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New Theory of the Aether. By *T. J. J. See.*

(Eighth Paper, A Supplement on the Discovery of the Cause of Gravitation.)

(With 8 Plates and 2 Portraits.)

Analysis of the Progress of the Problem.

In the preceding papers I have discussed somewhat carefully the nature of gravitation as a wave-phenomenon, and in the Seventh Paper, Oct. 22, 1922, finally connected this force with magnetism, under a law of nature, which is verified in the physical theory of the Earth and Sun. The remarkable result thus attained is highly satisfactory, especially in breaking down the historic isolation of gravitation, from the electrodynamic and related forces, which *Faraday* always held to be possible.

This question, however, long was neglected, very largely because the wave-fields of the aether are wholly invisible. Yet in dealing with those subtile phenomena of nature which are not disclosed to mortal sight our processes of investigation obviously ought to be penetrating and so exhaustive that nothing is left undone, even when it becomes advisable to invoke the calculus of probability.

It is well known that from the beginning of his great discoveries — such as the Invariability of the major axes and mean motion of the planets, 1773; the Dynamical Theory of the Tides, for the motions of our sea, 1777; the Invariable Plane of the solar system, and the Physical Explanation of the laws of the motions of Jupiter's satellites, 1784; the Cause of the Great Inequality of Jupiter and Saturn, 1785; the Cause of the secular acceleration of the Moon's mean motion, 1787 — *Laplace* always studied the phenomena of nature from the point of view of the theory of probability. His repeated triumphs where such great masters as *Euler, Clairault, D'Alembert*, and *Lagrange* had failed, are sufficient testimony to the value of the method. It is therefore not surprising to find the following reasoning in the Théorie Analytique des Probabilités:

»We are so far from knowing all the agencies of nature, and their various modes of action, that it would be unbecoming in a philosopher to deny the phenomena simply because they are inexplicable in the present state of our knowledge. Only we must examine them with an attention which is the more scrupulous the more difficult it appears to be to admit them. And the calculus of probabilities becomes indispensable for determining how far the observations must be multiplied in order to obtain in favour of the agents indicated by them a probability superior to the reasons one may have for rejecting them.«

Accordingly, in view of the difficulties continually encountered by investigators since the age of *Newton*, it seems to me that the hidden processes of gravitation offer an especially suitable field for the use of the method of the calculus of probability; and thus it is desirable to add a supplementary Eighth Paper, dealing comprehensively with this subject.

In the 240 years since the preparation of the Principia, 1684–86, two distinct epochs may be recognized:

(a) The Century of Verification of the Newtonian Law — including the celebrated analytical researches of *Clairault, Euler, D'Alembert, Lagrange* and *Laplace* — which may be said to close, exactly 100 years after the publication of the Principia, in *Laplace*'s triumphant discoveries of the cause of the great inequality of Jupiter and Saturn, 1785, and of the secular acceleration of the Moon's mean motion, 1787 — yet with the results given final form in the Mécanique Céleste vols. I and II, 1799, vol. III, 1801, vol. IV, 1805, vol. V, 1825.

This remarkable period of verification, which even later culminated in the greatest of all these triumphs, — the theoretical discovery of Neptune from the perturbations of Uranus, by *Adams* and *Leverrier*, 1846 — was characterized by a concentration of effort on the analytical consequences of the law of universal gravitation, with little or no thought of the physical cause which might underlie this great law of nature.

The general assumption of investigators found expression in the opening paragraph of the Mécanique Céleste, 1799, that »the nature of that singular modification, by means of which a body is transported from one place to another, is now and always will be unknown; it is designated by the name of Force. We can only ascertain its effects, and the laws of its action«.

(b) The Century of Electrodynamic Researches, beginning with the experiments of *Oersted*, 1819, which very soon found expression in the laws of *Ampère*, and led to the long series of experimental discoveries by *Faraday*, the mathematical theory of the Earth's magnetism, by *Gauss*, 1838, and the fundamental electrodynamic law, discovered by *Weber*, 1846 — the whole progress culminating in *Maxwell*'s attempt at a mathematical interpretation of the observations of *Faraday* during the last years (1860–1866) of that eminent investigator.

It is well known that during this later period *Faraday* was deeply impressed with a connection between electrodynamics, magnetism and gravitation. At his advanced age, however, the experimental attacks on the problem failed, and even *Maxwell*'s brilliant work was left incomplete at his premature death in 1879. Yet *Maxwell*'s celebrated Treatise on Electricity and Magnetism, 1873, became a standard work on the subject, and is not yet supplanted by a more comprehensive treatment of the whole question.

Turning aside from the promising start thus made by *Faraday* and *Maxwell*, a few investigators, some twenty years ago, apparently forgot the lessons of History, and began a movement for explaining many related phenomena of nature by the Theory of Relativity: yet it proved to be so clearly retrograde that it failed, and only led to the diffusion of a mass of errors which never should have been given currency in the literature of science. Such an example of retrogradation as the recent ill-advised attempt to modify the Newtonian law, without really understanding its geometrical, physical and historical significance, may serve to emphasize the importance of a more diligent search for the cause of gravitation, along the electrodynamic lines already begun by *Faraday*, when *Maxwell* was a young man, yet neglected for sixty years, till the appearance of the New Theory of the Aether, 1922.

Contemplating therefore the entire period of 240 years since the preparation of the Principia, it is apparent that the greatest early cultivator of the analytical theory of gravitation was *Euler*. Then came the complete restoration of the Newtonian law, under the profound physical intuition of *Laplace*, as finally set forth in the Mécanique Céleste.

When occupied with any very difficult problem it was *Laplace*'s habit to exclaim: »Read *Euler*! read *Euler*! he is our master!« Likewise, in comparatively recent times, Dr. *G. W. Hill* related that the germs of the method employed in his famous Researches in the Lunar Theory, 1877, already had found partial expression in a paper of *Euler*, whose Collected Works happily are reappearing, and again coming within the reach of the modern reader.

But great as was the analytical mastery of *Euler*, it is recognized that the physical intuition and dynamical judgement of *Laplace* was indispensable to the final triumph of the Newtonian law.

Accordingly in our day the great outstanding problem of Astronomy is the cause of universal gravitation. This is an inheritance from the great masters of former ages, and we cannot shirk it.

Although this problem of the cause of gravitation profoundly engaged the attention of *Newton*, a solution of it was not possible in his time; yet he correctly attributed gravitation to impulses (waves) in the aetherial medium, but it was exactly two centuries after the appearance of the 3rd edition of the Treatise on Opticks, 1721, before it became possible to resume *Newton*'s unfinished work, by developing the Kinetic Theory of the Aether.

In view of this lapse of time the thoughtful investigator should not hesitate to go back to the greatest outstanding mystery of the universe since the age of *Newton*. It relates to the nature of universal gravitation, under which the heavenly bodies everywhere are mutually drawn together in pairs, and often physically coupled into revolving systems of Binary Stars. The physical bond uniting our planets to the Sun, the satellites to their several planets, and the components of the physically coupled double stars, one to another, is the mysterious power of gravitation.

The attractive force is so stupendous as to exceed the tensile strength of millions of immense cables of the strongest steel, yet conveyed in right lines through the all-pervading medium of the aether. This subtile medium is wholly invisible, yet adequate to bind together the whole structure of the starry heavens, by wave-action everywhere pursuing minimum paths, — and thus exerting mutual tensions on the heavenly bodies, in lines as rectilinear as delicate spider webs joining them in pairs.

It was feared by Sir *Isaac Newton* that under the mutual gravitation of the heavenly bodies the existing order of the solar system eventually might be in danger of subversion, and require the exertion of a restorative power. By improving and extending the Newtonian theory, *Euler* was able to discover that the mutual gravitation of the planets itself is largely preservative of the order of the solar system; and when *Euler*'s processes were still further extended by *Lagrange*, *Laplace* and *Poisson*, it became known that the order of the solar system is essentially stable, if not forever, at least for millions of ages. Thus on June 15, 1829, *Fourier* could exclaim in his Eloge Historique de *Laplace*:

»In general nature holds in reserve conservative and ever-present forces which act as soon as the trouble begins and augment in proportion as the disturbance is greater. They quickly re-establish the accustomed order. We find in all parts of the universe this preservative power. The forms of the great planetary orbits and their inclinations vary and oscillate in the course of ages: but these changes are limited. The principal dimensions are maintained and this immense assemblage of heavenly bodies oscillates about a mean position toward which it is always carried. Everything is adjusted for order, perpetuity and harmony.« . . . .

»Whatever may have been the physical cause of the formation of the planets, it imparted to all the bodies a projectile motion in a common direction about an immense globe; in this way the solar system has been rendered stable. The same effect is produced in the systems of satellites and of rings. Order is there maintained by the power of the central mass. Hence, there is not, as *Newton* himself and *Euler* had suspected, an adventitious force which should one day repair or prevent the disturbance which time had wrought. This is the law of gravitation itself, which rules all, is sufficient for all, and maintains order and variety. Having proceeded once only from supreme wisdom, it has presided since the origin of time, and renders all disorder impossible. *Newton* and *Euler* did not yet recognize all the perfections of the universe. In general whenever any doubt has arisen in regard to the exactness of the Newtonian law, and, for the explanation of apparent irregularities, we have proposed the accession of a foreign cause, it has always happened, after a profound examination, that the original law has been verified. It explains to-day all known phenomena. The more precise the observations, the better do they conform to theory. Of all the great mathematicians, *Laplace* is the one who has penetrated most profoundly these great questions; he has, so to speak, settled them.«

The mystery surrounding the cause of universal gravitation in *Newton*'s time was so extreme, that although the illustrious author of the Principia ascribed gravitation to impulses (waves) in the subtile aethereal medium that is diffused over the universe, yet as he could not make out the mode of operation, he would frame no hypotheses. The cautious attitude of the founder of the theory of universal gravitation extended to his successors, and even to *Laplace*, the great restorer and perfector of the Newtonian theory.

After the researches of *Faraday* and *Maxwell*, however, a less cautious and less despairing attitude became allowable: so that in the researches on the Electrodynamic Wave-Theory of Physical Forces, begun in 1914, and further developed in the New Theory of the Aether, 1920–1922, I was able to recognize in wave-action the cause of universal gravitation, and to adduce satisfactory proof that this force is transmitted across space with the velocity of light.

This proof is based on a great mass of closely related phenomena — in astronomy, magnetism, electrodynamics, radio-telegraphy, magneto-optics, and other branches of physics; so that perhaps little doubt will exist in the mind of any experienced natural philosopher that the real cause of gravitation is correctly assigned.

Yet it may be well to elucidate the discussion still further, by the researches made during the past three years. Among the grounds for such procedure we cite the following papers by two celebrated German mathematicians:

1. *C. F. Gauss*, Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstoßungskräfte, Resultate aus den Beobachtungen des Magnetischen Vereins im Jahre 1839, Leipzig, 1840, reprinted in *Gauss'* Werke, Bd. 5, 197–242.

2. *P. G. Lejeune-Dirichlet*, Vorlesungen über die im umgekehrten Verhältnis des Quadrats der Entfernung wirkenden Kräfte, Leipzig, 1876.

These mathematical researches by *Gauss* and *Dirichlet* are of great elegance and extreme generality, and leave very little to be desired as respects the geometrical consequences of forces varying inversely as the square of the distances. Yet *Gauss* and *Dirichlet* are as silent as *Newton* himself in respect to the causes which might underly forces varying in the ratio of the inverse squares.

MacLaurin, who was a friend and disciple of *Newton*, says of his great master: »He found he was not able, from experiment and observation, to give a satisfactory account of this (subtile aetherial) medium, and the manner of its operation, in producing the chief phenomena of nature«. (Account of the Philosophical Discoveries of Sir *Isaac Newton*, London, 1748, p. 111.)

Accordingly, as *Newton* could not make out the manner of the operation of the impulses or waves in the aether, to which he ascribed gravitation, and *Gauss* and *Dirichlet* have treated almost wholly of the geometrical consequences of laws of the Newtonian type, $f = mm'/r^2$, yet have not attempted to assign the physical cause of such forces in nature, that unsolved problem obviously deserves increased attention. For if a careful analysis should definitely disclose the cause of great laws of nature, heretofore hidden from mortal eye for centuries, the result would be eminently worthy of the meditation of the geometer! No triumph could be more welcome to a philosopher occupied with the observed order of the heavens!

It has been customary, from the time of *Newton*, to ascribe the gravitation of the planets to some power emanating from the Sun, and to imagine that as the area of the sphere surfaces in space increases outwardly directly as the square of the distance, the intensity of the influence controlling the motion of the planet ought to decrease with increasing distance in the inverse ratio of the square of the distances.

Such general reasoning appears to be essentially sound, as noted by *Whewell* in his History of the Inductive Sciences, 3rd edition, 1857, vol. II, Lib. VII, ch. I and II, pp. 114–140. Yet as this subject very seldom is viewed in its true historical development, we shall quote *Whewell*'s discussion in sufficient detail to show how the law of the inverse squares first arose:

»The proposition that the attractive force of the sun varies inversely as the square of the distance from the centre, had already been divined, if not fully established. If the orbits of the planets were circles, this proportion of the forces might be deduced in the same manner as the propositions concerning circular motion, which *Huyghens* published in 1673; yet it does not appear that *Huyghens* made this application of his principles. *Newton*, however, had already made this step some years before this time. Accordingly, he says in a letter to *Halley*, on *Hooke*'s claim to this discovery (Biog. Brit., art. *Hooke*). 'When *Huygenius* put out his Horologium Oscillatorium, a copy being presented to me [1]), in my letter of thanks I gave those rules in the end thereof a particular commendation for their usefulness in computing the forces of the moon from the earth, and the earth from the sun'. He says, moreover, 'I am almost confident by circumstances, that Sir *Christopher Wren* knew the duplicate proportion when I gave him a visit; and then Mr. *Hooke*, by his book Cometa, will prove the last of us three that knew it'. *Hooke*'s Cometa was published in 1678. These inferences were all connected with *Kepler*'s law, that the times are in the sesquiplicate ratio of the major axes of the orbits. But *Halley* had also been led to the duplicate proportion by another train of reasoning, namely, by considering the force of the sun as an emanation, which must become more feeble in proportion to the increased spherical surface over which it is diffused, and therefore in the inverse proportion of the square of the distances. (*Bullialdus*, in 1645, had asserted that the force by which the sun 'prehendit et harpagat', takes hold of and grapples the planets, must be as the inverse square of the distance.) In this view of the matter, however, the difficulty was to determine what would be the motion of a body acted on by such a force, when the orbit is not circular, but oblong. The investigation of this case was a problem which, we can easily conceive, must have appeared of very formidable complexity while it was unsolved, and the first of its kind. Accordingly, *Halley*, as his biographer says, 'finding himself unable to make it out in any geometrical way, first applied to Mr. *Hooke* and Sir *Christopher Wren*, and meeting with no assistance from either of them, he went to Cambridge in August (1684), to Mr. *Newton*, who supplied him fully with what he had so ardently sought'.«

This record of the early history of the law of the inverse squares shows that the development of it extended from *Bullialdus*' work of 1645 to the confirmation and general proof of the law in the Principia, 1687, where *Newton* first showed that it explains the motions of the planets in elliptical orbits, as well as the motions of the comets in the three conic sections.

Whewell's citation of *Halley*'s early view that gravitation is an emanation, and therefore decreases with the geometrical

[1]) This original presentation copy, with *Huyghens*' autograph (Pour Monsieur *Newton*) still plainly legible, is now in the writer's library, — a confirmatory circumstance which increases the historical interest in *Newton*'s letter to *Halley*, with respect to the earliest formulation of the law of the inverse squares. (Note added July 2, 1925.)

expansion of the spherical surfaces over which it is diffused, as we recede from the Sun, is important, since in his perplexity, as to the law of nature, he visited *Newton* at Cambridge, August, 1684, and then induced that great philosopher to prepare the Principia, 1686, as finally published at *Halley*'s personal expense, 1687.

It is doubly gratifying to find that the views entertained by *Halley* and *Newton*, as *Whewell* explains, are justified — above all, because under contraction the cross sections of all waves receding from a centre are now shown, by new illustrations, to follow rigorously the law of the inverse squares. Gravitation therefore incontestably is a wave-phenomenon!

Our conclusion in confirmation of the wave-theory is the more satisfactory, since it is shown at the close of this paper, that the theory of ultra-mundane corpuscles, originally put forth by *Fatio de Duillier*, and Abbe *Varignon*, after communication with *Newton*, but re-invented half a century later by *Le Sage* of Geneva, and published by *Prevost* at Paris in 1818, is permanently overthrown, by definite criteria of undoubted validity. Hence the wave-theory alone survives, is sufficient to explain all phenomena, and therefore, attains a complete triumph!

Unfortunately at the present time it appears that very few astronomers or geometers have been trained to look upon the cause of universal gravitation as discoverable, whereas it seems that this discovery is attainable along the very lines of experimental and mathematical inquiry considered most suitable by *Halley* and Sir *Isaac Newton*, 1684.

And if the clear recognition of the cause of gravitation should make it possible for us to ascertain the reason for the celebrated theorems discovered by *Laplace*, 1773, that to the first power of the masses the major axes and mean motions of the planets are invariable, and the similar beautiful theorem discovered by *Poisson*, 1808, that the constancy of these elements holds also to the second powers of the masses — two theorems which we now generalize to all powers of the masses, so that we may declare the absolute invariability of the planetary axes and mean motions, so long as the masses are invariable — the outcome would be doubly interesting to all investigators of the geometry of the heavens.

No problem could be more worthy of the contemplation of astronomers and natural philosophers than the demonstrated cause of the forces on which the order and stability of the universe depends.

Part I. An Investigation of the Physical Cause underlying the Forces of Nature which follow the Law of the Inverse Squares.

1. The Geometrical Significance of the Function $f = k^2/r^n$, or $y = k^2/x^n$.

Apparently it is not generally realized that the Newtonian law can be expressed rigorously only by means of a sextuple integral, which takes account of every particle in each of the two attracting masses:

$$\begin{aligned} dm &= \sigma\, dx\, dy\, dz \qquad & m &= \iiint \sigma\, dx\, dy\, dz \\ dm' &= \sigma'\, dx'\, dy'\, dz' \qquad & m' &= \iiint \sigma'\, dx'\, dy'\, dz' \end{aligned} \tag{1}$$

at the squares of their several mutual distances,

$$r^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2 \tag{2}$$

thus:

$$\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int \frac{\sigma\, dx\, dy\, dz \cdot \sigma'\, dx'\, dy'\, dz'}{(x'-x)^2 + (y'-y)^2 + (z'-z)^2} = f = m\,m'/r^2. \tag{3}$$

The usual expression for the gravitational force is the latter simplified integral form, $f = m\,m'/r^2$, which is accurate only for spherical masses made up of concentric layers of uniform density, or for the individual particles regarded as points.

Under these restrictions the law of gravitation has the integral form, $f = m\,m'/r^2 = k^2/r^2$, where k is the Gaussian constant. For the sake of clearness and simplicity, let us represent this integral function graphically on the plane of xy. Then the general form will be

$$y = f = k^2/r^n = 1/x^n \tag{4}$$

when we put $k = 1$, which is the simplest case.

1. The case where $n = 1$, a rectangular hyperbola. If $n = 1$ we find $y = 1/x$, and thus the curve obviously is a rectangular hyperbola referred to its asymptotes. It passes through the point $p\,(x,y) = p\,(1,1)$, and is shown in the accompanying figure Plate II, by the line through the point a which is symmetrical with respect to the two axes.

2. Having found the geometrical representation of the function $y = 1/x$, we naturally extend the inquiry to other integral values of the exponent n, and thus take $n = 2$. The case $n = 0$ gives no curve, but only a horizontal line through the point $p\,(1,1)$, and may be dismissed from consideration.

The function $y = 1/x^2 = f = k^2/r^2$ is shown by the next curve through the point b, which cuts that of $y = 1/x$ in such a way that for values of y greater than 1, the curve is above the rectangular hyperbola referred to its asymptotes, and for values less than 1, it is below the rectangular hyperbola.

3. In a similar manner the function $y = k^2/r^3 = 1/x^3$ is shown by the next line indicated in the diagram. Thus this curve is similar to that of $y = 1/x^2$, but with the departure from the rectangular hyperbola more accentuated. A curve $y = 1/x^4$ would be yet more separated from the rectangular hyperbola, and so on, $n = 5, 6, \ldots\ldots n$. It is scarcely worth while to pursue this series of higher curves, since we see the tendency resulting from increasing n, from $n = 1$, to $n = 4$.

4. If we extend this reasoning to higher values of n, we find that, when n is very large, the function $y = k^2/r^n = 1/x^n$ is the limiting curve above the point $p\,(1,1)$, and below descends rapidly, as shown in the diagram. It must be conceived as very nearly parallel to the axis of y, and close to the x-axis, except near the point $p\,(1,1)$. For $n = \infty$ the curve would coincide in direction with the axes, except near the point $p(1,1)$.

