Mathematics Research Developments

Understanding Quaternions

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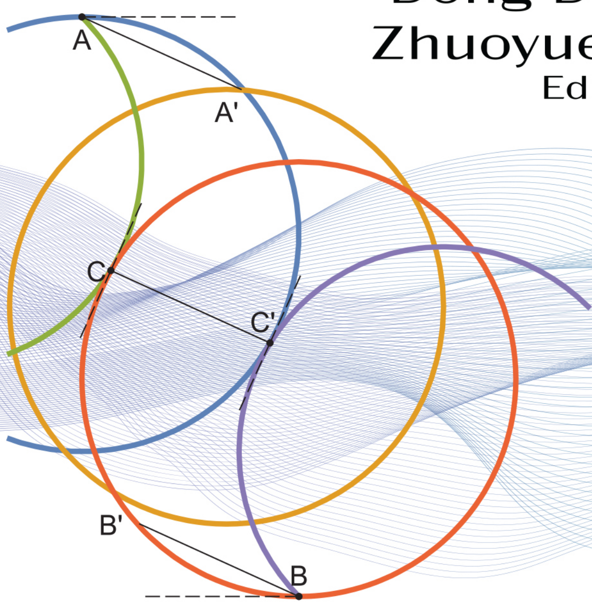
Editors



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UNDERSTANDING QUATERNIONS

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PREFACE

Quaternions are members of a noncommutative division algebra first invented by William Rowan Hamilton. They form an interesting algebra where each object contains 4 scalar variables, instead of Euler angles, which is useful to overcome the gimbal lock phenomenon when treating the rotation of objects.

This book is about the mathematical basics and applications of quaternions. The first four chapters mainly concerns the mathematical theories, while the latter three chapters are related with three application aspects. It is expected to provide useful clues for researchers and engineers in the related area. In detail, this book is organized as follows:

In Chapter 1, mathematical basics including the quaternion algebra and operations with quaternions, as well as the relationships of quaternions with other mathematical parameters and representations are demonstrated. In Chapter 2, how quaternions are formulated in Clifford Algebra, how it is used in explaining rotation group in symplectic vector space and parallel transformation in holonomic dynamics are presented. In Chapter 3, the wave equation for a spin 3/2 particle, described by 16-component vector-bispinor, is investigated in spherical coordinates. In Chapter 4, hyperbolic Lobachevsky and spherical Riemann models, parameterized coordinates with spherical and cylindric symmetry are studied. In Chapter 5, ship hydrodynamics with allowance of trim and sinkage is investigated and validated with experiments. In Chapter 6, the ballast flying phenomenon based on Discrete Discontinuous Analysis is presented. In Chapter 7, a numerical study is proposed to analyze the effect of the caisson sliding subjected to a hydrodynamic loading in the stability of the rear side of the rubble mound breakwater.

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Chapter 1

MATHEMATICAL BASICS AND APPLICATIONS OF QUATERNIONS

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ABSTRACT

The mathematical basics of quaternion is presented in this chapter which includes quaternion algebra and operations with quaternions, as well as the relationships of quaternions with other mathematical parameters and representations. The applications of quaternions and their advantages are described, taking into account the issues of quaternion estimation in guidance, navigation and control systems.

Keywords: quaternion algebra, quaternion applications, guidance, navigation and control

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INTRODUCTION

The Quaternions were introduced by W. R. Hamilton [1] in 1843, and the papers of many other scientists, such as L. Euler, O. Rodrigues, A. Cayley [2] allowed to develop the theoretical basis for applications of quaternions, especially in representations of a rigid body rotations.

Without the current level of understanding of vectors, spaces and other mathematical concepts these genius scientists created the fundaments of quaternions and their applications which are commonly and widely used now. In this chapter we are going to bring the way of understanding of quaternions using current level of physical and mathematical concepts to not pass the difficulties these genius scientists went through. We are going to make the concept of quaternions and their applications understandable as easy as possible, and first thing which will be helpful to focus on is that using visual imagination in understanding of abstract concepts can lead to the misinterpretations. Thus, to go forward sometimes we need to take concepts as abstractions and imagine them in forms of abstractions and not visualizations.

QUATERNION ALGEBRA

Basic Concepts

Let us consider a quaternion Q. It is a hypercomplex number and consists of one real element and three imaginary elements which are orthogonal to each other:

$$Q = q_0 \cdot 1 + q_1 \cdot \boldsymbol{i} + q_2 \cdot \boldsymbol{j} + q_3 \cdot \boldsymbol{k},$$

where q_0, q_1, q_2, q_3 – are the components of the quaternion Q, $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ – are bases of the hypercomplex space.

These bases are connected with each other in the following way:

$$\begin{cases} 1 \circ \boldsymbol{i} = \boldsymbol{i} \circ 1 = \boldsymbol{i}, 1 \circ \boldsymbol{j} = \boldsymbol{j} \circ 1 = \boldsymbol{j}, 1 \circ \boldsymbol{k} = \boldsymbol{k} \circ 1 = \boldsymbol{k}, \\ \boldsymbol{i} \circ \boldsymbol{i} = \boldsymbol{i}^2 = -1, \boldsymbol{j} \circ \boldsymbol{j} = \boldsymbol{j}^2 = -1, \boldsymbol{k} \circ \boldsymbol{k} = \boldsymbol{k}^2 = -1, \\ \boldsymbol{i} \circ \boldsymbol{j} = -\boldsymbol{j} \circ \boldsymbol{i} = \boldsymbol{k}, \boldsymbol{j} \circ \boldsymbol{k} = -\boldsymbol{k} \circ \boldsymbol{j} = \boldsymbol{i}, \boldsymbol{k} \circ \boldsymbol{i} = -\boldsymbol{i} \circ \boldsymbol{k} = \boldsymbol{j}, \end{cases} \tag{1}$$

where the operator "∘" denotes quaternion product. For the better understanding of the relations between these imaginary units, taking into account that the counter-clockwise rotations from the view-point are defined as positive ones in the mechanics, we will use the illustration in Figure1.

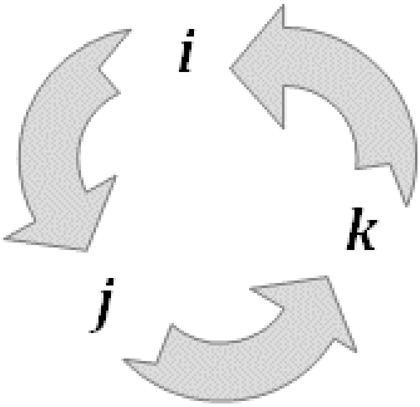


Figure 1. Relations between imaginary units.

Without deeper exploration of mathematical concepts of Group Theory and the relations of quaternions to it, the abstraction of quaternion product for imaginary units as the superposition of dot and cross products analogous to 3D vectors can be adopted for better understanding, for example:

$$\boldsymbol{i} \circ \boldsymbol{i} = \boldsymbol{i} * \boldsymbol{i} + \boldsymbol{i} \times \boldsymbol{i} = -1 + 0 = -1, \boldsymbol{i} \circ \boldsymbol{j} = \boldsymbol{i} * \boldsymbol{j} + \boldsymbol{i} \times \boldsymbol{j} = 0 + \boldsymbol{k},$$

where operator "∗" denotes dot product and "×" denotes cross product.

The quaternions are widely used in the representations of rotations nowadays [3]. From that point of view, it is appropriate to talk mostly about the quaternions used in the description of rotations.

Quaternion Operations

Let us consider the basic operations on quaternions. The addition of two quaternions $P = p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}$ and $Q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$ is the sum of their components:

$$P + Q = (p_0 + q_0) + (p_1 + q_1)\boldsymbol{i} + (p_2 + q_2)\boldsymbol{j} + (p_3 + q_3)\boldsymbol{k},$$

which is true for the subtraction of quaternions as well.

Multiplication of any scalar a with any quaternion Q is the same as their quaternion product and corresponds to the multiplication of every element of quaternion by that scalar:

$$aQ = a \circ Q = aq_0 + aq_1\boldsymbol{i} + aq_2\boldsymbol{j} + aq_3\boldsymbol{k}.$$

The multiplication of two quaternions P and Q refers to the operator of quaternion product and can be derived using expressions (1):

$$\begin{gathered}\Lambda = P \circ Q = (p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}) \circ (q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) = \\ p_0q_0 + p_0q_1\boldsymbol{i} + p_0q_2\boldsymbol{j} + p_0q_3\boldsymbol{k} + \\ p_1q_0\boldsymbol{i} - p_1q_1 + p_1q_2\boldsymbol{k} - p_1q_3\boldsymbol{j} + \\ p_2q_0\boldsymbol{j} - p_2q_1\boldsymbol{k} - p_2q_2 + p_2q_3\boldsymbol{i} + \\ p_3q_0\boldsymbol{k} + p_3q_1\boldsymbol{j} - p_3q_2\boldsymbol{i} - p_3q_3 = \\ (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3) + \\ (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2)\boldsymbol{i} + \\ (p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1)\boldsymbol{j} + \\ (p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0)\boldsymbol{k}.\end{gathered}$$

Taking into account that $\Lambda = \lambda_0 + \lambda_1\boldsymbol{i} + \lambda_2\boldsymbol{j} + \lambda_3\boldsymbol{k}$ is also a quaternion we can determine its components as:

$$\begin{cases}\lambda_0 = p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3, \\ \lambda_1 = p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2, \\ \lambda_2 = p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1, \\ \lambda_3 = p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0.\end{cases}$$

To avoid misinterpretation, the components of quaternions are better to not represent as a vector, as in case of 3D vectors it leads to the stack of imagination in quaternion product used in the descriptions of the rotations, moreover the vector is a general concept in mathematics. On the other hand, 3D vectors can be mapped to the hypercomplex space in the form of quaternions which is an abstraction and never needed to visualize for understanding.

Let us see what we obtain, when we change the order of multiplication:

$$
\begin{gathered}
M = Q \circ P = \ (q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) \circ (p_0 + p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}\,) = \\
q_0p_0 + q_0p_1\boldsymbol{i} + q_0p_2\boldsymbol{j} + q_0p_3\boldsymbol{k} + \\
q_1p_0\boldsymbol{i} - q_1p_1 + q_1p_2\boldsymbol{k} - q_1p_3\boldsymbol{j} + \\
q_2p_0\boldsymbol{j} - q_2p_1\boldsymbol{k} - q_2p_2 + q_2p_3\boldsymbol{i} + \\
q_3p_0\boldsymbol{k} + q_3p_1\boldsymbol{j} - q_3p_2\boldsymbol{i} - q_3p_3 = \\
(p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3) + \\
(p_0q_1 + p_1q_0 - p_2q_3 + p_3q_2)\boldsymbol{i} + \\
(p_0q_2 + p_1q_3 + p_2q_0 - p_3q_1)\boldsymbol{j} + \\
(p_0q_3 - p_1q_2 + p_2q_1 + p_3q_0)\boldsymbol{k}.
\end{gathered}
$$

As it can be seen $M = m_0 + m_1\boldsymbol{i} + m_2\boldsymbol{j} + m_3\boldsymbol{k}$ with components:

$$
\begin{cases}
m_0 = p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3, \\
m_1 = p_0q_1 + p_1q_0 - p_2q_3 + p_3q_2, \\
m_2 = p_0q_2 + p_1q_3 + p_2q_0 - p_3q_1, \\
m_3 = p_0q_3 - p_1q_2 + p_2q_1 + p_3q_0,
\end{cases}
$$

is not the same as quaternion Λ. Therefore, changing the order of multiplication of two quaternions changes the result of multiplication, and the operator of quaternion product is not commutative:

$$P \circ Q \neq Q \circ P.$$

However, the operator of quaternion product is associative and distributive over addition:

$$\Lambda \circ P \circ Q = \Lambda \circ (P \circ Q) = (\Lambda \circ P) \circ Q,$$

$$\Lambda \circ (P + Q) = \Lambda \circ P + \Lambda \circ Q.$$

It is advised to prove these properties of the quaternion product as an exercise using expressions (1).

The conjugate quaternion of Q is usually denoted as $\tilde{Q}$ and is defined as:

$$\tilde{Q} = q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}.$$

The conjugate of the quaternion product is the quaternion product of the conjugate quaternions in the product in the reverse order:

$$\tilde{\Lambda} = (\widetilde{P \circ Q}) = \tilde{Q} \circ \tilde{P},$$

which can be easily derived as an exercise based on (1).

The norm of any quaternion Q is the square root of its quaternion product by its conjugate:

$$|Q| = \sqrt{Q \circ \tilde{Q}} = \sqrt{\tilde{Q} \circ Q} =$$
$$\sqrt{(q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) \circ (q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k})},$$
$$|Q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

The inverse quaternion of Q is denoted as $Q^{-1}$and is defined as

$$Q \circ Q^{-1} = Q^{-1} \circ Q = 1,$$

thus,

$$Q^{-1} = \frac{\tilde{Q}}{|Q|^2}.$$

The normed quaternion or unit quaternion is a quaternion Q with norm equals 1, it is also called versor:

$$|Q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1.$$

For the unit quaternions their conjugates and inverses are the same. Unit quaternions are commonly used in the representations of rotations, and they have an interesting property which can lead to the simplifications of the kinematic equations of the rotations [4].

Let us consider the square of the norm of a unit quaternion:

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1.$$

Calculating the time derivative of the above expression, we get:

$$2q_0 \frac{\partial q_0}{\partial t} + 2q_1 \frac{\partial q_1}{\partial t} + 2q_2 \frac{\partial q_2}{\partial t} + 2q_3 \frac{\partial q_3}{\partial t} = 0,$$

which can be represented in more compact form:

$$q_0 \dot{q}_0 + q_1 \dot{q}_1 + q_2 \dot{q}_2 + q_3 \dot{q}_3 = 0,$$

where $\dot{q}_i = \partial q_i / \partial t\,, i = \overline{0,3}$.

QUATERNIONS AND OTHER MATHEMATICAL REPRESENTATIONS

Euler Angles and Quaternions

In the real world applications, in general, a rigid body has 6 degrees of freedom – ways to move: 3 independent linear movement directions and 3 independent axes of rotation. Euler angles are used in the representation of rotations of a rigid body and were introduced by Leonhard Euler. Currently widely used sequence and names of rotations to describe the orientation of a rigid body is following (Figure 2):

1. rotation around Z_0 axis, which is usually orthogonal to the local horizontal plane and points to down, by angle ψ – yaw,

2. rotation around the generated after the firs rotation Y' axis, which is in the local horizontal plane and deviates from the direction to the East by angle ψ, by angle ϑ – pitch,
3. rotation around generated after the previous rotation axis X_1, which forms right-handed coordinate frame, by angle γ – roll.

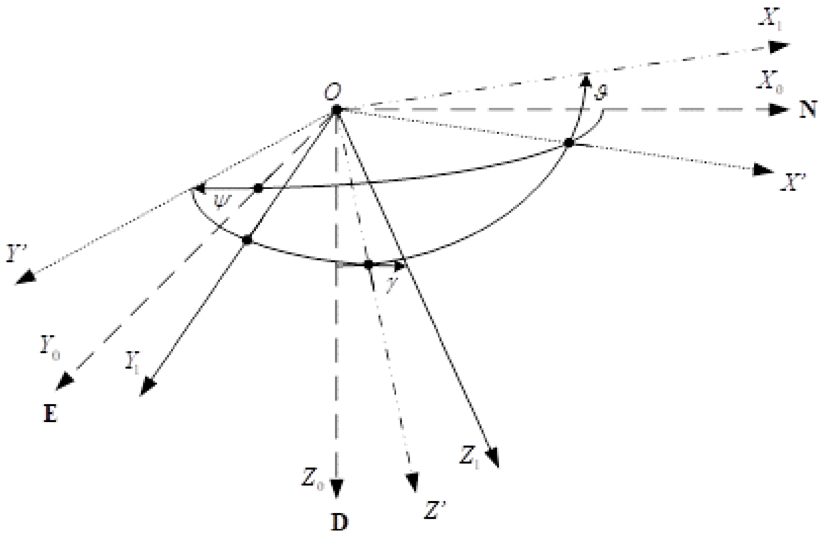


Figure 2. Rotations by Euler angles.

Let us consider a 3D vector $\vec{p}_0 = p_{x0}\vec{X}_0^0 + p_{y0}\vec{Y}_0^0 + p_{z0}\vec{Z}_0^0$, where unit vectors $\vec{X}_0^0, \vec{Y}_0^0, \vec{Z}_0^0$ are bases of the coordinate frame $OX_0Y_0Z_0$, p_{x0}, p_{y0}, p_{z0} are projections of vector $\vec{p}_0$ to the coordinate coordinate frame $OX_0Y_0Z_0$:

$$p_{x0} = \vec{p}_0 * \vec{X}_0^0,$$
$$p_{y0} = \vec{p}_0 * \vec{Y}_0^0,$$
$$p_{z0} = \vec{p}_0 * \vec{Z}_0^0,$$

where "$*$" denotes dot product. Considering the vector $\vec{p}_0$ attached to coordinate frame $OX_0Y_0Z_0$, i.e. its components p_{x0}, p_{y0}, p_{z0} are constant in $OX_0Y_0Z_0$, we are going to find its projections to the coordinate frame $OX_1Y_1Z_1$ which is rotated, for example, about Y_0 axis of $OX_0Y_0Z_0$ by a pitch angle ϑ (Figure 3):

$$\vec{p}_0 = p_{x1}\vec{X}_1^0 + p_{y1}\vec{Y}_1^0 + p_{z1}\vec{Z}_1^0,$$

where unit vectors $\vec{X}_1^0, \vec{Y}_1^0, \vec{Z}_1^0$ are bases of the coordinate frame $OX_1Y_1Z_1$, p_{x1}, p_{y1}, p_{z1} are projections of vector $\vec{p}_0$ to the coordinate frame $OX_1Y_1Z_1$. Remember, that all derivations using drawings are based on the schemes, where the coordinate frames are right-handed and rotations are positive (Figure 5), i.e. the rotations are counter clockwise (CCW) around their axes viewing from direction of increase their coordinates.

It is obvious that $p_{y1} = p_{y0}$.

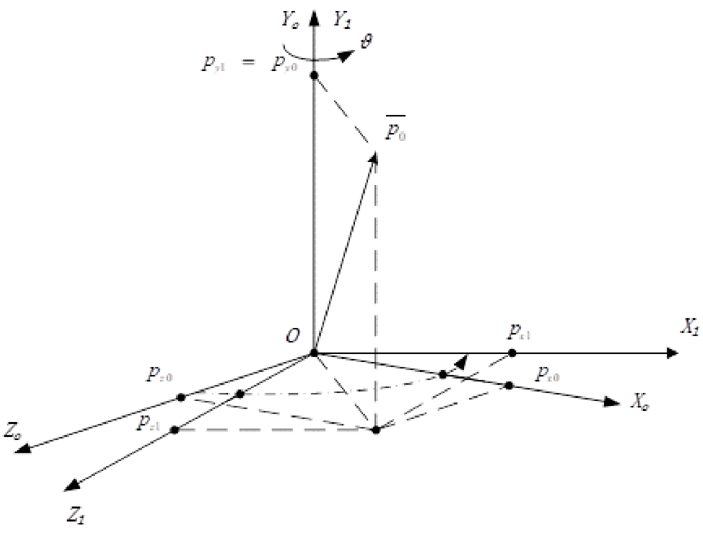


Figure 3. Projections of vector $\vec{p}_0$ to coordinate frames $OX_0Y_0Z_0$ and $OX_1Y_1Z_1$.

With more clear picture (Figure 4) the determination of other two projections becomes a simple geometrical task:

$$\frac{p_{x1}}{\cos\vartheta} = p_{x0} - p_{z0}\tan\vartheta,$$

$$\frac{p_{z1}}{\cos\vartheta} = p_{x0}\tan\vartheta + p_{z0},$$

which are correspondingly equal:

$$\begin{cases} p_{x1} = p_{x0}\cos\vartheta - p_{z0}\sin\vartheta\,, \\ p_{z1} = p_{x0}\sin\vartheta + p_{z0}\cos\vartheta\,. \end{cases} \tag{2}$$

To define p_{x0}, p_{z0} through p_{x1}, p_{z1} and angle ϑ we just need to multiply both sides of first equation of (2) by $\cos\vartheta$ and both sides of second equation of (2) by $\sin\vartheta$ and add them side-by-side for p_{x0}, and multiply both sides of second equation of (2) by $\cos\vartheta$ and both sides of first equation of (2) by $\sin\vartheta$ and subtract from second equation the first side-by-side for p_{z0}:

$$\begin{cases} p_{x0} = p_{x1}\cos\vartheta + p_{z1}\sin\vartheta\,, \\ p_{z0} = p_{z1}\cos\vartheta - p_{x1}\sin\vartheta\,. \end{cases} \tag{3}$$

Based on these expressions it is easy to find the solution of opposite task: finding the projections m_{x1}, m_{y1}, m_{z1} to the bases of initial coordinate $OX_0Y_0Z_0$ of a 3D vector $\vec{m}$ attached to the coordinate frame $OX_1Y_1Z_1$, rotating around $OX_0Y_0Z_0$. Rotating 3D vector can be expressed as $\vec{m} = m_{x0}\vec{X}_1^0 + m_{y0}\vec{Y}_1^0 + m_{z0}\vec{Z}_1^0$, where m_{x0}, m_{y0}, m_{z0} are constant in $OX_1Y_1Z_1$. We use notations m_{x0}, m_{y0}, m_{z0} because initially $OX_1Y_1Z_1$ and $OX_0Y_0Z_0$ were aligned and the projections of the vector $\vec{m}$ to the bases $OX_1Y_1Z_1$ and $OX_0Y_0Z_0$ at that initial moment were the same. To get the results for the m_{x1}, m_{y1}, m_{z1}, we just need to swap indices 0 and 1 in formula (3) and replace p with m, as in case of the rotating vector the projections presented in Figure 3 and 4 will be swapped and p will be replaced with m:

$$\begin{cases} m_{x1} = m_{x0}\cos\vartheta + m_{z0}\sin\vartheta\,, \\ m_{z1} = m_{z0}\cos\vartheta - m_{x0}\sin\vartheta\,. \end{cases} \tag{4}$$

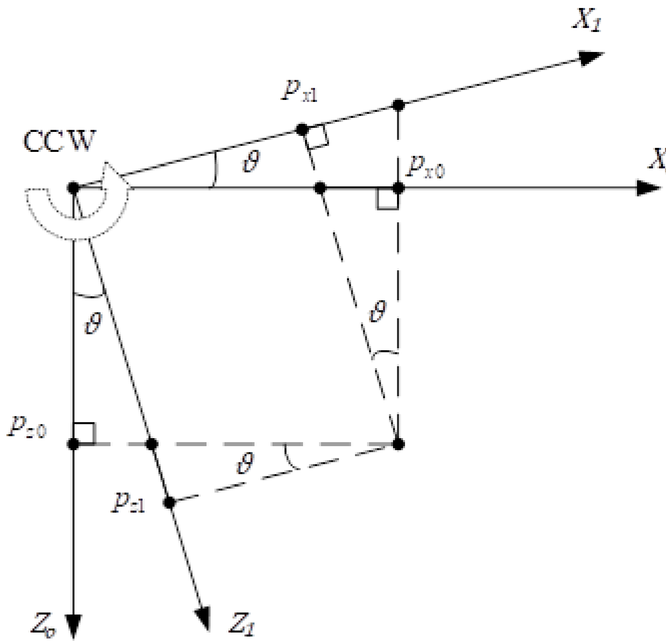


Figure 4. Projection plane.

In this case we do not even need to redraw the diagrams.

Introduction of rotation matrices helped to solve such and more complicated geometrical tasks simpler without any drawings. The idea here is to represent the connections between projections of the same vector in different coordinate frames in form of matrix-vector multiplication:

$$\begin{bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{bmatrix} = R \begin{bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{bmatrix},$$

where R – rotation matrix:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

Let us define rotation matrix for the case considered above as $R = R_Y(\vartheta)$.

$$\begin{bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{bmatrix} = \begin{bmatrix} r_{11}p_{x1} + r_{12}p_{y1} + r_{13}p_{z1} \\ r_{21}p_{x1} + r_{22}p_{y1} + r_{23}p_{z1} \\ r_{31}p_{x1} + r_{32}p_{y1} + r_{33}p_{z1} \end{bmatrix}.$$

Based on the above, taking into account the dependencies (3) of p_{x0}, p_{y0}, p_{z0} on p_{x1}, p_{y1}, p_{z1} and angle ϑ ($p_{y0} = p_{y1}$), we can find the elements of matrix $R_Y(\vartheta)$:

$$R_Y(\vartheta) = \begin{bmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{bmatrix}. \tag{5}$$

For the rotating vector $\vec{m}$ based on (4) we have:

$$\begin{bmatrix} m_{x1} \\ m_{y1} \\ m_{z1} \end{bmatrix} = R_Y(\vartheta) \begin{bmatrix} m_{x0} \\ m_{y0} \\ m_{z0} \end{bmatrix}. \tag{6}$$

So, we should remember the difference between operations of a fixed vector in initial frame and calculations of its projections to bases of the rotated coordinate frame and operations of a rotated vector and calculations of its projections to the bases of the fixed frame. These operations are opposite.

To get this matrix, based only on the Figure 3 without considering the projection plane and its geometry, it is helpful to create the following table and calculate cosines or dot products between the bases, where at the up row of the table the newly created bases are stored and at the right column the bases of the previous frame are presented:

$$\begin{array}{ccc} & \begin{array}{ccc} \vec{X}_1^0 & \vec{Y}_1^0 & \vec{Z}_1^0 \end{array} & * \\ R_Y(\vartheta) = & \begin{bmatrix} \cos\vartheta & \cos\frac{\pi}{2} & \cos\left(\frac{\pi}{2} - \vartheta\right) \\ \cos\frac{\pi}{2} & \cos 0 & \cos\frac{\pi}{2} \\ \cos\left(\frac{\pi}{2} + \vartheta\right) & \cos\frac{\pi}{2} & \cos\vartheta \end{bmatrix} & \begin{array}{c} \vec{X}_0^0 \\ \vec{Y}_0^0, \\ \vec{Z}_0^0 \end{array} \end{array}$$

or

$$R_Y(\vartheta) = \begin{array}{cc} \begin{array}{ccc} \vec{X}_1^0 & \vec{Y}_1^0 & \vec{Z}_1^0 \end{array} & * \\ \begin{bmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{bmatrix} & \begin{array}{c} \vec{X}_0^0 \\ \vec{Y}_0^0 \\ \vec{Z}_0^0 \end{array} \end{array},$$

which is the same as (5).

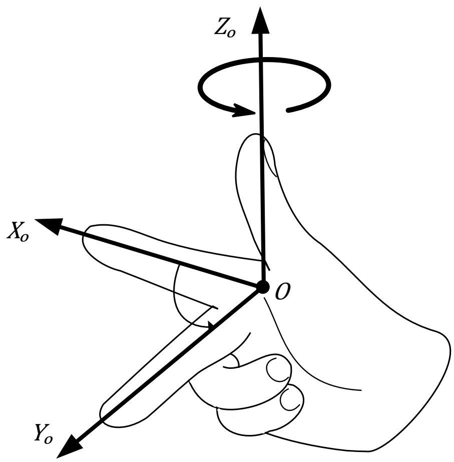


Figure 5. Right-handed coordinate frame and positive rotation.

Using such tables is very helpful to avoid misinterpretations in calculations of rotation matrices and vector projections as from it we can write down these expressions:

$$\begin{cases} \vec{X}_1^0 = \vec{X}_0^0 \cos\vartheta - \vec{Z}_0^0 \sin\vartheta\,, \\ \vec{Y}_1^0 = \vec{Y}_0^0, \\ \vec{Z}_1^0 = \vec{X}_0^0 \sin\vartheta + \vec{Z}_0^0 \cos\vartheta\,, \\ \vec{X}_0^0 = \vec{X}_1^0 \cos\vartheta + \vec{Z}_1^0 \sin\vartheta\,, \\ \vec{Y}_0^0 = \vec{Y}_1^0, \\ \vec{Z}_0^0 = -\vec{X}_1^0 \sin\vartheta + \vec{Z}_1^0 \cos\vartheta\,. \end{cases}$$

Let us calculate the product of $R_Y(\vartheta)$ with its transpose $R_Y^T(\vartheta)$:

$$R_Y(\vartheta)R_Y^T(\vartheta) = \begin{bmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{bmatrix} \begin{bmatrix} \cos\vartheta & 0 & -\sin\vartheta \\ 0 & 1 & 0 \\ \sin\vartheta & 0 & \cos\vartheta \end{bmatrix} =$$
$$\begin{bmatrix} \cos^2\vartheta + \sin^2\vartheta & 0 & -\cos\vartheta\sin\vartheta + \sin\vartheta\cos\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta\cos\vartheta + \cos\vartheta\sin\vartheta & 0 & \sin^2\vartheta + \cos^2\vartheta \end{bmatrix} =$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

where I is identity matrix. As we know the inverse R^{-1} of any square matrix R is defined as:

$$RR^{-1} = R^{-1}R = I.$$

Taking into account the above we can see that the transpose of the rotation matrix and its inverse are the same $R^{-1} = R^T$. The matrices whit such properties are called orthogonal matrices.

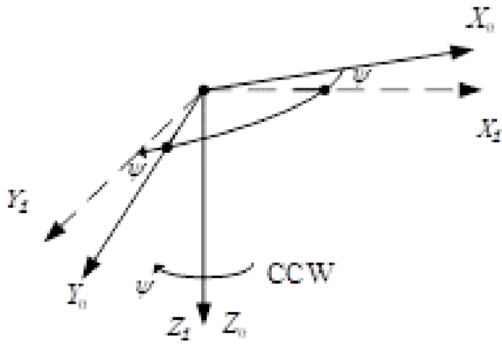
With these considerations in the mind, we can find p_{x1}, p_{y1}, p_{z1} based on the values p_{x0}, p_{y0}, p_{z0} and angle ϑ in the following form:

$$R^{-1}\begin{bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{bmatrix} = R^{-1}R\begin{bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{bmatrix} = I\begin{bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{bmatrix},$$

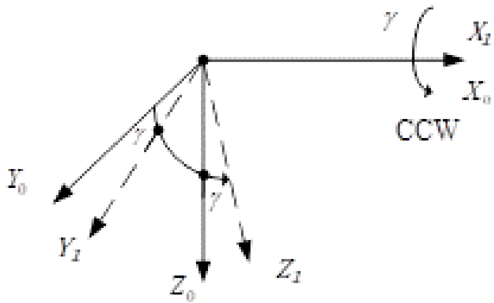
and, by reversing left and right sides with the use of $R^{-1} = R^T$, we get

$$\begin{bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{bmatrix} = R^T\begin{bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{bmatrix}.$$

The above mathematical operations are true for other 2 types of rotations. Let us get their rotation matrices using only their spatial drawings (Figure 6 *a)*, *b)*) and the table method proposed above.



a)



b)

Figure 6. Yaw and roll rotations.

Based on Figure 6 *a)* and 6 *b)* we can write down the rotation matrices for yaw and roll rotations correspondingly:

$$R_Z(\psi) = \begin{array}{cc} \begin{array}{ccc} \vec{X}_1^0 & \vec{Y}_1^0 & \vec{Z}_1^0 \end{array} & * \\ \begin{bmatrix} \cos\psi & \cos\left(\frac{\pi}{2}+\psi\right) & \cos\frac{\pi}{2} \\ \cos\left(\frac{\pi}{2}-\psi\right) & \cos\psi & \cos\frac{\pi}{2} \\ \cos\frac{\pi}{2} & \cos\frac{\pi}{2} & \cos 0 \end{bmatrix} & \begin{array}{c} \vec{X}_0^0 \\ \vec{Y}_0^0 \\ \vec{Z}_0^0 \end{array} \end{array},$$

$$R_Z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_X(\gamma) = \begin{array}{cc} \begin{array}{ccc} \vec{X}_1^0 & \vec{Y}_1^0 & \vec{Z}_1^0 \end{array} & * \\ \begin{bmatrix} \cos 0 & \cos\frac{\pi}{2} & \cos\frac{\pi}{2} \\ \cos\frac{\pi}{2} & \cos\gamma & \cos\left(\frac{\pi}{2}+\gamma\right) \\ \cos\frac{\pi}{2} & \cos\left(\frac{\pi}{2}-\gamma\right) & \cos\gamma \end{bmatrix} & \begin{array}{c} \vec{X}_0^0 \\ \vec{Y}_0^0 \\ \vec{Z}_0^0 \end{array} \end{array},$$

$$R_X(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}.$$

Based on all the above, let us consider the vector $\vec{p}_0$ attached to coordinate frame $OX_0Y_0Z_0$ which has projections p_{x0}, p_{y0}, p_{z0} to its bases, and coordinate frame $OX_1Y_1Z_1$ is rotated around $OX_0Y_0Z_0$, and frame $OX_2Y_2Z_2$ is rotated arround $OX_1Y_1Z_1$. These rotations are described by rotation matrices R_1 and R_2 correspondingly. And our task is to get the projections of the vector $\vec{p}_0$ on the bases of $OX_2Y_2Z_2$. We can write the following equations:

$$\begin{bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{bmatrix} = R_1 \begin{bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{bmatrix},$$
$$\begin{bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{bmatrix} = R_2 \begin{bmatrix} p_{x2} \\ p_{y2} \\ p_{z2} \end{bmatrix}.$$

This means that:

$$\begin{bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{bmatrix} = R_1 \begin{bmatrix} p_{x1} \\ p_{y1} \\ p_{z1} \end{bmatrix} = R_1R_2 \begin{bmatrix} p_{x2} \\ p_{y2} \\ p_{z2} \end{bmatrix} = R \begin{bmatrix} p_{x2} \\ p_{y2} \\ p_{z2} \end{bmatrix}.$$

So the resulting rotation matrix, in this case is $R = R_1R_2$. To find projections p_{x2}, p_{y2}, p_{z2} we just need to do the following:

$$\begin{bmatrix} p_{x2} \\ p_{y2} \\ p_{z2} \end{bmatrix} = R^T \begin{bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{bmatrix} = (R_1R_2)^T \begin{bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{bmatrix} = R_2^TR_1^T \begin{bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{bmatrix}.$$

And it is true for any number of rotations done: the resulting rotation matrix will be the matrix multiplication of the matrices of consecutive rotations.

How can we do the same operations using quaternions? Let us consider the example presented on Figure 3 and 4 and related to expression (5). To use quaternions, the hypercomplex mapping of the vector is introduced, such as if the vector $\vec{p}_0$ has projections p_{x0}, p_{y0}, p_{z0} on bases of coordinate frame $OX_0Y_0Z_0$, its hypercomplex mapping to $OX_0Y_0Z_0$ is $P_0 = 0 + p_{x0}\boldsymbol{i} + p_{y0}\boldsymbol{j} + p_{z0}\boldsymbol{k}$, which is a quaternion as well. The hypercomplex mapping of $\vec{p}_0$ to the rotated around $OX_0Y_0Z_0$ coordinate frame $OX_1Y_1Z_1$ is $P_1 = 0 + p_{x1}\boldsymbol{i} + p_{y1}\boldsymbol{j} + p_{z1}\boldsymbol{k}$. Taking into account, that the bases of the hypercomplex space don't change and this abstraction should not lead the reader to misinterpretation, as it would be in case of considering quaternions as a generalization of the 3D vectors. If quaternion $Q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$ describes the rotation of $OX_1Y_1Z_1$ around $OX_0Y_0Z_0$ then P_0 and P_1 are connected by the following equation:

$$P_0 = Q \circ P_1 \circ \tilde{Q}, \tag{7}$$

where $\tilde{Q} = q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}$ is conjugate quaternion of Q. There were many debates between the scientists of the 19th century on how to represent rotation using quaternions, even a version of $P_0 = Q \circ P_1$ was proposed, but it was not correct and without going through that debates we will just take into account the form (7) presented above.

Let us do the mathematical operations in (7) based on the quaternion algebra:

$$P_0 = Q \circ P_1 \circ \tilde{Q} = (q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}) \circ \left(0 + p_{x1}\boldsymbol{i} + p_{y1}\boldsymbol{j} + p_{z1}\boldsymbol{k}\right) \circ (q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k}) = 0 +$$
$$\boldsymbol{i}\left(p_{x1}(q_0^2 + q_1^2 - q_2^2 - q_3^2) + 2p_{y1}(q_1q_2 - q_0q_3) + 2p_{Z1}(q_0q_2 + q_1q_3)\right) +$$
$$\boldsymbol{j}\left(p_{y1}(q_0^2 - q_1^2 + q_2^2 - q_3^2) + 2p_{x1}(q_0q_3 + q_1q_2) + 2p_{Z1}(q_2q_3 - q_0q_1)\right) +$$
$$\boldsymbol{i}\left(p_{z1}(q_0^2 - q_1^2 - q_2^2 + q_3^2) + 2p_{x1}(q_1q_3 - q_0q_2) + 2p_{y1}(q_0q_1 + q_2q_3)\right).$$

Based on the above and equations (3):

$$p_{x0} = p_{x1}(q_0^2 + q_1^2 - q_2^2 - q_3^2) + 2p_{y1}(q_1q_2 - q_0q_3) +$$
$$+2p_{Z1}(q_0q_2 + q_1q_3) =$$
$$p_{x1}\cos\vartheta + p_{z1}\sin\vartheta,$$
$$p_{y0} = p_{y1}(q_0^2 - q_1^2 + q_2^2 - q_3^2) +$$
$$+2p_{x1}(q_0q_3 + q_1q_2) + 2p_{Z1}(q_2q_3 - q_0q_1) = p_{y1},$$
$$p_{z0} = p_{z1}(q_0^2 - q_1^2 - q_2^2 + q_3^2) +$$
$$+2p_{x1}(q_1q_3 - q_0q_2) + 2p_{y1}(q_0q_1 + q_2q_3) =$$
$$p_{z1}\cos\vartheta - p_{x1}\sin\vartheta.$$

Therefore,

$$q_0^2 + q_1^2 - q_2^2 - q_3^2 = \cos\vartheta,$$
$$q_1q_2 - q_0q_3 = 0,$$
$$2(q_0q_2 + q_1q_3) = \sin\vartheta,$$
$$q_0^2 - q_1^2 + q_2^2 - q_3^2 = 1,$$
$$q_0q_3 + q_1q_2 = 0,$$
$$q_2q_3 - q_0q_1 = 0,$$
$$q_0^2 - q_1^2 - q_2^2 + q_3^2 = \cos\vartheta,$$
$$2(q_1q_3 - q_0q_2) = -\sin\vartheta,$$
$$q_0q_1 + q_2q_3 = 0.$$

By addition of the left and right sides of the first and seventh equations from the above list, we get:

$$q_0^2 - q_2^2 = \cos\vartheta = \cos^2\frac{\vartheta}{2} - \sin^2\frac{\vartheta}{2}. \tag{8}$$

Subtracting the left and right sides of the eighth equation from third equation, we will have:

$$2q_0q_2 = \sin\vartheta = 2\cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2}. \tag{9}$$

From other equations of the above list it is easy to prove that $q_1 = q_3 = 0$, and from (8) and (9) it is obvious that

$$q_0 = \cos\frac{\vartheta}{2},$$

$$q_2 = \sin\frac{\vartheta}{2},$$

and $Q = \cos\frac{\vartheta}{2} + \sin\frac{\vartheta}{2}\boldsymbol{j}$ has norm $|Q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1$, so Q is a unit quaternion, which was discussed earlier on unit quaternions and their relations to the rotations.

Keeping this result in mind, we can follow to the other abstraction: any rotation by angle ϑ about any Y axis is represented by a quaternion of form $Q_Y(\vartheta) = \cos\frac{\vartheta}{2} + \sin\frac{\vartheta}{2}\boldsymbol{j}$. Similarly we can get the results for the rotations by any angle ψ around any Z axis $Q_Z(\psi) = \cos\frac{\psi}{2} + \sin\frac{\psi}{2}\boldsymbol{k}$ and by any angle γ around X axis $Q_X(\gamma) = \cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\boldsymbol{i}$.

We remember that for a unit quaternion its inverse and conjugate are the same, so to express P_1 through P_0 we can do these simple operations:

$$\tilde{Q} \circ P_0 \circ Q = \tilde{Q} \circ Q \circ P_1 \circ \tilde{Q} \circ Q = P_1,$$

from where

$$P_1 = \tilde{Q} \circ P_0 \circ Q.$$

If we have a sequence of rotations and their quaternions such as $P_0 = Q_1 \circ P_1 \circ \tilde{Q}_1$, $P_1 = Q_2 \circ P_2 \circ \tilde{Q}_2$, etc., the quaternion Q of the resulting rotation will be the product of sequential rotation quaternions:

$$P_0 = Q_1 \circ P_1 \circ \tilde{Q}_1 = P_0 = Q_1 \circ Q_2 \circ P_2 \circ \tilde{Q}_2 \circ \tilde{Q}_1,$$
$$Q = Q_1 \circ Q_2,$$
$$\tilde{Q} = (Q_1 \widetilde{\circ Q_2}) = \tilde{Q}_2 \circ \tilde{Q}_1.$$

Let us now consider the sequence of 3 rotations by Euler angles presented in Figure 2 and define the connections between Euler angles and quaternion.

Based on the concepts on rotation matrices the resulting rotation matrix R for the consecutive rotations by Euler angle can be expressed as:

$$R = R_Z(\psi)R_Y(\vartheta)R_X(\gamma).$$

It is advised to calculate the elements of R as an exercise and get the following result with denoting $\cos(\cdot) = \mathrm{c}(\cdot), \sin(\cdot) = \mathrm{s}(\cdot)$:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathrm{c}\psi\mathrm{c}\vartheta & \mathrm{c}\psi\mathrm{s}\vartheta\mathrm{s}\gamma - \mathrm{s}\psi\mathrm{c}\gamma & \mathrm{c}\psi\mathrm{s}\vartheta\mathrm{c}\gamma + \mathrm{s}\psi\mathrm{s}\gamma \\ \mathrm{s}\psi\mathrm{c}\vartheta & \mathrm{s}\psi\mathrm{s}\vartheta\mathrm{s}\gamma + \mathrm{c}\psi\mathrm{c}\gamma & \mathrm{s}\psi\mathrm{s}\vartheta\mathrm{c}\gamma - \mathrm{c}\psi\mathrm{s}\gamma \\ -\mathrm{s}\vartheta & \mathrm{c}\vartheta\mathrm{s}\gamma & \mathrm{c}\vartheta\mathrm{c}\gamma \end{bmatrix}.$$

For the resulting quaternion Q we can even do its derivation without any drawings just keeping in mind the abstractions discussed above:

$$Q = Q_Z(\psi) \circ Q_Y(\vartheta) \circ Q_X(\gamma) = \left(\cos\frac{\psi}{2} + \sin\frac{\psi}{2}\boldsymbol{k}\right) \circ \left(\cos\frac{\vartheta}{2} + \sin\frac{\vartheta}{2}\boldsymbol{j}\right) \circ \left(\cos\frac{\gamma}{2} + \sin\frac{\gamma}{2}\boldsymbol{i}\right). \tag{10}$$

Doing the derivation of the components of Q as an exercise will convince, that quaternion is easier to represent and it is more intuitive than rotation matrices. Here is the point where the abstractions take advantage on visual imagination.

Expanding (10) we will get:

$$\begin{cases} Q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}, \\ q_0 = \cos\frac{\psi}{2}\cos\frac{\vartheta}{2}\cos\frac{\gamma}{2} + \sin\frac{\psi}{2}\sin\frac{\vartheta}{2}\sin\frac{\gamma}{2}, \\ q_1 = \cos\frac{\psi}{2}\cos\frac{\vartheta}{2}\sin\frac{\gamma}{2} - \sin\frac{\psi}{2}\sin\frac{\vartheta}{2}\cos\frac{\gamma}{2}, \\ q_2 = \cos\frac{\psi}{2}\sin\frac{\vartheta}{2}\cos\frac{\gamma}{2} + \sin\frac{\psi}{2}\cos\frac{\vartheta}{2}\sin\frac{\gamma}{2}, \\ q_3 = -\cos\frac{\psi}{2}\sin\frac{\vartheta}{2}\sin\frac{\gamma}{2} + \sin\frac{\psi}{2}\cos\frac{\vartheta}{2}\cos\frac{\gamma}{2}. \end{cases}$$

Based on the above results and the fact that Q is a unit quaternion. i.e., $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$, it is easy to show that:

$$\begin{cases} \psi = \operatorname{atan2}\left(2(q_1q_2 + q_0q_3), 1 - 2(q_2^2 + q_3^2)\right), \\ \vartheta = \operatorname{asin}\big(2(q_0q_2 - q_1q_3)\big), \\ \gamma = \operatorname{atan2}\left(2(q_0q_1 + q_2q_3), 1 - 2(q_1^2 + q_2^2)\right). \end{cases} \tag{11}$$

By expressing elements of the vector $\vec{p}_0$ using rotation matrix and quaternion we can find that:

$$p_{x0} = r_{11}p_{x1} + r_{12}p_{y1} + r_{13}p_{z1} =$$
$$p_{x1}(q_0^2 + q_1^2 - q_2^2 - q_3^2) + 2p_{y1}(q_1q_2 - q_0q_3) + 2p_{Z1}(q_0q_2 + q_1q_3),$$
$$p_{y0} = r_{21}p_{x1} + r_{22}p_{y1} + r_{23}p_{z1} =$$
$$p_{y1}(q_0^2 - q_1^2 + q_2^2 - q_3^2) + 2p_{x1}(q_0q_3 + q_1q_2) + 2p_{Z1}(q_2q_3 - q_0q_1),$$
$$p_{z0} = r_{31}p_{x1} + r_{32}p_{y1} + r_{33}p_{z1} =$$
$$p_{z1}(q_0^2 - q_1^2 - q_2^2 + q_3^2) + 2p_{x1}(q_1q_3 - q_0q_2) + 2p_{y1}(q_0q_1 + q_2q_3).$$

The above allows us to define the following system of equations:

$$\begin{cases} r_{11} = q_0^2 + q_1^2 - q_2^2 - q_3^2, \\ r_{12} = 2(q_1q_2 - q_0q_3), \\ r_{13} = 2(q_0q_2 + q_1q_3), \\ r_{21} = 2(q_0q_3 + q_1q_2), \\ r_{22} = q_0^2 - q_1^2 + q_2^2 - q_3^2, \\ r_{23} = 2(q_2q_3 - q_0q_1), \\ r_{31} = 2(q_1q_3 - q_0q_2), \\ r_{32} = 2(q_0q_1 + q_2q_3), \\ r_{33} = q_0^2 - q_1^2 - q_2^2 + q_3^2. \end{cases} \tag{12}$$

By the additions of the left and right sides of the first, fifth and ninth equations of (12), keeping in mind $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$ we can find:

$$4q_0^2 - 1 = r_{11} + r_{22} + r_{33}.$$

Subtraction of the left and right sides of the sixth equation of (12) from the eighth equation of (12) we get:

$$4q_0 q_1 = r_{32} - r_{23}.$$

Considering third and seventh equations of (12), then fourth and second equations of (12) we can easily get:

$$4q_0 q_2 = r_{13} - r_{31},$$
$$4q_0 q_3 = r_{21} - r_{12}.$$

Based on the above we can express element of the quaternion through elements of the rotation matrix:

$$\begin{cases} q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}, \\ q_1 = \frac{1}{4q_0}(r_{32} - r_{23}), \\ q_2 = \frac{1}{4q_0}(r_{13} - r_{31}), \\ q_3 = \frac{1}{4q_0}(r_{21} - r_{12}). \end{cases} \tag{13}$$

Rodrigues's Rotation Formula and Quaternions

Let us remind several concepts from physics and mechanics. If a 3D vector $\vec{p} = p_x\vec{X}^0 + p_y\vec{Y}^0 + p_z\vec{Z}^0$ with constant length from its starting position $\vec{p}_0 = p_{0x}\vec{X}^0 + p_{0y}\vec{Y}^0 + p_{0z}\vec{Z}^0$ in the reference frame $OXYZ$ uniformly rotates around any axis $\hat{\omega}$ with angular velocity 1 radian/s, its time derivative is equal

$$\dot{\vec{p}}(t) = \vec{\omega} \times \vec{p}(t), \tag{14}$$

where $\vec{\omega} = \omega_x\vec{X}^0 + \omega_y\vec{Y}^0 + \omega_z\vec{Z}^0$ is the vector of angular velocity $|\vec{\omega}| = 1$, $\vec{X}^0, \vec{Y}^0, \vec{Z}^0$ are bases of coordinate frame $OXYZ$.

Equation (14) can be rewritten in the vector form using the representation of cross product by skew-symmetric matrix multiplication and notations $p(t) = [p_x \quad p_y \quad p_z]^T$, $p_0 = [p_{0x} \quad p_{0y} \quad p_{0z}]^T$:

$$\dot{p}(t) = \omega_\times p(t), \tag{15}$$

where $\omega_\times$ is a skew-symmetric matrix ($\omega_\times^T = -\omega_\times$) and is expressed as:

$$\omega_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}.$$

For a scalar $a(t)$ with initial value $a(t = 0) = a_0$ similar equation

$$\dot{a}(t) = Aa(t),$$

has solution in the form:

$$a(t) = e^{At}a_0.$$

The similar solution is applicable for the equation (15):

$$p(t) = e^{\omega_\times t}p_0. \tag{16}$$

Based on the formula (6) for the rotating vector we can state that the rotation matrix R is the same as the matrix exponential $e^{\omega_\times t}$ and the vector $\vec{\omega}$ is the unit vector of the axis $\widehat{\omega}$:

$$R = e^{\omega_\times t}.$$

The matrix exponential can be represented as the Maclaurin series of exponential function [5]:

$$e^{\omega_\times t} = I + \omega_\times t + \frac{(\omega_\times t)^2}{2!} + \frac{(\omega_\times t)^3}{3!} + \cdots, \tag{17}$$

where I is identity matrix of $\mathbb{R}^{3\times 3}$.

