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Ulrich Langer, Dirk Pauly,
Sergey I. Repin (Eds.)

MAXWELL'S EQUATIONS

ANALYSIS AND NUMERICS

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Edited by
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Preface

This volume contains 12 chapters that provide some recent developments in the analysis and numerics of Maxwell's equations. The contributions result from Workshop 1 on "Analysis and Numerics of Acoustic and Electromagnetic Problems" held at the Radon Institute for Computational and Applied Mathematics (RICAM) in Linz, Austria, October 17–22, 2016. This workshop was the first workshop within the Special Semester on "Computational Methods in Science and Engineering," which took place in Linz, October 10–December 16, 2016; see also the website:

https://www.ricam.oeaw.ac.at/specsem/specsem2016/

Maxwell's equations of electro-dynamics are of huge importance in mathematical physics, engineering, and especially in mathematics, leading since their discovery to interesting mathematical problems and even to new fields of mathematical research, particularly in the analysis and numerics of partial differential equations and applied functional analysis. The impact to science in general has been formulated by the famous physicist, RICHARD FEYNMAN:

> *From a long view of the history of mankind – seen from, say, ten thousand years from now – there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics. The American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade.*

The deep understanding of Maxwell's equations and the possibility of their numerical solution in complex geometries and different settings have led to very efficient and robust simulation methods in Computational Electromagnetics. Moreover, efficient simulation methods pave the way for optimizing electromagnetic devices and processes. Digital communication and e-mobility are two fields where simulation and optimization techniques that are based on Maxwell's equations play a deciding role.

More than 70 scientists from 14 countries participated in the workshop; see Figure 1. The workshop brought together different communities, namely people working in analysis of Maxwell's equations with those working in numerical analysis of Maxwell's equations and computational electromagnetics and acoustics. This collection of selected contributions contains original papers that are arranged in an alphabetical order. We are now going to give short description of these contributions.

In Chapter 1, Alonso Rodríguez, Bertolazzi, and Valli proposed and analyzed two variational saddle-point formulations of the curl-div system. Moreover, suitable Hilbert spaces and curl-free and divergence-free finite elements are employed. Finally, numerical tests illustrate the performance of the proposed approximation methods.

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Figure 1: Participants of the first workshop of the special semester 2016 at RICAM.

In Chapter 2, Bauer gives an asymptotic expansion of time dependent Maxwell's equations in terms of iterated div-curl systems in case that charge velocities are small in comparison with the speed of light.

In Chapter 3, Bauer, Pauly, and Schomburg prove that the space of differential forms with weak exterior- and co-derivative is compactly embedded into the space of square integrable forms. Mixed boundary conditions and weak Lipschitz domains are considered. Furthermore, canonical applications such as Maxwell estimates, Helmholtz decompositions, and static solution theories are shown.

In Chapter 4, Bonnet-Ben Dhia, Fliss, and Tjandrawidjaja considered the 2D Helmholtz equation with a complex wavenumber in the exterior of a convex polygonal obstacle with a Robin-type boundary condition using the principle of the half-space matching method. It is proved that this system is of Fredholm type and the theoretical results are supported by numerical experiments.

In Chapter 5, Cogar, Colton, and Monk present an approach to the problem of the possible non-uniqueness of solutions to inverse electromagnetic scattering problems in anisotropic media through the use of appropriate "target signatures," i. e., eigenvalues associated with the direct scattering problem that are accessible to measurement from a knowledge of the scattering data. In this contribution, three different sets of eigenvalues are utilized as target signatures.

In Chapter 6, Costabel and Dauge investigate Maxwell eigenmodes in three-dimensional bounded electromagnetic cavities that have the form of a product of lower dimensional domains in some system of coordinates such as Cartesian, cylindrical, and spherical variables. As application of their general formulas, explicit eigenpairs in a cuboid, in a circular cylinder, and in a cylinder with a coaxial circular hole are found.

In Chapter 7, Hiptmair and Pechstein show stable discrete regular decompositions for Nédélec's tetrahedral edge element spaces of any polynomial degree on a bounded Lipschitz domain. Such decompositions have turned out to be crucial in the numerical analysis of "edge" finite element methods for variational problems in computational electromagnetics. Key tools for these constructions are continuous regular decompositions, boundary-aware local co-chain projections, projection-based interpolations, and quasi-interpolations with low regularity requirements.

In Chapter 8, Kress presents a survey on uniqueness, that is, identifiability and on reconstruction issues for inverse obstacle scattering for time-harmonic acoustic and electromagnetic waves. New integral equation formulations for transmission eigenvalues that play an important role through their connections with the linear sampling method and the factorization method for inverse scattering problems for penetrable objects are given as well.

In Chapter 9, Nicaise and Tomezyk suggest a variational formulation of the time-harmonic Maxwell equation with impedance boundary conditions in polyhedral domains, and show existence and uniqueness of weak solutions by a compact perturbation argument. Corner and edge singularities are investigated and a wavenumber explicit error analysis is performed.

In Chapter 10, Osterbrink and Pauly investigate time-harmonic electro-magnetic scattering or radiation problems governed by Maxwell's equations in an exterior weak Lipschitz domain with mixed boundary conditions. A solution theory in terms of a Fredholm-type alternative using the framework of polynomially weighted Sobolev spaces, Eidus' principle of limiting absorption, and local compact embeddings is presented.

In Chapter 11, Picard considers a coupled system of Maxwell's equations and the equations of elasticity, where the coupling occurs not via material properties but through an interaction on an interface separating the two regimes. Evolutionary well-posedness in the sense of Hadamard well-posedness supplemented by causal dependence is shown for a natural choice of generalized interface conditions. The results are obtained in a Hilbert space setting (Picard's approach) incurring no regularity constraints on the boundary and the interface of the underlying regions.

In Chapter 12, Waurick addresses the continuous dependence of solutions to certain equations on the coefficients. Three examples are discussed: A homogenization problem for a Kelvin–Voigt model for elasticity, the discussion of continuous dependence of the coefficients for acoustic waves with impedance-type boundary conditions, and a singular perturbation problem for a mixed-type equation. By means of counterexamples optimality of these results are obtained.

The careful reviewing process was only possible with the help of the anonymous referees who did an invaluable work that helped the authors to improve their contributions. Furthermore, we would like to thank the administrative and technical staff of RICAM for their support during the special semester. Last but not least, we express our thanks to Apostolos Damialis and Nadja Schedensack from the Walter de Gruyter GmbH, Berlin/Boston, for continuing support and patience while preparing this volume.

Linz, Essen, St. Petersburg
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Contents

Ana Alonso Rodríguez, Enrico Bertolazzi, and Alberto Valli

1 The curl–div system: theory and finite element approximation

Abstract: We first propose and analyze two variational formulations of the curl–div system that rewrite it as a saddle-point problem. Existence and uniqueness results are then an easy consequence of this approach. Second, introducing suitable constrained Hilbert spaces, we devise other variational formulations that turn out to be useful for numerical approximation. Curl-free and divergence-free finite elements are employed for discretizing the problem, and the corresponding finite element solutions are shown to converge to the exact solution. Several numerical tests are also included, illustrating the performance of the proposed approximation methods.

Keywords: Curl–div system, well-posedness, finite element approximation

MSC 2010: 65N30, 35J56, 35Q35, 35Q60

1 Introduction

The curl–div system often appears in electromagnetism (electrostatics, magnetostatics) and in fluid dynamics (rotational incompressible flows, velocity–vorticity formulations). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain (i. e., a bounded, open and connected set): depending on the boundary condition, in its most basic form it reads

$$\begin{cases} \operatorname{curl} \mathbf{u} = \mathbf{J} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = f & \text{in } \Omega \\ \mathbf{u} \times \mathbf{n} = \mathbf{a} & \text{on } \partial\Omega \,, \end{cases} \tag{1.1}$$

or

$$\begin{cases} \operatorname{curl} \mathbf{u} = \mathbf{J} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = f & \text{in } \Omega \\ \mathbf{u} \cdot \mathbf{n} = b & \text{on } \partial\Omega \,, \end{cases} \tag{1.2}$$

with in addition some topological conditions assuring uniqueness.

The aim of this paper is two-fold: first, at the theoretical level, we present a couple of saddle-point variational formulations of the curl–div system and show that they are

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well-posed; second, focusing on discretization, we devise other non-standard variational formulations of this problem which lead to simple and efficient finite element schemes for its numerical approximation.

Concerning the second issue, the main novelty resides in the functional framework we adopt: we look for the solution in the spaces of curl-free or divergence-free vector fields. For the sake of implementation, we also describe in detail how to construct a simple finite element basis for these vector spaces; convergence of the finite element approximations is then shown easily. A key point of our approach is a suitable tree–cotree decomposition of the graph given by the nodes and the edges of the mesh.

The paper is organized as follows. In Section 2, after having recalled some classical results, by means of a saddle-point approach we show that the curl–div system has a unique solution, for both types of boundary condition. Sections 3 and 4 are devoted to devising two other new variational formulations, that will be used for numerical approximation, and to prove that they are well-posed. In Section 5, we give an overview of some previous results related to the discretization of the curl–div system. In Sections 6 and 7, the finite element numerical approximation of the curl–div system based on the new variational formulations is described and analyzed. In the last section, we finally present several numerical results that illustrate the performance of the proposed approximation methods.

2 Theoretical results

Let us start with some notation. Let Ω be a bounded domain of $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ and let $(\partial\Omega)_0, \dots, (\partial\Omega)_p$ be the connected components of $\partial\Omega$, $(\partial\Omega)_0$ being the external one. From the topological point of view, p is the rank of the second homology group of $\overline{\Omega}$, namely, the second Betti number $\beta_2(\Omega)$. The unit outward normal vector on $\partial\Omega$ is indicated by $\mathbf{n}$.

The space of infinitely differentiable functions with compact support in Ω is denoted by $C_0^\infty(\Omega)$. The classical Sobolev spaces are denoted by $H^s(\Omega)$ or $H^s(\partial\Omega)$, for $s \in \mathbb{R}$; for $s = 0$, we write $H^0(\Omega) = L^2(\Omega)$. The space of (essentially) bounded and measurable functions defined in Ω is denoted by $L^\infty(\Omega)$. Moreover, we define

$$\begin{aligned}
H(\mathrm{curl};\Omega) &= \{\mathbf{v} \in (L^2(\Omega))^3 \mid \mathrm{curl}\,\mathbf{v} \in (L^2(\Omega))^3\}\,,\\
H(\mathrm{curl}^0;\Omega) &= \{\mathbf{v} \in (L^2(\Omega))^3 \mid \mathrm{curl}\,\mathbf{v} = \mathbf{0} \text{ in } \Omega\}\,,\\
H(\mathrm{div};\Omega) &= \{\boldsymbol{\xi} \in (L^2(\Omega))^3 \mid \mathrm{div}\,\boldsymbol{\xi} \in L^2(\Omega)\}\,,\\
H(\mathrm{div}^0;\Omega) &= \{\boldsymbol{\xi} \in (L^2(\Omega))^3 \mid \mathrm{div}\,\boldsymbol{\xi} = 0 \text{ in } \Omega\}\,.
\end{aligned}$$

The space of traces on $\partial\Omega$ of functions ϕ belonging to $H^1(\Omega)$ is the space $H^{1/2}(\partial\Omega)$ (whose dual space is the space $H^{-1/2}(\partial\Omega)$); the space of normal traces $\boldsymbol{\xi} \cdot \mathbf{n}$ on $\partial\Omega$ of vector fields $\boldsymbol{\xi}$ belonging to $H(\mathrm{div};\Omega)$ is $H^{-1/2}(\partial\Omega)$; the space of tangential traces $\mathbf{v}\times\mathbf{n}$ on $\partial\Omega$ of vector fields $\mathbf{v}$ belonging to $H(\mathrm{curl};\Omega)$ is denoted by $H^{-1/2}(\mathrm{div}_\tau;\partial\Omega)$ (for the interested reader, an intrinsic characterization of this space can be found in Buffa and Ciarlet [24, 25]; see also Alonso Rodríguez and Valli [8, Section A1]).

In the following, we also need to consider a set closed curves in $\overline{\Omega}$, denoted by $\{\sigma_n\}_{n=1}^g$, that are representatives of a basis of the first homology group (whose rank is therefore equal to g, the first Betti number $\beta_1(\Omega)$): in other words, this set is a maximal set of non-bounding closed curves in $\overline{\Omega}$. Let us recall that an explicit and efficient construction of the closed curves $\{\sigma_n\}_{n=1}^g$ is given by Hiptmair and Ostrowski [39]. For a more detailed presentation of the homological concepts that are useful in this context, see, e. g., Bossavit [20, Chap. 5], Hiptmair [37, Section 2 and Section 3], Gross and Kotiuga [35, Chapter 1 and Chapter 3]; see also Benedetti et al. [13], Alonso Rodríguez et al. [4].

2.1 The curl–div system with assigned tangential component on the boundary

Let $\boldsymbol{\eta}$ be a symmetric matrix, uniformly positive definite in Ω, with entries belonging to $L^\infty(\Omega)$. Given $\mathbf{J} \in (L^2(\Omega))^3$, $f \in L^2(\Omega)$, $\mathbf{a} \in H^{-1/2}(\mathrm{div}_\tau;\partial\Omega)$, $\boldsymbol{\alpha} \in \mathbb{R}^p$, we look for $\mathbf{u} \in (L^2(\Omega))^3$ such that

$$\begin{cases} \mathrm{curl}(\boldsymbol{\eta}\mathbf{u}) = \mathbf{J} & \text{in } \Omega \\ \mathrm{div}\,\mathbf{u} = f & \text{in } \Omega \\ (\boldsymbol{\eta}\mathbf{u})\times\mathbf{n} = \mathbf{a} & \text{on } \partial\Omega \\ \displaystyle\int_{(\partial\Omega)_r} \mathbf{u}\cdot\mathbf{n} = \alpha_r & \text{for each } r = 1,\dots,p\,. \end{cases} \tag{2.1}$$

The data must satisfy the necessary conditions $\mathrm{div}\,\mathbf{J} = 0$ in Ω, $\int_\Omega \mathbf{J}\cdot\boldsymbol{\rho} + \int_{\partial\Omega} \mathbf{a}\cdot\boldsymbol{\rho} = 0$ for each $\boldsymbol{\rho} \in \mathcal{H}(m)$, where $\mathcal{H}(m)$ is the space of Neumann harmonic fields, namely,

$$\mathcal{H}(m) = \{\boldsymbol{\rho} \in (L^2(\Omega))^3 \mid \mathrm{curl}\,\boldsymbol{\rho} = \mathbf{0} \text{ in } \Omega, \mathrm{div}\,\boldsymbol{\rho} = 0 \text{ in } \Omega, \boldsymbol{\rho}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}\,, \tag{2.2}$$

whose dimension is known to be equal to g, the rank of the first homology group of $\overline{\Omega}$, and finally $\mathbf{J}\cdot\mathbf{n} = \mathrm{div}_\tau\,\mathbf{a}$ on $\partial\Omega$ (for a summary of the properties of the spaces of harmonic fields and for a definition of the tangential divergence operator div_τ; see, e. g., Alonso Rodríguez and Valli [8, Section A1 and Section A4]).

By means of a variational approach Saranen [59, 60] has shown that this problem has a unique solution (see also the results proved in Alonso Rodríguez and Valli [8, Section A3], and the more abstract approach by Picard [52, 53]). Let us briefly summarize the principal points of this procedure. The method is based on the Helmholtz decomposition, namely, a splitting of the solution in three terms, orthogonal with respect to the scalar product $\int_{\Omega} \boldsymbol{\eta}^{-1}\mathbf{v} \cdot \mathbf{w}$, that reads

$$\boldsymbol{\eta}\mathbf{u} = \boldsymbol{\eta} \operatorname{curl} \mathbf{q} + \operatorname{grad} \chi + \boldsymbol{\eta}\mathbf{h}\,.$$

Here, the vector field $\mathbf{q}$ satisfies $\operatorname{curl}(\boldsymbol{\eta} \operatorname{curl} \mathbf{q}) = \mathbf{J}$ in Ω and $(\boldsymbol{\eta} \operatorname{curl} \mathbf{q}) \times \mathbf{n} = \mathbf{a}$ on $\partial\Omega$; χ is the solution to $\operatorname{div}(\boldsymbol{\eta}^{-1} \operatorname{grad} \chi) = f$ in Ω and $\chi = 0$ on $\partial\Omega$; $\mathbf{h}$ is a generalized Dirichlet harmonic field, namely, it is an element of the finite dimensional vector space

$$\begin{aligned} \mathcal{H}_{\eta}(e) = \{\boldsymbol{\pi} \in (L^2(\Omega))^3 \mid \operatorname{curl}(\boldsymbol{\eta}\boldsymbol{\pi}) = \mathbf{0} \text{ in } \Omega, \operatorname{div} \boldsymbol{\pi} = 0 \text{ in } \Omega, \\ (\boldsymbol{\eta}\boldsymbol{\pi}) \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\,, \end{aligned} \tag{2.3}$$

whose dimension is known to be equal to p (precisely, $\mathbf{h}$ is the unique element of $\mathcal{H}_{\eta}(e)$ satisfying $\int_{(\partial\Omega)_r} \mathbf{h} \cdot \mathbf{n} = \alpha_r - \int_{(\partial\Omega)_r} \boldsymbol{\eta}^{-1} \operatorname{grad} \chi \cdot \mathbf{n}$ for each $r = 1, \ldots, p$).

Since a solution $\mathbf{q}$ to $\operatorname{curl}(\boldsymbol{\eta} \operatorname{curl} \mathbf{q}) = \mathbf{J}$ in Ω and $(\boldsymbol{\eta} \operatorname{curl} \mathbf{q}) \times \mathbf{n} = \mathbf{a}$ on $\partial\Omega$ is not unique ($\mathbf{q} + \operatorname{grad} \phi$ is still a solution), other equations have to be added. Typically, one imposes the gauge conditions $\operatorname{div} \mathbf{q} = 0$ in Ω, $\mathbf{q} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $\mathbf{q} \perp \mathcal{H}(m)$.

The approach we have just described has thus led to two variational problems: a standard Dirichlet boundary value problem for χ, and a constrained problem for $\mathbf{q}$ (the determination of the harmonic field $\mathbf{h}$ also needs some additional work, but it is an easy finite dimensional problem).

Numerical approaches for approximating these two problems are easily devised. In fact, the first one is a standard elliptic problem. Numerical approximation can be performed by scalar nodal elements in $H^1(\Omega)$, looking for the unknown χ and then computing its gradient, or by means of a mixed method in $H(\operatorname{div}; \Omega) \times L^2(\Omega)$, in which $\operatorname{grad} \chi \in H(\operatorname{div}; \Omega)$ is directly computed as an auxiliary unknown.

Concerning the problem related to the vector field $\mathbf{q}$, a first choice is to work in $H(\operatorname{curl}; \Omega) \cap H(\operatorname{div}; \Omega)$, hence with globally-continuous nodal finite elements for each component of $\mathbf{q}$; the drawback is that, in the presence of re-entrant corners, the solution is singular (it does not belong to $(H^1(\Omega))^3$) and $(H^1(\Omega))^3$ is a closed subspace of $H(\operatorname{curl}; \Omega) \cap H(\operatorname{div}; \Omega)$, hence in this case a finite element scheme cannot be convergent (see, e. g., Costabel et al. [29]).

An alternative method is to formulate the problem as a saddle-point problem for the vector field $\mathbf{q}$ in $H(\operatorname{curl}; \Omega)$, in which the divergence constraint is imposed in a week sense, introducing a scalar Lagrange multiplier; in this way the number of degrees of freedom is rather high, as, besides an edge approximation of the vector field $\mathbf{q}$, one has also to consider a nodal approximation of the scalar Lagrange multiplier. The resulting algebraic problem is associated to an indefinite matrix; however, for its resolution efficient regularization techniques are known (see Hiptmair [37, Section 6.1]).

A way for avoiding the introduction of a Lagrange multiplier is to solve the equation $\operatorname{curl}(\boldsymbol{\eta} \operatorname{curl} \mathbf{q}) = \mathbf{J}$ in Ω by using edge elements without any gauge. Though the matrix to deal with is singular, the conjugate gradient method is known to be a viable tool for solving the associated algebraic problem (see the theoretical result by Kaasschieter [40]; see also Bossavit [20, Section 6.2], Bíró [16]); however, the computation of the right-hand side should be done with particular care (see Fujiwara et al. [33], Bíró et al. [17], Ren [56]), and, for problems with a large number of unknowns, it is not easy to devise an efficient preconditioner.

Summing up, the most classical variational formulations of the curl–div system are not completely satisfactory when numerical approximation has to be performed. We will present in Section 3 a new variational formulation of problem (2.1) that looks much more suitable for finite element discretization.

However, before coming to this point, we want to put the problem on a solid foundation, providing in this and in the following section a proof of the well-posedness of the curl–div system. Instead of reporting the classical result obtained by Saranen [59, 60], we propose a saddle-point formulation that to our knowledge has not been considered yet. With this approach, one does not introduce the potentials $\mathbf{q}$ and χ, keeps the original unknown $\mathbf{u}$ and imposes the curl constraint by means of a Lagrange multiplier: it could be seen as a least-squares formulation with a constraint on the curl of $\mathbf{u}$, or similarly, a Lagrangian method for a constrained optimization problem.

Let us derive step by step the variational problem we are interested in. Taking the gradient of the second equation in (2.1) we obtain $\operatorname{grad}\operatorname{div}\mathbf{u} = \operatorname{grad} f$. Multiplying for a test vector field $\boldsymbol{\xi}$, integrating in Ω and integrating by parts we obtain

$$-\int_{\Omega}(\operatorname{div}\mathbf{u} - f)\operatorname{div}\boldsymbol{\xi} + \int_{\partial\Omega}(\operatorname{div}\mathbf{u} - f)\boldsymbol{\xi}\cdot\mathbf{n} = 0\,.$$

The integral on the boundary will be omitted in the variational formulation, in order to impose in a suitable weak sense the condition $\operatorname{div}\mathbf{u} - f = 0$ on $\partial\Omega$.

Multiplying the first equation in (2.1) by $\mathbf{v}$, integrating in Ω and integrating by parts we find

$$\int_{\Omega}\mathbf{J}\cdot\mathbf{v} = \int_{\Omega}\operatorname{curl}(\boldsymbol{\eta}\mathbf{u})\cdot\mathbf{v} = \int_{\Omega}\boldsymbol{\eta}\mathbf{u}\cdot\operatorname{curl}\mathbf{v} + \int_{\partial\Omega}\mathbf{n}\times\boldsymbol{\eta}\mathbf{u}\cdot\mathbf{v}\,,$$

hence

$$\int_{\Omega}\boldsymbol{\eta}\mathbf{u}\cdot\operatorname{curl}\mathbf{v} = \int_{\Omega}\mathbf{J}\cdot\mathbf{v} + \int_{\partial\Omega}\mathbf{a}\cdot\mathbf{v}\,.$$

Then, introducing a Lagrange multiplier $\mathbf{p}$, we are led to consider the problem

$$\int_{\Omega}\operatorname{div}\mathbf{u}\operatorname{div}\boldsymbol{\xi} + \int_{\Omega}\boldsymbol{\eta}\boldsymbol{\xi}\cdot\operatorname{curl}\mathbf{p} = \int_{\Omega}f\operatorname{div}\boldsymbol{\xi}$$

$$\int_{\Omega}\boldsymbol{\eta}\mathbf{u}\cdot\operatorname{curl}\mathbf{v} = \int_{\Omega}\mathbf{J}\cdot\mathbf{v} + \int_{\partial\Omega}\mathbf{a}\cdot\mathbf{v}\,.$$

Now the natural question is: which are the variational spaces for $\mathbf{u}$, $\boldsymbol{\xi}$, $\mathbf{p}$ and $\mathbf{v}$? Define the Hilbert spaces

$$\begin{aligned}\mathcal{W} &= \left\{\boldsymbol{\xi} \in H(\mathrm{div};\Omega) \mid \int_{(\partial\Omega)_r} \boldsymbol{\xi}\cdot\mathbf{n} = 0 \text{ for each } r = 1,\dots,p\right\}\\ \mathcal{Q} &= \left\{\mathbf{v} \in H(\mathrm{curl};\Omega) \mid \int_\Omega \mathbf{v}\cdot\mathbf{w} = 0 \text{ for each } \mathbf{w} \in H(\mathrm{curl}^0;\Omega)\right\}.\end{aligned} \tag{2.4}$$

We choose $\mathbf{u}, \boldsymbol{\xi} \in \mathcal{W}$ and $\mathbf{p}, \mathbf{v} \in \mathcal{Q}$. It is worth noting that the space $H(\mathrm{curl}^0;\Omega)$ can be described as

$$H(\mathrm{curl}^0;\Omega) = \mathrm{grad}\, H^1(\Omega) \overset{\perp}{\oplus} \mathcal{H}(m) \tag{2.5}$$

(see, e. g., Alonso Rodríguez and Valli [8, Section A3]). Therefore, by integration by parts, an element $\mathbf{v} \in \mathcal{Q}$ can be characterized as an element in $H(\mathrm{curl};\Omega)$ such that $\mathrm{div}\,\mathbf{v} = 0$ in Ω, $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$ and $\mathbf{v} \perp \mathcal{H}(m)$.

Summing up, our variational problem is

$$\begin{aligned}&\text{find } \mathbf{u} \in \mathcal{W}, \mathbf{p} \in \mathcal{Q} :\\ &\quad \int_\Omega \mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\boldsymbol{\xi} + \int_\Omega \boldsymbol{\eta}\boldsymbol{\xi}\cdot\mathrm{curl}\,\mathbf{p} = \int_\Omega f\,\mathrm{div}\,\boldsymbol{\xi}\\ &\quad \int_\Omega \boldsymbol{\eta}\mathbf{u}\cdot\mathrm{curl}\,\mathbf{v} = \int_\Omega \mathbf{J}\cdot\mathbf{v} + \int_{\partial\Omega} \mathbf{a}\cdot\mathbf{v}\\ &\text{for each } \boldsymbol{\xi} \in \mathcal{W}, \mathbf{v} \in \mathcal{Q}.\end{aligned} \tag{2.6}$$

Before analyzing this problem, we need an additional tool. It is known that it is possible to select a basis $\{\boldsymbol{\pi}_s^\eta\}_{s=1}^p$ of the space of harmonic fields $\mathcal{H}_\eta(e)$ defined in (2.3) with the properties

$$\int_{(\partial\Omega)_r} \boldsymbol{\pi}_s^\eta \cdot \mathbf{n} = \delta_{rs}$$

(see, e. g., Alonso Rodríguez and Valli [8, Section A4]; for $\boldsymbol{\eta} = \mathrm{Id}$ we simply write $\boldsymbol{\pi}_s$). Then, if $\mathbf{u}$ is a solution to problem (2.1) with $\alpha_r = 0$, $r = 1,\dots,p$, we check easily that $\mathbf{u} + \sum_{r=1}^p \alpha_r \boldsymbol{\pi}_r^\eta$ is a solution to problem (2.1) with given α_r.

This also says that a solution $\mathbf{u}$ of problem (2.1), if it exists, is unique. In fact, taking vanishing data, it follows from the first three equations that $\mathbf{u} \in \mathcal{H}_\eta(e)$, and consequently it can be written as $\mathbf{u} = \sum_{s=1}^p u_s \boldsymbol{\pi}_s^\eta$. Then, for each $r = 1,\dots,p$,

$$0 = \int_{(\partial\Omega)_r} \mathbf{u}\cdot\mathbf{n} = \sum_{s=1}^p u_s \int_{(\partial\Omega)_r} \boldsymbol{\pi}_s^\eta \cdot \mathbf{n} = u_r\,,$$

and in conclusion $\mathbf{u} = \mathbf{0}$.

Theorem 1. *If* $(\mathbf{u}, \mathbf{p})$ *is a solution to problem* (2.6) *then* $\mathbf{p} = \mathbf{0}$ *and* $\mathbf{u}$ *is a solution to problem* (2.1) *for* $\alpha_r = 0$, $r = 1, \ldots, p$.

Proof. By the Stokes theorem for closed surfaces, we know that $\operatorname{curl}\mathbf{v} \in \mathcal{W}$ for each $\mathbf{v} \in \mathcal{Q}$. Therefore, taking $\boldsymbol{\xi} = \operatorname{curl}\mathbf{p}$ in the first equation we find

$$\int_\Omega \boldsymbol{\eta} \operatorname{curl}\mathbf{p} \cdot \operatorname{curl}\mathbf{p} = 0\,,$$

hence $\operatorname{curl}\mathbf{p} = \mathbf{0}$; since the elements in $\mathcal{Q}$ are orthogonal to $H(\operatorname{curl}^0;\Omega)$ (with respect to the $L^2(\Omega)$-scalar product), it follows $\mathbf{p} = \mathbf{0}$.

Choosing $\boldsymbol{\xi} \in (C_0^\infty(\Omega))^3$ we find that $\operatorname{grad}(\operatorname{div}\mathbf{u} - f) = 0$ in Ω in the distributional sense, hence $(\operatorname{div}\mathbf{u} - f)$ is constant in Ω. Take $\widehat{\boldsymbol{\xi}} \in H(\operatorname{div};\Omega)$ and define $\widehat{\xi}_r = \int_{(\partial\Omega)_r} \widehat{\boldsymbol{\xi}} \cdot \mathbf{n}$. Then $\boldsymbol{\xi} = \widehat{\boldsymbol{\xi}} - \sum_{r=1}^p \widehat{\xi}_r \boldsymbol{\pi}_r^\eta$ belongs to $\mathcal{W}$ and satisfies $\operatorname{div}\boldsymbol{\xi} = \operatorname{div}\widehat{\boldsymbol{\xi}}$. Hence the first equation in problem (2.6) is satisfied for each $\widehat{\boldsymbol{\xi}} \in H(\operatorname{div};\Omega)$, and by integration by parts we find $\operatorname{div}\mathbf{u} - f = 0$ on $\partial\Omega$, hence $\operatorname{div}\mathbf{u} = f$ in Ω.

Let us prove that the second equation is indeed satisfied for each $\hat{\mathbf{v}} \in H(\operatorname{curl};\Omega)$. Let $P\hat{\mathbf{v}}$ be the $L^2(\Omega)$-orthogonal projection of $\hat{\mathbf{v}}$ on $H(\operatorname{curl}^0;\Omega)$. Then $P\hat{\mathbf{v}} = \operatorname{grad}\widehat{\omega} + \widehat{\boldsymbol{\rho}}$, with $\widehat{\omega} \in H^1(\Omega)$ and $\widehat{\boldsymbol{\rho}} \in \mathcal{H}(m)$, $\mathbf{v} = (\hat{\mathbf{v}} - P\hat{\mathbf{v}}) \in \mathcal{Q}$, and $\operatorname{curl}\mathbf{v} = \operatorname{curl}\hat{\mathbf{v}}$. Moreover,

$$\int_\Omega \mathbf{J} \cdot \mathbf{v} + \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{v} = \int_\Omega \mathbf{J} \cdot \hat{\mathbf{v}} + \int_{\partial\Omega} \mathbf{a} \cdot \hat{\mathbf{v}} - \int_\Omega \mathbf{J} \cdot P\hat{\mathbf{v}} - \int_{\partial\Omega} \mathbf{a} \cdot P\hat{\mathbf{v}}$$

and, by integrating by parts in Ω and on $\partial\Omega$,

$$\begin{aligned}\int_\Omega \mathbf{J} \cdot P\hat{\mathbf{v}} + \int_{\partial\Omega} \mathbf{a} \cdot P\hat{\mathbf{v}} &= \int_\Omega \mathbf{J} \cdot (\operatorname{grad}\widehat{\omega} + \widehat{\boldsymbol{\rho}}) + \int_{\partial\Omega} \mathbf{a} \cdot (\operatorname{grad}\widehat{\omega} + \widehat{\boldsymbol{\rho}}) \\ &= -\int_\Omega \operatorname{div}\mathbf{J}\widehat{\omega} + \int_{\partial\Omega} \mathbf{J} \cdot \mathbf{n}\widehat{\omega} + \int_\Omega \mathbf{J} \cdot \widehat{\boldsymbol{\rho}} - \int_{\partial\Omega} \operatorname{div}_\tau \mathbf{a}\widehat{\omega} + \int_{\partial\Omega} \mathbf{a} \cdot \widehat{\boldsymbol{\rho}} = 0\,,\end{aligned}$$

having used the compatibility conditions on the data $\mathbf{J}$ and $\mathbf{a}$.

Hence the second equation is satisfied for each $\hat{\mathbf{v}} \in H(\operatorname{curl};\Omega)$, and taking $\hat{\mathbf{v}} \in (C_0^\infty(\Omega))^3$ it follows $\operatorname{curl}(\boldsymbol{\eta}\mathbf{u}) = \mathbf{J}$ in Ω in the distributional sense. Repeating the same procedure for $\hat{\mathbf{v}} \in H(\operatorname{curl};\Omega)$, integration by parts gives $\boldsymbol{\eta}\mathbf{u} \times \mathbf{n} = \mathbf{a}$ on $\partial\Omega$. □

The existence of a solution to problem (2.1) is therefore reduced to the proof of the existence of a solution to problem (2.6). This is a consequence of well-known results for saddle-point problems (see, e. g., Boffi et al. [19, Section 4.2]). In fact, the following two propositions permit us to apply the general well-posedness theory.

Proposition 1. *The bilinear form* $a(\boldsymbol{\psi}, \boldsymbol{\xi}) = \int_\Omega \operatorname{div}\boldsymbol{\psi} \operatorname{div}\boldsymbol{\xi}$ *is coercive in the space* $\mathcal{B}_0 \times \mathcal{B}_0$, *where*

$$\mathcal{B}_0 = \left\{ \boldsymbol{\xi} \in \mathcal{W} \mid \int_\Omega \boldsymbol{\eta}\boldsymbol{\xi} \cdot \operatorname{curl}\mathbf{v} = 0 \textit{ for all } \mathbf{v} \in \mathcal{Q} \right\}.$$

Proof. Indeed, we have already seen that, if $\boldsymbol{\xi} \in \mathcal{B}_0$, then it follows that $\int_\Omega \boldsymbol{\eta\xi} \cdot \operatorname{curl} \mathbf{v} = 0$ for all $\mathbf{v} \in H(\operatorname{curl};\Omega)$. Therefore, by integration by parts we deduce at once that $\operatorname{curl}(\boldsymbol{\eta\xi}) = \mathbf{0}$ in Ω and $\boldsymbol{\eta\xi} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. Coercivity follows from the Friedrichs inequality: there exists a constant $C > 0$ such that for any vector field $\boldsymbol{\xi}$ belonging to $H(\operatorname{div};\Omega)$, with $\operatorname{curl}(\boldsymbol{\eta\xi}) \in (L^2(\Omega))^3$, $\boldsymbol{\eta\xi} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ and satisfying $\int_{(\partial\Omega)_r} \boldsymbol{\xi} \cdot \mathbf{n} = 0$ for each $r = 1, \dots, p$, it holds

$$\|\boldsymbol{\xi}\|_{L^2(\Omega)} \leq C\big(\|\operatorname{curl}(\boldsymbol{\eta\xi})\|_{L^2(\Omega)} + \|\operatorname{div} \boldsymbol{\xi}\|_{L^2(\Omega)}\big) .$$

This result can be shown by adapting in a straightforward way the proof presented, e. g., in Fernandes and Gilardi [32], using the fact that the space

$$\{\boldsymbol{\xi} \in H(\operatorname{div};\Omega) \mid \operatorname{curl}(\boldsymbol{\eta\xi}) \in \big(L^2(\Omega)\big)^3, \boldsymbol{\eta\xi} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}$$

is compactly imbedded in $(L^2(\Omega))^3$ (see, e. g., Weber [64], Picard [54]). □

Proposition 2. *The bilinear form* $b(\boldsymbol{\xi}, \mathbf{v}) = \int_\Omega \boldsymbol{\eta\xi} \cdot \operatorname{curl} \mathbf{v}$ *satisfies an inf–sup condition, namely, there exists* $\beta > 0$ *such that for each* $\mathbf{v} \in \mathcal{Q}$ *there exists* $\boldsymbol{\xi} \in \mathcal{W}$, $\boldsymbol{\xi} \neq \mathbf{0}$, *satisfying*

$$\int_\Omega \boldsymbol{\eta\xi} \cdot \operatorname{curl} \mathbf{v} \geq \beta \|\boldsymbol{\xi}\|_{\mathcal{W}} \|\mathbf{v}\|_{\mathcal{Q}} .$$

Proof. If $\operatorname{curl} \mathbf{v} = \mathbf{0}$ in Ω, nothing has to be proved. Then suppose that $\operatorname{curl} \mathbf{v} \neq \mathbf{0}$. We have already seen that $\operatorname{curl} \mathbf{v} \in \mathcal{W}$ for each $\mathbf{v} \in \mathcal{Q}$, and that any vector field $\mathbf{v} \in \mathcal{Q}$ satisfies $\operatorname{div} \mathbf{v} = 0$ in Ω, $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $\mathbf{v} \perp \mathcal{H}(m)$. The thesis follows by choosing $\boldsymbol{\xi} = \operatorname{curl} \mathbf{v}$, as $\operatorname{div} \boldsymbol{\xi} = 0$ in Ω and the Friedrichs inequality

$$\|\mathbf{v}\|_{L^2(\Omega)} \leq C \|\operatorname{curl} \mathbf{v}\|_{L^2(\Omega)}$$

is valid for $\mathbf{v} \in H(\operatorname{curl};\Omega) \cap H(\operatorname{div}^0;\Omega)$ satisfying $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $\mathbf{v} \perp \mathcal{H}(m)$ (see, e. g., Girault and Raviart [34, Section 3.5] if $\mathcal{H}(m) = \emptyset$, or Fernandes and Gilardi [32] if $\mathcal{H}(m) \neq \emptyset$). □

In conclusion, by means of these two propositions we have proved that the saddle-point problem (2.6) has a unique solution, and thus the same is true for problem (2.1).

2.2 The curl–div system with assigned normal component on the boundary

Let $\boldsymbol{\mu}$ be a symmetric matrix, uniformly positive definite in Ω, with entries belonging to $L^\infty(\Omega)$. Given $\mathbf{J} \in (L^2(\Omega))^3$, $f \in L^2(\Omega)$, $b \in H^{-1/2}(\partial\Omega)$, $\boldsymbol{\beta} \in \mathbb{R}^g$, we look for $\mathbf{u} \in (L^2(\Omega))^3$

such that

$$\begin{cases} \operatorname{curl} \mathbf{u} = \mathbf{J} & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{\mu}\mathbf{u}) = f & \text{in } \Omega \\ \boldsymbol{\mu}\mathbf{u} \cdot \mathbf{n} = b & \text{on } \partial\Omega \\ \oint_{\sigma_n} \mathbf{u} \cdot d\mathbf{s} = \beta_n & \text{for each } n = 1, \ldots, g\,, \end{cases} \tag{2.7}$$

where the data satisfy the necessary conditions $\operatorname{div} \mathbf{J} = 0$ in Ω, $\int_\Omega f = \int_{\partial\Omega} b$; moreover, since we need to give a meaning to the line integral of $\mathbf{u}$ on σ_n, we follow the arguments in Alonso Rodríguez et al. [7, Section 2] and we also assume that $\mathbf{J} \cdot \mathbf{n} = 0$ on $\partial\Omega$ (which is more restrictive than the necessary condition $\int_{(\partial\Omega)_r} \mathbf{J} \cdot \mathbf{n} = 0$ for each $r = 1, \ldots, p$).

The variational approach proposed by Saranen [59, 60] shows that this problem has a unique solution (see also Alonso Rodríguez and Valli [8, Section A3], and the results obtained by Picard [52, 53]). Again, the method is based on a orthogonal decomposition result, through which the solution is split as

$$\mathbf{u} = \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{q} + \operatorname{grad} \chi + \mathbf{h}\,,$$

where the vector field $\mathbf{q}$ is a solution to $\operatorname{curl}(\boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{q}) = \mathbf{J}$ in Ω and $\mathbf{q} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$; χ is the solution to $\operatorname{div}(\boldsymbol{\mu} \operatorname{grad} \chi) = f$ in Ω and $\boldsymbol{\mu} \operatorname{grad} \chi \cdot \mathbf{n} = b$ on $\partial\Omega$; $\mathbf{h}$ is a generalized Neumann harmonic field, namely, it is an element of the finite dimensional vector space

$$\begin{aligned} \mathcal{H}_\mu(m) = \{\boldsymbol{\rho} \in (L^2(\Omega))^3 \mid \operatorname{curl} \boldsymbol{\rho} = \mathbf{0} \text{ in } \Omega, \operatorname{div}(\boldsymbol{\mu}\boldsymbol{\rho}) = 0 \text{ in } \Omega, \\ \boldsymbol{\mu}\boldsymbol{\rho} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\,, \end{aligned} \tag{2.8}$$

whose dimension is known to be equal to g (precisely, $\mathbf{h}$ is the unique element of $\mathcal{H}_\mu(m)$ satisfying $\oint_{\sigma_n} \mathbf{h} \cdot d\mathbf{s} = \beta_n - \oint_{\sigma_n} \boldsymbol{\mu}^{-1} \operatorname{curl} \mathbf{q} \cdot d\mathbf{s}$ for each $n = 1, \ldots, g$).

Since a solution $\mathbf{q}$ to $\operatorname{curl}(\boldsymbol{\eta} \operatorname{curl} \mathbf{q}) = \mathbf{J}$ in Ω and $\mathbf{q} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ is not unique ($\mathbf{q} + \operatorname{grad} \phi$, with $\phi = 0$ on $\partial\Omega$, is still a solution), other equations have to be added. The standard gauge conditions are $\operatorname{div} \mathbf{q} = 0$ in Ω and $\mathbf{q} \perp \mathcal{H}(e)$, where $\mathcal{H}(e)$ is the space of Dirichlet harmonic vector fields, namely,

$$\begin{aligned} \mathcal{H}(e) = \{\boldsymbol{\pi} \in (L^2(\Omega))^3 \mid \operatorname{curl} \boldsymbol{\pi} = \mathbf{0} \text{ in } \Omega, \operatorname{div} \boldsymbol{\pi} = 0 \text{ in } \Omega, \\ \boldsymbol{\pi} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}\,. \end{aligned} \tag{2.9}$$

We do not specify the details of the proof of the existence of a solution $\mathbf{q}$ because here, as in the previous case, we base the theoretical analysis of the curl–div system (2.7) on a saddle-point variational formulation, quite close to that proposed by Kikuchi [42] (the limitations in that paper are that the domain has a simple topological shape, the boundary conditions are homogeneous and the coefficient $\boldsymbol{\mu}$ is a constant scalar parameter). With this approach, the introduction of the potentials $\mathbf{q}$ and χ is not needed, the original unknown $\mathbf{u}$ is kept and the equation related to the divergence is imposed by a means of Lagrange multiplier; more precisely, what we propose looks like a least-squares formulation with a constraint on the divergence of $\mathbf{u}$. Let us also point out that another variational formulation, more suitable for numerical approximation, will be introduced in Section 4.

We proceed as follows. Taking the curl of the first equation in (2.7) we obtain $\operatorname{curl}\operatorname{curl}\mathbf{u} = \operatorname{curl}\mathbf{J}$. Multiplying for a test vector field $\mathbf{v}$, integrating in Ω and integrating by parts we obtain

$$\int_\Omega (\operatorname{curl}\mathbf{u} - \mathbf{J}) \cdot \operatorname{curl}\mathbf{v} + \int_{\partial\Omega} \mathbf{n} \times (\operatorname{curl}\mathbf{u} - \mathbf{J}) \cdot \mathbf{v} = 0\,.$$

The integral on the boundary will be omitted in the variational formulation, in order to impose in a suitable weak sense the condition $\mathbf{n} \times (\operatorname{curl}\mathbf{u} - \mathbf{J}) = \mathbf{0}$ on $\partial\Omega$.

Multiplying the second equation in (2.7) by φ, integrating in Ω and integrating by parts we find

$$\int_\Omega f\varphi = \int_\Omega \operatorname{div}(\boldsymbol{\mu}\mathbf{u})\varphi = -\int_\Omega \boldsymbol{\mu}\mathbf{u} \cdot \operatorname{grad}\varphi + \int_{\partial\Omega} \boldsymbol{\mu}\mathbf{u} \cdot \mathbf{n}\varphi\,,$$

hence

$$\int_\Omega \boldsymbol{\mu}\mathbf{u} \cdot \operatorname{grad}\varphi = -\int_\Omega f\varphi + \int_{\partial\Omega} \boldsymbol{\mu}\mathbf{u} \cdot \mathbf{n}\varphi\,.$$

Then, introducing a Lagrange multiplier λ, we are led to consider the problem

$$\begin{aligned}
&\int_\Omega \operatorname{curl}\mathbf{u} \cdot \operatorname{curl}\mathbf{v} + \int_\Omega \boldsymbol{\mu}\mathbf{v} \cdot \operatorname{grad}\lambda = \int_\Omega \mathbf{J} \cdot \operatorname{curl}\mathbf{v} \\
&\int_\Omega \boldsymbol{\mu}\mathbf{u} \cdot \operatorname{grad}\varphi = -\int_\Omega f\varphi + \int_{\partial\Omega} b\varphi\,.
\end{aligned}$$

The variational spaces are

$$\begin{aligned}
\mathcal{V} &= \Big\{\mathbf{v} \in H(\operatorname{curl};\Omega) \mid \operatorname{curl}\mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \\
&\qquad\qquad \oint_{\sigma_n} \mathbf{v} \cdot d\mathbf{s} = 0 \text{ for each } n = 1, \ldots, g\Big\} \\
\mathcal{R} &= \Big\{\varphi \in H^1(\Omega) \mid \int_\Omega \varphi = 0\Big\}\,,
\end{aligned} \tag{2.10}$$

and the variational problem is

$$
\begin{aligned}
&\text{find } \mathbf{u} \in \mathcal{V}, \lambda \in \mathcal{R} : \\
&\qquad \int_\Omega \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} + \int_\Omega \boldsymbol{\mu} \mathbf{v} \cdot \operatorname{grad} \lambda = \int_\Omega \mathbf{J} \cdot \operatorname{curl} \mathbf{v} \\
&\qquad \int_\Omega \boldsymbol{\mu} \mathbf{u} \cdot \operatorname{grad} \varphi = -\int_\Omega f\varphi + \int_{\partial\Omega} b\varphi \\
&\text{for each } \mathbf{v} \in \mathcal{V}, \varphi \in \mathcal{R} .
\end{aligned}
\tag{2.11}
$$

Let us select a basis $\{\boldsymbol{\rho}_m^\mu\}_{m=1}^g$ of the space of harmonic fields $\mathcal{H}_\mu(m)$ defined in (2.8) with the properties

$$\oint_{\sigma_n} \boldsymbol{\rho}_m^\mu \cdot d\mathbf{s} = \delta_{nm}$$

(see, e. g., Alonso Rodríguez and Valli [8, Section A4]; for $\boldsymbol{\mu} = \mathrm{Id}$ we simply write $\boldsymbol{\rho}_m$). Then, if $\mathbf{u}$ is a solution to problem (2.7) with $\beta_n = 0$, $n = 1, \dots, g$, the vector field $\mathbf{u} + \sum_{n=1}^g \beta_n \boldsymbol{\rho}_n^\mu$ is a solution to problem (2.7) with assigned β_n.

A consequence of this remark is that a solution $\mathbf{u}$ of problem (2.7), if it exists, is unique. Taking in fact vanishing data, it follows from the first three equations that $\mathbf{u} \in \mathcal{H}_\mu(m)$, and thus it can be written as $\mathbf{u} = \sum_{n=1}^g u_n \boldsymbol{\rho}_n^\mu$. Then, for each $n = 1, \dots, g$,

$$0 = \oint_{\sigma_n} \mathbf{u} \cdot d\mathbf{s} = \sum_{m=1}^g u_m \oint_{\sigma_n} \boldsymbol{\rho}_m^\mu \cdot d\mathbf{s} = u_n ,$$

and in conclusion $\mathbf{u} = \mathbf{0}$.

Theorem 2. *If* $(\mathbf{u}, \lambda)$ *is a solution to problem* (2.11)*, then* $\lambda = 0$ *and* $\mathbf{u}$ *is a solution to problem* (2.7) *for* $\beta_n = 0$, $n = 1, \dots, g$.

Proof. For $\varphi \in \mathcal{R}$ it holds $\oint_{\sigma_n} \operatorname{grad} \varphi \cdot d\mathbf{s} = 0$ for each $n = 1, \dots, g$, hence $\operatorname{grad} \varphi \in \mathcal{V}$ for each $\varphi \in \mathcal{R}$. Therefore, taking $\mathbf{v} = \operatorname{grad} \lambda$ in the first equation we find

$$\int_\Omega \boldsymbol{\mu} \operatorname{grad} \lambda \cdot \operatorname{grad} \lambda = 0 ,$$

hence $\operatorname{grad} \lambda = \mathbf{0}$ and $\lambda = \text{const}$ in Ω; since the elements in $\mathcal{R}$ have zero mean, it follows $\lambda = 0$.

Choosing $\mathbf{v} \in (C_0^\infty(\Omega))^3$ we find that $\operatorname{curl}(\operatorname{curl} \mathbf{u} - \mathbf{J}) = \mathbf{0}$ in Ω in the distributional sense. Moreover, integrating by parts we also find

$$\int_{\partial\Omega} (\operatorname{curl} \mathbf{u} - \mathbf{J}) \cdot \mathbf{n} \times \mathbf{v} = 0$$

for each $\mathbf{v} \in \mathcal{V}$. Since $(\operatorname{curl} \mathbf{u} - \mathbf{J})$ is curl-free, from (2.5) we know that it can be written as

$$\operatorname{curl} \mathbf{u} - \mathbf{J} = \operatorname{grad} \chi + \sum_{n=1}^g \zeta_n \boldsymbol{\rho}_n$$

for $\chi \in H^1(\Omega)$ and $\zeta_n \in \mathbb{R}$. Thus we have

$$0 = \int_{\partial\Omega} (\operatorname{curl} \mathbf{u} - \mathbf{J}) \cdot \mathbf{n} \times \mathbf{v} = \int_{\partial\Omega} \operatorname{grad} \chi \cdot \mathbf{n} \times \mathbf{v} + \sum_{n=1}^{g} \zeta_n \int_{\partial\Omega} \boldsymbol{\rho}_n \cdot \mathbf{n} \times \mathbf{v} .$$

In addition, we recall from Buffa [23], Hiptmair et al. [38] that the tangential trace of $\mathbf{v} \in \mathcal{V}$ can be written on $\partial\Omega$ as

$$\mathbf{n} \times \mathbf{v} = \mathbf{n} \times \operatorname{grad} \vartheta + \sum_{m=1}^{g} \eta_m \mathbf{n} \times \boldsymbol{\rho}'_m ,$$

where $\vartheta \in H^1(\Omega)$, $\eta_m \in \mathbb{R}$ and the vector fields $\boldsymbol{\rho}'_m$ satisfy the relations

$$\int_{\partial\Omega} \boldsymbol{\rho}_n \cdot \mathbf{n} \times \boldsymbol{\rho}'_m = \delta_{nm}$$

(see Hiptmair et al. [38], Alonso Rodríguez et al. [7, Lemmas 4 and 5]). By integration by parts on $\partial\Omega$, we find

$$\int_{\partial\Omega} \operatorname{grad} \chi \cdot \mathbf{n} \times \mathbf{v} = -\int_{\partial\Omega} \chi \operatorname{div}_\tau (\mathbf{n} \times \mathbf{v}) = 0 ,$$

as $\operatorname{div}_\tau(\mathbf{n} \times \mathbf{v}) = -\operatorname{curl} \mathbf{v} \cdot \mathbf{n}$ on $\partial\Omega$; similarly, $\int_{\partial\Omega} \boldsymbol{\rho}_n \cdot \mathbf{n} \times \operatorname{grad} \vartheta = 0$ for each $n = 1, \dots, g$. In conclusion, we have obtained

$$0 = \int_{\partial\Omega} (\operatorname{curl} \mathbf{u} - \mathbf{J}) \cdot \mathbf{n} \times \mathbf{v} = \sum_{n,m=1}^{g} \zeta_n \eta_m \int_{\partial\Omega} \boldsymbol{\rho}_n \cdot \mathbf{n} \times \boldsymbol{\rho}'_m = \sum_{n=1}^{g} \zeta_n \eta_n .$$

Since η_n are arbitrary, it follows $\zeta_n = 0$ for each $n = 1, \dots, g$, and consequently $\operatorname{curl} \mathbf{u} - \mathbf{J} = \operatorname{grad} \chi$ in Ω. On the other hand, from the assumptions on the data, $\operatorname{div}(\operatorname{curl} \mathbf{u} - \mathbf{J}) = 0$ in Ω and $(\operatorname{curl} \mathbf{u} - \mathbf{J}) \cdot \mathbf{n} = 0$ on $\partial\Omega$, hence $\operatorname{grad} \chi = \mathbf{0}$ in Ω.

Let us prove now that the second equation is indeed satisfied for each $\widehat{\varphi} \in H^1(\Omega)$. Let $\widehat{\varphi}_\Omega = \frac{1}{\operatorname{meas} \Omega} \int_\Omega \widehat{\varphi}$. Then $\varphi = (\widehat{\varphi} - \widehat{\varphi}_\Omega) \in \mathcal{R}$ and $\operatorname{grad} \widehat{\varphi} = \operatorname{grad} \varphi$. Moreover,

$$\begin{aligned} -\int_\Omega f\widehat{\varphi} + \int_{\partial\Omega} b\widehat{\varphi} &= -\int_\Omega f\varphi + \int_{\partial\Omega} b\varphi - \widehat{\varphi}_\Omega \Big(-\int_\Omega f + \int_{\partial\Omega} b \Big) \\ &= -\int_\Omega f\varphi + \int_{\partial\Omega} b\varphi , \end{aligned}$$

having used the compatibility conditions on the data f and b.

Hence the second equation is satisfied for each $\widehat{\varphi} \in H^1(\Omega)$, and taking $\widehat{\varphi} \in C_0^\infty(\Omega)$ it follows $\operatorname{div}(\boldsymbol{\mu}\mathbf{u}) = f$ in Ω in the distributional sense. Repeating the same procedure for $\widehat{\varphi} \in H^1(\Omega)$, integration by parts gives $\boldsymbol{\mu}\mathbf{u} \cdot \mathbf{n} = b$ on $\partial\Omega$. □

As in the previous section, the existence of a solution to problem (2.7) is therefore reduced to the proof of the existence of a solution to a variational saddle-point problem, in this case problem (2.11). Applying the general theory reported, e. g., in Boffi et al. [19, Section 4.2], we prove that problem (2.11) has a unique solution. In fact, the following results hold true.

Proposition 3. *The bilinear form* $a(\mathbf{w}, \mathbf{v}) = \int_\Omega \operatorname{curl} \mathbf{w} \cdot \operatorname{curl} \mathbf{v}$ *is coercive in the space* $\mathcal{D}_0 \times \mathcal{D}_0$*, where*

$$\mathcal{D}_0 = \left\{ \mathbf{v} \in \mathcal{V} \mid \int_\Omega \boldsymbol{\mu}\mathbf{v} \cdot \operatorname{grad} \varphi = 0 \textit{ for all } \varphi \in \mathcal{R} \right\}.$$

Proof. Indeed, we already know that, if $\mathbf{v} \in \mathcal{D}_0$, then it holds $\int_\Omega \boldsymbol{\mu}\mathbf{v} \cdot \operatorname{grad} \varphi = 0$ for all $\varphi \in H^1(\Omega)$. Therefore, by integration by parts we deduce at once that $\operatorname{div}(\boldsymbol{\mu}\mathbf{v}) = 0$ in Ω and $\boldsymbol{\mu}\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Coercivity follows from the Friedrichs inequality

$$\|\mathbf{v}\|_{L^2(\Omega)} \leq C\big(\|\operatorname{curl} \mathbf{v}\|_{L^2(\Omega)} + \|\operatorname{div}(\boldsymbol{\mu}\mathbf{v})\|_{L^2(\Omega)}\big).$$

This inequality is valid for a vector field $\mathbf{v}$ belonging to $H(\operatorname{curl};\Omega)$, with $\operatorname{div}(\boldsymbol{\mu}\mathbf{v}) \in L^2(\Omega)$, $\boldsymbol{\mu}\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$, and satisfying $\operatorname{curl} \mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $\oint_{\sigma_n} \mathbf{v} \cdot d\mathbf{s} = 0$ for each $n = 1, \dots, g$. This result can be shown by adapting in a straightforward way the proof presented, e. g., in Fernandes and Gilardi [32] (see also Alonso Rodríguez et al. [7, Lemma 9]), using the fact that the space

$$\{\mathbf{v} \in H(\operatorname{curl};\Omega) \mid \operatorname{div}(\boldsymbol{\mu}\mathbf{v}) \in \big(L^2(\Omega)\big)^3, \boldsymbol{\mu}\mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

is compactly imbedded in $(L^2(\Omega))^3$ (see, e. g., Weber [64], Picard [54]). □

Proposition 4. *The bilinear form* $b(\mathbf{v}, \varphi) = \int_\Omega \boldsymbol{\mu}\mathbf{v} \cdot \operatorname{grad} \varphi$ *satisfies an inf–sup condition, namely, there exists* $\beta > 0$ *such that for each* $\varphi \in \mathcal{R}$ *there exists* $\mathbf{v} \in \mathcal{V}$, $\mathbf{v} \neq \mathbf{0}$*, satisfying*

$$\int_\Omega \boldsymbol{\mu}\mathbf{v} \cdot \operatorname{grad} \varphi \geq \beta \|\mathbf{v}\|_{\mathcal{V}} \|\varphi\|_{\mathcal{R}}.$$

Proof. We can suppose $\operatorname{grad} \varphi \neq \mathbf{0}$. The thesis follows by choosing $\mathbf{v} = \operatorname{grad} \varphi$, as $\operatorname{curl} \mathbf{v} = \mathbf{0}$ in Ω and the Poincaré inequality

$$\|\varphi\|_{L^2(\Omega)} \leq C\|\operatorname{grad} \varphi\|_{L^2(\Omega)}$$

is valid for $\varphi \in \mathcal{R}$ (see, e. g., Dautray and Lions [31, p. 127]). □

In conclusion, we have proved that the saddle-point problem (2.11) has a unique solution, and thus the same is true for problem (2.7).

Remark 1. The same existence result can be proved for the problem

$$\begin{cases} \operatorname{curl}\mathbf{u} = \mathbf{J} & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{\mu}\mathbf{u}) = f & \text{in } \Omega \\ \boldsymbol{\mu}\mathbf{u}\cdot\mathbf{n} = b & \text{on } \partial\Omega \\ \int_\Omega \boldsymbol{\mu}\mathbf{u}\cdot\boldsymbol{\rho}_n^\mu = \beta_n & \text{for each } n = 1,\dots,g\,, \end{cases} \tag{2.12}$$

where the field $\mathbf{J}$ is only required to satisfy the necessary compatibility conditions $\operatorname{div}\mathbf{J} = 0$ in Ω and $\int_{(\partial\Omega)_r}\mathbf{J}\cdot\mathbf{n} = 0$ for each $r = 1,\dots,p$ (namely, the more restrictive assumption $\mathbf{J}\cdot\mathbf{n} = 0$ on $\partial\Omega$ has been dropped).

In the variational formulation, one has only to replace the space $\mathcal{V}$ by

$$\mathcal{V}_\sharp = \Big\{\mathbf{v}\in H(\operatorname{curl};\Omega) \mid \int_\Omega \boldsymbol{\mu}\mathbf{v}\cdot\boldsymbol{\rho}_n^\mu = 0 \text{ for all } n = 1,\dots,g\Big\}\,,$$

keeping the other space $\mathcal{R}$ (that still satisfies $\operatorname{grad}\mathcal{R}\subset\mathcal{V}_\sharp$).

The proofs can be easily adapted: the only point that deserves some explanation is that now the variational solution $\mathbf{u}$ is shown to satisfy $\operatorname{curl}(\operatorname{curl}\mathbf{u}-\mathbf{J}) = \mathbf{0}$ in Ω, and moreover, $(\operatorname{curl}\mathbf{u}-\mathbf{J})\times\mathbf{n} = \mathbf{0}$ on $\partial\Omega$. This latter result follows from the fact that the first variational equation is indeed satisfied for all $\widehat{\mathbf{v}}\in H(\operatorname{curl};\Omega)$, and not only for $\mathbf{v}\in\mathcal{V}_\sharp$. In fact, let $P_\mu\widehat{\mathbf{v}}$ be the orthogonal projection of $\widehat{\mathbf{v}}$ on $\mathcal{H}_\mu(m)$ with respect to the scalar product $\int_\Omega \boldsymbol{\mu}\mathbf{v}\cdot\mathbf{w}$. Then $\mathbf{v} = (\widehat{\mathbf{v}}-P_\mu\widehat{\mathbf{v}})\in\mathcal{V}_\sharp$ and $\operatorname{curl}\mathbf{v} = \operatorname{curl}\widehat{\mathbf{v}}$, as the elements in $\mathcal{H}_\mu(m)$ are curl-free.

Thus $(\operatorname{curl}\mathbf{u}-\mathbf{J})\in\mathcal{H}(e)$, and the conditions $\int_{(\partial\Omega)_r}(\operatorname{curl}\mathbf{u}-\mathbf{J})\cdot\mathbf{n} = 0$ for each $r = 1,\dots,p$ permit to conclude that $\operatorname{curl}\mathbf{u}-\mathbf{J} = \operatorname{curl}\boldsymbol{\Phi}$ in Ω (see, e. g., Cantarella et al. [26]). Therefore,

$$\begin{aligned} \int_\Omega(\operatorname{curl}\mathbf{u}-\mathbf{J})\cdot(\operatorname{curl}\mathbf{u}-\mathbf{J}) &= \int_\Omega(\operatorname{curl}\mathbf{u}-\mathbf{J})\cdot\operatorname{curl}\boldsymbol{\Phi} \\ &= \int_\Omega \operatorname{curl}(\operatorname{curl}\mathbf{u}-\mathbf{J})\cdot\boldsymbol{\Phi} + \int_{\partial\Omega}(\operatorname{curl}\mathbf{u}-\mathbf{J})\cdot\mathbf{n}\times\boldsymbol{\Phi} = 0\,, \end{aligned}$$

namely, $\operatorname{curl}\mathbf{u} = \mathbf{J}$ in Ω.

3 A new variational formulation for problem (2.1)

The discussion at the beginning of Section 2.1 should have explained why our aim here is to find a different variational formulation for problem (2.1), a formulation that turns out to be more suitable for numerical approximation.

In our procedure, the first step is to find a vector field $\mathbf{u}^\star \in (L^2(\Omega))^3$ satisfying

$$\begin{cases} \operatorname{div} \mathbf{u}^\star = f & \text{in } \Omega \\ \int\limits_{(\partial\Omega)_r} \mathbf{u}^\star \cdot \mathbf{n} = \alpha_r & \text{for each } r = 1, \dots, p \,. \end{cases} \tag{3.1}$$

Such a vector field does exist: for instance, one can think to take $\mathbf{J} = \mathbf{0}$ and $\mathbf{a} = \mathbf{0}$ in (2.1), or any choice of $\mathbf{J}$ and $\mathbf{a}$ satisfying the compatibility conditions (indeed, we will not assume in the sequel that $\operatorname{curl}(\boldsymbol{\eta}\mathbf{u}^\star) = \mathbf{0}$ or $(\boldsymbol{\eta}\mathbf{u}^\star) \times \mathbf{n} = \mathbf{0}$).

The vector field $\mathbf{W} = \mathbf{u} - \mathbf{u}^\star$ satisfies

$$\begin{cases} \operatorname{curl}(\boldsymbol{\eta}\mathbf{W}) = \mathbf{J} - \operatorname{curl}(\boldsymbol{\eta}\mathbf{u}^\star) & \text{in } \Omega \\ \operatorname{div} \mathbf{W} = 0 & \text{in } \Omega \\ (\boldsymbol{\eta}\mathbf{W}) \times \mathbf{n} = \mathbf{a} - (\boldsymbol{\eta}\mathbf{u}^\star) \times \mathbf{n} & \text{on } \partial\Omega \\ \int\limits_{(\partial\Omega)_r} \mathbf{W} \cdot \mathbf{n} = 0 & \text{for each } r = 1, \dots, p \,, \end{cases} \tag{3.2}$$

and the second step of the procedure is finding a simple variational formulation of this problem.

Multiplying the first equation by a test function $\mathbf{v} \in H(\operatorname{curl};\Omega)$, integrating in Ω and integrating by parts, we find:

$$\begin{aligned} \int\limits_\Omega \mathbf{J} \cdot \mathbf{v} &= \int\limits_\Omega \operatorname{curl}[\boldsymbol{\eta}(\mathbf{W} + \mathbf{u}^\star)] \cdot \mathbf{v} \\ &= \int\limits_\Omega \boldsymbol{\eta}(\mathbf{W} + \mathbf{u}^\star) \cdot \operatorname{curl} \mathbf{v} - \int\limits_{\partial\Omega} [\boldsymbol{\eta}(\mathbf{W} + \mathbf{u}^\star) \times \mathbf{n}] \cdot \mathbf{v} \\ &= \int\limits_\Omega \boldsymbol{\eta}\mathbf{W} \cdot \operatorname{curl} \mathbf{v} + \int\limits_\Omega \boldsymbol{\eta}\mathbf{u}^\star \cdot \operatorname{curl} \mathbf{v} - \int\limits_{\partial\Omega} \mathbf{a} \cdot \mathbf{v} \,. \end{aligned}$$

Let us introduce the space

$$\mathcal{W}_0 = \Big\{ \boldsymbol{\xi} \in H(\operatorname{div};\Omega) \mid \operatorname{div} \boldsymbol{\xi} = 0 \text{ in } \Omega, \\ \int\limits_{(\partial\Omega)_r} \boldsymbol{\xi} \cdot \mathbf{n} = 0 \text{ for each } r = 1, \dots, p \Big\} \,. \tag{3.3}$$

Note that this space can be written as $\mathcal{W}_0 = \operatorname{curl}[H(\operatorname{curl};\Omega)]$: in fact, the inclusion $\operatorname{curl}[H(\operatorname{curl};\Omega)] \subset \mathcal{W}_0$ is obvious, while the inclusion $\mathcal{W}_0 \subset \operatorname{curl}[H(\operatorname{curl};\Omega)]$ is a classical result concerning vector potentials (see, e. g., Cantarella et al. [26]). The vector field $\mathbf{W}$ is thus a solution to

$$\mathbf{W} \in \mathcal{W}_0 \ : \ \int\limits_\Omega \boldsymbol{\eta}\mathbf{W} \cdot \operatorname{curl} \mathbf{v} = \int\limits_\Omega \mathbf{J} \cdot \mathbf{v} - \int\limits_\Omega \boldsymbol{\eta}\mathbf{u}^\star \cdot \operatorname{curl} \mathbf{v} \\ + \int\limits_{\partial\Omega} \mathbf{a} \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in H(\operatorname{curl};\Omega) \,. \tag{3.4}$$

More precisely, $\mathbf{W}$ is the *unique* solution of that problem: in fact, assuming $\mathbf{J} = \mathbf{u}^* = \mathbf{a} = \mathbf{0}$, and taking $\mathbf{v}$ such that $\operatorname{curl} \mathbf{v} = \mathbf{W}$, it follows at this point $\int_\Omega \boldsymbol{\eta} \mathbf{W} \cdot \mathbf{W} = 0$, hence $\mathbf{W} = \mathbf{0}$.

Let us remark at once that, due to the identity $\mathcal{W}_0 = \operatorname{curl}[H(\operatorname{curl}; \Omega)]$, an edge finite element scheme related to this variational formulation leads to a well-structured stiffness matrix: the one of the curl curl operator (for a suitable set of the basis functions, see (6.9) and Proposition 6).

Remark 2. Let us consider the electrostatic problem in a domain with simple topological shape, namely, problem (2.1) with $\mathbf{J} = \mathbf{0}$ in Ω, $\mathbf{a} = \mathbf{0}$ on $\partial\Omega$, and $p = 0$. We have already seen in Section 2.1 that $\boldsymbol{\eta}\mathbf{u} = \operatorname{grad} \chi$ in Ω, where the potential χ satisfies $\operatorname{div}(\boldsymbol{\eta}^{-1} \operatorname{grad} \chi) = f$ in Ω and $\chi = 0$ on $\partial\Omega$. In this situation, the simplest way for determining the approximate solution is clearly to solve this Dirichlet boundary value problem by using nodal finite elements.

4 A new variational formulation for problem (2.7)

The variational formulation of the curl–div system with assigned normal component on the boundary that we present here is similar to the one we have proposed in Alonso Rodríguez et al. [4] for the problem of magnetostatics. However, we think it can be interesting for its particular simplicity, as here we will formulate the problem in the space $\mathcal{V}_0 = \operatorname{grad}[H^1(\Omega)]$, while in [4] it was set in the space $H(\operatorname{curl}^0; \Omega)$, which in the general topological case is more complicated to discretize.

Also in this case, we need a preliminary step: to find a vector field $\mathbf{u}^* \in (L^2(\Omega))^3$ satisfying

$$\begin{cases} \operatorname{curl} \mathbf{u}^* = \mathbf{J} & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{u}^* \cdot d\mathbf{s} = \beta_n & \text{for each } n = 1, \ldots, g\,. \end{cases} \tag{4.1}$$

This vector field does exist: for instance, one can choose $f = 0$ and $b = 0$ in (2.7), or any choice of f and b satisfying the compatibility condition (indeed, we do not need to assume in the sequel that $\operatorname{div}(\boldsymbol{\mu}\mathbf{u}^*) = 0$ or $(\boldsymbol{\mu}\mathbf{u}^*) \cdot \mathbf{n} = 0$).

The vector field $\mathbf{V} = \mathbf{u} - \mathbf{u}^*$ satisfies

$$\begin{cases} \operatorname{curl} \mathbf{V} = \mathbf{0} & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{\mu}\mathbf{V}) = f - \operatorname{div}(\boldsymbol{\mu}\mathbf{u}^*) & \text{in } \Omega \\ (\boldsymbol{\mu}\mathbf{V}) \cdot \mathbf{n} = b - (\boldsymbol{\mu}\mathbf{u}^*) \cdot \mathbf{n} & \text{on } \partial\Omega \\ \oint_{\sigma_n} \mathbf{V} \cdot d\mathbf{s} = 0 & \text{for each } n = 1, \ldots, g\,, \end{cases} \tag{4.2}$$

and now we only have to find a variational formulation of this problem.

Multiplying the second equation by a test function $\phi \in H^1(\Omega)$, integrating in Ω and integrating by parts we find:

$$\begin{aligned}\int_\Omega f\phi &= \int_\Omega \operatorname{div}[\boldsymbol{\mu}(\mathbf{V}+\mathbf{u}^*)]\phi \\ &= -\int_\Omega \boldsymbol{\mu}(\mathbf{V}+\mathbf{u}^*)\cdot \operatorname{grad}\phi + \int_{\partial\Omega} [\boldsymbol{\mu}(\mathbf{V}+\mathbf{u}^*)\cdot\mathbf{n}]\phi \\ &= -\int_\Omega \boldsymbol{\mu}\mathbf{V}\cdot \operatorname{grad}\phi - \int_\Omega \boldsymbol{\mu}\mathbf{u}^*\cdot \operatorname{grad}\phi + \int_{\partial\Omega} b\phi\,.\end{aligned}$$

Let us introduce the space

$$\begin{aligned}\mathcal{V}_0 = \Big\{\mathbf{v}\in H(\operatorname{curl};\Omega)\mid \operatorname{curl}\mathbf{v} = \mathbf{0} \text{ in } \Omega,\\ \oint_{\sigma_n}\mathbf{v}\cdot d\mathbf{s} = 0 \text{ for each } n=1,\dots,g\Big\}\,.\end{aligned} \tag{4.3}$$

Note that this space can be written as $\mathcal{V}_0 = \operatorname{grad}[H^1(\Omega)]$: in fact, the inclusion $\operatorname{grad}[H^1(\Omega)] \subset \mathcal{V}_0$ is obvious, while the inclusion $\mathcal{V}_0 \subset \operatorname{grad}[H^1(\Omega)]$ is a classical result concerning scalar potentials (see, e. g., Cantarella et al. [26]). The vector field **V** is thus a solution to

$$\begin{aligned}\mathbf{V}\in\mathcal{V}_0 \ :\ \int_\Omega \boldsymbol{\mu}\mathbf{V}\cdot\operatorname{grad}\phi = -\int_\Omega f\phi - \int_\Omega \boldsymbol{\mu}\mathbf{u}^*\cdot\operatorname{grad}\phi \\ + \int_{\partial\Omega} b\phi \quad \forall\,\phi\in H^1(\Omega)\,.\end{aligned} \tag{4.4}$$

It is easy to see that **V** is indeed the *unique* solution of that problem: in fact, assuming $f = b = 0$, $\mathbf{u}^* = \mathbf{0}$, and taking ϕ such that $\operatorname{grad}\phi = \mathbf{V}$, it follows at once $\int_\Omega \boldsymbol{\mu}\mathbf{V}\cdot\mathbf{V} = 0$, hence $\mathbf{V} = \mathbf{0}$.

Also in this case we remark that, due to the identity $\mathcal{V}_0 = \operatorname{grad}[H^1(\Omega)]$, a nodal finite element scheme related to this variational formulation leads to a very simple and nice stiffness matrix: the one of the Laplace operator $-\Delta$ (for all the basis functions except one, see (7.7)).

5 Finite element approximation: generalities

Without pretending to be exhaustive, in this section we give a general overview of the methods that have been proposed for the finite element numerical approximation of the curl–div problem (mainly for the magnetostatic case given by (2.7) with $f = 0$ and

$b = 0$); our aim is simply to show here the advantage of the finite element methods we are going to introduce.

The magnetostatic problem has been considered since a long time, though very often in a simple topological situation, as it is probably the "most frequently encountered field problem in electrical engineering design" (see Chari et al. [28]).

A formulation in terms of a vector potential $\mathbf{A}$ such that curl $\mathbf{A} = \boldsymbol{\mu}\mathbf{u}$ is quite classical, and has been analyzed by Coulomb [30], Barton and Cendes [12], Preis et al. [55] (see also the more recent point of view involving mimetic finite differences presented in Brezzi and Buffa [22], Lipnikov et al. [43]). Since the unknown is a vector field, the computational cost is higher than that needed to solve problem (2.7), that, as we will see in (7.7), in our formulation is essentially a scalar problem. Moreover, the magnetic vector potential approach presents two additional disadvantages: firstly, the right-hand sides f and b must be vanishing, or, if this not the case, one has the additional step given by the identification of a scalar function Ψ such that $\operatorname{div}(\boldsymbol{\mu} \operatorname{grad} \Psi) = f$ in Ω and $\boldsymbol{\mu} \operatorname{grad} \Psi \cdot \mathbf{n} = b$ on $\partial\Omega$; secondly, the vector potential $\mathbf{A}$ needs a gauge condition, thus another scalar equation (and unknown) has to be introduced. The method we devise in Section 6 for solving problem (2.1) has two steps: the first one has the aim of simply reducing the problem to the search of a suitable magnetic vector potential, and the second step can be performed without introducing a differential gauge, so that the overall scheme is cheap and efficient.

The remark concerning the computational cost also holds for many methods formulated in terms of the field $\mathbf{u}$: let us mention the mixed methods proposed by Kikuchi [42], Kanayama et al. [41], the least-squares approaches by Chang and Gunzburger [27], Bensow and Larson [14], Bochev et al. [18], the negative-norm least-squares schemes by Bramble and Pasciak [21], the weak Galerkin formulations by Wang and Wang [63], and the even more expensive two field-based methods by Rikabi et al. [58], Perugia [49] and Alotto and Perugia [10].

The co-volume method proposed by Nicolaides and Wu [48] is based on a system of two orthogonal grids like the classical Voronoi–Delaunay mesh pair, and for this reason this approach is not completely general, as some restrictions on the primal mesh and on the topological properties of the computational domain are needed.

