impulses which occasioned the attractive force of matter, at least in our solar system. But we have strong reasons for supposing, in the present day, that in using the word impulses, the great geometer was thinking of the systematic ideas of *Varignon* and *Fatio de Duillier*, subsequently reinvented and perfected by *Le Sage*: these ideas, in effect, had been communicated to him before they were published to the world.«

»According to *Le Sage*, there are, in the regions of space, bodies moving in every possible direction, and with excessive rapidity. The author applied to these the name of Ultra-mundane Corpuscles. Their totality constituted the gravitative fluid, if, indeed, the designation of a fluid be applicable to an assemblage of particles having no mutual connexion.«

»A single body placed in the midst of such an ocean of movable particles, would remain at rest although it were impelled equally in every direction. On the other hand, two bodies ought to advance towards each other, since they would serve the purpose of mutual screens, since the surfaces facing each other would no longer be hit in the direction of their line of junction by the ultramundane particles, since there would then exist currents, the effect of which would no longer be neutralized by opposite currents. It will be easily seen, besides, that two bodies plunged into the gravitative fluid, would tend to approach each other with an intensity which would vary in the inverse proportion of the square of the distance.«

Arago's statement above, that »this force was then either the result of the tendency of an aethereal fluid to move from the free regions of space where its density is a maximum, towards the planetary bodies around which there exists a greater degree of rarefaction, or the consequence of the impulsive force of some fluid medium«, is especially worthy of attention. *Newton* had entertained the idea that the density of the aether was less towards the heavenly bodies, yet he could not make out why it is less. It was only when we developed the Wave-Theory, AN 5044, p. 57, that a clear demonstration became available, showing that just as the amplitudes of the waves increase towards the centre, under the law $A = k/r$, so also the density of the aether increases as we proceed outward in space, and hence the wave stress is towards the centre with a force $f = A^2 = k^2/r^2$, as observed in *Newton*'s law of gravitation.

Altogether the admirable summary by *Arago* is so luminous that I have thought it best to quote it in full, as the best analysis of the ideas transmitted to us by the philosophers who witnessed the progress since the foundation of the theory of universal gravitation, 1686.

This account therefore of the early speculations of *Varignon* and of *Fatio de Duillier*, and of their communication with *Newton*, prior to publishing their views on the cause of gravitation, is of decided historical interest. It is recalled that the friendship between *Newton* and *Varignon* was such that in *Newton*'s study, to the end of his life, there hung a portrait of *Varignon*, along with that of Lord *Halifax*, the life-long friend and patron of the great philosopher. And *Fatio de Duillier* was in *Newton*'s intimate friendship for forty years, 1687–1727 (cf. *Brewster*'s Life of *Newton*, 1855, vol. II, pp. 36–40; and as regards *Varignon*, pp. 70–74, and 290–295).

Notwithstanding the plausible considerations adduced by *Arago* above, and the similar arguments put forth by *Maxwell*, in regard to *Le Sage*'s theory, it is certain that these ultramundane corpuscles, as discussed by *Varignon*, *Fatio de Duillier*, *Newton*, *Halley* and *Le Sage*, are not the cause of gravitation, for the following reasons:

1. Gravitation is connected with magnetism, through the law $I/g = \eta^2(r^2/s^2 + r^2/s'^2)$, which is numerically verified in the magnetic and gravitational forces of our globe, as shown in the VII. Paper on the New Theory of the Aether, 1922. For since the ultramundane corpuscles will not explain magnetism, they necessarily fail to account for gravitation, which long stood in isolation, yet is now connected with another general force in nature characterized by a duality of powers, or attraction to two poles. The wave-theory, and the wave-theory alone, will explain these two great classes of phenomena.

2. The Fluctuations of the Moon's mean motion discovered by *Newcomb*, 1869–1909, can be accounted for only by the wave-theory, not by the theory of ultramundane corpuscles. For ultramundane corpuscles would not account for the decrease of the Moon's gravity to the earth at the time of lunar eclipses, while the wave-theory does so perfectly.

3. The theory of ultramundane corpuscles fails in electrodynamics, acoustic attraction, the motion of Mercury's perihelion, etc., and in general will not satisfy the great generalization known as the correlation of forces, and the conservation of energy, while the kinetic theory of the aether meets all these requirements, and is established by necessary and sufficient conditions drawn from the known laws of nature.

4. There is, however, a slight similarity between the free motion of the aetheron and that of the ultramundane corpuscle: yet in the New Theory of the Aether it is shown that wave-action underlies the forces of nature, and that they can be accounted for in no other way, as we see by the clear and unmistakable argument drawn from the phenomenon of acoustic attraction, where ultramundane corpuscles are not involved.

5. Accordingly, by considering phenomena of attraction, experimentally verified in the theory of sound, which do not involve the ultra-mundane corpuscles, we may get rid of the traditional theory of *Le Sage*. In its primitive form this theory of ultra-mundane corpuscles was early considered by *Varignon*, *Fatio de Duillier*, *Halley*, and *Newton*; yet two centuries ago these natural philosophers were unacquainted with acoustic attraction, and thus lacked a criterion for rejecting the theory of ultra-mundane corpuscles.

6. The wave-theory alone, therefore, accounts for the pressure of the aether towards the heavenly bodies, by the law of density, which necessarily follows in the propagation of waves from the atoms of matter: and such waves are actually observed to come from the Sun in an infinite variety of phenomena associated with sunspots, magnetic storms, aurorae, Earth currents, the magnetic tides of our globe, etc.

In addition to the above difficulties of the theory of ultra-mundane corpuscles, *Maxwell* and Lord *Kelvin* have both pointed out very serious objections based on the theory of energy, which have not been overcome in the half century since they were published.

The theory of ultra-mundane corpuscles therefore entirely fails, while the wave-theory of physical forces developed in the Kinetic Theory of the Aether incontestably triumphs!

22. The Historical Problem of the Velocity of the Propagation of Universal Gravitation across the Celestial Spaces.

One other question deserving of historical review is that relating to the velocity of propagation of gravitation. In the Exposition du Système du Monde, English translation by *Pond*, 1809, vol. 2, p. 230, *Laplace* enumerates five assumptions underlying the theory of universal gravitation, all duly verified by experience and observation during the first century of the Newtonian law:

»1. That gravitation takes place between the most minute particles of bodies.«

»2. That it is proportional to their masses.«

»3. That it is inversely as the squares of the distances.«

»4. That it is transmitted instantaneously from one body to another.«

»5. And it equally acts on bodies in a state of repose, and upon those which, moving in its direction, seem in part to withdraw themselves from its activity.«

In the discussion of the 4th proposition *Laplace* reasons as follows:

»We have no method of measuring the length of time in which gravity is propagated, because the action of the Sun having once attained the planets, it continues to act on them as if the attractive force was communicated instantaneously to the extremities of the system; we cannot therefore ascertain in how long a time it is transmitted to the Earth, no more than we could measure the velocity of light, were it not for the aberration and eclipses of Jupiter's satellites. But it is not the same with the small difference that may exist in the action of gravity upon bodies, according to the direction and quantity of their velocity. Analysis has shown me, that there should result an acceleration in the mean motions of the planets round the Sun, and in the mean motions of the satellites about their planets.«

»I had imagined this method of explaining the secular equation of the Moon, when I believed with other geometricians that it was inexplicable on the principle of universal gravitation. I found that if it arose from this cause, we must suppose in the Moon, in order to release it entirely from its gravity towards the Earth, a velocity in the centre of this planet at least six million times greater than that of light; the true cause of this equation being now known, we are certain that the activity of gravity is much greater than this. This force therefore acts with a velocity which we may consider as infinite, and we may conclude that the action of the Sun is transmitted in an indivisible instant to the extremities of the planetary system.«

Laplace's discussion of the velocity of gravitation is further elucidated in the introduction to volume IV of the Mécanique Céleste, 1805:

»As no variation in the mean motion of the earth is indicated by observation, we may infer: First, that the Sun, during the last two thousand years, has not lost a two millionth part of its substance; Second, that the effect of the impulsion of light upon the Moon's secular equation is insensible. The analysis of this effect may be applied to gravity, considered as being the result of the impulsion of a fluid producing the effect of gravity, by moving with extreme rapidity towards the attracting body. From this it follows, that, to satisfy the phenomena, we must suppose this fluid to have an excessively great velocity, at least one hundred million times greater than that of light. This velocity would be infinite in the hypotheses admitted by mathematicians relative to the action of gravity; these hypotheses may therefore be used without fear of any perceptible error.«

It will be noticed that he speaks as if the »gravific fluid« was conceived as moving towards the central masses with a velocity 100 million times greater than that of light. No reason is given why the »fluid« is so moving, nor is it explained how the motion is kept up: obviously the hypothesis is inadmissible to-day, yet it must have been very much in vogue a century ago, otherwise *Laplace* would not have given it such prominence in his leading works[1]).

Altogether this is an impressive presentation of the historical difficulty of the problem of the velocity of the force of gravitation. For a long time no progress could be made, as we see by this careful summary of the leading views transmitted to us from the age of *Newton*.

In the biography of *Laplace*, 1842, *Arago* has a very careful discussion of this question, which should also be cited:

»If attraction is the result of the impulse of a fluid, its action ought to employ a finite time in traversing the immense spaces which separate the celestial bodies. If the sun, then, were suddenly extinguished, the earth after the catastrophe would, mathematically speaking, still continue for some time to experience its attractive influence. The contrary would happen on the occasion of the sudden birth of a planet; a certain time would elapse before the attractive force of the new body would make itself felt on the Earth.«

»Several geometers of the last century were of opinion that the force of attraction is not transmitted instantaneously from one body to another; they even assigned to it a comparatively inconsiderable velocity of propagation. *Daniel Bernoulli*, for example, in attempting to explain how the spring tide arrives upon our coasts a day and a half after the syzygies, that is to say, a day and a half after the epochs when the sun and moon are most favourably situated for the production of this magnificent phenomenon, assumed that the disturbing force required all this time (a day and a half) for its propagation from the moon to the ocean. So feeble a velocity was inconsistent with the mechanical explanation of attraction of which we have just spoken. The explanation, in effect, necessarily supposes that the proper motions of the celestial bodies are insensible compared with the motion of the gravitative fluid.«

»After having discovered that the diminution of the eccentricity of the terrestrial orbit is the real cause of the observed acceleration of the motion of the moon, *Laplace*, on his part, endeavoured to ascertain whether this mysterious acceleration did not depend on the gradual propagation of attraction.«

»The result of calculation was at first favourable to the plausibility of the hypothesis. It showed that the gradual propagation of the attractive force would introduce into the movement of our satellite a perturbation proportional to the square of the time which elapsed from the commencement of any epoch; that in order to represent numerically the results of astronomical observations it would not be necessary to assign a feeble velocity to attraction; that a propagation eight millions of times more rapid than that of light would satisfy all the phenomena.«

[1]) In *Pierre Prevost*'s edition of *Le Sage*'s Theory of Ultramundane Corpuscles, Paris, 1818, pp. 21–23, the view is taken that the speed of the corpuscles much surpasses that of light, yet the largest multiplier named (p. 22) is 100000 (cent mille fois plus vite que la lumière), which is much less than the multiplier used by *Laplace*.