Let us consider the skew-symmetric matrix $\omega_\times$, and, taking into account that $|\vec{\omega}| = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} = 1$, do the following calculations:

$$\omega_\times^3 = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} =$$
$$\begin{bmatrix} -\omega_z^2 - \omega_y^2 & \omega_y\omega_x & \omega_z\omega_x \\ \omega_x\omega_y & -\omega_z^2 - \omega_x^2 & \omega_z\omega_y \\ \omega_x\omega_z & \omega_y\omega_z & -\omega_y^2 - \omega_x^2 \end{bmatrix}\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} =$$
$$\begin{bmatrix} \omega_y\omega_x\omega_z - \omega_z\omega_x\omega_y & \omega_z\left(\omega_z^2 + \omega_y^2\right) + \omega_z\omega_x^2 & -\omega_y\left(\omega_z^2 + \omega_y^2\right) - \omega_y\omega_x^2 \\ -\omega_z(\omega_z^2 + \omega_x^2) - \omega_z\omega_y^2 & -\omega_x\omega_y\omega_z + \omega_z\omega_y\omega_x & \omega_x\omega_y^2 + \omega_x(\omega_z^2 + \omega_x^2) \\ \omega_y\omega_z^2 + \omega_y\left(\omega_y^2 + \omega_x^2\right) & -\omega_x\omega_z^2 - \omega_x\left(\omega_y^2 + \omega_x^2\right) & \omega_x\omega_z\omega_y - \omega_y\omega_z\omega_x \end{bmatrix},$$

or

$$\omega_\times^3 = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} = -\omega_\times. \tag{18}$$

To do a rotation around axis $\hat{\omega}$ by angle θ radians, in this case we need time θ seconds as the angular velocity equals 1 radian/s. Based on (18) and the Maclaurin series of sine and cosine functions the expression (17) for the rotation by angle θ can be rewritten as:

$$R_{\hat{\omega}}(\theta) = e^{\omega_\times \theta} = I + \omega_\times\theta + \omega_\times^2\frac{\theta^2}{2!} - \omega_\times\frac{\theta^3}{3!} - \omega_\times^2\frac{\theta^4}{4!} + \omega_\times\frac{\theta^5}{5!} + \omega_\times^2\frac{\theta^6}{6!}\cdots =$$
$$I + \omega_\times\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) + \omega_\times^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!}\cdots\right),$$

or

$$R_{\widehat{\omega}}(\theta) = I + \sin\theta\,\omega_{\times} + (1 - \cos\theta)\omega_{\times}^2. \tag{19}$$

The above equation is also known as Rodrigues's formula which with the use of short notations $\sin\theta = \mathrm{s}\,\theta$ and $\cos\theta = \mathrm{c}\,\theta$ can be presented as:

$$R_{\widehat{\omega}}(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +$$
$$+ \begin{bmatrix} 0 & -\omega_z\,\mathrm{s}\,\theta & \omega_y\,\mathrm{s}\,\theta \\ \omega_z\,\mathrm{s}\,\theta & 0 & -\omega_x\,\mathrm{s}\,\theta \\ -\omega_y\,\mathrm{s}\,\theta & \omega_x\,\mathrm{s}\,\theta & 0 \end{bmatrix} +$$
$$\begin{bmatrix} \left(-\omega_z^2 - \omega_y^2\right)(1 - \mathrm{c}\,\theta) & \omega_y\omega_x(1 - \mathrm{c}\,\theta) & \omega_z\omega_x(1 - \mathrm{c}\,\theta) \\ \omega_x\omega_y(1 - \mathrm{c}\,\theta) & (-\omega_z^2 - \omega_x^2)(1 - \mathrm{c}\,\theta) & \omega_z\omega_y(1 - \mathrm{c}\,\theta) \\ \omega_x\omega_z(1 - \mathrm{c}\,\theta) & \omega_y\omega_z(1 - \mathrm{c}\,\theta) & \left(-\omega_y^2 - \omega_x^2\right)(1 - \mathrm{c}\,\theta) \end{bmatrix},$$

based on which we can write down the following system of equations:

$$\begin{cases} r_{11} = 1 - \left(\omega_z^2 + \omega_y^2\right)(1 - \mathrm{c}\,\theta), \\ r_{12} = -\omega_z\,\mathrm{s}\,\theta + \omega_y\omega_x(1 - \mathrm{c}\,\theta), \\ r_{13} = \omega_y\,\mathrm{s}\,\theta + \omega_z\omega_x(1 - \mathrm{c}\,\theta), \\ r_{21} = \omega_z\,\mathrm{s}\,\theta + \omega_x\omega_y(1 - \mathrm{c}\,\theta), \\ r_{22} = 1 - (\omega_z^2 + \omega_x^2)(1 - \mathrm{c}\,\theta), \\ r_{23} = -\omega_x\,\mathrm{s}\,\theta + \omega_z\omega_y(1 - \mathrm{c}\,\theta), \\ r_{31} = -\omega_y\,\mathrm{s}\,\theta + \omega_x\omega_z(1 - \mathrm{c}\,\theta), \\ r_{32} = \omega_x\,\mathrm{s}\,\theta + \omega_y\omega_z(1 - \mathrm{c}\,\theta), \\ r_{33} = 1 - \left(\omega_y^2 + \omega_x^2\right)(1 - \mathrm{c}\,\theta). \end{cases} \tag{20}$$

Thus, taking into account (13) and $1 - \mathrm{c}\,\theta = 1 - \mathrm{c}^2\frac{\theta}{2} + \mathrm{s}^2\frac{\theta}{2} = 2\,\mathrm{s}^2\frac{\theta}{2}$ we get:

$$\begin{cases} q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} = \frac{1}{2}\sqrt{4 - 2\,\mathrm{s}^2\frac{\theta}{2}\left(2\omega_x^2 + 2\omega_y^2 + 2\omega_z^2\right)}, \\ q_1 = \frac{1}{4q_0}(r_{32} - r_{23}) = \frac{1}{4q_0}2\omega_x\,\mathrm{s}\,\theta = \frac{1}{q_0}\omega_x\,\mathrm{s}\frac{\theta}{2}\mathrm{c}\frac{\theta}{2}, \\ q_2 = \frac{1}{4q_0}(r_{13} - r_{31}) = \frac{1}{4q_0}2\omega_y\,\mathrm{s}\,\theta = \frac{1}{q_0}\omega_y\,\mathrm{s}\frac{\theta}{2}\mathrm{c}\frac{\theta}{2}, \\ q_3 = \frac{1}{4q_0}(r_{21} - r_{12}) = \frac{1}{4q_0}2\omega_z\,\mathrm{s}\,\theta = \frac{1}{q_0}\omega_z\,\mathrm{s}\frac{\theta}{2}\mathrm{c}\frac{\theta}{2}, \end{cases}$$

or

$$\begin{cases} q_0 = \cos\frac{\theta}{2}, \\ q_1 = \omega_x \sin\frac{\theta}{2}, \\ q_2 = \omega_y \sin\frac{\theta}{2}, \\ q_3 = \omega_z \sin\frac{\theta}{2}. \end{cases} \tag{21}$$

According to Euler's rotation theorem any composition of rotations applied from the same origin can be represented as a single rotation around some axis applied at the same origin. In the case we considered above that axis is $\hat{\omega}$, and its unit vector $\omega = [\omega_x \quad \omega_y \quad \omega_z]^T$ with the angle θ are the parameters that lay under Euler axis and angle representation of rotations.

Euler Parameters and Quaternions

Euler parameters are one more model of the representation of rotations. They are introduced as a set of real numbers a, b, c, d such that:

$$a^2 + b^2 + c^2 + d^2 = 1.$$

These parameters are result of the formula we considered in the previous section, and defined as:

$$\begin{cases} a = \cos\frac{\theta}{2}, \\ b = \omega_x \sin\frac{\theta}{2}, \\ c = \omega_y \sin\frac{\theta}{2}, \\ d = \omega_z \sin\frac{\theta}{2}. \end{cases} \tag{22}$$

As we can see these parameters are identical to the components of the rotation quaternion $Q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$.

These parameters then are divided into two components: a scalar and $\omega = \left[\omega_x \sin\frac{\theta}{2} \quad \omega_y \sin\frac{\theta}{2} \quad \omega_z \sin\frac{\theta}{2}\right]^T$ vector to describe the rotation of the vector $p = [p_x \quad p_y \quad p_z]^T$ based on its initial projections on the reference frame $p_0 = [p_{0x} \quad p_{0y} \quad p_{0z}]^T$:

$$\vec{p} = \vec{p}_0 + 2a(\vec{\omega} \times \vec{p}_0) + 2\big(\vec{\omega} \times (\vec{\omega} \times \vec{p}_0)\big). \tag{23}$$

It is advised as an exercise to check that formula (23) and (16) taking into account (19) are the same.

Cayley-Klein Parameters and Quaternions

Cayley-Klein parameters are defined as a set of complex numbers $\alpha, \beta, \gamma, \delta$ such that [6]

$$\begin{cases} \alpha\tilde{\alpha} + \beta\tilde{\beta} = 1, \\ \alpha\tilde{\alpha} + \gamma\tilde{\gamma} = 1, \\ \beta\tilde{\beta} + \delta\tilde{\delta} = 1, \\ \tilde{\alpha}\beta + \tilde{\gamma}\delta = 0, \\ \alpha\delta - \beta\gamma = 1, \\ \delta = \tilde{\alpha}, \\ \beta = -\tilde{\gamma}. \end{cases} \tag{24}$$

The representation of α, β using their real e_0, e_2 and imaginary parts e_3, e_1 is shown below:

$$\alpha = e_0 + ie_3,$$

$$\beta = e_2 + ie_1,$$

where i is imaginary unit: $i^2 = -1$.

Based on the (24) it is easy to get

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1.$$

So, we have very familiar for us expression. To not go deeper into complex derivations and keep everything simple for understanding, we just can state that elements e_0, e_1, e_2, e_3 of Cayley-Klein parameters are the same as Euler parameters or components of rotation quaternion. Thus,

$$\begin{cases} \alpha = q_0 + iq_3, \\ \beta = q_2 + iq_1, \\ \gamma = -\tilde{\beta} = -q_2 + iq_1, \\ \delta = \tilde{\alpha} = q_0 - iq_3. \end{cases} \tag{25}$$

To use Cayley-Klein parameters let us consider a constant vector $p_0 = [p_{0x} \quad p_{0y} \quad p_{0z}]^T$ in reference frame $OX_0Y_0Z_0$ which is represented in the coordinate frame $OX_1Y_1Z_1$ rotated around $OX_0Y_0Z_0$ by components p_{1x}, p_{1y}, p_{1z}. The operators P_0 and P_1 are used to represent elements p_{0x}, p_{0y}, p_{0z} and p_{1x}, p_{1y}, p_{1z} of vector p_0 in 2D space with complex axes:

$$P_0 = \begin{bmatrix} p_{0z} & p_{0x} - ip_{0y} \\ p_{0x} + ip_{0y} & -p_{0z} \end{bmatrix},$$

$$P_1 = \begin{bmatrix} p_{1z} & p_{1x} - ip_{1y} \\ p_{1x} + ip_{1y} & -p_{1z} \end{bmatrix}.$$

The connection between these two operators is described by Cayley-Klein parameters in the following way:

$$P_0 = \mathbf{Q}P_1\mathbf{Q}^H, \tag{26}$$

where $\mathbf{Q}$ is a unitary matrix $\mathbf{Q}^H = \widetilde{\mathbf{Q}}^T$ is conjugate transpose or adjoint matrix of $\mathbf{Q}$, such that $\mathbf{Q}\mathbf{Q}^H = \mathbf{Q}^H\mathbf{Q} = I$, $|\mathbf{Q}||\mathbf{Q}^H| = 1$, and expressed as:

$$\mathbf{Q} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$
$$\mathbf{Q}^H = \begin{bmatrix} \tilde{\alpha} & \tilde{\gamma} \\ \tilde{\beta} & \tilde{\delta} \end{bmatrix} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}.$$

Let us rewrite the equation (26) in the following form:

$$\begin{bmatrix} p_{0z} & p_{0x} - ip_{0y} \\ p_{0x} + ip_{0y} & -p_{0z} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} p_{1z} & p_{1x} - ip_{1y} \\ p_{1x} + ip_{1y} & -p_{1z} \end{bmatrix} \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix},$$

from expanding it and taking into account expressions $p_{0x} = \frac{1}{2}\,(P_0(1,2) + P_0(2,1))$ $p_{0y} = \frac{1}{2} i(P_0(1,2) - P_0(2,1))$, where $P_0(m,n)$ is element of P_0 at m^th^ row and n^th^ column, we get

$$p_{0x} = p_{1x}\frac{1}{2}(\alpha^2 - \beta^2 - \gamma^2 + \delta^2) + p_{1y}\frac{i}{2}(-\alpha^2 - \beta^2 + \gamma^2 + \delta^2) +$$
$$p_{1z}(-\alpha\beta + \gamma\delta),$$
$$p_{0y} = p_{1x}\frac{i}{2}(\alpha^2 - \beta^2 + \gamma^2 - \delta^2) + p_{1y}\frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) +$$
$$p_{1z}i(-\alpha\beta - \gamma\delta),$$
$$p_{0z} = p_{1x}(\beta\delta - \alpha\gamma) + p_{1y}i(\alpha\gamma + \beta\delta) + p_{1z}(\alpha\delta + \beta\gamma).$$

Combining the above with the expressions obtained by rotation matrix:

$$\begin{bmatrix} p_{x0} \\ p_{y0} \\ p_{z0} \end{bmatrix} = \begin{bmatrix} r_{11}p_{x1} + r_{12}p_{y1} + r_{13}p_{z1} \\ r_{21}p_{x1} + r_{22}p_{y1} + r_{23}p_{z1} \\ r_{31}p_{x1} + r_{32}p_{y1} + r_{33}p_{z1} \end{bmatrix},$$

we can define the elements of rotation matrix based on Cayley-Klein parameters using the following system of equations:

$$\begin{cases} r_{11} = \frac{1}{2}(\alpha^2 - \beta^2 - \gamma^2 + \delta^2), \\ r_{12} = \frac{i}{2}(-\alpha^2 - \beta^2 + \gamma^2 + \delta^2), \\ r_{13} = (-\alpha\beta + \gamma\delta), \\ r_{21} = \frac{i}{2}(\alpha^2 - \beta^2 + \gamma^2 - \delta^2), \\ r_{22} = \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2 + \delta^2), \\ r_{23} = i(-\alpha\beta - \gamma\delta), \\ r_{31} = \beta\delta - \alpha\gamma, \\ r_{32} = i(\alpha\gamma + \beta\delta), \\ r_{33} = \alpha\delta + \beta\gamma. \end{cases} \tag{27}$$

It is advised to do an exercise and check the assumptions (25) based on (27), (13) and (24).

Modified Rodrigues Parameters and Quaternions

Rodrigues proposed using more compact form of parameters than Euler parameters or quaternions [7]:

$$r_i = \frac{q_i}{q_0}, i = \overline{1,3}.$$

The vector $r = [r_1 \quad r_2 \quad r_3]^T$ is the classical Rodrigues parameters vector or also called Gibbs vector. To get the elements of quaternions we need to do these transformations:

$$\begin{cases} q_0 = \frac{1}{\sqrt{1+r^T r}}, \\ q_i = \frac{r_i}{\sqrt{1+r^T r}}, i = \overline{1,3}. \end{cases} \tag{28}$$

Based on (22) we can express classical Rodrigues parameters vector through Euler angle and axis representation of rotation:

$$r = \tan\frac{\theta}{2}\omega.$$

As we can see the classical Rodrigues parameters have singularities at q_0 which is case when the resulting rotation angle θ equals $\pm\pi$, as $q_0 = \cos\frac{\theta}{2}$. The same singularity we have when we use quaternions or Euler parameters as the rotation by π radians around any axis leads to the same attitude as the rotation was done by $-\pi$ radians around the same axis. In case of quaternions or Euler parameters the singularity is not as critical as for classical Rodrigues parameters because with quaternions we have 2 correct solutions, while with classical Rodrigues parameters we have no solution for the rotations $\pm\pi$ radians.

Modified Rodrigues parameters are defined to overcome the singularities at $\pm\pi$ radians and using elements of quaternions can be written as [7, 8]:

$$\rho_i = \frac{q_i}{1+q_0}, i = \overline{1,3}. \tag{29}$$

The singularities in this case are at $q_0 = -1$, which corresponds to angles of rotations $\pm 2\pi$.

To express quaternion elements through modified Rodrigues parameters vector $\rho = [\rho_1 \quad \rho_2 \quad \rho_3]^T$ we just need to solve the system (29), taking into account that quaternion Q is a unit quaternion ($q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$):

$$\begin{cases} q_0 = \frac{1-\rho^T\rho}{1+\rho^T\rho}, \\ q_i = \frac{2\rho_i}{1+\rho^T\rho}, i = \overline{1,3}. \end{cases}$$

It is advised to get the above expressions as an exercise, and derive the simple identity presented below:

$$\frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2},$$

to express the elements of the modified Rodrigues parameters vector through Euler's rotation axis and angle using formula (22):

$$\rho = \tan\frac{\theta}{4}\omega.$$

APPLICATIONS OF QUATERNIONS

Advantages of the Application of Quaternions in Representations of Rotations

Quaternions are widely used in the representations of the rotations and they have many other applications, such as usage in solving 3D shape registration problems [9].

Rotation representations using Euler angles lead to the singularities known as gimbal lock (Figure 7), related to the coincidence of the rotation axes. Moreover, kinematic equations derived with the use of Euler angles contain many trigonometric functions whith costly calculations on the computers which was a problem in the past. Quaternions don't lead to singularities related to gimbal lock, and the kinematic equations of rotations derived using them do not contain trigonometric functions [10]:

$$\dot{Q} = \frac{1}{2} Q \circ \Omega_I, \tag{30}$$

where $\dot{Q}$ is time derivative of quaternion Q, Ω_I is hypercomplex mapping of angular velocity vector of rotation from initial coordinate frame to new coordinate frame $\boldsymbol{I}$, expressed in its projections to new coordinate frame:

$$\Omega_I = 0 + \omega_x^I \boldsymbol{i} + \omega_y^I \boldsymbol{j} + \omega_z^I \boldsymbol{k}.$$

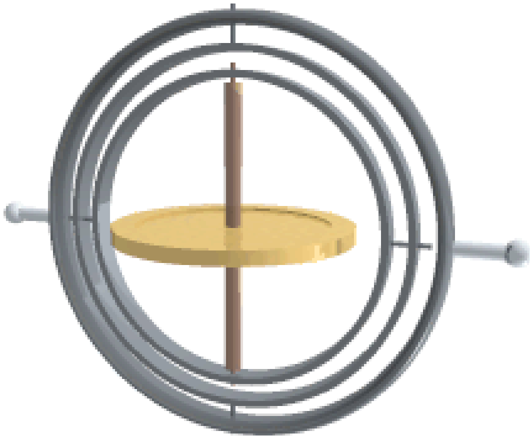


Figure 7. Gimbal lock.

Equation (30) can be rewritten as:

$$\begin{cases} \dot{q}_0 = -\frac{\omega_x^I q_1 + \omega_y^I q_2 + \omega_z^I q_3}{2}, \\ \dot{q}_1 = \frac{\omega_x^I q_0 - \omega_y^I q_3 + \omega_z^I q_2}{2}, \\ \dot{q}_2 = \frac{\omega_x^I q_3 + \omega_y^I q_0 - \omega_z^I q_1}{2}, \\ \dot{q}_3 = \frac{-\omega_x^I q_2 + \omega_y^I q_1 + \omega_z^I q_0}{2}. \end{cases}$$

It is advised to derive the above expressions from (30) based on the rules of quaternion product as an exercise.

Based on the above we can state that using quaternions leads to sufficient simplifications of mathematical representations of several problems as well as allows us to avoid many critical uncertainties or singularities. In many cases applications of quaternions are preferable as singularities related to them provide 2 correct solutions and never an empty solution.

Estimation of Quaternions

The direct physical measurement of quaternions is not implemented yet. So, the estimation of quaternions for solutions of many real-world tasks is an urgent problem.

Let us consider the kinematic equations of a rigid body rotation (31). They can be represented in the form [11]:

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_x^I & -\omega_y^I & -\omega_z^I \\ \omega_x^I & 0 & \omega_z^I & -\omega_y^I \\ \omega_y^I & -\omega_z^I & 0 & \omega_x^I \\ \omega_z^I & \omega_y^I & -\omega_x^I & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = A \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}, \tag{31}$$

and have solution in the discrete form:

$$\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}_{k+1} = \left[\cos\frac{\theta}{2} I + \frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} A dt\right] \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}_k ,$$

where k is step index, dt is sampling time, and angle $\theta = \sqrt{(\omega_x^I dt)^2 + \left(\omega_y^I dt\right)^2 + (\omega_z^I dt)^2}$.

Equation (31) can be considered as the process model, while the formula (11), defining the connections between the Euler angles and quaternions, can serve as the measurement model. In [11] it is well described how to create an unscented Kalman filter for estimation of quaternions. To the solution of the problem of orientation quaternion estimation using Kalman filtering many papers are dedicated now. In [12] a simpler approach is proposed. With two sources of quaternion data (30) and (10), we can observe interesting properties:

- solution of (30) contains low frequency errors, as integration of the sensor data flattens high-frequent fluctuations, but with an error at initial conditions it accumulates errors,

- quaternion calculations based on the (10) using sensor data, contain high frequency errors on the other hand.

Using these two sources a scheme of error compensation can be proposed using any simple low pass filter, for example an RC-filter (Figure 8).

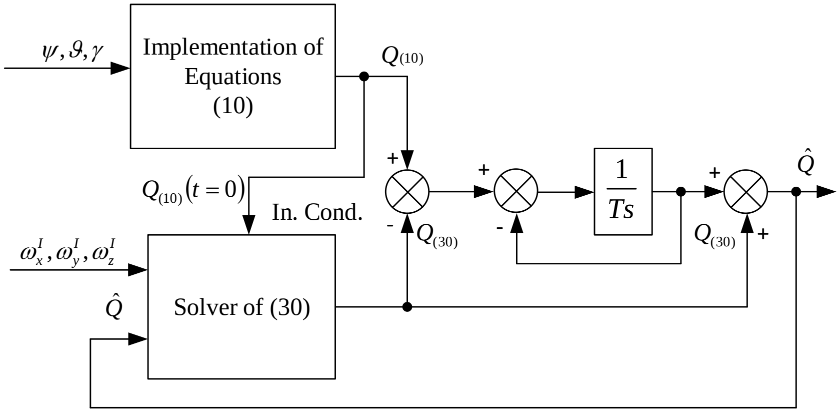


Figure 8. Simple version of quaternion estimation.

Based on the above diagram (Figure 8) the equations for the estimation of quaternion can be written as:

$$\begin{cases} \dot{\hat{q}}_0 = \dfrac{2q_0^{(10)} - 2\hat{q}_0 - \hat{q}_1\omega_x^I T - \hat{q}_2\omega_y^I T - \hat{q}_3\omega_z^I T}{2T}, \\ \dot{\hat{q}}_1 = \dfrac{2q_1^{(10)} - 2\hat{q}_1 + \hat{q}_0\omega_x^I T + \hat{q}_2\omega_z^I T - \hat{q}_3\omega_y^I T}{2T}, \\ \dot{\hat{q}}_2 = \dfrac{2q_2^{(10)} - 2\hat{q}_2 + \hat{q}_0\omega_y^I T - \hat{q}_1\omega_z^I T + \hat{q}_3\omega_x^I T}{2T}, \\ \dot{\hat{q}}_3 = \dfrac{2q_3^{(10)} - 2\hat{q}_3 + \hat{q}_0\omega_z^I T + \hat{q}_1\omega_y^I T - \hat{q}_2\omega_x^I T}{2T}, \end{cases}$$

where $\hat{q}_0, \hat{q}_1, \hat{q}_2, \hat{q}_3$ are estimates of quaternion components and $\dot{\hat{q}}_0, \dot{\hat{q}}_1, \dot{\hat{q}}_2, \dot{\hat{q}}_3$ are their time derivatives, $q_0^{(10)}, q_1^{(10)}, q_2^{(10)}, q_3^{(10)}$ are elements of quaternion $Q_{(10)}$ defined by formula (10) through Euler angles ψ, ϑ, γ, $Q_{(30)}$ is quaternion obtained through equation (30) using the measurements $\omega_x^I, \omega_y^I, \omega_z^I$ of the angular velocity of rotation in the rotating frame, T is time constant – a parameter which requires tuning based on spectrum characteristics of calculations obtained from (10).

The further exploration and analysis of quaternion estimation methods and their precisions are topics for research which is advised to do for the deeper understanding of quaternion applications in the real-world.

CONCLUSION

In this chapter the general concepts of quaternions, their algebra, connections with other mathematical representations, and their estimation problems were considered and described for their better understanding. The material from this chapter can be useful in exploration of the problems related to application of quaternions in the real world solutions, their simulation and estimation.

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Chapter 2

UNDERSTANDING QUATERNIONS FROM MODERN ALGEBRA AND THEORETICAL PHYSICS

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Abstract

Quaternions were appeared through Lagrangian formulation of mechanics in Symplectic vector space. Its general form was obtained from the Clifford algebra, and Frobenius' theorem, which says that "the only finite-dimensional real division algebra are the real field $\mathbf{R}$, the complex field $\mathbf{C}$ and the algebra $\mathbf{H}$ of quaternions" was derived. They appear also through Hamilton formulation of mechanics, as elements of rotation groups in the symplectic vector spaces.

Quaternions were used in the solution of 4-dimensional Dirac equation in QED, and also in solutions of Yang-Mills equation in QCD as elements of noncommutative geometry.

We present how quaternions are formulated in Clifford Algebra, how it is used in explaining rotation group in symplectic vector space and parallel transformation in holonomic dynamics. When a dynamical system

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has hysteresis, pre-symplectic manifolds and nonholonomic dynamics appear.

Quaternions represent rotation of 3-dimensional sphere $\mathbf{S}^3$. Artin's generalized quaternions and Rohlin-Pontryagin's embedding of quaternions on 4-dimensional manifolds, and Kodaira's embedding of quaternions on $\mathbf{S}^1 \times \mathbf{S}^3$ manifolds are also discussed.

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Keywords: quaternions, Frobenius' theorem, Clifford algebra, symplectic vector space, holonomy groups

1. Introduction

Quaternion $\mathbf{H}$ was discovered by Hamilton [1], and known as one real division algebra like real numbers $\mathbf{R}$ and complex number $\mathbf{C}$ which satisfies a quadratic equation $x^2 = \alpha e + \beta x$ ($\alpha, \beta \in \mathbf{R}$, e is unit element) [2]. Let Q_n denotes n dimensional vector space of $\mathbf{R}, \mathbf{C}, \mathbf{H}$, and G_n denotes the group of linear transformation that preserve inner products

$$y_i = \sum_{i=1}^{n} \sigma_{ij} x_j.$$

The condition $\sigma(x) \cdot \sigma(y) = x \cdot y$ leads $\sum_{i=1}^{n} \bar{\sigma}_{ij}\sigma_{ik} = \delta_{jk}$. G_n is called orthogonal, unitary or symplectic group accordig as scalars are $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ [2, 3]. Quarternions $\mathbf{H}$ are related to symplectic transformations, Complex numbers $\mathbf{C}$ are related to unitary transformations and Real numbers $\mathbf{R}$ are related to orthogonal transformations.

Properties of $\mathbf{H}$ can be studied in the general framework of symplectic vector spaces and Clifford algebra [4, 5, 6]. Artin [7] considered a generalized quaternion algebra in the framework of Clifford algebra and quadratic forms.

Usefulness of $\mathbf{H}$ in representing Lorentz transformations of the spinors was presented by Dirac [8]. Algebra using $\mathbf{R}, \mathbf{C}, \mathbf{H}$ is called K-theory and used in noncommutative geometry [9, 10, 11, 12], and applied in the standard model of elementary particles .

The Hopf lemma, which guaranties that "a commutative real algebra without divisor of zero, having quadratic mappings and every element is a square, the algebra is 2-dimensional", induced complex analytical manifold which is diffeomorphic to tensor product of a circle $\mathbf{S}^1$ and a sphere $\mathbf{S}^3$. Rotations of

$\mathbf{S}^3$ are expressed by $\mathbf{H}$, and Kodaira [13] clarified the structure of the Hopf manifold using complex projective space $\mathbf{P}^3(\mathbf{C})$, and Hopf algebra was used in quantum groups.

The structure of this article is as follows. In section 2, quaternions as bases of symplectic vector space is presented, and in section 3 formulation of Artin on generalized quaternions as bases of Clifford algebra is presented. In physical application, orthogonal groups are more familiar than symplectic groups. In section 4, Milnor's formulation [14] and Grassmann manifolds over quaternion fields are discussed. In section 5, application of quaternion in parallel transformations in holonomy group and Kodaira's Hopf space analysis on complex projective manifold [13] is presented. In section 6, Serre's formulation [15] of representing tensor products of $\mathbf{H}$ and $\mathbf{C}$ as $SL(2, \mathbf{F}_3)$, where $\mathbf{F}_3$ is free group on 3 elements, and its application to quantum group based on Hopf algebra are shown. Summary and discussion are given in section 7.

2. Quaternions and Symplectic Vector Space

In classical mechanics, equation of motion is

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}}{m}, \quad \frac{d\mathbf{r}}{dt} = \mathbf{v}$$

and when $\mathbf{F}$ is a derivative of a potential w, the Lagrangian $L = \frac{1}{2}m\mathbf{v}^2 - w$ is considered, and between arbitrary times t_0 and t_1, $\int_{t_0}^{t_1} Ldt$ is called Hamiltonian action [4]. When one defines a triplet $y =^t (t, \mathbf{r}, \mathbf{v})$ and

$$dy = \begin{pmatrix} dt \\ d\mathbf{r} \\ d\mathbf{v} \end{pmatrix}, \delta y = \begin{pmatrix} \delta t \\ \delta\mathbf{r} \\ \delta\mathbf{v} \end{pmatrix},$$

a Lagrangian form $\sigma(dy)(\delta y)$ is defined as

$$\langle m d\mathbf{v} - \mathbf{F}dt, \delta\mathbf{r} - \mathbf{v}\delta t\rangle - \langle m\delta\mathbf{v} - \mathbf{F}\delta t, d\mathbf{r} - \mathbf{v}dt\rangle$$

where $\langle , \rangle$ is the scalar product. The vector dy is tangent to a leaf x which passes y if

$$\sigma(dy)(\delta y) = 0 \quad \forall \delta y$$

The direction of the leaf is the kernel of the form σ.

Sympleqtic vector space is a real vector space E of n dimension on which 2-form σ with following properties are defined.

For two vectors X and Y of E, when $\sigma(X)(Y)$ is zero, X and Y are called orthogonal. Since $\sigma(Y)(X) = -\sigma(X)(Y)$, the relation is symmetric. The space $orth(H)$ is defined as a set of vectors X which are orthogonal to all $Y \in H$:

$$X \in orth(H) \Leftrightarrow \sigma(X)(Y) = 0, \quad \forall Y \in H$$

A vector subspace$H$ is called isotropic if $H \subset orth(H)$, and co-isotrope if $orth(H) \subset H$, respectively. If $H = orth(H)$, the vector space H is called self-orthogonal. For a base $S = [U_1, \cdots, U_p, V_1 \cdots V_p]$ when

$$\sigma(S(x))(S(y)) = \bar{x} \cdot J \cdot y, \quad x, y \in R^{2p}$$

where

$$J = \begin{pmatrix} & & & \vdots & 1 & & \\ & 0 & & \vdots & & \ddots & \\ & & & \vdots & & & 1 \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ -1 & & & \vdots & & & \\ & \ddots & & \vdots & & 0 & \\ & & -1 & \vdots & & & \end{pmatrix}$$

is satisfied, E is called symplectique of dimension $2p$. An element of general linear transformation A that satisfy

$$\sigma(A(X))(A(Y)) = \sigma(X)(Y)$$

is called symplectic group $Sp(E)$.

For a p dimensional vector field E, a manifold $V = E^* \times E$ is defined by

$$y \equiv \begin{pmatrix} P \\ Q \end{pmatrix} \quad P \in E^*, Q \in E.$$

When the Cartan form $\varpi(dy) \equiv P(dQ)$, is a potential, i.e. $\sigma = \nabla\varpi$,

$$\sigma(dy)(\delta y) = [dP](\delta Q) - [\delta P](dQ).$$

For $V' \subset V$, an induced form $\sigma_{V'}$ from the Lagrangian σ_V satisfy for $dy, \delta y \in H$

$$\sigma_{V'}(dy)(\delta y) = \sigma_V(dy)(\delta y)$$
$$ker(\sigma_{V'}) \equiv H \cap orth(H).$$

Souriau [4] defined a torus $T \ni z$, that defines $z = exp(is)$ ($s \in \mathbf{R}$, $mod 2\pi$), and the quantum manifold Y whose structure is defind by $\xi \to \varpi$. The exterior derivative $\nabla\varpi$ and ϖ satisfy

$$\begin{aligned} dim(ker\nabla\varpi)) &\equiv 1 \\ dim(ker\varpi \cap ker\nabla\varpi) &\equiv 0 \end{aligned}$$

In symplectic space, ϖ is 1-form associated with derivative operations P on Y, and $\sigma = \nabla\varpi$ is a 2-form. One introduces the vector $i_Y(\xi)$ which satisfies

$$\sigma(i_Y(\xi)) \equiv 0, \quad \varpi(i_Y(\xi)) = 1.$$

When $z = e^{\sqrt{-1}s} \in T$, a Lie group operation on z in the manifold Y is $\underline{z}_Y = exp(si_Y)$. Lagrangian σ is defined by the relation

$$\sigma(dx)(\delta x) \equiv (d\xi)(\delta\xi)$$

$ker(D(P)(\xi)$ for all $\xi \in Y$ is produced by $i_Y(\xi)$,

When $x \in U$, $P^{-1}(x)$ is an orbit of T, and $\sigma_Y(d\xi)(\delta\xi) = \sigma_U(dx)(\delta x)$

When U is quantizable, there exists a differential application $x \to z_{jk}$ such that

$$\varpi_k(dx) - \varpi_j(dx) \equiv \frac{dz_{jk}}{\sqrt{-1}z_{jk}}$$

vector field Z_V on manifolds V on which a group G operate as

$$Z_V(x) = D(\hat{x})(e)(Z)$$

where for $a = e \cdot da = Z \cdot dx = 0$, $Z_V(x) = d[a_V(x)]$.

Moment μ of the group G is on Y, the form ϖ,

$$\mu \cdot Z = \varpi(Z_Y(\xi)) \quad Z \in \mathcal{G}, \quad \delta_{\mu\cdot Z}\xi = Z_Y(\xi),$$

where $\xi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ and $\bar{\xi} = (\bar{z}_1, \bar{z}_2)$ where $z_1, z_2 \in \mathbf{C}$. For all elements a and x in the E

$$a_E(x) = a + x, \quad Z_E(x) = Z.$$

E has the quantification (Y, P):

- $Y = E \times T$
- $\varpi\delta\begin{pmatrix} x \\ z \end{pmatrix} \equiv \frac{\delta z}{\sqrt{-1}z} + \frac{1}{2}\sigma(x)(\delta x) \quad [z \in T, x \in E]$
- $P\begin{pmatrix} x \\ z \end{pmatrix} \equiv x$

If Y has the Lie group structure, the product has the Weyl group structure [16]

$$\begin{pmatrix} x \\ z \end{pmatrix} \times \begin{pmatrix} x' \\ z' \end{pmatrix} = \begin{pmatrix} x + x' \\ zz'e^{-\sqrt{-1}\sigma(x)(x')/2} \end{pmatrix}.$$

The real one $\varpi(\delta\xi) = \bar{\xi}\delta\xi/\sqrt{-1}$ and when $x \in \mathbf{S}^2$ and $Z \in \mathbf{R}^3$

$$\mu(j(Z)) = -\lambda\langle x, Z\rangle$$

$$\gamma(\Phi(a) \cdot Z) = a \cdot \gamma(Z) \cdot \bar{a}$$

When $G = SO(3)$, it is possible to choose

$$\sigma(dx)(\delta x) \equiv \lambda\langle x, dx \times \delta x\rangle$$

and define

$$j(z)(y) = z \times y \quad \forall y \in \mathbf{R}^3$$

and define the moment $\mu \equiv g(-\lambda x)j^{-1}$.

When $\lambda = 1/2$,

$$j(Z)_Y(\xi) = -\frac{\sqrt{-1}}{2}\gamma(Z)\xi \quad \forall \xi \in Y, \quad \gamma(Z) = \sum_{i=1}^{3} \sigma_j Z^j$$

where σ_i are $SL(2, \mathbf{C})$ Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}.$$

which is the rotation on $\mathbf{S}^2$ expressed by quaternions $\gamma(Z) = \sum_{i=1}^{3} \sigma_j Z^j$.

Quaternions $\mathbf{H}$ are the orthogonal bases of the Hilbert space [4, 6], consisting of elements $\mathbf{I}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ which satisfy

$$\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}$$
$$\mathbf{ji} = -\mathbf{k}, \quad \mathbf{kj} = -\mathbf{i}, \quad \mathbf{ik} = -\mathbf{j}.$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \sigma_1 \quad \mathbf{j} = \sigma_2, \quad \mathbf{k} = \sigma_3$$

and

$$\mathbf{H} = \mathbf{I}a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d = \begin{pmatrix} a + \sqrt{-1}d & -c + \sqrt{-1}b \\ c + \sqrt{-1}b & a - \sqrt{-1}d \end{pmatrix}, \quad a, b, c, d \in \mathbf{R}$$

and

$$\mathbf{H}^* = \mathbf{I}a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d.$$

We define $a + \sqrt{-1}d = \alpha$ and $c + \sqrt{-1}b = \beta$, $\alpha, \beta \in \mathbf{C}$, and express

$$\mathbf{H} = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \quad \mathbf{H}^* = \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix}.$$

The subset $\mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ is called $Pu(\mathbf{H})$.

Quaternion appeared in symplectic vector space in the representation of $SO(3)$ rotation group of a point $\mathbf{x}$ on $\mathbf{S}^2 = SO(3)/SO(2)$.

Instead of the real quaternion algebras, there are complex quaternion algebras [16] with elements

$$\mathbf{x} = \kappa\mathbf{I} + \lambda\mathbf{i}/\sqrt{-1} + \mu\mathbf{j}/\sqrt{-1} + \nu\mathbf{k}/\sqrt{-1}$$

for $\kappa, \lambda, \mu, \nu \in \mathbf{R}$. In section 5, we extend $\mathbf{x} \in \mathbf{S}^2$ to $\mathbf{x} \in \mathbf{S}^3 = SO(4)/SO(3) \sim O(4)/O(3)$.

3. Quaternions and Clifford Algebra

Frobenius' theorem says that the only finite-dimensional real division algebra are the real field $\mathbf{R}$, the complex field $\mathbf{C}$ and the algebra of Hamilton's quaternions $\mathbf{H}$. The theorem can be obtained by using the Clifford algebras, which is defined in $d-$dimensional vector field E with orthogonal bases $e_1, e_2, \cdots, e_d$

with quadratic form $q(x)$, for $x \in E$ [7]. A quadratic map $q(\mathbf{x}) = B(\mathbf{x}, \mathbf{x})$ satisfies $q(a\mathbf{x}) = a^2 q(\mathbf{x})$ and

$$2B(\mathbf{x}, \mathbf{y}) = q(\mathbf{x} + \mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y}).$$

An isometric mapping $\tau : V \to V'$, τ is a linear isomorphism expressed by $GL(V)$ that satisfies

$$B(\tau(\mathbf{x}), \tau(\mathbf{y})) = B(\mathbf{x}, \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in V.$$

A Clifford mapping $j : (E, q) \to A$, where vector space E has orthogonal bases $(e_1, e_2, \cdots, e_d)$ and algebra A is defined by elements $a_1, a_2, \cdots, a_d$, is a linear mapping $j(e_i) = a_i$ such that

- $1 \notin j(E)$
- $(j(x))^2 = -q(x)1 = -q(x)$ for all $x \in E$.

If additionally $j(E)$ generates A, then A is called Clifford algebra.

We define the set

$$E^+ = \{a \in E; a^2 \geq 0\} \quad \text{and} \quad E^- = \{a \in E; a^2 \leq 0\}$$

For $a, b \in E^-$, $\beta(a, b) = -(ab + ba)$ is a bilinear mapping of $E^- \times E^-$ into A. Since $\beta(a, b) = \frac{1}{2}(a^2 + b^2 - (a + b)^2)$ β takes values in $\mathbf{R}$. Depending on the univerality of the mapping [6] $A = span(1, E^-)$ is identical to $\mathbf{C}$ or $\mathbf{H}$.

Artin [7] defined in isotropic vector space V with orthogonal geometry, and A be any vector which is not contained in the line $\langle N \rangle$. He cosidered $V = \langle N, A \rangle$ i.e. images of N and A fills V, and hyperbolic plane $V = \langle N, M \rangle$, each N and M symplectic

$$N^2 = M^2 = 0, \quad NM = 1.$$

The subspace $U_0 = \langle N_1, N_2, \cdots, N_{r-1} \rangle \perp W$ is orthogonal to N_r, but does not contain N_r. $P_r = \langle N_r, M_r \rangle \subset U_0^*$.

A vector space over the field k of dimension 2^n with basis elements e_S for each subset S of M is defined as $C(V)$ in the space V.

When dim$V \leq 4$, in $C(V)$, A_i are orthogonal bases, and a multiplication is denoted by $\circ$.

$$(S_1 + S_2 + S_3) + S_4 = S_1 + S_2 + S_3 + S_4$$
$$(S_1 + S_2 + S_3) \cap T = (S_1 \cap T) + (S_2 \cap T) + (S_3 \cap T)$$

The basis elements e_S and its extension is

$$e_S \circ e_T = \prod_{s\in S, t\in T} (s,t) \cdot \prod_{i\in S\cap T} A_i^2 \cdot e_{S+T},$$

where (s,t) equals $+1$ for $s \leq t$, and -1 for $s > t$.

The associativity is checked by [7, 17, 25]

$$(e_S \circ e_T) \circ e_R = \prod_{s\in S, t\in T} (s,t) \cdot \prod_{j\in S+T, r\in R} (j,r) \prod_{i\in S\cap T} A_i^2 \cdot \prod_{\lambda\in (S+T)\cap R} A_\lambda^2 \cdot e_{S+T+R}$$

which becomes

$$\prod_{s\in S, t\in T} (s,t) \prod_{s\in S. r\in R} (s,r) \prod_{t\in T, r\in R} (t,r).$$

Product over A_ν^2 is for ν that appear in more than one of the sets S, T, R.

$C(V)$ has the unit element e_ϕ, and the vector A_i is identified with the vector $e_{[i]}$, where the set $\{i\}$ contain all the single element i.

$$A_i \circ A_i = e_{[i]} \circ e_{[i]} = (i,i) A_i^2 e_\phi = A_i^2.$$

If $i \neq j$ then

$$(A_i \circ A_j) + (A_j \circ A_i) = (i,j) e_{[i,j]} + (j,i) e_{[i,j]} = 0$$

When $r = 3$, $e_S = A_{i_1} \circ A_{i_2} \circ A_{i_3}$, where A_1, A_2, A_3 is an orthogonal basis of V,

$$\begin{aligned}
(i_1 \circ i_2) &= -(i_2 \circ i_1) = -A_3^2 i_3, \quad i_1^2 = -A_2^2 A_3^2, \\
(i_2 \circ i_3) &= -(i_3 \circ i_2) = -A_1^2 i_1, \quad i_2^2 = -A_3^2 A_1^2, \\
(i_3 \circ i_1) &= -(i_1 \circ i_3) = -A_2^2 i_2, \quad i_3^2 = -A_1^2 A_2^2.
\end{aligned}$$

Algebra of this form is called generalized quaternion algebra.

When

$$A_1^2 = 1, \quad A_2^2 = -1, \quad A_3^2 = a \in \text{scalar norm}\, k^*$$

one has

$$\begin{aligned}
(i_1 \circ i_2) &= -(i_2 \circ i_1) = -a i_3, \quad i_1^2 = +a \\
(i_2 \circ i_3) &= -(i_3 \circ i_2) = -a i_1, \quad i_2^2 = -a \\
(i_3 \circ i_1) &= -(i_1 \circ i_3) = -a i_2, \quad i_3^2 = +1
\end{aligned}$$

The multiplication rule can be represented by 2×2 matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i_1 = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}, \quad i_2 = \begin{pmatrix} 0 & a \\ -1 & 0 \end{pmatrix}, \quad i_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The space $C(V)$ consists of $C^+(V)$ and $C^-(V)$ of dimension 4. $C^+(V)$ is spanned by even number of quaternions $e_S = A_{i1} \circ A_{i2}$ or $A_{i1} \circ A_{i2} \circ A_{i3} \circ A_{i4}$, and $C^-(V)$ is spanned by odd number of quaternions $e_S = A_{i1}$ or $A_{i1} \circ A_{i2} \circ A_{i3}$.

$e_S \circ e_S = \prod_{s_1,s_2 \in S}(s_1, s_2) \cdot \prod_{i \in S} A_i^2 \in k^*$, where (s_1, s_2) is a sign which is $+1$ if $s_1 \leq s_2$ and -1 if $s_1 > s_2$, and A_i^2 is the ordinary square of the basis of a vector A.

For a given $S \neq \phi, M$, there is centralizer, that satisfy for $\alpha = \sum_S \gamma_S e_S$, $e_T \circ \alpha \circ e_T^{-1} = \alpha$,

$$C_0(V) = k + ke_M$$

Pure quaternions are related to three dimensional even algebra ${Cl_3}^+$

$$\mathbf{i} \equiv -e_{23}, \quad \mathbf{j} \equiv -e_{31}, \quad \mathbf{k} \equiv -e_{12}.$$

The element e_{123} which corresponds to the Pauli $SL(2, \mathbf{C})$ matrix form

$$\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}$$

commutes with $\sigma_1, \sigma_2, \sigma_3$ and is called the center element. Center elements in $\mathbf{H}$ are

$$\{w \in \mathbf{H} | wq = qw \quad \forall q \in \mathbf{H}\}.$$

In terms of $Cl_{0,3} \simeq \mathbf{H} \times \mathbf{H}$, for $\mathbf{a}, \mathbf{b} \in \mathbf{R}^{0.3} \subset Cl_{0,3}$

$$\mathbf{a} \times \mathbf{b} = \langle \mathbf{ab}(1 - e_{123}) \rangle_1.$$

When $\mathbf{a}, \mathbf{b} \in \mathbf{R}^3 \subset \mathbf{H}$, $\mathbf{a} \times \mathbf{b}$ is imaginary part of $\mathbf{ab}$, and when $\mathbf{a}, \mathbf{b} \in \mathbf{R}^3 \subset Cl_3$, $-\langle \mathbf{ab}e_{123} \rangle_1$

Euclidean vector space $\mathbf{R}^7$ can be defined by orthnormal bases $e_1, e_2, \cdots, e_7$ with antisymmetry $e_i \times e_j = -e_j \times e_i$, and $e_i \times e_{i+1} = e_{i+3}$, where indices are permuted cyclically and translated modulo 7 [17]. Product of two vectors $\mathbf{a}, \mathbf{b}$ in $\mathbf{R}^7$ defines $\mathbf{a} \times \mathbf{b} = \frac{1}{2}(\mathbf{a} \circ \mathbf{b} - \mathbf{b} \circ \mathbf{a})$ which are called Octonions $\mathbf{O} = \mathbf{R} \oplus \mathbf{R}^7$.

By using a new imaginary unit l ($l^2 = -1$), octonion can be constructed from quaternion by the Cayley-Dickson doubling process [17], $\mathbf{O} = \mathbf{H} \oplus \mathbf{H}l$, in analogy to $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}j$.

In Clifford algebra, products of octonions $a = \alpha + \mathbf{a} \in \mathbf{R} \oplus \mathbf{R}^7$ and $b = \beta + \mathbf{b} \in \mathbf{R} \oplus \mathbf{R}^7$ is

$$a \circ b = \alpha\beta + \alpha\mathbf{b} + \mathbf{a}\beta + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$$

and in $\mathbf{R} \oplus \mathbf{R}^{0,7}$

$$a \circ b = \langle ab(1 - \mathbf{v}) \rangle_{0,1}$$

where $\mathbf{v} = e_{124} + e_{235} + e_{346} + e_{457} + e_{561} + e_{672} + e_{713}$.

In the expression of $\mathbf{R} \oplus \mathbf{R}^{0,7}$, physical parametrization of the scalar part (time of the system) can be chosen uniquely.

4. Grassmann Manifolds over Quaternion Fields

For a complex n dimensional space V, Complex Grassmanian $G_k(V)$ is the set of subspaces of complex codimension k in V, which is sometimes called $(n-k)$-plane in V [24]. It is represented as

$$G_k(V) = \frac{U(n)}{U(k) \times U(n-k)}$$

For a real n dimensional space V, real Grassmanian $G_k(V)$ is the set of subspaces of real codimension k in V. It is represented as

$$G_k(V) = \frac{O(n)}{O(k) \times O(n-k).}$$

Complex structure on $\mathbf{R}^n$ defines a linear transformation J that satisfies $J^2 = -I$. Geodesics of curves from I to $-I$ on Orthogonal group $\mathbf{O}(n)$ is homeomorphic to complex structure space $\Omega_1(n)$. [14].

- $\Omega_1(n) = \Omega\mathbf{O}(n)$ is sets of complex structures.
- $\Omega_2(n)$ is sets of quaternion strutures on $\mathbf{C}^{n/2}$ or the vector space of $\mathbf{H}^{n/4}$
- $\Omega_3(16r)$ is all subsets of $\mathbf{H}^{4r}$, or Grassman manifolds over quaternion fields. When $V = V_1 + V_2$, $dim_{\mathbf{H}} V_1 = dim_{\mathbf{H}} V_2 = 2r$.
- $\Omega_4(16r)$ is sets of isometric operators from V_1 to V_2. It is isomorphic to symplectic group $\mathbf{Sp}(2r)$
- $\Omega_5(16r)$ is sets of vector space $W \in V_1$ such that 1) W is closed in J_1 and 2) $V_1 = W_1 \oplus J_2 W$.
- $\Omega_6(16r)$ is sets of real subsets $X \in W$, such that W can be decomposed as $X \oplus J_1 X$.
- $\Omega_7(16r)$ is sets of all real Grassmann manifolds of $X \simeq \mathbf{R}^{2r}$.
- $\Omega_8(16r)$ is sets of all real isometric operators from X_1 to X_2.