Finally, the methods based on a magnetic scalar "potential" (the so-called reduced scalar potential) require the preliminary determination of a source field $\mathbf{H}_e$. Doing this by means of the Biot–Savart formula is not cheap from the computational point of view, and sometimes it induces cancellation errors (see Simkin and Trowbridge [62], Balac and Caloz [11]). In Mayergoyz et al. [45], it was suggested how to avoid this drawback by introducing an additional scalar potential, thus proposing a more expensive scheme (a complete analysis of this more complex formulation is in Bermudez et al. [15]). The method we propose in Section 7 for solving problem (2.7) presents two steps: the first one leads to a problem where the unknown is essentially a magnetic scalar "potential," but this is done without using the Biot–Savart formula, and in the end it turns out to be cheap and reliable.

Our methods in Section 6 and Section 7 are related to the so-called tree–cotree gauge used for the numerical approximation of magnetostatic and eddy current problems (see, e. g., Albanese and Rubinacci [1, 2], Ren and Razek [57], Manges and Cendes [44]); it could be seen as a rigorous mathematical version of that approach.

Before going on, a few remarks are in order. The techniques based on a tree–cotree decomposition of the nodes and the edges of the mesh can have some drawbacks, both for the construction of scalar or vector potentials and for the determination of a finite element basis. In fact, the stability of the methods depends on the choice of the tree (see Hiptmair [36]), and a clear theoretical result concerning the best selection for numerical approximation is not known. In this paper, as well as in our previous experience (see Alonso Rodríguez et al. [4], Alonso Rodríguez et al. [5]), choosing a breadth-first spanning tree has shown to be suitable and has lead to efficient numerical schemes. However, there are no rigorous results on this subject, and a deeper analysis, that would be quite interesting, could be the topic of a future research.

Let us introduce now some notation. In the following sections, we assume that $\Omega \subset \mathbb{R}^3$ is a polyhedral bounded domain with Lipschitz boundary $\partial\Omega$. We consider a tetrahedral triangulation $\mathcal{T}_h = (V, E, F, T)$ of $\overline{\Omega}$, denoting by V the set of vertices, E the set of edges, F the set of faces and T the set of tetrahedra of $\mathcal{T}_h$.

We will use these spaces of finite elements (see Monk [46, Section 5.6, Section 5.5, Section 5.4 and Section 5.7] for a complete presentation): the space L_h of continuous piecewise-linear elements, with dimension n_v, the number of vertices in $\mathcal{T}_h$; the space N_h of Nédélec edge elements of degree 1, with dimension n_e, the number of edges in $\mathcal{T}_h$; the space RT_h of Raviart–Thomas elements of degree 1, with dimension n_f, the number of faces in $\mathcal{T}_h$; the space PC_h of piecewise-constant elements, with dimension n_t, the number of tetrahedra in $\mathcal{T}_h$.

The following inclusions are well known:

$$L_h \subset H^1(\Omega) \quad , \quad N_h \subset H(\mathrm{curl};\Omega) \quad , \quad \mathrm{RT}_h \subset H(\mathrm{div};\Omega) \quad \mathrm{PC}_h \subset L^2(\Omega)\,.$$

Moreover, $\operatorname{grad} L_h \subset N_h$, $\operatorname{curl} N_h \subset \mathrm{RT}_h$ and $\operatorname{div} \mathrm{RT}_h \subset \mathrm{PC}_h$. The basis of L_h is denoted by $\{\psi_{h,1}, \dots, \psi_{h,n_v}\}$, with $\psi_{h,i}(v_j) = \delta_{i,j}$ for $1 \le i,j \le n_v$; the basis of N_h is denoted by $\{\mathbf{w}_{h,1}, \dots, \mathbf{w}_{h,n_e}\}$, with $\int_{e_j} \mathbf{w}_{h,i} \cdot \boldsymbol{\tau} = \delta_{i,j}$ for $1 \le i,j \le n_e$; the basis of RT_h is denoted by $\{\mathbf{r}_{h,1}, \dots, \mathbf{r}_{h,n_f}\}$, with $\int_{f_m} \mathbf{r}_{h,l} \cdot \boldsymbol{\nu} = \delta_{l,m}$ for $1 \le l,m \le n_f$.

Fixing a total ordering $v_1, \dots, v_{n_v}$ of the elements of V, an orientation on the elements of E and F is induced: if the end points of e_j are v_a and v_b for some $a, b \in \{1, \dots, n_v\}$ with $a < b$, then the oriented edge e_j will be denoted by $[v_a, v_b]$, with unit tangent vector $\boldsymbol{\tau} = \frac{v_b - v_a}{|v_b - v_a|}$; if the face f_m has vertices v_a, v_b and v_c with $a < b < c$, the oriented face f_m will be denoted by $[v_a, v_b, v_c]$ and its unit normal vector $\boldsymbol{\nu} = \frac{(v_b - v_a)\times(v_c - v_a)}{|(v_b - v_a)\times(v_c - v_a)|}$ is obtained by the right-hand rule.

We have already introduced the set of closed curves $\{\sigma_n\}_{n=1}^{g}$. We recall here that indeed they can be constructed as 1-cycles in $\mathcal{T}_h$, therefore, they are suitable for being employed in finite element approximation (see Hiptmair and Ostrowski [39]; see also Alonso Rodríguez et al. [4]).

6 Finite element approximation of problem (3.4)

We are ready now for the presentation of our finite element approximation procedure of problem (2.1). It can be performed in two steps. The first one, that is quite cheap, is finding a finite element potential $\mathbf{u}_h^\star \in \mathrm{RT}_h$ such that

$$\begin{cases} \operatorname{div} \mathbf{u}_h^\star = f_h & \text{in } \Omega \\ \displaystyle\int_{(\partial\Omega)_r} \mathbf{u}_h^\star \cdot \mathbf{n} = \alpha_r & \text{for each } r = 1, \dots, p\,, \end{cases} \tag{6.1}$$

where $f_h \in \mathrm{PC}_h$ is the piecewise-constant interpolant $I_h^{\mathrm{PC}} f$ of f. This can be done by means of a simple and efficient algorithm as shown in Alonso Rodríguez and Valli [9].

The second step concerns the numerical approximation of problem (3.4). Here, the main issue is to determine a finite element subspace of $\mathcal{W}_0$, and a suitable finite element basis. The natural choice is clearly

$$\mathcal{W}_{0,h} = \Big\{ \boldsymbol{\xi}_h \in \mathrm{RT}_h \mid \operatorname{div} \boldsymbol{\xi}_h = 0 \text{ in } \Omega, \int_{(\partial\Omega)_r} \boldsymbol{\xi}_h \cdot \mathbf{n} = 0 \text{ for each } r = 1, \dots, p \Big\}\,. \tag{6.2}$$

For the ease of notation, let us set $n_Q = n_e - (n_v - 1)$. As proved in Alonso Rodríguez et al. [6], the dimension of $\mathcal{W}_{0,h}$ is equal to $n_Q - g$, and a basis is given by the curls of suitable Nédélec elements belonging to N_h.

To make clear this point, following Alonso Rodríguez et al. [6], some notation are necessary. As shown in Hiptmair and Ostrowski [39] (see also Alonso Rodríguez et al. [4]), it is possible to construct a set of 1-cycles $\{\sigma_n\}_{n=1}^g$, representing a basis of the first homology group $\mathcal{H}_1(\overline{\Omega}, \mathbb{Z})$, as a formal sum of edges in $\mathcal{T}_h$ with integer coefficients. More precisely, let us consider the graph given by the vertices and the edges of $\mathcal{T}_h$ on $\partial\Omega$. The number of connected components of this graph coincides with the number of connected components of $\partial\Omega$. For each $r = 0, 1, \dots, p$, let $S_{\partial\Omega}^r = (V_{\partial\Omega}^r, M_{\partial\Omega}^r)$ be a spanning tree of the corresponding connected component of the graph. Then consider the graph (V, E), given by all the vertices and edges of $\mathcal{T}_h$, and a spanning tree $S = (V, M)$ of this graph such that $M_{\partial\Omega}^r \subset M$ for each $r = 0, 1, \dots, p$. Let us order the edges in such a way that the edge e_l belongs to the cotree of S for $l = 1, \dots, n_Q$ and the edge e_{n_Q+i} belongs to the tree S for $i = 1, \dots, n_v - 1$. In particular, denote by e_q, $q = 1, \dots, 2g$, the set of edges of $\partial\Omega$, constructed by Hiptmair and Ostrowski [39], that have the following properties: they all belong to the cotree, and each one of them "closes" a 1-cycle γ_q that is a representative of a basis of the first homology group $\mathcal{H}_1(\partial\Omega, \mathbb{Z})$ (whose rank is indeed equal to $2g$). With this notation, we recall that the 1-cycles σ_n can be expressed as the formal sum

$$\sigma_n = \sum_{q=1}^{2g} A_{n,q}\gamma_q = \sum_{q=1}^{2g} A_{n,q}e_q + \sum_{i=n_Q+1}^{n_e} a_{n,i}e_i\,, \tag{6.3}$$

for suitable and explicitly computable integers $A_{n,q}$.

The idea that leads to the construction of the basis of $\mathcal{W}_{0,h}$ is now the following: first, consider the set

$$\{\operatorname{curl} \mathbf{w}_{h,l}\}_{l=2g+1}^{n_Q}\,,$$

Then look for g functions $\mathbf{z}_{h,\lambda} \in \mathrm{RT}_h$, $\lambda = 1, \ldots, g$, of the form

$$\mathbf{z}_{h,\lambda} = \sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \operatorname{curl} \mathbf{w}_{h,\upsilon}\,,$$

where the linearly independent vectors $\mathbf{c}^{(\lambda)} \in \mathbb{R}^{2g}$ are chosen in such a way that

$$\oint_{\sigma_n} \left(\sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \mathbf{w}_{h,\upsilon} \right) \cdot d\mathbf{s} = 0$$

for $n = 1, \ldots, g$. This can be done since σ_n is formed by the "closing" edges e_q, $q = 1, \ldots, 2g$, and by edges belonging to the spanning tree, so that

$$\oint_{\sigma_n} \left(\sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \mathbf{w}_{h,\upsilon} \right) \cdot d\mathbf{s} = \sum_{q=1}^{2g} A_{n,q} \int_{e_q} \left(\sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \mathbf{w}_{h,\upsilon} \right) \cdot \boldsymbol{\tau} = \sum_{q=1}^{2g} A_{n,q} c_q^{(\lambda)}\,,$$

and the matrix $A \in \mathbb{Z}^{g\times 2g}$ with entries $A_{n,q}$ has rank g (see Hiptmair and Ostrowski [39], Alonso Rodríguez et al. [4, Section 6]). Thus we only have to determine a basis $\mathbf{c}^{(\lambda)} \in \mathbb{R}^{2g}$ of the kernel of A, $\lambda = 1, \ldots, g$. An easy way for determining these vectors $\mathbf{c}^{(\lambda)}$ is presented in Alonso Rodríguez et al. [6].

Proposition 5. *The vector fields*

$$\{\operatorname{curl} \mathbf{w}_{h,l}\}_{l=2g+1}^{n_Q} \cup \left\{ \operatorname{curl} \left(\sum_{\upsilon=1}^{2g} c_\upsilon^{(\lambda)} \mathbf{w}_{h,\upsilon} \right) \right\}_{\lambda=1}^{g} \subset \mathcal{W}_{0,h}$$

are linearly independent and in particular they are a basis of $\mathcal{W}_{0,h}$.

Proof. The proof that these vector fields are linearly independent is in Alonso Rodríguez et al. [6, Proposition 2]. The second statement is then straightforward, as their number is $n_Q - g$, the dimension of $\mathcal{W}_{0,h}$. □

Let us denote this basis by $\{\operatorname{curl} \boldsymbol{\omega}_{h,l}\}_{l=g+1}^{n_Q}$, with

$$\boldsymbol{\omega}_{h,l} = \begin{cases} \mathbf{w}_{h,l} & \text{for } l = 2g+1, \dots, n_Q \\ \sum_{\upsilon=1}^{2g} c_\upsilon^{(l-g)} \mathbf{w}_{h,\upsilon} & \text{for } l = g+1, \dots, 2g\,. \end{cases} \tag{6.4}$$

Proposition 6. *The vector fields $\{\boldsymbol{\omega}_{h,l}\}_{l=g+1}^{n_Q}$ are linearly independent.*

Proof. Suppose we have $\sum_{l=g+1}^{n_Q} \theta_l \boldsymbol{\omega}_{h,l} = \mathbf{0}$ for some θ_l. This can be rewritten as

$$\begin{aligned} \mathbf{0} &= \sum_{l=2g+1}^{n_Q} \theta_l \mathbf{w}_{h,l} + \sum_{l=g+1}^{2g} \theta_l \Big(\sum_{\upsilon=1}^{2g} c_\upsilon^{(l-g)} \mathbf{w}_{h,\upsilon} \Big) \\ &= \sum_{l=2g+1}^{n_Q} \theta_l \mathbf{w}_{h,l} + \sum_{\upsilon=1}^{2g} \Big(\sum_{l=g+1}^{2g} \theta_l c_\upsilon^{(l-g)} \Big) \mathbf{w}_{h,\upsilon}\,, \end{aligned}$$

thus $\theta_l = 0$ for $l = 2g+1, \dots, n_Q$ and $\sum_{l=g+1}^{2g} \theta_l c_\upsilon^{(l-g)} = 0$ for $\upsilon = 1, \dots 2g$, as $\{\mathbf{w}_{h,l}\}_{l=1}^{n_Q}$ are linearly independent. Since the vectors $\mathbf{c}^{(l-g)} \in \mathbb{R}^{2g}$, $l = g+1, \dots, 2g$, are linearly independent, we also obtain $\theta_l = 0$ for $l = g+1, \dots, 2g$, and the result follows. □

We are now in a position to formulate the finite element approximation of (3.4), that reads as follows:

$$\begin{aligned} \mathbf{W}_h \in \mathcal{W}_{0,h} \ : \ \int_\Omega \boldsymbol{\eta} \mathbf{W}_h \cdot \operatorname{curl} \mathbf{v}_h &= \int_\Omega \mathbf{J} \cdot \mathbf{v}_h - \int_\Omega \boldsymbol{\eta} \mathbf{u}_h^\star \cdot \operatorname{curl} \mathbf{v}_h \\ &\quad + \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{v}_h \quad \forall\, \mathbf{v}_h \in N_h^\star\,, \end{aligned} \tag{6.5}$$

where

$$N_h^\star = \operatorname{span}\{\boldsymbol{\omega}_{h,l}\}_{l=g+1}^{n_Q}\,. \tag{6.6}$$

The corresponding algebraic problem is a square linear system of dimension $n_Q - g$, and it is uniquely solvable. In fact, we note that $\mathcal{W}_{0,h} = \operatorname{curl} N_h^\star$, hence we can choose $\mathbf{v}_h^\star \in N_h^\star$ such that $\operatorname{curl} \mathbf{v}_h^\star = \mathbf{W}_h$; from (6.5) we find at once $\mathbf{W}_h = \mathbf{0}$, provided that $\mathbf{J} = \mathbf{u}_h^\star = \mathbf{a} = \mathbf{0}$.

The convergence of this finite element scheme is easily shown by standard arguments. For the ease of reading, let us present the proof.

Theorem 3. *Let $\mathbf{W} \in \mathcal{W}_0$ and $\mathbf{W}_h \in \mathcal{W}_{0,h}$ be the solutions of problem* (3.4) *and* (6.5), *respectively. Set $\mathbf{u} = \mathbf{W} + \mathbf{u}^\star$ and $\mathbf{u}_h = \mathbf{W}_h + \mathbf{u}_h^\star$, where $\mathbf{u}^\star \in H(\mathrm{div};\Omega)$ and $\mathbf{u}_h^\star \in \mathrm{RT}_h$ are solutions to problem* (3.1) *and* (6.1), *respectively. Assume that $\mathbf{u}$ is regular enough, so* *that the interpolant $I_h^{\mathrm{RT}}\mathbf{u}$ is defined. Then the following error estimate holds:*

$$\|\mathbf{u}-\mathbf{u}_h\|_{H(\mathrm{div};\Omega)} \le c_0\big(\|\mathbf{u}-I_h^{\mathrm{RT}}\mathbf{u}\|_{L^2(\Omega)} + \|f - I_h^{\mathrm{PC}} f\|_{L^2(\Omega)}\big)\,. \tag{6.7}$$

Proof. Since $N_h^\star \subset H(\mathrm{curl};\Omega)$, we can choose $\mathbf{v}=\mathbf{v}_h \in N_h^\star$ in (3.4). By subtracting (6.5) from (3.4), we end up with

$$\int_\Omega \boldsymbol{\eta}[(\mathbf{W}+\mathbf{u}^\star)-(\mathbf{W}_h+\mathbf{u}_h^\star)]\cdot \mathrm{curl}\,\mathbf{v}_h = 0 \qquad \forall\, \mathbf{v}_h \in N_h^\star\,,$$

namely,

$$\int_\Omega \boldsymbol{\eta}(\mathbf{u}-\mathbf{u}_h)\cdot \mathrm{curl}\,\mathbf{v}_h = 0 \qquad \forall\, \mathbf{v}_h \in N_h^\star\,. \tag{6.8}$$

Then, recalling that $\mathcal{W}_{0,h} = \mathrm{curl}\, N_h^\star$, so that $\mathbf{W}_h = \mathrm{curl}\,\mathbf{v}_h^\star$ for a suitable $\mathbf{v}_h^\star \in N_h^\star$, using (6.8) we find

$$\begin{aligned} c_1\|\mathbf{u}-\mathbf{u}_h\|^2_{L^2(\Omega)} &\le \int_\Omega \boldsymbol{\eta}(\mathbf{u}-\mathbf{u}_h)\cdot(\mathbf{u}-\mathbf{u}_h)\\ &= \int_\Omega \boldsymbol{\eta}(\mathbf{u}-\mathbf{u}_h)\cdot(\mathbf{u}-\mathbf{W}_h-\mathbf{u}_h^\star)\\ &= \int_\Omega \boldsymbol{\eta}(\mathbf{u}-\mathbf{u}_h)\cdot(\mathbf{u}-\mathrm{curl}\,\mathbf{v}_h^\star-\mathbf{u}_h^\star)\\ &= \int_\Omega \boldsymbol{\eta}(\mathbf{u}-\mathbf{u}_h)\cdot(\mathbf{u}-\mathrm{curl}\,\mathbf{v}_h-\mathbf{u}_h^\star)\\ &\le c_2\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)}\|\mathbf{u}-\boldsymbol{\xi}_h-\mathbf{u}_h^\star\|_{L^2(\Omega)} \quad \forall\,\boldsymbol{\xi}_h \in \mathcal{W}_{0,h}\,. \end{aligned}$$

We can choose $\boldsymbol{\xi}_h = (I_h^{\mathrm{RT}}\mathbf{u} - \mathbf{u}_h^\star) \in \mathcal{W}_{0,h}$; in fact, $\mathrm{div}(I_h^{\mathrm{RT}}\mathbf{u}) = I_h^{\mathrm{PC}}(\mathrm{div}\,\mathbf{u}) = I_h^{\mathrm{PC}} f = f_h$ and $\int_{(\partial\Omega)_r} I_h^{\mathrm{RT}}\mathbf{u}\cdot\mathbf{n} = \int_{(\partial\Omega)_r}\mathbf{u}\cdot\mathbf{n} = \alpha_r$ for each $r=1,\dots,p$. Then it follows at once $\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} \le \frac{c_2}{c_1}\|\mathbf{u}-I_h^{\mathrm{RT}}\mathbf{u}\|_{L^2(\Omega)}$. Moreover, $\mathrm{div}(\mathbf{u}-\mathbf{u}_h) = f - f_h = f - I_h^{\mathrm{PC}} f$, and the thesis is proved. ☐

A sufficient condition for defining the interpolant of $\mathbf{u}$ is that $\mathbf{u} \in (H^{\frac{1}{2}+\delta}(\Omega))^3$, $\delta > 0$ (see Monk [46, Lemma 5.15]). This is satisfied if, e. g., $\boldsymbol{\eta}$ is a scalar Lipschitz function in $\overline{\Omega}$ and $\mathbf{a} \in (H^\gamma(\partial\Omega))^3$, $\gamma > 0$ (see Alonso and Valli [3]). Moreover, if $\mathbf{u} \in (H^1(\Omega))^3$ and $f \in H^1(\Omega)$ we have $\|\mathbf{u}-\mathbf{u}_h\|_{H(\mathrm{div};\Omega)} = O(h)$.

6.1 The algebraic problem

The solution $\mathbf{W}_h \in \mathcal{W}_{0,h}$ is given by $\mathbf{W}_h = \sum_{l=g+1}^{n_Q} W_l\, \mathrm{curl}\,\boldsymbol{\omega}_{h,l}$. Hence the finite dimensional problem (6.5) can be rewritten as

$$\sum_{l=g+1}^{n_Q} W_l \int_\Omega \boldsymbol{\eta} \operatorname{curl} \boldsymbol{\omega}_{h,l} \cdot \operatorname{curl} \boldsymbol{\omega}_{h,m} = \int_\Omega \mathbf{J} \cdot \boldsymbol{\omega}_{h,m} - \int_\Omega \boldsymbol{\eta} \mathbf{u}_h^\star \cdot \operatorname{curl} \boldsymbol{\omega}_{h,m} + \int_{\partial\Omega} \mathbf{a} \cdot \boldsymbol{\omega}_{h,m} \, , \tag{6.9}$$

for each $m = g+1, \dots, n_Q$.

Theorem 4. *The matrix* $\mathbf{K}^\star$ *with entries*

$$K^\star_{ml} = \int_\Omega \boldsymbol{\eta} \operatorname{curl} \boldsymbol{\omega}_{h,l} \cdot \operatorname{curl} \boldsymbol{\omega}_{h,m}$$

is symmetric and positive definite.

Proof. It is enough to recall that the vector fields $\{\operatorname{curl} \boldsymbol{\omega}_{h,l}\}_{l=g+1}^{n_Q}$ are linearly independent (see Proposition 5). More precisely, they are a basis of $\mathcal{W}_{0,h}$, hence $\mathbf{K}^\star$ is the mass matrix in $\mathcal{W}_{0,h}$ with weight $\boldsymbol{\eta}$. □

7 Finite element approximation of problem (4.4)

Similar to the previous case, also the finite element approximation of problem (2.7) involves two steps. The first one is finding a finite element potential $\mathbf{u}_h^* \in N_h$ such that

$$\begin{cases} \operatorname{curl} \mathbf{u}_h^* = \mathbf{J}_h & \text{in } \Omega \\ \oint_{\sigma_n} \mathbf{u}_h^* \cdot d\mathbf{s} = \beta_n & \text{for each } n = 1, \dots, g \, , \end{cases} \tag{7.1}$$

where $\mathbf{J}_h \in \mathrm{RT}_h$ is the Raviart–Thomas interpolant $I_h^{\mathrm{RT}}\mathbf{J}$ of $\mathbf{J}$ (we therefore assume that $\mathbf{J}$ is so regular that its interpolant $I_h^{\mathrm{RT}}\mathbf{J}$ is defined; for instance, as already recalled, it is enough to assume $\mathbf{J} \in (H^{\frac{1}{2}+\delta}(\Omega))^3$, $\delta > 0$: see Monk [46, Lemma 5.15]). An efficient algorithm for computing $\mathbf{u}_h^*$, based on a tree–cotree decomposition of the mesh, is described in Alonso Rodríguez and Valli [9].

The second step is related to the numerical approximation of problem (4.4). It is quite easy to find a finite element subspace of $\mathcal{V}_0$ and a suitable finite element basis. The natural choice is clearly

$$\mathcal{V}_{0,h} = \Big\{ \mathbf{v}_h \in N_h \mid \operatorname{curl} \mathbf{v}_h = \mathbf{0} \text{ in } \Omega, \oint_{\sigma_n} \mathbf{v}_h \cdot d\mathbf{s} = 0 \text{ for each } n = 1, \dots, g \Big\} \, , \tag{7.2}$$

which can be rewritten as $\mathcal{V}_{0,h} = \operatorname{grad} L_h$. Since the dimension of this space is $n_v - 1$, a finite element basis is determined by taking $\operatorname{grad} \psi_{h,i}$, $i = 1, \dots, n_v - 1$, $\psi_{h,i}$ being the basis functions of the finite element space L_h.

The finite element approximation of (4.4) is easily obtained:

$$\mathbf{V}_h \in \mathcal{V}_{0,h} \ : \quad \int_\Omega \boldsymbol{\mu}\mathbf{V}_h \cdot \operatorname{grad}\phi_h = -\int_\Omega f\phi_h - \int_\Omega \boldsymbol{\mu}\mathbf{u}_h^* \cdot \operatorname{grad}\phi_h + \int_{\partial\Omega} b\phi_h \quad \forall\, \phi_h \in L_h^* \ , \tag{7.3}$$

where

$$L_h^* = \operatorname{span}\{\psi_{h,i}\}_{i=1}^{n_v-1} = \{\phi_h \in L_h \,|\, \phi_h(v_{n_v}) = 0\}\,. \tag{7.4}$$

The corresponding algebraic problem is a square linear system of dimension $n_v - 1$, and it is uniquely solvable. In fact, since $\mathcal{V}_{0,h} = \operatorname{grad} L_h^*$, we can choose $\phi_h^* \in L_h^*$ such that $\operatorname{grad}\phi_h^* = \mathbf{V}_h$; from (7.3) we find at once $\mathbf{V}_h = \mathbf{0}$, provided that $f = b = 0$, $\mathbf{u}_h^* = \mathbf{0}$.

The convergence of this finite element scheme is easily proved by following the arguments previously presented.

Theorem 5. *Let* $\mathbf{V} \in \mathcal{V}_0$ *and* $\mathbf{V}_h \in \mathcal{V}_{0,h}$ *be the solutions of problem* (4.4) *and* (7.3), *respectively. Set* $\mathbf{u} = \mathbf{V} + \mathbf{u}^*$ *and* $\mathbf{u}_h = \mathbf{V}_h + \mathbf{u}_h^*$, *where* $\mathbf{u}^* \in H(\operatorname{curl};\Omega)$ *and* $\mathbf{u}_h^* \in N_h$ *are solutions to problem* (4.1) *and* (7.1), *respectively. Assume that* $\mathbf{u}$ *and* $\mathbf{J}$ *are regular enough, so that the interpolants* $I_h^N\mathbf{u}$ *and* $I_h^{\mathrm{RT}}\mathbf{J}$ *are defined. Then the following error estimate holds:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\operatorname{curl};\Omega)} \le c_0\big(\|\mathbf{u} - I_h^N\mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{J} - I_h^{\mathrm{RT}}\mathbf{J}\|_{L^2(\Omega)}\big)\,. \tag{7.5}$$

Proof. Since $L_h^* \subset H^1(\Omega)$, we can choose $\phi = \phi_h \in L_h^*$ in (4.4). By subtracting (7.3) from (4.4), we end up with

$$\int_\Omega \boldsymbol{\mu}[(\mathbf{V} + \mathbf{u}^*) - (\mathbf{V}_h + \mathbf{u}_h^*)] \cdot \operatorname{grad}\phi_h = 0 \qquad \forall\, \phi_h \in L_h^*\,,$$

namely,

$$\int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot \operatorname{grad}\phi_h = 0 \qquad \forall\, \phi_h \in L_h^*\,. \tag{7.6}$$

Then, since $\mathcal{V}_{0,h} = \operatorname{grad} L_h^*$ and thus $\mathbf{V}_h = \operatorname{grad}\phi_h^*$ for a suitable $\phi_h^* \in L_h^*$, from (7.6) we find

$$\begin{aligned} c_1\|\mathbf{u} - \mathbf{u}_h\|^2_{L^2(\Omega)} &\le \int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{u}_h) \\ &= \int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{V}_h - \mathbf{u}_h^*) \\ &= \int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \operatorname{grad}\phi_h^* - \mathbf{u}_h^*) \end{aligned}$$

$$= \int_\Omega \boldsymbol{\mu}(\mathbf{u} - \mathbf{u}_h) \cdot (\mathbf{u} - \operatorname{grad} \phi_h - \mathbf{u}_h^*)$$

$$\leq c_2 \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \|\mathbf{u} - \mathbf{v}_h - \mathbf{u}_h^*\|_{L^2(\Omega)} \quad \forall\, \mathbf{v}_h \in \mathcal{V}_{0,h}\,.$$

We can choose $\mathbf{v}_h = (I_h^N \mathbf{u} - \mathbf{u}_h^*) \in \mathcal{V}_{0,h}$; in fact, $\operatorname{curl}(I_h^N \mathbf{u}) = I_h^{\mathrm{RT}}(\operatorname{curl} \mathbf{u}) = I_h^{\mathrm{RT}} \mathbf{J} = \mathbf{J}_h$ and $\oint_{\sigma_n} I_h^N \mathbf{u} \cdot d\mathbf{s} = \oint_{\sigma_n} \mathbf{u} \cdot d\mathbf{s} = \beta_n$ for each $n = 1, \ldots, g$. Then we find at once $\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \leq \frac{c_2}{c_1} \|\mathbf{u} - I_h^N \mathbf{u}\|_{L^2(\Omega)}$. Moreover, $\operatorname{curl}(\mathbf{u} - \mathbf{u}_h) = \mathbf{J} - \mathbf{J}_h = \mathbf{J} - I_h^{\mathrm{RT}} \mathbf{J}$, and the assertion follows. □

Sufficient conditions for defining the interpolants of $\mathbf{u}$ and $\mathbf{J} = \operatorname{curl} \mathbf{u}$ are that they both belong to $(H^{\frac{1}{2}+\delta}(\Omega))^3$, $\delta > 0$ (see Monk [46, Lemma 5.15 and Theorem 5.41]). This is for instance satisfied if $\boldsymbol{\mu}$ is a scalar Lipschitz function in $\overline{\Omega}$ and $b \in H^\gamma(\partial\Omega)$, $\gamma > 0$ (see Alonso and Valli [3]). Moreover, if $\mathbf{u} \in (H^1(\Omega))^3$ and $\mathbf{J} \in (H^1(\Omega))^3$ we have $\|\mathbf{u} - \mathbf{u}_h\|_{H(\operatorname{curl};\Omega)} = O(h)$.

7.1 The algebraic problem

The solution $\mathbf{V}_h \in \mathcal{V}_{0,h}$ is given by $\mathbf{V}_h = \sum_{i=1}^{n_v - 1} V_i \operatorname{grad} \psi_{h,i}$. Hence the finite dimensional problem (7.3) can be rewritten as

$$\sum_{i=1}^{n_v-1} V_i \int_\Omega \boldsymbol{\mu} \operatorname{grad} \psi_{h,i} \cdot \operatorname{grad} \psi_{h,j} = -\int_\Omega f \psi_{h,j} - \int_\Omega \boldsymbol{\mu} \mathbf{u}_h^* \cdot \operatorname{grad} \psi_{h,j} + \int_{\partial\Omega} b \psi_{h,j}\,, \tag{7.7}$$

for each $j = 1, \ldots, n_v - 1$.

We have at once the following.

Theorem 6. *The matrix $\mathbf{K}^*$ with entries*

$$K_{ji}^* = \int_\Omega \boldsymbol{\eta} \operatorname{grad} \psi_{h,i} \cdot \operatorname{grad} \psi_{h,j}$$

is symmetric and positive definite.

8 Numerical results

In this section, we present some numerical experiments with the aim of illustrating the effectiveness of the two proposed formulations and the behavior of their finite element approximation.

All the numerical computations have been performed by means of a MacBook Pro, with a processor 2.9 GHz Intel Core i7, 16 GB 2133 MHz RAM. We have used Netgen (see

[61]) to construct the meshes, and the package Pardiso (see [51, 50]) to solve the linear systems by means of a direct method (thus circumventing possible conditioning problems).

A peculiar point of our procedure is the choice of a suitable spanning tree of the graph given by the nodes and the edges of the mesh. As we have already noted, the stability of the method depends on this choice, in a way that is not completely clarified at the theoretical level. In our computations, we have systematically chosen a breadth-first spanning tree; this, together with the use of direct solvers for the algebraic systems, has always provided good numerical results. Breadth-first spanning trees have also shown to be an efficient choice in Alonso Rodríguez et al. [4], Alonso Rodríguez et al. [5].

We consider different test cases for each one of the two proposed formulations. For both formulations, the first test case is a problem with a known analytical solution. In this way, we can validate the code and illustrate the convergence properties of the finite element discretization. In the second test case, the data are very similar to those of the first test case, the difference only being a concentrated perturbation of the datum at the right-hand side of the divergence equation. We expect a solution that mainly differs from the solution of the first test case in a neighborhood of the support of the perturbation. For the problem in which the tangential component of the velocity is assigned, we present the computations for two different topological situations, in order to show that the approximation method is insensitive to the shape of the computational domain. In the third test case, the computational domain is similar to that of problem number 13 in the TEAM workshop (see [47]). The aim of this test case is to check the behaviour of the methods in a more realistic setting.

8.1 Numerical results for the problem with assigned tangential component on the boundary

Let us recall the system of equations that we consider:

$$\begin{cases} \operatorname{curl}(\boldsymbol{\eta}\mathbf{u}) = \mathbf{J} & \text{in } \Omega \\ \operatorname{div}\mathbf{u} = f & \text{in } \Omega \\ (\boldsymbol{\eta}\mathbf{u}) \times \mathbf{n} = \mathbf{a} & \text{on } \partial\Omega \\ \displaystyle\int_{(\partial\Omega)_r} \mathbf{u}\cdot\mathbf{n} = \alpha_r & \text{for each } r = 1, \dots, p\,. \end{cases}$$

For the sake of simplicity, in the sequel we will take $\boldsymbol{\eta}$ equal to the identity.

The data of the first test are such that the vector field $\mathbf{u} = [-x_1x_2, x_1x_2, 0]^T$ is the exact solution, hence in particular we have $\mathbf{J} = [0, 0, x_2 + x_1]^T$ and $f = x_1 - x_2$.

The computational domain Ω is a cylinder with a cavity. The cylinder has a vertical axis, height equal to $H = 100$, and the cross section given by the circle centered at the origin and of radius $R = 60$. The cavity is a similar cylinder but with height $h = 60$ and cross section of radius $r = 30$. The boundary of Ω has therefore two connected components. We include the Netgen file describing the geometry.

```
algebraic3d

solid cyl1 = cylinder(0,0,0;0,0,1; 60.)
         and plane( 0, 0, 50 ; 0, 0, 1 )
         and plane( 0, 0,-50 ; 0, 0, -1 );

solid cyl2 = cylinder(0,0,0;0,0,1; 30.)
         and plane( 0, 0, 30 ; 0, 0, 1 )
         and plane( 0, 0,-30 ; 0, 0, -1 );

solid cyl_in_cyl = cyl1 and not cyl2;

tlo cyl_in_cyl;
```

To check that the convergence rate is linear as expected, we solve the problem with five different meshes, described in Table 1.1.

Table 1.1: Description of the five meshes for the problem with assigned tangential component on the boundary (first test case, simply-connected domain).

	Elements	Faces	Edges	Vertices	DOF
Mesh 1	538	1246	886	180	707
Mesh 2	4304	9288	6048	1066	4983
Mesh 3	34 432	71 584	44 264	7114	37 151
Mesh 4	275 456	561 792	337 712	51 378	286 335
Mesh 5	2 203 648	4 450 816	2 636 256	389 090	2 247 167

The relative error is computed in the following way:

$$\mathrm{RE}(h) = \frac{\sqrt{\sum_{t\in T} |t|(u_{|t} - u_{h|t})^2}}{\sqrt{\sum_{t\in T} |t|(u_{|t})^2}}, \qquad (8.1)$$

being T the set of tetrahedra of the mesh and $|t|$ the volume of the tetrahedron t.

The convergence rate is estimated comparing the error for two different meshes:

$$\text{Estimated Rate} = \frac{\log[\mathrm{RE}(h_1)/\mathrm{RE}(h_2)]}{\log(h_1/h_2)}. \qquad (8.2)$$

Table 1.2: Relative error, mesh size, convergence rate and computational cost for the problem with assigned tangential component on the boundary (first test case, simply-connected domain).

	Relative error	h	Rate	CPU [ms]
Mesh 1	0.216	41.99		≈ 14
Mesh 2	0.131	31.68	1.657	≈ 62
Mesh 3	0.068	16.30	0.969	≈ 707
Mesh 4	0.034	8.16	0.998	≈ 11 161
Mesh 5	0.017	4.09	1.009	≈ 407 829

The results are reported in Table 1.2.

In the second test case, we consider a perturbed problem, namely, a problem with the same values of $\mathbf{J}$, $\mathbf{a}$, and α_r for each $r = 1, \dots, p$, but with a new value for the divergence, given by $f_\epsilon = f + \epsilon$, where $\epsilon = 1000$ in the ball of radius 10 centered at the point $[45, 0, 0]^T$ and $\epsilon = 0$ otherwise. In Figure 1.1, one can compare the solutions of the first test case and of the second test case (namely, of the problem with a known analytical solution and of the perturbed problem). We are not showing the whole computational domain but only a cut along the plane $x_2 = 10$.

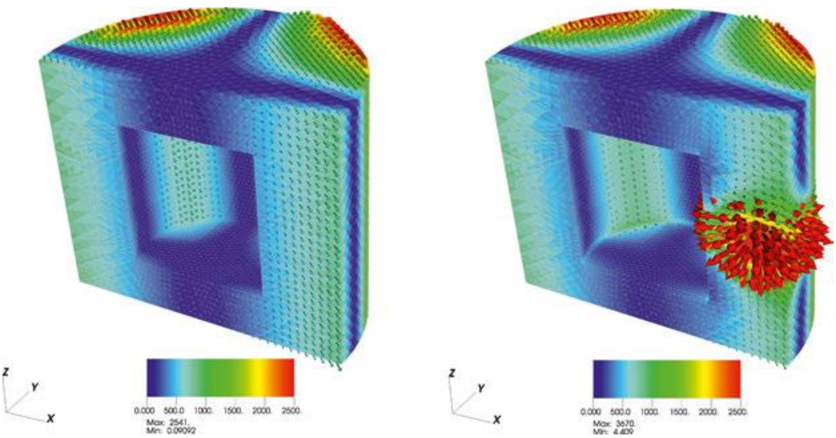

Figure 1.1: The solution $\mathbf{u}$ of the test problem in a simply-connected domain with a known analytical solution (left) and with a perturbed value for the divergence (right). In the figures, the domain is cut along the plane $x_2 = 10$.

In order to show the proposed method is also working for a domain with a more general topological shape, we have solved the problem for a toroidal domain with a concentric toroidal cavity. The connected components of the boundary are two and also the first Betti number of the computational domain is equal to two. More precisely, the computational domain Ω is the subtraction two domains: the larger one is the cylinder of height 2 with circular cross section of radius 1.2 minus the cylinder with the same height and cross section of radius 0.4; the cavity is the cylinder of height 1.6 with circular cross section of radius 1 minus the cylinder of the same height and cross section of radius 0.6. All the mentioned cylinders have their axis coincident with the x_3-axis.

For completeness, we include the Netgen file describing the geometry:

```
algebraic3d

solid cyl1a = cylinder(0,0,0;0,0,1; 1.2)
          and plane( 0, 0, 1 ; 0, 0, 1 )
          and plane( 0, 0,-1 ; 0, 0, -1 );

solid cyl1b = cylinder(0,0,0;0,0,1; 0.4)
          and plane( 0, 0, 1 ; 0, 0, 1 )
          and plane( 0, 0,-1 ; 0, 0, -1 );

solid cyl2a = cylinder(0,0,0;0,0,1; 1.)
          and plane( 0, 0, 0.8 ; 0, 0, 1 )
          and plane( 0, 0,-0.8 ; 0, 0, -1 );

solid cyl2b = cylinder(0,0,0;0,0,1; 0.6)
          and plane( 0, 0, 0.8 ; 0, 0, 1 )
          and plane( 0, 0,-0.8 ; 0, 0, -1 );

solid cyl1 = cyl1a and not cyl1b;
solid cyl2 = cyl2a and not cyl2b;

solid cyl_in_cyl = cyl1 and not cyl2;

tlo cyl_in_cyl;
```

The data of this test are such that the exact solution is $\mathbf{u} = [x_3x_1, x_3x_2, x_3^2]^T$, hence in particular we have $\mathbf{J} = [-x_2, x_1, 0]^T$ and $f = 4x_3$.

Again we have solved the problem with five different meshes, described in Table 1.3. The results are reported in Table 1.4.

In this case, the related perturbed problem has this form: we have kept the same values of $\mathbf{J}$, $\mathbf{a}$ and α_r for each $r = 1, \ldots, p$, just modifying the datum at the right-hand side of the divergence equation, now given by $\widehat{f}_\epsilon = f + \epsilon$, with $\epsilon = -15$ in the ball centered at the point $[-0.8, 0, 0.9]^T$ and radius 0.1, $\epsilon = 15$ in the ball centered at the point $[0.8, 0, 0.9]^T$ and radius 0.1 and $\epsilon = 0$ otherwise.

Table 1.3: Description of the five meshes for the problem with assigned tangential component on the boundary (first test case, non-simply connected domain).

	Elements	Faces	Edges	Vertices	DOF
Mesh 1	1358	3169	2264	453	1810
Mesh 2	10 864	23 540	15 393	2717	12 675
Mesh 3	86 912	181 072	112 270	18 110	94 159
Mesh 4	695 296	1 419 584	854 668	130 380	724 287
Mesh 5	5 562 368	11 240 704	6 663 384	985 048	5 678 335

Table 1.4: Relative error, mesh size, convergence rate and computational cost for the problem with assigned tangential component on the boundary (first test case, non-simply connected domain).

	Relative error	*h*	Rate	CPU [ms]
Mesh 1	0.685	0.101		≈80
Mesh 2	0.498	0.060	1.634	≈170
Mesh 3	0.250	0.031	0.956	≈1931
Mesh 4	0.125	0.015	1.008	≈ 34 779
Mesh 5	0.063	0.008	1.012	≈2 481 110

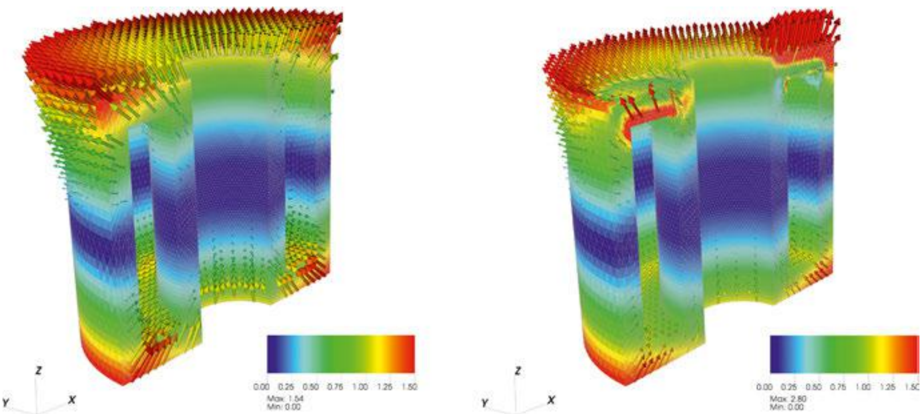

Figure 1.2: The solution **u** of the test problem in a non-simply connected domain with a known analytical solution (left) and with a perturbed value for the divergence (right). In the figures, only half of the domain is drawn.

In Figure 1.2, one can compare the solutions of the problem with a known analytical solution and of the perturbed problem. We are showing only half of the computational domain.

Let us note that we have not indeed constructed the basis described in Proposition 5, as we have used the set of generators $\{\operatorname{curl} \mathbf{w}_{h,l}\}_{l=1}^{n_Q}$, that in the case of a non-simply connected domain, are not linearly independent (the dimension of $\mathcal{W}_{0,h}$ is $n_Q - g$). In this case, the associated linear system is singular, but it is possible to find a solution in an efficient way (for instance, using the package Pardiso).

In the third test problem, the domain is the box $(-300\,300) \times (-300\,300) \times (-250\,250)$ (in mm), with three cavities corresponding to two channels and a plate (see Figure 1.3). The geometry is inspired to the problem number 13 in the TEAM workshop (see [47]). The thickness of the channels and the plate is $\delta = 3.2$ mm, the width $w = 50$ mm and the height $l = 126.4$ mm (so the plate is the hexaedron $(-1.6, 1.6) \times (-25, 25) \times (-63.2, 63.2)$). The distance between the plate and the channels is 0.5 mm, while the distance between the channels and the plane $x_2 = 0$ is 15 mm. The datum $\mathbf{J}$ is supported in a coil placed between the channels and the plate. More precisely, its support is the cylinder of height 100 mm with circular cross section centered at the origin and of radius 120 mm minus the analogous cylinder of the same height and cross section of radius 30 mm. Within the coil, we have $\mathbf{J} = [-x_2, x_1, 0]^T$, while $\mathbf{J}$ is zero outside the coil. All the other data, namely, f, $\mathbf{a}$ and α_r for $r = 1, \ldots, p$, are equal to zero.

The Netgen description of the geometry is the following.

```
algebraic3d

solid m1 = orthobrick(4.2,15,60;122.2,65,63.2);
solid m2 = orthobrick(4.2,15,-63.2;122.2,65,-60);
solid m3 = orthobrick(122.2,15,-63.2;125.4,65,63.2);

solid n1 = orthobrick(-122.2,-65,60;-4.2,-15,63.2);
solid n2 = orthobrick(-122.2,-65,-63.2;-4.2,-15,-60);
solid n3 = orthobrick(-125.4,-65,-63.2;-122.2,-15,63.2);

solid s = orthobrick(-1.6,-25,-63.2;1.6,25,63.2);

solid m    = m1 or m2 or m3;
solid n    = n1 or n2 or n3;
solid hole = m or n or s;

solid box = orthobrick(-300,-300,-250;300\,300\,250);

solid cyla = cylinder(0,0,0;0,0,1; 120.)
         and plane( 0, 0, 50 ; 0, 0, 1 )
         and plane( 0, 0,-50 ; 0, 0, -1 );

solid cylb = cylinder(0,0,0;0,0,1; 30.)
```

```
        and plane( 0, 0, 50 ; 0, 0, 1 )
        and plane( 0, 0,-50 ; 0, 0, -1 );

solid cyl = cyla and not cylb;

solid mat1 = box and not (hole or cyl);

tlo cyl;
tlo mat1;
```

In Figure 1.3, we show the computational domain and the datum **J**. A description of the used mesh is in Table 1.5. Figure 1.4 shows the solution **u** of the third test problem.

We also show in Figure 1.5 four level sets of the solution and in Figure 1.6 ten different level sets from $|\mathbf{u}| = 1000$ to $|\mathbf{u}| = 3000$.

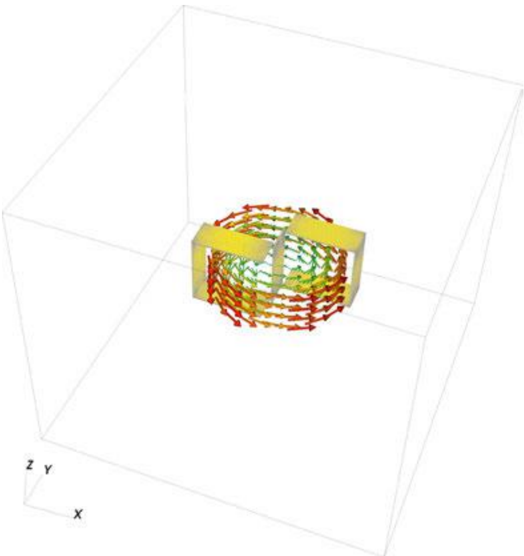

Figure 1.3: The computational domain and the datum of the third test problem.

Table 1.5: Description of the mesh for the third test problem.

Elements	Faces	Edges	Vertices	DOF
2 070 592	4 171 728	2 461 752	360 620	2 101 133

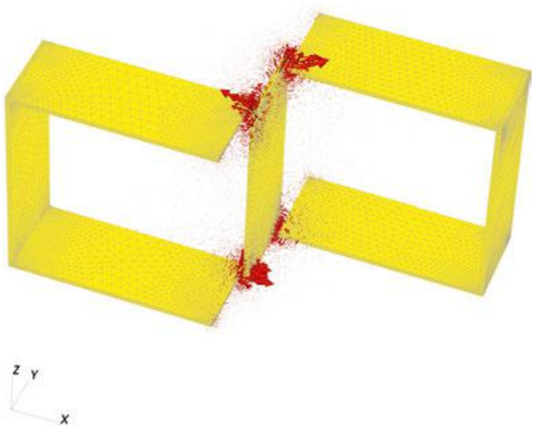

Figure 1.4: The solution **u** of the third test problem.

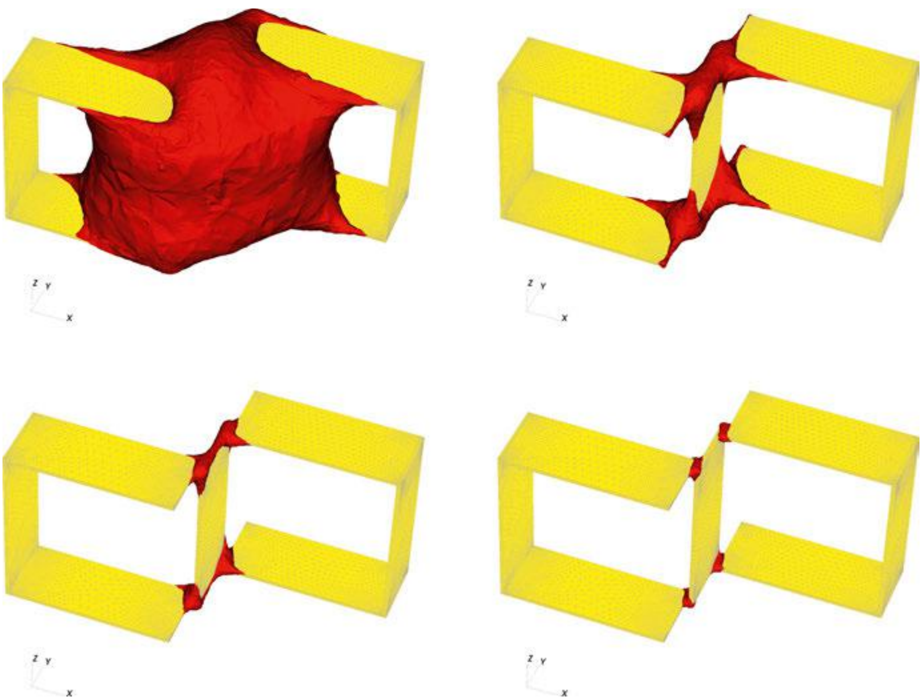

Figure 1.5: Four level sets of the solution **u** of the third test problem: $|\mathbf{u}| = 300$ (top-left), $|\mathbf{u}| = 1200$ (top-right), $|\mathbf{u}| = 2500$ (bottom-left), $|\mathbf{u}| = 5000$ (bottom-right).

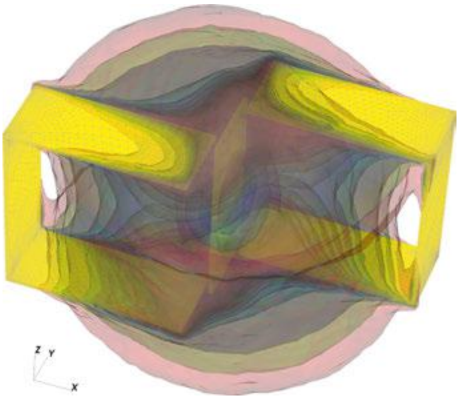

Figure 1.6: A single figure with ten level sets of the solution **u**, from |**u**| = 1000 to |**u**| = 6000.

8.2 Numerical results for the problem with assigned normal component on the boundary

We recall the system of equations:

$$\begin{cases} \operatorname{curl} \mathbf{u} = \mathbf{J} & \text{in } \Omega \\ \operatorname{div}(\boldsymbol{\mu}\mathbf{u}) = f & \text{in } \Omega \\ \boldsymbol{\mu}\mathbf{u} \cdot \mathbf{n} = b & \text{on } \partial\Omega \\ \oint_{\sigma_n} \mathbf{u} \cdot d\mathbf{s} = \beta_n & \text{for each } n = 1, \dots, g \,, \end{cases}$$

and, for the sake of simplicity, in the sequel we will take $\boldsymbol{\mu}$ equal to the identity.

In the first and second test case, the computational domain is the toroidal domain with a concentric toroidal cavity that we have considered in the previous section. The data of the first test are again such that the exact solution is $\mathbf{u} = [x_3x_1, x_3x_2, x_3^2]^T$. In Table 1.6, we report the data of the meshes used for estimating the convergence rate, already presented in Table 1.3 but now including the number of degrees of freedom of this specific formulation.

Table 1.6: Description of the five meshes for the problem with assigned normal component on the boundary.

	Elements	Faces	Edges	Vertices	DOF
Mesh 1	1358	3169	2264	453	452
Mesh 2	10 864	23 540	15 393	2717	2716
Mesh 3	86 912	181 072	112 270	18 110	18 109
Mesh 4	695 296	1 419 584	854 668	130 380	130 379
Mesh 5	5 562 368	11 240 704	6 663 384	985 048	985 047

Table 1.7: Relative error, mesh size, convergence rate and computational cost for the problem with assigned normal component on the boundary.

	Relative error	h	Rate	CPU [ms]
Mesh 1	0.126	0.685		≈ 25
Mesh 2	0.084	0.498	1.234	≈ 59
Mesh 3	0.050	0.250	0.748	≈506
Mesh 4	0.027	0.125	0.881	≈5188
Mesh 5	0.014	0.063	0.952	≈113 083

The relative error and the convergence rate are computed as in (8.1) and (8.2), respectively. The results are reported in Table 1.7.

In the second test problem, we consider a perturbed problem with the same values of $\mathbf{J}$, b, and β_n for each $n = 1, \dots, g$, just modifying the datum at the right-hand side of the divergence equation, setting $\widehat{f}_\epsilon = f + \epsilon$, with $\epsilon = -15$ in the ball centered at the point $[-0.8, 0, 0.9]^T$ and radius 0.1, $\epsilon = 15$ in the ball centered at the point $[0.8, 0, 0.9]^T$ and radius 0.1 and $\epsilon = 0$ otherwise. (Note that $\int_\Omega \epsilon = 0$, hence the compatibility condition $\int_\Omega \widehat{f}_\epsilon = \int_{\partial\Omega} b$ is satisfied.) In Figure 1.7, one can compare the solutions of the problem with a known analytical solution and of the perturbed problem. Only half of the computational domain is shown.

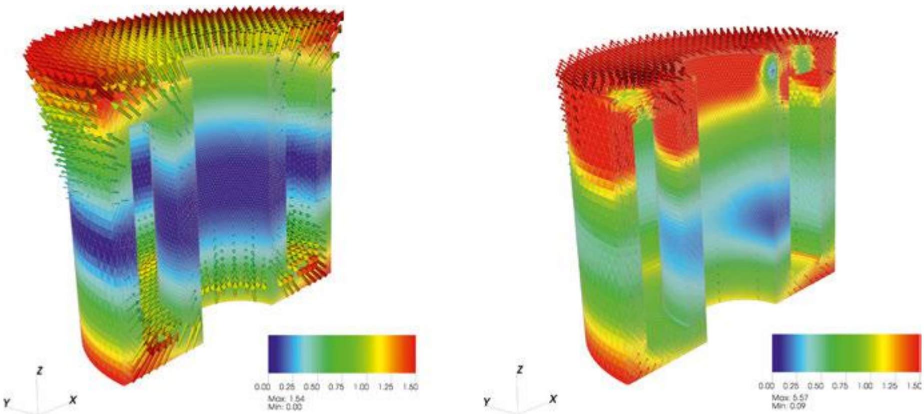

Figure 1.7: The solution **u** of the test problem with a known analytical solution (left) and with a perturbed value for the divergence (right). In the figures, only half of the domain is drawn.

In the third test case, the geometry of the problem is again inspired to that of problem number 13 in the TEAM workshop. However, in order to have a non-simply connected computational domain, we have slightly modified it, as the plate now has thickness equal to 4.2 mm (instead of 3.2 mm), and thus touches the channels. From the topological point of view now, we have only one cavity, precisely, a 2-torus, and thus also the computational domain Ω is a 2-torus.

We include the Netgen file describing the geometry:

```
algebraic3d

solid m1 = orthobrick(  -1.6, 15,   60; 122.2, 65, 63.2);
solid m2 = orthobrick(  -1.6, 15,-63.2; 122.2, 65,  -60);
solid m3 = orthobrick(122.2, 15,-63.2; 125.4, 65, 63.2);

solid n1 = orthobrick( -122.2, -65,   60;    1.6, -15,63.2);
solid n2 = orthobrick( -122.2, -65,-63.2;    1.6, -15,-60);
solid n3 = orthobrick( -125.4, -65,-63.2; -122.2, -15,63.2);

solid s = orthobrick(-1.6,-25,-63.2;1.6,25,63.2);

solid m    = m1 or m2 or m3;
solid n    = n1 or n2 or n3;
solid hole = m or n or s;

solid box = orthobrick(-300,-300,-250;300\,300\,250);

solid cyla = cylinder(0,0,0;0,0,1; 120.)
         and plane( 0, 0, 50 ; 0, 0, 1 )
         and plane( 0, 0,-50 ; 0, 0, -1 );

solid cylb = cylinder(0,0,0;0,0,1; 30.)
         and plane( 0, 0, 50 ; 0, 0, 1 )
         and plane( 0, 0,-50 ; 0, 0, -1 );

solid cyl = cyla and not cylb;

solid mat1 = box and not (hole or cyl);

tlo cyl;
tlo mat1;
```

The structure of the computational mesh is reported in Table 1.8.

As before, the datum **J** is supported in the cylinder of height 100 mm and circular cross section of radius 120 mm minus the cylinder of the same height and cross section of radius 30 mm (both cylinders have their axis coincident with the x_3-axis). In this

Table 1.8: Description of the mesh for the third test problem (one 2-torus cavity).

Elements	Faces	Edges	Vertices	DOF
2 075 264	4 181 824	2 468 392	361 832	361 831

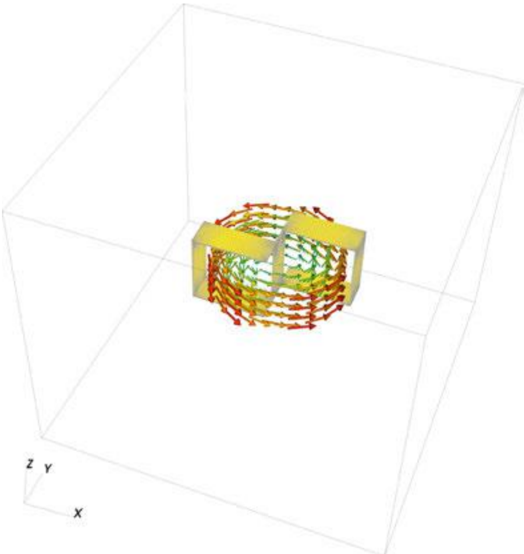

Figure 1.8: The computational domain and the datum of the third test problem (one 2-torus cavity).

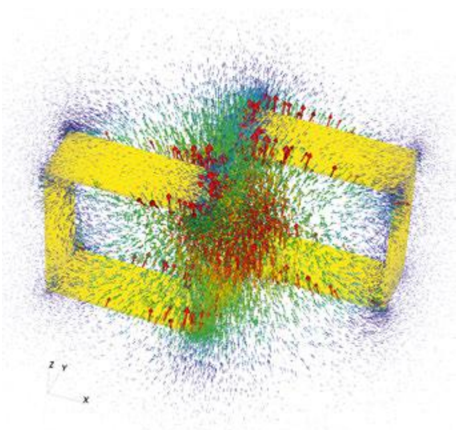

Figure 1.9: The solution **u** of the third test problem (one 2-torus cavity).

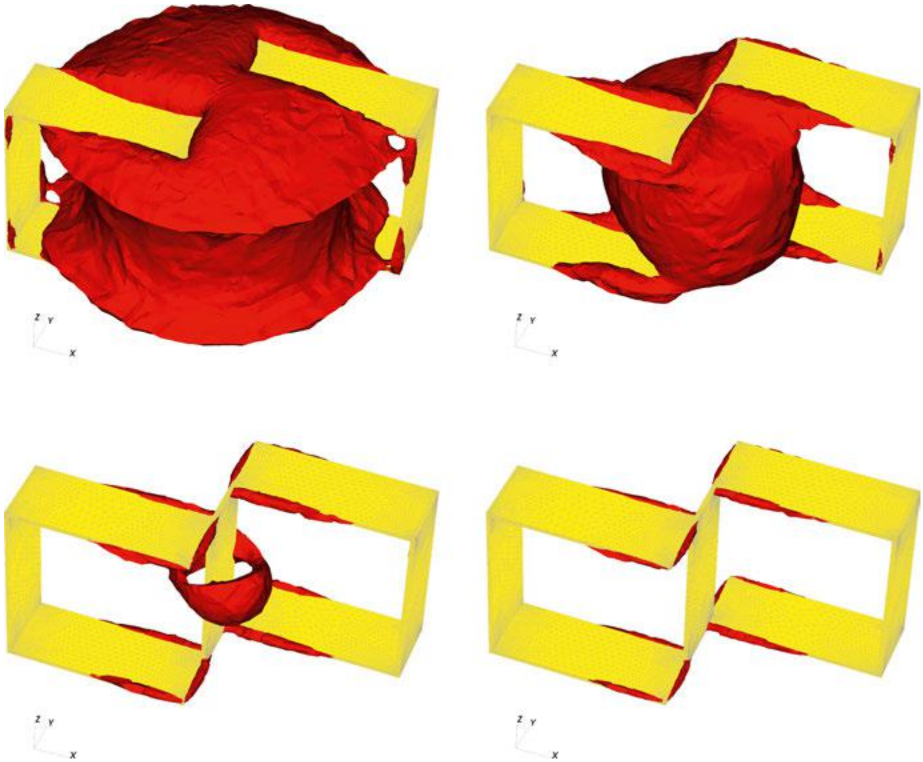

Figure 1.10: Four level sets of the solution **u** of the third test problem (one 2-torus cavity): $|\mathbf{u}| = 1500$ (top-left), $|\mathbf{u}| = 2500$ (top-right), $|\mathbf{u}| = 3500$ (bottom-left), $|\mathbf{u}| = 4000$ (bottom-right).

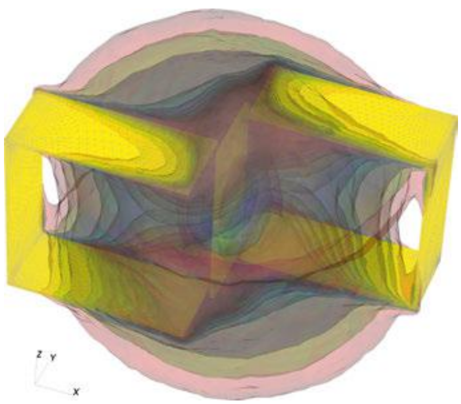

Figure 1.11: A single figure with ten level sets of the solution **u**, from $|\mathbf{u}| = 1000$ to $|\mathbf{u}| = 6000$ (third test problem with one 2-torus cavity).

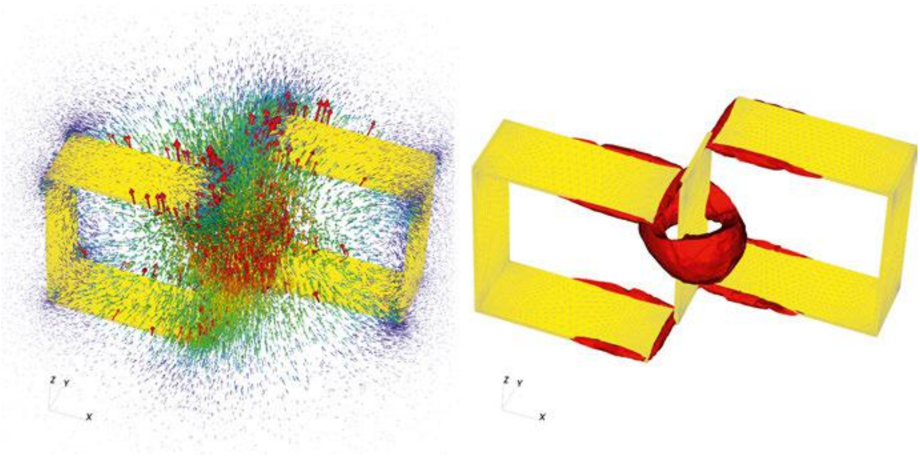

Figure 1.12: The solution (left) and the level set $|\mathbf{u}| = 3500$ (right) of the third test problem. The computational domain is simply-connected and has three cavities as in problem number 13 of the TEAM workshop.

coil, we have $\mathbf{J} = [-x_2, x_1, 0]^T$, while $\mathbf{J}$ is zero outside. All the other data, namely, f, b and β_n for $n = 1, \dots, g$, are equal to zero. In Figure 1.8, we show the computational domain and the datum $\mathbf{J}$.

Figure 1.9 shows the solution $\mathbf{u}$ of the third test problem. We also show in Figure 1.10 four level sets of the solution and in Figure 1.11 ten different level sets from $|\mathbf{u}| = 1000$ to $|\mathbf{u}| = 6000$.

For permitting a comparison, we also present some results for the problem with the same data but with three cavities, namely, for the computational domain of the problem number 13 of the TEAM workshop (in particular, this computational domain Ω is simply-connected). The mesh is the same than in Table 1.5, except for the number of degrees of freedom, that now is coincident with the number of vertices less one, namely, 360 619.

In Figure 1.12, we report the solution and just one level set, the one corresponding to $|\mathbf{u}| = 3500$, because the results are very similar to those in Figures 1.9 and 1.10.

Bibliography

[1] R. Albanese and G. Rubinacci, Magnetostatic field computations in terms of two-components vector potentials, *Int. J. Numer. Methods Eng.*, **29** (1990), 515–532.

[2] R. Albanese and G. Rubinacci, Finite element methods for the solution of 3D eddy current problems, *Adv. Imaging Electron Phys.*, **102** (1998), 1–86.

[3] A. Alonso and A. Valli, An optimal domain decomposition preconditioner for low-frequency time-harmonic Maxwell equations, *Math. Comput.*, **68** (1999), 607–631.

[4] A. Alonso Rodríguez, E. Bertolazzi, R. Ghiloni, and A. Valli, Construction of a finite element basis of the first de Rham cohomology group and numerical solution of 3D magnetostatic problems, *SIAM J. Numer. Anal.*, **51** (2013), 2380–2402.

[5] A. Alonso Rodríguez, E. Bertolazzi, R. Ghiloni, and A. Valli, Finite element simulation of eddy current problems using magnetic scalar potentials, *J. Comput. Phys.*, **294** (2015), 503–523.

[6] A. Alonso Rodríguez, J. Camaño, R. Ghiloni, and A. Valli, Graphs, spanning trees and divergence-free finite elements in domains of general topology, *IMA J. Numer. Anal.*, **37** (2017), 1986–2003.

[7] A. Alonso-Rodríguez, J. Camaño, R. Rodríguez, A. Valli, and P. Venegas, Finite Element Approximation of the Spectrum of the Curl Operator in a Multiply Connected Domain, *Found. Comput. Math.*, **18** (2018), 1493–1533.

[8] A. Alonso Rodríguez and A. Valli, *Eddy Current Approximation of Maxwell Equations*, Springer-Verlag Italia, Milan, 2010.

[9] A. Alonso Rodríguez and A. Valli, Finite element potentials, *Appl. Numer. Math.*, **95** (2015), 2–14.

[10] P. Alotto and I. Perugia, A field-based finite element method for magnetostatics derived from an error minimisation approach, *Int. J. Numer. Methods Eng.*, **49** (2000), 573–598.

[11] S. Balac and G. Caloz, The reduced scalar potential in regions with permeable materials: Reasons for loss of accuracy and cancellation, *Int. J. Numer. Model.*, **20** (2007), 163–180.

[12] M. L. Barton and Z. J. Cendes, New vector finite elements for three-dimensional magnetic field computation, *J. Appl. Phys.*, **61** (1987), 3919–3921.

[13] R. Benedetti, R. Frigerio, and R. Ghiloni, The topology of Helmholtz domains, *Expo. Math.*, **30** (2012), 319–375.

[14] R. Bensow and M. G. Larson, Discontinuous least-squares finite element method for the div–curl problem, *Numer. Math.*, **101** (2005), 601–617.

[15] A. Bermúdez, R. Rodríguez, and P. Salgado, A finite element method for the magnetostatic problem in terms of scalar potentials, *SIAM J. Numer. Anal.*, **46** (2008), 1338–1363.

[16] O. Bíró, Edge element formulations of eddy current problems, *Comput. Methods Appl. Mech. Eng.*, **169** (1999), 391–405.

[17] O. Bíró, K. Preis, and K. R. Richter, On the use of the magnetic vector potential in the nodal and edge finite element analysis of 3D magnetostatic problems, *IEEE Trans. Magn.*, **32** (1996), 651–654.

[18] P. B. Bochev, K. Peterson, and C. M. Siefert, Analysis and computation of compatible least-squares methods for div-curl equations, *SIAM J. Numer. Anal.*, **49** (2011), 159–181.

[19] D. Boffi, F. Brezzi, and M. Fortin, *Mixed Finite Element Methods and Applications*, Springer, Heidelberg, 2013.

[20] A. Bossavit, *Computational Electromagnetism*, Academic Press, San Diego, 1998.

[21] J. H. Bramble and J. E. Pasciak, A new approximation technique for div–curl systems, *Math. Comput.*, **73** (2004), 1739–1762.

[22] F. Brezzi and A. Buffa, Innovative mimetic discretizations for electromagnetic problems, *J. Comput. Appl. Math.*, **234** (2010), 1980–1987.

[23] A. Buffa, Hodge decompositions on the boundary of nonsmooth domains: the multi-connected case, *Math. Models Methods Appl. Sci.*, **11** (2001), 1491–1503.

[24] A. Buffa and P. Ciarlet Jr., On traces for functional spaces related to Maxwell's equations. I. An integration by parts formula in Lipschitz polyhedra, *Math. Methods Appl. Sci.*, **24** (2001), 9–30.

[25] A. Buffa and P. Ciarlet Jr., On traces for functional spaces related to Maxwell's equations. II. Hodge decompositions on the boundary of Lipschitz polyhedra and applications, *Math. Methods Appl. Sci.*, **24** (2001), 31–48.

[26] J. Cantarella, D. DeTurck, and H. Gluck, Vector calculus and the topology of domains in 3-space, *Am. Math. Mon.*, **109** (2002), 409–442.
[27] C.-L. Chang and M. D. Gunzburger, A finite element method for first order elliptic systems in three dimensions, *Appl. Math. Comput.*, **23** (1987), 171–184.
[28] M. V. K. Chari, G. Bedrosian, J. D'Angelo, and A. Konrad, Finite element applications in electrical engineering, *IEEE Trans. Magn.*, **29** (1993), 1306–1314.
[29] M. Costabel, M. Dauge, and S. Nicaise, Corner singularities of Maxwell interface and eddy current problems, in *Operator Theoretical Methods and Applications to Mathematical Physics* (I. Gohberg, A. F. dos Santos, F.-O. Speck, F. S. Teixeira, and W. Wendland, eds.), Birkhäuser Verlag, Basel, 2004, pp. 241–256.
[30] J.-L. Coulomb, Finite element three dimensional magnetic field computation, *IEEE Trans. Magn.*, **MAG-17** (1981), 3241–3246.
[31] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology. Volume 2: Functional and Variational Methods*, Springer, Berlin, 1992.
[32] P. Fernandes and G. Gilardi, Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions, *Math. Models Methods Appl. Sci.*, **7** (1997), 957–991.
[33] K. Fujiwara, T. Nakata, N. Takahashi, and H. Ohashi, On the continuity of the magnetizing current density in 3-D magnetic field analysis with edge element, *IEEE Trans. Magn.*, **31** (1995), 1364–1367.
[34] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier–Stokes Equations*, Springer-Verlag, Berlin, 1986.
[35] P. W. Gross and P. R. Kotiuga, *Electromagnetic Theory and Computation: A Topological Approach*, Cambridge University Press, New York, 2004.
[36] R. Hiptmair, Multilevel gauging for edge elements, *Computing*, **64** (2000), 97–122.
[37] R. Hiptmair, Finite elements in computational electromagnetism, *Acta Numer.*, **11** (2002), 237–339.
[38] R. Hiptmair, P. R. Kotiuga, and S. Tordeux, Self-adjoint curl operators, *Ann. Mat. Pura Appl. (4)*, **191** (2012), 431–457.
[39] R. Hiptmair and J. Ostrowski, Generators of $H_1(\Gamma_h, \mathbb{Z})$ for triangulated surfaces: construction and classification, *SIAM J. Comput.*, **31** (2002), 1405–1423.
[40] E. F. Kaasschieter, Preconditioned conjugate gradients for solving singular systems, *J. Comput. Appl. Math.*, **24** (1988), 265–275.
[41] H. Kanayama, H. Motoyama, K. Endo, and F. Kikuchi, Three-dimensional magnetostatic analysis using Nédélec's elements, *IEEE Trans. Magn.*, **26** (1990), 682–685.
[42] F. Kikuchi, Mixed formulations for finite element analysis of magnetostatic and electrostatic problems, *Jpn. J. Appl. Math.*, **6** (1989), 209–221.
[43] K. Lipnikov, G. Manzini, F. Brezzi, and A. Buffa, The mimetic finite difference method for the 3D magnetostatic field problems on polyhedral meshes, *J. Comput. Phys.*, **230** (2011), 305–328.
[44] J. B. Manges and Z. J. Cendes, A generalized tree-cotree gauge for magnetic field computation, *IEEE Trans. Magn.*, **31** (1995), 1342–1347.
[45] I. D. Mayergoyz, M. V. K. Chari, and J. D'Angelo, A new scalar potential formulation for three-dimensional magnetostatic problems, *IEEE Trans. Magn.*, **23** (1987), 3889–3894.
[46] P. Monk, *Finite Element Methods for Maxwell's Equations*, Oxford University Press, Oxford, 2003.
[47] T. Nakata and K. Fujiwara, Summary of results for benchmark problem 13 (3-D nonlinear magnetostatic model), *Compel*, **11** (1992), 345–369.
[48] R. A. Nicolaides and X. Wu, Covolume solutions of three-dimensional div–curl equations, *SIAM J. Numer. Anal.*, **34** (1997), 2195–2203.

[49] I. Perugia, A mixed formulation for 3D magnetostatic problems: theoretical analysis and face-edge finite element approximation, *Numer. Math.*, **84** (1999), 305–326.

[50] C. G. Petra, O. Schenk, and M. Anitescu, Real-time stochastic optimization of complex energy systems on high-performance computers, *Comput. Sci. Eng.*, **16** (2014), 32–42.

[51] C. G. Petra, O. Schenk, M. Lubin, and K. Gärtner, An augmented incomplete factorization approach for computing the Schur complement in stochastic optimization, *SIAM J. Sci. Comput.*, **36** (2014), C139–C162.

[52] R. Picard, Randwertaufgaben in der verallgemeinerten Potentialtheorie, *Math. Methods Appl. Sci.*, **3** (1981), 218–228.

[53] R. Picard, On the boundary value problems of electro- and magnetostatics, *Proc. R. Soc. Edinb., Sect. A*, **92** (1982), 165–174.

[54] R. Picard, An elementary proof for a compact imbedding result in generalized electromagnetic theory, *Math. Z.*, **187** (1984), 151–164.

[55] K. Preis, I. Bardi, O. Bîró, C. Magele, G. Vrisk, and K. R. Richter, Different finite element formulations of 3D magnetostatic fields, *IEEE Trans. Magn.*, **28** (1992), 1056–1059.

[56] Z. Ren, Influence of the R. H. S. on the convergence behaviour of the curl–curl equation, *IEEE Trans. Magn.*, **32** (1996), 655–658.

