X

46 r

DEC 20 1922

ASTRONOMISCHE NACHRICHTEN.

Band 217. Sondernummer.

LIBRARY OF THE UNIVERSITY OF CALIFORNIA

New Theory of the Aether. By *T. J. J. See.*

(Seventh Paper.) (With 7 Plates and 2 Portraits.)

1. Discovery of the Cause of Magnetism and of a Remarkable Connection between Magnetism and Universal Gravitation.

(i) Introductory remarks and definitions.

In this seventh paper we demonstrate the cause of magnetism, and outline certain remarkable discoveries in connection therewith, more especially a new physical law now finally established between the magnetism of the earth and terrestrial gravitation. Some of these discoveries have been known to me since 1914, and are recognized in the third paper (AN 5079), and in volume I of the Electrodynamic Wave-Theory of Physical Forces, 1917, but the proofs now available are made so much more complete, that it is advisable to re-examine the whole subject somewhat briefly.

The analytical expressions for the accelerating forces under *Newton*'s law of gravitation, and under the new law for the total intensity of the earth's magnetic forces, are similar in form, except for the two poles in the case of the magnetic action, each of which is a centre of attraction, exerting its appropriate stress upon the aether in the wave-field of the globe:

$$g = mm'/r^2; \text{ for the sun } G = k^2/r^2, \text{ astron. units;} \quad (1)$$

$$I = \mu\mu'/s^2 + \mu\mu'/s'^2 . \quad (2)$$

Here the gravitational masses are m and m', at the distance r, the radius of the earth, and the acceleration g; μ is the pole strength of the earth's magnetism, and μ' that of a standard steel bar magnet of weight $^1/_2$ kilogram, as used by *Gauss* (Allgemeine Theorie des Erdmagnetismus, p. 46), $I =$ the total intensity of the earth's magnetism, $s =$ the length of the curved line of magnetic force, obtained by integrating along the curved path ds between the place of observation o and the pole p, $s = \int_o^p ds$, in the solid globe of the earth, $s' = \int_o^{p'} ds'$, being the corresponding curved line of magnetic force to the other pole.

It is recognized that a magnetic bar upon the earth is under a dual system of forces — the gravitational and the magnetic. *Gauss* showed (p. 46) that the pole strength of the earth $\mu = 8464 \cdot 10^{18} \mu'$; and he calculated that on the average, under uniform distribution of these standard steel bar magnets, with parallel axes, each cubic metre of the matter of the globe would have within it an amount of magnetism equivalent to 7.831 of these bars.

The average cubic metre of the earth's matter, with density 5.5, weighs 5500 kilograms; and thus the ratio of the magnetic matter of the globe to the whole of it is:

$$\eta = 3.9155/5500 = 1/1404.674 . \quad (3)$$

A more exact value of this constant probably is

$$\eta = 1/1408.12 \quad (4)$$

yet from the considerations indicated in deriving equation (49), it is possible that the value may be as small as $\eta = 1/1414.213$, with the results shown in equation (115) below.

From the above equations (1) and (2) we obtain by division:

$$1/g = \mu\mu'/mm' \cdot (r^2/s^2 + r^2/s'^2) . \quad (5)$$

It is shown in the first paper, AN 5044, p. 54, that under tri-dimensional expansion in free space, the wave amplitude $A = k/r$, and that the energy of the waves, from which the forces arise, becomes $f = A^2 = k^2/r^2$, exactly as in the equations (1) and (2) above. The potential is a state of stress of the aether due to the integration for the waves of the several atoms at their respective distances r, s or s' as the case may be.

Thus for gravitation we have:

$$V = \int 1/r \cdot dm =$$

$$= \iiint \sigma[(x-x')^2 + (y-y')^2 + (z-z')^2]^{-1/2} \, dx \, dy \, dz . \quad (6)$$

And for magnetism, having regard to the two poles, we have likewise:

$$\Omega = \int 1/s \cdot d\mu + \int 1/s' \cdot d\mu =$$

$$= \iiint^{\mu} \sigma/s \cdot dx \, dy \, dz + \iiint^{\mu} \sigma/s' \cdot dx \, dy \, dz . \quad (7)$$

This magnetic potential is subject to the *Gauss*ian equation of condition for the whole earth as a magnet:

$$\int d\mu = 0 \quad (8)$$

because, on the two-fluid theory, there must be in the entire mass exactly as much positive as negative magnetism, so that the integral is zero. The wave-theory leads to an identical result.

Accordingly the magnetic forces acting on another magnet of pole strengths μ', μ' become in the integral form:

$$I = f + f' = \mu\mu'/s^2 + \mu\mu'/s'^2 \quad (9)$$

as in equation (2) above.

For in the wave-theory not only gravitation, but also magnetism is due to the energies of the atomic waves, which yield appropriate forces proportional to the squares of the amplitudes of the vibrations. Now, as found above, the atoms with magnetic properties, due to the concerted way in which they oscillate in parallel planes, are to the whole of the atoms of the earth as 1:1408.12. And therefore the corresponding forces I and g, resulting from the integrals for the magnetic and gravitational masses respectively, must both involve the squares of the amplitudes, since, as shown in AN 5044, this occurs for the gravitational force,

$$g = A^2 = k^2/r^2 = -dV/dr . \quad (10)$$

Moreover, since the complete differentiation of the magnetic potential, by (7), leads to:

$$d\Omega/ds = \int 1/s^2 \cdot d\mu + \int 1/s'^2 \cdot d\mu =$$
$$= \iiint^{\mu} (\partial\Omega/\partial s \cdot \partial s/\partial x \cdot dx + \partial\Omega/\partial s \cdot \partial s/\partial y \cdot dy + + \partial\Omega/\partial s \cdot \partial s/\partial z \cdot dz) \qquad (11)$$
$$+ \iiint^{\mu} (\partial\Omega/\partial s' \cdot \partial s'/\partial x \cdot dx + \partial\Omega/\partial s' \cdot \partial s'/\partial y \cdot dy + + \partial\Omega/\partial s' \cdot \partial s'/\partial z \cdot dz)$$

we have for the integral of the forces between two magnets referred to their poles:

$$I = f + f' = \mu' \int 1/s^2 \cdot d\mu + \mu' \int 1/s'^2 \cdot d\mu = \mu\mu'/s^2 + \mu\mu'/s'^2 . \qquad (12)$$

The ratio I/g thus necessarily involves the square of η, which is common to the two forces depending on wave-action:[1]

$$\eta^2 = (1/1408.12)^2 . \qquad (13)$$

Accordingly equation (5) yields the harmonic law connecting the magnetism of the globe with terrestrial gravitation:

$$I/g = \eta^2 (r^2/s^2 + r^2/s'^2) = 1/1982802 \cdot (r^2/s^2 + r^2/s'^2) \qquad (14)$$

This remarkable law holds true throughout the terrestrial spheroid, so far as the magnetism is regular. It fulfills, for example, exact criteria at the poles, the equator, and in intermediate latitudes. And therefore, with an appropriate value for η, a similar law will hold true for the sun, Jupiter or any other planet exhibiting cosmical magnetism admitting of measurement of the intensity. We show hereafter that in the case of the sun $\eta = 1/157$.

It appears in this paper that the magnetic forces always act in curved lines, instead of in the straight lines assumed by *Gauss* and all previous investigators. Such a fundamental change in mathematico-physical theory requires us to investigate carefully the physical cause of magnetism. This cause is now definitely assigned to wave-action, by an argument which appears to be so convincing as to be incontestable.

(ii) Definition of magnetic lines of force and of the line integral.

As we have to deal very frequently with the lines of magnetic force, we remark that the differential equation of such a line, at any point (x, y, z), is

$$X\,dx/ds = Y\,dy/ds = Z\,dz/ds \qquad (15)$$

where X, Y, Z are the components of the vector R, or total directed force, parallel to the axes, and dx, dy, dz are the projections on the axes of the spacial element of the curve ds. Thus the line-integral between the points o and p becomes:

$$L = \int_o^p (X\,dx/ds + Y\,dy/ds + Z\,dz/ds)\,ds \qquad (16)$$
$$d\psi = (X\,dx/ds + Y\,dy/ds + Z\,dz/ds) .$$

Now let the expression under the integral be an exact differential, as in (16), then the value of L is the same for any two forms of the path between o and p, provided one form of the path may be changed into the other by continuous motion without passing out of this region. Accordingly, the difference of magnetic potential is given by the integral along the path ds:

$$L = \int_o^p d\psi\,ds = \psi_o - \psi_p . \qquad (17)$$

As two forms of a curve through two terminal points can be changed into each other under continuous motion chiefly by rotation depending on symmetry, it follows that the line of magnetic force admits of the integral in (17) when the force system is symmetrical about an axis.

In the harmonic law and new theory here outlined therefore we always have in view a magnet possessing symmetry. Hence the cosmical globe here considered is not our actual earth, with its irregular distribution of magnetism, and unsymmetrical axis, but a homogeneous uniformly magnetized sphere. This restriction in our premises, however, is only for reasons of simplicity in establishing the rigorous validity of the harmonic law above formulated.

In dealing with such a compound heterogeneous mass as our actual earth, it is necessary to have recourse to an expansion in spherical harmonics, arranged to converge for internal points, as employed by *Gauss*, 1838. Yet even here the new harmonic law will give a surprising approximation to the mean law of the intensity of terrestrial magnetism found in our globe as a whole.

It was first recognized by *Humboldt* in 1804, from measurements of intensity made during his American voyage (1798–1804), that the total intensity is 1.000 at the magnetic equator, and increases somewhat steadily towards the magnetic poles, where *Gauss* afterwards made the average intensity to be 1.977. The increase of intensity with higher latitudes presents many irregularities, and this makes it necessary to resort to spherical harmonics to give the law of intensity over the globe.

(iii) Method for constructing the lines of magnetic force.

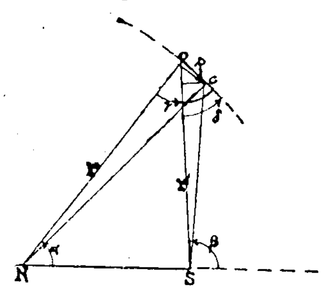

Fig. 1. Resolution of the forces to the two poles.

Before preceeding with the theory of magnetism, we shall show how to draw the lines of force, as by the system of rulers devised by Dr. *Roget* near the middle of the 19th century. Let rectilinear radii vectores be drawn from any point on the line of force to the two magnetic poles N and S, as shown in the figure 1.

[1] Remembering the dual system of forces acting on a bar magnet at the earth's surface, we have another way of reaching the same result, as follows:

1. The pole strength of the earth, $\mu = 8464 \cdot 10^{18} \mu'$, where μ' is the pole strength of *Gauss'* unit bar magnet. The quantity of matter generating the magnetism μ is 1/1408 of the earth's mass.

2. The gravitational mass of the unit bar magnet $m' = 1/(8464 \cdot 10^{18})$ of the matter producing the magnetism of the earth, which may be taken as unity in the divisor.

3. Then the gravitational mass of the earth, $m = (1408)(8464 \cdot 10^{18})$, the magnetic unit of mass again being that of the *Gauss*ian unit bar.

4. Accordingly, replacing the numbers in units of magnetism $(\mu' \cdot \mu')$ with others representing the matter which produces the magnetism, in order to get the ratio $\mu\mu'/mm'$, we have:

$$\mu\mu'/mm' = [(8464 \cdot 10^{18})\mu'\mu']/[(1408)(8464 \cdot 10^{18}) \cdot \mu'\mu'/(8464 \cdot 10^{18})] = (1/1408)/1408, \text{ or } \eta^2 = 1/(1408)^2.$$

Here the two points o and o' are taken to be indefinitely near each other, and thus oo' is tangential to the curve, in the direction of the magnetic force.

Then obviously the sines of the angles γ and δ are inversely as the forces emanating from the corresponding poles, and therefore directly as the squares of the corresponding distances, thus:

$$\sin\gamma/r^2 = \sin\delta/r'^2 \,. \tag{18}$$

But by geometry,

$$\sin\gamma = r\,d\alpha/ds \qquad \sin\delta = r'\,d\beta/ds \tag{19}$$

and therefore by substituting in (18) we get:

$$d\alpha/r = d\beta/r' \,. \tag{20}$$

It is obvious from the figure 1 that

$$r\sin\alpha = r'\sin\beta \tag{21}$$

and on multiplying equation (20) by (21) the product yields

$$\sin\alpha\,d\alpha = \sin\beta\,d\beta$$

which admits of integration in the simple form:

$$\cos\alpha = \cos\beta + C \,. \tag{22}$$

To construct the lines of force by the mechanical means devised by Dr. *Roget*, let two rulers of equal length revolve around their ends fixed in N and S as centres, while the moveable vertical boundary, taken to be the side of a right triangle ABC, with base on NS prolonged, slides along.

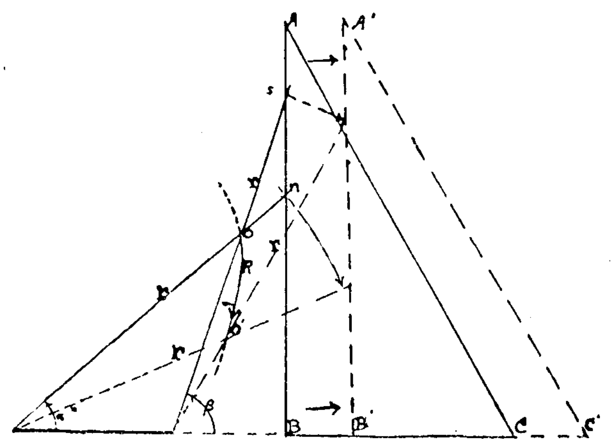

Fig. 2. Dr. *Roget*'s mechanical method for constructing the magnetic lines of force.

The intersection of the rulers resting against the sliding triangle ABC will fulfill the conditions specified in equation (22). For nB is the cosine of α to the radius r, and sB in like manner is the cosine of β to the radius r, and $NS = C$.

Hence the point of a pencil, held at this intersection o, will at once describe a line of the magnetic force, oo'. In the present paper the lines of force for the earth's field have not been drawn by such mechanical magnetic line apparatus, yet the accuracy is sufficient for all purposes here in view, which is chiefly the exposition of valid physical theory.

(iv) The physical nature of the magnetic lines of force investigated by means of an experimentum crucis, which shows that these lines represent the axial rotations of the wave-elements.

1. It is now shown that the stress in the aether along the line of force directed to the pole is due to waves flat in the plane of the equator of the magnet, parallel to which the elements of aether or aetherons rotate, about the lines of force, and thereby generate the tension along these lines which *Faraday* recognized as causing the lines to tend to shorten themselves. Towards the ends of the magnet there is a gradual change in direction of the stress, due to the absence of wave-emitting atoms, all vibrating in concert; and therefore the wave movement turns around and the lines reenter the magnetic poles, as illustrated in the following figure 3 from *Newton*'s Principia, 1687, lib. II, section VIII, prop. XLI, theorem XXXII, where the curvature of the waves to a new radius is exhibited after the movement from the centre A passes through the orifice BC.

2. Now in magnetism each atom emits its own waves, and as the motion spreads with increased facility along the axis of the magnet, owing to the increased stress due to waves crowding in that direction, the lines of force are nearly straight at the centre of the magnet, and curve most rapidly around after passing the poles.

This simple view of the dynamics of the magnetic lines of force scarcely seems to require further explanation, as I have treated it briefly in AN 5048, p. 162, in connection with *Dolbear*'s experiment, based on an observable mechanical model shown in practice to give the curvature and dynamical properties of the lines of force. Nevertheless the accompanying figure will enable us to see more clearly why the lines have near the ends of the magnet the increased curvature actually observed. The extension of the *Newton*ian figure is authorized by this example in vector composition.

3. Accordingly we shall illustrate the flatness of the lines of force near the centre of the bar magnet, compared to the rapid curvature towards either end, by another figure deduced from that given by *Newton* in the Principia, 1687, as follows:

a) Imagine adjacent additional centres of disturbance, A', A'', $A'''\cdots$, and A_1, A_2, $A_3\cdots$, all in vertical line with the centre A. And make additional orifices above and below BC, as $B'C'$, $B''C''$, $B'''C'''\cdots$, B_1C_1, B_2C_2, $B_3C_3\cdots$, through which the wave disturbances may pass.

b) Then the waves in the same phase will everywhere mutually support each other: the disturbing centres being in the same parallel line, the wave fronts will become straightened by the mutual support of the separate independent disturbances.

c) Now imagine the orifices brought closer and closer together, yet maintained as distinct centres of disturbances: we see that beyond the line BC, prolonged in both directions, the wave fronts will become quite straight in the centre, but will curve around rapidly only near the end of the extreme orifice $B^{\mathrm{v}}C^{\mathrm{v}}$, $B_{\mathrm{v}}C_{\mathrm{v}}$, above and below respectively. This is exactly what occurs in magnetism: the lines of force curve around conspicuously as we approach the ends of the magnet

d) The poles, in fact, are the centres of the reacting stress in the medium when agitated by all the atoms vibrating in concert, and emitting waves of the kind here described. The lines of force being axes of rotations for the aetherons, as the waves move along, there is a tendency in these lines to shorten themselves, as in *Dolbear*'s experiment; the result is tension along the lines, and as they are of minimum length, they tend to keep straight near the centre of the magnet, and to curve sensibly only near the ends of the bar, just as in the water-wave experiment above described from *Newton*'s diagram of 1687.

4. The experiments here described are accurate and can be verified by actual trial for water waves, which are simple and easily understood. They disclose to us the true nature of magnetism, for the following reasons:

a) The results conform to *Dolbear*'s experiment, where the dynamical influences at work are easily understood, and admit of but one interpretation.

b) They are verified in the actual movement of water, the waves of which also have tension along their axes and tend to straighten themselves to a minimum axis $s = \int_{o}^{p} ds$ on the principle of Least Action.