Bott showed that homotopy group $\pi_i\mathbf{O}$ is isomorphic to $\pi_{i+8}\mathbf{O}$

van Baal et al [31, 32, 33, 34] defined $\bar\sigma_\mu = (\mathbf{I}, \sqrt{-1}\mathbf{i}, \sqrt{-1}\mathbf{j}, \sqrt{-1}\mathbf{k})$, $\sigma_\mu = (\mathbf{I}, -\sqrt{-1}\mathbf{i}, -\sqrt{-1}\mathbf{j}, -\sqrt{-1}\mathbf{k})$ whose products are expressed by

$$\sigma_\mu\bar\sigma_\nu = \eta^{\mu\nu}_\alpha \bar\sigma_\alpha, \quad \bar\sigma_\mu\sigma_\nu = \bar\eta^{\mu\nu}_\alpha \sigma_\alpha,$$

where $\eta^{\mu\nu}_\alpha = \bar\eta^{\mu\nu}_\alpha$ are real 't Hooft symbol [35],

$$\begin{aligned} \eta^{\mu\nu}_\alpha &= \epsilon_{\mu\nu a}, \quad \eta^{4\nu}_\alpha = -\delta^\nu_\alpha \quad (\mu, \nu, \alpha = 1, 2, 3), \\ \eta^{\mu 4}_\alpha &= \delta^\mu_\alpha, \qquad (\mu, \alpha = 1, 2, 3), \quad \eta^{44}_\alpha = 0 \end{aligned}$$

and $Q = q_\mu\bar\sigma_\mu$ are quaternions.

Dirac [8] applied quaternions to Lorentz transformations. He considered a quaternion

$$\mathbf{q} = \mathbf{I}q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$$

and a vector A_μ in space-time, whose square is

$$A_0^2 - A_1^2 - A_2^2 - A_3^2.$$

q is expressed as the ratio of quaternions $q = uv^{-1}$, and Lorentz group was defined by introducing three quantities

$$Q_1 = u\bar{v}, \quad Q_2 = u\bar{u}, \quad Q_3 = v\bar{v}$$

When u and v are replaced by λu and λv,

$$\begin{aligned} u\lambda(\bar{\lambda v}) &= u\lambda\bar{\lambda}\bar{v} = Q_1\lambda\bar{\lambda} \\ Q_1 &= X_0 + \mathbf{i}X_1 + \mathbf{j}X_2 + \mathbf{k}X_3 \end{aligned} \tag{1}$$

Q_2 and Q_3 were put

$$Q_2 = X_4 - X_0, \quad Q_3 = X_4 + X_5$$

Dirac showed that $\xi_\nu = X_\nu/X_0$ and $\eta_\nu = X_\nu/X_5$ $(\nu = 1, 2, 3, 4)$ transform like space-time vectors. For $f = (1/2)(1 + \mathbf{I})$, the spinorial basis is defined by fX_0 and fX_5 [19].

Lorentz transformation was performed by

$$q^* = (aq \pm \mu a)(-\mu aq \pm a)^{-1}$$

where a is an arbitrary quaternion and μ is a pure imaginary quaternion.

Quaternions represent rotation of 3-dimensional sphere $\mathbf{S}^3$. The manifold $\mathbf{S}^3 \times \mathbf{R}$, where $\mathbf{R}$ represents periodic and anti-periodic time variables is ihomeomorphic to T^4. Applications of quaternions as rotation operator are discussed in [36, 37, 38]. We see in the next section that the manifold of $\mathbf{S}^1 \times \mathbf{S}^3$ has different Riemann surface structure than that of T^4.

5. Holonomy Group and Quaternion

When self-orthogonal subspace of E is H and the Lagrangean form is defined by σ, $ker(\sigma) \subset H$, the base of $ker(\sigma)$ is written as T and the base of H is written as $[T, U]$. In this case the transformation matrix considered in section 2, $S = [U_1, \cdots, U_p, V_1, \cdots, V_p]$ changes to $S = [T_1, \cdots, T_q, U_1, \cdots, U_p, V_1, \cdots, V_p]$ and the matrix of σ components becomes

$$J = \left(\begin{array}{ccc|ccc|ccc} & & & & & & & & \\ & 0 & & & 0 & & & 0 & \\ & & & & & & & & \\ \hline & & & & & & 1 & & \\ & 0 & & & 0 & & & \ddots & \\ & & & & & & & & 1 \\ \hline & & & -1 & & & & & \\ & 0 & & & \ddots & & & 0 & \\ & & & & & -1 & & & \end{array}\right)$$

where we assume $q = dim(ker(\sigma)); q{+}p = dim(H); q{+}2p = n$. It is possible to choose another base $S' = [T, U, V']$ of E and matrices $M'_{kj} = \sigma(V'_k)(U_j)$, and $V'' = V' \cdot M^{-1}$.

V'' is a new base of V' that satisfy

$$\sigma(U_j)(V''_k) = \begin{cases} 1 & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k \end{cases}$$

One takes

$$V_k = V''_k + \frac{1}{2}\sum_j U_j \sigma(V''_j)(V''_k) \quad k = 1, 2 \cdots p.$$

The space E is called pre-symplectic.

We consider complex manifolds $M = \mathbf{C}^4$, $S = (0,0,0,0)$ and their submanifold without singular points $\bar{S} = \mathbf{P}^3$, which is complex projective space covered by $u_j \in \mathbf{C}^3 : \mathbf{P}^3 = \sum_{j=0}^{3} u_j$. Points ζ on $\mathbf{P}^3$ and t on $\mathbf{R}$ define maps

$$\Phi : (t, \zeta_1.\zeta_2) \to (z_1, z_2) = (\zeta_1 e^{t\beta_1}, \zeta_2 e^{f\beta_2}),$$

which is 1 to 1 mapping from $\mathbf{R} \times \mathbf{S}^3 \to W$ defined by z_1, z_2. For maps $C^* : (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \to (g\zeta_1, g\zeta_2, g\zeta_3, g\zeta_4)$, $W/C^* = \mathbf{P}^3$ and $g^m : (t, \zeta_1, \zeta_2) \to (t + m, \zeta_1, \zeta_2), m \in \mathbf{Z}$. Therefore $W/G = \mathbf{R}/\mathbf{Z} \times \mathbf{S}^3 = \mathbf{S}^1 \times \mathbf{S}^3$ [13].

A Lagrangian system with linear constraints are holonomic or non-holonomic [39] if imposed constraints are integrable or not. Condition is characterized by the parallel transformation of the solution curves in complex plane or quaternion tangent plane.

Consider a group G_n which is transitive on the unit sphere S, there is the map

$$p_1: \quad G_n \to S$$

Let M_n be the projective space associated with vector fields Q_n. When non-zero elements of Q_n, x and y have relation $y = xq$, $(q \in Q)$, they belong to equivalence classes, and there is a projection

$$p: \quad S \to M_n$$

If H is the subgroup of G_n which has a fixed point $z_0 = p(x_0)$ and there is a map

$$p_2: \quad G_n \to M_n, \quad M_n = G_n/H,$$

then H can be expressed by the direct product $H = Q' \times G_{n-1}$, where Q' is the $\mathbf{S}^3$, and vector fields consist of quaternions of $|q| = 1$ [3].

Choosing a complex analytic space $W = \mathbf{C}^2 - \{0\}$ and mappings

$$g: z(z_1, z_2) \to g(z) = (\alpha_1 z_1 \alpha_2 z_2)$$

and $G = g^m|_{m\in\mathbf{Z}}$, where $|\alpha_1| > 1, |\alpha_2| > 1$ are constants. The manifold W/G is called Hopf manifold, which is diffeomorphic to $\mathbf{S}^1 \times \mathbf{S}^3$.

In the real case $\mathbf{S}^3 \to \mathbf{R}P^3$ mapping is possible, and analogously in the complex case $\mathbf{S}^1 \to \mathbf{S}^3 \to \mathbf{C}P^3$ is possible [48]. Since $\mathbf{C}P^1 = \mathbf{S}^2$, $\mathbf{S}^3$ is a complete image of $\mathbf{S}^2$, and the map $\mathbf{S}^3 \to \mathbf{S}^2$ is called the Hopf map [3, 47, 48].

Kodaira [13] and Kodaira and Spencer [22] used complex projective line $\mathbf{P}^1 = \mathbf{C} \cup \infty = U_1 \cup U_2$, where $U_1 = \mathbf{C}$, $U_2 = \mathbf{P}^1 - \{0\}$, and defined complex analytical functions in

$$M_t = U_1 \times \mathbf{P}^1 \cup U_2 \times \mathbf{P}^1, \quad t \in \mathbf{C}.$$

When $(z_1, \zeta_1) \in U_1 \times \mathbf{P}^1$ and $(z_2, \zeta_2) \in U_2 \times \mathbf{P}^1$ satisfy

$$z_1 \cdot z_2 = 1, \quad \zeta_1 = z_2^2 \zeta_2 + t z_2$$

the two points are regarded as identical.

He used conformal mappings of

$$W_j = \{(z_j, s)||z_j| < 1, |s| < 1\} = U_j \times D$$

where $D = \{s \in \mathbf{C}||s| < 1\}$, and considered when the condition

$$z_j = f_{jk}(z_k, s) = (f^1_{jk}(z_k, t), f^2_{jk}(z_k, t))$$

is satisfied (z_j, s) and (z_k, s) are identical in the projected space.

He showed that dimention of cohomology $dim H^1(M_t, \Theta_t)$, where Θ_t is the sheaf of germs of conformal tangent vector space, is 1 for $t = 0$ and 0 for $t \neq 0$. The discontinuity suggests validity of the difference of pre-symplectic structure of quantum dynaics and symplectic structure of classical dynamics. General summary of works of Kodaira and Spencer are given in [23].

Using $\phi : A \otimes A \to A$ and its dual ψ, Hopf algebra (A, ϕ, ψ) defines algebra of complex projective spaces. $\mathbf{P}^1(\mathbf{C})$ consists of $\zeta = (1, z_1) = (1, \zeta_1/\zeta_0)$, $\zeta_0 \neq 0$. Mapping from $\mathbf{P}^1(\mathbf{C})$ to a 3−dimesional projective plane $\mathbf{P}^3(\mathbf{C})$: $\mathbf{P}^1 \to \mathbf{P}^3$ were expressed by Cavalieri and Miles [18] by

$$\phi([S : T]) = [S^3 : S^2T : ST^2 : T^3] \equiv [X : Y : Z : W]$$

which defines a twisted cubic $V(\mathbf{P})$ in $\mathbf{P}^3$.

Let P_1, P_2 be homogenious polynomials in 4 variables $P_1 = XW - YZ, P_2 = XZ - Y^2$, for example, and define

$$V(\mathbf{P}) = \{[X, Y, Z, W]|P_1(X, Y, Z, W) = P_2(X, Y, Z, W) = 0\} \subset \mathbf{P}^3(\mathbf{C}).$$

For any point p on a three dimensional real sphare $\mathbf{S}^3 \subset \mathbf{C}^2$, there is a unique complex line l_p, or there is a function H such that

$$\begin{aligned} H : \quad & \mathbf{S}^3 \to \mathbf{P}^1(\mathbf{C}) \\ & P \to l_P. \end{aligned}$$

has the image of Riemann surface isomorphic to $\mathbf{P}(\mathbf{C})$ which is called a twisted circle in $\mathbf{P}^3(\mathbf{C})$.

In [18], locus of $P_3 = YW - Z^2$ is also considered and the vanishing locus of any two of the three polynomials is strictly larger than the rest polynomial.

When we replace $\mathbf{C}$ to $\mathbf{H}$ and consider the mapping $\mathbf{S}^3 \to \mathbf{S}^7 \to \mathbf{S^4} = \mathbf{H}P^1$ [48]

$$\begin{array}{l}
\mathbf{C}^4 \setminus \{(0,0,0,0)\} \xrightarrow{\mathrm{P}} \mathbf{C}j \\
\downarrow \pi \\
\mathbf{P}^3(\mathbf{C}j)
\end{array}$$

where $\mathbf{C}j$ is $Pu(\mathbf{H})$, we obtain

$$\begin{aligned}
P_1 &= S^3T^3 - S^2TST^2 = ST - TS = 2ST, \\
P_2 &= S^3ST^2 - S^2TS^2T = S^2 - T^2, \\
P_3 &= S^2TT^3 - ST^2ST^2 = T^2 - S^2,
\end{aligned}$$

in which $TS = -ST, S^2 = -1$ and $T^2 = -1$ are used. Locus of P_2 and P_3 give no additional conditions on twisted cubic in $\mathbf{P}^3(\mathbf{C})$, but that of P_1 defines a plane.

Mathematically, structure of complex manifolds of complex dimension 2, $\mathbf{C}^2 \sim \mathbf{R}^4$ was studied by Russian groups [41, 42] and Japanese groups. Kodaira [21, 13] showed that the $\mathbf{S}^1 \times \mathbf{S}^3$, where $\mathbf{S}^1$ represents holonomy group of quaternion variables defined by $\mathbf{S}^3$, is homeomorphic to Hopf space and have 2 linearly independent vectors.

Rotations of $\mathbf{S}^3$ is expressed by quaternions, whose operations are given by symplectic groups. Quaternions can be expressed by Pauli matrices which follow orthogonal groups $O(3)$, which allowes rotations of $\mathbf{S}^2$ locally. However the local infinitesimal quantomorphism in [4] is too restrictive, when one patches conformally transformed curves globally.

Motions of particles in space-time represented by Lagrangian $L(t, \mathbf{q}, \dot{\mathbf{q}})$ are related by holonomy curves, which characterize the structure of the manifold on which curves are defined. In case of magnetic interactions the magnetic field $\mathbf{B}$ whose coordinate is chosen along the $y-$axis, is a function of the magnetizing field $\mathbf{H} = \mathbf{B} - \mathbf{M}/\epsilon_0 c^2$ whose coordinate is chosen along the $x-$axis, shows hysteresis curves, which indicates that nonholonomic effects appear in Nature.

In Hamiltonian formulation of classical mechanics, change of coordinate from $(\mathbf{p}, \mathbf{q}, t)$,

$$z_k = (p_k + iq^k)/\sqrt{2}, \quad \bar{z}_k = (p_k - iq^k)/\sqrt{2}$$

defines a 2-form $\omega = -i \sum_k d\bar{z}_k \wedge dz_k$.

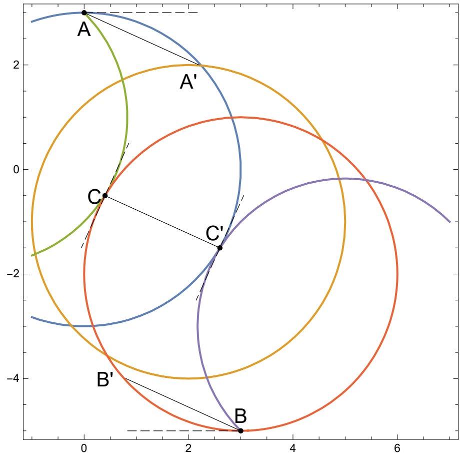

Figure 1. Holonomy curve $AC'B + BCA$, and $(CB')(BC') + (C'A')(AC)$. The parallel transformation of the former curve have the direction parallel to the x axis, as shown by dashed lines from A and B. The parallel transformations of the latter curve have the direction parallel to CC', from A and B respectively, as AA' and BB'. The system is holonomic if quaternion vectors e_x and $e_{CC'}$ are parallel. Curves on the manifold of $\mathbf{S}^1 \times \mathbf{S}^3$ is non-holonomic, since one $\mathbf{P}(\mathbf{C})$ of $\mathbf{P}^3(\mathbf{C})$ can be interchanged with $\mathbf{P}(\mathbf{C})$ of $\mathbf{S}^1$ in complex projective spaces. Figure copied from the reference [43].

The assumption of maximum conformality allows to express the solution by patching local solutions of evolution equations.

Extension to $K(\mathbf{P}, \mathbf{Q}, t) = H(\mathbf{p}, \mathbf{q}, t) + \frac{\partial S}{\partial t}$ is performed by choosing

$$\begin{aligned} S(\mathbf{p}_1, \mathbf{q}_1, t) &= \int_{(\mathbf{p}_0,\mathbf{q}_0,t)}^{(\mathbf{p}_1,\mathbf{q}_1,t)} \mathbf{p}d\mathbf{q} - \int_{(\mathbf{p}'_0,\mathbf{q}'_0,t)}^{(\mathbf{p}'_1,\mathbf{q}'_1,t)} \mathbf{P}d\mathbf{Q} \\ &= \int_G dS - \int_{G'} dS \neq 0 \end{aligned}$$

when there are hysteresis. In Fig.2 CC' is the parallel transformation by quaternion $e_{CC'}$. On the curve G, line-integral $\int_G dS$ becomes 0 due to symplectic structure of the bases, while on the curve G', although $ddS = 0$, the Stokes

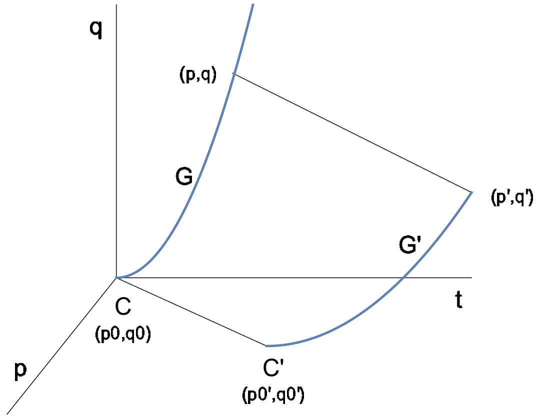

Figure 2. Lowering the order of a Hamiltonian system. The curve in space-time G' from (p'_0, q'_0) to (p', q') in pre-symplectic form is projected to the curve G from (p_0, q_0) to (p, q) in symplectic form. (p_0, q_0) is obtained from (p'_0, q'_0) by a parallel transformation, and (p, q) is obtained from (p', q') by the parallel transformation of the $Pu(\mathbf{H}) : e_{\mathbf{H}}$. Actions along G' and along G are not the same, when the system is non-holonomic.

theorem says for an area σ surrounded by $(CB'BC') + (C'A'AC)$ that

$$\int_{(CB'BC')+(C'A'AC)} \mathbf{P}d\mathbf{Q} - Hdt = \int_{\partial\sigma} \mathbf{P}d\mathbf{Q} - Hdt = \int_{\sigma} d(\mathbf{P}d\mathbf{Q} - Hdt)$$

is not necessary 0.

The Hamiltonian $K(\mathbf{P}, \mathbf{Q}, t) = H(\mathbf{p}, \mathbf{q}, t)$ satisfies the canonical form

$$\frac{d\mathbf{P}}{dt} = -\frac{\partial K}{\partial \mathbf{Q}}, \quad \frac{d\mathbf{Q}}{dt} = \frac{\partial K}{\partial \mathbf{P}}.$$

Arnoldt assumes that the equation $h = H(p_1, \cdots, p_n, q_1, \cdots, q_n)$ can be solved for $p_1 = K(\mathbf{P}, \mathbf{Q}, T; h)$, where $\mathbf{P} = (p_2, \cdots, p_n)$, $\mathbf{Q} = (q_2, \cdots, q_n)$, $T = -q_1$. Then [39]

$$\mathbf{p}d\mathbf{q} - Hdt = \mathbf{P}d\mathbf{Q} - KdT - d(Ht) + tdH.$$

For a physically distinguished configuration space Y and Hamiltonian vector fields ξ_f, ξ_g,

$$[\xi_f, \xi_g] = -\xi_{[f,g]},$$

and the Lagrange bracket is $\omega = d\theta_Y$. θ_Y is canonical 1-form [4, 5]. In terms of local chart $(\pi^{-1}(U), q^1. \cdots, q^4, p_1, \cdots .p_4)$, the symplectic form for a charge e and pullback 2-form f, $\omega_e = d\theta_Y + e\pi^* f$ is

$$\omega_e|_{\pi^{-1}(U)} = \sum_{i=1}^{4} dp_i \wedge dq^i + e/2 \sum_{ij} (f_{ij} \circ \pi) dq^i \wedge dq^j$$

6. Quaternions in Non-Commutative Geometry and Quantum Groups

Serre [15] defined a group G as a union of subgroup of 8 elements

$$E = \pm\mathbf{I}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k} \tag{2}$$

and 16 elements

$$(\pm\mathbf{I} \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})/2, \tag{3}$$

where $\mathbf{I}, \mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are bases of quaternions. G is called an entire quaternion of Hurwitz [15]. Relations between Hurwitz number and Riemann space are written in [18].

A group G is called resoluble, if there is a series

$$\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$$

and G_{i-1} different from G_i and G_i/G_{i-1} is commutative for $1 \leq i \leq n$. Similarly G is called hyper-resoluble if G_i/G_{i-1} is cyclic.

The group G is invertible with quaternion ring, and is isomorphic to the special linear transformation $SL(2, \mathbf{F}_3)$, where $\mathbf{F}_3$ are free group on 3 elements [15]. Tensor products of quaternions $\mathbf{H}$ and complex numbers $\mathbf{C}$

$$\mathbf{H} \otimes_{\mathbf{R}} \mathbf{C} = \mathbf{M}_2(\mathbf{C})$$

define the group G.

This construction has an application in quantum groups, which are defined on algebraic Hopf group., elements of which are not necessarily commutative/noncommutative [49]. A representation of Hopf algebra (π, V) consists of a vector space V and algebraic endomorphism $\pi : A \to End(V)$. The algebra A contains coproduct

$$\Delta : A \to A \otimes A$$

and coalgebra morphism

$$\epsilon : A \to \mathbf{C}.$$

Universal enveloping algebra $U_q(sl(2,\mathbf{C}))$ of Jimbou [49] and Drinfeld [50] contains generators X^+, X^-, K, K^{-1}, where $X^+ = Max(X,0)$ and $X^- = Max(-X,0)$ are Riesz space variables [4], with the following relations

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1 \\ KX^{\pm}K^{-1} &= q^{\pm 2}X^{\pm} \\ ([X^+, X^-]) &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

There are algebraic morphism from X^+, X^-, K, K^{-1} of $U_q(sl(2,\mathbf{C}))$ to $X_i^+, X_i^-, K_i, K_i^{-1}$ $(1 \le i \le l)$ of $U_q(\hat{sl}(2,\mathbf{C}))$ that satisfy [49]

$$\begin{aligned} \phi({X_0}^{\pm}) &= X^{\mp}, \quad \phi({X_1}^{\pm}) = X^{\pm} \\ \phi(K_0) &= K^{-1}, \quad \phi(K_1) = K \end{aligned}$$

$$\begin{aligned} K_i K_i^{-1} &= K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i \\ K_i {X_j}^{\pm} {K_i}^{-1} &= {q_i}^{\pm a_{ij}} {X_j}^{\pm} \\ ([{X_i}^+, {X_j}^-]) &= \delta_{ij}\frac{K_i - {K_i}^{-1}}{q_i - {q_i}^{-1}} \end{aligned}$$

where a_{ij} is the element of the General Cartan Matrix. Derivation of $U_q(\mathcal{G})$ from $U_q(\hat{\mathcal{G}})$ is satisfied for a_{ij} of $\hat{\mathcal{G}} \in A_l$ type Lie group [49].

One can construct $SL(2,\mathbf{F}_3)$ from $\{K_0, K_1, X_1^{\pm}\}$ using quaternions, but ϕ does not preserve coproducts. One can define automorphism T_λ $(\lambda \in \mathbf{C})$ such that

$$T_\lambda({X_0}^{\pm}) = \lambda^{\pm}X_0^{\pm}, \quad T_\lambda(X) = X \quad (X = X_1^{\pm}, K_0, K_1)$$

and derive representations of $U_q(sl(2,\mathbf{C}))$ depending on the parameter λ [49]. Hence quaternions define a morphism of Hopf algebra which is dependent on a parameter λ.

Differential calculus on the real quantum plane and mapping of quantum groups on non-commutative differential manifolds did not show qualitative difference of Stokes' theorem from that of classical groups [51]. An extension to complex quantum plane was discussed in [52].

7. Summary and Discussion

We started from 2-form in symplectic space and from invariance of Lagrangian 2-forms, derived linear combination of vectors with Pauli $SL(2,\mathbf{C})$ matrices as bases and obtained pure quaternions. From Clifford algebra with quadratic forms, we obtained generalized quaternions, and found that in addition to real numbers $\mathbf{R}$ and complex numbers $\mathbf{C}$, quaternions $\mathbf{H}$ form finite division algebra.

In variational analysis of dynamics, centralizer of dynamical transformations is important, and dynamics follows the $SO(3)$ transformation, parallel transformations by quaternion vectors play important role in non-holonomic systems. In standard holonomic systems, parallel transformations by real vectors are considered.

Guillou and Marin [40] explained the work of Rohlin [41] and Pontryagin [42]. Pontryagin showed in mappings of $(n+2)-$dimensional sphere Σ^{n+2} into an $n-$dimensional sphere $\mathbf{S}^n$, there are exactly two classes of mappings, when $n \geq 2$.

The theorem can be applied to the existence of holonomy curves α : $ACB \oplus BC'A$ and $\beta : ACB' \oplus BC'A'$, shown in the Figure 1. (B and B' are identified by the parallel transformation and A and A' are identified similarly.) The index of intersection [42] $J(\alpha_i, \alpha_j) = J(\beta_i, \beta_j) = 0$, but $J(\alpha_i, \beta_j) = \delta_{ij}$ where i, j for the curve α indicates indices of the set $\{AC, CB, BC', C'A\}$ and for the curve β indicates indices of the set $\{AC, CB', BC', C'A'\}$. The pre-symplectic structure of the vector space E, which allows presence of different kernel space of parallel transformations is essential for the quantum dynamics, while classical dynamics can be understood by symplectic structure of the vector space.

On the manifold of $\mathbf{S}^1 \times \mathbf{S}^3$, we consider exactly two mappings from the sphere Σ^5 into $\mathbf{S}^3$ [42], where $\mathbf{S}^3$ is defined by complex variables

$$(z_1, z_2 | z_1\bar{z}_1 + z_2\bar{z}_2 = 1)$$

which has the same number of degrees of freedom as that of quaternions, and $\mathbf{S}^1$ of quaternions defines a group $\mathbf{g} \in G$, satisfying $\psi_g(\mathbf{x}) = \mathbf{g}\mathbf{x}\mathbf{g}^{-1}$ is chosen to satisfy $\psi_g(\mathbf{i}) = \mathbf{i}$, or $\mathbf{g} = \cos\theta + \mathbf{i}\sin\theta$.

The pre-symplectic form vector space is important in propagation of solitons in hysteretic media [44]. When one discretizes time in evolution equations, one obtains recursion operator. In the case of viscous Burger's equation [45]

$$u_t + u\partial u/\partial x = \mu\partial^2 u/\partial x^2; \quad t \geq 0; \quad u(0,x) = u_0(x); \quad \mu > 0,$$

the equation can be written as

$$\partial u/\partial t + \partial f(u)/\partial x = 0, \quad u(0,x) = u_0(x),$$

whose discretized form is

$$u_n^{k+1} - (u_{n+1}^k + u_{n-1}^k)/2)/h + (f(u_{n+1}^k) - f(u_{n-1}^k))/2l = 0$$

where grid points of half plane $\bar{\mathbf{R}}_+ \times \mathbf{R}$ are $\{(kh, nl); k, n \in \mathbf{Z}, k \geq 0\}$. The first term contains a term

$$(2u_n^k - u_{n+1}^k - u_{n-1}^k)/2h.$$

Discretization in time affects symplectic structure of the vector space of solutions.

Dosch et al [53, 54, 55] studied spectrum of mesons and baryons in supersymmetric light front QCD embedded in AdS_5/CFT_4 space. Conformal algebra alows patching analytical solutions of different times, and the mass spectra of mesons and baryons show specific proportionalities.

Understanding quaternions is relevant to understanding non-commutative geometry, and via Lie algebra, it is related to understanding Heisenberg's hamiltonian and quantum groups.

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Chapter 3

SOLUTIONS WITH SPHERICAL SYMMETRY OF THE EQUATION FOR A SPIN 3/2 PARTICLE

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Abstract

The wave equation for a spin 3/2 particle, described by 16-component vector-bispinor, is investigated in spherical coordinates. In the frame of Pauli–Fierz approach, the complete equation is split into the main equation, and two additional constraints, algebraic and differential ones. There are constructed solutions on which 4 operators are diaginalized: energy, square and third projection of the total angular momentum, and spacial reflection, they correspond to quantum numbers $\{\epsilon, j, m, P\}$. After separating the variables, we derive the main system of 8 radial first-order equations and additional 2 algebraic and 2 differential constraints. Solutions of the radial equations are constructed as linear combinations of Bessel functions. With the use of the known properties of the Bessel functions, the system of differential equations is transformed to the form of purely algebraic equations with respect to three quantities a_1, a_2, a_3. Its solutions may be chosen in various ways by resolving the simple linear condition $A_1 a_1 + A_2 a_2 + A_3 a_3 = 0$, where coefficients A_i are expressed trough the quantum numbers ϵ, j. Any two linearly independent sets a_1, a_2, a_3 determine quantum states (1) and (2) of the spin 3/2 particle. Two most

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simple and symmetric solutions $a_i^{(1)}$ and $a_i^{(2)}$ have been chosen. Thus, at fixed quantum numbers $\{\epsilon, j, m, P\}$ there exists double-degeneration of the quantum states. Explicit form of the operator associated with such a degeneration is not found.

Keywords: spin 3/2 particle, degrees of freedom, spherical symmetry, exact solutions, Bessel functions, degeneration of quantum states

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1. INTRODUCTION

The theory of spin 3/2 particle is attracted steady interest after the seminal investigation by Pauli and Fierz – see [1–14]. Let us recall the most significant aspect of spin 3/2 particle theory. First of all, it is the problem of choosing an initial system of equations. The most consistent is an approach based on Lagrangian formalism and a correct first order equation for multi-component wave function which are based on the general theory of 1-st order relativistic wave equations.

However investigations are based on the use of 2-nd order equations. Such an choice is of prime importance when we take into account the present of external electromagnetic (or gravitational) fields. Applying the first order approach ensures correct solving the problem of independent degrees of freedom in presence of external fields; for instance see in [15].

The great attention was given to existence in this theory solutions which correspond of a particle moving with velocity greater than the light velocity. Finally a separate interest has a massless case for spin 3/2 field, when – as shown by Pauli and Fierz – there exists specific gauge symmetry: the 4-gradient of arbitrary bispinor function $\Phi(x)$ provides us with solution for the massless field equation (for instance, see in [15]).

In the present paper we examine the problem of degree of freedom for a massive spin 3/2 particle specified for solutions with spherical symmetry. For simplicity we restrict ourselves to Minkowski space-time model.

2. System of Equations and Spherical Symmetry

The wave equation for such a particle may be presented in the form (we assume the use of the tetrad formalism, see [15])

$$\begin{gathered}[\, i\gamma^{\beta}(x)(\nabla_{\beta}+\Gamma_{\beta}(x))-m\,]\,\Psi_{\alpha}=0\,,\\ \gamma^{\alpha}(x)\Psi_{\alpha}=0\,,\;\;(\nabla_{\alpha}+\Gamma_{\alpha})\Psi^{\alpha}=0\,.\end{gathered} \tag{1}$$

The mass parameter is designated as $m=\frac{Mc}{\hbar}$; the wave function $\Psi_{\alpha}(x)$ behaves as a bispinor with respect to tetrad transformations, and as a generally covariant vector with respect to coordinate transformations; note the notation $\gamma^{\beta}(x)=e^{\beta}_{(a)}(x)\gamma^{a}$, the symbol $\Gamma_{\beta}(x)$ designates the bispinor connection

$$\Gamma_{\beta}(x)=\frac{1}{2}(\sigma^{ab})_{k}^{\;l}e^{\beta}_{(a)}(\nabla_{\alpha}e_{(b)\beta})\,. \tag{2}$$

Below, it will be convenient to use the wave function with tetrad-vector index $\Psi_{l}(x)$, it relates to the previous function $\Psi_{\alpha}(x)$ in accordance with the rule

$$\Psi_{l}(x)=e^{\beta}_{(l)}(x)\Psi_{\beta}(x),\quad \Psi_{\beta}(x)=e^{(l)}_{\beta}(x)\Psi_{l}(x). \tag{3}$$

With (3) in mind, the first equation in (1) is transformed to the form

$$\{i\gamma^{\alpha}[(\partial_{\alpha}+\Gamma_{\alpha})\delta^{l}_{k}+e^{\beta}_{(k)}e^{(l)}_{\beta;\alpha}]-m\delta^{l}_{k}\}\Psi_{l}=0 \tag{4}$$

(the symbol $;\alpha$ in $e^{(l)}_{\beta;\alpha}$ stands for covariant derivative ∇_{α}), where

$$e^{\beta}_{(k)}e^{(l)}_{\beta;\alpha}=(L_{\alpha})_{k}^{\;l}=\frac{1}{2}(j^{ab})_{k}^{\;l}e^{\beta}_{(a)}(\nabla_{\alpha}e_{(b)\beta})\,,(j^{ab})_{k}^{\;l}=\delta^{a}_{k}\;g^{bl}-\delta^{b}_{k}g^{al}\,. \tag{5}$$

Generators J^{ab} for spin-vector representation are $J^{ab}=\sigma^{ab}\otimes I+I\otimes j^{ab}$. Taking in mind (5), equation (4) is transformed to the form

$$[i\gamma^{\alpha}(x)(\partial_{\alpha}+B_{\alpha})-m]\,\Psi)=0\,,B_{\alpha}=\Gamma_{\alpha}\otimes I+I\otimes L_{\alpha}\,. \tag{6}$$

While using the wave function $\Psi_{l}(x)$ (see (3)), additional constraints in (1) read as follows:

$$\gamma^{l}\Psi_{l}=0\,,\quad [e^{(l)\alpha}\partial_{\alpha}+e^{(l)\alpha})+e^{(l)\alpha}\Gamma_{\alpha}\,]\,\Psi_{l}=0\,. \tag{7}$$

We will specify equations in spherical coordinates $x^\alpha = (t,\ r,\ \theta,\ \phi)$:

$$dS^2 = dt^2\ -\ dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{8}$$

Taking the spherical tetrad as in [16], for $\gamma^\alpha(x)$ and $B_\alpha(x)$ we get expressions (we write down only non-vanishing connections)

$$\begin{aligned} \gamma^\alpha &= (\gamma^0, \gamma^3, \gamma^1/r, \gamma^2/r\sin\theta)\,, \\ \Gamma_\theta = \sigma^{31}, &\ \ \Gamma_\phi = \sin\theta\sigma^{32} + \cos\theta\sigma^{12}, \\ L_\theta = j^{31}, &\ \ L_\phi = \sin\theta j^{32} + \cos\theta j^{12}, \\ B_\theta = J^{31}, &\ \ B_\phi = \sin\theta J^{32} + \cos\theta J^{12}. \end{aligned} \tag{9}$$

Correspondingly, Eq. (6) takes the form

$$\left[i\gamma^0\partial_0 + i\gamma^3\partial_r + \frac{i\gamma^1 J^{31} + i\gamma^2 J^{32}}{r} + \frac{1}{r}\Sigma_{\theta,\phi} - m \right] \Psi = 0\ . \tag{10}$$

The (θ, ϕ)-dependent operator $\Sigma_{\theta,\phi}$ is determined by relation

$$\Sigma_{\theta,\phi} = i\gamma^1\partial_\theta + \gamma^2 \frac{i\partial_\phi + (i\sigma^{12}\otimes I + I\otimes ij^{12})\cos\theta}{\sin\theta}\ . \tag{11}$$

Having in mind identity

$$\frac{i\gamma^1 J^{31} + i\gamma^2 J^{32}}{r} = \frac{i\gamma^3}{r} + \frac{\gamma^1\otimes T^2 - \gamma^2\otimes T^1}{r},$$

we change the above equation to the form

$$\left[i\gamma^0\partial_0 + i\gamma^3(\partial_r + \frac{1}{r}) + \frac{\gamma^1\otimes T^2 - \gamma^2\otimes T^1}{r} + \frac{1}{r}\Sigma_{\theta,\phi} - m \right] \Psi = 0\ . \tag{12}$$

Searching solutions of Eq. (12) in the spherically symmetric form, we are to diagonalize operators of the square and third projection of the total angular momentum. In Cartesian basis, the components of the total angular momentum are defined as follows

$$J_i^{Cart} = l_i\ +\ S_i\ , \quad S_1 = iJ^{23}\ , \quad S_2 = iJ^{31}\ , \quad S_3 = iJ^{12}\ . \tag{13}$$

Using spinor representation for Dirac matrices and taking in mind the above definition for generators (5), we obtain

$$S_i = \frac{1}{2}\,\Sigma_i\otimes I\ +\ I\otimes T_i\ , \qquad T_i = \left| \begin{array}{cc} 0 & 0 \\ 0 & \tau_i \end{array} \right|. \tag{14}$$

Correspondingly, the wave function has two indices, of bispinor A and of 4-vector (l):

$$\Psi_{A(l)} = \begin{vmatrix} \Psi_{1(0)} & \Psi_{1(1)} & \Psi_{1(2)} & \Psi_{1(3)} \\ \Psi_{2(0)} & \Psi_{2(1)} & \Psi_{2(2)} & \Psi_{2(3)} \\ \Psi_{3(0)} & \Psi_{3(1)} & \Psi_{3(2)} & \Psi_{3(3)} \\ \Psi_{4(0)} & \Psi_{4(1)} & \Psi_{4(2)} & \Psi_{4(3)} \end{vmatrix} . \tag{15}$$

The wave functions for the particle in Cartesian and spherical tetrads are related by (θ, ϕ)-dependent gauge transformation:

$$\Psi^{sph} = S\ \Psi^{Cart}\ , \quad S = \begin{vmatrix} U_2 & 0 \\ 0 & U_2 \end{vmatrix} \otimes \begin{vmatrix} 1 & 0 \\ 0 & O_3 \end{vmatrix} , \tag{16}$$

where $U_2(\theta, \phi)$ and $O_3(\theta, \phi)$ are as shown below

$$U_2 = \begin{vmatrix} \cos\theta/2\ e^{i\phi/2} & \sin\theta/2\ e^{-i\phi/2} \\ -\sin\theta/2\ e^{i\phi/2} & \cos\theta/2\ e^{-i\phi/2} \end{vmatrix} ,$$

$$(L_a{}^b) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta\ \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & \sin\theta\ \cos\phi & \sin\theta\ \sin\phi & \cos\theta \end{vmatrix} . \tag{17}$$

After performing needed calculation, $J_i^{Cart} = l_i + S_i\ ,\ J_i = S\ J_i^{Cart}\ S^{-1}$, for operators J_i in spherical tetrad basis we obtain [16] other expressions

$$J_1 = l_1 + S_3 \frac{\sin\phi}{\sin\theta}\ , \quad J_2 = l_2 + S_3 \frac{\cos\phi}{\sin\theta}\ , \quad J_3 = l_3 = -i\frac{\partial}{\partial\phi}\ . \tag{18}$$

The problem of searching eigenvectors of two operators $\vec{J}^2$ and J_3:

$$\vec{J}^2\ \psi(\theta, \phi) = j(j+1)\ \psi(\theta, \phi)\ , \quad J_3\ \psi(\theta, \phi) = m\ \psi(\theta, \phi) \tag{19}$$

(the eigenvalues m of J_3 should not be confused with the mass parameter $m = \frac{Mc}{\hbar}$) reduces to combining them in terms of Wigner's D-functions [17]. To this end, it is convenient to operate with a diagonal matrix $S_3 = ij^{12}$, however we have

$$S_3 = \frac{1}{2} \begin{vmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \otimes I\ +\ I \otimes \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} .$$

So we make transformation to cyclic basis (in vector space) $\tilde{\Psi} = (I \otimes U)\ \Psi$:

$$\left| \begin{array}{c} \tilde{\Psi}_{(0)} \\ \tilde{\Psi}_{(1)} \\ \tilde{\Psi}_{(2)} \\ \tilde{\Psi}_{(3)} \end{array} \right| = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1/\sqrt{2} & i/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & +1/\sqrt{2} & i/\sqrt{2} & 0 \end{array} \right| \left| \begin{array}{c} \Psi_{(0)} \\ \Psi_{(1)} \\ \Psi_{(2)} \\ \Psi_{(3)} \end{array} \right| , \qquad (20)$$

$$\left| \begin{array}{c} \Psi_{(0)} \\ \Psi_{(1)} \\ \Psi_{(2)} \\ \Psi_{(3)} \end{array} \right| = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 0 & 1 & 0 \end{array} \right| \left| \begin{array}{c} \tilde{\Psi}_{(0)} \\ \tilde{\Psi}_{(1)} \\ \tilde{\Psi}_{(2)} \\ \tilde{\Psi}_{(3)} \end{array} \right| . \qquad (21)$$

This results in

$$\tilde{S}_3 = \frac{1}{2} \left| \begin{array}{cccc} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right| \otimes I \ + \ I \otimes \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right| . \qquad (22)$$

In cyclic basis, the main equation (6) formally preserves its form

$$\Big[i\gamma^{\alpha}(x)(\partial_{\alpha} + \Gamma_{\alpha}(x) \otimes I + I \otimes \tilde{L}_{\alpha}(x)) - m \Big]\, \tilde{\Psi}(x) = 0 \, . \qquad (23)$$

Additional constraints in the basis $\tilde{\Psi}_l(x)$ read as follows

$$\gamma^{l}(U^{-1})_{lk} \tilde{\Psi}_k(x) = 0 \, , \qquad (24)$$

$$\Big[e^{(l)\alpha}\partial_{\alpha} + e^{(l)\alpha}(x) + e^{(l)\alpha}(x)\Gamma_{\alpha}(x) \Big]\, U^{-1}_{lk} \tilde{\Psi}_k(x) = 0 \, . \qquad (25)$$

Below we need transformed generators (see (10))

$$i\tilde{j}^{23} = \frac{1}{\sqrt{2}} \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right| = \tilde{T}^1 \, , i\tilde{J}^{31} = \frac{1}{\sqrt{2}} \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & +i & 0 & 0-i \\ 0 & 0 & +i & 0 \end{array} \right| = \tilde{T}^2 \, ,$$

$$i\tilde{j}^{12} = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right| = \tilde{T}^3 \, .$$

Starting with Eq. (12), after separating a simple multiplier in the wave function, $\tilde{\Psi}(x) = (\epsilon^{-i\epsilon t}/r)\ \tilde{\Phi}(r,\theta,\phi)$, we obtain

$$\left[\gamma^0\epsilon + i\gamma^3\partial_r + \frac{\gamma^1\otimes\tilde{T}_2 - \gamma^2\otimes\tilde{T}_1}{r} + \frac{1}{r}\tilde{\Sigma}_{\theta,\phi} - m\right]\tilde{\Phi}(r,\theta,\phi) = 0\ , \tag{26}$$

$$\tilde{\Sigma}_{\theta,\phi} = i\gamma^1\partial_\theta + \gamma^2\frac{i\partial_\phi + i\tilde{S}_3\cos\theta}{\sin\theta}\,. \tag{27}$$

There exist 16 eigenvectors of the matrix $\tilde{S}_3$; therefore we have 16 eigenvectors of operators $\vec{J}^2,\ J_3$:

$$\vec{J}^2\ \psi(\theta,\phi) = j(j+1)\ \psi(\theta,\phi)\,,\quad J_3\ \psi(\theta,\phi) = m\ \psi(\theta,\phi)\,.$$

So the most general substitution for the wave function with quantum numbers j,m is (vector-bispinor is divided in two vector-spinors):

$$\tilde{\xi}_l = \left|\begin{array}{c} f_0\ \delta_i^0\ D_{-1/2}\ +\ f_1\ \delta_i^1\ D_{-3/2}\ +\ f_2\ \delta_i^2\ D_{-1/2}\ +\ f_3\ \delta_i^3\ D_{+1/2} \\ g_0\ \delta_i^0\ D_{+1/2}\ +\ g_1\ \delta_i^1\ D_{-1/2}\ +\ g_2\ \delta_i^2\ D_{+1/2}\ +\ g_3\ \delta_i^3\ D_{+3/2} \end{array}\right|; \tag{28}$$

the second vector-spinor $\tilde{\eta}_i$ has the similar structure but with other radial functions:

$$f_i(r)\ \Longrightarrow\ h_i(r)\,,\qquad g_i(r)\ \Longrightarrow\ \nu_i(r)\,. \tag{29}$$

It should be noted that for the minimal value $j = 1/2$ we must take a more simple initial wave function: $j = \frac{1}{2},\ f_1 = 0,\ h_1 = 0,\ g_3 = 0,\ \nu_3 = 0.$

Equation (26) may be written in splintered form

$$\left[\epsilon + i\sigma_3\partial_r + \frac{\sigma_1\otimes\tilde{T}_2 - \sigma_2\otimes\tilde{T}_1}{r} + \frac{1}{r}\tilde{\Sigma}_{\theta,\phi}\right]\tilde{\xi} = m\tilde{\eta}, \tag{30}$$

$$\left[\epsilon - i\sigma_3\partial_r - \frac{\sigma_1\otimes\tilde{T}_2 - \sigma_2\otimes\tilde{T}_1}{r} - \frac{1}{r}\ \tilde{\Sigma}_{\theta,\phi}\right]\tilde{\eta} = m\ \tilde{\xi}, \tag{31}$$

where

$$\tilde{\Sigma}_{\theta,\phi} = i\sigma_1\partial_\theta\ +\ \sigma_2\frac{i\partial_\phi\ +\ (1/2\sigma_3\otimes I\ +\ I\otimes\tilde{T}_3)\cos\theta}{\sin\theta}\,.$$

In addition to operators $i\partial_t,\ \vec{J}^2,\ J_3$, we will diagonalize also the operator of spacial reflection. We start with the Cartesian P-operators in bispinor and vector spaces, after transforming them to spherical-cyclic basis, we get

$$(\tilde{\Pi}_k{}^l) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \quad \Pi = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}. \tag{32}$$

From eigenvalue equation $[\ (\ \Pi \otimes \tilde{\Pi}_k{}^l\ )\ \hat{P}\ ]\ \tilde{\Phi}_l(r,\theta,\phi) = P\ \tilde{\Phi}_k(r,\theta,\phi)$ we derive two eigenvalues and respective sets of restrictions on radial functions:

$$\begin{aligned} &\nu_0 = \delta\ f_0\ ,\quad \nu_1 = \delta\ f_3\ ,\quad \nu_2 = \delta\ f_2\ ,\quad \nu_3 = \delta\ f_1\ , \\ &h_0 = \delta\ g_0\ ,\quad h_1 = \delta\ g_3\ ,\quad h_2 = \delta\ g_2\ ,\quad h_3 = \delta\ g_1\ , \end{aligned} \tag{33}$$

where $\delta = +1$ corresponds to the parity $P = (-1)^{J+1}$, and $\delta = -1$ corresponds to $P = (-1)^J$.

3. Separating the Variables

Having performed all needed calculations for separating the variables in equations (30)–(31), with the use of the known recurrent formulas for Wigner D-functions [17]

$$\begin{aligned} \partial_\theta D_{+1/2} &= \frac{1}{2}\left(aD_{-1/2} - bD_{+3/2}\right), \\ [\sin^{-1}\theta(-m - \frac{1}{2}\cos\theta)]D_{+1/2} &= \frac{1}{2}\left(-aD_{-1/2} - bD_{+3/2}\right), \\ \partial_\theta D_{-1/2} &= \frac{1}{2}\left(bD_{-3/2} - aD_{+1/2}\right), \\ [\sin^{-1}\theta(-m + \frac{1}{2}\cos\theta)]D_{-1/2} &= \frac{1}{2}\left(-bD_{-3/2} - aD_{+1/2}\right), \\ \partial_\theta D_{+3/2} &= \frac{1}{2}\left(bD_{+1/2} - cD_{+5/2}\right), \\ [\sin^{-1}\theta(-m - \frac{3}{2}\cos\theta)]D_{+3/2} &= \frac{1}{2}\left(-bD_{+1/2} - cD_{+5/2}\right), \\ \partial_\theta D_{-3/2} &= \frac{1}{2}\left(cD_{-5/2} - bD_{-1/2}\right), \\ [\sin^{-1}\theta(-m + \frac{3}{2}\cos\theta)]D_{-3/2} &= \frac{1}{2}\left(-cD_{-5/2} - bD_{-1/2}\right), \end{aligned} \tag{34}$$

where

$$a = j + 1/2 \, , b = \sqrt{(j-1/2)(j+3/2)} \, , c = \sqrt{(j-3/2)(j+5/2)} \, , \quad (35)$$

we find 8 radial equations

$P = (-1)^{j+1}$,

$$\begin{gathered} \epsilon g_0 - i\frac{d}{dr} g_0 - i\frac{a}{r} f_0 = m \ f_0 \ , \epsilon f_0 + i\frac{d}{dr} f_0 + i\frac{a}{r} g_0 = m \ g_0 \ , \\ \epsilon g_3 - i\frac{d}{dr} g_3 - i\frac{b}{r} f_3 = m \ f_1 \ , \epsilon f_3 + i\frac{d}{dr} f_3 + i\frac{\sqrt{2}}{r} g_2 + i\frac{b}{r} g_3 = m \ g_1 \ , \\ \epsilon g_2 - i\frac{d}{dr} g_2 - i\frac{\sqrt{2}}{r} f_3 - i\frac{a}{r} f_2 = m \ f_2 \ , \\ \epsilon f_2 + i\frac{d}{dr} f_2 + i\frac{\sqrt{2}}{r} g_1 + i\frac{a}{r} g_2 = m \ g_2 \ , \\ \epsilon g_1 - i\frac{d}{dr} g_1 - i\frac{\sqrt{2}}{r} f_2 - i\frac{b}{r} f_1 = m \ f_3 \ , \epsilon f_1 + i\frac{d}{dr} f_1 + i\frac{b}{r} g_1 = m \ g_3 \ ; \end{gathered} \quad (36)$$

to obtain similar equations for states with parity $P = (-1)^j$ it suffices to make formal change m to $(-m)$.