[57] Z. Ren and A. Razek, New technique for solving three-dimensional multiply connected eddy-current problems, *IEE Proc. A*, **137** (1990), 135–140.

[58] J. Rikabi, C. F. Bryant, and E. M. Freeman, An error-based approach to complementary formulations of static field solutions, *Int. J. Numer. Methods Eng.*, **26** (1963–1987), 1988.

[59] J. Saranen, On generalized harmonic fields in domains with anisotropic nonhomogeneous media, *J. Math. Anal. Appl.*, **88** (1982), 104–115.

[60] J. Saranen, Erratum: “On generalized harmonic fields in domains with anisotropic nonhomogeneous media”, *J. Math. Anal. Appl.*, **91** (1983), 300.

[61] J. Schöberl, NETGEN An advancing front 2D/3D-mesh generator based on abstract rules, *Comput. Vis. Sci.*, **1** (1997), 41–52.

[62] J. Simkin and C. W. Trowbridge, Three-dimensional nonlinear electromagnetic field computations, using scalar potentials, *IEE Proc. B*, **127** (1980), 368–374.

[63] C. Wang and J. Wang, Discretization of div–curl systems by weak Galerkin finite element methods on polyhedral partitions, *J. Sci. Comput.*, **68** (2016), 1144–1171.

[64] C. Weber, A local compactness theorem for Maxwell’s equations, *Math. Methods Appl. Sci.*, **2** (1980), 12–25.

Sebastian Bauer

2 Darwin and higher order approximations to Maxwell's equations in $\mathbb{R}^3$

Abstract: This contribution is concerned with an asymptotic expansion of Maxwell's equations in case that charge velocities are small in comparison with the speed of light. In every order of expansion, two curl–div systems have to be solved in which solutions of the previous order enter on the right-hand side. It is proved that in case of a bounded underlying domain Ω in every order k of expansion solutions are well-defined and give an approximation of solutions of Maxwell's equations with a L^2 error bound $\mathcal{O}((v/c)^{k+1})$ if initial values of the electromagnetic fields are suitably adapted. In case of $\Omega = \mathbb{R}^3$, weighted L^2 spaces are used for solving curl–div systems. It is shown that solutions of the approximation are only L^2, if certain derivatives of the multipole expansion of the sources vanish. For that reason, a careful analysis of mapping properties of vector differential operators in weighted L^2 spaces is given which might be of interest in its own right.

Keywords: Asymptotic expansion, Maxwell's equations, Darwin approximation, weighted L^2 spaces, exterior domain, spherical vector harmonics

MSC 2010: 35C20, 35J46, 35Q60, 41A60, 78A25, 78A30

1 Introduction

Roughly speaking, Maxwell's equations were the classical culmination point of a number of earlier systems of equations systems attempting to describe electromagnetic phenomena. Here, we shall mention three historical milestones on the way to Maxwell's equations. The first is given by the equations of electro and magnetostatics

$$\begin{aligned} \operatorname{div} E &= \frac{\varrho}{\varepsilon_0}\,, \qquad & \operatorname{curl} B &= \mu_0 j\,, \\ \operatorname{curl} E &= 0\,, & \operatorname{div} B &= 0\,, \end{aligned} \tag{1}$$

established by Coulomb, Lagrange and Gauss. The second set of equations involves Faraday's law of induction and is sometimes referred to as the eddy current model:

$$\begin{aligned} \operatorname{div} E &= \frac{\varrho}{\varepsilon_0}\,, \qquad & \operatorname{curl} B &= \mu_0 j\,, \\ \operatorname{curl} E &= -\partial_t B\,, & \operatorname{div} B &= 0\,. \end{aligned} \tag{2}$$

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Finally, Maxwell added the so-called displacement-current and ended with a Lorentz-invariant set of equations which contains two dynamic and two constraint equations:

$$\begin{aligned} \operatorname{div} E &= \frac{\varrho}{\varepsilon_0}\,, \qquad \operatorname{curl} B = \mu_0 j + \frac{1}{c^2}\partial_t E\,, \\ \operatorname{curl} E &= -\partial_t B\,, \qquad \operatorname{div} B = 0\,. \end{aligned} \tag{3}$$

Here, all equations are written in SI units, ϱ and j are given source distributions, ε_0 is the dielectric constant, μ_0 the magnetic permeability and c the light velocity, where all three constants relate to vacuum and we have $c^2\varepsilon_0\mu_0 = 1$. Local charge conservation, i. e., $\partial_t\varrho + \operatorname{div} j = 0$, is built-in in (3), but not in (2). For another set of equations, known as Darwin equations, the electric field $E = E^L + E^T$ is formally split into a curl-free part E^L, $\operatorname{curl} E^L = 0$ and a divergence-free part E^T, $\operatorname{div} E^T = 0$. In contrast to Maxwell's equations, the time derivative of the transversal part is neglected:

$$\begin{aligned} \operatorname{div} E^L &= \frac{\varrho}{\varepsilon_0}\,, \qquad \operatorname{curl} B = \mu_0 j + \frac{1}{c^2}\partial_t E^L\,, \qquad \operatorname{curl} E^T = -\partial_t B\,, \\ \operatorname{curl} E^L &= 0\,, \qquad \operatorname{div} B = 0\,, \qquad \operatorname{div} E^T = 0\,. \end{aligned} \tag{4}$$

While (3) is a hyperbolic system, the equation systems (1), (2) and (4) are of elliptic type. In (1), two independent curl–div systems have to be solved, in (2) two curl–div systems have to be solved successively and the solution of the first system enters as source term into the second system. Finally, in (4) we have three curl–div systems, the solution of the first enters as source into the second and the solution of the second enters as source into the third system.

It is well known (see, e. g., [10, 28, 4]) that systems (1), (2) and (4) can formally be derived from Maxwell's equations as zeroth-, first- and second-order approximations, respectively, in an asymptotic limit of small charge velocities if compared to speed of light. We shall give the corresponding scaling in Section 2 and the formal asymptotic expansion of Maxwell's equations is given in (8), (9). This formal expansion leads to two main questions: First, are the equations of the asymptotic expansion well-posed? Secondly, do the solutions of the asymptotic expansion give a good approximation of solutions of Maxwell's equations with a suitable error bound?

With regard to initial boundary value problems of Maxwell's equations in bounded domains of $\mathbb{R}^3$, both questions have been tackled successfully, in [10] for natural boundary conditions and in [24, 25] for Silver–Müller absorbing boundary conditions on an artificial outer part of the boundary. For numerical implementations, see [8, 7]. We give a self-contained presentation of the bounded domain case with natural boundary conditions using a simple solution theory of curl–div systems going back to [23] and [26, 27] in Section 5. For the convenience of the reader, the usual proof of the approximation property is given before in Section 4.

However, our main interest here is to understand the case of unbounded domains in $\mathbb{R}^3$, particularly exterior domains. Some results on Darwin systems and asymptotic expansions in exterior domains of $\mathbb{R}^3$, which we have occasion to return to in Section 11, are given in [17, 11, 18]. For sake of simplicity and in order to focus on the effects of the unboundedness of the domains and to be able to ignore the influence of boundary effects, we shall restrict our attention to the free space case, $\Omega = \mathbb{R}^3$.

From studies of the asymptotic expansion of coupled systems of charged matter and electromagnetic fields in [16, 15, 14, 3, 5, 1, 2], we are used to the fact that well-posedness and the approximation property of the asymptotic expansion break down in order $(\frac{|v|}{c})^3$ due to effects of so-called "radiation damping," which have to be taken into account by means of some extra radiation terms. Coupled systems here means that charged matter, which enters Maxwell's equations by means of source terms, is also a degree of freedom governed, e. g., by Newton's equation of motion if matter is modeled by point particles or governed by a Vlasov equation if matter is modeled by a kinetic approach. The electromagnetic field enters as force term the equations describing the matter. In both cases, particle and kinetic, the extra radiation term is built using the third time derivative of the dipole moment of the zeroth-order contribution ϱ_0 of the charge distribution, i. e., $\int_{\mathbb{R}^3} x\ddot{\varrho}_0(t,x)\,dx$; see [16] for a particle model and [1] for a kinetic model.

The studies mentioned above rely on classical solutions and pointwise estimates and investigate coupled systems. Nevertheless, we shall show in this paper that effects due to radiation of multipoles also enter asymptotic expansions of Maxwell's equations with prescribed sources in an L^2 setting; see Section 10. The occurring mathematical difficulties in unbounded domains mostly rely on the fact that for those domains differential operators acting on suitable L^2 spaces do not have closed ranges. This problem is often remedied by using polynomially weighted L^2 spaces. With $w(x) = (1+|x|^2)^{1/2}$ and for $s \in \mathbb{R}$, we say $f \in L^2_s$ if and only if $f \in L^2_{\mathrm{loc}}$ and $w^s f \in L^2$; the L^2_s norm of f is given by the L^2-norm of $w^s f$. We use weighted Sobolev spaces with growing weights, e. g., $E \in H_s(\mathrm{curl})$ if and only if $E \in L^2_s$ and $\mathrm{curl}\,E \in L^2_{s+1}$. With the corresponding scalar products, all of these spaces are Hilbert spaces. For precise definitions, see Section 9. The key ingredient of this paper is a careful analysis of the usual vector differential operators grad, div and curl acting on weighted Sobolev spaces of $\mathbb{R}^3$. Actually, this analysis has already been done in [31] in $\mathbb{R}^n$ and has been generalized to exterior domains of $\mathbb{R}^n$ with inhomogeneous and anisotropic media in [20] using differential forms and exterior derivatives. Furthermore, results of [20], relevant for this paper, have been translated into the language of vector calculus in [21, Appendix B]. For the convenience of the reader, we shall give a self-contained presentation of the reasoning and results of [31] for $\mathbb{R}^3$ using vector calculus instead of calculus of differential forms from the beginning, thereby reproving some results from [21]; see Sections 7, 8 and 9. For this purpose, we first give a representation of vector differential operators in spherical coordinates; see Section 7. Secondly, harmonic homogeneous polynomials play a major role in characterizing kernels and defects of differential operators in weighted Sobolev spaces. For the Laplacian, this has been shown in [19]. We establish a fine-structure of harmonic homogeneous vector polynomials using vector spherical harmonics (see Section 8), which we thirdly use in Section 9 to exactly characterize defects and kernels of vector differential operators in weighted Sobolev spaces; see Theorem 4. In Section 10, we use this result to analyze the well-posedness of the iteration scheme used in the asymptotic expansion. In short, we shall show that assuming smooth charge and current distributions in L^2_s for sufficiently large s the solution of the Darwin approximation is in (unweighted) L^2 if and only if the second time derivative of the dipole moment of the charge distribution vanishes, i. e., $\int x\ddot{\varrho}(x,t)\,dx = 0$. Otherwise, the solutions are only in L^2_s for all $s < -1/2$ and in particular not in L^2; see Subsection 10.3. The usual proof of the approximation property (see [10] or Section 4) gives L^2-error bounds and essentially relies on L^2 approximations. In order to get this proof working, we also need the third-order approximation in (unweighted) L^2, which is the case if and only if the third time derivative of the quadrupole moment of the charge distribution and the third time derivative of some moment of the current distribution vanishes; see Subsection 10.4.

2 Scalings

We are looking for physical situations in which the equation systems (1), (2) and (4) might be good approximations of Maxwell's equations. To this end, we introduce characteristic values for the physical quantities and derive dimensionless Maxwell's equations. From given sources ϱ and j, we take a typical length $\bar{l}$, e. g., the diameter of support of the source distribution, and a typical time $\bar{t}$, e. g., the time a typical charged particle needs to travel the distance $\bar{l}$. From this, we have the typical velocity $\bar{v} = \frac{\bar{l}}{\bar{t}}$ and we assume that $\eta := \bar{v}/c \ll 1$. With scales $\bar{E}, \bar{B}, \bar{j}$ of the electromagnetic fields and the current density, unspecified until now, we define dimensionless quantities $x = \bar{l}x'$, $t = \bar{t}t'$, $E = \bar{E}E'$, $B = \bar{B}B'$, $\varrho = \bar{\varrho}\varrho'$, $j = \bar{j}j'$, $E'(t') = \frac{E(\bar{t}t')}{\bar{E}}$ .... A simple computation gives dimensionless Maxwell's equations

$$\frac{\bar{v}\bar{E}}{c^2\bar{B}}\partial_{t'}E' - \operatorname{curl}' B' = -\mu_0\frac{\bar{j}\bar{l}}{\bar{B}}j', \qquad \frac{\varepsilon_0\bar{E}}{\bar{l}\bar{\varrho}}\operatorname{div}' E' = \varrho',$$

$$\frac{\bar{v}\bar{B}}{\bar{E}}\partial_{t'}B' + \operatorname{curl}' E' = 0, \qquad \operatorname{div}' B' = 0$$

and dimensionless charge conservation

$$\frac{\bar{\varrho}\bar{v}}{\bar{j}}\partial_{t'}\varrho' + \operatorname{div}' j' = 0.$$

In the literature, there are two different scalings leading to slightly different asymptotic expansion. In this contribution, we use the scales from [10]:

$$\bar{E} = \frac{\bar{l}\bar{\varrho}}{\varepsilon_0}, \quad \bar{B} = \frac{\bar{E}}{c}, \quad \bar{j} = c\bar{\varrho}. \tag{5}$$

This choice leads to the following set of equations (for simplicity, we shall suppress all primes):

$$\eta\partial_t E - \operatorname{curl} B = -j\,, \qquad \operatorname{div} E = \varrho\,,$$
$$\eta\partial_t B + \operatorname{curl} E = 0\,, \qquad \operatorname{div} B = 0\,,$$

together with the charge conservation equation

$$\eta\partial_t \varrho + \operatorname{div} j = 0\,.$$

In connection with the Vlasov–Maxwell system, the scaling is not made explicit, but some computations show that the scalings in, e. g., [28, 3] differ from the scaling above by the definition of $\bar{j}$. In the latter two publication $\bar{j} = \bar{v}\bar{\varrho}$ is used, leading to

$$\eta\partial_t E - \operatorname{curl} B = -\eta j\,, \qquad \operatorname{div} E = \varrho\,,$$
$$\eta\partial_t B + \operatorname{curl} E = 0\,, \qquad \operatorname{div} B = 0\,,$$

and charge conservation equation

$$\partial_t \varrho + \operatorname{div} j = 0\,.$$

Of course, this leads to slightly different asymptotic expansions.

3 Formal asymptotic expansions

We now formally expand the resulting dimensionless Maxwell's equations with natural boundary conditions in powers of η, where ν is the outward unit normal on the boundary of the underlying domain. For the scaling (5), we have

$$\begin{aligned} &\eta\partial_t E^\eta - \operatorname{curl} B^\eta = -j^\eta\,, && \operatorname{div} E^\eta = \varrho^\eta\,, && E^\eta \wedge \nu = 0\,, && E^\eta(0,\cdot) = E_0^\eta\,,\\ &\eta\partial_t B^\eta + \operatorname{curl} E^\eta = 0\,, && \operatorname{div} B^\eta = 0\,, && \partial_t B^\eta\cdot\nu = 0\,, && B^\eta(0,\cdot) = B_0^\eta\,. \end{aligned} \tag{6}$$

We make the formal Ansatz

$$\begin{aligned} E^\eta &= E^0 + \eta E^1 + \eta^2 E^2 + \cdots\,, \qquad & B^\eta &= B^0 + \eta B^1 + \eta^2 B^2 + \cdots\,,\\ \varrho^\eta &= \varrho^0 + \eta\varrho^1 + \eta^2\varrho^2 + \cdots\,, \qquad & j^\eta &= j^0 + \eta j^1 + \eta^2 j^2 + \cdots\,, \end{aligned} \tag{7}$$

equate in every order of η and find the resulting equations

$$\begin{aligned} &\operatorname{curl} E^0 = 0\,, && \operatorname{div} E^0 = \varrho^0\,, && E^0\wedge\nu = 0\,,\\ &\operatorname{curl} B^0 = j^0\,, && \operatorname{div} B^0 = 0\,, && \partial_t B^0\cdot\nu = 0 \end{aligned} \tag{8}$$

for $k = 0$ and

$$\begin{aligned} \operatorname{curl} E^k &= -\partial_t B^{k-1}, & \operatorname{div} E^k &= \varrho^k, & E^k \wedge \nu &= 0, \\ \operatorname{curl} B^k &= \partial_t E^{k-1} + j^k, & \operatorname{div} B^k &= 0, & \partial_t B^k \cdot \nu &= 0 \end{aligned} \tag{9}$$

for $k \geq 1$. In the k-th step approximation, we have to solve curl–div systems, where the given sources and the solutions of the $k-1$ step enter as right-hand sides. While this is rather simple for a bounded domain (see Section 6), it is rather involved for exterior domains, since in that case solutions of a curl–div systems with right-hand sides in L^2 are generally not in L^2 but only in some weighted L^2 spaces. Therefore, iterating becomes difficult; see Section 10.

4 Proof of the approximation property

In this section, we repeat the proof that if the iteration scheme (8), (9) is well-defined with solutions in L^2, these solutions give an approximation of solutions of Maxwell's equations (6) with a suitable error bound provided suitably adapted initial values of (6). We shall use the classical Sobolev spaces of vector analysis $H(\operatorname{grad}, \Omega)$, $H(\operatorname{curl}, \Omega)$, $H(\operatorname{div}, \Omega)$ as well as $H(\mathring{\operatorname{grad}}, \Omega)$, $H(\mathring{\operatorname{curl}}, \Omega)$, $H(\mathring{\operatorname{div}}, \Omega)$, the latter generalizing the associated natural boundary conditions in a weak sense; see, e. g., Section 5. For sake of brevity, we set, e. g., $H(\operatorname{curl}, \operatorname{div}, \Omega) = H(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)$ and so on. For a function or field $f = f(t, x)$, we set $f(t) = f(t, \cdot)$ and $\| \cdot \|$ denotes the usual L^2 norm over Ω

We start with a classical result on solutions of (6) which might be obtained by using for instance semi-group theory.

Proposition 1. *For a given domain $\Omega \subset \mathbb{R}^3$, $\eta > 0$, $T > 0$ and given source distributions ϱ^η and j^η with*

$$\begin{aligned} &\varrho^\eta \in C^1([0,T]; L^2(\Omega)) \\ &j^\eta \in C^1([0,T]; L^2(\Omega; \mathbb{R}^3)) \cap C^0([0,T]; H(\operatorname{div}; \Omega)) \\ &0 = \eta \partial_t \varrho^\eta + \operatorname{div} j^\eta \quad \textit{charge conservation} \end{aligned} \tag{10}$$

and given initial fields

$$E_0^\eta \in H(\mathring{\operatorname{curl}}, \operatorname{div}, \Omega) \quad \textit{and} \quad B_0^\eta \in H(\operatorname{curl}, \operatorname{div}, \Omega)$$

satisfying the constraints

$$\operatorname{div} E_0^\eta = \varrho^\eta(0) \quad \textit{and} \quad \operatorname{div} B_0^\eta = 0,$$

the Maxwell initial boundary value problem (6) *(boundary conditions in weak sense) admits a unique solution (E^η, B^η) with*

$$\begin{aligned} E^\eta &\in C^1([0,T]; L^2(\Omega; \mathbb{R}^3)) \cap C^0([0,T]; H(\mathring{\operatorname{curl}}, \operatorname{div}, \Omega)) \quad \textit{and} \\ B^\eta &\in C^1([0,T]; L^2(\Omega; \mathbb{R}^3)) \cap C^0([0,T]; H(\operatorname{curl}, \operatorname{div}, \Omega)). \end{aligned}$$

The following a priori estimate is also classical; a sketch of the proof can be found in [10].

Proposition 2. *For a domain $\Omega \subset \mathbb{R}^3$, $T > 0$ and fields*

$$e \in C^1([0,T]; L^2(\Omega; \mathbb{R}^3)) \cap C^0([0,T]; H(\mathring{\text{curl}}, \Omega)) \quad \text{and}$$
$$b \in C^1([0,T]; L^2(\Omega; \mathbb{R}^3)) \cap C^0([0,T]; H(\text{curl}, \Omega))$$

define

$$k = \eta \partial_t e - \operatorname{curl} b\,, \qquad l = \eta \partial_t b + \operatorname{curl} e\,,$$
$$\mathcal{E} = \left(\int_\Omega e^2 + b^2 \, dx \right)^{1/2}, \qquad m = \left(\int_\Omega k^2 + l^2 \, dx \right)^{1/2}.$$

Then we have for all $0 \le t \le T$

$$\mathcal{E}(t) \le \mathcal{E}(0) + \frac{2}{\eta} \int_0^t m(s)\, ds\,.$$

Now we can give the approximation property theorem.

Theorem 1. *Let $k \in \mathbb{N}$, $T > 0$ and*

$$\varrho^\eta = \varrho^0 + \eta \varrho^1 + \cdots + \eta^{k+1} \varrho^{k+1}\,,$$
$$j^\eta = j^0 + \eta j^1 + \cdots + \eta^{k+1} j^{k+1}\,,$$

such that ϱ^η and j^η satisfy (10) *for all $\eta \ge 0$. Assume that there are fields $E^0, \ldots, E^{k+1}$ and $B^0, \ldots, B^{k+1}$, such that*

$$E^l \in C^1([0,T]; L^2(\Omega; \mathbb{R}^3)) \cap C^0([0,T]; H(\mathring{\text{curl}}, \text{div}, \Omega)) \quad \text{and}$$
$$B^l \in C^1([0,T]; L^2(\Omega; \mathbb{R}^3)) \cap C^0([0,T]; H(\text{curl}, \text{div}, \Omega))\,,$$

for $l = 0, \ldots, k+1$, satisfying the iteration equations (8) *and* (9)*, respectively. Let (E^η, B^η) be the solution of* (6) *according to Proposition* 1 *subject to initial values*

$$E_0^\eta = E^0(0) + \eta E^1(0) + \cdots + \eta^k E^k(0) + \eta^{k+1} E^{\text{free}}$$
$$B_0^\eta = B^0(0) + \eta B^1(0) + \cdots + \eta^k B^k(0) + \eta^{k+1} B^{\text{free}}$$

with

$$E^{\text{free}} \in H(\mathring{\text{curl}}, \text{div}, \Omega)\,, \qquad \operatorname{div} E^{\text{free}} = \varrho^{k+1}\,,$$
$$B^{\text{free}} \in H(\text{curl}, \text{div}, \Omega)\,, \qquad \operatorname{div} B^{\text{free}} = 0\,.$$

Then the following estimate holds for all $0 \le t \le T$ *and all* $0 \le \eta$:

$$\left\| E^\eta(t) - \sum_{i=0}^{k} E^k(t) \right\| \le \eta^{k+1} \Big(\left\| E_0^{\text{free}} - E^{k+1}(0) \right\| + \left\| B_0^{\text{free}} - B^{k+1}(0) \right\| + \left\| E^{k+1}(t) \right\| + 2 \int_0^t \left\| \partial_t E^{k+1}(s) \right\| + \left\| \partial_t B^{k+1}(s) \right\| \, ds \Big)$$

$$\left\| B^\eta(t) - \sum_{i=0}^{k} B^k(t) \right\| \le \eta^{k+1} \Big(\left\| E_0^{\text{free}} - E^{k+1}(0) \right\| + \left\| B_0^{\text{free}} - B^{k+1}(0) \right\| + \left\| B^{k+1}(t) \right\| + 2 \int_0^t \left\| \partial_t E^{k+1}(s) \right\| + \left\| \partial_t B^{k+1}(s) \right\| \, ds \Big)$$

Proof. Setting

$$e = E^\eta - \sum_{i=0}^{k+1} E^i \qquad b = B^\eta - \sum_{i=0}^{k+1} B^i$$

we only have to compute

$$\eta \partial_t e - \operatorname{curl} b = -\eta^{k+2} \partial_t E^{k+1} ,$$
$$\eta \partial_t b + \operatorname{curl} e = -\eta^{k+2} \partial_t B^{k+1} ,$$

apply Proposition 2 and are done. □

Remark 1. In contrast to [10], we allow for higher order contributions to the initial electromagnetic fields. For this reason, we can also admit for current densities varying fast in a neighborhood of $t = 0$, compare [10, (3.29)]. For a discussion of non-adapted initial electromagnetic fields, see [10, pp. 41–42].

5 Curl–div systems in bounded domains

First, we give a well-known and very simple solution theory of curl–div systems using L^2-decompositions. For an arbitrary domain $\Omega \subset \mathbb{R}^3$, let $\mathring{C}^\infty(\Omega)$ denote the space of smooth functions or fields with compact support. We denote the curl operator acting on test-fields by $\underline{\operatorname{curl}} : \mathring{C}^\infty(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$. Then $H(\operatorname{curl}; \Omega)$, as set, is the domain of the adjoint operator $\underline{\operatorname{curl}}^*$. In this section, we denote with curl the usual curl operator acting as unbounded operator in L^2, i. e., $\operatorname{curl} : H(\operatorname{curl}) \subset L^2(\Omega) \to L^2(\Omega)$. Finally, we define $\mathring{\operatorname{curl}}$ as the adjoint of curl and $H(\mathring{\operatorname{curl}}, \Omega)$ as domain of $\mathring{\operatorname{curl}}$. It is well known that for sufficiently smooth domains Ω, such that a tangential trace may be defined,

$H(\mathring{\mathrm{curl}},\Omega)$ is just the kernel of the tangential trace operator. Thus, $E \in H(\mathring{\mathrm{curl}},\Omega)$ is a generalization of the boundary condition $E \wedge \nu = 0$ on $\partial\Omega$. We immediately get the following orthogonal L^2-decomposition (the bar denotes closure in L^2):

$$L^2(\Omega;\mathbb{R}^3) = \overline{\mathring{\mathrm{curl}}\, H(\mathring{\mathrm{curl}},\Omega)} \oplus H_0(\mathrm{curl},\Omega) = \overline{\mathrm{curl}\, H(\mathrm{curl},\Omega)} \oplus H_0(\mathring{\mathrm{curl}},\Omega)\,,$$

where $E \in H_0(\mathrm{curl},\Omega)$ if and only if $E \in H(\mathrm{curl},\Omega)$ and $\mathrm{curl}\, E = 0$ and so on. In the same manner, we define $\underline{\mathrm{grad}}$, $\mathrm{div} := -\underline{\mathrm{grad}}^*$, $\mathring{\mathrm{grad}} := -\mathrm{div}^*$ and $\underline{\mathrm{div}}$, $\mathrm{grad} := -\underline{\mathrm{div}}^*$ and finally $\mathring{\mathrm{div}} := -\mathrm{grad}^*$ leading to the following decompositions:

$$L^2(\Omega;\mathbb{R}^3) = \overline{\mathring{\mathrm{grad}}\, H(\mathring{\mathrm{grad}},\Omega)} \oplus H_0(\mathrm{div},\Omega) = \overline{\mathrm{grad}\, H(\mathrm{grad},\Omega)} \oplus H_0(\mathring{\mathrm{div}},\Omega)\,,$$

$$L^2(\Omega) = \overline{\mathrm{div}\, H(\mathrm{div},\Omega)} = \overline{\mathring{\mathrm{div}}\, H(\mathring{\mathrm{div}},\Omega)} \oplus \mathrm{Lin}\{1\}\,.$$

Note that $E \in H(\mathring{\mathrm{div}},\Omega)$ is just the generalization of vanishing normal trace. If the underlying domain is bounded and sufficiently smooth (e. g., weakly Lipschitz is enough), then the embeddings

$$H(\mathring{\mathrm{curl}},\Omega) \cap H(\mathrm{div},\Omega) \hookrightarrow L^2(\Omega) \quad \text{and} \quad H(\mathrm{curl},\Omega) \cap H(\mathring{\mathrm{div}},\Omega) \hookrightarrow L^2(\Omega) \tag{11}$$

are compact (the spaces on the left-hand side are equipped with their graph norms); see [30, 29, 22, 32] and [13, 6] for results with mixed boundary conditions. If these embeddings are compact, it is easy to see that Poincaré-type estimates hold and the ranges are already closed. We define so-called Dirichlet and Neumann fields:

$$\mathcal{H}_{\mathrm{D}} = H_0(\mathring{\mathrm{curl}},\Omega) \cap H_0(\mathrm{div},\Omega) \quad \text{and} \quad \mathcal{H}_{\mathrm{N}} = H_0(\mathrm{curl},\Omega) \cap H_0(\mathring{\mathrm{div}},\Omega).$$

Using compactness, it is easy to see that both spaces are finite dimensional. Actually, also in rather non-smooth settings the dimensions of $\mathcal{H}_{\mathrm{D}}$ and $\mathcal{H}_{\mathrm{N}}$ are given by the first and second Betti number of the underlying domain; see [23]. Using compactness, the inclusions $\mathring{\mathrm{curl}}\, H(\mathring{\mathrm{curl}},\Omega) \subset H_0(\mathring{\mathrm{div}},\Omega)$ and $\mathring{\mathrm{grad}}\, H(\mathring{\mathrm{grad}},\Omega) \subset H_0(\mathring{\mathrm{grad}},\Omega)$ as well as the decompositions itself we get refined decompositions:

$$\begin{aligned}
L^2(\Omega;\mathbb{R}^3) &= H_0(\mathring{\mathrm{div}},\Omega) \oplus \mathrm{grad}\, H(\mathrm{grad},\Omega)\\
&= \mathring{\mathrm{curl}}\, H(\mathring{\mathrm{curl}},\Omega) \oplus \mathcal{H}_{\mathrm{N}} \oplus \mathrm{grad}\, H(\mathrm{grad},\Omega)\\
&= \mathring{\mathrm{curl}}\, \big(H(\mathring{\mathrm{curl}},\Omega) \cap H_0(\mathrm{div},\Omega)\big) \oplus \mathcal{H}_{\mathrm{N}} \oplus \mathrm{grad}\, H(\mathrm{grad},\Omega)\\
&= \mathring{\mathrm{curl}}\, H(\mathring{\mathrm{curl}},\Omega) \oplus H_0(\mathrm{curl},\Omega) \qquad (12)\\
L^2(\Omega;\mathbb{R}^3) &= \mathrm{curl}\, H(\mathrm{curl},\Omega) \oplus H_0(\mathring{\mathrm{curl}},\Omega)\\
&= \mathrm{curl}\, H(\mathrm{curl},\Omega) \oplus \mathcal{H}_{\mathrm{D}} \oplus \mathring{\mathrm{grad}}\, H(\mathring{\mathrm{grad}},\Omega)
\end{aligned}$$

$$\begin{aligned}
&= \operatorname{curl}(H(\operatorname{curl},\Omega)\cap H_0(\mathring{\operatorname{div}},\Omega)) \oplus \mathcal{H}_{\mathrm{D}} \oplus \mathring{\operatorname{grad}}\, H(\mathring{\operatorname{grad}},\Omega)\\
&= H_0(\operatorname{div},\Omega) \oplus \mathring{\operatorname{grad}}\, H(\mathring{\operatorname{grad}},\Omega)\\
L^2(\Omega) &= \operatorname{div}(H(\operatorname{div},\Omega)\cap H_0(\mathring{\operatorname{curl}},\Omega))\\
&= \mathring{\operatorname{div}}\,(H(\mathring{\operatorname{div}},\Omega)\cap H_0(\operatorname{curl},\Omega)) \oplus \operatorname{Span}\{1\}
\end{aligned}$$

From (12), one can easily conclude the following proposition (with boundary conditions in a generalized sense).

Proposition 3. *Let $\Omega \subset \mathbb{R}^3$ be a domain with compactness property* (11). *For $F \in L^2(\Omega;\mathbb{R}^3)$, $f \in L^2(\Omega)$ and $G \in L^2(\Omega;\mathbb{R}^3)$, $g \in L^2(\Omega)$ the curl–div systems*

$$\begin{cases} \operatorname{curl} E = F\\ \operatorname{div} E = f\\ E\wedge \nu = 0\\ E \perp \mathcal{H}_{\mathrm{D}} \end{cases} \quad \text{and} \quad \begin{cases} \operatorname{curl} B = G\\ \operatorname{div} B = g\\ B\cdot \nu = 0\\ B \perp \mathcal{H}_{\mathrm{N}} \end{cases} \tag{13}$$

are uniquely solvable in $H(\operatorname{curl};\Omega)\cap H(\operatorname{div},\Omega)$ if and only if $F \in H_0(\mathring{\operatorname{div}},\Omega)$, $F \perp \mathcal{H}_{\mathrm{N}}$ and $G \in H_0(\operatorname{div},\Omega)$, $G \perp \mathcal{H}_{\mathrm{D}}$, $\int_\Omega g\,dx = 0$, respectively. In these cases, the solution operators are continuous.

6 Well-posedness of the iteration scheme in bounded domains

For $F \in H_0(\mathring{\operatorname{div}},\Omega)$, $f \in L^2(\Omega)$ with $F \perp \mathcal{H}_{\mathrm{N}}$ and $G \in H_0(\operatorname{div},\Omega)$, $g \in L^2(\Omega)$ with $G \perp \mathcal{H}_{\mathrm{D}}$, $\int_\Omega g\,dx = 0$, we denote the solutions of the curl–div systems (13) according to Proposition 3 by $\Gamma_{\mathrm{D}}(F,f)$ and $\Gamma_{\mathrm{N}}(G,g)$, respectively. With $\Pi_{\mathcal{H}_{\mathrm{D}}}$ and $\Pi_{\mathcal{H}_{\mathrm{N}}}$, we denote the orthogonal projectors on $\mathcal{H}_{\mathrm{D}}$ and $\mathcal{H}_{\mathrm{N}}$, respectively. By straightforward computations using the decompositions (12), we get the following result on well-defined iterations in L^2.

Theorem 2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with compactness property* (11), *$k \in \mathbb{N}_0$, $T > 0$ and*

$$\begin{aligned}
\varrho^\eta &= \varrho^0 + \eta\varrho^1 + \cdots + \eta^{k+1}\varrho^{k+1},\\
j^\eta &= j^0 + \eta j^1 + \cdots + \eta^{k+1} j^{k+1},
\end{aligned}$$

such that ϱ^η and j^η satisfy (10) *for all $\eta \geq 0$.*

Then the iteration scheme (8), (9) has solutions

$$\begin{aligned}
E^l &\in C^1([0,T];L^2(\Omega;\mathbb{R}^3)) \cap C^0([0,T];H(\mathring{\operatorname{curl}},\operatorname{div},\Omega)) \quad \text{and}\\
B^l &\in C^1([0,T];L^2(\Omega;\mathbb{R}^3)) \cap C^0([0,T];H(\operatorname{curl},\operatorname{div},\Omega))
\end{aligned}$$

for $l = 0, \dots, k+1$ if and only if $j^0 \perp \mathcal{H}_{\mathrm{D}}$. In this case, all solutions are given by ($l = 1, \dots, k$)

$$\begin{aligned}
E^0(t) &= \Gamma_{\mathrm{D}}\big(0, \varrho^0(t)\big) + \varphi_0^0 - \int_0^t \Pi_{\mathcal{H}_{\mathrm{D}}} j^1(s)\, ds \\
B^0(t) &= \Gamma_{\mathrm{N}}\big(j^0(t), 0\big) + \gamma_0^0 \\
E^l(t) &= \Gamma_{\mathrm{D}}\big(-\partial_t B^{l-1}(t), \varrho^l(t)\big) + \varphi_0^l - \int_0^t \Pi_{\mathcal{H}_{\mathrm{D}}} j^{l+1}(s)\, ds \\
B^l(t) &= \Gamma_{\mathrm{N}}\big(j^l(t) + \partial_t E^{l-1}(t), 0\big) + \gamma_0^l \\
E^{k+1}(t) &= \Gamma_{\mathrm{D}}\big(-\partial_t B^k(t), \varrho^{k+1}(t)\big) + \varphi^{k+1}(t) \\
B^{k+1}(t) &= \Gamma_{\mathrm{N}}\big(j^{k+1}(t) + \partial_t E^k(t), 0\big) + \gamma_{k+1}(t)
\end{aligned}$$

with time-independent fields $\varphi_0^0, \varphi_0^l \in \mathcal{H}_{\mathrm{D}}$ and $\gamma_0^0, \gamma_0^l \in H_0(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega)$ and possibly time-depending fields

$$\varphi^{k+1}(t) \in C^1\big([0,T]; \mathcal{H}_{\mathrm{D}}\big) \quad \text{and} \quad \gamma^{k+1}(t) \in C^1\big([0,T]; H_0(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega)\big)$$

to be chosen freely.

Remark 2. Combining Theorem 1 and Theorem 2, we notice that in order to have the approximation property we need additional constraint equations on the initial electromagnetic fields: Decomposing E_0^η according to the seventh line of (12) in $E_0^\eta = E_0^{\eta,1} + E_0^{\eta,2} + E_0^{\eta,3}$ with

$$E_0^{\eta,1} \in \mathrm{curl}(H(\mathrm{curl}, \Omega)), \quad E_0^{\eta,2} \in \mathcal{H}_{\mathrm{D}}, \quad E_0^{\eta,3} \in \mathring{\mathrm{grad}}\, H(\mathring{\mathrm{grad}}, \Omega)$$

we see that $E_0^{\eta,3}$ is fixed by the usual constraint equation $\mathrm{div}\, E_0^\eta = \varrho^\eta(0)$, $E_0^{\eta,1}$ is determined up to relevant order by adaption to the asymptotic expansion and $E_0^{\eta,2}$ remains to be chosen freely.

With regard to the initial magnetic field, we consider a non-standard decomposition of B_0^η according to

$$L^2(\Omega; \mathbb{R}^3) = \mathring{\mathrm{curl}}\, (H(\mathring{\mathrm{curl}}, \Omega)) \oplus H_0(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega) \oplus \mathring{\mathrm{grad}}\, H(\mathring{\mathrm{grad}}, \Omega)$$

with

$$B_0^{\eta,1} \in \mathring{\mathrm{curl}}\, (H(\mathring{\mathrm{curl}}, \Omega)), \quad B_0^{\eta,2} \in H_0(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega), \quad B_0^{\eta,3} \in \mathring{\mathrm{grad}}\, H(\mathring{\mathrm{grad}}, \Omega)\,.$$

$B_0^{\eta,3}$ is fixed by the usual constraint equation $\mathrm{div}\, B_0^\eta = 0$, $B_0^{\eta,1}$ is determined up to relevant order by adaption to the asymptotic expansion and $B_0^{\eta,2}$ remains to be chosen freely.

7 Spherical vector calculus

We introduce the usual orthonormal basis vector fields of spherical coordinates $\mathbf{e}_r$, $\mathbf{e}_\vartheta$ and $\mathbf{e}_\phi$. A vector field E may be decomposed in its three parts:

$$E = E^r \mathbf{e}_r + E^\vartheta \mathbf{e}_\vartheta + E^\phi \mathbf{e}_\phi$$

and we call $E_r := E^r \mathbf{e}_r$ the radial part of E and $E_s := E^\vartheta \mathbf{e}_\vartheta + E^\phi \mathbf{e}_\phi$ the spherical part of E. Furthermore, we make use of the following operator notation: D is the derivative w. r. t. r and M is multiplication with r. The usual differential operators grad, curl and div have the following representation in spherical coordinates, where we use the usual convention for matrix multiplication and $\wedge$ is the usual vector product in $\mathbb{R}^3$, particularly $\mathbf{e}_r \wedge \mathbf{e}_\vartheta = \mathbf{e}_\phi$, $\mathbf{e}_\vartheta \wedge \mathbf{e}_\phi = \mathbf{e}_r$ and $\mathbf{e}_\phi \wedge \mathbf{e}_r = \mathbf{e}_\vartheta$:

$$\begin{aligned}
\operatorname{grad} u &= (\mathbf{e}_r, 1) \begin{pmatrix} \mathrm{D}u \\ \mathrm{M}^{-1} \operatorname{Grad} u \end{pmatrix} \\
\operatorname{curl} E &= (\mathbf{e}_r, 1) \begin{pmatrix} 0 & \mathrm{M}^{-1} \operatorname{Curl} \\ -\mathrm{M}^{-1} \mathbf{e}_r \wedge \operatorname{Grad} & \mathrm{M}^{-1} \mathrm{D} \mathrm{M} \mathbf{e}_r \wedge \end{pmatrix} \begin{pmatrix} E^r \\ E_s \end{pmatrix} \\
\operatorname{div} E &= (\mathrm{M}^{-2} \mathrm{D} \mathrm{M}^2,\ \mathrm{M}^{-1} \operatorname{Div}) \begin{pmatrix} E^r \\ E_s \end{pmatrix}
\end{aligned} \tag{14}$$

where the spherical differential operators Grad, Div and Curl in spherical coordinates are given by

$$\begin{aligned}
\operatorname{Grad} u &= (\mathbf{e}_\vartheta, \mathbf{e}_\phi) \begin{pmatrix} \partial_\vartheta u \\ (\sin \vartheta^{-1}) \partial_\phi u \end{pmatrix} \\
\operatorname{curl} E_s &= (\sin \vartheta)^{-1} (\partial_\vartheta (\sin \vartheta E^\phi) - \partial_\phi E^\vartheta) \\
\operatorname{Div} E_s &= (\sin \vartheta)^{-1} (\partial_\vartheta (\sin \vartheta E^\vartheta) + \partial_\phi E^\phi) ;
\end{aligned}$$

see, e. g., [12]. In this notation, the operator D does not act on the basis vector-fields $\mathbf{e}_r$, $\mathbf{e}_\vartheta$, $\mathbf{e}_\phi$ but only on the vector component functions, e. g., $\mathrm{D}(E^\phi \mathbf{e}_\phi) := (\partial_r E^\phi) \mathbf{e}_\phi$. Now we can represent the higher order differential operators (using, e. g., that Curl Grad $= 0$ and $\mathbf{e}_r \wedge (\mathbf{e}_r \wedge E_s) = -E_s$):

$$\begin{aligned}
\operatorname{curl} \operatorname{curl} E &= (\mathbf{e}_r, 1) \begin{pmatrix} -\mathrm{M}^{-2} \operatorname{Curl} \mathbf{e}_r \wedge \operatorname{Grad} & \mathrm{M}^{-2} \mathrm{D} \mathrm{M} \operatorname{Curl} \mathbf{e}_r \wedge \\ \mathrm{M}^{-1} \mathrm{D} \operatorname{Grad} & -\mathrm{M}^{-2} \mathbf{e}_r \wedge \operatorname{Grad} \operatorname{Curl} - \mathrm{M}^{-1} \mathrm{D}^2 \mathrm{M} \end{pmatrix} \begin{pmatrix} E^r \\ E_s \end{pmatrix} \\
\operatorname{grad} \operatorname{div} E &= (\mathbf{e}_r, 1) \begin{pmatrix} \mathrm{D} \mathrm{M}^{-2} \mathrm{D} \mathrm{M}^2 & \mathrm{D} \mathrm{M}^{-1} \operatorname{Div} \\ \mathrm{M}^{-3} \mathrm{D} \mathrm{M}^2 \operatorname{Grad} & \mathrm{M}^{-2} \operatorname{Grad} \operatorname{Div} \end{pmatrix} \begin{pmatrix} E^r \\ E_s \end{pmatrix}
\end{aligned}$$

For later purposes, we need some commutators with radial functions. Let $\psi = \psi(r)$ smooth. For a differential operator $\mathcal{D}$, we define the commutator with multiplication with ψ by $C_{\mathcal{D},\psi} = \mathcal{D}\psi - \psi\mathcal{D}$. Elementary computations yield

$$\begin{aligned} C_{\mathrm{grad},\psi} u &= \psi' u \mathbf{e}_r \\ C_{\mathrm{curl},\psi} E &= \psi' \mathbf{e}_r \wedge E_s \\ C_{\mathrm{div},\psi} E &= \psi' E^r . \end{aligned} \tag{15}$$

Combining these formulas and employing $\Delta = \operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div}$ for vector fields and $\Delta = \operatorname{div}\operatorname{grad}$ for functions, we find

$$C_{\Delta,\psi} E = (-\psi' \mathrm{M}^{-2}\mathrm{D}\mathrm{M}^2 - \psi'')E \tag{16}$$

8 Spherical harmonics and spherical vector harmonics

We introduce the well-known spherical harmonics Y_n^m, $n \in \mathbb{N}_0$, $-n \le m \le n$ as the complete set of L^2-orthonormal eigenfunctions of the scalar Laplace–Beltrami operator Div Grad on the unit sphere S^2; see, e. g., [9]. We have

$$\operatorname{Div}\operatorname{Grad} Y_n^m + n(n+1)Y_n^m = 0 \quad \text{and} \quad \int_{S^2} Y_n^m Y_{n'}^{m'} = \delta_{nn'}\delta_{mm'} , \tag{17}$$

where δ_{ij} is Kronecker's symbol. Then with

$$U_n^m := \frac{1}{\sqrt{n(n+1)}} \operatorname{Grad} Y_n^m \quad \text{and} \quad V_n^m := \mathbf{e}_r \wedge U_n^m \tag{18}$$

the set $\{U_n^m, V_l^k \ : \ n, l \in \mathbb{N}\,,\ -n \le m \le n\,,\ -l \le k \le l\}$ is a complete L^2-orthonormal set of eigenvector functions of the vector Laplace–Beltrami operator

$$\operatorname{Grad}\operatorname{Div} + \mathbf{e}_r \wedge \operatorname{Grad}\operatorname{Curl} \quad \text{on } L^2(S^2).$$

In particular, we have

$$\begin{aligned} &\operatorname{Curl} U_n^m = 0, \quad \operatorname{Div} U_n^m = -\sqrt{n(n+1)}Y_n^m, \qquad \operatorname{Grad}\operatorname{Div} U_n^m = -n(n+1)U_n^m, \\ &\operatorname{Div} V_n^m = 0, \quad \operatorname{Curl} V_n^m = -\sqrt{n(n+1)}Y_n^m, \quad \mathbf{e}_r \wedge \operatorname{Grad}\operatorname{Curl} V_n^m = -n(n+1)V_n^m . \end{aligned} \tag{19}$$

For every smooth vector field E on $\mathbb{R}^3$ or $\dot{\mathbb{R}}^3 = \mathbb{R}^3 \setminus \{0\}$, there are unique expansions

$$E = \sum_{n=0}^{\infty} \sum_{|m|\le n} y_{n,m} Y_n^m \mathbf{e}_r + \sum_{n=1}^{\infty} \sum_{|m|\le n} u_{n,m} U_n^m + \sum_{n=1}^{\infty} \sum_{|m|\le n} v_{n,m} V_n^m , \tag{20}$$

$$\Delta E = \sum_{n=0}^{\infty} \sum_{|m|\le n} \tilde{y}_{mn} Y_n^m \mathbf{e}_r + \sum_{n=1}^{\infty} \sum_{|m|\le n} \tilde{u}_{n,m} U_n^m + \sum_{n=1}^{\infty} \sum_{|m|\le n} \tilde{v}_{n,m} V_n^m , \tag{21}$$

where $y_{n,m}$, $\tilde{y}_{mn}$, $u_{n,m}$, $\tilde{u}_{n,m}$, $v_{n,m}$ and $\tilde{v}_{n,m}$ are smooth functions of r alone. Now let us assume that E is a smooth harmonic vector-field on $\mathbb{R}^3$ or $\dot{\mathbb{R}}^3$, i. e., $\Delta E = 0$. We expand E and ΔE according to (20) and (21). Using $\Delta E = 0$ and the convergence properties of the expansion, we get the following set of differential equations for $y_{n,m}$, $u_{n,m}$ and $v_{n,m}$: For $n = 0$, we have

$$0 = \tilde{y}_{0,0} = -\mathrm{DM}^{-2}\mathrm{DM}^2 y_{0,0} \tag{22}$$

For $n \in \mathbb{N}$, we use the abbreviation $\lambda_n = n(n+1)$ and compute

$$0 = \tilde{v}_{n,m} = (\mathrm{M}^{-2}\lambda_n - \mathrm{M}^{-1}\mathrm{D}^2\mathrm{M})v_{n,m} \tag{23}$$

$$0 = \tilde{y}_{n,m} = (\lambda_n \mathrm{M}^{-2} - \mathrm{DM}^{-2}\mathrm{DM}^2)y_{n,m} - 2\sqrt{\lambda_n}\mathrm{M}^{-2}u_{n,m} \tag{24}$$

$$0 = \tilde{u}_{n,m} = -2\sqrt{\lambda_n}\mathrm{M}^{-2}y_{n,m} + (\lambda_n \mathrm{M}^{-2} - \mathrm{M}^{-1}\mathrm{D}^2\mathrm{M})u_{n,m} \tag{25}$$

Equations (22) and (23) have each two fundamental solutions:

$$\begin{aligned} y_{0,0} &= r \quad \text{and} \quad y_{0,0} = r^{-2} \\ v_{n,m} &= r^n \quad \text{and} \quad v_{n,m} = r^{-n-1} \end{aligned}$$

The equations (24) and (25) are coupled and have four fundamental solutions:

$$\begin{aligned} u_{n,m} &= \sqrt{n}r^{-n-2} & y_{n,m} &= -\sqrt{n+1}r^{-n-2}, \\ u_{n,m} &= \sqrt{n+1}r^{n-1} & y_{n,m} &= \sqrt{n}r^{n-1}, \\ u_{n,m} &= \sqrt{n+1}r^{-n} & y_{n,m} &= \sqrt{n}r^{-n}, \\ u_{n,m} &= \sqrt{n}r^{n+1} & y_{n,m} &= -\sqrt{n+1}r^{n+1}. \end{aligned}$$

From these fundamental solutions, we obtain the well-known family of homogeneous harmonic functions in $\mathbb{R}^3$ and $\dot{\mathbb{R}}^3$, respectively: For $n \in \mathbb{N}_0$ and $-n \le m \le n$, we define

$$p_n^m := r^n Y_n^m, \qquad q_n^m := r^{-2n-1} p_n^m .$$

Now we create a suitable system of harmonic homogeneous vector fields. Two single homogeneous harmonics needs an extra place in this system:

$$P_{1,0}^1 := rY_0^0 \mathbf{e}_r \qquad Q_{1,0}^1 = r^{-2} Y_0^0 \mathbf{e}_r = r^{-3} P_{1,0}^1$$

In Cartesian coordinates, these fields are simply

$$P_{1,0}^1(x) = \frac{1}{4\pi} x \quad \text{and} \quad Q_{1,0}^1(x) = \frac{1}{4\pi} \frac{x}{|x|^3} .$$

For $n \in \mathbb{N}$ and $-n \le m \le n$, we define

$$\begin{aligned} P_{n,m}^2 &= r^n V_n^m \\ Q_{n,m}^2 &= r^n V_n^m = r^{-2n-1} P_{n,m}^2 \end{aligned}$$

$$\begin{aligned} P^3_{n+1,m} &= r^{n+1}(-\sqrt{n+1}Y^m_n\mathbf{e}_r + \sqrt{n}U^m_n) \\ Q^3_{n+1,m} &= r^{-n-2}(-\sqrt{n+1}Y^m_n\mathbf{e}_r + \sqrt{n}U^m_n) = r^{-2n-3}P^3_{n+1,m} \\ P^4_{n-1,m} &= r^{n-1}(\sqrt{n+1}Y^m_n\mathbf{e}_r + \sqrt{n}U^m_n) \\ Q^4_{n-1,m} &= r^{-n}(\sqrt{n+1}Y^m_n\mathbf{e}_r + \sqrt{n}U^m_n) = r^{-2n+1}P^4_{n-1,m} . \end{aligned} \tag{26}$$

Therefore, $p^{\cdot}_n$, $P^{\cdot}_{n,\cdot}$ are harmonic on $\mathbb{R}^3$ and homogeneous of degree n, while $q^{\cdot}_n$, $Q^{\cdot}_{n,\cdot}$ are harmonic on $\dot{\mathbb{R}}^3$ and homogeneous of degree $-n-1$. For notational convenience, for $\sigma \in \mathbb{N}_0$, $m \in \mathbb{Z}$ and $l = 1,\dots,4$ we set $p^m_\sigma = q^m_\sigma = 0$ and $P^l_{\sigma,m} = Q^l_{\sigma,m} = 0$ if still undefined. Every field E on $\mathbb{R}^3$ with $\Delta E = 0$ can be written as

$$E = \sum_{l=1}^{4}\sum_{n,m} p_{n,m,l}P^l_{n,m}$$

with $e_{n,m,l} \in \mathbb{C}$ while a field E on $\dot{\mathbb{R}}^3$ with ΔE can be written as

$$E = \sum_{l=1}^{4}\sum_{n,m} (p_{n,m,l}P^l_{n,m} + q_{n,m,l}Q^l_{n,m})$$

with $p_{n,m,l}, q_{n,m,l} \in \mathbb{C}$. By direct computations, the following equations can be verified:

$$\begin{aligned} &\operatorname{grad} p^0_0 = 0 && \operatorname{grad} p^m_n = \sqrt{n}P^4_{n-1,m}, \quad n \geq 1 \\ &\operatorname{div} P^1_{1,0} = 3p_{0,0} && \operatorname{curl} P^1_{1,0} = 0 \\ &\operatorname{div} P^2_{n,m} = 0 && \operatorname{curl} P^2_{n,m} = -\sqrt{n+1}P^4_{n-1,m} \\ &\operatorname{div} P^3_{n+1,m} = -2(n+3)\sqrt{n+1}p^m_n && \operatorname{curl} P^3_{n+1,m} = 2(n+3)\sqrt{n}P^2_{n,m} \\ &\operatorname{div} P^4_{n-1,m} = 0 && \operatorname{curl} P^4_{n-1,m} = 0 \end{aligned} \tag{27}$$

The analogue relations for the qs and Qs are

$$\begin{aligned} &\operatorname{grad} q^0_0 = -Q^1_{1,0} && \operatorname{grad} q^m_n = \sqrt{n+1}Q^3_{n+1,m}, \quad n \geq 1 \\ &\operatorname{div} Q^1_{1,0} = 0 && \operatorname{curl} Q^1_{1,0} = 0 \\ &\operatorname{div} Q^2_{n,m} = 0 && \operatorname{curl} Q^2_{n,m} = \sqrt{n}Q^3_{n+1,m} \\ &\operatorname{div} Q^3_{n+1,m} = 0 && \operatorname{curl} Q^3_{n+1,m} = 0 \\ &\operatorname{div} Q^4_{n-1,m} = (1-2n)\sqrt{n}q^m_n && \operatorname{curl} Q^4_{n-1,m} = (1-2n)\sqrt{n+1}Q^2_{n,m} \end{aligned} \tag{28}$$

For $\sigma \in \mathbb{N}_0$ and $l = 1,2,3,4$, we define

$$\begin{aligned} \mathcal{P}^l_\sigma &:= \operatorname{Lin}\{P^l_{\sigma,m} \ : m \in \mathbb{Z}\} \\ \mathcal{Q}^l_\sigma &:= \operatorname{Lin}\{Q^l_{\sigma,m} \ : m \in \mathbb{Z}\} \\ \mathbf{p}_\sigma &:= \operatorname{Lin}\{p_{\sigma,m} \ : \ m \in \mathbb{Z}\} \\ \mathbf{q}_\sigma &:= \operatorname{Lin}\{q_{\sigma,m} \ : \ m \in \mathbb{Z}\} \end{aligned}$$

and

$$\mathcal{P}^l = \bigcup_{\sigma\in\mathbb{N}_0} \mathcal{P}^l_\sigma, \quad \mathcal{Q}^l = \bigcup_{\sigma\in\mathbb{N}_0} \mathcal{Q}^l_\sigma, \quad \mathbf{p} = \bigcup_{\sigma\in\mathbb{N}_0} \mathbf{p}_\sigma, \quad \mathbf{q} = \bigcup_{\sigma\in\mathbb{N}_0} \mathbf{q}_\sigma$$

and

$$\mathcal{P}^l = \bigcup_{\sigma\in\mathbb{N}_0} \mathcal{P}^l_\sigma, \quad \mathcal{Q}^l = \bigcup_{\sigma\in\mathbb{N}_0} \mathcal{Q}^l_\sigma, \quad \mathbf{p} = \bigcup_{\sigma\in\mathbb{N}_0} \mathbf{p}_\sigma, \quad \mathbf{q} = \bigcup_{\sigma\in\mathbb{N}_0} \mathbf{q}_\sigma$$

Note that the spaces $\mathcal{P}^1_0$, $\mathcal{P}^2_0$, $\mathcal{P}^3_0$, $\mathcal{P}^3_1$ and $\mathcal{Q}^1_0$, $\mathcal{Q}^2_0$, $\mathcal{Q}^3_0$, $\mathcal{Q}^3_1$ and $\mathcal{P}^1_\sigma$, $\mathcal{Q}^1_\sigma$ for all $\sigma \geq 2$ are trivial. In the following diagram $\hookrightarrow$ indicates a bijection. Using (27) and (28), for $\sigma \geq 1$ we conclude the first four lines and for $\sigma = 0$ we find the four last lines:

$$\begin{array}{ccccccc}
\mathcal{P}^3_{\sigma+1} & \overset{\text{curl}}{\hookrightarrow} & \mathcal{P}^2_\sigma & \overset{\text{curl}}{\hookrightarrow} & \mathcal{P}^4_{\sigma-1} & \overset{\text{curl}}{\rightarrow} & \{0\} \\
\mathcal{P}^3_{\sigma+1} & \overset{\text{div}}{\hookrightarrow} & \mathbf{p}_\sigma & \overset{\text{grad}}{\hookrightarrow} & \mathcal{P}^4_{\sigma-1} & \overset{\text{div}}{\rightarrow} & \{0\} \\
\mathcal{Q}^4_{\sigma-1} & \overset{\text{curl}}{\hookrightarrow} & \mathcal{Q}^2_\sigma & \overset{\text{curl}}{\hookrightarrow} & \mathcal{Q}^3_{\sigma+1} & \overset{\text{curl}}{\rightarrow} & \{0\} \\
\mathcal{Q}^4_{\sigma-1} & \overset{\text{div}}{\hookrightarrow} & \mathbf{q}_\sigma & \overset{\text{grad}}{\hookrightarrow} & \mathcal{Q}^3_{\sigma+1} & \overset{\text{div}}{\rightarrow} & \{0\} \\
\mathcal{P}^1_1 & \overset{\text{curl}}{\rightarrow} & \{0\} & & & & \\
\mathcal{P}^1_1 & \overset{\text{div}}{\hookrightarrow} & \mathbf{p}_0 & \overset{\text{grad}}{\rightarrow} & \{0\} & & \\
 & & & & \mathcal{Q}^1_1 & \overset{\text{curl}}{\rightarrow} & \{0\} \\
 & & \mathbf{q}_0 & \overset{\text{grad}}{\hookrightarrow} & \mathcal{Q}^1_1 & \overset{\text{div}}{\rightarrow} & \{0\}
\end{array} \tag{29}$$

Therefore, the vector field $P^1_{1,0}$ is special: It is rotation-free, and thus a gradient, namely in Cartesian coordinates $P^1_{1,0}(x) = \operatorname{grad}(x^2/8\pi)$, but not a gradient of a harmonic function. On the other hand, $Q^1_{1,0}$ is special, as it has vanishing divergence in $\dot{\mathbb{R}}^3$, but is not a rotation of a harmonic vector field. Note that q^0_0 is just Green's function for the Laplace operator, and in Cartesian coordinates, we have $q^0_0(x) = \frac{1}{4\pi|x|}$.

9 Vector differential operators in weighted Sobolev spaces

We use the following radial weight function:

$$w(x) = \left(1 + |x|^2\right)^{1/2}$$

and for $s \in \mathbb{R}$ we define the following Hilbert spaces:

$$\begin{aligned}
L^2_s &:= L^2_s(\mathbb{R}^3) := \{f \in L^2_{\text{loc}}(\mathbb{R}^3) \,:\, w^s f \in L^2(\mathbb{R}^3)\} \\
L^2_s &:= L^2_s(\mathbb{R}^3;\mathbb{R}^3) := \{E \in L^2_{\text{loc}}(\mathbb{R}^3;\mathbb{R}^3) \,:\, w^s E \in L^2(\mathbb{R}^3;\mathbb{R}^3)\} \\
H^m_s &:= H^m_s(\mathbb{R}^3) := \{f \in H^m_{\text{loc}}(\mathbb{R}^3) \,:\, \partial^\alpha f \in L^2_{s+|\alpha|}(\mathbb{R}^3) \quad \text{for all} \quad |\alpha| \leq m\}
\end{aligned}$$

$$
\begin{aligned}
H_s^m &:= H_s^m(\mathbb{R}^3;\mathbb{R}^3) := \{E \in H_{\mathrm{loc}}^m(\mathbb{R}^3;\mathbb{R}^3) \,:\, \partial^\alpha E \in L^2_{s+|\alpha|}(\mathbb{R}^3) \quad \text{for all} \quad |\alpha| \le m\}\\
H_s(\mathrm{curl}) &:= \{E \in L^2_s \,:\, \mathrm{curl}\, E \in L^2_{s+1}\}\\
H_{s,0}(\mathrm{curl}) &:= \{E \in L^2_s \,:\, \mathrm{curl}\, E = 0\}\\
H_s(\mathrm{div}) &:= \{E \in L^2_s \,:\, \mathrm{div}\, E \in L^2_{s+1}\}\\
H_{s,0}(\mathrm{div}) &:= \{E \in L^2_s \,:\, \mathrm{div}\, E = 0\}
\end{aligned}
$$

together with natural scalar-products, e. g.,

$$\langle f, g\rangle_{L^2_s} = \int_{\mathbb{R}^3} w^{2s} f \bar{g} \,.$$

Note that the usual L^2-scalar-product

$$\langle f, g\rangle := \int_{\mathbb{R}^3} f\bar{g} \quad \text{and} \quad \langle F, G\rangle := \int_{\mathbb{R}^3} F \cdot \overline{G}$$

gives dualities between L^2_s and L^2_{-s}. We consider the following operators:

$$
\begin{aligned}
\mathrm{curl}_s \;:\; H_{s-1}(\mathrm{curl}) &\longrightarrow L^2_s\\
E &\longmapsto \mathrm{curl}\, E\\
\mathrm{div}_s \;:\; H_{s-1}(\mathrm{div}) &\longrightarrow L^2_s\\
E &\longmapsto \mathrm{div}\, E\\
\mathrm{grad}_s \;:\; H^1_{s-1} &\longrightarrow L^2_s\\
u &\longmapsto \mathrm{grad}\, u
\end{aligned}
\tag{30}
$$

Remark 3. Let $s \in \mathbb{R}$.

(i) By a standard approximation argument, we have

$$
\begin{aligned}
0 &= \langle E, \mathrm{curl}\, F\rangle - \langle \mathrm{curl}\, E, F\rangle \quad \text{for all} \quad E \in H_{s-1}(\mathrm{curl}), F \in H_{-s}(\mathrm{curl})\\
0 &= \langle f, \mathrm{div}\, E\rangle + \langle \mathrm{grad}\, f, E\rangle \quad \text{for all} \quad f \in H^1_{s-1}, E \in H_{-s}(\mathrm{div})
\end{aligned}
$$

(ii) For later reference, we note that for all $m \in \mathbb{N}$ the following three statements are equivalent:

$$
\begin{aligned}
&P^l_{n,m} \subset L^2_{-s},\\
&P^l_{n,m} \subset H_{-s}(\mathrm{curl}) \cap H_{-s}(\mathrm{div}) \cap H^m_{-s},\\
&n < -3/2 + s\,.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&p^m_n \subset L^2_{-s},\\
&p^m_n \subset H^m_{-s},\\
&n < -3/2 + s
\end{aligned}
$$

are equivalent.

The starting point is a result by McOwen about the Laplace operator in weighted L^p spaces, here specialized to L^2.

Lemma 1 (McOwen [19]). *Let* $J := \mathbb{R} \setminus \{1/2 + m \,:\, m \in \mathbb{Z}\}$.

$$\begin{array}{cccc} \Delta_s & : & H^2_{s-2} & \longrightarrow & L^2_s \\ & & u & \longmapsto & \Delta u \end{array}$$

defines a Fredholm operator if and only if $s \in J$. *In that case, its kernel and range are given by*

$$\ker(\Delta_s) = \bigcup_{n<1/2-s} \mathbf{p}_n$$

$$\operatorname{im}(\Delta_s) = \left\{ f \in L^2_s \,:\, \langle f, p \rangle = 0 \quad \textit{for all} \quad p \in \bigcup_{n<s-3/2} \mathbf{p}_n \right\}$$

If the operator $\Delta = \operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div}$ on vector fields is represented in Cartesian coordinates, it is just the scalar Δ-operator on the Cartesian coefficients. Therefore, this result is immediately generalized to vector fields.

Lemma 2.

$$\begin{array}{ccccc} \Delta_s & : & H^2_{s-2} & \longrightarrow & L^2_s \\ & & u & \longmapsto & (\operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div})E \end{array}$$

defines a Fredholm operator if and only if $s \in J$. *In that case, its kernel and range are given by*

$$\ker(\Delta_s) = \bigcup_{n<1/2-s} \bigcup_{l=1}^{4} \mathcal{P}^l_\sigma$$

$$\operatorname{im}(\Delta_s) = \left\{ F \in L^2_s \,:\, \langle F, P \rangle = 0 \quad \textit{for all} \quad P \in \bigcup_{n<s-3/2} \bigcup_{l=1}^{4} \mathcal{P}^l_\sigma \right\}$$

In particular, for $s \in J$ we have that Δ_s is injective for $s > -1/2$ and surjective for $s < 3/2$, and thus bijective in $(-1/2, 3/2) \cap J$. In the next step, we characterize the harmonic vector-fields by means of certain harmonics.

Lemma 3. *For* $s \in J$, *we have*

$$H_{s,0}(\operatorname{curl}) \cap H_{s,0}(\operatorname{div}) = \bigcup_{\sigma<-3/2-s} \mathcal{P}^4_\sigma \tag{31}$$

Proof. If $E \in \bigcup_{\sigma<-3/2-s} \mathcal{P}^4_\sigma$, then $E \in H_{s,0}(\operatorname{curl}) \cap H_{s,0}(\operatorname{div})$. On the other hand, if $E \in H_{s,0}(\operatorname{curl}) \cap H_{s,0}(\operatorname{div})$ then $E \in \ker(\Delta_{s+2})$; therefore, $E \in \bigcup_{\sigma<-3/2-s} \bigcup_{l=1}^4 \mathcal{P}^l_\sigma$ and we can write

$$E = \sum_{\sigma<-3/2-s} \sum_{l=1}^{4} P^l_\sigma .$$

Using diagram (29) and orthogonality on sphere, ω denoting the integration variable on S^2, we find

$$\int_{S^2} P^l_{n,m}(1,\omega)\overline{P^{l'}_{n',m'}}(1,\omega) = \delta_{nn'}\delta_{mm'}\delta_{l,l'} \quad \text{and} \quad \int_{S^2} p^m_n(1,\omega)\overline{p^{m'}_{n'}}(1,\omega) = \delta_{nn'}\delta_{mm'}\,,$$

and conclude $P^1_\sigma = P^2_\sigma = P^3_\sigma = 0$ for all $\sigma < -3/2 - s$. □

We introduce finite dimensional subspaces to complete $\operatorname{im}(\Delta_{s-2})$ in L^2_s. To this end, we define dual bases to the homogeneous harmonic functions and vector fields: We chose a smooth function ψ in $\mathbb{R}^3$ such that $0 \le \psi \le 1$, $\psi = 0$ on $B_1(0)$ and $\psi = 1$ on $\mathbb{R}^3 \setminus B_2(0)$. Then some elementary computations using the introduced calculus (see (16)), and recall $\mathrm{C}_{\Delta,\psi} = \Delta\psi - \psi\Delta$ as well as the orthonormality of $\{Y^m_n\}$ and $\{U^m_n, V^m_n\}$ on S^2:

$$\begin{aligned} \left\langle \mathrm{C}_{\Delta,\psi} q^m_n, p^{m'}_{n'} \right\rangle &= -3\delta_{nn'}\delta_{mm'} \\ \left\langle \mathrm{C}_{\Delta,\psi} Q^k_{n,m}, P^{l'}_{n',m'} \right\rangle &= \delta_{nn'}\delta_{mm'}\delta_{kl} c(n) \end{aligned} \tag{32}$$

with constants $c(n) \neq 0$. In the following, the notation $\bigoplus$ indicates an algebraical and topological direct sum. From (32) and Lemma 2, we conclude the next result.

Lemma 4. *For $s \in J$, we have in the vector case*

$$L^2_s = \operatorname{im}(\Delta_{s-2}) \bigoplus \mathrm{C}_{\Delta,\psi}\mathcal{Q}_{\sigma<s-3/2}\,,$$

where

$$\mathcal{Q}_{\sigma<s-3/2} := \bigoplus_{l=1}^{4} \bigoplus_{\sigma<s-3/2} \mathcal{Q}^l_\sigma\,,$$

and in the scalar case

$$L^2_s = \operatorname{im}(\Delta_{s-2}) \bigoplus \mathrm{C}_{\Delta,\psi}\mathbf{q}_{\sigma<s-3/2}\,,$$

where

$$\mathbf{q}_{\sigma<s-3/2} := \bigoplus_{\sigma<s-3/2} \mathbf{q}_\sigma\,.$$

Remark 4. We have

$$\mathrm{C}_{\Delta,\psi}\mathcal{Q}_{\sigma<s-3/2} = \bigoplus_{\sigma<s-3/2} \mathrm{C}_{\Delta,\psi}\mathcal{Q}_\sigma \quad \text{and} \quad \mathrm{C}_{\Delta,\psi}\mathcal{Q}_\sigma = \bigoplus_{l=1}^{4} \mathrm{C}_{\Delta,\psi}\mathcal{Q}^l_\sigma$$

and

$$\mathrm{C}_{\Delta,\psi}\mathcal{Q}_{\sigma<\varrho} \subset C^\infty_0(\dot{\mathbb{R}}^3)\,.$$

We define the annihilator of the $-s$ integrable harmonic polynomials:

$$\mathcal{A}_s^4 := \left\{F \in L_s^2 \,:\, \langle F, P\rangle = 0 \text{ for all } P \in \bigcup_{\sigma<s-3/2} \mathcal{P}_\sigma^4\right\}$$

and for $s > 3/2$

$$L_{s,0}^2 := \{f \in L_s^2 \,:\, \langle f, 1\rangle = 0\}$$

Note that $\operatorname{Lin}\{1\} = \mathbf{p}_{\sigma=0}$.

Lemma 5. *Let $s \in J$. Then*

$$\operatorname{div} H_{s-1}(\operatorname{div}) = L_s^2 \quad \textit{if} \quad s < 3/2 \tag{i}$$

$$\operatorname{div} H_{s-1}(\operatorname{div}) = L_{s,0}^2 \quad \textit{if} \quad s > 3/2 \tag{ii}$$

$$L_s^2 = L_{s,0}^2 \bigoplus \mathrm{C}_{\Delta,\psi}\mathbf{q}_{\sigma=0} \quad \textit{if}\; s > 3/2 \tag{iii}$$

$$\operatorname{curl} H_{s-1}(\operatorname{curl}) + \operatorname{grad} H_{s-1}^1 = \mathcal{A}_s^4 \tag{iv}$$

$$L_s^2 = \mathcal{A}_s^4 \bigoplus \mathrm{C}_{\Delta,\psi}\mathcal{Q}_{<s-3/2}^4 \tag{v}$$

Proof. Let $s \in J$. We start with the scalar case and equation (iii): The inclusion "$\supset$" is obvious. In order to show the inclusion "$\subset$" let $f \in L_s^2$. According to Lemma 4, we have

$$f = \operatorname{div}\operatorname{grad} u + \sum_{\sigma<3/2-s} \mathrm{C}_{\Delta,\psi} q_\sigma$$

with $u \in H_{s-2}^2$ and $q_\sigma \in \mathbf{q}_\sigma$. Now we have to show that for $\sigma > 0$ functions $\mathrm{C}_{\Delta,\psi} q_\sigma$ are divergences of some $H_{s-1}(\operatorname{div})$ vector-fields. A simple computation yields

$$\mathrm{C}_{\Delta,\psi} q_\sigma = \operatorname{div}(\mathrm{C}_{\operatorname{grad},\psi} q_\sigma) + \mathrm{C}_{\operatorname{div},\psi} \operatorname{grad} q_\sigma\,.$$

Clearly, the first term is in $\operatorname{div} H_{s-1}(\operatorname{div})$ as it is smooth and has compact support. Using (28), we have $\operatorname{grad} q_\sigma^m = \sqrt{\sigma+1} Q_{\sigma+1,m}^3$ for $\sigma \geq 1$. (For $\sigma = 0$, we recall $\operatorname{grad} q_0^0 = -Q_{1,0}^1$.) Employing (15), (19) and (26), we find

$$\mathrm{C}_{\operatorname{div},\psi} Q_{\sigma+1,m}^3 = -\sqrt{\sigma+1}\,\psi' r^{-\sigma-2} Y_\sigma^m = \operatorname{div}\left(\frac{1}{\sqrt{\sigma}}\psi' r^{-\sigma-1} U_\sigma^m\right),$$

being divergence of a smooth field with compact support. Therefore, equations (iii) and (i) are shown.

Since $\mathbf{p}_{\sigma=0} = \operatorname{Lin}\{1\}$ partial integration yields $\operatorname{div} H_{s-1}(\operatorname{div}) \subset L_{s,0}^2$ for $s > 3/2$ and according to (iii) the spaces $\operatorname{div} H_{s-1}(\operatorname{div})$ and $L_{s,0}^2$ have same co-dimension in L_s^2 and are thus the same, which is equation (ii).

Next, we turn to the vector case and equation (v). Again the inclusion "$\supset$" is obvious. For the other inclusion, let $F \in L_s^2$. According to Lemma 4, we may decompose F into

$$F = \operatorname{curl}\operatorname{curl} H - \operatorname{grad}\operatorname{div} H + \sum_{l=1}^{4} \sum_{\sigma<s-3/2} \mathrm{C}_{\Delta,\psi} Q_\sigma^l$$

with some $H \in H^2_{s-2}$ and $Q^l_\sigma \in \mathcal{Q}^l_\sigma$. We now show that for $l = 1, 2, 3$,

$$\mathrm{C}_{\Delta,\psi} Q^l_\sigma \in \mathcal{Z} := \operatorname{curl} C_0^\infty(\dot{\mathbb{R}}^3) + \operatorname{grad} C_0^\infty(\dot{\mathbb{R}}^3)\,.$$

A simple computation gives

$$\mathrm{C}_{\Delta,\psi} Q^l_\sigma = \operatorname{curl} \mathrm{C}_{\mathrm{curl},\psi} Q^l_\sigma + \mathrm{C}_{\mathrm{curl},\psi} \operatorname{curl} Q^l_\sigma - \operatorname{grad} \mathrm{C}_{\mathrm{div},\psi} Q^l_\sigma - \mathrm{C}_{\mathrm{grad},\psi} \operatorname{div} Q^l_\sigma\,. \tag{33}$$

The first and the third term are fine, being curl or gradient of a smooth compactly supported field or function, respectively. Now we consider the second term $\mathrm{C}_{\mathrm{curl},\psi} \operatorname{curl} Q^l_\sigma$. For $l = 1$ and $l = 3$, we have $\operatorname{curl} Q^l_\sigma = 0$; see (28). In the remaining case $l = 2$, we have using in this order (28), (26), (15), (18), (14) and (18):

$$\begin{aligned}
\mathrm{C}_{\mathrm{curl},\psi} \operatorname{curl} Q^2_{\sigma,m} &= \mathrm{C}_{\mathrm{curl},\psi} \sqrt{\sigma} Q^3_{\sigma+1,m} \\
&= \mathrm{C}_{\mathrm{curl},\psi} \sqrt{\sigma} r^{-\sigma-2}(-\sqrt{\sigma+1} Y^m_\sigma \mathbf{e}_r + \sqrt{\sigma} U^m_\sigma) \\
&= \psi' \mathbf{e}_r \wedge \sigma r^{-\sigma-2} U^m_\sigma \\
&= \psi' \sigma r^{-\sigma-2} V^m_\sigma \\
&= \operatorname{curl}\Big(-\sqrt{\frac{\sigma}{\sigma+1}} \psi' r^{-\sigma-1} Y^m_\sigma \mathbf{e}_r\Big)
\end{aligned}$$

being curl of a smooth compactly supported vector-field. Now we consider the fourth term $\mathrm{C}_{\mathrm{grad},\psi} \operatorname{div} Q^l_\sigma$ in (33). Since $\operatorname{div} Q^l_\sigma = 0$ for $l = 1, 2, 3$ (see (28)), here is nothing to do. All together equation (v) is shown. In order to prove equation (iv) using partial integration, we have $\operatorname{curl} H_{s-1}(\mathrm{curl}) + \operatorname{grad} H^1_{s-1} \subset \mathcal{A}^4_s$ and according to equation (v) both spaces have same co-dimension in L^2_s and are, therefore, the same. □

As an easy consequence of Lemma 5, we obtain decompositions of Helmholtz–Weyl type of weighted L^2-vector-spaces.