Le Sage notes that light comes from the Sun in 8 minutes, which is 32 million leagues (1 league = 3 miles) distant, and then computes that it would traverse 2000000000000000000 leagues in 10000 years: hence he concludes that the distance whence the ultramundane corpuscles were projected for the ten thousand years of creation was two thousand trillions of leagues, a trillion being 1000000 billions, etc.

The principle of using the supposed velocity of the gravific corpuscles for computing the extent of the universe, from the assumed duration of the world, evidently was too hazardous to appeal to *Laplace*.

»Although the true cause of the acceleration of the moon is now well known, the ingenious calculation of which I have just spoken does not the less on that account maintain its place in science. In a mathematical point of view, the perturbation depending on the gradual propagation of the attractive force which this calculation indicates has a certain existence. The connection between the velocity of perturbation and the resulting inequality is such that one of the two quantities leads to a knowledge of the numerical value of the other. Now, upon assigning to the inequality the greatest value which is consistent with the observations after they have been corrected for the effect due to the variation of the eccentricity of the terrestrial orbit, we find the velocity of the attractive force to be fifty millions of times the velocity of light!«

»If it be borne in mind, that this number is an inferior limit, and that the velocity of the rays of light amounts to 77000 leagues (192000 English miles) per second, the philosophers who profess to explain the force of attraction by the impulsive energy of a fluid, will see what prodigious velocities they must satisfy.«

The researches of *Tisserand* on the electrodynamic law of *Weber*, Comptes Rendus, Sept. 30, 1872, p. 760–763, and Mécanique Céleste, Tome IV, Chapter XXVII, pp. 499–508, proved a useful guide in my early calculations (Electrodynamic Wave-Theory of Physical Forces, vol. I, 1917, pp. 142–151). But we point out that this question did not admit of solution prior to the development of the Wave-Theory of Physical Forces, 1914–1925.

Even after the proofs developed in the Wave-Theory, some authorities continued to adhere to the views of *Laplace*. This habit, however, was true only of those who lose sight of contemporary progress, and still think in terms of traditions coming from the 18^{th} century, and therefore it is not to be taken seriously. It is certain that universal gravitation is due to wave-action and therefore the chief force of nature is propagated with the velocity of light!

About the only celestial effect of the propagation of gravitation, with the velocity of light, and of electrodynamic waves in general, is a small secular progression of the perihelion. In the case of Mercury, $\delta\varpi = +14''.51$; Jupiter's V^{th} Satellite, $\delta\varpi = 4''.23$; *Encke*'s Comet, $\delta\varpi = 0''.62$; the double stars: β Persei, $\delta\varpi = 3291''.9$, α Aurigae, $\delta\varpi = 65''.7$, α Centauri, $\delta\varpi = 0''.0018$, per century (cf. Electrodynamic Wave-Theory of Physical Forces, vol. I, 1917, pp. 142–151, and AN 5048, p. 137–138).

If we contemplate the results of the progress here outlined, which was begun by *Tisserand*, Comptes Rendus, Sept. 30, 1872, and concluded by me, in the publications just cited, (1914–1922), we shall not fail to be impressed by the progress made since the age of *Newton* and *Laplace*. *Newton* and his contemporaries assumed the action of gravitation to be instantaneous at all distances, and *Laplace* tried to find an observational criterion, from the outstanding part of the Moon's secular acceleration, by which the velocity of gravitation could be concluded.

Among the reasons why the illustrious author of the Mécanique Céleste failed in his attempt, may be mentioned: 1. He lived before the discovery that the velocity of electric currents on wires is essentially identical with that of light: which was experimentally confirmed after the development of *Maxwell*'s electromagnetic theory of light, some 60 years ago. 2. Although *Gauss* had considered electrodynamic action as perhaps propagated in time, like light, as early as 1835, yet he did not bring his work to a conclusion until *Weber*'s law had been formulated in 1846; and therefore in *Laplace*'s days the idea of instantaneous action, current since the days of *Newton*, and implied in the form of *Newton*'s Law, still held in the minds of most philosophers. 3. *Laplace* failed to note that the retarded distance, due to the finite velocity of gravitation, $r(1-v_r)$ is, to the first order, equal to the unretarded distance r, with the result that the force is directed approximately to the contemporary position of the attracting body.

This last consideration was before *Tisserand*, in his early researches on *Weber*'s electrodynamic law, C. R., Sept. 30, 1872, and myself, in generalizing and extending the use of *Weber*'s Law, 1915, (Electrodynamic Wave-Theory of Physical Forces, vol. 1, 1917, pp. 142–151). After the advances made in the New Theory of the Aether, 1922, all these results seem very obvious; yet they were far from plain in 1914, and in the time of *Laplace* could not even have been anticipated.

23. *Newcomb*'s Discussion of the Cause of Gravitation, 1883: His Objections overcome, under the Valid Conditions fulfilled by the Wave-Theory.

It is recognized that the late Professor *Simon Newcomb* not only was one of the greatest mathematical astronomers of his age, but also endowed with an equally remarkable physical intuition, so that he was capable of going directly to the root of any problem in science. Thus it is worth while to examine his discussion of the cause of gravitation, which fortunately we have found clearly outlined in the fourth edition, revised, of the Astronomy, by *Newcomb* and *Holden*, American Science Series, Henry Holt and Company New York, 1883. Chapter V, pp. 131–151 is on Universal Gravitation, and of more than usual interest as reflecting the mature philosophic opinions of this great investigator.

Newcomb dwells on the validity and permanency of the Newtonian theory, but remarks that gravitation »may yet be shown to be the result of some more general law« — apparently in allusion to *Weber*'s electrodynamic law, already considered by him in the American Ephemeris Paper on the motion of Mercury's perihelion, 1881. He adds, however, that »thus far no theory of the subject having the slightest probability in its favor has been propounded.« This shows that up to 1883 the electrodynamic theory was not satisfactorily developed.

Newcomb then explains *Le Sage*'s theory of ultramundane corpuscles, and the mutual screening effect claimed for two bodies, under the slight excess of corpuscles acting on their outside. He does not seem to favor *Le Sage*'s theory any more than Sir *W. Thomson* (Lord *Kelvin*) had done (Proceedings, Roy. Soc. Edinburgh, 1872), or *Maxwell*, in the article Atom, Encyc. Brit., 9^{th} ed., 1875.

Newcomb's final paragraph is the most significant:

»One of the commonest conceptions to account for gravitation is that of a fluid, or ether, extending through all space, which is supposed to be animated by certain vibrations, and forms a vehicle, as it were, for the transmission of gravitation. This and all other theories of the kind are subject to the fatal objection of proposing complicated systems to account for the most simple and elementary facts. If, indeed, such systems were otherwise known to exist, and if it could be shown that they really would produce the effect of gravitation, they would be entitled to reception. But since, they have been imagined only to account for gravitation itself, and since there is no proof of their existence except that of accounting for it, they are not entitled to any weight whatever. In the present state of science, we are justified in regarding gravitation as an ultimate principle of matter, incapable of alteration by any transformation to which matter can be subjected. The most careful experiments show that no chemical process to which matter can be subjected either increases or diminishes its gravitating principle in the slightest degree. We cannot therefore see how this principle can ever be referred to any more general cause.«

This is a good summary of *Newcomb*'s views, as known to me by correspondence and later personal contact with him, which extended over nearly 25 years, 1884–1909. It will be noticed that he alludes to the »fluid or ether extending through all space, which is supposed to be animated by certain vibrations.« This is all the attention he gives to the wave-theory, — which shows that in his time it was not so developed as to command confidence. He alludes to *W. B. Taylor*'s memoir in the Smithsonian Report for 1876, entitled: Kinetic Theories of Gravitation, which includes many descriptive sketches, yet nothing convincing.

Accordingly, the only part of *Newcomb*'s discussion calling for special consideration is an admission, which we put in italics:

»If, indeed, such systems were otherwise known to exist, and if it could be shown that they really would produce the effect of gravitation, they would be entitled to reception.«

It is scarcely necessary to point out that these conditions have now been fully complied with in the New Theory of the Aether, and thus it is entitled to favorable reception by the scientific world

The present proof of *Weber*'s electrodynamic law is absolutely overwhelming, the wave-theory of magnetism independently established by the laws of the lines of force, and by *Faraday*'s rotation of the beam of polarized light, 1845, while the connection with gravitation is numerically verified in the observed theory of the Earth and Sun, VII paper, 1922.

The new theory of acoustic attraction makes visible to our eyes some of the deepest secrets of wave motion, in traversing the dense globes of the heavenly bodies. This deflection of the waves also finds confirmation in *Newcomb*'s Fluctuations of the Moon's Mean Motion, 1909, and in the bending of radio waves about the terrestrial spheroid, (AN 5044 and AN 5317).

Finally, in regard to *Newcomb*'s remark that the theory of an aether animated by certain vibrations is too complicated a system for explaining the most simple and elementary facts, we need only point out that the wave-theory is not more complex than the forces of gravitation themselves are. For the potential due to an integration of the waves from all the particles, according to their several amplitudes, at distances r,

$$A = k/r = k/\sqrt{[(a-x)^2+(b-y)^2+(c-z)^2]^{1/2}}$$

$$V = \int 1/r \cdot \mathrm{d}m = \iiint [(a-x)^2+(b-y)^2+(c-z)^2]^{-1/2}\,\sigma\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \\ = \iiint [(a-x)^2+(b-y)^2+(c-z)^2]^{-1/2}\,k\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad (222)$$

yields the force of gravitation acting along the x-axis upon a unit mass:

$$X = -\partial V/\partial a = \\ = \iiint [(a-x)^2+(b-y)^2+(c-z)^2]^{-3/2}\,\sigma(a-x)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad (223)$$

with similar expressions for the forces Y and Z.

The Newtonian law of the attraction between any two masses is given by a sextuple integral which often takes the simpler form for particles shown in the second term:

$$f = \iiiint\!\!\iint [(x'-x)^2+(y'-y)^2+(z'-z)^2]^{-1} \times \\ \times \sigma\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \cdot \sigma'\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' = m\,m'/r^2\,. \qquad (224)$$

Therefore the wave-theory is just as complex as *Newton*'s Law, and not an iota more so: the perfect identity of the constant k with the particles σ in $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in the above integrals (222) yields the final triumph of the wave-theory!