For minimal value $j = 1/2$, we get more simple system (formally it suffices to set $f_1 = 0, \ g_3 = 0, \ b = 0, \ a = 1$):

$P = (-1)^{j+1}, \ j = 1/2$,

$$\begin{gathered} (\epsilon - i\frac{d}{dr}) g_0 = (m + \frac{i}{r}) f_0, \quad (\epsilon + i\frac{d}{dr}) f_0 = (m - \frac{i}{r}) g_0, \\ (\epsilon + i\frac{d}{dr}) f_3 + i\frac{\sqrt{2}}{r} g_2 = m g_1, \quad (\epsilon - i\frac{d}{dr}) g_2 - i\frac{\sqrt{2}}{r} f_3 = (m + \frac{i}{r}) f_2, \\ (\epsilon + i\frac{d}{dr}) f_2 + i\frac{\sqrt{2}}{r} g_1 = (m - \frac{i}{r}) g_2, \quad (\epsilon - i\frac{d}{dr}) g_1 - i\frac{\sqrt{2}}{r} f_2 = m f_3. \end{gathered} \quad (37)$$

4. Separating the Variables and Additional Constraints

Now we are to separate the variables in the constraints

$$\begin{gathered} \gamma^l (U^{-1})_{lk} \ \tilde{\Psi}_k(x) = 0 \, , \\ \left[e^{(l)\alpha} \partial_\alpha + e^{(l)\alpha}_{\ ;\alpha}(x) + e^{(l)\alpha}(x) \Gamma_\alpha(x) \right] (U^{-1})_{lk} \tilde{\Psi}_k(x) = 0 \, . \end{gathered} \quad (38)$$

After simple calculation, from the first relation in (38) we obtain 2 algebraic constraints (they are the same for both parity values; also they are valid for the case $j = 1/2$)

$$\delta = \pm 1, \quad f_0 - \sqrt{2} g_1 + f_2 = 0 \ , \quad g_0 + \sqrt{2} f_3 - g_2 = 0 \ . \tag{39}$$

Now we turn to differential constraint. Using explicit form of the components $\tilde{\Psi}_l$:

$$\tilde{\Psi}_0 = \left| \begin{array}{c} f_0 D_{-1/2} \\ g_0 D_{+3/2} \\ h_0 D_{-1/2} \\ \nu_0 D_{+3/2} \end{array} \right| , \qquad \tilde{\Psi}_3 = \left| \begin{array}{c} f_2 D_{-1/2} \\ g_2 D_{+1/2} \\ h_2 D_{-1/2} \\ \nu_2 D_{+1/2} \end{array} \right| ,$$

$$\tilde{\Psi}_1 = \frac{1}{\sqrt{2}} \left| \begin{array}{c} -f_1 D_{-3/2} + f_3 D_{+1/2} \\ -g_1 D_{-1/2} + g_3 D_{+3/2} \\ -h_1 D_{-3/2} + h_3 D_{+1/2} \\ -\nu_1 D_{-1/2} + \nu_3 D_{+3/2} \end{array} \right| ,$$

$$\tilde{\Psi}_2 = \frac{1}{\sqrt{2}} \left| \begin{array}{c} -i f_1 D_{-3/2} - i f_3 D_{+1/2} \\ -i g_1 D_{-1/2} - i g_3 D_{+3/2} \\ -i h_1 D_{-3/2} - i h_3 D_{+1/2} \\ -i \nu_1 D_{-1/2} - i \nu_3 D_{+3/2} \end{array} \right| .$$

we subsequently find three terms in (52). For the first term we get (remembering on the multiplier r^{-1} at the wave function¡ we make the change $\partial_r \Longrightarrow \partial_r - \frac{1}{r}$)

$$e^{(l)\alpha} \partial_\alpha \ \tilde{\Psi}_l = e^{(0)0} \ \partial_0 \tilde{\Psi}_0 + e^{(3)r} \ (\partial_r - \frac{1}{r}) \tilde{\Psi}_3 + e^{(1)\theta} \ \partial_\theta \tilde{\Psi}_1 + e^{(2)\phi} \ \partial_\phi \tilde{\Psi}_2$$

$$= -i\epsilon \left| \begin{array}{c} f_0 D_{-1/2} \\ g_0 D_{+1/2} \\ h_0 D_{-1/2} \\ \nu_0 D_{+1/2} \end{array} \right| - \frac{1}{\sqrt{2}\, r} \partial_\theta \left| \begin{array}{c} -f_1 D_{-3/2} + f_3 D_{+1/2} \\ -g_1 D_{-1/2} + g_3 D_{+3/2} \\ -h_1 D_{-3/2} + h_3 D_{+1/2} \\ -\nu_1 D_{-1/2} + \nu_3 \ D_{+3/2} \end{array} \right|$$

$$-(\partial_r - \frac{1}{r}) \left| \begin{array}{c} f_2 D_{-1/2} \\ g_2 D_{+1/2} \\ h_2 D_{-1/2} \\ \nu_2 D_{+1/2} \end{array} \right| - \frac{im}{\sqrt{2}\, r \sin\theta} \left| \begin{array}{c} -i f_1 D_{-3/2} - i f_3 D_{+1/2} \\ -i g_1 D_{-1/2} - i g_3 D_{+3/2} \\ -i h_1 D_{-3/2} - i h_3 D_{+1/2} \\ -i \nu_1 D_{-1/2} - i \nu_3 D_{+3/2} \end{array} \right| . \tag{40}$$

Taking in mind relations

$$e^{(0)\alpha}_{\ ;\alpha} = 0 \ , \quad e^{(1)\alpha}_{\ ;\alpha} = -\frac{1}{r} \ \mathrm{ctg}\, \theta \ , \quad e^{(2)\alpha}_{\ ;\alpha} = 0 \ , \quad e^{(3)\alpha}_{\ ;\alpha} = -\frac{2}{r} \ ,$$

we find the term $e^{(l)\alpha}_{;\alpha}\tilde{\Psi}_l$:

$$e^{(l)\alpha}_{\ ;\alpha}\ \tilde{\Psi}_l = +\frac{\text{ctg}\ \theta}{\sqrt{2}\ r}\left| \begin{array}{c} f_1\ D_{-3/2} - f_3\ D_{+1/2} \\ g_1\ D_{-1/2} - g_3\ D_{+3/2} \\ h_1\ D_{-3/2} - h_3\ D_{+1/2} \\ \nu_1\ D_{-1/2} - \nu_3\ D_{+3/2} \end{array} \right| - \frac{2}{r}\left| \begin{array}{c} f_2\ D_{-1/2} \\ g_2\ D_{+1/2} \\ h_2\ D_{-1/2} \\ \nu_2\ D_{+1/2} \end{array} \right| . \tag{41}$$

For the term $e^{(l)\alpha}\ \Gamma_\alpha\ \tilde{\Psi}_l$, taking in mind expressions for connection $\Gamma_\alpha(x)$, we derive

$$e^{(l)\alpha}\Gamma_\alpha\tilde{\Psi}_l = e^{(1)\theta}\Gamma_\theta\tilde{\Psi}_1 + e^{(2)\phi}\Gamma_\phi\tilde{\Psi}_2 = -\frac{1}{r}(\sigma^{31}\tilde{\Psi}_1 + \sigma^{32}\tilde{\Psi}_2) - \frac{\text{ctg}\ \theta}{r}\sigma^{12}\tilde{\Psi}_2\,,$$

and further find

$$e^{(l)\alpha}\Gamma_\alpha\ \tilde{\Psi}_l = -\frac{1}{\sqrt{2}\ r}\left| \begin{array}{c} g_1 D_{-1/2} \\ f_3\ D_{+1/2} \\ \nu_1 D_{-1/2} \\ h_3 D_{+1/2} \end{array} \right| + \frac{\text{ctg}\ \theta}{\sqrt{2}\ r}\left| \begin{array}{c} +\frac{1}{2}f_1 D_{-3/2} + \frac{1}{2}f_3 D_{+1/2} \\ -\frac{1}{2}g_1 D_{-1/2} - \frac{1}{2}g_3 D_{+3/2} \\ +\frac{1}{2}h_1 D_{-3/2} + \frac{1}{2}h_3 D_{+1/2} \\ -\frac{1}{2}\nu_1 D_{-1/2} - \frac{1}{2}\nu_3 D_{+3/2} \end{array} \right| . \tag{42}$$

Summing (40), (41), (42), we get the following form for differential constraint in (52):

$$-i\epsilon\left| \begin{array}{c} f_0 D_{-1/2} \\ g_0 D_{+1/2} \\ h_0 D_{-1/2} \\ \nu_0 D_{+1/2} \end{array} \right| - (\partial_r + \frac{1}{r})\left| \begin{array}{c} f_2 D_{-1/2} \\ g_2 D_{+1/2} \\ h_2 D_{-1/2} \\ \nu_2 D_{+1/2} \end{array} \right| - \frac{1}{\sqrt{2}r}\left| \begin{array}{c} g_1 D_{-1/2} \\ f_3 D_{+1/2} \\ \nu_1 D_{-1/2} \\ h_3 D_{+1/2} \end{array} \right|$$

$$+\frac{1}{\sqrt{2}\ r}\partial_\theta\left| \begin{array}{c} f_1 D_{-3/2} - f_3 D_{+1/2} \\ g_1 D_{-1/2} - g_3 D_{+3/2} \\ h_1 D_{-3/2} - h_3 D_{+1/2} \\ \nu_1 D_{-1/2} - \nu_3 D_{+3/2} \end{array} \right|$$

$$+\frac{1}{\sqrt{2}\ r}\left| \begin{array}{c} f_1\frac{-m+3/2\cos\theta}{\sin\theta}D_{-3/2} + f_3\frac{-m-1/2\cos\theta}{\sin\theta}D_{+1/2} \\ g_1\frac{-m+1/2\cos\theta}{\sin\theta}D_{-1/2} + g_3\frac{-m-3/2\cos\theta}{\sin\theta}D_{+3/2} \\ h_1\frac{-m+3/2\cos\theta}{\sin\theta}D_{-3/2} + h_3\frac{-m-1/2\cos\theta}{\sin\theta}D_{+3/2} \\ \nu_1\frac{-m+1/2\cos\theta}{\sin\theta}D_{-1/2} + \nu_3\frac{-m-3/2\cos\theta}{\sin\theta}D_{+3/2} \end{array} \right| = 0\,.$$

Whence, transforming two last terms with the help of the consequences of recurrent formulas (34):

$$
\partial_\theta D_{+1/2} - \frac{-m - 1/2\ \cos\theta}{\sin\theta} D_{+1/2} = +aD_{-1/2}\,,
$$
$$
\partial_\theta D_{-1/2} + \frac{-m + 1/2\ \cos\theta}{\sin\theta} D_{-1/2} = -aD_{+1/2}\,,
$$

$$
\partial_\theta D_{+3/2} - \frac{-m - 3/2\ \cos\theta}{\sin\theta} D_{+3/2} = +bD_{+1/2}\,,
$$
$$
\partial_\theta D_{-3/2} + \frac{-m + 3/2\ \cos\theta}{\sin\theta} D_{-3/2} = -bD_{-1/2}\,,
$$

we produce

$$
-i\epsilon \left| \begin{array}{c} f_0 D_{-1/2} \\ g_0 D_{+1/2} \\ h_0 D_{-1/2} \\ \nu_0 D_{+1/2} \end{array} \right| - (\frac{d}{dr} + \frac{1}{r}) \left| \begin{array}{c} f_2 D_{-1/2} \\ g_2 D_{+1/2} \\ h_2 D_{-1/2} \\ \nu_2 D_{+1/2} \end{array} \right|
$$

$$
-\frac{1}{\sqrt{2}r} \left| \begin{array}{c} g_1 D_{-1/2} \\ f_3 D_{+1/2} \\ \nu_1 D_{-1/2} \\ h_3 D_{+1/2} \end{array} \right| - \frac{1}{\sqrt{2}\ r} \left| \begin{array}{c} (b\ f_1 + a\ f_3)\ D_{-1/2} \\ (a\ g_1 + b\ g_3)\ D_{-1/2} \\ (b\ h_1 + a\ h_3)\ D_{-1/2} \\ (a\ \nu_1 + b\ \mu_3)\ D_{+1/2} \end{array} \right| .
$$

Thus, we derive four differential equations in radial variable, which after taking into account the P-parity restrictions lead to only two constrains (they are valid for both values of parity)

$$
\begin{aligned}
&-i\epsilon f_0 - (\frac{d}{dr} + \frac{1}{r}) f_2 - \frac{1}{\sqrt{2}\ r} g_1 - \frac{1}{\sqrt{2}\ r}(b\ f_1 + a\ f_3) = 0\ , \\
&-i\epsilon g_0 - (\frac{d}{dr} + \frac{1}{r}) g_2 - \frac{1}{\sqrt{2}\ r} f_3 - \frac{1}{\sqrt{2}\ r}(a\ g_1 + b\ g_3) = 0\ .
\end{aligned} \tag{43}
$$

For minimal value $j = 1/2$, equations (43) become simpler

$$
\begin{aligned}
&-i\epsilon f_0 - (\frac{d}{dr} + \frac{1}{r}) f_2 - \frac{1}{\sqrt{2}r} g_1 - \frac{1}{\sqrt{2}\ r} f_3 = 0\,, \\
&-i\epsilon g_0 - (\frac{d}{dr} + \frac{1}{r}) g_2 - \frac{1}{\sqrt{2}\ r} f_3 - \frac{1}{\sqrt{2}\ r} g_1 = 0\,.
\end{aligned} \tag{44}
$$

5. Solving Equations for Functions f_0, g_0

First we are to solve equations for functions f_0, g_0 (see two first equations in (37)). Summing and subtracting these two equations we get

$$(\frac{d}{dr} + \frac{a}{r})F_0 = i(\epsilon + m)G_0\,, \quad (\frac{d}{dr} - \frac{a}{r})G_0 = i(\epsilon - m)F_0\,, \tag{45}$$

where $F_0 = f_0 + g_0\,,\ G_0 = f_0 - g_0$. From (45) follow 2-nd order equations for separate functions:

$$\begin{gathered}\left(\frac{d^2}{dr^2} - \frac{a^2 + a}{r^2} + \epsilon^2 - m^2\right) F_0 = 0\,,\ l = a\,;\\ \left(\frac{d^2}{dr^2} - \frac{(a-1)a}{r^2} + \epsilon^2 - m^2\right) G_0 = 0\,,\ l' = a - 1\,.\end{gathered} \tag{46}$$

They reduce to Bessel functions in the variable $x = \sqrt{\epsilon^2 - m^2}r$:

$$\left(\frac{d^2}{dx^2} + 1 - \frac{l(l+1)}{x^2}\right) F_0(x) = 0\,, \left(\frac{d^2}{dx^2} + 1 - \frac{l'('l+1)}{x^2}\right) G_0(x) = 0\,. \tag{47}$$

Let it be

$$F_0(x) = a_0\sqrt{x}\ Z_l(x),\ G_0(x) = b_0\sqrt{x}\ Z_{l'}(x),$$

then Eqs. (47) yield

$$\begin{gathered}Z_l'' + \frac{1}{x}\,Z_l' + 1 - \frac{(l+1/2)^2}{x^2}Z_l = 0\,, \quad l + 1/2 = j + 1 = p\,,\\ Z_{l'}'' + \frac{1}{x}\,Z_{l'}' + 1 - \frac{(l-1/2)^2}{x^2}Z_{l'} = 0\,, \quad l' + 1/2 = j = p - 1\,.\end{gathered} \tag{48}$$

Thus, functions F_0 and G_0 are constructed through Bessel functions.

Let us turn back to first order equations (45) for functions F_0, G_0 written in the variable x:

$$\begin{gathered}(\frac{d}{dx} + \frac{p}{x})Z_p = \sqrt{\frac{\epsilon + m}{\epsilon - m}}\ (i\frac{b_0}{a_0})\ Z_{p-1}\,,\\ (\frac{d}{dx} - \frac{p-1}{x})Z_{p-1} = \sqrt{\frac{\epsilon - m}{\epsilon + m}}\ (i\frac{a_0}{b_0})\ Z_p\,.\end{gathered} \tag{49}$$

Due to the known formulas

$$(\frac{d}{dx} + \frac{p}{z})Z_p = Z_{p-1}\,, \quad (\frac{d}{dx} - \frac{p}{z})Z_p = -Z_{p+1} = 0\,, \tag{50}$$

we find the relative coefficient between a_0 and b_0:

$$\sqrt{\epsilon + m}\, b_0 = -i\,\sqrt{\epsilon - m}\, a_0. \tag{51}$$

6. The Matrix Form of the Main System

Now, we turn to 6 equations from (36):

$$\begin{aligned}
(\epsilon + i\frac{d}{dr})f_1 + i\frac{b}{r}g_1 &= m\ g_3, \\
(\epsilon - i\frac{d}{dr})g_3 - i\frac{b}{r}f_3 &= m\ f_1, \\
(\epsilon + i\frac{d}{dr})f_2 + i\frac{\sqrt{2}}{r}g_1 &= (m - i\frac{a}{r})g_2, \\
(\epsilon - i\frac{d}{dr})g_2 - i\frac{\sqrt{2}}{r}f_3 &= (m + i\frac{a}{r})f_2, \\
(\epsilon + i\frac{d}{dr})f_3 + i\frac{\sqrt{2}}{r}g_2 + i\frac{b}{r}g_3 &= m\ g_1, \\
(\epsilon - i\frac{d}{dr})g_1 - i\frac{\sqrt{2}}{r}f_2 - i\frac{b}{r}f_1 &= m\ f_3.
\end{aligned} \tag{52}$$

It is convenient to employ the following variables:

$$\begin{aligned}
f_1 + g_3 = F_1\,,\ \ f_1 - g_3 = G_1\,, \\
f_2 + g_2 = F_2\,,\ \ f_2 - g_2 = G_2\,, \\
f_3 + g_1 = F_3\,,\ \ f_3 - g_1 = G_3\,.
\end{aligned} \tag{53}$$

Summing and subtracting equations in (52) within each pair, we produce

$$\begin{aligned}
\frac{d}{dr}G_1 + i(m-\epsilon)F_1 &= +\frac{b}{r}G_3\,, \\
\frac{d}{dr}F_1 - i(m+\epsilon)G_1 &= -\frac{b}{r}F_3\,; \\
(\frac{d}{dr} - \frac{a}{r})G_2 + i(m-\epsilon)F_2 &= +\frac{\sqrt{2}}{r}G_3\,, \\
(\frac{d}{dr} + \frac{a}{r})F_2 - i(m+\epsilon)G_2 &= -\frac{\sqrt{2}}{r}F_3\,; \\
\frac{d}{dr}G_3 - \frac{\sqrt{2}}{r}G_2 + i(m-\epsilon)F_3 &= +\frac{b}{r}G_1\,, \\
\frac{d}{dr}F_3 + \frac{\sqrt{2}}{r}F_2 - i(m+\epsilon)G_3 &= -\frac{b}{r}F_1\,.
\end{aligned} \tag{54}$$

This system may be presented in a matrix form

$$\frac{d}{dr}\left|\begin{array}{c} F_1 \\ F_2 \\ F_3 \\ G_1 \\ G_2 \\ G_3 \end{array}\right|$$

$$= \left|\begin{array}{cccccc} 0 & 0 & -\frac{b}{r} & i(\epsilon+m) & 0 & 0 \\ 0 & -\frac{a}{r} & -\frac{\sqrt{2}}{r} & 0 & i(\epsilon+m) & 0 \\ -\frac{b}{r} & -\frac{\sqrt{2}}{r} & 0 & 0 & 0 & i(\epsilon+m) \\ i(\epsilon-m) & 0 & 0 & 0 & 0 & \frac{b}{r} \\ 0 & i(\epsilon-m) & 0 & 0 & \frac{a}{r} & \frac{\sqrt{2}}{r} \\ 0 & 0 & i(\epsilon-m) & \frac{b}{r} & \frac{\sqrt{2}}{r} & 0 \end{array}\right| \left|\begin{array}{c} F_1 \\ F_2 \\ F_3 \\ G_1 \\ G_2 \\ G_3 \end{array}\right| . \qquad (55)$$

For the case of minimal value $j = 1/2$ $(a = 1, b = 0, F_1 = 0, G_1 = 0)$, the corresponding matrix equation becomes simpler

$$\frac{d}{dr}\left|\begin{array}{c} F_2 \\ F_3 \\ G_2 \\ G_3 \end{array}\right| = \left|\begin{array}{cccc} -\frac{1}{r} & -\frac{\sqrt{2}}{r} & i(\epsilon+m) & 0 \\ -\frac{\sqrt{2}}{r} & 0 & 0 & i(\epsilon+m) \\ i(\epsilon-m) & 0 & \frac{1}{r} & \frac{\sqrt{2}}{r} \\ 0 & i(\epsilon-m) & \frac{\sqrt{2}}{r} & 0 \end{array}\right| \left|\begin{array}{c} F_2 \\ F_3 \\ G_2 \\ G_3 \end{array}\right| . \qquad (56)$$

The system (56) may be presented in (2+2)-form (F and G are 2-components columns)

$$(\frac{d}{dr} + \frac{\alpha}{r})F = i(\epsilon+m)G, \quad (\frac{d}{dr} - \frac{\alpha}{r})G = i(\epsilon-m)F, \qquad (57)$$

where the matrix α is

$$\alpha = \left|\begin{array}{cc} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{array}\right| .$$

Applying the exclusion method, we derive 2-nd order separate equations:

$$
\begin{aligned}
&(\frac{d^2}{dr^2} + \epsilon^2 - m^2)F = \frac{1}{r^2}(\alpha^2 + \alpha)F\,, \\
&(\frac{d^2}{dr^2} + \epsilon^2 - m^2)G = \frac{1}{r^2}(\alpha^2 - \alpha)G\,.
\end{aligned}
\tag{58}
$$

Similar approach may be applied to the case of arbitrary j. The system (55) is presented in (3+3)-form as follows

$$
(\frac{d}{dr} + \frac{A}{r})F = i(\epsilon + m)G\,, \quad (\frac{d}{dr} - \frac{A}{r})G = i(\epsilon - m)F\,,
\tag{59}
$$

where

$$
A = \left| \begin{array}{ccc} 0 & 0 & b \\ 0 & a & \sqrt{2} \\ b & \sqrt{2} & 0 \end{array} \right| .
$$

Applying the exclusion method, we derive 2-nd order separate equations:

$$
\begin{aligned}
&(\frac{d^2}{dr^2} + \epsilon^2 - m^2)F = \frac{1}{r^2}(A^2 + A)F\,, \\
&(\frac{d^2}{dr^2} + \epsilon^2 - m^2)G = \frac{1}{r^2}(A^2 - A)G\,.
\end{aligned}
\tag{60}
$$

7. The Case of Minimal Value $j = 1/2$

For the minimal value $j = 1/2$ we have equations

$$
\begin{gathered}
(\frac{d^2}{dr^2} + \epsilon^2 - m^2)F = \frac{1}{r^2}(\alpha^2 + \alpha)F\,, \\
(\frac{d^2}{dr^2} + \epsilon^2 - m^2)G = \frac{1}{r^2}(\alpha^2 - \alpha)G\,, \\
\alpha = \left| \begin{array}{cc} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{array} \right| , \quad \alpha^2 = \left| \begin{array}{cc} 3 & \sqrt{2} \\ \sqrt{2} & 2 \end{array} \right| , \\
\alpha^2 + \alpha = \left| \begin{array}{cc} 4 & 2\sqrt{2} \\ 2\sqrt{2} & 2 \end{array} \right| , \ \alpha^2 - \alpha = \left| \begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array} \right| .
\end{gathered}
\tag{61}
$$

Equations for G_2, G_3 turn out to be separated:

$$
\left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 - \frac{2}{r^2} \right) G_2 = 0\,, \quad \left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 - \frac{2}{r^2} \right) G_3 = 0\,;
$$

their solutions are expressed in terms of Bessel functions

$$G_2 = b_2\sqrt{x}\, Z_{3/2}(x)\,,\ G_3 = b_3\sqrt{x}\, Z_{3/2}(x)\,.$$

The system for F_2, F_3 reads

$$\left(\frac{d^2}{dr^2} + \epsilon^2 - m^2\right)\begin{vmatrix} F_2 \\ F_3 \end{vmatrix} = \frac{2}{r^2}\begin{vmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{vmatrix}\begin{vmatrix} F_2 \\ F_3 \end{vmatrix}, \quad \Delta F = \frac{2}{r^2} T F\,. \tag{62}$$

Applying linear transformation we will reduce the mixing matrix to a diagonal form

$$\bar{F} = SF, \quad \Delta \bar{F} = \frac{2}{r^2}(STS^{-1})\,\bar{F}\,, \quad STS^{-1} = T_0 = \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix}. \tag{63}$$

So we get algebraic equation

$$\begin{vmatrix} (2-\lambda_2) & \sqrt{2} \\ \sqrt{2} & (1-\lambda_2) \end{vmatrix}\begin{vmatrix} s_{11} \\ s_{12} \end{vmatrix} = 0\,, \begin{vmatrix} (2-\lambda_3) & \sqrt{2} \\ \sqrt{2} & (1-\lambda_3) \end{vmatrix}\begin{vmatrix} s_{21} \\ s_{22} \end{vmatrix} = 0\,.$$

Solutions exist if the determinant of the system vanishes, so we find two roots $\lambda_2 = 0,\ \lambda_3 = 3$. Further, we determine the elements of the matrix by solving linear equations

$$2s_{11} + \sqrt{2}s_{12} = 0 \quad \Longrightarrow \quad s_{12} = -\sqrt{2}\, s_{11} \ \ (\text{let } s_{11} = 1)\,,$$
$$2s_{21} + \sqrt{2}s_{22} = 3s_{21} \quad \Longrightarrow \quad s_{21} = +\sqrt{2}\, s_{22} \ \ (\text{let } s_{22} = 1\,.$$

Thus, the needed transformation (one of many possible) is

$$\begin{gathered} S = \begin{vmatrix} 1 & -\sqrt{2} \\ +\sqrt{2} & 1 \end{vmatrix}, \ S^{-1} = \frac{1}{3}\begin{vmatrix} 1 & +\sqrt{2} \\ -\sqrt{2} & 1 \end{vmatrix}, \\ \bar{F} = S\,F\,, \begin{vmatrix} \bar{F}_2 \\ \bar{F}_3 \end{vmatrix} = \begin{vmatrix} 1 & -\sqrt{2} \\ +\sqrt{2} & 1 \end{vmatrix}\begin{vmatrix} F_2 \\ F_3 \end{vmatrix}. \end{gathered} \tag{64}$$

In this way we arrive at two separate equations for $\bar{F}_2$ and $\bar{F}_3$:

$$\left(\frac{d^2}{dr^2} + \epsilon^2 - m^2\right)\bar{F}_2 = 0\,, \quad \left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 - \frac{6}{r^2}\right)\bar{F}_3 = 0\,;$$

their solutions are: $\bar{F}_2 = a_2\sqrt{x}\, Z_{1/2}\,,\ \bar{F}_3 = a_3\sqrt{x}\, Z_{5/2}$.

Now we are to consider additional constrains

$$\begin{gathered} g_1 = \frac{1}{\sqrt{2}}(f_2 + f_0)\,, \quad f_3 = \frac{1}{\sqrt{2}}(g_2 - g_0)\,, \\ -i\epsilon f_0 - \left(\frac{d}{dr} + \frac{1}{r}\right)f_2 = \frac{1}{\sqrt{2}\,r}(g_1 + f_3) = 0\,, \\ -i\epsilon g_0 - \left(\frac{d}{dr} + \frac{1}{r}\right)g_2 = \frac{1}{\sqrt{2}\,r}(f_3 + g_1) = 0\,. \end{gathered} \tag{65}$$

In other variables

$$f_0 + g_0 = F_0\,, \quad f_0 - g_0 = G_0\,,$$
$$f_2 + g_2 = F_2\,, \quad f_2 - g_2 = G_2\,,$$
$$f_3 + g_1 = F_3\,, \quad f_3 - g_1 = G_3\,,$$

they read

$$F_3 = \frac{1}{\sqrt{2}}(F_2 + G_0)\,, \quad G_3 = -\frac{1}{\sqrt{2}}(G_2 + F_0)\,,$$
$$-i\epsilon F_0 - (\frac{d}{dr} + \frac{1}{r})F_2 = \frac{1}{\sqrt{2}r}\,2F_3\,, \quad -i\epsilon G_0 - (\frac{d}{dr} + \frac{1}{r})G_2 = 0\,. \tag{66}$$

Taking into account the formulas

$$F_2 = \frac{1}{3}a_2\sqrt{x}\;Z_{1/2} + \frac{\sqrt{2}}{3}a_3\sqrt{x}\;Z_{5/2}\,,$$
$$F_0 = a_0\sqrt{x}\;Z_{3/2}\,, \quad G_0 = b_0\sqrt{x}\;Z_{1/2}\,,$$
$$F_3 = -\frac{\sqrt{2}}{3}a_2\sqrt{x}\;Z_{1/2} + \frac{a_3}{3}\sqrt{x}\;Z_{5/2}\,, \tag{67}$$
$$G_2 = b_2\sqrt{x}\;Z_{3/2}\,, \quad G_3 = b_3\sqrt{x}\;Z_{3/2}\,,$$

we can transform eqs. (66) to Bessel functions. First consider algebraic relations, they give

$$-\frac{\sqrt{2}}{3}a_2\sqrt{x}\;Z_{1/2} + \frac{a_3}{3}\sqrt{x}\;Z_{5/2}$$
$$= \frac{1}{\sqrt{2}}\left\{\frac{1}{3}a_2\sqrt{x}\;Z_{1/2} + \frac{\sqrt{2}}{3}a_3\sqrt{x}\;Z_{5/2} + b_0\sqrt{x}\;Z_{1/2}\right\},$$
$$b_3\sqrt{x}\;Z_{3/2} = -\frac{1}{\sqrt{2}}\left\{b_2\sqrt{x}\;Z_{3/2} + a_0\sqrt{x}\;Z_{3/2}\right\}\;;$$

whence follow linear relations for numerical coefficients

$$-\frac{\sqrt{2}}{3}a_2 = \frac{1}{\sqrt{2}}(\frac{1}{3}a_2 + b_0), b_3 = -\frac{1}{\sqrt{2}}(b_2 + a_0) \quad \Longrightarrow$$
$$b_0 + a_2 = 0\,, \quad a_0 + b_2 + \sqrt{2}b_3 = 0\,. \tag{68}$$

Now consider differential constrains in the variable x:

$$-(\frac{d}{dx} + \frac{1}{x})(\frac{1}{3}a_2\sqrt{x}Z_{1/2} + \frac{\sqrt{2}}{3}a_3\sqrt{x}Z_{5/2})$$

$$= \frac{i\epsilon}{\sqrt{\epsilon^2 - m^2}} a_0 \sqrt{x} Z_{3/2} + \frac{1}{x}\sqrt{2}(-\frac{\sqrt{2}}{3} a_2 \sqrt{x} Z_{1/2} + \frac{a_3}{3}\sqrt{x} Z_{5/2}),$$

$$-(\frac{d}{dx} + \frac{1}{x})\, b_2 \sqrt{x}\; Z_{3/2} = \frac{i\epsilon}{\sqrt{\epsilon^2 - m^2}} b_0 \sqrt{x}\; Z_{1/2}\,.$$

Whence, taking in mind the identity

$$\frac{d}{dx}\sqrt{x} = \sqrt{x}\;(\frac{d}{dx} + \frac{1/2}{x})\,,$$

we derive

$$-\frac{1}{3} a_2\;(\frac{d}{dx} + \frac{3/2}{x}) Z_{1/2} - \frac{\sqrt{2}}{3} a_3\;(\frac{d}{dx} + \frac{3/2}{x}) Z_{5/2}$$
$$= \frac{i\epsilon a_0}{\sqrt{\epsilon^2 - m^2}} Z_{3/2} - \frac{2}{3} a_2 \frac{1}{x} Z_{1/2} + a_3 \frac{\sqrt{2}}{3}\frac{1}{x} Z_{5/2}\,,$$
$$(\frac{d}{dx} + \frac{3/2}{x})\; b_2\; Z_{3/2} = -\frac{i\epsilon b_0}{\sqrt{\epsilon^2 - m^2}}\; Z_{1/2}\,.$$

Further, after re-grouping the terms in the first equation we obtain

$$-\frac{1}{3} a_2\;(\frac{d}{dx} - \frac{1/2}{x}) Z_{1/2} - \frac{\sqrt{2}}{3} a_3\;(\frac{d}{dx} + \frac{5/2}{x}) Z_{5/2} = \frac{i\epsilon a_0}{\sqrt{\epsilon^2 - m^2}} Z_{3/2}\,,$$
$$(\frac{d}{dx} + \frac{3/2}{x})\; b_2\; Z_{3/2} = -\frac{i\epsilon b_0}{\sqrt{\epsilon^2 - m^2}}\; Z_{1/2}\,. \qquad (69)$$

Now, taking into account the properties of Bessel functions (50), we derive linear relations for coefficients (for convenience let us write down also the consequences of algebraic constraints)

$$a_2 - \sqrt{2} a_3 = 3\frac{i\epsilon}{\sqrt{\epsilon^2 - m^2}} a_0\,, \quad b_2 = -\frac{i\epsilon}{\sqrt{\epsilon^2 - m^2}} b_0\,,$$
$$b_0 + a_2 = 0\,, \quad a_0 + b_2 + \sqrt{2} b_3 = 0\,. \qquad (70)$$

With the help of two last equations in (70), we exclude the variables a_0, b_0 from two first equations:

$$a_2 - \sqrt{2} a_3 = -3\frac{i\epsilon}{\sqrt{\epsilon^2 - m^2}}\;(b_2 + \sqrt{2} b_3)\,, \quad b_2 = \frac{i\epsilon}{\sqrt{\epsilon^2 - m^2}}\; a_2. \qquad (71)$$

Besides, from the above established relation $\sqrt{\epsilon + m}\; b_0 = -i\,\sqrt{\epsilon - m}\; a_0$ it follows

$$-\sqrt{\epsilon + m}\; a_2 = +i\,\sqrt{\epsilon - m}\;(b_2 + \sqrt{2} b_3)\,.$$

Therefore, Eqs. (71) reduce to the form

$$(2\epsilon + m)\, a_2 + (\epsilon - m)\, \sqrt{2} a_3 = 0 \,, \quad m\, b_2 - \epsilon\, \sqrt{2} b_3 = 0 \,. \tag{72}$$

Now, we are to take into account that the columns $\bar{F}(x)$ and $\bar{G}(x)$ cannot be considered as independent, instead they must obey the differential condition

$$\left(\frac{d}{dr} + \frac{\alpha}{r} \right) F = i(\epsilon + m) G \implies \left(\frac{d}{dr} + \frac{S \alpha S^{-1}}{r} \right) \bar{F} = i(\epsilon + m)\, S\bar{G} \,, \tag{73}$$

where

$$\bar{F} = \left| \begin{array}{c} a_2 \sqrt{x}\; Z_{1/2} \\ a_3 \sqrt{x}\; Z_{5/2} \end{array} \right| , \quad \bar{G} = \left| \begin{array}{c} b_2 \sqrt{x}\; Z_{3/2} \\ b_3 \sqrt{x}\; Z_{3/2} \end{array} \right| . \tag{74}$$

In the variable x, Eq. (73) has the form

$$\left(\frac{d}{dx} + \frac{S \alpha S^{-1}}{x} \right) \bar{F}(x) = i \sqrt{\frac{\epsilon + m}{\epsilon - m}}\; SG(x) \,. \tag{75}$$

The left-hand and right-hand sides of Eq. (75) are

$$\left| \begin{array}{c} (d/dx - 1/x)\; a_2 \sqrt{x}\; Z_{1/2} \\[2mm] (d/dx + 2/x)\; a_3 \sqrt{x}\; Z_{5/2} \end{array} \right| ,$$

$$i \sqrt{\frac{\epsilon + m}{\epsilon - m}}\; SG(x) = i \sqrt{\frac{\epsilon + m}{\epsilon - m}} \left| \begin{array}{c} (b_2 - \sqrt{2} b_3)\; \sqrt{x}\; Z_{3/2} \\[2mm] (\sqrt{2} b_2 + b_3)\; \sqrt{x}\; Z_{3/2} \end{array} \right| .$$

Therefore, we conclude that Eq. (75) gives

$$(\frac{d}{dx} - \frac{1/2}{x}) a_2 Z_{1/2} = i \sqrt{\frac{\epsilon + m}{\epsilon - m}} (b_2 - \sqrt{2} b_3) Z_{3/2},$$

$$(\frac{d}{dx} + \frac{5/2}{x}) a_3 Z_{5/2} = i \sqrt{\frac{\epsilon + m}{\epsilon - m}} (\sqrt{2} b_2 + b_3) Z_{3/2} \,;$$

whence we derive linear relations between coefficients:

$$a_2 \; = -i \sqrt{\frac{\epsilon + m}{\epsilon - m}}\; (b_2 - \sqrt{2} b_3) \,, \quad a_3 = +i \sqrt{\frac{\epsilon + m}{\epsilon - m}}\; (\sqrt{2} b_2 + b_3) \,. \tag{76}$$

They can be readily resolved with respect to b_2, b_3:

$$b_2 = +i \sqrt{\frac{\epsilon - m}{\epsilon + m}} \frac{1}{3}\; (a_2 - \sqrt{2}\, a_3) \,, \quad b_3 = -i \sqrt{\frac{\epsilon - m}{\epsilon + m}} \frac{1}{3}\; (\sqrt{2} a_2 + a_3) \,. \tag{77}$$

Let us collect together all independent equations for unknown coefficients

$$(2\epsilon + m)\, a_2 + (\epsilon - m)\, \sqrt{2} a_3 = 0\,, \quad m\, b_2 - \epsilon\, \sqrt{2} b_3 = 0\,,$$
$$b_2 = +i \sqrt{\frac{\epsilon - m}{\epsilon + m}} \frac{1}{3}\, (a_2 - \sqrt{2}\, a_3)\,, \quad b_3 = -i \sqrt{\frac{\epsilon - m}{\epsilon + m}} \frac{1}{3}\, (\sqrt{2} a_2 + a_3)\,. \tag{78}$$

We readily verify that substituting two last equations into the second one, we obtain the first equation

$$m\, (a_2 - \sqrt{2}\, a_3) + \epsilon\, \sqrt{2} (\sqrt{2} a_2 + a_3) = 0 \quad \Longrightarrow \quad (2\epsilon + m)\, a_2 + (\epsilon - m)\, \sqrt{2} a_3 = 0\,.$$

This means that in (78) we have only three independent relations:

$$a_2 = -\frac{\epsilon - m}{2\epsilon + m}\, \sqrt{2} a_3\,, \quad b_2 = -i \sqrt{\frac{\epsilon - m}{\epsilon + m}}\, \frac{1}{3}\, \frac{3\epsilon}{2\epsilon + m}\, \sqrt{2} a_3\,,$$
$$b_3 = -i \sqrt{\frac{\epsilon - m}{\epsilon + m}}\, \frac{1}{3}\, \frac{3m}{2\epsilon + m}\, a_3\,,$$

they determine completely solutions for the case of minimal value $j = 1/2$.

8. Studying General Case $j = 3/2, 5/2, ...$

Two relevant systems are

$$r^2 \left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 \right) \begin{vmatrix} F_1 \\ F_2 \\ F_3 \end{vmatrix} = \begin{vmatrix} b^2 & \sqrt{2} b & b \\ \sqrt{2} b & a^2 + a + 2 & \sqrt{2}(a + 1) \\ b & \sqrt{2}(a + 1) & b^2 + 2 \end{vmatrix} \begin{vmatrix} F_1 \\ F_2 \\ F_3 \end{vmatrix}\,, \tag{79}$$

$$r^2 \left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 \right) \begin{vmatrix} G_1 \\ G_2 \\ G_3 \end{vmatrix} = \begin{vmatrix} b^2 & \sqrt{2} b & -b \\ \sqrt{2} b & a^2 - a + 2 & \sqrt{2}(a - 1) \\ -b & \sqrt{2}(a - 1) & b^2 + 2 \end{vmatrix} \begin{vmatrix} G_1 \\ G_2 \\ G_3 \end{vmatrix}\,. \tag{80}$$

Symbolically Eqs. (79)–(80) read

$$\Delta F = T F\,, \quad \Delta G = T' G\,, \quad \Delta = r^2 (\frac{d^2}{dr^2} + \epsilon^2 - m^2)\,. \tag{81}$$

We are to find linear transformations over F and G, which change the mixing matrices T and T' to diagonal form:

$$\bar{F} = SF, \quad \Delta \bar{F} = (S T S^{-1})\, \bar{F}\,, \quad S T S^{-1} = T_0 = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix}; \tag{82}$$

$$\bar{G} = SG, \quad \Delta \bar{G} = (S' T' S'^{-1})\, \bar{G}\,, \quad S' T' S'^{-1} = T'_0 = \begin{vmatrix} \lambda'_1 & 0 & 0 \\ 0 & \lambda'_2 & 0 \\ 0 & 0 & \lambda'_3 \end{vmatrix}. \tag{83}$$

Let us consider the first equation $ST = T_0 S$, it yields

$$\left| \begin{array}{ccc} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{array} \right| \left| \begin{array}{ccc} b^2 & \sqrt{2}b & b \\ \sqrt{2}b & a^2+a+2 & \sqrt{2}(a+1) \\ b & \sqrt{2}(a+1) & b^2+2 \end{array} \right|$$

$$= \left| \begin{array}{ccc} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{array} \right| \left| \begin{array}{ccc} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{array} \right| ,$$

whence we obtain three sub-systems

$$\begin{aligned} s_{11}b^2 + s_{12}\sqrt{2}b + s_{13}b &= \lambda_1 s_{11}\,, \\ s_{11}\sqrt{2}b + s_{12}(a^2+a+2) + s_{13}\sqrt{2}(a+1) &= \lambda_1 s_{12}\,, \\ s_{11}b + s_{12}\sqrt{2}(a+1) + s_{13}(b^2+2) &= \lambda_1 s_{13}\,; \end{aligned}$$

$$\begin{aligned} s_{21}b^2 + s_{22}\sqrt{2}b + s_{23}b &= \lambda_2 s_{21}\,, \\ s_{21}\sqrt{2}b + s_{22}(a^2+a+2) + s_{23}\sqrt{2}(a+1) &= \lambda_2 s_{22}\,, \\ s_{21}b + s_{22}\sqrt{2}(a+1) + s_{23}(b^2+2) &= \lambda_2 s_{23}\,; \end{aligned}$$

$$\begin{aligned} s_{31}b^2 + s_{32}\sqrt{2}b + s_{33}b &= \lambda_3 s_{31}\,, \\ s_{31}\sqrt{2}b + s_{32}(a^2+a+2) + s_{33}\sqrt{2}(a+1) &= \lambda_3 s_{32}\,, \\ s_{31}b + s_{32}\sqrt{2}(a+1) + s_{33}(b^2+2) &= \lambda_3 s_{33}\,. \end{aligned}$$

Here there arise three eigenvalue problems

$$\begin{aligned} &\left| \begin{array}{ccc} b^2-\lambda_1 & \sqrt{2}b & b \\ \sqrt{2}b & (a^2+a+2)-\lambda_1 & \sqrt{2}(a+1) \\ b & \sqrt{2}(a+1) & b^2+2-\lambda_1 \end{array} \right| \left| \begin{array}{c} s_{11} \\ s_{12} \\ s_{13} \end{array} \right| = 0\,, \\ &\left| \begin{array}{ccc} b^2-\lambda_2 & \sqrt{2}b & b \\ \sqrt{2}b & (a^2+a+2)-\lambda_2 & \sqrt{2}(a+1) \\ b & \sqrt{2}(a+1) & b^2+2-\lambda_2 \end{array} \right| \left| \begin{array}{c} s_{21} \\ s_{22} \\ s_{23} \end{array} \right| = 0\,, \\ &\left| \begin{array}{ccc} b^2-\lambda_3 & \sqrt{2}b & b \\ \sqrt{2}b & (a^2+a+2)-\lambda_3 & \sqrt{2}(a+1) \\ b & \sqrt{2}(a+1) & b^2+2-\lambda_3 \end{array} \right| \left| \begin{array}{c} s_{31} \\ s_{32} \\ s_{33} \end{array} \right| = 0\,. \end{aligned} \tag{84}$$

From the very beginning, it should be noted that the lines in the matrices (84) can be fixed only up to arbitrary numerical factors (which correlates with linearity of differential equations).

Nontrivial solutions exist only if the determinant of the matrices equal to zero:

$$\det \begin{vmatrix} b^2 - \lambda & \sqrt{2}b & b \\ \sqrt{2}b & (a^2+a+2)-\lambda & \sqrt{2}(a+1) \\ b & \sqrt{2}(a+1) & b^2+2-\lambda \end{vmatrix} = 0\,, \tag{85}$$

the roots are (note that one root is double degenerated)

$$\lambda_1 = \lambda_3 = ((j-1/2)(j+1/2) = j'(j'+1)\,, \qquad j' = j - 1/2\,; \tag{86}$$

$$\lambda_2 = (j+3/2)(j+5/2) = j'(j'+1)\,, \qquad j' = j + 3/2\,. \tag{87}$$

After performing needed calculation we find explicit form for transformation matrix S (a fixed variant from various ones: below we need also the inverse matrix)

$$S = \begin{vmatrix} \sqrt{2j+3} & 0 & -\sqrt{2j-1} \\ \sqrt{2j-1} & \sqrt{2}\sqrt{2j+3} & \sqrt{2j+3} \\ \sqrt{2}\sqrt{2j+3} & -\sqrt{2j-1} & 0 \end{vmatrix} ,$$

$$S^{-1} = \frac{1}{8(j+1)\sqrt{2j-1}}$$

$$\times \begin{vmatrix} \sqrt{(2j+3)(2j-1)} & (2j-1) & \sqrt{2}\sqrt{(2j+3)(2j-1)} \\ \sqrt{2}(2j+3) & \sqrt{2}\sqrt{(2j+3)(2j-1)} & -2(2j+1) \\ -6j-5 & \sqrt{(2j+3)(2j-1)} & \sqrt{2}(2j+3) \end{vmatrix} . \tag{88}$$

Diagonalization of the matrix T' is done by similar scheme. Equation for $\lambda'_1, \lambda'_2, \lambda'_1$ is

$$\det \begin{vmatrix} b^2 - \lambda' & \sqrt{2}b & -b \\ \sqrt{2}b & (a^2-a+2)-\lambda' & \sqrt{2}(a-1) \\ -b & \sqrt{2}(a-1) & b^2+2-\lambda' \end{vmatrix} = 0\ , \tag{89}$$

and the roots are (one root is double degenerated)

$$\lambda'_1 = \lambda'_3 = (j+1/2)(j+3/2) = j'(j'+1), \qquad j' = j + 1/2\,; \tag{90}$$

$$\lambda'_2 = (j-3/2)(j-1/2) = j'(j'+1), \qquad j' = j - 3/2\,. \tag{91}$$

In this case, the needed transformation S' may be chosen in the form

$$S' = \begin{vmatrix} \sqrt{j-1/2} & 0 & -\sqrt{j+3/2} \\ \sqrt{j+3/2} & -\sqrt{2}\,\sqrt{j-1/2} & \sqrt{j-1/2} \\ \sqrt{2}\,\sqrt{j-1/2} & \sqrt{j+3/2} & 0 \end{vmatrix} , \tag{92}$$

$$S'^{-1} = -\frac{1}{4j\sqrt{j+3/2}}$$

$$\left| \begin{array}{ccc} -\sqrt{(j+3/2)(j-1/2)} & -(j+3/2) & -\sqrt{2}\sqrt{(j-1/2)(j+3/2)} \\ \sqrt{2}(j-1/2) & \sqrt{2}\sqrt{(j+3/2)(j-1/2)} & -(2j+1) \\ 3j+1/2 & -\sqrt{(j-1/2)(j+3/2)} & -\sqrt{2}(j-1/2) \end{array} \right| .$$

Taking into account expressions for the roots, we can write down equations for 8 separate functions:

$$\begin{gathered} \left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 - \frac{j'(j'+1)}{r^2} \right) F_0 = 0\,, \\ j' = j + 1/2\,, \quad F_0 = a_0 f_{j+1/2} = a_0 \sqrt{x} Z_{j+1}\ ; \\ \left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 - \frac{j'(j'+1)}{r^2} \right) G_0 = 0\,, \\ j' = j - 1/2\,, G_0 = b_0 f_{j-1/2} = b_0 \sqrt{x} Z_j\ ; \end{gathered} \tag{93}$$

$$\begin{gathered} \left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 - \frac{j'(j'+1)}{r^2} \right) \bar{F}_1 = 0\,, \\ j' = j - 1/2\,, \quad \bar{F}_1 = a_1 f_{j-1/2} = a_1 \sqrt{x} Z_j\ ; \\ \left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 - \frac{j'(j'+1)}{r^2} \right) \bar{G}_1 = 0\,, \\ j' = j + 1/2\,, \quad \bar{G}_1 = b_1 f_{j+1/2} = b_1 \sqrt{x} Z_{j+1}\ ; \end{gathered} \tag{94}$$

$$\begin{gathered} \left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 - \frac{j'(j'+1)}{r^2} \right) \bar{F}_2 = 0\,, \\ j' = j + 3/2\,, \quad \bar{F}_2 = a_2 f_{j+3/2} = a_2 \sqrt{x} Z_{j+2}\ ; \\ \left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 - \frac{j'(j'+1)}{r^2} \right) \bar{G}_2 = 0\,, \\ j' = j - 3/2\,, \quad \bar{G}_2 = b_2 f_{j-3/2} = b_2 \sqrt{x} Z_{j-1}\ ; \end{gathered} \tag{95}$$

$$\begin{gathered} \left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 - \frac{j'(j'+1)}{r^2} \right) \bar{F}_3 = 0\,, \\ j' = j - 1/2\,, \quad \bar{F}_3 = a_3 f_{j-1/2} = a_3 \sqrt{x} Z_j\ ; \\ \left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 - \frac{j'(j'+1)}{r^2} \right) \bar{G}_3 = 0\,, \\ j' = j + 1/2\,, \quad \bar{G}_3 = b_3 f_{j+1/2} = b_3 \sqrt{x} Z_{j+1}\ . \end{gathered} \tag{96}$$

They reduce to Bessel's type. All solutions are determined for the time present up to arbitrary numerical factors.