5. From this reasoning it will be seen that a series of lines may be drawn in the plane (x,y), for the function of the force varying with the inverse powers of the distances, $f = k^2/r^n$. For the integers $n = 1, 2, 3, \ldots . n$, we get the isolated curves shown. They are separated by finite intervals at all points of the plane (x,y), except at the point of intersection $p\,(1,1)$, and are easily understood from the accompanying figure Plate II. The lines thus intersect at one point only, $p\,(1,1)$, but are otherwise spread over the plane (x,y), within a certain

region S, lying above the rectangular hyperbola for $y=1$, to $y=\infty$; and below the said hyperbola S' is the region for values of y less than 1.

6. Now let our function $y=k^2/x^n$ be given the more general form

$$y = k^2/x^{n+\nu} \qquad (5)$$

where ν is a decimal fraction, $\nu=0$, to $\nu=1$. This will give continuity to the function between the integral values $n=1, 2, 3, 4, \ldots n=\infty$.

Consider then the decimal fractional exponent ν, $\nu=0.0000 \ldots 01$, $\nu=0.9999 \ldots 9$, $\nu=0$, $\nu=1$. By making ν vary continuously from $\nu=0$, to $\nu=1$, we shall obviously fill in the finite spaces between the curves corresponding to integral values of n.

7. Thus the function of the forces $y=k^2/r^{n+\nu}$ will be continuous, since the expression $y=1/x^n \cdot 1/x^\nu$, $n=1, 2, 3, \ldots n$, $\nu=0$, $\nu=1$, will hold at any point of the region under discussion. The whole series of curves obviously involves a double summation:

$$\sum_{n=0}^{n=n}\sum_{\nu=0}^{\nu=1} 1/x^n \cdot 1/x^\nu = P_{(n,\nu)} \qquad (6)$$

over the space representing the function $y=1/x^n \cdot 1/x^\nu$, which is outlined in the above illustration. For any one value of y, the values of the function of the forces is restricted to a horizontal line, yet $P_{(n,\nu)}$ may be at any point on the line, corresponding to different laws of force $f=k^2/r^{n+\nu}$.

8. Accordingly the function $f=k^2/r^n$, or $y=1/x^n \cdot 1/x^\nu$ is continuous over the space included between the extreme curves: and the sum total of the curves which could be traced would be the doubly infinite series,

$$y = S_{i,j} = \sum_{n=0}^{n=\infty}\sum_{\nu=0}^{\nu=1} 1/x^n \cdot 1/x^\nu \qquad (7)$$

depending on the laws by which the force varies with the distance.

9. Within the limit $n=0$, $n=\infty$, there are an infinity of curves: and between the limits $\nu=0$, $\nu=1$, there is a second infinity of curves. Hence in the space $S_{i,j}$, over the area described above, we have the integral:

$$S = \sum_{n=0}^{n=\infty}\sum_{\nu=0}^{\nu=1} \Delta y/x^n \cdot \Delta x/x^\nu = \int_{n=0}^{n=\infty}\int_{\nu=0}^{\nu=1} f(x,y)\,dx\,dy = A\,. \qquad (8)$$

10. In conclusion, we see that $y=1/x^n$, for $n=0$ is no curve, but only a straight line parallel to the x-axis, yet for all integral values, $n=1, 2, \ldots n$, there are isolated lines, everywhere separated by finite intervals, except near the point $p(1,1)$. By introducing the fractional exponent ν, $y=1/x^n \cdot 1/x^\nu$, $\nu=0 \ldots \nu=1$, we get the continuous series of lines for filling in the plane between the integral lines, which gives an area $S_{i,j}$ covered by the curves $y=\Phi(x,y)$:

$$S_{i,j} = \int_{n=0}^{n=\infty}\int_{\nu=0}^{\nu=1} f(x,y)\,dx\,dy = A\,. \qquad (9)$$

11. It appears that the expression for the gravitational forces which govern the motions of the planets in their orbits, namely: $f=m\,m'/r^2$, has the form here treated of,

$$f = y = k^2/r^n = 1/r^2 \qquad (10)$$

if the gravitational constant of *Gauss*, $k=0.017202098\,14$, be taken as unity, which we can do in studying the function of the attraction in a single system.

12. The function of the planetary forces, $f=y=k^2/r^2=1/x^2$ thus corresponds to a single line in a bounded region of a plane. And the question naturally arises: why is Nature's law restricted to this one choice? Are not an infinity of other laws possible by a mere change of the exponent from 2 to $n+\nu$?

13. (a) It is certainly established by observation that no integral exponent can explain the planetary motions, except $n=2$.

(b) It is certain that no fractional exponent, $\nu=0$, to $\nu=1$, will hold for the solar system, unless it is $\nu=0.0000001046$, or smaller, corresponding to the motion of Mercury. For this value will explain the outstanding anomaly in the motion of Mercury's perihelion, if any such anomaly really exists. (cf. A.N. 5048, p. 143.)

14. Thus the restriction of Nature's law to the integral number 2 is an extraordinary phenomenon: there is too small a probability, one out of a double infinity, to allow us to ascribe this result to chance. It is therefore worthy of the meditation of philosophers why nature prefers such restriction when a double infinity of other laws are mathematically possible, yet we know by observation that only $n=2$ really exists.

2. Geometrical and Physical Interpretation of the Law of *Newton*.

As shown in the passage above cited, from *Whewell*'s History of the Inductive Sciences, the reasoning of *Halley*, August, 1684, just prior to his visit to *Newton* at Cambridge, which led to the preparation of the Principia, 1684–1686, was to the following effect:

1. Gravity is an emanation from the sun, and decreases inversely as the square of the distance, because the sphere surface over which the emanation is diffused in space increases directly as the square of the distance. Thus two sphere surfaces at distances r and R are respectively:

$$s = 4\pi\, r^2 \qquad S = 4\pi\, R^2\,. \qquad (11)$$

Accordingly, we have from the proportion between these surfaces:

$$S = s\; R^2/r^2\,. \qquad (12)$$

And if $R=2r$, the result is $S=4s$, which puts *Halley*'s geometrical reasoning in the simplest algebraic form.

2. *Halley*'s reasoning is also illustrated by the following geometrical figure, which shows graphically how the sphere surface is quadrupled at double the distance.

The law of decrease for a physical emanation should, however, have a physical ground for following the law of apparent angular magnitude, which is the inverse of the geometrical expansion, and is developed in paragraph 4 below.

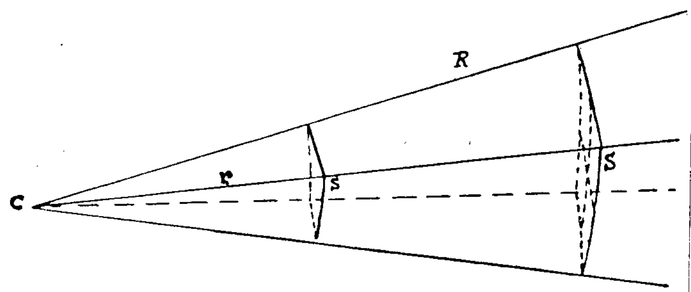

Fig. 1. Illustrating the increase of the base in a pyramidal element of a sphere surface, at distances r and R. At a distance $R=2r$, the element of the surface $S=4s$, which expanse of the sphere surface was imagined by *Halley*, 1684, to weaken the emanation of the Sun. The real physical reason for this weakening of the emanation at increased distance, however, is the decreased cross section, under the square of the amplitude of the waves, shown in Plate I above, making the wave energy follow the law $f=k^2/r^2$, not the purely geometrical ground originally assigned.

3. Accordingly, it appears by the spherical pyramid shown in Fig. 1 that the sphere surface is quadrupled at the original double distance; and therefore the whole sphere surfaces of radii r and R, drawn about a common centre, stand in the ratio: $s:S=4\pi r^2:4\pi R^2 \quad S=s\,(R/r)^2$

whatever be R and r.

It is true, therefore, as *Halley* conceived in August, 1684, before his visit to *Newton* at Cambridge, which resulted in the preparation of the Principia, that a geometrical emanation from the Sun would become diffused, so as to decrease inversely as the square of the distances: yet the question remains as to the physical nature of this emanation, and why it follows a law the inverse of the geometrical increase of the sphere surface.

4. In AN 5044, p. 54, by rigorous reasoning, in respect to waves proceeding from a centre, we find that the law of amplitude for the waves is $A=k/r$; and the law of force, given by the action of the waves, under steady flow, is $f=k^2/r^2$, as confirmed by observation in the gravitation of the heavenly bodies. It thus appears that the formula $y=1/x^2$ points to wave-action as the law of nature. The amplitudes of the waves at different distances are illustrated in Figure 2, which shows how the aether is churned up by the increase of amplitude towards the centre, and thereby rendered of heterogeneous density

$$\sigma=\nu r\,. \qquad (13)$$

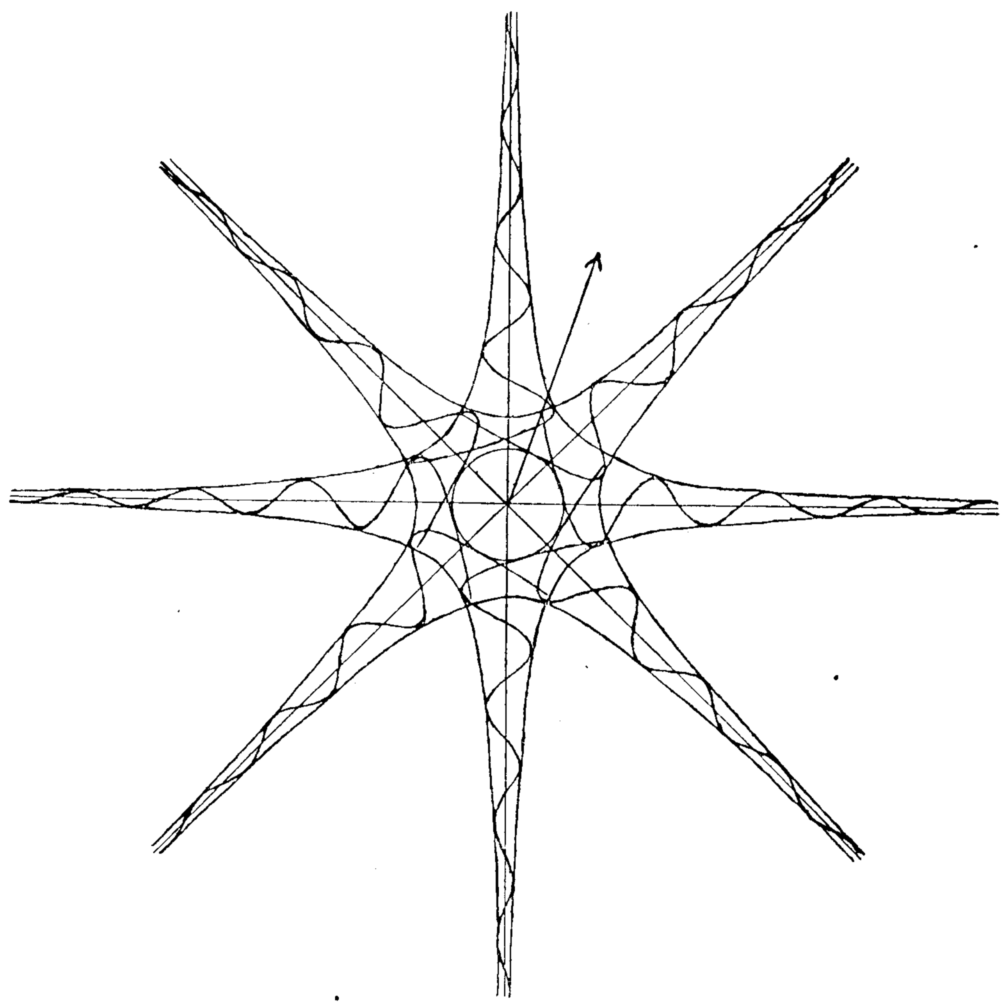

Fig. 2. Illustrating the asymptotic increase of the amplitude of the waves towards the centre, which renders the aether heterogeneous. The arrow shows the proper motion of the star carrying the wave-field with it.

5. This result is so important in showing that the aether about the Sun, under a steady flow[1]) of waves from that centre, cannot be of homogeneous density, but must decrease in density towards that centre of disturbance, owing to the increasing amplitude of the waves, that we repeat part of the discussion, AN 5044, p. 54.

The displacement of any particle of a medium due to wave motion, of a given wave length, is independent of the periodic time, and since the oscillatory orbits of the particles are described in equal times, under continuous flow of the waves, these orbits will be proportional to the displacements or other homologous lines pertaining to the periodic paths of the particles. Let the velocities of the moving particles be v, and m their mass; then their kinetic energies will be represented by $\frac{1}{2}mv^2$. In the spherical expansion of the aether waves there will be no loss of energy in free space; hence on two successive sphere surfaces of thickness dr, the energies are equal, so that we have:

$$4\pi r^2 \cdot \tfrac{1}{2}m v^2 = 4\pi r'^2 \cdot \tfrac{1}{2}m v'^2$$

or

$$v^2 : v'^2 = r'^2 : r^2 . \tag{14}$$

The kinetic energy of the vibrating molecules varies inversely as the square of the distance. But the velocity varies also as the amplitude, in simple harmonic motion: therefore for the amplitudes A' and A'', corresponding to the radii r' and r'', we have, by taking the square root in equation (14):

$$A' : A'' = r'' : r' \tag{15}$$

$$A'' = A' r'/r'' = k''/r'' . \tag{16}$$

Accordingly the amplitude or side displacement becomes,

$$A = k/r \tag{17}$$

and

$$V = M/r = \int\int\int \frac{\sigma\, dx\, dy\, dz}{\sqrt{[(a-x)^2+(b-y)^2+(c-z)^2]}} \tag{18}$$

which is the law of the potential first used by *Laplace* in 1782 (Oeuvres Complètes de *Laplace*, Tome X, pp. 348–352). Thus it appears that if there be aether waves propagated outwardly from any molecule of matter, the amplitude, or maximum side displacement of the oscillating particles of the aether, will vary inversely as the radius of the spherical wave-surface.

6. The theory of the amplitude of steady wave motion from the Sun's centre being thus established, on valid geometrical and physical grounds, it only remains to inquire into the law of energy for this wave action, at any distance. As a particle in simple harmonic motion has the velocity

$$v = 2\pi\, A/t = (2\pi/t)\, k/r \tag{19}$$

and the energy is

$$E = \tfrac{1}{2}m v^2 = 2m\pi^2/t^2 \cdot k^2/r^2$$

the energy being proportional to $\frac{1}{2}mv^2$, we have for the energy of the action in any unit of time, the simple expression

$$f = k^2/r^2 \tag{20}$$

which is the well known formula for the central force of gravitation under the Newtonian law.

7. We come therefore to the obvious conclusion, that if gravitational waves in the aether be receding from the Sun, under steady flow — as they certainly do in the form of light and heat — then the aether will react towards that centre with the force $f = k^2/r^2$, as shown for gravitation by *Newton* in the Principia, 1687.

(a) In paragraph 4 above, we show that the amplitude of the waves increases towards that centre, inversely as the radius, $A = k/r$, and thus the aether is heterogeneous, with many of the properties of an aeolotropic infinite elastic solid (*Kelvin*), yet really a world gas in kinetic equilibrium.

(b) The energy of wave action is as the square of the amplitude, $f = k^2/r^2$, as observed in the law of gravitation, and discussed above in paragraph 6.

(c) Is such an emanation conformable to the reasoning of *Halley*, August, 1684, when he discussed the problem with *Newton*, and brought about the preparation of the Principia, 1686? This is an important question, owing to *Newton*'s view that gravity is due to impulses (waves) in the aethereal medium.

(d) We answer that the waves in the aether here described correspond exactly with what *Newton* called impulses in the aether, and fulfill the law of the inverse squares, entertained by *Halley*, August, 1684, but already known some time before to Sir *Christopher Wren* and *Newton*.

8. The waves, as a steady flow of emanation directed from the Sun, but reacting towards that centre, fulfill the law of the intensity required geometrically by Plate I and illustrated more generally by Fig. 2 above. The solid angle subtended by a body at any distance is $\omega = 4\pi/n$, where n is any number representing the fractional part of the sphere surface, and hence the formula $S = s\,(R/r)^2$, gives for a greater distance R, $\omega' = 4\pi/n'$, the equation: $\omega' R^2 = \omega r^2$, whence

$$\omega' = \omega\,(r/R)^2 \tag{21}$$

which is the formula for the solid angle, in place of the surfaces, Fig. 1.

In this way, by the decreasing solid angle, one could explain the emanation entertained by *Halley*, August, 1684, under the law of the inverse squares already known to *Wren* and *Newton* some time before.

9. But it is very important to show that there is a physical as well as a geometrical way of looking at the expression for the square of the amplitude of the waves as follows. Let us remember that the amplitude $A = k/r$ is the side motion depicted in the plane of the wave-field shown above in paragraph 4: and hence that the square of the amplitude $A^2 = f = k^2/r^2$ is the relative solid angle or conical element of the sphere surface $d\omega'$ occupied by the contracting waves at any greater distance R, so that:

$$d\omega' = d\omega\,(r/R)^2 \tag{22}$$

[1]) It is remarkable that the wave-theory as now developed leads to the celebrated theory of the Music of the Spheres held by *Pythagoras*, *Philolaos* and other Greek philosophers. For us mortals ordinary music is wave vibration in the coarse medium of the air, repeated in such periods as to furnish harmony and rhythm. Naturally if there be Celestial Music proceeding from the infinitely complex gravitational and magnetic wave-fields about the planets, the waves in the aether, yielding the divine melody, would be heard only by the gods, with senses analogous to radio receiving sets, appropriate to beings living in the aether, αἰθέρι ναίων (*Homer*, Iliad, IV, 166). At suitable distances therefore the distinctive wave-fields accompanying the several planets might produce a superfine celestial harmony and rhythm not inappropriately called the Music of the Spheres. For a vision of the infinitely complex nature of the wave-fields of the planets the reader is referred to equation (71a).

The energy of the waves is as the square of the amplitude; but the contracted area of the sphere surface on which this same wave energy acts is also proportional to $(r/R)^2$ at the distances r and R. Hence as the energy in wave action is as the square of the amplitude, and the element of the solid angle of the sphere surface $d\omega'$, on which the contracted wave acts, follows the same law of distance inversely, there is a double proof that the law of the inverse squares should hold true in nature. It results geometrically and also physically!

10. In visible confirmation of this result we may imagine a point on the asymptotic tangent, of any one of the wave-rays, above depicted, to revolve about the central axis of the wave-ray, so as to generate in space the base of a tapering cone: then the cross section of the cone at any distance r will be proportional to k^2/r^2. This is the surface on which the receding waves act, and conforms to the law of universal gravitation. If the waves therefore exist, as postulated, there can be no doubt that the Newtonian law should have the form $f = k^2/r^2$.

We see also by this inward increase of the amplitude $A = k/r$, that under the steady flow of waves, the aether is unequally churned up, so that medium cannot be homogeneous, but must be of less density towards the centre, where the amplitudes are greater; the density of this kinetic medium therefore increases outwardly directly as the radius, and thus it presses steadily inward towards the centre from which the waves are receding. This reaction of the outflowing waves, corresponding to the law $f = k^2/r^2$, which at the same time keeps the aether heterogeneous, and under steady inward pressure, is the cause of universal gravitation, the law of amplitude, $A = k/r$, yielding the central force $f = k^2/r^2$.

From the considerations adduced in dealing with Plate II above it would appear that nature might choose for the intersecting path of her law that of any curve of the type $y = k^2/r^{n+\nu}$, cutting the line ac in any one of the infinite number of points included between a and c; yet she restricts her choice rigorously to the one point b, corresponding to $n = 2$, $\nu = 0$.

The true explanation of this severe restriction, when an infinity of points on the line ac are available, thus follows incontestably. It points to the wave-theory of physical forces as the only conception fulfilling the necessary and sufficient conditions imposed by geometry and by the observed laws of nature!

3. Observational Proof of the Existence of Electrodynamic Waves from the Sun, Moon and Planets.

The data furnished by observation may be grouped under a number of distinct headings, but they severally confirm the existence of waves proceeding from the Sun, Moon and other heavenly bodies.

1. It is found by observation, as reported in AN 5140, p. 130, footnote, that long waves, up to 2000 metres in length, actually do come to the earth from certain sunspot areas: that is, irregularities in the Earth's electromagnetic wave-field are definitely associated with commotions in the Sun, incident to the development of sunspots. On this point the experimental proof is overwhelming, and absolutely beyond controversy. Moreover, the following is historically true:

(a) Ever since 1852 it has been recognized that the »Magnetic Storms« observed upon the Earth are associated with the development of the sunspots, and by similarity of curves follow the same mathematical law. The identical curves for the two phenomena matured by *Rudolf Wolf* of Zürich, 1875, are fully verified and beyond question, now for $^3/_4$ of a century.