In the present VIII Paper the whole argument is carefully re-examined in the light of the calculus of probability, which in the hands of *Laplace* rendered such signal service to Science. And we venture to think that if *Laplace*'s arguments were considered powerful, in case of the great problems with which he dealt, the strength of the present arguments will be found still greater, at least in proportion to the difficulty of the problem of the cause of gravitation, compared to such problems as the motion of Jupiter and Saturn, and the Moon's secular acceleration.

For my part I confess to as extreme a surprise at the manifold infinite strength of the present argument, as at the bewildering weakness of the arguments of the earlier investigators, in not seeing the inconceivable and overwhelming strength of the wave-theory.

24. The Order of Nature indicated by the Calculus of Probability: Summary and Conclusions.

1. In explanation of the use of the Calculus of Probability, for inferring the actual processes of nature, we remark at the outset that a law of nature represents a vast mass of phenomena observed to occur in a certain order, both in time and space: the series of phenomena thus represented depend on a physical cause constantly acting in a fixed way: the number of phenomena conforming to the law may be, and usually is, infinitely great.

For example, in the phenomena of light, for all practical purposes, an infinity of waves pass in a second, and certainly in a year of time. And if magnetism be a wave-phenomenon, as shown in the VII Paper, then the waves of magnetism acting on *Faraday*'s rotated beam of polarized light, in the celebrated experiment of 1845, also represents a series of two infinities, — one of the waves of polarized light, the plane of which is to be rotated, thus yielding an observed order that is both constant and dependable; the other of the magnetic waves acting on the light, in a fixed order, to produce the constancy of the rotation of the plane of the polarized beam observed.

Similar reasoning applies to all the great groups of related or associated phenomena. Accordingly, when we calculate the probability of the observed order in nature, from one group of phenomena, we do not duplicate the probability from the associated or related phenomena, but merely take account of the independent groups which may properly represent both classes of phenomena in combination.

The total probability calculated by this method therefore is a minimum!

For example in equation (A) (140) section **15**, we considered merely the coincidence of the observed surfaces with the theoretical surfaces throughout space, resulting from the wave-theory. We did not integrate for the infinities of waves from the two centres[1]), in the three coordinates, nor for the infinity of time involved in the constant action of gravitation, and yet all of these latter considerations would have been allowable.

Thus instead of a probability of $8\infty^3$, we now introduce the additional multipliers: $8\infty^3 \times 8\infty^3 \times 2\infty = 128\infty^7$, yielding in all

$$N = 1024\infty^{10} \qquad (226)$$

instead of $8\infty^3$, as given in equation (140) (A). Moreover, a volume integration is required for each mass, as follows. We must triply integrate for all the vibrating atoms of each body, in r, ϑ, φ, as in equation (126), and thus we find that three more infinities arise from the new limits incident to the volume and density of the masses. Our final value of N therefore becomes for all the atomic waves:

$$N_1 = 1024\infty^{13} \qquad (227)$$

2. Accordingly when we compute the total probability of the wave-theory, we have to take account not only of geometrical concordances, but also of the physical laws verified by unlimited numbers of observed phenomena, often in other sciences, such as electrodynamics, magnetism, acoustic attraction, sunspots, magnetic storms, aurorae, Earth currents, the semi-diurnal magnetic tide, etc. After considerable investigation of the relative values of these several lines of evidence, drawn from the observed order of nature, I have labored to reduce them to calculation by the following considerations of the appropriate grouping and values.

For example the science of Electrodynamics represents an unlimited mass of ordered phenomena, following definite and exact laws, verified by careful observation, such as:

(a) *Biot* and *Savart*'s law for the intensity of a current in a straight wire, yielding coincidences with the wave-theory at all points of the plane of xy normal to z, the axis of the wire, and extending indefinitely along that axis. In addition to this argument drawn from geometry, under the concordances at all points of space, (x, y, z), equivalent to $8\infty^3$, we have the argument drawn from the physical grounds, equivalent to $8\infty^3 \times 2\infty = 16\infty^4$: so that the total chances become not less than $c_1 = 128\infty^7$.

(b) *Ohm*'s law for variable resistance, or variable electromotive force, at constant distance, in like manner, would yield, from similar considerations of geometry and steady physical action, constant in time, a total chance not less than

$$c_2 = 128\infty^7 .$$

(c) *Oersted*'s experiment of 1819, which yields ordered orientation phenomena, for magnets of all lengths, at all distances, — in like manner, would give from geometrical considerations, and physical action, constant in time, a total chance not less than $c_3 = 128\infty^7$.

(d) The observed circularity of the lines of force shown in iron filings free to arrange themselves in winrows, owing to the opposite polarity of the adjacent particles in any winrow, presents one of the most impressive groups of ordered phenomena in physical science. It occurs at all points of space, at any distance (x, y) from the wire, and everywhere along the axis z. Accordingly for reasons similar to the above, the total chance of this order is not less than

$$c_4 = 128\infty^7 .$$

The arguments thus adduced in favor of the wave-theory from but four chief groups of the ordered phenomena, in electrodynamics, therefore, accumulates to the enormous compound number:

$$N_2 = c_1\, c_2\, c_3\, c_4 = (128)^4\,(\infty^7)^4 = 268\,435\,456\infty^{28} . \qquad (228)$$

We recall, by the calculus of probability, that the chances favorable to the wave-theory are as N_2 to 1, and hence we need not extend the discussion to the consideration of other related, but less uniquely defined phenomena. Our number for the chances N_2 therefore is a minimum. It is sufficient, however, to enable us to conclude that the best defined phenomena in electrodynamics cannot be explained except by the wave-theory. And we easily come to that conclusion without dealing with any of the more involved phenomena, in which the interpretation might be obscure or doubtful. The best and most conclusive evidence alone is sought, and that comes from the simplest laws admitting of a unique interpretation.

3. The phenomena of magnetism offer a field for the study of ordered phenomena scarcely less rich and varied than that of electrodynamics proper, to which magnetism is related. It was by the action of moving currents (wave-fields) that *Ampère* developed a practical process for making artificial magnets, in the years immediately following *Oersted*'s experiment of 1819.

If we attempt to compute the chances that magnetism points to wave action, we should consider such phenomena as the following:

(a) The shape of the lines of magnetic force, at the various points of space, and their uniform coincidence with the theoretical forms they should have under the wave-theory, which is accorded a physical basis by the experiment of *Dolbear* who rotated a series of discs on a flexible axis, and confirmed by

[1]) This would involve a sextuple integral of the type:

$$\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty} dx\, dy\, dz\, dx'\, dy'\, dz' = 64\infty^6 . \qquad (225)$$

Usually we write it simply as in the second member, to avoid repetition of the integrals which already are sufficiently explained.

Faraday's experiment of 1845, on the rotation of the beam of polarized light by magnetism. The chance that this vast mass of ordered phenomena, physically confirmed through all space and steady in time, represents the wave-theory is not less than

$$c_1 = 128\,\infty^7 .$$

(b) The connection of magnetism with gravitation, discovered by me in 1922, and confirmed by the numbers representing *Gauss*' theory throughout our globe, thus explaining the unequal depths of the magnetic poles in the two terrestrial hemispheres, with the geometrical and physical basis thus shown to exist and to be constant in time, yields a chance in favor of the wave-theory not less than

$$c_2 = 128\,\infty^7 .$$

(c) The connections between sunspots, magnetic storms, aurorae, Earth currents, have been known since 1850· this connection is illustrated and verified throughout our globe, in the three coordinates (x, y, z) with three types of phenomena in each coordinate, all steady in time. The chance that this vast mass of ordered phenomena represents the wave-theory certainly is not less than $c_3 = 128\,\infty^7$

and might be held to be much greater yet.

(d) Then again we might consider the following additional masses of ordered phenomena:

(α) The increase of the total intensity towards the poles,

(β) The semi-diurnal magnetic tides, which so puzzled *Airy* and Sir *John Herschel*,

(γ) The observed magnetism of the Sun and the form of the solar corona, the lines of which coincide with the lines of force in terrestrial magnetism.

These resulting laws follow both geometrical and physical laws applicable to all space, and steady in time, and therefore would yield chances favoring the wave-theory certainly not less than $c_4 = 128\,\infty^7$.

Without considering other associated phenomena, or further mutual relationships between these, we may conclude that the total chances favorable to the wave-theory in magnetism certainly is not less than N_3 to 1, where

$$N_3 = c_1\, c_2\, c_3\, c_4 = 268\,435\,456\,\infty^{28} . \qquad (229)$$

4. We shall now consider the related phenomena of wave-bending or wave deflection, in such related groups as:

(α) Acoustic attraction,

(β) The propagation of radio waves about our globe,

(γ) The assimilation of both types of waves to light.

(a) Here the coincidences of the observed result with the indications of the wave-theory occur throughout space (x, y, z); they have a physical as well as a geometrical basis, and the concordance is steady in time. Hence on these grounds it seems certain that our minimum value is at least:

$$c_1 = 128\,\infty^7 .$$

(b) But such wave-action accords with the gravitational requirements of increased pressure beyond the heavenly bodies, and increased tension between them: such results have both a geometrical and physical basis and are steady in time. From these considerations therefore c_2 is not less than:

$$c_2 = 128\,\infty^7 .$$

(c) Associated phenomena of the following kind:

(α') The outstanding motion of Mercury's perihelion,

(β') The fluctuations of the moon's mean motion,

(γ') The loss of energy in radio transmission,

which have a geometrical and a physical basis, steady in time, yield us from similar considerations:

$$c_3 = 128\,\infty^7 .$$

(d) A final argument may be drawn from such phenomena as:

(α'') The correlation of forces,

(β'') The conservation of energy,

(γ'') The nature of the potential and the constancy of the planetary major axes.

The aether and the steady flow of waves in this elastic medium explains all these ordered phenomena, geometrically, physically and in time. Accordingly, in like manner,

$$c_4 = 128\,\infty^7 .$$

From all these considerations, it follows therefore that we may take N_4 as not less than

$$N_4 = c_1\, c_2\, c_3\, c_4 = 268\,435\,456\,\infty^{28} . \qquad (230)$$

Summary and Conclusions drawn from the Calculus of Probability.