9. FURTHER SPECIFYING SOLUTIONS

It should be emphasized that parameters a_1, a_2, a_3 and b_1, b_2, b_3 cannot be considered as independent, because there exists the first-order differential equation which relates F and G:

$$(\frac{d}{dr} + \frac{A}{r})F = i(m+\epsilon)G\,; \tag{97}$$

with the use of the formulas $F = S^{-1}\bar{F}\,,\ G = S^{'-1}\bar{G}$ it may be presented as

$$(\frac{d}{dr} + \frac{A}{r})S^{-1}\bar{F} = i(m+\epsilon)S^{'-1}\bar{G} \quad \Longrightarrow$$
$$(\frac{d}{dr} + \frac{SAS^{-1}}{r})\bar{F} = i(m+\epsilon)SS^{'-1}\bar{G}\,.$$

Allowing for the identity

$$SAS^{-1} = \left| \begin{array}{ccc} -(j+3/2) & 0 & \sqrt{2} \\ 0 & j+3/2 & 0 \\ -\sqrt{2}(j+1/2) & 0 & j+1/2 \end{array} \right| , \tag{98}$$

we derive

$$(\frac{d}{dr} + \frac{SAS^{-1}}{r}) \left| \begin{array}{c} a_1 f_{j-1/2} \\ a_2 f_{j+3/2} \\ a_3 f_{j-1/2} \end{array} \right|$$
$$= \left| \begin{array}{c} (\frac{d}{dr} - \frac{j+3/2}{r})a_1 f_{j-1/2} + \frac{\sqrt{2}}{r} a_3 f_{j-1/2} \\ (\frac{d}{dr} + \frac{j+3/2}{r})a_2 f_{j+3/2} \\ -\sqrt{2}\frac{j+1/2}{r} a_1 f_{j-1/2} + (\frac{d}{dr} + \frac{j+1/2}{r})a_3 f_{j-1/2} \end{array} \right| . \tag{99}$$

Taking in mind the identity

$$SS^{'-1} = \left| \begin{array}{ccc} \sqrt{2}\sqrt{\frac{j-1/2}{j+3/2}}\,\frac{j+1/2}{j} & \frac{1}{\sqrt{2}\,j} & \frac{1}{j}\sqrt{\frac{j-1/2}{j+3/2}} \\ -\sqrt{2} & 0 & 2 \\ \sqrt{\frac{j-1/2}{j+3/2}}\frac{j+1/2}{j} & \frac{j+1/2}{j} & \frac{1}{\sqrt{2}}\frac{1}{j}\sqrt{\frac{j-1/2}{j+3/2}} \end{array} \right| , \tag{100}$$

we derive

$$SS^{'-1}\left|\begin{array}{c} b_1 f_{j+1/2} \\ b_2 f_{j-3/2} \\ b_3 f_{j+1/2} \end{array}\right|$$

$$=\left|\begin{array}{c} \sqrt{\frac{j-1/2}{j+3/2}}(\frac{j+1/2}{j}\sqrt{2}b_1+\frac{1}{j}b_3)f_{j+1/2}+\frac{1}{j}\frac{1}{\sqrt{2}}b_2 f_{j-3/2} \\ (-\sqrt{2}b_1+2b_3)f_{j+1/2} \\ \frac{1}{\sqrt{2}}\sqrt{\frac{j-1/2}{j+3/2}}(\frac{j+1/2}{j}\sqrt{2}b_1+\frac{1}{j}b_3)f_{j+1/2}+\frac{j+1/2}{j}b_2 f_{j-3/2} \end{array}\right| . \qquad (101)$$

Thus, we have three differential constraints

$$(\frac{d}{dr}+\frac{j+3/2}{r})a_2 f_{j+3/2}=i(\epsilon+m)(-\sqrt{2}b_1+2b_3)f_{j+1/2}\ , \qquad (102)$$

$$[(\frac{d}{dr}-\frac{j+3/2}{r})a_1+\frac{\sqrt{2}}{r}a_3]f_{j-1/2}=i(\epsilon+m)$$
$$\times\left[\sqrt{\frac{j-1/2}{j+3/2}}(\frac{j+1/2}{j}\sqrt{2}b_1+\frac{1}{j}b_3)f_{j+1/2}+\frac{1}{j}\frac{1}{\sqrt{2}}b_2 f_{j-3/2}\right], \qquad (103)$$

$$[-\frac{2j+1}{r}a_1+(\frac{d}{dr}+\frac{j+1/2}{r})\sqrt{2}a_3]f_{j-1/2}=i(\epsilon+m)$$
$$\times\left[\sqrt{\frac{j-1/2}{j+3/2}}(\frac{j+1/2}{j}\sqrt{2}b_1+\frac{1}{j}b_3)f_{j+1/2}+\frac{2j+1}{j}\frac{1}{\sqrt{2}}b_2 f_{j-3/2}\right]. \qquad (104)$$

Subtracting Eq. (104) from Eq. (103), we get

$$\left(\frac{d}{dr}-\frac{j+3/2}{r}+\frac{2j+1}{r}\right)a_1 f_{j-1/2}-\left(\frac{d}{dr}+\frac{j+1/2}{r}-\frac{1}{r}\right)\sqrt{2}a_3 f_{j-1/2}$$
$$=-i(\epsilon+m)\ 2\frac{b_2}{\sqrt{2}}f_{j-3/2}\,.$$

Now, let us multiply Eq. (103) by $(2j+1)$, and from the result subtract Eq. (104), so produce

$$(2j+1)\left(\frac{d}{dr}-\frac{j+3/2}{r}+\frac{1}{r}\right)a_1 f_{j-1/2}$$
$$-\left(\frac{d}{dr}+\frac{j+1/2}{r}-\frac{2j+1}{r}\right)\sqrt{2}a_3 f_{j-1/2}$$

$$= i(\epsilon + m) \sqrt{\frac{j-1/2}{j+3/2}} \; [\; (2j+1)\sqrt{2}b_1 + 2b_3 \;] f_{j+1/2} \, .$$

Therefore, we have three equations

$$(\frac{d}{dr} + \frac{j+3/2}{r}) a_2 f_{j+3/2} = i(\epsilon + m)(-\sqrt{2}b_1 + 2b_3) f_{j+1/2} \; ,$$

$$(\frac{d}{dr} + \frac{j-1/2}{r}) [a_1 - \sqrt{2}a_3] f_{j-1/2} = -i(\epsilon + m) \, 2\frac{b_2}{\sqrt{2}} f_{j-3/2} \, ,$$

$$[(2j+1)a_1 - \sqrt{2}a_3] \; (\frac{d}{dr} - \frac{j+1/2}{r}) f_{j-1/2}$$

$$= i(\epsilon + m) \sqrt{\frac{j-1/2}{j+3/2}} \; [\; (2j+1)\sqrt{2}b_1 + 2b_3 \;] f_{j+1/2} \, .$$

Transforming them to the terms of Bessel functions

$$f_{j+1/2} = \sqrt{x} Z_{j+1} \; , \; f_{j-1/2} = \sqrt{x} Z_j \; , \; f_{j+3/2} = \sqrt{x} Z_{j+2} \; , \; f_{j-3/2} = \sqrt{x} Z_{j-1} \; ,$$

we obtain

$$(\frac{d}{dx} + \frac{j+2}{x}) a_2 Z_{j+2} = i\sqrt{\frac{\epsilon + m}{\epsilon - m}} (-\sqrt{2}b_1 + 2b_3) Z_{j+1} \; ,$$

$$(\frac{d}{dx} + \frac{j}{x}) [a_1 - \sqrt{2}a_3] Z_j = -i\sqrt{\frac{\epsilon + m}{\epsilon - m}} \, 2\frac{b_2}{\sqrt{2}} Z_{j-1} \, ,$$

$$[(2j+1)a_1 - \sqrt{2}a_3] (\frac{d}{dx} - \frac{j}{x}) Z_j$$

$$= i\sqrt{\frac{\epsilon + m}{\epsilon - m}} \sqrt{\frac{j-1/2}{j+3/2}} \; [\; (2j+1)\sqrt{2}b_1 + 2b_3 \;] Z_{j+1} \, .$$

Whence, applying the known properties of Bessel functions, we arrive at linear relations for numerical coefficients:

$$a_2 = i\sqrt{\frac{\epsilon + m}{\epsilon - m}} (-\sqrt{2}b_1 + 2b_3), \quad a_1 - \sqrt{2}a_3 = -i\sqrt{\frac{\epsilon + m}{\epsilon - m}} 2\frac{b_2}{\sqrt{2}},$$
$$(2j+1)a_1 - \sqrt{2}a_3 = -i\sqrt{\frac{\epsilon + m}{\epsilon - m}} \sqrt{\frac{j-1/2}{j+3/2}} \; [(2j+1)\sqrt{2}b_1 + 2b_3] \, . \tag{105}$$

They may be resolved under the variables a_i and under the b_i:

$$a_1 = i\sqrt{\frac{\epsilon + m}{\epsilon - m}} \left\{ -\sqrt{2} \, \frac{j+1/2}{j} \frac{\sqrt{4j-2}}{\sqrt{4j+6}} b_1 + \; \frac{\sqrt{2}}{2} \frac{1}{j} b_2 - \frac{\sqrt{4j-2}}{\sqrt{4j+6}} \frac{1}{j} b_3 \right\} ,$$

$$a_2 = i\sqrt{\frac{\epsilon + m}{\epsilon - m}} \left\{ -\sqrt{2}\ b_1 + 2\, b_3 \right\} , \tag{106}$$

$$a_3 = i\sqrt{\frac{\epsilon + m}{\epsilon - m}} \left\{ -\frac{j+1/2}{j} \frac{\sqrt{4j-2}}{\sqrt{4\,j+6}} b_1 + \frac{1}{2} \frac{2j+1}{j} b_2 - \frac{\sqrt{2}}{2} \frac{\sqrt{4\,j-2}}{\sqrt{4j+6}} \frac{1}{j} b_3 \right\} ;$$

and

$$b_1 = i\sqrt{\frac{\epsilon - m}{\epsilon + m}} \left\{ \frac{\sqrt{2}}{4} \frac{2j+1}{j+1} \frac{\sqrt{4j+6}}{\sqrt{4j-2}} a_1 + \frac{\sqrt{2}}{4} \frac{1}{j+1} a_2 - \frac{\sqrt{4j+6}}{\sqrt{4\,j-2}} \frac{1}{2\,(j+1)} a_3 \right\} ,$$

$$b_2 = i\sqrt{\frac{\epsilon - m}{\epsilon + m}} \left\{ \frac{\sqrt{2}}{2}\ a_1 -\ a_3 \right\} , \tag{107}$$

$$b_3 = i\sqrt{\frac{\epsilon - m}{\epsilon + m}} \left\{ \frac{1}{4} \frac{2j+1}{j+1} \frac{\sqrt{4j+6}}{\sqrt{4\,j-2}} a_1 - \frac{2j+1}{4\,(j+1)} a_2 - \sqrt{2} \frac{\sqrt{4j+6}}{\sqrt{4j-2}} \frac{1}{4\,(j+1)} a_3 \right\} .$$

10. Accounting for Algebraic and Differential Constraints

The above constraints (39) and (43) being transformed to the variables F_a, G_a will read:

$$\begin{gathered} F_2 + G_0 = \sqrt{2} F_3 \, , \quad G_2 + F_0 = -\sqrt{2} G_3 \, , \\ -i\epsilon F_0 - (\frac{d}{dr} + \frac{1}{r}) F_2 = \frac{1}{\sqrt{2}r} [b\ F_1 + (a+1)\ F_3] \, , \\ -i\epsilon G_0 - (\frac{d}{dr} + \frac{1}{r}) G_2 = \frac{1}{\sqrt{2}r} [b\ G_1 + (a-1)\ G_3]. \end{gathered} \tag{108}$$

We are to transform Eqs. (108) to the variables

$$F_0, \bar{F}_1, \bar{F}_2, \bar{F}_3,\ G_0, \bar{G}_1, \bar{G}_2, \bar{G}_3.$$

With the use of the formulas $F = S^{-1} \bar{F},\ G = S'^{-1} \bar{G}$, we get expressions for F_i, G_i:

$$F_1 = \frac{1}{8(j+1)\sqrt{2j-1}} \left\{ \sqrt{(2j+3)(2j-1)}\ a_1 f_{j-1/2} \right.$$

$$\left. +(2j-1) a_2 f_{j+3/2} + \sqrt{(2j+3)(2j-1)} \sqrt{2} a_3 f_{j-1/2} \right\} ,$$

$$F_2 = \frac{1}{8(j+1)\sqrt{2j-1}} \left\{ \sqrt{2}(2j+3) a_1 f_{j-1/2} \right.$$

$$\left. + \sqrt{2}\sqrt{(2j+3)(2j-1)} a_2 f_{j+3/2} - 2(2j+1) a_3 f_{j-1/2} \right\},$$

$$F_3 = \frac{1}{8(j+1)\sqrt{2j-1}} \left\{ (-6j-5) a_1 f_{j-1/2} \right.$$

$$\left. + \sqrt{(2j+3)(2j-1)} a_2 f_{j+3/2} + \sqrt{2}(2j+3) a_3 f_{j-1/2} \right\},$$

and

$$G_1 = -\frac{1}{4j\sqrt{j+3/2}} \{ -\sqrt{(j+3/2)(j-1/2)} b_1 f_{j+1/2} - (j+3/2) b_2 f_{j-3/2} -$$

$$- \sqrt{2}\sqrt{(j-1/2)(j+3/2)} b_3 f_{j+1/2} \},$$

$$G_2 = -\frac{1}{4j\sqrt{j+3/2}} \left\{ \sqrt{2}(j-1/2) b_1 f_{j+1/2} \right.$$

$$\left. + \sqrt{2}\sqrt{(j+3/2)(j-1/2)} b_2 f_{j-3/2} - (2j+1) b_3 f_{j+1/2} \right\},$$

$$G_3 = -\frac{1}{4j\sqrt{j+3/2}} \left\{ (3j+1/2) b_1 f_{j+1/2} \right.$$

$$\left. - \sqrt{(j-1/2)(j+3/2)} b_2 f_{j-3/2} - \sqrt{2}(j-1/2) b_3 f_{j+1/2} \right\}.$$

It is convenient to introduce shortening notations

$$\alpha = (\det S)^{-1} = \frac{1}{8(j+1)\sqrt{2j-1}},$$

$$\beta = (\det S')^{-1} = -\frac{1}{4j\sqrt{j+3/2}}.$$

Let us substitute expressions for F_0, F_i and G_0, G_i:

$$F_0 = a_0 f_{j+1/2},$$

$$F_1 = \alpha \left\{ \sqrt{(2j+3)(2j-1)} a_1 f_{j-1/2} \right.$$

$$\left. + (2j-1) a_2 f_{j+3/2} + \sqrt{(2j+3)(2j-1)}\sqrt{2} a_3 f_{j-1/2} \right\},$$

$$F_2 = \alpha \left\{ \sqrt{2}(2j+3) a_1 f_{j-1/2} \right.$$

$$\left. + \sqrt{2}\sqrt{(2j+3)(2j-1)} a_2 f_{j+3/2} - 2(2j+1) a_3 f_{j-1/2} \right\},$$

$$F_3 = \alpha \Big\{ (-6j-5) a_1 f_{j-1/2}$$
$$+ \sqrt{(2j+3)(2j-1)} a_2 f_{j+3/2} + \sqrt{2}(2j+3) a_3 f_{j-1/2} \Big\} ,$$

and

$$G_0 = b_0 f_{j-1/2},$$
$$G_1 = \beta \Big\{ -\sqrt{(j+3/2)(j-1/2)} b_1 f_{j+1/2}$$
$$-(j+3/2) b_2 f_{j-3/2} - \sqrt{2}\sqrt{(j-1/2)(j+3/2)} b_3 f_{j+1/2} \Big\} ,$$
$$G_2 = \beta \Big\{ \sqrt{2}(j-1/2) b_1 f_{j+1/2}$$
$$+ \sqrt{2}\sqrt{(j+3/2)(j-1/2)} b_2 f_{j-3/2} - (2j+1) b_3 f_{j+1/2} \Big\} ,$$
$$G_3 = \beta \big\{ (3j+1/2) b_1 f_{j+1/2}$$
$$-\sqrt{(j-1/2)(j+3/2)} b_2 f_{j-3/2} - \sqrt{2}(j-1/2) b_3 f_{j+1/2} \Big\}$$

into algebraic equations in (108), this results in

$$\alpha \Big\{ \sqrt{2}(2j+3) a_1 f_{j-1/2} + \sqrt{2}\sqrt{(2j+3)(2j-1)} a_2 f_{j+3/2}$$
$$- 2(2j+1) a_3 f_{j-1/2} \big\} + b_0 f_{j-1/2}$$
$$= \sqrt{2}\, \alpha \Big\{ (-6j-5) a_1 f_{j-1/2} + \sqrt{(2j+3)(2j-1)} f_{j+3/2} a_2$$
$$+ \sqrt{2}(2j+3) a_3) f_{j-1/2} \Big\} ,$$
$$\beta \Big\{ \sqrt{2}(j-1/2) b_1 f_{j+1/2} + \sqrt{2}\sqrt{(j+3/2)(j-1/2)} b_2 f_{j-3/2}$$
$$-(2j+1) b_3 f_{j+1/2} \big\} + a_0 f_{j+1/2}$$
$$= -\sqrt{2}\, \beta \Big\{ (3j+1/2) b_1 f_{j+1/2} - \sqrt{(j-1/2)(j+3/2)} b_2 f_{j-3/2}$$
$$- \sqrt{2}(j-1/2) b_3 f_{j+1/2} \Big\} .$$

After needed re-grouping the terms, we obtain linear constraints for numerical coefficients

$$8\alpha\, (j+1)(\sqrt{2}\, a_1 - a_3) + b_0 = 0 \, , \quad 4\beta j\, (\sqrt{2}\, b_1 - b_3) + a_0 = 0 \, . \tag{109}$$

Whence, allowing for expressions for α and β, we get

$$\sqrt{2}\, a_1 - a_3 = -\sqrt{2}\sqrt{j-1/2}\, b_0 \, , \quad \sqrt{2}\, b_1 - b_3 = +\sqrt{j+3/2}\, a_0 \, . \tag{110}$$

The third and fourth equations in (108) give respectively

$$-(\frac{d}{dr}+\frac{1}{r})\alpha\left\{\sqrt{2}(2j+3)\ a_1 f_{j-1/2}\right.$$

$$\left.+\sqrt{2}\sqrt{(2j+3)(2j-1)}a_2 f_{j+3/2}-2(2j+1)a_3 f_{j-1/2}\right\}$$

$$=+i\epsilon a_0 f_{j+1/2}+\frac{1}{r}\frac{j+3/2}{\sqrt{2}}\alpha\left\{(-6j-5)\ \ a_1 f_{j-1/2}\right.$$

$$\left.+\sqrt{(2j+3)(2j-1)}a_2 f_{j+3/2}+\sqrt{2}(2j+3)a_3 f_{j-1/2}\right\}$$

$$+\frac{\sqrt{(j-1/2)(j+3/2)}}{\sqrt{2}r}\alpha\left\{\sqrt{(2j+3)(2j-1)}a_1 f_{j-1/2}\right.$$

$$\left.+(2j-1)a_2 f_{j+3/2}+\sqrt{(2j+3)(2j-1)}\sqrt{2}a_3 f_{j-1/2}\right\}$$

and

$$-(\frac{d}{dr}+\frac{1}{r})\beta\left\{\sqrt{2}(j-1/2)\ \ b_1 f_{j+1/2}\right.$$

$$\left.+\sqrt{2}\sqrt{(j+3/2)(j-1/2)}\ \ b_2 f_{j-3/2}-(2j+1)\ \ b_3 f_{j+1/2}\right\}$$

$$=+i\epsilon b_0 f_{j-1/2}+\frac{1}{r}\frac{j-1/2}{\sqrt{2}}\beta\left\{(3j+1/2)b_1 f_{j+1/2}\right.$$

$$\left.-\sqrt{(j-1/2)(j+3/2)}b_2 f_{j-3/2}-\sqrt{2}(j-1/2)b_3 f_{j+1/2}\right\}$$

$$+\frac{\sqrt{(j-1/2)(j+3/2)}}{\sqrt{2}r}\left\{-\sqrt{(j+3/2)(j-1/2)}b_1 f_{j+1/2}\right.$$

$$\left.-(j+3/2)b_2 f_{j-3/2}-\sqrt{2}\sqrt{(j-1/2)(j+3/2)}b_3 f_{j+1/2}\right\}.$$

After elementary manipulation we arrive at

$$\left[-(j+3/2)\sqrt{2}a_1+(2j+1)a_3\right]\left(\frac{d}{dr}+\frac{1}{r}\right)f_{j-1/2}$$

$$-\sqrt{2}\ \sqrt{(j+3/2)(j-1/2)}\left(\frac{d}{dr}+\frac{j+3/2}{r}\right)a_2 f_{j+3/2}$$

$$=+i\epsilon\frac{a_0}{2\alpha}f_{j+1/2}+\frac{1}{r}\frac{j+3/2}{\sqrt{2}}[(-2j-3)a_1+(2j+1)\sqrt{2}a_3]\ f_{j-1/2}\,, \qquad (111)$$

$$\left[-\sqrt{2}(j-1/2)b_1+(2j+1)b_3\right]\left(\frac{d}{dr}+\frac{1}{r}\right)f_{j+1/2}$$

$$= \sqrt{2}\sqrt{(j+3/2)(j-1/2)}\left(\frac{d}{dr} - \frac{j-1/2}{r}\right) b_2 f_{j-3/2}$$

$$= +i\epsilon\, \frac{b_0}{\beta} f_{j-1/2} + \frac{1}{r}\frac{j-1/2}{\sqrt{2}}\; [(2j-1)\; b_1 - (2j+1)\sqrt{2}b_3]\; f_{j+1/2}\,. \qquad (112)$$

Now, transform the last equations to Bessel functions (let it be $p = j+1$):

$$f_{j+1/2} = \sqrt{x}\; Z_{j+1} = \sqrt{x}\; Z_p\,, \quad f_{j+3/2} = \sqrt{x}\; Z_{j+2} = \sqrt{x}\; Z_{p+1}\,,$$
$$f_{j-1/2} = \sqrt{x}\; Z_j = \sqrt{x}\; Z_{p-1}\,, \quad f_{j-3/2} = \sqrt{x}\; Z_{j-1} = \sqrt{x}\; Z_{p-2}\,. \qquad (113)$$

In this way we reduce differential constraints to the form

$$[-(j+3/2)\sqrt{2}a_1 + (2j+1)a_3](\frac{d}{dx} + \frac{1}{x})\sqrt{x}Z_{p-1}$$
$$-\sqrt{2}\sqrt{(j+3/2)(j-1/2)}(\frac{d}{dx} + \frac{j+3/2}{x})a_2\sqrt{x}Z_{p+1}$$
$$= +i\Gamma\frac{a_0}{2\alpha}\sqrt{x}Z_p + \frac{1}{x}\frac{j+3/2}{\sqrt{2}}[(-2j-3)a_1 + (2j+1)\sqrt{2}a_3]\sqrt{x}Z_{p-1}\,, \qquad (114)$$

$$[-\sqrt{2}(j-1/2)b_1 + (2j+1)b_3](\frac{d}{dx} + \frac{1}{x})\sqrt{x}Z_p$$
$$-\sqrt{2}\sqrt{(j+3/2)(j-1/2)}(\frac{d}{dx} - \frac{j-1/2}{x})b_2\sqrt{x}Z_{p-2}$$
$$= +i\Gamma\frac{b_0}{\beta}\sqrt{x}\; Z_{p-1} + \frac{1}{x}\frac{j-1/2}{\sqrt{2}}[(2j-1)b_1 - (2j+1)\sqrt{2}b_3]\sqrt{x}Z_p\,; \qquad (115)$$

where the notation $\Gamma = \epsilon/\sqrt{\epsilon^2 - m^2}$ is used. Further, applying the above commutation rule, we derive

$$[-(j+3/2)\sqrt{2}a_1 + (2j+1)a_3](\frac{d}{dx} + \frac{3/2}{x})Z_{p-1}$$
$$-\sqrt{2}\sqrt{(j+3/2)(j-1/2)}(\frac{d}{dx} + \frac{j+2}{x})a_2 Z_{p+1}$$
$$= +i\Gamma\frac{a_0}{2\alpha}Z_p + \frac{1}{x}\frac{j+3/2}{\sqrt{2}}[(-2j-3)a_1 + (2j+1)\sqrt{2}a_3]Z_{p-1}\,, \qquad (116)$$

$$[-\sqrt{2}(j-1/2)b_1 + (2j+1)b_3](\frac{d}{dx} + \frac{3/2}{x})Z_p$$
$$-\sqrt{2}\sqrt{(j+3/2)(j-1/2)}(\frac{d}{dx} - \frac{j-1}{x})b_2 Z_{p-2}$$
$$= i\Gamma\frac{b_0}{\beta}Z_{p-1} + \frac{1}{x}\frac{j-1/2}{\sqrt{2}}[(2j-1)b_1 - (2j+1)\sqrt{2}b_3]Z_p\,. \qquad (117)$$

The second equation it is convenient to re-write with the use of primed parameter:

$$p = j+1, p-1 = j, p+1 = j+2\,, \quad p-1 = p' = j\,, p'+1 = j+1, p'-1 = j-1\,,$$

so we arrive at the two equations

$$[-(j+3/2)\sqrt{2}a_1 + (2j+1)a_3](\frac{d}{dx} - \frac{p-1}{x})\; Z_{p-1}$$
$$-\sqrt{2}\;\sqrt{(j+3/2)(j-1/2)}(\frac{d}{dx} + \frac{p+1}{x})\; a_2\; Z_{p+1} = i\Gamma \frac{a_0}{2\alpha}\; Z_p\,,$$

$$[-(j-1/2)\sqrt{2}b_1 + (2j+1)b_3](\frac{d}{dx} + \frac{p'+1}{x})\; Z_{p'+1}$$
$$-\sqrt{2}\sqrt{(j+3/2)(j-1/2)}\left(\frac{d}{dx} - \frac{p'-1}{x}\right) b_2\; Z_{p'-1} = i\Gamma \frac{b_0}{\beta}\; Z_{p'}\,.$$

Whence, applying the known properties of Bessel functions, we derive two linear equations for numerical coefficients (additionally we write down the above two constraints as well)

$$\begin{aligned} \sqrt{2}\; a_1 - a_3 &= -\sqrt{j-1/2}\;\sqrt{2}b_0\,, \\ \sqrt{2}\; b_1 - b_3 &= +\sqrt{j+3/2}\; a_0\,, \\ i\frac{\Gamma}{2\alpha}\; a_0 &- (j+3/2)\;\sqrt{2}a_1 \\ +\sqrt{(j+3/2)(j-1/2)}\;\sqrt{2}a_2 &+ (2j+1)\; a_3 = 0\,, \\ i\frac{\Gamma}{\beta}\; b_0 &+ (j-1/2)\;\sqrt{2}b_1 \\ -\sqrt{(j+3/2)(j-1/2)}\;\sqrt{2}b_2 &- (2j+1)\; b_3 = 0\,. \end{aligned} \tag{118}$$

Let us remind that we have to remember the previously derived conditions

$$b_0 = -i\frac{\sqrt{\epsilon - m}}{\sqrt{\epsilon + m}}\; a_0\;, \frac{1}{2\alpha} = 4(j+1)\sqrt{2j-1}\;, \frac{1}{\beta} = -4j\sqrt{j+3/2}\,.$$

First, with the use of

$$a_0 = \frac{\sqrt{2}}{\sqrt{j+3/2}}b_1 - \frac{1}{\sqrt{j+3/2}}b_3\,, b_0 = -\frac{a_1}{\sqrt{j-1/2}} + \frac{1}{\sqrt{2}\sqrt{j-1/2}}a_3\,, \tag{119}$$

let us exclude the variables a_0 and b_0 from two last equations in (118):

$$
\begin{aligned}
& i\frac{\Gamma}{2\alpha}\frac{1}{\sqrt{j+3/2}}(\sqrt{2}b_1 - b_3) - (j+3/2)\sqrt{2}a_1 \\
& + \sqrt{(j+3/2)(j-1/2)}\sqrt{2}a_2 + (2j+1)\, a_3 = 0\,, \\
& i\frac{\Gamma}{\beta}\frac{1}{\sqrt{j-1/2}}(-a_1 + \frac{a_3}{\sqrt{2}}) + (j-1/2)\sqrt{2}b_1 \\
& - \sqrt{(j+3/2)(j-1/2)}\sqrt{2}b_2 - (2j+1)b_3 = 0\,.
\end{aligned} \tag{120}
$$

In turn, due do the known relationship $\sqrt{\epsilon - m}\; a_0 = i\sqrt{\epsilon + m}\; b_0$ we get

$$
\frac{\sqrt{\epsilon - m}}{\sqrt{j+3/2}}\;(\;\sqrt{2}b_1 - b_3\;) = \frac{i\sqrt{\epsilon+m}}{\sqrt{j-1/2}}\;(\;-a_1 + \frac{a_3}{\sqrt{2}}\;)\,. \tag{121}
$$

This identity permits us reduce Eqs. in (120) to the form, when the first contains only the variables a_i, and the second – only the variables b_i:

$$
\begin{aligned}
& \frac{4\epsilon(j+1)}{\epsilon - m}(a_1 - \frac{a_3}{\sqrt{2}}) - (j+\frac{3}{2})a_1 \\
& + (2j+1)\frac{a_3}{\sqrt{2}} + \sqrt{(j+\frac{3}{2})(j-\frac{1}{2})}a_2 = 0\,, \\
& \frac{4\epsilon j}{\epsilon + m}(b_1 - \frac{b_3}{\sqrt{2}}) - (j-\frac{1}{2})b_1 + (2j+1)\frac{b_3}{\sqrt{2}} \\
& + \sqrt{(j+\frac{3}{2})(j-\frac{1}{2})}b_2 = 0\,;
\end{aligned} \tag{122}
$$

remembering that coefficient a_0, b_0 may be found by means of (119). The general structure of Eqs. (122) is

$$
A_1 a_1 + A_2 a_2 + A_3 a_3 = 0\,, \qquad B_1 b_1 + B_2 b_2 + B_3 b_3 = 0\,, \tag{123}
$$

where

$$
A_1 = \frac{4(j+1)\epsilon - (j+3/2)(\epsilon - m)}{\epsilon - m},
$$

$$
A_3 = -\frac{1}{\sqrt{2}}\frac{4(j+1)\epsilon - (2j+1)(\epsilon - m)}{\epsilon - m},
$$

$$
A_2 = \sqrt{(j-1/2)(j+3/2)},
$$

$$
B_1 = \frac{4\epsilon j - (j-1/2)(\epsilon + m)}{\epsilon + m}\;,
$$

$$B_3 = -\frac{1}{\sqrt{2}} \frac{4\epsilon j - (2j+1)(\epsilon + m)}{\epsilon + m} ,$$

$$B_2 = \sqrt{(j-1/2)(j+3/2)} .$$

Having used the above established expressions for coefficients b_i through a_i, we may transform the second linear relation in (123) to the variables a_i:

$$\frac{(6\,\epsilon\,j + 2\,jm + 5\,\epsilon + 3\,m)}{2(\epsilon - m)}\; a_1 - \frac{(2\,\epsilon\,j + 2\,jm + 3\,\epsilon + m)}{2(\epsilon - m)}\; \sqrt{2} a_3$$

$$+ \frac{1}{4}\,\sqrt{4\,j + 6}\,\sqrt{4\,j - 2}\; a_2 = 0 .$$

this equation coincides with the first one in (123). Therefore, in (122) we have only one independent linear constraint

$$A_1 a_1 + A_2 a_2 + A_3 a_3 = 0 . \qquad (124)$$

We are to fix two independent solutions of the main linear constraint for coefficients a_i. In this way obtain two independent solutions of the equations for spin 3/2 particle, evidently there exist many possibilities for that choosing. By simplicity reason, we may set

$$I, \quad a_1 = +\sqrt{2} a_3 , \frac{2j\epsilon + (\epsilon + m)}{\epsilon - m}\; a_1 + \sqrt{(j+3/2)(j-1/2)}\; a_2 = 0 ,$$
$$II, a_1 = -\sqrt{2} a_3 , \frac{(2j+2)(\epsilon + m)}{\epsilon - m}\; a_1 + \sqrt{(j+3/2)(j-1/2)}\; a_2 = 0 .$$

Let us fix the quantity $a_2 = -[\sqrt{(j+3/2)(j-1/2)}]^{-1}$, then the above formulas become shorter:

$$\begin{aligned} I, a_1 &= \frac{\epsilon - m}{2j\epsilon + (\epsilon + m)} , \sqrt{2} a_3 = +a_1 , a_2 = -\frac{1}{\sqrt{(j+3/2)(j-1/2)}} , \\ II, a_1 &= \frac{\epsilon - m}{(2j+2)(\epsilon + m)} , \sqrt{2} a_3 = -a_1 , a_2 = -\frac{1}{\sqrt{(j+3/2)(j-1/2)}} . \end{aligned} \qquad (125)$$

11. CONCLUSION

The wave equation for a spin 3/2 particle, described by 16-component vector-bispinor, is investigated in spherical coordinates. The complete equation is split into the main equation, and two additional constraints, algebraic and differential ones. There are constructed solutions on which 4 operators are diaginalyzed: they correspond to quantum numbers $\{\epsilon, j, m, P\}$. After separating the variables, we derive the main system of 8 radial first order equations and additional 2 algebraic and 2 differential constraints. Solutions of the radial equations are constructed as linear combinations of a number of Bessel functions. With the use of the known properties of the Bessel functions, the system of differential equations is transformed to an algebraic linear constraint for three numerical coefficients a_1, a_2, a_3. Its solutions may be chosen in various ways by resolving the simple linear condition $A_1 a_1 + A_2 a_2 + A_3 a_3 = 0$, where coefficients A_i are expressed trough the quantum numbers ϵ, j. Two most simple and symmetric solutions have been chosen. Thus, at fixed quantum numbers $\{\epsilon, j, m, P\}$ there exists double-degeneration of the quantum states.

Closely related extensions of that problem might be: particle in presence of the Coulomb field, and particle in curved space-time background, for instance of de Sitter type.

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Chapter 4

Spinor Maxwell Equations in Riemannian Space-Time and Modeling Constitutive Relations

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Abstract

It is known that vacuum Maxwell equations being considered on the background of any pseudo-Riemannin space-time may be interpreted as Maxwell equations in Minkowski space but specified in some effective medium, which constitutive relations are determined by metric of the curved space-time. In that context, we will consider space-time models

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with event horizon. All of them have a metric of one the same structure, we restrict ourselves to spherically symmetric case, and consider de Sitter, anti de Sitter, and Schwarzschild models. Also we will study hyperbolic Lobachevsky and spherical Riemann models, parameterized coordinates with spherical and cylindric symmetry. We will prove that in all examined cases, effective tensors and of of electric permittivity (ϵ_{ij}) and magnetic permeability (ϵ_{ij}) obey one the same condition: $\epsilon_{ij}(x)\,\mu_{jk}(x) = \delta_{ik}$. Simplicity of expressions for these tensors $\epsilon_{ij}(x)$ and $\mu_{jk}(x)$ is misleading, for each of curved space-time model we are to solve Maxwell equations separately and anew. We will construct these solutions explicitly, applying Maxwell equations in spinor form:

Keywords: Maxwell equations, spinor formalism, Riemannian space-time, constitutive relations, spherical and cylindric symmetry

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1. INTRODUCTION

It is known that vacuum Maxwell equations being considered on the background of any pseudo-Riemannin space-time may be interpreted as Maxwell equations in Minkowski space but specified in some effective medium, which constitutive relations are determined by metric of the curved space-time [1]–[45]. Detailed treatment of such a possibility for quasi-Cartesian coordinates was given in [46].

Recall the general approach. Let us start with Maxwell equations in some medium when using some curvilinear coordinates $(x = x^{\sigma})$ with relevant metric $G_{\alpha\beta}(x)$. So we have

$$\partial_{\alpha}\, F_{\beta\gamma} + \partial_{\beta}\, F_{\gamma\alpha} + \partial_{\gamma}\, F_{\alpha\beta} = 0\,, \quad \frac{1}{\sqrt{-G}}\,\partial_{\beta}\sqrt{-G}\,H^{\alpha\beta} = J^{\alpha}\,, \tag{1}$$

where $G = \det[G_{\alpha\beta}(x)]$ stands for a determinant of the metric tensor. We are to use two electromagnetic tensors, $F_{\alpha\beta}(x)$ and $H^{\alpha\beta}(x)$, related to each other by means of some constitutive equations. Also let us assume that a certain Riemannian space-time model is parameterized by formally similar coordinates (x^{σ}) with and defined by metric $g_{\alpha\beta}(x)$. Maxwell vacuum equations in that space-time have the form

$$\partial_{\alpha} f_{\beta\gamma} + \partial_{\beta} f_{\gamma\alpha} + \partial_{\gamma} f_{\alpha\beta} = 0\,, \quad \frac{1}{\sqrt{-g}}\,\partial_{\beta}\sqrt{-g}\,h^{\beta\alpha} = j^{\alpha}\,, \tag{2}$$

where

$$h_{\alpha\beta}(x) = \epsilon_0 f_{\alpha\beta}(x)\,, \quad h^{\alpha\beta}(x) = \epsilon_0 g^{\alpha\rho}(x) g^{\beta\sigma}(x)\, f_{\rho\sigma}(x)\,. \tag{3}$$

In (3) we have specified vacuum constitutive relations, note that $g = \det[g_{\alpha\beta}(x)]$. The second equation in (2) may be re-written as

$$\frac{\sqrt{-G}}{\sqrt{-g}}\,\frac{1}{\sqrt{-G}}\,\partial_\beta \sqrt{-G}\frac{\sqrt{-g}}{\sqrt{-G}}\,h^{\beta\alpha} = j^\alpha\,. \tag{4}$$

We may define new field variables

$$F_{\alpha\beta}(x) = f_{\alpha\beta}(x)\,, \;\; H^{\beta\alpha}(x) = \frac{\sqrt{-g}}{\sqrt{-G}}\,h^{\beta\alpha}(x)\,, \;\; J^\alpha(x) = \frac{\sqrt{-g(x)}}{\sqrt{-G(x)}}\,j^\alpha(x)\,, \tag{5}$$

then eqs. (2) may be interpreted as Maxwell equations of the form (1) in flat space-time :

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0\,, \quad \frac{1}{\sqrt{-G}}\,\partial_\beta \sqrt{-G}\,H^{\alpha\beta} = J^\alpha\,; \tag{6}$$

and relationship between two electromagnetic tensors is determined by the formula

$$H^{\alpha\beta}(x) = \epsilon_0\,\frac{\sqrt{-g(x)}}{\sqrt{-G(x)}}\,g^{\alpha\rho}(x) g^{\beta\sigma}(x)\;F_{\rho\sigma}(x)\,. \tag{7}$$

In that context, let us consider space-time models with event horizon (in absence of the external currents vanish). All of them have a metric of one the same structure, we restrict ourselves to spherically symmetric case, and consider de Sitter, anti de Sitter, and Schwarzschild models. Also we will study hyperbolic Lobachevsky and spherical Riemann models, parameterized coordinates with spherical and cylindric symmetry.

We will prove that in all examined cases, effective tensors (ϵ_{ij}) and (ϵ_{ij}) of of electric permittivity and magnetic permeability obey one the same condition:

$$\epsilon_{ij}(x)\,\mu_{jk}(x) = \delta_{ik}\,. \tag{8}$$

Simplicity of relations (7)–(8) is misleading, in fact for each of curved space-time model we are to solve Maxwell equations separately and anew. We will construct these solutions explicitly, applying Maxwell equations in spinor form:

$$i\sigma^{\alpha}(x)\ [\,\partial_{\alpha} + \Sigma_{\alpha}(x) \otimes I + I \otimes \Sigma_{\alpha}(x)\,]\ \xi(x) = 0\,, \tag{9}$$

where local Pauli matrices and 2-spinor connection are used, the quantity $\xi(x)$ designates symmetric electromagnetic spinor.

2. Maxwell Equations, Spinors and Quaternions

To introduce spinor notations, let us start with the ordinary Dirac equation

$$(i\gamma^{a}\partial_{a} - m)\Psi = 0, \gamma^{a} = \begin{vmatrix} 0 & \bar{\sigma}^{a} \\ \sigma^{a} & 0 \end{vmatrix}, \Psi = \begin{vmatrix} \xi^{\alpha} \\ \eta_{\dot{\alpha}} \end{vmatrix}, \{\alpha, \dot{\alpha}\} = 1, 2\,; \tag{10}$$

$\sigma^{a} = (I, \sigma^{j}),\ \bar{\sigma}^{a} = (I, -\sigma^{j})$. In 2-spinor form we have two equations

$$i\sigma^{a}\partial_{a}\xi = m\eta\,, \quad i\bar{\sigma}^{a}\partial_{a}\eta = m\xi\,. \tag{11}$$

It is convenient to attach spinor indices to Pauli matrices: $\sigma^{a} = (\sigma^{a})_{\dot{\beta}\alpha}$, $\bar{\sigma}^{a} = (\bar{\sigma}^{a})^{\beta\dot{\alpha}}$, then eqs. (11) read

$$i(\sigma^{a}\partial_{a})_{\dot{\beta}\alpha}\,\xi^{\alpha} = m\eta_{\dot{\beta}}, \quad i(\bar{\sigma}^{a}\partial_{a})^{\beta\dot{\alpha}}\,\eta_{\dot{\alpha}} = m\xi^{\beta}\,. \tag{12}$$

Electromagnetic tensor is equivalent to a pair of symmetrical 2-rank spinors: $F_{mn} \longleftrightarrow \{\xi^{\alpha\beta}, \eta_{\dot{\alpha}\dot{\beta}}\}$; correspondingly, 8 Maxwell equations are presented as follows

$$(\sigma^{a}\partial_{a})_{\dot{\rho}\alpha}\,\xi^{\alpha\beta} = (\sigma^{b})_{\dot{\rho}\alpha}\omega^{\alpha\beta}\,J_{b}\,, \quad (\bar{\sigma}^{a}\partial_{a})^{\rho\dot{\alpha}}\,\eta_{\dot{\alpha}\dot{\beta}} = (\bar{\sigma}^{b})^{\rho\dot{\alpha}}\omega_{\dot{\alpha}\dot{\beta}}\,J_{b}\,; \tag{13}$$

the second equation is conjugate to the first. In (13) we use spinor metrical matrices

$$(\omega_{\alpha\beta}) = i\sigma^{2}\,, \quad (\omega^{\alpha\beta}) = -i\sigma^{2}\,; \quad (\omega_{\dot{\alpha}\dot{\beta}}) = i\sigma^{2}\,, \quad (\omega^{\dot{\alpha}\dot{\beta}}) = -i\sigma^{2}\,. \tag{14}$$

To prove equivalence of the spinor form (13) to ordinary Maxwell equation in vector notations let us apply notations without spinor indices. To this end, we take into account identities

$$(\xi^{\alpha\beta}) = \Sigma^{mn}F_{mn}\sigma^{2}\,, \quad (\eta_{\dot{\alpha}\dot{\beta}}) = -\bar{\Sigma}^{mn}F_{mn}\sigma^{2}\,, \tag{15}$$

where

$$\Sigma^{mn} = \frac{1}{4}(\bar{\sigma}^m \sigma^n - \bar{\sigma}^n \sigma^m) \,, \quad \bar{\Sigma}^{mn} = \frac{1}{4}(\sigma^m \bar{\sigma}^n - \sigma^n \bar{\sigma}^m) \,. \tag{16}$$

Then, eqs. (13) may be re-written as

$$\sigma^a \partial_a \, \Sigma^{mn} F_{mn} = -\sigma^b J_b \,, \qquad \bar{\sigma}^a \partial_a \, \bar{\Sigma}^{mn} F_{mn} = -\bar{\sigma}^b J_b \,. \tag{17}$$

We are to take into account identities

$$\Sigma^{mn} F_{mn} = \sigma^1 (F_{01} - iF_{23}) + \sigma^2 (F_{02} - iF_{31}) + \sigma^3 (F_{03} - iF_{12}) \,,$$

$$\bar{\Sigma}^{mn} F_{mn} = \sigma^1 (-F_{01} - iF_{23}) + \sigma^2 (-F_{02} - iF_{31}) + \sigma^3 (-F_{03} - iF_{12}) \,;$$

with notations

$$F_{01} = -E^1 \,, F_{02} = -E^2 \,, F_{03} = -E^3 \,, F_{23} = B^1 \,, F_{31} = B^2 \,, F_{12} = B^3 \tag{18}$$

they read

$$\begin{gathered} \Sigma^{mn} F_{mn} = -\sigma^1 (E^1 + iB^1) - \sigma^1 (E^2 + iB^2) - \sigma^1 (E^3 + iB^3) = -\sigma^j a_j , \\ \bar{\Sigma}^{mn} F_{mn} = \sigma^1 (E^1 - iB^1) + \sigma^1 (E^2 - iB^2) + \sigma^1 (E^3 - iB^3) = +\sigma^j b_j , \end{gathered} \tag{19}$$

and

$$(\xi^{\alpha\beta}) = \Sigma^{mn} F_{mn} \sigma^2 = \left| \begin{array}{cc} -i(a_1 - ia_2) & ia_3 \\ ia_3 & +i(a_1 + ia_2) \end{array} \right| ,$$

$$(\eta_{\dot{\alpha}\dot{\beta}}) = -\bar{\Sigma}^{mn} F_{mn} \sigma^2 = \left| \begin{array}{cc} -i(b_1 - ib_2) & ib_3 \\ ib_3 & i(b_1 + ib_2) \end{array} \right| .$$

Taking into account (19), Maxwell equations (17) may be presented in the form

$$\sigma^a \partial_a \, \sigma^j a_j = \sigma^b J_b \,, \quad \bar{\sigma}^a \partial_a \, \sigma^j b_j = -\bar{\sigma}^b J_b \,, \tag{20}$$

or

$$(\partial_0 + \sigma^l \partial_l) \, (\sigma^k a_k) = J_0 + \sigma^j J_j \,, \quad (\partial_0 - \sigma^l \partial_l) \, (\sigma^k b_k) = -J_0 + \sigma^j J_j \,. \tag{21}$$

From (21) we derive

$$\sigma^n \partial_0 a_n + (\delta_{lk} + i\omega_{nlk} \sigma^n) \partial_l a_k \,, = J_0 + \sigma^n J_n$$

$$\sigma^n \partial_0 b_n - (\delta_{lk} + i\omega_{nlk}\sigma^n)\partial_l b_k = -J_0 + \sigma^n J_n \,.$$

Therefore, we have four equations

$$\partial_l a_l = J_0 \,, \partial_0 a_n + i\omega_{nlk}\partial_l a_k = J_n \,, \partial_l b_l = J_0 \,, \partial_0 b_n - i\omega_{nlk}\partial_l b_k = J_n \,,$$

or differently

$$\begin{array}{ll} (1) & \partial_l(E^l + iB^l) = J_0 \,, \\ (2) & \partial_0(E^l + iB^l) + i\omega_{nlk}\partial_l(E^k + iB^k) = J_n \,, \\ (1') & \partial_l(E^l - iB^l) = J_0 \,, \\ (2') & \partial_0(E^l - iB^l) - i\omega_{nlk}\partial_l(E^k - iB^l) = J_n \,. \end{array}$$

Summing and subtracting equations within each pair, we obtain

$$1 + 1', \quad \partial_l E^l = J_0 \,, \quad 1 - 1', \quad \partial_l B^l = 0 \,,$$

$$2 + 2', \quad \partial_0 E^n - \omega_{nlk}\partial_l B^k = J_k \,, \quad 2 - 2', \quad \partial_0 B^n + \omega_{nlk}\partial_l E^k = 0 \,;$$

they may be identified with Maxwell equations in vector form

$$\mathrm{div}\,\mathbf{E} = J^0, \quad \mathrm{div}\,\mathbf{B} = 0 \,, \quad \mathrm{rot}\,\mathbf{B} = \partial_0\mathbf{E} + \mathbf{J} \,, \quad \mathrm{rot}\,\mathbf{E} = -\partial_0\mathbf{B} \,, \qquad (22)$$

where

$$\mathbf{E} = (E^n) \,, \; \mathbf{B} = (B^n) \,, \quad J^0 = J_0 \,, \; \mathbf{J} = (J^n) = (-J_n).$$

3. Separating the Variables in de Sitter Like Models

Generally covariant Maxwell equations in spinor form can be found by the same method which is used for generalizing the Dirac equation

$$\begin{array}{c} i\sigma^\alpha(x)\,[\partial_\alpha + \Sigma_\alpha(x)]\,\xi(x) = m\,\eta(x)\,, \\ i\bar{\sigma}^\alpha(x)\,\left[\partial_\alpha + \bar{\Sigma}_\alpha(x)\right]\eta(x) = m\,\xi(x)\,. \end{array} \qquad (23)$$

So, the Maxwell equations in spinor form are to generalized as follows [46]

$$\begin{array}{c} i\sigma^\alpha(x)\,[\partial_\alpha + \Sigma_\alpha(x) \otimes I + I \otimes \Sigma_\alpha(x)]\,\xi(x) = \sigma^\beta(x)(-i\sigma^2)J_\beta(x)\,, \\ i\bar{\sigma}^\alpha(x)\,\left[\partial_\alpha + \bar{\Sigma}_\alpha(x) \otimes I + I \otimes \bar{\Sigma}_\alpha(x)\right]\eta(x) = \sigma^\beta(x)(+i\sigma^2)J_\beta(x)\,, \end{array} \qquad (24)$$

where (see in [46]):

$$\sigma^{\alpha}(x) = \sigma^{a} e^{\alpha}_{(a)}(x)\,, \quad \bar{\sigma}^{\alpha}(x) = \bar{\sigma}^{a} e^{\alpha}_{(a)}(x)\,,$$
$$\Sigma_{\alpha}(x) = \frac{1}{2}\Sigma^{ab} e^{\beta}_{(a)} \nabla_{\alpha}(e_{(b)\beta}), \quad \bar{\Sigma}_{\alpha}(x) = \frac{1}{2}\bar{\Sigma}^{ab} e^{\beta}_{(x)} \nabla_{\alpha}(e_{(b)\beta})\,, \tag{25}$$
$$\Sigma^{ab} = \frac{1}{4}(\,\bar{\sigma}^{a}\sigma^{b} - \bar{\sigma}^{b}\,\sigma^{a}\,), \quad \bar{\Sigma}^{ab} = \frac{1}{4}(\sigma^{a}\bar{\sigma}^{b} - \sigma^{b}\bar{\sigma}^{a}).$$

As in Minkowski space, the second equation is conjugate to the first, so it suffices to study only the first one (further we follow the case without current)

$$\sigma^{\alpha}(x)\,[\partial_{\alpha} + \Sigma_{\alpha}(x) \otimes I + I \otimes \Sigma_{\alpha}(x)]\,\xi(x) = 0\,. \tag{26}$$