(b) The aurora upon the Earth is also periodic and associated with these commotions in the Sun, because it follows the same law as the sunspots and magnetic storms. All these phenomena are associated wave-phenomena, and depend on moving waves in the aether, proceeding from the Sun and passing the Earth with the velocity of light. These waves incident to the commotions in the Sun cause the magnetic needle to tremble: the »Magnetic Storms« are due to gusts of waves, not unlike our gusts of wind, except that they act chiefly by variable wave action or induction. This causes the needle to tremble, and hence the cycle of tremors corresponds perfectly to the cycles of spot development in the Sun, as graphically illustrated by *Rudolf Wolf*, and other investigators.

(c) It has been known since the middle of the 19th century that the Earth currents of our globe, which affect the electric state of telegraph and cable lines, are also directly associated with the sunspots, aurorae and magnetic storms. In regard to these electric disturbances, Sir *George Airy*, in his Treatise on Magnetism, 1870, p. 204, says:

»They are not connected with thunder-storms or any other known disturbance of the atmosphere; but they are invariably connected with exhibitions of Aurora Borealis, and with spontaneous galvanic currents in the ordinary telegraph-wires: and this connection is found to be so certain, that upon remarking the display of one of the three classes of phenomena, we can at once assert that the other two are observable (the Aurora Borealis sometimes not visible here, but certainly visible in a more northern latitude).«

2. The above discussion of the sunspots, aurorae and magnetic storms and Earth currents gives good evidence that a wave-field with variable inductive action is passing the Earth, the magnetism of which at times should be conspicuously disturbed as a whole. This is definitely and visibly demonstrated by the following phenomena which show simultaneity of action in all parts of the globe:

(a) In the Phil. Trans. of the R. Soc. for 1892, A, Plate 8, Professor *W. Grylls Adams* has given a graphical record of simultaneous disturbances of the magnetic needle throughout the world, in the great »Magnetic Storm« of June 25, 1885. The plate is here shown, with all the separate records reduced to Greenwich Mean Time, and speaks for itself (s. Plate IV).

It was correctly interpreted for the first time in my Electrodynamic Wave-Theory of Physical Forces, vol. I, 1917, pp. 33–55, and still further verified and extended in the New Theory of the Aether, Seventh Paper, AN **217**, Oct. 22, 1922, to which latter paper the reader is referred for proof of the cause of the semi-diurnal tide in the magnetism of the Earth.

3. The variable inductive action of the Sun's rotating magnetic poles upon the Earth's rotating magnetic poles explains the semi-diurnal tide in the magnetism of the Earth.

Just as the motion of a magnet near a wire was found by *Faraday*, 1834, to give rise to the induction of a current in the wire, which could be measured by the galvanometer, and thus led to the discovery of the dynamo, so also my discovery and proof of the wave-theory of magnetism, VII. Paper on the New Theory of the Aether, AN **217**, Oct. 22, 1922, led to the correct explanation of the semi-diurnal tide in the Earth's magnetism, which so greatly perplexed Sir *George Airy*, Sir *John Herschel*, and Dr. *Humphrey Lloyd*, 1870.

(a) The formula for the total magnetic intensity, discovered by me in Feb., 1922, involves two terms, but otherwise has a form similar to that of gravitation:

$$\left.\begin{array}{ll} I = \mu\mu'/s^2 + \mu\mu'/s'^2 \quad . \quad . \quad . \quad . & \textit{See}, 1922 \\ g = f = m\,m'/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . & \textit{Newton}, 1686 \end{array}\right\} \quad (23)$$

where μ is the magnetic pole strength of the Earth, and μ' that of the unit bar magnet or suspended needle, and s and s' are the distances to the magnetic poles in our globe, measured along the curved lines of magnetic force. If the magnetic action of the Sun varies, owing to the relative motion of the poles of the Earth or Sun, then the terrestrial pole strength μ will be variable, by induction, and we shall have a true tide in the magnetism of the Earth.

(b) The magnetic tide thus introduced will take the form therefore of a cyclical change in μ which becomes variable by an amount $\pm\nu$ in half a day. Accordingly we have:

$$I = \mu\,(1 \pm \nu)\,\mu'/s^2 + \mu\,(1 \mp \nu)\,\mu'/s'^2 \,. \qquad (24)$$

The magnetic tide will pull back and forth, in the line from the Red Sea to Hudson's Bay, changing its direction in periodic cycles, like the ebb and flow of the sea, twice daily: for at such intervals are the Earth's opposite poles brought nearest the Sun by the Earth's rotation.

In the well known equations for the tidal analysis of our sea we may put for the semi-diurnal tide:

$$\delta h = A \cos 2n\,t + B \sin 2n\,t \,. \qquad (25)$$

By similarity, in the magnetic tide, we may put

$$\nu = \delta\mu = \alpha \cos 2n\,t + \beta \sin 2n\,t \qquad (26)$$

and hence we could write the above equation in the periodic form:

$$I = \mu/s^2 \cdot \{1 \pm (\alpha \cos 2n\,t + \beta \sin 2n\,t)\}\,\mu' + \\ + \mu/s'^2 \cdot \{1 \mp (\alpha \cos 2n\,t + \beta \sin 2n\,t)\}\,\mu' \,. \qquad (27)$$

It is unnecessary to comment on this result, except to say it is one of the most impressive arguments for the wave-theory of physical forces. Just as the approach of the waters of our sea to the point beneath the Moon, and the anti-lunar point, is followed by two tides of the sea daily, so also the movement of the two magnetic poles of the Earth, under the rotation of our globe, gives rise to semi-diurnal inductive action in the magnetism of our globe, under the powerful magnetic field of the Sun, and therefore to a true semi-diurnal tide in the magnetism of the Earth.

4. The Wave-Theory of Physical Forces confirmed by the Discovery of the Wave-Theory of Magnetism, and the Connection of Magnetism with Gravitation.

1. In 1922 I established the New Law:

$$I/g = \eta^2\,(r^2/s^2 + r^2/s'^2) \quad . \qquad (28)$$

connecting magnetism with gravitation; and having shown that magnetism is a wave-phenomenon, was able to declare definitely that gravitation also is a wave-phenomenon.

(a) This inference is theoretical, or mathematical, yet confirmed by *Faraday*'s experiment of 1845, on the rotation of a beam of plane polarized light by magnetism. Thus it is deduced mathematically, but verified by most conclusive experiments, and numerically confirmed in the magnetic and gravitational forces of the Earth.

(b) If the observations, prior to 1922, confirmed the passage by the Earth of aether waves, some of them as much as 2000 metres in length, and definitely known to proceed from certain disturbed areas in the Sun, — this is all the observational proof we could desire for the wave-theory of gravitation, which conforms to the wave-theory of magnetism, the aurora, magnetic storms, etc.

(c) The law $f = k^2/r^2$ points to waves of the type observed: they are shown to proceed from the Sun, and are therefore the sole cause of universal gravitation. Out of all the waves passing the Earth only a few will be observable, chiefly the irregular magnetic waves connected with changing sunspots; yet the proof of the few establishes the existence of the whole, and thus of the waves of gravitation, through the magnetic and electric commotions associated with disturbances in the Sun.

The argument here outlined gives continuity to the theories of magnetism, electrodynamics, and magneto-optics, by connecting all of these phenomena with universal gravitation, in the cases of the Earth, Sun and Moon, from which two latter bodies semi-diurnal magnetic tides are found to proceed. This is the only possible way of explaining the magnetic tides of our globe, which so greatly puzzled and surprised Sir *George Airy*, Sir *John Herschel*, and Dr. *Humphrey Lloyd*, 1870.

2. The wave-theory of magnetism is so clearly proved in the VII. Paper on the New Theory of the Aether, AN **217**, Oct. 22, 1922, that we recall part of that discussion:

(a) Imagine adjacent additional centres of disturbance, A', A'', A''' . . . . , and A_1, A_2, A_3 . . . . , all in vertical line with the centre A. And make additional orifices above and below BC, as $B'C'$, $B''C''$, $B'''C'''$ . . . . , B_1C_1, B_2C_2, B_3C_3 . . . . , through which the wave disturbances may pass.

(b) Then the waves in the same phase will everywhere mutually support each other: the disturbing centres being in the same parallel line, the wave fronts will become straightened by the mutual support of the separate independent disturbances.

(c) Now imagine the orifices brought closer and closer together, yet maintained as distinct centres of disturbances: we see that beyond the line BC, prolonged in both directions, the wave fronts will become quite straight in the centre, but will curve around rapidly only near the end of the extreme orifice $B^\nu C^\nu$, $B_\nu C_\nu$, above and below respectively. This is exactly what occurs in magnetism: the lines of force curve around conspicuously as we approach the ends of the magnet.

(d) The poles, in fact, are the centres of the reacting stress in the medium when agitated by all the atoms vibrating in concert, and emitting waves of the kind here described. The lines of force being axes of rotations for the aetherons, as the waves move along, there is a tendency in these lines to shorten themselves, as in *Dolbear*'s experiment: the result is tension along the lines; and as they are of minimum length, they tend to keep straight near the centre of the magnet, and to curve sensibly only near the ends of the bar, just as in the water-wave experiment above described from *Newton*'s diagram of 1687.

3. The diagrams for illustrating the true nature of magnetism are so suitable for this purpose that we reproduce them without further comment. Cf. Plate IV.

4. The experiments here described are accurate and can be verified by actual trial for water waves, which are simple and easily understood. They disclose to us the true nature of magnetism, for the following reasons:

(a) The results conform to *Dolbear*'s experiment, where the dynamical influences at work are easily understood, and admit of but one interpretation.

(b) They are verified in the actual movement of water, the waves of which also have tension along their axes and tend to straighten themselves to a maximum in the rotational motion of the filaments about their axes,

$$s = \int_0^p ds \tag{29}$$

on the principle of Least Action.

By actual experiment, 1845, *Faraday* found that the plane of a beam of polarized light was rotated when passed along the line of force, through heavy glass, carbon disulphide and similar substances, and the more rotated the longer the paths. This fact shows clearly that aether waves of the type here described underlie magnetism. They are proved to exist by the practical experiments with water waves, by *Dolbear*'s experiment, on tangible models, and by *Faraday*'s celebrated experiment on the rotation of the plane of polarization by magnetism.

There is one other experiment which equally supports the above conclusion, namely the revolution of a flexible hoop set loosely on an axis, in the apparatus commonly used to show the effects of centrifugal force. When the hoop is spun rapidly about its axis, it becomes of oval shape, bulged out at the equator and drawn in at the poles of rotation, like the figures of the planets, which it is used to illustrate.

Now imagine a series of such hoops mounted side by side, and tied together mutually along the axis. Then, when the rotation develops, the whole line of connected hoops will shorten itself, under the centrifugal force, just as in *Faraday*'s lines of force. It is impossible to imagine a more convincing proof than that here suggested.

The argument here cited establishes the wave-theory of magnetism upon a permanent basis. And as magnetism is connected with gravitation through the mathematical law of 1922,

$$I = \mu\mu'/s^2 + \mu\mu'/s'^2 \tag{23}$$

which I discovered by extending *Gauss*' General Theory of Terrestrial Magnetism, 1838, it follows from this verified mathematical theory that gravitation is a wave-phenomenon — as held by *Newton* as far back as 1678.

5. The wave-theory of gravitation is also strikingly confirmed by the close agreement of the calculated with the observed periods in the Fluctuations of the Moon's mean motion, established by *Newcomb*, 1909, after researches extending over 40 years (cf. AN 5048, pp. 155–161), and by the simple explanation given in the New Theory of the Aether, II. Paper, AN 5048, p. 143, of the small outstanding anomaly of 28″ per century in the progression of Mercury's perihelion.

There is no other explanation than that here cited of the Fluctuations of the Moon's mean motion. Moreover, the theory of the bending of the waves in passing through the Earth so as to partly release the Moon, is verified by the theory of light, and by a large body of phenomena in radio telegraphy, showing that the waves travel more slowly in the globe than in the air and free space above the Earth, and thereby the wave-front in radio telegraphy is bent around the globe, even to the antipodes, as in the French observations of the signals from Bordeaux at Chatham Island, East of New Zealand, (cf. AN 5317, Sept., 1924).

The Theory of Relativity has been completely overthrown, and thus, owing to its failure to account for the motion of Mercury's perihelion — by postulating 43″ per century, when only 28″ or less is available — the wave-theory alone holds its place to-day. And as it corresponds to a small deviation from the law of gravitation $f = mm'/r^2$, yielding

$$f = mm'/r^{2.0000001046} \tag{30}$$

such as *Newton* foresaw, in the Principia, 1686, there can be no doubt of the validity of the wave-theory, under *Weber*'s Electrodynamic Law, now established as the more general form of *Newton*'s Law.

5. The Geometrical and Physical Interpretation of the Electrodynamic Law of *Biot* and *Savart*, 1820.

1. *Biot* and *Savart*'s law of 1820 for the intensity of the field about a straight wire.

In the III. Paper on the New Theory of the Aether, AN 5070, pp. 255–258, we have discussed with care and accuracy the geometrical and physical significance of the law of *Biot* and *Savart*, 1820, for a straight wire. It is shown by a precise analysis of the known facts that the only admissible conclusions are the following:

(a) The straight wire is surrounded by a wave-field, as shown also by *Oersted*'s experiment of 1819, for the action of a current upon a magnetic needle. The waves are generated by the surge of the electric disturbance from side to side, on all sides of the wire, like the oscillatory surge observed in the discharge of a Leyden jar: but as the waves cannot expand freely in tridimensional space, because of their simultaneous origin at every point of the wire's cylindrical surface, the expansion can only be cylindrical in character; and hence there results the formula derived from observations by *Biot* and *Savart*, 1820, namely:

$$I = kH/r \tag{31}$$

where I is the intensity, k a constant, H the current strength, and r the distance (cf. *Biot* and *Savart*, Ann. Chim. Phys. **15**, p. 222, 1820).

(b) This law is verified by observation at all distances r, and for any fixed current strength H. It gives a curve of the following type, which is a rectangular hyperbola referred to its asymptotes.

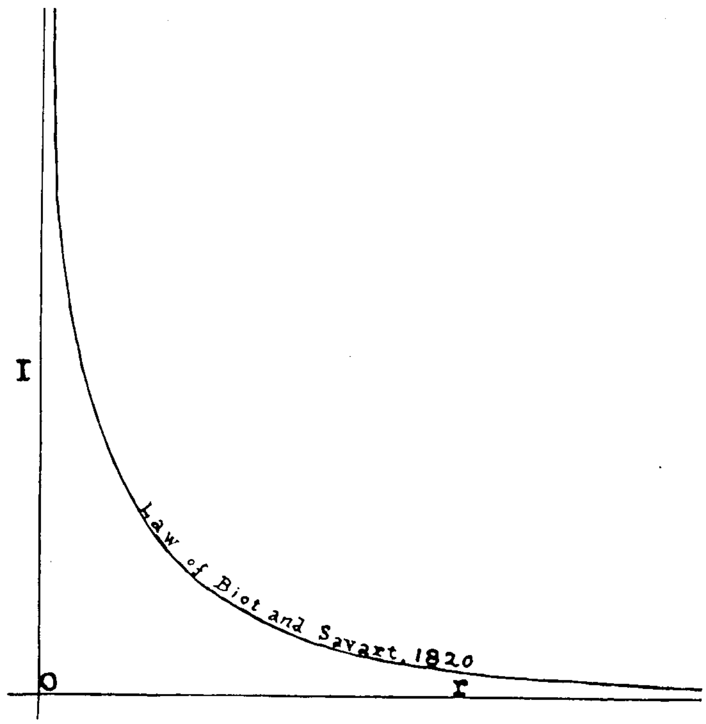

Fig. 3. Illustration of the law of *Biot* and *Savart*, 1820, for the intensity of a current in a straight wire. The curve is a rectangular hyperbola referred to its asymptotes, $I = ki/r$, $y = 1/x$; and the physical basis for such a law is explained by the action of waves restricted in their expansion, as illustrated in figure 4 below.

(c) In the expanding cylindrical surface about the wire, the element of surface is

$$ds = dl \cdot r\, d\omega \qquad (32)$$

where dl is the element of length parallel to the axis of the cylinder, and $r d\omega$ is the product of the distance into the element of the angle $d\omega$, which gives the element of the arc around the axis. When the intensity of the wave-action decreases it must be inversely as r — the other elements dl and $d\omega$ being constant.

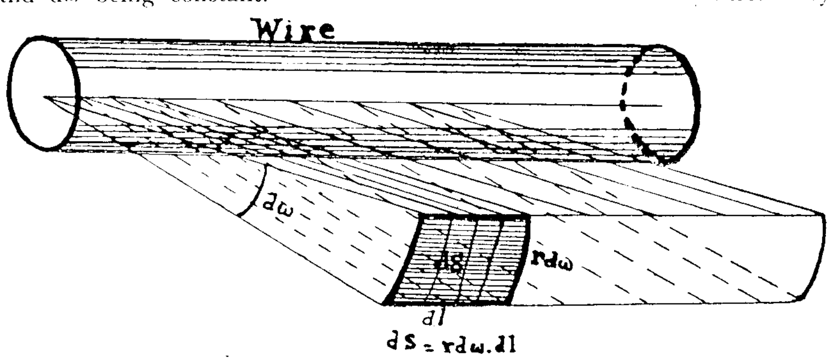

Fig. 4. Illustration of the law of *Biot* and *Savart*, in which the waves cannot expand endwise, along the cylindrical axis of the wire, because of other waves originating at every point, and therefore are restricted to expansion in the circle $r d\omega$, as shown in the figure. Waves, under such restricted expansion, are the only possible explanation of this simple law, $I = ki/r$, which therefore becomes a very powerful argument for the wave-theory.

This remarkable electrodynamic law of 1820 establishes the dependence of current action on waves expanding about the cylinder, in the one possible dimension r, and therefore the intensity following *Biot* and *Savart*'s simple law of the inverse distances.

(d) This law is different from most of the other laws of nature, which follow the inverse square of the distances; but the reason for it is made plain in a way which admits of no dispute. One interpretation of *Biot* and *Savart*'s law is possible, and only one: the variation in the intensity therefore is a direct result of the only possible expansion of the waves about the wire as the distance r increases.

A single equation defines perfectly a single unknown: for any values, $I_1, I_2, I_3, \ldots\ldots I_\nu$, there are the corresponding inverse distances $r_1, r_2, r_3 \ldots\ldots r_\nu$.

(e) The law of *Biot* and *Savart* holds rigorously at an infinity of points, $\nu = 1, \ldots\ldots \nu = \infty$; and as the path of the law therefore is a true rectangular hyperbola referred to its asymptotes, the probability of this event, without a geometrical and physical cause, is only $p = 1 : \infty$.

The chances are thus rigorously infinity to one that *Biot* and *Savart*'s law would not be verified by observation unless the wave theory were true. See section **24** for final calculations in the theory of probability.

2. The wave-theory confirmed by the form of arrangement which iron filings take in the field about a straight wire.

The argument for the wave-theory deduced from the wonderfully simple law of *Biot* and *Savart*, 1820, is in the highest degree satisfactory; yet if an independent argument, confirming the law of *Biot* and *Savart* and the wave-theory, could be deduced from other observed phenomena, it would materially increase the probability of the wave-theory as representing the true order of nature. What do we find pointing to such independent confirmation?

(a) In AN 5079, pp. 259–260, Fig. 2, Plate 5, of the III. Paper on the New Theory of the Aether, we give a simple and obvious explanation of the circles taken by iron filings in the field about a wire bearing a current. When the current flows each little filing becomes a magnet; and the filings therefore form into concentric circles about the axis of the wire. They act like the magnetic needle in *Oersted*'s experiment of 1819, by setting their axes at right angles to the direction of the current in the wire.

(b) This situation assumed by the filings, under the inductive action of the waves, makes the north pole of one filing adhere to the south of the next adjacent filing, and vice versa. Thus the filings form into winrows, or concentric circles, about the axis of the wire bearing the current; and the more the plate on which the filings rest is gently tapped, the more perfect becomes the circular winrows of filings.

(c) The ordered phenomena exhibited by such filings on a plate of glass involves thousands, yea millions of small pieces of iron; they all obey the circular law, with the north pole of one joined to the south pole of the other, and vice versa.