5. By actual experiment, 1845, *Faraday* found that the plane of a beam of polarized light was rotated when passed along the line of force, through heavy glass, carbon disulphide and similar substances, and the more rotated the longer the path s. This fact shows clearly that aether waves of the type here described underlie magnetism. They are proved to exist by the practical experiments with water waves, by *Dolbear*'s experiment, on tangible models, and by *Faraday*'s celebrated experiment on the rotation of the plane of polarization by magnetism.

6. There is one other experiment which equally supports the above conclusion, namely the revolution of a flexible hoop set loosely on an axis, in the apparatus commonly used to show the effects of centrifugal force. When the hoop is spun rapidly about its axis, it becomes of oval shape, bulged out at the equator and drawn in at the poles of rotation, like the figures of the planets which it is used to illustrate.

Now imagine a series of such hoops mounted side by side, and tied together mutually along the axis. Then, when the rotation develops, the whole line of connected hoops will shorten itself, under the centrifugal force, just as in *Faraday*'s lines of force. It is impossible to imagine a more convincing proof than that here suggested.

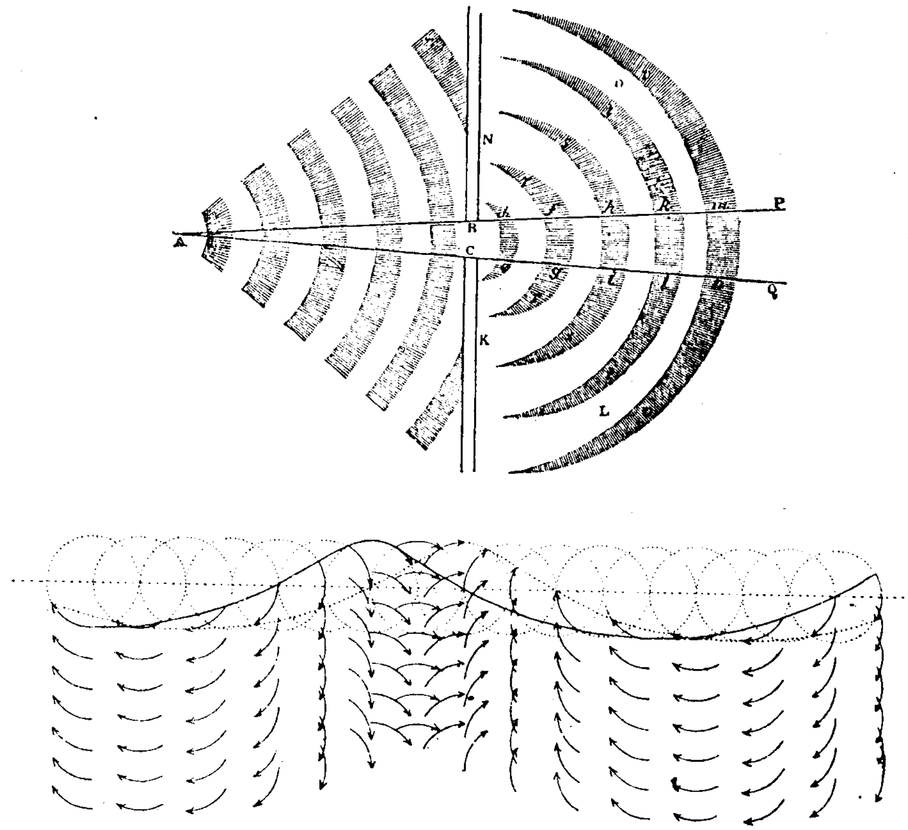

Fig. 3. *Newton*'s diagram of the spread of waves to a new radius, after passing through an orifice *BC*; with *Airy*'s illustration of the nature of the wave-motion, below (cf Tides and Waves, 1845).

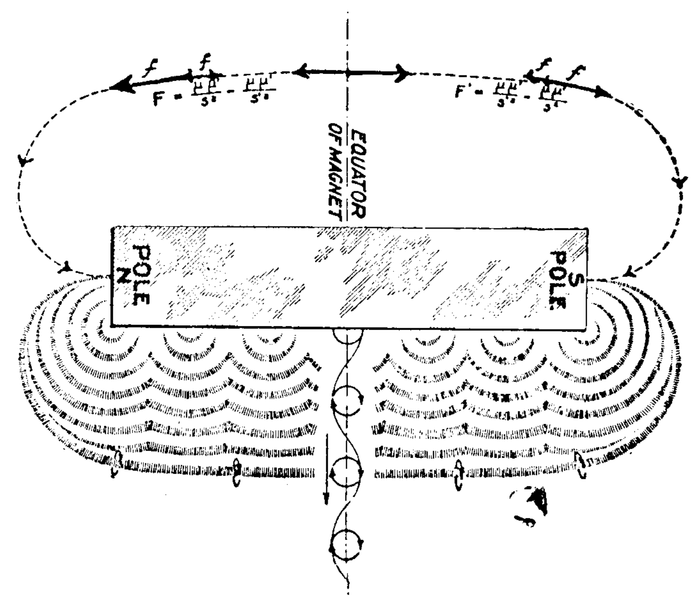

Fig 4. Illustration of the simultaneous compounding of wave motions from closely adjacent orifices, by extension of *Newton*'s theory. As the rotations about the axes are parallel, the tension in each wave disturbance tends to shorten the arc of the whole wave filament, and make the wave front a minimum. We thus get magnetic lines of force nearly straight in the equatorial regions of the magnet, with rapid curvature towards the poles. This diagram, in connection with Fig. 3 above, completely explains the observed phenomena in magnetism, and assures us that the wave theory assigns the true cause of magnetism. Drawn by *T. F. Greathead.*

2. New General Formulae for the Intensity of the Ponderomotive Force in every Part of the Magnetic Field: the New Theory rigorously verified by the Law of *Biot* for a Short Magnet deep in the Interior of the Globe as the Simplest Basis of the Earth's Magnetism, 1816.

(i) Law of the aether stress along the line of magnetic force from pole to pole.

When we have a new theory of any phenomenon in nature nothing is more satisfactory than its expression in the form of a geometrical law which enables us to confirm the mathematical rigor of the theory for every part of space. Thus we need the means for exploring the entire magnetic field, from pole to pole, to see if our harmonic law is everywhere rigorously fulfilled.

If this law is found to be exact throughout the whole field of the magnet, this property of mathematical accuracy alone will constitute an overwhelming argument for the validity of the new theory. Indeed, unless a contradiction can be established, we may safely conclude that the law as formulated is a true law of nature.

1. We begin with the magnetic equator, since in this region the two equal poles are equally remote; and as s and s' are equal, the two terms for the aether stress are equal, and the oppositely directed forces perfectly balanced, thus:

$$I = f + f' = \mu' \int 1/s^2 \cdot d\mu + \mu' \int 1/s'^2 \cdot d\mu = \mu\mu'/s^2 + \mu\mu'/s'^2 \quad (23)$$

where μ and μ are the strengths of the poles of the greater magnet (the earth for instance), μ' and μ' the strengths of the poles of the smaller magnet, or needle, which we may regard as suspended by a vertical thread attached to its centre of gravity, while the needle itself assumes the horizontal position.

This fact was first carefully observed by *Humboldt* when he crossed the earth's magnetic equator in northern Peru 7° 1′ south latitude, 313° 41′ east longitude, between the silver mining town of Micuipampa and Caxamarca, where the elevation is about 12000 feet (Cosmos, vol. 1, *Bohn* translation, p. 177). The true laws of the earth's magnetism are so important that we shall be justified in deriving general formulae and comparing them with the results of observations in both terrestrial hemispheres. What is true of the earth's magnetism is even more rigorously true of a symmetrical magnetic bar, because a good artificial magnet is much more regular and exact in its laws of attraction than the earth, which is made up of many lesser magnets very irregularly arranged into one large globular magnet.

2. As we go towards either pole the force pulling towards the other pole weakens, leaving unbalanced the two terms f and f', which are oppositely directed, thereby yielding the aether tension I, yet becoming more and more unequal towards the poles.

a) If we go towards the north pole, $f = \mu\mu'/s^2$ increases, because s steadily decreases: on the other hand, f' decreases, because s' quite as steadily increases. Hence in the northern hemisphere the difference of stress will be:

$$F = \mu\mu'/s^2 - \mu\mu'/s'^2 \quad (24)$$

where F is the ponderomotive force pulling the small magnet (with poles μ' and μ') bodily towards the north pole of the large magnet, with poles μ and μ, which may be the earth.

b) If we go towards the south pole, the force f pulling towards the north pole weakens, leaving the pull towards the south pole correspondingly predominant. Hence as $f = \mu\mu'/s^2$ decreases, while $f' = \mu\mu'/s'^2$ increases, we have the corresponding inequality of stress:

$$F' = \mu\mu'/s'^2 - \mu\mu'/s^2 . \quad (25)$$

This is therefore the ponderomotive force pulling the small magnet towards the south pole of the large magnet, which may be the earth. In view of these equations, the reader may now advantageously refer again to figure 4 above, which gives a connected representation of all the wave phenomena in the field about a magnet, together with the forces thereby generated. It is well also to refer to the photograph of magnetic action reproduced in plate 2, fig. 1, which places the new theory beyond controversy, because the effects of the ponderomotive forces are rendered directly visible to the eye of the reader. This is the first photograph of the kind ever taken for illustrating the mutual actions of two magnets, the smaller magnets being shown in four leading positions.

3. In view of the above considerations, our harmonic law

$$I/g = \eta^2 r^2 (1/s^2 + 1/s'^2) \quad (26)$$

may be written for the earth's magnetic and gravitational actions in the form:

$$I/g = \mu\mu'/mm' \cdot (r^2/s^2 + r^2/s'^2) = \eta^2 r^2 (1/s^2 + 1/s'^2) . \quad (27)$$

Here, in the right member of the equation, we have merely replaced I by the above magnetic expression (23), and for g substituted the familiar gravitational formula, $g = mm'/r^2$. In fact equation (27) may be said to have the following meaning:

The action of the large magnet, such as the uniformly magnetized sphere of the earth, upon the small magnet at the distance of the two poles, from which its stresses are exerted, is to the gravitative action of the earth towards its centre, as

$$\mu\mu'(1/s^2 + 1/s'^2) : (mm'/r^2) = \eta^2 (r^2/s^2 + r^2/s'^2 : 1.) \quad (28)$$

Thus at unit distance, $r = 1$, the magnetic force is to the gravitational force as 1 to 1960000, if $\eta = 1/1400$.

In dealing with the actual earth elsewhere, we find from the total intensities at the poles calculated by *Gauss* that the theoretical ratio is slightly different, namely: 1 to 1982802. We need not here inquire into the theoretical sources of this trifling difference, as that would raise too many difficult analytical questions relative to *Gauss'* theory. The difference itself, moreover, is very trifling.

4. Accordingly, if I denote the total force, or aether stress along the line of force, so that $I = \mu\mu' (1/s^2 + 1/s'^2)$, we have for any place on the globe:

$$I/g = \eta^2 r^2 (1/s^2 + 1/s'^2) \quad (29)$$

or

$$[\mu\mu' (1/s^2 + 1/s'^2)] / [mm'/r^2] = \eta^2 (r^2/s^2 + r^2/s'^2) . \quad (30)$$

5. This means that the earth's magnetic part acting at the distance to the two foci, s and s', is to the gravity of the whole mass m, acting at the distance to the centre r, as

$$\eta^2 (r^2/s^2 + r^2/s'^2) : 1 = 1/1960000 \cdot (r^2/s^2 + r^2/s'^2)$$

if

$$\eta = 1/1400 .$$

a) As shown above in (23) the stress in the aether along the line of force is perfectly balanced only at the equator, because there $\mu\mu'/s^2 = \mu\mu'/s'^2$. The unbalanced tension along the line of force at any point whatever is equivalent to the effect of each pole pulling unequally on its own end of the needle. The old doctrine of repulsion need not be considered at all.

b) North or south of the equator therefore the stress is unbalanced, because we have:

$\mu\mu'/s^2 - \mu\mu'/s'^2 = F =$ ponderomotive force in the northern hemisphere

$\mu\mu'/s'^2 - \mu\mu'/s^2 = F' =$ ponderomotive force in the southern hemisphere.

This unbalancing of the stresses in either hemisphere is easily confirmed by observation. A small magnet suspended by a thread easily is seen to be swung bodily and end-on towards the nearest pole. If the proper pole of the suspended magnet be not presented to that of the larger magnet, the smaller magnet will quickly reverse itself, and then swing over, deflecting the thread about its centre visibly from the vertical direction of gravitation, as shown in plate 2, fig. 1.

6. It is the increased power of the term due to the nearer pole, and decreased power of the term due to the opposite pole which throws the aether stress along the line of force out of balance; and this lack of balance gives therefore the difference, or visible ponderomotive forces:

$$F = \mu\mu'/s^2 - \mu\mu'/s'^2 \text{, in the northern hemisphere}$$
$$F' = \mu\mu'/s'^2 - \mu\mu'/s^2 \text{, in the southern hemisphere.} \quad (31)$$

At the poles it is obvious that we shall have:

$$\text{North Pole, } F = \mu\mu'/s^2 \text{, because } \mu\mu'/s'^2 = 0$$
$$\text{South Pole, } F' = \mu\mu'/s'^2 \text{, because } \mu\mu'/s^2 = 0 . \quad (32)$$

The results here developed contain the whole theory of the mutual actions of two magnets upon each other. But as such a theory does not exist today in any book in the world, we have felt authorized to explain the new theory in sufficient detail to assure the reader of its entire rigor. The following development by *Biot*, who reached similar results from another point of view, will also prove of interest to the student of this subject.

The problem of magnetism has been so unsatisfactorily treated heretofore that it is highly advisable to examine it from several aspects. Thus *Biot*'s formula gives identical results at the two poles, yet no physical cause is assignable from his reasoning why such a result should follow, whilst on the wave-theory of magnetism we have a very tangible physical cause always before our minds, and generating the ponderomotive force exactly as observed in nature.

(ii) Outline of the simple theory of terrestrial magnetism proposed by *Biot* in 1816.

In his Traité de Physique, 1816, tome III, p. 139, the celebrated *J. B. Biot* developed an hypothesis which gives a simple view of the earth's magnetism. It had been somewhat outlined by *Tobias Mayer* of Göttingen about the middle of the 18[th] century, but *Biot* gave it a form of much greater mathematical elegance and rigor. He imagines a single magnet whose axis passes through the centre of the earth, in a direction perpendicular to the magnetic equator, but of small length compared to the earth's radius.

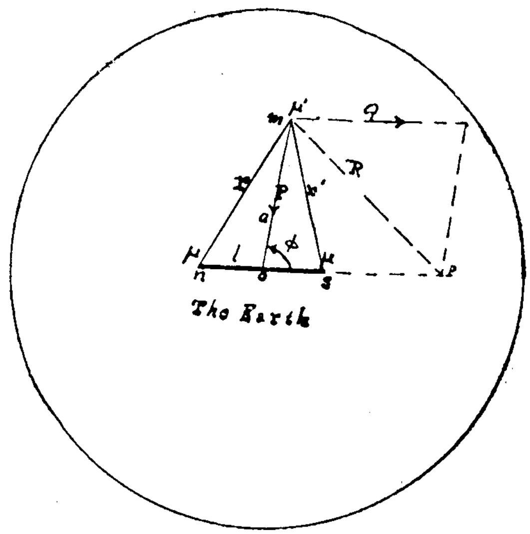

Fig. 5. Illustration of *Biot*'s theory of a short magnet near the earth's centre.

In order to give a distinct analysis of *Biot*'s theory we derive first the expressions for the force exerted by a bar magnet upon a unit pole, as ordinarily given. Let the line ns denote the distance $2l$ between the two poles of a magnet, each of pole strength μ, acting upon a unit pole at m of strength μ'. If μ' be austral (south seeking), the forces exerted by the poles n and s will become:

$$f = -\mu\mu'/r^2 \qquad f' = +\mu\mu'/r'^2 . \quad (33)$$

These forces may be resolved into two, namely P in the direction mo, and Q in the direction ns, — the former somewhat small, the latter much larger:

$$P = \mu\mu' a (1/r'^3 - 1/r^3)$$
$$Q = \mu\mu' l (1/r'^3 + 1/r^3) . \quad (34)$$

Let mp be the direction of the resultant R of the two forces; then as the sides of the triangle mop are proportional to the forces in their directions we have:

$$mo : op = P : Q = a(r^3 - r'^3) : [l(r^3 + r'^3)] \quad (35)$$
$$op = l(r^3 + r'^3)/(r^3 - r'^3) . \quad (36)$$

But by trigonometry we have

$$r^2 = a^2 + l^2 + 2al\cos\Phi \qquad r'^2 = a^2 + l^2 - 2al\cos\Phi . \quad (37)$$

Now l is taken to be small compared to a, and under these conditions we may expand these expressions in series, neglecting terms of l/a higher than the first.

$$r^{-3} = a^{-3}[1 - 3(l/a)\cos\Phi]$$
$$r'^{-3} = a^{-3}[1 + 3(l/a)\cos\Phi] . \quad (38)$$

Wherefore, by addition and subtraction respectively, we get:

$$r'^{-3} + r^{-3} = 2a^{-3}, \quad r'^{-3} - r^{-3} = 2a^{-3} \cdot 3(l/a)\cos\Phi . \quad (39)$$

Substituting these expressions in (34) above, we find, on putting for $2\mu l$ the magnetic moment $m = 2l\mu$:

$$P = m\mu' \, 3\cos\Phi/a^3, \qquad Q = m\mu'/a^3$$
$$P = Q\cdot 3\cos\Phi. \tag{40}$$

The magnitude of the resultant force R is given by the equation for the composition of the vectors P and Q:

$$R^2 = P^2+Q^2-2PQ\cos\Phi$$
$$= Q^2\,[(3\cos\Phi)^2+1-2\cdot 3\cos\Phi\cos\Phi] \tag{41}$$

whence

$$R = m\mu'/a^3\cdot(1+3\cos^2\Phi)^{1/2}. \tag{42}$$

If therefore the point m is on the prolongation of the axis of the magnet, we have: $\Phi = 0°$, and:

$$R = 2\,m\mu'/a^3. \tag{43}$$

But if the point m is in the equator, $\Phi = 90°$, and we have:

$$R = m\mu'/a^3. \tag{44}$$

The simple equations (43) and (44) represent *Biot*'s celebrated result, that if the earth's magnetism be due to a short magnet in the centre of the globe, the total magnetic force at the poles ought to be exactly twice as great as at the equator.