In (26), the quantity $\xi(x)$ stands for a symmetric 2-rank spinor, it can be treated as a (2×2)-matrix function. Eq. (26) may be presented with the help of Ricci rotation coefficients $\gamma_{abc}(x)$ as follows

$$\left[\sigma^{c} e^{\alpha}_{(c)}(x)\partial_{\alpha} + \sigma^{c}(\frac{1}{2}\Sigma^{ab} \otimes I + I \otimes \frac{1}{2}\Sigma^{ab})\gamma_{abc}(x)\right]\xi(x) = 0\,;$$
$$\gamma_{abc} = +e_{(a)\beta;\alpha} e^{\beta}_{(b)} e^{\alpha}_{(c)}\,, \quad \Sigma^{ab} = \frac{1}{4}(\bar{\sigma}^{a}\sigma^{b} - \bar{\sigma}^{b}\sigma^{a})\,. \tag{27}$$

Let us specify eq. (26) in static de Sitter coordinates:

$$dS^2 = (1 - r^2/\rho^2)c^2 dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{dr^2}{1 - r^2/\rho^2}\,, \tag{28}$$

where ρ is the curvature radius; below we will apply dimensionless coordinates $ct/\rho; \Longrightarrow t,\ r/\rho \Longrightarrow r$. We use diagonal spherical tetrad

$$x^{\alpha} = (t,\ \theta,\ \phi,\ \ r)\,, \ \ \varphi = 1 - r^2\,, \ \ \varphi' = \frac{d\varphi}{dr}\,,$$
$$e_{(0)\alpha} = (\frac{1}{\sqrt{\varphi}},\ 0,\ 0,\ 0)\,, \quad e_{(3)\alpha} = (0,\ 0,\ 0,\ \ \sqrt{\varphi})\,, \tag{29}$$
$$e_{(1)\alpha} = (0,\ \frac{1}{r},\ 0,\ 0)\,, \quad e_{(2)\alpha} = (0, 0,\ \frac{1}{r\sin\theta},\ 0)\,.$$

Local Pauli matrices are

$$\sigma^{\alpha}(x) = (\frac{1}{\sqrt{\varphi}}, -\frac{\sigma^1}{r}, -\frac{\sigma^2}{r\sin\theta}, -\sqrt{\varphi}\sigma^3), \bar{\sigma}^{\alpha}(x) = (\frac{1}{\sqrt{\varphi}}, \frac{\sigma^1}{r}, \frac{\sigma^2}{r\sin\theta}, \sqrt{\varphi}\sigma^3)\,.$$

Taking in mind general formulas

$$\sigma^{ab} = \frac{1}{4} \left| \begin{array}{cc} \bar{\sigma}^a \sigma^b - \bar{\sigma}^b \sigma^a & 0 \\ 0 & \sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a \end{array} \right| = \left| \begin{array}{cc} \Sigma^{ab} & 0 \\ 0 & \bar{\Sigma}^{ab} \end{array} \right| ,$$

$$\Gamma_\alpha(x) = \frac{1}{2} \left| \begin{array}{cc} \Sigma^{ab} e^\beta_{(a)} \nabla_\alpha(\, e^\alpha_{(b);\beta} & 0 \\ 0 & \bar{\Sigma}^{ab} e^\beta_{(a)} \nabla_\alpha (e^\alpha_{(b);\beta} \end{array} \right| = \left| \begin{array}{cc} \Sigma_\alpha(x) & 0 \\ 0 & \bar{\Sigma}_\alpha(x) \end{array} \right|$$

and the known expressions [47] for bispinor connection in de Sitter space, we find bispinor connections

$$\Gamma_t = \left| \begin{array}{cc} \Sigma_t & 0 \\ 0 & \bar{\Sigma}_t \end{array} \right| = \frac{\varphi'}{2} \, \sigma^{03} = \frac{\varphi'}{2} \left| \begin{array}{cc} \sigma^3/2 & 0 \\ 0 & -\sigma^3/2 \end{array} \right| , \quad \Gamma_r = 0 \, ,$$

$$\Gamma_\theta = \left| \begin{array}{cc} \Sigma_\theta & 0 \\ 0 & \bar{\Sigma}_\theta \end{array} \right| = \sqrt{\varphi} \, \sigma^{31} = \sqrt{\varphi} \left| \begin{array}{cc} -i\sigma^2/2 & 0 \\ 0 & -i\sigma^2/2 \end{array} \right| , \tag{30}$$

$$\Gamma_\phi = \left| \begin{array}{cc} \Sigma_\phi & 0 \\ 0 & \bar{\Sigma}_\phi \end{array} \right| = \sqrt{\varphi} \sin\theta \left| \begin{array}{cc} i\sigma^1/2 & 0 \\ 0 & i\sigma^1/2 \end{array} \right| + \cos\theta \left| \begin{array}{cc} -i\sigma^3/2 & 0 \\ 0 & -i\sigma^3/2 \end{array} \right| .$$

Therefore, eq. (26) takes the form

$$\{\sigma^t (\partial_t + \Sigma_t \otimes I + I \otimes \Sigma_t) + \sigma^r \partial_r$$

$$+ \sigma^\theta (\partial_\theta + \Sigma_\theta \otimes I + I \otimes \Sigma_\theta) + \sigma^\phi (\partial_\phi + \Sigma_\phi \otimes I + I \otimes \Sigma_\phi)\} \, \xi(x) = 0 \, ,$$

that is

$$\left\{ \frac{1}{\sqrt{\varphi}} \left[\partial_t + \frac{\varphi'}{4} (\sigma^3 \otimes I + I \otimes \sigma^3) \right] + \sigma^3 \sqrt{\varphi} \partial_r + \frac{\sigma^1}{r} \left[\partial_\theta - i \frac{\sqrt{\varphi}}{2} (\sigma^2 \otimes I + I \otimes \sigma^2) \right] \right.$$
$$\left. + \frac{\sigma^2}{r \sin\theta} \left[\partial_\phi + i \frac{\sqrt{\varphi}}{2} \sin\theta (\sigma^1 \otimes I + I \otimes \sigma^1) - i \frac{\cos\theta}{2} (\sigma^3 \otimes I + I \otimes \sigma^3) \right] \right\} \xi = 0 \, .$$

It is convenient to re-group the terms differently

$$\left\{ \frac{\partial}{\partial t} + \varphi [\sigma^3 \frac{\partial}{\partial r} + \frac{i}{r} (-\sigma^1 \frac{\sigma^2 \otimes I + I \otimes \sigma^2}{2} + \sigma^2 \frac{\sigma^1 \otimes I + I \otimes \sigma^1}{2}) \right.$$

$$+ \frac{\varphi'}{2\varphi} \frac{\sigma^3 \otimes I + I \otimes \sigma^3}{2}]$$

$$+\frac{\sqrt{\varphi}}{r}\left[\sigma^1\partial_\theta F - i\sigma^2\frac{i\partial_\phi + \cos\theta(\sigma^3\otimes I + I\otimes\sigma^3)/2}{\sin\theta}\right]\Bigg\}\,\xi = 0\,. \qquad (31)$$

Let us search solutions with spherical symmetry, by diagonalizing operators of total angular momentum. In this tetrad basis it has Schrödinger-like structure

$$J_1 = l_1 + \frac{\cos\phi}{\sin\theta}S_3, \quad J_2 = l_2 + \frac{\sin\phi}{\sin\theta}S_3, \quad J_3 = l_3 = -i\frac{\partial}{\partial\phi}\,, \qquad (32)$$

where

$$S_3 = ij^{12} = \frac{1}{2}(\sigma^3\otimes I + I\otimes\sigma^3), \quad \sigma^3 = \left|\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right|. \qquad (33)$$

We are to find eigenstates of the operator S_3:

$$\frac{1}{2}(\sigma^3\otimes I + i\otimes\sigma^3)\left|\begin{array}{cc} a & b \\ b & c \end{array}\right| = \sigma\left|\begin{array}{cc} a & b \\ b & c \end{array}\right|. \qquad (34)$$

Explicitly, this equation reads

$$\left|\begin{array}{cc} a & b \\ -b & -c \end{array}\right| + \left|\begin{array}{cc} a & -b \\ b & -c \end{array}\right| = 2\sigma\left|\begin{array}{cc} a & b \\ b & c \end{array}\right|,$$

whence it follows the linear system with three different solutions:

$$\left\{\begin{array}{l} a = \sigma a \\ 0 = \sigma b \\ c = -\sigma c \end{array}\right. \quad \Longrightarrow \quad \begin{array}{ll} \sigma = +1, & a = 1,\ b = 0,\ c = 0\,; \\ \sigma = 0, & a = 0,\ b = 1,\ c = 0\,; \\ \sigma = -1, & a = 0,\ b = 0,\ c = 1\,. \end{array} \qquad (35)$$

So we have three eigenvalues $\sigma = -1, 0, +1$ and corresponding three eigenstates:

$$\sigma = +1,\left|\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right|, \quad \sigma = 0,\left|\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right|, \quad \sigma = -1,\left|\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right|. \qquad (36)$$

In accordance with general theory [47], we construct solutions obeying two equations

$$(J_1^2 + J_2^2 + J_3^2)\,\xi(x) = j(j+1)\,\xi(x), \quad J_3\,\xi(x) = m\,\xi(x)\,,$$

and $\xi(x)$ should have the form (we apply the known Wigner functions [47])

$$\xi(x) = e^{-i\omega t}\left|\begin{array}{cc} f(r)\,D_{-1} & h(r)\,D_0 \\ h(r)\,D_0 & g(r)\,D_{+1} \end{array}\right|, \quad D_\sigma = D^j_{-m,-\sigma}(\phi,\theta,0)\,; \qquad (37)$$

where $f,\ g,\ h$ stand for unknown radial functions; indices j, m at Wigner functions are omitted. The substitution (35) is correct only for the following j, m:

$$j = 1, 2, 3, ...; \quad m = -j, -j+1, ..., j-1, j \ .$$

For states with $j = 0$, the initial substitution is different

$$j = 0, \qquad \xi(x) = e^{-i\omega t} \left| \begin{array}{cc} 0 & h(r) \\ h(r) & 0 \end{array} \right| . \tag{38}$$

Now we should separate the variables in eq. (31). First, we find (the factor $e^{-i\omega t}$ is omitted for brevity)

$$\xi(x) = \left| \begin{array}{cc} f(r)\, D_{-1} & h(r)\, D_0 \\ h(r)\, D_0 & g(r)\, D_{+1} \end{array} \right| ,$$

$$\frac{\partial}{\partial t}\xi = \left| \begin{array}{cc} -i\omega f\, D_{-1} & -i\omega h\, D_0 \\ -i\omega h\, D_0 & -i\omega g\, D_{+1} \end{array} \right| , \quad \sigma^3 \frac{\partial}{\partial r}\xi = \left| \begin{array}{cc} f' D_{-1} & h' D_0 \\ -h' D_0 & -g' D_{+1} \end{array} \right| ,$$

$$\frac{i}{r}\left[-\sigma^1 \frac{\sigma^2 \otimes I + I \otimes \sigma^2}{2} + \sigma^2 \frac{\sigma^1 \otimes I + I \otimes \sigma^1}{2}) \right] \xi$$

$$= \frac{1}{r} \left| \begin{array}{cc} f(r) D_{-1} & 2h(r) D_0 \\ -2h(r) D_0 & -g(r) D_{+1} \end{array} \right| ,$$

$$\frac{\sigma^3 \otimes I + I \otimes \sigma^3}{2}\xi = \left| \begin{array}{cc} f(r)\, D_{-1} & 0 \\ 0 & -g(r)\, D_{+1} \end{array} \right| .$$

Now, we find the action of angular operator $\Sigma_{\theta,\phi}$ on spinor $\xi(x)$ (taking in mind that $i\partial_\phi D^j_{-m,\sigma} = -m\ D^j_{-m,\sigma}$):

$$\Sigma_{\theta,\phi}\xi(x) = \sigma^1 \partial_\theta \xi - i\sigma^2 \frac{-m + \cos\theta(\sigma^3 \otimes I + I \otimes \sigma^3)/2}{\sin\theta}\xi$$

$$= \left| \begin{array}{cc} h\ \partial_\theta D_0 & g\ \partial_\theta D_{+1} \\ f\ \partial_\theta D_{-1} & h\ \partial_\theta D_0 \end{array} \right| + \frac{1}{\sin\theta} \left| \begin{array}{cc} h\ m D_0 & g\ (m + \cos\theta) D_{+1} \\ f\ (-m + \cos\theta) D_{-1} & -h\ m D_0 \end{array} \right| ,$$

or

$$\Sigma_{\theta,\phi}\xi(x)$$

$$= \left| \begin{array}{cc} h\ (\partial_\theta + m \sin^{-1})\ D_0 & g\ [\partial_\theta + (m + \cos\theta)\sin^{-1}\theta] D_{+1} \\ f\ [\partial_\theta + (-m + \cos\theta)\sin^{-1}\theta]\ D_{-1} & h\ (\partial_\theta - m\sin^{-1})\ D_0 \end{array} \right| .$$

With the use of the known recurrent formulas for Wigner functions [47]

$$\partial_\theta D_{-1} = \frac{1}{2}(b\ D_{-2} - a\ D_0)\,, (-m + \cos\theta)\sin^{-1}\theta\ D_{-1} = \frac{1}{2}(-b\ D_{-2} - a\ D_0)\,,$$

$$\partial_\theta D_{+1} = \frac{1}{2}(a\ D_0 - b\ D_{+2})\,, (m + \cos\theta)\sin^{-1}\theta\ D_{+1} = \frac{1}{2}(a\ D_0 + b\ D_{+2})\,,$$

$$\partial_\theta D_0 = \frac{1}{2}(a\ D_{-1} - a\ D_{+1})\,, m\sin^{-1}\theta\ D_0 = \frac{1}{2}(a\ D_{-1} + a\ D_{+1})\,,$$

$$a = \sqrt{j(j+1)}\,, \quad b = \sqrt{(j-1)(j+1)}\,,$$

we derive

$$\Sigma_{\theta,\phi}\xi = a\left|\begin{array}{cc} h\ D_{-1} & g\ D_0 \\ -f\ D_0 & -h\ D_{+1} \end{array}\right|. \tag{39}$$

Therefore, eq. (31) takes the form

$$\left|\begin{array}{cc} -i\omega f\, D_{-1} & -i\omega h\, D_0 \\ -i\omega h\, D_0 & -i\omega g\, D_{+1} \end{array}\right| + \varphi\left\{\left|\begin{array}{cc} f' D_{-1} & h' D_0 \\ -h' D_0 & -g' D_{+1} \end{array}\right|\right.$$

$$\left. + \frac{1}{r}\left|\begin{array}{cc} f\ D_{-1} & 2h\ D_0 \\ -2h\ D_0 & -g\ D_{+1} \end{array}\right| + \frac{\varphi'}{2\varphi}\left|\begin{array}{cc} f\ D_{-1} & 0 \\ 0 & -g\ D_{+1} \end{array}\right|\right\}$$

$$+ \frac{\sqrt{\varphi}}{r}a\left|\begin{array}{cc} h\ D_{-1} & g\ D_0 \\ -f\ D_0 & -h\ D_{+1} \end{array}\right| = 0\,,$$

whence it follows the system of 4 radial equations:

$$\begin{gathered} -i\omega f + \varphi(\frac{d}{dr} + \frac{1}{r} + \frac{\varphi'}{2\varphi})f + a\frac{\sqrt{\varphi}}{r}h = 0\,, \\ +i\omega g + \varphi(\frac{d}{dr} + \frac{1}{r} + \frac{\varphi'}{2\varphi})g + a\frac{\sqrt{\varphi}}{r}h = 0\,, \\ -i\omega h + \varphi(\frac{d}{dr} + \frac{2}{r})h + a\frac{\sqrt{\varphi}}{r}g = 0\,, \\ +i\omega h + \varphi(\frac{d}{dr} + \frac{2}{r})h + a\frac{\sqrt{\varphi}}{r}f = 0\,; \end{gathered} \tag{40}$$

remembering that $a = \sqrt{j(j+1)}$.

Equations for the case $j = 0$ follow from (40) by setting $f = 0, g = 0$ and $a = 0$:

$$0 = 0, \quad 0 = 0\,, -i\omega h + \varphi(\frac{d}{dr} + \frac{2}{r})h = 0\,, +i\omega h + \varphi(\frac{d}{dr} + \frac{2}{r})h = 0\,; \tag{41}$$

There exists only one and trivial solution: $h(r) = 0$, which means that Maxwell equations do not have solutions with $j = 0$.

Turning to (40), let us sum and subtract equations 3 and 4, this leads to

$$2\varphi(\frac{d}{dr} + \frac{2}{r})h + a\frac{\sqrt{\varphi}}{r}(f+g) = 0\,, \quad h = \frac{ia}{2\omega}\frac{\sqrt{\varphi}}{r}(f-g)\,. \tag{42}$$

It is readily checked that the first equation in (42) turns out to be an identity $0 = 0$ by substituting from equations 3 and 4 the variables $f(r)$ and $g(r)$ expressed through $h(r)$. This means that we have only three independent equations

$$\begin{gathered} h = \frac{ia}{2\omega}\frac{\sqrt{\varphi}}{r}(f-g)\,, \\ -i\omega f + \varphi(\frac{d}{dr} + \frac{1}{r} + \frac{\varphi'}{2\varphi})f + a\frac{\sqrt{\varphi}}{r}h = 0\,, \\ +i\omega g + \varphi(\frac{d}{dr} + \frac{1}{r} + \frac{\varphi'}{2\varphi})g + a\frac{\sqrt{\varphi}}{r}h = 0\,. \end{gathered} \tag{43}$$

Excluding the variable h, we get

$$\begin{gathered} (\frac{d}{dr} + \frac{1}{r} + \frac{\varphi'}{2\varphi} - \frac{i\omega}{\varphi})f + \frac{ia^2}{2\omega r^2}(f-g) = 0\,, \\ (\frac{d}{dr} + \frac{1}{r} + \frac{\varphi'}{2\varphi} + \frac{i\omega}{\varphi})g + \frac{ia^2}{2\omega r^2}(f-g) = 0\,. \end{gathered} \tag{44}$$

Let us sum and subtract equations in (44), in the same time introducing new variables, $f + g = F,\ f - g = G$, this results in

$$(\frac{d}{dr} + \frac{1}{r} + \frac{\varphi'}{2\varphi})F - \frac{i\omega}{\varphi}G + \frac{ia^2}{\omega r^2}G = 0, (\frac{d}{dr} + \frac{1}{r} + \frac{\varphi'}{2\varphi})G - \frac{i\omega}{\varphi}F = 0\,. \tag{45}$$

The system (45) is simplified by substitutions $F = (r\sqrt{\varphi})^{-1}\bar{F}\,,\quad G = (r\sqrt{\varphi})^{-1}\bar{G}$; so we obtain

$$i\omega\frac{d}{dr}\bar{F} + (\frac{\omega^2}{\varphi} - \frac{a^2}{r^2})\bar{G} = 0\,, \quad \varphi\frac{d}{dr}\bar{G} = i\omega\bar{F}\,, \tag{46}$$

whence it follows a 2-nd order equation for main function $\bar{G}$:

$$\left(\frac{d^2}{dr^2} + \frac{\varphi'}{\varphi}\frac{d}{dr} + \frac{\omega^2}{\varphi^2} - \frac{j(j+1)}{r^2\varphi} \right) \bar{G} = 0 \, , \quad \bar{F}(r) = \frac{\varphi(r)}{i\omega}\frac{d}{dr}\bar{G}(r) \, . \tag{47}$$

4. Solutions in Minkowski Space

Let us briefly consider the simplest variant of equation (47) for Minkowski space:

$$\left(\frac{d^2}{dr^2} + \omega^2 - \frac{j(j+1)}{r^2} \right) \bar{G} = 0 \, . \tag{48}$$

we have an equation with regular point $r = 0$ and irregular point $r = \infty$ of the rank 2, so it belongs to confluent hypergeometric type. Possible asymptotic behavior for solutions is as follows

$$r \to 0, \; \bar{G} \sim r^{j+1}, \; r^{-j} \, , \quad r \to \infty, \; \bar{G} \sim e^{-i\omega r}, e^{+i\omega r} \, . \tag{49}$$

With the help of substitution

$$\bar{G} = r^a r^{br} g(r), \quad a = j+1, \; -j \, , \quad b = \pm i\omega \, , \tag{50}$$

we get the following equation (in the variable $x = -2br$)

$$x \, \frac{d^2 g}{dx^2} + (2a - x) \, \frac{dg}{dx} - a \, g = 0 \, , \tag{51}$$

which is identified with confluent hypergeometric equation

$$x \, \frac{d^2 F}{dx^2} + (c - x) \, \frac{dF}{dx} - a \, F = 0 \, , \quad c = 2a \, . \tag{52}$$

Let us fix parameters a and b:

$$a = j+1, \quad b = i\omega \, , \quad x = -2br = -2i\omega r \, . \tag{53}$$

then the regular in the point $r = 0$ solution is (see notations in [...])

$$\bar{G}_1(x) = x^{j+1} e^{-x/2} \Phi(a, c; x) \, , \quad a = j+1 \, , c = 2a \, . \tag{54}$$

Taking in mind the known Kummer identity $\Phi(a,c;x) = e^{x}\Phi(c-a,c;-x)$,, we readily prove that solution is real in all points (up to the factor ± 1):

$$\bar{G}_1(x) = x^{j+1}e^{-x/2}\Phi(j+1,2j+2;x) = \\ = (-1)^{j+1}(x^*)^{j+1}e^{-x^*/2}\Phi(j+1,2j+2;x^*), \quad x^* = (-x)\,. \tag{55}$$

Because the second parameter $c = 2(j+1)$ takes on integer values, the singular near the point $r = 0$ is given by the function (see in [48]):

$$g(x) = x^{a}e^{-x/2}\Psi(a,c;x)\,, \quad a = -j\,, c = -2j\,; \\ g(x \to 0) = x^{-j}\cdot\frac{\Gamma(1-c)}{\Gamma(a-c+1)} = x^{-j}\cdot\frac{\Gamma(1+2j)}{\Gamma(j+1)}\,. \tag{56}$$

Description becomes more symmetric after transforming the main equation (48) to Bessel form (let $z = \omega r$,):

$$\bar{G}(r) = \sqrt{r}g(r)\,, \ \frac{d^2 g}{dz^2} + \frac{1}{z}\frac{dg}{dz} + \left(1 - \frac{p^2}{z^2}\right)g = 0, \ p = (j+1/2)\,. \tag{57}$$

Two independent solutions $J_p(z)$ and $J_{-p}(z)$ are referred to confluent hypergeometric functions:

$$J_{\pm p}(z) = (\frac{z}{2})^{\pm p}\frac{e^{iz}}{\Gamma(1\pm p)}\Phi(\pm p + 1/2, \pm 2p + 1; -2iz)\,; \tag{58}$$

also there are known relations

$$J_{\pm p}(z) = (\frac{z}{2})^{\pm p}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+1\pm p)}\,(\frac{iz}{2})^{2n}\,, \\ J_{\pm p}(|z| \to \infty) = \sqrt{\frac{2}{\pi z}}\,\cos[z - (\frac{1}{2}\pm p)\frac{\pi}{2}]\,. \tag{59}$$

5. SOLUTIONS IN DE SITTER SPACE

Let us study he main radial equation for $\bar{G}(r)$ in de Sitter model:

$$\left(\frac{d^2}{dr^2} + \frac{\varphi'}{\varphi}\frac{d}{dr} + \frac{\omega^2}{\varphi^2} - \frac{j(j+1)}{r^2\varphi}\right)\bar{G} = 0\,, \tag{60}$$

explicitly in reads

$$\left(\frac{d^2}{dr^2} - \frac{2r}{1-r^2}\frac{d}{dr} + \frac{\omega^2}{(1-r^2)^2} - \frac{j(j+1)}{r^2(1-r^2)}\right)\bar{G} = 0\,. \tag{61}$$

In the variable $z = r^2$, we have

$$\Big[\frac{d^2}{dz^2} + (\frac{1}{z-1} + \frac{1}{2z})\,\frac{d}{dz} + \\ + \frac{\omega^2}{4(-1+z)^2} - \frac{\omega^2}{4(-1+z)} + \frac{\omega^2}{4z} + \frac{j+j^2}{4(z-1)} - \frac{j+j^2}{4z^2} - \frac{j+j^2}{4z}\Big]\,\bar{G} = 0\,. \tag{62}$$

Here we have an equation of hypergeometric type with tree regular points. Behavior of solutions in vicinity of singular points is

$$z \to 0, \quad \bar{G} = z^a, a = \frac{j+1}{2}, -\frac{j}{2}; \qquad z \to 1, \quad \bar{G} = (1-z)^b, \qquad b = \pm\frac{i\,\omega}{2}\,.$$

Searching complete solutions in the form $\bar{G} = z^a\,(1-z)^b\,H(z)$, after performing the needed calculation we arrive at

$$4z(1-z)H'' + [8a(1-z) - 8bz + \, 2(1-3z)]\,H' + \\ + \big[[4a(a-1) + 2a - j(j+1)]\frac{1}{z} + [4b(b-1) + 4b + \omega^2]\frac{1}{1-z} - \\ -4a(a-1) - 8ab - 4b(b-1) - 6a - 6b\big]\,H = 0\,.$$

Equating coefficients at z^{-1} and $(z-1)^{-1}$ to zero, we find yet known restrictions on parameters a and b, and obtain more simple equation

$$z(1-z)H'' + [2a + \frac{1}{2} - (2a+2b+3/2)z]H' - (a+b)(a+b+1/2)H = 0, \tag{63}$$

which is identified with hypergeometric equation with parameters

$$\alpha = a + b\,, \quad \beta = a + b + \frac{1}{2}\,, \quad \gamma = 2a + \frac{1}{2}\,. \tag{64}$$

In order to find asymptotic behavior of basic solutions $F(z) = u_1(z)$ at $z \to 1$, we should apply the Kummer relation [48]

$$u_1(z) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}u_2 + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}u_6\,, \tag{65}$$

where

$$u_1 = H(\alpha, \beta; \gamma; z)\,, \quad u_2 = H(\alpha, \beta; \alpha+\beta+1-\gamma; 1-z)\,,$$
$$u_6 = (1-z)^{\gamma-\alpha-\beta} H(\gamma-\alpha, \gamma-\beta; \gamma+1-\alpha-\beta; 1-z)\,. \tag{66}$$

When $z \to 1$, relation (65) gives

$$F_1(z \to 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}\,.$$

Therefore, the complete solution $\bar{G}$ at $z \to 1$ behaves as follows (taking in mind $\gamma-\alpha-\beta = -2b$, and $b = \pm i\,\omega/2$)

$$\bar{G}_1(z \to 1) = \Gamma(\gamma)\left[\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}(1-z)^{b} + \frac{\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{-b}\right].$$

Due to identities

$$(\gamma-\alpha-\beta) = -2b, \quad \alpha+\beta-\gamma = +2b = (\gamma-\alpha-\beta)^*;$$
$$(\gamma-\alpha) = a + \frac{1}{2} - b = \beta^*, \quad (\gamma-\beta) = a - b = \alpha^*,$$

we may conclude that the function $\bar{G}(z \to 1)$ is real. It is readily proved that the complete function $\bar{G}_1(z)$ is real in the whole region of variable z. To this end, we apply the Kummer identity

$$u_1 = F(\alpha, \beta; \gamma; z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta; \gamma; z)\,,$$

which provides us with e two apparently different representations for one the same function

$$z^a (1-z)^b F(\alpha, \beta; \gamma; z) = z^a (1-z)^b (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta; \gamma; z)\,.$$

which can be re-written as follows (remembering that a is real)

$$\bar{G}_1(z) = z^a (1-z)^b F(\alpha, \beta; \gamma; z) = z^a (1-z)^{b^*} F(\beta^*, \alpha^*; \gamma; z)\,. \tag{67}$$

so $\bar{G}_1(z) = [\bar{G}_1(z)]^*$.

Now, it is convenient to fix parameters, $a = (j+1)/2\,,\ b = +i\omega/2$; this choice corresponds to regular at $z = 0$ solution . Singular solution refers to the function $u_5(z)$:

$$u_5(z) = z^{1-\gamma} F(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; z)\,; \tag{68}$$

for corresponding complete $\bar{G}_5$ we find the needed asymptotic

$$\bar{G}_5(z) = z^a z^{1-\gamma} = z^{(j+1)/2} z^{-j-1/2} = z^{-j/2} \,. \tag{69}$$

For u_5 there are possible two different representations

$$u_5(z) = z^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma; 2 - \gamma; z) =$$
$$z^{1-\gamma} (1-z)^{\gamma-\alpha-\beta} F(1-\alpha, 1-\beta; 2-\gamma; z) \,;$$

so we have two representations for the complete solutions

$$\bar{G}_5(z) = z^{-j/2} (1-z)^{i\omega/2} F(-j/2 + i\omega/2, 1/2 - j/2 + i\omega/2; 1/2 - j; z) =$$
$$= z^{-j/2} (1-z)^{-i\omega/2} F(1/2 - j/2 - i\omega/2, -j/2 - i\omega/2; 1/2 - j; z) \,,$$

so that $\bar{G}_5(z) = [\bar{G}_5(z)]^*$. In order to construct complex and conjugate solutions with the given behavior at $z \to 1$:

$$u_2 \sim (1-z)^b = (1-z)^{+i\omega/2} \,, \quad u_6 \sim (1-z)^{-b} = (1-z)^{-i\omega/2} \,,$$

we are to apply Kummer solutions $u_2(z)$ and $u_6(z)$.

In order to clarify additionally the physical meaning of arising mathematical task, we turn bach to eq. (60)

$$\varphi \left(\frac{d}{dr} \varphi \frac{d}{dr} + \omega^2 - \frac{j(j+1)}{r^2} \varphi \right) \bar{G} = 0 \tag{70}$$

and transform it to a new variable r_*:

$$\begin{gathered} \varphi \frac{d}{dr} = \frac{d}{dr_*} \quad \Longrightarrow \quad dr_* = \frac{dr}{\varphi(r)} = \frac{dr}{1-r^2} \,; \\ r_* = \frac{1}{2} \ln \frac{1+r}{1-r} \,; \quad r \to 0, r_* \to 0 \,; \quad r \to 1, r_* \to +\infty \,. \end{gathered} \tag{71}$$

Correspondingly, eq. (89) reads

$$[\frac{d^2}{dr_*^2} + \omega^2 - \frac{j(j+1)}{r^2}(1-r^2)]\bar{G} = 0, r \to 0, \; [\frac{d^2}{dr_*^2} - \frac{j(j+1)}{r^2}]\bar{G} = 0;$$
$$r \to +1, \quad [\frac{d^2}{dr_*^2} + \omega^2]\bar{G} = 0, \quad \bar{G} = e^{\pm i\omega r_*} = (\cos\omega r_* \pm i \sin\omega r_*) \,.$$

Near the horizon (at $r \to 1$), solutions behave as massless harmonic waves. Eq. (72) may ve treated as Schrödinger-like equation with effective potential

$$[\frac{d^2}{dr_*^2} + \omega^2 - U(r_*)]\bar{G} = 0\,, \quad U(r) = \frac{j(j+1)}{r^2}(1-r^2)\,; \tag{72}$$

we should take in mind relations

$$r = \frac{e^{2r_*}-1}{e^{2r_*}+1}, \quad U(r_*) = (j+1)(\frac{1}{r^2}-1) = \frac{4j(j+1)e^{2r_*}}{(e^{2r_*}-1)^2}\,. \tag{73}$$

6. Solutions in Anti de Sitter Model

The study from previous section may be extended to anti de Sitter space-time:

$$dS^2 = \varphi dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{dr^2}{\varphi},\ \varphi = 1 + r^2,\ r \in (0, +\infty)\,. \tag{74}$$

We do not need to repeat the most of the above calculation, and may start with (eq. 47)

$$\left(\frac{d^2}{dr^2} + \frac{\varphi'}{\varphi}\frac{d}{dr} + \frac{\omega^2}{\varphi^2} - \frac{j(j+1)}{r^2\varphi}\right)\bar{G} = 0\,, \tag{75}$$

now it reads

$$\left(\frac{d^2}{dr^2} + \frac{2r}{1+r^2}\frac{d}{dr} + \frac{\omega^2}{(1+r^2)^2} - \frac{j(j+1)}{r^2(1+r^2)}\right)\bar{G} = 0\,. \tag{76}$$

Transforming it to a new variable $r^2 = y,\ y \in (0, +\infty)$, we get

$$\left(\frac{d^2}{dy^2} + P^2\right)\bar{G} = 0\,, \quad P^2(y) = \frac{[\omega^2 - j(j+1)]\,y - j(j+1)}{4y^2(1+y)^2}\,. \tag{77}$$

We may interpret it as an equation of Scrödinger type with an effective linear momentum $P^2(y)$, its behavior at singular points is

$$\begin{gathered} y \to 0, \quad P^2 \sim \frac{-j(j+1)}{4y^2} \to -\infty\,; \\ y \to \infty, \quad P^2 \sim \frac{\omega^2 - j(j+1)}{4y^3} = \begin{cases} +0, & \omega^2 - j(j+1) > 0\ (A)\,, \\ -0, & \omega^2 - j(j+1) < 0\ (B)\,. \end{cases} \end{gathered} \tag{78}$$

In the quantum-mechanical context we have easily interpretable only the case (A), when $\omega^2 > j(j+1)$; the situation (B) is anomalous, for instance, a corresponding classical cannot be moving with such parameters.

Let us transform eq. (77) to a new variable, $y = -z, z = -r^2, z \in (-\infty, 0)$:

$$\left(\frac{d^2}{dz^2} + \frac{1-3z}{2z(1-z)}\frac{d}{dz} - \frac{\omega^2}{4z(1-z)^2} - \frac{j(j+1)}{4z^2(1-z)} \right) \bar{G} = 0 \,. \qquad (79)$$

Applying the substituion $\bar{G} = z^a\,(1-z)^b\,H(z)$; we derive an equation for $H(z)$ (see result (63) with the change ω^2 to $-\omega^2$)

$$\begin{gathered} 4z(1-z)H'' + [8a(1-z) - 8bz + 2(1-3z)]\,H' + \\ + \big\{ [4a(a-1) + 2a - j(j+1)]\frac{1}{z} + [4b(b-1) + 4b - \omega^2]\frac{1}{1-z} - \\ -4a(a-1) - 8ab - 4b(b-1) - 6a - 6b \big\} H = 0 \,. \qquad (80) \end{gathered}$$

Impose evident restrictions $a = (j+1)/2, -j/2\,,\ b = \pm\omega/2$; then fix parameters as follows

$$a = \frac{j+1}{2}\,,\ b = -\frac{\omega}{2} < 0\,, \quad \bar{G}(z) = z^{(j+1)/2}(1-z)^{-\omega/2}H(z)\,. \qquad (81)$$

All possible functions $H(z)$ must be solutions of hypergeometric equation

$$z(1-z)F'' + [\,\gamma - (\alpha+\beta+1)\,z\,F' - \alpha\beta\,F = 0$$

with parameters

$$\alpha = \frac{j+1-\omega}{2}\,,\ \beta = \alpha + \frac{1}{2}\,,\ \gamma = 2a + \frac{1}{2} = j + 3/2\,. \qquad (82)$$

Taking $F(z)$ as the Kummer solution $u_1(z)$ (see notations in [48])

$$u_1(z) = F(\alpha,\beta,\gamma;z) = F(\frac{j+1-\omega}{2}, \frac{j+2-\omega}{2}, +\frac{3}{2}; z)\,, \qquad (83)$$

we get situation, when it is possible to obtain solutions in polynomials, $\alpha = -n,\ n = 0, 1, 2, ...$:

$$\omega = 2n + j + 1\,, \quad u_1(z) = F(-n, -n + \frac{1}{2}, j + 3/2; z)\,. \qquad (84)$$

The corresponding complete solution is given by

$$\bar{G}_1(z) = z^{(j+1)/2}(1-z)^{-n-(j+1)/2}\,(1 + c_1 z + ... + c_m z^n)\ ;$$

at $z \to -\infty$ we have

$$\bar{G}_1(z \to -\infty) = z^{(j+1)/2}(-z)^{-n-(j+1)/2}\,(1 + c_1 z + ... + c_n z^n) \to \text{const}\,.$$

Thus, we have constructed solutions $\bar{G}_1(z)$ in quasi-polynomial form, finite at two singular points, $r = 0$ and $r = \infty$; the quantization of parameter ω is $\omega = 2n + j + 1, \quad n = 0, 1, 2, ...$

Let us study the case of singular solutions, when $\bar{G} \sim z^{-j/2}$. To get it, we should use another Kummer solution

$$u_5(z) = z^{1-\gamma} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma; z) = \\ = z^{-j-1/2} F(\frac{-j-\omega}{2}, \frac{-j-\omega+1}{2}, -j + \frac{1}{2}; z)\,; \tag{85}$$

the respective complete solution is

$$\bar{G}_5(z) = z^{-j/2} z^{-\omega/2} F(\frac{-j-\omega}{2}, \frac{-j-\omega+1}{2}, -j + \frac{1}{2}; z)\,. \tag{86}$$

In fact, also we can apply some quantization condition

$$\frac{-j+1-\omega}{2} = -n', \quad \omega = 2n' - j + 1\,, \\ \bar{G}_5(z) = z^{-j/2} z^{-n+j/2} F(-n - \frac{1}{2}, -n, -j + \frac{1}{2}; z)\,. \tag{87}$$

The structure of this spectrum is substantially different from previous one: in particular, at each j there are possible a number of negative values for ω. To find behavior of that solution at infinity, we apply the following Kummer formula

$$u_5(z) = \frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\beta+1-\gamma)} e^{i\pi(1-\gamma)}\, u_3(z) + \\ + \frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(1-\beta)\Gamma(\alpha+1-\gamma)} e^{i\pi(1-\gamma)} u_4(z)\,;$$

at $z \to -\infty$ it gives

$$u_5(z \to -\infty) = \frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\beta+1-\gamma)} e^{i\pi(1-\gamma)}\,(-z)^{-\alpha} +$$

$$+\frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(1-\beta)\Gamma(\alpha+1-\gamma)}e^{i\pi(1-\gamma)}(-z)^{-\beta}\,;$$

so the corresponding complete solution is

$$\bar{G}_5(z\to-\infty)=\frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\beta+1-\gamma)}e^{i\pi(1-\gamma)}\,z^{-j/2}z^{-n+j/2}(-z)^{+n}+$$

$$+\frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(1-\beta)\Gamma(\alpha+1-\gamma)}e^{i\pi(1-\gamma)}z^{-j/2}z^{-n+j/2}(-z)^{+n-1/2}\,,$$

whence ignoring the second term we arrive at

$$\bar{G}_5(z\to-\infty)=\frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\beta+1-\gamma)}e^{i\pi(1-\gamma)}\cdot 1\,. \tag{88}$$

Thus, solution $u_5(z)$ leads to complete solution $\bar{G}_5(z)$ with quasi-polynomial structure, which is singular at $z=0$ and regular at infinity $z=-\infty$; the corresponding quantization rule is $\omega=2n'-j$. This type of solutions is hardly of physical interest.

In order to clarify physical sense of arising problem, let us turn back to eq. (75), written in the form

$$\varphi\left(\frac{d}{dr}\varphi\frac{d}{dr}+\omega^2-\frac{j(j+1)}{r^2}\varphi\right)\bar{G}=0\,; \tag{89}$$

and transform it to a new variable

$$\varphi\frac{d}{dr}=\frac{d}{dr_*}\quad\Longrightarrow\quad dr_*=\frac{dr}{\varphi(r)}=\frac{dr}{1+r^2}\,;$$
$$r_*=\arctan r\,,\quad \tan r_*=r\,;\;r\to 0,r_*\to 0\,;r\to+\infty,r_*\to+\frac{\pi}{2}\,. \tag{90}$$

Eq. (89) in this variable reads

$$\left[\frac{d^2}{dr_*^2}+\omega^2-j(j+1)(1+\frac{1}{\tan^2 r_*})\right]\bar{G}=0\,,\;r_*\in(0,\frac{\pi}{2})\,. \tag{91}$$

Here we have Schrödinger type equation

$$\left[\frac{d^2}{dr_*^2}+\omega^2-U(r_*)\right]\bar{G}=0,\quad U=j(j+1)(1+\frac{1}{\tan^2 r_*})\,,\,. \tag{92}$$

This problem is easily interpretable in quantum mechanics if the following inequality is valid

$$\omega^2 > j(j+1) \iff \Omega^2 > \frac{c^2}{\rho^2} j(j+1)\,. \tag{93}$$

In this point we should recall that related to solutions $\bar{G}_1(z)$ spectrum for ω satisfies this requirement

$$\omega^2 - j(j+1) = 4n^2 + (4n+1)(j+1) > 0\,. \tag{94}$$

Let us consider from this point of view the spectrum related to $\bar{G}_5(z)$:

$$\omega = k - j\,, \quad \text{where} \quad k = (2n'+1) \in \{1, 3, 5, ...\}\,. \tag{95}$$

From (95) it follows

$$\omega^2 - j(j+1) = k^2 - 2kj - j\ .$$

Taking in mind the roots

$$k_1 = j - j\sqrt{1+1/j}\,,\ -1 < k_1 < 0\,; \qquad k_2 = j + j\sqrt{1+1/j}\,,\ k_2 > 2j\,,$$

we conclude

$$\begin{aligned} &\omega^2 - j(j+1) < 0\,, \quad \text{when} \quad (2n'+1) < j + j\sqrt{1+1/j}\,, \\ &\omega^2 - j(j+1) > 0\,, \quad \text{when} \quad (2n'+1) > j + j\sqrt{1+1/j}\,; \end{aligned} \tag{96}$$

solutions of the type $\bar{G}_5(z)$ are relevant just to the situation badly interpretable from physical point of view.

7. Maxwell Equation in Schwarzschild Metric

We may start with eq. (40), specifying it to Schwarzschild space-time with $\varphi = 1 - \frac{1}{r}$. The main equation formally is the same

$$\left(\frac{d^2}{dr^2} + \frac{\varphi'}{\varphi}\frac{d}{dr} + \frac{\omega^2}{\varphi^2} - \frac{j(j+1)}{r^2\varphi} \right) \bar{G} = 0\,, \tag{97}$$

or explicitly

$$\left(\frac{d^2}{dr^2} + \frac{1}{r(r-1)}\frac{d}{dr} + \frac{\omega^2 r^2}{(r-1)^2} - \frac{j(j+1)}{r^2}\frac{r}{r-1}\right)\bar{G} = -0\,. \qquad (98)$$

Here we have equation with tree singular points, the points $r = 0, 1$ are regular, the point $r = \infty$ is irregular of the rank 2; this is the class of confluent Heun functions. Equation (97) becomes more understandable after transforming in to other variable:

$$\left(\varphi\frac{d}{dr}\varphi\frac{d}{dr} + \omega^2 - \frac{j(j+1)}{r^2}\varphi\right)\bar{G} = 0\,, dr_* = \frac{dr}{\varphi} = dr(1 + \frac{1}{r-1})\,,$$
$$r_* = r + \ln(r-1), r \to \infty, r_* \to +\infty; r \to 1+0, r_* \to -\infty\,, \qquad (99)$$
$$\left(\frac{d^2}{dr_*^2} + \omega^2 - U(r_*)\right)\bar{G} = 0\,,\ U(r_*) = \omega^2 - \frac{j(j+1)}{r^2}\varphi\,.$$

Let us specify behavior of the effective potential at two infinities

$$U(r_* \to +\infty) = \frac{j(j+1)}{r^2}\frac{r-1}{r} = +0\,,$$
$$U(r_* \to -\infty) = \frac{j(j+1)}{r^2}\frac{r-1}{r} = +0\,; \qquad (100)$$

this means that here we have an effective potential of barrier type. tenting to zero both at $r \to 1, r_* \to -\infty$) and at infinity ($r \to \infty, r_* \to +\infty$).