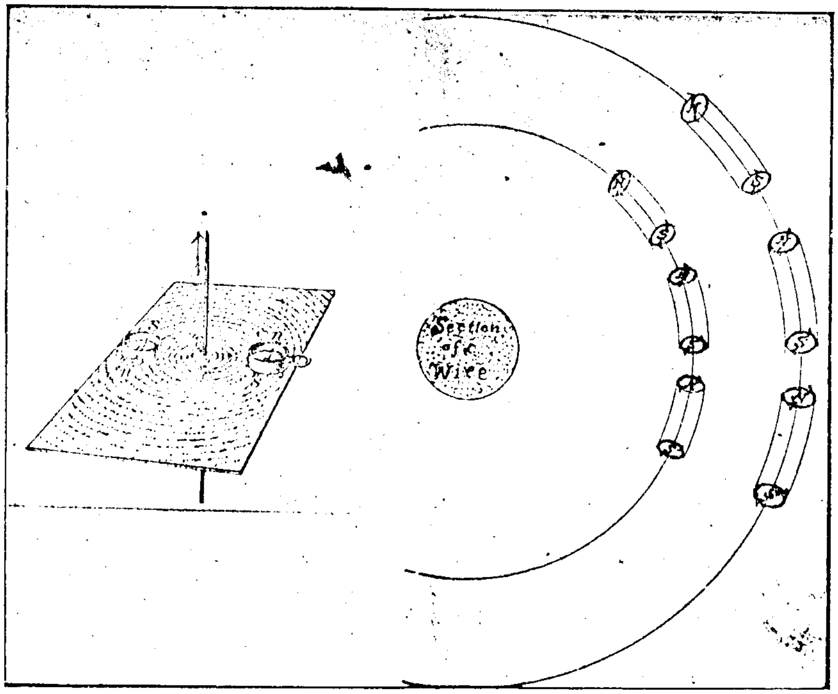

Fig. 5. Illustration of a magnetic whirl about a wire, with wave-theory of this whirl, on the right. Each little piece of iron filing becomes a small magnet, and they are drawn together by the attractions of their opposite poles. The filings also tend to form winrows, when they are disturbed by tapping, because each pole slightly repels its neighbor of the same polarity in the next row.

The law therefore is rigorous and always fulfilled — a true law of nature!

We may explain all this orderly arrangement by assigning to the aether waves rotations in the plane through the axis of the wire bearing the current. This, and this alone, will account for the observed phenomena! The explanation is unique, and confirmed by necessary and sufficient conditions: it is the only explanation possible, and the chances are infinity to one that it represents the true laws of nature.

6. The Proper Interpretation of *Ohm*'s Law, 1826, and of *Oersted*'s Experiment, 1819.

1. *Ohm*'s Law of Resistance, and its relationship to *Biot* and *Savart*'s Law.

In the III. Paper on the New Theory of the Aether, AN 5079, pp. 258–259, we have considered the geometrical and physical significance of *Ohm*'s Law of 1826, namely,

$$I = H/R \qquad (33)$$

the intensity I, or electromotive force, is equal to a constant H divided by the resistance R.

This law is shown to be identical with *Biot* and *Savart*'s law at a fixed distance: because at such fixed distance the only change that will be felt will be that depending on I or R. When the resistance R is fixed, the only change will be incident to change of the electromotive force, I. If I is constant, a change may occur by varying R, the resistance, and vice versa. No other changes are possible at a fixed distance.

2. It thus appears that *Ohm*'s law is a special case of *Biot* and *Savart*'s law of 1820. It is what follows under variable I or R. In AN 5079, p. 258, the following summary of conclusions occurs:

»(a) In *Biot* and *Savart*'s law we vary the distance, with fixed electromotive force, and observe the change in the intensity: the observed result confirms the wave-theory.«

»(b) In *Ohm*'s law we also deal with a current in a wire, or wires, and when the electromotive force is fixed, we study the law of resistance (R), or intensity of action (I), at a fixed distance, where the needle of the galvanometer may be located.«

»(c) Thus *Biot* and *Savart*'s Law, with a fixed steady current, serves for calculating the varying intensity at any distance, in accordance with the requirements of the wave-theory. In the same way, *Ohm*'s law, when H is constant, but with varying resistance, R, serves for calculating the intensity at a fixed distance.«

3. The two laws are brought into mutual relationship as follows:

»We may write *Biot* and *Savart*'s law in the form: $I = KH/r$ and *Ohm*'s law in the form: $I = H/R$.«

»Accordingly on combining the two expressions, which we may do by equating the identical intensity at any point, we obtain

$$KH/r = H/R \quad \text{or} \quad K = r/R\,. \qquad (34)$$

»Therefore we find on substituting for K its value, for any value H and r,

$$I = r\,H/R\,r = H/R \qquad (33)$$

which again yields *Ohm*'s law, in the form which holds for any fixed distance.«

»These two laws therefore confirm the wave-theory of the entire field about a wire bearing a steady current. *Ohm*'s law implies a cylindrical wave-field — the resistance and intensity being the axes of a rectangular hyperbola referred to its asymptotes — *Biot* and *Savart*'s law also represents a rectangular hyperbola of the same type, but with r varying instead of R.«

4. This theory of magnetism confirmed by *Oersted*'s experiment, 1819.

Oersted's celebrated experiment of 1819 throws a much clearer light on the nature of magnetism than most persons suppose.

(a) When the current in the wire flows, the needle sets itself at right angles to the axis of the wire. If therefore we could prove the nature of the magnetic field by some other process, as by the winrowing of the iron filings formed on a plate pierced by a vertical wire — the filings becoming minute magnets with opposite ends joined, in the concentric circles of the winrows — then we should have good proof as to why the needle sets itself at right angles to the axis of the wire bearing the current.

(b) The theory of solenoids and of electromagnets verifies this theory of magnetism. *Ampère* first made artificial magnets by the use of solenoids, in which steady currents were made to flow. This confirmed the nature of solenoidal magnets, and brought such magnetism into direct relationship to the winrows of iron filings, when such filings are sprinkled upon a plate normal to a vertical wire bearing a current.

(c) Without such waves of magnetism with rotations of the aether filaments about the lines of force, *Faraday*'s experiment of 1845 for rotating the beam of polarized light would not be possible. As the rotation of the plane of polarized light is an observed fact, magnets must emit waves of the type above explained: and nothing but such rotations of the elements of the aether about the lines of force will account for the tension of these lines.

(d) Thus one argument confirms another, and the whole of them are so interlocked mutually that any breaking down of the connection is impossible. The proof is based on necessary and sufficient conditions: one cause and only one, namely, the rotation of the aether about the lines of force, will explain magnetism and electrodynamics.

Part II. The Geometrical and Physical Significance of *Laplace*'s Potential Function *V*: *Lagrange*'s Force Function *U*, for *Newton*'s Law of the Mutual Attractions of the Heavenly Bodies in Pairs, Explained by Wave-Action, giving Tension on the Aether pulling in Straight Lines between the Bodies: Illustrations of the Known Processes of the Wave-Theory in Acoustic Attraction, and in *Newcomb*'s Fluctuations of the Moon's Mean Motion.

7. Critical Examination into the Significance of the Potential, as found in the Earliest Usages of Geometers.

The expression for the potential of a planetary body of mass

$$M = \iiint \sigma \, dx \, dy \, dz \tag{35}$$

was first used by *Laplace* in 1782 (Oeuvres Complètes de *Laplace*, Tome X, pp. 348–353)

$$V = \int \frac{dm}{r} = \int\int\int \frac{\sigma \, dx \, dy \, dz}{\sqrt{[(a-x)^2 + (b-y)^2 + (c-z)^2]}} = \frac{M}{r} \tag{36}$$

(cf. AN 5044, pp. 68–70). This indicates that we take a summation of every atom of matter at its respective distance, $V = \int 1/r \cdot dm$, which corresponds to an integration of the wave-effects from the several particles under the law of wave-amplitude, $A = k/r$, known to exist in nature. The potential thus depends on the mass, and inverse distance, because this is the law of amplitude; the value $V = M/r$ for a spherical body being of the well known form given by the above triple integral $= M/r$.

The potential is thus a state of stress in the aether, incident to the mass of a body, and its inverse distance, and therefore depends on waves proceeding from the several particles of matter, under the law of amplitude $A = k/r$.

In AN 5044, p. 70, a brief summary of the mathematical reasons for this conclusion occurs.

Taking the expressions for two independent curves, the amplitude and the potential, we have:

$$A = y = k/x \qquad V = y = M/x \,. \tag{37}$$

It will be noticed that they belong to the same geometrical species — both being rectangular hyperbolas referred to their asymptotes — and can be made identical throughout, from $x = 0$, to $x = \infty$, by introducing a summation Σ, such that $\Sigma k = M = \int dm$,

$$\iiint \sigma/r \cdot dx \, dy \, dz = \iiint k/r \cdot dx \, dy \, dz \,. \tag{38}$$

Accordingly it appears that, by the mere variation of a parameter, the curves are made to coincide rigorously point by point, from $x = 0$, to $x = \infty$. Therefore the chances against such a rigorous coincidence accidentally occurring throughout infinite space, $x = 0$, to $x = \infty$, becomes infinity to one, or,

$$c = \int_0^\infty dx = \infty \tag{39}$$

and thus its actual occurrence points unmistakably to a true law of nature.

It seems therefore certain and incontestable that the potential represents geometrically and physically the total accumulated stress due to the whole mass under the average wave amplitude of the field about the attracting body in question.

In order, however, to remove forever the possibility of contesting the foregoing interpretation of the geometrical and physical significance of the potential, we shall now traverse, as an addendum to this section 7, the history of the subject from its first introduction by *Laplace*, 1782.

1. *Laplace*'s analysis. In the Mém. de l'Acad. Roy. des Sciences de Paris, 1782; 1785, (reprinted in Oeuvres Complètes de *Laplace*, Tome X, pp. 341–419), *Laplace* has a celebrated memoir entitled: »Théorie des Attractions des Sphéroïdes et de la Figure des Planètes.« After putting for the element of mass of a planet, $dM = r^2 \, dr \, dp \, dq \sin p$, so that in the polar coordinates the mass

$$M = \iiint r^2 \, dr \, dp \, dq \sin p \tag{40}$$

Laplace says:

»Si l'on nomme V la somme de toutes les molécules du sphéroïde divisées par leurs distances à un point extérieur, on aura

$$V = \int 1/r \cdot dM = \iiint r \, dr \, dp \, dq \sin p = \\ = \tfrac{1}{2} \iint (r'^2 - r^2) \, dp \, dq \sin p \,.\text{«} \tag{41}$$

This latter result he then writes in a slightly different notation. This earliest definition shows the sense in which V originally was used.

On page 352–3 of the same memoir *Laplace* comes to the following results:

»Si l'on désigne par V la somme de toutes les molécules du sphéroïde, divisées par leurs distances respectives au point attiré, que l'on nomme x, y, z les coordonnées d'une molécule dM du sphéroïde, et a, b, c celles du point attiré, on aura

$$V = \int [(a-x)^2 + (b-y)^2 + (c-z)^2]^{-1/2} \, dM \,.\text{«} \tag{42}$$

»En désignant ensuite par A, B, C les attractions du sphéroïde, parallèlement aux axes des x, des y et des z, on aura

$$A = \int [(a-x)^2 + (b-y)^2 + (c-z)^2]^{-3/2} (a-x)\, dM = -\partial V/\partial a \quad (43)$$

on aura pareillement

$$B = -\partial V/\partial b \qquad c = -\partial V/\partial c \quad (44)$$

d'où il suit généralement que, si l'on connaît V, il sera facile d'en conclure, par la seule différentiation, l'attraction du sphéroïde, parallèlement à une droite quelconque a, en considérant cette droite comme une des coordonnées rectangles du point attiré.«

»La valeur précédente de V, réduite en série, devient

$$V = \int \frac{dM}{\sqrt{(a^2+b^2+c^2)}} \left[1 + \tfrac{1}{2}\, \frac{2ax + 2by + 2cz - x^2 - y^2 - z^2}{a^2+b^2+c^2} + \tfrac{3}{8}\, \frac{(2ax + 2by + 2cz - x^2 - y^2 - z^2)^2}{(a^2+b^2+c^2)^2} + \cdots \right] \quad (45)$$

cette suite est ascendante relativement aux dimensions du sphéroïde et descendante relativement aux coordonnées du point attiré, et si l'on n'a égard qu'à son premier terme, ce qui suffit lorsque le point attiré est à une très grande distance, on aura $V = M(a^2+b^2+c^2)^{-1/2}$, M étant la masse entière du sphéroïde. Cette expression sera plus exacte encore si l'on place l'origine des coordonnées au centre de gravité du sphéroïde, car on a, par la propriété de ce centre,

$$\int x\, dM = 0 \qquad \int y\, dM = 0 \qquad \int z\, dM = 0 \quad (46)$$

en sorte que, si l'on considère le rapport des dimensions du sphéroïde à sa distance au point attiré comme une très petite quantité du premier ordre, l'équation

$$V = M(a^2+b^2+c^2)^{-1/2} \quad (47)$$

sera exacte aux quantités près du troisième ordre.«

These passages give a clear and simple explanation of the definitions employed by *Laplace* in the earliest paper dealing with the subject. The definition here cited gives no physical significance to the function V, except to define it as the sum of all the molecules of the spheroid divided by their respective distances from the attracted point:

$$V = \int 1/r \cdot dM = \iiint 1/r \cdot r^2\, dr\, dp\, dq \sin p \qquad r = \sqrt{[(a-x)^2 + (b-y)^2 + (c-z)^2]} \quad (48)$$

so that by differentiating V we get the force acting in any direction. In this respect V resembles the Force Function, U, introduced by *Lagrange* for dealing with the mutually attracting planets of the Solar System.

2. The Analysis of *George Green*, 1828.

The well known paper entitled: »An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism«, was first published at Nottingham, 1828, and reprinted in *Green*'s Mathematical Papers, edited by *Ferrers*, 1870. In the preface to this essay, pages 37–38, of this reprint, *Green* remarks on the use of analysis in physical science: »M. *Fourier*, by his investigations relative to Heat, has not only discovered the general equations on which its motion depends, but has likewise been led to new analytical formulae, by whose aid MM. *Cauchy* and *Poisson* have been enabled to give the complete theory of the motion of waves in an indefinitely extended fluid.«

In dealing with the action of electricity (p. 19), *Green* uses V to denote the sum of all the electric particles acting on a point divided by their respective distances from this point. If x', y', z' be the coordinates of the electrified particle and x, y, z the coordinates of an exterior point, so that the distance r' is defined by the equation

$$r' = \sqrt{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]} \quad (49)$$

then *Green* puts:

$$V = \int 1/r' \cdot \varrho'\, dx'\, dy'\, dz' \quad (50)$$

the integral comprehending every particle in the electrified mass under consideration, ϱ' being the density of the electricity in this particle, $dx'\, dy'\, dz'$ being the element of volume, and therefore $\varrho'\, dx'\, dy'\, dz'$ the quantity of electricity in the element of volume.

The function V thus defined, and shown to fulfill the differential equations of *Laplace*, for an exterior point

$$\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0 \quad (51)$$

and of *Poisson* for an interior point

$$\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = -4\pi\varrho \quad (52)$$

Green calls the Potential Function, (p. 27). This is the earliest use of the term, and preceded *Helmholtz*'s theory of the Conservation of Energy, 1847, by some twenty years.

From this outline it will be seen that *Green* follows the definitions of *Laplace*, though he extends the researches on electricity and magnetism, not only to volumes of matter but also to surface action, and therefore develops the celebrated transformation known as *Green*'s Theorem (p. 23).

3. Analysis of *Gauss*, 1839. The results of *Gauss*' researches on forces varying inversely as the square of the distance, are given in Resultate aus den Beobachtungen des Magnetischen Vereins im Jahre 1839, Leipzig, 1840, (reprinted in *Gauss*' Werke, Band V, pp. 197–242). On page 199 of *Gauss*' Werke, Band V, we find that this celebrated geometer also follows *Laplace* very strictly, putting for the potential:

$$V = \mu^0/r^0 + \mu'/r' + \mu''/r'' + \text{etc.} = \sum \mu/r \quad (53)$$

$$r = \sqrt{[(a-x)^2 + (b-y)^2 + (c-z)^2]} \quad (54)$$

whence the expression for V takes the form:

$$V = \int_0^{\pi} \int_0^{2\pi} \int_0^{r} 1/r \cdot k\, r^2 \sin u\, du\, d\lambda\, dr \quad (55)$$

$k\, dt = \sigma\, dx\, dy\, dz$ being the mass in an element of volume

$$dt = r^2 \sin u\, du\, d\lambda\, dr \quad (56)$$

in polar coordinates.

Accordingly it appears that *Gauss* does not depart from the definitions or mathematical analysis of *Laplace*, *Poisson* and *Green*, though his results are more general and comprehensive than those of the earlier investigators, who founded the theory of the potential function, and first introduced the use of it into physical science. As before remarked, *Laplace*'s potential function V for the particles of an attracting mass is exactly analogous to *Lagrange*'s force function U, as applied to the mutually attracting planets, in that a simple differentiation of the expression in respect to the coordinates gives the force resolved in any direction.

4. Analysis of *Dirichlet*, 1856–57, and of *Riemann*, 1861.

The lectures of *Dirichlet* at Göttingen for the winter semester of 1856–57, were finally published by *Grube*, under the title: Vorlesungen über die im umgekehrten Verhältnis des Quadrats der Entfernung wirkenden Kräfte von *P. G. Le Jeune-Dirichlet*, Leipzig, 1876. *Dirichlet* takes the Potential as the sum of all active mass-particles, each divided by its distance from the attracted point, thus (p. 4):

$$V = \sum 1/r \cdot m = m/r + m'/r' + m''/r'' \cdots. \qquad (57)$$

Accordingly, as in the works of *Gauss* and *Laplace*. this is also written by *Dirichlet* (pp. 69–70):

$$V = \int 1/r \cdot k \, ds = \iiint 1/r \cdot k \, dx \, dy \, dz \qquad (58)$$
$$k = \text{density}, \quad r = \sqrt{[(a-x)^2 + (b-y)^2 + (c-z)^2]}.$$

If we introduce polar coordinates,

$$\begin{aligned} a &= R\cos\vartheta' & x &= \varrho\cos\vartheta \\ b &= R\sin\vartheta'\cos\varphi' & y &= \varrho\sin\vartheta\cos\varphi \\ c &= R\sin\vartheta'\sin\varphi' & z &= \varrho\sin\vartheta\sin\varphi \end{aligned} \qquad (59)$$

the expression for r becomes

$$\begin{aligned} r &= \sqrt{\{R^2 + \varrho^2 - 2R\varrho\,[\cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi-\varphi')]\}} \\ \cos\omega &= \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi-\varphi') \end{aligned} \qquad (60)$$

where ω is the angle between the radii vectores R, ϱ, referred to the origin at the centre of the attracting mass.

And the Potential Function becomes:

$$V = R^2\int_0^{2\pi} d\varphi' \int_0^{\pi} k'\,(R^2 - 2R\varrho\cos\omega + \varrho^2)^{-1/2}\sin\vartheta'\,d\vartheta' \qquad (61)$$

which admits of development in converging series of spherical harmonics, having the form

$$V = R\sum_0^\infty (\varrho/R)^n \iint k'\,P_n(\cos\omega)\sin\vartheta'\,d\vartheta'\,d\varphi' \qquad (62)$$

for an inner point, $\varrho < R$,

$$V = R\sum_0^\infty (R/\varrho)^{n+1} \iint k'\,P_n(\cos\omega)\sin\vartheta'\,d\vartheta'\,d\varphi' \qquad (63)$$

for an outer point, $\varrho > R$.

The Vorlesungen delivered by *Riemann* at Göttingen, 1861, and closely related to those of *Dirichlet*, were published by *Hattendorff*, under the title: Schwere, Elektrizität und Magnetismus, Hannover, 1875. *Riemann*'s definition of the Potential follows the standard form:

$$\begin{aligned} V &= \iiint 1/r \cdot \varrho \, da \, db \, dc \\ r &= \sqrt{[(a-x)^2 + (b-y)^2 + (c-z)^2]} \end{aligned} \qquad (64)$$

Accordingly it appears that the function of the potential defined by *Laplace* in 1782 as

$$V = \iiint 1/r \cdot \sigma \, dx \, dy \, dz \qquad (65)$$

was adopted by his successors without change. The occasion for the geometrical significance of the function, in connection with the physical basis upon which it rests, was not carefully inquired into until the problem was taken up by the present writer in the New Theory of the Aether, 1920, AN 5044.

It is now obvious that the integral of the density into the element of volume, divided by the distance:

$$V = \iiint \sigma\,[(a-x)^2 + (b-y)^2 + (c-z)^2]^{-1/2}\,dx\,dy\,dz \qquad (66)$$

means physically that each atom sends out a wave of amplitude $A = k/r$, and thus the potential is the integral of these superposed wave amplitudes, for every element of mass, at its appropriate distance. The accumulated stress of the aether is therefore proportional to the superposed effects of all the atoms, and thus proportional to the whole mass, and inversely as the distance,

$$V = \int \frac{dM}{r} = \iiint \frac{\sigma\,dx\,dy\,dz}{\sqrt{[(a-x)^2 + (b-y)^2 + (c-z)^2]}}$$

and the forces resolved parallel to the coordinate axes:

$$\begin{aligned} X &= -\frac{\partial V}{\partial a} = \iiint \frac{\sigma(a-x)\,dx\,dy\,dz}{[(a-x)^2 + (b-y)^2 + (c-z)^2]^{3/2}} \\ Y &= -\frac{\partial V}{\partial b} = \iiint \frac{\sigma(b-y)\,dx\,dy\,dz}{[(a-x)^2 + (b-y)^2 + (c-z)^2]^{3/2}} \\ Z &= -\frac{\partial V}{\partial c} = \iiint \frac{\sigma(c-x)\,dx\,dy\,dz}{[(a-x)^2 + (b-y)^2 + (c-z)^2]^{3/2}}. \end{aligned} \qquad (67)$$

The resultant total force therefore is

$$R = \sqrt{[X^2 + Y^2 + Z^2]}. \qquad (68)$$

Final conclusion regarding the geometrical and physical significance of the potential.