Theorem 3. *For $s \in J$, we have:*

(i) *for* $s < -3/2$

$$\begin{aligned}
L^2_s &= H_{s,0}(\mathrm{curl}) + H_{s,0}(\mathrm{div}) \\
\bigcup_{\sigma < -3/2-s} \mathcal{P}^4_\sigma &= H_{s,0}(\mathrm{curl}) \cap H_{s,0}(\mathrm{div})
\end{aligned}$$

(ii) *for* $-3/2 < s < 3/2$,

$$L^2_s = H_{s,0}(\mathrm{curl}) \oplus H_{s,0}(\mathrm{div})$$

(iii) *for* $s > 3/2$,

$$L^2_s = H_{s,0}(\mathrm{curl}) \oplus H_{s,0}(\mathrm{div}) \oplus \mathrm{C}_{\Delta,\psi} \mathcal{Q}^4_{<s-3/2}$$

Proof. Notice that $\operatorname{curl} H_{s-1}(\operatorname{curl}) \subset H_{s,0}(\operatorname{div})$ and $\operatorname{grad} H^1_{s-1} \subset H_{s,0}(\operatorname{curl})$. The second line of (i) is just a repetition of Lemma 3. Therefore, using Lemma 5, the only things which remain to prove are the inclusions $H_{s,0}(\operatorname{curl}) \subset \mathcal{A}^4_s$ and $H_{s,0}(\operatorname{div}) \subset \mathcal{A}^4_s$. These inclusions are a direct consequence of the first and second line in (29). □

Now we consider restrictions of the curl_s and div_s operators defined in (30) to spaces with vanishing divergence and rotation, respectively, which leads to Fredholm operators. In a slight abuse of notation, these restricted operators are called curl_s and div_s again.

Theorem 4. *Let $s \in J$.Then*
(i)

$$\begin{array}{rcccc} \operatorname{grad}_s & : & H^1_{s-1} & \longrightarrow & H_{s,0}(\operatorname{curl}) \\ & & u & \longmapsto & \operatorname{grad} u \end{array}$$

is a surjective Fredholm operator with

$$\ker(\operatorname{grad}_s) = \begin{cases} \{0\} & \text{for } s < 3/2 \\ \mathbf{p}_{\sigma=0} = \operatorname{Lin}\{1\} & \text{for } s > 3/2 \end{cases}.$$

(ii)

$$\begin{array}{rcccc} \operatorname{curl}_s & : & H_{s-1}(\operatorname{curl}) \cap H_{s-1,0}(\operatorname{div}) & \longrightarrow & L^2_s \cap H_{s,0}(\operatorname{div}) \\ & & E & \longmapsto & \operatorname{curl} E \end{array}$$

is a Fredholm operator with

$$\ker(\operatorname{curl}_s) = \bigcup_{\sigma < -1/2-s} \mathcal{P}^4_\sigma$$

and

$$\operatorname{im}(\operatorname{curl}_s) = \left\{ F \in H_{s,0}(\operatorname{div}) \ :\ \langle F, P \rangle = 0 \quad \text{for all} \quad P \in \bigcup_{\sigma < s-3/2} \mathcal{P}^2_\sigma \right\}$$

(iii)

$$\begin{array}{rcccc} \operatorname{div}_s & : & H_{s-1}(\operatorname{div}) \cap H_{s-1,0}(\operatorname{curl}) & \longrightarrow & L^2_s \\ & & E & \longmapsto & \operatorname{div} E \end{array}$$

is a Fredholm operator with

$$\ker(\operatorname{div}_s) = \bigcup_{\sigma < -1/2-s} \mathcal{P}^4_\sigma$$

and

$$\operatorname{im}(\operatorname{div}_s) = \left\{ f \in L^2_{s,0} \ :\ \langle f, p \rangle = 0 \quad \text{for all} \quad p \in \bigcup_{\sigma < s-3/2} \mathbf{p}_\sigma \right\}$$

We shall prove the second and third statement of Theorem 4 together with the following theorem, which states that we can enforce surjectivity by augmenting the operators curl_s and div_s in a suitable way.

Theorem 5. *Let $s \in J$ and $s > 3/2$. Then*
(i)

$$\begin{array}{rcccc} \widehat{\operatorname{curl}_s} & : & H_{s-1}(\operatorname{curl}) \cap H_{s-1,0}(\operatorname{div}) \oplus \mathrm{C}_{\Delta,\psi}\,\mathcal{Q}^4_{\sigma<s-5/2} & \longrightarrow & L^2_s \cap H_{s,0}(\operatorname{div}) \\ & & E & \longmapsto & \operatorname{curl} E \end{array}$$

is a topological isomorphism
(ii)

$$\begin{array}{rcccc} \widehat{\operatorname{div}_s} & : & H_{s-1}(\operatorname{div}) \cap H_{s-1,0}(\operatorname{curl}) \oplus \mathrm{C}_{\Delta,\psi}\,\mathcal{Q}^4_{\sigma<s-5/2} & \longrightarrow & L^2_s \\ & & E & \longmapsto & \operatorname{div} E \end{array}$$

is a topological isomorphism

Proof. First, we prove (i) of Theorem 4. The statement on the kernel is obvious. In order to prove surjectivity, let $F \in H_{s,0}(\operatorname{curl})$. According to Lemma 5 and Theorem 3, we have $F = \operatorname{curl} E + \operatorname{grad} u$ with some $E \in H_{s-1}(\operatorname{curl})$ and $u \in H^1_{s-1}$. Using $0 = \operatorname{curl} F = \operatorname{curl}\operatorname{curl} E + \operatorname{curl}\operatorname{grad} u$, we find $\operatorname{curl}\operatorname{curl} E = 0$; therefore, $\operatorname{curl} E \in H_{s,0}(\operatorname{curl}) \cap H_{s,0}(\operatorname{div})$. Employing Lemma 3 $\operatorname{curl} E$ can be represented by certain vector harmonics

$$\operatorname{curl} E = \sum_{\sigma<-3/2-s} P^4_\sigma \quad \text{with some } P^4_\sigma \in \mathcal{P}^4_\sigma .$$

Utilizing the second line of (29), there are $p_{\sigma+1} \in \mathbf{p}_{\sigma+1}$ such that $\operatorname{grad} p_{\sigma+1} = P^4_\sigma$. Since $\mathbf{p}_{\sigma+1} \subset H^1_{s-1}$ for $\sigma < -3/2 - s$, this yields $\operatorname{curl} E \in \operatorname{grad} H^1_{s-1}$. Hence, (i) is proved.

Now we proceed to Theorem 4 (ii) which we shall prove together with Theorem 5 (i). The statement on the kernel follows by Lemma 3. For the statement on the range, let $F \in H_{s,0}(\operatorname{div})$. We decompose F according to Lemma 5. Since $\langle F, P^4\rangle = 0$ for all $P^4 \in \bigcup_{\sigma<s-3/2} \mathcal{P}^4_\sigma$ we have $F = \operatorname{curl} E + \operatorname{grad} u$ with some $E \in H_{s-1}(\operatorname{curl})$ and $u \in H^1_{s-1}$. Using $\operatorname{div} F = 0$, we conclude $\operatorname{div}\operatorname{grad} u = 0$ and using Lemma 3 $\operatorname{grad} u \in \bigcup_{\sigma<s-3/2} \mathcal{P}^4_\sigma$. With the first line of (29), we have $\operatorname{grad} u = \operatorname{curl} P^2$ with some $P^2 \in \bigcup_{\sigma<s-5/2} \mathcal{P}^2_\sigma \subset L^2_{s-1}$. Summarizing, we have $F = \operatorname{curl} E$ with some $E \in H_{s-1}(\operatorname{curl})$. The next step is to show whether we can replace this potential E by some potential with vanishing divergence. To this end, we again decompose E according to Lemma 5 into

$$E = \operatorname{curl} E_1 + \sum_{\sigma<s-5/2} \mathrm{C}_{\Delta,\psi} Q^4_\sigma \quad \text{with } E_1 \in H_{s-2}(\operatorname{curl}) \text{ and } Q^4_\sigma \in \mathcal{Q}^4_\sigma .$$

(The gradient-term in the decomposition can be dropped since we are only interested in $\operatorname{curl} E$.) Clearly, $\operatorname{div}\operatorname{curl} E_1 = 0$ but $\operatorname{div} \sum_{\sigma<s-5/2} \mathrm{C}_{\Delta,\psi} Q^4_\sigma \neq 0$ in general. We shall look for some projectors that indicates the contribution to F coming from $\sum_{\sigma<s-5/2} \mathrm{C}_{\Delta,\psi} Q^4_\sigma$.

To this end, we define $P^4_\sigma = r^{2\sigma+1}Q^4_\sigma \in \mathcal{P}^4_\sigma$, $P^2_{\sigma+1} = \operatorname{curl}^{-1} P^4_\sigma \in \mathcal{P}^2_{\sigma+1}$ and $p_{\sigma+1} = \operatorname{grad}^{-1} P^4_\sigma$; see the first two lines of (29). For $\sigma < s - 3/2$, we have $P^2_{\sigma+1} \in L^2_{-s}$, $p_{\sigma+1} \in L^2_{-s}$. Let $\tilde{E} \in H_{s-1}(\mathrm{curl}) \cap H_{s-1,0}(\mathrm{div})$, we compute

$$\langle \operatorname{curl} \tilde{E}, P^2_{\sigma+1}\rangle = \langle \tilde{E}, P^4_\sigma\rangle = \langle \tilde{E}, \operatorname{grad} p_{\sigma+1}\rangle = -\langle \operatorname{div} \tilde{E}, p_{\sigma+1}\rangle = 0$$
$$\langle \operatorname{curl} \mathrm{C}_{\Delta,\psi} Q^4_\sigma, P^2_{\sigma+1}\rangle = \langle \mathrm{C}_{\Delta,\psi} Q^4_\sigma, P^4_\sigma\rangle \neq 0.$$

Therefore, Theorem 4(ii) and Theorem 5(i) are proven.

Now we turn to Theorem 4(iii) and Theorem 5(ii). Again the statement on the kernel follows with Lemma 3. For the statement on the range, let $f \in L^2_{s,0}$. Using Lemma 5, we have $f = \operatorname{div} E$ with some $E \in H_{s-1}(\mathrm{div})$. Using Lemma 5 again, we decompose E into

$$E = \operatorname{grad} h + \sum_{\sigma<s-5/2} \mathrm{C}_{\Delta,\psi} Q^4_\sigma \quad \text{with } \in H^1_{s-1} \text{ and } Q^4_\sigma \in \mathcal{Q}^4_\sigma$$

(here we can drop the rotation term) and define $p_{\sigma+1}$ as above. Then we have for all $\tilde{E} \in H_{s-1}(\mathrm{div}) \cap H_{s-1,0}(\mathrm{curl})$

$$\langle \operatorname{div} \tilde{E}, p_{\sigma+1}\rangle = -\langle \tilde{E}, P^4_\sigma\rangle = -\langle \tilde{E}, \operatorname{curl} P^2_{\sigma+1}\rangle = 0$$
$$\langle \operatorname{div} \mathrm{C}_{\Delta,\psi} Q^4_\sigma, p_{\sigma+1}\rangle = -\langle \mathrm{C}_{\Delta,\psi} P^4_\sigma, P^4_\sigma\rangle \neq 0.$$

Hence, everything is proved. □

10 Well-posedness of the iteration scheme in $\mathbb{R}^3$

Now we use the mapping properties of div and curl in weighted L^2-spaces to investigate the well-posedness of our iteration scheme defined in (8), (9). For sake of simplicity, we assume

$$\varrho^\eta = \varrho_0 \tag{34}$$
$$j^\eta = j^0 + \eta j^1 \tag{35}$$

with $\operatorname{div} j^0 = 0$ and $\partial_t \varrho_0 + \operatorname{div} j^1 = 0$, but the analysis can be easily generalized to the case of general asymptotic expansion like in (7). Furthermore, we shall assume that all sources are sufficiently smooth, say C^∞, and decay sufficiently fast at infinity, such that the sources together with all time derivatives are in L^2_s for sufficiently large s.

For convenience, we introduce some additional notation: For $\hat{s} \in \mathbb{R}$ and a function or a vector-field u, we say $u \in L^2_{<\hat{s}}$ if and only if $u \in L^2_s$ for all $s < \hat{s}$; we say $u \in L^2_{>\hat{s}}$ if and only if $u \in L^2_s$ for some $s > \hat{s}$. In addition for $t \in \mathbb{R}$ and some set $V \subset L^2_t$, let $V^\perp$ denote the annihilator of V in L^2_{-t} with regard to the $L^2_{-t} - L^2_t$ duality $\langle\cdot,\cdot\rangle$, i. e.,

$$V^\perp = \{w \in L^2_{-t} \mid \langle w, v\rangle = 0 \text{ for all } v \in V\}.$$

For $s \in \mathbb{R}$, we set

$$\mathcal{P}^2_{\sigma<s} := \bigcup_{\sigma<s} \mathcal{P}^2_\sigma \quad \text{and} \quad \mathbf{p}_{\sigma<s} := \bigcup_{\sigma<s} \mathbf{p}_\sigma \,.$$

10.1 Zeroth order: Electro and magneto statics

In this order, we have to solve the following curl–div systems:

$$\begin{aligned} \operatorname{curl} E^0 &= 0, \qquad & \operatorname{div} E^0 &= \varrho^0, \\ \operatorname{curl} B^0 &= j^0, \qquad & \operatorname{div} B^0 &= 0. \end{aligned} \tag{36}$$

Using Theorem 4, we have for $s \in J$:

$$\begin{aligned} E^0 \in L^2_{s-1} &\Leftrightarrow \varrho^0 \in L^2_s \quad \wedge \quad \varrho^0 \in (\mathbf{p}_{\sigma<s-3/2})^\perp \\ B^0 \in L^2_{s-1} &\Leftrightarrow j^0 \in H_{s,0}(\operatorname{div}) \quad \wedge \quad j^0 \in (\mathcal{P}^2_{\sigma<s-3/2})^\perp \end{aligned}$$

The solutions E^0 and B^0 are uniquely defined in $L^2_{>-3/2}$ by (36). Without any further assumptions on the source distributions, we have $E^0 \in L^2_{<1/2}$. If the monopole contribution $\langle \varrho^0, p_{0,0} \rangle$ vanishes, we conclude $E^0 \in L^2_{<3/2}$. Since $\mathcal{P}^2_{\sigma=0}$ is trivial, we have $B^0 \in L^2_{<3/2}$. Hence, the zeroth-order approximation is well-defined in L^2.

10.2 First order: eddy-current approximation

With (E^0, B^0), the unique solutions of (36) in $L^2_{<1/2} \times L^2_{<3/2}$ we have to solve the following curl–div systems:

$$\begin{aligned} \operatorname{curl} E^1 &= -\partial_t B^0, \qquad & \operatorname{div} E^1 &= 0, \\ \operatorname{curl} B^1 &= j_1 + \partial_t E^0, \qquad & \operatorname{div} B^1 &= 0\,. \end{aligned} \tag{37}$$

Using that $\partial_t B^0$ is the unique solution in $L^2_{<3/2}$ of the problem $\operatorname{curl} \partial_t B^0 = \partial_t j^0$, $\operatorname{div} \partial_t B^0 = 0$ we have

$$\begin{aligned} E^1 \in L^2_{s-1} &\Leftrightarrow \partial_t B^0 \in H_{s,0}(\operatorname{div}) \quad \wedge \quad \underbrace{\partial_t B^0 \in (\mathcal{P}^2_{\sigma<s-3/2})^\perp}_{=:(1)} \\ &\Leftrightarrow \partial_t j^0 \in L^2_{s+1} \quad \wedge \quad \partial_t j^0 \in (\mathcal{P}^2_{\sigma<s-1/2})^\perp \quad \wedge \quad (1) \end{aligned}$$

With regard to B^1, we notice that $\partial_t E^0 \in L^2_{<1/2}$ is the unique solution of $\operatorname{curl} \partial_t E^0 = 0$, $\operatorname{div} \partial_t E^0 = \partial_t \varrho^0$. Using $\partial_t \varrho^0 + \operatorname{div} j^1$, we compute

$$B^1 \in L_{s-1} \Leftrightarrow j^1 + \partial_t E^0 \in H_{s,0}(\mathrm{div}) \wedge \underbrace{j^1 + \partial_t E^0 \in (\mathcal{P}^2_{\sigma<s-3/2})^{\perp}}_{:=(2)}$$
$$\Leftrightarrow \partial_t \varrho^0 \in L^2_{s+1} \quad \text{and} \quad \partial_t \varrho^0 \in (\mathbf{p}_{\sigma<s-1/2})^{\perp} \ \wedge \ (2)$$

Due to our general assumption $\partial_t \varrho + \operatorname{div} j^1 = 0$, we have local, and thus global charge conservation, that is, $\langle \partial_t \varrho^0, p_{0,0} \rangle = 0$. Thus, without further assumptions on the sources $E^1 \in L^2_{<3/2}$ and $B^1 \in L^2_{<1/2}$ and the first-order approximation is well-defined in L^2.

10.3 Second order, Darwin approximation

With $(E^0, B^0) \in L^2_{<1/2} \times L^2_{<3/2}$ and $(E^1, B^1) \in L^2_{<3/2} \times L^2_{<1/2}$ the unique solutions of (36) and (37), respectively, we have to solve the following curl–div systems:

$$\begin{aligned} \operatorname{curl} E^2 &= -\partial_t B^1, & \operatorname{div} E^2 &= 0, \\ \operatorname{curl} B^2 &= \partial_t E^1, & \operatorname{div} B^2 &= 0. \end{aligned} \tag{38}$$

Utilizing that $\partial_t B^1 \in L^2_{<1/2}$ is the unique solution of $\operatorname{curl} \partial_t B^1 = \partial_t j^1 + \partial_t^2 E^0$, $\operatorname{div} \partial_t B^1 = 0$ and $\partial_t^2 E^0 \in L^2_{<1/2}$ is the unique solution of $\operatorname{curl} \partial_t^2 E^2 = 0$, $\operatorname{div} \partial_t^2 E^0 = \partial_t^2 \varrho^0$ we find for $s \in J$

$$\begin{aligned} E^2 \in L^2_{s-1} &\Leftrightarrow \partial_t B^1 \in \operatorname{im}(\operatorname{curl}_s) \\ &\Leftrightarrow \partial_t B^1 \in H_{s,0}(\mathrm{div}) \wedge \underbrace{\partial_t B^1 \in (\mathcal{P}^2_{\sigma<s-3/2})^{\perp}}_{(1)} \\ &\Leftrightarrow \partial_t j^1 + \partial_t^2 E^0 \in \operatorname{im}(\operatorname{curl}_{s+1}) \wedge (1) \\ &\Leftrightarrow \partial_t j^1 + \partial_t^2 E^0 \in H_{s+1,0}(\mathrm{div}) \wedge \underbrace{\partial_t j^1 + \partial_t^2 E^0 \in (\mathcal{P}^2_{\sigma<s-1/2})^{\perp}}_{(2)} \wedge (1) \\ &\Leftrightarrow \partial_t^2 \varrho^0 \in \operatorname{im}(\operatorname{div}_{s+2}) \wedge (2) \wedge (1) \\ &\Leftrightarrow \partial_t^2 \varrho^0 \in (\mathbf{p}_{\sigma<s+1/2})^{\perp} \wedge (2) \wedge (1) \end{aligned}$$

- Since, without further assumptions on the sources, only the second time derivative of the monopole moment has to vanish, i. e., $\langle \partial_t^2 \varrho^0, p_{0,0} \rangle = 0$, we can only conclude $E^2 \in L^2_{<-1/2}$.
- If the second time derivative of the dipole moment does not vanish, i. e., if there is any $i\{\pm 1, 0\}$ with $\langle \partial_t^2 \varrho^0, p_{1,i} \rangle \neq 0$, then $E^2 \notin L^2_{>-1/2}$.
- If the second time derivative of the dipole moment vanishes, we conclude $E^2 \in L^2_{<1/2}$, note that conditions (2) and (1) are void for $s < 3/2$.

Hence, $E^2 \in L^2_{<1/2}$, and thus in L^2, if and only if the second time derivative of the dipole moment of the charge distribution vanishes.

Now we turn to B^2: Using that $\partial_t E^1 \in L^2_{<3/2}$ is the unique solution of $\operatorname{curl} \partial_t E^1 = -\partial_t^2 B^0$, $\operatorname{div} \partial_t E^1 = 0$ and $\partial_t^2 B^0 \in L^2_{<3/2}$ is the unique solution of $\operatorname{curl} \partial_t^2 B^0 = \partial_t^2 j^0$, $\operatorname{div} \partial_t^2 B^0 = 0$ we conclude for $s \in J$:

$$\begin{aligned}
B^2 \in L^2_{s-1} &\Leftrightarrow \partial_t E^1 \in \operatorname{im}(\operatorname{curl}_s)\\
&\Leftrightarrow \partial_t E^1 \in H_{s,0}(\operatorname{div}) \wedge \underbrace{\partial_t E^1 \in (\mathcal{P}^2_{\sigma<s-3/2})^\perp}_{(3)}\\
&\Leftrightarrow \partial_t^2 B^0 \in \operatorname{im}(\operatorname{curl}_{s+1}) \wedge (3)\\
&\Leftrightarrow \partial_t^2 B^0 \in H_{s+1,0}(\operatorname{div}) \wedge \underbrace{\partial_t^2 B^0 \in (\mathcal{P}^2_{\sigma<s-1/2})^\perp}_{(4)} \wedge (3)\\
&\Leftrightarrow \partial_t^2 j^0 \in \operatorname{im}(\operatorname{curl}_{s+2}) \wedge (4) \wedge (3)\\
&\Leftrightarrow \partial_t^2 j^0 \in H_{s+2,0}(\operatorname{div}) \wedge \partial_t^2 j^0 \in (\mathcal{P}^2_{\sigma<s+1/2})^\perp \wedge (4) \wedge (3)
\end{aligned}$$

- Without further assumptions on the current distribution, we can only conclude $B^2 \in L^2_{<-1/2}$.
- If the second time derivative of the "first current moment" does not vanish, i. e., if there is any $i\{\pm 1, 0\}$ with $\langle \partial_t^2 j^0, P^2_{1,i}\rangle \neq 0$, then $B^2 \notin L^2_{>-1/2}$.
- If the second time derivative of the first current moment vanishes, we conclude $B^2 \in L^2_{<1/2}$, note that conditions (4) and (3) are void for $s < 3/2$.

Hence, $B^2 \in L^2_{<1/2}$, and thus in L^2, if and only if the second time derivative of the first current moment of the zeroth-order current distribution vanishes.

10.4 Third order, radiation order

With $(E^0, B^0) \in L^2_{<1/2} \times L^2_{<3/2}$, $(E^1, B^1) \in L^2_{3/2} \times L^2_{<1/2}$ and $(E^2, B^2) \in L^2_{<-1/2} \times L^2_{<1/2}$ the unique solutions of (36), (37) and (38), respectively, we have to solve the following curl–div systems:

$$\begin{aligned}
\operatorname{curl} E^3 &= -\partial_t B^2, \qquad & \operatorname{div} E^2 &= 0,\\
\operatorname{curl} B^3 &= \partial_t E^2, \qquad & \operatorname{div} B^3 &= 0.
\end{aligned} \tag{39}$$

In the same manner as in the previous orders, we find

$$\begin{aligned}
E^3 \in L^2_{s-1} &\Leftrightarrow \partial_t B^2 \in \operatorname{im}(\operatorname{curl}_s)\\
&\Leftrightarrow \partial_t B^2 \in H_{s,0}(\operatorname{div}) \wedge \underbrace{\partial_t B^2 \in (\mathcal{P}^2_{\sigma<s-3/2})^\perp}_{(1)}\\
&\Leftrightarrow \partial_t^2 E^1 \in \operatorname{im}(\operatorname{curl}_{s+1}) \wedge (1)\\
&\Leftrightarrow \partial_t^2 E^1 \in H_{s+1,0}(\operatorname{div}) \wedge \underbrace{\partial_t^2 E^1 \in (\mathcal{P}^2_{\sigma<s-1/2})^\perp}_{(2)} \wedge (1)
\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \partial_t^3 B^0 \in \operatorname{im}(\operatorname{curl}_{s+2}) \wedge (2) \wedge (1)\\
&\Leftrightarrow \partial_t^3 B^0 \in H_{s+2,0}(\operatorname{div}) \wedge \underbrace{\partial_t^3 B^0 \in (\mathcal{P}^2_{\sigma<s+1/2})^{\perp}}_{(3)} \wedge (2) \wedge (1)\\
&\Leftrightarrow \partial_t^3 j^0 \in \operatorname{im}(\operatorname{curl}_{s+3}) \wedge (3) \wedge (2) \wedge (1)\\
&\Leftrightarrow \partial_t^3 j^0 \in (\mathcal{P}^2_{\sigma<s+3/2})^{\perp} \wedge (3) \wedge (2) \wedge (1)
\end{aligned}$$

- In general, we only have $E^3 \in L^2_{<-3/2}$, and thus the solution E^3 is only determined by (39) modulo a time-dependent spatially constant vector.
- If, in addition to our general assumptions, $\partial_t^3 j^0 \in (\mathcal{P}^2_1)^{\perp}$, since conditions (3), (2), (1) are void for $s < 1/2$, we have $E^3 \in L^2_{<-1/2}$ and E^3 is unique.
- If, in addition to our general assumptions, $\partial_t^3 j^0 \in (\mathcal{P}^2_1 \cup \mathcal{P}^2_2)^{\perp}$ then $E^3 \in L^2_{<1/2}$. Note that in this case for $s < 3/2$ conditions (2) and (1) are void and for $1/2 < s < 3/2$ condition (3) is satisfied: Since now $\partial_t^3 j^0 \in \operatorname{im}(\operatorname{curl}_{s+3})$ we have $\partial_t^3 B^0 \in L^2_{<5/2}$ and have to show $\partial_t^3 B^0 \in (\mathcal{P}^2_1)^{\perp}$. We already know that $E^1 \in L^2_{<3/2}$, and thus $\partial_t^2 E^1 \in L^2_{<3/2}$. Because of $\operatorname{curl} \partial_t^2 E^1 = \partial_t^3 B^0 \in L^2_{<5/2}$ we conclude $\partial_t^2 E^1 \in H_{3/2}(\operatorname{curl})$ and using Remark 3 as well as (27) we compute for $i \in \{\pm 1, 0\}$

$$\begin{aligned}
\langle \partial_t^3 B^0, P^2_{1,i} \rangle &= \langle \operatorname{curl} \partial_t^2 E^1, P^2_{1,i} \rangle = \langle \partial_t^2 E^1, \operatorname{curl} P^2_{1,i} \rangle\\
&= -\sqrt{2} \langle \partial_t^2 E^1, P^4_{0,i} \rangle
\end{aligned}$$

Now we utilize $\partial_t^2 E^1 = \operatorname{curl} \partial_t B^2 \in L^2_{<3/2}$ and $\partial_t B^2 \in L^2_{<1/2}$, thus $\partial_t B^2 \in H_s(\operatorname{curl})$ for all $s < 1/2$ and we continue using Remark 3 and (27) again

$$\langle \partial_t^2 E^1, P^4_{0,i} \rangle = \langle \operatorname{curl} \partial_t B^2, P^4_{0,i} \rangle = \langle \operatorname{curl} \partial_t B^2, \operatorname{curl} P^4_{0,i} \rangle = 0\,.$$

Summarizing, we have $E^3 \in L^2$ if and only if $\partial_t^3 j^0 \in (\mathcal{P}^2_1 \cup \mathcal{P}^2_2)^{\perp}$.

Now we shall proceed with B^3:

$$\begin{aligned}
B^3 \in L^2_{s-1} &\Leftrightarrow \partial_t E^2 \operatorname{im}(\operatorname{curl}_s)\\
&\Leftrightarrow \partial_t E^2 \in H_{s,0}(\operatorname{div}) \wedge \underbrace{\partial_t E^2 \in (\mathcal{P}^2_{\sigma<s-3/2})^{\perp}}_{(4)}\\
&\Leftrightarrow \partial_t^2 B^1 \in \operatorname{im}(\operatorname{curl}_{s+1}) \wedge (4)\\
&\Leftrightarrow \partial_t^2 B^1 \in H_{s+1,0}(\operatorname{div}) \wedge \underbrace{\partial_t^2 B^1 \in (\mathcal{P}^2_{\sigma<s-1/2})^{\perp}}_{(5)} \wedge (4)\\
&\Leftrightarrow \partial_t^2 j^1 + \partial_t^3 E^0 \in \operatorname{im}(\operatorname{curl}_{s+2}) \wedge (5) \wedge (4)\\
&\Leftrightarrow \partial_t^3 E^0 \in L^2_{s+2} \wedge \underbrace{\partial_t^2 j^1 + \partial_t^3 E^0 \in (\mathcal{P}^2_{\sigma<s+1/2})^{\perp}}_{(6)} \wedge (5) \wedge (4)\\
&\Leftrightarrow \partial_t^3 \varrho^0 \in \operatorname{im}(\operatorname{curl}_{s+3}) \wedge (6) \wedge (5) \wedge (4)\\
&\Leftrightarrow \partial_t^3 \varrho^0 \in (\mathbf{p}_{\sigma<s+3/2})^{\perp} \wedge (6) \wedge (5) \wedge (4)
\end{aligned}$$

- If the third time derivate of the dipole moment of the charge distribution does not vanish, then we only have $B^3 \in L^2_{<-3/2}$, and thus the solution B^3 is only determined by (39) modulo a time-dependent spatially constant vector.
- If the third time derivate of the dipole moment vanishes, we have $B^3 \in L^2_{<-1/2}$, and thus B^3 is uniquely defined by (39), note that conditions (6), (5), (4) are void for $s < 1/2$.
- If the third time derivate of the dipole moment vanishes, we have $B^3 \in L^2_{<-1/2}$. If in addition, the third time derivative of the quadrupole moment vanishes, i. e., $\langle \partial_t^3 \varrho^0, p_{2,i} \rangle = 0$ for all $i \in \{\pm 2, \pm 1, 0\}$, then $B^3 \in L^2_{<1/2}$. Note that in this case for $1/2 < s < 3/2$ conditions (5) and(4) are void and condition (6) is satisfied: Since $\partial_t^2 j^1 + \partial_t^3 E^0 \in L^2_{<5/2}$ in this case, using Remark 3 and (27), we can compute for all $i \in \{\pm 1, 0\}$:

$$\begin{aligned} \langle \partial_t^2 j^1 + \partial_t^3 E^0, P^2_{1,i} \rangle &= -\langle \operatorname{curl} \partial_t^2 B^1, P^2_{1,i} \rangle = \langle \partial_t^2 B^1, \operatorname{curl} P^2_{1,i} \rangle \\ &= -\sqrt{2} \langle \partial_t^2 B^1, P^4_{0,i} \rangle = \sqrt{2} \langle \operatorname{curl} \partial_t E^2, P^4_{0,i} \rangle \\ &= -\sqrt{2} \langle \partial_t B^2, \operatorname{curl} P^4_{0,i} \rangle = 0 \,. \end{aligned}$$

Summarizing, we have $B^3 \in L^2$ if and only if the third time derivatives of the dipole and of the quadrupole moment vanish.

11 Discussion

In this contribution, we proved that the asymptotic expansion of electromagnetic fields up to order two (Darwin order) gives approximation fields in L^2 if and only if the first time derivative of the monopole moment and the second time derivative of the dipole moment of the charge distribution as well as the second time derivative of the first current moment $\langle j^0, P^2_{1,i} \rangle$, $i = -1, 0, 1$ of the zeroth-order contribution to the current density vanish.

The third-order approximation is in L^2 if and only if in addition the third time derivative of the quadrupole moment of the charge distribution and the third time derivative of the second current moment $\langle j^0, P^2_{2,i} \rangle$, $i = -2, \dots, 2$, of the zeroth-order contribution to the current distribution vanish. In all other cases, approximations of second and third order are only given in weighted L^2 spaces and the approximation property can only be expected with respect to weighted L^2 norms. In particular, without these assumptions on the sources, it is not possible to use initial values of the approximation fields as initial data for the Maxwell fields and to solve Maxwell's equations by usual L^2-theory at the same time.

The results obtained here are in good accordance with usual multipole expansion of radiation fields in physics textbooks (see, e. g., [12]) but are in contradiction to [18], where for the second-order approximation both L^2-fields and the approximation property are claimed without assumptions on the multipole contributions to the sources.

In [11], Darwin systems are studied for exterior domains of $\mathbb{R}^3$ with boundary (see also [17] for the two-dimensional case) proving well-posedness in L^2_{-1} (in our notation). We expect that these results can be sharpened by generalizing the approach of the paper at hand to exterior domains with boundaries using results from [20, 21] and their vector calculus equivalents.

Bibliography

[1] S. Bauer, Post-Newtonian dynamics at order 1.5 in the Vlasov-Maxwell system. https://arxiv.org/abs/math-ph/0602031ArXive-prints, February 2006.

[2] S. Bauer, A non-relativistic model of plasma physics containing a radiation reaction term, *Kinet. Relat. Models*, **11**(1) (2018), 25–42.

[3] S. Bauer and M. Kunze, The Darwin approximation of the relativistic Vlasov–Maxwell system, *Ann. Henri Poincaré*, **6**(2) (2005), 283–308.

[4] S. Bauer and M. Kunze, Radiative friction for charges interacting with the radiation field: classical many-particle systems, in *Analysis, Modeling and Simulation of Multiscale Problems*, pp. 531–551, Springer, Berlin, 2006.

[5] S. Bauer, M. Kunze, G. Rein, and A. D. Rendall, Multipole radiation in a collisionless gas coupled to electromagnetism or scalar gravitation, *Commun. Math. Phys.*, **266**(1) (2006), 267–288.

[6] S. Bauer, D. Pauly, and M. Schomburg, The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions, *SIAM J. Math. Anal.*, **48**(4) (2016), 2912–2943.

[7] P. Ciarlet Jr. and E. Sonnendrücker, A decomposition of the electromagnetic field—application to the Darwin model, *Math. Models Methods Appl. Sci.*, **7**(8) (1997), 1085–1120.

[8] P. Ciarlet Jr. and J. Zou, Finite element convergence for the Darwin model to Maxwell's equations, *RAIRO Modél. Math. Anal. Numér.*, **31**(2) (1997), 213–249.

[9] R. Courant and D. Hilbert, *Methods of Mathematical Physics. Vol. I*, Interscience Publishers, Inc., New York, N.Y., 1953.

[10] P. Degond and P. Raviart, An analysis of the Darwin model of approximation to Maxwell's equations, *Forum Math.*, **4** (1992), 13–44.

[11] N. Fang and L. A. Ying, Three-dimensional exterior problem of Darwin model and its numerical computation, *Math. Models Methods Appl. Sci.*, **18**(10) (2008), 1673–1701.

[12] J. D. Jackson, *Classical Electrodynamics*, 3rd edition, Wiley, New York, 1998.

[13] F. Jochmann, A compactness result for vector fields with divergence and curl in $L^q(\Omega)$ involving mixed boundary conditions, *Appl. Anal.*, **66**(1–2) (1997), 189–203.

[14] M. Kunze and A. D. Rendall, Simplified models of electromagnetic and gravitational radiation damping, *Class. Quantum Gravity*, **18**(17) (2001), 3573–3587.

[15] M. Kunze and A. D. Rendall, The Vlasov–Poisson system with radiation damping, *Ann. Henri Poincaré*, **2**(5) (2001), 857–886.

[16] M. Kunze and H. Spohn, Post-Coulombian dynamics at order c^{-3}, *J. Nonlinear Sci.*, **11**(5) (2001), 321–396.

[17] F. Li and L. A. Ying, Exterior problem of the Darwin model and its numerical computation, *M2AN Math. Model. Numer. Anal.*, **37**(3) (2003), 515–532.

[18] C. Liao and L. A. Ying, Exterior problem of magnetic field in Darwin model and its numerical solution, *Numer. Methods Partial Differ. Equ.*, **24**(2) (2008), 418–434.
[19] R. C. McOwen, The behavior of the Laplacian on weighted Sobolev spaces, *Commun. Pure Appl. Math.*, **32**(6) (1979), 783–795.
[20] D. Pauly, Generalized electro-magneto statics in nonsmooth exterior domains, *Analysis (Munich)*, **27**(4) (2007), 425–464.
[21] D. Pauly, Hodge–Helmholtz decompositions of weighted Sobolev spaces in irregular exterior domains with inhomogeneous and anisotropic media, *Math. Methods Appl. Sci.*, **31** (2008), 1509–1543.
[22] R. Picard, An elementary proof for a compact imbedding result in generalized electromagnetic theory, *Math. Z.*, **187**(2) (1984), 151–164.
[23] R. Picard, On the boundary value problems of electro- and magnetostatics, *Proc. R. Soc. Edinb. A*, **92**(1–2) (1982), 165–174.
[24] P. A. Raviart and E. Sonnendrücker, Approximate models for the Maxwell equations, in *International Symposium on Mathematical Modelling and Computational Methods Modelling 94 (Prague, 1994) J. Comput. Appl. Math.*, **63**(1–3) (1995), 69–81.
[25] P. A. Raviart and E. Sonnendrücker, A hierarchy of approximate models for the Maxwell equations, *Numer. Math.*, **73**(3) (1996), 329–372.
[26] J. Saranen, On generalized harmonic fields in domains with anisotropic nonhomogeneous media, *J. Math. Anal. Appl.*, **88**(1) (1982), 104–115.
[27] J. Saranen, On electric and magnetic problems for vector fields in anisotropic nonhomogeneous media, *J. Math. Anal. Appl.*, **91**(1) (1983), 254–275.
[28] J. Schaeffer, The classical limit of the relativistic Vlasov–Maxwell system, *Commun. Math. Phys.*, **104**(3) (1986), 403–421.
[29] C. Weber, A local compactness theorem for Maxwell's equations, *Math. Methods Appl. Sci.*, **2**(1) (1980), 12–25.
[30] N. Weck, Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries, *J. Math. Anal. Appl.*, **46** (1974), 410–437.
[31] N. Weck and K. J. Witsch, Generalized spherical harmonics and exterior differentiation in weighted Sobolev spaces, *Math. Methods Appl. Sci.*, **17**(13) (1994), 1017–1043.
[32] K. J. Witsch, A remark on a compactness result in electromagnetic theory, *Math. Methods Appl. Sci.*, **16**(2) (1993), 123–129.

Sebastian Bauer, Dirk Pauly, and Michael Schomburg

3 Weck's selection theorem: The Maxwell compactness property for bounded weak Lipschitz domains with mixed boundary conditions in arbitrary dimensions

Abstract: It is proved that the space of differential forms with weak exterior and co-derivative, is compactly embedded into the space of square integrable differential forms. Mixed boundary conditions on weak Lipschitz domains are considered. Furthermore, canonical applications such as Maxwell estimates, Helmholtz decompositions and a static solution theory are proved. As a side product and crucial tool for our proofs, we show the existence of regular potentials and regular decompositions as well.

Keywords: Maxwell compactness property, weak Lipschitz domain, Maxwell estimate, Helmholtz decomposition, electro-magneto statics, mixed boundary conditions, vector potentials

MSC 2010: 35A23, 35Q61

1 Introduction

The aim of this contribution is to prove a compact embedding, so called "Weck's selection theorem" or (generalized) Maxwell compactness property [28, 29, 24], of differential q-forms with weak exterior and co-derivative into the space of square integrable q-forms subject to mixed boundary conditions on bounded weak Lipschitz domains $\Omega \subset \mathbb{R}^N$, i. e.,

$$\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \hookrightarrow \mathsf{L}^{2,q}(\Omega)$$

is compact. The main result is given by Theorem 4.8. Here, $N \geq 2$ and $0 \leq q \leq N$ are natural numbers, the dimension of the domain Ω and the rank of the differential

Note: In memoriam of our dear friend and mentor, Karl-Josef (Charlie) Witsch (1948–2017).
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forms, respectively. This generalises the results from [1], where bounded weak Lipschitz domains in the classical setting of $\mathbb{R}^3$ were considered. In fact, the results from [1] can be recovered by setting $N = 3$ and $q = 1$ or $q = 2$.

Similar results for strong Lipschitz domains in three dimensions can be found in [11, 8]. For a historical overview of the mathematical treatment of Weck's selection theorem (Maxwell compactness property), see [1, 13, 25] and the literature cited therein. In particular, let us mention the important papers [28, 27, 24, 3, 32, 11, 25]. We emphasise that in [32] Witsch was able to go even beyond Lipschitz regularity (p-cusps). In [30], Weck applied Witsch's ideas to the theory of elasticity.

The central role of compact embeddings of this type can, for example, be seen in connection with Hilbert space complexes, where the compact embeddings immediately provide closed ranges, solution theories by continuous inverses, Friedrichs/Poincaré-type estimates, and access to Hodge–Helmholtz-type decompositions, Fredholm theory, div–curl-type lemmas, and a posteriori error estimation; see [21, 20, 22]. In exterior domains, where local versions of the compact embeddings hold, one obtains radiation solutions (scattering theory) with the help of Eidus' limiting absorption principle [5–7]; see [14–16, 18, 17, 19]. We elaborate on some of these applications in our Section 5.

Finally, we note that by the same arguments as in [24] our results extend to Riemannian manifolds.

2 Notation, preliminaries and outline of the proof

Let $\Omega \subset \mathbb{R}^N$ be a bounded weak Lipschitz domain. For a precise definition of weak Lipschitz domains, see Definitions 2.3 and 2.5. In short, Ω is an N-dimensional $\mathsf{C}^{0,1}$-submanifold of $\mathbb{R}^N$ with boundary, i. e., a manifold with Lipschitz atlas. Let $\Gamma := \partial\Omega$, which is itself an $(N-1)$-dimensional Lipschitz-manifold without boundary, consist of two relatively open subsets Γ_τ and Γ_ν such that $\overline{\Gamma}_\tau \cup \overline{\Gamma}_\nu = \Gamma$ and $\Gamma_\tau \cap \Gamma_\nu = \emptyset$. The separating set $\overline{\Gamma}_\tau \cap \overline{\Gamma}_\nu$ (interface) will be assumed to be a, not necessarily connected, $(N-2)$-dimensional Lipschitz-submanifold of Γ. We shall call (Ω, Γ_τ) a weak Lipschitz pair.

We will be working in the framework of alternating differential forms; see, for example, [10]. The vector space $\mathring{\mathsf{C}}^{\infty,q}(\Omega)$ is defined as the subset of $\mathsf{C}^{\infty,q}(\Omega)$, the set of smooth alternating differential forms of rank q, having compact support in Ω. Together with the inner product,

$$\langle E, H\rangle_{\mathsf{L}^{2,q}(\Omega)} := \int_\Omega E \wedge \star H$$

it is an inner product space.[1] We may then define $\mathsf{L}^{2,q}(\Omega)$ as the completion of $\mathring{\mathsf{C}}^{\infty,q}(\Omega)$ with respect to the corresponding norm. $\mathsf{L}^{2,q}(\Omega)$ can be identified with those q-forms having L^2-coefficients with respect to any coordinate system. Using the weak version of Stokes' theorem,

$$\langle \mathrm{d}\, E, H\rangle_{\mathsf{L}^{2,q+1}(\Omega)} = -\langle E, \delta\, H\rangle_{\mathsf{L}^{2,q}(\Omega)}, \qquad E \in \mathring{\mathsf{C}}^{\infty,q}(\Omega),\ H \in \mathring{\mathsf{C}}^{\infty,q+1}(\Omega), \tag{1}$$

weak versions of the exterior derivative and co-derivative can be defined. Here, d is the exterior derivative, $\delta = (-1)^{N(q-1)} \star \mathrm{d} \star$ the co-derivative and $\star$ the Hodge-star-operator on Ω. We thus introduce the Sobolev (Hilbert) spaces (equipped with their natural graph norms)

$$\mathsf{D}^q(\Omega) := \{E \in \mathsf{L}^{2,q}(\Omega) : \mathrm{d}\, E \in \mathsf{L}^{2,q+1}(\Omega)\}, \quad \Delta^q(\Omega) := \{E \in \mathsf{L}^{2,q}(\Omega) : \delta\, E \in \mathsf{L}^{2,q-1}(\Omega)\}$$

in the distributional sense. It holds

$$\star\mathsf{D}^q(\Omega) = \Delta^{N-q}(\Omega), \qquad \star\Delta^q(\Omega) = \mathsf{D}^{N-q}(\Omega).$$

We further define the test forms

$$\mathring{\mathsf{C}}^{\infty,q}_{\Gamma_\tau}(\Omega) := \{\varphi|_\Omega \ : \ \varphi \in \mathring{\mathsf{C}}^{\infty,q}(\mathbb{R}^N),\ \operatorname{dist}(\operatorname{supp}\varphi, \Gamma_\tau) > 0\}$$

and note that $\mathring{\mathsf{C}}^{\infty,q}_{\emptyset}(\Omega) = \mathsf{C}^{\infty,q}(\overline{\Omega})$. We now take care of boundary conditions. First, we introduce strong boundary conditions as closures of test forms by

$$\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) := \overline{\mathring{\mathsf{C}}^{\infty,q}_{\Gamma_\tau}(\Omega)}^{\mathsf{D}^q(\Omega)}, \quad \mathring{\Delta}^q_{\Gamma_\nu}(\Omega) := \overline{\mathring{\mathsf{C}}^{\infty,q}_{\Gamma_\nu}(\Omega)}^{\Delta^q(\Omega)}. \tag{2}$$

For the full boundary case $\Gamma_\tau = \Gamma$ (resp., $\Gamma_\nu = \Gamma$), we set

$$\mathring{\mathsf{D}}^q(\Omega) := \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega), \quad \mathring{\Delta}^q(\Omega) := \mathring{\Delta}^q_{\Gamma_\nu}(\Omega).$$

Furthermore, we define weak boundary conditions in the spaces

$$\begin{aligned} \mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega) &:= \{E \in \mathsf{D}^q(\Omega) \ : \ \langle E, \delta\,\varphi\rangle_{\mathsf{L}^{2,q}(\Omega)} = -\langle \mathrm{d}\, E, \varphi\rangle_{\mathsf{L}^{2,q+1}(\Omega)} \text{ for all } \varphi \in \mathring{\mathsf{C}}^{\infty,q+1}_{\Gamma_\nu}(\Omega)\}, \\ \mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Omega) &:= \{H \in \Delta^q(\Omega) \ : \ \langle H, \mathrm{d}\,\varphi\rangle_{\mathsf{L}^{2,q}(\Omega)} = -\langle \delta\, H, \varphi\rangle_{\mathsf{L}^{2,q-1}(\Omega)} \text{ for all } \varphi \in \mathring{\mathsf{C}}^{\infty,q-1}_{\Gamma_\tau}(\Omega)\}, \end{aligned} \tag{3}$$

and again for $\Gamma_\tau = \Gamma$ (resp., $\Gamma_\nu = \Gamma$), we set

$$\mathring{\mathbf{D}}^q(\Omega) := \mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega), \quad \mathring{\mathbf{\Delta}}^q(\Omega) := \mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Omega).$$

We note that in Definitions (1) and (2), the smooth test forms can by mollification be replaced by their respective Lipschitz continuous counterparts, e. g., $\mathring{\mathsf{C}}^{\infty,q}_{\Gamma_\tau}(\Omega)$ can be

1 For simplicity, we work in a real Hilbert space setting.

replaced by $\mathring{\mathsf{C}}^{0,1,q}_{\Gamma_\tau}(\Omega)$. Similarly, in Definition (3) the smooth test forms can by completion be replaced by their respective closures, i. e., $\mathring{\mathsf{C}}^{\infty,q+1}_{\Gamma_\nu}(\Omega)$ and $\mathring{\mathsf{C}}^{\infty,q-1}_{\Gamma_\tau}(\Omega)$ can be replaced by $\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)$ and $\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)$, respectively. In (2) and (3), homogeneous tangential and normal traces on Γ_τ, respectively Γ_ν, are generalised. Clearly,

$$\mathring{\mathsf{D}}^{q}_{\Gamma_\tau}(\Omega) \subset \mathring{\mathbf{D}}^{q}_{\Gamma_\tau}(\Omega), \quad \mathring{\Delta}^{q}_{\Gamma_\nu}(\Omega) \subset \mathring{\boldsymbol{\Delta}}^{q}_{\Gamma_\nu}(\Omega)$$

and it will later be shown that in fact equality holds under our regularity assumptions on the boundary. In case of full boundary conditions, the equality even holds without any assumptions on the regularity of the boundary, as can be seen by a short functional analytic argument (see [1]) but which is unavailable for the mixed boundary case.

We define the closed subspaces

$$\mathsf{D}^q_0(\Omega) := \{E \in \mathsf{D}^q(\Omega) \,:\, \mathrm{d}\,E = 0\}, \quad \Delta^q_0(\Omega) := \{E \in \Delta^q(\Omega) \,:\, \delta\,E = 0\}$$

as well as $\mathring{\mathsf{D}}^{q}_{\Gamma_\tau,0}(\Omega) := \mathring{\mathsf{D}}^{q}_{\Gamma_\tau}(\Omega) \cap \mathsf{D}^q_0(\Omega)$ and $\mathring{\Delta}^{q}_{\Gamma_\nu,0}(\Omega) := \mathring{\Delta}^{q}_{\Gamma_\nu}(\Omega) \cap \Delta^q_0(\Omega)$. Analogously, for the weak spaces,

$$\mathring{\mathbf{D}}^{q}_{\Gamma_\tau,0}(\Omega) := \mathring{\mathbf{D}}^{q}_{\Gamma_\tau}(\Omega) \cap \mathsf{D}^q_0(\Omega), \quad \mathring{\boldsymbol{\Delta}}^{q}_{\Gamma_\nu,0}(\Omega) := \mathring{\boldsymbol{\Delta}}^{q}_{\Gamma_\nu}(\Omega) \cap \Delta^q_0(\Omega).$$

In addition to the latter canonical Sobolev spaces, we will also need the classical Sobolev spaces for the Euclidean components of q-forms. Note that Ω, together with the global identity chart, is a N-dimensional Riemannian manifold. In particular, q-forms $E \in \mathsf{L}^{2,q}(\Omega)$ can be represented globally in Cartesian coordinates by their components E_I, i. e., $E = \sum_I E_I \mathrm{d}x^I$. Here, we use the ordered multi-index notation $\mathrm{d}x^I = \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_q}$ for $I = (i_1, \dots, i_q) \in \{1, \dots, N\}^q$. The inner product for $E, H \in \mathsf{L}^{2,q}(\Omega)$ is given by

$$\langle E, H\rangle_{\mathsf{L}^{2,q}(\Omega)} = \int_\Omega E \wedge \star H = \sum_I \int_\Omega E_I H_I = \sum_I \langle E_I, H_I\rangle_{\mathsf{L}^2(\Omega)} = \langle \vec{E}, \vec{H}\rangle_{\mathsf{L}^2(\Omega)},$$

where we introduce the vector proxy notation:

$$\vec{E} = [E_I]_I \in \mathsf{L}^2(\Omega; \mathbb{R}^{N_q}), \quad N_q := \binom{N}{q}.$$

For $k \in \mathbb{N}$, we can now define the Sobolev space $\mathsf{H}^{k,q}(\Omega)$ as the subset of $\mathsf{L}^{2,q}(\Omega)$ having each component E_I in $\mathsf{H}^k(\Omega)$. In these cases, we have for $|\alpha| \le k$,

$$\partial^\alpha E = \sum_I \partial^\alpha E_I \mathrm{d}x^I \quad \text{and} \quad \langle E, H\rangle_{\mathsf{H}^{k,q}(\Omega)} := \sum_{0 \le |\alpha| \le k} \langle \partial^\alpha E, \partial^\alpha H\rangle_{\mathsf{L}^{2,q}(\Omega)}$$

and we use the vector proxy notation also for the gradient, i. e.,

$$\nabla \vec{E} = [\partial_n E_I]_{n,I} = [\dots \nabla E_I \dots]_I \in \mathsf{L}^2(\Omega; \mathbb{R}^{N \times N_q}).$$

In particular, for $E, H \in \mathsf{H}^{1,q}(\Omega)$,

$$\begin{aligned}\langle E, H\rangle_{\mathsf{H}^{1,q}(\Omega)} &= \langle E, H\rangle_{\mathsf{L}^{2,q}(\Omega)} + \sum_{n=1}^{N}\langle \partial_n E, \partial_n H\rangle_{\mathsf{L}^{2,q}(\Omega)} = \sum_I \Big(\int_\Omega E_I H_I + \sum_n \int_\Omega \partial_n E_I \partial_n H_I\Big)\\ &= \sum_I \big(\langle E_I, H_I\rangle_{\mathsf{L}^2(\Omega)} + \langle \nabla E_I, \nabla H_I\rangle_{\mathsf{L}^2(\Omega)}\big)\\ &= \langle \vec{E}, \vec{H}\rangle_{\mathsf{L}^2(\Omega)} + \langle \nabla\vec{E}, \nabla\vec{H}\rangle_{\mathsf{L}^2(\Omega)} = \langle \vec{E}, \vec{H}\rangle_{\mathsf{H}^1(\Omega)}.\end{aligned}$$

Boundary conditions for $\mathsf{H}^{1,q}(\Omega)$-forms can again be defined strongly and weakly, i. e., by closure

$$\mathring{\mathsf{H}}^{1,q}_{\Gamma_\tau}(\Omega) := \overline{\mathring{\mathsf{C}}^{\infty,q}_{\Gamma_\tau}(\Omega)}^{\mathsf{H}^{1,q}(\Omega)}$$

and by integration by parts

$$\mathbf{\mathring{H}}^{1,q}_{\Gamma_\tau}(\Omega) := \{E \in \mathsf{H}^{1,q}(\Omega) \,:\, \langle E_I, \partial_n \phi\rangle_{\mathsf{L}^2(\Omega)} = -\langle \partial_n E_I, \phi\rangle_{\mathsf{L}^2(\Omega)} \text{ for all } n, I \text{ and all } \phi \in \mathring{\mathsf{C}}^{\infty}_{\Gamma_\nu}(\Omega)\},$$

respectively. Let us also introduce the following Sobolev type spaces:

$$\begin{aligned}\mathsf{D}^{k,q}(\Omega) :&= \{E \in \mathsf{H}^{k,q}(\Omega) \,:\, \mathrm{d}\,E \in \mathsf{H}^{k,q+1}(\Omega)\},\\ \Delta^{k,q}(\Omega) :&= \{E \in \mathsf{H}^{k,q}(\Omega) \,:\, \delta\,E \in \mathsf{H}^{k,q-1}(\Omega)\}.\end{aligned}$$

Remark 2.1. We emphasise that by switching Γ_τ and Γ_ν we can define the respective boundary conditions on the other part of the boundary as well. Moreover, all definitions of our spaces extend literally to any open subset $\Omega \subset \mathbb{R}^N$ and any relatively open complementary boundary pairs Γ_τ and Γ_ν.

Finally, we introduce our transformations ε.

Definition 2.2. A transformation $\varepsilon : \mathsf{L}^{2,q}(\Omega) \to \mathsf{L}^{2,q}(\Omega)$ will be called admissible, if ε is bounded, symmetric, and uniformly positive definite. More precisely, ε is a self-adjoint operator on $\mathsf{L}^{2,q}(\Omega)$ and there exists $\underline{\varepsilon}, \overline{\varepsilon} > 0$ such that for all $E \in \mathsf{L}^{2,q}(\Omega)$

$$\underline{\varepsilon}|\varepsilon E|_{\mathsf{L}^{2,q}(\Omega)} \le |E|_{\mathsf{L}^{2,q}(\Omega)} \le \overline{\varepsilon}\sqrt{\langle \varepsilon E, E\rangle_{\mathsf{L}^{2,q}(\Omega)}}.$$

2.1 Lipschitz domains

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary $\Gamma := \partial\,\Omega$. We introduce the setting we will be working in. Define (cf. Figure 3.2)

$$\begin{aligned}I := (-1,1), \quad B := I^N \subset \mathbb{R}^N, \qquad B_\pm &:= \{x \in B \,:\, \pm x_N > 0\}, \qquad B_0 := \{x \in B \,:\, x_N = 0\},\\ B_{0,\pm} &:= \{x \in B_0 \,:\, \pm x_1 > 0\}, \quad B_{0,0} := \{x \in B_0 \,:\, x_1 = 0\}.\end{aligned}$$

Definition 2.3 (Weak Lipschitz domain). Ω is called weak Lipschitz, if the boundary Γ is a Lipschitz submanifold of the manifold $\overline{\Omega}$, i. e., there exist a finite open covering $U_1, \dots, U_K \subset \mathbb{R}^N$ of Γ and vector fields $\phi_k : U_k \to B$, such that for $k = 1, \dots, K$

(i) $\phi_k \in \mathsf{C}^{0,1}(U_k, B)$ is bijective and $\psi_k := \phi_k^{-1} \in \mathsf{C}^{0,1}(B, U_k)$;

(ii) $\phi_k(U_k \cap \Omega) = B_-$

hold.

Remark 2.4. For $k = 1, \dots, K$, we have $\phi_k(U_k \setminus \overline{\Omega}) = B_+$ and $\phi_k(U_k \cap \Gamma) = B_0$.

Definition 2.5 (Weak Lipschitz domain and weak Lipschitz interface). Let Ω be weak Lipschitz. A relatively open subset Γ_τ of Γ is called weak Lipschitz, if Γ_τ is a Lipschitz submanifold of Γ, i. e., there are an open covering $U_1, \dots, U_K \subset \mathbb{R}^N$ of Γ and vector fields $\phi_k := U_k \to B$, such that for $k = 1, \dots, K$ and in addition to (i), (ii) in Definition 2.3 one of

(iii) $U_k \cap \Gamma_\tau = \emptyset$;

(iii′) $U_k \cap \Gamma_\tau = U_k \cap \Gamma \ \Rightarrow\ \phi_k(U_k \cap \Gamma_\tau) = B_0$;

(iii″) $\emptyset \neq U_k \cap \Gamma_\tau \neq U_k \cap \Gamma \ \Rightarrow\ \phi_k(U_k \cap \Gamma_\tau) = B_{0,-}$

holds. We define $\Gamma_\nu := \Gamma \setminus \overline{\Gamma}_\tau$ to be the relatively open complement of Γ_τ.

Definition 2.6 (Weak Lipschitz pair). A pair (Ω, Γ_τ) conforming to Definitions 2.3 and 2.5 will be called weak Lipschitz.

Remark 2.7. If (Ω, Γ_τ) is weak Lipschitz, so is (Ω, Γ_ν). Moreover, for the cases (iii), (iii′) and (iii″) in Definition 2.5 we further have

(iii) $U_k \cap \Gamma_\tau = \emptyset \ \Rightarrow\ U_k \cap \Gamma_\nu = U_k \cap \Gamma \ \Rightarrow\ \phi_k(U_k \cap \Gamma_\nu) = B_0$;

(iii′) $U_k \cap \Gamma_\tau = U_k \cap \Gamma \ \Rightarrow\ U_k \cap \Gamma_\nu = \emptyset$;

(iii″) $\emptyset \neq U_k \cap \Gamma_\tau \neq U_k \cap \Gamma \ \Rightarrow\ \emptyset \neq U_k \cap \Gamma_\nu \neq U_k \cap \Gamma \ \Rightarrow\ \phi_k(U_k \cap \Gamma_\nu) = B_{0,+}$ and $\phi_k(U_k \cap \overline{\Gamma}_\tau \cap \overline{\Gamma}_\nu) = B_{0,0}$.

In the literature, the notion of a Lipschitz domain $\Omega \subset \mathbb{R}^N$ is often used for a strong Lipschitz domain. For this, let us define for $x \in \mathbb{R}^N$,

$$x' := (x_1, x_2, \dots, x_{N-1}), \quad x'' := (x_2, \dots, x_{N-1}).$$

Definition 2.8 (Strong Lipschitz domain). Ω is called strong Lipschitz, if there are an open covering $U_1, \dots, U_K \subset \mathbb{R}^N$ of Γ, rigid body motions $R_k = A_k + a_k$, $A_k \in \mathbb{R}^{N \times N}$ orthogonal, $a_k \in \mathbb{R}^N$ and $\xi_k \in \mathsf{C}^{0,1}(I^{N-1}, I)$, such that for $k = 1, \dots, K$

(i) $R_k(U_k \cap \Omega) = \{x \in B : x_N < \xi_k(x')\}$.

Remark 2.9. For $k = 1, \dots, K$, we have

$$R_k(U_k \setminus \overline{\Omega}) = \{x \in B : x_N > \xi_k(x')\}, \quad R_k(U_k \cap \Gamma) = \{x \in B : x_N = \xi_k(x')\}.$$

Definition 2.10 (Strong Lipschitz domain and strong Lipschitz interface). Let Ω be strong Lipschitz. A relatively open subset Γ_τ of Γ is called strong Lipschitz, if there exist an open covering $U_1, \ldots, U_K \subset \mathbb{R}^N$ of Γ, rigid body motions R_k, and $\xi_k \in \mathsf{C}^{0,1}(I^{N-1}, I)$, $\zeta_k \in \mathsf{C}^{0,1}(I^{N-2}, I)$, such that for $k = 1, \ldots, K$ and in addition to (i) in Definition 2.8 one of
(ii) $U_k \cap \Gamma_\tau = \emptyset$;
(ii′) $U_k \cap \Gamma_\tau = U_k \cap \Gamma \;\Rightarrow\; R_k(U_k \cap \Gamma_\tau) = \{x \in B : x_N = \xi_k(x')\}$;
(ii″) $\emptyset \neq U_k \cap \Gamma_\tau \neq U_k \cap \Gamma \;\Rightarrow\; R_k(U_k \cap \Gamma_\tau) = \{x \in B : x_N = \xi_k(x'),\ x_1 < \zeta_k(x'')\}$

holds. We define $\Gamma_\nu := \Gamma \setminus \overline{\Gamma}_\tau$ to be the relatively open complement of Γ_τ.

Definition 2.11 (Strong Lipschitz pair). A pair (Ω, Γ_τ) conforming to Definitions 2.8 and 2.10 will be called strong Lipschitz.

Remark 2.12. If (Ω, Γ_τ) is strong Lipschitz, so is (Ω, Γ_ν). Moreover, for the cases (ii), (ii′) and (ii″) in Definition 2.10 we further have
(ii) $U_k \cap \Gamma_\tau = \emptyset \;\Rightarrow\; U_k \cap \Gamma_\nu = U_k \cap \Gamma \;\Rightarrow\; R_k(U_k \cap \Gamma_\nu) = \{x \in B : x_N = \xi_k(x')\}$;
(ii′) $U_k \cap \Gamma_\tau = U_k \cap \Gamma \;\Rightarrow\; U_k \cap \Gamma_\nu = \emptyset$;
(ii″) $\emptyset \neq U_k \cap \Gamma_\tau \neq U_k \cap \Gamma \;\Rightarrow\; \emptyset \neq U_k \cap \Gamma_\nu \neq U_k \cap \Gamma \;\Rightarrow$

$$R_k(U_k \cap \Gamma_\nu) = \{x \in B : x_N = \xi_k(x'),\ x_1 > \zeta_k(x'')\},$$
$$R_k(U_k \cap \overline{\Gamma}_\tau \cap \overline{\Gamma}_\nu) = \{x \in B : x_N = \xi_k(x'),\ x_1 = \zeta_k(x'')\}.$$

Remark 2.13. The following holds:
(i) Ω strong Lipschitz $\Rightarrow$ Ω weak Lipschitz
(ii) (Ω, Γ_τ) strong Lipschitz pair $\Rightarrow$ (Ω, Γ_τ) weak Lipschitz pair

For a proof just define $\phi_k := \varphi_k \circ R_k$ with $\varphi_k : U_k \to B$ given by

$$\varphi_k(x) := \begin{bmatrix} x_1 - \zeta_k(x'') \\ x'' \\ x_N - \xi_k(x') \end{bmatrix}.$$

Note that the contrary does not hold as the implicit function theorem is not available for Lipschitz maps.

For later purposes, we introduce special notation for the half-cube domain

$$\Xi := B_-, \quad \gamma := \partial\Xi \tag{4}$$

and its relatively open boundary parts γ_τ and $\gamma_\nu := \gamma \setminus \overline{\gamma_\tau}$. We will only consider the cases

$$\gamma_\nu = \emptyset, \quad \gamma_\nu = B_0, \quad \gamma_\nu = B_{0,+} \tag{5}$$

and we note that Ξ and γ, γ_τ, γ_ν are strong Lipschitz, see Figure 3.1.

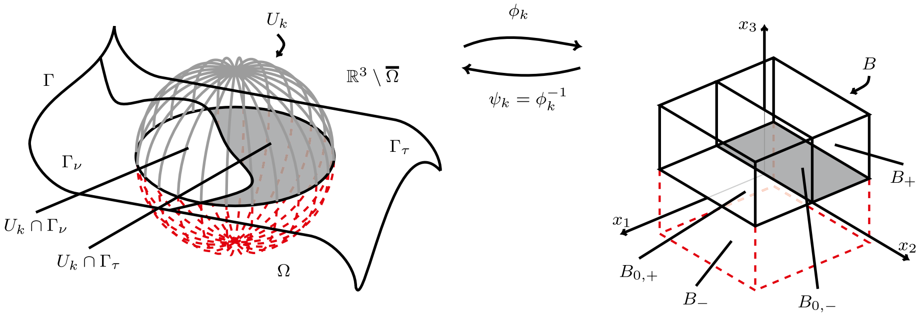

Figure 3.1: Mappings ϕ_k and ψ_k between a ball U_k and the cube B.

2.2 Outline of the proof

Let (Ω, Γ_τ) be a weak Lipschitz pair for a bounded domain $\Omega \subset \mathbb{R}^N$.

– As a first step, we observe $\mathring{\mathsf{H}}^{1,q}_{\Gamma_\tau}(\Omega) = \mathring{\mathbf{H}}^{1,q}_{\Gamma_\tau}(\Omega)$, i. e., for the $\mathsf{H}^{1,q}(\Omega)$-spaces the strong and weak definitions of the boundary conditions coincide; see Lemma 2.14.
– In the second and essential step, we construct various regular $\mathsf{H}^{1,q}$-potentials on simple domains, mainly for the half-cube Ξ from (4) with the special boundary constellations (5), i. e.,

$$\mathring{\mathbf{D}}^q_{\Gamma_\nu,0}(\Xi) = \mathring{\mathsf{D}}^q_{\gamma_\nu,0}(\Xi) = \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\gamma_\nu}(\Xi), \quad \mathring{\mathbf{\Delta}}^q_{\Gamma_\nu,0}(\Xi) = \mathring{\Delta}^q_{\Gamma_\nu,0}(\Xi) = \delta\,\mathring{\mathsf{H}}^{1,q+1}_{\gamma_\nu}(\Xi);$$

see Section 3. Potentials of this type are called regular potentials.
– In the third step, Section 3.3, it is shown that the strong and weak definitions of the boundary conditions coincide on the half-cube Ξ from (4) with the special boundary constellation (5), i. e.,

$$\mathring{\mathbf{D}}^q_{\Gamma_\nu}(\Xi) = \mathring{\mathsf{D}}^q_{\Gamma_\nu}(\Xi), \quad \mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Xi) = \mathring{\Delta}^q_{\Gamma_\nu}(\Xi). \tag{6}$$

– The fourth step proves the compact embedding on the half-cube Ξ from (4) with the special boundary constellations (5), i. e.,

$$\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Xi) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Xi) \hookrightarrow \mathsf{L}^{2,q}(\Xi) \tag{7}$$

is compact; see Section 4.1.
– In the fifth step, Theorem 4.7, (6) is established for weak Lipschitz domains, i. e.,

$$\mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega) = \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega), \quad \mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Omega) = \mathring{\Delta}^q_{\Gamma_\nu}(\Omega).$$

– In the last step, we finally prove the compact embedding (7) for weak Lipschitz pairs, i. e.,

$$\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \hookrightarrow \mathsf{L}^{2,q}(\Omega)$$

is compact; see our main result Theorem 4.8.

2.3 Some important results

Within our proofs, we need a few important technical lemmas. First, the strong and weak definitions of the boundary conditions coincide for $\mathsf{H}^{1,q}(\Omega)$-forms, which is a density result for $\mathsf{H}^{1,q}(\Omega)$-forms. This is an immediate consequence of the corresponding scalar result, whose proof can be found in [11, Lemma 2, Lemma 3] and with a simplified proof in [1, Lemma 3.1].

Lemma 2.14 (Weak and strong boundary conditions coincide for $\mathsf{H}^{1,q}(\Omega)$). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let (Ω, Γ_τ) be a weak Lipschitz pair as well as*

$$\underset{\sim}{\mathring{\mathsf{H}}}{}^{1,q}_{\Gamma_\tau}(\Omega) := \{u \in \mathsf{H}^{1,q}(\Omega) \,:\, u|_{\Gamma_\tau} = 0\}$$

in the sense of traces. Then $\mathring{\mathbf{H}}{}^{1,q}_{\Gamma_\tau}(\Omega) = \underset{\sim}{\mathring{\mathsf{H}}}{}^{1,q}_{\Gamma_\tau}(\Omega) = \mathring{\mathsf{H}}{}^{1,q}_{\Gamma_\tau}(\Omega)$.

Another crucial tool in our arguments is a universal extension operator for the Sobolev spaces $\mathsf{D}^{k,q}(\Omega)$ and $\Delta^{k,q}(\Omega)$ given in [9], which is based on the universal extension operator for standard Sobolev spaces $\mathsf{H}^k(\Omega)$ introduced by Stein in [26]. "Universality" in this context means that the operator, which is given by a single formula, is able to extend all orders of Sobolev spaces simultaneously. More precisely, the following theorem, which is taken from [9, Theorem 3.6], holds.

Lemma 2.15 (Stein's extension operator). *Let $\Omega \subset \mathbb{R}^N$ be a bounded strong Lipschitz domain. Then for $k \in \mathbb{N}_0$ and $0 \le q \le N$ there exists a (universal) linear and continuous extension operator*

$$\mathcal{E} \,:\, \mathsf{D}^{k,q}(\Omega) \to \mathsf{D}^{k,q}(\mathbb{R}^N).$$

More precisely, $\mathcal{E}$ satisfies $\mathcal{E}E = E$ a.e. in Ω and there exists $c > 0$ such that for all $E \in \mathsf{D}^{k,q}(\Omega)$

$$|\mathcal{E}E|_{\mathsf{D}^{k,q}(\mathbb{R}^N)} \le c|E|_{\mathsf{D}^{k,q}(\Omega)}.$$

Furthermore, $\mathcal{E}$ can be chosen such that $\mathcal{E}E$ has a fixed compact support in $\mathbb{R}^N$ for all $E \in \mathsf{D}^{k,q}(\Omega)$.

Our third lemma summarises well-known and fundamental results for the theory of Maxwell's equations from [23, 24]. For this, we denote orthogonality and the orthogonal sum in $\mathsf{L}^{2,q}(\Omega)$ by $\perp$ and $\oplus$, respectively, and introduce the harmonic Dirichlet and Neumann forms

$$\mathcal{H}^q_D(\Omega) := \mathring{\mathsf{D}}{}^q_0(\Omega) \cap \Delta^q_0(\Omega), \qquad \mathcal{H}^q_N(\Omega) := \mathsf{D}^q_0(\Omega) \cap \mathring{\Delta}{}^q_0(\Omega),$$

respectively.

Lemma 2.16 (Picard's generalisation of Weck's selection theorem, Helmholtz decompositions and Maxwell estimates). *Let $\Omega \subset \mathbb{R}^N$ be a bounded weak Lipschitz domain. Then the embeddings*

$$\mathring{\mathsf{D}}^q(\Omega) \cap \Delta^q(\Omega) \hookrightarrow \mathsf{L}^{2,q}(\Omega), \qquad \mathsf{D}^q(\Omega) \cap \mathring{\Delta}^q(\Omega) \hookrightarrow \mathsf{L}^{2,q}(\Omega)$$

are compact and $\mathcal{H}^q_D(\Omega)$, $\mathcal{H}^q_N(\Omega)$ are finite-dimensional. Moreover, the Helmholtz decompositions

$$\begin{aligned} \mathsf{L}^{2,q}(\Omega) &= \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Omega) \oplus \Delta^q_0(\Omega) & \mathsf{L}^{2,q}(\Omega) &= \mathrm{d}\,\mathsf{D}^{q-1}(\Omega) \oplus \mathring{\Delta}^q_0(\Omega) \\ &= \mathring{\mathsf{D}}^q_0(\Omega) \oplus \delta\,\Delta^{q+1}(\Omega) & &= \mathsf{D}^q_0(\Omega) \oplus \delta\,\mathring{\Delta}^{q+1}(\Omega) \\ &= \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Omega) \oplus \mathcal{H}^q_D(\Omega) \oplus \delta\,\Delta^{q+1}(\Omega), & &= \mathrm{d}\,\mathsf{D}^{q-1}(\Omega) \oplus \mathcal{H}^q_N(\Omega) \oplus \delta\,\mathring{\Delta}^{q+1}(\Omega) \end{aligned}$$

are valid. In particular, all ranges are closed subspaces of $\mathsf{L}^{2,q}(\Omega)$ and

$$\begin{aligned} \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Omega) &= \mathring{\mathsf{D}}^q_0(\Omega) \cap \mathcal{H}^q_D(\Omega)^\perp, & \mathrm{d}\,\mathsf{D}^{q-1}(\Omega) &= \mathsf{D}^q_0(\Omega) \cap \mathcal{H}^q_N(\Omega)^\perp, \\ \delta\,\Delta^{q+1}(\Omega) &= \Delta^q_0(\Omega) \cap \mathcal{H}^q_D(\Omega)^\perp, & \delta\,\mathring{\Delta}^{q+1}(\Omega) &= \mathring{\Delta}^q_0(\Omega) \cap \mathcal{H}^q_N(\Omega)^\perp. \end{aligned}$$

Furthermore, there exists $c > 0$ such that

$$c|E|_{\mathsf{L}^{2,q}(\Omega)} \leq |\mathrm{d}\,E|_{\mathsf{L}^{2,q+1}(\Omega)} + |\delta\,E|_{\mathsf{L}^{2,q-1}(\Omega)}$$

holds for all $E \in \mathring{\mathsf{D}}^q(\Omega) \cap \Delta^q(\Omega) \cap \mathcal{H}^q_D(\Omega)^\perp$ and all $E \in \mathsf{D}^q(\Omega) \cap \mathring{\Delta}^q(\Omega) \cap \mathcal{H}^q_N(\Omega)^\perp$, i. e., the Maxwell (or Friedrichs–Poincaré-type) estimates are valid.