According to the scale of intensity formerly employed *Humboldt* found that at the magnetic equator, — between Micuipampa and Caxamarca, Peru, at an elevation of nearly 12000 feet in the Andes, — the intensity was 1.000; and at the poles *Gauss* calculated the average intensity to be 1.977. This confirms *Biot*'s law quite accurately; for half of the mean polar intensity, found by *Gauss*' profound theory, is 0.9885, in perfect accord with the mean of many determinations along the magnetic equator, as shown by the following table:

Sixteen widely separated determinations of the total intensity near the magnetic equator.

Name of Place	Latitude	East Longitude	I = Total Magnetic Intensity
Cape of Good Hope	−34° 11′	18° 26′	1.014
Mauritius	−20 9	57 31	1.144
Madras	+13 4	80 17	1.031
Otaheite	−17 29	210 30	1.094
Galapagos Islands	− 0 50	270 23	1.069
Magnetic equator near Caxamarca	− 7 2	281 12	1.000
Quito	0 0	281 15	1.067
Plateau of Antisana	− 0 25	281 20	1.068
Montevideo	−34 53	303 47	1.060
Rio de Janeiro	−22 55	316 51	0.878
Bahia	−12 59	321 30	0.871
Minimum of faint zone east of Brazil	−19 59	322 36	0.706
Pernambuco	− 8 4	325 9	0.914
Porto Praya	+14 54	336 30	1.156
Ascension	− 7 56	345 36	0.873
St. Helena	−15 55	354 17	0.836

Mean value of total intensity . . . . . . $I_0 =$ 0.9863
Half of *Gauss*' mean value at the poles $\frac{1}{2}I_c =$ 0.9885
Difference . . . . = +0.0022.

3. Detailled Analysis of the Law connecting the Mean Total Intensity of the Magnetism of the earth with Terrestrial Gravitation.

(i) Analysis and application of the harmonic law, $I/g = \eta^2\,(r^2/s^2+r^2/s'^2)$.

Let I denote the mean total intensity of magnetism at any point of the earth's surface; and let g denote the acceleration of terrestrial gravity, ordinarily taken as 981 cm C. G. S. It is usual to designate the value of the horizontal component of the earth's magnetism by γ, and to express this force in 10^{-5} of a C. G. S.-unit. Thus at Cheltenham, Md., the Magnetic Observatory of the U. S. Coast Survey in 1906 found the value of γ to be 20 (New International Encycl., New York, 1916, vol. 22, article Terrestrial Magnetism, p. 121: or *L. A. Bauer*, United States Magnetic Tables and Magnetic Charts for 1905, Washington, 1908).

$$\gamma = 0.00020 \text{ C. G. S.}$$

As the magnetic declination for 1906 was 70° 27′, we find

$$I = \gamma\sec\theta = 0.00020\cdot 2.98838 = 0.000597676$$
$$= 0.0006, \text{ nearly.} \tag{45}$$

But since the declination changes slowly, and also the value of γ varies progressively from year to year, we need not dwell on the higher decimal places; for although accurate values for the Observatory are available, yet other places in the same community would have slightly different values. Thus we use the round number $6\cdot 10^{-4}$ as representing the mean value of I at Cheltenham for the year 1906.

Accordingly we have by observation:

$$I/g = 6/9810000 = 1/1635000. \tag{46}$$

It is to be observed that at the equator the harmonic law gives the ratio $I\eta^2 = 1/1982802$, and at the mean poles $I\eta^2 = 1/991401$, exactly double the equatorial value, as under *Biot*'s law. And similar confirmations of the harmonic law will be found at various stations in middle latitude, so far as the earth's magnetism is regular, in undergoing steady increase towards the poles.

In the applications of the harmonic law

$$I/g = \eta^2\,(r^2/s^2+r^2/s'^2)$$

it is to be noticed that at the magnetic pole in Boothia Felix the line of magnetic force s' running to the pole in the Antarctic becomes infinite, so that the second term in the above equation vanishes, and we must therefore use only the first term in our calculations.

In like manner, when we apply the formula to the pole in the Antarctic, s becomes infinite, and the first term vanishes, so that only the second term remains for use in calculations.

At the magnetic equator on the other hand, the two terms become of exactly equal value, and the curved lines to the poles attain maximum values $s = \sqrt{2} = 1.4142$, $s' = \sqrt{2} = 1.4142$, so that the sum $r^2/s^2+r^2/s'^2 = 1.000$. Also, by observation, each of the equatorial terms is approximately $\frac{1}{2}$, as we find by actual integration $s = \int_0^p ds$, $s' = \int_0^{p'} ds'$.

Towards the poles, on the other hand, the divisor s or s' of the chief term becomes small, augmenting the intensity at the poles according to the data of observation and *Gauss'* theory. The harmonic law therefore is extremely simple, and the force changes steadily, yielding the required increase of intensity towards the poles, and only half of this mean intensity at the equator.

Now it was shown by *Gauss* (Allgemeine Theorie des Erdmagnetismus, 1838, p. 46), as more fully explained below, that on the average there are in each cubic metre of the earth's matter the equivalent of 7.831 bar magnets each weighing one German pound, all of them weighing 3.9155 kg. And as the density of the earth is taken to be 5.5, the average cubic metre of the earth's matter will weigh 5500 kg. Dividing 3.9155 by 5500, we find from *Gauss'* figures that:

$$\eta = 3.9155/5500 = 1 : 1404.674^{\text{th}} \qquad (47)$$

part of the earth is magnetic, like the perfectly saturated steel bars used in the Observatory at Göttingen, 1833–38.

These figures imply that the distribution of the bar magnets within the earth is uniform, whereas a more natural hypothesis would be to take the density of the bars to be proportional to the density of the matter in the different spherical shells of which the earth is made up.

On calculating the weight of the average cubic metre of the earth's matter, on this hypothesis, we find it comes out 5524.13 kg, instead of 5500 kg, as previously assumed on the theory of homogeneity. The ratio of increase is 1.004387 to 1. Since the external action of the earth's magnetism was found to correspond to 7.831 standard bars, in the observations used by *Gauss*, and this observed datum cannot be increased by any alteration of our hypothesis, we can adjust the difference only by taking somewhat fewer bars in each cubic metre, 7.831/1.004387 = 7.7968, in place of 7.831. The result is:

$$\eta = 3.8984/5500 = 1/1410.837\,. \qquad (48)$$

Giving this value a weight of 3, and the above value a weight of 2, we get as our mean result:

$$\eta = 1/1408.372\,. \qquad (49)$$

The result is thus very near the value $\eta = 1 : 1408.12$ previously indicated, equation (4), and thus we adhere to that value as the most probable.

It should be explained that the curved line of magnetic force, along which magnetic stress towards either pole acts, is to be integrated between the place of observation and the pole properly located in the earth. Here we come to a new property of magnetism as distinguished from gravitation. For gravity acts in straight lines, while magnetism acts in curved lines, along the lines of magnetic force directed to either pole. It is directed along the tangents of these curves towards the nearer pole, and always is positive, as I have found by careful experiments with soft iron, and also with freely suspended small magnetic needles.

The small suspended magnetic needles are magnets, free to turn the appropriate end to the nearer pole; and when so suspended by a thread they behave exactly as soft iron in which magnetism is induced by the waves in the field of the larger magnet. It is easy to find by trial of this simple experiment, that just as soft iron filings when laid upon a glass plate and jarred, will arrange themselves along the magnetic line of force, so also freely suspended needles will show exactly the same tendency. The ponderomotive force or unbalanced stress in the aether is along the curved paths to the nearest pole.

Hence in considering the earth's magnetism we have to take the integrals:

$$\begin{aligned} s &= \int_0^p ds = \int_0^p [1+(dy/dx)^2+(dz/dx)^2]^{1/2}\,dx \\ s' &= \int_0^{p'} ds' = \int_0^{p'} [1+(dy/dx)^2+(dz/dx)^2]^{1/2}\,dx \end{aligned} \qquad (50)$$

as explained above.

In general the curves of the magnetic lines of force in the earth's field are curves of double curvature; so that the rigorous integration of (50) is difficult, because we do not know the equations of the curves by which we might compute dy/dx and dz/dx. Thus we have to consider the curvature along any path whatever in finding $s = \int_0^p ds$, $s' = \int_0^{p'} ds'$.

In its most general form the equation for the curvature along any path in space has the form

$$1/\varrho = [(d^2x/ds^2)^2+(d^2y/ds^2)^2+(d^2z/ds^2)^2]^{1/2} \qquad (51)$$

where ϱ is the variable radius of curvature.

But in our present practical calculations, it suffices to take the curved line s as lying in the plane through the magnetic poles of the earth, which are found to be at depths of $0.766r$ for the north pole, and of $0.666r$ for the south pole. With these slight restrictions we have calculated the length of s and s' for Cheltenham, Maryland, namely, $s = 0.913\,r$, $s' = 3.361\,r$, $(1/s^2+1/s'^2) = 1.2114$, whence we find the theoretical ratio to be:

$$I/g = 1.2114/1982802 = 1/1636810\,. \qquad (52)$$

Table of observed values of the total intensity at various places and of the product $I\eta^2 = 1 : 1960000$, which increases towards the poles:

Place	Latitude	East Longitude	I	$I\eta^2 =$ 1 : 1960000 *)
Spitzbergen	+79° 50′	11° 40′	1.562	1 : 1254802
Cape of Good Hope	−34 11	18 26	1.014	1 : 1930940
Hammerfest	+70 40	23 46	1.506	1 : 1306575
Mauritius	−20 9	57 31	1.144	1 : 1711300
King George's Sound	−35 2	117 56	1.709	1 : 1146872
Hobartown	−42 53	147 24	1.817	1 : 1078700
Sydney	−33 51	151 17	1.685	1 : 1163205
South magn. pole	−72 35	152 30	2.253	1 : 869951
New Zealand	−35 16	174 0	1.591	1 : 1231930
Otaheite	−17 29	210 30	1.094	1 : 1791600
San Francisco	+37 49	237 35	1.591	1 : 1231930
North magn. pole	+73 35	264 21	1.701	1 : 1152264
Cheltenham Md.	+38 44	283 10	1.738	1 : 1127400
Conception	−36 42	286 50	1.218	1 : 1609195
Valparaiso	−33 2	288 19	1.176	1 : 1666666
Falkland Isl.	−51 32	301 53	1.367	1 : 1433800
Montevideo	−34 53	303 47	1.060	1 : 1847313
Rio de Janeiro	−22 55	316 51	0.878	1 : 2232330
St. Helena	−15 55	354 17	0.836	1 : 2339105

*) In this table η is taken as 1 : 1400, but it is not worth while to recompute it.

The close agreement of this ratio with the observed value given in (46) above is so remarkable that the application of the new formula requires no comment. It is evident that the new formulae

$$I/g = (1/1408.12)^2\,[(r^2/s^2)+(r^2/s'^2)]$$

$$s = \int_0^p ds \qquad s' = \int_0^{p'} ds'$$

will hold for any part of the globe. For what will apply at the equator, at the pole, and at a typical station in middle latitude will apply generally to any part of the earth's surface. But it is evident that we must expect the test to be fulfilled only for the mean value of I, since the calculated action from the distant pole can take no account of local magnetic attractions, which often are of sensible magnitude.

(ii) The magnetic lines of force are paths of least action for the aether stress.

The unexpected result above brought out, that the stress in the aether which gives rise to the magnetic forces, is exerted along the curved line of magnetic force and thus is tangential to the line of force at every point, requires something more than passing notice. Such a result has hardly been considered in the science of dynamics as handed down by the great classic authorities, — such as *Newton*, *Euler*, *Lagrange*, *Laplace*, *Gauss*, *Jacobi*, *Hamilton*. Yet we must remember that these classic authorities were occupied chiefly with gravitational forces, which act in right lines; and if they dealt with other forces occasionally, it was always assumed that the stress from which the forces arise act in right lines, like gravitation. When we come to magnetism, however, the case is different: we have a 'duality of powers', and stresses acting in curved lines.

It is admitted by the most eminent mathematicians, that in the operations of nature the changes take place according to the principle of Least Action. Already *Fermat* had established by rigorous test the fact of action in Least Time for such forces as light:

$$\tau = \int 1/v \cdot ds\,.$$

In later time this geometric condition was generalized for the other forces of nature also.

In applying the above formula (49) to the magnetism of the earth, we notice that as magnetism is a stress, and directed along the line of force, we must assume this curved path to be the path of least action for the operation of the stress in the aether called magnetism. The distance s therefore is a curved path

$$s = \int_0^p ds = \int_0^p [1+(dy/dx)^2+(dz/dx)^2]^{1/2}\,dx$$

$$s' = \int_0^{p'} ds' = \int_0^{p'} [1+(dy/dx)^2+(dz/dx)^2]^{1/2}\,dx$$

where the limits are the place of observation and the nearer pole located at the proper depth in the globe of the earth.

Let us examine into these lines of force, in the hope of finding special cases of straight lines, as at the two surface magnetic poles of the globe, where the direction of the total intensity of the magnetic force is vertical, and there should be no curvature of this special path either above or below the earth's surface, except for local inequalities of magnetism, which in this general theory of the mean total intensity of the total force is left out of account.

According to *Gauss*, (p. 46), the total intensity of the magnetism at the north pole is:

$$I = 1.701\,. \qquad (53)$$

At the south pole likewise *Gauss* finds:

$$I' = 2.253\,. \qquad (54)$$

Restricting ourselves to the consideration of the mean total force of the magnetic intensity, and disregarding local influences altogether, it is evident that in the region of the poles, the curved line of magnetic force s becomes straight, while $s' = \infty$, and therefore we can write the harmonic law in the form:

$$I/g = 1/1982802 \cdot (r/s)^2 \qquad (55)$$

where r is the earth's radius, at which g is determined, and $s = \int_0^p ds$, also is expressed in units of the earth's radius.

Using the values found by *Gauss*, in (53) and (54) above, we find for the depths of the two poles as more fully discussed hereafter:

$$\text{North pole,}\quad s = 0.766\,r \qquad \text{South pole,}\quad s' = 0.666\,r\,. \qquad (56)$$

The results here brought out are quite remarkable. As s at the poles is a straight line, we find that the pole is located much nearer the surface in the southern than in the northern hemisphere. Hence the mean total intensity at the south pole is a maximum, 2.253, while in the northern hemisphere the pole is much deeper down and the total force correspondingly weaker, $I = 1.701$. This very simple deduction throws light upon the asymmetry of the earth's magnetic system, long recognized, but heretofore not understood.

Hence we may be sure that the harmonic law will hold for the entire arc, from the magnetic equator to the magnetic poles. It is certain that the magnetic force or stress from either pole not only varies inversely as the square of the distance s, along the curved line of force, but also renders the pole a true centre of attraction, as so long held in the theory of magnetism. This is what *Airy* calls the »duality of powers«, (Treatise on Magnetism, 1870, p. 10).

This confirmation of the law of inverse squares, thus verifies the wave-theory of physical forces. It was noticed by *Faraday* that lines of force tend to shorten themselves, (Experimental Researches in Electricity, no. 3269), which led him to the theory of tension along the lines of force. We have explained this mechanical tendency by waves with rotations about the lines of force. Hence these lines of force are minimum paths for the whirling filaments, with tension along then directed to the poles.

(iii) Determination of the depth of the magnetic poles.

1. *Gauss* shows that for the two poles the mean value of the total magnetic force is:

$$I = 1.977\,. \qquad (57)$$

2. To get the depth of the mean pole we have therefore to solve the equation:

$$1/n = 1.977/1982802 = 1/1002935\,. \qquad (58)$$

This is the ratio of magnetic force at the pole to gravity.

3. And we then introduce the factor $1/s^2$, with the condition that when the whole force is exerted from the pole at a certain depth $s<r$, we have the observed ratio indicated in the second number:

$$1/1982802\cdot(r/s)^2 = 1/1002935 \qquad (59)$$

which gives:

$$s = r(1002935/1982802)^{1/2} = 0.71128\,r\,. \qquad (60)$$

4. Accordingly, we find by a very simple process that the pole placed at a depth of $0.71128\,r$, will generate the increase of force noticed at the pole above the part 1/1982802, which would correspond to the distance unity, in this case the radius of the earth.

5. Since the intensity of the total magnetic force is observed to increase from the equator to the pole, according to the general law of terrestrial magnetism discovered by *Humboldt* 1799–1804, and first announced by him to the Paris Academy, An XIII, 26th Frimaire (Jan. 16, 1805), in a joint paper with M. *Biot* (Cosmos, vol. I, pp. 179–181, *Bohn* Translation), we naturally attribute this increase of the magnetic force to the smaller distance at which the stress is exerted, by the nearer pole, that to the other pole decreasing correspondingly.

6. It must be remembered that in magnetism the pole is a real centre of attraction, corresponding to the centre of gravity of a heavenly body, for purely gravitational forces; and consequently our reference of magnetic forces must be to the poles by which they are exerted. *Airy* justly says that magnetism is characterized by a »Duality of Powers« (Treatise on Magnetism, 1870, p. 10); hence we must not on that account fail to refer the forces to their appropriate centres. And when we do this all the chief phenomena of terrestrial magnetism may be explained by the laws of attraction for forces varying inversely as the square of the distance, which is another most impressive proof of the connection of gravity with magnetism, and of magnetism with gravitation — both of these forces being due to wave-action, following the same laws, yet exerted along rectilinear and curvilinear paths respectively.