Maxwell's celebrated Treatise on Electricity and Magnetism, 1873, contains exact definitions and suggestive discussion of the geometrical and physical significance of the potential: in fact *Maxwell* is the best guide I know of, in dealing with this hitherto obscure subject. But as he says in section 863 that he could not solve the mystery from *C. Neumann*'s latest treatment, I finally was led to the theory developed in this paper.

After studying *Maxwell*'s critical discussion, which is condensed in Chapter 23, Sections 846–866, I conclude that the whole case may be summed up as follows:

1. The validity of *Weber*'s fundamental electrodynamic law of 1846 is admitted, and that implies wave action propagated in time, and therefore generating inductions, by the action of the waves, — not only electric and magnetic, but also gravitational. Gravitational waves are similar to the magnetic waves, except that they are not polarized, and thus cannot be used in a dynamo, nor can their effects be perceived, except by the deflections, etc., in transmission through the Earth, at the time of lunar eclipses — such as occur in *Newcomb*'s Fluctuations of the Moon's Mean Motion.

2. The criticism of *Clausius* (*Poggendorff*'s Annalen, Band 135, p. 612) correctly shows that *Riemann*'s electrodynamic formula is inadmissible, in that it implies a propagation of the potential, like light, under the modified form of *Poisson*'s equation:

$$d^2V/dx^2 + d^2V/dy^2 + d^2V/dz^2 + 4\pi\varrho = 1/a^2 \cdot d^2V/dt^2 \qquad (69)$$
$$a = \text{Velocity}$$

and otherwise does not conform to *Weber*'s law, and other known electrodynamic laws, such as those of *Ampère*. *Maxwell* notes the equally valid criticism of *Clausius* against *C. Neumann*'s theory of the transmission of the potential from particle to particle, but in a manner different from that of light. *Maxwell* says he could not form from *Neumann*'s theory the »Construirbare Vorstellung« of the process of transmission which appeared necessary to *Gauss*, in the letter to *Weber*, March 19, 1845.

3. All such theories, implying that the potential is transmitted in time, manifestly are untenable. The potential is a state of wave-stress in the aether, depending for gravitational bodies upon the triple integral

$$V=\int\int\int\sigma\left[(a-x)^2+(b-y)^2+(c-z)^2\right]^{-1/2}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \quad (70)$$

and thus a function of the mass and inverse distance, because each atom emits characteristic waves, and their amplitudes vary inversely as the distances. This state of wave-stress varies with the mass and distance simply, $V=\int 1/r\cdot\mathrm{d}m$, but does not depend on the time, except in moving bodies as r changes.

For electrical and magnetic bodies corresponding potentials exist, as defined by *Maxwell*: but these potentials too are states of stress in the aether due to waves. The waves produce forces, which themselves are transmitted in time like light, while the potentials are superposed states of stress, not agencies of action. All hypotheses of the transmission of the potential in time, as if representing action, are misleading!

Perhaps we ought to add that the forces, which are transmitted in time, could be changed only by change of mass: in that sense the potential also would be changed correspondingly, and the change would come about with the velocity of light. But whilst the forces act upon bodies with the velocity of light, the potential is merely a superposed wave-state, making possible the stresses for generating the forces observed in physical actions. These latter act upon bodies and thus do work, whilst the potential is a fixed state of stress making possible the action of the forces.

4. It is remarkable that the simple theory of the potential set forth in the New Theory of the Aether so long escaped the notice of geometers and natural philosophers. It is directly deducible from *Laplace*'s definition of $V=\int 1/r\cdot\mathrm{d}m$, 1782, yet the natural development from the amplitude of the waves was overlooked for about 140 years. As the laws of *Weber*, *Ampère*, *Biot* and *Savart*, *Ohm*, *Oersted*, etc., are harmonized by this simple theory, we have no hesitation in affirming that the wave-theory represents the true order of nature. Any attempt to deny this obvious conclusion is equivalent to the contradiction of the plainest truths of geometry, and the rejection of the best experimental evidence of our senses in the vast science of Electrodynamics.

Interpretation of *Lagrange*'s force function U.

We may now point out that in the force function, introduced by *Lagrange* for the mutual potential of a system of bodies, taken in pairs, $m_i m_j$, at mutual distances Δ_{ij}, the very form of the expression for this function, namely:

$$\begin{aligned}U = m_0 m_1/\Delta_{0.1} + m_0 m_2/\Delta_{0.2} + m_0 m_3/\Delta_{0.3} + \cdots + m_0 m_n/\Delta_{0.n} \quad & i=0,\ j=1,\cdots j=n\\ + m_1 m_2/\Delta_{1.2} + m_1 m_3/\Delta_{1.3} + \cdots + m_1 m_n/\Delta_{1.n} \quad & i=1,\ j=2,\cdots j=n\\ + m_2 m_3/\Delta_{2.3} + \cdots + m_2 m_n/\Delta_{2.n} \quad & i=2,\ j=3,\cdots j=n \qquad (71)\\ \cdots\cdots\cdots\cdots \quad & \cdots\cdots\\ + m_{n-1} m_n/\Delta_{n-1,n} \quad & i=n-1,\ j=n\end{aligned}$$

$$=\sum_{i=0}^{i=n-1}\sum_{j=1}^{j=n} m_i m_j/\Delta_{ij},\quad \Delta_{ij}^2=(\xi_i-\xi_j)^2+(\eta_i-\eta_j)^2+(\zeta_i-\zeta_j)^2 \quad (72)$$

shows that the gravitational forces act in right lines between the bodies, severally connecting them in pairs, and thus consist in tensions or stretching of the aethereal medium filling the celestial spaces.

Accordingly all the foregoing discussion applies to *Lagrange*'s force function U, for a system of separate bodies in space, at mutual distances Δ_{ij}, as well as to *Laplace*'s function V, for the particles of a single mass like the Earth or Sun. We notice merely that in *Lagrange*'s force function U the bodies are treated as particles of mass $m_i m_j$ acting from their several centres of gravity. Thus *Lagrange*'s force function has a finite algebraic form, and does not involve integrals — which greatly simplifies many calculations in the Theory of universal gravitation.

In forming a more critical judgement, however, of *Lagrange*'s Function, it should be borne in mind that the terms of the planetary potential do really involve sextuple integrals, the differentiation of which gives the Newtonian forces pulling in right lines between the bodies in pairs. Owing to limitations of space we shall here write only six terms of the semi-quadrilateral of integrals:

$$\begin{aligned}U=&\int\int\int\int\int\int\frac{\sigma_0\,\sigma_1\,\mathrm{d}x_0\,\mathrm{d}x_1\,\mathrm{d}y_0\,\mathrm{d}y_1\,\mathrm{d}z_0\,\mathrm{d}z_1}{\sqrt{[(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2]}}+\int\int\int\int\int\int\frac{\sigma_0\,\sigma_2\,\mathrm{d}x_0\,\mathrm{d}x_2\,\mathrm{d}y_0\,\mathrm{d}y_2\,\mathrm{d}z_0\,\mathrm{d}z_2}{\sqrt{[(x_2-x_0)^2+(y_2-y_0)^2+(z_2-z_0)^2]}}+\int\int\int\int\int\int\frac{\sigma_0\,\sigma_3\,\mathrm{d}x_0\,\mathrm{d}x_3\cdots}{\sqrt{[(x_3-x_0)^2+\cdots}}+\cdots\\ &+\int\int\int\int\int\int\frac{\sigma_1\,\sigma_2\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}z_1\,\mathrm{d}z_2}{\sqrt{[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]}}+\int\int\int\int\int\int\frac{\sigma_1\,\sigma_3\,\mathrm{d}x_1\,\mathrm{d}x_3\cdots}{\sqrt{[(x_3-x_1)^2+\cdots}}+\cdots\\ &+\int\int\int\int\int\int\frac{\sigma_2\,\sigma_3\,\mathrm{d}x_2\,\mathrm{d}x_3\cdots}{\sqrt{[(x_3-x_2)^2+\cdots}}+\cdots\\ &+\cdots \quad (71\,\mathrm{a})\end{aligned}$$

$$=\sum_{i=0}^{i=i}\sum_{j=1}^{j=j}\int\int\int\int\int\int\frac{\sigma_i\,\mathrm{d}x_i\,\mathrm{d}y_i\,\mathrm{d}z_i\,\sigma_j\,\mathrm{d}x_j\,\mathrm{d}y_j\,\mathrm{d}z_j}{\sqrt{[(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2]}}. \quad (72\,\mathrm{a})$$

It will be seen therefore that *Lagrange*'s function in reality is made up of a series of sextuple integrals: yet we often use the finite algebraic form for the terms when we treat the heavenly bodies as having their masses collected at their centres of gravity, which at a distance is very nearly correct, owing to their figures, under universal gravitation, taking the form of spheroids made up of concentric spheroidal shells of uniform density. The above integrals (71a) are well adapted for showing the complexity of the planetary forces, under the wavefields of the different bodies, which, if spaced at suitable intervals, might give rise to the Music of the Spheres, as explained in the footnote of section **2**, between equations (13) and (14).

8. The Integrals required for the Evaluation of the Mutual Interaction of two Infinite Interpenetrating Masses.

In taking up the more general problem of attraction we shall begin with the integrals which arise in the evaluation of the mutual interactions of ponderable bodies, when the bodies interpenetrate and are of unlimited extent, so as to occupy all space.

Let us imagine the universe filled with any one kind of matter, such as Hydrogen, of density σ, so that the element of mass becomes

$$dm = \sigma\, dx\, dy\, dz\,. \tag{73}$$

The integral for all such matter in the Universe is

$$M = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sigma\, dx\, dy\, dz\,. \tag{74}$$

If there be another kind of matter, such as Helium, spread over the universe, of density σ', and with the element

$$dm' = \sigma'\, dx'\, dy'\, dz' \tag{75}$$

then the integral for all such matter in the universe is

$$M' = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \sigma'\, dx'\, dy'\, dz'\,. \tag{76}$$

Matter of the type of Hydrogen and Helium would freely interpenetrate, owing to the movement of the molecules; and thus to compute the mutual potential energy of two such infinite masses, we should have to evaluate a sextuple integral, which not only includes every particle of each mass, but gives the potential of each element, $dm = \sigma\, dx\, dy\, dz$ in respect to every element of the other body $dm' = \sigma'\, dx'\, dy'\, dz'$ at the infinitely varying distances r, $r = \sqrt{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]}$, which mutually connect these two elements in pairs throughout all space.

The mutual potential energy of two infinite interpenetrating masses can therefore be expressed only by means of a sextuple integral of the following form:

$$\Omega = \int\int\int\int\int\int \frac{\sigma\, dx\, dy\, dz\, \sigma'\, dx'\, dy'\, dz'}{\sqrt{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]}}\,. \tag{77}$$

For if the potential due to one body in respect to unit mass be

$$V = \int\int\int \frac{\sigma\, dx\, dy\, dz}{\sqrt{[(a-x)^2 + (b-y)^2 + (c-z)^2]}} \tag{77a}$$

the forces resolved along the coordinate axes for an indefinitely extended mass would be:

$$X = -\partial V/\partial a = \int_{-\infty}^{+\infty}\int\int \frac{\sigma(a-x)\, dx\, dy\, dz}{[(a-x)^2 + (b-y)^2 + (c-z)^2]^{3/2}}$$

$$Y = -\partial V/\partial b = \int_{-\infty}^{+\infty}\int\int \frac{\sigma(b-y)\, dx\, dy\, dz}{[(a-x)^2 + (b-y)^2 + (c-z)^2]^{3/2}} \tag{77b}$$

$$Z = -\partial V/\partial c = \int_{-\infty}^{+\infty}\int\int \frac{\sigma(c-z)\, dx\, dy\, dz}{[(a-x)^2 + (b-y)^2 + (c-z)^2]^{3/2}}\,.$$

And as the total force is

$$R = \sqrt{[X^2 + Y^2 + Z^2]} \tag{78}$$

so that the components are:

$$\begin{aligned} -\partial R/\partial x &= X[X^2 + Y^2 + Z^2]^{-1/2} \\ -\partial R/\partial y &= Y[X^2 + Y^2 + Z^2]^{-1/2} \\ -\partial R/\partial z &= Z[X^2 + Y^2 + Z^2]^{-1/2} \end{aligned} \tag{79}$$

we easily verify the form of (77) above, when the mutual potential energy is sought between both masses, by means of the sextuple integration. It must include each element of either mass with the corresponding element of the other at all distances.

Accordingly, it appears that a triple integral, of the form of (74) above,

$$M = \int_{-\infty}^{+\infty}\int\int \sigma\, dx\, dy\, dz$$

includes all points of space; and since the integral (76) also includes another independent summation of all points of space, it follows that the sextuple integral (77) includes a summation of all points of space in respect to all other points of space at their several distances $r = \sqrt{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]}$. This conception is very important not only in the theory of the potential and attraction of gravitating matter, but also in the theory of the velocity-potential Φ, which comes up in the Theory of the Aether, in which the forces are explained by wave-action.

Before leaving the problem of potential energy, we may cite a practical example given by Lord *Kelvin*, Baltimore Lectures, 1904, p. 270, as follows.

If R denote the resultant force acting on any particle $dm = \sigma\, dx\, dy\, dz$ at any point $p(x, y, z)$, the exhaustion of gravitational energy produced by bringing a vast number N of equal masses from rest at an infinite distance to an equally-spaced distribution through a sphere of radius r is easily shown to be

$$E = 1/8\pi \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R^2\, dx\, dy\, dz = \tfrac{3}{10} F r \tag{80}$$

where F denotes the resultant force of the attraction of the N equal masses on a material point of mass M equal to the sum of their masses, $M = N\mu$, where μ is the mass of any one of the globes, placed at the spherical surface of radius r.

By such methods it is possible to calculate the resultant force of an indefinitely expanded nebulous mass upon a particle, within or without, and we may also study the mutual action of two infinite interpenetrating nebulous masses, such as the hydrogen and helium of the sidereal universe. But in view of our ignorance of the laws of density, and of the distances r, at which the elements dm, dm' act, there are limits on the practical usefulness of the calculations, owing to the uncertainty of the hypotheses underlying the elements of the integral and therefore affecting its evaluation.

9. The Mutual Actions of Masses are to be grouped in Pairs, made up in Accordance with the Terms of *Lagrange*'s Force Function.

If a system of bodies be subjected to the mutual action of their gravitational forces, we see by the form of *Lagrange*'s force function, equation (71) above, that the grouping of the masses is in pairs, the general form for the potential being

$$U = \sum_{i=0}^{i=i} \sum_{j=1}^{j=j} m_i m_j / \Delta_{i,j} . \quad (81)$$

For the forces are found by differentiating U in respect to the coordinates, since that gives the components of the stress acting along the axes.

1. For just two masses the potential, like the force, is confined to a single term of the same integral form as that for the Newtonian law, $V = mm'/r$:

$$U^{(2)} = m_0 m_1 / \Delta_{0.1} ; \quad (82)$$

while for three masses there are two additional terms, namely:

$$U^{(3)} = m_0 m_1 / \Delta_{0.1} + m_0 m_2 / \Delta_{0.2} + m_1 m_2 / \Delta_{1.2} . \quad (83)$$

And in general for n bodies, each increase in n adds the column of terms to the right, in the semi-quadrilateral form outlined in equation (71). Thus the mutual interactions of the parts of the system become very complex with increase in the number of bodies composing it.

As we see by equation (77) above the mutual potential energy of two infinite interpenetrating masses is given by the sextuple integral:

$$E_1^{(0\cdot 1)} = \int\int\int_{-\infty}^{+\infty}\int\int\int \frac{\sigma_0\, dx_0\, dy_0\, dz_0 \cdot \sigma_1\, dx_1\, dy_1\, dz_1}{\sqrt{[(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2]}} \quad (84)$$

in which $dm_0 = \sigma_0\, dx_0\, dy_0\, dz_0$ is the element of the first mass, and $dm_1 = \sigma_1\, dx_1\, dy_1\, dz_1$ that of the second mass, and the distances are

$$\Delta_{i,j} = \sqrt{[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]} \quad (85)$$

for the mutual distances of the elements dm and dm_1.

If there were conceived a third infinite mass, with element $dm_2 = \sigma_2\, dx_2\, dy_2\, dz_2$ the mutual potential energy relative to dm_0 would involve two additional integrals: namely that relative to dm_2

$$E_2^{(0\cdot 2)} = \int\int\int_{-\infty}^{+\infty}\int\int\int \frac{\sigma_0\, dx_0\, dy_0\, dz_0 \cdot \sigma_2\, dx_2\, dy_2\, dz_2}{\sqrt{[(x_2-x_0)^2+(y_2-y_0)^2+(z_2-z_0)^2]}} . \quad (86)$$

And that relative to dm_1

$$E_3^{(1\cdot 2)} = \int\int\int_{-\infty}^{+\infty}\int\int\int \frac{\sigma_1\, dx_1\, dy_1\, dz_1 \cdot \sigma_2\, dx_2\, dy_2\, dz_2}{\sqrt{[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]}} . \quad (87)$$

If there be a fourth infinite mass $dm_3 = \sigma_3\, dx_3\, dy_3\, dz_3$, we should have to add in like manner three additional integrals:

$$E_4^{(0\cdot 3)} = \int\int\int_{-\infty}^{+\infty}\int\int\int \frac{\sigma_0\, dx_0\, dy_0\, dz_0 \cdot \sigma_3\, dx_3\, dy_3\, dz_3}{\sqrt{[(x_3-x_0)^2+(y_3-y_0)^2+(z_3-z_0)^2]}} \quad (88)$$

$$E_5^{(1\cdot 3)} = \int\int\int_{-\infty}^{+\infty}\int\int\int \frac{\sigma_1\, dx_1\, dy_1\, dz_1 \cdot \sigma_3\, dx_3\, dy_3\, dz_3}{\sqrt{[(x_3-x_1)^2+(y_3-y_1)^2+(z_3-z_1)^2]}} \quad (89)$$

$$E_6^{(2\cdot 3)} = \int\int\int_{-\infty}^{+\infty}\int\int\int \frac{\sigma_2\, dx_2\, dy_2\, dz_2 \cdot \sigma_3\, dx_3\, dy_3\, dz_3}{\sqrt{[(x_3-x_2)^2+(y_3-y_2)^2+(z_3-z_2)^2]}} . \quad (90)$$

However high the order of the system may be, the integrals will follow the form of the algebraic terms in *Lagrange*'s force function, and thus may be summed up like the algebraic terms in equation (71).

In passing from the algebraic terms of (71) to the integrals (84)–(90) inclusive, we go from an action conceived as concentrated in the centres of gravity of the mass, at their respective inverse distances, to an action which must be integrated as operating between the particles throughout space, and cannot be conceived as concentrated anywhere.

Now if the diffuse matter of a nebula may be conceived as diffused indefinitely through the universe, and two such interpenetrating masses could mutually interact, according to the Newtonian law of universal gravitation, still more is such interaction conceivable for wave-disturbances, proceeding from two separate centres, with the waves so interpenetrating at all points of space, that the mutual actions of the waves must be integrated throughout all space. We shall investigate the effects of this wave-interpenetration after we have defined the Fourier-Poisson Function Φ, known as the velocity potential, section 13, but meanwhile we may contemplate the wave movement from the physical point of view.

10. How the Wave-Effects spread from two Centres of Disturbance, to give Tension between the Bodies and Increase of Pressure beyond them.

In explanation of the earlier researches on the wave-theory of gravitation, which took definite direction since 1914, it may be added that for many years I had been familiar with the optical interference patterns, formed when polarized light is transmitted through uniaxial and biaxial crystals, illustrated in *Ganot*'s Physics, 14th edition, 1893, Art. 667, Plate II, figs. 4, 5, 6; and in Sir John *Herschel*'s great Treatise on Light, Encyc. Metrop., London, 1849, Figs. 176–183, Arts. 892–902, pp. 518–519. Cf. Table V.