It appears from the above considerations that the total chance of the wave-theory in comparison with any other theory, -- as judged by the infinitudes of concordances, observed throughout nature to support the wave-theory,— becomes the product of the separate numbers N_1, N_2, N_3, N_4, as above estimated:

$$H = 1024(268435456)^3\,\infty^{97} . \qquad (231)$$

In Part I above we have carefully investigated the significance of the law of the inverse squares, and found that in theory a double infinity of other laws could occur, if nature had a free hand in varying the exponents in the law of force:

$$f = k^2/r^{n+\nu} = y = k^2/x^{n+\nu} \qquad (232)$$

$$n = 0 \cdots n = \infty \qquad \nu = 0 \cdots \nu = 1.000000000 .$$

But in fact Nature has no such freedom, and the observed restriction of $n+\nu$ to 2.000000000 ······ thus adds chances equivalent to an infinity of the second order to the indications favorable to the wave-theory. Moreover the above coefficient, $1024(268435456)^3$, adds practically a Third Infinity, $1024(268435456)^3 = \infty$, and therefore our final calculation of the chances favorable to the wave-theory, becomes ϖ to 1, where ϖ is

$$\varpi = 1024(268435456)^3\,\infty^{99} . \qquad (233)$$

An Infinity of the hundredth order!

$$\varpi = \infty^{100} \qquad (234)$$

This calculation of the total chances favorable to the wave-theory involves no violent hypothesis, nor extreme assumption, but is based on concordances everywhere observed, and where the phenomena of nature are so simple and well defined that the interpretation is unique. A result such as the above for ϖ is hopelessly and forever beyond conceiving even by a geometer. For the value of ϖ is equivalent to an integral of the hundredth order, between infinite limits:

$$\varpi = \int_{-\infty}^{+\infty}\!\!\int\!\!\int \int_{-\infty}^{+\infty}\!\!\int\!\!\int \int_{-\infty}^{+\infty}\!\!\int\!\!\int \cdots \int_{-\infty}^{+\infty}\!\!\int\!\!\int dx_1\, dy_1\, dz_1 \cdot dx_2\, dy_2\, dz_2 \times \times dx_3\, dy_3\, dz_3 \cdots dx_i\, dy_i\, dz_i \quad (235)$$

where $i = 100$.

Now it is well known that a triple integral between infinite limits is equivalent to an infinity of the third order and represents a summation of all the points in space, taken singly. An infinity of the sixth order represents a summation of all the points in space taken singly, and then each point again combined with every other point, thus grouping all points in pairs. Thus a triple integration into a triple integration leads to the sextuple integral, with infinite limits.

This type of non-nuple integration is an infinity of the ninth order, equivalent to repeating again the above sextuple integration for each point of space.

A duo-decimmuple integration is equivalent to combining again the non-nuple integral with every point in space, or repeating again, in combination with the second sextuple integration, every step in the first sextuple integration. For in the duo-decimmuple integration there results four successive integrations for every point in the immensity of space!

Let us now examine these impressive results philosophically, in the light of experience, and the habits, judgement and usage of the greatest mathematicians of former ages. In the time of *Newton*, the doctrine of chances was not yet reduced to a scientific basis, and thus the author of the Principia did not avail himself of such aids to the human mind. But from beginnings at the epoch of *Newton* the calculus of probability took definite scientific form a century later, especially in *Laplace*'s Théorie Analytique des Probabilités, 1812, which fixed the principles of the Science.

An infinity of the first order, yielding a probability of infinity to one in favor of any Theory, was justly held by *Laplace*, the founder of the theory of probability, to be conclusive and definite ground for the conviction that the theory under investigation represents a true law of nature.

I concur in this conclusion of the illustrious author of the Mécanique Céleste, since no ground is left for supporting any other theory.

Yet as the above calculated infinity of the 100^{th} order results naturally from the increasing combinations of separate and independent ordered phenomena supporting the wave-theory, and there is no ground for questioning the legitimacy of the increase of the total probability with the multiplication of infinities of higher order, we conclude that the total chance in favor of the wave-theory is of the order of ∞^{100} to 1, or an Infinity of the hundredth order!

It seems therefore certain and incontestable that the wave-theory corresponds to the true laws of nature, and all other theories are forever barred from the consideration of geometers and natural philosophers.

Concluding Note on the Lunar Theory.

Hitherto we have considered carefully certain aspects of the planetary theory, and many other large groups of physical phenomena, yet have given somewhat slight attention to the lunar theory, which is especially suitable for exhibiting the gravitational theory of forces as directed magnitudes depending on wave-action.

If M be the mass of the Sun, acting at a distance ϱ, and ψ the angle of elongation of the Moon from the Sun, r the Moon's distance, and m its mass, (the Earth's mass being unity); then the components of the Sun's disturbing force, which act on the motion of the Moon, to change its path in space, in respect to the centre of the Earth, are:

The Tangential Component $= T = \frac{3}{2} M\, m\, r/\varrho^3 \cdot \sin 2\psi$ (236)

The Radial Component $= R = M\, m\, r/\varrho^3 \cdot (1 - 3\cos^2\psi)$ (237)

The Perpendicular Component $= P =$

$= 3M\, m\, r/\varrho^3 \cdot \cos\psi \sin \Omega \sin I$ (238)

the total disturbing force being the vector magnitude,

$$F = \sqrt{(T^2 + R^2 + P^2)}\,. \quad (239)$$

As the total disturbing force F acts in various directions depending on the variability of r, ϱ, ψ, Ω, and I, the Moon's path is continually changing. We shall analyse briefly a few of these changes:

(a) The tangential component T acts constantly towards the line of syzygies. In the First Quarter it tends to retard the orbital motion, but really accelerates it, by decreasing the centrifugal force and letting the Moon drop nearer the Earth; in the Second Quarter the component T tends to accelerate the orbital motion, but by increasing the centrifugal force lengthens the radius vector, and thereby really retards the Moon's motion: in the Third Quarter the effect is similar to that in the First Quarter: and in the Fourth Quarter the effect is similar to that in the Second Quarter.

The most celebrated lunar inequality due to the tangential disturbing force is the Variation, which was discovered from observation by *Tycho Brahe* about A. D. 1590, but had been known to the Arabian astronomer *Aboul Wefa* as early as A. D. 980. At its maximum, in the octants, $\psi = 45^\circ$, 135°, 235°, 325°, the Variation amounts to about $32'$ or a full diameter of the Moon, corresponding to an orbital motion of over an hour. It was first explained theoretically by *Newton* in the Principia, 1686.

(b) The radial component R acts outwardly, decreasing the Moon's gravity to the Earth, from $\psi = 0^\circ$ to $\psi = 55^\circ$, and inwardly from $\psi = 55^\circ$ to $\psi = 125^\circ$; and then again outwardly, $\psi = 125^\circ$ to $\psi = 235^\circ$; inwardly, $\psi = 235^\circ$ to $\psi = 305^\circ$; and then again outwardly, $\psi = 305^\circ$ to $\psi = 360^\circ$. The inward force at the Quadratures, $\cos\psi = 0$, is $R = M\, m\, r/\varrho^3$, while at the Syzygies, $\cos\psi = \pm 1$, it is $R = -2\, M\, m\, r/\varrho^3$. As the arc ψ on which the outward force acts is about 220°, in a revolution, while the inward force extends over an arc of about 140°, and moreover the outward force is twice as powerful as the inward force, the average effect for the entire revolution is to decrease the Moon's gravity by $1/358^{th}$ part of the whole, and thus lengthen the Moon's period by an amount corresponding to a decreased angular motion of $1/179^{th}$ part, or about 4 hours.

This does not alter the sector described by the radius vector, since the component acts in the direction of the radius; yet the Moon's real velocity and angular velocity are diminished, owing to the effect of placing the Moon at a greater distance. The centrifugal force is equal to its gravity diminished by the action of the Sun, and the radius vector describes the same sector that it would describe without this action, — the radius vector, however, being augmented by $1/358^{th}$ part of the whole, and the angular motion diminished by a 179^{th} part.

(c) The Annual Equation results from the annual change in ϱ^3, yielding radial disturbing forces greater by a 20^{th} part at perihelion, thus retarding the Moon by about 13′, and less by a similar amount at aphelion, which correspondingly accelerates the Moon's motion.

(d) The Evection, discovered by *Ptolemy* 140 A. D., is a distortion of the form of the lunar orbit, under the action of the disturbing forces, by which the eccentricity is increased or diminished by about $1/5^{th}$ of the whole. It depends on the return of the Sun to the same position relative to the perigee. The Moon's longitude may be increased or decreased by about 1°15′, and thus the evection is by far the largest of all the Lunar Inequalities, the eccentricity being greatest when the line of apses passes through the Sun, and least when the line of apses is at right angles to the line joining the Earth and Sun.

(e) It remains to notice the progression of the perigee, which also depends mainly on the radial component. This is the most difficult of all the lunar calculations, and is very celebrated as confirming finally the rigor of the Newtonian law. In his address to the British Association in Australia, 1914, Professor *E. W. Brown* estimated the exponent in the law not to differ from 2 by more than 1 part in 400000000, or $2.000000000 \pm 0.000000005$.

(f) The perpendicular component P accounts for the regression of the Moon's nodes in 18.6 years: this component acts towards the plane of the ecliptic, and as the Moon revolves forward, the node or intersection point moves backward, the inclination also varying between certain limits, according to the power of the force P.

(g) Without dwelling on the other chief inequalities in the lunar motions, — such as the secular acceleration, the inequalities in latitude and longitude depending on the figure of the Earth, the parallactic inequality, the inequalities due to the action of the principal planets, etc., — we merely remark that they all conform strictly to gravitational principles, and thus the disturbing forces represent vector magnitudes, corresponding to wave-action.

It is obvious that these varied lunar inequalities, depending on the respective disturbing forces, all conform to the Newtonian law, not only at all points of space in the three coordinates (x, y, z) but also for all lengths of the three resolved components (T, R, P), and moreover throughout all time — for 2645 years at least, since the lunar eclipses recorded at Babylon in the reign of Mardocempad, 720 B. C.

It is easy to see how many powerful arguments for the wave-theory could be based on these concordances. From some preliminary considerations I believe that if we include likewise the corresponding inequalities of the eight Major Planets, about half of which may be large enough to be found by observation, it might easily be shown that we could calculate for the wave-theory the additional chances, $c_1 = \infty^{10}$, $c_2 = \infty^{10}$, $\cdots\cdots c_{10} = \infty^{10}$, and thus put at the minimum:

$$N_5 = c_1 \cdot c_2 \cdot c_3 \cdots\cdots c_{10} = \infty^{100}$$

making $$\varpi = \infty^{200}\,. \qquad (240)$$

It is, however, unnecessary to accumulate further arguments for the wave-theory; and therefore we merely remark that this theory represents the true order of nature, as certainly as Newtonian gravitation governs the illimitable operations of the starry heavens!

In his reflections on the law of universal gravitation, Exposition du Système du Monde, Liv. IV, chap. XV, *Laplace* notes how the theory of gravitation has confirmed the motion of the Earth, already evident from its simplicity; and then sagaciously remarks: »We may increase the probability of a theory, either by diminishing the number of hypotheses on which it rests, or by augmenting the number of phenomena which it explains.«

He then shows that all the celestial phenomena follow from the principle of gravitation, by which they are connected into a true science of the heavens, with concordances of the highest accuracy: and remarks, that if we consider that there does not exist a single phenomenon which cannot be referred to the law of gravitation, we shall have no reason to fear that any one will question its truth in consequence of phenomena not heretofore observed.