7. Having found the average depth of the mean magnetic poles, it will now be in order to determine the depth of the actual north and south magnetic poles in the solid globe of the earth. At the northern magnetic pole *Gauss*' theory gives the total intensity as

$$I = 1.701$$

wherefore we find

$$1/n = 1.701/1982802 = 1/1165668\,. \qquad (61)$$

And

$$s = r(1165668/1982802)^{1/2} = 0.76674\,r\,. \qquad (62)$$

8. For the actual south pole, we have likewise

$$I' = 2.253$$

$$1/n' = 1/880072 \qquad (63)$$

whence

$$s' = (880072/1982802)^{1/2} = 0.66622\,r\,. \qquad (64)$$

9. The southern magnetic pole is quite appreciably nearer the surface than the northern. In fact the difference in the depth of the two poles amounts to almost exactly one-tenth of the terrestrial radius, or

$$\int_0^p ds - \int_0^{p'} ds' = s - s' = 0.10052\,r\,. \qquad (65)$$

This is a very remarkable feature of the magnetism of the globe, and so far as I can find out it has scarcely been considered by previous investigators. Yet such a lopsided position of the two magnetic poles, — the southern being one-tenth of a terrestrial radius nearer the Antarctic Continent, — must have some meaning in the physical constitution of our planet.

10. Perhaps the phenomenon of this notable magnetic asymmetry here brought to light is too novel to justify as yet any satisfactory discussion. But we think it worth while to point out that the magnetic asymmetry corresponds closely to the land and water hemispheres, the origin of which I have treated in AN 484-445, 1916.

If this coincidence in position is accidental it is quite remarkable. On the other hand, if there be a real physical connection of the pole nearest the earth's surface with the great briny ocean which overlies half the world, we might explain it by the greater conductivity of salt water for the electrical wave-action, on which the magnetism of the earth so essentially depends. Whether such a secular asymmetry of the magnetic system of the earth could develop with the lapse of the billions of years involved in the growth of the earth is a question which must be left to the future researches of natural philosophers.

At any rate I deem it desirable to direct attention to the only known surface cause of such asymmetry, and the singular coincidence in the positions of the two systems — the magnetic system being bodily displaced $0.05\,r = 200$ miles towards the ocean hemisphere. Whatever conclusions may be developed, these two remarkable asymmetries, — one relating to the ocean and the other to the magnetic system, — are the greatest outstanding physical features of the globe, and their essential coincidence therefore is the more extraordinary. It certainly must appear to philosophers very surprising that such vast outstanding features have received little or no study in the researches heretofore made on the origin of the globe, and the distribution of the magnetism in the two hemispheres.[1])

It only remains to add that as the difference of the depths of the magnetic poles from the surface of the earth is 0.10 of the radius or 637.8 kms, about 400 miles, the north pole is displaced downward 318 kms, or 200 miles, while the south pole is displaced upward, towards the Antarctic, by an equal amount. The absolute amount of this displacement thus is very large.

[1]) Mr. *E. F. Wesley*, of *Wheldon* and *Wesley*, London, was able to place in my hands a full set of the great series of memoirs on terrestrial magnetism by General Sir *Edward Sabine*. They had been presented to Sir *John Herschel*, as they successively appeared in the Philosophical Transactions, and finally purchased by Mr. *Wesley*, with the *Herschel* library. Without this valuable *Herschel* collection my labors would have encountered increased difficulty.

4. The Harmonic Law affords an Experimentum Crucis as to the Nature of Magnetism.

(i) The aether stress arising under the harmonic law gives forces directed towards the nearer pole.

If we examine the second member of the equation for the harmonic law connecting the total intensity of the earth's magnetism with terrestrial gravitation, namely:

$$I/g = \eta^2 (r^2/s^2 + r^2/s'^2) \qquad (66)$$

we perceive that at the magnetic equator the two terms are exactly equal, while as we approach either pole, the term becomes largest for the pole which is nearest, while the other term vanishes. This equation therefore represents a stress in the aether in the form of an unbalanced tension.

At the magnetic equator the two oppositely directed stresses exactly balance. Accordingly at this place there is no force, because the balanced tension acts in the tangent, and therefore is precisely parallel to the axis of the magnet. Likewise at any other point of the magnetic line of force the tension is in the direction of the tangent, yet on either side of the magnetic equator the term corresponding to the remoter pole decreases, while that directed to the nearer pole increases. And as the stress therefore is no longer exactly balanced, that directed to the nearer pole becomes predominant. It is this outstanding unbalanced stress which appears as a force directed to the nearer pole, along the curved line s or s' as the case may be.

This is the most remarkable physical characteristic of magnetism, and heretofore it has not been well understood. The theory of the action of a magnet upon a unit pole is essentially defective and misleading. For if one pole, say austral, is presented, it tends to move one way along the line of magnetic force; while if the opposite pole, say the boreal, is presented, it tends to move the other way along the magnetic lines of force.

Now in nature there is no such thing as the separation of the two poles. As pointed out in AN 5079, p. 247, one pole cannot exist and act separately, any more than one side of the human body. However short be the pieces into which a magnet is broken, the two poles still persist, even to dust-like or molecular dimensions. Hence the conclusion that magnetism is a property inherent in the molecules or atoms.

About the year 1820 the celebrated French physicist *Ampère* reached the conclusion from the action of galvanic currents in producing artificial magnets that magnetism consisted essentially in the circulation of elementary electric currents about the atoms. A very similar view was taken by *Gauss* (Allgemeine Theorie des Erdmagnetismus, 1838, p. 49; *Gauss* Werke 5.168) who reasons as follows:

»In unserer Theorie ist angenommen, daß in jedem meßbaren magnetisierten Teile des Erdkörpers genau eben so viel positives wie negatives Fluidum enthalten sei. Hätten die magnetischen Flüssigkeiten gar keine Realität, sondern wären sie nur ein fingiertes Substitut für galvanische Ströme in den kleinsten Teilen der Erde, so ist jene Gleichheit schon von selbst an die Befugnis zu dieser Substitution geknüpft; legt man hingegen den magnetischen Flüssigkeiten wirkliche Realität bei, so könnte man ohne Ungereimtheit die vollkommene Gleichheit der Quantitäten beider Flüssigkeiten in Zweifel ziehen. In Beziehung auf einzelne magnetische Körper (natürliche oder künstliche Magnete) ließe sich die Frage, ob in ihnen ein merklicher Überschuß der einen oder der andern Flüssigkeit enthalten sei, oder nicht, leicht durch sehr scharfe Versuche entscheiden, da im erstern Falle ein mit einem solchen Körper belasteter Lotfaden eine Abweichung von der vertikalen Lage zeigen müßte (und zwar in der Richtung des magnetischen Meridians). Wenn dergleichen Versuche, mit vielen künstlichen Magneten in einem von Eisen hinlänglich entfernten Lokale angestellt, niemals die geringste Abweichung zeigen sollten (wie wohl zu vermuten steht), so würde allerdings jene Gleichheit auch für die ganze Erde mit größter Wahrscheinlichkeit anzunehmen sein, immer aber doch die Möglichkeit einiger Ungleichheit noch nicht ganz ausgeschlossen.«

In the Electr. Wave-Theory of Phys. Forc., vol. I, 1917, p. 20, and AN 5079, pp. 261-262, it is clearly shown that *Ampère*'s theory of galvanic currents about the atoms, — to which *Gauss* strongly inclines in the above passage, — is identical with the wave-theory. Thus the two theories are one and the same. The existence of electric currents about the atoms implies waves emitted by the atoms which are flat in the planes of their equators. It is waves propagated from the wire bearing a galvanic current that calls forth the magnetic property in iron, steel, nickel or other substances subjected to such action.

Thus by the demonstrated identity of effects the wave-theory has the sanction of *Ampère* and *Gauss*, though it was not developed in their time, nor stated in the way which has developed since the memorable triumph of the undulatory theory under the analysis of *Fourier* and *Poisson*.

(ii) At either pole of the earth, the magnet stands vertical, because the tension to the other pole along the line s or s' vanishes.

By an examination of the above equation (66) we perceive that near the north pole of the earth, the line of force s' running away to the other pole is of infinite length, and the term depending on r^2/s'^2 therefore vanishes. A corresponding result happens at the south pole of our globe, where the term depending on r^2/s^2 disappears, owing to the infinite distance to the north pole along the curved line s.

In fact this property of magnets, by which the lines of the earth's magnetic force at the poles become very straight — corresponding to a very flat field — offers very serious practical difficulty to polar explorers. As far back as Feb. 17, 1841, difficulty was experienced by Sir *James Ross*, when he attempted to judge from the observed dip of 88°40′ how far away the southern magnetic pole would be. His observations showed that the direction of the dip from the vertical was only 80′, ordinarily corresponding to 80 miles, yet he estimated the distance to the pole as about 160 nautical miles — multiplying the normal change of dip by two.

When *Shackleton*'s party — Dr. *Mackay*, Professor *David*, Sir *Douglas Mawson* — approached the south magnetic pole in 1909, they found that the dip changed very slowly, — evidently owing to the flatness of the magnetic field — and they were nearly at the end of their vital resources before they came near the region of the pole. On the evening of

Jan. 15, 1909, the dip was observed to be 89°48′, and Sir *Douglas Mawson*, at that time somewhat inexperienced, estimated that the pole was distant only 12 or 13 miles. On Jan. 16, they reached the estimated spot, by forced marches, yet the point of verticity probably was still quite a distance away, for reasons which now seem fairly obvious.

For in 1912, Sir *Douglas Mawson* again sought to reach the pole from Commonwealth Bay, on the other side (Home of the Blizzard, 2 vols., Lippincott, Phil., 1914), and found by measurement that when he was at dip 89°43′.5, only 16′.5 from the vertical, the rate of change was so slow that he had to travel three or four nautical miles to effect a change of a single minute in the dip. Thus in this last effort he did not reach the south magnetic pole, but got only within an estimated distance of some 50 or 60 miles of it. Probably it was near *Gauss*' calculated place, two or three times this distance.

It is a curious fact that *Gauss*' calculated position of the pole lies almost half way between the positions attained by *Mawson* in 1909, and 1912, as shown on the map given in plate 3, from *Shackleton*'s report on the Geology of the Antarctic. Hence in view of *Mawson*'s experience of 1912, when the magnetic field was found to be so very flat that he had to go three or four nautical miles to effect a change in the dip of only a single minute, I believe the southern magnetic pole has not yet been attained by any explorer.

It would appear to be very near the position assigned by *Gauss*' profound theory, namely: 72°35′ south latitude, 152°30′ east longitude. The *Shackleton* party in 1909 got within about 80 miles of this site, and *Mawson* in 1912 was within about 130 miles of it; yet the untraversed south magnetic polar area of elliptical form, with centre near *Gauss*' position, was still at least 160 miles long and about 100 miles wide. The centre of this elliptical area has never yet been explored, and thus *Gauss*' calculated position still is in a veritable Terra Incognita.

Returning now to the above formula for the harmonic law, we see that the lines of magnetic force at the poles become excessively straight and parallel. And hence, just as two parallel lines meet only at infinity, so also the returning branch of a very straight closed line can reach the other pole only by traversing an infinite distance in its circuit.

Accordingly, it is true that at the poles of a magnet the conjugate term in the harmonic law becomes rigorously zero. The magnetic attraction on the vertical needle therefore is wholly downward, and the curved line s or s' to the pole becomes rigorously a right line, as assumed in the foregoing theory, for calculating the depths of the poles below the earth's surface. This mathematical method for locating the depth of the poles in the earth is therefore entirely rigorous; and the only uncertainty which can arise is from some physical modification of magnetic wave action, such as the absorption studied by *Majorana* at Turin, 1919, in his researches on the absorption of gravitation (cf. Phil. Mag., May, 1920).

(iii) Photographic illustration of the directions of the forces exerted in magnetism.

In view of the considerable confusion of thought on the subject now prevalent it is very important to have a convincing demonstration of the true nature of magnetism. Thus it appears well to illustrate an easy experiment by a photograph admitting of accurate reproduction.

1. We suspend by threads four small magnetic needles, and so space them about the large magnet as shown in fig. 1, plate 2. It will be seen from the photograph that in all cases the magnet exerts a very sensible pull on the small needles. They are therefore bodily drawn away from the vertical as shown in the photograph here reproduced.

2. The statement so often made that a magnet exerts only a directive action on a magnetic needle, therefore, is not generally true. In the case of the earth, with the poles almost infinitely distant, the action is indeed mainly directive; yet there is always a slight bodily pull on the needle, northward in our hemisphere, and southward in the southern hemisphere.

3. With the photograph of the effect of the forces acting on the four needles, herewith reproduced in fig. 1, plate 2, our theory of the nature of magnetism is completely demonstrated. The argument underlying the harmonic law is seen to be a fact. It is impossible to claim that a similar theory ever before was proposed by any other investigator. And as the wave-theory now is definitely proved for the first time it would appear that *Helmholtz* was not far wrong when he said that our failure to discover the cause of magnetism was the disgrace of the 19th century.

4. It seems likely that the cause never could have been discovered by reasoning based on the theory of the action on a unit north pole — a half magnet! when no such thing exists in nature! — and hence I have examined the problem from the ground up. In the unpublished Preliminary Paper which I sent to the Royal Society in 1914, it was shown conclusively that a needle suspended by a thread is bodily attracted to a wire bearing a steady galvanic current, the ponderomotive force being

$$F = \mu' z i (1/r + 1/r') \qquad (67)$$

as shown in section 12 (ii) below.

The treatises on physics, indeed, give no clear statement as to what happens in this case. They simply evade the difficulty cleverly, and sometimes vaguely. Even *Maxwell* declared in his address on Action at a Distance (Scient. Pap., vol. 2, p. 317), that »the most obvious deduction from this new fact (*Oersted*'s experiment) was that the action of the current on the magnet was not a push-and-pull force, but a rotatory force, and accordingly many minds were set a speculating on vortices and streams of aether whirling round the current.«

5. Just before making this amazing announcement, showing that he had never tried the experiment which I carried out in 1914, *Maxwell* in this address said:

»We have now arrived at the great discovery by *Oersted* of the connection between electricity and magnetism. *Oersted* found that an electric current acts on a magnetic pole, but that it neither attracts nor repels it, but causes it to revolve round the current. He expresses this by saying that 'the electric conflict acts in a revolving manner'.«

6. Evidently the great *Maxwell* believed that the magnet is not bodily attracted to a wire bearing a current. An error authorized by so eminent an authority as *Maxwell*

naturally would be constantly copied by the less cautious investigators. And thus to this day there is no clear statement in any standard work issued prior to the Electr. Wave-Theory of Phys. Forc., vol. 1, 1917, in which I explained that the needle is bodily attracted to the wire, by wave-action, just as it is also bodily attracted to the pole of another magnet, by stresses along the lines of force, as above deduced from the harmonic law,

$$I/g = \eta^2 (r^2/s^2+r^2/s'^2) = 1/1982802 \cdot (r^2/s^2+r^2/s'^2) \quad (68)$$

which is fundamental in the theory of cosmical magnetism.

7. The magnetic vector component $\eta = 1:1408$ depends on the earth's constitution, as does also $g = 981$ cm. But if we pass to any other planet as Venus, Mars, Jupiter or the sun, it is evident that whilst the numerical value of I and of g will be changed, as well as the vector component η, yet another formula of the same type will hold. Thus for the various planets we could write the following series of equations:

$$\begin{aligned} I_1/g_1 &= \eta_1^2 (r_1^2/s_1^2+r_1^2/s_1'^2) \\ I_2/g_2 &= \eta_2^2 (r_2^2/s_2^2+r_2^2/s_2'^2) \\ &\cdots \\ I_\nu/g_\nu &= \eta_\nu^2 (r_\nu^2/s_\nu^2+r_\nu^2/s_\nu'^2)\,. \end{aligned} \quad (69)$$

(iv) Calculation of the magnetic vector component for the sun shows that 1 : 157th part of the solar mass is magnetic.

1. The formula which connects terrestrial magnetism with gravitation upon the earth is therefore of general validity. Gravitational action from the centre of a spherical planet, — the integral action of all the particles under haphazard arrangement, whether the mass be homogeneous or made up of concentric shells of uniform density — would have a mean value at the surface. And the vector component η, — representing the fractional part of the planet's mass which is magnetic, — when we take its square for the composition, according to the law for directed magnitudes, would lead to equations of the form given above.

2. The force g is compounded for the single distance r, upon which gravity depends, while the magnetic force I depends on the »Duality of Powers«, as *Airy* calls them, and therefore has to be calculated from both foci of the magnetic planet. This explains the theory in a simple way; yet in practice we can not find η by observation, except perhaps in the case of the sun, the magnetism of which is very powerful, about 80 times that of the earth's magnetism, according to the observers at the Mt. Wilson Solar Observatory.

3. In general if we had any standard of force, as at the surface of a spherical shell concentric with the centre of an ordinary magnet, like the surfaces of the above spheres for the magnetic planets and sun, we could write similar equations for the forces of magnets under experiment in our laboratories. Thus the assigned cause of magnetism is general, and the harmonic law of universal validity.

4. The harmonic law

$$I/g = \eta^2 (r^2/s^2+r^2/s'^2)$$

may be put in a somewhat different form:

$$g\,\eta^2 r^2 = I/(1/s^2+1/s'^2)\,. \quad (70)$$

But since

$$M/r^2 \cdot \eta^2 r^2 = M/1982802 \quad (71)$$

we perceive that the gravitational force attracting to the centre of the earth, is at that distance 1982802 times more powerful than the total magnetic intensity I, acting at such distance that $(r^2/s^2+r^2/s'^2) = 1$, which is near the magnetic equator.

These considerations show how tangible is the connection now established between magnetism representing a fraction of the mass η attracting to two centres, and gravitation directed to a single centre. Gravitation is the mean action incident to the haphazard arrangement of the planes of the atoms in a non-magnetic body; while in a magnetic body the planes of the atoms take on parallelism, and the attraction a »Duality of Powers«, as if the forces come from the two poles.