Sir John *Herschel* experimentally tested the form of this pattern by comparing it, as observed, with the curve known as the lemniscate, so carefully studied by *Gauss*. *Herschel*'s discussion runs thus:

»This variation of form, as well as the general figure of the curves, bears a perfect resemblance to what obtains in the curve well known to geometers under the name of the lemniscate, whose general equation is

$$(x^2 + y^2 + a^2)^2 = a^2 (b^2 + 4x^2), \qquad (91)$$

when the parameter b gradually diminishes from infinity to zero; $2a$ representing the constant distance between the poles.«

»The apparatus just described affords a ready and very accurate method of comparing the real form of the rings with this or any other proposed hypothesis. If fixed against an opening in the shutter of a darkened room, with the lens H outwards, and a beam of solar light be thrown on the latter, parallel to the axis of the apparatus, the whole system of rings will be seen finely projected against a screen held at a moderate distance from E. Now, if this screen be of good smooth paper tightly stretched on a frame, the outlines of the several rings may easily be traced with a pencil on it, and the poles being in like manner marked, we have a faithful representation of the rings, which may be compared at leisure with a system of lemniscates, or any other curve graphically constructed, so as to pass through points in them chosen where the tint is most decided. This has accordingly been done, and it has been found that lemniscates so constructed coincide throughout their whole extent to minute precision, with the outlines of the rings so traced, the points graphically laid down falling on the pencilled outlines. The graphical construction of these curves is rendered easy by the well known property of the lemniscate, in which the rectangle under two lines PA, $P'A$ drawn from the poles to any point A in the periphery is invariable throughout the whole curve. This is easily shown from the above equation, and the value of this constant rectangle in any one curve is represented by $a \times b$.«

»When we shift from one ring to another, a remains the same, because the poles are the same for all. To determine the variation of b, let the rings be illuminated with homogeneous light (or viewed through a red glass), and outlined by projection, as above. Then, if we determine the actual value of ab by measuring the lengths of two lines PA, $P'A$ drawn from P, P' to any point of each curve; and, calculating their product, (to which ab is equal), it will be found that this product, and therefore the parameter b, increases in the arithmetical progression 0, 1, 2, 3, 4, etc. for the several dark intervals of the rings beginning at the pole, and in the progression 1/2, 3/2, 5/2, etc. for the brightest intermediate spaces. To ensure accuracy, the mean of a number of values of PA, $P'A$, at different points of the periphery, may be taken to obviate the effect of any imperfection in the crystal.«

Already in 1914, when studying Sir John *Herschel*'s Treatise on Light I became greatly impressed[1]) with the work of *Gauss*, Allgemeine Theorie des Erdmagnetismus, 1838, which includes the following figure (*Gauss*' Werke, Band V, p. 176).

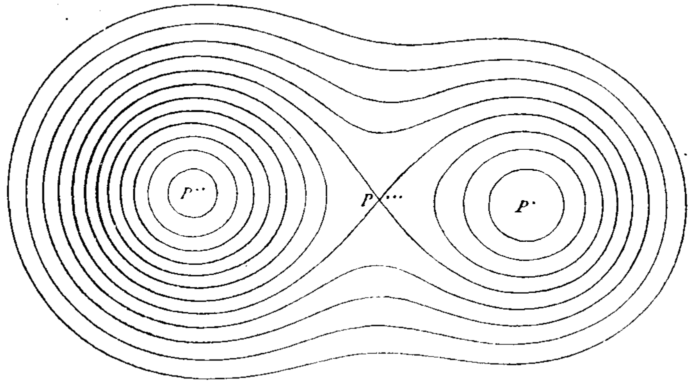

Fig. 6. *Gauss*' illustration of the magnetic equipotential surfaces about two neighboring points P^* and P^{**}, which have maximum values of the potential V, namely V^* and V^{**}, like two adjacent poles. The potential for any point between them, as P^{***}, has a smaller value V^{***}. In general the value of V decreases in the outer closed lines, as in the gravitational equipotential surfaces drawn by *Thomson* and *Tait*, 1873, Plate VII; all of which is found to be the best possible argument for the wave-theory of magnetism, gravitation, and similar physical forces.

Owing to constant familiarity with the nature of the attraction about two masses, which I had studied for 25 years, the forms of these surfaces, and their conformation about the two centres, — corresponding to the two axes in crystals around which *Newton*'s rings in the interference pattern are formed, when light is transmitted through thin layers of the material, as described in *Herschel*'s experiments above cited, — at once impressed me as furnishing an accurate picture of the wave-operations underlying universal gravitation, magnetism, etc.

Accordingly, in the Electrodynamic Wave-Theory of Physical Forces, vol. I, 1917, pp. 137–139, I discussed the equipotential surfaces about two attracting masses, — one case representing a pair of equal stars, the other case having masses in the ratio of 10 to 1, as treated also somewhat elaborately in my Researches on the Evolution of the Stellar Systems, vol. II, 1910 — and showed that at every point of the surface of the hour-glass figure about the two centres, »the total resultant force is normal to the surface and directed inward. There is thus a process of constriction at work, under the gravitational action of the two bodies, which is less powerful as the surfaces become more distant from the centres. The constant of *Jacobi* also correspondingly decreases, as the surfaces recede and are less constricted.«

»It is justly observed that the narrowing of the surface into an hour-glass form between the two bodies is a visible and obvious effect of the tension in the medium.«

»The pressure in the medium towards the separate centres, due to the waves propagated from those centres,

[1]) It was of course recognized from the first that the transmission of polarized light through thin plates of crystal, with the formation of the interference pattern analogous to *Newton*'s rings, but resembling the lemniscate of *Bernoulli*, is not quite identical physically with the problem of the equipotential surfaces under the free waves of gravitation proceeding from two centres. Yet as the surface forms due to interference, in the transmission of the light waves through the crystal, are similar to the equipotential surfaces about two equal stars, as drawn by Lord *Kelvin*, 1873, the problem of the equipotential surfaces came to be more and more studied, and the closeness of the analogy between the light wave interference patterns and the gravitational equipotential surfaces confirmed. In both cases there is a radiation of waves taking form about two centres; and thus the surfaces and patterns may properly be compared, — even if the movement of the gravitational waves in space is free, while in the crystal the waves are polarized and the movement so restricted that only interference patterns result about the axial centres of the crystal.

thus gives tension between the bodies, tends to narrow the surfaces of constant relative energy, or equipotential surfaces, and gives them the forms here shown.«

The subject here treated of in 1917 is discussed with more detail in the VI. Paper on the New Theory of the Aether, AN 5140, pp. 103–110, to which the reader is especially referred. The figures and vectorial analysis with geometrical composition given in AN 5140 (cf. Plate VII below), afford perfectly conclusive evidence for the wave-theory of universal gravitation. Every possible combination of the separate vectors, directed to the two centres of attraction, is treated of in AN 5140, pp. 105–106, Plate VII below, and all the phenomena confirm the wave-theory.

In fact it may be said that if in any case waves are known to exist, as recorded by observation in solar disturbances, so systematically studied in magnetic storms, aurorae, Earth currents, — then the mere chance that such ordered concentric surfaces could exist about two attracting masses, without depending on waves from these centres, would exceed all the points in infinite space to one, or

$$c_1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx\,dy\,dz : 1 \qquad 8\infty^3 : 1 = c_1 \qquad (92)$$

an infinity of the third order!

11. New Diagram for illustrating the Motion of the Molecules of Air in the Phenomenon of Acoustic Attraction: Confirmation of the Wave-Theory of Universal Gravitation by Definite Laboratory Experiments admitting of but one Interpretation.

The diagram Plate VI illustrates in considerable detail the wave-theory of acoustic attraction worked out by me in 1917, but not published until the appearance of the V. and VI. Papers on the Aether, AN 5130, 5140, Nov. 1921; Jan., 1922.

From the study of this figure we may make out the nature of the disturbances of the air particles in acoustic attraction, which are exactly analogous to the disturbances of the aetherons in the wave-theory of gravitation.

1. It will be seen that as the sound waves from either centre, S or S', advance through the balloons filled with carbon dioxide, a gas heavier than air, the wave in the air outruns that proceeding through the heavier gaseous medium filling the balloons. Thus in any phase of the wave the disturbance outside either globe is ahead of that passing through that inert mass: the result is a series of uncompensated impulses exerted in rapid succession on the rear of the balloons, which are elastic membranes yielding physical reactions on the whole masses.

2. These balloons react to the uncompensated agitations or impulses in their rears: and the rebound of either balloon is towards the other! Thus the pressure is increased beyond either balloon, by the waves from the other working the air particles out from between the balloons, and gradually transferring them around to the rear, where slight impulses are developed with the passing of each wave of sound. Accordingly, by this process of wave agitation, a partial rarefaction develops between the balloons, yielding a tension in the line SS', and an increased pressure beyond the balloons, on the same axis prolonged.

3. The increase of pressure beyond, with simultaneous development of tension in the line SS' connecting the bodies, is precisely the mechanical effect required for explaining the observed facts of universal gravitation. All the known gravitational phenomena would be perfectly accounted for if this could be proved to occur in nature. And such proof is forthcoming not only in the air, from the experiments on acoustic attraction, by *Guyot*, *Schellbach*, *Guthrie* and Sir *W. Thomson*, reported in the Phil. Magaz. for June, 1871, but also in the aether from the present known facts of radio wave-transmission about the Earth (cf. AN 5317, Sept., 1924), in which wave-bending is exactly similar to that here described.

4. The bending of the radio-wave about the Earth is shown to extend to the very antipodes, in the careful experiments conducted by the French from the towers at Bordeaux, when the waves were noted at Chatham Island, some 300 miles East of New Zealand. It has been recognized by me since 1914 (cf. AN 5044, pp. 70–71), that the radio-wave bends around the Earth, for the same reason that light bends in passing through a prism: namely, because the velocity is less in the dense mass, owing to the wave motion being there resisted by the ponderable matter in the path, whereby the normal to the wave front is tilted towards the denser body. This correct theory of 1920 has been demonstrated more and more, as by concerts heard in deep mines, and in remote continents. Thus it follows that the aether waves in radio telegraphy do pass through the globe. This is also required by the enormous elasticity of the aether, $\epsilon = 689321600000$ times greater than that of air in proportion to its density, which does not permit a break in the continuity of that medium.

5. A vast mass of observed phenomena connected with sunspots, magnetic storms, aurorae and Earth currents point to electrodynamic waves from the Sun. The waves are almost incessantly passing the Earth, and thus are simultaneously observed at all points of our globe. These waves also are bent about the Earth as indicated in the celebrated Fluctuations of the Moon's Mean Motion, first formulated by *Newcomb*, 1909, but first explained by the present writer, Dec. 10, 1916, (Electrodynamic Wave-Theory of Physical Forces, vol. I, 1917).

6. Now the above celestial phenomena involving waves in the aether correspond exactly with the wave-theory of acoustic attraction, as illustrated above. And since acoustic attraction is a tangible phenomenon of our laboratories, which may be dealt with definitely, in precise measurements, admitting of not the slightest doubt, it is justly cited as an objective demonstration of the true behavior of the aether waves passing through and refracted about the heavenly bodies. The wave-theory of universal gravitation therefore is amply confirmed by experiments of undoubted validity, which admit of one interpretation and only one.

12. The Fluctuations of the Moon's Mean Motion established by *Newcomb*, 1909, point to the Wave-Theory.

This subject has been discussed at length in volume I of my Electrodynamic Wave-Theory of Physical Forces, 1917;

but at that time the proofs of the wave-theory were much less complete than they are now, and therefore we shall bring the argument up to date, by a few considerations, which should be borne in mind at the present day.

1. It is now fully recognized that radio waves of the length of thousands of metres, which are employed in long distance transmission, go through the Earth itself. Musical concerts conveyed by such waves are heard in deep mines, and even in distant continents — a Pittsburg concert heard in Calcutta, a Chicago concert in Melbourne, etc. And finally signals from the Lafayette Towers at Bordeaux, using wave lengths of 22.5 km, were heard at the antipodes, near Chatham Island, some 300 miles east of New Zealand.

2. The only way we can explain these phenomena is by recognizing that the waves travel through the Earth itself, but at a slower pace than through the air, owing to the increased resistance of the solid globe, and thereby the wave-front is bent around the Earth, even to the antipodes, at Chatham Island. The theory of this movement is exactly similar to that of light through a prism, and was first published by me in AN 5044, p. 51, May, 1920. Accordingly if gravitation be due to waves, these waves will also be bent in passing through the Earth, and thus the Sun's action on the Moon will be less when our satellite is in or near the shadow of the Earth, as at the time of lunar eclipses.

3. This subject was first touched upon by Dr. *K. F. Bottlinger*, in a crowned prize Inaugural Dissertation, at the University of Munich, 1912, printed in Freiburg, 1912.

But although *Bottlinger* first treated the problem carefully, he did not find the result he expected, not being able to discover from the Moon's motion the theoretical eclipse cycles corresponding to the observed fluctuations. In my work of 1917 I was more successful, as shown by the following table of the observed fluctuations and theoretical fluctuations depending on eclipse cycles.

Periods of the observed fluctuations, according to *Newcomb E. W. Brown*, and others.	Theoretical fluctuations deduced from the theory of lunar eclipses, 1917.	Differences, Obs. Calc.
275 years	277.59 years	2.59 years
60 years	61.7006 years	1.7006 years
—	18.03 years	

4. The eclipse cycles employed by me have been in use since the time of *Ptolemy*, and were deduced by the Greeks from observations recorded in the chronology of the era of *Nabonassar*. Modern investigators therefore are unable to deny the validity of their observational foundation, based on the records of 2500 years. The only way they can evade the argument built up in 1916–17 is to hold that the Moon's motion has no regular periodicity. *Newcomb*, *Brown* and others are on record, however, as verifying the periodicity, by their researches in the lunar theory, with the cycles above cited. They spoke confidently of these observed fluctuations a few years before I discovered the theoretical cycles of eclipses on which they depend.

5. If now Professor *Brown* hesitates to uphold a theory which he indicated, yet did not deduce,[1]) we may well dismiss the discussion as a case of *Brown*, 1914, versus *Brown*, 1920.

The Moon continues to depart a little from its predicted place, — as at the recent eclipse in New York, Jan. 24, 1925, when it proved to be about 4″ behind of its calculated place. This result shows that the old theory is defective. The only new hope therefore lies in the lunar eclipse cycles, with the theoretical fluctuations in 18.03, 61.7006, and 277.59 years, worked out by me in 1917.

6. The theoretical fluctuations thus deduced leave no outstanding anomaly between the years 1829 and 1909 — 80 years of the best modern records — which is much in excess of 1″ or at most 2″. I cannot believe that the later records will be materially different, if the Lunar calculations were properly carried out. It is most improbable that the Moon would suddenly depart from cycles shown by occultations since 1680.

7. Professor *E. W. Brown*'s New Lunar Tables, 1922, give only very slight differences from *Hansen*'s Tables of the Moon, 1857. Neither of these tables really represent the actual motion of the Moon. As far back as 1878 *Newcomb* had to introduce an empirical correction of 16″, with period of 275 years, and before 1907–1909 he discovered another of 60 year period, with coefficient of 3″. Both of these periods are distinctly indicated, and some irregularities already were known to *Laplace* in 1802. Now the periods are found from the lunar theory, in eclipse cycles used since the age of the Greeks and Babylonians. It is not easy to see how such a result, referred to true physical causes, can be evaded by investigators interested in the progress of celestial mechanics. The very essence of the Newtonian theory in astronomy is to refer observed phenomena to known physical causes.

8. To get at the basis of this reasoning we notice that just as the long wireless wave from Bordeaux, France, is bent

[1]) In the Address to the British Association in Australia, Aug. 22. 1914, Report, pp. 318–319, Professor *Brown* first points out that the outstanding residuals in the Moon's motion »consist mainly of long-period fluctuations in the mean longitude« »These unexplained differences between theory and observation may be separated into two parts. First, *Newcomb*'s term of period between 250 and 300 years and coefficient 13″, and, second, the fluctuations which appear to have an approximate period of 60 to 70 years. The former appears to be more important than the latter, but from the investigator's point of view it is less so. The force depends on the degree of inclination of the curve to the zero line or on the curvature, according to the hypothesis made. In either case the shorter period term is much more striking, and, as I have pointed out on several occasions, it is much more likely to lead to the sources of these terms than the longer period. It is also, at least for the last sixty years, much better determined from observation, and is not likely to be confounded with unknown secular changes.«

»Various hypotheses have been advanced within the last few years to account for these terms.« . . . . »The shading of gravitation by interposing matter, e. g. at the time of eclipses, has been examined by *Bottlinger*. For one reason alone, I believe this is very doubtful. It is difficult to see how new periodicities can be produced; the periods should be combinations of those already present in the moon's motion. The sixty to seventy years' fluctuation stands out in this respect because its period is not anywhere near any period present in the moon's motion or any probable combination of the moon's periods. Indeed Dr. *Bottlinger*'s curve shows this: there is no trace of the fluctuation.«

around to the antipodes, near New Zealand, so also the gravitational waves from the Sun should be bent towards the axis of the Earth's shadow, as shown in the following figure, 1917, with wave front put in 1920:

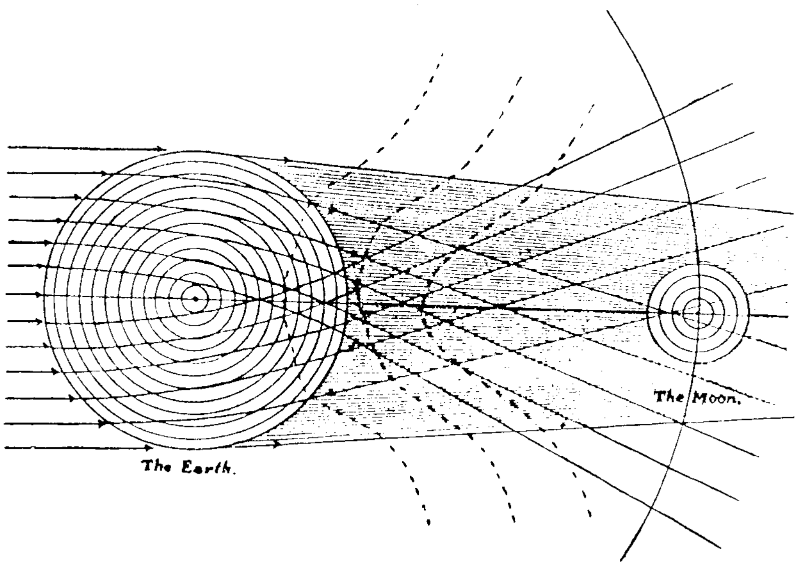

Fig. 7. Illustration of the bending of the Sun's gravitational waves in propagation through the Earth, at the time of lunar eclipses. This slightly releases the Sun's control of our Satellite, and it tends to fly the tangent. The result is the fluctuations in the Moon's mean motion with periods of 18.03, 61.7 and 277.59 years, as found by the researches of *Newcomb*, 1869 1909. Radio Telegraphy sends long waves to the antipodes, and this bending of the radio wave likewise is due to the slower propagation in the solid globe of the Earth, just as in the case of the Sun's gravitational waves here illustrated.

It is easy to see that the verified radio wave bending absolutely confirms this theory: it is therefore verified by the best observational data in radio telegraphy and by the whole wave-theory of light. Better proof of the refraction, deflection, and absorption of gravitational wave-energy in passing through our globe could not be desired.

9. Therefore since the waves of gravitation, which may be of any length between a metre and a thousand kilometres, would be bent as shown above, it is clear that the attraction of the Sun upon the Moon will vary a little in different parts of this shadow. It would be a miracle if the interposition of the Earth's solid mass in the path of the Sun's gravitational waves did not decrease slightly the solar pull on the Moon, by which our satellite would tend to fly the tangent near the time of lunar eclipses. From the movement of the sound waves about the carbon dioxide balloons, the pressure is increased beyond them: therefore the pressure in the aether beyond the Earth would be increased, and the Sun's tension on the Moon relaxed, by the interposition of the Earth's globe. The analogy is perfectly plain, and this conclusion is incontestable.

10. *Bottlinger*'s method of analysis is sufficiently condensed in vol. 1 of the Electrodynamic Wave-Theory of Physical Forces, pp. 114–115. He considers the interaction between a mass-point and a sphere made up of homogeneous concentric layers. If μ be the mass of the mass-point, M the mass of the sphere, R its radius, and $\sigma=\sigma(r)$ its density; then the attraction on μ is the integral of the acceleration vector to the elements of mass of the sphere:

$$z=\int B\,\mathrm{d}m \tag{93}$$

where

$$B=k^2/\varrho^2\cdot e^{-\varkappa\int_0^s\sigma\,\mathrm{d}\rho} \tag{94}$$

k^2 being the Gaussian constant of attraction, $\varkappa$ the absorption constant, and φ the angle about the axis of symmetry, and the other quantities as illustrated in the figure 8.

Bottlinger's theory of a slight absorption of a ray of gravitation in traversing the Earth.