So it is with the New Theory of the Aether! It explains not only the cause of gravitation, and connects this force with electrodynamic forces, including molecular and atomic forces: but also shows the way in which these various forces arise from wave-action, so that the myriad-fold order of vast independent groups of phenomena, connected by the New Theory of the Aether, are completely explained.

After this extensive investigation of the indications of the calculus of probabilities, it only remains to emphasize the distinctive features in the progress of the theory of universal gravitation during the past 250 years. This great advance consisted essentially in the verification of the Newtonian law, during the first century; and the development of electrodynamic laws during the second century. This latter progress, however, stopped short of real fruition, in default of the Kinetic Theory of the Aether, which was begun indeed by Sir *Isaac Newton*, 1678–1721, yet not given definite formulation till the appearance of this series of Papers, 1920–1925.

Such a brief outline of the leading facts obviously justifies a discerning investigator in departing from the stereotyped usage of the period since the close of the researches of *Laplace*, 1827, which became largely stationary, in that it neglected the unfinished researches of *Newton* relative to the cause of attractive forces.

To-day it no longer suffices merely to verify the theory of gravitation, as before the publication of the Mécanique Céleste, 1799–1825: we must earnestly resume the unfinished researches of Sir *Isaac Newton*, definitely assign and prove the cause of universal gravitation!

After being concealed from mortal sight for centuries, this sublime physical agency becomes to-day the greatest outstanding mystery of the universe! And therefore it calls for the resumption of the work of discovery, as in other creative epochs of Science.

As the supplement here brought out was the occasion of developing the theory of the Infinite Integral, by which alone the extreme complexity of the interpenetrating wave-fields of gravitation could be adequately treated, according to the methods of *Fourier*, I have dedicated the frontispiece of this paper to that unrivaled mathematician.

In view of the course of my own researches during the past forty years, no doubt it will seem very natural to add also the portrait of *Laplace*, in appreciation of the inspiration and guidance afforded in so many studies by the Mécanique Céleste, and the Théorie Analytique des Probabilités. For the mystery of the force of gravitation, so profoundly treated of in the works of *Newton* and *Laplace*, has been the subject of the writer's meditation for nearly half a century.

In reviewing the progress of these researches, I recall especially how much I was impressed by the conclusions of *Newton*, in the General Scholium to the Principia, 1713, that the force of gravitation penetrates to the very centres of the solid masses of the Sun and planets, without suffering the least diminution of its force; and by *Laplace*'s analogous reasoning that the Sun's gravitational action is transmitted undiminished through the great spheroidal mass of Jupiter, — otherwise very noticeable inequalities would be introduced into the laws for the motions of the satellites, when almost incessantly eclipsed in the shadow of that giant planet.

It was not, therefore, until Dec. 10, 1916, that I was able to break down the traditions of two centuries, by detecting periodicities in the lunar eclipse cycles, which indicated a slight screening effect due to the transmission of the Sun's gravitative force through the Earth's mass. It thus became possible to assign the physical cause of *Newcomb*'s unexplained fluctuations of the moon's mean motion (Electrodynamic Wave-Theory of Physical Forces, vol. I, 1917), which opened the way to a definite assignment of the cause of universal gravitation, more fully developed in the New Theory of the Aether, 1920–22, and now finally verified by the most incontestable proofs in this Eighth Paper.

In the earliest studies of the Principia, about 1885, it was noticed that the philosophy of *Newton* is concerned chiefly with forces, as we see clearly from his Preface written in 1686:

»For all the difficulty of (natural) philosophy seems to consist in this, from the phenomena of motions to investigate the forces of Nature, and then from these forces to demonstrate the other phenomena. And to this end, the general propositions in the first and second books are directed. In the third book we give an example of this in the explication of the system of the World. For by the propositions mathematically demonstrated in the first books, we there derive from the celestial phenomena, the forces of Gravity with which bodies tend to the Sun and the several Planets. Then from these forces by other propositions, which are also mathematical, we deduce the motions of the Planets, the Comets, the Moon, and the Sea. I wish we could derive the rest of the phenomena of Nature by the same kind of reasoning from mechanical principles. For I am induced by many reasons to suspect that they may all depend upon certain forces by which the particles of bodies, by some causes hitherto unknown, are either mutually impelled towards each other and cohere in regular figures, or are repelled and recede from each other; which forces being unknown, philosophers have hitherto attempted the search of Nature in vain.«

This last remark shows how deeply *Newton* was occupied with the laws of gravitational and even of molecular forces, and how he searched for the various causes involved. The Wave-Theory is the only valid explanation of physical forces, and of their modes of action, but it was very slow in being developed, for several reasons which merit careful analysis. The solid foundation which made the wave-theory possible was *Maxwell*'s masterly proof of the existence of an aetherial medium, in the celebrated Treatise on Electricity and Magnetism, 1873; yet after *Maxwell*'s premature death, 1879, the advance ceased, and about 1916–20, the movement temporarily became retrograde, under the pernicious influence of the doctrines of Relativity, which increased the obscurity and bewilderment when real light was needed.

In the Introduction of this Paper we have pointed out the difficulties experienced by *Newton* and his immediate successors, in attacking the problem of the cause of universal gravitation, and shown why the question was not discussed by *Laplace*, a century later — namely, because the illustrious author of the Mécanique Céleste was concerned chiefly with the verification of the Newtonian law, and its rigorous re-establishment for the great mass of phenomena made known by the progress of physical astronomy during the 18th century.

Now that about another full century has elapsed, since the researches of *Laplace*, is it not a little remarkable, in a matter affecting the progress of illumination, — tout ce qui peut contribuer au progrès des lumières, as *Laplace* says to *Napoleon*, in the dedication of the Théorie Analytique des Probabilités, 1812, — that among hundreds of investigators during this period the author of the New Theory of the Aether alone has attacked seriously *Newton*'s unfinished problem of the cause of universal gravitation?

It is noted that about two centuries have passed since the last labors of the founder of the theory of gravitation, yet it is remarkable that until the present effort, the problem of the cause of gravitation scarcely has been touched by a single person, in a way at all worthy of the dignity of the subject.

We may therefore appropriately conclude this paper by the remarkable observation of *Archimedes*, in a letter to *Dositheus*:

Πόσα γὰρ τῶν ἐν Γεωμετρίᾳ θεωρημάτων οὐκ εὐμέθοδ[…] ἐν ἀρχᾷ φανέντα, χρόνῳ τὰν ἐξεργασίαν λαμβάνοντ[…] Κόνων μὲν οὐχ ἱκανὸν λαβὼν ἐς τὰν ματεύσιν αὐτ[…] χρόνον, μετάλλαξεν τὸν βίον, καὶ ἄδηλα ἐποίησεν· κ[…] ταῦτα πάντα εὑρὼν, καὶ ἄλλα πολλὰ ἐξευρὼν, καὶ […] πλεῖον προάγαγε Γεωμετρίαν. Ἐπιστάμεθα γὰρ ὑπάρξασ[…] αὐτῷ σύνεσιν οὐ τὰν τυχοῦσαν περὶ τὸ μάθημα, κ[…] φιλοπονίαν ὑπερβάλλουσαν. Μετὰ δὲ τὰν Κόνων[…] τελευτὰν πολλῶν ἐτέων ἐπιγεγενημένων, οὐδ' ὑφ' ἑ[…] οὐδὲν τῶν προβλημάτων αἰσθανόμεθα κεκινημένον.

»In fact, how many theorems in Geometry which h[…] seemed at first impracticable are in time successfully wor[…] out! Now *Conon* died before he had sufficient time to inv[…] gate the theorems referred to; otherwise he would h[…] discovered and made manifest all these things, and would h[…] enriched Geometry by many other discoveries besides. I know well that it was no common ability that he brou[…] to bear on Mathematics, and that his industry was ex[…] ordinary. But, though many years have elapsed since *Con*[…]

death, I do not find that any one of the problems has been stirred by a single person.« (Sir *T. L. Heath*'s English edition of the Works of *Archimedes*, p. 151).

Apparently the tendency to follow the beaten path of Least Resistance applies not only to the physical systems of the universe, but also to the progress of science, both in the age of *Conon* and *Archimedes*, and throughout all time — except under the creative efforts of the most extraordinary discoverers during the past 2200 years!

Accordingly, the philosopher who dares to leave the smooth but barren path, for the untrodden, yet fruitful road to discovery; and in the difficult quest for illuminating the great secrets of nature, finally persists until he attains those summits near the stars, already is beyond all praise! Even the noblest verses for eternal work scarcely honor sufficiently the exalted merits of the discoverer of the highest laws:

Exegi monumentum aere perennius,
regalique situ Pyramidum altius. (Horace, Carmen XXX).

For as regards the sublimity of such research, and the dignity of the discovery of the laws of the universe, Dr. *Edmund Halley* justly declares in the Hexameter Verses prefixed to *Newton*'s Principia:

Nec fas est proprius mortali attingere divos.
Nor is it lawful for mortals to approach nearer the Deity.

Very grateful acknowledgements are due to several sympathetic associates, without whose kind encouragement it would have been difficult to complete this Supplement, with the detailed proofs and careful confirmation of the authorities quoted during the past 250 years: Rear Admiral *John H. Dayton*, U. S. N., Commandant at Mare Island; Dr. *C. B. Camerer*, U. S. N., whose words of encouragement proved as helpful as his eminent medical skill; Mr. *A. E. Axlund*, experienced Civil Engineer, who has read the whole paper with an interest appropriate to our frequent conferences during the past 17 years; Professor *J. S. Ricard*, S. J., Director of the Observatory, Santa Clara, who for years has followed closely all the researches made at Mare Island; Mr. *W. S. Trankle*, Assistant, whose timely aid in the preparation of the Manuscript has been almost invaluable; and above all to Mrs. *See*, who has contributed most to lengthen those hours of quiet study without which it would have been impossible to record the traces of light as it came traveling downward from the stars.

Vallejo, California, June 25, 1925.

T. J. J. See.

Ausgegeben 1926 Febr. 27.