Corollary 2.17 (Refined Helmholtz decompositions). *Let $\Omega \subset \mathbb{R}^N$ be a bounded weak Lipschitz domain. Then*

$$\begin{aligned} \mathring{\mathsf{D}}^q(\Omega) &= \mathring{\mathsf{D}}^q_0(\Omega) \oplus \big(\mathring{\mathsf{D}}^q(\Omega) \cap \delta\,\Delta^{q+1}(\Omega)\big), & \mathrm{d}\,\mathring{\mathsf{D}}^q(\Omega) &= \mathrm{d}\big(\mathring{\mathsf{D}}^q(\Omega) \cap \delta\,\Delta^{q+1}(\Omega)\big), \\ \mathsf{D}^q(\Omega) &= \mathsf{D}^q_0(\Omega) \oplus \big(\mathsf{D}^q(\Omega) \cap \delta\,\mathring{\Delta}^{q+1}(\Omega)\big), & \mathrm{d}\,\mathsf{D}^q(\Omega) &= \mathrm{d}\big(\mathsf{D}^q(\Omega) \cap \delta\,\mathring{\Delta}^{q+1}(\Omega)\big), \\ \Delta^q(\Omega) &= \big(\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Omega) \cap \Delta^q(\Omega)\big) \oplus \Delta^q_0(\Omega), & \delta\,\Delta^q(\Omega) &= \delta\big(\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}(\Omega) \cap \Delta^q(\Omega)\big), \\ \mathring{\Delta}^q(\Omega) &= \big(\mathrm{d}\,\mathsf{D}^{q-1}(\Omega) \cap \mathring{\Delta}^q(\Omega)\big) \oplus \mathring{\Delta}^q_0(\Omega), & \delta\,\mathring{\Delta}^q(\Omega) &= \delta\big(\mathrm{d}\,\mathsf{D}^{q-1}(\Omega) \cap \mathring{\Delta}^q(\Omega)\big). \end{aligned}$$

Let $\pi_{q,\Omega} : \mathsf{L}^{2,q}(\Omega) \to \delta\,\mathring{\Delta}^{q+1}(\Omega)$ be the orthonormal Helmholtz projector onto $\delta\,\mathring{\Delta}^{q+1}(\Omega)$. By the latter corollary, $\pi_{q,\Omega}$ maps $\mathsf{D}^q(\Omega)$ to

$$\mathsf{D}^q(\Omega) \cap \delta\,\mathring{\Delta}^{q+1}(\Omega) = \mathsf{D}^q(\Omega) \cap \mathring{\Delta}^q_0(\Omega) \cap \mathcal{H}^q_N(\Omega)^\perp.$$

Corollary 2.18 (Maxwell estimate for d and Neumann boundary condition). *Assume $\Omega \subset \mathbb{R}^N$ to be a bounded weak Lipschitz domain. Then for all $E \in \mathsf{D}^q(\Omega)$ it holds $\pi_{q,\Omega}E \in \mathsf{D}^q(\Omega) \cap \delta\,\mathring{\Delta}^{q+1}(\Omega)$ and $\mathrm{d}\,\pi_{q,\Omega}E = \mathrm{d}\,E$ as well as*

$$c|\pi_{q,\Omega}E|_{\mathsf{L}^{2,q}(\Omega)} \leq |\mathrm{d}\,E|_{\mathsf{L}^{2,q+1}(\Omega)},$$

with c from Lemma 2.16.

If $\Omega = \mathbb{R}^N$, a similar theory holds true utilising polynomially weighted Sobolev spaces; see [23] for details. Let $\pi_{q,\mathbb{R}^N} : \mathsf{L}^{2,q}(\mathbb{R}^N) \to \Delta_0^q(\mathbb{R}^N)$ be the orthonormal Helmholtz projector onto $\Delta_0^q(\mathbb{R}^N)$.

Lemma 2.19 (Helmholtz decompositions and Maxwell estimate for d in the whole space). *It holds* $\mathcal{H}_N^q(\mathbb{R}^N) = \mathcal{H}_D^q(\mathbb{R}^N) = \{0\}$ *and*

$$\mathsf{L}^{2,q}(\mathbb{R}^N) = \mathsf{D}_0^q(\mathbb{R}^N) \oplus \Delta_0^q(\mathbb{R}^N), \quad \mathsf{D}^q(\mathbb{R}^N) = \mathsf{D}_0^q(\mathbb{R}^N) \oplus (\mathsf{D}^q(\mathbb{R}^N) \cap \Delta_0^q(\mathbb{R}^N)).$$

Moreover, for all $E \in \mathsf{D}^q(\mathbb{R}^N)$ *it holds* $\pi_{q,\mathbb{R}^N} E \in \mathsf{D}^q(\mathbb{R}^N) \cap \Delta_0^q(\mathbb{R}^N)$ *and* $\mathrm{d}\,\pi_{q,\mathbb{R}^N} E = \mathrm{d}\,E$ *as well as*

$$|\pi_{q,\mathbb{R}^N} E|_{\mathsf{D}^q(\mathbb{R}^N)} \leq |E|_{\mathsf{D}^q(\mathbb{R}^N)}.$$

Regularity in the whole space (see, e. g., [12, (4.7) or Lemma 4.2(i)]) shows the following result.

Lemma 2.20 (Regularity in the whole space). $\mathsf{D}^q(\mathbb{R}^N) \cap \Delta^q(\mathbb{R}^N) = \mathsf{H}^{1,q}(\mathbb{R}^N)$ *with equal norms. More precisely,* $E \in \mathsf{D}^q(\mathbb{R}^N) \cap \Delta^q(\mathbb{R}^N)$ *if and only if* $E \in \mathsf{H}^{1,q}(\mathbb{R}^N)$ *and*

$$|E|^2_{\mathsf{H}^{1,q}(\mathbb{R}^N)} = |E|^2_{\mathsf{L}^{2,q}(\mathbb{R}^N)} + |\mathrm{d}\,E|^2_{\mathsf{L}^{2,q+1}(\mathbb{R}^N)} + |\delta\,E|^2_{\mathsf{L}^{2,q-1}(\mathbb{R}^N)}.$$

3 Regular potentials

As one of our main steps (step 4), in Section 4.1 the compact embedding is proved on the half-cube $\Xi \subset \mathbb{R}^N$. This will be achieved (in step 2) by constructing regular $\mathsf{H}^1(\Xi)$-potentials for d-free and δ-free $\mathsf{L}^{2,q}(\Xi)$-forms, which will then enable us to use Rellich's selection theorem. This section is devoted to the construction and existence of these regular potentials, i. e., to step 2.

3.1 Regular potentials without boundary conditions

Let us recall

$$\mathrm{d}\,\mathsf{D}^{q-1}(\Omega) = \mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp, \quad \delta\,\Delta^{q+1}(\Omega) = \Delta_0^q(\Omega) \cap \mathcal{H}_D^q(\Omega)^\perp$$

from Lemma 2.16. The next two lemmas ensure the existence of $\mathsf{H}^{1,q}(\Omega)$-potentials without boundary conditions for strong Lipschitz domains.

Lemma 3.1 (Regular potential for d without boundary condition). *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded strong Lipschitz domain. Then there exists a continuous linear operator*

$$\mathcal{T}_{\mathrm{d}} : \mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp \to \mathsf{H}^{1,q-1}(\mathbb{R}^N) \cap \Delta_0^{q-1}(\mathbb{R}^N)$$

such that for all $E \in \mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp$

$$\mathrm{d}\,\mathcal{T}_\mathrm{d} E = E \quad \textit{in}\ \Omega.$$

Especially,

$$\mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp = \mathrm{d}\,\mathsf{H}^{1,q-1}(\Omega) = \mathrm{d}\big(\mathsf{H}^{1,q-1}(\Omega) \cap \Delta_0^{q-1}(\Omega)\big)$$

and the regular potential depends continuously on the data. Particularly, these are closed subspaces of $\mathsf{L}^{2,q}(\Omega)$ *and* $\mathcal{T}_\mathrm{d}$ *is a right inverse to* d. *By a simple cut-off technique* $\mathcal{T}_\mathrm{d}$ *may be modified to*

$$\mathcal{T}_\mathrm{d} : \mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp \to \mathsf{H}^{1,q-1}(\mathbb{R}^N)$$

such that $\mathcal{T}_\mathrm{d} E$ *has a fixed compact support in* $\mathbb{R}^N$ *for all* $E \in \mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp$.

Proof. Suppose $E \in \mathsf{D}_0^q(\Omega) \cap \mathcal{H}_N^q(\Omega)^\perp$. By Lemma 2.16, there exists $H \in \mathsf{D}^{q-1}(\Omega)$ with $\mathrm{d}\,H = E$ in Ω. Applying Corollary 2.18, we get $\pi_{q-1,\Omega} H \in \mathsf{D}^{q-1}(\Omega) \cap \delta\,\mathring{\Delta}^q(\Omega)$ with $\mathrm{d}\,\pi_{q-1,\Omega} H = \mathrm{d}\,H = E$ and

$$|\pi_{q-1,\Omega} H|_{\mathsf{D}^{q-1}(\Omega)} \le c|E|_{\mathsf{L}^{2,q}(\Omega)}.$$

Note that $\pi_{q-1,\Omega} H$ is uniquely determined. By the Stein extension operator $\mathcal{E} : \mathsf{D}^{0,q-1}(\Omega) \to \mathsf{D}^{0,q-1}(\mathbb{R}^N)$ from Lemma 2.15, we have $\mathcal{E}\pi_{q-1,\Omega} H \in \mathsf{D}^{0,q-1}(\mathbb{R}^N)$ with compact support. Projecting again, now with Lemma 2.19 onto $\Delta_0^{q-1}(\mathbb{R}^N)$, we obtain $\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H \in \mathsf{D}^{q-1}(\mathbb{R}^N) \cap \Delta_0^{q-1}(\mathbb{R}^N)$ (again uniquely determined) with $\mathrm{d}\,\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H = \mathrm{d}\,\mathcal{E}\pi_{q-1,\Omega} H$ and

$$|\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H|_{\mathsf{D}^{q-1}(\mathbb{R}^N)} \le |\mathcal{E}\pi_{q-1,\Omega} H|_{\mathsf{D}^{q-1}(\mathbb{R}^N)} \le c|\pi_{q-1,\Omega} H|_{\mathsf{D}^{q-1}(\Omega)}.$$

Lemma 2.20 shows $\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H \in \mathsf{H}^{1,q-1}(\mathbb{R}^N) \cap \Delta_0^{q-1}(\mathbb{R}^N)$ with

$$|\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} = |\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H|_{\mathsf{D}^{q-1}(\mathbb{R}^N)}.$$

Finally, $\mathcal{T}_\mathrm{d} E := \pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H \in \mathsf{H}^{1,q-1}(\mathbb{R}^N) \cap \Delta_0^{q-1}(\mathbb{R}^N)$ meets our needs as

$$|\mathcal{T}_\mathrm{d} E|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} \le c|E|_{\mathsf{L}^{2,q}(\Omega)}$$

and $\mathrm{d}\,\mathcal{T}_\mathrm{d} E = \mathrm{d}\,\pi_{q-1,\mathbb{R}^N} \mathcal{E}\pi_{q-1,\Omega} H = \mathrm{d}\,\mathcal{E}\pi_{q-1,\Omega} H = \mathrm{d}\,\pi_{q-1,\Omega} H = \mathrm{d}\,H = E$ in Ω. □

By Hodge-$\star$-duality, we get a corresponding result for the δ-operator.

Lemma 3.2 (Regular potential for δ without boundary condition). *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded strong Lipschitz domain. Then there exists a continuous linear operator,*

$$\mathcal{T}_\delta : \Delta_0^q(\Omega) \cap \mathcal{H}_D^q(\Omega)^\perp \to \mathsf{H}^{1,q+1}(\mathbb{R}^N) \cap \mathsf{D}_0^{q+1}(\mathbb{R}^N),$$

such that for all $E \in \Delta_0^q(\Omega) \cap \mathcal{H}_D^q(\Omega)^\perp$

$$\delta\, \mathcal{T}_\delta E = E \quad \text{in } \Omega.$$

Especially,

$$\Delta_0^q(\Omega) \cap \mathcal{H}_D^q(\Omega)^\perp = \delta\, \mathsf{H}^{1,q+1}(\Omega) = \delta\big(\mathsf{H}^{1,q+1}(\Omega) \cap \mathsf{D}_0^{q+1}(\Omega)\big)$$

and the regular potential depends continuously on the data. In particular, these are closed subspaces of $\mathsf{L}^{2,q}(\Omega)$ *and* $\mathcal{T}_\delta$ *is a right inverse to* δ. *By a simple cut-off technique* $\mathcal{T}_\delta$ *may be modified to*

$$\mathcal{T}_\delta : \Delta_0^q(\Omega) \cap \mathcal{H}_D^q(\Omega)^\perp \to \mathsf{H}^{1,q+1}(\mathbb{R}^N)$$

such that $\mathcal{T}_\delta E$ *has a fixed compact support in* $\mathbb{R}^N$ *for all* $E \in \Delta_0^q(\Omega) \cap \mathcal{H}_D^q(\Omega)^\perp$.

3.2 Regular potentials with boundary conditions for the half-cube

Now we start constructing $\mathsf{H}^{1,q}(\Xi)$-potentials on Ξ with boundary conditions. Let us recall our special setting on the half-cube

$$\Xi = B_- \quad \text{and} \quad \gamma_\nu = \emptyset, \quad \gamma_\nu = B_0 \quad \text{or} \quad \gamma_\nu = B_{0,+}.$$

Furthermore (cf. Figure 3.2), we extend Ξ over γ_ν by

$$\tilde{\Xi} = \operatorname{int}(\overline{\Xi} \cup \overline{\hat{\Xi}}), \quad \hat{\Xi} := \begin{cases} \{x \in B : x_N > 0\} = B_+, & \text{if } \gamma_\nu = B_0, \\ \{x \in B : x_N, x_1 > 0\} = \{x \in B_+ : x_1 > 0\} =: B_{+,+}, & \text{if } \gamma_\nu = B_{0,+}. \end{cases}$$

Lemma 3.3 (Regular potential for d with partial boundary condition on the half-cube). *There exists a continuous linear operator*

$$\mathcal{S}_\mathrm{d} : \mathring{\mathbf{D}}^q_{\gamma_\nu,0}(\Xi) \to \mathsf{H}^{1,q-1}(\mathbb{R}^N) \cap \mathring{\mathsf{H}}^{1,q-1}_{\gamma_\nu}(\Xi),$$

such that for all $H \in \mathring{\mathbf{D}}^q_{\gamma_\nu,0}(\Xi)$

$$\mathrm{d}\, \mathcal{S}_\mathrm{d} H = H \quad \text{in } \Xi.$$

Especially,

$$\mathring{\mathbf{D}}^q_{\gamma_\nu,0}(\Xi) = \mathring{\mathsf{D}}^q_{\gamma_\nu,0}(\Xi) = \mathrm{d}\, \mathring{\mathsf{H}}^{1,q-1}_{\gamma_\nu}(\Xi) = \mathrm{d}\, \mathring{\mathsf{D}}^{q-1}_{\gamma_\nu}(\Xi) = \mathrm{d}\, \mathring{\mathbf{D}}^{q-1}_{\gamma_\nu}(\Xi)$$

and the regular $\mathring{\mathsf{H}}^{1,q-1}_{\gamma_\nu}(\Xi)$*-potential depends continuously on the data. In particular, these spaces are closed subspaces of* $\mathsf{L}^{2,q}(\Xi)$ *and* $\mathcal{S}_\mathrm{d}$ *is a right inverse to* d. *Without loss of generality,* $\mathcal{S}_\mathrm{d}$ *maps to forms with a fixed compact support in* $\mathbb{R}^N$.

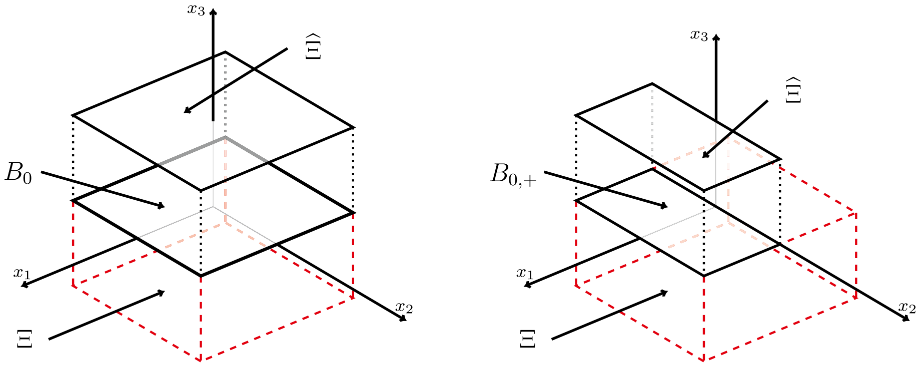

Figure 3.2: The half-cube $\Xi = B_{-}$, extended by $\widehat{\Xi}$ to the polygonal domain $\widetilde{\Xi}$, and the rectangles $\gamma_\nu = B_0$ and $\gamma_\nu = B_{0,+}$.

Proof. The case $\gamma_\nu = \emptyset$ is done in Lemma 3.1. Hence let $\gamma_\nu = B_0$ or $\gamma_\nu = B_{0,+}$. Suppose $H \in \mathring{\mathsf{D}}^q_{\gamma_\nu,0}(\Xi)$ and define $\widetilde{H} \in \mathsf{L}^{2,q}(\widetilde{\Xi})$ as extension of H by zero to $\widehat{\Xi}$ by

$$\widetilde{H} := \begin{cases} H & \text{in } \Xi, \\ 0 & \text{in } \widehat{\Xi}. \end{cases} \tag{8}$$

By definition of $\mathring{\mathsf{D}}^q_{\gamma_\nu,0}(\Xi)$ (definition of the weak boundary condition), it follows $\mathrm{d}\,\widetilde{H} = 0$ in $\widetilde{\Xi}$, i. e., $\widetilde{H} \in \mathsf{D}^q_0(\widetilde{\Xi})$. Because $\widetilde{\Xi}$ is strong Lipschitz and topologically trivial, especially $\mathcal{H}^q_N(\widetilde{\Xi}) = \{0\}$, Lemma 3.1 yields a regular potential $E = \mathcal{T}_{\mathrm{d}}\widetilde{H} \in \mathsf{H}^{1,q-1}(\mathbb{R}^N) \cap \mathsf{D}^{q-1}_0(\mathbb{R}^N)$ with $\mathrm{d}\,E = \widetilde{H}$ in $\widetilde{\Xi}$ and

$$|E|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} \le c|\widetilde{H}|_{\mathsf{L}^{2,q}(\widetilde{\Xi})} \le c|H|_{\mathsf{L}^{2,q}(\Xi)}.$$

In particular, $E \in \mathsf{H}^{1,q-1}(\widehat{\Xi})$ and $\mathrm{d}\,E = 0$ in $\widehat{\Xi}$, i. e., $E \in \mathsf{H}^{1,q-1}(\widehat{\Xi}) \cap \mathsf{D}^{q-1}_0(\widehat{\Xi})$. Using Lemma 3.1 again, this time in $\widehat{\Xi}$, we obtain $F = \mathcal{T}_{\mathrm{d}}E \in \mathsf{H}^{1,q-2}(\mathbb{R}^N) \subset \mathsf{H}^{1,q-2}(\widehat{\Xi})$ with $\mathrm{d}\,F = E$ in $\widehat{\Xi}$ and

$$|F|_{\mathsf{H}^{1,q-2}(\mathbb{R}^N)} \le c|E|_{\mathsf{L}^{2,q}(\widehat{\Xi})}.$$

Since $E \in \mathsf{H}^{1,q-1}(\widehat{\Xi})$, we have $F \in \mathsf{D}^{1,q-2}(\widehat{\Xi})$. Let $\mathcal{E} : \mathsf{D}^{1,q-2}(\widehat{\Xi}) \to \mathsf{D}^{1,q-2}(\mathbb{R}^N)$ be the Stein extension operator from Lemma 2.15. Then

$$\begin{array}{cccc} \mathcal{S}_{\mathrm{d}} : & \mathring{\mathsf{D}}^q_{\gamma_\nu,0}(\Xi) & \longrightarrow & \mathsf{H}^{1,q-1}(\mathbb{R}^N) \\ & H & \longmapsto & E - \mathrm{d}(\mathcal{E}F) \end{array}$$

is linear and continuous as

$$\begin{aligned} |\mathcal{S}_{\mathrm{d}}H|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} &\le |E|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} + |\mathcal{E}F|_{\mathsf{D}^{1,q-2}(\mathbb{R}^N)} \\ &\le |E|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} + |F|_{\mathsf{D}^{1,q-2}(\widehat{\Xi})} \le |E|_{\mathsf{H}^{1,q-1}(\mathbb{R}^N)} \le c|H|_{\mathsf{L}^{2,q}(\Xi)}. \end{aligned}$$

Since $\mathcal{S}_{\mathrm{d}}H = 0$ in $\widehat{\Xi}$, we have $\mathcal{S}_{\mathrm{d}}H|_{\gamma_\nu} = 0$, which means $\mathcal{S}_{\mathrm{d}}H \in \underset{\sim}{\mathring{\mathsf{H}}}{}^{1,q-1}_{\gamma_\nu}(\Xi)$. Therefore, by Lemma 2.14 we see $\mathcal{S}_{\mathrm{d}}H \in \mathring{\mathsf{H}}^{1,q-1}_{\gamma_\nu}(\Xi) \subset \mathring{\mathsf{D}}^{q-1}_{\gamma_\nu}(\Xi) \subset \mathring{\mathbf{D}}^{q-1}_{\gamma_\nu}(\Xi)$. Moreover, $\mathrm{d}(\mathcal{S}_{\mathrm{d}}H) = \mathrm{d}\,E = \widetilde{H}$ in $\widetilde{\Xi}$, especially $\mathrm{d}(\mathcal{S}_{\mathrm{d}}H) = H$ in Ξ. Finally, we note

$$\mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\gamma_\nu}(\Xi) \subset \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\gamma_\nu}(\Xi) \subset \mathring{\mathsf{D}}^{q}_{\gamma_\nu,0}(\Xi),\ \mathrm{d}\,\mathring{\mathbf{D}}^{q-1}_{\gamma_\nu}(\Xi) \subset \mathring{\mathbf{D}}^{q}_{\gamma_\nu,0}(\Xi) \subset \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\gamma_\nu}(\Xi),$$

completing the proof. □

Again by Hodge-$\star$-duality, we obtain the following.

Lemma 3.4 (Regular potential for δ with partial boundary condition on the half-cube). *There exists a continuous linear operator*

$$\mathcal{S}_\delta : \mathring{\mathbf{\Delta}}^{q}_{\gamma_\nu,0}(\Xi) \to \mathsf{H}^{1,q+1}(\mathbb{R}^N) \cap \mathring{\mathsf{H}}^{1,q+1}_{\gamma_\nu}(\Xi),$$

such that for all $H \in \mathring{\mathbf{\Delta}}^{q}_{\gamma_\nu,0}(\Xi)$

$$\delta\,\mathcal{S}_\delta H = H \quad \text{in } \Xi.$$

Especially

$$\mathring{\mathbf{\Delta}}^{q}_{\gamma_\nu,0}(\Xi) = \mathring{\Delta}^{q}_{\gamma_\nu,0}(\Xi) = \delta\,\mathring{\mathsf{H}}^{1,q+1}_{\gamma_\nu}(\Xi) = \delta\,\mathring{\Delta}^{q+1}_{\gamma_\nu}(\Xi) = \delta\,\mathring{\mathbf{\Delta}}^{q+1}_{\gamma_\nu}(\Xi)$$

and the regular $\mathring{\mathsf{H}}^{1,q+1}_{\gamma_\nu}(\Xi)$-potential depends continuously on the data. In particular, these spaces are closed subspaces of $\mathsf{L}^{2,q}(\Xi)$ and $\mathcal{S}_\delta$ is a right inverse to δ. Without loss of generality, $\mathcal{S}_\delta$ maps to forms with a fixed compact support in $\mathbb{R}^N$.

3.3 Weak and strong boundary conditions coincide for the half-cube

Now the two main density results immediately follow. We note that this has already been proved for the $\mathsf{H}^{1,q}(\Omega)$-spaces in Lemma 2.14, i. e., $\mathring{\mathbf{H}}^{1,q}_{\Gamma_\tau}(\Omega) = \mathring{\mathsf{H}}^{1,q}_{\Gamma_\tau}(\Omega)$.

Lemma 3.5 (Weak and strong boundary conditions coincide for the half-cube).

$$\mathring{\mathbf{D}}^{q}_{\gamma_\nu}(\Xi) = \mathring{\mathsf{D}}^{q}_{\gamma_\nu}(\Xi) \quad \textit{and} \quad \mathring{\mathbf{\Delta}}^{q}_{\gamma_\nu}(\Xi) = \mathring{\Delta}^{q}_{\gamma_\nu}(\Xi).$$

Proof. Suppose $E \in \mathring{\mathbf{D}}^{q}_{\gamma_\nu}(\Xi)$, and therefore $\mathrm{d}\,E \in \mathring{\mathbf{D}}^{q+1}_{\gamma_\nu,0}(\Xi)$. By Lemma 3.3, there exists $H = \mathcal{S}_{\mathrm{d}}\,\mathrm{d}\,E \in \mathring{\mathsf{H}}^{1,q}_{\gamma_\nu}(\Xi)$ with $\mathrm{d}\,H = \mathrm{d}\,E$. By Lemma 3.3, we get $E - H \in \mathring{\mathbf{D}}^{q}_{\gamma_\nu,0}(\Xi) = \mathring{\mathsf{D}}^{q}_{\gamma_\nu,0}(\Xi)$, and hence $E \in \mathring{\mathsf{D}}^{q}_{\gamma_\nu}(\Xi)$. □

4 Weck's selection theorem

4.1 The compact embedding for the half-cube

First, we show the main result on the half-cube $\Xi = B_{-}$ with the special boundary patches

$$\gamma_\nu = \emptyset, \quad \gamma_\nu = B_0 \quad \text{or} \quad \gamma_\nu = B_{0,+}$$

from the latter section. To this end, let ε be an admissible transformation on $\mathsf{L}^{2,q}(\Xi)$ and let us consider the densely defined and closed (unbounded) linear operator

$$\mathrm{d}_\tau^{q-1} : \mathring{\mathsf{D}}_{\gamma_\tau}^{q-1}(\Xi) \subset \mathsf{L}^{2,q-1}(\Xi) \to \mathsf{L}_\varepsilon^{2,q}(\Xi); \quad E \mapsto \mathrm{d}\, E$$

together with its (Hilbert space) adjoint

$$-\,\delta_\nu^q\,\varepsilon := \left(\mathrm{d}_\tau^{q-1}\right)^* : \varepsilon^{-1}\mathring{\boldsymbol{\Delta}}_{\gamma_\nu}^q(\Xi) \subset \mathsf{L}_\varepsilon^{2,q}(\Xi) \to \mathsf{L}^{2,q-1}(\Xi); \quad H \mapsto -\,\delta\,\varepsilon H.$$

Note that by Lemma 3.5 we have $\mathring{\boldsymbol{\Delta}}_{\gamma_\nu}^q(\Xi) = \mathring{\Delta}_{\gamma_\nu}^q(\Xi)$. Here, $\mathsf{L}_\varepsilon^{2,q}(\Xi)$ denotes $\mathsf{L}^{2,q}(\Xi)$ equipped with the inner product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}_\varepsilon^{2,q}(\Xi)} := \langle \varepsilon\,\cdot\,,\,\cdot\,\rangle_{\mathsf{L}^{2,q}(\Xi)}$. Let $\oplus_\varepsilon$ denote the orthogonal sum with respect to the $\mathsf{L}_\varepsilon^{2,q}$-scalar product. The projection theorem yields immediately.

Lemma 4.1 (Regular Helmholtz decompositions for the half-cube). *The Helmholtz decompositions*

$$\mathsf{L}_\varepsilon^{2,q}(\Xi) = \mathring{\mathsf{D}}_{\gamma_\tau,0}^q(\Xi) \oplus_\varepsilon \varepsilon^{-1}\mathring{\Delta}_{\gamma_\nu,0}^q(\Xi), \qquad \mathring{\mathsf{D}}_{\gamma_\tau,0}^q(\Xi) = \mathrm{d}\,\mathring{\mathsf{H}}_{\gamma_\tau}^{1,q-1}(\Xi), \qquad \mathring{\Delta}_{\gamma_\nu,0}^q(\Xi) = \delta\,\mathring{\mathsf{H}}_{\gamma_\nu}^{1,q+1}(\Xi)$$

hold. Moreover, the refined Helmholtz decompositions

$$\begin{aligned}
\mathring{\mathsf{D}}_{\gamma_\tau}^q(\Xi) &= \mathrm{d}\,\mathring{\mathsf{H}}_{\gamma_\tau}^{1,q-1}(\Xi) \oplus_\varepsilon \left(\mathring{\mathsf{D}}_{\gamma_\tau}^q(\Xi) \cap \varepsilon^{-1}\,\delta\,\mathring{\mathsf{H}}_{\gamma_\nu}^{1,q+1}(\Xi)\right),\\
\varepsilon^{-1}\mathring{\Delta}_{\gamma_\nu}^q(\Xi) &= \left(\mathrm{d}\,\mathring{\mathsf{H}}_{\gamma_\tau}^{1,q-1}(\Xi) \cap \varepsilon^{-1}\mathring{\Delta}_{\gamma_\nu}^q(\Xi)\right) \oplus_\varepsilon \varepsilon^{-1}\,\delta\,\mathring{\mathsf{H}}_{\gamma_\nu}^{1,q+1}(\Xi),\\
\mathring{\mathsf{D}}_{\gamma_\tau}^q(\Xi) \cap \varepsilon^{-1}\mathring{\Delta}_{\gamma_\nu}^q(\Xi) &= \left(\mathrm{d}\,\mathring{\mathsf{H}}_{\gamma_\tau}^{1,q-1}(\Xi) \cap \varepsilon^{-1}\mathring{\Delta}_{\gamma_\nu}^q(\Xi)\right) \oplus_\varepsilon \left(\mathring{\mathsf{D}}_{\gamma_\tau}^q(\Xi) \cap \varepsilon^{-1}\,\delta\,\mathring{\mathsf{H}}_{\gamma_\nu}^{1,q+1}(\Xi)\right)
\end{aligned}$$

are valid, and the respective regular potentials, given by the operators $\mathcal{S}_\mathrm{d}$ and $\mathcal{S}_\delta$ from Lemma 3.3 and Lemma 3.4, respectively, depend continuously on the data.

Proof. The projection theorem yields $\mathsf{L}_\varepsilon^{2,q}(\Xi) = \overline{\mathrm{d}\,\mathring{\mathsf{D}}_{\gamma_\tau}^{q-1}(\Xi)} \oplus_\varepsilon \varepsilon^{-1}\mathring{\boldsymbol{\Delta}}_{\gamma_\nu,0}^q(\Xi)$. Furthermore,

$$\overline{\mathrm{d}\,\mathring{\mathsf{D}}_{\gamma_\tau}^{q-1}(\Xi)} = \mathrm{d}\,\mathring{\mathsf{D}}_{\gamma_\tau}^{q-1}(\Xi) = \mathrm{d}\,\mathring{\mathsf{H}}_{\gamma_\tau}^{1,q-1}(\Xi) = \mathring{\mathsf{D}}_{\gamma_\tau,0}^q(\Xi)$$

by Lemma 3.3 and

$$\mathring{\boldsymbol{\Delta}}_{\gamma_\nu,0}^q(\Xi) = \mathring{\Delta}_{\gamma_\nu,0}^q(\Xi) = \delta\,\mathring{\mathsf{H}}_{\gamma_\nu}^{1,q+1}(\Xi)$$

by Lemma 3.4. The other assertions follow immediately. □

Lemma 4.2 (Weck's selection theorem for the half-cube). *The embedding*

$$\mathring{\mathsf{D}}^q_{\gamma_\tau}(\Xi) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\gamma_\nu}(\Xi) \hookrightarrow \mathsf{L}^{2,q}_\varepsilon(\Xi)$$

is compact.

Proof. Let $(H_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathring{\mathsf{D}}^q_{\gamma_\tau}(\Xi)\cap\varepsilon^{-1}\mathring{\Delta}^q_{\gamma_\nu}(\Xi)$. By Lemma 4.1, we can decompose

$$H_n = H_n^{\mathrm{d}} + H_n^{\delta} = \mathrm{d}\,E_n^{\mathrm{d}} + \varepsilon^{-1}\,\delta\,E_n^{\delta} \in \big(\mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\gamma_\tau}(\Xi) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\gamma_\nu}(\Xi)\big) \oplus_\varepsilon \big(\mathring{\mathsf{D}}^q_{\gamma_\tau}(\Xi) \cap \varepsilon^{-1}\,\delta\,\mathring{\mathsf{H}}^{1,q+1}_{\gamma_\nu}(\Xi)\big),$$

with $E_n^{\mathrm{d}} = \mathcal{S}_{\mathrm{d}} H_n^{\mathrm{d}}$ and $E_n^{\delta} = \mathcal{S}_{\delta} H_n^{\delta}$. Then $\mathrm{d}\,H_n^{\delta} = \mathrm{d}\,H_n$ and $\delta\,\varepsilon H_n^{\mathrm{d}} = \delta\,\varepsilon H_n$ as well as

$$\begin{aligned}
|E_n^{\mathrm{d}}|_{\mathsf{H}^{1,q-1}(\Xi)} &\le c\,|H_n^{\mathrm{d}}|_{\mathsf{L}^{2,q}(\Xi)} \le c\,|H_n|_{\mathsf{L}^{2,q}_\varepsilon(\Xi)},\\
|E_n^{\delta}|_{\mathsf{H}^{1,q+1}(\Xi)} &\le c\,|H_n^{\delta}|_{\mathsf{L}^{2,q}(\Xi)} \le c\,|H_n|_{\mathsf{L}^{2,q}_\varepsilon(\Xi)}.
\end{aligned}$$

By Rellich's selection theorem and without loss of generality, (E_n^{d}) and (E_n^{δ}) converge in $\mathsf{L}^{2,q-1}(\Xi)$ and $\mathsf{L}^{2,q+1}(\Xi)$, respectively. Moreover,

$$\begin{aligned}
|H_n^{\mathrm{d}} - H_m^{\mathrm{d}}|^2_{\mathsf{L}^{2,q}_\varepsilon(\Xi)} &= \langle H_n^{\mathrm{d}} - H_m^{\mathrm{d}}, \mathrm{d}(E_n^{\mathrm{d}} - E_m^{\mathrm{d}})\rangle_{\mathsf{L}^{2,q}_\varepsilon(\Xi)}\\
&= -\langle \delta\,\varepsilon(H_n^{\mathrm{d}} - H_m^{\mathrm{d}}), E_n^{\mathrm{d}} - E_m^{\mathrm{d}}\rangle_{\mathsf{L}^{2,q-1}(\Xi)} \le c\,|E_n^{\mathrm{d}} - E_m^{\mathrm{d}}|_{\mathsf{L}^{2,q-1}(\Xi)},\\
|H_n^{\delta} - H_m^{\delta}|^2_{\mathsf{L}^{2,q}_\varepsilon(\Xi)} &= \langle H_n^{\delta} - H_m^{\delta}, \varepsilon^{-1}\,\delta(E_n^{\delta} - E_m^{\delta})\rangle_{\mathsf{L}^{2,q}_\varepsilon(\Xi)}\\
&= -\langle \mathrm{d}(H_n^{\delta} - H_m^{\delta}), E_n^{\delta} - E_m^{\delta}\rangle_{\mathsf{L}^{2,q+1}(\Xi)} \le c\,|E_n^{\delta} - E_m^{\delta}|_{\mathsf{L}^{2,q+1}(\Xi)}.
\end{aligned}$$

Thus (H_n^{d}) and (H_n^{δ}) converge in $\mathsf{L}^{2,q}_\varepsilon(\Xi)$ and altogether (H_n) converges in $\mathsf{L}^{2,q}_\varepsilon(\Xi)$ as well. □

Remark 4.3. The use of Helmholtz decompositions and regular potentials in the proof of Lemma 4.2 demonstrates the main idea behind an elegant proof of a compact embedding. This general idea carries over to proofs of compact embeddings related to other kinds of Hilbert complexes as well, arising, e. g., in elasticity, general relativity or biharmonic problems; see, for example, [22].

4.2 The compact embedding for weak Lipschitz domains

The aim of this section is to transfer Lemma 4.2 to arbitrary weak Lipschitz pairs (Ω, Γ_τ). To this end, we will employ a technical lemma, whose proof is sketched in [24, Section 3] and [31, Remark 2]. We give a detailed proof in the Appendix. Let us consider the following situation: Let $\Theta, \widetilde{\Theta}$ be two bounded domains in $\mathbb{R}^N$ with boundaries $\Upsilon := \partial\,\Theta$, $\widetilde{\Upsilon} := \partial\,\widetilde{\Theta}$ and let $\Upsilon_0 \subset \Upsilon$ be relatively open. Moreover, let

$$\phi : \Theta \to \widetilde{\Theta}, \qquad \psi := \phi^{-1} : \widetilde{\Theta} \to \Theta$$

be Lipschitz diffeomorphisms, this is, $\phi \in \mathsf{C}^{0,1}(\Theta, \widetilde{\Theta})$ and $\psi = \phi^{-1} \in \mathsf{C}^{0,1}(\widetilde{\Theta}, \Theta)$. Then $\widetilde{\Theta} = \phi(\Theta)$, $\widetilde{\Upsilon} = \phi(\Upsilon)$ and we define $\widetilde{\Upsilon}_0 := \phi(\Upsilon_0)$.

Lemma 4.4 (Pull-back lemma for Lipschitz transformations). *Let $E \in \mathring{\mathbf{D}}^q_{\Upsilon_0}(\Theta)$, respectively, $E \in \mathring{\mathsf{D}}^q_{\Upsilon_0}(\Theta)$ and $H \in \varepsilon^{-1}\mathring{\mathbf{\Delta}}^q_{\Upsilon_0}(\Theta)$, respectively, $H \in \varepsilon^{-1}\mathring{\Delta}^q_{\Upsilon_0}(\Theta)$ for an admissible transformation ε on $\mathsf{L}^{2,q}(\Theta)$. Then*

$$\begin{aligned} &\psi^* E \in \mathring{\mathbf{D}}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta}), \text{ resp., } \mathring{\mathsf{D}}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta}) && \text{and} && \mathrm{d}\,\psi^* E = \psi^*\,\mathrm{d}\,E, \\ &\psi^* H \in \mu^{-1}\mathring{\mathbf{\Delta}}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta}), \text{ resp., } \mu^{-1}\mathring{\Delta}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta}) && \text{and} && \delta\,\mu\psi^* H = \pm \star \mathrm{d}\,\psi^* \star \varepsilon H = \pm \star \psi^* \star \delta\,\varepsilon H, \end{aligned}$$

where $\mu := (-1)^{qN-1} \star \psi^* \star \varepsilon\phi^*$ is an admissible transformation. Moreover, there exists $c > 0$, independent of E and H, such that

$$|\psi^* E|_{\mathsf{D}^q(\widetilde{\Theta})} \leq c|E|_{\mathsf{D}^q(\Theta)}, \qquad |\psi^* H|_{\mu^{-1}\Delta^q(\widetilde{\Theta})} \leq c|H|_{\varepsilon^{-1}\Delta^q(\Theta)}.$$

Let (Ω, Γ_τ) be a bounded weak Lipschitz pair as introduced in Definitions 2.3 and 2.5. We adjust Lemma 4.4 to our situation: Let $U_1, \ldots, U_K$ be an open covering of Γ according to Definitions 2.3 and 2.5 and set $U_0 := \Omega$. Therefore, $U_0, \ldots, U_K$ is an open covering of $\overline{\Omega}$. Moreover, let $\chi_k \in \mathring{\mathsf{C}}^\infty(U_k)$, $k \in \{0, \ldots, K\}$, be a partition of unity subordinate to the open covering $U_0, \ldots, U_K$. Now suppose $k \in \{1, \ldots, K\}$. We define

$$\begin{aligned} &\Omega_k := U_k \cap \Omega, && \Gamma_k := U_k \cap \Gamma, && \Gamma_{\tau,k} := U_k \cap \Gamma_\tau, && \Gamma_{\nu,k} := U_k \cap \Gamma_\nu, \\ &\widehat{\Gamma}_k := \partial\Omega_k, && \Sigma_k := \widehat{\Gamma}_k \setminus \Gamma, && \widehat{\Gamma}_{\tau,k} := \operatorname{int}(\Gamma_{\tau,k} \cup \overline{\Sigma}_k), && \widehat{\Gamma}_{\nu,k} := \operatorname{int}(\Gamma_{\nu,k} \cup \overline{\Sigma}_k), \\ & && \sigma := \gamma \setminus \overline{B}_0, && \widehat{\gamma}_\tau := \operatorname{int}(\gamma_\tau \cup \overline{\sigma}), && \widehat{\gamma}_\nu := \operatorname{int}(\gamma_\nu \cup \overline{\sigma}). \end{aligned}$$

Lemma 4.4 will from now on be used with

$$\Theta := \Omega_k, \quad \widetilde{\Theta} := \Xi, \qquad \phi := \phi_k : \Omega_k \to \Xi, \quad \psi := \psi_k : \Xi \to \Omega_k$$

and with one of the following cases:

$$\Upsilon_0 := \Gamma_{\tau,k}, \quad \Upsilon_0 := \widehat{\Gamma}_{\tau,k}, \quad \Upsilon_0 := \Gamma_{\nu,k}, \quad \Upsilon_0 := \widehat{\Gamma}_{\nu,k}.$$

Then $\Upsilon = \widehat{\Gamma}_k$ and $\widetilde{\Upsilon} = \phi_k(\widehat{\Gamma}_k) = \gamma$ as well as (depending on the respective case)

$$\begin{aligned} &\widetilde{\Upsilon}_0 = \phi_k(\Gamma_{\tau,k}) = \gamma_\tau, && \widetilde{\Upsilon}_0 = \phi_k(\widehat{\Gamma}_{\tau,k}) = \widehat{\gamma}_\tau, && \gamma_\tau \in \{\emptyset, B_0, B_{0,-}\}, && \gamma_\nu = \gamma \setminus \overline{\gamma}_\tau, \\ &\widetilde{\Upsilon}_0 = \phi_k(\Gamma_{\nu,k}) = \gamma_\nu, && \widetilde{\Upsilon}_0 = \phi_k(\widehat{\Gamma}_{\nu,k}) = \widehat{\gamma}_\nu, && \gamma_\nu \in \{\emptyset, B_0, B_{0,+}\}, && \gamma_\tau = \gamma \setminus \overline{\gamma}_\nu. \end{aligned}$$

Remark 4.5. Lemmas 3.3, 3.4, 3.5, 4.1, 4.2 hold for $\gamma_\nu = B_{0,-}$ without any (substantial) modification as well.

It is straightforward to show the following.

Lemma 4.6 (Localisation). *Let (Ω, Γ_τ) be a bounded weak Lipschitz pair and let k be in $\{1, \ldots, K\}$. Then for $E \in \mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega)$, respectively, $E \in \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega)$ and $H \in \mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Omega)$, respectively, $H \in \mathring{\Delta}^q_{\Gamma_\nu}(\Omega)$ it holds*

$$
\begin{aligned}
&E \in \mathring{\mathbf{D}}^q_{\Gamma_{\tau,k}}(\Omega_k), \qquad \chi_k E \in \mathring{\mathbf{D}}^q_{\widehat{\Gamma}_{\tau,k}}(\Omega_k), \qquad H \in \mathring{\mathbf{\Delta}}^q_{\Gamma_{\nu,k}}(\Omega_k), \qquad \chi_k H \in \mathring{\mathbf{\Delta}}^q_{\widehat{\Gamma}_{\nu,k}}(\Omega_k),\\
&E \in \mathring{\mathsf{D}}^q_{\Gamma_{\tau,k}}(\Omega_k), \qquad \chi_k E \in \mathring{\mathsf{D}}^q_{\widehat{\Gamma}_{\tau,k}}(\Omega_k), \qquad H \in \mathring{\Delta}^q_{\Gamma_{\nu,k}}(\Omega_k), \qquad \chi_k H \in \mathring{\Delta}^q_{\widehat{\Gamma}_{\nu,k}}(\Omega_k).
\end{aligned}
$$

Theorem 4.7 (Weak and strong boundary conditions coincide). *Let the pair (Ω, Γ_τ) be bounded and weak Lipschitz. Then $\mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega) = \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega)$ and $\mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Omega) = \mathring{\Delta}^q_{\Gamma_\nu}(\Omega)$.*

Proof. Suppose $E \in \mathring{\mathbf{D}}^q_{\Gamma_\tau}(\Omega)$. Then we see $\chi_0 E \in \mathring{\mathsf{D}}^q(\Omega) \subset \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega)$ by mollification. Let $k \in \{1, \ldots, K\}$. Then $\chi_k E \in \mathring{\mathbf{D}}^q_{\widehat{\Gamma}_{\tau,k}}(\Omega_k)$ by Lemma 4.6. Lemma 4.4, Lemma 3.5 (with $\gamma_\nu := \gamma_\tau$) and Remark 4.5 yield

$$
\psi_k^*(\chi_k E) \in \mathring{\mathbf{D}}^q_{\widehat{\gamma}_\tau}(\Xi) = \mathring{\mathsf{D}}^q_{\widehat{\gamma}_\tau}(\Xi), \qquad \widehat{\gamma}_\tau = \phi_k(\widehat{\Gamma}_{\tau,k}), \qquad \gamma_\tau \in \{\emptyset, B_0, B_{0,-}\}.
$$

Then $\chi_k E = \chi_k \phi_k^* \psi_k^* E \in \mathring{\mathsf{D}}^q_{\widehat{\Gamma}_{\tau,k}}(\Omega_k) \subset \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega)$ follows by Lemma 4.4. Therefore, we obtain $E = \sum_k \chi_k E \in \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega)$. $\mathring{\mathbf{\Delta}}^q_{\Gamma_\nu}(\Omega) = \mathring{\Delta}^q_{\Gamma_\nu}(\Omega)$ follows analogously or by Hodge-$\star$-duality. □

Now the compact embedding for bounded weak Lipschitz pairs (Ω, Γ_τ) can be proved.

Theorem 4.8 (Weck's selection theorem). *Let (Ω, Γ_τ) be a bounded weak Lipschitz pair and let ε be an admissible transformation on $\mathsf{L}^{2,q}(\Omega)$. Then the embedding*

$$
\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1} \mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \hookrightarrow \mathsf{L}^{2,q}_\varepsilon(\Omega)
$$

is compact.

Proof. Suppose (E_n) is a bounded sequence in $\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1} \mathring{\Delta}^q_{\Gamma_\nu}(\Omega)$. Then by mollification

$$
E_{0,n} := \chi_0 E_n \in \mathring{\mathsf{D}}^q(\Omega) \cap \varepsilon^{-1} \mathring{\Delta}^q(\Omega)
$$

and $E_{0,n}$ even has compact support in Ω. By classical results (see [28, 29, 24]), $(E_{0,n})$ contains a subsequence, which is again denoted by $(E_{0,n})$, converging in $\mathsf{L}^{2,q}_\varepsilon(\Omega)$. Let $k \in \{1, \ldots, K\}$. By Lemma 4.6,

$$
E_{k,n} := \chi_k E_n \in \mathring{\mathsf{D}}^q_{\widehat{\Gamma}_{\tau,k}}(\Omega_k), \qquad \varepsilon E_{k,n} \in \mathring{\Delta}^q_{\widehat{\Gamma}_{\nu,k}}(\Omega_k),
$$

and the sequence $(E_{k,n})$ is bounded in $\mathring{\mathsf{D}}^q_{\widehat{\Gamma}_{\tau,k}}(\Omega_k) \cap \varepsilon^{-1} \mathring{\Delta}^q_{\widehat{\Gamma}_{\nu,k}}(\Omega_k)$ by the product rule. By Lemma 4.4, we have $\psi_k^* E_{k,n} \in \mathring{\mathsf{D}}^q_{\widehat{\gamma}_\tau}(\Xi)$ and

$$
|\psi_k^* E_{k,n}|_{\mathsf{D}^q(\Xi)} \le c |E_{k,n}|_{\mathsf{D}^q(\Omega_k)},
$$

showing that $(\psi_k^* E_{k,n})$ is bounded in $\mathring{\mathsf{D}}^q_{\widehat{\gamma}_\tau}(\Xi)$. Analogously, $(\psi_k^* E_{k,n}) \subset \mu_k^{-1}\mathring{\Delta}^q_{\widehat{\gamma}_\nu}(\Xi)$ is bounded in $\mu_k^{-1}\mathring{\Delta}^q_{\widehat{\gamma}_\nu}(\Xi)$ with the admissible transformation $\mu_k := (-1)^{qN-1} \star \psi_k^* \star \varepsilon\phi_k^*$. Thus $(\psi_k^* E_{k,n})$ is bounded in

$$\mathring{\mathsf{D}}^q_{\widehat{\gamma}_\tau}(\Xi) \cap \mu_k^{-1}\mathring{\Delta}^q_{\widehat{\gamma}_\nu}(\Xi) \subset \mathring{\mathsf{D}}^q_{\widehat{\gamma}_\tau}(\Xi) \cap \mu_k^{-1}\mathring{\Delta}^q_{\gamma_\nu}(\Xi), \quad \gamma_\nu \in \{\emptyset, B_0, B_{0,+}\}, \quad \widehat{\gamma}_\tau = \gamma \setminus \overline{\gamma}_\nu.$$

Thus, by Lemma 4.2 and without loss of generality, $(\psi_k^* E_{k,n})$ is a Cauchy sequence in $\mathsf{L}^{2,q}(\Xi)$. Now

$$E_{k,n} = \phi_k^* \psi_k^* E_{k,n} \in \mathsf{L}^{2,q}(\Omega_k)$$

and Lemma 4.4 yields

$$|E_{k,n} - E_{k,m}|_{\mathsf{L}^{2,q}(\Omega_k)} \leq c|\psi_k^* E_{k,n} - \psi_k^* E_{k,m}|_{\mathsf{L}^{2,q}(\Xi)}.$$

Hence $(E_{k,n})$ is a Cauchy sequence in $\mathsf{L}^{2,q}(\Omega_k)$ and so in $\mathsf{L}^{2,q}_\varepsilon(\Omega)$ for their extensions by zero to Ω. Finally, extracting convergent subsequences for $k = 1, \ldots, K$, we see that

$$(E_n) = \left(\sum_{k=0}^{K} \chi_k E_n\right) = \left(\sum_{k=0}^{K} E_{k,n}\right)$$

is a Cauchy sequence in $\mathsf{L}^{2,q}_\varepsilon(\Omega)$. □

Remark 4.9 (Independence of the transformation). By standard techniques, it can be shown that Weck's selection theorem is independent of the transformation ε, i. e., the compactness of the embedding in Theorem 4.8 does not depend on ε. For details, see [2].

5 Applications

From now on, let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let (Ω, Γ_τ) be a weak Lipschitz pair as well as $\varepsilon : \mathsf{L}^{2,q}(\Omega) \to \mathsf{L}^{2,q}(\Omega)$ be admissible. Then by Theorem 4.8, the embedding

$$\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \hookrightarrow \mathsf{L}^{2,q}(\Omega) \tag{9}$$

is compact. The results of this section immediately follow in the framework of a general functional analytic toolbox; see [21, 20, 22]. For details, see also the proofs in [1] for the classical case of vector analysis.

5.1 The Maxwell estimate

A first consequence of (9) is that the space of so-called "harmonic" Dirichlet–Neumann forms

$$\mathcal{H}^q_\varepsilon(\Omega) := \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega)$$

is finite-dimensional, as the unit ball in $\mathcal{H}^q_\varepsilon(\Omega)$ is compact by (9). Using the Helmholtz projections of Theorem 5.2, we see that the dimension of $\mathcal{H}^q_\varepsilon(\Omega)$ does not depend on ε, in particular $\dim \mathcal{H}^q_\varepsilon(\Omega) = \mathcal{H}^q(\Omega)$. By a standard indirect argument, (9) immediately implies the so-called Maxwell estimate.

Theorem 5.1 (Maxwell estimate). *There exists a positive constant c_{m}, such that for all $E \in \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp_\varepsilon}$*

$$|E|_{\mathsf{L}^{2,q}_\varepsilon(\Omega)} \le c_{\mathrm{m}} \left(|\,\mathrm{d}\,E|^2_{\mathsf{L}^{2,q+1}(\Omega)} + |\,\delta\,\varepsilon E|^2_{\mathsf{L}^{2,q-1}(\Omega)} \right)^{1/2}.$$

Here, we denote by $\perp_\varepsilon$ orthogonality with respect to the $\mathsf{L}^{2,q}_\varepsilon(\Omega)$-inner product.

5.2 Helmholtz decompositions

Applying the projection theorem to the densely defined and closed (unbounded) linear operators,

$$\mathrm{d}^{q-1}_\tau : \mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \subset \mathsf{L}^{2,q-1}(\Omega) \to \mathsf{L}^{2,q}_\varepsilon(\Omega)\,; \quad E \mapsto \mathrm{d}\,E$$

with (Hilbert space) adjoint (see Theorem 4.7)

$$-\,\delta^q_\nu\,\varepsilon := \left(\mathrm{d}^{q-1}_\tau\right)^* : \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \subset \mathsf{L}^{2,q}_\varepsilon(\Omega) \to \mathsf{L}^{2,q-1}(\Omega)\,; \quad H \mapsto -\,\delta\,\varepsilon H$$

and

$$-\varepsilon^{-1}\,\delta^{q+1}_\nu : \varepsilon^{-1}\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega) \subset \mathsf{L}^{2,q+1}(\Omega) \to \mathsf{L}^{2,q}_\varepsilon(\Omega)\,; \quad H \mapsto -\varepsilon^{-1}\,\delta\,H$$

with adjoint (see Theorem 4.7)

$$\mathrm{d}^q_\tau := \left(-\varepsilon^{-1}\,\delta^{q+1}_\nu\right)^* : \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \subset \mathsf{L}^{2,q}_\varepsilon(\Omega) \to \mathsf{L}^{2,q+1}(\Omega)\,; \quad E \mapsto \mathrm{d}\,E$$

we obtain the Helmholtz decompositions

$$\mathsf{L}^{2,q}_\varepsilon(\Omega) = \overline{\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)} \oplus_\varepsilon \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega), \tag{10}$$

$$\mathsf{L}^{2,q}_\varepsilon(\Omega) = \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \oplus_\varepsilon \overline{\varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)}. \tag{11}$$

Therefore, $\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) = \overline{\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)} \oplus_\varepsilon \mathcal{H}^q_\varepsilon(\Omega)$ and, altogether, we get the refined Helmholtz decomposition

$$\mathsf{L}^{2,q}_\varepsilon(\Omega) = \overline{\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)} \oplus_\varepsilon \mathcal{H}^q_\varepsilon(\Omega) \oplus_\varepsilon \overline{\varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)}. \tag{12}$$

Theorem 5.2 (Helmholtz decompositions). *The orthonormal decompositions*

$$\mathsf{L}^{2,q}_\varepsilon(\Omega) = \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \oplus_\varepsilon \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega)$$

$$
\begin{aligned}
&= \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \oplus_\varepsilon \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)\\
&= \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \oplus_\varepsilon \mathcal{H}^q_\varepsilon(\Omega) \oplus_\varepsilon \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)
\end{aligned}
$$

hold. Furthermore,

$$
\begin{aligned}
\mathrm{d}\,\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) &= \mathrm{d}\big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)\big) = \mathrm{d}\big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp_\varepsilon}\big),\\
\delta\,\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) &= \delta\big(\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \varepsilon\,\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)\big) = \delta\big(\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \varepsilon\big(\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp_\varepsilon}\big)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) &= \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp_\varepsilon}, &\qquad \delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega) &= \mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp},\\
\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) &= \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \oplus_\varepsilon \mathcal{H}^q_\varepsilon(\Omega), &\qquad \mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega) &= \delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega) \oplus_{\varepsilon^{-1}} \varepsilon\,\mathcal{H}^q_\varepsilon(\Omega).
\end{aligned}
$$

The ranges $\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)$ *and* $\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)$ *are closed subspaces of* $\mathsf{L}^{2,q}_\varepsilon(\Omega)$*. Moreover, the* d-*, respectively,* δ*-potentials are uniquely determined in* $\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp_\varepsilon}$ *and* $\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \varepsilon(\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp_\varepsilon})$*, respectively, and depend continuously on their respective images.*

Proof. For $\varepsilon = \mathrm{id}$, (10) and (11) yield

$$
\begin{aligned}
\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) &= \big(\overline{\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)} \cap \mathring{\Delta}^q_{\Gamma_\nu}(\Omega)\big) \oplus \mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega),\\
\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) &= \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \oplus \big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \overline{\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)}\big)
\end{aligned}
$$

and thus with (10), (11) and (12)

$$
\begin{aligned}
\delta\,\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) &= \delta\big(\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \overline{\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)}\big) = \delta\big(\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \mathcal{H}^q(\Omega)^\perp\big),\\
\mathrm{d}\,\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) &= \mathrm{d}\big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \overline{\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)}\big) = \mathrm{d}\big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega) \cap \mathcal{H}^q(\Omega)^\perp\big).
\end{aligned}
$$

Now Theorem 5.1 implies the closedness of the ranges and the continuity of the potentials. The other assertions follow immediately. □

Corollary 5.3 (Refined Helmholtz decompositions). *It holds*

$$
\begin{aligned}
\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) &= \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \oplus_\varepsilon \big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega)\big)\\
&= \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \oplus_\varepsilon \big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)\big)\\
&= \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \oplus_\varepsilon \mathcal{H}^q_\varepsilon(\Omega) \oplus_\varepsilon \big(\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)\big),\\
\varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) &= \big(\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega)\big) \oplus_\varepsilon \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega)\\
&= \big(\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega)\big) \oplus_\varepsilon \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega)\\
&= \big(\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega)\big) \oplus_\varepsilon \mathcal{H}^q_\varepsilon(\Omega) \oplus_\varepsilon \varepsilon^{-1}\,\delta\,\mathring{\Delta}^{q+1}_{\Gamma_\nu}(\Omega).
\end{aligned}
$$

5.3 Static solution theory

As a further application, we turn to the boundary value problem of generalized electro and magnetostatics with mixed boundary values: Let $F \in \mathsf{L}^{2,q+1}(\Omega)$, $G \in \mathsf{L}^{2,q-1}(\Omega)$, E_τ, $E_\nu \in \mathsf{L}^{2,q}_\varepsilon(\Omega)$ and let ε be admissible. The problem is to find $E \in \mathsf{D}^q(\Omega) \cap \varepsilon^{-1}\Delta^q(\Omega)$ such that

$$\begin{aligned} \mathrm{d}\, E &= F, \\ \delta\, \varepsilon E &= G, \\ E - E_\tau &\in \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega), \\ \varepsilon(E - E_\nu) &\in \mathring{\Delta}^q_{\Gamma_\nu}(\Omega). \end{aligned} \tag{13}$$

For uniqueness, we require the additional conditions

$$\langle \varepsilon E, D_\ell \rangle_{\mathsf{L}^{2,q}_\varepsilon(\Omega)} = \alpha_\ell \in \mathbb{R}, \quad \ell = 1, \ldots, d, \tag{14}$$

where d is the dimension and $\{D_\ell\}$ an ε-orthonormal basis of $\mathcal{H}^q_\varepsilon(\Omega)$. The boundary values on Γ_τ and Γ_ν, respectively, are realised by the given volume forms E_τ and E_ν, respectively.

Theorem 5.4 (Static solution theory). (13) *admits a solution, if and only if*

$$E_\tau \in \mathsf{D}^q(\Omega), \quad E_\nu \in \varepsilon^{-1}\Delta^q(\Omega),$$

and

$$F - \mathrm{d}\, E_\tau \perp \mathring{\Delta}^{q+1}_{\Gamma_\nu,0}(\Omega), \quad G - \delta\, \varepsilon E_\nu \perp \mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau,0}(\Omega). \tag{15}$$

The solution $E \in \mathsf{D}^q(\Omega) \cap \varepsilon^{-1}\Delta^q(\Omega)$ can be chosen in a way such that condition (14) *with $\alpha \in \mathbb{R}^d$ is fulfilled, which then uniquely determines the solution. Furthermore, the solution depends linearly and continuously on the data.*

Note that (15) is equivalent to

$$F - \mathrm{d}\, E_\tau \in \mathrm{d}\, \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega), \qquad G - \delta\, \varepsilon E_\nu \in \delta\, \mathring{\Delta}^q_{\Gamma_\nu}(\Omega).$$

For homogeneous boundary data, i. e., $E_\tau = E_\nu = 0$, the latter theorem immediately follows from a functional analytic toolbox (see [21, 20, 22]), which even states a sharper result: The linear static Maxwell-operator

$$\begin{aligned} M \quad : \quad \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) &\longrightarrow \mathrm{d}\, \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \times \delta\, \mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \times \mathbb{R}^d \\ E &\longmapsto (\mathrm{d}\, E, \delta\, \varepsilon E, (\langle \varepsilon E, D_\ell \rangle_{\mathsf{L}^{2,q}_\varepsilon(\Omega)})_{\ell=1}^d) \end{aligned}$$

is a topological isomorphism. Its inverse M^{-1} maps not only continuously onto its domain of definition $\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega)$, but also compactly into $\mathsf{L}^{2,q}_\varepsilon(\Omega)$ by (9). For homogeneous kernel data, i. e., for

$$\begin{aligned} M_0 \quad : \quad \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \cap \varepsilon^{-1}\mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \cap \mathcal{H}^q_\varepsilon(\Omega)^{\perp_\varepsilon} &\longrightarrow \mathrm{d}\, \mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \times \delta\, \mathring{\Delta}^q_{\Gamma_\nu}(\Omega) \\ E &\longmapsto (\mathrm{d}\, E, \delta\, \varepsilon E) \end{aligned},$$

we have $|M_0^{-1}| \le (c_{\mathrm{m}}^2 + 1)^{1/2}$. For details and a proof of Theorem 5.4 in the classical setting of vector analysis, see [1].

5.4 General regular potentials and decompositions

A closer inspection of the proof of Lemma 3.3 shows that Lemma 3.3 and Lemma 3.4 hold for more general situations. Using the partition of unity from Section 4.2 and the concept of extendable strong Lipschitz pairs, we can even generalise Lemma 3.3 and Lemma 3.4 to general strong Lipschitz pairs. Note that by Theorem 5.2

$$\mathrm{d}\,\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) = \mathring{\mathsf{D}}^{q+1}_{\Gamma_\tau,0}(\Omega) \cap \mathcal{H}^{q+1}_\varepsilon(\Omega)^{\perp_\varepsilon}, \qquad \mathring{\mathsf{D}}^{q+1}_{\Gamma_\tau,0}(\Omega) = \mathrm{d}\,\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \oplus_\varepsilon \mathcal{H}^{q+1}_\varepsilon(\Omega). \tag{16}$$

Theorem 5.5 (Regular potentials and decompositions for strong Lipschitz domains). *Let $\Omega \subset \mathbb{R}^N$ and let (Ω, Γ_τ) be a bounded strong Lipschitz pair.*

(i) *There exists a continuous linear operator*

$$\mathcal{S}^q_{\mathrm{d}} : \mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) \to \mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega),$$

such that $\mathrm{d}\,\mathcal{S}^q_{\mathrm{d}} = \mathrm{id}|_{\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega)}$. *Especially,*

$$\mathrm{d}\,\mathring{\mathsf{D}}^{q-1}_{\Gamma_\tau}(\Omega) = \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega)$$

and the regular $\mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega)$-potential depends continuously on the data. In particular, these spaces are closed subspaces of $\mathsf{L}^{2,q}(\Omega)$ and $\mathcal{S}^q_{\mathrm{d}}$ is a right inverse to d.

(ii) *The regular decompositions*

$$\begin{aligned}\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) &= \mathring{\mathsf{H}}^{1,q}_{\Gamma_\tau}(\Omega) + \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega)\\ &= \mathcal{S}^{q+1}_{\mathrm{d}}\,\mathrm{d}\,\mathring{\mathsf{D}}^q_{\Gamma_\tau}(\Omega) \dotplus \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega),\end{aligned}$$

$$\begin{aligned}\mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) &= \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega) + \big(\mathring{\mathsf{H}}^{1,q}_{\Gamma_\tau}(\Omega) \cap \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega)\big)\\ &= \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega) \oplus \mathcal{H}^q(\Omega)\\ &= \mathrm{d}\,\mathring{\mathsf{H}}^{1,q-1}_{\Gamma_\tau}(\Omega) \oplus_\varepsilon \mathcal{H}^q_\varepsilon(\Omega)\end{aligned}$$

hold with linear and continuous regular decomposition, respectively, potential operators, which can be defined explicitly by the orthonormal Helmholtz projectors and the operators $\mathcal{S}^q_{\mathrm{d}}$. Note that $\mathcal{H}^q(\Omega)$ is a subspace of smooth forms, i. e., it holds $\mathcal{H}^q(\Omega) = \mathring{\mathsf{D}}^q_{\Gamma_\tau,0}(\Omega) \cap \mathring{\Delta}^q_{\Gamma_\nu,0}(\Omega) \cap \mathsf{C}^{\infty,q}(\Omega)$.

Hodge-⋆-duality yields the corresponding results for the co-derivative δ.

For details, see [2]. In the case of no or full boundary conditions, related results on regular potentials and regular decompositions are presented in [4].

Appendix A. Proof of Lemma 4.4 (pull-back lemma for Lipschitz transformations)

We start out by proving the assertions for the exterior derivative.

A.1 Without boundary conditions

Let $E = \sum_I E_I \,\mathrm{d}\, x^I \in \mathsf{D}^q(\Theta)$. We have to show $\psi^* E \in \mathsf{D}^q(\widetilde{\Theta})$ with $\mathrm{d}\,\psi^* E = \psi^* \,\mathrm{d}\, E$.

(i) Let us first consider $\Phi = \sum_I \Phi_I \,\mathrm{d}\, x^I \in \mathsf{C}^{0,1,q}(\Theta)$, i. e., $\Phi_I \in \mathsf{C}^{0,1}(\Theta)$ for all I. In the following, we denote by $\widetilde{\cdot}$ the composition with ψ. We have

$$\mathrm{d}\,\psi_j = \sum_i \partial_i \psi_j \,\mathrm{d}\, x^i, \qquad \psi^* \Phi = \sum_I \widetilde{\Phi}_I \psi^* \,\mathrm{d}\, x^I = \sum_I \widetilde{\Phi}_I (\mathrm{d}\,\psi_{i_1}) \wedge \cdots \wedge (\mathrm{d}\,\psi_{i_q}),$$
$$\mathrm{d}\,\Phi = \sum_{I,j} \partial_j \Phi_I (\mathrm{d}\, x_j) \wedge (\mathrm{d}\, x^I).$$

By Rademacher's theorem, $\widetilde{\Phi}_I = \Phi_I \circ \psi$ and ψ_j belong to $\mathsf{C}^{0,1}(\widetilde{\Theta}) \subset \mathsf{H}^1(\widetilde{\Theta})$ and the chain rule holds, i. e., $\partial_i \widetilde{\Phi}_I = \sum_j \widetilde{\partial_j \Phi_I} \partial_i \psi_j$. As $\psi_j \in \mathsf{H}^1(\widetilde{\Theta})$ we get $\mathrm{d}\,\psi_j \in \mathsf{D}^1_0(\widetilde{\Theta})$ by

$$\langle \mathrm{d}\,\psi_j, \delta\,\varphi \rangle_{\mathsf{L}^{2,1}(\widetilde{\Theta})} = -\langle \psi_j, \delta\,\delta\,\varphi \rangle_{\mathsf{L}^{2,0}(\widetilde{\Theta})} = 0$$

for all $\varphi \in \mathring{\mathsf{C}}^{\infty,2}(\widetilde{\Theta})$. Thus by definition, we see

$$\begin{aligned}
\mathrm{d}\,\psi^* \Phi &= \sum_I (\mathrm{d}\,\widetilde{\Phi}_I) \wedge (\mathrm{d}\,\psi_{i_1}) \wedge \cdots \wedge (\mathrm{d}\,\psi_{i_q}) = \sum_{I,i} \partial_i \widetilde{\Phi}_I (\mathrm{d}\, x^i) \wedge (\mathrm{d}\,\psi_{i_1}) \wedge \cdots \wedge (\mathrm{d}\,\psi_{i_q}) \\
&= \sum_{I,i,j} \widetilde{\partial_j \Phi_I} \partial_i \psi_j (\mathrm{d}\, x^i) \wedge (\mathrm{d}\,\psi_{i_1}) \wedge \cdots \wedge (\mathrm{d}\,\psi_{i_q}) \\
&= \sum_{I,j} \widetilde{\partial_j \Phi_I} (\mathrm{d}\,\psi_j) \wedge (\mathrm{d}\,\psi_{i_1}) \wedge \cdots \wedge (\mathrm{d}\,\psi_{i_q}).
\end{aligned}$$

On the other hand, it holds

$$\psi^* \,\mathrm{d}\,\Phi = \sum_{I,j} \widetilde{\partial_j \Phi_I} (\psi^* \,\mathrm{d}\, x_j) \wedge (\psi^* \,\mathrm{d}\, x^I) = \sum_{I,j} \widetilde{\partial_j \Phi_I} (\mathrm{d}\,\psi_j) \wedge (\mathrm{d}\,\psi_{i_1}) \wedge \cdots \wedge (\mathrm{d}\,\psi_{i_q}).$$

Therefore, $\psi^* \Phi \in \mathsf{D}^q(\widetilde{\Theta})$ and $\mathrm{d}\,\psi^* \Phi = \psi^* \,\mathrm{d}\,\Phi$.

(ii) For general $E \in \mathsf{D}^q(\Theta)$, we pick $\Phi \in \mathring{\mathsf{C}}^{\infty,q+1}(\widetilde{\Theta})$. Note $\operatorname{supp}\Phi \subset\subset \widetilde{\Theta} = \phi(\Theta)$. Replacing ψ by ϕ in (i) we have $\phi^* \star \Phi \in \mathsf{D}^{N-q-1}(\Theta)$ with $\mathrm{d}\,\phi^* \star \Phi = \phi^* \,\mathrm{d} \star\Phi$ and, since $\phi^* \star \Phi = \sum_I \widetilde{(\star\Phi)_I} \phi^* \,\mathrm{d}\, x^I$ holds, $\operatorname{supp}\phi^* \star \Phi \subset\subset \Theta$. By standard mollification, we obtain a sequence $(\Psi_n) \subset \mathring{\mathsf{C}}^{\infty,N-q-1}(\Theta)$ with $\Psi_n \to \phi^* \star \Phi$ in $\mathsf{D}^{N-q-1}(\Theta)$. Furthermore, $\star\Psi_n \in \mathring{\mathsf{C}}^{\infty,q+1}(\Theta)$. Then

$$\langle \psi^* E, \delta\,\Phi \rangle_{\mathsf{L}^{2,q}(\widetilde{\Theta})} = \int_{\widetilde{\Theta}} \psi^* E \wedge \star\,\delta\,\Phi = \pm \int_{\widetilde{\Theta}} \psi^* E \wedge \psi^* \phi^* \,\mathrm{d} \star\Phi = \pm \int_{\widetilde{\Theta}} \psi^* (E \wedge \phi^* \,\mathrm{d} \star\Phi)$$

$$= \pm \int_{\Theta} E \wedge \phi^* \,\mathrm{d} \star \Phi = \pm \int_{\Theta} E \wedge \mathrm{d}\, \phi^* \star \Phi \leftarrow \pm \int_{\Theta} E \wedge \mathrm{d}\, \Psi_n$$
$$= \pm \int_{\Theta} E \wedge \star \star \mathrm{d} \star \star \Psi_n = \pm \langle E, \delta \star \Psi_n \rangle_{\mathsf{L}^{2,q}(\Theta)}$$
$$= \pm \langle \mathrm{d}\, E, \star \Psi_n \rangle_{\mathsf{L}^{2,q+1}(\Theta)} \to \pm \big\langle \mathrm{d}\, E, \star \phi^* \star \Phi \big\rangle_{\mathsf{L}^{2,q+1}(\Theta)} = \pm \int_{\Theta} \mathrm{d}\, E \wedge \phi^* \star \Phi$$
$$= \pm \int_{\widetilde{\Theta}} \psi^* (\mathrm{d}\, E \wedge \phi^* \star \Phi) = \pm \int_{\widetilde{\Theta}} (\psi^* \,\mathrm{d}\, E) \wedge \star \Phi = -\big\langle \psi^* \,\mathrm{d}\, E, \Phi \big\rangle_{\mathsf{L}^{2,q+1}(\widetilde{\Theta})}$$

and hence $\psi^* E \in \mathsf{D}^q(\widetilde{\Theta})$ with $\mathrm{d}\, \psi^* E = \psi^* \,\mathrm{d}\, E$.

(iii) Let $E \in \mathsf{D}^q(\Theta)$. By (ii), we know $\psi^* E \in \mathsf{D}^q(\widetilde{\Theta})$ with $\mathrm{d}\, \psi^* E = \psi^* \,\mathrm{d}\, E$. Hence

$$|\psi^* E|^2_{\mathsf{L}^{2,q}(\widetilde{\Theta})} = \int_{\widetilde{\Theta}} \psi^* E \wedge \star \psi^* E = \int_{\Theta} \phi^* \psi^* E \wedge \phi^* \star \psi^* E$$
$$= \pm \int_{\Theta} E \wedge \star (\star \phi^* \star \psi^*) E \leq c |E|^2_{\mathsf{L}^{2,q}(\Theta)}$$

and

$$|\mathrm{d}\, \psi^* E|_{\mathsf{L}^{2,q+1}(\widetilde{\Theta})} = |\psi^* \,\mathrm{d}\, E|_{\mathsf{L}^{2,q+1}(\widetilde{\Theta})} \leq c |\mathrm{d}\, E|_{\mathsf{L}^{2,q+1}(\Theta)}.$$

A.2 With strong boundary condition

Let $E \in \mathring{\mathsf{D}}^q_{\Upsilon_0}(\Theta)$ and $(E_n) \subset \mathring{\mathsf{C}}^{\infty,q}_{\Upsilon_0}(\Theta)$ with $E_n \to E$ in $\mathsf{D}^q(\Theta)$. By Appendix A.1(ii), we know $\psi^* E_n, \psi^* E \in \mathsf{D}^q(\widetilde{\Theta})$ with $\mathrm{d}\, \psi^* E_n = \psi^* \,\mathrm{d}\, E_n$ as well as $\mathrm{d}\, \psi^* E = \psi^* \,\mathrm{d}\, E$. Furthermore, $\psi^* E_n$ has compact support away from $\widetilde{\Upsilon}_0$. Using standard mollification, we obtain $\psi^* E_n \in \mathring{\mathsf{D}}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta})$. Moreover, by A.1(iii), $\psi^* E_n \to \psi^* E$ in $\mathsf{D}^q(\widetilde{\Theta})$. Therefore, $\psi^* E \in \mathring{\mathsf{D}}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta})$ with $\mathrm{d}\, \psi^* E = \psi^* \,\mathrm{d}\, E$.

A.3 With weak boundary condition

Let $E \in \mathring{\mathbf{D}}^q_{\Upsilon_0}(\Theta)$ and $\Phi \in \mathring{\mathsf{C}}^{\infty,q+1}_{\widetilde{\Upsilon}_1}(\widetilde{\Theta})$, where $\Upsilon_1 = \Upsilon \setminus \overline{\Upsilon}_0$. By Appendix A.1(ii), we again know $\psi^* E \in \mathsf{D}^q(\widetilde{\Theta})$ with $\mathrm{d}\, \psi^* E = \psi^* \,\mathrm{d}\, E$. Moreover, by Appendix A.2, we have $\phi^* \star \Phi \in \mathring{\mathsf{D}}^{N-q-1}_{\Upsilon_1}(\Theta)$, and hence $\star \phi^* \star \Phi \in \mathring{\Delta}^{q+1}_{\Upsilon_1}(\Theta)$. We repeat the calculation from Appendix A.1(ii) to arrive at

$$\big\langle \psi^* E, \delta\, \Phi \big\rangle_{\mathsf{L}^{2,q}(\widetilde{\Theta})} = \int_{\widetilde{\Theta}} \psi^* E \wedge \star\, \delta\, \Phi = \pm \big\langle E, \star \phi^* \,\mathrm{d} \star \Phi \big\rangle_{\mathsf{L}^{2,q}(\Theta)}$$
$$= \pm \big\langle E, \star \,\mathrm{d}\, \phi^* \star \Phi \big\rangle_{\mathsf{L}^{2,q}(\Theta)} = \pm \big\langle E, \delta \star \phi^* \star \Phi \big\rangle_{\mathsf{L}^{2,q}(\Theta)}$$

$$= \pm\langle \mathrm{d}\, E, \star\phi^* \star \Phi\rangle_{\mathsf{L}^{2,q+1}(\Theta)} = -\langle \psi^* \,\mathrm{d}\, E, \Phi\rangle_{\mathsf{L}^{2,q+1}(\widetilde{\Theta})} = -\langle \mathrm{d}\, \psi^* E, \Phi\rangle_{\mathsf{L}^{2,q+1}(\widetilde{\Theta})}$$

and, therefore, $\psi^* E \in \mathring{\mathbf{D}}^q_{\widetilde{\Upsilon}_0}(\widetilde{\Theta})$.