5. Now as for applying these formulae to the sun, we notice that the Mt. Wilson observers found the sun's polarity similar to that of the earth's, yet the intensity of magnetization about 80 times greater. Hence if η_s denote the part of the sun's mass which is magnetic, we have by observation the following equation:

$$\eta_s^2 = 80\,\eta_\varepsilon^2 \quad (72)$$

where $\eta_\varepsilon = 1/1408$, as deduced from the researches of *Gauss*, and used throughout our theory of the earth's magnetism.

6. It would thus appear that globe for globe the part of the sun's mass which is magnetic is $1/1408 \cdot \sqrt{80} = 1/157.42$ of the whole of that immense mass of flaming fluid.

The total intensity of solar magnetism being as the square of this fraction, we have

$$\eta^2_s = (1/157.42)^2 = 1/24781\,. \quad (73)$$

And our equation for the harmonic law as applied to the sun becomes:

$$I/g = \eta^2 (r^2/s^2+r^2/s'^2) = r^2/24781 \cdot (1/s^2+1/s'^2)\,. \quad (74)$$

Since the force of gravity at the solar surface is 27301.6 cm (AN 3992, p. 134), we find that at the part of the sun where $(r^2/s^2+r^2/s'^2) = 1$, which is near the solar equator, the value of I would be:

$$I = 1.1017 \text{ cm}\,. \quad (75)$$

It thus appears that at the sun's equator the balanced stress, represented by magnetic forces, if unbalanced, could produce an acceleration of over one centimetre per second, and at the solar poles over two centimetres. This force is not large absolutely, yet on matter suspended by repulsive forces it operates powerfully in generating the lines of the coronal streamers visible during total solar eclipses; and in cycles of the sun spot period produces stupendous electric luminosity effects somewhat analogous to a solar Aurora Borealis and Aurora Australis, which become sensible in droughts and heat waves felt upon our globe.

We pause here to recall *Gauss*' result, for the earth, and our extension of it to the sun, in terms of other units. In the Allgemeine Theorie des Erdmagnetismus, 1838, p. 46 (*Gauss* Werke 5.165), *Gauss* shows that the total number of magnets, each weighing one German pound, which it would be necessary to distribute throughout the globe to account for the observed magnetism of the earth is

$$N = 8464 \cdot 10^{18}\,.$$

The values which I use are slightly different from those employed by *Gauss* eighty four years ago. With my constants the value of N comes out:

$$N = 8454.457 \cdot 10^{18}$$
$$= 8454457000000000000000. \quad (76)$$

To explain the significance of this immense number of magnets for the entire earth, *Gauss* remarks that in order to obtain a substitute for the action of the globe in outer space, we should have to assume, under uniform distribution with parallel magnetic axes, nearly eight such magnets to each cubic metre of the earth's mass (more exactly 7.831).

Such a result, $8.454457 \cdot 10^{21}$ one-pound magnets, everywhere with parallel magnetic axes, is impressive enough an illustration of the magnetism of our globe; yet for the sun the number of such one pound magnets would be enormously greater. In fact the magnetic part of the sun is 1/157 of the whole, and since the sun is 330000 more massive than the earth, this is equivalent to 2102 times the total mass of our globe.

Accordingly for the sun we should have the higher number:

$$N = 8.454457 \cdot 10^{21} \cdot 2102 = 17.77127 \cdot 10^{24}$$
$$= 17771270000000000000000000. \quad (77)$$

To convey a clear idea of this effect we may imagine our earth to have the property of perfect magnetism. Then if all its particles were reduced to magnetic bars with parallel axes, it would require 2102 such perfect magnetic globes, like the hypothetical substitute for the earth, when uniformly distributed throughout the sun's mass, to give the magnetic field which actually surrounds our sun and acts on the planets as they revolve in their orbits.

Since the solar magnetic field is rendered variable by the outbursts of sunspots, with their increased emission of magnetic waves, we need not be surprised at the earth currents (really eddy currents) 'Magnetic Storms' and Aurorae observed upon our globe.

5. Investigation of the Supposed Motion of the Magnetic Poles in the Earth.

(i) Sir *James Clark Ross'* attempts to reach the south magnetic pole, 1841.

In order to obtain a clear view of the supposed motion of the magnetic poles in the solid globe of the earth, we must first review the locations assigned to the poles by leading explorers at different epochs. We shall then be able to judge if there is evidence of a progressive motion of the poles, and, if so, to fix its character as accurately as possible.

On June 1, 1831, the north magnetic pole was located by Sir *James Ross* in 70°5′ North Latitude, 263°14′ East Longitude, where the dip was observed to be 89°59′. This observation no doubt was as accurate as could be expected, but since the north magnetic pole does not seem to have been located by actual observation at any later period, we have no observational data to enable us to judge of its supposed motion since 1831.

We shall therefore first examine the problem of the supposed motion of the south magnetic pole, where approximate data are furnished by Sir *James Ross'* observations of 1841, and by Sir *Douglas Mawson*'s observations of 1909 and 1912. The south magnetic pole is the more instructive also because of the way *Gauss'* calculated position fits in with the recently observed places.

On February 17, 1841, Sir *James Ross* discovered Cape Gauss, near 76°12′ South Latitude, 164° East Longitude. Here the vertical walls of ice stopped the westward cruise of the Erebus and Terror; but from measures taken on the ice he observed the dip to be 88°40′, or 80′ from magnetic verticity, »so that the pole was only 160 miles distant«.

The place of *Ross'* ship is indicated on the accompanying map (plate 3); and as he found the variation there to be 109°24′ east, I have accurately charted his calculated position of the south magnetic pole, at this nearest approach to it. The places commonly assigned to *Ross'* estimate of the place of the pole, frequently are so inexact that it is necessary to exercise caution, to avoid being misled. Thus in the article Polar Regions, Encycl. Brit., 9th ed., 1885, vol. 19, p. 330, it is stated that »the south magnetic pole was calculated to be in 76° S. and 145°20′ E., or about 500 miles southwest from the ship's position«. There is no good authority for this statement, and it cannot be correct. The place laid down on the accompanying map is from *Ross'* observations, and he expressly declares that »the pole was only 160 miles distant«.

Ross believed the pole to be in the snow capped mountains, slightly to the north of west from his position, and as the summits were over 10000 feet high, he could behold from the sea the range of mountains in which the pole is placed, and yet he could not reach it, owing to the indefinitely extended vertical walls of ice.

In his account he adds: »The range of mountains in the extreme west, which, if they be of an equal elevation with Mount Erebus, were not less than fifty leagues distant (150 nautical miles), and therefore undoubtedly the seat of the southern magnetic pole, was distinguished by the name of His Royal Highness, Prince *Albert*.«

It is worthy of notice that *Ross* seems to have been aware of the flatness of the earth's field, near the magnetic pole; for when the dip is only 80′ from verticity, he estimates the pole to be distant 160 miles — so that twice as many miles would have to be traversed to produce the required change of dip. This same problem arises with Sir *Douglas Mawson*, 1912, as shown below, but it was not given much attention in the dash for the magnetic pole made Jan. 16, 1909.

(ii) Sir *Douglas Mawson*'s search for the south magnetic pole, 1909.

1. In *Shackleton*'s Heart of the Antarctic (2 vols., Lippincott, Phila., 1909) a detailed account is given of the search for the south magnetic pole. The *Shackleton* party seems to have believed that the pole was in rapid movement. Thus on p. 383, they say:

»In the interval between 1841, when these observations were made, and 1902, when the Discovery Expedition again located the south magnetic pole, it had moved about two hundred geographical miles to the eastward.«

2. This statement as to the motion of the pole since 1841 must be received with great reserve for the following reasons:

a) On p. 177 it is stated in the record for Jan. 12, 1909, that on carefully analysing the results of the advance copy of the Discovery Expedition Magnetic Report, Mr. *Mawson* decided that »the magnetic pole, instead of moving easterly, as it had done in the interval between *Sabine*'s observations in 1841, and the time of the Discovery Expedition in 1902, was likely now to be traveling somewhat to the northwest«. This was of course on the supposition that the pole moves quite rapidly. Such reasoning, however, was not justified, because the pole had never yet been accurately located.

b) On Jan. 15, *Mawson* got a good latitude observation, 72°42′, and twenty minutes before noon found by the dip circle that the dip was 89°45′, so that they had at length approached very near to the south magnetic pole. That evening the dip was again found to be 89°48′, the day's march having been 14 miles. Having calculated that the pole was not over 13 miles away, they rested till early next morning. Thus they made an early start for the spot fixed upon for the pole, 72°25′ South Latitude, 155°16′ East Longitude, and reached it by great effort at $3^h 30^m$ p. m., — guiding their course by vertical marks erected every two miles or so, as they traveled, the compass now being useless, on account of the great proximity to the pole.

3. Owing to the extreme weariness of the party and their shortage of food they took no further observations, merely raising the British Flag, and taking possession of the plateau about the pole, and retracing their steps with all haste to the little depot where the dip was 89°48′, which they reached at 10^h p. m. (p. 182). On Jan. 15, *Mawson* estimated (p. 180) that »in order to accurately locate the mean position (of the pole) possibly a month of continuous observation would be needed, but that the position he indicated was now as close as he could locate it.«

The dash to the pole place fixed upon by the apparent rapid change in the dip was thus all that was attempted. Apparently they did not even count the number of oscillations of the needle in 10 minutes, which would have given a measure of the total intensity at the pole, and been valuable scientific data. The journey had, however, proved to be much longer than had been expected, which again emphasizes the flatness of the magnetic field near the pole.

In 1841 *Ross* had taken the distance to be two miles to a single minute of change in the dip. By actual journey the *Shackleton* party found the distance there and back at least 500 miles. If we take the single distance at 240 miles on a great circle, the change of 80′ in the dip from *Ross*' place in 1841, would imply an average multiplication by three miles for each minute of the change in dip. The distance, however, from *Ross*' place at the sea to the actual pole probably was about 320 miles, which would make the average multiplier of the change of dip four instead of three.

(iii) The expedition to the south magnetic pole, by *Mawson*, *Bage* and *Webb*, from Commonwealth Bay, 1912.

In the Home of the Blizzard (Lippincott, Phila., 1914, 2 vols.) we find a detailed account of the approach to the south magnetic pole from the other side, the base being in Commonwealth Bay. The following is a brief summary of the chief measurements of dip, and other phenomena noted.

On Nov. 10, 1912, the party started for the south magnetic pole, the journey being mainly to the south, and slightly to the east. On Nov. 20, the dip was 87°27′, requiring a change of 153′ to reach the pole; but by Nov. 27, the dip had changed to 88°54′, yet as the change was somewhat sudden it was thought to be »too large«, (p. 287), — perhaps the reading should have been 88°24′. The dip continued to decrease slowly, and on Dec. 3, it was steady at 88°30′ — a result showing that the above value for Nov. 27 was an error.

As the party sped on they seemed to find the dip nearly stationary for a time, 89°11′, — what it had been since leaving the station at 150 miles. Sixty five miles more appeared to yield little change in the dip. On Dec. 17 they passed 70° south latitude — making about 14 miles a day, and the dip was found to be 89°25′. On Dec. 19, the dip was 89°35′, and at 256 miles the altitude of the plateau was 5600 feet, while on Dec. 21, their sledge-meter showed 301 miles.

On page 296 *Bage* describes the difficulties of making accurate observations:

»Magnetic work under these conditions is an extremely uncomfortable operation. Even a light wind will eddy round the break-wind, and it is wind which makes low temperatures formidable. Nearly all the work has to be done with bare fingers or thin instrument-gloves, and the time taken is far greater than in temperate climates, owing to the fingers constantly 'going' and because of the necessity of continually freeing the instrument from the condensed moisture of the breath. Considering that the temperature was — 12° F. when he had finished his four hours' work, it may be imagined that *Webb* was ready for his hot tea. The dip proved to be 89°43′.5, that is: sixteen and half minutes from the vertical. The altitude was just over five thousand nine hundred feet, in latitude 70°36′.5 south and longitude 148°10′ east.«

The party was now within 175 miles of where *David*, *Mawson* and *Mackay* had stopped in 1909. They had to turn back after getting within 16′.5 of the 90° dip at the pole. *Bage*'s diary says:

»We have now been exactly six weeks on the tramp and somehow feel rather sad at turning back, even though it has not been quite a sunday school picnic all along. It is a great disappointment not to see a dip of 90°, but the time is too short with this 'climate'. It was higher than we expected to get, after the unsatisfactory dips obtained near the two-hundred-mile depot. The rate of increase since that spot has been fairly uniform and indicates that 90° might be reached in another fifty to sixty miles, if the same rate held, and that means at least another week. It's no good thinking about it, for 'orders are orders'. We 'll have our work cut out to get back as it is. Twenty-five days till we are overdue. Certainly we have twenty-three days' food, eight days' with us, ten days' at two hundred miles, and five days' at sixty-seven miles; so with luck we should not go hungry, but *Webb* wants to get five more full sets of dips if possible on the way back, and this means two and a half days.«

This southward journey of *Mawson*'s party from Commonwealth Bay is full of instruction, like that of the *Shackleton* party from Ross Sea in 1909. In both explorations the observers found the pole with 90° dip was much further away than they had at first expected. This was due to the extreme flatness of the magnetic field near the pole.

(iv) True place of the south magnetic pole probably is within 30 miles of the position calculated by *Gauss*, 1838.

After some study of the effect of the increasing flatness of the magnetic field, as we near the pole, I have ventured to construct a table of distances from the pole, with changing rate of increase for a given increment of dip ($\Delta\theta = 3'$) towards the magnetic pole.

This table may not be complete, but it is very instructive, as affording a simple means of harmonizing the conflicting estimates of the observers who have tried to locate the pole from opposite sides. It seems to show that the true pole is very near the point located by *Gauss* in 1838, and almost certainly not over 30 miles away.

Number of circle about magnetic pole i	Differences of dip from the pole $\Delta\theta_i - \Delta\theta_{i+1}$	m_i = multiplier for 1′ of dip to give equivalent distance in naut. miles.	$m_i(\Delta\theta_i - \Delta\theta_{i+1})$ = radial width of circles in naut. miles.
1	16′.5 — 12′	4	18
2	12 — 9	5	15
3	9 — 6	6	18
4	6 — 3	7	21
5	3 — 0	8	24

$$\sum_{i=1}^{i=5} m_i(\Delta\theta_i - \Delta\theta_{i+1}) = 96 \text{ miles}$$

Estimated total distance of *Mawson*'s party in 1912 from pole . . . . = 106 miles
Estimated distance of *Shackleton*'s party of 1909 from pole . . . . = 70 miles.

In fact it is about 70 miles northwest of the furtherest point reached by *Shackleton*'s party in 1909, and about 105 miles southeast of where *Mawson*'s party halted in 1912. The star on the map shows where we locate the south magnetic pole, after a careful study of all the evidence furnished by the parties of *Shackleton* and *Mawson*.

This very accurate confirmation of the position of the south magnetic pole indicated by *Gauss* in 1838, is well calculated to impress us with the rigor of the method of calculation used by that great mathematician. Observation has not yet been able to improve on the results of the mathematical calculations made before any explorer had visited the Antarctic Continent!

Gauss' method rests on 24 constants, and thus requires measures of the three independent magnetic elements of dip, declination and total intensity, at eight places of observation. The method does not require observations in the southern hemisphere, yet the more symmetrically the stations are distributed about the earth the better. Above all, great accuracy is required in the magnetic measurements, and as *Gauss* was a great master in this line of research, very little improvement has ever been made on his original constants of 1838.

(v) Is there any evidence of the motion of the earth's magnetic poles?

This is a question which has been long debated, yet from the above analysis of the evidence it would seem as if motion is not definitely proved. Before making the above careful analysis of the records I had inclined to the impression of a counter-clockwise rotation of the south magnetic pole, about the *Gauss*ian position (M) of the Maximum Magnetic Moment for the earth. But in this early survey of the data I had relied upon the pole being not over 160 miles from the position of *Ross*' ship Erebus, Feb. 17, 1841. We have seen above that *Ross* held the pole to be distant only 160 miles, yet in so doing he took the multiplier for converting changes of dip into nautical miles to be only 2, whereas the above table based upon *Mawson*'s observational experience showed that the average multiplier for *Ross*' distance should have been at least 3, perhaps 4.

We find in the records of *Shackleton*'s party, 1909, abundant evidence of a current view that the magnetic pole is near the coast, and even moving eastward! As they travel inland they are surprised at the slowness of the change of dip, and conclude that it has suddenly moved westward. As a matter of fact they had not sufficiently allowed for the flatness of the field very near the pole. And these errors of reckoning were not only current among the explorers, but are also repeated in the article Terrestrial Magnetism, Encycl. Brit., 11th ed., by such a scientific authority as Dr. *Charles Chree*, director of the Kew Observatory.

Dr. *Chree* points out that *Sabine*'s Chart (1841) gave for the south magnetic pole 73° 30′ South Latitude, 147° 30′ East Longitude. He says Professor *J. C. Adams* in his researches reached the coordinates 73° 40′ S., and 147° 7′ E. *Chree* then gives the following table as a summary of the chief data.

(A) Southern Cross Expedition, 72° 40′ S., 152° 30′ E.
(B) The Voyage of the »Discovery«, 1902-3 72 51 S., 156 25 E.
(C) *Shackleton*'s Expedition, 1908-9 72 25 S., 155 16 E.

In conclusion Dr. *Chree* thinks »there is at least moderate probability that a considerable movement towards the northeast has taken place during the last seventy years.«

Unfortunately for Dr. *Chree*'s argument we see from the foregoing investigation that there is not the slightest evidence of any such motion. If any motion at all is shown, it is to the westward. For *Gauss*' theoretical determination of the position of the pole is our only trustworthy guide. It is valid for the epoch 1830, and the observational work of 1909 and 1912 show the pole still so very near *Gauss*' original position that we cannot be sure that any motion at all has occurred in the intervening 80 years. The star (*) on the map is the most probable place indicated by the available observations, yet being only about 3′ from the pole of *Gauss* or 25 miles, we can not safely conclude that any motion at all has occurred. Under the most favorable conditions *Gauss*' calculated place may be uncertain by 2′; and at least 1′ of uncertainty always exists in the observations of 1909 and 1912, because of the difficulty in setting and reading the dip circle accurately while traveling in so severe a climate.