T. J. J. See. New Theory of the Aether.

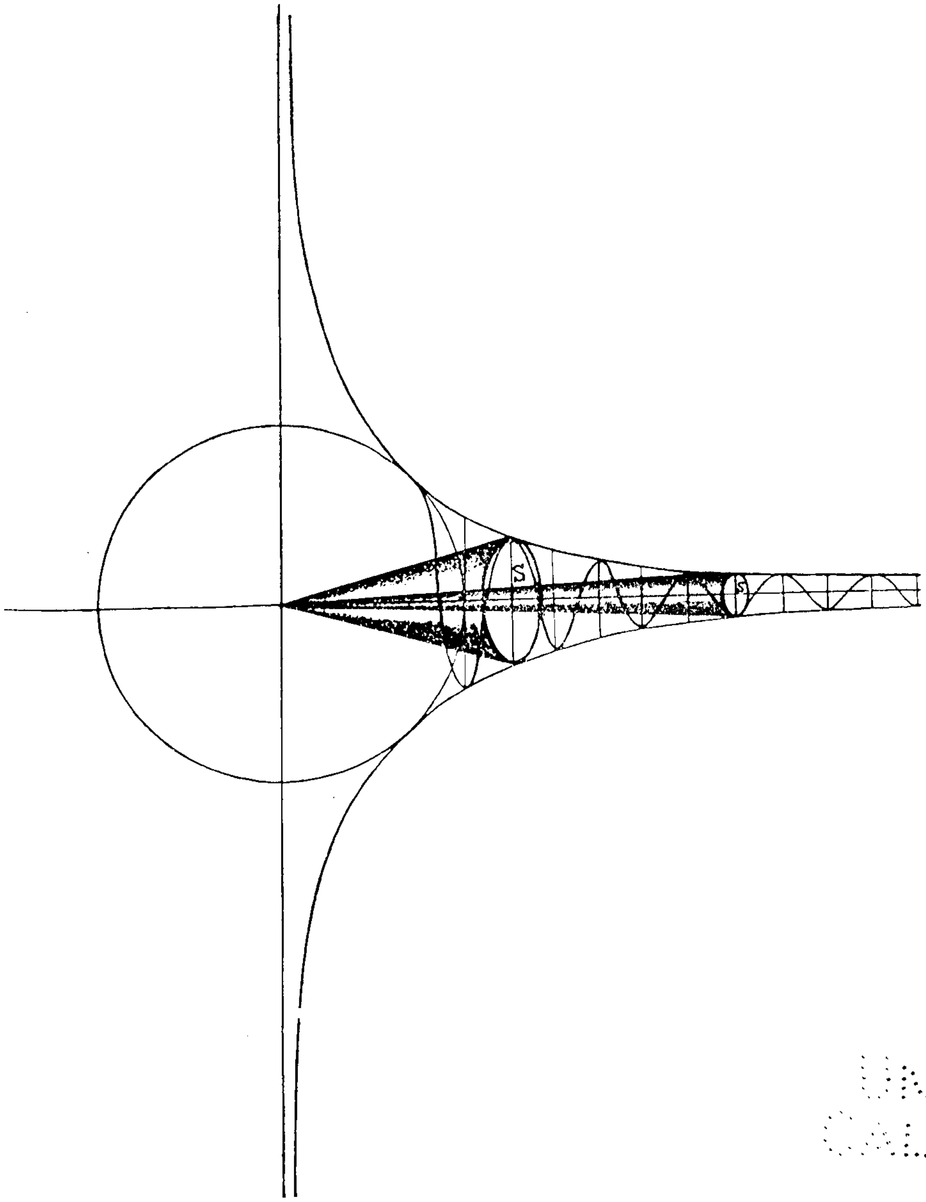

Plate I. Illustration of the Law of Amplitude $A=k/r$, and of the diminishing cross sections of the waves, shown by the bases of the cones S and s, which give the relative energies, $f=k^2/r^2$, of the waves at all distances. These contracting cones indicate the Geometrical and Physical Basis of the Newtonian Law much more clearly than is shown by the considerations employed by *Halley* and *Newton*, 1684; and point unmistakably to the Wave-Theory of Gravitation, which is now proved to have chances in its favor equivalent to ∞^{100} to 1, compared to any other conceivable cause.

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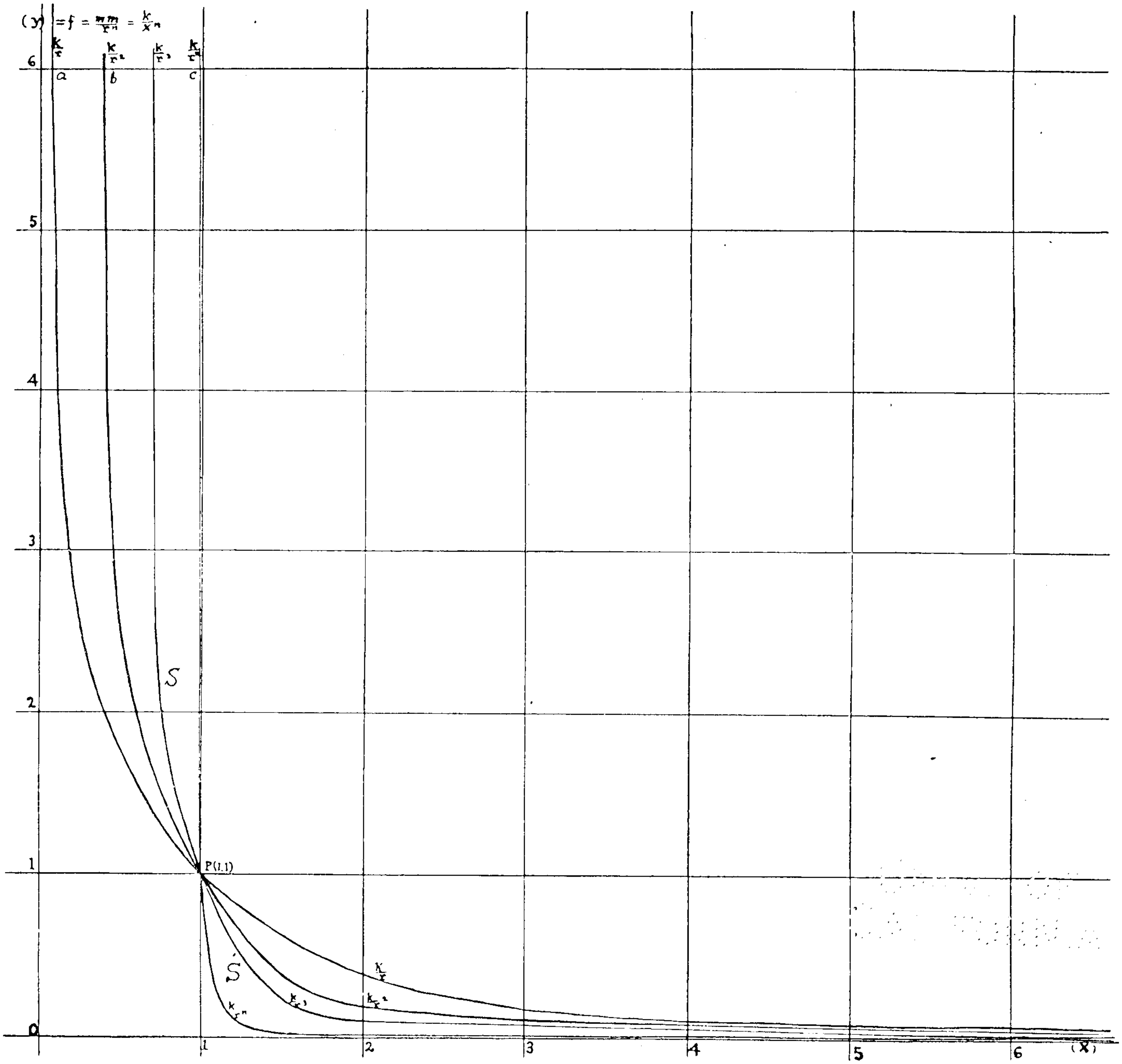

Plate II.

Illustration by curves of the Doubly Infinite Variety of the Functions $f=k^2/r^{n+\nu}$ which might be imagined to represent Forces varying as the inverse powers of the distances. In Nature we find $n+\nu=2.0000000000\cdots$, and therefore we here examine the occasion for the restriction of the Forces to this one integral inverse power, $n=2$.

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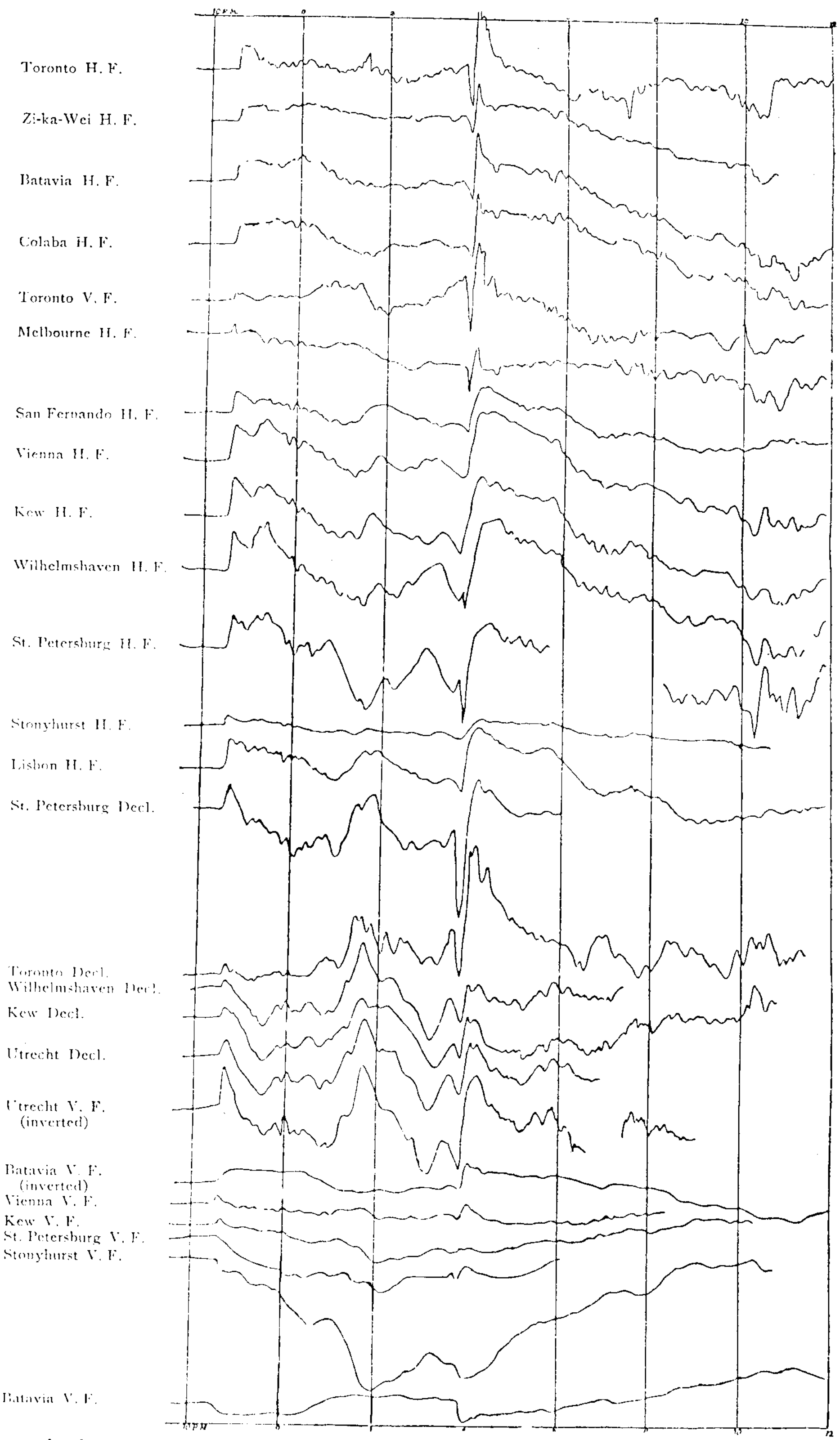

Plate III. Graphical record of the simultaneous disturbances of the Earth's magnetism, at different observatories, in the great »Magnetic Storm« of June 25, 1885, reduced to Greenwich Mean Time, by Professor *W. Grylls Adams*, Phil. Trans., 1892, A, Plate 8.