Accordingly, the attraction becomes

$$z=\int B\cos\alpha\,\mathrm{d}m= \\ =k^2\iiint\sigma\,\mathrm{d}\varrho\,\mathrm{d}\alpha\,\mathrm{d}\varphi\sin\alpha\cos\alpha\, e^{-\varkappa\int_0^s\sigma\,\mathrm{d}\rho}\,. \tag{95}$$

If we introduce the angle β, the integration being between 0 and $\frac{1}{2}\pi$, we have

$$\sin\alpha=R/A\cdot\sin\beta \qquad \cos\alpha\,\mathrm{d}\alpha=R/A\cdot\cos\beta\,\mathrm{d}\beta\,. \tag{96}$$

Then, the definite integral for the attraction becomes:

$$z=k^2R^2/A^2\cdot\int_0^{1/2\pi}(\beta)\int_0^{2\pi}(\varphi)\int_0^{2r\cos\beta}(\varrho)\times \\ \times\sigma\,\mathrm{d}\beta\,\mathrm{d}\varrho\,\mathrm{d}\varphi\sin\beta\cos\beta\, e^{-\varkappa\int_0^s\sigma\,\mathrm{d}\rho} \tag{97}$$

where

$$\sigma=\sigma(r)=\sigma\,[R^2+S^2-2R\,S\cos\beta]^{1/2}\,. \tag{98}$$

Bottlinger points out that in this expression the integral is independent of A, the distance of the point μ from the sphere, and the attraction of the sphere of concentric layers reciprocally as the square of the distance, as in the Newtonian law. The constant of attraction alone remains indeterminate and a special function of the property of the sphere. He then proceeds to show that a factor should be introduced of the form:

$$(1-\tfrac{3}{4}\varkappa\,\sigma\,R) \tag{99}$$

the fraction $\frac{3}{4}\varkappa\,\sigma\,R$ representing the loss of gravitational

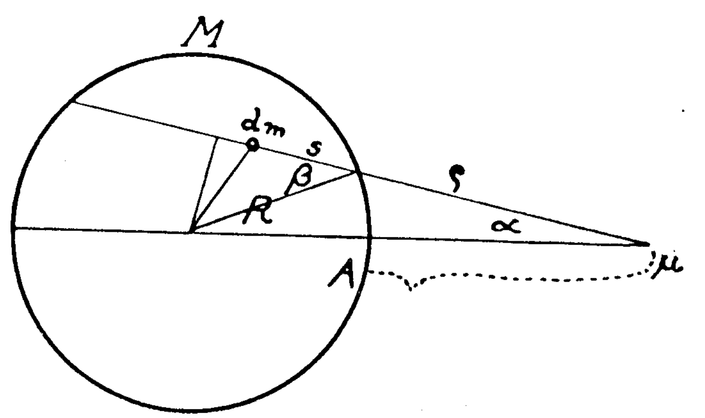

Fig. 8.

energy due to absorption. He calls this factor h, and deduces for the Earth,

$$h\,♁ = 1/160000\,. \qquad (100)$$

11. This explains the method of procedure. The absorption coefficient is $\varkappa$, and from what we know of wave motion in general, throughout nature, this coefficient is not zero, though it may be very small.

Let us examine the problem in another way. Is the Sun's gravitational action on the Moon constant, in spite of the bending of the gravitational waves in passing through the Earth's mass? Does not the interposition of the Earth's mass in the path of the Sun's waves, with their refraction, deflection, absorption, necessarily decrease their action on the Moon at the time of eclipses, — as first considered by *Bottlinger*, 1912, yet first proved by me, 1917, from the careful study of eclipse cycles of the required periods?

It seems difficult if not impossible to deny the variability of the integral for the attraction: and thus we may put, as confirmed by the eclipse cycles:

$$\delta(z)/\delta t = \frac{\delta}{\delta t}\left[k^2R^2/A^2\cdot\int_0^{\frac{1}{2}\pi}(\beta)\int_0^{2\pi}(\varphi)\int_0^{2r\cos\beta}(\varrho)\times \times\sigma\,d\beta\,d\varrho\,d\varphi\,\sin\beta\cos\beta\, e^{-\varkappa\int_0^{s}\sigma\,d\rho}\right] = \tfrac{3}{4}\varkappa\sigma R\,. \qquad (101)$$

12. The wave process at work in acoustic attraction lends support to the above equation. $\delta(z)/\delta t$ necessarily is variable, and confirmed by the eclipse cycles in theoretical periods of 18.03 years, 61.7006 years, and 277.59 years! Otherwise how are we to explain such close accordance between observation and theory?

No amount of caution will justify an investigator in trying to deny the cause of the lunar fluctuations, thus accounted for on valid eclipse cycles dating back to the age of the Greeks, and confirmed by laboratory experiments in sound, which any one can verify and illustrate by diagrams making the bending of the waves visible to the eye, as in the case of the radio waves observed to be bent around the globe to the antipodes. To deny such evidence is to reject the best experimental data in light, radio telegraphy, and the theory of sound: and thus to deny the plainest evidence of our senses, when the indications are supported by several independent sciences!

Part III. The Velocity-Potential Φ, with Solution of the *Fourier-Poisson* Partial Differential Equation in the Case of Waves from more than one Centre: the State of the Aether under the unlimited series of Waves from the Stars of the Sidereal Universe leads to an Infinite Integral, which may be integrated by the Process of Successive Approximations.

13. The Physical Significance of the Velocity-Potential Φ, in Accordance with *Lagrange*'s Theorem, $u\,dx + v\,dy + w\,dz = d\Phi$ for the Motion of Fluids.

A remarkable theorem applicable to the motions of fluids was first enunciated by *Lagrange*. Suppose that for any part of a fluid mass, we express the velocity components of the particle in terms of the coordinates x, y, z and the time t. Then if after a small interval dt, a new particle has reached x, y, z, we have for the excess of its velocity over the first particle the components:

$$du/dt\cdot dt \qquad dv/dt\cdot dt \qquad dw/dt\cdot dt \qquad (102)$$

while the components of the change of velocity of the original particle are

$$\delta u/\delta t\cdot dt \qquad \delta v/\delta t\cdot dt \qquad \delta w/\delta t\cdot dt\,. \qquad (103)$$

The change in the position of the new particle, in space, is expressed by d/dt, and remains invariable, while the change of velocity of the original particle $\delta/\delta t$ is in respect to a particle of the fluid relatively to another particle. Hence we have in brief (cf. *Maxwell*, Scientific Papers, vol. II, *p.* 468), for any component of the velocity u:

$$\delta u/\delta t = du/dt + u\,du/dx + v\,du/dy + w\,du/dz\,; \qquad (104)$$

and if the motion is very small, the terms $u\,du/dx$, $v\,du/dy$, $w\,du/dz$, etc., diminish in relative importance, and ultimately

$$\delta u/\delta t = du/dt\,. \qquad (105)$$

Now *Lagrange* enunciated the important theorem that if in any part of a fluid mass, for any moment, a perfect differential exists, such that

$$u\,dx + v\,dy + w\,dz = d\Phi \qquad (106)$$

then this perfect differential will hold for all subsequent time.

This condition ensures a steady state of the fluid mass, suitable to the uniform propagation of waves, under *Poisson*'s equation

$$d\Phi/dt = a^2\,(\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2)\,. \qquad (107)$$

This condition may be otherwise expressed in the form of the invariability of the line-integral for the circulation:

$$\delta\left[\int(u\,dx + v\,dy + w\,dz)\right]\Big/\,\delta t = 0 \qquad (108)$$

or the line-integral of the tangential component velocity around any closed curve of a moving fluid remains constant throughout all time. This line-integral is known as the circulation, and hence in any closed line moving with the fluid the circulation remains constant, or $\delta\int d\Phi = 0$.

Accordingly, when the exact differential exists,

$$u\,dx + v\,dy + w\,dz = d\Phi \qquad (106)$$

the velocity in any direction is expressed by the corresponding rate of change of Φ, which is called the velocity-potential. Therefore for such conservative movement, under steady conditions, as in wave propagation:

$$du/dx + dv/dy + dw/dz = \partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2\,. \qquad (109)$$

When the fluid is incompressible this integral round a closed circuit is evanescent, and the momentum, like the circulation, is zero; but for a compressible fluid, the existence of a velocity-potential Φ does not imply evanescence of the integral for the momentum round a closed circuit (cf. Lord *Rayleigh*, Theory of Sound, 2nd ed., 1896, vol. 2, pp. 8–9).

In the case of the aether, however, the fluid is so nearly absolutely incompressible that the above theorems will hold, and we may take $d\Phi$ to be essentially an exact differential; so that the velocity in any direction is expressed by the corresponding rate of change of Φ, and therefore

$$du/dx + dv/dy + dw/dz = \partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2\,. \qquad (110)$$

Let us now consider, as in AN 5085, pp. 391–392, the motion of a wave across a closed surface, such as a sphere of radius r, $S=4\pi r^2$. Then the rate of the flow of the fluid outward, across the element dS, becomes: $dS\cdot d\Phi/dn$. And when the density is constant, the total loss of fluid in time dt is given by the double integral:

$$\delta(\tfrac{4}{3}\pi\sigma r^3)/\delta t=\iint d\Phi/dn\cdot dS\,dt \qquad (111)$$

where the integration is to be extended over the entire surface $S=4\pi r^2$.

Now when the sphere surface S is full both at the beginning and at the end of dt, the loss of fluid vanishes, so that

$$\delta(\tfrac{4}{3}\pi\sigma r^3)/\delta t=\iint d\Phi/dn\cdot dS\,dt=0\,. \qquad (112)$$

The equation of continuity, for an incompressible fluid deduced from the spacial element $dx\,dy\,dz$, under this condition of no loss of fluid across the boundary, is

$$\partial^2\Phi/\partial x^2+\partial^2\Phi/\partial y^2+\partial^2\Phi/\partial z^2=0 \qquad (113)$$

or briefly $\nabla^2\Phi=0$.

And as *Poisson*'s equation of wave motion is

$$\partial^2\Phi/\partial t^2=a^2\nabla^2\Phi \qquad (114)$$

we see that $\nabla^2\Phi=0$ excludes the existence of waves, if this condition held rigorously for the time dt.

Wherefore we conclude that in traversing the surface S, the condition in (112) will hold for the wave from the centre at the beginning and also at the end of the time dt, corresponding to the propagation of a wave through all its phases, over the wave-length λ, which represents a complete oscillation of the fluid.

But for shorter intervals, the equation (112) will not hold rigorously; so that temporarily, over an interval less than the wave frequency $\tau=2\pi/\nu=\lambda/V$, there is both slight compressibility and a flow of the fluid across the boundary $S=4\pi r^2$, and, for $dt<\tau$ we have:

$$\delta(\tfrac{4}{3}\pi\sigma r^3)/\delta t=\iint d\Phi/dn\cdot dS\,dt=\pm dm \qquad (115)$$

where dm is the total fluid temporarily lost, an infinitesimal mass positive or negative.

Accordingly, in the wave motion of the aether, there is slight compressibility,[1]) and a minute temporary radial motion of the fluid does take place. Hence we cannot have purely transverse motion, as assumed in the traditional form of the wave-theory of light due to *Fresnel* and *Cauchy*.

This discussion comprehends the points of chief interest in respect to the aether, which has an elasticity 689321600000 times greater than that of air in proportion to its density. It is in this ultimate medium that the waves of gravitation, magnetism, electrodynamics, light and radiant heat are transmitted. No other medium underlies the aether, and thus the energy can not be lost, but only transformed by wave-changes.

A definite proof that energy is not lost in the aether is furnished by the law of the inverse squares: for observation shows that this law holds rigorously true between the Sun and a body at all distances. It would not do so, however, if any energy were lost in transmission to increased distance.

Proof of this may be drawn from the motion of certain comets, such as *Halley*'s. This body has made nearly 30 revolutions since it was first observed before the beginning of the Christian Era. The records of the early revolutions, to be sure, are too vague to give reliable data on the orbital motion, yet since the time of *Newton*, when exact data became available, three revolutions have been carefully followed, as the comet receded away to great depths in space, — without disclosing the smallest deviation of the attractive force from the law of the inverse squares. The same law is tested in the motion of the great comets of 1680, 1843 and 1882, which had unusually small perihelion distances.

And as was pointed out by *Newton* and *Laplace*, the fixity of the perihelion of a planet of appreciable eccentricity is the most delicate test of the rigor of the Newtonian law. As the outstanding progression of the perihelion of Mercury is perfectly explained by the wave-theory (AN 5048), we have in the three centuries of observations of this rapidly revolving body, which also travels in the most eccentric orbit of any planet, the most delicate test of the rigor of the law of the inverse squares. In view of the lunar fluctuations, when the Sun's gravitational waves are transmitted through the Earth, as at the time of lunar eclipses, the law of *Newton* perfectly accounts for the motion of Mercury's perihelion (cf. AN 5048, p. 143).

14. Integration of the Differential Equation for Φ in the Theory of the Velocity-Potential for Waves in the Aether.

In the New Theory of the Aether, AN 5130, pp. 283–286, we show that the treatment of the velocity-potential Φ takes the form used by *Poisson* in the Theory of Sound and other elastic media:

$$\partial^2\Phi/\partial t^2=a^2\{\partial^2\Phi/\partial x^2+\partial^2\Phi/\partial y^2+\partial^2\Phi/\partial z^2\} \qquad (116)$$
$$\Phi=\Omega(x,y,z,t)\quad \Phi=\Omega(x,y,z)\quad t=0\,.$$

or the equivalent form used by *Fourier*, in the Théorie Analytique de la Chaleur, 1821:

$$\partial^2\Theta/\partial t^2=a^2\{\partial^2\Theta/\partial x^2+\partial^2\Theta/\partial y^2+\partial^2\Theta/\partial z^2\} \qquad (117)$$
$$\Theta=f(x,y,z,t)\quad \Theta=f(x,y,z)\quad t=0\,.$$

In the theory of waves, we have for plane waves along the x-axis:

$$y=A\sin[2\pi/\lambda\cdot(Vt-x)+\alpha] \qquad (118)$$

[1]) Because the elasticity of the aether, $\varepsilon=689\,321\,600\,000$ times greater than that of air in proportion to its density, enormous as it is, is not really infinite. Thus the aether can sustain the action of any forces, however great, yet it undergoes some compression and rarefaction about great binary stars, and about any star fulfills the law of density $\sigma=\nu r$, owing to the central increase of the wave amplitudes. And the waves are everywhere propagated with the enormous, yet finite velocity, $V=300\,000$ km per second, the velocity of the aetheron being 471 239 km.

In AN 5370, May 28, 1925, Professor *S. Mohorovičič* has a suggestive paper which was presented to the St. Stephan Acad. of Sciences, Budapest, Febr. 27, 1925. He quotes my value $v=\frac{1}{2}\pi c_0=1.57\,V=471\,239$ km·sec^{-1}, as conforming to the researches of *C. A. Mebius* (Über die Dichte des Aethers und ihre Beziehung zur *Planck*schen Konstante, S. 8 Göt. kungl. Vet. och Vitt.-Samh. Handl. F. F. XXVIII: 5 Göteborg 1924), yet by somewhat different premises himself deduces $v=\pm c_0\sqrt{2}=424\,263$ km·sec^{-1}. These newer researches show notable progress since the publication of the New Theory of the Aether, 1920–22.

But in tri-dimensional space, the disturbance spreads in all directions with the velocity

$$V = at, \quad (at)^2 = x^2 + y^2 + z^2 \tag{119}$$

and from any point $P(x, y, z)$, the sphere surface becomes:

$$(at)^2 = (x + at\cos\theta)^2 + (y + at\sin\theta\sin\omega)^2 + \\ + (z + at\sin\theta\cos\omega)^2 \tag{120}$$

which we have treated in previous papers.

In the treatment of *Poisson*'s equation of wave motion,

$$\partial^2\Phi/\partial t^2 = a^2\,\{\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2\} \tag{121}$$
$$\Phi = \Omega(x, y, z) \quad t = 0$$

we have found (AN 5048) that for three variables (x, y, z)

$$\Phi = \Omega(x, y, z) = (1/8\pi^3)\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty}\Omega(\xi, \eta, \zeta)\cos\xi(x-\lambda)\cos\eta(y-\mu)\cos\zeta(z-\nu)\,d\xi\,d\eta\,d\zeta\,d\lambda\,d\mu\,d\nu \tag{122}$$

in which ξ, η, ζ and λ, μ, ν extend from $-\infty$ to $+\infty$.

This may be transformed into

$$\Phi = \Omega(x, y, z) = (1/8\pi^3)\cdot\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty}\Omega(\xi, \eta, \zeta)\cos\lambda(\xi-x)\cos\mu(\eta-y)\cos\nu(\zeta-z)\,d\xi\,d\eta\,d\zeta\,d\lambda\,d\mu\,d\nu \tag{123}$$

$$= (1/8\pi^3)\cdot\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty}\Omega(\xi, \eta, \zeta)\,e^{-[\lambda(\xi-x)+\mu(\eta-y)+\nu(\zeta-z)]\sqrt{-1}}\cdot d\xi\,d\eta\,d\zeta\,dx\,dy\,dz\,. \tag{124}$$

By including the factor $1/8\pi^3$ in the arbitrary function, this may be written in the well known form of the expression for any time t,

$$\Phi = \Omega(x, y, z, t) = \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty} e^{(A+Bht)\sqrt{-1}}\cdot\psi_1(\xi, \eta, \zeta)\,d\xi\,d\eta\,d\zeta\,d\lambda\,d\mu\,d\nu\,. \tag{125}$$

And finally, in the IV. Paper, AN 5085, we have reached *Poisson*'s double integral: $\Phi = \Phi' + \Phi''$

$$= 1/4\pi\cdot\int_0^{\pi}\!\!\int_0^{2\pi} F\{x + at\cos\theta,\ y + at\sin\theta\sin\omega,\ z + at\sin\theta\cos\omega\}\,t\sin\theta\,d\theta\,d\omega$$

$$+ 1/4\pi\cdot\frac{\partial}{\partial t}\int_0^{\pi}\!\!\int_0^{2\pi} H\{x + at\cos\theta,\ y + at\sin\theta\sin\omega,\ z + at\sin\theta\cos\omega\}\,t\sin\theta\,d\theta\,d\omega\,. \tag{126}$$

This expression for the velocity potential Φ, will hold rigorously for the waves emanating from any mathematical point $P(x, y, z)$ and traversing all space from that centre of disturbance. But in nature the waves proceed from all atoms of a mass, and thus we must extend the integral of *Poisson* by taking the triple integral for the volume and density:

$$\Phi = \int_0^{r}\!\!\int_0^{\pi}\!\!\int_0^{2\pi}(\sigma/4\pi)\int_0^{\pi}\!\!\int_0^{2\pi} F\{x + at\cos\theta,\ y + at\sin\theta\sin\omega,\ z + at\sin\theta\cos\omega\}\,r^2\sin\theta\,dr\,d\theta\,d\omega\cdot t\sin\theta\,d\theta\,d\omega$$

$$+ \int_0^{r}\!\!\int_0^{\pi}\!\!\int_0^{2\pi}(\sigma/4\pi)\cdot\frac{\partial}{\partial t}\int_0^{\pi}\!\!\int_0^{2\pi} H\{x + at\cos\theta,\ y + at\sin\theta\sin\omega,\ z + at\sin\theta\cos\omega\}\,r^2\sin\theta\,dr\,d\theta\,d\omega\cdot t\sin\theta\,d\theta\,d\omega\,. \tag{127}$$

This is a double quintuple integral, and by referring to the equations (122) or (125) above we see that (127) corresponds to a single non-nuple integral in the original form of these equations, because the disturbances must be conceived to proceed from each atom of the mass,

$$m = \int_0^{r}\!\!\int_0^{\pi}\!\!\int_0^{2\pi}\sigma\,r^2\sin\theta\,dr\,d\theta\,d\omega\,. \tag{128}$$

Now in the physical universe, such independent gravitational waves must be imagined to proceed from the several atoms of all bodies whatsoever, just as light waves do from each atom of the self-luminous gases of the stars. Accordingly, such integration has to be extended to the waves from all masses severally; and as there is an infinitude of bodies, the result is an integral infinitely repeated, or an infinite integral, though the value of the disturbance remains finite at every point of space.

Compounding of the velocity-potentials for two wave-centres, like a Sun and planet, planet and satellite, or a binary star.

The velocity-potential for a single centre of disturbance has the form:

$$\Phi = \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty} e^{(A+Bht)\sqrt{-1}}\cdot\psi_1(\xi, \eta, \zeta)\,d\xi\,d\eta\,d\zeta\,d\lambda\,d\mu\,d\nu\,. \tag{129}$$

It is evident that for two such centres, the waves of which mutually interpenetrate, the velocity-potentials would be of the types

$$\Phi = \Omega(x, y, z, t) \qquad \Phi' = \Omega'(x', y', z', t) \tag{130}$$

or $\Phi = \Omega(x + at\cos\theta,\ y + at\sin\theta\sin\omega,\ z + at\sin\theta\cos\omega, t)$

$\Phi' = \Omega'(x' + at\cos\theta,\ y' + at\sin\theta\sin\omega,\ z' + at\sin\theta\cos\omega, t)$.