Now we are to construct formal solutions of eq. (98):

$$\left[\frac{d^2}{dr^2} + (\frac{1}{r-1} - \frac{1}{r})\frac{d}{dr} + \omega^2(1 + \frac{2}{r-1} + \right.$$
$$\left. + \frac{1}{(r-1)^2}) + j(j+1)(\frac{1}{r} - \frac{1}{r-1})\right]\bar{G} = -0\,. \qquad (101)$$

In (non-physical) singular point $r = 0$ solutions behave as

$$G'' - \frac{1}{r}G' + \frac{j(j+1)}{r}G = 0\,, \quad G \sim r^c, \quad c = 0,\ c = 2\,. \qquad (102)$$

Near the point $r = 1$ we have

$$\left[\frac{d^2}{dr^2} + \frac{1}{r-1}\frac{d}{dr} + \frac{\omega^2}{(r-1)^2}\right]\bar{G} = -0\,, \quad G \sim (r-1)^a\,, \quad a = \pm i\omega\,. \qquad (103)$$

To find asymptotic at infinity, we transform equation the the variable $x = 1/r$:

$$\left[\frac{d^2}{dx^2} + (\frac{2}{x} - \frac{1}{1-x})\frac{d}{dx} + \frac{\omega^2}{x^4(1-x)^2} - \frac{j(j+1)}{x^2(1-x)}\right] G = 0\,.$$

it becomes simpler near the point $x = 0$:

$$\left[\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + \frac{\omega^2}{x^4} - \frac{j(j+1)}{x^2}\right] G = 0\,, \quad G \sim x^A e^{B/x}\,,$$

further we derive

$$-\frac{AB}{x^3} - \frac{B(A-2)}{x^3} - \frac{2B}{x^3} + \frac{B^2}{x^4} + \frac{\omega^2}{x^4} = 0\,,$$

whence it follows

$$\begin{gathered} B^2 + \omega^2 \quad \Longrightarrow \quad B = \pm i\omega\,, \\ AB + B(A-2) + 2B = 0\,, \quad 2AB = 0\,, \quad A = 0\,. \end{gathered} \tag{104}$$

Therefore, general solution of equation (98) may be searched in the form $\bar{G} = r^c (r-1)^a e^{br}\ g(r)$. After needed calculation we arrive at

$$g'' + \left(\frac{2c}{r} + \frac{2a}{r-1} + 2b + \frac{1}{r(r-1)}\right) g' +$$
$$+\left[\frac{c(c-1)}{r^2} + \frac{a(a-1)}{(r-1)^2} + \frac{2ca}{r-1} - \frac{2ca}{r} + \frac{2cb}{r} + \frac{2ab}{r-1} + b^2 +\right.$$
$$+\frac{c}{r-1} - \frac{c}{r^2} - \frac{c}{r} + \frac{a}{r} + \frac{a}{(r-1)^2} - \frac{a}{r-1} + \frac{b}{r-1} - \frac{b}{r} +$$
$$\left. +\omega^2 + \frac{2\omega^2}{r-1} + \frac{\omega^2}{(r-1)^2} + \frac{j(j+1)}{r} - \frac{j(j+1)}{r-1}\right] g = 0\,.$$

Imposing evident restrictions, we get 8 variants of parameters a, b, c:

$$a = \pm i\omega\,, \quad c = 0, 2\,, \quad b = \pm i\omega\,.$$

So resulting equation becomes simpler

$$g'' + \left(\frac{2a+1}{r-1} + \frac{2c-1}{r} + 2b\right) g' +$$
$$+\left[\frac{2ca + 2ab + c - a + b - j(j+1) + 2\omega^2}{r-1} +\right.$$
$$\left. + \frac{-2ca + 2cb - c + a - b + j(j+1)}{r}\right] g = 0\,. \tag{105}$$

Because the physical region of radial variable is the interval $r \in (1, +\infty)$, the most interesting wold be a series in the variable $x = r - 1$. Transformed to this variable x equation reads (the prime designates d/dx):

$$g'' + (p + \frac{p_1}{x} + \frac{p_2}{x+1})g' + (\frac{q_1}{x} + \frac{q_2}{x+1})g = 0 \,, \quad x \in (0, +\infty) \,. \tag{106}$$

Its solutions may be constructed as power series: $g(x) = \sum_{k=0}^{\infty} c_k x^k$, After performing needed calculation we derive recurrent formulas

$$\begin{gathered} n = 0, \quad q_1\, c_0 + p_1\, c_1 = 0 \,; \\ n = 1, \quad (q_1 + q_2)\, c_0 + (p + p_1 + p_2 + q_1)\, c_1 + (2 + 2p_1)\, c_2 = 0 \,; \\ n = 2, 3, ..., \qquad [p(n-1) + q_1 + q_2]\, c_{n-1} + \\ +[n(n-1) + (p + p_1 + p_2)n + q_1]\, c_n + [(n+1)n + p_1(n+1)]\, c_{n+1} = 0 \,. \end{gathered}$$

Possible convergence radii are found by Poincaré – Perrone method: dividing the last relation by $n^2 c_{n-1}$

$$\frac{1}{n^2}[p(n-1) + q_1 + q_2] +$$

$$+\frac{1}{n^2}[n(n-1)+(p+p_1+p_2)n+q_1]\,\frac{c_n}{c_{n-1}}+\frac{1}{n^2}[(n+1)n+p_1(n+1)]\,\frac{c_{n+1}}{c_n}\,\frac{c_n}{c_{n-1}} = 0$$

and tending $n \to \infty$, we obtain algebraic equation which determines possible convergence radii:

$$R = \lim_{n\to\infty} \frac{c_n}{c_{n-1}} = R \,, \quad R + R^2 = 0 \,, \quad R_{conv} = \frac{1}{|R|} = 1, \infty \,.$$

Recall that complete solutions have the structure

$$\begin{gathered} \bar{G}(r) = r^c (r-1)^a e^{br} g(r) \quad \Longrightarrow \quad \bar{G}(x) = (1+x)^c x^a e^{b(1+x)} g(x) \,, \\ c = 0, 2 \,; \quad a = -i\omega, +i\omega \,; \quad b = -i\omega, +i\omega \,; \quad x \in (0, +\infty) \,; \end{gathered} \tag{107}$$

below we list 8 variants of solutions (they are collected in pairs of conjugate ones)

$$\begin{array}{llll} c = 0, & a = +i\omega, & b = +i\omega, & \bar{G}_1 = x^{+i\omega} e^{+i\omega(1+x)} g_1(x) \,, \\ c = 0, & a = -i\omega, & b = -i\omega, & \bar{G}_1^* = x^{-i\omega} e^{-i\omega(1+x)} g_1^*(x) \,; \\ c = 0, & a = +i\omega, & b = -i\omega, & \bar{G}_2 = x^{+i\omega} e^{-i\omega(1+x)} g_2(x) \,, \\ c = 0, & a = -i\omega, & b = +i\omega, & \bar{G}_2^* = x^{-i\omega} e^{+i\omega(1+x)} g_2^*(x) \,; \end{array} \tag{108}$$

$$
\begin{array}{llll}
c = 2, & a = +i\omega, & b = +i\omega, & \bar{G}_3 = (1+x)^2 x^{+i\omega} e^{+i\omega(1+x)} g_3(x)\,, \\
c = 2, & a = +i\omega, & b = -i\omega, & \bar{G}_4 = (1+x)^2 x^{+i\omega} e^{-i\omega(1+x)} g_4(x)\,; \\
c = 2, & a = -i\omega, & b = +i\omega, & \bar{G}_4^* = (1+x)^2 x^{-i\omega} e^{+i\omega(1+x)} g_4^*(x)\,, \\
c = 2, & a = -i\omega, & b = -i\omega. & \bar{G}_3^* = (1+x)^2 x^{-i\omega} e^{-i\omega(1+x)} g_x^3(x)\,.
\end{array}
\tag{109}
$$

8. Solutions in Spherical Riemann Space

Now we consider Maxwell equations in spherical Riemann model:

$$
dS^2 = dt^2 - dr^2 - \sin^2 r d\theta^2 - \sin^2\theta d\phi^2\,,
$$
$$
x^\alpha = (t, r, \theta, \phi)\,, \quad g_{\alpha\beta} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\sin^2 r & 0 \\ 0 & 0 & 0 & -\sin^2 r \sin^2\theta \end{array} \right| . \tag{110}
$$

We will use the following terad

$$
\begin{array}{c}
e^\alpha_{(0)} = (1, 0, 0, 0)\,, \quad e^\alpha_{(3)} = (0, 1, 0, 0)\,, \\
e^\alpha_{(1)} = (0, 0, \dfrac{1}{\sin r}, 0)\,, \quad e^\alpha_{(2)} = (1, 0, 0, \dfrac{1}{\sin r \sin\theta})\,;
\end{array}
\tag{111}
$$

by changing the numeration for coordinates $x^\alpha = (t, r, \theta, \phi) \Longrightarrow x^\alpha = (t, \theta, \phi, r)$ the tetrad (111) becomes diagonal. Ricci rotations coefficient equal

$$
\gamma_{ab0} = 0, \quad \gamma_{ab1} = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\tan r} \\ 0 & 0 & 0 & 0 \\ 0 & +\frac{1}{\tan r} & 0 & 0 \end{array} \right| ,
$$
$$
\gamma_{ab2} = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & +\frac{1}{\tan\theta \sin r} & 0 \\ 0 & -\frac{1}{\tan\theta\sin r} & 0 & -\frac{1}{\tan r} \\ 0 & 0 & +\frac{1}{\tan r} & 0 \end{array} \right| , \quad \gamma_{ab3} = 0\,.
$$

Starting with general spinor equation

$$
\left[\sigma^c e^\alpha_{(c)}(x)\partial_\alpha + \sigma^c (\frac{1}{2}\Sigma^{ab} \otimes I + I \otimes \frac{1}{2}\Sigma^{ab}) \gamma_{abc}(x) \right] \xi(x) = 0\,, \tag{112}
$$

we arrive at

$$\Big[\partial_t + \Big\{\sigma^3\partial_r + \frac{i}{\tan r}\Big(-\sigma^1\frac{\sigma^2\otimes I + I\otimes\sigma^2}{2} + \sigma^2\frac{\sigma^1\otimes I + I\otimes\sigma^1}{2}\Big)\Big\} +$$
$$+\frac{1}{\sin r}\Big\{\sigma^1\partial_\theta - i\sigma^2\frac{i\partial_\varphi + \cos\theta(\sigma^3\otimes I + I\otimes\sigma^3)/2}{\sin\theta}\Big\}\Big]\,\xi = 0\,.$$

comparing it with eq. (31), we write down radial equations by formal changes in the system (40):

$$\begin{aligned} -i\omega f + (\frac{d}{dr} + \frac{1}{\tan r})f + \frac{a}{\sin r}h = 0\,, \\ i\omega g + (\frac{d}{dr} + \frac{1}{\tan r})g + \frac{a}{\sin r}h = 0\,, \\ -i\omega h + (\frac{d}{dr} + \frac{2}{\tan r})h + \frac{a}{\sin r}g = 0\,, \\ i\omega h + (\frac{d}{dr} + \frac{2}{\tan r})h + \frac{a}{\sin r}f = 0\,. \end{aligned} \tag{113}$$

Summing and subtracting third and fourth equations, we derive

$$2(\frac{d}{dr} + \frac{2}{\tan r})h + \frac{a}{\sin r}(f+g) = 0\,, \quad 2i\omega h + \frac{a}{\sin r}(f-g) = 0\,. \tag{114}$$

It is readily checked that the first equation in (113) is a result of combining 3 remaining ones. Therefore we have only three independent equations

$$\begin{aligned} h = -\frac{a}{2i\omega\sin r}(f-g)\,, \\ -i\omega f + (\frac{d}{dr} + \frac{1}{\tan r})f + \frac{a}{\sin r}h = 0\,, \\ i\omega g + (\frac{d}{dr} + \frac{1}{\tan r})g + \frac{a}{\sin r}h = 0\,. \end{aligned} \tag{115}$$

Excluding the variable $h(r)$ we obtain:

$$\begin{aligned} (\frac{d}{dr} + \frac{1}{\tan r} - i\omega)f + \frac{ia^2}{2\omega\sin^2 r}(f-g) = 0\,, \\ (\frac{d}{dr} + \frac{1}{\tan r} + i\omega)g + \frac{ia^2}{2\omega\sin^2 r}(f-g) = 0\,. \end{aligned} \tag{116}$$

Summing and subtracting these two equations, and using new variables, $f+g = F,\ f-g = G$, we arrive at the system

$$(\frac{d}{dr} + \frac{1}{\tan r})F - i\omega G + \frac{ia^2}{\omega\sin^2 r}G = 0, (\frac{d}{dr} + \frac{1}{\tan r})G - i\omega F = 0. \tag{117}$$

System (117) may be simplified by separating multipliers, $F = \sin^{-1} r\,\bar{F}\,,\ G = \sin^{-1} r\,\bar{G}$, in this way we get

$$\frac{d}{dr} i\omega \bar{F} + (\omega^2 - \frac{a^2}{\sin^2 r})\bar{G} = 0\,, \quad \frac{d}{dr}\bar{G} = i\omega \bar{F}\,. \tag{118}$$

whence it follows an equation for main function

$$\left(\frac{d^2}{dr^2} + \omega^2 - \frac{a^2}{\sin^2 r} \right) \bar{G} = 0\,. \tag{119}$$

In the new variable, $y = \frac{1-\cos r}{2}$, the last equation reads

$$\left[y(1-y)\frac{d^2}{dy^2} + (\frac{1}{2} - y)\frac{d}{dy} + \omega^2 - \frac{a^2}{4y(1-y)} \right] \bar{G} = 0\,. \tag{120}$$

Its solution are searched in the form $\bar{G} = y^A\,(1-y)^B\,g(y)$; this results in

$$y(1-y)g'' + [2A + 1/2 - (2A + 2B + 1)y]\,g' +$$

$$+\frac{1}{y}\,[A(A-1) + \frac{1}{2}A - \frac{a^2}{4}]\,g + \frac{1}{1-y}\,[B(B-1) + \frac{1}{2}B - \frac{a^2}{4}]\,g +$$

$$+[\ \omega^2 - 2AB - A(A-1) - B(B-1) - A - B\ ]\ g = 0\,.$$

Equating coefficients at y^{-1} and $(1-y)^{-1}$ to zero, we get $A = (j+1)/2,\ -j/2; B = (j+1)/2,\ -j/2$. The above equation for $g(y)$ is simplified and and recognized as hypergeometric equation with parameters

$$\gamma = 2A + 1/2\,, \quad \alpha = A + B - \omega\,, \quad \beta = A + B + \omega\,. \tag{121}$$

Let us fix parameters A and B: $A = (j+1)/2\,,\ B = (j+1)/2$, so obtaining

$$\gamma = j + 3/2\,, \quad \alpha = j + 1 - \omega\,, \quad \beta = j + 1 + \omega\,. \tag{122}$$

We get polynomials imposing evident restriction

$$\alpha = -n, \quad n = 1, 2, 3, \ldots\,, \quad \omega = n + j + 1\,; \tag{123}$$

corresponding complete solution has the structure

$$\bar{G}(y) = y^{(j+1)/2}\,(1-y)^{(j+1)/2}\,F(-n,\,n+2j+2,\,j+3/2;\ y)\,, \tag{124}$$

it equals to zero at the points $y \to 0,\ y \to 1\ \ (r \to 0,\ r \to \pi)$.

9. Solutions in Lobachevsky Space

The main radial equation reads

$$\left(\frac{d^2}{dr^2} + \omega^2 - \frac{a^2}{\sinh^2 r}\right)\bar{G} = 0\,, \quad r \in (0,\infty)\,. \tag{125}$$

In a new variable $y = \frac{1-\cosh r}{2}$ it takes the form

$$\left[y(y-1)\frac{d^2}{dy^2} + (y-1/2)\frac{d}{dy} + \omega^2 - \frac{a^2}{4y(y-1)}\right]\bar{G} = 0\,. \tag{126}$$

Formally, this equation differs from that used in previous section only in sign at ω^2. Substitution for $\bar{G}(y)$ is the same

$$\bar{G} = y^A\,(1-y)^B\,g(y), \quad A = \frac{j+1}{2}, -\frac{j}{2}; \quad B = \frac{j+1}{2}, -\frac{j}{2}\,; \tag{127}$$

for $g(y)$ we get an equation of hypergeometric type

$$y(1-y)g'' + \left[2A + \frac{1}{2} - (2A+2B+1)y\right]\,g' - \left[(A+B)^2 + \omega^2\right]g = 0$$

with parameters

$$\gamma = 2A + 1/2\,, \quad \alpha = A + B - i\omega\,, \quad \beta = A + B + i\omega\,. \tag{128}$$

Let us fix parameters as follows (negative B ensures the term $(1-y)^B$ tending to zero when $y \to \infty$):

$$A = \frac{j+1}{2},\; B = -\frac{j}{2};\; \gamma = j + 3/2,\; \alpha = 1/2 - i\omega\,,\; \beta = 1/2 + i\omega, \tag{129}$$

thus we have constructed the needed solution (see notations in [48])

$$\bar{G}_1(y) = y^{(j+1)/2}\,(1-y)^{-j/2}\,u_1(y), \quad u_1(y) = F(\alpha,\beta,\gamma;y); \tag{130}$$

it tends to zero at the points $y = 0$ $(r = 0)$. The singular point $y = 1$ does not belong to physical region. To find behavior of this solution in infinity, we should apply the Kummer formulas

$$u_1(y) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)}\,u_3(y) + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\alpha)}\,u_4(y)\,, \tag{131}$$

where

$$u_3(y) = (-y)^{-\alpha} F(\alpha, \alpha + 1 - \gamma, \alpha + 1 - \beta; \frac{1}{y})$$

$$= (-y)^{(-1/2+i\omega)} F(1/2 - i\omega, -i\omega - j, 1 - 2i\omega; \frac{1}{y}) \,,$$

$$u_4(y) = (-y)^{-\beta} F(\beta + 1 - \gamma, \beta, \beta + 1 - \alpha; \frac{1}{y})$$

$$= (-y)^{(-1/2-i\omega)} F(i\omega - j, (1/2 + i\omega), 1 + 2i\omega; \frac{1}{y}) \,.$$

As $y \to +\infty$, the last formula gives

$$u_1(y) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)} (-y)^{(-1/2+i\omega)} + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\alpha)} (-y)^{(-1/2-i\omega)} \,.$$

Therefore, the complete solution behaves as follows

$$\bar{G}_1 \; (y \to \infty) = (-1)^{-(j+1)/2}$$
$$\times \left\{ \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)} (-y)^{i\omega} + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\alpha)} (-y)^{-i\omega} \right\} . \qquad (132)$$

Taking in mind identities

$$\beta - \alpha = 2i\omega \,, \; \alpha - \beta = -2i\omega \,, \; \gamma - \alpha = j + i\omega + 1 \,, \; \gamma - \beta = j - i\omega + 1 \,,$$

we conclude that $\bar{G}_1(y \to \infty)$ is real up to simple phase factor. In initial variable r, asymptotic (132) is determined by the formulas

$$(-y)^{i\omega} \approx (\frac{1}{4})^{i\omega} e^{i\omega r} \,, \; \bar{G}_1 \; (r \to \infty) = M e^{i\omega r} + M^* e^{-i\omega r}, \qquad (133)$$

where

$$M = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)} \, (\frac{1}{4})^{i\omega} \,.$$

It is readily proved that when using Kummer solutions u_3 and u_4, their corresponding complete solutions $\bar{G}_3$ and $\bar{G}_4$ are conjugate to each other and have the asymptotic $u_3 \sim$ const $e^{i\omega r}$, $u_4 \sim$ const $e^{-i\omega r}$.

10. Solutions with Cylindric Symmetry in Spherical Riemann Space

Let us consider spinor Maxwell equations in cylindric coordinates of the spherical Riemann model, it is specified by the formulas

$$
\begin{gathered}
u_1 = \sin r \cos\phi\,, \quad u_2 = \sin r \sin\phi\,, \quad u_3 = \cos r \sin z\,, \\
u_0 = \cos r \cos z\,, \quad u_0^2 + u_1^2 + u_2^2 + u_3^2 = 1\,, \\
dS^2 = dt^2 - dr^2 - \sin^2 r\, d\phi^2 - \cos^2 r\, dz^2\,, \; x^\alpha = (t, r, \phi, z)\,, \\
e^{\beta}_{(a)}(x) = \left| \begin{array}{llll} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sin^{-1} r & 0 \\ 0 & 0 & 0 & \cos^{-1} r \end{array} \right|,
\end{gathered}
\tag{134}
$$

these coordinates belong to: $r \in [0,\ +\pi/2],\ \phi \in [-\pi,\ +\pi],\ z \in [-\pi,\ +\pi]$.

Ricci rotation coefficients are (we write down only non-vanishing ones)

$$
\gamma_{ab0} = 0\,, \quad \gamma_{ab1} = 0\,, \quad \gamma_{122} = +\frac{\cos r}{\sin r}\,, \quad \gamma_{313} = +\frac{\sin r}{\cos r}\,. \tag{135}
$$

Starting with general spinor form of Maxwell equations

$$
\begin{gathered}
\left[\sigma^c e^{\alpha}_{(c)}(x)\partial_\alpha + \sigma^c(\frac{1}{2}\Sigma^{ab} \otimes I + I \otimes \frac{1}{2}\Sigma^{ab})\gamma_{abc}(x) \right] \xi(x) = 0\,, \\
\Sigma^{0j} = \frac{1}{2}\sigma^j, \quad \Sigma^{12} = -\frac{i}{2}\sigma^3, \quad \Sigma^{23} = -\frac{i}{2}\sigma^1, \quad \Sigma^{31} = -\frac{i}{2}\sigma^2,
\end{gathered}
\tag{136}
$$

we obtain

$$
\begin{gathered}
[\ \partial_t + \sigma^1 \partial_r + \frac{\sigma^2}{\sin r}\partial_\phi + \frac{\sigma^3}{\cos r}\partial_z \\
+\sigma^2(\Sigma^{12} \otimes I + I \otimes \Sigma^{12})\gamma_{122} + \sigma^3(\Sigma^{31} \otimes I + I \otimes \Sigma^{31})\gamma_{313}]\xi(x) = 0\,,
\end{gathered}
$$

that is

$$
\begin{gathered}
\Big[\partial_t + \sigma^1 \partial_r - \frac{i\sigma^2}{2}(\sigma^3 \otimes I + I \otimes \sigma^3)\frac{\cos r}{\sin r} \\
-\frac{i\sigma^3}{2}(\sigma^2 \otimes I + I \otimes \sigma^2)\frac{\sin r}{\cos r} + \frac{\sigma^2}{\sin r}\partial_\phi + \frac{\sigma^3}{\cos r}\partial_z \Big]\, \xi = 0\,.
\end{gathered}
\tag{137}
$$

The structure of this equation allows the following substitution for electromagnetic spinor

$$\xi(t, r, \phi, z) = e^{-i\omega t}\, e^{im\phi}\, e^{ikz} \left| \begin{array}{cc} f(r) & h(r) \\ h(r) & g(r) \end{array} \right| , \tag{138}$$

so we derive

$$\Big\{ -i\omega I + \sigma^1 \partial_r - \frac{i\cos r}{2\sin r}\sigma^2(\sigma^3 \otimes I + I \otimes \sigma^3) - \frac{i\sin r}{2\cos r}\sigma^3(\sigma^2 \otimes I + I \otimes \sigma^2)$$

$$+\frac{im}{\sin r}\sigma^2 + \frac{ik}{\cos r}\sigma^3 \Big\} \left| \begin{array}{cc} f(r) & h(r) \\ h(r) & g(r) \end{array} \right| = 0 \,, \tag{139}$$

and further we find the system of 4 equations:

$$\begin{gathered}
(\frac{d}{dr} + \frac{m}{\sin r} - \frac{\sin r}{\cos r})h + (-i\omega + \frac{ik}{\cos r})f = 0 \,, \\
(\frac{d}{dr} - \frac{m}{\sin r} - \frac{\sin r}{\cos r})h + (-i\omega - \frac{ik}{\cos r})g = 0 \,; \\
(\frac{d}{dr} + \frac{m}{\sin r} + \frac{\cos r}{\sin r} - \frac{1}{2}\frac{\sin r}{\cos r})g + \frac{1}{2}\frac{\sin r}{\cos r}f + (-i\omega + \frac{ik}{\cos r})h = 0, \\
(\frac{d}{dr} - \frac{m}{\sin r} + \frac{\cos r}{\sin r} - \frac{1}{2}\frac{\sin r}{\cos r})f + \frac{1}{2}\frac{\sin r}{\cos r}g + (-i\omega - \frac{ik}{\cos r})h = 0 \,.
\end{gathered} \tag{140}$$

Summing and subtracting equations in each pair, we obtain (let it be $F = f + g\,, G = f - g$)

$$\begin{gathered}
\frac{ik}{\cos r}\,F - i\omega G + \frac{2m}{\sin r}h = 0 \,, \\
\frac{ik}{\cos r}\,G - i\omega F + 2(\frac{d}{dr} - \frac{\sin r}{\cos r})h = 0 \,, \\
-\frac{2ik}{\cos r}\,h - \frac{m}{\sin r}F + (\frac{d}{dr} + \frac{\cos r}{\sin r} - \frac{\sin r}{\cos r})G = 0 \,, \\
-2i\omega\, h - \frac{m}{\sin r}G + (\frac{d}{dr} + \frac{\cos r}{\sin r})F = 0 \,.
\end{gathered} \tag{141}$$

Let us express from 1-st, 2-nd, and 4-th equations the variables $\omega G,\ \omega F,\ 2i\omega h$, and substitute them into the third equation, this results in the identity $0 \equiv 0$.

Therefore, only three equations in (141) are independent:

$$\begin{gathered}\frac{ik}{\cos r}\,F - i\omega G + \frac{2m}{\sin r}h = 0\,,\\ \frac{ik}{\cos r}\,G - i\omega F + 2(\frac{d}{dr} - \frac{\sin r}{\cos r})h = 0\,,\\ -2i\omega\,h - \frac{m}{\sin r}G + (\frac{d}{dr} + \frac{\cos r}{\sin r})F = 0\,.\end{gathered} \tag{142}$$

Taking into account identities

$$F = \frac{1}{\sin r}\bar{F}\,,\ (\frac{d}{dr} + \frac{\cos r}{\sin r})F = \frac{1}{\sin r}\frac{d\bar{F}}{dr}\,,$$

$$h = \frac{1}{\cos r}\bar{h}\,,\ (\frac{d}{dr} - \frac{\sin r}{\cos r})h = \frac{1}{\cos r}\frac{d\bar{h}}{dr}\,,$$

we may simplify equations (142):

$$\begin{gathered}\frac{ik}{\cos r}\,\frac{1}{\sin r}\bar{F} - i\omega G + \frac{2m}{\sin r}\frac{1}{\cos r}\bar{h} = 0\,,\\ \frac{ik}{\cos r}\,G - i\omega\frac{1}{\sin r}\bar{F} + \frac{2}{\cos r}\frac{d\bar{h}}{dr} = 0\,,\\ -2i\omega\,\frac{1}{\cos r}\bar{h} - \frac{m}{\sin r}G + \frac{1}{\sin r}\frac{d\bar{F}}{dr} = 0\,.\end{gathered} \tag{143}$$

Let it be $2i\bar{h} = \bar{H}$. Then the last system is presented as follows

$$\omega G = \frac{k\bar{F} - m\bar{H}}{\cos r\sin r}\,,$$

$$\frac{k}{\cos r}\,\omega G - \frac{\omega^2}{\sin r}\bar{F} - \frac{\omega}{\cos r}\frac{d\bar{H}}{dr} = 0\,, -\omega^2\,\frac{1}{\cos r}\,\bar{H} - \frac{m}{\sin r}\,\omega\,G + \frac{\omega}{\sin r}\frac{d\bar{F}}{dr} = 0\,.$$

Excluding the function G, we derive

$$\begin{gathered}\frac{1}{\cos r}t(\omega\frac{d}{dr} + \frac{km}{\cos r\sin r}t)\bar{H} + \frac{1}{\sin r}(\omega^2 - \frac{k^2}{\cos^2 r})\bar{F} = 0\,,\\ \frac{1}{\sin r}(\omega\frac{d}{dr} - \frac{km}{\sin r\cos r})\bar{F} + \frac{1}{\cos r}(-\omega^2 + \frac{m^2}{\sin^2 r})\bar{H} = 0\,.\end{gathered} \tag{144}$$

Let us transform the system to new variable $\sin r = \sqrt{z}\,,\ z \in [0,1]$, then we arrive at

$$\begin{aligned} &\Big([2\omega \frac{d}{dz} + \frac{km}{z(1-z)}\Big] \bar{H} + \frac{\omega^2 - k^2 - \omega^2 z}{z(1-z)}\, \bar{F} = 0\,, \\ &\Big[2\omega \frac{d}{dz} - \frac{km}{z(1-z)}\Big] \bar{F} + \frac{m^2 - \omega^2 z}{z(1-z)}\, \bar{H} = 0\,. \end{aligned} \tag{145}$$

Note that from (145) straightforwardly follow two differential equations with 4 singular points:

$$\bar{H}, \quad z = 0, 1, \infty, (1 - \frac{k^2}{\omega^2})\,; \qquad \bar{F}, \quad z = 0, 1, \infty, \frac{m^2}{\omega^2}\,.$$

There exists possibility to reduce the problem to equations with three singular points. Indeed, let us define new variables, $\bar{H} = V + W,\ \bar{F} = V - W$, then the system (145) reads

$$\Big[2\omega \frac{d}{dz} + \frac{km}{z(1-z)}\Big] (V + W) + \frac{\omega^2 - k^2 - \omega^2 z}{z(1-z)}\, (V - W) = 0\,,$$

$$\Big[2\omega \frac{d}{dz} - \frac{km}{z(1-z)}\Big] (V - W) + \frac{m^2 - \omega^2 z}{z(1-z)}\, (V + W) = 0\,.$$

Summing and subtracting these equations we get

$$\begin{aligned} &\Big[4\omega \frac{d}{dz} + \frac{\omega^2 - k^2 + m^2 - 2\omega^2 z}{z(1-z)}\Big] V - \frac{\omega^2 - (k+m)^2}{z(1-z)}\, W = 0\,, \\ &\Big[4\omega \frac{d}{dz} W - \frac{\omega^2 - k^2 + m^2 - 2\omega^2 z}{z(1-z)}\Big] W + \frac{\omega^2 - (k-m)^2}{z(1-z)}\, V = 0\,. \end{aligned} \tag{146}$$

We readily derive a 2-nd order equation for $W(z)$:

$$z(z-1)W'' + (2z-1)W' + \Big\{ -\frac{\omega(\omega+2)}{4} - \frac{k^2}{4(z-1)} + \frac{m^2}{4z} \Big\} W = 0\,. \tag{147}$$

Near the points $z = 0, 1$, solutions behave as

$$z \to 0, W = z^A, A = \pm\frac{|m|}{2}; z \to 1, W = (z-1)^B,\ B = \pm\frac{|k|}{2}. \tag{148}$$

In all region of z, solutions are searched in the form $W(z) = z^A(z-1)^B\bar{W}(z)$. After needed calculation we arrive at

$$(z-1)z\bar{W}'' + [2A(z-1) + 2Bz + (2z-1)]\,\bar{W}'$$

$$+\left((A+B)(A+B+1) - \frac{\omega(\omega+2)}{4} - \frac{k^2}{4(z-1)} + \frac{B^2}{z-1} + \frac{m^2}{4z} - \frac{A^2}{z}\right)\bar{W} = 0\,.$$

Imposing yet known restrictions (148), we obtain

$$z(1-z)\bar{W}'' + [2A+1-(2A+2B+2)z)]\bar{W}'$$
$$-[(A+B)(A+B+1) - \frac{1}{4}\omega(\omega+2)]\,\bar{W} = 0, \qquad (149)$$

which is identified with the equation of hypergeometric type

$$z(1-z)\,\frac{d^2F}{dz^2} + [\gamma - (\alpha+\beta+1)\gamma]\,\frac{dF}{dz} - \alpha\beta\,F = 0\,.$$

Let us fix parameters A and B so that solutions be finite at the points $z = 0,\ 1$:

$$A = +\frac{|m|}{2}\,,\ B = +\frac{|k|}{2}\,,\ \gamma = |m| + 1\,,$$
$$\alpha = \frac{|k| + |m| - \omega}{2}\,,\ \beta = \frac{|k| + m| + \omega}{2} + 1\,, \qquad (150)$$

and accept the standard requirement for polynomials:

$$\alpha = -n,\ \ \omega = 2n + |k| + |m|\,,\ \ \beta = n + 1 + |m| + |k|\,,$$
$$n = 0, 1, 2, ...,\quad W(z) = z^{|m|/2}(z-1)^{|k|/2}F(\alpha,\beta,\gamma;z)\,. \qquad (151)$$

Now, let us turn to equation for the second function $V(z)$. There exist symmetry between two equations in (146): the system is invariant under the formal changes

$$V \Longleftrightarrow W,\quad \omega \Longleftrightarrow -\omega,\quad m \Longleftrightarrow -m. \qquad (152)$$

Therefore, from the 2-nd order equation (147) for $W(z)$, without any calculation we obtain a respective equation for $V(z)$:

$$z(z-1)V'' + (2z-1)W' + \left[-\frac{\omega(\omega-2)}{4} - \frac{k^2}{4(z-1)} + \frac{m^2}{4z}\right]V = 0\,. \qquad (153)$$

We are to apply the same substitution $V(z) = z^A(z-1)^B\bar{V}(z)$. After the needed calculation we get an equation for $\bar{V}(z)$:

$$(z-1)z\bar{V}'' + [2A(z-1) + 2Bz + (2z-1)]\,\bar{V}'$$
$$+[\,(A+B)^2 + A + B - \frac{1}{4}\omega(\omega-2) - \frac{k^2}{4(z-1)} + \frac{B^2}{z-1} + \frac{m^2}{4z} - \frac{A^2}{z}\,]\,\bar{V} = 0\,.$$

Imposing evident restrictions on A B, we arrive at an equation of hypergeometric type

$$z(1-z)\bar{V}'' + [2A + 1 - (2A + 2B + 2)z)]\,\bar{V}'$$
$$-[\,(A+B)(A+B+1) - \frac{1}{4}\omega(\omega-2)\,]\,\bar{V} = 0. \tag{154}$$

with parameters

$$A = +\frac{|m|}{2}\,,\; B = +\frac{|k|}{2}\,,\; \gamma' = |m| + 1\,,$$
$$;\alpha' = \frac{|k| + |m| + \omega}{2}\,,\; \beta' = \frac{|k| + m| - \omega}{2} + 1\,. \tag{155}$$

Further, applying polynomial condition, we find needed solutions

$$\beta' = -n',\;\; \omega = 2(n'+1) + |k| + |m|\,,\;\; \alpha' = n' + 1 + |m| + |k|\,,$$
$$n' = 0, 1, 2, ...,\quad V(z) = z^{|m|/2}(z-1)^{|k|/2}F(\alpha', \beta', \gamma'; z)\,. \tag{156}$$

A relative coefficient between two functions, $\bar{W}(z)$ and $\bar{V}(z)$, may be found with the use of first order relations, related these function.

In similar way, we could study the spinor Maxwell equations also in hyperbolic Lobachevsky space, being parameterized by cylindric coordinates according to the formulas:

$$u_1 = \sinh r \cos\phi\,,\;\; u_2 = \sinh r \sin\phi\,,\;\; u_3 = \cosh r \sinh z\,,$$
$$u_0 = \cosh r \cosh z\;,\quad u_0^2 - u_1^2 - u_2^2 - u_3^2 = 1,\quad u_0 > +1\,;$$
$$dS^2 = dt^2 - dr^2 - \sinh^2 r\, d\phi^2 - \cosh^2 r\, dz^2\,,\quad x^\alpha = (t, r, \phi, z)\,,$$
$$e^{\beta}_{(a)}(x) = \left|\begin{array}{llll} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sinh^{-1} r & 0 \\ 0 & 0 & 0 & \cosh^{-1} r \end{array}\right|. \tag{157}$$

where $r \in [0,\; +\infty)$, $\phi \in [-\pi,\; +\pi]$, $z \in (-\infty,\; +\infty)$. The treatment would be similar and it does not require new ideas.

11. Riemannian Geometry and Modeling the Constitutive Relations for Special Media

It was noted in Introduction that vacuum Maxwell equations being considered on the background of pseudo-Riemannin space-time may be interpreted as Maxwell equations in Minkowski space but specified in some effective media which constitutive relation are determined by metric of Riemannian space-time. and relationship between two electromagnetic tensors is governed by the formula

$$H^{\alpha\beta}(x) = \epsilon_0 \, \frac{\sqrt{-g(x)}}{\sqrt{-G(x)}} \, g^{\alpha\rho}(x) g^{\beta\sigma}(x) \; F_{\rho\sigma}(x) \, . \tag{158}$$

In that context, let us consider space-time models with event horizon (let the external currents vanish); all of them have a metric of one the same structure (we restrict ourselves to spherically symmetric case; for definiteness we take in mind de Sitter, anti de Sitter, and Schwarzschild models)

$$\begin{gathered} dS^2 = \varphi dt^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 - \varphi^{-1} dr^2 \, , \\ \text{Minkowski space} \;\; dS_0^2 = dt^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 - dr^2 \, . \end{gathered} \tag{159}$$

Because the metric determinant of the models is the same, $G(x) = g(x)$, effective constitutive relations (159) become simpler

$$H^{\alpha\beta}(x) = \epsilon_0 \, g^{\alpha\rho}(x) g^{\beta\sigma}(x) \; F_{\rho\sigma}(x) \, . \tag{160}$$

Taking into account explicit form of metric tensor (numerating coordinates as follows $x^\alpha = (t, \theta, \phi, r)$):

$$g^{\beta\alpha} = \left| \begin{array}{cccc} 1/\varphi & 0 & 0 & 0 \\ 0 & -1/r^2 & 0 & 0 \\ 0 & 0 & -1/r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & -\varphi \end{array} \right| , \tag{161}$$

we get constitutive relations modified by Riemannian metric

$$H^{0j} = \epsilon_0 g^{00} g^{jj} F_{0j} \, ,$$

$$H^{01} = \frac{1}{\varphi} \left(-\frac{\epsilon_0}{r^2} , \quad F_{01} \right) , \quad H^{02} = \frac{1}{\varphi} \left(-\frac{\epsilon_0}{r^2 \sin^2\theta} F_{02} \right) , \quad H^{03} = -\epsilon_0 F_{03} \, ,$$

$$H^{ij} = \epsilon_0 g^{ii} g^{jj} F_{ij} \, ,$$

$$H^{23} = \varphi \left(\frac{\epsilon_0}{r^2 \sin^2\theta} F_{23} \right) , \quad H^{31} = \varphi \left(\frac{\epsilon_0}{r^2} F_{31} \right) , \quad H^{12} = \left(\frac{\epsilon_0}{r^4 \sin^4\theta} F_{12} \right) .$$

The last formulas may be re-written differently

$$
\begin{aligned}
D^{\theta} = \frac{1}{\varphi}\epsilon_0 \frac{E_\theta}{r^2}, \quad D^{\phi} = \frac{1}{\varphi}\epsilon_0 \frac{E_\phi}{r^2 \sin^2\theta}, \quad D^r = \epsilon_0 E_r \,; \\
H^{\theta} = \varphi \frac{1}{\mu_0} \frac{B_\theta}{r^2 \sin^2\theta}, \quad H^{\phi} = \varphi \frac{1}{\mu_0} \frac{B_\phi}{r^2}, \quad H^r = \frac{1}{\mu_0} \frac{B_r}{r^4 \sin^4\theta};
\end{aligned}
\tag{162}
$$

were we have used identity $c^2 = 1/\epsilon_0\mu_0$ and definitions for two tensors

$$
\begin{gathered}
(F_{ab}) = \left| \begin{array}{cccc} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & +cB^2 \\ E^2 & +cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & +cB^1 & 0 \end{array} \right| , \\
(H^{ab}) = \left| \begin{array}{cccc} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & -H^3/c & +H^2/c \\ D^2 & +H^3/c & 0 & -H^1/c \\ D^3 & -H^2/c & +H^1/c & 0 \end{array} \right| .
\end{gathered}
\tag{163}
$$

The constitutive relations (162) can be presented with the help of electric permittivity and magnetic permeability tensors:

$$
\begin{gathered}
D^i(x) = \epsilon_0 \epsilon_{ij}(x) E^{(j)}(x)\,, \quad H^i(x) = \frac{1}{\mu_0}\mu_{ij}(x) B^{(j)}(x)\,, \\
[\epsilon_{ij}(r)] = \left| \begin{array}{ccc} \varphi^{-1}(r) & 0 & 0 \\ 0 & \varphi^{-1}(r) & 0 \\ 0 & 0 & 1 \end{array} \right| , \quad [\mu_{ij}(r)] = \left| \begin{array}{ccc} \varphi(r) & 0 & 0 \\ 0 & \varphi(r) & 0 \\ 0 & 0 & 1 \end{array} \right| ,
\end{gathered}
\tag{164}
$$

where

$$
E^{(j)} = \frac{E_j}{h_j}, \quad B^{(1)} = \frac{B_1}{h_2 h_3}, \quad B^{(2)} = \frac{B_3}{h_3 h_1}, \quad B^{(3)} = \frac{B_3}{h_1 h_2}\,, \tag{165}
$$

and h_j are determined by Minkowski metric (159) in spherical coordinates:

$$
dS_0^2 = dt^2 - h_1 d\theta^2 - h_2 d\phi^2 - h_3 dr^2\,. \tag{166}
$$

Simplicity of relations (164) is misleading, in fact for each of curved space-time model we are to solve Maxwell equations separately and anew.

Let us compare the effective constitutive equations for four models:

Minkowski ($r \in (0, +\infty)$),

$$\epsilon = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \mu = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix};$$

de Sitter ($r \in (0, +1)$),

$$\epsilon = \begin{vmatrix} (1-r^2)^{-1} & 0 & 0 \\ 0 & (1-r^2)^{-1} & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \mu = \begin{vmatrix} (1-r^2) & 0 & 0 \\ 0 & (1-r^2) & 0 \\ 0 & 0 & 1 \end{vmatrix};$$

anti de Sitter ($r \in (0, +\infty)$),

$$\epsilon = \begin{vmatrix} (1+r^2)^{-1}(r) & 0 & 0 \\ 0 & (1+r^2)^{-1} & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \mu = \begin{vmatrix} (1+r^2) & 0 & 0 \\ 0 & (1+r^2) & 0 \\ 0 & 0 & 1 \end{vmatrix};$$

Schwarzschild ($r \in (1, +\infty)$),

$$\epsilon = \begin{vmatrix} (1-1/r)^{-1} & 0 & 0 \\ 0 & (1-1/r)^{-1} & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \mu = \begin{vmatrix} (1-1/r) & 0 & 0 \\ 0 & (1-1/r) & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Direct comparison is possible only for Minkowski and anti de Sitter models, due to the same region for radial coordinates.

That interpretation is possible also for other space-time models. Let us discuss hyperbolic Lobachevsky and spherical Riemann models being compared with Minkowski one:

$$\begin{aligned} dS_0^2 &= dt^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 - dr^2, \\ dS^2 &= dt^2 - \sinh^2 r\, d\theta^2 - \sinh^2 r\, \sin^2\theta\, d\phi^2 - dr^2, \\ dS^2 &= dt^2 - \sin^2 r\, d\theta^2 - \sin^2 r \sin^2\theta\, d\phi^2 - dr^2. \end{aligned} \tag{167}$$

For Lobachevsky space we have

$$H^{0j} = \frac{\sinh^2 r}{r^2} \cdot \epsilon_0 g^{jj} F_{0j}\,, \quad H^{03} = -\frac{\sinh^2 r}{r^2} \epsilon_0 F_{03},$$

$$H^{01} = -\frac{\sinh^2 r}{r^2} \frac{\epsilon_0}{\sinh^2 r} F_{01}, H^{02} = -\frac{\sinh^2 r}{r^2} \frac{\epsilon_0}{\sinh^2 r \sin^2\theta} F_{02},$$

$$H^{ij} = \frac{\sinh^2 r}{r^2}\, \epsilon_0 g^{ii} g^{jj} F_{ij}\,, \quad H^{12} = \frac{\sinh^2 r}{r^2} \frac{\epsilon_0}{\sinh^4 r \sin^4\theta} F_{12},$$

$$H^{23} = \frac{\sinh^2 r}{r^2} \frac{\epsilon_0}{\sinh^2 r \sin^2\theta} F_{23}, H^{31} = \frac{\sinh^2 r}{r^2} \frac{\epsilon_0}{\sinh^2 r} F_{31},$$

or differently

$$\begin{gathered} H^{01} = -\frac{\epsilon_0}{r^2} F_{01}\,,\ H^{02} = -\frac{\epsilon_0}{r^2\sin^2\theta} F_{02}\,,\ H^{03} = -\frac{\sinh^2 r}{r^2} \epsilon_0 F_{03}\,, \\ H^{23} = \frac{\epsilon_0}{r^2\sin^2\theta} F_{23}\,,\ H^{31} = \frac{\epsilon_0}{r^2} F_{31}\,,\ H^{12} = \frac{r^2}{\sinh^2 r} \frac{\epsilon_0}{r^4 \sin^4\theta} F_{12}\,, \end{gathered} \tag{168}$$

which may be re-written in terms of effective tensors as follows

$$\begin{gathered} D^i(x) = \epsilon_0 \epsilon_{ij}(x) E^{(j)}(x)\,, \quad H^i(x) = \frac{1}{\mu_0} \mu_{ij}(x) B^{(j)}(x)\,, \\ [\epsilon_{ij}(r)] = \left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sinh^2 r / r^2 \end{array} \right|, \quad [\mu_{ij}(r)] = \left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 / \sinh^2 r \end{array} \right|. \end{gathered} \tag{169}$$

For spherical model we have similar result with evident modifications

$$\sinh^2 r, r \in (0, +\infty) \quad \Longrightarrow \quad \sin^2 r, r \in (0, \pi)\,.$$

Let us consider examples with cylindric symmetry. For flat Minkowski space we have (numerate coordinates as $x^\alpha = (t, r, \phi, z)$)

$$G_{\alpha\beta} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right|; \tag{170}$$

in spherical Riemann space an analogous metric is

$$g_{\alpha\beta} = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\sin^2 r & 0 \\ 0 & 0 & 0 & -\cos^2 z \end{array} \right|,\ r \in (0, \frac{\pi}{2}),\ z \in (-\frac{\pi}{2}, +\frac{\pi}{2}). \tag{171}$$

Generated by the last metric constitutive relations have the form

$$H^{\alpha\beta} = \epsilon_0 \frac{\sin r \cos z}{r} g^{\alpha\rho} g^{\beta\sigma} F_{\rho\sigma} \, , \tag{172}$$

that is

$$H^{0j} = \frac{\sin r \cos z}{r} \epsilon_0 g^{jj} F_{0j} \, , \; H^{03} = -\frac{\sin r \cos z}{r} \frac{\epsilon_0}{\cos^2 z} F_{03},$$
$$H^{01} = -\frac{\sin r \cos z}{r} \epsilon_0 F_{01}, \; H^{02} = -\frac{\sin r \cos z}{r} \frac{\epsilon_0}{\sin^2 r} F_{02},$$
$$H^{ij} = \frac{\sin r \cos z}{r} \epsilon_0 g^{ii} g^{jj} F_{ij} \, , \; H^{12} = \frac{\sin r \cos z}{r} \frac{\epsilon_0}{\sin^2 r} F_{12},$$
$$H^{23} = \frac{\sin r \cos z}{r} \frac{\epsilon_0}{\sin^2 r \cos^2 z} F_{23} \, , \; H^{31} = \frac{\sin r \cos z}{r} \frac{\epsilon_0}{\cos^2 r} F_{31} \, .$$

This result may be presented with the help of two effective tensors as follows

$$D^i(x) = \epsilon_0 \epsilon_{ij}(x) E^{(j)}(x), [\epsilon_{ij}(r,z)] = \left| \begin{array}{ccc} \frac{\sin r \cos z}{r} & 0 & 0 \\ 0 & \frac{r \cos z}{\sin r} & 0 \\ 0 & 0 & \frac{\sin r}{r \cos z} \end{array} \right| ,$$
$$H^i(x) = \frac{1}{\mu_0} \mu_{ij}(x) B^{(j)}(x), [\mu_{ij}(r,z)] = \left| \begin{array}{ccc} \frac{r}{\sin r \cos z} & 0 & 0 \\ 0 & \frac{\sin r}{r \cos z} & 0 \\ 0 & 0 & \frac{r \cos z}{\sin r} \end{array} \right| , \tag{173}$$

where referring to cylindric coordinates in Minkowski space electromagnetic components are

$$E^{(j)} = \frac{E_j}{h_j}, \quad B^{(1)} = \frac{B_1}{h_2 h_3}, \quad B^{(2)} = \frac{B_3}{h_3 h_1}, \quad B^{(3)} = \frac{B_3}{h_1 h_2} \, ,$$
$$dt^2 - dr^2 - r^2 d\phi^2 - dz^2 = dt^2 - h_1 dr^2 - h_2 d\phi^2 - h_3 dz^2 \, .$$

In the case of hyperbolic Lobachevsky space we have similar results with evident modifications

$$D^i(x) = \epsilon_0 \epsilon_{ij}(x) E^{(j)}(x), \quad [\epsilon_{ij}(r,z)] = \left| \begin{array}{ccc} \frac{\sinh r \cosh z}{r} & 0 & 0 \\ 0 & \frac{r \cosh z}{\sinh r} & 0 \\ 0 & 0 & \frac{\sinh r}{r \cosh z} \end{array} \right| ,$$
$$H^i(x) = \frac{1}{\mu_0} \mu_{ij}(x) B^{(j)}(x), \quad [\mu_{ij}(r,z)] = \left| \begin{array}{ccc} \frac{r}{\sinh r \cosh z} & 0 & 0 \\ 0 & \frac{\sinh r}{r \cosh z} & 0 \\ 0 & 0 & \frac{r \cosh z}{\sinh r} \end{array} \right| .$$

We can see that for all examples, effective tensors (ϵ_{ij}) and (ϵ_{ij}) obey one the same condition: $\epsilon_{ij}(x) \, \mu_{jk}(x) = \delta_{ik}$.

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Chapter 5

APPLICATIONS FOR RIGID BODY MOTION PREDICTIONS WITH CFD

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Abstract

A six-Dof motion solver based on unit quaternions is implemented for rigid body motion predictions in the open source code OpenFoam. The implementing process is explained in detail. The six-Dof module is then tested with two standard cases. The first case is the water entry phenomenon of a free falling sphere. The displacement history and impacting forces are analyzed. The second case is a KCS (KRISO container ship) model with the allowances of sinkage and trim. The results are validated with experimental data. These models can be used for simulating six-Dof motions of rigid bodies in marine engineering.

Keywords: confined waterway, resistance, ship-generated waves, inland vessel

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1. Introduction

Ship hydrodynamics simulations based on CFD (Computational Fluid Dynamics) has been widely used in recent years. Potential flow theory is a great success and has provided reliable results in this area. The fluids are assumed to be inviscid and irrotational, which simplifies the original NS (Navier Stokes) equations. Efforts have been continuously contributed to directly solving the NS equations, so that more flow details can be captured. Currently, considering both the accuracy and computational cost, RANS (Reynolds-Averaged Navier Stokes) based approaches have been adopted in many commercial and open source codes. Among them, OpenFoam is outstanding which provides multiple functionalities for ship simulations, including different turbulence models, mesh generation methods for complex geometries, mesh moving methods, etc. Other abilities can also be implemented for various purposes.

To simulate the six-Dof motions of a ship, a set of rigid body equations should be solved. Two coordinate systems are normally used, an earth-fixed one and a ship-fixed one [1, 3, 4]. An important concept is the transformation between the two coordinate systems. The Euler angle-based approach has been used in many works and is intuitive to understand [1, 8]. However, it fails when certain angles reach 90°. This is referred as the gimbal lock phenomenon which limits the rigid body motions [10]. The quaternion-based method can be adopted to avoid this problem [6], which is emphasized in this work. OpenFoam uses a symplectic method originated from molecular dynamics, which rotates the rigid body a half of the Euler angle each time to avoid the gimbal lock phenomenon [5].

In this study, a six-Dof motion solver based on unit quaternions is implemented in OpenFoam. Water entry cases of a free falling sphere and a ship with free trim and sinkage are simulated and validated with experimental data. These models are useful for rigid body predictions in ship hydrodynamics simulations.

2. Computational Methods

2.1. Governing Equations

Since DNS (Direct Numerical Simulation) and LES (Large-Eddy Simulation) are computationally expensive, Reynolds decomposition is always used to separate the velocity fluctuations from the mean flow velocity. The Reynolds-

Averaged Navier-Stokes (RANS) equations can then be obtained [2]:

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla \cdot (\rho(\mathbf{u} - \mathbf{u}_g)\mathbf{u}) = -\nabla p^* + \nabla \cdot (\mu_{eff}\nabla\mathbf{u}) + \rho\mathbf{g} + \mathbf{f}_\sigma \tag{2}$$

where $\mathbf{u}$, ρ, p and $\mathbf{g}$ are the velocity, fluid density, pressure and gravitation. $\mu_{eff} = \rho(\nu + \nu_t)$ is the effective dynamic viscosity. The eddy viscosity ν_t is obtained from a turbulence model. $\mathbf{f}_\sigma$ is the surface tension term. The grid velocity $\mathbf{u}_g$ is used to account for the mesh motion.

2.2. Multiphase Method

The VOF (Volume-of-fluid) approach is used for multiphase flow simulations, together with an artificial compression term. The transport equation reads:

$$\frac{\partial\alpha}{\partial t} + \nabla \cdot [\alpha(\mathbf{u} - \mathbf{u}_g)] + \nabla \cdot [\alpha(1-\alpha)\mathbf{u}_r] = 0 \tag{3}$$

where α is the phase fraction, which takes values within the range $0 \le \alpha \le 1$. $\alpha = 0$ and $\alpha = 1$ correspond with gas and liquid respectively. It can be seen that the compression term (the last term on the left-hand side) only takes effect within the interface. This term is able to compress the free surface towards a sharper one. $\mathbf{u} = \alpha\mathbf{u}_w + (1-\alpha)\mathbf{u}_a$ is the effective velocity, and $\mathbf{u}_r = \mathbf{u}_w - \mathbf{u}_a$ is the relative velocity between the two phases, where the subscripts 'w' and 'a' denote water and air respectively. The density and dynamic viscosity are calculated according to:

$$\begin{aligned} \rho &= \alpha\rho_w + (1-\alpha)\rho_a \\ \mu &= \alpha\mu_w + (1-\alpha)\mu_a \end{aligned} \tag{4}$$

The surface tension term $\mathbf{f}_\sigma$ is calculated as:

$$\mathbf{f}_\sigma = \sigma\kappa\nabla\alpha \tag{5}$$

where σ is the surface tension coefficient (0.07 kg/s^2 in water). κ is the curvature of the free surface interface, determined from the volume of fraction by $\kappa = -\nabla \cdot (\nabla\alpha/|\nabla\alpha|)$.

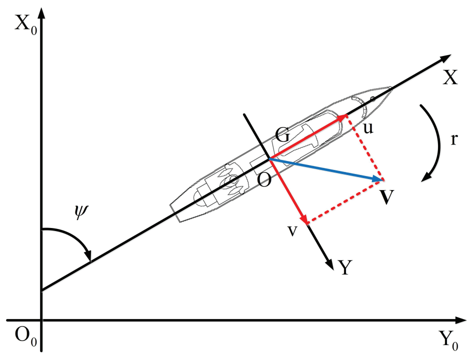

Figure 1. Coordinate systems of the 6Dof motion solver.

2.3. 6Dof Motion Solver

To calculate the 6Dof motions of the ship, a set of rigid body equations should be solved, where two coordinate systems are used (Figure 1). In the earth-fixed system, the 6Dofs of a ship, including surge, sway, heave, roll, pitch and yaw, can be denoted as $\mathbf{u} = (\mathbf{x}_1, \mathbf{x}_2) = (x, y, z, \phi, \theta, \psi)$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the translational and rotational motions respectively. The linear and angular velocities are represented as $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) = (u, v, w, p, q, r)$.