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Fig. 176. Art. 892.

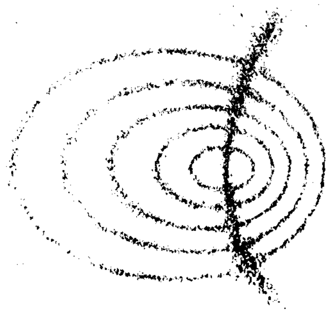

Fig. 177. Art. 893.

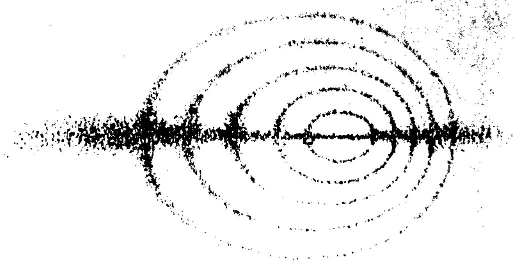

Fig. 179. Art. 900.

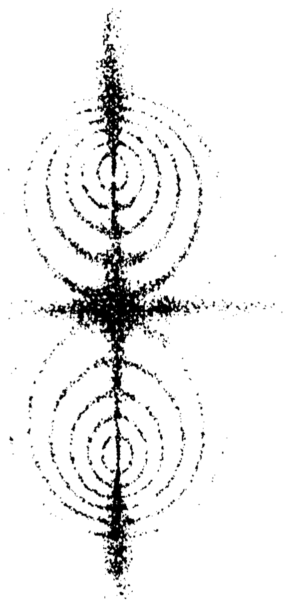

Fig. 180. Art. 901.

Fig. 181. Art. 900.

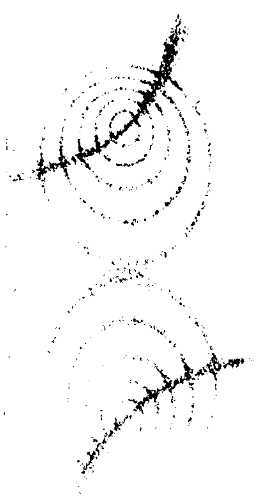

Fig. 182. Art. 900.

Fig. 183. Art. 902.

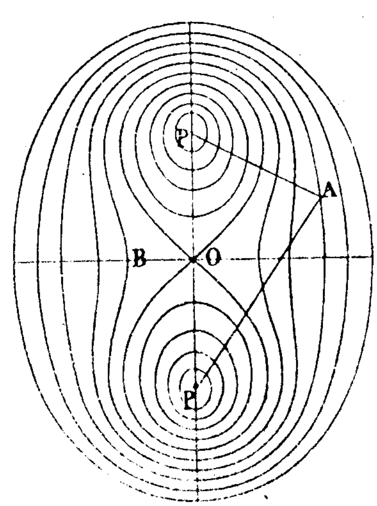

Plate V.

Illustrations of the Lemniscate of *Bernoulli*, as used by Sir *John Herschel* in the study of certain crystals under Polarized Light, for confirming the accuracy of the Wave-Theory. As the optical patterns of the Interference Fringes conform exactly to the Curve of the Lemniscate. *Herschel* held that the observations fulfilled all requirements of the Theory of Undulations, which confirmed the researches of *Fresnel*.

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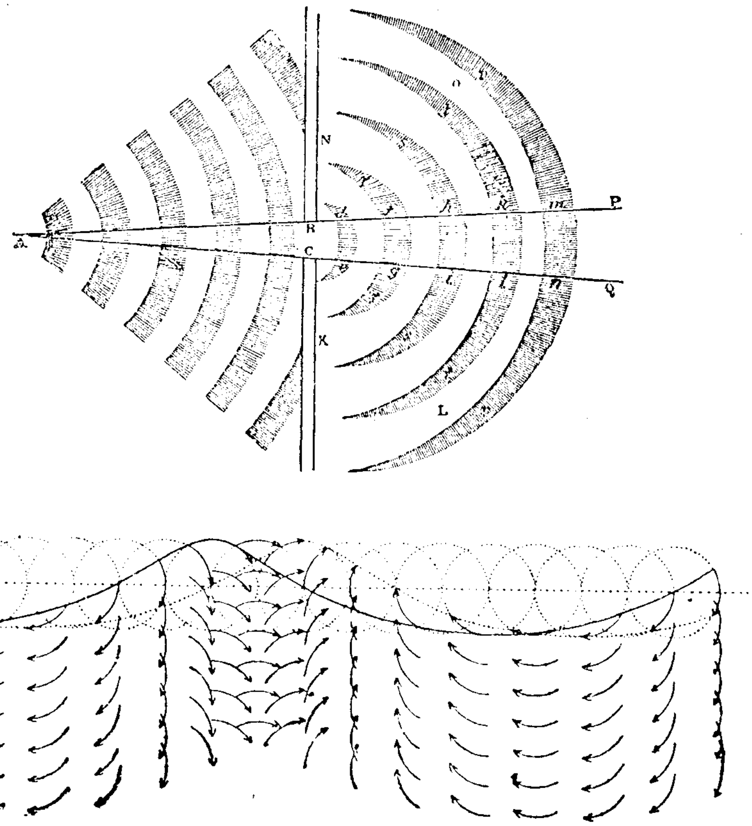
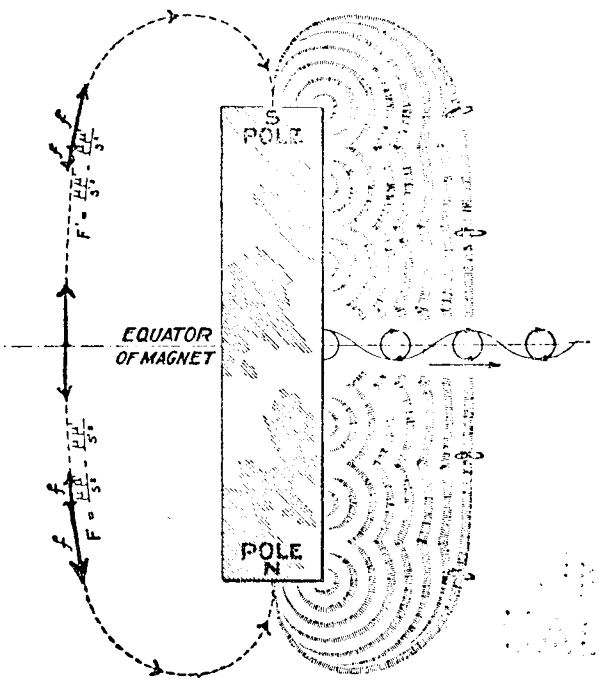


(a)



(b)

Plate IV. (a) *Newton*'s diagram of the spread of waves to a new radius, after passing through an orifice *B C*; with *Airy*'s illustration of the nature of the wave-motion below (cf. Tides and Waves, 1845).

(b) Illustration of the simultaneous compounding of wave motions from closely adjacent orifices, by extension of *Newton*'s theory. As the rotations about the axes are parallel, the tension in each wave-disturbance tends to shorten the arc of the whole wave filament, and make the wave front a minimum. We thus get magnetic lines of force nearly straight in the equatorial regions of the magnet, with rapid curvature towards the poles. This lower diagram (b), in connection with the diagram (a) above, completely explains the observed phenomena in magnetism, and assures us that the wave-theory assigns the true cause of magnetism. Drawn by *J. F. Greathead*.

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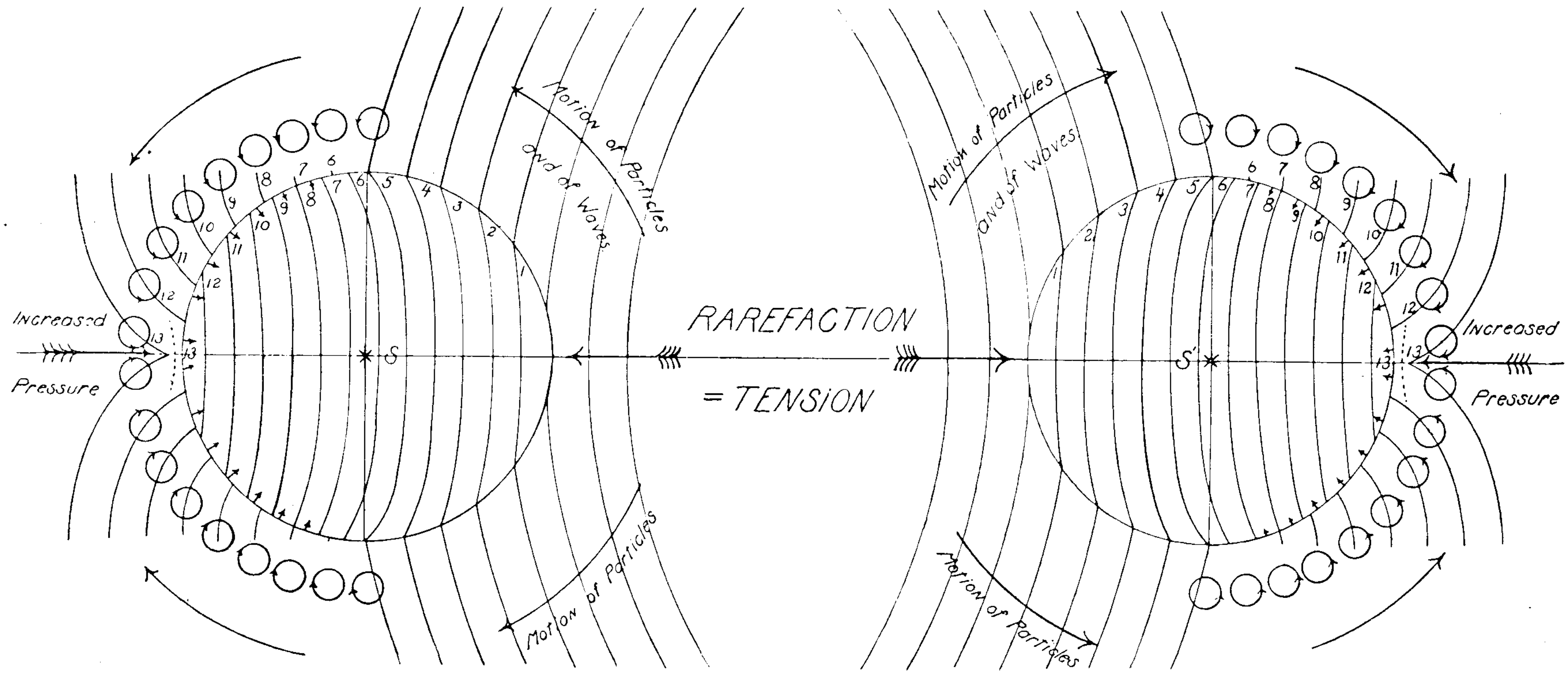

Wave-Theory of Accoustic Attraclion.