A.4 Assertions for the co-derivative

It holds by Appendix A.1(ii),

$$\varepsilon H \in \Delta^q(\Theta) \;\Leftrightarrow\; \star\varepsilon H \in \mathsf{D}^{N-q}(\Theta) \;\Leftrightarrow\; \psi^* \star \varepsilon\phi^*\psi^* H \in \mathsf{D}^{N-q}(\widetilde{\Theta}) \;\Leftrightarrow\; \mu\psi^* H \in \Delta^q(\widetilde{\Theta}).$$

Moreover, using Appendix A.1(iii) μ is admissible since for all $H \in \mathsf{L}^{2,q}(\widetilde{\Theta})$,

$$\begin{aligned}
\langle \mu H, H\rangle_{\mathsf{L}^{2,q}(\widetilde{\Theta})} &= \pm\langle \star\psi^* \star \varepsilon\phi^* H, H\rangle_{\mathsf{L}^{2,q}(\widetilde{\Theta})} = \pm\langle \psi^* \star \varepsilon\phi^* H, \star H\rangle_{\mathsf{L}^{2,N-q}(\widetilde{\Theta})} \\
&= \pm\int_{\widetilde{\Theta}} \psi^* \star \varepsilon\phi^* H \wedge H = \pm\int_{\Theta} \star\varepsilon\phi^* H \wedge \star\star\phi^* H \\
&= \pm\langle \varepsilon\phi^* H, \phi^* H\rangle_{\mathsf{L}^{2,q}(\Theta)} \geq c|\phi^* H|^2_{\mathsf{L}^{2,q}(\Theta)} \geq c|H|^2_{\mathsf{L}^{2,q}(\widetilde{\Theta})}.
\end{aligned}$$

Furthermore,

$$\delta\, \mu\psi^* H = \pm \star \mathrm{d}\, \psi^* \star \varepsilon H = \pm \star \psi^* \star \delta\, \varepsilon H.$$

The remaining assertions now follow by Appendix A.1–A.3 and Hodge-$\star$-duality.

Bibliography

[1] S. Bauer, D. Pauly, and M. Schomburg, The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions, *SIAM J. Math. Anal.*, **48**(4) (2016), 2912–2943.

[2] S. Bauer, D. Pauly, and M. Schomburg, Weck's selection theorem: The Maxwell compactness property for bounded weak Lipschitz domains with mixed boundary conditions in arbitrary dimensions. https://arxiv.org/abs/1809.01192, 2018.

[3] M. Costabel, A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains, *Math. Methods Appl. Sci.*, **12**(4) (1990), 365–368.

[4] M. Costabel and A. McIntosh, On Bogovskii and regularized Poincare integral operators for de Rham complexes on Lipschitz domains, *Math. Z.*, **265**(2) (2010), 297–320.

[5] D. M. Eidus, On the principle of limiting absorption, *Mat. Sb. (N.S.)*, **57**(99) (1962), 13–44.

[6] D. M. Eidus, On the limiting amplitude principle, *Dokl. Akad. Nauk SSSR*, **158** (1964), 794–797.

[7] D. M. Eidus, The principle of limiting amplitude, *Usp. Mat. Nauk*, **24**(3(147)) (1969), 91–156.

[8] P. Fernandes and G. Gilardi, Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions, *Math. Models Methods Appl. Sci.*, **7**(7) (1997), 957–991.

[9] R. Hiptmair, J. Li, and J. Zou, Universal extension for Sobolev spaces of differential forms and applications, *J. Funct. Anal.*, **263** (2012), 364–382.

[10] K. Jänich, *Vector Analysis*, Springer, New York, 2001.
[11] F. Jochmann, A compactness result for vector fields with divergence and curl in $L^q(\Omega)$ involving mixed boundary conditions, *Appl. Anal.*, **66** (1997), 189–203.
[12] P. Kuhn and D. Pauly, Regularity results for generalized electro-magnetic problems, *Analysis (Munich)*, **30**(3) (2010), 225–252.
[13] R. Leis, *Initial Boundary Value Problems in Mathematical Physics*, Teubner, Stuttgart, 1986.
[14] D. Pauly, *Niederfrequenzasymptotik der Maxwell-Gleichung im inhomogenen und anisotropen Außengebiet*. Dissertation, Universität Duisburg-Essen, Fachbereich Mathematik, http://duepublico.uni-duisburg-essen.de, 2003.
[15] D. Pauly, Low frequency asymptotics for time-harmonic generalized Maxwell equations in nonsmooth exterior domains, *Adv. Math. Sci. Appl.*, **16**(2) (2006), 591–622.
[16] D. Pauly, Generalized electro-magneto statics in nonsmooth exterior domains, *Analysis (Munich)*, **27**(4) (2007), 425–464.
[17] D. Pauly, Complete low frequency asymptotics for time-harmonic generalized Maxwell equations in nonsmooth exterior domains, *Asymptot. Anal.*, **60**(3–4) (2008), 125–184.
[18] D. Pauly, Hodge–Helmholtz decompositions of weighted Sobolev spaces in irregular exterior domains with inhomogeneous and anisotropic media, *Math. Methods Appl. Sci.*, **31** (2008), 1509–1543.
[19] D. Pauly, On polynomial and exponential decay of eigen-solutions to exterior boundary value problems for the generalized time-harmonic Maxwell system, *Asymptot. Anal.*, **79**(1–2) (2012), 133–160.
[20] D. Pauly, A global div–curl-lemma for mixed boundary conditions in weak Lipschitz domains and a corresponding generalized A_0^*-A_1-lemma in Hilbert spaces, *Analysis (Munich)*, https://arxiv.org/abs/1707.00019, 2019.
[21] D. Pauly, Solution theory, variational formulations, and functional a posteriori error estimates for general first order systems with applications to electro-magneto statics and more, *Numer. Funct. Anal. Optim.*, https://arxiv.org/abs/1611.02993, 2019.
[22] D. Pauly and W. Zulehner, The divDiv-complex and applications to biharmonic equations, *Appl. Anal.*, https://arxiv.org/abs/1609.05873, 2019.
[23] R. Picard, Randwertaufgaben der verallgemeinerten Potentialtheorie, *Math. Methods Appl. Sci.*, **3** (1981), 218–228.
[24] R. Picard, An elementary proof for a compact imbedding result in generalized electromagnetic theory, *Math. Z.*, **187** (1984), 151–164.
[25] R. Picard, N. Weck, and K.-J. Witsch, Time-harmonic Maxwell equations in the exterior of perfectly conducting, irregular obstacles, *Analysis (Munich)*, **21** (2001), 231–263.
[26] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey, 1970.
[27] C. Weber, A local compactness theorem for Maxwell's equations, *Math. Methods Appl. Sci.*, **2** (1980), 12–25.
[28] N. Weck, Über die Kompaktheit des Lösungsoperators zu den Maxwellschen Gleichungen, *B. I. – Hochschultaschenbücher, Bibliographisches Inst., Mannheim*, **725** (1972), 41–55.
[29] N. Weck, Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries, *J. Math. Anal. Appl.*, **46** (1974), 410–437.
[30] N. Weck, Local compactness for linear elasticity in irregular domains, *Math. Methods Appl. Sci.*, **17** (1994), 107–113.
[31] N. Weck, Traces of differential forms on Lipschitz boundaries, *Analysis (Munich)*, **24** (2004), 147–169.
[32] K.-J. Witsch, A remark on a compactness result in electromagnetic theory, *Math. Methods Appl. Sci.*, **16** (1993), 123–129.

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4 Numerical analysis of the half-space matching method with Robin traces on a convex polygonal scatterer

Abstract: We consider the 2D Helmholtz equation with a complex wavenumber in the exterior of a convex polygonal obstacle, with a Robin-type boundary condition. Using the principle of the half-space matching method, the problem is formulated as a system of coupled Fourier-integral equations, the unknowns being the Robin traces on the infinite straight lines supported by the edges of the polygon. We prove that this system is a Fredholm equation of the second kind, in a L^2 functional framework. The truncation of the Fourier integrals and the finite element approximation of the corresponding numerical method are also analyzed. The theoretical results are supported by various numerical experiments.

Keywords: Helmholtz equation, Fourier-integral operators, Fredholm equation, Mellin transform, error estimates

MSC 2010: 35J05, 31A10, 65N12, 65R20, 35S30

1 Introduction

1.1 Motivation

This study takes place in the general framework of the development of numerical methods for the simulation and the optimization of ultrasonic Non-Destructive Testing (NDT) experiments. NDT consists of detecting defects in an elastic structure by measuring the ultrasonic echoes produced by these defects, when they are illuminated by some incident ultrasonic wave. In particular, one needs to simulate the interaction of a given incident wave with a compactly supported defect in an infinite medium. When

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this medium is homogeneous and isotropic, there exist several efficient methods to solve this problem, like perfectly matched layers or integral equations. However, difficulties arise in more complex configurations [5, 7, 21]. Among them, one important case which remains unsolved is the case where the infinite medium is an infinite elastic plate made of an anisotropic homogeneous material.

A new method called the Half-Space Matching (HSM) method (inspired by [13]) has been introduced recently (see [20]) in view of tackling this problem. As a first step, the method has been applied in [8] to the acoustic scalar problem in $\mathbb{R}^2$, showing that anisotropy can be taken into account easily, without any additional cost. The method mainly relies on a decomposition of the infinite domain, exterior to the obstacle, into the union of several overlapping half-spaces, where a Fourier-integral representation of the solution is available.

In this first version of the method, the unknowns of the Fourier-integral equations are the Dirichlet traces of the field on the boundaries of the different half-spaces. But with this choice, the method cannot be extended to the case of the elastic plate, where both the traces of the displacement and of the normal stress are required to derive the half-plate representations. This is a first motivation of the present paper where we consider still a scalar problem but with different types of traces, including Neumann, Dirichlet, and Robin traces.

The content of the present paper is the following. We first derive the HSM formulation for general types of traces. Then we prove the well-posedness of the continuous problem by adapting the arguments used in [8]. More importantly, the main contribution of this paper is the numerical analysis of the discretized formulation which is not straightforward and has never been addressed in previous works.

The model problem that we consider is presented in the next subsection. The corresponding HSM formulation is the object of Section 2. Section 3 is devoted to the theoretical analysis of the formulation: we use Fredholm theory, the main tools being the Mellin transform [12, 17] and Hilbert–Schmidt operators [19, p. 210]. The discretization aspects are detailed in Section 4 and error estimates are derived for an appropriate Fourier discretization. Some numerical results are finally presented in Section 5.

1.2 The model problem

The problem that we consider is the 2D Helmholtz equation in the exterior of a compact convex polygonal obstacle $\mathcal{O}$, with a boundary condition of Robin type. More precisely, the problem takes the following form where ω, α, and β are some complex constants whose characteristics are specified below, ν denotes the outgoing normal to $\mathcal{O}$ and the data is a given function g defined on the boundary of the obstacle $\partial\mathcal{O}$:

$$\left|\begin{array}{ll} \Delta p + \omega^2 p = 0 & \text{in } \Omega = \mathbb{R}^2 \backslash \mathcal{O}, \\ \alpha p + \beta \dfrac{\partial p}{\partial \nu} = g & \text{on } \partial\mathcal{O}. \end{array}\right. \tag{1}$$

In the sequel, we will use the following assumptions:

$$\operatorname{Im}\omega > 0, \quad \beta \neq 0, \quad \operatorname{Im}\left(\frac{\alpha}{\beta\omega}\right) \geq 0, \quad \text{and} \quad g \in L^2(\partial\mathcal{O}), \tag{2}$$

which lead to several results as follows:

1. Since $\operatorname{Im}\omega > 0$ (which can be justified in a dissipative medium), we will look for a solution p which belongs to $H^1(\Omega)$. More precisely, p is exponentially decaying at infinity and satisfies

$$\forall\varepsilon < \operatorname{Im}(\omega), \quad (x,y) \mapsto p(x,y)e^{\varepsilon\sqrt{x^2+y^2}} \in H^1(\Omega). \tag{3}$$

   However, we emphasize that the numerical method also works in the non-dissipative case, that is when $\omega \in \mathbb{R}^+$. In this latter case, p is chosen as the outgoing solution of (1) (defined as the unique solution satisfying the Sommerfeld condition).

2. As $\beta \neq 0$, the problem (1) admits the following variational formulation:

$$\left|\begin{array}{l}\text{Find } p \in H^1(\Omega) \text{ such that for all } q \in H^1(\Omega)\\ \displaystyle\int_\Omega \nabla p \cdot \overline{\nabla q} - \omega^2 \int_\Omega p\bar{q} - \frac{\alpha}{\beta}\int_{\partial\mathcal{O}} p\bar{q} = \frac{-1}{\beta}\int_{\partial\mathcal{O}} g\bar{q}.\end{array}\right. \tag{4}$$

   Using the fact that for $p \in H^1(\Omega)$:

$$\begin{aligned}&\operatorname{Im}\left(\frac{-1}{\omega}\left(\int_\Omega |\nabla p|^2 - \omega^2|p|^2 - \frac{\alpha}{\beta}\int_{\partial\mathcal{O}} |p|^2\right)\right)\\ &= \frac{\operatorname{Im}(\omega)}{|\omega|^2}\int_\Omega |\nabla p|^2 + \operatorname{Im}(\omega)\int_\Omega |p|^2 + \operatorname{Im}\left(\frac{\alpha}{\beta\omega}\right)\int_{\partial\mathcal{O}} |p|^2,\end{aligned}$$

   one deduces, due to the assumption that $\operatorname{Im}(\frac{\alpha}{\beta\omega}) \geq 0$, that the bilinear form is coercive. Then the problem is well-posed by the Lax–Milgram theorem.

Remark 1. For the data g on the boundary, we make the assumption $g \in L^2(\partial\mathcal{O})$, which is convenient for our approach, and which differs from the natural one ($g \in H^{-\frac{1}{2}}(\partial\mathcal{O})$) that would be used in a variational approach. In particular, since $g \in L^2(\partial\mathcal{O})$, we know from classical regularity results [15] that $p \in H^{3/2}(\Omega)$.

Remark 2 (The Dirichlet case). Taking $\beta = 0$ and $\alpha \neq 0$ in (1), one simply recovers a Dirichlet boundary condition (a case which has been already treated in [8]). In that case, the natural hypothesis in a variational approach would be $g \in H^{\frac{1}{2}}(\partial\mathcal{O})$. We point out that our approach allows to consider more general Dirichlet data which are only in $L^2(\partial\mathcal{O})$. As a consequence, the solution may not be in H^1 up to the boundary (see [3] for a similar problem). Note that the numerical analysis performed in Section 4 is also valid in the Dirichlet case, which is illustrated numerically in Section 5.3.1.

2 The half-space matching formulation

The half-space matching method consists in coupling several analytical representations of the solution in half-planes surrounding the obstacle.

2.1 Geometry and notation

Let us consider a convex polygon $\mathcal{O}$ with n edges $\Sigma^j_{\mathcal{O}}$, $j = 0, \dots, n-1$. For convenience, we introduce $\mathbb{Z}/n\mathbb{Z}$ the ring of integers modulo n. For $j \in \mathbb{Z}/n\mathbb{Z}$, the angle between $\Sigma^j_{\mathcal{O}}$ and $\Sigma^{j+1}_{\mathcal{O}}$ is denoted as $\theta^{j,j+1}$ or equivalently $\theta^{j+1,j}$. Because of the convexity, one has

$$0 < \theta^{j,j+1} < \pi. \tag{5}$$

To define the half-spaces, we introduce several local coordinate systems (x^j, y^j). The origin of all of them is the centroid O of the polygon $\mathcal{O}$. We choose the reference Cartesian coordinate system $(O, \mathbf{e}^0_x, \mathbf{e}^0_y)$ such that $\mathbf{e}^0_x$ is orthogonal to $\Sigma^0_{\mathcal{O}}$ and oriented to the exterior of the polygon, while the axis $\mathbf{e}^0_y$ is $\pi/2$ counter clockwise to $\mathbf{e}^0_x$. The other local coordinate systems $(O, \mathbf{e}^j_x, \mathbf{e}^j_y)$ are defined recursively as follows:

$$\forall j \in \mathbb{Z}/n\mathbb{Z}, \quad \left| \begin{aligned} \mathbf{e}^{j+1}_x &= -\cos\theta^{j,j+1}\mathbf{e}^j_x + \sin\theta^{j,j+1}\mathbf{e}^j_y, \\ \mathbf{e}^{j+1}_y &= -\sin\theta^{j,j+1}\mathbf{e}^j_x - \cos\theta^{j,j+1}\mathbf{e}^j_y. \end{aligned} \right. \tag{6}$$

If we define l^j as the distance of the centroid of the polygon to the edge $\Sigma^j_{\mathcal{O}}$, each half-plane Ω^j is defined in the local coordinate system $(O, \mathbf{e}^j_x, \mathbf{e}^j_y)$ as

$$\forall j \in \mathbb{Z}/n\mathbb{Z},, \quad \Omega^j = \{x^j \geq l^j\} \times \{y^j \in \mathbb{R}\},$$

and its boundary denoted by Σ^j is given by

$$\Sigma^j = \{x^j = l^j\} \times \{y^j \in \mathbb{R}\}.$$

All these notations are summarized in Figure 4.1 for three examples of polygon.

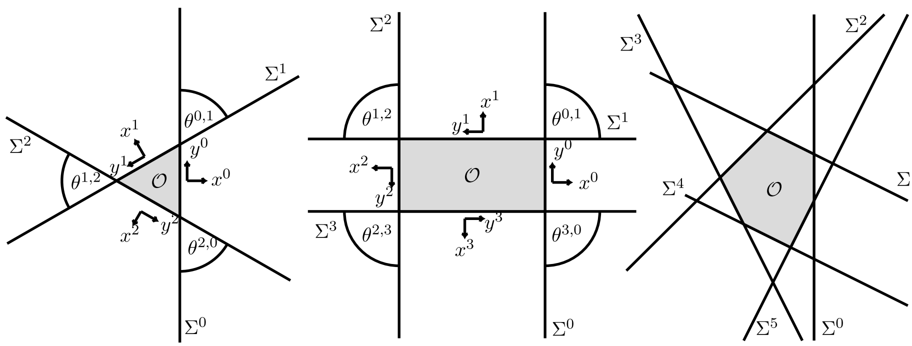


Figure 4.1: Examples of polygons $\mathcal{O}$ for $n = 3, 4$, and 6 and associated notation.

2.2 Half-space problems

The jth half-space problem is defined as follows: given $\psi \in L^2(\Sigma^j)$, $P^j(\psi)$ is the unique solution in $H^1(\Omega^j)$ of

$$\left| \begin{aligned} &\Delta P^j + \omega^2 P^j = 0, && \text{in } \Omega^j, \\ &\alpha P^j + \beta \frac{\partial P^j}{\partial x^j} = \psi && \text{on } \Sigma^j. \end{aligned} \right. \tag{7}$$

This problem is well-posed under assumptions (2) for the same reasons than the ones detailed in Section 1.2. Remark again that in the usual framework, we would take $\psi^j \in H^{-1/2}(\Sigma^j)$, but here we take $\psi^j \in L^2(\Sigma^j)$. Applying the Fourier transform in y^j defined as

$$\forall \psi^j \in L^2(\Sigma^j), \quad \hat{\psi}^j(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi^j(y^j) e^{-\imath \xi y^j} \mathrm{d}y^j, \tag{8}$$

we obtain the following ordinary differential equation in x^j, parametrized by the Fourier variable ξ:

$$\left| \begin{aligned} \frac{\partial^2 \hat{P}^j}{(\partial x^j)^2} + (\omega^2 - \xi^2)\hat{P}^j &= 0, && x^j > l^j, \\ \alpha \hat{P}^j + \beta \frac{\partial \hat{P}^j}{\partial x^j} &= \hat{\psi}^j, && x^j = l^j, \end{aligned} \right. \tag{9}$$

whose unique L^2 solution is given by

$$\hat{P}^j(x^j, \xi) = A(\xi) e^{\imath \sqrt{\omega^2 - \xi^2}(x^j - l^j)}, \tag{10}$$

where $\operatorname{Im} \sqrt{\omega^2 - \xi^2} > 0$ and

$$(\alpha + \imath\beta\sqrt{\omega^2 - \xi^2})A(\xi) = \hat{\psi}(\xi). \tag{11}$$

One can check that, thanks to assumptions (2), the quantity $\alpha + \imath\beta\sqrt{\omega^2 - \xi^2}$ never vanishes for $\xi \in \mathbb{R}$. Finally, by taking the inverse Fourier transform, the solution $P^j(\psi)$ of (7) is given by

$$P^j(x^j, y^j) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\hat{\psi}(\xi)}{\alpha + \imath\beta\sqrt{\omega^2 - \xi^2}} e^{\imath \sqrt{\omega^2 - \xi^2}(x^j - l^j)} e^{i\xi y^j} \mathrm{d}\xi. \tag{12}$$

2.3 Half-space matching integral equations

For the solution p of problem (1), let us define the Robin traces

$$\forall j \in \mathbb{Z}/n\mathbb{Z}, \quad \varphi^j := \left(\alpha p + \beta \frac{\partial p}{\partial x^j} \right)\Big|_{\Sigma^j}. \tag{13}$$

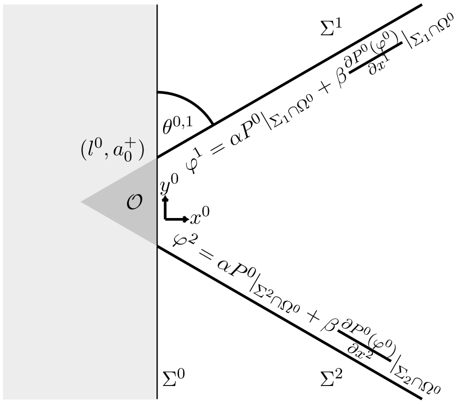


Figure 4.2: Compatibility conditions on $\Sigma^1 \cap \Omega^0$ and $\Sigma^2 \cap \Omega^0$.

Note that $\varphi^j \in L^2(\Sigma^j)$ since $p \in H^{3/2}(\Omega)$. Our objective is to derive integral equations linking the φ^j by using half-space representations of Section 2.2 and the fact that the half-spaces Ω^j overlap. First, the restriction of p in Ω^j is the solution of (7) for $\psi = \varphi^j$. By uniqueness,

$$p|_{\Omega^j} = P^j(\varphi^j). \tag{14}$$

Then the quantity

$$\left(\alpha p + \beta \frac{\partial p}{\partial x^{j\pm1}}\right)\bigg|_{\Sigma^{j\pm1}\cap\Omega^j} \tag{15}$$

is equal both to

$$\varphi^{j\pm1}|_{\Sigma^{j\pm1}\cap\Omega^j} \quad \text{(by definition of } \varphi^{j\pm1}\text{)}$$

and to

$$\alpha P^j(\varphi^j) + \beta \frac{\partial P^j(\varphi^j)}{\partial x^{j\pm1}}\bigg|_{\Sigma^{j\pm1}\cap\Omega^j} \quad \text{(by (14))}.$$

This provides the compatibility relations (see Figure 4.2)

$$\varphi^{j\pm1} = \alpha P^j(\varphi^j) + \beta \frac{\partial P^j(\varphi^j)}{\partial x^{j\pm1}} \quad \text{on } \Sigma^{j\pm1} \cap \Omega^j, \quad \forall j \in \mathbb{Z}/n\mathbb{Z}. \tag{16}$$

Remark 3.
- Such compatibility relations have been firstly introduced in [6, 13] for Dirichlet traces in the case of periodic media.

– Here, we have used the overlap of two consecutive half-spaces Ω^j and $\Omega^{j\pm1}$. This will be sufficient for our formulation, even for polygons with more than four edges where non-consecutive half-spaces may overlap (see Figure 4.1 on the right).

This leads to introducing the following Fourier integral operator:

$$D^{j,j\pm1} : L^2(\Sigma^j) \to L^2(\Sigma^{j\pm1} \cap \Omega^j) \tag{17}$$

$$\psi \to \alpha P^j(\psi) + \beta \frac{\partial P^j(\psi)}{\partial x^{j\pm1}}\bigg|_{\Sigma^{j\pm1}\cap\Omega^j}, \tag{18}$$

which can be expressed, using (12), as a kernel operator acting on the Fourier transform

$$[D^{j,j\pm1}\psi](r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} k^{j,j\pm1}(r,\xi)\hat{\psi}(\xi)\mathrm{d}\xi, \quad r \geq 0 \tag{19}$$

with the following kernel:

$$k^{j,j\pm1}(r,\xi) = \frac{\alpha + \beta(-\cos(\theta^{j,j\pm1})\imath\sqrt{\omega^2-\xi^2} + \sin(\theta^{j,j\pm1})\imath\xi)}{\alpha + \imath\beta\sqrt{\omega^2-\xi^2}} e^{\imath\sqrt{\omega^2-\xi^2}r\sin(\theta^{j,j\pm1})} e^{\imath\xi(a_j^{\pm}+r\cos(\theta^{j,j\pm1}))}. \tag{20}$$

Here, $a_j^{\pm}$ denotes the ordinate of the intersection point of Σ^j and $\Sigma^{j\pm1}$ in (x^j, y^j) local coordinates and r is the radial variable of the polar coordinates centered at this intersection point. If $\theta^{j,j+1} = \pi/2$ (which holds for instance for all j when $\mathcal{O}$ is a rectangle), the previous operator has the simpler form

$$[D^{j,j\pm1}\psi](r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\alpha + \imath\beta\xi}{\alpha + \imath\beta\sqrt{\omega^2-\xi^2}} e^{\imath\sqrt{\omega^2-\xi^2}r} e^{\imath\xi a_j^{\pm}} \hat{\psi}(\xi)\mathrm{d}\xi. \tag{21}$$

It is not so difficult to see that the operator $D^{j,j\pm1}$ is continuous from $L^2(\Sigma^j)$ to $L^2(\Omega^j \cap \Sigma^{j\pm1})$. Indeed, if $\psi \in L^2(\Sigma^j)$, we can show that $P^j(\psi)$, the solution of the half-space problem (7) in Ω^j, is in $H^{3/2}(\Omega^j)$. It suffices then to use the continuity of the trace operators. Let us remark that it is less obvious when using directly the expression (19)–(20) of $D^{j,j\pm1}$, but this will be a by-product of the next section.

Summing up, we have finally the following system of coupled equations satisfied by the φ^j's:

$$\varphi^j = \begin{cases} D^{j-1,j}\varphi^{j-1} & \text{on } \Sigma^j \cap \Omega^{j-1} \\ g & \text{on } \Sigma^j_{\mathcal{O}} \\ D^{j+1,j}\varphi^{j+1} & \text{on } \Sigma^j \cap \Omega^{j+1}, \end{cases} \qquad \forall j \in \mathbb{Z}/n\mathbb{Z} \tag{22}$$

where we have used the boundary condition satisfied by p on $\partial\mathcal{O}$. The system of equations (22) can be written in a matricial form as

$$(\mathbb{I} - \mathbb{D})\Phi = G, \tag{23}$$

where

$$\Phi \in V := \left\{ (\varphi^0, \varphi^1, \dots, \varphi^{n-1}) \in \prod_{j=0}^{n-1} L^2(\Sigma^j) \right\}, \tag{24}$$

$\mathbb{I}$ corresponds to the identity operator and $\mathbb{D}$ is given by

$$\mathbb{D} := \begin{bmatrix} 0 & D^{1,0} & 0 & \dots & 0 & D^{n-1,0} \\ D^{0,1} & 0 & D^{2,1} & \dots & 0 & 0 \\ 0 & D^{1,2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & D^{n-1,n-2} \\ D^{0,n-1} & 0 & 0 & \dots & D^{n-2,n-1} & 0 \end{bmatrix}, \tag{25}$$

where for all $j \in \mathbb{Z}/n\mathbb{Z}$ we have identified a function of $L^2(\Sigma^j \cap \Omega^{j\pm1})$ to a function of $L^2(\Sigma^j)$ by extending it by 0. Remark then that for all Φ in V, $\mathbb{D}\Phi$ is in $\tilde{V}$ where

$$\tilde{V} := \{\tilde{\Phi} = (\tilde{\varphi}^0, \tilde{\varphi}^1, \dots, \tilde{\varphi}^{n-1}) \in V,\ \tilde{\varphi}^j = 0 \text{ on } \Sigma^j_{\mathcal{O}}\ \forall j \in \mathbb{Z}/n\mathbb{Z}\}. \tag{26}$$

Remark 4. If we want to make the extension by 0 explicit, we have to replace in $\mathbb{D}$, $D^{j,j\pm1}$ by $E^{j,j\pm1}D^{j,j\pm1}$ where

$$\begin{aligned} E^{j,j\pm1} : L^2(\Sigma^{j\pm1} \cap \Omega^j) &\to L^2(\Sigma^{j\pm1}) \\ \psi &\mapsto E^{j,j\pm1}\psi \end{aligned} \tag{27}$$

with

$$E^{j,j\pm1}\psi = \begin{cases} \psi & \text{on } \Sigma^{j\pm1} \cap \Omega^j \\ 0 & \text{on } \Sigma^{j\pm1} \setminus (\Sigma^{j\pm1} \cap \Omega^j). \end{cases}$$

All the properties of $D^{j,j\pm1}$ also hold trivially for $E^{j,j\pm1}D^{j,j\pm1}$. In order to enhance readability, we have chosen to drop these extension operators.

Lemma 5 (Equivalence). *Let $g \in L^2(\partial\mathcal{O})$. If $p \in H^1(\Omega)$ is solution of* (1), *then $\Phi = (\varphi^0, \varphi^1, \dots, \varphi^{n-1})$ where φ^j is defined by* (13) *belongs to V and is a solution of* (23).

Conversely, if $\Phi \in V$ is a solution of (23), *then p satisfying* (14) *for all $j \in \mathbb{Z}/n\mathbb{Z}$ is a function defined "unequivocally" in Ω. Moreover, $p \in H^1(\Omega)$ and is solution of* (1).

Proof. The first assertion is true by construction. Conversely, suppose that $\Phi = (\varphi^0, \dots, \varphi^{n-1}) \in V$ is a solution of (23). This implies that the φ^j's satisfy the system of coupled equations (22). Now, let us introduce $P^j(\varphi^j) \in H^1(\Omega^j)$ for all $j \in \mathbb{Z}/n\mathbb{Z}$, the solution of the half-space problem (7) with $\psi = \varphi^j$. By definition,

$$\varphi^j = \alpha P^j(\varphi^j) + \beta \left.\frac{\partial P^j(\varphi^j)}{\partial x^j}\right|_{\Sigma^j}. \tag{28}$$

Because the φ^j's satisfy the first set of equations of (22), we have by definition of $D^{j\pm1,j}$ that

$$\varphi^j = \alpha P^{j\pm1}(\varphi^{j\pm1}) + \beta \frac{\partial P^{j\pm1}(\varphi^{j\pm1})}{\partial x^j} \quad \text{on } \Sigma^j \cap \Omega^{j\pm1}, \quad \forall j \in \mathbb{Z}/n\mathbb{Z}. \tag{29}$$

From (28) and (29), we have that

$$\alpha P^j(\varphi^j) + \beta \frac{\partial P^j(\varphi^j)}{\partial x^j}\bigg|_{\Sigma^j\cap\Omega^{j\pm1}} = \alpha P^{j\pm1}(\varphi^{j\pm1}) + \beta \frac{\partial P^{j\pm1}(\varphi^{j\pm1})}{\partial x^j}\bigg|_{\Sigma^j\cap\Omega^{j\pm1}}.$$

In particular, the previous relations for $j = 0$ and $\pm = +$ and for $j = 1$ and $\pm = -$ yield to

$$\alpha P^0(\varphi^0) + \beta \frac{\partial P^0(\varphi^0)}{\partial x^0}\bigg|_{\Sigma^0\cap\Omega^1} = \alpha P^1(\varphi^1) + \beta \frac{\partial P^1(\varphi^1)}{\partial x^0}\bigg|_{\Sigma^0\cap\Omega^1}$$

and

$$\alpha P^1(\varphi^1) + \beta \frac{\partial P^1(\varphi^1)}{\partial x^1}\bigg|_{\Sigma^1\cap\Omega^0} = \alpha P^0(\varphi^0) + \beta \frac{\partial P^0(\varphi^0)}{\partial x^1}\bigg|_{\Sigma^1\cap\Omega^0}.$$

Let

$$Q = P^0(\varphi^0) - P^1(\varphi^1) \quad \text{in } \Omega^0 \cap \Omega^1.$$

Because $P^0(\varphi^0)$ and $P^1(\varphi^1)$ satisfy the same Helmholtz equation and because of the previous relations, Q satisfies the problem

$$\left|\begin{aligned} &\Delta Q + \omega^2 Q = 0 && \text{in } \Omega^0 \cap \Omega^1, \\ &\alpha Q + \beta \frac{\partial Q}{\partial \nu} = 0 && \text{on } \partial(\Omega^0 \cap \Omega^1), \end{aligned}\right.$$

where ν is the interior normal to $\Omega^0 \cap \Omega^1$. This problem is well-posed under assumptions (2) for the same reasons as the ones detailed in Section 1.2. So $Q = 0$ in $\Omega^0 \cap \Omega^1$ which means that $P^0(\varphi^0)$ and $P^1(\varphi^1)$ coincide in the overlapping zone $\Omega^0 \cap \Omega^1$.

Similar arguments enable us to show that for all $j \in \mathbb{Z}/n\mathbb{Z}$, $P^j(\varphi^j)$ and $P^{j+1}(\varphi^{j+1})$ coincide in the overlapping zone $\Omega^j \cap \Omega^{j+1}$. We can then define unequivocally a function p by

$$\forall j \in \mathbb{Z}/n\mathbb{Z}, \quad p|_{\Omega^j} = P^j(\varphi^j).$$

Because the half-space solutions coincide two by two in the overlapping zones, the function p is in $H^1(\Omega)$ and is solution of the Helmholtz equation in Ω. Moreover, by definition

$$\forall j \in \mathbb{Z}/n\mathbb{Z}, \quad \left(\alpha p + \beta \frac{\partial p}{\partial x^j}\right)\bigg|_{\Sigma^j_{\mathcal{O}}} = \left(\alpha P^j(\varphi^j) + \beta \frac{\partial P^j(\varphi^j)}{\partial x^j}\right)\bigg|_{\Sigma^j_{\mathcal{O}}} = \varphi^j|_{\Sigma^j_{\mathcal{O}}} = g,$$

where the last equality is obtained by using the second set of equations of (22). Hence, the function p is then solution of (1). □

Finally,

$$G := (g^0, g^1, \dots, g^{n-1}) \in V \quad \text{where } g^j = \begin{cases} g & \text{on } \Sigma^j_{\mathcal{O}}, \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

Writing

$$\Phi = G + \tilde{\Phi} \tag{31}$$

where $\tilde{\Phi}$ is in $\tilde{V}$, we obtain an equivalent system

$$(\mathbb{I} - \mathbb{D})\tilde{\Phi} = \mathbb{D}G. \tag{32}$$

This system constitutes the half-space matching formulation which will be analyzed in Section 3.

3 Analysis of the continuous formulation

In this section, we consider the general problem

$$\text{Find } \tilde{\Phi} \in \tilde{V}, \quad (\mathbb{I} - \mathbb{D})\tilde{\Phi} = F, \tag{33}$$

where $\tilde{V}$ is defined in (26), $\mathbb{D}$ is defined in (25), and $F \in \tilde{V}$. Denoting $\mathcal{L}(A)$ as the set of bounded linear operators of a vector space A, we show in this section the following main results.

Theorem 6. *The operator* $(\mathbb{I} - \mathbb{D}) \in \mathcal{L}(\tilde{V})$ *is the sum of a coercive operator and a compact one. Moreover, Problem* (33) *is well-posed.*

A naive idea would be that $\mathbb{D} \in \mathcal{L}(\tilde{V})$ is compact, but it is not. However, it can be decomposed as the sum of an operator of norm strictly less than 1 and a compact operator. This decomposition is linked to a similar decomposition of the operators $D^{j,j\pm 1}$. Inspired by the proofs for the Dirichlet case shown in [8], we prove the properties of the operators for the Robin case in Section 3.1 and finally show the theorem in Section 3.2.

3.1 Properties of the operators $D^{j,j\pm 1}$

Let us concentrate first on the operator $D^{0,1}$ and similar properties will be given, without proof, for all the operators $D^{j,j\pm 1}$ at the end of this section. To simplify the notation, we denote in this section, $D^{0,1} = D$, $x^0 = x$, $y^0 = y$. We will identify, when necessary, Σ^0 to $\mathbb{R}$, its upper part $\Sigma^0 \cap \Omega^1$ to $(a_0^+, +\infty)$, its lower part $\Sigma^0 \cap \Omega^{n-1}$ to $(-\infty, a_0^-)$ and finally $\Sigma^1 \cap \Omega^0$ to $\mathbb{R}^+$. Let us also introduce for any open interval I included in J, an open interval of $\mathbb{R}$, the restriction operator χ_I

$$\begin{array}{rcl} \chi_I : L^2(J) & \rightarrow & L^2(J) \\ \varphi & \mapsto & \left| \begin{array}{l} \chi_I \varphi = \varphi \text{ on } I \\ \chi_I \varphi = 0 \text{ on } J \setminus I \end{array} \right. \end{array}$$

In the sequel, we are going to decompose the operator D progressively in order to isolate a compact part and a part for which we get the norm explicitly.

First, from the definition (17), we can decompose simply D as

$$D = D_D + D_N \tag{34}$$

where

$$\begin{array}{rcl} D_D : L^2(\Sigma^0) & \rightarrow & L^2(\Sigma^1 \cap \Omega^0) \\ \psi & \mapsto & \alpha P^0(\psi)|_{\Sigma^1 \cap \Omega^0} \end{array} \quad \text{and} \quad \begin{array}{rcl} D_N : L^2(\Sigma^0) & \rightarrow & L^2(\Sigma^1 \cap \Omega^0) \\ \psi & \mapsto & \beta \left. \dfrac{\partial P^0(\psi)}{\partial x^1} \right|_{\Sigma^1 \cap \Omega^0} \end{array}. \tag{35}$$

Lemma 7. *The operator $D_D : L^2(\Sigma^0) \rightarrow L^2(\Sigma^1 \cap \Omega^0)$ is compact.*

Proof. By definition of D_D, we have

$$\forall \psi \in L^2(\Sigma^0), \quad D_D \psi(r) = \int_{\mathbb{R}} k_D(\xi, r) \hat{\psi}(\xi) \, \mathrm{d}\xi$$

with

$$k_D(\xi, r) = \frac{\alpha}{\alpha + \imath \beta \sqrt{\omega^2 - \xi^2}} e^{\imath \sqrt{\omega^2 - \xi^2} r \sin \theta^{0,1}} e^{\imath \xi (a_0^+ + r \cos \theta^{0,1})}.$$

Using Fubini's theorem, we obtain

$$\begin{aligned} \|k_D(\xi, r)\|^2_{L^2(\mathbb{R} \times \mathbb{R}^+)} &= \int_{\mathbb{R}} \int_{\mathbb{R}^+} \frac{|\alpha|^2}{|\alpha + \imath \beta \sqrt{\omega^2 - \xi^2}|^2} e^{-2 \operatorname{Im}(\sqrt{\omega^2 - \xi^2}) r \sin \theta^{0,1}} \mathrm{d}r \mathrm{d}\xi \\ &= \int_{\mathbb{R}} \frac{|\alpha|^2}{2 \operatorname{Im}(\sqrt{\omega^2 - \xi^2}) \sin \theta |\alpha + \imath \beta \sqrt{\omega^2 - \xi^2}|^2} \mathrm{d}\xi \\ &< +\infty. \end{aligned}$$

This proves that D_D is the composition of the Fourier operator $\psi \mapsto \hat{\psi}$ and of a Hilbert–Schmidt operator. The lemma follows. □

Let us focus now on D_N. For all ψ, $D_N \psi$ is, up to the parameter β, the normal trace on $\Sigma^1 \cap \Omega^0$ of the half-space solution $P^0(\psi)$ in Ω^0 with a Robin data ψ on the boundary Σ^0. Because the half-line $\Sigma^1 \cap \Omega^0$ touches Σ^0, the operator D_N is not compact. The lack of compactness is precisely due to the intersection point. So let us isolate the intersection point by decomposing D_N thanks to restriction operators (see Figure 4.3):

$$D_N = \chi_{(0,b)} D_N + \chi_{(b,+\infty)} D_N, \quad \text{with } b > 0. \tag{36}$$

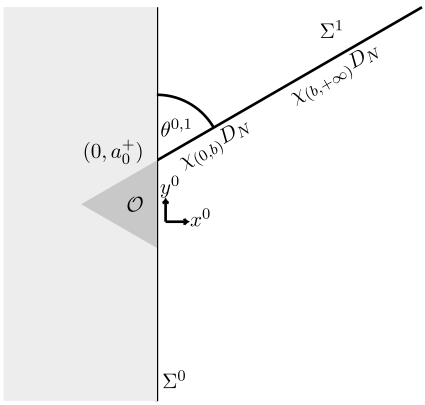


Figure 4.3: Decomposition of the operator D_N into $\chi_{(0,b)} D_N$ and $\chi_{(b,+\infty)} D_N$.

Lemma 8. *For any $b > 0$, the operator $\chi_{(b,+\infty)} D_N : L^2(\Sigma^0) \to L^2(\Omega^0 \cap \Sigma^1)$ is compact.*

Proof. By definition of D_N, we have

$$\forall \psi \in L^2(\Sigma^0), \quad D_N \psi(r) = \int_{\mathbb{R}} k_N(\xi, r) \hat{\psi}(\xi)\, \mathrm{d}\xi$$

with

$$k_N(\xi, r) = \frac{\beta(-\imath\sqrt{\omega^2 - \xi^2} \cos\theta^{0,1} + \imath\xi \sin\theta^{0,1})}{\alpha + \imath\beta\sqrt{\omega^2 - \xi^2}} e^{\imath\sqrt{\omega^2-\xi^2} r \sin\theta^{0,1}} e^{\imath\xi(a_0^+ + r\cos\theta^{0,1})}. \tag{37}$$

Again by Fubini's theorem, we get for $b > 0$

$$\begin{aligned} &\|k_N(\xi, r)\|^2_{L^2(\mathbb{R}\times(b,+\infty))} \\ &= \int_{\mathbb{R}} \int_b^{+\infty} \frac{|\beta|^2 | -\imath\sqrt{\omega^2 - \xi^2} \cos\theta^{0,1} + \imath\xi \sin\theta^{0,1}|^2}{|\alpha + \imath\beta\sqrt{\omega^2 - \xi^2}|^2} e^{-2(\sqrt{\omega^2-\xi^2}) r \sin\theta^{0,1}} \mathrm{d}r\mathrm{d}\xi \end{aligned}$$

$$= \int_{\mathbb{R}} \frac{|\beta|^2| - \imath\sqrt{\omega^2 - \xi^2}\cos\theta^{0,1} + \imath\xi\sin\theta^{0,1}|^2}{2\,\mathrm{Im}(\sqrt{\omega^2 - \xi^2})\sin\theta^{0,1}|\alpha + \imath\beta\sqrt{\omega^2 - \xi^2}|^2} e^{-2\,\mathrm{Im}(\sqrt{\omega^2-\xi^2})b\sin\theta^{0,1}} \mathrm{d}\xi$$
$$< +\infty.$$

We conclude as in the proof of Lemma 7. □

As you can notice, this proof requires that $b > 0$. To analyse the non-compact part $\chi_{(0,b)}D_N$, inspired by the Dirichlet case [8] and more generally by the singularity theory [17], we decompose finally $\chi_{(0,b)}D_N$ as

$$\chi_{(0,b)}D_N = \chi_{(0,b)}L_N + \chi_{(0,b)}(D_N - L_N)$$

where L_N is obtained by taking $\omega = 0$ in the expression (37) of $k_N(\xi, r)$

$$L_N\psi(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\psi}(\xi)(-\cos\theta^{0,1} - \imath\,\mathrm{sgn}(\xi)\sin\theta^{0,1})e^{-|\xi|r\sin\theta^{0,1}} e^{\imath\xi(a_0^+ + r\cos\theta^{0,1})}\mathrm{d}\xi, \quad r > 0. \tag{38}$$

The operator L_N is similar to D_N, but it is associated with the Laplace operator. Indeed, it can also be defined as

$$\begin{aligned} L_N &: L^2(\Sigma^0) \to L^2(\Sigma^1 \cap \Omega^0), \\ L_N\psi &:= \beta \left.\frac{\partial}{\partial x^1} v(\psi)\right|_{\Sigma^1\cap\Omega^0}, \end{aligned} \tag{39}$$

where, for all $\psi \in L^2(\Sigma^0)$, $v(\psi)$ is the solution (at least in the distributional sense) to

$$\left| \begin{aligned} &-\Delta v = 0 && \text{in } \Omega^0, \\ &\beta\frac{\partial v}{\partial x} = \psi && \text{on } \Sigma^0. \end{aligned} \right.$$

We refer to the Appendix for the precise definition of the appropriate functional framework for this problem.

Lemma 9. *The operator $\chi_{(0,b)}(D_N - L_N)$ is compact.*

Proof. It is a kernel operator whose kernel is given by

$$k(\xi, r) = (c_1(\xi)e^{\imath\sqrt{\omega^2-\xi^2}r\sin\theta^{0,1}} - c_2(\xi)e^{-|\xi|r\sin\theta^{0,1}})e^{\imath\xi(a^+ + r\cos\theta^{0,1})}$$

where from (37) and (38), we have that

$$c_1, c_2, 1/c_2 \in L^\infty(\mathbb{R}), \quad \text{and} \quad \left|\frac{c_1(\xi)}{c_2(\xi)}\right| \to 1 \quad \text{when } \xi \to +\infty$$

Consequently,

$$|k(\xi, r)| = |c_2(\xi)|e^{-|\xi|r\sin\theta^{0,1}} \left|\frac{c_1(\xi)}{c_2(\xi)} e^{-q(\xi)r\sin\theta^{0,1}} - 1\right|$$

where

$$q(\xi) = \sqrt{\xi^2 - \omega^2} - |\xi| = \frac{-\omega^2}{\sqrt{\xi^2 - \omega^2} + |\xi|} = \mathcal{O}\left(\frac{1}{\xi}\right).$$

We deduce that

$$\left|k(\xi, r)\right| \leq C|q(\xi)|e^{-|\xi| r \sin\theta^{0,1}}$$

which enables us to conclude as in the proof of Lemma 7. □

Finally, let us focus on the properties of L_N which are summarized in this fundamental lemma.

Lemma 10. *The operator L_N is continuous from $L^2(\Sigma^0)$ to $L^2(\Sigma^1 \cap \Omega^0)$ and its norm is bounded by 1. Moreover, we have:*
- $\exists\, C \in (0,1), \forall\, \varphi \in L^2(\Sigma^0), \|L_N \chi_{(a_0^+,+\infty)}\varphi\| \leq C\|\chi_{(a_0^+,+\infty)}\varphi\|$;
- $L_N \chi_{(-\infty, a_0^-)}$ *is a compact operator.*

We give the proof which is quite technical in Appendix A. As we will see in the proof of Theorem 6, it is not sufficient to know that the norm of L_N is bounded by 1. This is the second part of the lemma which will enable us to conclude that $\mathbb{I} - \mathbb{D}$ is a sum of a coercive operator and a compact one. As indicated in the Appendix, the constant C is linked to the angle $\theta^{0,1}$ between Σ^0 and $\Sigma^1 \cap \Omega^0$:

$$C = \cos(\theta^{0,1}/2).$$

When $\theta^{0,1}$ tends to 0, this constant tends to 1.

Gathering all the results of this section, we can show the following properties of D.

Proposition 11. *The operator D is such that $D - L$ is a compact operator from $L^2(\Sigma^0)$ to $L^2(\Sigma^1 \cap \Omega^0)$ where L is a continuous operator from $L^2(\Sigma^0)$ to $L^2(\Sigma^1 \cap \Omega^0)$ which satisfies:*
- $\exists\, C \in (0,1), \forall\, \varphi \in L^2(\Sigma^0), \|L\chi_{(a_0^+,+\infty)}\varphi\| \leq C\|\chi_{(a_0^+,+\infty)}\varphi\|$;
- $L\chi_{(-\infty, a_0^-)}$ *is a compact operator.*

Proof. The operator L is nothing else but $\chi_{(0,b)} L_N$. Indeed, using all the operators introduced in this section, we write

$$D - \chi_{(0,b)} L_N = \chi_{(0,b)}(D_N - L_N) + \chi_{(b,+\infty)} D_N + D_D.$$

From Lemmas 7, 8, and 9, we have that the operator $D - \chi_{(0,b)} L_N$ is compact. As L_N satisfies Lemma 10, the operator $\chi_{(0,b)} L_N$ inherits similar properties. □

Finally, we have obviously similar results for all the operators $D^{j,j\pm1}$ for $j \in \mathbb{Z}/n\mathbb{Z}$. Again, we will identify, when necessary, Σ^j to $\mathbb{R}$, its upper part $\Sigma^j \cap \Omega^{j+1}$ to $(a_j^+, +\infty)$, its lower part $\Sigma^j \cap \Omega^{j-1}$ to $(-\infty, a_j^-)$. Finally, in order to give a short statement of the following theorem, we use the notation

$$\forall j, \quad (a_j^-, -\infty) = (-\infty, a_j^-).$$

Theorem 12. *The operator $D^{j,j\pm1}$ is such that $D^{j,j\pm1} - L^{j,j\pm1}$ is a compact operator from $L^2(\Sigma^j)$ to $L^2(\Sigma^{j\pm1} \cap \Omega^j)$ where $L^{j,j\pm1}$ is a continuous operator from $L^2(\Sigma^j)$ to $L^2(\Sigma^{j\pm1} \cap \Omega^j)$ which satisfies:*
- $\exists C^{j,j\pm1} \in (0,1), \forall \varphi \in L^2(\Sigma^j), \|L^{j,j\pm1}\chi_{(a_j^\pm, \pm\infty)}\varphi\| \leq C^{j,j\pm1}\|\chi_{(a_j^\pm, \pm\infty)}\varphi\|$;
- $L^{j,j\pm1}\chi_{(a_j^\mp, \mp\infty)}$ *is a compact operator from* $L^2(\Sigma^j)$ *to* $L^2(\Sigma^{j\pm1} \cap \Omega^j)$.

Remark 13. The constant $C^{j,j\pm1}$ is linked to the angle $\theta^{j,j\pm1}$ between Σ^j and $\Sigma^{j\pm1} \cap \Omega^j$. More precisely, we can show, as in Appendix A, that

$$C^{j,j\pm1} = \cos(\theta^{j,j\pm1}/2).$$

Remark 14. Theorem 12 has links with classical analysis for second kind boundary equations on non-smooth domains. Indeed, using the notation $\mathbf{x}^j = (x^j, y^j)$, $D^{j,j\pm1}$ can be written in layer-potential form as

$$D^{j,j\pm1}\psi^j(\mathbf{x}) = -\frac{1}{\beta}\int_{\Sigma^j}\left(\alpha G(\mathbf{x}, \mathbf{x}^j) + \beta\frac{\partial G}{\partial x^{j\pm1}}(\mathbf{x}, \mathbf{x}^j)\right)\psi^j(\mathbf{x}^j)\mathrm{d}\mathbf{x}^j, \quad \mathbf{x} \in \Sigma^{j\pm1} \cap \Omega^j, \tag{40}$$

where G is the Green's function for the Helmholtz equation satisfying the Robin condition on Σ^j (see for instance [10] for a characterization of G). Formula (19) is nothing else but the Plancherel equality applied to (40). As G is a smooth perturbation of the Green's function of the Laplace equation with Neumann boundary condition on Σ^j, and as $\mathrm{Im}(\omega) > 0$, the properties of $D^{j,j\pm1}$ can be deduced from those of the double layer potential operator defined by

$$\int_{\Sigma^j}\frac{\partial G_0}{\partial x^{j\pm1}}(\mathbf{x}, \mathbf{x}^j)\psi^j(\mathbf{x}^j)\mathrm{d}\mathbf{x}^j, \quad \mathbf{x} \in \Sigma^{j\pm1} \cap \Omega^j,$$

where

$$G_0(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi}\log(|\mathbf{x} - \mathbf{x}'|).$$

This operator, as an operator acting on L^2 functions on the sides of a bounded polygon has been discussed and analyzed in [4, 9]. Let us mention that in [9, Lemma 1], the same bound for the norm of the operator $L^{j,j\pm1}$ has been found.

3.2 Proof of Theorem 6

Let us prove now that the operator $\mathbb{I}-\mathbb{D}$ is the sum of a coercive operator and a compact one in $\tilde{V}$. Using Theorem 12, and the following obvious decomposition

$$\forall\tilde{\Phi} = (\tilde{\varphi}^0, \tilde{\varphi}^1, \dots, \tilde{\varphi}^{n-1}) \in \tilde{V}, \quad \forall j \in \mathbb{Z}/n\mathbb{Z},\ \tilde{\varphi}^j = \chi_{(-\infty, a_j^-)}\tilde{\varphi}^j + \chi_{(a_j^+, +\infty)}\tilde{\varphi}^j$$

we can decompose the operator $\mathbb{D}$ as follows:

$$\forall\tilde{\Phi} \in \tilde{V}, \quad \mathbb{D}\tilde{\Phi} = \mathbb{L}\tilde{\Phi} + \mathbb{K}\tilde{\Phi}, \tag{41}$$

where

$$\mathbb{L} := \begin{bmatrix} 0 & L^{1,0}\chi_{(-\infty,a_1^-)} & 0 & \dots & 0 & L^{n-1,0}\chi_{(a_{n-1}^+,+\infty)} \\ L^{0,1}\chi_{(a_0^+,+\infty)} & 0 & L^{2,1}\chi_{(-\infty,a_2^-)} & \dots & 0 & 0 \\ 0 & L^{1,2}\chi_{(a_1^+,+\infty)} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & L^{n-1,n-2}\chi_{(-\infty,a_{n-1}^-)} \\ L^{0,n-1}\chi_{(-\infty,a_0^-)} & 0 & 0 & \dots & L^{n-2,n-1}\chi_{(a_{n-2}^+,+\infty)} & 0 \end{bmatrix}, \tag{42}$$

and

$$\mathbb{K} := \begin{bmatrix} 0 & K^{1,0} & 0 & \dots & 0 & K^{n-1,0} \\ K^{0,1} & 0 & K^{2,1} & \dots & 0 & 0 \\ 0 & K^{1,2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & K^{n-1,n-2} \\ K^{0,n-1} & 0 & 0 & \dots & K^{n-2,n-1} & 0 \end{bmatrix}, \tag{43}$$

with

$$\forall j \in \mathbb{Z}/n\mathbb{Z}, \quad K^{j\pm1,j} = (D^{j\pm1,j} - L^{j\pm1,j}) + L^{j\pm1,j}\chi_{(a_{j\pm1}^{\pm}, \pm\infty)} \tag{44}$$

From Theorem 12, we get easily that the operator $\mathbb{K}$ is compact in $\tilde{V}$.

Moreover, by Theorem 12, for all $j \in \mathbb{Z}/n\mathbb{Z}$, we have that for all $\tilde{\varphi}^j \in L^2(\Sigma^j)$ such that $\tilde{\varphi}^j|_{\Sigma^j_{\mathcal{O}}} = 0$

$$\begin{aligned}\|L^{j,j-1}\chi_{(-\infty,a_j^-)}\tilde{\varphi}^j\|^2 + \|L^{j,j+1}\chi_{(a_j^+,+\infty)}\tilde{\varphi}^j\|^2 &\le C_j^2[\|\chi_{(-\infty,a_j^-)}\tilde{\varphi}^j\|^2 + \|\chi_{(a_j^+,+\infty)}\tilde{\varphi}^j\|^2] \\ &= C_j^2\|\tilde{\varphi}^j\|^2_{L^2(\Sigma^j)},\end{aligned}$$

where $C_j = \max(\cos(\theta^{j,j-1}/2), \cos(\theta^{j,j+1}/2))$. Consequently, the norm of the operator $\mathbb{L}$ is strictly less than 1. This implies that the operator $\mathbb{I} - \mathbb{L}$ is coercive in $\tilde{V}$, its coercivity constant being given by

$$\alpha = 1 - \max_{j\in\mathbb{Z}/n\mathbb{Z}} \cos \frac{\theta^{j,j+1}}{2}. \tag{45}$$

Let us now show that Problem (33) is well posed. Since it is Fredholm of index 0, it is sufficient to show the uniqueness. We will suppose that $\mathbb{F} = 0$ and show that the corresponding solution $\tilde{\Phi} \in \tilde{V}$ necessarily vanishes. By Lemma 5, we can define unequivocally a function p satisfying (14) for all $j \in \mathbb{Z}/n\mathbb{Z}$. Moreover, $p \in H^1(\Omega)$ and is solution of (1) with $g = 0$. Problem (1) being well posed, $p = 0$ and then $P^j(\varphi^j) = 0$ for all j. Consequently, $\varphi^j = 0$ for all $j \in \mathbb{Z}/n\mathbb{Z}$.

4 Discretization

4.1 The discrete problem

To get a discrete problem that we can solve numerically, we use three main ingredients:

1. We truncate the integrals which appear in the definition of the integral operators $D^{j,j\pm1}$: the integral for $\xi \in \mathbb{R}$ is replaced by an integral for $|\xi| \le \hat{T}$ for some $\hat{T} \in \mathbb{R}^+$.
2. Then we introduce finite dimensional subspaces $\tilde{V}_{\mathbf{h}}$ of $\tilde{V}$ on which a Galerkin approximation is computed. To define the space $\tilde{V}_{\mathbf{h}}$, we truncate the infinite lines Σ^j as follows:

$$\Sigma^j_T = \{(x^j = l^j, y^j), -T_j < y^j < T_j\} \tag{46}$$

and we mesh these truncated lines into segments $[M^j_i, M^j_{i+1}]$, $i \in \{1, \dots, N_j\}$ whose maximum length is h_j. Let $T = \min_j T_j$ and $h = \max_j h_j$. Finally, the space $V_{\mathbf{h}}$ with $\mathbf{h} = (T, h)$ built with Lagrange finite elements of degree l ($l \in \mathbb{N}^*$) is given by

$$\{(\psi^0_{\mathbf{h}}, \dots, \psi^{n-1}_{\mathbf{h}}) \in V,\ \forall j,\ \psi^j_{\mathbf{h}} \text{ is polynomial of degree } l \text{ on } [M^j_i, M^j_{i+1}],\ i \in \{1, \dots, N_j\}\}, \tag{47}$$

and $\tilde{V}_{\mathbf{h}} = V_{\mathbf{h}} \cap \tilde{V}$. Let us emphasize that of course

$$\forall \tilde{\Psi} \in V, \quad \inf_{\psi_{\mathbf{h}}\in V_{\mathbf{h}}} \|\Psi - \Psi_{\mathbf{h}}\| \xrightarrow[\mathbf{h}\to(+\infty,0)]{} 0. \tag{48}$$

3. Finally, quadrature formulae have to be used to evaluate the Fourier integrals which appear in the variational formulation.

In what follows, we will study the error due to points 1 and 2 but not the quadrature formulae.

For this purpose, we consider the three following variational problems:

0. The exact problem:
   Find $\tilde{\Phi} \in \tilde{V}$ such that

$$(\mathbb{B}\tilde{\Phi}, \tilde{\Psi}) = (\mathbb{D}G, \tilde{\Psi}), \quad \forall \tilde{\Psi} \in \tilde{V}, \tag{49}$$

   where $\mathbb{B} = \mathbb{I} - \mathbb{D}$, $\mathbb{D}$ is defined by (25), $\tilde{V}$ by (26), G by (30), and $(\cdot,\cdot)$ denotes the L^2 scalar product in V. We have the expression

$$\forall\,(\Phi,\Psi) \in V, \quad (\mathbb{D}\Phi,\Psi) = \sum_{j\in\mathbb{Z}/n\mathbb{Z}} \Bigg(\int\limits_{\Sigma^j\cap\Omega^{j+1}} [D^{j+1,j}\varphi^{j+1}]\psi^j + \int\limits_{\Sigma^j\cap\Omega^{j-1}} [D^{j-1,j}\varphi^{j-1}]\psi^j \Bigg), \tag{50}$$

   with

$$\forall j \in \mathbb{Z}/n\mathbb{Z}, \quad \forall \psi \in L^2(\Sigma^j), \quad [D^{j\pm1,j}\psi](r) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} k^{j\pm1,j}(r,\xi)\hat{\psi}(\xi)\,\mathrm{d}\xi, \quad r \geq 0,$$

   and $k^{j\pm1,j}(r,\xi)$ is given by (20).
1. The semi-discrete problem (truncation of the integrals):
   Find $\tilde{\Phi}_{\hat{T}} \in \tilde{V}$ such that

$$(\mathbb{B}_{\hat{T}}\tilde{\Phi}_{\hat{T}}, \tilde{\Psi}) = (\mathbb{D}_{\hat{T}}G, \tilde{\Psi}), \quad \forall \tilde{\Psi} \in \tilde{V}, \tag{51}$$

   where $\mathbb{B}_{\hat{T}} = \mathbb{I} - \mathbb{D}_{\hat{T}}$ and $\mathbb{D}_{\hat{T}}$ is defined by

$$\forall\,(\Phi,\Psi) \in V, \quad (\mathbb{D}_{\hat{T}}\Phi,\Psi) = \sum_{j\in\mathbb{Z}/n\mathbb{Z}} \Bigg(\int\limits_{\Sigma^j\cap\Omega^{j+1}} [D_{\hat{T}}^{j+1,j}\varphi^{j+1}]\psi^j + \int\limits_{\Sigma^j\cap\Omega^{j-1}} [D_{\hat{T}}^{j-1,j}\varphi^{j-1}]\psi^j \Bigg), \tag{52}$$

   where

$$[D_{\hat{T}}^{j\pm1,j}\psi](r) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\hat{T}}^{\hat{T}} k^{j\pm1,j}(r,\xi)\hat{\psi}(\xi)\mathrm{d}\xi, \quad r \geq 0. \tag{53}$$

2. The discrete problem (truncation of the infinite lines Σ^j and meshing):
   Find $\tilde{\Phi}_{\hat{T},\mathbf{h}} \in \tilde{V}_{\mathbf{h}}$ such that

$$(\mathbb{B}_{\hat{T}}\tilde{\Phi}_{\hat{T},\mathbf{h}}, \tilde{\Psi}_{\mathbf{h}}) = (\mathbb{D}_{\hat{T}} \quad G_{\mathbf{h}}, \tilde{\Psi}_{\mathbf{h}}), \quad \forall \tilde{\Psi}_{\mathbf{h}} \in \tilde{V}_{\mathbf{h}}, \tag{54}$$

   where $G_{\mathbf{h}} \in V_{\mathbf{h}}$ is the interpolate of G.

Our first objective is to prove that for $\hat{T}$ and T large enough, and for h small enough, the above discrete problem is well posed. The second objective is to prove that the error $\|\Phi - \Phi_{\hat{T},\mathbf{h}}\|$ (where $\Phi_{\hat{T},\mathbf{h}} = \tilde{\Phi}_{\hat{T},\mathbf{h}} + G_{\mathbf{h}}$) tends to 0 when $\hat{T} \to +\infty$, $T \to +\infty$, and $h \to 0$. And finally if the φ^j's are regular enough (whose precise definition will be given later), we will also estimate the convergence rate.

Remark 15. As in [18], the difficulty of the numerical analysis comes from the fact that the operator appearing in the discrete problem is the sum of a coercive part and a compact part, which both depend on $\hat{T}$.

As a first step, we will derive the same type of result but only for the semi-discrete problem.

4.2 Numerical analysis of the semi-discrete problem

For $\hat{T} > 0$, we denote by $\Pi_{\hat{T}}$ the projection operator on $L^2(\mathbb{R})$ defined by

$$\forall \psi \in L^2(\mathbb{R}), \quad \Pi_{\hat{T}}\psi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\hat{T}}^{\hat{T}} \hat{\psi}(\xi) e^{i\xi y} \mathrm{d}\xi. \tag{55}$$

In other words,

$$\forall \psi \in L^2(\mathbb{R}), \quad \widehat{\Pi_{\hat{T}}\psi}(\xi) = \chi_{[-\hat{T},\hat{T}]}(\xi)\hat{\psi}(\xi). \tag{56}$$

Then we denote by $\mathbb{\Pi}_{\hat{T}}$ the projection operator on V defined by

$$\forall \Phi = (\varphi^0, \dots, \varphi^{n-1}) \in V, \quad \mathbb{\Pi}_{\hat{T}}\Phi = (\Pi_{\hat{T}}\varphi^0, \dots, \Pi_{\hat{T}}\varphi^{n-1}). \tag{57}$$

Using Plancherel and Lebesgue theorems, one can easily check the following properties that will be used in the sequel:

$$\|\mathbb{\Pi}_{\hat{T}}\Phi\|_V \leq \|\Phi\|_V \tag{58}$$

$$\forall \Phi, \Psi \in V, \quad (\mathbb{\Pi}_{\hat{T}}\Phi, \Psi) = (\Phi, \mathbb{\Pi}_{\hat{T}}\Psi) \tag{59}$$

$$\forall \Phi \in V, \quad \|\mathbb{\Pi}_{\hat{T}}\Phi - \Phi\|_V \to 0 \quad \text{when } \hat{T} \to +\infty \tag{60}$$

Using this definition, we have $\mathbb{D}_{\hat{T}} = \mathbb{D}\mathbb{\Pi}_{\hat{T}}$, where $D_{\hat{T}}$ is defined by (52). The main results of this section are given in the following theorem.

Theorem 16.
1. **[Stability]** *There exists* $\hat{T}_{\min}$ *such that the semi-discrete problem* (51) *is well posed for* $\hat{T} \geq \hat{T}_{\min}$.
2. **[Convergence]** *The solution* $\tilde{\Phi}_{\hat{T}}$ *of the semi-discrete problem* (51) *tends to the exact solution* $\tilde{\Phi}$ *of* (49) *when* $\hat{T}$ *tends to infinity.*
3. **[Error estimates]** *Let* $\Phi = \tilde{\Phi} + G = (\varphi^0, \dots, \varphi^{n-1})$, *where* $\tilde{\Phi}$ *is the solution of* (49). *If there exists* $s > 0$ *such that for all* $j \in \mathbb{Z}/n\mathbb{Z}$, $\varphi^j \in H^s(\Sigma^j)$, *we have*

$$\|\tilde{\Phi} - \tilde{\Phi}_{\hat{T}}\|_V \leq \frac{C}{\hat{T}^s} \sum_{j \in \mathbb{Z}/n\mathbb{Z}} \left(\frac{1}{\sqrt{\sin(\theta^{j,j+1})}} \|\varphi^{j+1}\|_{H^s(\Sigma^{j+1})} + \frac{1}{\sqrt{\sin(\theta^{j,j-1})}} \|\varphi^{j-1}\|_{H^s(\Sigma^{j-1})} \right). \tag{61}$$

The rest of this section is dedicated to the proof of this theorem. To do so, we will need several lemmas:

- the two first lemmas give properties of the operators $L^{j\pm1,j}\Pi_{\hat{T}}$ for the $L^{j\pm1,j}$ appearing in (42). These results are the equivalent of the two properties stated in Theorem 12, but they are not a straightforward consequence of this theorem. The difficulty comes from the fact that, in general, for a function ψ, the support of $\Pi_{\hat{T}}\psi$ is not the same than the one of ψ. These lemmas enable us to deduce properties of the operator $\mathbb{D}\mathbb{\Pi}_{\hat{T}}$, used in Lemma 20 as a basic tool for the stability and the convergence result.
- To establish the error estimates, we will use finally Lemma 21.

Lemma 17. *For all $j \in \mathbb{Z}/n\mathbb{Z}$, the operator $L^{j\pm1,j}$ appearing in Theorem* 12 *satisfies*

$$\exists \tilde{C}^{j\pm1,j} \in (0,1),\ \forall \varphi \in L^2(\Sigma^{j\pm1}),\quad \|L^{j\pm1,j}\Pi_{\hat{T}}\chi_{(a^{\mp}_{j\pm1},\mp\infty)}\varphi\| \le \tilde{C}^{j\pm1,j}\|\chi_{(a^{\mp}_{j\pm1},\mp\infty)}\varphi\|;$$

where $\tilde{C}^{j\pm1,j} = \max(\sin(\theta^{j\pm1,j}/2), \cos(\theta^{j\pm1,j}/2))$.

Proof. As explained in the proof of Lemma 11, $L^{0,1} = \chi_{(0,b)}L_N$ where L_N is defined by (39). It suffices then to show that

$$\forall \hat{T} > 0,\ \forall \psi \in L^2(\Sigma^0)\quad \|L_N\Pi_{\hat{T}}\chi_{(a_0^+,+\infty)}\psi\| \le C\|\chi_{(a_0^+,+\infty)}\psi\| \tag{62}$$

with

$$C = \max(\sin(\theta^{0,1}/2), \cos(\theta^{0,1}/2))$$

to obtain the result for $L^{0,1}$. A similar proof can be applied to other $L^{j\pm1,j}$.

We stress again that (62) is not a direct consequence of Lemma 10 since $\Pi_{\hat{T}}\chi_{(a_0^+,+\infty)}\psi$ is not supported in $(a_0^+,+\infty)$.

We introduce the linear operators S and A of $\mathcal{L}(L^2(\mathbb{R}))$ defined by

$$\forall \psi \in L^2(\mathbb{R}),\quad S\psi(y) = \frac{1}{2}(\psi(y) + \psi(2a_0^+ - y)) \quad \text{and } A\psi(y) = \frac{1}{2}(\psi(y) - \psi(2a_0^+ - y))$$

We have obviously $S + A = \mathrm{Id}$.

The key point is that $\Pi_{\hat{T}}$ commutes with S and A. Indeed, from

$$\widehat{S\psi}(\xi)e^{\imath 2a_0^+\xi} = \widehat{S\psi}(-\xi),$$

we deduce that $\Pi_{\hat{T}}S\psi$ is symmetric with respect to a_0^+:

$$\begin{aligned}\Pi_{\hat{T}}S\psi(2a_0^+ - y) &= \frac{1}{\sqrt{2\pi}}\int_{-\hat{T}}^{\hat{T}} \widehat{S\psi}(\xi)e^{i(2a_0^+-y)\xi}\,\mathrm{d}\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\hat{T}}^{\hat{T}} \widehat{S\psi}(-\xi)e^{-iy\xi}\,\mathrm{d}\xi\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\hat{T}}^{\hat{T}} \widehat{S\psi}(\xi) e^{iy\xi} \mathrm{d}\xi$$
$$= \Pi_{\hat{T}} S\psi(y).$$

Similarly, we prove that $\Pi_{\hat{T}} A\psi$ is anti-symmetric with respect to a_0^+:

$$\Pi_{\hat{T}} A\psi(2a_0^+ - y) = -\Pi_{\hat{T}} A\psi(y).$$

Finally, gathering all these properties, we get

$$\begin{aligned} S\Pi_{\hat{T}}\psi(y) &= \frac{1}{2}(\Pi_{\hat{T}}\psi(y) + \Pi_{\hat{T}}\psi(2a_0^+ - y)) \\ &= \frac{1}{2}(\Pi_{\hat{T}} S\psi(y) + \Pi_{\hat{T}} A\psi(y) + \Pi_{\hat{T}} S\psi(2a_0^+ - y) + \Pi_{\hat{T}} A\psi(2a_0^+ - y)) \\ &= \Pi_{\hat{T}} S\psi(y), \end{aligned}$$

and the same result can be obtained for A. To summarize, we have

$$\Pi_{\hat{T}} S\psi = S\Pi_{\hat{T}}\psi \text{ and } \Pi_{\hat{T}} A\psi = A\Pi_{\hat{T}}\psi. \tag{63}$$

Now let us apply all these properties to our purpose. Since $S + A = \mathrm{Id}$, $S^2 = S$, and $A^2 = A$:

$$\begin{aligned} L_N \Pi_{\hat{T}} &= L_N \Pi_{\hat{T}}(S + A) \\ &= L_N \Pi_{\hat{T}}(S^2 + A^2) \\ &= L_N S \Pi_{\hat{T}} S + L_N A \Pi_{\hat{T}} A \end{aligned}$$

so that

$$\forall \psi \in L^2(\Sigma^0), \quad \|L_N \Pi_{\hat{T}} \chi_{(a_0^+,+\infty)}\psi\| \le \|L_N S\| \|\Pi_{\hat{T}}\| \|S\chi_{(a_0^+,+\infty)}\psi\| + \|L_N A\| \|\Pi_{\hat{T}}\| \|A\chi_{(a_0^+,+\infty)}\psi\|. \tag{64}$$

Moreover, since for any $\psi \in L^2(\mathbb{R})$,

$$\|\Pi_{\hat{T}}\| \le 1 \quad \text{and} \tag{65}$$
$$\|S\chi_{(a_0^+,+\infty)}\psi\|_{L^2(\mathbb{R})} = \|A\chi_{(a_0^+,+\infty)}\psi\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2}} \|\chi_{(a_0^+,+\infty)}\psi\|_{L^2(\mathbb{R})}, \tag{66}$$

we get

$$\|L_N \Pi_{\hat{T}} \chi_{(a_0^+,+\infty)}\psi\| \le \frac{1}{\sqrt{2}}(\|L_N S\| + \|L_N A\|) \|\chi_{(a_0^+,+\infty)}\psi\|. \tag{67}$$

Finally, the estimates (90) and (91) proven in the Appendix enables us to show (62).

□

Remark 18. Let us emphasize that the key property (63) is true because in the definition of $\Pi_{\hat{T}}$, the Fourier integral has been truncated to a symmetric interval $[-\hat{T}, \hat{T}]$.

Lemma 19. *For all $\hat{T}$, the operator $L^{j\pm1,j}\Pi_{\hat{T}}\chi_{(a^{\pm}_{j\pm1},\pm\infty)}$ is compact.*

Moreover, let ψ_n be a sequence of $L^2(\Sigma^{j\pm1})$, for $\hat{T}_n \to +\infty$ and for $j \in \mathbb{Z}/n\mathbb{Z}$ such that ψ_n converges weakly to 0 in $L^2(\Sigma^{j\pm1})$, then

$$L^{j\pm1,j}\Pi_{\hat{T}_n}\chi_{(a^{\pm}_{j\pm1},\pm\infty)}\psi_n \to 0 \quad \text{in } L^2(\Sigma^j \cap \Omega^{j\pm1}).$$

Proof. As explained in the proof of the previous lemma, it suffices to show the result replacing $L^{j\pm1,j}$ by L_N to deduce the one for $L^{0,1}$ and in a similar way the one for the other $L^{j\pm1,j}$.

Again, the difficulty is that, in general, for a function ψ, $\Pi_{\hat{T}}\chi_{(-\infty,a_0^-)}\psi$ is not supported in $(-\infty, a_0^-)$, so that the results cannot be deduced from the second point of Lemma 10.