For it is clear physically and geometrically that t is common to both expressions, while the coordinates (x, y, z)

and (x', y', z') are different in the two wave-fields, owing to the different origins of the disturbances, at points p and p'. Therefore, as t is simultaneous, the two equations for the separate velocity-potentials are connected by a relation for denoting the double wave-fields which interpenetrate:

$$\Phi = \Omega(x, y, z, t)\,[1 + \Omega'(x', y', z', t)] = \Phi(1 + \Phi'). \qquad (131)$$

Likewise, for the secondary wave centre,

$$\Phi' = \Omega'(x', y', z', t)\,[1 + \Omega(x, y, z, t)] = \Phi'(1 + \Phi). \qquad (132)$$

The integral form for the velocity-potential in a double wave-field thus becomes:

$$\Phi = \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty} e^{(A+Bht)\sqrt{-1}} \cdot \psi_1(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta\, d\lambda\, d\mu\, d\nu \left\{1 + \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty} e^{(A'+B'h't)\sqrt{-1}} \cdot \psi_2(\xi',\eta',\zeta')\, d\xi'\, d\eta'\, d\zeta'\, d\lambda'\, d\mu'\, d\nu'\right\} \qquad (133)$$

the integration to be extended over the two fields one after the other, which double sextuple integration will include all the coordinates and their simultaneous changes incident to the double interpenetration of the waves.

If the centre of disturbance be of finite dimensions included within a surface of sensible dimensions, (r, θ, ω), we must moreover extend the integration for the volume of the vibrating matter:

$$\Phi = \int_0^r\!\!\int_0^\pi\!\!\int_0^{2\pi}\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty} [\sigma\, r^2\, dr \sin\theta\, d\theta\, d\omega]\, e^{(A+Bht)\sqrt{-1}} \cdot \psi_1(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta\, d\lambda\, d\mu\, d\nu \times$$

$$\times \left\{1 + \int_0^{r'}\!\!\int_0^\pi\!\!\int_0^{2\pi}\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{+\infty} [\sigma'\, r'^2\, dr'\, d\theta'\, d\omega']\, e^{(A'+B'h't)\sqrt{-1}} \cdot \psi_2(\xi',\eta',\zeta')\, d\xi'\, d\eta'\, d\zeta'\, d\lambda'\, d\mu'\, d\nu'\right\} \qquad (134)$$

so that we should have a double non-nuple integral of the above form, from extending the integration to the elements of the masses m and m':

$$m = \int_0^r\!\!\int_0^\pi\!\!\int_0^{2\pi} \sigma\, dr \cdot r\, d\theta \cdot r \sin\theta\, d\omega \qquad m' = \int_0^{r'}\!\!\int_0^\pi\!\!\int_0^{2\pi} \sigma'\, dr' \cdot r'\, d\theta' \cdot r' \sin\theta'\, d\omega' \qquad (135)$$

which, by *Poissen*'s reduction, may be transformed into a double quintuple integral.

It thus appears that the wave-field about even two bodies is highly complex. The complication seems less formidable, however, if we recall that each centre of disturbance sends out the spherical waves: $\Phi = \Omega(x, y, z, t)$ and $\Phi' = \Omega(x', y', z', t)$; so that under the interpenetration of these waves we get as above, when t is common to both fields:

$$\begin{aligned} \Phi &= \Omega(x, y, z, t)\,\{1 + \Omega'(x', y', z', t)\} \\ \Phi &= \Omega'(x', y', z', t)\,\{1 + \Omega(x, y, z, t)\}. \end{aligned} \qquad (136)$$

This expression holds true rigorously only for unique point-wave commotions, which must be extended by integration to all the matter in each of the two stars, if the whole of the waves are to be included in our investigation.

The compound velocity-potential Φ or Φ' naturally corresponds to the composition of stresses with resulting forces in the aether about the two stars. These forces are indicated in the accompanying diagram (Plate VII), by the geometrical composition shown in the outer part, and by the visible stretching and distortion of the surfaces between the stars, and the crowding together of the surfaces in the regions of increased pressure beyond them, on the extremities of the horizontal axis.

The equipotential surfaces here shown were first given in *Thomson* and *Tait*'s Treatise on Natural Philosophy, 1st edition, 1873. They are drawn from *Newton*'s law of gravitation, treated simply as a fact, without regard to the cause underlying that force. Yet the fact that the vectorial stresses in the aether are geometrically compoundable, so as to give these distorted surfaces, points incontestably to the wave-theory of gravitation. In receding from each of the stars, the waves necessarily would react towards those centres, with forces proportional to the masses, into the square of the amplitude of the waves, as shown by *Laplace*'s definition of the potential, 1782:

$$V = \int 1/r \cdot dM = \int\!\!\int\!\!\int [(a-x)^2 + (b-y)^2 + (c-z)^2]^{-1/2}\, \sigma\, dx\, dy\, dz$$

which yields the force:

$$\partial V/\partial r = -M/r^2 = \int\!\!\int\!\!\int [(a-x)^2 + (b-y)^2 + (c-z)^2]^{-1}\, \sigma\, dx\, dy\, dz\,.$$

In the case of two stars, the resultant force R is not directed towards either body, except at the ends of the longer axis of the figure, where the directions of the forces coincide with the line connecting the bodies. In all other points R points between the masses, as shown by the vectors, as geometrically compounded in the diagram, Plate VII.

15. Some Geometrical and Physical Grounds for holding that the Wave-Theory represents the True Order of Nature.

First: we point out that the law of the inverse squares holds for the gravitational waves of each body at all points of space, and therefore in geometrical composition wherever these waves mutually interpenetrate.

The illustration for the cross section of the wave motion at the surfaces S and s, already discussed above, may yield us a more general inference which is of such high importance as to be worthy of the careful attention of geometers.

1. We see that the cross section of the amplitude of any wave whatever, at distances R and r, R being the solar radius, or any convenient multiple thereof, and r any larger planetary distance, follows the law $s = S(R/r)^2$, whatever be the distances separating the wave surfaces in free space. Accordingly the waves from every centre conform to this law of the inverse squares, which holds rigorously true throughout the universe of free aether.

2. Now let us imagine waves mutually interpenetrating from, say, opposite directions: then it is evident, from this rigorous geometrical law of wave-expansion, that as each wave will separately follow the law, the resultant of the combinations of wave motions throughout space, likewise will conform to the laws of wave expansion from the separate centres.

3. The waves are everywhere mutually in combination, according to the law of the inverse squares. The geometrical components, yielding the two vector magnitudes, representing the forces directed to the two bodies m and m' are of the type:

$$f = k^2 m/r^2 \qquad f' = k^2 m'/r'^2 \tag{137}$$

$$\begin{aligned} f &= -\partial V/\partial r = \\ &= \iiint [(a-x)^2+(b-y)^2+(c-z)^2]^{-1}\,\sigma\,dx\,dy\,dz \\ f' &= -\partial V'/\partial r' = \\ &= \iiint [(a'-x')^2+(b'-y')^2+(c'-z')^2]^{-1}\,\sigma'\,dx'\,dy'\,dz'\,. \end{aligned} \tag{138}$$

And the geometrical composition follows the law (cf. AN 5140, p. 105):

$$\begin{aligned} R^2 &= (k^2m/r^2)^2+(k^2m'/r'^2)^2+2(k^2m/r^2)\,(k^2m'/r'^2)\cos\chi \\ \cos\chi &= (xx'+yy'+zz')/rr' \end{aligned} \tag{139}$$

which is illustrated above, in the foregoing figure, Plate VII, χ being the angle between the radii vectores drawn from P to the two stars.

4. The geometrical composition just described is known to take place with rigorous accuracy. The equipotential surfaces computed from the law of gravitation by Lord *Kelvin*, 1873, and minutely verified by the law of vector-composition, shows the coincidence of these two results at all points of space. This coincidence is an overwhelming argument for the wave-theory of gravitation.

The wave-theory, and no other theory whatever, will explain the perfect coincidence of the surfaces above deduced from the law of gravitation, (without regard to the cause involved), with the surfaces which would result from the law of vector composition. Under the wave-theory here set forth, as the physical cause of universal gravitation, this coincidence is perfect throughout all space, the extent of which is:

$$S = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx\,dy\,dz = 8\infty^3\,. \qquad (A)\ldots(140)$$

And therefore this verified triply extended concordance at every point of space, an infinity of the third order to one, is the chance that the wave-theory represents the true order of nature, compared to any other theory which might be suggested.

Second: In view of the geometrical foundations already explained as pointing unmistakably to the wave-theory, it only remains to assign some physical grounds why the theory of waves in the aether is especially suited for explaining the tremendous forces operating to bind together the heavenly bodies, by an unseen mechanism, which holds the comets, satellites, planets and stars in their orbits.

(a) These stupendous forces are equivalent to the tensile strength of millions of immense cables of the strongest steel. To pull the Moon daily about our Earth requires the exertion of the full power of 5000000000000 cables of steel a foot in diameter, when each square inch of it would lift 30 tons. And to hold our Earth in its orbit, there would have to be an 11-inch cable of such steel on each square foot of a cross section of our globe, — thus practically covering our Earth over solid with such a tightly crowded forest of steel cables — all stretched to the limit of their tensile strength!

(b) Now one of the most remarkable facts known to us is that although we see the physical world by means of the aether waves entering our eyes, yet the aether itself is perfectly invisible. The medium pulling between the planets is less visible to mortal sight than spider webs stretched across the planetary spaces! That could hold true only if the aether were enormously elastic, capable of renewing its stresses instantly, and yet so superfine that its waves penetrate the heavenly bodies, without resisting their orbital motions. And the aether is shown to have these properties! This renders it wholly invisible and also non-disruptible by any forces to which it may be subjected under the attraction of the giant masses of the stars.

(c) The elasticity of the aether is 689321600000 times greater than that of air in proportion to its density. We notice that even the sluggish air is not easily disrupted, since to start violent waves in it we require the action of a high explosive. In view of this enormous elasticity it is therefore almost infinitely more difficult to start violent waves in the aether than in our air. But, on the other hand, when waves once exist in the aether they cannot be destroyed, since no finer medium underlies the aether. The waves may be transformed and renewed, in this wave medium, yet they travel incessantly, as *Fourier* says of radiant heat, and thereby are always generating and sustaining the forces observed to be at work in nature.

(d) The velocity of the wave-motion in free space is 300000 km per second, and that of the aetherons $1.57\,V = 471239$ km, or 294000 miles. With this enormous speed of the aetheron we see that the aether always maintains itself in a state of kinetic equilibrium. The medium cannot be disrupted, owing to the enormous velocity of the aetherons. If the medium were really displaced, it would restore itself in a millionth of a second, and yet such an adjustment would be wholly invisible. In the case of lightning discharge, which represents the release of accumulated wave stresses on rain drops, when their size is growing, the jarring of the Earth indicates how truly the aether is in commotion throughout our physical world.

(e) Moreover, the high velocity of the aetherons above mentioned renders the aether non-resisting to the planets and similar uniformly moving bodies. But as the planets have wave-fields about them, which are carried along in their orbital motion, there arises a readjustment of the wave-field when the velocity changes, and this readjustment of the motion thus gives rise to inertia, and momentum. The inertia of a body is due to the adaptation of the wave-field about it to the new state of motion: and the momentum of a body is due to the adaptation of the wave-field of a moving body to a state of rest.

From the time of *Newton* it has been difficult for philosophers to explain the inertia and momentum of bodies. In his day *Newton* took these laws as facts, without assigning any explanation of such phenomena, — probably because he could not make out clearly the mode of operation underlying inertia and momentum; just as he made no hypotheses regarding the cause of gravitation, because he could not discover the processes underlying this chief force of nature.

(f) From the definition by *Laplace*, 1782, of what is now called the potential:

$$V = \int 1/r \cdot \mathrm{d}M = \iiint [(a-x)^2 + (b-y)^2 + (c-z)^2]^{-1/2} \sigma \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

we have an integral for the whole mass of a body, divided by the distance of each particle from a fixed point (a, b, c). Accordingly the triple integral of the potential is really a summation of the stresses due to the amplitudes of the waves from all the particles. For the law of amplitude, $A = k/r$, applied to each atom at its appropriate distance, gives the above triple integration as extended to the whole mass.

(g) And in proportion as the total mass increases, the accumulated wave stress also increases, when the distance r from the attracted point is constant. Hence the potential or accumulated wave stress of fixed amplitude is proportional to the mass directly. And at different distances we see from the same formula that the potential or accumulated wave stress varies as the distance inversely, which is a proof that the amplitude of the wave is the other chief variable factor.

This analysis proves definitely that the potential is nothing but an expression for the wave stress appropriate to the mass and inverse distance, or amplitude. The potential is therefore uniquely identified as an integration for the stresses due to superposed waves. The true nature of the potential as an integration for the wave-stress depending on the mass for the number of the waves superposed, and on the amplitudes for inverse distances, is thus made clear and beyond controversy.

(h) However great a body may become the aether about it can take on a few more vibrations, as the mass is correspondingly increased: and thus with the increase of the mass, the potential augments, and the force of gravitation augments correspondingly, according to the law of the square of the amplitude. This superposition of waves increases the stress, proportionally to the mass: and as the aether is a highly elastic kinetic medium, it cannot be disrupted by the attractions of the heavenly bodies, however great may be their masses.

(i) Accordingly it is gratifying to know that without the slightest danger of disruption, the aether can support any force arising in nature. This is one of the chief problems to be overcome in solving the problem of the cause of universal gravitation.

Therefore since the physical grounds assigned for the wave-theory, from the New Theory of the Aether, are in every way as probable as the geometrical grounds, and such waves are everywhere observed in nature, we conclude that the wave-theory assigns the true cause of universal gravitation. No other theory is admissible, when the wave-theory is proved by an accumulated probability, section 24, not less than $\varpi = \infty^{100}$ to one!

In conclusion it only remains to add a final word in support of the wave-theory and of Sir *Isaac Newton*'s geometrical methods, which lie at the basis of the Principia. *Newton* always used geometrical methods, because the gravitational forces are directed magnitudes, corresponding to the wave-theory; and such forces have to be compounded geometrically, — by completing the parallelogram of forces, and thereby finding the magnitude and direction of the disturbing force.

1. Thus the whole of *Newton*'s geometrical method in the Principia is an incomparable argument by that great geometer, for the wave-theory of gravitation.

2. To overthrow the wave-theory, thus unconsciously supported by *Newton*, throughout the Principia, by the most masterly geometrical methods, it would in fact be necessary to overthrow the Principia, the highest development in the physical sciences, which *Laplace* justly declared will always have an interest surpassing that of any other achievement of the human intellect.

3. Such is the incomparable, though hitherto unsuspected, strength of the wave-theory of universal gravitation, shown in this paper to have a probability in its favor of more than ∞^{100} to 1, compared to any other theory which could be conceived, and therefore it is incontestably established as the law of nature.

16. Investigation of an Infinite Integral arising in the Theory of the Aether as applied to the Stars of the Sidereal Universe.

The theory of the mutual attractions of the heavenly bodies, under the Newtonian law of universal gravitation, has led to many notable advances, not only in the sublime science of celestial mechanics, but also in the theory of differential equations and other important domains of higher analysis. Thus the mathematical requirements of astronomy have enriched geometry; and in turn the improvements in the processes of pure analysis have made possible those penetrating physical researches, on the mutual actions of the heavenly bodies, which justly rank the recondite theories of celestial mechanics among the most impressive monuments of the human intellect. The rigorous development of the theory of universal gravitation by *Newton* and *Euler*, *Lagrange* and *Laplace*, *Gauss* and *Hansen*, *Leverrier* and *Newcomb*, *Adams* and *Darwin*, *Hill* and *Poincaré*, make the creations of celestial mechanics the wonder of succeeding ages!

In the course of this historic development it is remarked that the analytical processes brought to light, and at length perfected are of extreme generality, and well adapted to the solution of the problems presented by the solar system. It is true that the general problem of three bodies remains insoluble, owing to the existence of infinite series of solutions for different dynamical conditions; yet the methods developed by *Newton*, *Euler*, *Lagrange* and their successors, apply to the double, triple, and multiple stars, and even give indications of solutions for sidereal systems of higher order, such as the globular clusters observed in the Milky Way and other portions of the immensity of space (cf. Dynamical Theory of the Globular Clusters, etc., Proc. Am. Phil. Soc., 1912).

Notwithstanding this brilliant development and its wide applicability to the systems of the sidereal universe, an extension of method has become necessary, in dealing with the cause of universal gravitation, under the wave-theory of physical forces, where we have to avail ourselves of the partial differential equations, and the processes of integration discovered by *Fourier* and *Poisson*.

The need has thus arisen for an investigation of the existence of an infinite integral encountered in the Theory of the Aether which heretofore has received but slight attention from geometers.

It turns out, however, that the processes of approximation met with universally in the integrations of the New Theory of the Aether are somewhat known, in a much more restricted form, in the development of the planetary and lunar theory, and many other branches of mathematical and physical science. We shall therefore study several of these earlier processes in sufficient detail to enable us to unfold by analogy the newer extension of analysis in the higher order of development required in the Kinetic Theory of the Aether.

It is only by such an analysis of the great historic processes of dynamics that we may hope to lend continuity to the successive steps in astronomy and mathematical analysis. And even then a few examples must serve for the various methods of approximation, which may be attained by many processes, and thus are as limitless as the literature of mathematical and physical science.

1. As shown above, the sextuple integrals of the *Fourier-Poisson* differential equation,

$$\partial^2\Phi/\partial t^2 = a^2(\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial y^2 + \partial^2\Phi/\partial z^2) \qquad (141)$$
$$\Phi = \Omega(x,y,z,t) = \Omega(x,y,z) \qquad t=0$$

may be reduced to the double integrals obtained by *Poisson* (AN 5130, p. 286, eq. 13).

For brevity, however, we adhere to the standard sextuple form (AN 5048, p. 166, eq. 46).

$$\Phi = \Omega(x,y,z,t) = \int\int\int\int\int\int_{-\infty}^{+\infty} e^{(A+Bht)\sqrt{-1}} \cdot \psi_1(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta\, d\lambda\, d\mu\, d\nu \qquad (142)$$

and thus avoid the longer form of the double quintuple integral under *Poisson*'s reduction, or the unreduced nonnuple integral which results when we extend the integration for the volume and density (AN 5130, p. 285, eq. 14), so as to include the waves from each atom of matter. Thus we treat the stars as points, — great centres of disturbance, with waves proceeding therefrom under the form $\Phi = \Omega(x,y,z,t)$ defined by the above sextuple integral.

Any other star will send out waves independent of the first, and thus the series of velocity-potentials will be indefinitely increased:

$$\Phi,\ \Phi',\ \Phi'',\ \Phi''',\ \cdots\ \Phi^{\nu} \qquad \nu=0,\ \cdots\ \nu=\infty\,. \qquad (143)$$

2. In the Principia, 1686, (Lib. III, prop. VI) *Newton* shows clearly, when treating of the motion of the Moon and of the satellites of Jupiter and Saturn, that the attractive force of the Sun or of any planet penetrates right through the sphere of attraction of any other body: and that their several attractions must be calculated independently and then compounded, to get the forces at work on the satellites, comets, or other bodies.

This is as if the forces depended on waves or impulses in the aether, which was the opinion held by *Newton* for the last 50 years of his life, (1677–1727), as clearly appears by *MacLaurin*'s statement that he early held this view, in his correspondence with *Boyle*, 1678. (*MacLaurin*'s Account of the Philosophical Discoveries of Sir *Isaac Newton*, London, 1748, p. 111).

In the General Scholium to the Principia, 1713, *Newton*'s mature opinion is expressed as follows:

»Hitherto we have explained the phenomena of the heavens and of our sea by the power of gravity, but have not yet assigned the cause of this power. This is certain, that it must proceed from a cause that penetrates to the very centres of the sun and planets, without suffering the least diminution of its force; that operates not according to the quantity of the surfaces of the particles upon which it acts (as mechanical causes use to do), but according to the quantity of the solid matter which they contain, and propagates its virtue on all sides to immense distances, decreasing always in the duplicate proportion of the distances. Gravitation towards the sun is made up out of the gravitations towards the several particles of which the body of the sun is composed; and in receding from the sun decreases accurately in the duplicate proportion of the distances as far as the orb of Saturn, as evidently appears from the quiescence of the aphelia of the planets; nay, and even to the remotest aphelia of the comets, if those aphelia are also quiescent. But hitherto I have not been able to discover the cause of these properties of gravity from phenomena, and I frame no hypotheses.«

3. Accordingly, it appears that *Newton* expressly emphasizes the penetrating power of gravitation: »This is certain, that it must proceed from a cause that penetrates to the very centres of the sun and planets, without suffering the least diminution of its force.« Thus he recognized not only that the action of one body would penetrate the sphere of action of another, but that gravitation is even transmitted through solid masses like the Sun, Earth, Jupiter and Saturn, without sensible diminution of its force, — otherwise considerable irregularities would be introduced into the motions of the satellites when immersed in the shadows of their primary planets.

How *Newton* conceived the aether, in which he held that the gravitational impulses are transmitted, is shown by the letter to *Boyle*, Feb. 28, 1678–1679:

»1. And, first, I suppose that there is diffused through all places an aethereal substance, capable of contraction or dilatation, strongly elastic; and, in a word, much like air in all respects, but far more subtile.«

»2. I suppose this aether pervades all gross bodies, but yet so as to stand rarer in their pores than in free spaces; and so much the rarer, as their pores are less.« (*Newton*'s Opera, edition *Horseley*, vol. IV, 1782, pp. 385–386).