If the southern magnetic pole gives little or no evidence of motion in 80 years, it would be natural to hold that the north magnetic pole is correspondingly fixed. Unfortunately the observational evidence is even more incomplete than that for the south pole. When only 30 years of age Sir *James Ross* was in the icefields with his uncle Sir *John Ross*, and as they could only move some eight miles in two years, yet they were near the north magnetic pole, the young man in sheer desperation finally got ashore and having found the dip to be 89° 59′ located the pole June 1, 1831, at

70° 5′ 17″ N., 263° 13′ 15″ E.

So far as I can learn no other explorer ever was able to reach the north magnetic pole. The Norwegian expedition of 1905–07, under *Roald Amundsen*, was to search for this spot, but although he traversed the Northwest Passage and came to San Francisco, with his vessel, and I conversed with him at Mare Island, I never heard of his being near the north magnetic pole. Probably the ice blocked the way in the channels to the south, as it did with the *Rosses* in 1831, and *Amundsen* had to steer a more northerly course. Accordingly it appears that Sir *James Ross* alone attained the north magnetic pole or got within 10 miles of it. The only other indications of value are drawn from *Gauss*' theory; but even here a contradiction arises, probably from a systematic bias at some unknown source.

For *Gauss* himself calculated the north magnetic pole to be at 73° 35′ N, 264° 21′ E. Longitude, which was 3°.5 from *Ross*' observed latitude. This considerable difference proved puzzling to *Gauss*, who says, (p. 44):

»Nach *Ross*'s Beobachtung fällt der nördliche magnetische Pol um 3° 30′ südlicher als nach unserer Rechnung, und letztere gibt, wie aus unsrer Vergleichungstafel ersichtlich ist, eine um 1° 12′ fehlerhafte Richtung der magnetischen Kraft an jenem Platze. Beim südlichen magnetischen Pole wird man eine bedeutend größere Verschiebung zu erwarten haben. Da in Hobarttown, als dem demselben am nächsten liegenden Beobachtungsorte, die berechnete Inklination, ohne Rücksicht auf das Zeichen, von der Rechnung um 3° 38′ zu klein angegeben wird, insofern man sich auf die Beobachtungen verlassen kann, so wird der wirkliche südliche magnetische Pol wahrscheinlich bedeutend nördlicher liegen als ihn unsere Rechnung angibt, und möchte derselbe etwa in der Gegend von 66° Breite und 146° Länge zu suchen sein.«

Accordingly it appears that in view of the observed difference at the north magnetic pole, *Gauss* was in doubt of the accuracy of the calculated place of the south magnetic pole; yet observations over 70 years later verified the true place of the south pole to be very near that assigned by the great mathematician.

As Dr. *Chree* has recently discussed the problem of the motion of the north magnetic pole from another point of view, we shall quote his summary in the Encycl. Brit., article Terrestrial Magnetism, 11th ed., 1911, page 382:

»Table XLV, Axis and Moment of First Order *Gauss*ian Coefficients.

Epoch	Authority	N. Lat.	W. Long.	M/R^3 in cgs. units.
1650	*H. Fritsche*	82° 50′	42° 55′	0.3260
1836	»	78 27	63 35	0.3260
1845	*J. C. Adams*	78 44	64 20	0.3282
1880	»	78 24	68 4	0.3234
1885	*Neumayer, Peterson, Bauer*	78 3	67 3	0.3230
1885	*Neumayer, Schmidt*	78 34	68 31	0.3230

»49. The first order *Gauss*ian constants have a simple physical meaning. The terms containing them represent the potential arising from the uniform magnetization of a sphere parallel to a fixed axis, the moment M of the spherical magnet being given by

$$M = R^3 [(g_1{}^0)^2 + (g_1{}^1)^2 + (h_1{}^1)^2]^{1/2}$$

where R is the earth's radius. The position of the north end of the axis of this uniform magnetization and the values of M/R^3 derived from the more important determinations of the *Gauss*ian constants are given in Table XLV. The data for 1650 are of somewhat doubtful value. If they were as reliable as the others, one would feel greater confidence in the reality of the apparent movement of the north end of the axis from east to west. The table also suggests a slight diminution in M since 1845, but it is open to doubt whether the apparent change exceeds the probable error in the calculated values.«

Accordingly, it thus appears that Dr. *Chree* is very doubtful of the supposed motion of the north magnetic pole to the westward.

Yet reasoning on the basis of the observed secular motion of the magnetic meridians, *Airy* remarks in his Treatise on Magnetism, 1870, p. 53:

»The system of magnetic meridians has undergone considerable changes in the times of modern accurate science. The southern point of Africa received from the Portuguese voyagers in the fifteenth century the name of L'Agulhas (the needle), because the direction of the compass-needle, or the local magnetic meridian, coincided there with the geographical meridian: it now makes with it an angle of about 30°. In the sixteenth century, the compass-needle in Britain pointed east of north: it now points from 20° to 30° (in different parts of the British isles) west of north. At the present time, a change of the opposite character is going on: in 1819 the westerly declination at Greenwich was about 24° 23′, which was probably its maximum; in the last thirty years it has diminished from 23°.5 to 20°, nearly. It is believed that the magnetic poles are rotating around the geographical poles from east to west.«

Great and impressive as are the surface changes here pointed out by *Airy*, it can scarcely be held that any regular cyclical motion of the magnetic poles are in progress. This could not be so and leave either pole fixed in its place, as we have shown is true of the south magnetic pole. Nor, on the other hand, can we adopt the hypothesis of motion and assign to the south magnetic pole a cycle of oscillation enabling it to present the same position in 1910 as 1830. We are thus forced to admit the most extensive secular motions of the magnetic meridians, yet compelled at the same time to deny any sensible motion of the magnetic poles in the solid globe of the earth.

The only way we could account for such a motion of the magnetic meridians without motion of the poles would be to assign the meridional shifts to superficial effects, perhaps due to Eddy-Currents in the globe, like those disturbances which manifest themselves chiefly in Earth Currents and Aurorae, and depend on the action of the sun and moon as explained hereafter. Perhaps the periodicity of the secular changes and their differences in local surface areas of the globe, could be explained by the mutual interactions of the various segments of the earth under the incessant magnetic disturbances of the heavenly bodies, especially the sun and moon.

Under the influence of cosmically induced eddy currents, depending on the sun and moon, and the sunspot cycle with its suddenly varying magnetic field, there could arise not only the uncompensated electric disturbances, and their varying dissipation, with auroral displays in the atmosphere of the higher latitudes, but also, from the way these disturbances are reflected and compensated within a globe so heterogeneous as our earth, a mass of progressive secular oscillations in the magnetic field near the surface. This is the most probable explanation of the secular changes in the earth's magnetism: for we must not regard our globe as one homogeneous mass, but a series of masses acting on each other mutually, and all under changing action depending on the sun, moon and sunspot cycles.

6. Theory of the Earth's Magnetic Moment, with *Gauss*' Explanation of his Method of Calculation.

(i) The constant part of the earth's magnetism depends on internal causes, and its potential may be developed in a convergent series of spherical harmonics.

It has long been recognized that the earth's magnetism may be separated into two parts:

1. A constant part, depending on internal causes, namely plane magnetic waves emitted from the atoms so lined up as to have their equatorial planes parallel to a common plane.

2. A periodic part, depending on fluctuating magnetic disturbances due to the sun, moon, and sunspot changes.

Let H denote the horizontal component of the magnetic force at any point of the earth's field; then the force usually is resolved into the components:

$$\begin{aligned}&\text{Towards the north, } X = H\cos\delta = -1/r\cdot\partial\Omega/\partial u\\ &\text{Towards the west, } Y = H\sin\delta = -1/(r\sin u)\cdot\partial\Omega/\partial\lambda \quad (78)\\ &\text{Vertically downward, } Z = H\,\mathrm{tg}\,\theta = \partial\Omega/\partial r\end{aligned}$$

where $\delta =$ the magnetic declination; $\theta =$ angle of the dip; and $\Omega =$ the potential due to the earth's field at a point of polar distance u and longitude λ, distant r from the centre of the globe.

If α, β, γ be the components of the magnetic force at any point (x, y, z), we shall have

$$\Omega = -\int 1/\varrho\cdot d\mu$$
$$\varrho = \{r^2 - 2rr_0[\cos u\cos u_0 + \sin u\sin u_0\cos(\lambda-\lambda_0)] + r_0^2\}^{1/2} \quad (79)$$

$$\Omega = -\int_{\infty}^{x,y,z}(\alpha\,dx + \beta\,dy + \gamma\,dz) = \int_{\infty}^{x,y,z}(\partial\Omega/\partial x\cdot dx + \partial\Omega/\partial y\cdot dy + \partial\Omega/\partial z\cdot dz) \quad (80)$$

in which Ω fulfills *Laplace*'s equation for every point of free space:

$$\partial^2\Omega/\partial x^2 + \partial^2\Omega/\partial y^2 + \partial^2\Omega/\partial z^2 = 0$$

or in polar coordinates (r, u, λ):

$$1/r^2\cdot\partial(r^2\cdot\partial\Omega/\partial r)/\partial r + 1/(r^2\sin u)\cdot\partial(\sin u\cdot\partial\Omega/\partial u)/\partial u + 1/(r^2\sin^2 u)\cdot\partial^2\Omega/\partial\lambda^2 = 0. \quad (81)$$

Therefore the potential of the earth's field may be expanded in convergent series of the form:

$$\Omega = (S_1/r^2 + S_2/r^3 + S_3/r^4 + \cdots) + S_0' + S_1'r + S_2'r^2 + S_3'r^3 + \cdots \quad (82)$$

in which $S_1, S_2, S_3\cdots S_0', S_1', S_2', S_3'\cdots$ are surface harmonics of the degree indicated by the subscripts.

Since the surface harmonic S_n can be expanded in the form

$$S_n = \sum P_n^m(\cos u)(A_{n,m}\cos m\lambda + B_{n,m}\sin m\lambda) \quad (83)$$

we have

$$\Omega = \sum_{n=0}^{n=\infty}\sum_{m=0}^{m=n}\{P_n^m(\cos u)(A_{n,m}\cos m\lambda + B_{n,m}\sin m\lambda) + r_n P_n^m(\cos u)(A_{n,m}\cos m\lambda + B_{n,m}\sin m\lambda)\} \quad (84)$$

The latter series, depending on periodic planetary influences outside of the earth, usually is separated from the other, because it is very small, and in fact was not included in *Gauss*' theory for the non-periodic part of the earth's magnetism. Thus we have for the principal development of the magnetic potential:

$$\Omega = S_1/r^2 + S_2/r^3 + S_3/r^4 + \cdots = \sum_{n=1}^{n=\infty}\sum_{m=0}^{m=n}\left\{\frac{P_n^m(\cos u)}{r^{n+1}}(A_{n,m}\cos m\lambda + B_{n,m}\sin m\lambda)\right\} \quad (85)$$

If we ignore all the hormonics beyond the first, we obtain:

$$\Omega = 1/r^2\cdot\{A_{1,0}P_1(\cos u) + P_1^1(\cos u)\times(A_{1,1}\cos\lambda + B_{1,1}\sin\lambda)\} \quad (86)$$
$$= 1/r^2\cdot\{0.3157\cos u + \sin u\times(0.0248\cos\lambda - 0.0603\sin\lambda)\}. \quad (87)$$

This last expression is a biaxial harmonic of order unity, (cf. *J. H. Jeans*, Mathematical Theory of Electricity and Magnetism, 3rd ed., 1915, p. 403), and is easily shown to be equal to $0.3224\cos\chi$, where χ is the angular distance of the point (u, λ) from the pole of a uniformly magnetized sphere with axis through

$$\begin{aligned}&\text{Lat. } 78^\circ 20'\text{ N., and Long. } 67^\circ 17'\text{ W.}\\ &u = 11^\circ 40', \quad \lambda = 292^\circ 43'\end{aligned} \quad (88)$$

as discussed by *Chree*, near the end of section 5 above.

Accordingly the potential is

$$\Omega = 0.3224\cos\chi./r^2 \quad (89)$$

which is the potential of a uniformly magnetized sphere having as direction of magnetization the radius through the point defined in equation (88). It is sometimes defined as equivalent to the potential of a single magnetic particle of appropriate magnetic mass at the centre of the earth, yet with the axis of the magnetic particle pointing in this same direction.

(ii) The calculation of the magnetic moment of a magnet explained more in detail.

In all the older theories it was recognized that in a magnet we have to imagine as much negative as positive magnetism, so that, as *Gauss* expressed it $\int d\mu = 0$, when the integral is extended throughout the whole body. We shall now examine the basis of this reasoning.

Let μ be a small quantity of magnetism, f the resultant magnetic force at the point (x, y, z), then the magnetic force exerted on a quantity μ of magnetism concentrated there is μf. Now for a whole magnet $\int d\mu = 0$, and in order to fulfill this condition for the opposite magnetisms we have:

$$\Sigma\mu = -\Sigma\mu' . \tag{90}$$

Let N be the centre of mass of the positive, S the centre of mass of the negative magnetism characteristic of the two poles of the magnet; then obviously the coordinates of N and S are:

$$\begin{aligned} \bar{x} &= \Sigma\mu x/\Sigma\mu, \quad \bar{y} = \Sigma\mu y/\Sigma\mu, \quad \bar{z} = \Sigma\mu z/\Sigma\mu; \\ \bar{x}' &= \Sigma\mu' x'/\Sigma\mu', \quad \bar{y}' = \Sigma\mu' y'/\Sigma\mu', \quad \bar{z}' = \Sigma\mu' z'/\Sigma\mu'. \end{aligned} \tag{91}$$

Now if l be the distance between N and S, whose coordinates are given in (91), we have:

$$l = [(\bar{x}-\bar{x}')^2+(\bar{y}-\bar{y}')^2+(\bar{z}-\bar{z}')^2]^{1/2}. \tag{92}$$

Then, there is a certain constant in the theory of a magnet, known as the magnetic moment, which we shall call K:

$$K = l\Sigma\mu = -l\Sigma\mu'. \tag{93}$$

The magnetic moment therefore is equal to the product of the length between the two poles by the amount of magnetism at either pole. If the magnet be placed in a uniform field, and the axis of the line NS makes the angle χ with the strength f, a directed magnitude defining the field, the whole couple becomes:

$$f\cdot(\Sigma\mu\,{}^1/_2 l - \Sigma\mu'\,{}^1/_2 l)\sin\chi = f\Sigma\mu l \sin\chi . \tag{94}$$

It follows from the definition in (93) that the magnetic moment is a positive magnitude $K = l\Sigma\mu$, and since $l = NS$, the pole is the centre of mass of the magnetic forces, just as the centre of oscillation gives the centre of gravity for the gravitative forces at work in the motion of a compound pendulum, of length l.

Now let ξ, η, ζ be the direction cosines of the magnetic axis of any element $dv = dx\,dy\,dz$ of a magnet, and $\mathfrak{J}$ such a quantity that $\mathfrak{J}dv$ is the magnetic moment of the element: then $\mathfrak{J}$ is the intensity of magnetization, at the point (x, y, z), where the element $dv = dx\,dy\,dz$ is taken, and we have

$$\mathfrak{J}\xi = A \qquad \mathfrak{J}\eta = B \qquad \mathfrak{J}\zeta = C \tag{95}$$

$\mathfrak{J}$ being a vector or directed magnitude, tangential to the line of magnetization, and A, B, C the ideal equivalent magnets with axes parallel to the coordinate axes.

If p, q, r be the direction cosines of the axis of the whole magnet, we have

$$\begin{aligned} p &= (\Sigma\mu x/\Sigma\mu - \Sigma\mu' x'/\Sigma\mu')\cdot 1/l = \Sigma[l\mu\cdot(x-x')/l]/(l\Sigma\mu) \\ &= \Sigma\,\delta K\,\xi/K = \Sigma\,\mathfrak{J}\xi\,dv/K = \Sigma A\,dv/K . \end{aligned} \tag{96}$$

Upon integration for the volume $dv = dx\,dy\,dz$, we may therefore write:

$$\begin{aligned} K &= 1/p\cdot\iiint A\,dx\,dy\,dz = 1/q\cdot\iiint B\,dx\,dy\,dz \\ &= 1/r\cdot\iiint C\,dx\,dy\,dz . \end{aligned} \tag{97}$$

The condition that K should be a maximum for a body like the earth evidently is that $R = (A^2+B^2+C^2)^{1/2}$ should be a maximum. Hence we rotate the axes to such a position that $\delta R = 0$, or

$$\delta K = \delta\iiint(A^2+B^2+C^2)^{1/2}\,dx\,dy\,dz = 0 \tag{98}$$

as explained in the following calculation.

(iii) *Gauss'* calculation of the magnetic moment of the earth.

In the 31[st] section of his Allgemeine Theorie *Gauss* proceeds to calculate the magnetic moment of the earth. He first remarks that it would be a misconception to attribute any significance to the mere surface location of the poles, or the chord joining them, if one were to call this line the magnetic axis of the earth.