We decompose

$$L_N\Pi_{\hat{T}}\chi_{(-\infty,a_0^-)} = L_N\chi_{(-\infty,a_0^-)} + L_N(\Pi_{\hat{T}} - I)\chi_{(-\infty,a_0^-)}$$

From Lemma 10, we know that $L_N\chi_{(-\infty,a_0^-)}$ is compact. We show in the rest of the proof that $L_N(\Pi_{\hat{T}} - I)\chi_{(-\infty,a_0^-)}$ is a Hilbert–Schmidt operator and that $L_N(\Pi_{\hat{T}} - I)\chi_{(-\infty,a_0^-)} \to 0$ when $\hat{T} \to +\infty$, and the results of the lemma follow.

Using the expression of L_N in (38), we get

$$L_N(\Pi_{\hat{T}} - I)\chi_{(-\infty,a_0^-)} = R^+_{\hat{T}}e^{\imath\theta^{0,1}} + R^-_{\hat{T}}e^{-\imath\theta^{0,1}},$$

where, for $\psi \in L^2(\Sigma^0)$,

$$R^+_{\hat{T}}\psi(r) = \frac{1}{2\pi}\int\limits_{\xi>\hat{T}} \mathrm{d}\xi \int\limits_{-\infty}^{a_0^-} \mathrm{d}y\, e^{-\xi r\sin\theta^{0,1}} e^{\imath\xi(a_0^+ - y + r\cos\theta^{0,1})}\psi(y),$$

$$R^-_{\hat{T}}\psi(r) = \frac{1}{2\pi}\int\limits_{\xi<-\hat{T}} \mathrm{d}\xi \int\limits_{-\infty}^{a_0^-} \mathrm{d}y\, e^{\xi r\sin\theta^{0,1}} e^{\imath\xi(a_0^+ - y + r\cos\theta^{0,1})}\psi(y).$$

By Fubini's theorem, we deduce that

$$R^{\pm}_{\hat{T}}\psi(r) = \frac{1}{2\pi}\int\limits_{-\infty}^{a_0^-} k^{\pm}_{\hat{T}}(r,y)\psi(y)\,\mathrm{d}y$$

with

$$k^{\pm}_{\hat{T}}(r,y) = \frac{e^{-\hat{T}(r\sin\theta^{0,1} \pm \imath(a_0^+ - y + r\cos\theta^{0,1}))}}{r\sin\theta^{0,1} \mp \imath(a_0^+ - y + r\cos\theta^{0,1})}.$$

Since the denominator never vanishes for $r \geq 0$ and $y < a_0^- < a_0^+$, one can easily check that

$$\int_0^{+\infty} \int_{-\infty}^{a_0^-} |k_{\hat{T}}^{\pm}(r,y)|^2 \mathrm{d}y\, \mathrm{d}r$$

is finite and tends to 0 when $\hat{T} \to +\infty$. □

From these two lemmas, we deduce the following result.

Lemma 20. *For all $\hat{T} > 0$, the operator $\mathbb{B}_{\hat{T}}$ is the sum of a coercive operator and a compact operator in $\tilde{V}$.*

Moreover, there exists $\gamma > 0$ and $\hat{T}_{min}$ such that for $\hat{T} \geq \hat{T}_{min}$

$$\forall \tilde{\Phi} \in \tilde{V} \quad \|\mathbb{B}_{\hat{T}} \tilde{\Phi}\| \geq \gamma \|\tilde{\Phi}\| \tag{68}$$

Proof. Let $\hat{T} > 0$. Let us remind that $\mathbb{B}_{\hat{T}} = \mathbb{I} - \mathbb{D}\mathbb{\Pi}_{\hat{T}}$. By the definitions (25) of $\mathbb{D}$ and (57) of $\mathbb{\Pi}_{\hat{T}}$, the operator $\mathbb{D}\mathbb{\Pi}_{\hat{T}}$ is nothing else but (25) with the terms $D^{j\pm1,j}$ replaced by $D^{j\pm1,j}\Pi_{\hat{T}}$. Finally, as in (41–42–43), we have the decomposition

$$\mathbb{B}_{\hat{T}} = \mathbb{I} - \mathbb{D}\mathbb{\Pi}_{\hat{T}} \quad \text{where } \mathbb{D}\mathbb{\Pi}_{\hat{T}} = \mathbb{L}_{\hat{T}} + \mathbb{K}_{\hat{T}}$$

with

- $\mathbb{L}_{\hat{T}}$ having the form of (42) where the terms $L^{j\pm1,j}\chi_{(a_{j\pm1}^{\mp},\mp\infty)}$ are replaced by $L^{j\pm1,j}\Pi_{\hat{T}}\chi_{(a_{j\pm1}^{\mp},\mp\infty)}$. Using Lemma 17 and the same arguments as in Section 3.2, we can show that the norm of $\mathbb{L}_{\hat{T}}$ is strictly less than 1, the norm being independent of $\hat{T}$. Therefore $\mathbb{I} - \mathbb{L}_{\hat{T}}$ is coercive in $\tilde{V}$ with a coercive constant $\tilde{\gamma}$ independent of $\hat{T}$.
- $\mathbb{K}_{\hat{T}}$ has the form of (43) where the terms $K^{j\pm1,j}$ are replaced by $K_{\hat{T}}^{j\pm1,j}$ with

$$K_{\hat{T}}^{j\pm1,j} = (D^{j\pm1,j} - L^{j\pm1,j})\Pi_{\hat{T}} + L^{j\pm1,j}\Pi_{\hat{T}}\chi_{(a_{j\pm1}^{\pm},\pm\infty)}. \tag{69}$$

  By using Theorem 12 and Lemma 19, $K_{\hat{T}}^{j\pm1,j}$ is compact. The operator $\mathbb{K}_{\hat{T}}$ is then also compact in $\tilde{V}$.

We have then proven the first part of the theorem. We show the second part of the theorem by contradiction. We suppose the existence of a sequence $\tilde{\Phi}_n \in \tilde{V}$ and a sequence $\hat{T}_n \to +\infty$ such that $\|\tilde{\Phi}_n\|_V = 1$ and $\mathbb{B}_{\hat{T}_n}\tilde{\Phi}_n \to 0$ in $\tilde{V}$. Using the first part of the proof, we have

$$\mathbb{B}_{\hat{T}_n} = (\mathbb{I} - \mathbb{L}_{\hat{T}_n}) - \mathbb{K}_{\hat{T}_n},$$

where the operator $(\mathbb{I} - \mathbb{L}_{\hat{T}_n})$ is coercive with a coercivity constant $\tilde{\gamma}$ independent of n and $\mathbb{K}_{\hat{T}_n}$ is compact. Rearranging the terms and taking the scalar product, we have

$$(\mathbb{B}_{\hat{T}_n}\tilde{\Phi}_n, \tilde{\Phi}_n) + (\mathbb{K}_{\hat{T}_n}\tilde{\Phi}_n, \tilde{\Phi}_n) = \big((\mathbb{I} - \mathbb{L}_{\hat{T}_n})\tilde{\Phi}_n, \tilde{\Phi}_n\big) \geq \tilde{\gamma} > 0, \tag{70}$$

and we will show that the left-hand side tends to 0 with n to establish the contradiction.

Since $\tilde{\Phi}_n$ is bounded in the Hilbert space $\tilde{V}$, it admits a weakly convergent subsequence that we denote also by $\tilde{\Phi}_n$: $\tilde{\Phi}_n \rightharpoonup \tilde{\Phi}$ in $\tilde{V}$. By (59) and (60),

$$\forall \tilde{\Psi} \in \tilde{V}, \quad (\mathbb{\Pi}_{\hat{T}_n} \tilde{\Phi}_n, \tilde{\Psi}) = (\tilde{\Phi}_n, \mathbb{\Pi}_{\hat{T}_n} \tilde{\Psi} - \tilde{\Psi}) + (\tilde{\Phi}_n, \tilde{\Psi}) \to (\tilde{\Phi}, \tilde{\Psi}),$$

which means that $\mathbb{\Pi}_{\hat{T}_n} \tilde{\Phi}_n \rightharpoonup \tilde{\Phi}$. As a consequence,

$$\mathbb{B}_{\hat{T}_n} \tilde{\Phi}_n = \tilde{\Phi}_n - \mathbb{D}\mathbb{\Pi}_{\hat{T}_n} \tilde{\Phi}_n \rightharpoonup \tilde{\Phi} - \mathbb{D}\tilde{\Phi} = \mathbb{B}\tilde{\Phi}.$$

Since by hypothesis, $\mathbb{B}_{\hat{T}_n} \tilde{\Phi}_n \to 0$, we conclude that $\mathbb{B}\tilde{\Phi} = 0$ which implies $\tilde{\Phi} = 0$ because $\mathbb{B}$ is invertible (see Theorem 6).

On the other hand, as written in (69), each operator $K_{\hat{T}_n}^{j\pm1,j}$ involved in the definition of $\mathbb{K}_{\hat{T}_n}$ is the sum of two operators such that:

- $(D^{j\pm1,j} - L^{j\pm1,j})\Pi_{\hat{T}_n} \tilde{\varphi}_n \to 0$ when $\tilde{\varphi}_n \rightharpoonup 0$ since $\Pi_{\hat{T}_n} \tilde{\varphi}_n \rightharpoonup 0$ because of (60) and $(D^{j\pm1,j} - L^{j\pm1,j})$ is compact because of Theorem 12;
- $L^{j\pm1,j} \Pi_{\hat{T}_n} \chi_{(a^{\pm}_{j\pm1}, \pm\infty)} \tilde{\varphi}_n \to 0$ when $\tilde{\varphi}_n \rightharpoonup 0$ because of Lemma 19.

Consequently, as $\tilde{\Phi}_n \rightharpoonup 0$, we have $\mathbb{K}_{\hat{T}_n} \tilde{\Phi}_n \to 0$ in $\tilde{V}$ when n tends to $+\infty$.

Gathering all these results, we have $(\mathbb{B}_{\hat{T}_n} \tilde{\Phi}_n, \tilde{\Phi}_n) + (\mathbb{K}_{\hat{T}_n} \tilde{\Phi}_n, \tilde{\Phi}_n)$ tends to 0 with n. This contradiction completes the proof. □

To establish the error estimates of Theorem 16, we need the following lemma.

Lemma 21. *Let $s > 0$ and $\psi \in H^s(\Sigma^j)$. There exists a constant $C > 0$ independent of ψ and $\hat{T}$ such that*

$$\|D^{j,j\pm1}(I - \Pi_{\hat{T}})\psi\|_{L^2} \leq \frac{C}{\hat{T}^s \sqrt{\sin(\theta^{j,j\pm1})}} \|\psi\|_{H^s(\Sigma^j)}. \tag{71}$$

Proof. By definition (19)–(20) of $D^{j,j\pm1}$, we have, by Cauchy–Schwarz inequality, Fubini's theorem, and by the Fourier definition of the Sobolev spaces [1]:

$$\|D^{j,j\pm1}(I - \Pi_{\hat{T}})\psi\|^2 \leq \|\psi\|^2_{H^s(\Sigma^j)} \int_{|\xi|>\hat{T}} \int_0^{+\infty} \frac{|k^{j,j\pm1}(r,\xi)|^2}{(1+\xi^2)^s} \mathrm{d}\xi \, \mathrm{d}r. \tag{72}$$

Moreover, an easy calculation gives

$$\int_{|\xi|>\hat{T}} \int_0^{+\infty} \frac{|k^{j,j\pm1}(r,\xi)|^2}{(1+\xi^2)^s} \mathrm{d}\xi \, \mathrm{d}r = \int_{|\xi|>\hat{T}} F(\xi) \mathrm{d}\xi,$$

where

$$F(\xi) = \frac{|\alpha + \beta(-\cos(\theta^{j,j\pm1}) i\sqrt{\omega^2 - \xi^2} + \sin(\theta^{j,j\pm1}) i\xi)|^2}{2|\alpha + i\beta\sqrt{\omega^2 - \xi^2}|^2 \operatorname{Im}(\sqrt{\omega^2 - \xi^2}) \sin(\theta^{j,j\pm1})(1 + \xi^2)^s}$$

is such that

$$F(\xi) \leq \frac{C}{\xi^{2s+1} \sin(\theta^{j,j\pm1})}$$

for some constant C depending only on α, β, ω, and s. The result follows. □

Proof of Theorem 16.

1. By Lemma 20, $\mathbb{B}_{\hat{T}}$ is the sum of a coercive and a compact operators. By Fredholm alternative, it is invertible if and only if it is injective. Again by Lemma 20, we have that there exists $\hat{T}_{\min}$ such that for $\hat{T} \geq \hat{T}_{\min}$, $\mathbb{B}_{\hat{T}}$ is injective.
2. From $\mathbb{B}\tilde{\Phi} = \mathbb{D}G$ and $\mathbb{B}_{\hat{T}}\tilde{\Phi}_{\hat{T}} = \mathbb{D}\Pi_{\hat{T}}G$, we deduce:

$$\begin{aligned}\mathbb{B}_{\hat{T}}(\tilde{\Phi} - \tilde{\Phi}_{\hat{T}}) &= \mathbb{B}\tilde{\Phi} - (\mathbb{B} - \mathbb{B}_{\hat{T}})\tilde{\Phi} - \mathbb{B}_{\hat{T}}\tilde{\Phi}_{\hat{T}} \\ &= \mathbb{D}(\mathbb{I} - \Pi_{\hat{T}})(\tilde{\Phi} + G) \\ &= \mathbb{D}(\mathbb{I} - \Pi_{\hat{T}})\Phi\end{aligned}$$

   which tends to 0 when $\hat{T}$ tends to $+\infty$ by (60). Lemma 20 then implies that

$$\|\tilde{\Phi} - \tilde{\Phi}_{\hat{T}}\| \leq \frac{1}{\gamma}\|\mathbb{B}_{\hat{T}}(\tilde{\Phi} - \tilde{\Phi}_{\hat{T}})\|,$$

   which proves that $\tilde{\Phi}_{\hat{T}}$ tends to $\tilde{\Phi}$ when $\hat{T}$ tends to $+\infty$.
3. The previous step provides also the following inequality:

$$\|\tilde{\Phi} - \tilde{\Phi}_{\hat{T}}\| \leq \frac{1}{\gamma}\|\mathbb{D}(\mathbb{I} - \Pi_{\hat{T}})\Phi\|. \tag{73}$$

   Combined with Lemma 21, we get the estimate (61). □

4.3 Error estimate for the discrete problem

The main result of this section is given in the following theorem.

Theorem 22.

1. *There exist* $\hat{T}_{\min}$, $T_{\min}$ *and* $h_{\max}$ *such that the discrete problem* (54) *is well posed for* $\hat{T} \geq \hat{T}_{\min}$, $T \geq T_{\min}$, *and* $h \leq h_{\max}$.
2. *The solution* $\tilde{\Phi}_{\hat{T},\mathbf{h}}$ *of the discrete problem* (54) *tends to the exact solution* $\tilde{\Phi}$ *of* (49) *when* $\hat{T} \to +\infty$ *and* $\mathbf{h} = (T, h) \to (+\infty, 0)$.

3. *If* $\Phi = \tilde{\Phi} + G$ *is such that* $\varphi^j \in H^s(\Sigma^j)$ *for* $j \in \mathbb{Z}/n\mathbb{Z}$ *with* $s > 0$*, there exists* $C > 0$ *such that*

$$\|\tilde{\Phi} - \tilde{\Phi}_{\hat{T},\mathbf{h}}\| \leq \frac{C}{\hat{T}^s} + Ce^{-\varepsilon T} + Ch^{\min(s,l+1)} \tag{74}$$

where ε *is given by* (3).

To show this theorem, we will use the following lemma (which is the discrete equivalent of Lemma 20).

Lemma 23. *There exists* $\gamma' > 0$, $\hat{T}_{\min}$, $T_{\min}$, *and* $h_{\max}$ *such that for* $\hat{T} \geq \hat{T}_{\min}$, $T \geq T_{\min}$, *and* $h \leq h_{\max}$,

$$\forall \tilde{\Phi}_{\mathbf{h}} \in \tilde{V}_{\mathbf{h}} \qquad \sup_{\tilde{\Psi}_{\mathbf{h}} \in \tilde{V}_{\mathbf{h}}, \tilde{\Psi}_{\mathbf{h}} \neq 0} \frac{|(\mathbb{B}_{\hat{T}} \tilde{\Phi}_{\mathbf{h}}, \tilde{\Psi}_{\mathbf{h}})|}{\|\tilde{\Psi}_{\mathbf{h}}\|} \geq \gamma' \|\tilde{\Phi}_{\mathbf{h}}\|.$$

Proof. We proceed as in the proof of Lemma 20 and prove the result by contradiction. We consider a sequence $h_n, h_n \to 0$, a sequence $T_n, T_n \to +\infty$, a sequence $\hat{T}_n, \hat{T}_n \to +\infty$, and a sequence $\tilde{\Phi}_{\hat{T}_n,\mathbf{h}_n} \in \tilde{V}_{\hat{T}_n,\mathbf{h}_n}$, $\mathbf{h}_n = (h_n, T_n)$ such that

$$\|\tilde{\Phi}_{\hat{T}_n,\mathbf{h}_n}\| = 1 \quad \text{and} \quad \forall \tilde{\Psi}_{\mathbf{h}_n} \in \tilde{V}_{\mathbf{h}_n}, \ \left|(\mathbb{B}_{\hat{T}_n} \tilde{\Phi}_{\hat{T}_n,\mathbf{h}_n}, \tilde{\Psi}_{\mathbf{h}_n})\right| \leq \frac{1}{n} \|\tilde{\Psi}_{\mathbf{h}_n}\|.$$

Since $\tilde{\Phi}_{\hat{T}_n,\mathbf{h}_n}$ is bounded in $\tilde{V}$, it admits a weakly convergent subsequence that we denote also by $\tilde{\Phi}_{\hat{T}_n,\mathbf{h}_n}$: $\tilde{\Phi}_{\hat{T}_n,\mathbf{h}_n} \rightharpoonup \tilde{\Phi}$. Moreover, for all $\tilde{\Psi} \in \tilde{V}$ and all $\tilde{\Psi}_{\mathbf{h}_n} \in \tilde{V}_{\mathbf{h}_n}$ we have

$$\begin{aligned} \left|(\mathbb{B}_{\hat{T}_n} \tilde{\Phi}_{\hat{T}_n,\mathbf{h}_n}, \tilde{\Psi})\right| &\leq \left|(\mathbb{B}_{\hat{T}_n} \tilde{\Phi}_{\hat{T}_n,\mathbf{h}_n}, \tilde{\Psi}_{\mathbf{h}_n})\right| + \left|(\mathbb{B}_{\hat{T}_n} \tilde{\Phi}_{\hat{T}_n,\mathbf{h}_n}, \tilde{\Psi} - \tilde{\Psi}_{\mathbf{h}_n})\right| \\ &\leq \frac{1}{n} \|\tilde{\Psi}_{\mathbf{h}_n}\| + \|\mathbb{B}_{\hat{T}_n}\| \|\tilde{\Psi} - \tilde{\Psi}_{\mathbf{h}_n}\|. \end{aligned}$$

Since $\|\mathbb{B}_{\hat{T}_n}\|$ is bounded by a constant independent of n, we deduce from (48) that

$$\mathbb{B}_{\hat{T}_n} \tilde{\Phi}_{\hat{T}_n,\mathbf{h}_n} \rightharpoonup 0 \quad \text{in } \tilde{V}.$$

We can then continue the proof as in Lemma 20 which results in the contradiction. □

Proof of Theorem 22.
1. This is a direct consequence of Lemma 23.
2. Let $\tilde{\Phi}$ be the solution of the original problem (49), $\tilde{\Phi}_{\hat{T}}$ the solution of the semi discrete problem (51) and $\tilde{\Phi}_{\hat{T},\mathbf{h}}$ the solution of the discrete problem (54). We have that

$$\forall \tilde{\Upsilon}_{\mathbf{h}} \in \tilde{V}_{\mathbf{h}}, \quad \|\tilde{\Phi} - \tilde{\Phi}_{\hat{T},\mathbf{h}}\| \leq \|\tilde{\Phi} - \tilde{\Upsilon}_{\mathbf{h}}\| + \|\tilde{\Upsilon}_{\mathbf{h}} - \tilde{\Phi}_{\hat{T},\mathbf{h}}\| \tag{75}$$

For all $\tilde{\Upsilon}_{\mathbf{h}} \in \tilde{V}_{\mathbf{h}}$ and all $\tilde{\Psi}_{\mathbf{h}} \in \tilde{V}_{\mathbf{h}}$, we have

$$(\mathbb{B}_{\hat{T}}(\tilde{\Upsilon}_{\mathbf{h}} - \tilde{\Phi}_{\hat{T},\mathbf{h}}), \tilde{\Psi}_{\mathbf{h}}) = (\mathbb{B}_{\hat{T}}(\tilde{\Upsilon}_{\mathbf{h}} - \tilde{\Phi}_{\hat{T}}), \tilde{\Psi}_{\mathbf{h}}) + (\mathbb{D}_{\hat{T}}(G - G_{\mathbf{h}}), \tilde{\Psi}_{\mathbf{h}}).$$

By Lemma 23 and by the continuity of $\mathbb{D}_{\hat{T}}$ and $\mathbb{B}_{\hat{T}}$, we get

$$\gamma' \|\tilde{\Upsilon}_{\mathbf{h}} - \tilde{\Phi}_{\hat{T},\mathbf{h}}\| \leq C(\|G - G_{\mathbf{h}}\|_V + \|\tilde{\Phi}_{\hat{T}} - \tilde{\Upsilon}_{\mathbf{h}}\|). \tag{76}$$

Gathering (75)–(76), we deduce that there exists $C > 0$, such that

$$\|\tilde{\Phi} - \tilde{\Phi}_{\hat{T},\mathbf{h}}\| \leq C\Big(\|\tilde{\Phi} - \tilde{\Phi}_{\hat{T}}\| + \|G - G_{\mathbf{h}}\|_V + \inf_{\tilde{\Upsilon}_{\mathbf{h}} \in \tilde{V}_{\mathbf{h}},\, \tilde{\Upsilon}_{\mathbf{h}} \neq 0} \|\tilde{\Phi} - \tilde{\Upsilon}_{\mathbf{h}}\|\Big). \tag{77}$$

By Theorem 16, the first term of the right-hand side tends to 0. $G_{\mathbf{h}}$ being the interpolant of G in $\tilde{V}_{\mathbf{h}}$, (48) ensures that the two last terms tend to 0 when $\mathbf{h} \to (+\infty, 0)$.

3. Let now suppose that $\Phi = \tilde{\Phi} + G = (\varphi^0, \dots, \varphi^{n-1})$ the solution of (49) is such that for all $j \in \mathbb{Z}/n\mathbb{Z}$, $\varphi^j \in H^s(\Sigma^j)$ for a certain $s > 0$. Then we deduce from Theorem 16 an estimation of the first term of the right-hand side of (77). For the second term, it suffices to use classical results of the interpolation error for Lagrange FE of order l:

$$\exists C > 0, \quad \|G - G_{\mathbf{h}}\|_V \leq C h^{\min(s,l+1)}.$$

Finally, for the last term, let us introduce the function $\tilde{\Phi}_T \in \tilde{V}$ defined by

$$\tilde{\Phi}_T = \Phi_T - G \quad \text{where } \Phi_T = (\chi_{(-T_0,T_0)}\varphi^0, \dots, \chi_{(-T_{n-1},T_{n-1})}\varphi^{n-1}).$$

We get

$$\inf_{\tilde{\Upsilon}_{\mathbf{h}} \in \tilde{V}_{\mathbf{h}},\, \tilde{\Upsilon}_{\mathbf{h}} \neq 0} \|\tilde{\Phi} - \tilde{\Upsilon}_{\mathbf{h}}\| \leq \|\tilde{\Phi} - \tilde{\Phi}_T\| + \inf_{\tilde{\Upsilon}_{\mathbf{h}} \in \tilde{V}_{\mathbf{h}},\, \tilde{\Upsilon}_{\mathbf{h}} \neq 0} \|\tilde{\Phi}_T - \tilde{\Upsilon}_{\mathbf{h}}\|$$

where using (3), we can show that

$$\|\tilde{\Phi} - \tilde{\Phi}_T\| \leq C e^{-\varepsilon T}$$

and using again the results on interpolation error of Lagrange FE

$$\inf_{\tilde{\Upsilon}_{\mathbf{h}} \in \tilde{V}_{\mathbf{h}},\, \tilde{\Upsilon}_{\mathbf{h}} \neq 0} \|\tilde{\Phi}_T - \tilde{\Upsilon}_{\mathbf{h}}\| \leq C h^{\min(s,l+1)}.$$

This ends the proof of the theorem. □

Remark 24. This error estimate has been obtained for simple regular mesh. A more sophisticated discretization method could be used as done in [11] for scattering problems.

5 Numerical results

The numerical results presented in this section are obtained using the Finite Element library XLiFE++ [16].

5.1 Qualitative validation of the method

In order to validate the method, we consider a particular data on a triangle given by

$$g = \frac{1}{4\imath}\left(\alpha H(\omega\sqrt{x^2+y^2}) + \beta\frac{\partial H}{\partial x^j}(\omega\sqrt{x^2+y^2})\right)\Big|_{\Sigma^j_{\mathcal{O}}}$$

with $\omega = 1 + 0.1\imath$, $\alpha = 2$, $\beta = -0.5$ and $H(\cdot)$ denotes the zeroth Hankel function of the first kind [2]. The exact solution of this problem is

$$p = \frac{1}{4\imath}H(\omega\sqrt{x^2+y^2}).$$

On Figure 4.4, we represent on the interval $(-T, T)$ the real and imaginary parts of the exact solution φ^0 (blue line) and of the solution $\varphi^0_{\hat{T},\mathbf{h}}$ (red dots) computed by using P1 finite elements with $h = 0.1$, $T = 20$, $\hat{T} = 10$, and a third-order Gauss quadrature with 1000 intervals. We get a L^2 relative error:

$$\frac{\|\varphi^0 - \varphi^0_{\hat{T},\mathbf{h}}\|_{L^2(\Sigma^0_T)}}{\|\varphi^0\|_{L^2(\Sigma^0_T)}}$$

of 0.090 %. On Figure 4.5 (left), we represent the Fourier transform of the computed solution. Remark that the behavior of this Fourier transform justifies the truncation of the Fourier integral and requires a precise quadrature especially near $\xi = \omega$.

Once we obtained the $\varphi^j_{\hat{T},\mathbf{h}}$'s, we can reconstruct an approximation of the solution p of (1) in each Ω^j by Formula (12). Here, we compute the solution in the domain Ω' represented in Figure 4.5 (right), where the white lines represent the position of the Σ^j. In the overlapping zones, we can choose indifferently one of the available half-plane representations, since they coincide up to the discretization error. Remark that although the solutions φ^j are not close to zero at $y^j = \pm T$, the reconstructed solution is accurate, with an $L^2(\Omega')$ relative error equal to 0.030 %.

The same results can also be obtained when the obstacle is a rectangle or a pentagon. The reconstruction results are shown in Figure 4.6. For a rectangle obstacle, the L^2 relative error for the lines is 0.042 % and the L^2 error on the reconstructed domain is 0.043 %, while for a pentagon, we get 0.074 % L^2 relative error on the lines and 0.054 % L^2 relative error on the reconstructed domain.

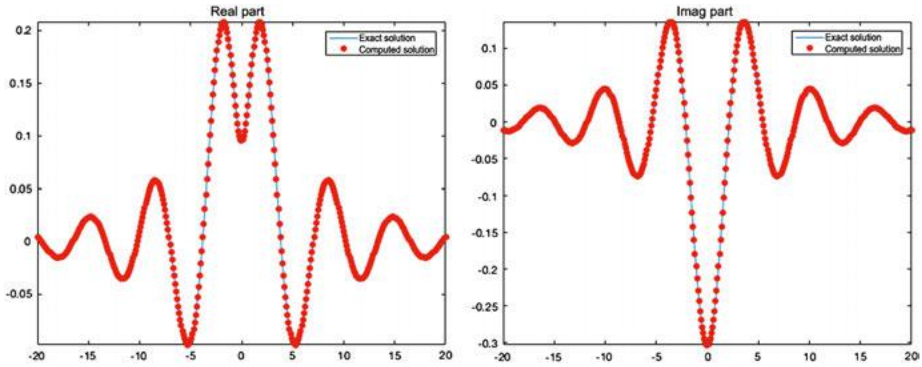


Figure 4.4: Real (left) and imaginary (right) part of the computed solution $\varphi^0_{\hat{T},\mathbf{h}}$ (red points) and the exact solution (blue line) on Σ^0.

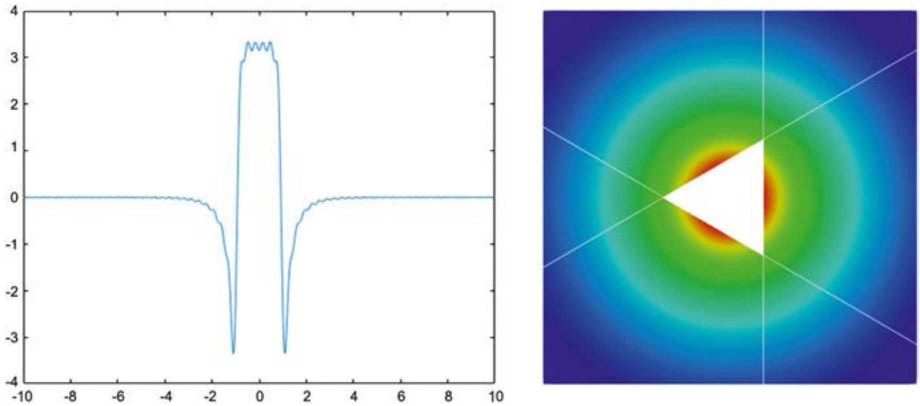


Figure 4.5: On the left: real part of the Fourier transform $\hat{\varphi}^0_{\hat{T},\mathbf{h}}$. On the right: reconstruction of the solution in Ω'.

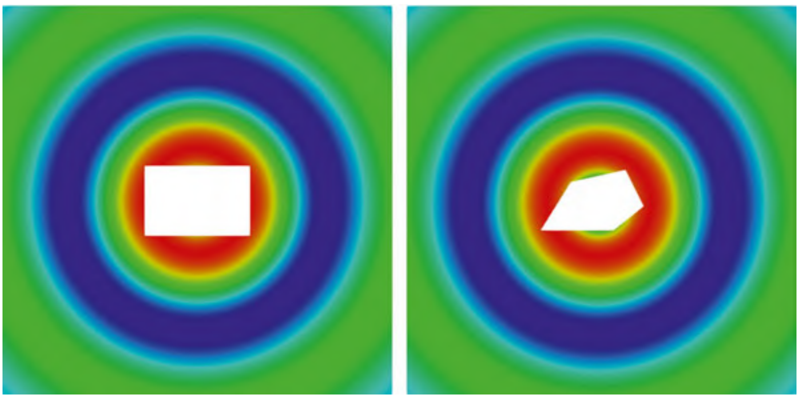


Figure 4.6: Reconstruction of the solution in Ω with rectangle and pentagon.

5.2 Quantitative validation of the error estimation

After this qualitative validation, we validate the error estimation derived in Section 4 by studying the influence of the different parameters. Since the triangle is regular, it suffices, by symmetry, to only consider the error on Σ^0. We still consider $\alpha = 2, \beta = -0.5$, and except in Section 5.2.1, $\omega = 1 + 0.1\imath$.

5.2.1 Influence of the length of the lines (parameter *T*)

From (74), we expect that the error will decay like $e^{-\varepsilon T}$, where ε is the imaginary part of the frequency. That is why, in this section (and only in this section), we consider different values of $\varepsilon \in \{0.05, 0.1, 0.2\}$. We fix the other parameters to $h = 0.025$, $\hat{T} = 10$, and use a third-order Gauss quadrature with 1000 intervals.

In Figure 4.7, we represent $\log(\varphi^0 - \varphi^0_{\hat{T},\mathbf{h}})$ as a function of T. The errors $(\varphi^0 - \varphi^0_{\hat{T},\mathbf{h}})$ decrease exponentially, depending on ε with the following behavior:

$$\text{err} := \|\varphi^0 - \varphi^0_{T,\mathbf{h}}\|_{L^2(\Sigma^0_T)} \sim e^{-\varepsilon T},$$

before finally becoming constant, which is due to the other discretization parameters.

5.2.2 Influence of the discretization in space (parameter *h*)

We plot the error $\log(\varphi^0 - \varphi^0_{\hat{T},\mathbf{h}})$ as a function of $\log h$. We use the P1 and P2 finite elements and the following parameters:

$$T = 40, \ \hat{T} = 10,$$

and a third-order quadrature with 1000 intervals.

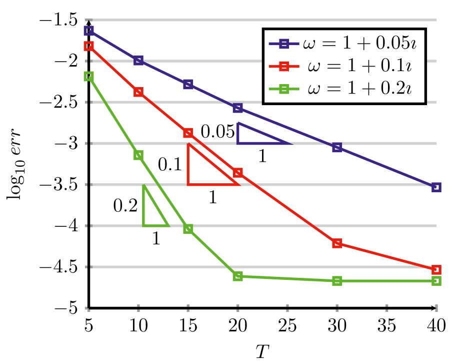


Figure 4.7: Influence of the length of the lines T for various values of $\varepsilon = \operatorname{Im}(\omega)$.

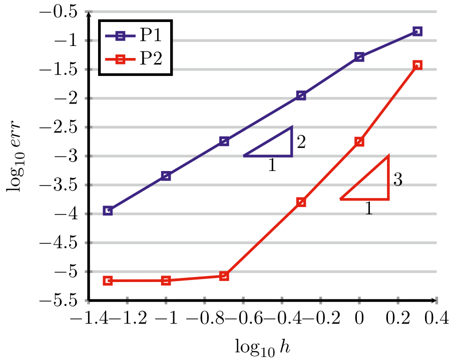


Figure 4.8: Influence of the space discretization h.

Figure 4.8 shows that the error decreases following:

$$\text{err} \sim h^{l+1},$$

before becoming constant because of the other discretization parameters.

5.2.3 Influence of the truncation of the Fourier integrals (parameter $\hat{T}$)

Finally, we plot the error $\log(\varphi^0 - \varphi^0_{\hat{T},\mathbf{h}})$ with respect to $\hat{T}$ and we use $T = 40$, $h = 0.1$, and a third-order quadrature with $100 \times \hat{T}$ intervals.

From Figure 4.9, we see that the error decreases exponentially due to the $\mathcal{C}^\infty$ regularity of the Hankel function.

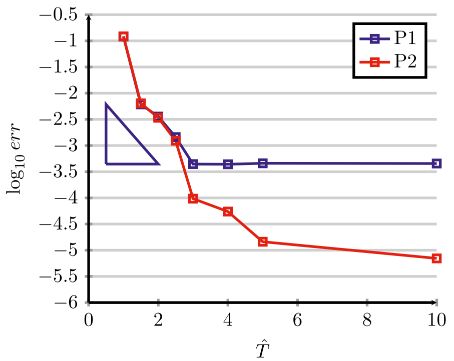


Figure 4.9: Influence of the length of the Fourier integral $\hat{T}$.

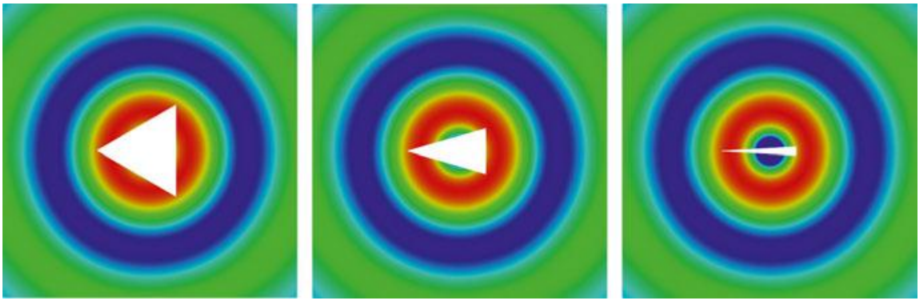


Figure 4.10: Reconstruction of the solution in Ω with triangles that becomes more and more flat.

5.2.4 The influence of the angles of the polygon

Referring to the Theorem 12, we investigate the influence of the angles of the polygon on the computation of the solution. Remember that the coercivity constant tends to zero when one of the angles tends to zero (see (45)).

We represent in Figure 4.10 the reconstruction of the solution around three different triangles with one angle becoming smaller and smaller ($\min(\theta^{j,j+1}) = 0.33\pi, 0.16\pi, 0.03\pi$). Qualitatively, the results look similar and the L^2 relative error are of the same order (resp., 1.01 %, 0.88 %, and 1.23 %). The condition number of the finite element matrices are 1617.27, 2482.05, and 4647.19, respectively, meaning that it is only slightly affected by the smallness of one of the angles.

5.3 Extension cases

5.3.1 Non-regular Dirichlet data

In this section, we consider the Dirichlet case, namely (1) with $\alpha = 1, \beta = 0$ and we use the half-space matching formulation (22) where the φ^j's correspond to the Dirichlet traces of p on the Σ^j's. As mentioned in Section 1.2, our formulation allows to consider a data $g \in L^2(\partial\mathcal{O})$ but $g \notin H^{1/2}(\partial\mathcal{O})$. In the following test, we take

$$g = \begin{cases} 1 & \text{if } x^0 = 0, y^0 > 0 \text{ or } x^1 = 0, y^1 < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{78}$$

We use P1 discontinuous finite elements since we have a discontinuous boundary condition on $\partial\mathcal{O}$. The real part of the $\varphi^0_{T,\mathbf{h}}$ and the Fourier transform are given in Figure 4.11. As the data is less regular than the previous example, the Fourier transform $\hat{\varphi}^0_{T,\mathbf{h}}$ decays more slowly than in the previous example (pay attention to the scale).

The reconstruction in Ω is shown in Figure 4.12. The result is good as there is no visible jump on different reconstructions from different φ^j.

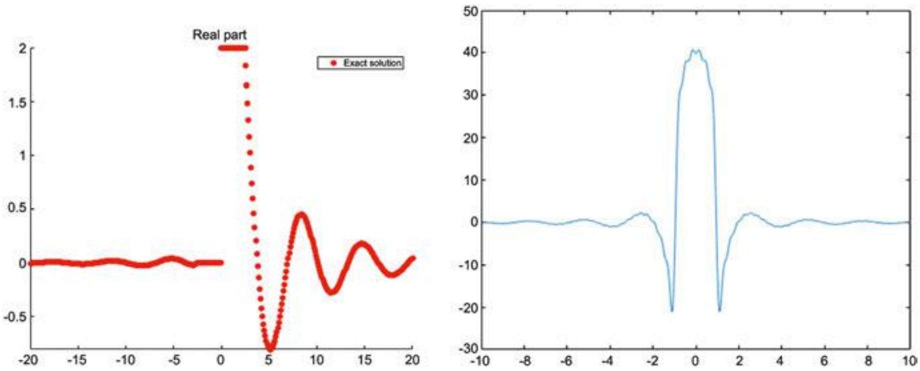


Figure 4.11: On the left: real part of the computed solution φ_h^0 (red points) on Σ^0. On the right: the Fourier transform $\hat{\varphi}^0$ for g defined in (78).

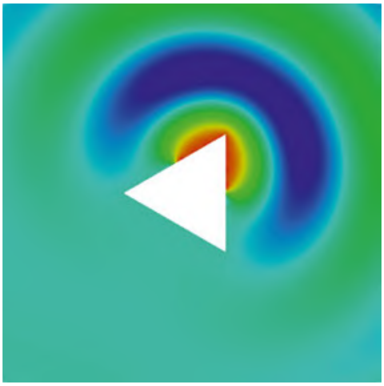


Figure 4.12: Real part of p with $\alpha = 1$, $\beta = 0$, $\omega = 1$, with g given defined in (78).

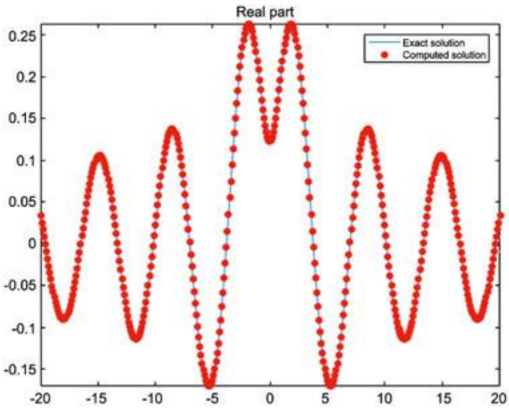
5.3.2 Non-dissipative case

Finally, remark that our theoretical results are established only for $\mathrm{Im}(\omega) > 0$. However, the numerical method works for the case without dissipation, provided that we use the representation of the outgoing solution in (12) for each half-space, which means that

$$\sqrt{\omega^2 - \xi^2} = \begin{cases} \sqrt{|\omega^2 - \xi^2|} & \text{for } \xi^2 < \omega^2, \\ \imath\sqrt{|\xi^2 - \omega^2|} & \text{for } \xi^2 > \omega^2. \end{cases}$$

To illustrate this, we once again validate the method by using the Hankel function on the boundary of the polygon with $\omega = 1$.

In Figure 4.13 on the left, as expected, we see that the solution decreases more slowly compared to the case with dissipation. The computed solution matches the



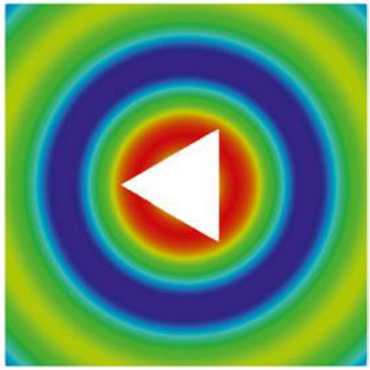


Figure 4.13: On the left: real part of the computed solution φ_h^0 (red points) on Σ^0. On the right: the reconstruction of the solution with $\omega = 1$.

exact solution well and the L^2 relative error on the lines is 1.50 %, which is higher than the case with dissipation ($\operatorname{Im}\omega > 0$). The reconstructed solution on the domain has an L^2 relative error of 0.79 %.

We also show here the solution of a scattering problem with an incident wave

$$p_{\text{inc}} = e^{\imath\omega(x\cos\gamma + y\sin\gamma)},$$

with $\gamma = 3\pi/4$. The scattered field is solution of (1) with the boundary data is

$$g = -\alpha p_{\text{inc}} - \beta\frac{\partial p_{\text{inc}}}{\partial n} \text{ on } \partial\mathcal{O}.$$

We consider the solution obtained with the parameters $T = 40$, $h = 0.05$, and $\hat{T} = 10$ as the "exact solution" (represented on Figure 4.14 (right)) and we plot the error for different value of $\hat{T}$ between 1 and 8. According to error estimate (74), we expect a behavior like

$$\text{err} \sim \frac{1}{\hat{T}^{s_0}},$$

where s_0 is the supremum of s values such that all traces φ^j belong to $H^s(\Sigma^j)$. Here, the theory of singularities [14] shows that $p \in H^{8/5}(\Omega)$, so that, taking its normal derivative, we get $s_0 = 1/10$. In fact, we observe on Figure 4.14 (left) that the error decreases more rapidly like

$$\text{err} \sim \frac{1}{\hat{T}^2}.$$

It is probably due to the discretization in space that cannot capture the singularity at the corner.

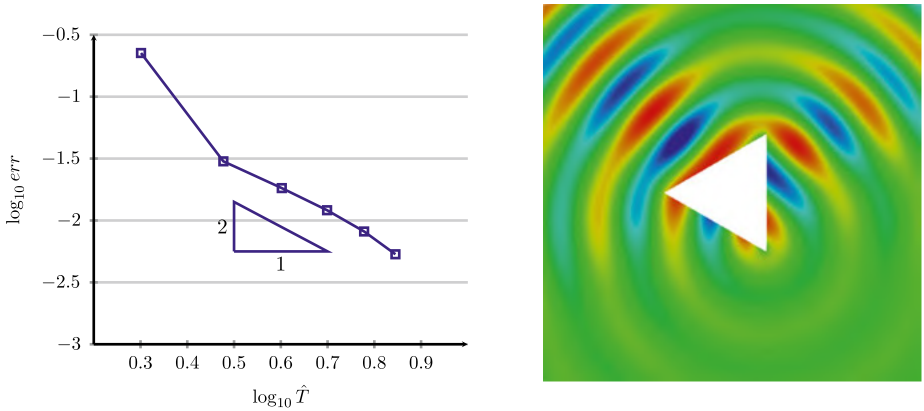


Figure 4.14: On the left: Influence of the length of the Fourier integral $\hat{T}$ in the scattering problem. On the right: real part of the scattered field with $\hat{T} = 10$.

Appendix A. Proof of Lemma 10

Let us remind the definition of the operator L_N. For all $\psi \in L^2(\Sigma^0)$, we consider the problem

$$\left| \begin{aligned} &-\Delta v = 0 && \text{in } \Omega^0, \\ &\beta \frac{\partial v}{\partial x} = \psi && \text{on } \Sigma^0. \end{aligned} \right. \tag{79}$$

This problem has a unique solution v in the following weighted Sobolev space (see, for instance, [17, Chapter 6]):

$$\left\{ u \in L^2_{\text{loc}}(\Omega^0) \;\middle|\; \frac{1}{r^{3/2}} u \in L^2(\Omega^0), \frac{1}{\sqrt{r}} \nabla u \in L^2(\Omega^0) \right\}.$$

The operator L_N is defined as

$$\begin{aligned} L_N &: L^2(\Sigma^0) \to L^2(\Sigma^1 \cap \Omega^0), \\ L_N \psi &:= \beta \frac{\partial}{\partial x^1} v(\psi) \Big|_{\Sigma^1 \cap \Omega^0}. \end{aligned}$$

We want to show in this Appendix that:

1. The operator L_N is continuous from $L^2(\Sigma^0)$ in $L^2(\Sigma^1 \cap \Omega^0)$ and its norm is bounded by 1;
2. $\exists C \in (0, 1), \forall \psi \in L^2(\Sigma^0), \|L_N \chi_{(a_0^+, +\infty)} \psi\| \le C \|\chi_{(a_0^+, +\infty)} \psi\|$;
3. $L_N \chi_{(-\infty, a_0^-)}$ is a compact operator from $L^2(\Sigma^0)$ in $L^2(\Sigma^1 \cap \Omega^0)$.

Let us begin the proof which is based on Mellin techniques.

1. Denoting (r,θ) the polar coordinates (the center is the intersection point between Σ^0 and $(\Sigma^1 \cap \Omega^0)$) defined by

$$\left| \begin{aligned} r &= \sqrt{(x-l^0)^2 + (y-a_0^+)^2} \in (0,+\infty), \\ \theta &= \frac{\pi}{2} - \arctan\left(\frac{y-a_0^+}{x-l^0}\right) \in (0,\pi). \end{aligned} \right. \tag{80}$$

We introduce the function w defined for almost everywhere $(t,\theta) \in \mathcal{B} \equiv \mathbb{R} \times (0,\pi)$ by $w(t,\theta) = v(x,y)$ where $t = \ln r$ and (r,θ) is defined in (80). It is the solution of

$$\left| \begin{aligned} &-\Delta_{t,\theta} w = 0 \quad \text{in } \mathcal{B}, \\ &\beta \frac{\partial w}{\partial \theta}(t,0) = \psi_0(t) := e^t \psi(e^t + a_0^+), \\ &-\beta \frac{\partial w}{\partial \theta}(t,\pi) = \psi_\pi(t) := e^t \psi(-e^t + a_0^+). \end{aligned} \right. \tag{81}$$

We can show by a simple change of variable that

$$\psi \in L^2(\Sigma^0) \quad \Rightarrow \quad t \mapsto e^{-t/2}\psi_0 \in L^2(\mathbb{R}) \quad \text{and} \quad t \mapsto e^{-t/2}\psi_\pi \in L^2(\mathbb{R}), \tag{82}$$

and

$$\|e^{-t/2}\psi_0\|_{L^2(\mathbb{R})} = \|\psi\|_{L^2(a_0^+,+\infty)} \quad \text{and} \quad \|e^{-t/2}\psi_\pi\|_{L^2(\mathbb{R})} = \|\psi\|_{L^2(-\infty,a_0^+)}. \tag{83}$$

It is possible to compute explicitly w by applying the Fourier–Laplace transform which is defined as

$$\check{u}(\lambda) \equiv [\mathcal{M}_{t\to\lambda}](\lambda) := \int_{\mathbb{R}} e^{-\lambda t} u(t) \mathrm{d}t. \tag{84}$$

It is an isomorphism between $\{u, e^{\gamma t}u \in L^2(\mathbb{R})\}$ and $L^2(l_{-\gamma})$ where $l_{-\gamma} = \{\lambda = -\gamma + is, s \in \mathbb{R}\}$, for all $\gamma \in \mathbb{R}$ and we have the Plancherel formula

$$\int_{\mathbb{R}} e^{2\gamma t} |u(t)|^2 \mathrm{d}t = \frac{1}{2\pi i} \int_{l_{-\gamma}} |\check{u}(\lambda)|^2 \mathrm{d}\lambda := \|\check{u}\|^2_{L^2(l_{-\gamma})}. \tag{85}$$

We have in particular thanks to (82–83)

$$\begin{aligned} &\lambda \mapsto \check{\psi}_0(\lambda) \in L^2(l_{1/2}), \quad \|\check{\psi}_0\|_{L^2(l_{1/2})} = \|\psi\|_{L^2(a_0^+,+\infty)}, \\ &\lambda \mapsto \check{\psi}_\pi(\lambda) \in L^2(l_{1/2}), \quad \|\check{\psi}_\pi\|_{L^2(l_{1/2})} = \|\psi\|_{L^2(-\infty,a_0^+)}. \end{aligned} \tag{86}$$

Applying the Fourier–Laplace transform to w, we have $\check{w}(\bullet,\theta) = \mathcal{M}_{t\mapsto\lambda} w(\bullet,\theta)$ which satisfies

$$\left| \begin{aligned} &-\lambda^2 \check{w}(\lambda,\theta) - \frac{\partial^2 \check{w}}{\partial \theta^2}(\lambda,\theta) = 0, \quad \forall \lambda \in \mathbb{C} \\ &\beta \frac{\partial \check{w}}{\partial \theta}(\lambda,0) = \check{\psi}_0(\lambda), \\ &-\beta \frac{\partial \check{w}}{\partial \theta}(\lambda,\pi) = \check{\psi}_\pi(\lambda), \end{aligned} \right.$$

and we can easily find the solution of this equation. We obtain for $\lambda \notin \mathbb{Z}$

$$\check{w}(\lambda,\theta) = \frac{\cos(\lambda(\pi-\theta))}{\beta\lambda\sin(\lambda\pi)}\check{\psi}_0(\lambda) + \frac{\cos(\lambda\theta)}{\beta\lambda\sin(\lambda\pi)}\check{\psi}_\pi(\lambda),$$

which, for $\lambda \notin \mathbb{Z}$, leads to

$$\beta\frac{\partial\check{w}}{\partial\theta}(\lambda,\theta) = A(\lambda,\pi-\theta)\check{\psi}_0(\lambda) - A(\lambda,\theta)\check{\psi}_\pi(\lambda),$$

where

$$A(\lambda,\theta) = \frac{\sin(\lambda\theta)}{\sin(\lambda\pi)}.$$

Moreover, we can show that for all $\theta \in (0,\pi)$, $s \mapsto |A(1/2+\imath s,\theta)|$ is in $L^\infty(\mathbb{R})$ and its supremum is attained at $s = 0$ and it is equal to $\sin(\theta/2)$. Using the Cauchy–Schwarz inequality $(ab+cd)^2 \le (a^2+c^2)(b^2+d^2)$, we have then, for all $\theta \in (0,\pi)$

$$\begin{aligned}\left\|\beta\frac{\partial\check{w}}{\partial\theta}(\lambda,\theta)\right\|^2_{L^2(l_{1/2})} &\le \left(\cos(\theta/2)\|\check{\psi}_0\|_{L^2(l_{1/2})} - \sin(\theta/2)\|\check{\psi}_\pi\|_{L^2(l_{1/2})}\right)^2\\ &\le \left(\|\check{\psi}_0\|^2_{L^2(l_{1/2})} + \|\check{\psi}_\pi\|^2_{L^2(l_{1/2})}\right) = \|\psi\|^2_{L^2(\Sigma^0)}.\end{aligned}$$

The last inequality for $\theta = \theta^{0,1}$, after the change of variable $r = e^t$, yields to

$$\beta\frac{\partial v}{\partial x^1}\bigg|_{\Sigma^1\cap\Omega^0} \in L^2(\Sigma^1\cap\Omega^0)$$

and

$$\left\|\beta\frac{\partial v}{\partial x^1}\bigg|_{\Sigma^1\cap\Omega^0}\right\|_{L^2(\Sigma^1\cap\Omega^0)} \le \|\psi\|_{L^2(\Sigma^0)}. \tag{87}$$

We have shown that the operator L_N is continuous from $L^2(\Sigma^0)$ to $L^2(\Sigma^1\cap\Omega^0)$ and its norm is bounded by 1.

2. The norm of $L_N\chi_{(a^0,+\infty)}$ can be deduced from the previous computation by taking $\psi_{(-\infty,a_0^+)} = 0$ or equivalently $\check{\psi}_\pi = 0$. We get for all $\theta \in (0,\pi)$:

$$\left\|\beta\frac{\partial\check{w}}{\partial\theta}(\lambda,\theta)\right\|_{L^2(l_{1/2})} \le \cos(\theta/2)\left\|\check{\psi}_0(\lambda)\right\|_{L^2(l_{1/2})} = \cos(\theta/2)\|\psi\|_{L^2(a_0^+,+\infty)}, \tag{88}$$

from where we conclude that the norm of the operator $L_N\chi_{(a^0,+\infty)}$ is bounded by $\cos(\theta^{0,1}/2)$.

3. Finally, let us consider the previous computation with $\psi = 0$ on $(a_0^-,+\infty)$. This corresponds to take $\psi_0 = 0$ and $\psi_\pi = e^t\psi(-e^t+a_0^+)$. Since ψ vanishes on $(a_0^-,+\infty)$, we have $e^{-\gamma t}\psi_\pi$ is in $L^2(\mathbb{R})$ for any $\gamma > 1$ and so, by (85), $\check{\psi}_\pi$ is in $L^2(l_\gamma)$ for all $\gamma > 1$. The previous computation yields to

$$\forall\lambda \notin \mathbb{Z},\quad \beta\frac{\partial\check{w}}{\partial\theta}(\lambda,\theta^{0,1}) = -A(\lambda,\theta^{0,1})\check{\psi}_\pi(\lambda).$$

We can show that

$$\forall\gamma > 1,\ \gamma \notin \mathbb{N}, \quad \sup_{\lambda\in l_\gamma} \lambda A(\lambda, \theta^{0,1}) < +\infty,$$

which enables us to deduce that

$$\forall\gamma > 1,\ \gamma \notin \mathbb{N}, \quad \lambda \frac{\partial \check{w}}{\partial\theta}(\lambda, \theta^{0,1}) \in L^2(l_\gamma).$$

By applying the inverse Laplace–Fourier transform, we have then

$$\forall\gamma > 1,\ \gamma \notin \mathbb{N}, \quad e^{-\gamma t} \frac{\partial^2 \check{w}}{\partial t\partial\theta}(t, \theta^{0,1}) \in L^2(\mathbb{R}),$$

and by change of variable,

$$\forall\gamma > 1,\ \gamma \notin \mathbb{N}, \quad r^{-\gamma+3/2}\partial_r\left(\frac{1}{r}\frac{\partial \check{w}}{\partial\theta}(r, \theta^{0,1})\right) \in L^2(\mathbb{R}^+).$$

If we choose $\gamma = 3/2$, the operator $L_N\chi_{(-\infty,a_0^-)} \in H^1(0,+\infty)$. By compact embedding of $H^1(0,b)$ in $L^2(0,b)$ for any $b > 0$, we show that $\chi_{(0,b)}L_N\chi_{(-\infty,a_0^-)}$ is compact. It suffices to use similar argument as in the proof of Lemma 8 to show that $\chi_{(b,+\infty)}L_N\chi_{(-\infty,a_0^-)}$ is a Hilbert–Schmidt operator.

Let us now give other properties of L_N which will be useful for the numerical analysis (see Section 4). We remind the definition of the symmetric and anti-symmetric operators, defined in the proof of Lemma 17, S and $A \in \mathcal{L}(L^2(\mathbb{R}))$

$$\forall\psi \in L^2(\mathbb{R}), \quad S\psi(y) = \frac{1}{2}(\psi(y) + \psi(2a_0^+ - y)), \quad \text{and} \quad A\psi(y) = \frac{1}{2}(\psi(y) - \psi(2a_0^+ - y))$$

We remind that $S + A = \mathrm{Id.}$, $S^2 = S$, $A^2 = A$, $\|S\| \le 1$, $\|A\| \le 1$ and for any $\psi \in L^2(\mathbb{R})$ such that $\psi_{(-\infty,a_0^+)} = 0$, we have

$$\|S\psi\|_{L^2(\mathbb{R})} = \|A\psi\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2}}\|\psi\|_{L^2(\mathbb{R})}. \tag{89}$$

Let us now study the norm of $L_N S\psi$ and $L_N A\psi$ for any $\psi \in L^2(\mathbb{R})$. By reproducing the previous calculations, we have easily that, writing $\widetilde{(S\psi)}_0 = \widetilde{(S\psi)}_\pi$,

$$\|L_N S\psi\|_{L^2(\Sigma^1\cap\Omega^0)} \le \sup_{\lambda\in l_{1/2}} \left|A(\lambda, \pi - \theta^{0,1}) - A(\lambda, \theta^{0,1})\right| \|S\psi\|_{L^2(a,+\infty)}$$

where we remind that

$$\forall\theta \in (0,\pi),\ \forall\lambda \notin \mathbb{Z}, \quad A(\lambda,\theta) = \frac{\sin(\lambda\theta)}{\sin(\lambda\pi)}.$$

We can show that the supremum is attained at $\lambda = 1/2$ and then

$$\sup_{\lambda \in l_{1/2}} \left|A(\lambda, \pi - \theta^{0,1}) - A(\lambda, \theta^{0,1})\right| = \left|\cos(\theta^{0,1}/2) - \sin(\theta^{0,1}/2)\right|.$$

Using that $\sqrt{2}\|S\psi\|_{L^2(a,+\infty)} = \|S\psi\|_{L^2(\mathbb{R})} \leq \|\psi\|_{L^2(\mathbb{R})}$, we obtain

$$\|L_N S\psi\|_{L^2(\Sigma^1 \cap \Omega^0)} \leq \frac{|\cos(\theta^{0,1}/2) - \sin(\theta^{0,1}/2)|}{\sqrt{2}} \|\psi\|_{L^2(\mathbb{R})}. \tag{90}$$

Similarly, we get

$$\|L_N A\psi\|_{L^2(\Sigma^1 \cap \Omega^0)} \leq \frac{\cos(\theta^{0,1}/2) + \sin(\theta^{0,1}/2)}{\sqrt{2}} \|\psi\|_{L^2(\mathbb{R})}. \tag{91}$$

Moreover, let us remark, that gathering these inequalities, we obtain an inequality which comparing to (87), is not optimal:

$$\begin{aligned} \|L_N \psi\|_{L^2(\Sigma^1 \cap \Omega^0)} &\leq \|L_N S\psi\|_{L^2(\Sigma^1 \cap \Omega^0)} + \|L_N A\psi\|_{L^2(\Sigma^1 \cap \Omega^0)} \\ &\leq \sqrt{2} \max(\cos(\theta^{0,1}/2), \sin(\theta^{0,1}/2)) \|\psi\|_{L^2(\mathbb{R})}. \end{aligned} \tag{92}$$

Moreover, for $\psi \in L^2(\mathbb{R})$ such that $\psi_{(-\infty, a_0^+)} = 0$, using (89), we obtain

$$\|L_N \psi\|_{L^2(\Sigma^1 \cap \Omega^0)} \leq \|L_N S\| \|S\psi\|_{L^2(\mathbb{R})} + \|L_N A\| \|A\psi\|_{L^2(\mathbb{R})} \leq C' \|\psi\|_{L^2(\mathbb{R})}$$

where

$$C' = \max(\cos(\theta^{0,1}/2), \sin(\theta^{0,1}/2)) \in (0, 1).$$

This result is then not optimal for $\theta^{0,1} \in (0, \pi/2)$ (compared to (88)) but the constantC', obtained that way, is still in $(0, 1)$.

Bibliography

[1] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, Elsevier Science, 2003.

[2] D. E. Amos, *A Subroutine Package for Bessel Functions of a Complex Argument and Nonnegative Order*, Sandia National Laboratory Report, SAND85-1018, 1985.

[3] T. Apel, S. Nicaise, and J. Pfefferer, *Discretization of the Poisson equation with non-smooth data and emphasis on non-convex domains*, arXiv, 2015.

[4] K. E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, 1997.

[5] E. Bécache, S. Fauqueux, and P. Joly, Stability of perfectly matched layers, group velocities and anisotropic waves, *J. Comput. Phys.*, **188**(2) (2003), 399–433, DOI: 10.1016/S0021-9991(03)00184-0.

[6] C. Besse, J. Coatléven, S. Fliss, I. Lacroix-Violet, and K. Ramdani, Transparent boundary conditions for locally perturbed infinite hexagonal periodic media, *Commun. Math. Sci.*, **11**(4) (2013), 907–938.

[7] A.-S. Bonnet-Ben Dhia, C. Chambeyron, and G. Legendre, On the use of perfectly matched layers in the presence of long or backward guided elastic waves, *Wave Motion*, **51**(2) (2014), 266–283, DOI: 10.1016/j.wavemoti.2013.08.001.

[8] A.-S. Bonnet-Ben Dhia, S. Fliss, and A. Tonnoir, The halfspace matching method: a new method to solve scattering problem in infinite media, *J. Comput. Appl. Math.*, **338** (2018), 44–68.

[9] G. A. Chandler, Galerkin's method for boundary integral equations on polygonal domains, *J. Aust. Math. Soc. Ser. B*, **26** (1984), 1–13.

[10] S. N. Chandler-Wilde and D. C. Hothersall, Efficient calculation of the green function for acoustic propagation above a homogeneous impedance plane, *J. Sound Vib.*, **180**(5) (1995), 705–724.

[11] S. N. Chandler-Wilde, S. Langdon, and M. Mokgolele, A high frequency boundary element method for scattering by convex polygons with impedance boundary conditions, *Commun. Comput. Phys.* ISSN 1991-7120, **11**(2) (2012), 573–593.

[12] M. Dauge, *Elliptic Boundary Value Problems on Corner Domains: Smoothness and Asymptotics of Solutions*, Springer Berlin Heidelberg, 2006.

[13] S. Fliss and P. Joly, Exact boundary conditions for time-harmonic wave propagation in locally perturbed periodic media, *Appl. Numer. Math.*, **59**(9) (2009), 2155–2178.

[14] P. Grisvard, *Singularities in Boundary Value Problems*, Springer Berlin Heidelberg, 1992.

[15] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Society for Industrial and Applied Mathematics, vol. 69, 2011.

[16] N. Kielbasiewicz and E. Lunéville, *User Documentation of XLiFE++*, POEMS, ENSTA, 2016.

[17] V. A. Kozlov, V. G. Mazia, and J. Rossmann, *Elliptic Boundary Value Problems in Domains with Point Singularities*, American Mathematical Soc., vol. 52, 1997.

[18] M. Lenoir and A. Tounsi, The localized finite element method and its application to the two-dimensional sea-keeping problem, *SIAM J. Numer. Anal.*, **25**(4) (1988).

[19] M. Reed and B. Simon, *I: Functional Analysis, vol. 1*, Elsevier, 1980.

[20] A. Tonnoir, *Transparent Conditions for the Diffraction of Elastic Waves in Anisotropic Media*, Ecole Polytechnique, PhD Thesis, 2015.

[21] A. Velichko and P. D. Wilcox, Modeling the excitation of guided waves in generally anisotropic multilayered media, *J. Acoust. Soc. Am.*, **121**(1) (2007), 60–69.

Samuel Cogar, David Colton, and Peter Monk

5 Eigenvalue problems in inverse electromagnetic scattering theory

Abstract: The inverse electromagnetic scattering problem for anisotropic media in general does not have a unique solution. A possible approach to this problem is through the use of appropriate "target signatures," i. e., eigenvalues associated with the direct scattering problem that are accessible to measurement from a knowledge of the scattering data. In this paper, we shall consider three different sets of eigenvalues that can be used as target signatures: (1) eigenvalues of the electric far field operator, (2) transmission eigenvalues, and (3) Stekloff eigenvalues.

Keywords: Inverse scattering, nondestructive testing, transmission eigenvalues, Stekloff eigenvalues, eigenvalues of the far field operator

MSC 2010: 35J25, 35P05, 35P25, 35R30

1 Introduction

An important unresolved problem in electromagnetic inverse scattering theory is how to detect flaws or changes in the constitutive parameters in an inhomogeneous anisotropic medium. Such a problem presents itself, for example, in efforts to detect structural changes in airplane canopies due to prolonged exposure to ultraviolet radiation and is currently resolved by simply discarding canopies every few months and replacing them with new ones. The difficulties in using electromagnetic waves to interrogate anisotropic media is due to the fact that the corresponding inverse scattering problem no longer has a unique solution even if multiples frequencies and multiple sources are used [11]. Hence alternate approaches to the nondestructive testing of anisotropic materials need to be developed.

A possible approach to the target identification problem for anisotropic materials is through the use of appropriate "target signatures," i. e., eigenvalues associated with the direct scattering problem that are accessible to measurement from a knowledge of the scattering data. The earliest attempt to do this was based on the use of

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so-called “scattering resonances” corresponding to the complex poles of the scattering operator. Such an approach appeared particularly fruitful since there is a deep and well-developed theory of such resonances that is readily available to the practitioner [16]. However, the use of scattering resonances as target signatures ultimately proved unsuccessful in electromagnetic interrogation due to the difficulty in determining the location of the complex resonances from measured scattering data which is known only for real values of the wave number.

A second attempt to determine target signatures from the far field data was to use the eigenvalues of either the electric or magnetic far field operator for this purpose. We will present this approach in Section 3 with numerical examples given in Section 6.1. A major drawback of this approach is the lack of any theory relating changes in the eigenvalues to changes in the material properties of the scatterer.

A more recent effort to determine appropriate target signatures for anisotropic materials is based on the use of transmission eigenvalues [2, 4]. As opposed to scattering resonances, for dielectrics these eigenvalues are real and can be readily determined from the scattering data. In view of their potential in the nondestructive testing of dielectric materials, we will present the basic theory of transmission eigenvalues in the next two sections of our paper and refer the reader to the two monographs [2] and [4] for further details. In contrast to the theory of scattering resonances, the theory of transmission eigenvalues is of more recent origin with many questions unanswered. In particular, it has been shown in special cases that complex transmission eigenvalues exist for dielectric materials but whether such eigenvalues exist in general and what their physical meaning is remains an open question.

There are two main problems with using transmission eigenvalues as target signatures. The first of these is that such an approach is only applicable to dielectric materials. The second is that one must interrogate the material over a range of frequencies centered at a transmission eigenvalue, i. e., one is forced to use multi-frequency data over a predetermined range of frequencies. A method to overcome both of these difficulties has recently been proposed that is based on using a modified far field operator instead of the standard far field operator that is used to determine both scattering resonances and transmission eigenvalues. In this new approach, the frequency is held fixed and a new artificial eigenparameter is introduced which can be determined from measured scattering data. In one version of this approach, the new artificial eigenparameter turns out to be an electromagnetic version of the classical Stekloff eigenvalue problem for elliptic equations and we will discuss this specific class of target signatures in Section 3 of this paper [3, 7].

The plan of our paper is as follows. In the next section (Section 2), we shall present the basic theory of transmission eigenvalues for Maxwell’s equations in an anisotropic medium and their use as target signatures. This is followed, in Section 3, by a discussion of the eigenvalues of the electric and magnetic far field operators. Next, in Section 4, we summarize two methods for determining transmission eigenvalues using the magnetic far field equation (one could also use the electric far field equation).

Then, in Section 5, we will show that, through the use of a modified far field operator, Stekloff eigenvalues corresponding to the anisotropic index of refraction can be determined from the measured scattering data. In Section 6, we present some numerical examples illustrating our results. Finally, in Section 7, we draw some conclusions and suggest some future directions of research.

2 Transmission eigenvalues

We begin by formulating the direct electromagnetic scattering problem that we will refer to throughout this paper. Let E^i, H^i be an incident field that is scattered by an inhomogeneous object occupying the domain D, where we assume that D has smooth boundary ∂D. The corresponding scattered field is denoted by E^s, H^s and $E = E^i + E^s$, $H = H^i + H^s$ is the total field. Then the (normalized) Maxwell's equations are

$$\left.\begin{aligned} \operatorname{curl} E - ikH &= 0 \\ \operatorname{curl} H + ikN(x)E &= 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3 \tag{2.1}$$

where $k > 0$ is the wave number, $x \in \mathbb{R}^3$, $N(x)$ is the symmetric matrix index of refraction with entries in $C^1(\overline{D})$ and E^s, H^s satisfy the Silver–Müller radiation condition

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0 \tag{2.2}$$

where $r = |x|$. We will assume that the incident field E^i, H^i is given by

$$\begin{aligned} E^i(x) &= E^i(x; d, p) = \frac{i}{k} \operatorname{curl}\operatorname{curl} p e^{ikx\cdot d} \\ H^i(x) &= H^i(x; d, p) = \operatorname{curl} p e^{ikx\cdot d} \end{aligned} \tag{2.3}$$

where $d \in \mathbb{R}^3$, $|d| = 1$, is the direction of the incident wave and $p \in \mathbb{R}^3$ is the polarization. Under the assumption that

$$\begin{aligned} &\overline{\xi} \cdot \operatorname{Re} N(x)\xi \ge \alpha|\xi|^2 \\ &\overline{\xi} \cdot \operatorname{Im} N(x)\xi \ge 0 \end{aligned} \tag{2.4}$$

for $x \in D$, $\xi \in \mathbb{C}^3$ and some constant $\alpha > 0$ it can be shown that there exists a unique solution $E, H \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3)$ of (2.1)–(2.3) [14].

From (2.1)–(2.3), it is easy to show [10] that the scattered electric field $E^s(x) = E^s(x; d, p)$ has the asymptotic behavior

$$E^s(x; d, p) = \frac{e^{ik|x|}}{|x|} \left\{ E_\infty(\hat{x}; d, p) + O\left(\frac{1}{|x|}\right) \right\} \tag{2.5}$$

as $|x| \to \infty$ where $\hat{x} = \frac{x}{|x|}$ and E_∞ is the *electric far field pattern* of the scattered wave. If we define

$$L_t^2(\mathbb{S}^2) := \{g: \mathbb{S}^2 \to \mathbb{C}^3 : g \in L^2(\mathbb{S}^2), g \cdot \nu = 0\},$$

where $\mathbb{S}^2$ is the unit sphere with unit outward normal ν, the *electric far field operator* $F_e: L_t^2(\mathbb{S}^2) \to L_t^2(\mathbb{S}^2)$ is given by

$$(F_e g)(\hat{x}) := \int_{\mathbb{S}^2} E_\infty(\hat{x}; d, g(d))\, ds(d). \tag{2.6}$$

It can easily be seen that F_e is compact [10].

Of central importance to the inverse scattering problem is the characterization of the null space of the electric far field operator. To this end, we define an *electromagnetic Herglotz pair* (E, H) to be a solution of Maxwell's equations

$$\begin{aligned} \operatorname{curl} E - ikH &= 0, \\ \operatorname{curl} H + ikE &= 0, \end{aligned} \tag{2.7}$$

of the form

$$\begin{aligned} E(x) &:= \int_{\mathbb{S}^2} E^i(x; d, g(d))\, ds(d), \\ H(x) &:= \int_{\mathbb{S}^2} H^i(x; d, g(d))\, ds(d), \end{aligned} \tag{2.8}$$

with kernel $g \in L_t^2(\mathbb{S}^2)$. The proof of the following theorem can be found in [4].

Theorem 2.1. *The electric far field operator $F_e: L_t^2(\mathbb{S}^2) \to L_t^2(\mathbb{S}^2)$ corresponding to the scattering problem* (2.1)–(2.3) *is injective with dense range if and only if there does not exist a nontrivial solution to the* transmission eigenvalue problem

$$\begin{aligned} &\left.\begin{aligned} \operatorname{curl}\operatorname{curl} E - k^2 N(x) E &= 0 \\ \operatorname{curl}\operatorname{curl} E_0 - k^2 E_0 &= 0 \end{aligned}\right\} \quad \text{in } D \\ &\left.\begin{aligned} \nu \times E &= \nu \times E_0 \\ \nu \times \operatorname{curl} E &= \nu \times \operatorname{curl} E_0 \end{aligned}\right\} \quad \text{on } \partial D \end{aligned} \tag{2.9}$$

where ν is the outward unit normal to ∂D and $E_0 := E_g$, $H_0 := H_g$ are an electromagnetic Herglotz pair with kernel ikg.

Values of k for which there exist nontrivial solutions to (2.9) are called *transmission eigenvalues*. Transmission eigenvalues play an important role in the theory of inverse scattering. In particular, as we shall see, these eigenvalues can be determined from the far field data and give qualitative information on the anisotropic index of refraction. As noted in the Introduction, this is of particular importance in the inverse scattering problem for anisotropic media since the anisotropic material parameters are not uniquely determined from the far field data. The mathematical theory of transmission eigenvalues is based on the following two fundamental results due to Cakoni, Gintides, and Haddar [6] (see also [4]), where for real $N(x)$ we define

$$n_* := \inf_{x\in D} \inf_{\|\xi\|=1} \overline{\xi} \cdot N(x)\xi, \quad n^* := \sup_{x\in D} \sup_{\|\xi\|=1} \overline{\xi} \cdot N(x)\xi.$$

Theorem 2.2. *Assume that for every $\xi \in \mathbb{C}^3$, $|\xi| = 1$, and some constants $\alpha > 0, \beta > 0$ one of the following inequalities is valid:*
1) $1 + \alpha \le n_* \le \overline{\xi} \cdot N(x)\xi \le n^* < \infty, x \in D;$
2) $0 < n_* \le \overline{\xi} \cdot N(x)\xi \le n^* \le 1 - \beta, x \in D.$

Then there exists an infinite countable set of positive transmission eigenvalues corresponding to (2.9) *with* $+\infty$ *as the only accumulation point.*

Note that, in contrast to scattering resonances, the above theorem says that for real $N(x)$ there exist positive transmission eigenvalues and, as we shall see in the next section, these can be determined from measured far field data and thus can be used as target signatures. It can be shown (cf. Theorem 8.12 of [10]) that if $N(x)$ is not real-valued then positive transmission eigenvalues do not exist.

Theorem 2.3. *Let $k_{1,D,N(x)}$ be the first positive transmission eigenvalue for* (2.9) *and let α and β be positive constants. Denote by k_{1,D,n_*} and k_{1,D,n^*} the first positive transmission eigenvalue of* (2.9) *for $N = n_* I$ and $N = n^* I$, respectively, and let $\|\cdot\|_2$ denote the Euclidean operator norm.*
1) *If* $\|N(x)\|_2 \ge \alpha > 1$, *then* $0 < k_{1,D,n^*} \le k_{1,D,N(x)} \le k_{1,D,n_*}$.
2) *If* $0 < \|N(x)\|_2 \le 1 - \beta$, *then* $0 < k_{1,D,n_*} \le k_{1,D,N(x)} \le k_{1,D,n^*}$.

Assuming that $k_{1,D,N(x)}$ can be computed from the far field measurements, Theorem 2.3 provides an approach to obtaining qualitative information on $N(x)$ by computing a constant n such that $k_{1,D,N(x)}$ is the first positive transmission eigenvalue corresponding to (2.9) with $N := nI$ for this n. The above theorem then implies that $n_* \le n \le n^*$. Since $N(x)$ is positive definite, $n_* = \lambda_1$ and $n^* = \lambda_3$ where λ_1 is the smallest and λ_3 is the largest eigenvalue of $N(x)$.