T. J. J. See, 1917.

Plate VI.

Diagram showing the working of the Wave-Theory, for the case of Acoustic Attraction. The Sound Waves go out from each centre, and are propagated through the other balloon filled with Carbon Dioxide Gas, but advance more rapidly through the Air outside, and thus the wave-disturbance is bent around to the rear of each balloon, before it reaches there through the CO_2. Two remarkable results follow: 1. Under the waves many particles of Air are worked out from between the balloons, and this develops a Tension, drawing the balloons together. 2. The waves operate to work particles of Air into the rear, beyond either balloon; and the outcome is Increase of Pressure there, as each wave passes and reacts on the elastic surface of the balloon, by a Wave Impulse from behind — a vis à tergo, a shove from behind, as it were. The development of Tension between, and Impulsive Increase of Pressure behind, explains Acoustic Attraction, and also the Cause of Universal Gravitation, in which two stars are under similarly working Aether Waves and identical Forces.

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T. J. J. See. New Theory of the Aether.

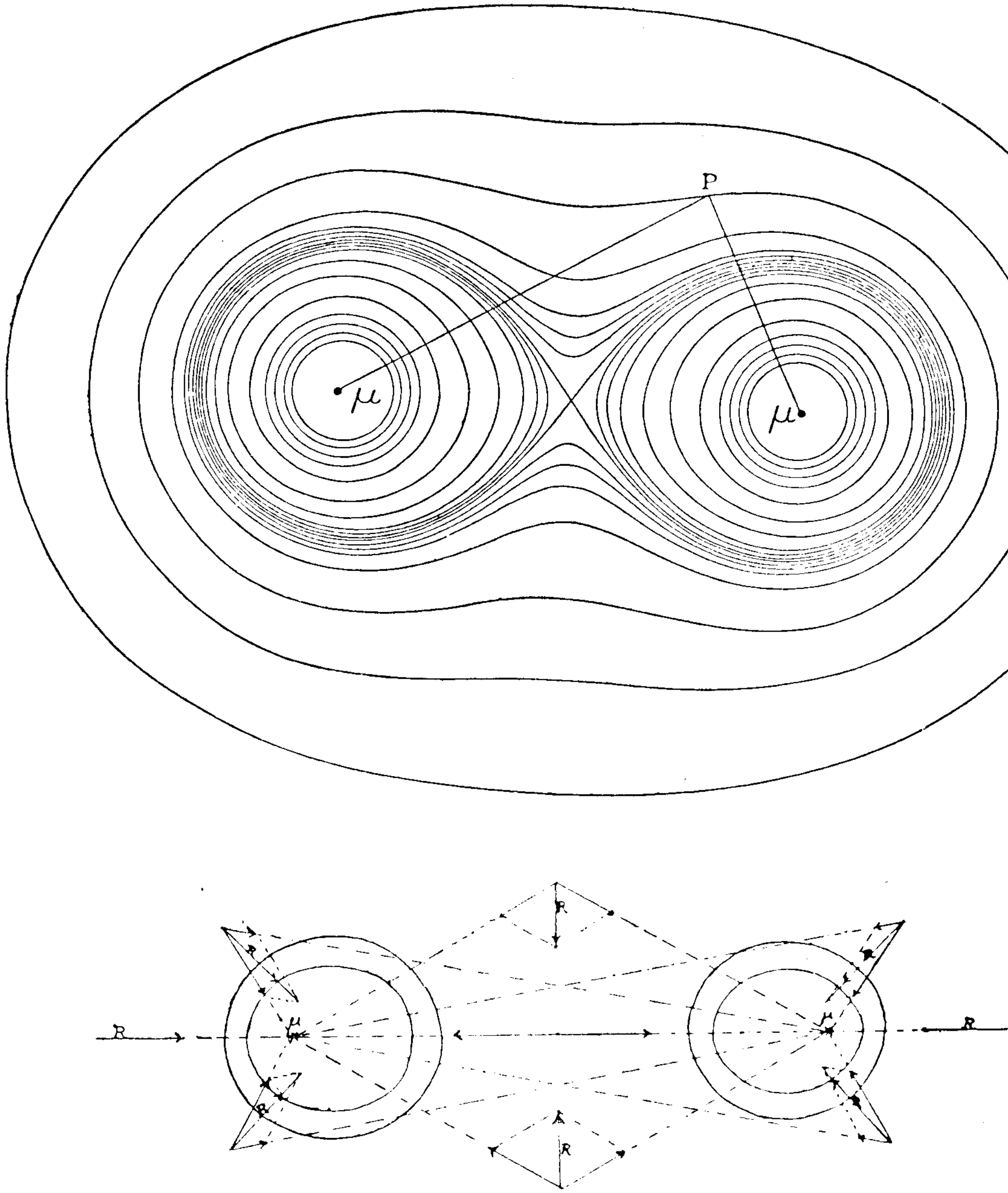

Plate VII. Diagram of the equipotential surfaces about two equal masses, μ and μ, originally given in *Thomson* and *Tait*'s Treatise on Natural Philosophy, 1st ed., 1873. Without regard to the cause involved this upper diagram represents the actual surfaces which exist under the potential of gravitation to two equal stars; but in the light of the New Theory of the Aether we may now interpret the meaning of the distortions of the surfaces shown, which were first published about half a century ago.

In the lower figure we see how the vectors directed to the two equal stars are compounded geometrically, in every part of the diagram, according to the law expressed in equation (139). This illustrates the whole theory of the equipotential surfaces, about two equal masses, and constitutes as triumphant a verification of the wave-theory as the incomparable Geometry of *Newton*'s Principia does for the law of universal gravitation, now referred at last to its true cause.

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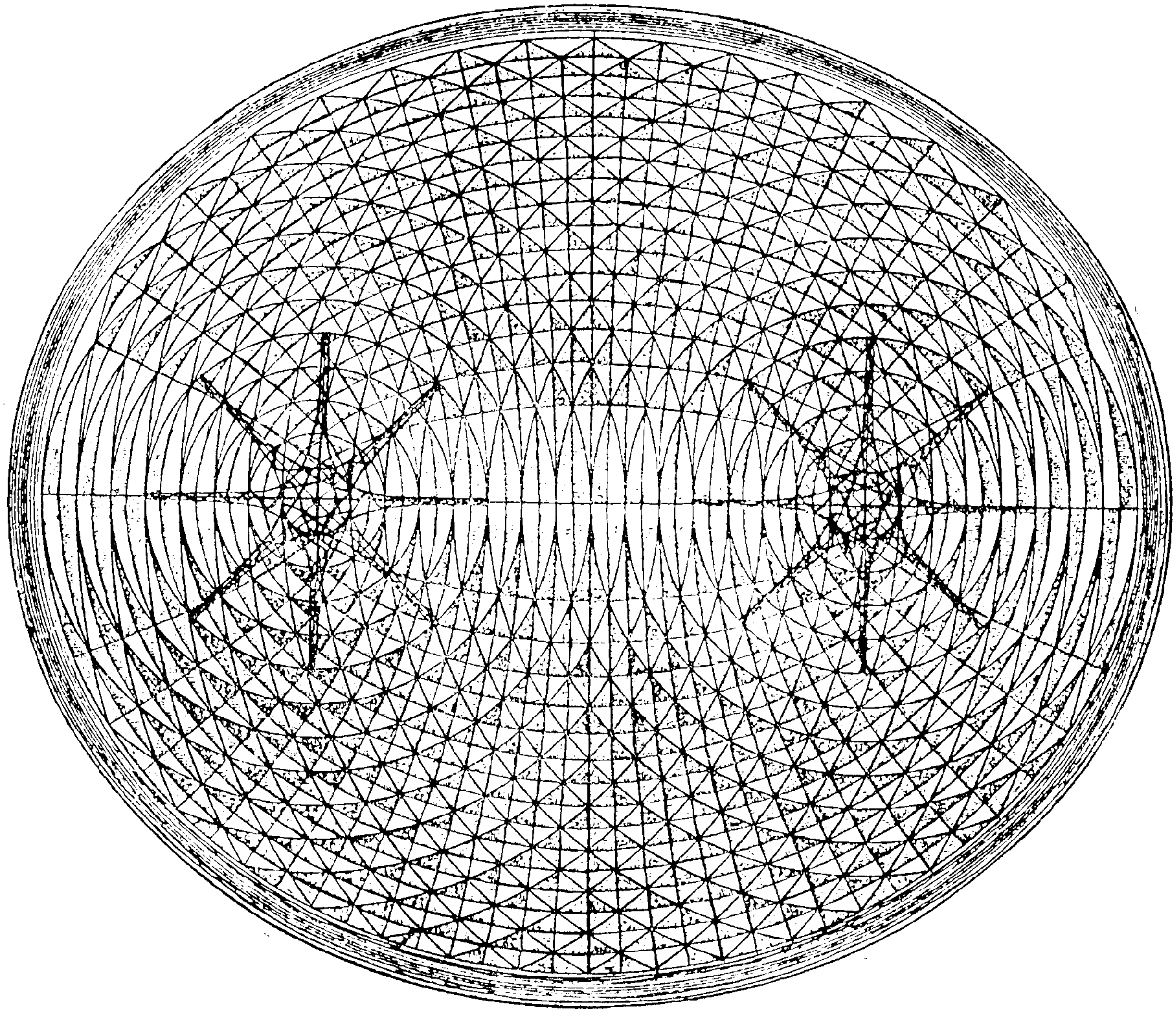

Plate VIII. Geometrical illustration of the wave-field about two equal stars. The wave-amplitudes increase asymptotically towards either body, which renders the aether of variable density $\sigma = \nu r$, while the wave-motion in concentric spheres, when reflected from the surfaces of the confocal ellipsoids, yields stresses along the tangents to the hyperboloids, which intersect the ellipsoids at right angles and with them constitute the system of confocal conics.

In nature the aether waves from the two centres are not reflected by the ellipsoidal surfaces, but proceed onward into infinite space; yet the reaction of the medium gives the stresses along the tangents to the hyperboloids exactly the same as if the waves were perfectly reflected by the confocal ellipsoids, and thus the state of wave-motion is rendered perpetual.

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