»The one way«, says *Gauss,* »in which we can give the conception of the magnetic axis of a body a general validity is that set forth in article 5 of the »Intensitas Vis Magneticae«, according to which we understand a straight line, in respect to which the moment of the free magnetism contained in the body is a maximum. To determine the position of the magnetic axis of the earth in this sense and at the same time the moment of the earth's magnetism in respect to the same, as already remarked above in art. 17, we require only a knowledge of the terms of the first order. According to our elements in art. 26, we have:

$$\begin{aligned} P' = &+925.782\cos u + 89.024\sin u\cos\lambda + \\ &-178.744\sin u\sin\lambda \end{aligned} \tag{99}$$

in which $-925.782\,R^3$, $-89.024\,R^3$, $+178.744\,R^3$ are the moments of the earth's magnetism in respect to the earth's axis, and the two earth-radii for the longitude 0° and 90°. By the earth's axis we are to understand the direction to the north pole, and the negative sign of the corresponding moment indicates that the magnetic axis makes an obtuse angle with it, that the magnetic north pole is turned to the south. The direction of the magnetic axis resulting from this is parallel to the diameter of the earth at 77° 50′ north latitude, 296° 29′ longitude (east), 77° 50′ south latitude, 116° 29′ longitude (east), and the magnetic moment in respect to the same $= 947.08\,R^3$.«

»In respect to this last result, we are to remember, that our elements are based upon a unit of intensity which is a thousandth part of that commonly used. In order to make the reduction to the absolute unit established in the Intensitas Vis Magneticae, we remark that in the latter the horizontal intensity of Göttingen, 1834, July 19, was found = 1.7748, from which with the inclination 68° 1′ the total intensity = 4.7414 follows, whereas by the above unit this was taken to be = 1357. The reduction factor is therefore = 0.0034941, and consequently the magnetic moment of the earth in absolute units

$$= 3.3092\,R^3 .\text{«} \tag{100}$$

»In this absolute unit for the terrestrial magnetic force the millimetre is taken as the unit of length, and therefore R must also be expressed in millimetres, whereby — since the ellipticity of the earth is entirely neglected — it is sufficient to consider R as the radius of a circle whose circumference amounts to 40000 million millimetres. According to this, the above magnetic moment is expressed by a number whose logarithm = 29.93136 or by 853800 quadrillions. According to the same absolute unit the magnetic moment of one of the sensitive magnetic bars experimented upon in 1832 (Intensitas, Art. 21) is = 100877000; and the magnetic moment of the earth is therefore 8464 trillion times larger.«

»Therefore 8464 trillion such magnetic bars, with parallel axes, would be required to replace the magnetic action of the earth in outer space; which in a uniform distribution throughout the whole bodily space of the earth would amount to nearly eight bars (more exactly 7.831) to each cubic metre.«

(iv) The fractional part of the earth which is magnetic found to be $\eta = 1/1404.674$.

The above analysis discloses how *Gauss* reached his celebrated result, that 7.831 parallel saturated steel bar magnets, each weighing one German pound, all of them 3.9155 kilograms, would be required in each cubic metre of the earth's mass to give externally the magnetic field actually observed at the surface of our globe.

After ariving at this striking practical result *Gauss* continues:

»When so stated, this result still retains its meaning even if we do not consider the earth a real magnet, but would ascribe terrestrial magnetism merely to purely galvanic streams in the earth. If, however, we consider the earth a real magnet, we are compelled to ascribe to each part of the same, which is one eighth cubic metre in size, on the average at least[1]), a magnetization quite as powerful as that bar contains, — a result indeed which will be unexpected by the physicist.«

In section 17 *Gauss* gives the following explanation of the analysis by which the above value of V is calculated:

»We choose r for the distance to the centre of the earth, and u for the angle which r makes with the northern part of the earth's axis, while λ denotes the angle of the plane through r and the earth's axis and a fixed meridian, reckoned positive to the east. Let V be a function developed in a series proceeding according to powers of r, which we give the following form:

$$V = R^2P^0/r + R^3P'/r^2 + R^4P''/r^3 + R^5P'''/r^4 + \text{etc.} \qquad (101)$$

The coefficients P^0, P', P'', etc., are here functions of u and λ; in order to discern how they depend on the internal distribution of the magnetic fluid in the interior of the earth, let $d\mu$ be an element of this magnetic fluid, ϱ its distance from O, and for $d\mu$ let r_0, u_0, λ_0, denote the same which r, u, λ, are for O. We have therefore expanded

$$V = -\int 1/\varrho \cdot d\mu \qquad (102)$$

through all $d\mu$. Furthermore

$$\varrho = \{r^2 - 2rr_0\,[\cos u \cos u_0 + \sin u \sin u_0 \cos(\lambda - \lambda_0)] + r_0^2\}^{1/2} \qquad (103)$$

and if we develop $1/\varrho$ in a series:

$$1/\varrho = 1/r \cdot (T^0 + T' r_0/r + T'' r_0^2/r^2 + \text{etc.}) \qquad (104)$$

we have

$$R^2P^0 = -\int T^0\, d\mu, \qquad R^3P' = -\int T' r_0\, d\mu, \quad R^4P'' = -\int T'' r_0^2\, d\mu. \qquad (105)$$

»Since $T^0 = 1$, it will follow by means of the fundamental assumption, that the mass of the positive and negative fluid in each measurable part of its conductor, and therefore in the whole earth, is equally large, or that

$$\int d\mu = 0, \qquad P^0 = 0; \qquad (106)$$

or the first term of our series for V falls away. We see further, that P' has the form

$$\begin{aligned} R^3P' &= \alpha \cos u + \beta \sin u \cos\lambda + \gamma \sin u \sin\lambda \\ \alpha &= -\int r_0 \cos u_0\, d\mu \\ \beta &= -\int r_0 \sin u_0 \cos\lambda_0\, d\mu \\ \gamma &= -\int r_0 \sin u_0 \sin\lambda_0\, d\mu. \end{aligned} \qquad (107)$$

»Thus, according to the explanation in Art. 5 of the Intensitas Vis Magneticae, $-\alpha$, $-\beta$, $-\gamma$ are the moments of the terrestrial magnetism in respect to three rectangular axes, of which the first is the earth's axis, the second and third are the equatorial radii for the longitude 0° and 90°.«

»The general formulae for all coefficients of the series for $1/\varrho$ we may assume as known; for our purpose it is merely necessary to remark that in respect to u and λ the coefficients are rational integral functions of $\cos u$, $\sin u \cos\lambda$, $\sin u \sin\lambda$, and T'' of the second order, T''' of the third order, etc. The same holds therefore also for the coefficients P'', P''', etc. The series for $1/\varrho$ and for V converge, so long as r is not smaller than R, or rather, not smaller than the radius of a sphere which encloses the whole of the magnetic parts of the earth.«

We have already remarked that in his treatment of magnetism, *Gauss* takes the action to be in right lines, whereas in nature the lines of magnetic force are curved. The corrections for this defect will be considered hereafter.

The above is the process of reasoning by which *Gauss* derives 7.831 parallel bar magnets each weighing a German pound, all of them 3.9155 kg for each cubic metre of the globe of the average weight = 5500 kg, and hence the mass component with the properties of a vector which I was led to deduce becomes

$$\eta = 3.9155/5500 = 1/1404.674. \qquad (108)$$

Hence taking account of *Laplace*'s law of density, we get the value finally adopted, namely:

$$\eta = 1/1408.12.$$

(v) Calculation of the fractional part of the earth which is magnetic, on the simple hypothesis that on the average the magnetism in the different spherical shells is proportional to the density of the matter in these shells.

Profound as is the analysis underlying *Gauss*' theory of the earth's magnetism, there remains in the theory one

[1]) »Insofar as we are not obliged to assume, in all magnetic parts of the earth, parallel magnetic axes throughout. The more such parallelism fails, the stronger must be the average magnetization of the parts, in order to bring forth the total magnetic moment.«

unnecessary element of weakness, which can be removed; and as the correction conforms to *Gauss*' mathematical criteria, yet brings the analysis of the theory down to a better physical basis, it is worthy of careful attention.

In section 32 of the Allgemeine Theorie *Gauss* says that the manner of the actual distribution of the magnetic fluid in the earth remains necessarily indeterminate; and then proceeds to point out that instead of any arbitrary distribution of the magnetic fluid within, we may always substitute a superficial distribution of magnetism which will give exactly the same effect in external space. This results from *Poisson*'s theorem on the volume and surface distribution of magnetism (Memoires de l'Institut tome 5, 1822). Wherefore *Gauss* concludes that a given action in external space may result from an indefinite number of different distributions of the magnetic fluid within.

From a purely mathematical point of view, this theorem is valid, yet from a physical point of view it fails entirely, because we know that in magnetism, as in gravitation, the forces acting around attracting bodies, depend on the matter within them, not upon their surfaces, or any mere mathematical abstraction.

Thus the one fatal weakness of the Theory of Relativity arose from the absurd claim that »Gravity is not a force, but a property of space« (*de Sitter*, MN 76, 1916, p. 702). Such a view is wholly untenable, because the force of gravity is proportional to the mass, and acts in right lines towards it: therefore gravity is a force depending on matter, and directly proportional to the amount of it gathered into the central attracting body, and in no sense is a property of space. Physically such claims are absurd!

In the same way, we know very well that magnetism depends on the atoms within the magnet, which have magnetic properties. For example, leaving out of account a slight change due to mere form of the bar, if we double the number of such atoms, by taking a magnet of double the mass, we practically double the intensity of the magnetic force in the field about it. Thus magnetism depends upon matter: it is a physical force!

From these considerations we see that whilst the mathematical possibility exists of a given action in external space resulting from many different mathematical distributions of the magnetism within, there is no such physical possibility. And as magnetism is a physical phenomenon, we are restricted in our choices of magnetic distribution to those which are consistent with the possible distribution of the matter. This leads to the physical theory that the magnetism of the earth depends on the density of the concentric spherical shells of which the globe is made up.

Accordingly, in the preceding section we have developed a criterion for reducing *Gauss*' infinite number of possible solutions for the distribution of the magnetism within the earth to a unique solution, with the density of the magnetic fluid everywhere proportional to the density of the matter within the earth. As *Gauss* took the distribution of the magnetism to be uniform in the evaluation above given, 7.831 one-pound bar magnets to each cubic metre of matter, whatever be its density, it seems advisable to consider the effect of the increase of density towards the centre, and relative decrease of density near the surface, according to *Laplace*'s law.

The magnetic moment found by *Gauss* can be adapted to this condition by the following process of calculation. The integral for the mass in any spherical layer of the globe of radius ϱ is (cf. AN 3992, p. 127):

$$\int_a^b dm = 4\pi \int_a^b \varrho^2 \, d\varrho \, \sigma_0 \cdot \sin(q x) ./ q x \qquad x = \varrho/r \qquad d\varrho = r\,dx\,. \quad (109)$$

And the total mass M

$$M = (4\pi r^3 \sigma_0/q) \int_0^x x \sin(q x)\, dx$$
$$= (4\pi r^3 \sigma_0/q^3) \cdot \{\sin(q x) - q x \cos(q x)\} \quad (110)$$

where x is the fraction of the earth's radius, $x = \varrho/r$.

In our present problem it suffices to use a numerical ratio

$$N = [1/(\sigma_1 r^3)] \sum_{i=0}^{i=1} \sigma_i (r_i^3 - r^3_{i-1}) \quad (111)$$

where the volumes of the shells $^4/_3\pi (r_i^3 - r^3_{i-1})$ are easily calculated, and the density σ_i is already determined (c. f. AN 3992).

For it is not known that *Laplace*'s law is rigorously true, and owing to the improvements which may ultimately be possible in the elements of *Gauss*' theory; attempts at extreme refinement are not justifiable. Accordingly we use the following table:

r_i	σ_i	$^1/_2(\sigma_i+\sigma_{i-1})$	$r_i^3-r^3_{i-1}$	$^1/_2(\sigma_i+\sigma_{i-1})\cdot(r_i^3-r^3_{i-1})$
10	2.55	3.15	271	853.66
9	3.75	4.37	217	948.29
8	4.99	5.60	169	946.40
7	6.21	6.80	127	863.60
6	7.38	7.92	91	720.72
5	8.46	8.93	61	544.73
4	9.40	9.76	37	361.12
3	10.12	10.43	19	198.17
2	10.74	10.90	7	76.30
1	11.07	11.14	1	11.14
0	11.215			

$$\sum_{i=0}^{i=10} = 5524.13$$

As the mean density of the earth is $\sigma_1 = 5.5$, $r = 10$, we have $\sigma_1 r^3 = 5500$; and the numerical reduction factor is

$$N = 5524.13/5500 = 1.004387\,. \quad (112)$$

Dividing the value 7.831 by this number we find that the average number of one-pound magnets required to produce the observed magnetism of the earth, if they were everywhere so distributed as to be proportional to the density of the matter, is

$$7.831/1.004387 = 7.7968. \quad (113)$$

The fractional part of the earth's mass which would be magnetic, under the hypothesis that the density of the magnets is everywhere proportional to the density of the matter, thus proves to be, as in equation (48):

$$\eta = 1/1410.837\,. \quad (114)$$

Hence for the reasons assigned in deriving (49) we use $\eta = 1/1408.12$.

As the connection between magnetism and gravitation was discovered from an extension of the researches of *Gauss*, we have, for the sake of uniformity, adhered closely to his constants. Yet it may be that at some future time values slightly different from those now in use may come to be preferred. For example, if $\eta = 1/1414.213 = 1/\sqrt{2000000}$, which involves only a slight change, then we should have:

$$I/g = 1/2000000, \text{ at the magnetic equator,} \qquad (115)$$
$$I/g = 1/1000000, \text{ at the poles.}$$

These round numbers are accurate enough for all practical purposes, and are easily remembered. They conform rigorously to *Biot*'s law of 1816, which represents the larger phenomena of the earth's magnetism disclosed by *Humboldt*'s law of 1804. The harmonic law, it should be noted, gives a physical basis for the laws of *Humboldt* and *Biot*, which heretofore has been wanting, and thus will prove extremely useful to investigators.

7. Outlines of *Gauss*' General Theory of the Earth's Magnetism.

We have seen, that *Gauss* takes r to denote the distance of any element of magnetism $d\mu$ from the centre of the earth, while u denotes the angle between r and the earth's north polar axis, and λ the angle, (reckoned positive to the east), between the plane containing r and the earth's axis and a fixed meridian. Thus let r_0, u_0, λ_0 be coordinates of the element $d\mu$ in the globe, r, u, λ those of a point considered as lying anywhere in space: then the radius vector connecting them is defined by the relation

$$\varrho^2 = r^2 - 2rr_0[\cos u \cos u_0 + \sin u \sin u_0 \cos(\lambda - \lambda_0)] + r_0^2. \quad (116)$$

And for the potential we have the integral:

$$\Omega = -\int\{r^2 - 2rr_0[\cos u \cos u_0 + \sin u \sin u_0 \cos(\lambda - \lambda_0)] + r_0^2\}^{-1/2} d\mu \qquad (117)$$

for the elements of magnetism $d\mu$ throughout the globe. This expression for Ω may be expanded into a converging series of solid spherical harmonics, involving sines and cosines of u and λ, and the ratio between the radius of the earth (R) and that of the external point (r).

We shall now enter at some length into *Gauss*' Allgemeine Theorie des Erdmagnetismus, 1838, because without this outline of *Gauss*' work, it will be difficult to interpret his results or to recognize their bearing upon the present problem of the law connecting magnetism with gravitation.

In the measurement of magnetic fluid *Gauss* takes as unit the quantity of boreal fluid, which, acting on a similar quantity of the same kind of fluid, at assumed unit distance, exerts a moving force equal to unity. If μ be the mass of fluid, at the distance ϱ, the magnetic force exerted is taken to be:

$$f = \pm\mu/\varrho^2 \qquad (118)$$

repulsive or attractive, in the direction of the line ϱ, according as μ is positive or negative.

Putting $d\mu$ for the mass of the magnetic fluid in any differential element dx, dy, dz, we have:

$$\Omega = -\int 1/\varrho \cdot d\mu = -\iiint^{\mu} 1/\varrho \cdot \sigma\, dx\, dy\, dz \qquad (119)$$

and the components of the earth's total magnetic force I, making the angle θ with any plane become:

$$\xi = \partial\Omega/\partial x \qquad \eta = \partial\Omega/\partial y \qquad \zeta = \partial\Omega/\partial z$$
$$I = [\xi^2 + \eta^2 + \zeta^2]^{1/2}$$
$$d\Omega = \partial\Omega/\partial x \cdot dx + \partial\Omega/\partial y \cdot dy + \partial\Omega/\partial z \cdot dz$$
$$= \xi\, dx + \eta\, dy + \zeta\, dz = I \cos\theta\, ds. \qquad (120)$$

Accordingly, the basis of the general theory is the equations:

$$X = -1/R \cdot \partial\Omega/\partial u$$
$$Y = -1/(R \sin u) \cdot \partial\Omega/\partial\lambda \qquad (121)$$
$$Z = -\partial\Omega/\partial R.$$

Then it follows that the magnetic potential Ω may be expanded:

$$\Omega = R(P_0 \cdot R/r + P_1 \cdot R^2/r^2 + P_2 \cdot R^3/r^3 + P_3 \cdot R^4/r^4 + \cdots). \quad (122)$$

The functions P_0, P_1, P_2 etc. are spherical surface harmonics, of degree indicated by the subscripts, depending on the angles u and λ, which will be more fully explained below.