The linear velocities in the body-fixed frame can be transformed to the earth-fixed frame by:

$$\dot{\mathbf{x}}_1 = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} cos\theta cos\psi & sin\phi sin\theta cos\psi - cos\phi sin\psi & cos\phi sin\theta cos\psi + sin\phi sin\psi \\ cos\theta sin\psi & sin\phi sin\theta sin\psi + cos\phi cos\psi & cos\phi sin\theta sin\psi - sin\phi cos\psi \\ -sin\theta & sin\phi cos\theta & cos\phi cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{J}_1 \mathbf{v}_1 \quad (6)$$

The angular velocities can also be transformed as:

$$\dot{\mathbf{x}}_2 = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & sin\phi tan\theta & cos\phi tan\theta \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi / cos\theta & cos\phi / cos\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{J}_2 \mathbf{v}_2 \quad (7)$$

Notice that this equation fails when $\theta = 90°$, which is known as the gimbal lock phenomenon. To avoid the singularity problem caused by this phenomenon, another approach based on the unit quaternions is adopted. A unit quaternion is defined as a complex number with one real part η and three imaginary parts $\boldsymbol{\epsilon} = [\epsilon_1, \epsilon_2, \epsilon_3]$, which satisfies:

$$\eta^2 + \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 = 1 \quad (8)$$

The transformation for linear and angular velocities can be realized by:

$$\dot{\mathbf{x}}_1 = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 1-2(\epsilon_2^2+\epsilon_3^2) & 2(\epsilon_1\epsilon_2-\epsilon_3\eta) & 2(\epsilon_1\epsilon_3+\epsilon_2\eta) \\ 2(\epsilon_1\epsilon_2+\epsilon_3\eta) & 1-2(\epsilon_1^2+\epsilon_3^2) & 2(\epsilon_2\epsilon_3-\epsilon_1\eta) \\ 2(\epsilon_1\epsilon_3-\epsilon_2\eta) & 2(\epsilon_2\epsilon_3+\epsilon_1\eta) & 1-2(\epsilon_1^2+\epsilon_2^2) \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{R}_1\mathbf{v}_1 \quad (9)$$

$$\dot{\mathbf{x}}_2 = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\epsilon_1 & -\epsilon_2 & -\epsilon_3 \\ \eta & -\epsilon_3 & \epsilon_2 \\ \epsilon_3 & \eta & -\epsilon_1 \\ -\epsilon_2 & \epsilon_1 & \eta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{R}_2\mathbf{v}_2 \quad (10)$$

Quaternions and Euler angles can be translated to each other [6]. The forces and moments can be projected from the earth system into the body system in a similar way:

$$\mathbf{F} = (X, Y, Z) = \mathbf{J}_1^{-1} \cdot \mathbf{F}_e \quad (11)$$

$$\mathbf{M} = (K, M, N) = \mathbf{J}_1^{-1} \cdot \mathbf{M}_e \quad (12)$$

After calculating the forces and moments on the rigid body, the accelerations can be obtained by solving the following equations:

$$\begin{aligned}
m[\dot{u} - vr + wq - x_g(q^2+r^2) + y_g(pq-\dot{r}) + z_g(pr+\dot{q})] &= X \\
m[\dot{v} - wp + ur - y_g(r^2+p^2) + z_g(qr-\dot{p}) + x_g(qp+\dot{r})] &= Y \\
m[\dot{w} - uq + vp - z_g(p^2+q^2) + x_g(rp-\dot{q}) + y_g(rq+\dot{p})] &= Z \\
I_x\dot{p} + (I_z - I_y)qr - (\dot{r}+pq)I_{xz} + (r^2-q^2)I_{yz} + (pr-\dot{q})I_{xy} & \\
+m[y_g(\dot{w} - uq + vp) - z_g(\dot{v} - wp + ur)] &= K \quad (13) \\
I_y\dot{q} + (I_x - I_z)rp - (\dot{p}+qr)I_{xy} + (p^2-r^2)I_{zx} + (qp-\dot{r})I_{yz} & \\
+m[z_g(\dot{u} - vr + wq) - x_g(\dot{w} - uq + vp)] &= M \\
I_z\dot{r} + (I_y - I_x)pq - (\dot{q}+rp)I_{yz} + (q^2-p^2)I_{xy} + (rq-\dot{p})I_{zx} & \\
+m[x_g(\dot{v} - wp + ur) - y_g(\dot{u} - vr + wq)] &= N
\end{aligned}$$

where (x_g, y_g, z_g) is the vector from the center of rotation to the center of gravity. I_{xx}, I_{yy} and I_{zz} are the moments of inertia around the center of rotation. The accelerations can then be integrated to obtain the velocities and further the ship displacements.

To diffuse the movements of the rigid body, the mesh deformation technique by solving the Laplace equation is adopted:

$$\nabla \cdot (\gamma \nabla \mathbf{u}_g) = 0 \quad (14)$$

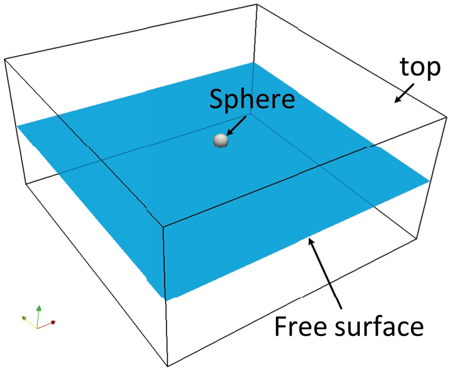

Figure 2. Computational domain of the water entry case.

where γ is the diffusivity coefficient. The grid velocity $\mathbf{u}_g$ can be obtained and the mesh information is updated in the computational domain.

3. Numerical Results and Discussions

3.1. Water Entry of a Free Falling Sphere

The water entry phenomenon of a free falling sphere is first simulated to validate the six-Dof motion solver. The computational domain can be found in Figure 2, with the sphere placing 1 m above the free surface. The density of the sphere is 500 kg/m^3 (half of the water density). The total grid number is 665,409. Six outer correctors and four pressure correctors are used since transient behaviors are important in this case, with five different time steps 1×10^{-2}, 5×10^{-3}, 1×10^{-3}, 5×10^{-4}, 1×10^{-4} s. No turbulence model is adopted in this case. Since no wave absorption techniques are used, the boundaries are placed far from the sphere, and the mesh size are incremented from the sphere so that waves can be diffused.

The displacement and vertical force histories are compared with [9] (Figures 3-4). The overall results agree with the work of [9]. The converge histories of the displacement and vertical force for the time steps 1×10^{-3}, 5×10^{-4} and 1×10^{-4} are close. The impact forces when the sphere first touches the free surface are predicted close to the work of Shen [9]. Discrepancies can be seen after about 4 s, where our results diffuse more. This may be caused by the fact that [9] used sponge layers around the sphere, which would have less influences of wave reflections. Although the mesh used here is designed specifically to reduce this phenomenon, the waves cannot be fully diffused by this method. However, both works converge to the same values at last. Small time steps ($\leq 1 \times 10^{-3}$ in this case) are better to be used for transient simulations.

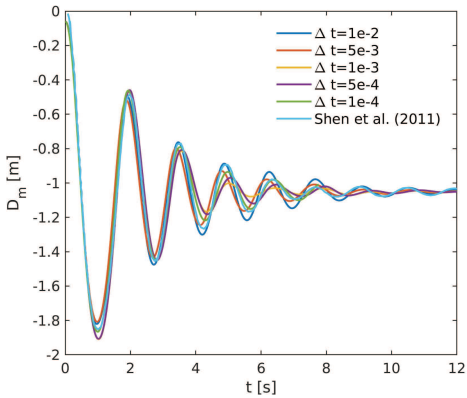

Figure 3. Comparison of the displacement history of the free falling sphere [9].

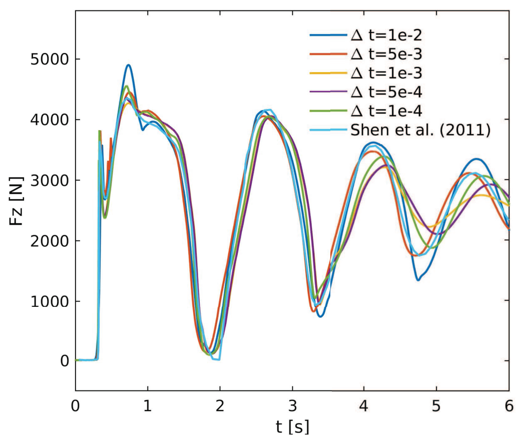

Figure 4. Comparison of the vertical force history of the free falling sphere [9].

3.2. Trim and Sinkage Prediction

A KCS model with free sinkage and trim is simulated in this case. The case setups are shown in Table 1, where six different ship velocities are designed. The boundary conditions are shown in Table **??**. The SST k-ω model is used in this case. The grid number is 1,074,514. The numerical schemes, corrector numbers and convergence criterion are the same as the water entry case. The resistance coefficient (C_T), sinkage (σ) and trim (τ) are calculated and compared with experiments [7]. The resistance coefficient is calculated as:

$$C_T = \frac{X_T}{\frac{1}{2}\rho U_{in}^2 S_{w0}} \tag{15}$$

where X_T, U_{in} and S_{w0} are the total resistance, ship speed and wetted surface area at rest. In Figure 5, the resistance coefficients are underestimated for $Fr \leq 0.260$. The sinkage is underestimated for $Fr \leq 0.195$ and overestimated for $Fr \geq 0.260$. The trim prediction is fairly well, with only the case $Fr = 0.108$ underestimated and $Fr = 0.282$ overestimated. The largest error of the three parameters are about 3.0%, 6.4% and 7.1% respectively. Overall, the implemented six-Dof model can predict the ship resistance and motion states with satisfactory accuracy.

Table 1. Case setups of the trim and sinkage prediction for the KCS model

Case No.	**1**	**2**	**3**	**4**	**5**	**6**
U_{in} $[m/s]$	0.914	1.281	1.647	1.922	2.196	2.380
Fr	0.108	0.152	0.195	0.227	0.260	0.282
Re	6.66×10^6	9.32×10^6	1.20×10^7	1.40×10^7	1.60×10^7	1.73×10^7

CONCLUSION

A six-Dof motion solveris implemented into the open source code OpenFoam. The six-Dof motion solver is based on unit quaternions, which can avoid the gimbal lock phenomenon limiting the rigid body motions. A water entry problem of a free falling sphere and a KCS model with six different ship velocities are used for validating the performance of the six-Dof motion solver. For the water entry case, the displacement and vertical force on the sphere are analyzed and compared with former researches. For the KCS case, the resistance

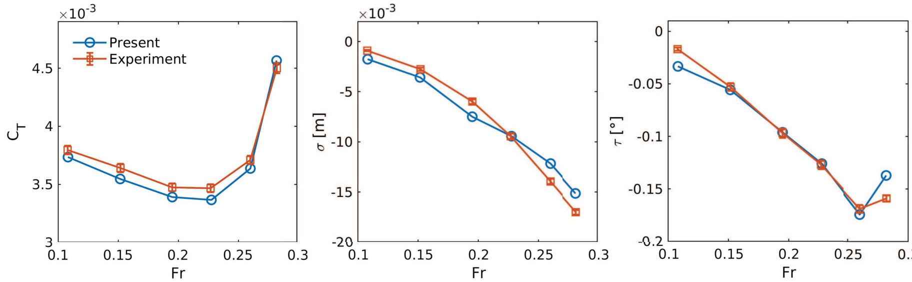

Figure 5. Comparison of resistance coefficient (C_T), sinkage (σ) and trim (τ) with experimental data [7].

coefficient, sinkage and trim are validated using experimental data. The ship hydrodynamics is harder to predict when the speed is low. This model can predict rigid body motions with no spatial limitations.

ACKNOWLEDGMENT

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Chapter 6

APPLICATIONS FOR THE BALLAST-FLIGHT

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Abstract

The frozen snow / ice blocks drop at high speed from train causing the ballast to fly up and may damage the car body. Thus, in this paper, we propose a numerical model based on a discrete element method to study the ballast flight caused by dropping snow / ice block in high-speed railways, and also to analyze the dynamic behavior of ballast particles during their collision with a snow / ice block. The results show that the number and maximum displacement of ballast particles increase as the train speed increases and that the incident angle greatly affects the movement direction of ballast particles. Results also show that the shape of the ice block affects the amount and extent of ballast flight.

Keywords: confined waterway, resistance, ship-generated waves, inland vessel

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1. INTRODUCTION

The phenomenon of ballast flight is one of the major problems in high speed ballasted track, which has resulted in major maintenance costs and safety concerns. Flying ballast particles may hit the rail, the train body or the waiting for passengers at through stations. Furthermore, small particles of ballast may come to rest between the railhead and the wheels of rail vehicles, which cause substantial local bending damage to the rail [1, 2]. The aerodynamic effect is commonly regarded as the main cause of ballast flight [3, 4]. However, according to the reported ballast projection-ballast flight incidents by Aerodynamic in Open Air project (AOA within the DEUFRAKO project), more than 50% of ballast flights happen in winter conditions, which is induced by the impact of falling ice [5] The effects of ballast projection caused by dropping snow/ice may lead to serious catastrophic consequences due to the high initial velocity, large mass, and chain reaction of the falling ice. This takes place as high-speed trains run through snowfall and blow up the snow which sticks to the underfloor equipment and freezes rapidly into ice. Then, the frozen snow and ice drops at high speeds from trains due to temperature changes, train vibration and heat from the brakes, causing the ballast to fly up and seriously damage the car body and the environment along the track [6], as shown in Fig. 1.

Figure 1. Snow fly and stick to the underfloor equipment.

Snow / ice dropping and influencing factors of ballast flight have been studied in many recent studies. Loponen *et al* [6] studied the amount of the excitation required to drop snow from the train underframe. They used a simplified equilibrium equation based on the adhesive forces to indicate that the snow dropping requires an acceleration amplitude of approximately 20-2000

g depending on the characteristics of the snow mass. Kawashima *et al* et al. [7] carried out an experimental study using air cannon tests to investigate the ballast-flight phenomenon caused by the dropping of accreted ice, and gave a relationship between the number of flying ballast stones and the mass, shape, speed, and angle of the ice. Furthermore, computational fluid dynamics (CFD) methods have been applied to the study of the flow between the train underbody and the track bed around the bogie area and its impact on the ballast flight [8, 9, 10]. Xie *et al* [11] used a 3-D numerical model based on the coupling between Navier-Stokes equations based model and a discrete phase model to investigate the flow field in the presence of snow accumulation in the train underframe. The numerical result showed that the snow particles accumulated and moved on the train bogies by the high-speed air.

The main purpose of this paper is to determine the influence factors of ballast flight and to analyze the dynamic behavior of ballast particles during their collision with a snow / ice block. The performance of the ballast flight is evaluated by considering the velocity, the shape and the incident angle of snow/ice blocks. The numerical results show that these factors influence significantly the ballast particles dynamics and also their flight as well as the collision between ballast particles.

2. GOVERNING EQUATIONS

In order to investigate the rigid motion of ballast blocks, the displacement (u, v) at any point (x, y) of a block i can be described by its first order approximation [12].

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 & -(y-y_0) & (x-x_0) & 0 & (y-y_0) \\ 0 & 1 & (x-x_0 & 0 & (y-y_0) & (x-x_0) \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ \gamma_0 \\ \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} \tag{1}$$

Where u_0 and v_0 are the translations of the block gravity center in x and y directions, and γ_0 is the rotation around (x_0,y_0), and ε_{xx}, ε_{yy} and ε_{xy} are the normal and shear strains.

The total potential energy Π_p of the block i is the summation over all the potential energy sources, which includes the elastic strain energy, initial stress potential energy, body force potential energy, and inertial energy, is given by:

$$\Pi_p = \int_{A_i} \frac{1}{2}\boldsymbol{\varepsilon_i}^T \boldsymbol{\sigma_i} dA_i + \int_{A_i} \boldsymbol{\varepsilon_i}^T \boldsymbol{\sigma_0} dA_i - \int_{A_i} \boldsymbol{U_i}^T \boldsymbol{f}_b dA_i + \int_{A_i} \boldsymbol{U_i}^T m \boldsymbol{T_i}^T \ddot{\boldsymbol{D}}_{\boldsymbol{i}} dA_i \quad (2)$$

where $\boldsymbol{\varepsilon_i}$ and $\boldsymbol{\sigma_i}$ are the strain and stress of block i , $\boldsymbol{\sigma_0}$ is the initial stress. $\boldsymbol{U_i} = (u, v)^T$ is the displacement of block i, $\boldsymbol{f_b}$ is the body forces applied on a block i, and m is the block mass per unit area, $\ddot{\boldsymbol{D}}_{\boldsymbol{i}}$ is the acceleration of block i.

Based on the minimized potential energy approach we can write:

$$\frac{\partial \Pi_p}{\partial \boldsymbol{D_i}} = 0 \Rightarrow \boldsymbol{KD} = \boldsymbol{F} \quad (3)$$

Consequently, we get the global matrix:

$$\begin{pmatrix} \boldsymbol{K_{11}} & \boldsymbol{K_{12}} & \boldsymbol{K_{13}} & \ldots & \boldsymbol{K_{1n}} \\ \boldsymbol{K_{21}} & \boldsymbol{K_{22}} & \boldsymbol{K_{23}} & \ldots & \boldsymbol{K_{2n}} \\ \boldsymbol{K_{31}} & \boldsymbol{K_{32}} & \boldsymbol{K_{33}} & \ldots & \boldsymbol{K_{3n}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{K_{n1}} & \boldsymbol{K_{n2}} & \boldsymbol{K_{n3}} & \ldots & \boldsymbol{K_{nn}} \end{pmatrix} \begin{pmatrix} \boldsymbol{D_1} \\ \boldsymbol{D_2} \\ \boldsymbol{D_3} \\ \vdots \\ \boldsymbol{D_n} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F_1} \\ \boldsymbol{F_2} \\ \boldsymbol{F_3} \\ \vdots \\ \boldsymbol{F_n} \end{pmatrix} \quad (4)$$

where $\boldsymbol{F_i}$ and $\boldsymbol{D_i}$ are the sub-matrices of force and displacement, which are 6×1 sub-matrices. The $\boldsymbol{K_{ij}}$ are 6×6 sub-matrices. Sub-matrices $\boldsymbol{K_{ij}}$ $(i = j)$ are determined by the block material properties, whereas $\boldsymbol{K_{ij}}$ $(i \neq j)$ are related to the contacts between blocks.

In the present study, using the penalty method which describes the surface contact constraints, is equivalent to place a spring between the two blocks [12, 13]. The strain energy of the contact spring is:

$$\Pi_c = \frac{1}{2} k \delta^2 \quad (5)$$

where k is the spring stiffness, while δ is the penetration overlap which should also be minimized; therefore, the penalty number k should be very large to achieve an accurate result.

3. Simulation Results

3.1. Velocity of Ice Block

In cross-section calculation, the ballasted track bed model was built as shown in Fig. 2. The distance between ice block and the top surface of the ballast bed was set to 600 mm, which is approximate equal to the height of the rail plus half the height of the bogie. Adding the unit mass, elastic modulus, Poisson ratio, friction angle, etc, to the stiffness matrix in accordance with the physical characteristics of ballast and ice [14, 15], as shown in Table. 1.

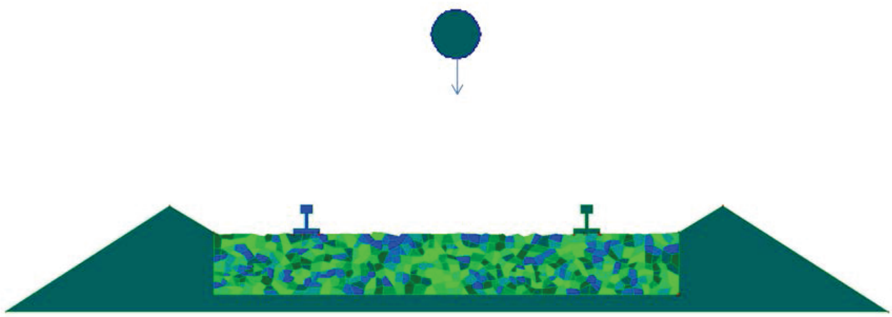


Figure 2. Track bed model.

Table 1. The material parameters of ice and ballast

–	Unit mass	Elastic modulus	Poisson ratio	Friction angle
Ice	$934kg/m^3$	$10GPa$	0.06	–
Ballast	$2600kg/m^3$	$50GPa$	0.20	45^o

We took five vertical speeds from 0 m/s to 20 m/s with an interval of 5 m/s. Overall, the simulation results presented below show that ballast flight responses to increased speed varied significantly. There are three key phenomena, are shown in Fig. 3: (1) the movement trajectory of ice block; (2) the evolution of marked ballast displacement; (3) the responses of ballast bed. The phenomenon showed that the ice block after the collision first rebounded and then rolled. When the speed was large than 10 m/s, the ice hit the rail (see Fig.

3(c) and 3(d)) or flied out of the range of the track bed (see Fig. 3(e)). The higher the speed, the higher rebound of marked ballast ejected, the severity of ballast bed responses generated, and the greater the number of ballast particles ejected. When the vertical speed exceeded 15 m/s, the vertical displacement of the ballast particles higher than 330 mm which is of importance as this is the distance from the ballast to the train underframe [7]. If the ballast particles reach the height of vehicles underside, they may be accelerated significantly due to the collision with the vehicle or may detach further ice blocks.

From Fig. 4, the numerical results show that the number of flying ballast particles and their vertical displacement increases from 0 m/s to 20 m/s. When the vertical speed of ice block is in the range of 10 m/s to 15 m/s, the results exhibit a peak of 40 % and 43 % of the total number of flying ballast, which corresponds respectively to the number of flying ballast particles and their vertical displacement. Above 15 m/s, this process evolution increases gradually until a plateau is reached (see Fig. 4). This non-uniform evolution may be related to the gradation of ballast composed of different particle sizes. As the speed increases, most of the small size and a part of the middle size of ballast particles in the impacted area fly to the above of the bed surface. When the speed excesses 15 m/s, the amount of ballast particles which can still be moved is about 29 % of the total number of flying ballast particles, which corresponds to big sizes particles.

Fig. 5 shows the displacement of the ice block and marked ballast. The analysis of the ice blocks trajectories and displacement show that their movement may be divided into three phases. Taking the speed of 10 m/s as an example, the first phase sited between t=0 s and t= 0.06 s corresponds to the free fall of the ice block. The second phase sited between t= 0.06 s and t =0.32 s, corresponds to the process of impact, rebound and fall. The third phase sited between t= 0.32 s and t =0.5 s, corresponds to ice block rolls on the surface of the track bed and gradually stabilizes. Likewise, we divide the movement of the marked ballast into three phases: stable state, fly up and fly back after the collision, and roll to the stable. It should mention that the severity and the duration of every phase strongly depend on the initial speed.

3.2. Shape of Ice Block

In the second simulation, circular, triangular and square ice blocks with a diameter/side length of 100 mm were placed 600 mm above the ballast bed surface

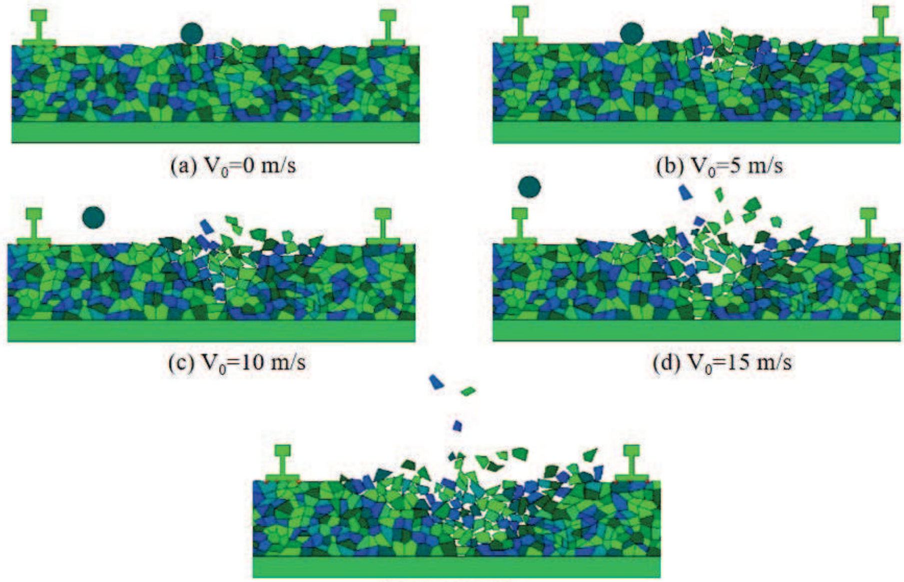
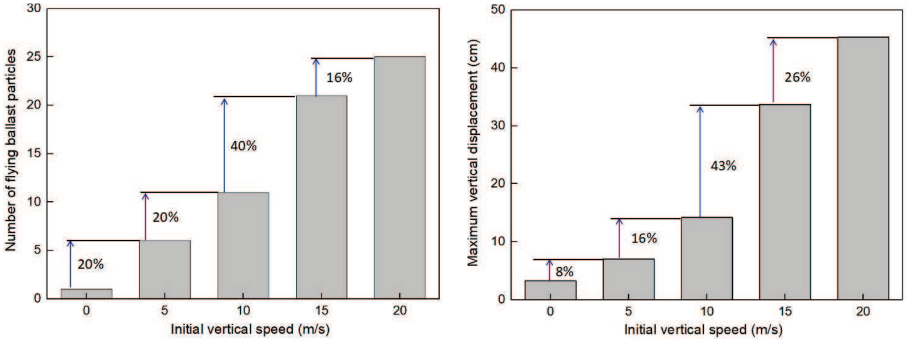


Figure 3. Computed ballast flight depending on the initial vertical velocity: (a) V_y= 0 m/s (free fall); (b) V_y= 5 m/s; (c) V_y= 10 m/s.

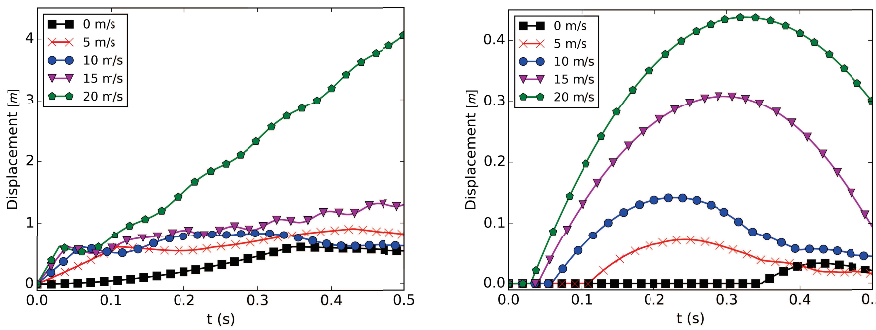
at the center of the track. With the initial vertical velocity of 10 m/s, the impact results are shown in Fig. 6.

Three shapes of ice block trajectories are significantly different. The triangular ice intrudes the ballast bed directly after colliding, while the square ice bounces off. The trajectory of the circular bounced and hit the trackbed twice. Furthermore, according to the response of the ballast bed after collision, the triangular ice ejected the greatest number of particles and caused the greatest height of ejection, on account of its contact area with the track bed being the smallest, resulting in maximum pressure. The square ice has the widest and the deepest impact among the three, due to its larger contact area and greater mass. The results of the displacement of the ice block and the marked ballast are shown in Fig. 7. After the collision, the triangular ice has the smallest displacement and the square ice has the largest rebound. The triangular ice causes the maximum displacement of ballast particles, followed by the circular ice and then the square ice.



(a) (b)

Figure 4. Relationship between (a) number of flying ballast particles and initial vertical speed of ice block (When the initial vertical speed is 0 m/s, it represents free fall motion.); (b) maximum displacement and initial vertical speed of ice block.



(a) (b)

Figure 5. The displacement of (a) ice block and (b) marked ballast for different speed in cross section.

Conclusion

This study took into account the shapes of the ice blocks and the contacts between ballast particles, where we assumed that contact constraints were imposed through the penalty method. The ballast flight induced by the dropping snow/ice with some variations in intensity depends on the velocity, the incident angle, and

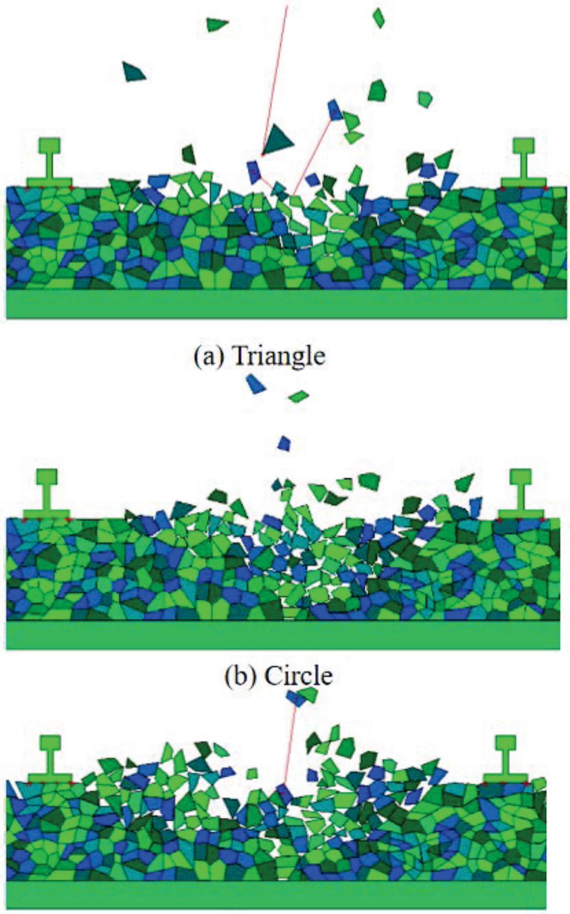
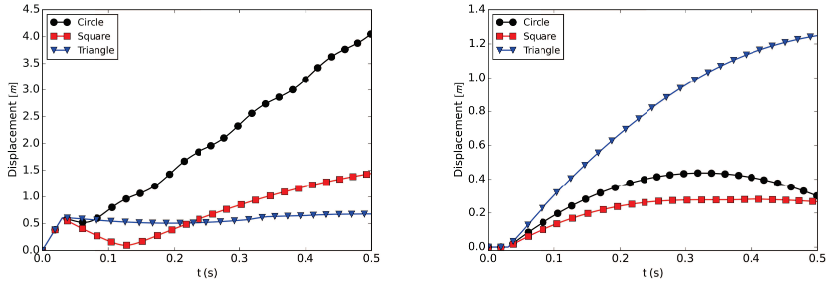

(a) Triangle
(b) Circle
(c) Square

Figure 6. Relationship between ballast flight and ice block shape: (a) Triangle; (b) Circle; (c) Square.

shapes of the ice blocks. The main findings derived from the numerical simulation may be summarized as follows:

- The speed of the snow/ice block, which directly depends on the speed of the train, has a great impact on the ballast flight. In the cross-section, the number of flying ballast particles and their vertical displacement in-



(a) (b)

Figure 7. The displacement of (a) ice block and (b) marked ballast of different shapes.

creases from 0 m/s to 20 m/s. When the vertical speed of ice block is in the range of 10 m/s to 15 m/s, the results exhibit a peak of 40 % and 43 % of the total number of flying ballast, which corresponds respectively to the number of flying ballast particles and their vertical displacement. Above 15 m/s, this process evolution increases gradually until a plateau is reached. Hence, setting the maximum operating speed according to the weather conditions is an effective measure to reduce serious consequences.

- The shape of the snow/ice block affects the extent of ballast flight. The triangular, circular and square ice blocks resulted in maximum vertical flying heights of 1.058 m, 0.435 m and 0.214 m and impact length of 1.10 m, 1.02 m and 1.25 m, respectively. Furthermore, when the impact position is outside the rail, the ballast flight will lead to more serious consequences all around the impact location.

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Chapter 7

APPLICATIONS FOR THE STABILITY OF CAISSON-TYPE BREAKWATERS

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Abstract

Rubble mound breakwaters consist of a vertical concrete caisson and rubble mound structure in which the sliding of the caisson is a potential cause of their failure. A numerical study is proposed to analyze the effect of the caisson sliding subjected to a hydrodynamic loading in the stability of the rear side of the rubble mound breakwater. The study takes into account the slope inclination of the breakwater as well as the contact between the blocks constituting the back of the breakwater, where the contact stresses are imposed through a penalty method. The developed model carried out by considering three different breakwaters constituted of blocks of accropodes, tetrapods and cubes forms. Results showed that the tetrapod blocks are the most stable, followed by accropode blocks and by cubic blocks.

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Keywords: confined waterway, resistance, ship-generated waves, inland vessel

1. INTRODUCTION

Breakwaters are used for protection of harbors and beaches against wave action, and are composed of the concrete caisson and the rubble mound structure, as shown in Fig. 1. Traditionally, the empirical formula was widely used in the design of breakwaters [1]. However, some essential factors are not taken into account, for example,the wave and structure interaction, and contacts between shaped blocks. Furthermore, the breakwater is subjected to three possible failure modes: the motion of caissons, global stability, and scouring of the rubble mound [2, 3].

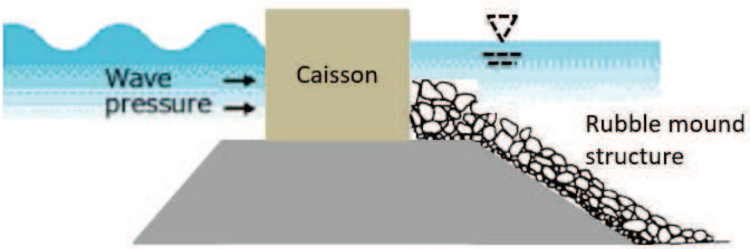

Figure 1. Schematic illustration of the studied breakwater. It consists of a vertical caisson and a rubble mound structure.

Various computational approaches have been proposed to analyze the failure mechanisms and the stability of breakwater, in particular with regard to the effect of the wave [4, 5]. H. Takahashi [6] conducted a finite element method (FEM) to study the bearing capacity characteristics of the breakwater under the influence of the tsunami. The horizontal loading, uplift force and seepage force are taken into consideration, but the specific displacement of a breakwater caisson and mound structures are not computed. Y. Sawada [7] carried out a study based on the Discrete Element Method (DEM) to examine the stability of a caisson breakwater. However, the model of the caisson, established by particles and beam elements, does not explicitly represent the shape of the blocks. Furthermore, DEM has a limitation because of its large consuming computation time which scales as the order of the number of particles N, since every particle and their interactions are tracked over time. J.P Latham [8] combined DEM and computational fluid dynamics (CFD) to analyze the wave dynamics of the granular solid skeleton. The model is suitable to study the wave-structure interaction of breakwater. However, the shape of rubble mound units are inaccurate, which can not show the interlock of rubble mound structures well.

In this paper, a numerical study is presented to assess the rear-side stability of the breakwater under hydrodynamic loading and under the caisson sliding. This study takes into account the shapes of the blocks, of the slope inclination and of contacts between shaped blocks, where we assume that contact constraints are imposed through the penalty method. The method has been evaluated by considering three breakwaters consisting of cubic, accropode and tetrapod blocks which are simulated under hydrodynamic forces.

2. MATHEMATICAL FORMULATION

In order to investigate the movement of caisson and discrete blocks, the first order approximation of the displacement (u, v) at any point (x, y) of a block i is interpolated as:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 & -(y-y_0) & (x-x_0) & 0 & (y-y_0) \\ 0 & 1 & -(x-x_0) & 0 & (y-y_0) & (x-x_0) \end{pmatrix} \cdot \{D\} = (T) \cdot \{D\} \quad (1)$$

where $\{D\}$ is the vector of variables corresponding to an individual block comprising the rigid body translations and rotation at the centroid of the block

$$\{D\} = \{\mu_0 \quad \nu_0 \quad \gamma_0 \quad \varepsilon_x \quad \varepsilon_y \quad \varepsilon_{xy}\}^T \quad (2)$$

where $(\mu_0, \nu_0, \gamma_0\ \varepsilon_x, \varepsilon_y\ , \varepsilon_{xy})$ are the normal and shear strain vectors of block centroid.

The equilibrium formulation is provided by the principle of potential energy minimization. The total potential energy π_i is given by

$$\pi_i = [D_i]^T(\frac{1}{2}[K_{ij}][D_i] - [F_i]) \quad (3)$$

where $[K_{ij}]$ are the material stiffness matrices; $[F_i]$ is the external force on the block i.

For more information regarding the derivation of these contributions, the reader is referred to Shi [9] and Shi and Goodman[10]. Minimization of the above energy potential yields

$$\frac{\partial \pi_i}{\partial D_{ri}} = -[F_i] + [K_{ii}][D_i] = 0 \quad (4)$$

Consequently, Equation 4 takes the partitioned form:

$$\begin{pmatrix} [K_{11}] & [K_{12}] & [K_{13}] & \dots & [K_{1n}] \\ [K_{21}] & [K_{22}] & [K_{23}] & \dots & [K_{2n}] \\ [K_{31}] & [K_{32}] & [K_{33}] & \dots & [K_{3n}] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ [K_{n1}] & [K_{n2}] & [K_{n3}] & \dots & [k_{nn}] \end{pmatrix} \begin{pmatrix} [D_1] \\ [D_2] \\ [D_3] \\ \vdots \\ [D_2] \end{pmatrix} = \begin{pmatrix} [F_1] \\ [F_2] \\ [F_3] \\ \vdots \\ [F_n] \end{pmatrix} \tag{5}$$

where $[F_i]$ is a 6 $\times$1 vector of forces acting on block i and $[D_i]$ contains the variables associated with block i . The off-diagonal submatrices $[K_{ij}](i \neq j)$ contain the stiffness components associated with the contact between block i and j , and $[K_{ii}]$ refers to the components of the material stiffness of block.

3. Simulation Results

3.1. Solution Behavior with Breakwater Slope

In order to analyze the influence of the slope of breakwater, three different slopes($\sqrt{3}$:1, 1:1 and 1:2) were investigated. In these three cases, the rubble mound structures of breakwater are composed of cubic blocks with the side length of 2.0 m. Six blocks, numbered from 1 to 6, were marked (see Fig. 2). The material parameters for the simulation are given in Table. 1. In this simulation, the contact behavior is defined by a friction angle of 45^o [11], and the horizontal load is equal to 1000 kN. The displacement of caisson and cubic rubble mound blocks in three slopes after 60-time steps of 0.05 seconds, is shown in Fig. 3 and 4.

Table 1. Material parameters used for simulation

Young's modulus E	Poisson's ratio	unit weight	Penalty spring constant
50 GPa	0.30	2600 kg/m^3	2$\times$10^8 N/m

Fig. 3 and Fig. 4 show the displacement of the caisson for three slopes. The displacement of the slope of $\sqrt{3}$: 1 is significantly larger than those of the two other slopes. The slope of 1:1 and 1:2 have roughly the same displacement from time t=0 s to t=1.5 s. The displacement for the slope 1:2 becomes steady after

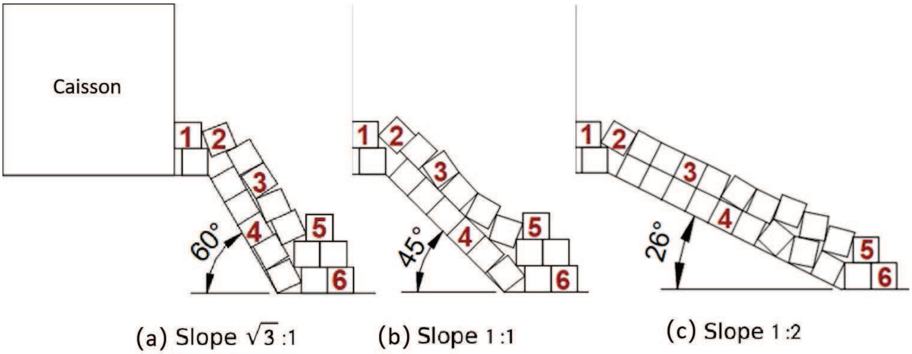

Figure 2. Slope of breakwater: (a) slope=$\sqrt{3}$:1, (b) slope=1:1, (c) slope=1:2.

t=1.5 s. The final caisson displacement for the three slopes is: $D_{\sqrt{3}:1}$ =0.501 m, $D_{1:1}$ =0.380 m, $D_{1:2}$ =0.238 m, respectively. Therefore, the flatter is the slope, the smaller the displacement of the blocks is, the steadier the breakwater is. The reason is that the rubble mound blocks in the flatter slope have large lateral resistance to the caisson. Block 2 and block 3 have larger vertical displacement than others. Block 1, 5 and 6 have large horizontal displacement. It also should be noted that the block 6 moves very little for the flatter slope while it is the one that moves the most for the steep slope. The standard deviation of the displacement or rotation of the rubble mound blocks at t=3.0 s is calculated for each slope by:

$$s = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N}} \tag{6}$$

where x_N is the displacement or rotation value of N blocks at t=3.0 s, $\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N}$ is the mean displacement or rotation value and $N = 6$. The standard deviations of horizontal displacement, vertical displacement and rotation is shown in Table. 2. For the displacement and rotation of rubble mound blocks, we all have $s_{\sqrt{3}:1} > s_{1:1} > s_{1:2}$. The flatter slope was lower standard deviation which indicates that the displacement and rotation of blocks tends to be close to the mean. Therefore, the breakwater is more stable when the slope is flatter.

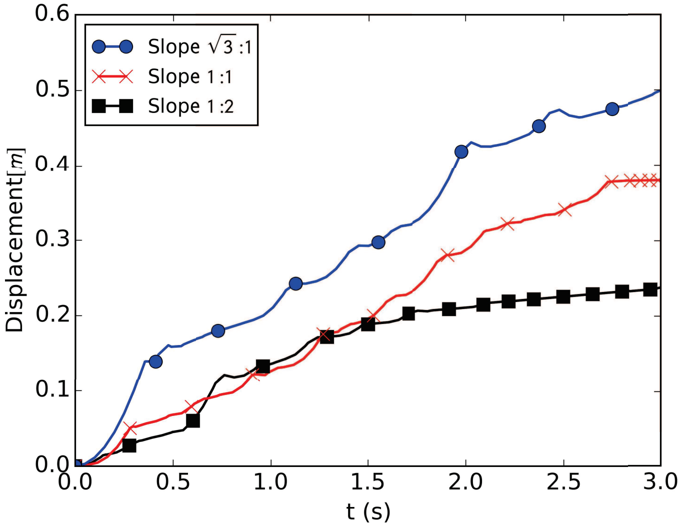

Figure 3. The displacement of caisson for cubic blocks for various slopes.

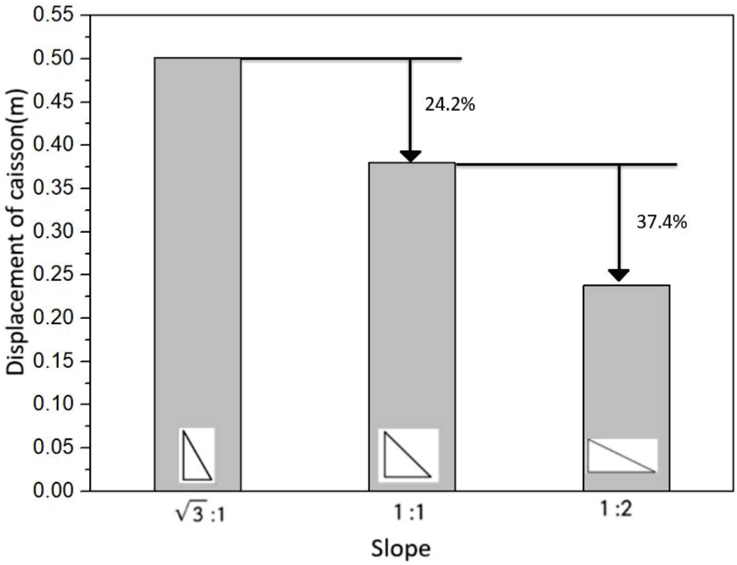

Figure 4. The displacement of caisson for various slopes(t=3.0s).

Table 2. Standard deviations for three slopes

Slope inclination	$\sqrt{3}$:1	1:1	1:2
Horizontal displacement(m)	0.231	0.161	0.121
Vertical displacement(m)	0.594	0.549	0.206
Rotation(rad)	0.069	0.058	0.043

3.2. Solution Behavior with Blocks Shapes

In this second simulation, three typical blocks of rubble mound structures [12] are modeled and placed on the side of the caisson breakwater. Block units are optimally arranged to make sure the initial position stability, as shown in Fig. 5. The displacement of the caisson and specific shaped blocks in three shapes after 60-time steps of 0.05 seconds, is shown in Fig. 6 and 7 .

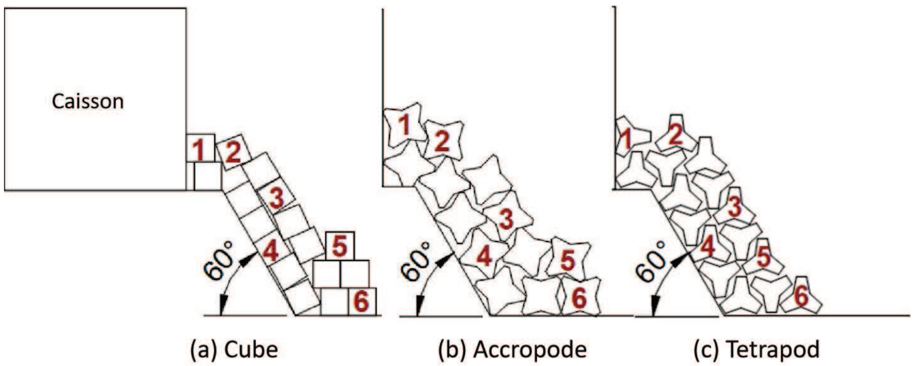

Figure 5. Shapes of breakwater blocks: (a) Cube, (b) Accropode, (c) Tetrapod.

Fig. 6 and Fig. 7 show that the displacement of the caisson hindered by the cubic rubble mound blocks is significantly larger than the displacement of the other two shapes. The displacement of caisson hindered by tetrapod shaped blocks, is almost zero and stable. From t=0 s to t=0.5 s, the displacement of caisson hindered by accropode blocks and tetrapod-shaped blocks are near and small. From t=0.5 s to t=1 s, the displacement of the caisson hindered by accropode blocks increases rapidly, and then it increase more slowly from t=1 s to t=3 s. This jump in the graphic can be explained by accropode blocks' rearrangement. The tetrapod blocks do not need to rearrange themselves because their geometry allows them a better nesting.

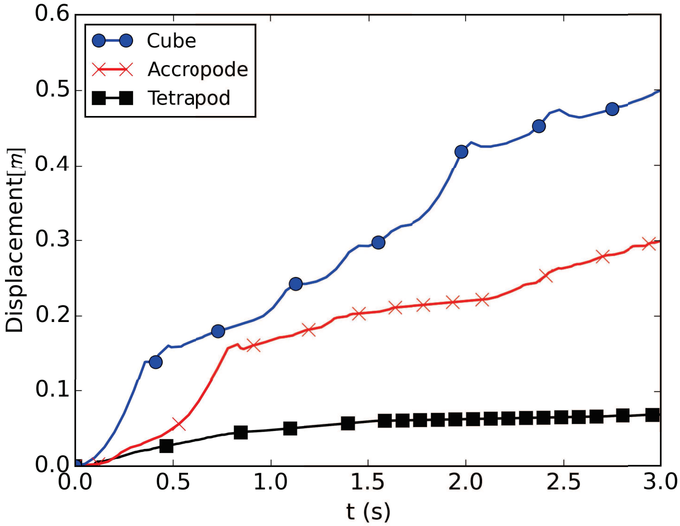

Figure 6. The displacement of caisson for various shapes.

The standard deviations of horizontal displacement, vertical displacement and rotation were calculated by Eq. (6) (see Table. 3). For the displacement, we have $s_{cube} > s_{accropod} > s_{tetrapod}$. However, for the rotation, the accropode blocks significantly greater than the two others. It also can be explained by the rearrangement of accropode blocks. Furthermore, the tetrapod shaped blocks have a very small displacement which induces to the smallest rotation among the three. The cubic blocks have larger displacement. But most of the displacement of blocks is the translation along the slope. It should be noted that although the accropode blocks have a larger rotation, it is very stable after rearrangement. Therefore, the tetrapod blocks are the most stable, followed by accropode and cubic rubble mound blocks.

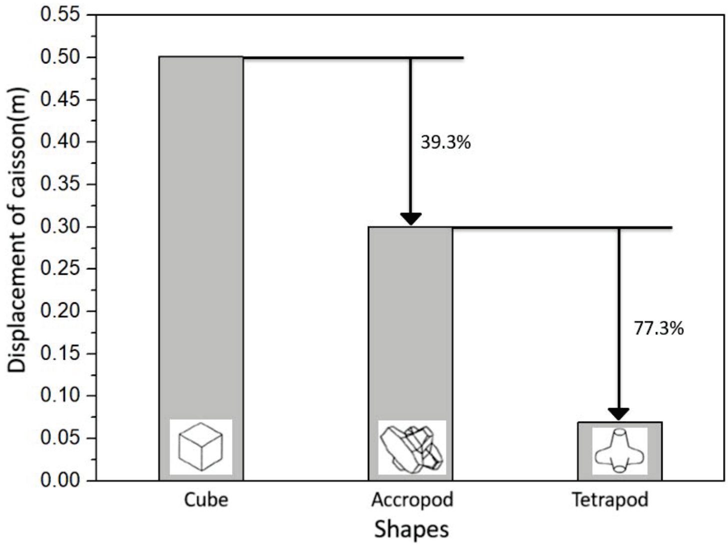

Figure 7. The displacement of caisson for various shapes(t=3.0s).

Table 3. Standard deviations for three shapes

shapes	Cube	Accropod	Tetrapod
Horizontal displacement(m)	0.231	0.197	0.025
Vertical displacement(m)	0.594	0.267	0.061
Rotation(rad)	0.069	0.148	0.028

CONCLUSION

A numerical model is proposed to study rear-side stability of the breakwater under hydrodynamic loading and under the caisson sliding. This study takes into account of the blocks shapes, the slope inclination and contacts between shaped blocks, where we assume that contact constraints are imposed through the penalty method. This paper demonstrated that the hydrodynamic load drives the caisson and the displacement of blocks with variations depending on breakwater slope, on the shape and on the cohesion between blocks. Major findings emerging from the simulation may be summarized as follows:

- The breakwater is more stable when the slope is flatter. The reason is that the blocks in the flatter slope have large lateral resistance to the caisson.
- The tetrapod blocks are the most stable, followed by accropode and by cubic blocks. Cubic rubble mound blocks bring resistance to breakwater by the mass whereas tetrapod and accropode blocks bring resistance through the mass and blocks interlock forces.

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UNDERSTANDING QUATERNIONS

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