If therefore we extend the integration through all elements of the magnetic fluid, we shall have:

$$\Omega = -\int 1/\varrho \cdot d\mu$$
$$= -\int\{r^2 - 2rr_0[\cos u \cos u_0 + \sin u \sin u_0 \cos(\lambda - \lambda_0) + r_0^2]\}^{-1/2} \cdot d\mu \qquad (123)$$

where

$$\varrho^2 = r^2 - 2rr_0 \cos(r, r_0) + r_0^2$$
$$\cos(r, r_0) = \cos u \cos u_0 + \sin u \sin u_0 \cos(\lambda - \lambda_0) \qquad (124)$$

and thus

$$\Omega = \int_0^{\mu}\int_0^{r}\int_0^{\pi}\int_0^{2\pi}\{r^2 - 2rr_0[\cos u \cos u_0 + \sin u \sin u_0 \cos(\lambda - \lambda_0) + r_0^2]\}^{-1/2} \cdot \sigma\, dr \cdot r \sin u\, du \cdot r \sin u\, d\lambda. \qquad (125)$$

As the development of this function depends on $1/\varrho$, we may put:

$$1/\varrho = 1/r \cdot (T_0 + T_1 \cdot r_0/r + T_2 \cdot r_0^2/r^2 + T_3 \cdot r_0^3/r^3 + \cdots) \quad (126)$$

Wherefore, since $\Omega = -\int 1/\varrho \cdot d\mu$, we have from equations (122), (123), (126)

$$P_0 \cdot R^2/r + P_1 \cdot R^3/r^2 + P_2 \cdot R^4/r^3 + \cdots =$$
$$-\{1/r \cdot \int T_0\, d\mu + 1/r^2 \cdot \int T_1 r_0\, d\mu + 1/r^3 \cdot \int T_2 r_0^2\, d\mu + \cdots\} \quad (127)$$

Equating like powers of r in this identify, we have:

$$P_0 R^2 = -\int T_0\, d\mu \qquad P_3 R^5 = -\int T_3 r_0^3\, d\mu$$
$$P_1 R^3 = -\int T_1 r_0\, d\mu \qquad P_4 R^6 = -\int T_4 r_0^4\, d\mu \qquad (128)$$
$$P_2 R^4 = -\int T_2 r_0^2\, d\mu \qquad \text{etc.}$$

By *Gauss*' fundamental assumption the mass of positive and negative magnetic fluid in every part of the earth is of equal magnitude, and thus also in the whole globe; so that we have

$$\int d\mu = 0. \qquad (129)$$

Therefore, since the function $T_0 = 1$, the equation $P_0 = -1/R^2 \cdot \int T_0\, d\mu$, leads to the result

$$P_0 = 0. \qquad (130)$$

And the series becomes

$$P_1 \cdot R^3/r^2 + P_2 \cdot R^4/r^3 + P_3 \cdot R^5/r^4 + P_4 \cdot R^6/r^5 + \cdots =$$
$$= -\{1/r^2 \cdot \int T_1 r_0\, d\mu + 1/r^3 \cdot \int T_2 r_0^2\, d\mu + 1/r^4 \cdot \int T_3 r_0^3\, d\mu + \cdots\} \quad (131)$$

$$P_1 R^3 = -\int T_1 r_0\, d\mu \qquad P_2 R^4 = -\int T_2 r_0^2\, d\mu$$
$$P_3 R^5 = -\int T_3 r_0^3\, d\mu \qquad P_4 R^6 = -\int T_4 r_0^4\, d\mu \qquad (132)$$

The first equation yields:

$$P_1 R^3 = -\int T_1 r_0 \, d\mu = \alpha \cos u + \beta \sin u \cos\lambda + \gamma \sin u \sin\lambda$$

where

$$\alpha = -\int r_0 \cos u_0 \, d\mu \qquad (133)$$
$$\beta = -\int r_0 \sin u_0 \cos\lambda_0 \, d\mu$$
$$\gamma = -\int r_0 \sin u_0 \sin\lambda_0 \, d\mu .$$

The coefficients $-\alpha$, $-\beta$, $-\gamma$ are explained by *Gauss*, in § 15 of his investigation Intensitas Vis Magneticae, as moments of the earth's magnetism, in respect to the three rectangular coordinate axes; the first being in respect to the earth's axis, and the two latter in respect to equatorial radii for longitudes $\lambda_0 = 0°$, $\lambda_0 = 90°$.

In the expansion of $1/\varrho$, we need only remark that as respects u and λ, the coefficients are rational integral functions of $\cos u$, $\sin u \cos\lambda$, $\sin u \sin\lambda$; in the case of T_2, they are of the second order; in the case of T_3 of the third order, etc. The same rule holds for P_2, P_3, etc.

The series for $1/\varrho$, and Ω converge so long as r is not smaller than R, or the observed point is external to the surface of the earth, in which the magnetic fluid acts to develop magnetism.

The function of the magnetic potential Ω for the point O satisfies *Laplace*'s equation:

$$\partial^2\Omega/\partial x^2 + \partial^2\Omega/\partial y^2 + \partial^2\Omega/\partial z^2 = 0 \qquad (134)$$

which may also be transformed into the spherical coordinates:

$$r \cdot \partial^2(r\,\Omega)/\partial r^2 + \partial^2\Omega/\partial u^2 + \operatorname{ctg} u \cdot \partial\Omega/\partial u + 1/\sin^2 u \cdot \partial^2\Omega/\partial\lambda^2 = 0 . \qquad (135)$$

The former expressions for the components of the magnetic force now become

north, $$X = -1/r \cdot \partial\Omega/\partial u = -R^3/r^3 \{\partial P_1/\partial u + R/r \cdot \partial P_2/\partial u + R^2/r^2 \cdot \partial P_3/\partial u + \cdots\} \qquad (136)$$

west, $$Y = -1/(r \sin u) \cdot \partial\Omega/\partial\lambda = -R^3/(r^3 \sin u) \cdot \{\partial P_1/\partial\lambda + R/r \cdot \partial P_2/\partial\lambda + R^2/r^2 \cdot \partial P_3/\partial\lambda + \cdots\} \qquad (137)$$

downward, $$Z = -\partial\Omega/\partial r = R^3/r^3 \cdot \{2P_1 + 3R\,P_2/r + 4R^2\,P_3/r^2 + \cdots\} . \qquad (138)$$

For points at the surface of the earth, $r = R$, and these expressions may be written

$$X = -\{\partial P_1/\partial u + \partial P_2/\partial u + \partial P_3/\partial u + \cdots\} \qquad (139)$$
$$Y = -1/\sin u \cdot \{\partial P_1/\partial\lambda + \partial P_2/\partial\lambda + \partial P_3/\partial\lambda + \cdots\} \qquad (140)$$
$$Z = 2P_1 + 3P_2 + 4P_3 + \cdots \qquad (141)$$

After deriving these expressions *Gauss* remarks that if we combine with these equations the well known theorem that every function of λ and u which has a definite finite value for all values of λ from 0° to 360°, and for u from 0° to 180°, can be developed in a series of the form $P_0 + P_1 + P_2 + P_3 + \cdots$ etc;

of which the general term P_n satisfies *Laplace*'s differential equation, that such a development is possible in only one definite way. Proceeding in this way *Gauss* is led to four theorems of which the following is the most remarkable:

1. The knowledge of the value of Ω in all points of the earth's surface suffices for deriving the general expression of Ω for the whole infinite space external to the earth's surface, and thereby the determination of the forces X, Y, Z, not only on the earth's surface, but also for the whole infinite space outside of it. To this end it suffices to develop Ω/R in a series, as shown hereafter.

The coefficient P_n satisfies the partial differential equation

$$n\,(n+1)\,P_n + \partial^2 P_n/\partial u^2 + \operatorname{ctg} u\,\partial P_n/\partial u + 1/\sin^2 u \cdot \partial^2 P_n/\partial\lambda^2 = 0 . \qquad (142)$$

If we designate by $P_{n,m}$ the following function of u only:

$$P_{n,m} = \left\{\cos^{n-m} u - \frac{(n-m)(n-m-1)}{2\,(2n-1)}\cos^{n-m-2} u + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2 \cdot 4\,(2n-1)(2n-3)}\cos^{n-m-4} u - \text{etc.}\right\}\sin^m u \qquad (143)$$

it follows that P_n has the form of an aggregate of $2n+1$ parts, as follows:

$$P_n = g_{n,0}\,P_{n,0} + (g_{n,1}\cos\lambda + h_{n,1}\sin\lambda)\,P_{n,1} + (g_{n,2}\cos 2\lambda + h_{n,2}\sin 2\lambda)\,P_{n,2} + \cdots + (g_{n,n}\cos n\lambda + h_{n,n}\sin n\lambda)\,P_{n,n} \qquad (144)$$

where $g_{n,0}$, $g_{n,1}$, $h_{n,1}$, $g_{n,2}$, $h_{n,2}$, etc., are definite numerical coefficients, of which the table calculated by *Gauss* is given below.

From this general formula it follows that P_1 has 3 indeterminate coefficients, P_2 has 5; P_3, 7; P_4, 9; the full expressions being:

$$P_1 = g_{1,0} + (g_{1,1}\cos\lambda + h_{1,1}\sin\lambda)\,P_{1,1}$$
$$P_2 = g_{2,0}\,P_{2,0} + (g_{2,1}\cos\lambda + h_{2,1}\sin\lambda)\,P_{2,1} + (g_{2,2}\cos 2\lambda + h_{2,2}\sin 2\lambda)\,P_{2,2}$$
$$P_3 = g_{3,0}\,P_{3,0} + (g_{3,1}\cos\lambda + h_{3,1}\sin\lambda)\,P_{3,1} + (g_{3,2}\cos 2\lambda + h_{3,2}\sin 2\lambda)\,P_{3,2} + (g_{3,3}\cos 3\lambda + h_{3,3}\sin 3\lambda)\,P_{3,3} \qquad (145)$$
$$P_4 = g_{4,0}\,P_{4,0} + (g_{4,1}\cos\lambda + h_{4,1}\sin\lambda)\,P_{4,1} + (g_{4,2}\cos 2\lambda + h_{4,2}\sin 2\lambda)\,P_{4,2} + (g_{4,3}\cos 3\lambda + h_{4,3}\sin 3\lambda)\,P_{4,3} + (g_{4,4}\cos 4\lambda + h_{4,4}\sin 4\lambda)\,P_{4,4} .$$

Already, in section 22, p. 26, *Gauss* has considered the probable distribution of the magnetism of the globe in respect to the radius, remarking that the series would converge rapidly if it is concentrated towards the centre, but less rapidly if more diffuse and irregular in its distribution. He then adds, as shown above, that the coefficient P_1 has three coefficients; P_2, five; P_3, seven; P_4, nine — making 24 constants for the first four terms of the series. As each complete observation of X, Y, Z gives three constants, he adds that exact observations at eight places would be theoretically sufficient to determine all the coefficients for the general theory of the earth's magnetism. But in practice a larger number of observations are necessary; and he reduces his equations to the following form for points on parallel circles of latitude:

$$X = k + k'\cos\lambda + K'\sin\lambda + k''\cos 2\lambda + K''\sin 2\lambda + + k'''\cos 3\lambda + K'''\sin 3\lambda + \text{etc.} \quad (146)$$

$$Y = l + l'\cos\lambda + L'\sin\lambda + l''\cos 2\lambda + L''\sin 2\lambda + + l'''\cos 3\lambda + L'''\sin 3\lambda + \text{etc.} \quad (147)$$

$$Z = m + m'\cos\lambda + M'\sin\lambda + m''\cos 2\lambda + M''\sin 2\lambda + m'''\cos 3\lambda + M'''\sin 3\lambda + \text{etc.} \quad (148)$$

Here it is assumed that the eight points, separated by convenient arcs, lie upon a great circle; and *Gauss* notes that there will be as many values of k, l, m, k', etc., as there are parallel circles adapted to this treatment. On each circle, according to theory $l = 0$, and from the values of l found by calculation we have a measure of the inadmissibility of the numbers adopted in the theory of any parallel.

The above expressions give for the coefficients the following equations:

$$k = -g_{1,0}\cdot dP_{1,0}/du - g_{2,0}\cdot dP_{2,0}/du + -g_{3,0}\cdot dP_{3,0}/du - \text{etc.} \quad (149)$$

$$m = 2g_{1,0}\,P_{1,0} + 3g_{2,0}\,P_{2,0} + 4g_{3,0}\,P_{3,0} + \text{etc.}$$

the number being twice as great as that of the number of parallel circles. On substituting in $dP_{1,0}/du$, $dP_{2,0}/du$, etc., and in $P_{1,0}$, $P_{2,0}$, etc., the numerical values of u, the coefficients $g_{1,0}$, $g_{2,0}$, $g_{3,0}$, etc., may be determined by the method of least squares.

In the same way we have for the determination of the coefficients $g_{1,1}$, $g_{2,1}$, $g_{3,1}$, etc.:

$$-k' = g_{1,1}\cdot dP_{1,1}/du + g_{2,1}\cdot dP_{2,1}/du + + g_{3,1}\cdot dP_{3,1}/du + \text{etc.}$$

$$L' = g_{1,1}\cdot P_{1,1}/\sin u + g_{2,1}\cdot P_{2,1}/\sin u + + g_{3,1}\cdot P_{3,1}/\sin u + \text{etc.} \quad (150)$$

$$m' = 2g_{1,1}\,P_{1,1} + 3g_{2,1}\,P_{2,1} + 4g_{3,1}\,P_{3,1} + \text{etc.}$$

the number of which is three times that of the parallel circles.

And likewise for the coefficients $h_{1,1}$, $h_{1,2}$, $h_{1,3}$, etc.:

$$-K' = h_{1,1}\cdot dP_{1,1}/du + h_{2,1}\cdot dP_{2,1}/du + + h_{3,1}\cdot dP_{3,1}/du + \text{etc.}$$

$$-l' = h_{1,1}\cdot P_{1,1}/\sin u + h_{2,1}\cdot P_{2,1}/\sin u + + h_{3,1}\cdot P_{3,1}/\sin u + \text{etc.} \quad (151)$$

$$M' = 2h_{1,1}\,P_{1,1} + 3h_{2,1}\,P_{2,1} + 4h_{3,1}\,P_{3,1} + \text{etc.}$$

In like manner we have for the determination of $g_{2,2}$, $g_{3,2}$, $g_{4,2}$, and $h_{2,2}$, $h_{3,2}$, $h_{4,2}$, the following equations:

$$-k'' = g_{2,2}\cdot dP_{2,2}/du + g_{3,2}\cdot dP_{3,2}/du + + g_{4,2}\cdot dP_{4,2}/du + \text{etc.}$$

$$L'' = 2g_{2,2}\cdot P_{2,2}/\sin u + 2g_{3,2}\cdot P_{3,2}/\sin u + + 2g_{4,2}\cdot P_{4,2}/\sin u + \text{etc.}$$

$$m'' = 3g_{2,2}\,P_{2,2} + 4g_{3,2}\,P_{3,2} + 5g_{4,2}\,P_{4,2} + \text{etc.}$$

$$-K'' = h_{2,2}\cdot dP_{2,2}/du + h_{3,2}\cdot dP_{3,2}/du + + h_{4,2}\cdot dP_{4,2}/du + \text{etc.} \quad (152)$$

$$-l'' = h_{2,2}\cdot P_{2,2}/\sin u + h_{3,2}\cdot P_{3,2}/\sin u + + h_{4,2}\cdot P_{4,2}/\sin u + \text{etc.}$$

$$M'' = 3h_{2,2}\,P_{2,2} + 4h_{3,2}\,P_{3,2} + 5h_{4,2}\,P_{4,2} + \text{etc.}$$

And so on, to higher orders of terms, as far as required.

Selecting the best data available in his time *Gauss* found for the 24 largest coefficients of g and h the following tabular values:

$g_{1,0} = +925.782$	$g_{2,2} = +0.493$
$g_{2,0} = -22.059$	$g_{3,2} = -73.193$
$g_{3,0} = -18.868$	$g_{4,2} = -45.791$
$g_{4,0} = -108.855$	$h_{2,2} = -39.010$
$g_{1,1} = +89.024$	$h_{3,2} = -22.766$
$g_{2,1} = -144.913$	$h_{4,2} = +42.573$
$g_{3,1} = +122.936$	$g_{3,3} = +1.396$
$g_{4,1} = -152.589$	$g_{4,3} = +19.774$
$h_{1,1} = -178.744$	$h_{3,3} = -18.750$
$h_{2,1} = -6.030$	$h_{4,3} = -0.178$
$h_{3,1} = +47.794$	$g_{4,4} = +4.127$
$h_{4,1} = +64.112$	$h_{4,4} = +3.175$ (153)

Gauss considers these coefficients as the Elements of the theory of the earth's magnetism; and thence collects his results into the following formula, putting for brevity e in place of $\cos u$, and f in place of $\sin u$:

$$\begin{aligned} \Omega/R = &-1.977 + 937.103e + 71.245e^2 - 18.868e^3 - 108.855e^4 \\ &+ (64.437 - 79.518e + 122.936e^2 + 152.589e^3)f\cos\lambda \\ &+ (-188.303 - 33.507e + 47.794e^2 + 64.112e^3)f\sin\lambda \\ &+ (7.035 - 73.193e - 45.791e^2)f^2\cos 2\lambda \\ &+ (-45.092 - 22.766e - 42.573e^2)f^2\sin 2\lambda \\ &+ (1.396 + 19.774e)f^3\cos 3\lambda \\ &+ (-18.750 - 0.178e)f^3\sin 3\lambda \\ &+ 4.127f^4\cos 4\lambda + 3.175f^4\sin 4\lambda\,. \end{aligned} \quad (154)$$

For the magnetic poles of the globe *Gauss* finds:

1. North pole: North latitude 73° 35′
   East longitude 264 21 from Greenwich
   Total intensity $I = 1.701$, in ordinary units
   $\Omega/R = +895.86\,.$ (155)

2. South pole: South latitude 72° 35′
   East longitude 152 30
   Total intensity $I' = 2.253$
   $\Omega/R = -1030.24\,.$ (156)

Gauss remarks that Sir *James Ross* found the north magnetic pole 3° 30′ further south than is given by this calculation; and that at the south pole the deviation between theory and observation may be yet more considerable, owing to certain defects in the observations at Hobarttown. He thinks the south magnetic pole probably lies appreciably further north than is given by the above calculation, and assigns south latitude 66° and east longitude 146° as its probable location.

It will be seen from the differences shown in the following table that *Gauss*' elements are comparatively very exact. In order to give to the mind a connected view of them, *Gauss* induced Dr. *Goldschmidt* to represent his results graphically. The excellence of the agreement is also shown by comparing these calculated charts with the recently observed charts.

Owing to the great importance of a correct understanding of the high accuracy of *Gauss*' theory, in the present investigation for connecting the magnetism of the earth with terrestrial gravitation, we reproduce a slightly modified form of *Gauss*' table for comparing his theory with observations throughout the globe. The table as here given has been condensed as much as possible, yet it is so impressive that it cannot fail to interest the modern reader.