As an example, consider an orthotropic medium that is translation invarient in the x_3 direction where $x = (x_1, x_2, x_3)^T$ [5] with

$$N = \begin{pmatrix} n_{1,1} & n_{1,2} & 0 \\ n_{2,1} & n_{2,2} & 0 \\ 0 & 0 & n_{3,3} \end{pmatrix}.$$

It is then possible to reduce the problem to two dimensions. In this example $D = (0,1)\times(0,1)$ and the 2×2 block of the matrix N is given by

$$\begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix} = \begin{pmatrix} 1/6 & 0 \\ 0 & 1/8 \end{pmatrix}.$$

Then $\lambda_1 = 0.125$, $\lambda_2 = 0.166$, and the computed $n = 0.135$. For details, see [5].

3 Eigenvalues of the far field operator

We shall now show that the electric (or magnetic) far field operators possess discrete eigenvalues which can then be approximated directly using scattering data. To this end, we need the following theorem from [9] (the result in [9] assumed that $N(x)$ was a scalar but the same proof is valid for $N(x)$ a symmetric matrix satisfying the assumption (2.4)).

Theorem 3.1. *Let E_g^i, H_g^i, and E_h^i, H_h^i be electromagnetic Herglotz pairs with kernels $g, h \in L_t^2(\mathbb{S}^2)$, respectively, and let E_g and E_h be the solutions of* (2.1)–(2.3) *with E^i, H^i replaced by E_g^i, H_g^i and E_h^i, H_h^i, respectively. Then*

$$k \iint_D \operatorname{Im} N(x) E_g \cdot \overline{E_h}\, dx = -2\pi(F_e g, h) - 2\pi(g, F_e h) - (F_e g, F_e h) \tag{3.1}$$

where $(\cdot,\cdot)$ denotes the inner product on $L_t^2(\mathbb{S}^2)$.

If $\operatorname{Im} N(x) = 0$, then it is an easy consequence of this theorem that the compact operator F_e is normal, and hence has an infinite number of eigenvalues [9]. In this case, it can also easily be seen from (3.1) that if $F_e g = \lambda g$ then

$$0 = -2\pi(\lambda g, g) - 2\pi(g, \lambda g) - |\lambda|^2 (g, g)$$

which implies that

$$|\lambda + 2\pi| = 2\pi \tag{3.2}$$

i. e., the eigenvalues of the electric far field operator all lie on the circle (3.2). A similar calculation can be done if, instead of using the electric far field operator, we use the *magnetic far field operator*, i. e., if

$$H^s(x; d, p) = \frac{e^{ik|x|}}{|x|}\left\{H_\infty(\hat{x}; d, p) + O\left(\frac{1}{|x|}\right)\right\} \tag{3.3}$$

and the magnetic far field operator $F_m\colon L_t^2(\mathbb{S}^2) \to L_t^2(\mathbb{S}^2)$ is defined by

$$(F_m g)(\hat{x}) := \int_{\mathbb{S}^2} H_\infty(\hat{x}; d, g(d))\, ds(d). \tag{3.4}$$

It is again easily seen that F_m is compact. In a similar manner to the electric far field operator, it can be shown that F_m is injective with dense range provided k is not an eigenvalue of the interior transmission problem

$$\begin{aligned} \operatorname{curl}(N(x)^{-1}\operatorname{curl} H) - k^2 H = 0 \\ \operatorname{curl}\operatorname{curl} H_0 - k^2 H_0 = 0 \end{aligned} \quad \text{in } D$$
$$\begin{aligned} \nu \times E = \nu \times E_0 \\ N(x)^{-1}(\nu \times \operatorname{curl} E) = \nu \times \operatorname{curl} E_0 \end{aligned} \quad \text{on } \partial D \tag{3.5}$$

and that if $N(x)$ is real then the compact operator F_m is normal, and hence has an infinite number of eigenvalues. An identity analogous to (3.1) can also be established for the magnetic far field operator F_m [12] and used to show that the eigenvalues of F_m all lie on the circle

$$\left|\lambda - \frac{2\pi i}{k}\right| = \frac{2\pi}{k}. \tag{3.6}$$

If $N(x)$ is not real, then we may still establish the existence of infinitely many eigenvalues of F_e and F_m using Lidski's theorem [10], as we show in the following theorem. We first remark that both F_e and F_m are trace-class operators, as can be seen by considering truncated spherical harmonic expansions of the kernel of each operator.

Theorem 3.2. *If* $\operatorname{Im} N(x)$ *is positive on a nonempty open set in* D*, then* F_e *has infinitely many eigenvalues.*

Proof. Since F_e is a trace-class operator, by Lidski's theorem it remains to show that F_e has a finite-dimensional nullspace and an imaginary part which is nonnegative. Unfortunately, the formula (3.1) does not provide the second requirement, and we instead show it for a slightly modified operator $\tilde{F}_e$. In order to prove the first part, we show that under our assumption on N no real transmission eigenvalues can exist, from which Theorem 2.6 implies that F_e is injective. Indeed, if E, E_0 satisfies the homogeneous interior transmission problem (2.9), then we see from the equation for E_0 in D and the integration by parts formula for the curl operator that

$$\int_{\partial D} [(\operatorname{curl} E_0 \times \overline{E_0}) \cdot \nu - (\operatorname{curl} \overline{E_0} \times E_0) \cdot \nu] ds = 0.$$

Applying the vector identity $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$, the boundary conditions, and the same vector identity again yields

$$\int_{\partial D} [(\operatorname{curl} E \times \overline{E}) \cdot \nu - (\operatorname{curl} \overline{E} \times E) \cdot \nu] ds = 0,$$

and it follows from another application of the integration by parts formula that

$$0 = \iint_D (\operatorname{curl}\operatorname{curl} E \cdot \overline{E} - E \cdot \operatorname{curl}\operatorname{curl} \overline{E}) dx$$

$$= 2ik^2 \iint_D \operatorname{Im} N(x)|E|^2 dx.$$

Thus, we observe that $E = 0$ on the open set $D_0 := \{x \in D : \operatorname{Im} N(x) = 0\}$, and by the unique continuation principle it follows that $E = 0$ in all of D. This result implies that $E_0 = 0$ as well, and we conclude that k is not a transmission eigenvalue.

In order to prove the second part, we rewrite (3.1) in terms of $\tilde{F}_e := -ikF_e$ as

$$ik^2 \iint_D \operatorname{Im} N(x) E_g \cdot \overline{E_h}\, dx = 2\pi(\tilde{F}_e g, h) - 2\pi(g, \tilde{F}_e h) - \frac{i}{k}(\tilde{F}_e g, \tilde{F}_e h), \tag{3.7}$$

from which it follows that for all $g \in L_t^2(\mathbb{S}^2)$ we have

$$\begin{aligned} \operatorname{Im}(\tilde{F}_e g, g) &= \frac{1}{2i}[(\tilde{F}_e g, g) - (g, \tilde{F}_e g)] \\ &= \frac{1}{4\pi i}\left[ik^2 \iint_D \operatorname{Im} N(x)|E_g|^2 dx + \frac{i}{k}\|\tilde{F}_e g\|^2\right] \\ &\geq 0. \end{aligned}$$

Therefore, the assumptions of Lidski's theorem are satisfied for the operator $\tilde{F}_e := -ikF_e$, and we conclude that $\tilde{F}_e$, and hence F_e has infinitely many eigenvalues. □

A similar computation establishes the result for the magnetic far field operator F_m. Note that the definition of the electric and magnetic far field operators in [12] differ by a factor of 4π from the ones that we are using.

4 Measurement of transmission eigenvalues

We will now consider the problem of determining transmission eigenvalues from the measured scattering data. In particular, we will assume that the index of refraction is real-valued and make use of Theorems 2.1 and 2.2. In particular, we present two methods for determining transmission eigenvalues from the measured scattering data. We first note that the transmission eigenvalue problems (2.9) and (3.5) are seen to be equivalent by a simple change of dependent variables, and hence have the same eigenvalues. Hence there is no ambiguity in simply referring to the eigenvalues of (2.9) and (3.5) as transmission eigenvalues. We will restrict our attention to considering $H_\infty(\hat{x}; d, p)$. We always assume that $\operatorname{Im} N = 0$ and that D is known (D can be determined by using the linear sampling method; cf. [4]).

We first show how transmission eigenvalues can be determined from the magnetic far field operator F_m.

Definition 4.1. If the solution E_0 of (2.9) is the electric field of an electromagnetic Herglotz pair then we call the transmission eigenvalue k a *nonscattering wave number.*

It is clear that the concept of nonscattering wave numbers is far more restrictive than the concept of transmission eigenvalues. Indeed, the only case known to date when a transmission eigenvalue is a nonscattering wave number is the case when D is a ball and $N(x) = n(|x|)I$. We define

$$H_{e,\infty}(\hat{x}; z, p) := \frac{ik}{4\pi}(\hat{x} \times p)e^{-ik\hat{x}\cdot z} \tag{4.1}$$

where $z \in \mathbb{R}^3$ and note that the right-hand side of (4.1) is the far field pattern of the magnetic field of an electric dipole. We now let $g_z^\alpha \in L_t^2(\mathbb{S}^2)$ be the Tikhonov regularized solution of the magnetic far field equation

$$(F_m g)(\hat{x}) = H_{e,\infty}(\hat{x}; z, p) \tag{4.2}$$

i. e., g_z^α is the solution to

$$(\alpha I + F_m^* F_m)g_z^\alpha = F_m^* H_{e,\infty}. \tag{4.3}$$

We then have the following result (cf. Theorem 4.44 of [2] for the scalar case; the proof in the vector case proceeds in the same manner).

Theorem 4.2. *Assume that D is simply connected and that $N(x)$ satisfies one of the two conditions stated in Theorem 2.2. Assume further that k is not a nonscattering wave number and let Hg_z^α denote the magnetic field of the electromagnetic field defined by (2.8). Then for any ball $B \subset D$, $\|Hg_z^\alpha\|_{L^2(D)}$ is bounded as $\alpha \to 0$ for almost every $z \in B$ if and only if k is not a transmission eigenvalue.*

In particular, if one plots k versus $\|g_z^\alpha\|_{L^2(\mathbb{S}^2)}$ for several choices of points z, then the location of transmission eigenvalues will appear as sharp peaks in the graph (for the scalar case, see Figure 4.2 of [2]).

We now turn our attention to a second method for determining transmission eigenvalues from the magnetic far field operator F_m which is based on the behavior of the phase of the eigenvalues of the compact normal operator F_m. To this end, we recall that if $k > 0$ is not a transmission eigenvalue then $F_m = F_{m,k}$ is injective where we have explicitly noted the dependence of F_m on k. Hence if $k > 0$ is not a transmission eigenvalue, we have the existence of a complete orthonormal basis $(g_j(k))_{j=1}^\infty$ of $L^2(\mathbb{S}^2)$ such that

$$F_{m,k} g_j(k) = \lambda_j(k) g_j(k) \tag{4.4}$$

where $\lambda_j(k) \neq 0$ forms a sequence of complex numbers that goes to zero as $j \to \infty$. Define

$$\hat{\lambda}_j(k) := \frac{\lambda_j(k)}{|\lambda_j(k)|}. \tag{4.5}$$

We then have the following theorem due to Lechleiter and Rennoch [13].

Theorem 4.3. *Assume that Condition 1 (resp., Condition 2) of Theorem 2.2 is valid. Let $k_0 > 0$ and let (k_ℓ) be a sequence of positive numbers converging to k_0 as $\ell \to \infty$. Assume there exists a sequence $(\hat{\lambda}_\ell) = \hat{\lambda}_{j_\ell}(k_\ell)$ for some index j_ℓ such that $\hat{\lambda}_\ell \to -1$ (resp., $\hat{\lambda}_\ell \to +1$) as $\ell \to \infty$. Then k_0 is a transmission eigenvalue.*

Note that since $F_{m,k}$ is compact and all the eigenvalues lie on the circle (3.6), the only possible accumulation points of the sequence $\hat{\lambda}_\ell$ are −1 and +1.

The criterion of Theorem 4.3 can be used as an indicator of transmission eigenvalues. However, the hard part is to prove that it occurs for every transmission eigenvalue. We refer the reader to [13] for a further discussion on this issue.

5 Stekloff eigenvalues

So far we have seen two families of eigenvalues that can be determined from scattering data:

Eigenvalues of the electric far field operator: These can be computed directly from the far field pattern using single frequency data. However, it is not easy to determine how changes in the material properties of the object (i. e., $N(x)$) perturb the eigenvalues.

Transmission eigenvalues: These have a direct relation to $N(x)$ as shown in Theorem 2.3. However, they have to be computed using multi-frequency data and can only be determined for dielectric scatterers.

We shall now introduce a family of eigenvalues from [7] that can be computed from the far field pattern at a single frequency, and for which a simple perturbation theory is known. This is achieved by constructing a modified far field operator using an auxiliary problem which includes an appropriate eigenparameter.

To define this problem choose a domain B such that either (1) $B = D$ or (2) B is a ball containing $\overline{D}$ in its interior. We also need an operator $S : L^2_t(\partial B) \to L^2_t(\partial B)$ such that S is self-adjoint, bounded, and

$$\langle Su, u\rangle \geq 0 \quad \text{for all } u \in L^2_t(\partial B),$$

where $\langle \cdot, \cdot \rangle$ is the L^2 inner product on ∂B. Next we define, for any sufficiently smooth vector field w, the tangential component of w on ∂B by

$$w_T = (\nu \times w) \times \nu \quad \text{on } \partial B.$$

Finally, we need to choose an impedance parameter $\lambda \in \mathbb{R}$ with $\lambda > 0$ (note that a standard impedance parameter would typically be complex). Now we can define the solution E_S of the following generalized impedance problem:

$$\operatorname{curl}\operatorname{curl} E_S - k^2 E_S = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B}, \tag{5.1}$$

$$\nu \times \operatorname{curl} E_S = \lambda S E_{S,T} \quad \text{on } \partial B, \tag{5.2}$$

$$E^i + E_S^s = E_S \quad \text{in } \mathbb{R}^3 \setminus B, \tag{5.3}$$

$$\lim_{r\to\infty} (\operatorname{curl} E_S^s \times x - ikrE_S^s) = 0. \tag{5.4}$$

This scattering problem has a unique solution for any $k > 0$ as shown in [7] (for any λ, any solution is always unique).

Then, as usual for a scattering problem, the scattered field E_S^s has the asymptotic expansion

$$E_S^s(x) = \frac{\exp(ikr)}{r} E_{S,\infty}(\hat{x}, d; p) + O\left(\frac{1}{r^2}\right) \quad \text{as } r \to \infty,$$

and we can then define the impedance far field operator by

$$(F_S g)(\hat{x}) = \int_{\mathbb{S}^2} E_{S,\infty}(\hat{x}; d, g(d))\, ds(d).$$

The modified far field operator is then defined by

$$F_M = F_m - F_S.$$

We can see a link between the modified far field operator and the interior Stekloff eigenvalue problem as argued in [7]. There it is shown that F_M is injective with dense range provided λ is not a generalized Stekloff eigenvalue of the problem

$$\operatorname{curl}\operatorname{curl} w - k^2 N w = 0 \quad \text{in } B, \tag{5.5}$$

$$\nu \times \operatorname{curl} w - \lambda S w_T = 0 \quad \text{on } \partial B. \tag{5.6}$$

It is then necessary to analyze the existence of generalized Stekloff eigenvalues, and this analysis depends on the choice of S. The most obvious choice corresponding to the standard impedance boundary condition is $S = I$. Unfortunately, direct calculation of the eigenvalues in the case when $N = 1$ and B is a ball shows that there are two families of eigenvalues having different accumulation points (one at infinity and one at zero). Indeed in this case, assuming $N = 1$, if λ is an eigenvalue then so is $-k^2/\lambda$. Thus they cannot be analyzed as the eigenvalues of a compact operator.

Instead, in [7] we make the choice of S as follows. Let $u \in L_t^2(\partial B)$ and define $q \in H^1(\partial B)/\mathbb{R}$ by solving

$$\Delta_{\partial B} q = \operatorname{curl}_{\partial B} u.$$

Note that this assumes that if $B = D$ then ∂D has just one connected component. Then $Su = \vec{\operatorname{curl}}_{\partial B} u$. Here $\Delta_{\partial B}$ is the Laplace–Beltrami operator on ∂B, and $\operatorname{curl}_{\partial B}$ and $\vec{\operatorname{curl}}_{\partial B}$ are the scalar and vector surface curls, respectively.

Using S, we can now write the generalized Stekloff eigenvalue problem as an operator equation. We introduce the operator $T : H(\mathrm{div}^0_{\partial B}, \partial B) \to H(\mathrm{div}^0_{\partial B}, \partial B)$ where

$$H(\mathrm{div}^0_{\partial B}, \partial B) = \{u \in L^2_t(\partial B) \mid \nabla_{\partial B} \cdot u = 0 \text{ on } \partial B\}$$

defined as follows. For given $f \in H(\mathrm{div}^0_{\partial B}, \partial B)$, we define w to be the weak solution of

$$\begin{aligned} \operatorname{curl}\operatorname{curl} w - k^2 \epsilon_r w &= 0 \quad \text{in } B \\ \nu \times \operatorname{curl} w &= -f \quad \text{on } \partial B. \end{aligned}$$

This is a well-posed problem provided k^2 is not an interior Neumann eigenvalue for the curl-curl operator. These eigenvalues form a discrete set and from now on we assume $k^2 > 0$ is not such an eigenvalue. Then

$$Tf = Sw_T \quad \text{on } \partial B.$$

The fact that Sw_T is surface divergence-free can be used to show that Tf is actually in $(H^{1/2}(\partial B))^3$, and hence the operator T is compact. Furthermore, it is self-adjoint, and consequently there exist infinitely many eigenvalues μ with associated eigenfunction $u \neq 0$ for the problem

$$Tu = \mu u.$$

Considering the definition of T, we see that if μ is an eigenvalue for T then $\lambda = -1/\mu$ is a generalized Stekloff eigenvalue. Thus we conclude the following.

Theorem 5.1 (Theorem 3.6 of [7]). *When ϵ_r is real, and k^2 is not an interior Neumann eigenvalue for the curl-curl operator, there exists a countable set of real generalized Stekloff eigenvalues that accumulate at infinity.*

Supposing now that we can measure generalized Stekloff eigenvalues, we can assume that changes in these eigenvalues can give information about changes in $N(x)$ as is the case for the Helmholtz equation [3]. To see this, suppose (w, λ), $w \neq 0$ is a generalized Stekloff eigenpair for permittivity $N(x)$ and that $(w_\delta, \lambda_\delta)$ is the corresponding eigenpair for $N(x) + \delta N(x)$ where $\|\delta N\|_{L^\infty}$ is small. Then assuming that $w \approx w_\delta$ (e. g., when the eigenvalue is simple and the perturbation δN is small), we have, neglecting quadratic terms, that

$$\lambda - \lambda_\delta \approx -k^2 \frac{(\delta N w, w)}{\langle Sw_T, Sw_T \rangle} \tag{5.7}$$

where $\langle \cdot, \cdot \rangle$ is the L^2 inner product on ∂B and $(\cdot, \cdot)$ is the L^2 inner product on B.

The main question now is how to determine generalized Stekloff eigenvalues (or at least a few of them) from far field scattering data. As in the case of transmission eigenvalues, this involves the far field equation, and this time we use the electric far field equation. The outgoing electric field due to a point dipole at position z with polarization q in free space is [10]

$$E_e = \frac{i}{k} \operatorname{curl}_x \operatorname{curl}_x \big(q\Phi(x,z)\big)$$

where curl_x denotes the curl with respect to x and Φ is the fundamental solution of the Helmholtz equation

$$\Phi(x,y) = \frac{\exp(ik|x-y|)}{4\pi|x-y|}, \quad x \neq y.$$

The far field pattern due to the dipole source is then given by

$$E_{e,\infty}(\hat{x},z;q) = \frac{ik}{4\pi}(\hat{x}\times q)\times\hat{x}\exp(-ik\hat{x}\cdot z)$$

where $\hat{x} \in \mathbb{S}^2$ is the observation direction (for comparison, see (3.3) for the definition of the magnetic far field pattern, and (4.1) for the magnetic far field pattern of a dipole source).

For generalized Stekloff eigenvalues, the far field equation corresponding to (4.2) is then to seek $g_{z,q} \in L^2_t(\mathbb{S}^2)$ such that

$$(F_M g_{z,q})(\hat{x}) = E_{e,\infty}(\hat{x},z;q) \quad \text{for all } \hat{x} \in \mathbb{S}^2. \tag{5.8}$$

As in the case of transmission eigenvalues, we actually solve a Tikhonov regularized version of this problem by choosing a regularization parameter $\alpha > 0$ and solving

$$(\alpha I + F_M^* F_M) g_{z,q,\alpha} = F_M^* E_{e,\infty}$$

where F_M^* is the L^2 adjoint of F_M. Note that F_M depends on the Stekloff parameter λ, so $g_{z,q,\alpha}$ is also dependent on λ. As λ varies, we can use $\|g_{z,q,\alpha}\|_{L^2_t(\mathbb{S}^2)}$ as an indicator function for Stekloff eigenvalues. Although we cannot prove that this is an appropriate indicator function, we can prove that there is an approximate solution of (5.8) that does have this property. All numerical tests suggest that the solution of the above regularized problem can indeed serve as an indicator function.

In a similar way to the proof of Theorem 4.2, we can now prove the analogous result for Stekloff eigenvalues. To do this, we need to recall the definition of the electric Herglotz wave function

$$v_g(x) = -ik \int_{\mathbb{S}^2} g(d)\exp(-ikx\cdot d)\,ds_d.$$

Theorem 5.2 (Theorem 4.2 of [7]). *Assume λ is not a Stekloff eigenvalue and k^2 is not an interior Neumann eigenvalue for the curl-curl problem. Let $z \in D$ and q be fixed. Then for every $\epsilon > 0$ there exists a function $g_\epsilon \in L^2_t(\mathbb{S}^2)$ that satisfies*

$$\lim_{\epsilon\to 0} \|F_M g_\epsilon - E_{e,\infty}(\cdot,z;q)\|_{L^2_t(\mathbb{S}^2)} = 0$$

and such that $\|v_{g_\epsilon}\|_{L^2_t(B)}$ is bounded as $\epsilon \to 0$.

Conversely, we can show that if λ is a Stekloff eigenvalue, $\|v_{g_\epsilon}\|_{L^2_t(B)}$ cannot remain bounded as $\epsilon \to 0$ for almost every $z \in D$. These results suggest that a graph of $\|v_{g_\epsilon}\|_{L^2_t(\mathbb{S}^2)}$ against λ will show peaks at the Stekloff eigenvalues (provided we sample several points $z \in D$). In practice, we do not use $\|v_{g_\epsilon}\|_{L^2_t(\mathbb{S}^2)}$ to detect eigenvalues because it is somewhat expensive to compute (we would replace g_ϵ by $g_{z,q,\alpha}$). Instead we use as a surrogate $\|g_{z,q,\alpha}\|_{L^2_t(\mathbb{S}^2)}$.

6 Numerical examples

Numerous examples of the computation of transmission eigenvalues exist in the literature (cf. [4]) and so we will not present more here. Instead we will focus on the two sets of eigenvalues discussed in this paper that can be computed at a single frequency: (1) eigenvalues of the electric far field operator and (2) generalized Stekloff eigenvalues.

Our numerical examples are all computed using synthetic far field data. This data is computed using the Netgen [15] finite element library using second-order edge elements and a fifth-order approximation to curved surfaces. We use a spherical perfectly matched layer, at a distance of half a wavelength from the circumscribing sphere for B, of thickness one quarter of a wavelength. The PML parameter is chosen to give approximately 0.6 % relative error in the computed far field pattern for scattering by a penetrable sphere of unit radius (measured in the L^2 norm). In all of the calculations, the wave number is chosen to be $k = 1$ so the wavelength in free space is 2π.

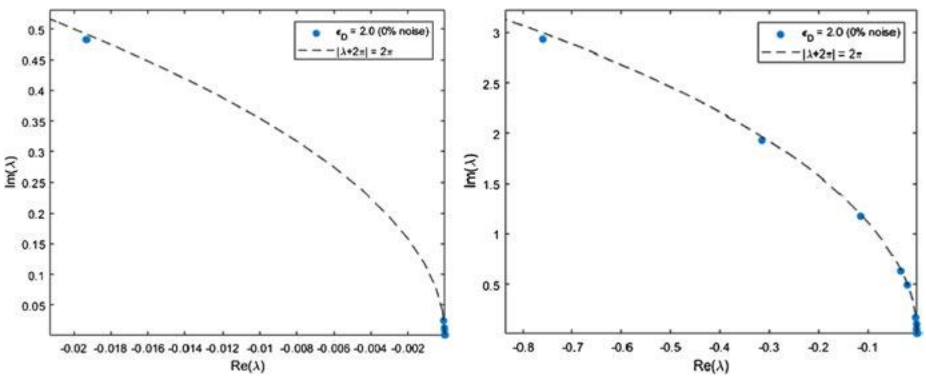
The far field pattern F_S of the generalized Stekloff scattering problem needed for the solution of (5.8) is computed by the same code with the addition of the calculation of an approximation to the operator S computed using third-order finite elements in $H^1(\partial B)$. Generalized Stekloff eigenvalues for arbitrary structures are computed using the same finite elements but now on a bounded domain as described in [7].

The far field operators are discretized by quadrature on the unit sphere. We use a finite element grid on the unit sphere having 99 nodes (made by Netgen) and use vertex based quadrature on each element to calculate the weights for each vertex value of the far field pattern.

Two domains are considered for the scatterer. The first is the unit cube, and the second is the (hockey) puck which is a circular cylinder of radius 3/2 and unit height centered at the origin. The latter scatterer has been suggested as a good experimental model, being dielectric and which can easily be damaged by drilling out portions. Experimental results are not considered here.

6.1 Eigenvalues of the far field operator

In this section, we investigate the use of eigenvalues of the electric far field operator as a target signature. Due to the ease of computing such eigenvalues, they seem to be a natural choice for this purpose, but a significant drawback is the lack of theory concerning their response to changes in the material parameters of an inhomogeneous medium. Thus, our study is confined to a collection of numerical examples, and to facilitate a direct comparison we perform the same numerical experiments as we will for Stekloff eigenvalues. In order to compute the eigenvalues of the electric far field operator F_e, we first discretize the operator using quadrature to obtain a matrix A. When we investigate the effect of noisy data, we obtain a noisy far field matrix A^{ε} by multiplying each component of the far field data by $1+\varepsilon\frac{\zeta+i\mu}{\sqrt{2}}$, where $\varepsilon > 0$ is a fixed parameter and ζ, μ are both uniformly distributed random numbers in $[-1, 1]$ computed using the `rand` command in `MATLAB`. The eigenvalues of A^{ε} are then computed using the `eig` command in `MATLAB`. In Figure 5.1, we see that the eigenvalues of the far field operator for both the unit cube and the puck lie on the circle $|\lambda + 2\pi| = 2\pi$ as implied by Theorem 3.1.

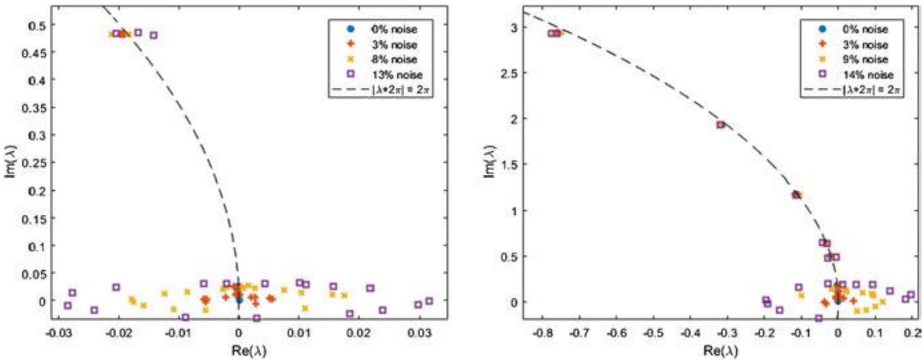


(a) unit cube (b) puck

Figure 5.1: The computed eigenvalues of the electric far field operator with $\epsilon_D = 2$ and no noise. The eigenvalues lie on the circle $|\lambda + 2\pi| = 2\pi$ and appear to converge to zero as predicted.

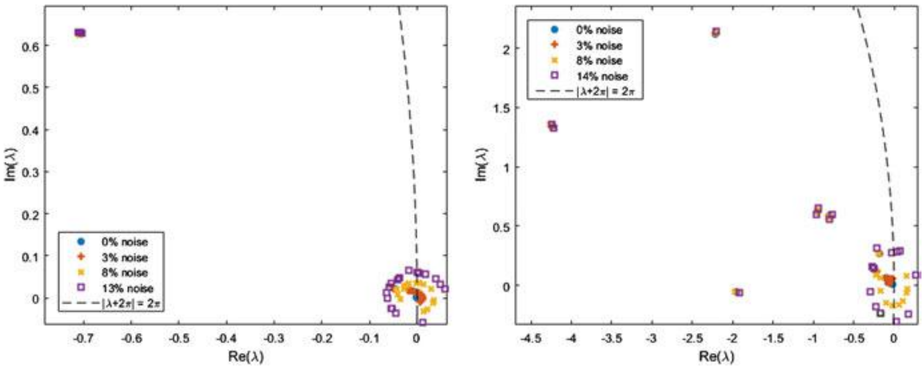
An important property of a target signature is that it is stable in the presence of noise. In Figure 5.2, we plot the eigenvalues of the far field operator for both the unit cube and puck with $\epsilon_D = 2$ for different amounts of noise, and in Figure 5.3 we perform the same test with $\epsilon_D = 2 + 2i$. In the presence of absorption (complex ϵ_D), the eigenvalues move inside the circle $|\lambda + 2\pi| = 2\pi$.

We remark that although the eigenvalues near the origin are highly sensitive to noise, the eigenvalues with larger magnitude tend to remain localized. This stability



(a) unit cube (b) puck

Figure 5.2: The computed eigenvalues of the electric far field operator with $\epsilon_D = 2$ and various levels of noise. The eigenvalues of larger magnitude remain stable in the presence of noise, whereas those near the origin are highly unstable.

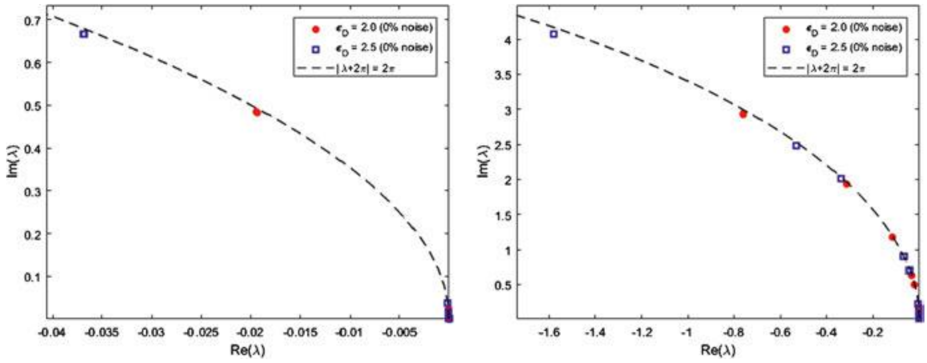


(a) unit cube (b) puck

Figure 5.3: The computed eigenvalues of the electric far field operator with $\epsilon_D = 2 + 2i$ and various levels of noise. The eigenvalues of larger magnitude remain stable in the presence of noise, whereas those near the origin are highly unstable.

is promising, and the distribution of the eigenvalues near the origin may even provide some measure of the noise level.

Of course, our primary point of inquiry is whether the eigenvalues of the far field operator reliably shift due to a change in an inhomogeneous medium. In Figure 5.4, we plot the eigenvalues corresponding to $\epsilon_D = 2$ and $\epsilon_D = 2.5$ for both the unit cube and puck. We remark that the eigenvalues with larger magnitude exhibit a noticeable shift due to this change, which are precisely the eigenvalues that remained stable in the presence of noise in our previous test.



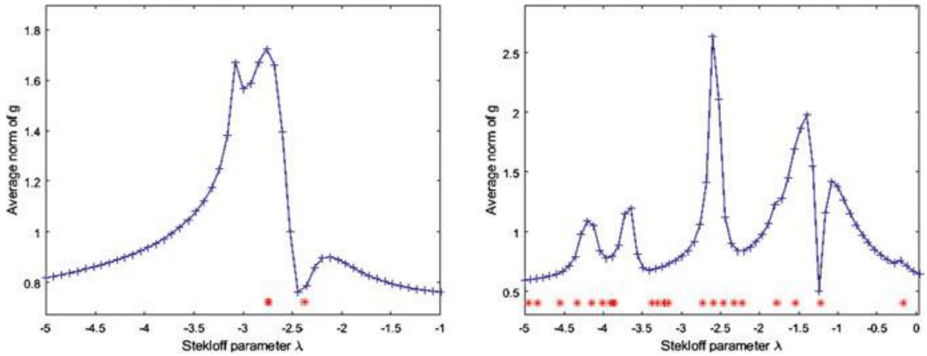
(a) unit cube (b) puck

Figure 5.4: The computed eigenvalues of the electric far field operator with $\epsilon_D = 2$ and $\epsilon_D = 2.5$, where no noise has been added. The eigenvalues shift due to the overall change in ϵ_D, and a greater shift is exhibited by eigenvalues of larger magnitude.

6.2 Stekloff eigenvalues

We now perform numerical tests for generalized Stekloff eigenvalues. In order to compute an approximate solution to the electric far field equation (5.8), we use the same matrix A described for the computation of eigenvalues of the electric far field operator, and we add noise in the same manner. We first comment on the choice of the domain B for both the unit cube and puck. The only requirement is that each scatterer is contained in B, but a natural choice is to choose B to be a ball centered at the origin. We remark that when we solve the far field equation for each sampled value of λ, we do so for 10 randomly chosen z in a ball (of radius 1/4 for the cube and 1/3 for the puck) contained inside D and average the norms of the solutions to serve as our indicator function. In Figures 5.5 and 5.6, we plot the average norm of g, the solution obtained from applying Tikhonov regularization to (5.8), against the Stekloff parameter λ for the cases in which $B = D$ and B is a ball, respectively. We see that the peaks in the plot approximate the first couple of eigenvalues well for both the unit cube and the puck when B is chosen to be a ball, but it is difficult to detect any eigenvalues reliably when $B = D$.

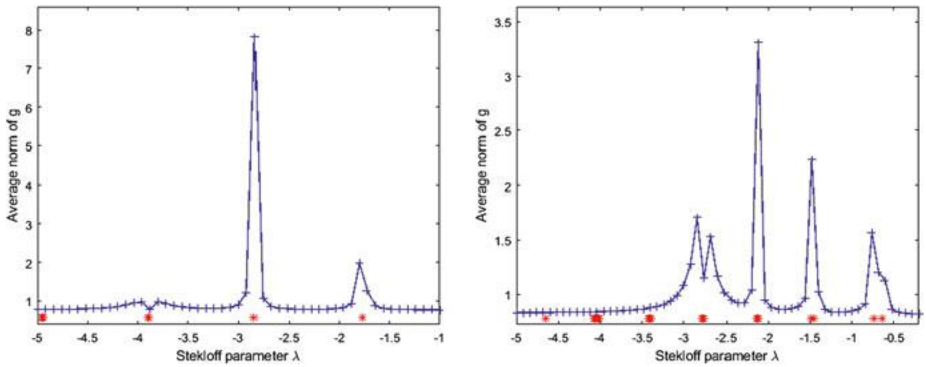
In Figures 5.7 and 5.8, we provide the same plots as in Figures 5.5 and 5.6, respectively, for various levels of noise. For the case $B = D$, the plot for the cube exhibits a peak in the presence of noise which does not coincide with any of the eigenvalues, and a similar peak appears in the plot for the puck near the eigenvalue of smallest magnitude. For the case $B \neq D$, we observe that only a couple of the smallest eigenvalues in magnitude remain detectable in the presence of noise for both the unit cube and the puck, and the noise seems to reduce the prominence of the peaks rather than shift them.



(a) unit cube, $B = D$

(b) puck, $B = D$

Figure 5.5: A plot of the average norm of g against the Stekloff parameter λ with $\epsilon_D = 2.0$ and no noise, where $B = D$. The stars represent the exact eigenvalues computed using finite elements. We observe the difficulty in reliably detecting any eigenvalues.

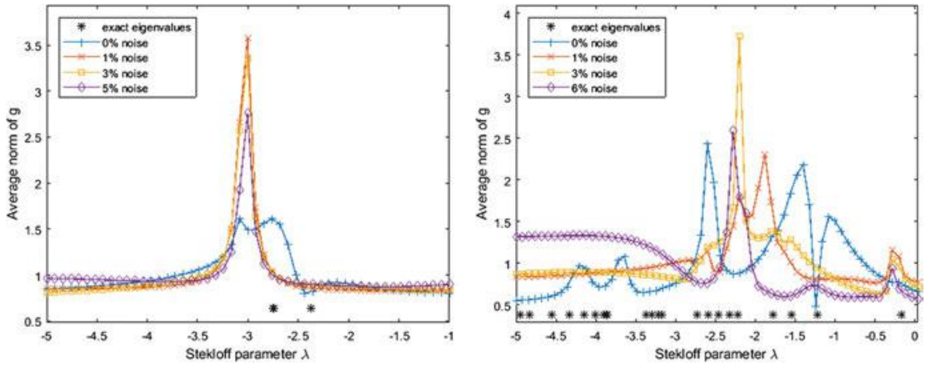


(a) unit cube, B is the unit ball

(b) puck, B is the ball of a radius 1.7

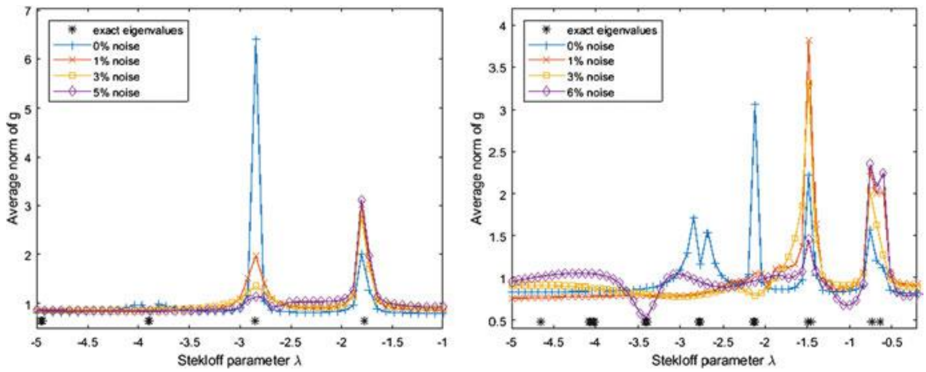
Figure 5.6: A plot of the average norm of g against the Stekloff parameter λ with $\epsilon_D = 2.0$ and no noise, where B is chosen to be a ball centered at the origin. The stars represent the exact eigenvalues computed using finite elements. We observe that the first couple of eigenvalues are detected in each case.

In Figures 5.9 and 5.10, we investigate the shift of generalized Stekloff eigenvalues due to an overall change in ϵ_D from 2 to 2.5. For the case $B = D$, we see that the exact eigenvalues shift and that there is some difference in the plot of the average norm of g, but since these two do not correspond well, it is difficult to make any definite conclusions about their usefulness in detecting changes in ϵ_D. The case $B \neq D$ displays a reduced sensitivity in the eigenvalues, with only the smallest eigenvalues for the puck exhibiting any noticeable shift. However, this choice of B improves the ability to detect eigenvalues, and consequently this shift may be seen in the peaks of the plot of the average norm of g.



(a) unit cube, $B = D$ (b) puck, $B = D$

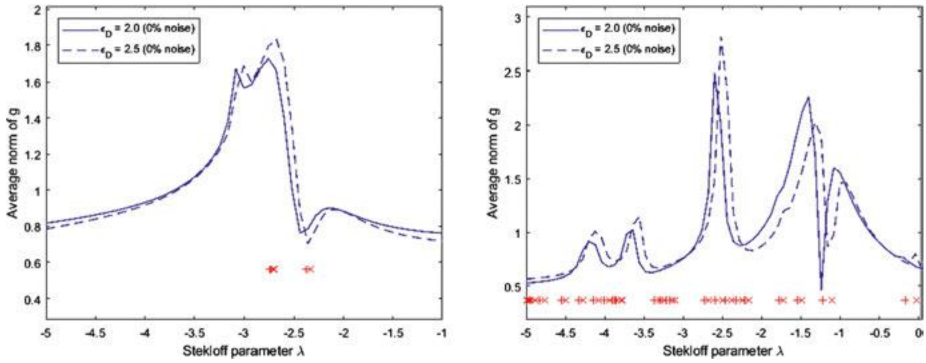
Figure 5.7: A plot of the average norm of g against the Stekloff parameter λ with $\epsilon_D = 2.0$ and $B = D$ for various levels of noise. The stars represent the exact eigenvalues computed using finite elements. Though some prominent peaks appear in the presence of noise for both scatterers, they do not correspond reliably to any of the eigenvalues.



(a) unit cube, B is the unit ball (b) puck, B is the ball of a radius 1.7

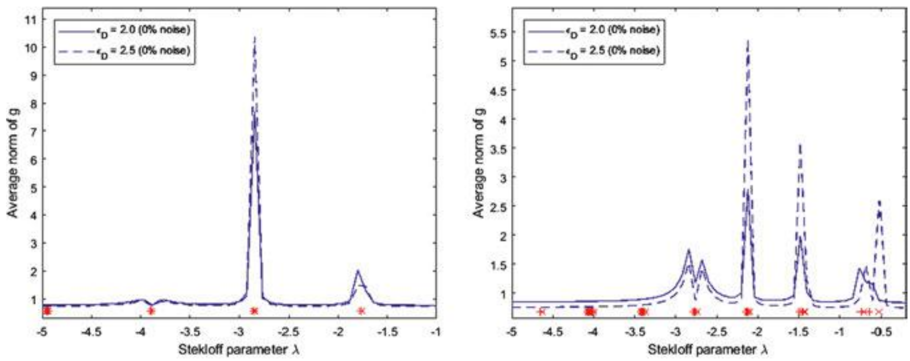
Figure 5.8: A plot of the average norm of g against the Stekloff parameter λ with $\epsilon_D = 2.0$ and $B \neq D$ for various levels of noise. The stars represent the exact eigenvalues computed using finite elements. Only a couple of eigenvalues remain detectable in the presence of noise.

The perturbation estimate (5.7) suggests that the shift of a Stekloff eigenvalue due to a change in ϵ_D is related to the magnitude of a corresponding eigenfunction in a neighborhood of the change, and in Figures 5.11 and 5.12 we plot a cross-section of an eigenfunction corresponding to the cube and puck, respectively. In Figure 5.11b, we see that D is disjoint from the regions in which the eigenfunction w is greatest, which suggests that an overall change in ϵ_D for the unit cube will not result in a large shift in the corresponding eigenvalue, as we observed. In contrast, we see in Figure 5.12b that D intersects with the regions of large magnitude of w and explains the observed shift of the corresponding eigenvalue for the puck in Figure 5.10. Though precise knowledge



(a) unit cube, $B = D$ (b) puck, $B = D$

Figure 5.9: A plot of the average norm of g against the Stekloff parameter λ with $\epsilon_D = 2.0, 2.5$, and no noise. The symbols "+" and "×" represent the exact eigenvalues computed using finite elements for $\epsilon_D = 2.0$ and $\epsilon_D = 2.5$, respectively. The exact eigenvalues clearly shift and there is some difference in the plot of the indicator function due to the overall change in ϵ_D.

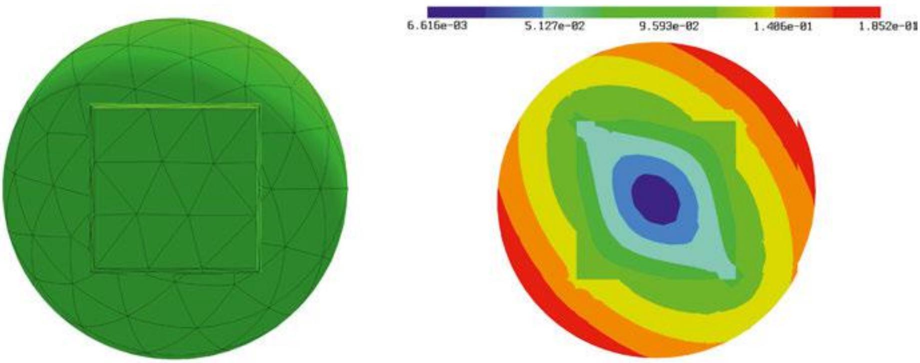


(a) unit cube, B is the unit ball (b) puck, B is the ball of a radius 1.7

Figure 5.10: A plot of the average norm of g against the Stekloff parameter λ with $\epsilon_D = 2.0, 2.5$, and no noise. The symbols "+" and "×" represent the exact eigenvalues computed using finite elements for $\epsilon_D = 2.0$ and $\epsilon_D = 2.5$, respectively. We observe no noticeable shift in the eigenvalues for the unit cube, but we do observe a shift in the smallest eigenvalues for the puck.

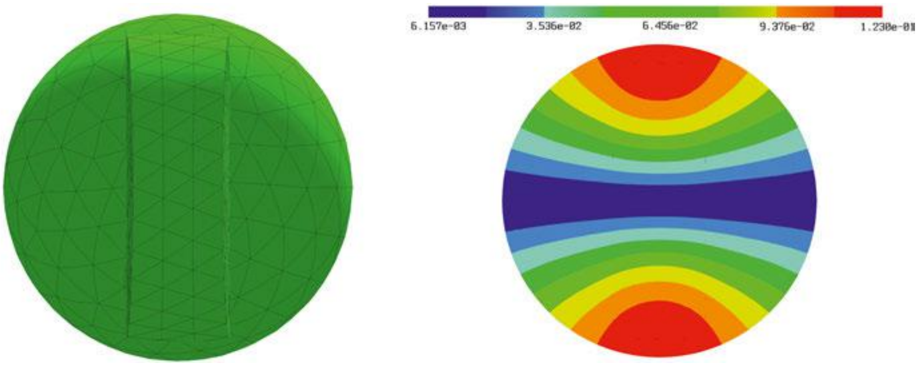
of the geometry and material properties of the scatterer must be known in order to take advantage of this information, this relationship between the eigenfunctions and the material properties may be highly useful in nondestructive testing of materials. In particular, it might allow for the localization of flaws in a material by observing which eigenvalues shift and which do not.

An important advantage of Stekloff eigenvalues over transmission eigenvalues is that Stekloff eigenvalues may in principle be computed for absorbing media, i. e., when ϵ_D has a nonzero imaginary part. Though the present theory does not include a



(a) Cross section of unit cube (b) Cross section of eigenfunction

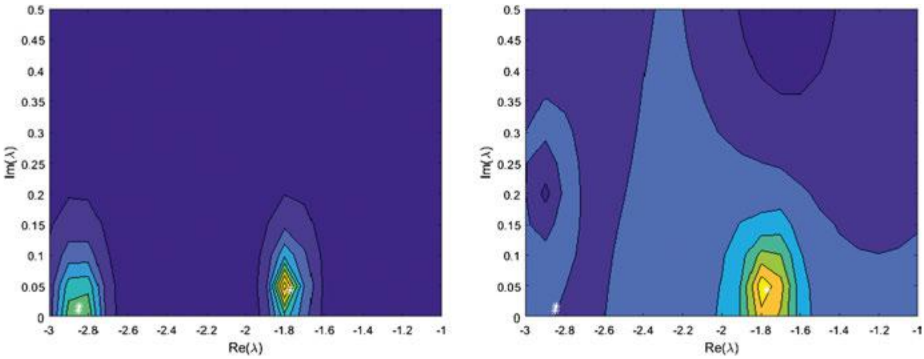
Figure 5.11: A cross-section of the unit cube surrounded by a ball and the corresponding cross-section of an eigenfunction.



(a) Cross section of puck (b) Cross section of eigenfunction

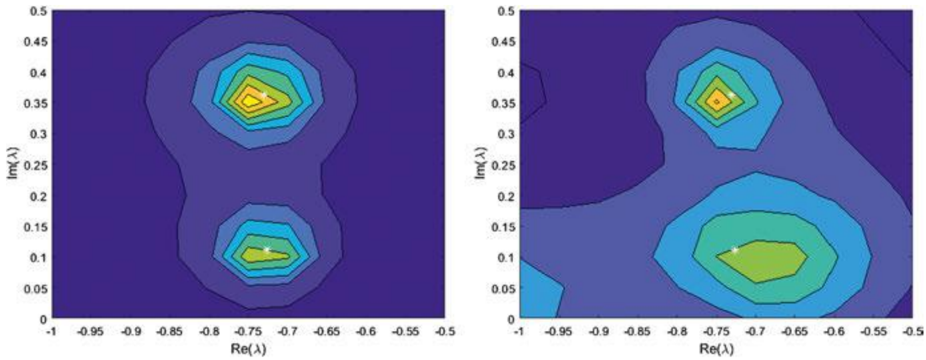
Figure 5.12: A cross-section of the puck surrounded by a ball and the corresponding cross-section of an eigenfunction.

proof of existence of electromagnetic Stekloff eigenvalues in this case, in Figures 5.13 and 5.14 we present an example of their computation for the unit cube and the puck when $\epsilon_D = 2+2i$ and B is chosen to be a ball. In these examples, we have paired the plot for each scatterer with its noisy counterpart in order to obtain a more direct measure of the effect of noise. We observe that all of the eigenvalues in this sampling region are detected when no noise is present, and one remains detectable to a reasonable degree of accuracy in the presence of 7 % noise. It should be noted that the computational expense is greatly increased by the necessity to sample in a region of the complex plane rather than in an interval on the real line. However, as in the previous examples for real ϵ_D, the computation of the modified Stekloff problems may be performed ahead of time for a given region B and applied to any case in which $D \subseteq B$.



(a) unit cube, no noise (b) unit cube, 7% noise

Figure 5.13: A base 10 contour plot of the average norm of g against the Stekloff parameter λ in the complex plane for the unit cube with $\epsilon_D = 2 + 2i$ and two different noise levels. Here, we choose B to be the unit ball. The white stars represent the exact eigenvalues computed using finite elements. We observe that all of the eigenvalues in this region are detected when no noise is present, and one remains detectable with 7% noise.



(a) puck, no noise (b) puck, 7% noise

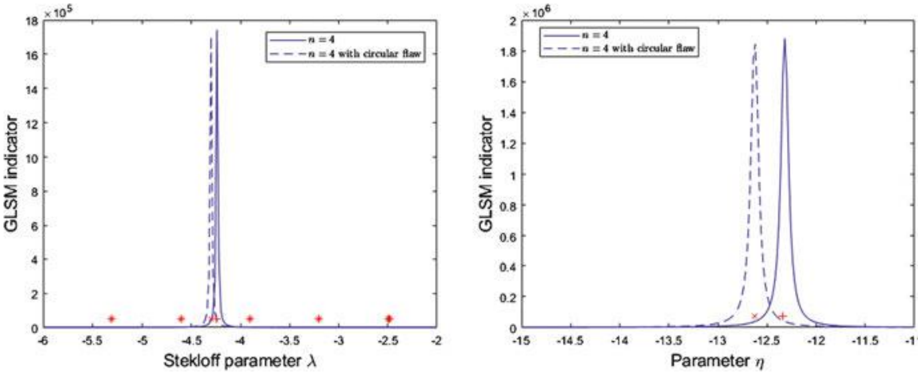
Figure 5.14: A base 10 contour plot of the average norm of g against the Stekloff parameter λ in the complex plane for the puck with $\epsilon_D = 2 + 2i$ and two different noise levels. Here, we choose B to be the unit ball. The white stars represent the exact eigenvalues computed using finite elements. We observe that all of the eigenvalues in this region are detected when no noise is present, and one remains detectable with 7% noise.

7 Conclusion and open problems

The fact that the electric far field data does not uniquely determine the material properties of an anisotropic medium presents many difficulties in the detection of changes in the material properties of a medium, and we have seen that various approaches using the idea of a target signature are available. An important question is which of these target signatures should be chosen for a given application, and unfortunately the answer is not entirely straightforward. Although the theory of transmission eigenvalues is applicable to dielectric media, the use of target signatures for absorbing media at this time rests with either the eigenvalues of the electric far field operator or generalized Stekloff eigenvalues, a pair with complementary strengths and weaknesses. On one hand, we have observed a noticeable shift in the eigenvalues of the electric far field operator due to an overall change in ϵ_D, whereas Stekloff eigenvalues do not appear to shift as reliably. On the other hand, the relationship between Stekloff eigenvalues and the permittivity ϵ_D is apparent in the variational formulation and lends itself to investigation by standard techniques in the theory of partial differential equations, whereas little is known about the eigenvalues of the electric far field operator beyond their distribution in the complex plane. In addition, the use of Stekloff eigenvalues requires some decision-making on the choice of B: choosing $B = D$ often improves sensitivity at the expense of reliable detection of eigenvalues, and choosing $B \neq D$ improves the detection of eigenvalues while reducing their sensitivity to changes in the medium. Thus, any attempt to use these methods would require some experimentation to determine the best choice, and there are multiple trade-offs to consider.

However, the story likely does not end with this rather disappointing observation, as these are not the only target signatures under current study. In particular, there are a number of possible ways in which the electric far field operator can be modified. An example in acoustic scattering modifies the far field operator with that corresponding to scattering by an auxiliary homogeneous medium, and the eigenparameter of interest η is the index of refraction of the auxiliary medium [1, 8]. An important advantage of this method is that the auxiliary scattering problem also depends on an additional parameter γ which may be tuned to improve the sensitivity of the eigenvalues to changes in the material properties, thus overcoming the loss of sensitivity resulting from the choice $B \neq D$.

In Figure 5.15, we show a direct comparison between Stekloff eigenvalues and these so-called modified transmission eigenvalues for acoustic scattering of a L-shaped domain, where we have used the recently developed generalized linear sampling method (cf. [1]) in order to detect the eigenvalues from far field data. This domain has been used for numerical testing of Stekloff eigenvalues and modified transmission eigenvalues previously (cf. [3] and [8], resp.), and we see that the shift in the eigenvalues due to a circular flaw located at $(x_c, y_c) = (0.1, 0.4)$ of radius $r_c = 0.05$ is much more pronounced for modified transmission eigenvalues than Stekloff eigenvalues. It should be noted that for the case of Stekloff eigenvalues there exist peaks in the GLSM indicator corresponding to some of the other exact eigenvalues shown, but the height of these peaks is considerably less than the one visible. We remark that the modified transmission eigenvalues correspond to the choice $\gamma = 0.5$ in [8] and that instead using $\gamma = 2$ produces poor results.



(a) Stekloff eigenvalues (b) Modified transmission eigenvalues

Figure 5.15: A direct comparison of Stekloff eigenvalues and modified transmission eigenvalues (with $\gamma = 0.5$) for acoustic scattering by a L-shaped domain. The shift in the eigenvalues due to a circular flaw located at $(x_c, y_c) = (0.1, 0.4)$ of radius $r_c = 0.05$ is much more pronounced for modified transmission eigenvalues than Stekloff eigenvalues. The red "+" symbol represents the exact eigenvalues for the unflawed domain, and the red "×" symbol represents the exact eigenvalues for the domain with a circular flaw.

This example indicates that, at least for acoustic scattering and with a proper choice of γ, modified transmission eigenvalues provide more information about the material properties of the scatterer than Stekloff eigenvalues. This observation is not too surprising, as can be seen from the fact that for spherically stratified media there exists a single Stekloff eigenvalue corresponding to a spherically symmetric eigenfunction, whereas there exist infinitely many such modified transmission eigenvalues. Extending this approach to Maxwell's equations is the focus of our current research.

Bibliography

[1] L. Audibert, F. Cakoni, and H. Haddar, New sets of eigenvalues in inverse scattering for inhomogeneous media and their determination from scattering data, *Inverse Probl.*, **33**(12) (2017), 125011.

[2] F. Cakoni, D. Colton, and H. Haddar, *Inverse Scattering Theory and Transmission Eigenvalues*, volume 88 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2016.

[3] F. Cakoni, D. Colton, S. Meng, and P. Monk, Stekloff eigenvalues in inverse scattering, *SIAM J. Appl. Math.*, **76**(4) (2016), 1737–1763.

[4] F. Cakoni, D. Colton, and P. Monk, *The Linear Sampling Method in Inverse Electromagnetic Scattering*, volume 80 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.

[5] F. Cakoni, D. Colton, P. Monk, and J. Sun, The inverse electromagnetic scattering problem for anisotropic media, *Inverse Probl.*, **26**(7) (2010), 074004, 14.

[6] F. Cakoni, D. Gintides, and H. Haddar, The existence of an infinite discrete set of transmission eigenvalues, *SIAM J. Math. Anal.*, **42**(1) (2010), 237–255.
[7] J. Camaño, C. Lackner, and P. Monk, Electromagnetic Stekloff eigenvalues in inverse scattering, *SIAM J. Math. Anal.*, **49**(6) (2017), 4376–4401.
[8] S. Cogar, D. Colton, S. Meng, and P. Monk, Modified transmission eigenvalues in inverse scattering theory, *Inverse Probl.*, **33**(12) (2017), 125002.
[9] D. Colton and R. Kress, Eigenvalues of the far field operator and inverse scattering theory, *SIAM J. Math. Anal.*, **26**(3) (1995), 601–615.
[10] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, volume 93 of *Applied Mathematical Sciences*, 3rd edition, Springer, New York, 2013.
[11] F. Gylys-Colwell, An inverse problem for the Helmholtz equation, *Inverse Probl.*, **12**(2) (1996), 139–156.
[12] A. Kirsch and N. Grinberg, *The Factorization Method for Inverse Problems*, volume 36 of *Oxford Lecture Series in Mathematics and its Applications*, Oxford University Press, Oxford, 2008.
[13] A. Lechleiter and M. Rennoch, Inside-outside duality and the determination of electromagnetic interior transmission eigenvalues, *SIAM J. Math. Anal.*, **47**(1) (2015), 684–705.
[14] P. Monk, *Finite Element Methods for Maxwell's Equations*, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2003.
[15] J. Schöberl, Netgen/Ngsolve, https://sourceforge.net/projects/netgen-mesher/, 2016, Version: NGSolve-6.2.1710-102-g4094509.
[16] M. Zworski, Mathematical study of scattering resonances, *Bull. Math. Sci.*, **7**(1) (2017), 1–85.

Martin Costabel and Monique Dauge

6 Maxwell eigenmodes in product domains

Abstract: This paper is devoted to Maxwell modes in three-dimensional bounded electromagnetic cavities that have the form of a product of lower dimensional domains in some systems of coordinates. The boundary conditions are those of the perfectly conducting or perfectly insulating body. The main case of interest is products in Cartesian variables. Cylindrical and spherical variables are also addressed. We exhibit common structures of polarization type for eigenmodes. In the Cartesian case, the cavity eigenvalues can be obtained as sums of Dirichlet or Neumann eigenvalues of positive Laplace operators and the corresponding eigenvectors have a tensor product form. We compare these descriptions with the spherical wave function Ansatz for a ball and show why the cavity eigenvalue of the ball are also Dirichlet or Neumann eigenvalues of some scalar operators. As application of our general formulas, we find explicit eigenpairs in a cuboid, in a circular cylinder, and in a cylinder with a coaxial circular hole. This latter example exhibits interesting "TEM" eigenmodes that have a one-dimensional vibrating string structure, and contribute to the least energy modes if the cylinder is long enough.

Keywords: Electromagnetic cavity, perfectly conducting cavity, Maxwell equations, short-circuit electric or magnetic eigenfunctions, TE or TM polarization, Debye potential

MSC 2010: 78A25, 35Q60, 35J05

1 Introduction

A domain Ω of $\mathbb{R}^n$ is called a product domain if for a choice of Cartesian coordinates $\boldsymbol{x} = (\boldsymbol{y}, \boldsymbol{z})$ in $\mathbb{R}^n$, the domain Ω coincides with the product $\mathcal{Y} \times \mathcal{Z}$ in the sense that

$$\boldsymbol{x} \in \Omega \quad \Longleftrightarrow \quad \boldsymbol{y} \in \mathcal{Y} \quad \text{and} \quad \boldsymbol{z} \in \mathcal{Z}.$$

In the three-dimensional space ($n = 3$), we may assume without restriction that $\mathcal{Y}$ has the dimension 2, and $\mathcal{Z}$, dimension 1, hence is an interval. Such a domain may also be called a cylinder. The main motivation of this work is to exhibit for electromagnetic

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cavity problems in a cylinder with arbitrary cross-section similar properties as those, well known, for acoustic modal problems.

Most of the results we present are not new and have their roots in the pioneering works by Mie (1908) and Debye (1909). Expressions for cavity modes in cylinders can be found in [10] and in balls in [9]. Our aim is to adopt a synthetic presentation that clearly links Laplace or Laplace-like eigenvectors to electromagnetic eigenmodes via TE (transverse electric) and TM (transverse magnetic) vector wave functions: The Laplace eigenvectors appear as Debye potentials. In particular, we carefully address the case when the cross-section ω of Ω contains holes (modelling, for instance, metallic wires) and prove the completeness of a system of TE, TM, and TEM modes. The TEM eigenmodes that enjoy both features of transverse electric and magnetic polarizations, often contribute the lowest frequencies, and this can be precisely quantified. This case was the first motivation for the present investigation.

The knowledge of Maxwell eigenmodes to an applied mathematics audience has some importance. Our results can be used as benchmarks for numerical methods for the computation of cavity modes. Also for transmission problems, our description of the interior eigenmodes may be useful, since there exist standard numerical methods that fail if the frequency coincides with an interior eigenfrequency.

1.1 The case of acoustics: The Dirichlet–Laplacian

The Laplace operator Δ in $\mathbb{R}^n$ is expressed in variables $\boldsymbol{x} = (x_1, \dots, x_n)$ as $\Delta = \sum_{1\le j\le n} \partial^2_{x_j}$ and it is the sum of the two Laplace operators in variables $\boldsymbol{y}$ and $\boldsymbol{z}$

$$\Delta = \Delta_{\boldsymbol{y}} + \Delta_{\boldsymbol{z}} .$$

The Sobolev space $H^1(\Omega)$ on the product domain $\Omega = \mathcal{Y} \times \mathcal{Z}$ can be written as

$$H^1(\Omega) = L^2(\mathcal{Y}, H^1(\mathcal{Z})) \cap H^1(\mathcal{Y}, L^2(\mathcal{Z})).$$

Likewise, the closure $H^1_0(\Omega)$ in $H^1(\Omega)$ of smooth functions with compact support in Ω satisfies

$$H^1_0(\Omega) = L^2(\mathcal{Y}, H^1_0(\mathcal{Z})) \cap H^1_0(\mathcal{Y}, L^2(\mathcal{Z})).$$

As a direct consequence we find, for any bounded product domain Ω, the full spectral description of the Dirichlet–Laplacian.

Theorem 1.1. *Let $(\lambda_j, v_j)_{j\ge1}$ and $(\mu_m, w_m)_{m\ge1}$ be the spectral sequences of $-\Delta_{\boldsymbol{y}}$ on $H^1_0(\mathcal{Y})$ and of $-\Delta_{\boldsymbol{z}}$ on $H^1_0(\mathcal{Z})$, respectively. This means that*

$$\lambda_1 < \lambda_2 \le \cdots$$

is the eigenvalue sequence of $-\Delta_{\boldsymbol{y}}$ *and* $(v_j)_{j\geq 1}$ *is an associated orthonormal basis, and the same for* $-\Delta_{\boldsymbol{z}}$.

Then the set of eigenvalues of $-\Delta$ *on* $H_0^1(\mathcal{Y}\times\mathcal{Z})$ *is*

$$\{\lambda_j + \mu_m,\ j \geq 1,\ m \geq 1\}$$

and the tensor functions $u_{j,m} := v_j \otimes w_m$, *i. e., defined as*

$$u_{j,m}(\boldsymbol{x}) = v_j(\boldsymbol{y})w_m(\boldsymbol{z}),$$

are orthonormal associated eigenvectors, and they form a basis of $H_0^1(\Omega)$.

Of course, a similar result holds for Neumann boundary conditions. This holds also for mixed Dirichlet–Neumann problems of the type

$$(\text{Dirichlet on } \partial\mathcal{Y}\times\mathcal{Z}) \quad \text{and} \quad (\text{Neumann on } \mathcal{Y}\times\partial\mathcal{Z})$$

for which the (λ_j, v_j) are still the Dirichlet eigenpairs on $\mathcal{Y}$, but the (μ_m, w_m) have to be taken as the Neumann eigenpairs on $\mathcal{Z}$. Finally, the Dirichlet and Neumann conditions can be also be swapped between $\mathcal{Y}$ and $\mathcal{Z}$.

1.2 The case of electromagnetism: The Maxwell system

From now on, the space dimension is $n = 3$. Let Ω be a domain in $\mathbb{R}^3$, representing a cavity filled by an homogeneous dielectric medium. We assume that the boundary of Ω represents *perfectly conducting* walls. After normalization, the cavity resonator problem is to find the frequencies $k \in \mathbb{R}$ and the nonzero electromagnetic fields $(\mathbf{E}, \mathbf{H})$ in $L^2(\Omega)^6$ such that

$$\begin{cases} \operatorname{curl}\mathbf{E} - ik\mathbf{H} = 0 & & \text{in } \Omega, \\ \operatorname{curl}\mathbf{H} + ik\mathbf{E} = 0 & & \text{in } \Omega, \\ \operatorname{div}\mathbf{E} = 0 \quad \text{and} & \operatorname{div}\mathbf{H} = 0 & \text{in } \Omega, \\ \mathbf{E}\times\mathbf{n} = 0 \quad \text{and} & \mathbf{H}\cdot\mathbf{n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$. The gauge conditions on the divergence are a consequence of the first two equations if $k \neq 0$. Nevertheless, we look for solutions of (1.1) including $k = 0$. The occurrence of $k = 0$ happens if and only if the domain Ω is topologically nontrivial, i. e., if Ω is not simply connected, or if $\partial\Omega$ is not connected; see Propositions 3.14 and 3.18 in [1].

Definition 1.2. The triples $(k, \mathbf{E}, \mathbf{H})$ solution of (1.1) with $(\mathbf{E}, \mathbf{H}) \neq 0$ are called Maxwell eigenmodes, k is called eigenfrequency, k^2 eigenvalue, and $\mathbf{E}$, $\mathbf{H}$ electric and magnetic eigenvectors.

Let Ω be a bounded product domain in $\mathbb{R}^3$. This means that

$$\Omega = \omega \times I, \quad \omega \subset \mathbb{R}^2, \quad I \text{ interval in } \mathbb{R}. \tag{1.2}$$

We denote correspondingly Cartesian coordinates in Ω by

$$x = (x_1, x_2, x_3) = (x_\perp, x_3), \quad x_\perp \in \omega \quad \text{and} \quad x_3 \in I.$$

We assume that ω is a bounded Lipschitz domain. We note that the boundary of Ω is connected but, if ω is not simply connected, the same holds for Ω.

Notation 1.3.
1. Denote by $\Delta_\perp = \partial_1^2 + \partial_2^2$ the Laplace operator in the variables $x_\perp$.
2. Let $(\lambda_j^{\mathsf{dir}}, v_j^{\mathsf{dir}})_{j\geq 1}$ be the eigenpair sequence of the Dirichlet problem in ω for the operator $-\Delta_\perp$.
3. Let $(\lambda_j^{\mathsf{neu}}, v_j^{\mathsf{neu}})_{j\geq 0}$ be the eigenpair sequence of the Neumann problem in ω for the operator $-\Delta_\perp$, with $\lambda_0^{\mathsf{neu}} = 0$ and $v_0^{\mathsf{neu}} = 1$.
4. Let $(\mu_m^{\mathsf{dir}}, w_m^{\mathsf{dir}})_{m\geq 1}$ be the eigenpair sequence of the Dirichlet problem in I for the operator $-\partial_3^2$.
5. Let $(\mu_m^{\mathsf{neu}}, w_m^{\mathsf{neu}})_{m\geq 0}$ be the eigenpair sequence of the Neumann problem in I for the operator $-\partial_3^2$, with $\mu_0^{\mathsf{neu}} = 0$ and $w_0^{\mathsf{neu}} = 1$.

One of the results of this paper is (see Theorem 3.6) the following.

Theorem 1.4. *Assume that ω is simply connected. Then the Maxwell eigenvalues k^2 span the set*

$$\{\lambda_j^{\mathsf{dir}} + \mu_m^{\mathsf{neu}},\ j \geq 1,\ m \geq 0\} \cup \{\lambda_j^{\mathsf{neu}} + \mu_m^{\mathsf{dir}},\ j \geq 1,\ m \geq 1\}. \tag{1.3}$$

(*including repetition according to multiplicities*).

In the sequel, we describe a corresponding basis of eigenvectors, constructed on the model of vector wave functions, according to the widely used **M** and **N** ansatz (Debye potentials). We include the case when ω is multiply connected: In this case, the relevant parameter is the number D of connected components of $\partial\omega$ and to the set (1.3), we have to add all the μ_m^{dir}, each of them with multiplicity $D - 1$, corresponding to the number of holes contained in ω. The corresponding modes are the TEM modes that have no component in the direction x_3.

This paper is organized as follows. In Section 2, we introduce general principles for the description of the Maxwell cavity modes. In Section 3, we give formulas for the electric eigenmodes $(\kappa^2, \mathbf{E})$ in the case when Ω has the cylindric form $\omega \times I$ with $\omega \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$, separating the modes according to their polarization in TE, TM, and TEM types. In Section 4, we deduce the structure of magnetic cavity modes and synthesize results in Table 6.1. In Section 5, we mention generalizations to special combinations of conducting and insulating boundary conditions.

As an application of our formulas, we consider in Section 6 the case when Ω is a cube (or, more generally, a cuboid), and in Section 7 the case when Ω is axisymmetric: Then Ω is a circular cylinder, or a circular cylinder with a coaxial cylindrical hole. We bring special attention to the latter case. Then the TEM modes appear in the explicit form (7.8).

We address the situation when Ω is a ball of radius R in Section 8. The analysis is in the same spirit and exhibits a close relation with a scalar Laplace-like operator in the "cylinder" $\mathbb{S}^2 \times (0, R)$.

Finally, in Section 9, again for product domains, we investigate the variable coefficient case, namely when ε is varying transversally, i. e., independently of the axial variable x_3. Then the TE and TM structures are no longer a valid Ansatz, in general. In replacement, we obtain wave guide formulations with separation of variables and tensor product form for eigenmodes.

2 Preliminaries

2.1 Electric and magnetic formulations for the Maxwell spectrum

We first recall the definition of the standard functional spaces associated with Maxwell equations on a domain $\Omega \subset \mathbb{R}^3$. The curl in 3D is defined as

$$\operatorname{curl} \mathbf{u} = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix} \quad \text{for } \mathbf{u} = (u_1, u_2, u_3)$$

and $\mathbf{H}(\operatorname{curl}, \Omega)$ is the space of $L^2(\Omega)$ fields with curl in $L^2(\Omega)$, while $\mathbf{H}_0(\operatorname{curl}, \Omega)$ is the subspace of $\mathbf{H}(\operatorname{curl}, \Omega)$ with perfectly conducting electric boundary condition $\mathbf{u} \times \mathbf{n} = 0$.

The divergence in 3D is defined as

$$\operatorname{div} \mathbf{u} = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 \quad \text{for } \mathbf{u} = (u_1, u_2, u_3)$$

and $\mathbf{H}(\operatorname{div}, \Omega)$ is the space of $L^2(\Omega)$ fields with divergence in $L^2(\Omega)$, and $\mathbf{H}_0(\operatorname{div}, \Omega)$ the subspace of $\mathbf{H}(\operatorname{div}, \Omega)$ with perfectly conducting magnetic boundary conditions $\mathbf{u} \cdot \mathbf{n} = 0$.

It is well known that the system of equations (1.1) can be formulated with $\mathbf{E}$ only (electric formulation) or $\mathbf{H}$ only (magnetic formulation). Each time a vector Helmholtz equation is found. Convenient functional spaces for the electric and magnetic variational formulations are

$$\mathbf{X}_\mathsf{N}(\Omega) := \mathbf{H}_0(\operatorname{curl}, \Omega) \cap \mathbf{H}(\operatorname{div}, \Omega) \quad \text{and} \quad \mathbf{X}_\mathsf{T}(\Omega) := \mathbf{H}(\operatorname{curl}, \Omega) \cap \mathbf{H}_0(\operatorname{div}, \Omega) \,.$$

In these spaces, *regularized* formulations make sense. This means that, introducing a parameter

$$s \geq 0$$

we introduce the electric variational formulations:

Find the eigenpairs $(\Lambda = \kappa^2, \mathbf{u})$ *with* $\mathbf{u} \neq 0$ *in* $\mathbf{X}_N(\Omega)$ *such that*

$$\int_\Omega \operatorname{curl}\mathbf{u}\ \operatorname{curl}\mathbf{v} + s\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\,d\mathbf{x} = \Lambda\int_\Omega \mathbf{u}\cdot\mathbf{v}\,d\mathbf{x}, \quad \forall\mathbf{v}\in\mathbf{X}_N(\Omega), \tag{2.1}$$

while magnetic formulations are the following.

Find the eigenpairs $(\Lambda = \kappa^2, \mathbf{u})$ *with* $\mathbf{u} \neq 0$ *in* $\mathbf{X}_T(\Omega)$ *such that*

$$\int_\Omega \operatorname{curl}\mathbf{u}\ \operatorname{curl}\mathbf{v} + s\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\,d\mathbf{x} = \Lambda\int_\Omega \mathbf{u}\cdot\mathbf{v}\,d\mathbf{x}, \quad \forall\mathbf{v}\in\mathbf{X}_T(\Omega). \tag{2.2}$$

Relying on [4, Theorem 1.1], we know that the eigenpairs of (2.1) split in two families:

a) the Maxwell eigenvalues, independent of s, for which the eigenvectors are divergence-free;
b) the gradients of the Dirichlet eigenvectors for $-\Delta$ on Ω, associated with eigenvalues $s\lambda_\Omega^{\mathsf{dir}}$.

Thus the regularization by $s\operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}$ makes the problem elliptic as soon as $s > 0$ and gives a description of the infinite dimensional kernel of the curl curl operator. The gauge conditions $\operatorname{div}\mathbf{E} = 0$ and $\operatorname{div}\mathbf{H} = 0$ in (1.1) ensure that we are always in case a). We can state the following.

Lemma 2.1.

1. *Let* $(k, \mathbf{E}, \mathbf{H})$ *be a Maxwell eigenmode solution of* (1.1). *Set* $\Lambda = k^2$. *Then, if* $\mathbf{E} \neq 0$, *it is solution of* (2.1) *for any* $s \geq 0$, *and if* $\mathbf{H} \neq 0$, *it is solution of* (2.2) *for any* $s \geq 0$.
2. *Let* $s \geq 0$. *If* $\Lambda \neq 0$ *and* $\mathbf{u}$ *is solution of* (2.1) *with* $\operatorname{div}\mathbf{u} = 0$, *then setting* $k = \pm\sqrt{\Lambda}$, $\mathbf{E} = \mathbf{u}$, *and* $\mathbf{H} = \frac{1}{ik}\operatorname{curl}\mathbf{E}$, *we obtain an eigenmode of* (1.1).
3. *Let* $s \geq 0$. *If* $\Lambda \neq 0$ *and* $\mathbf{u}$ *is solution of* (2.2) *with* $\operatorname{div}\mathbf{u} = 0$, *then setting* $k = \pm\sqrt{\Lambda}$, $\mathbf{H} = \mathbf{u}$, *and* $\mathbf{E} = -\frac{1}{ik}\operatorname{curl}\mathbf{H}$, *we obtain an eigenmode of* (1.1).

2.2 Product domain

Let $\Omega \subset \mathbb{R}^3$ be of product form $\omega\times I$, with $\omega \subset \mathbb{R}^2$ and an interval I. We denote Cartesian coordinates and component of vectors as

$$x = (x_1, x_2, x_3) = (x_\perp, x_3) \quad \text{and} \quad \mathbf{u} = (u_1, u_2, u_3) = (\mathbf{u}_\perp, u_3).$$

Likewise, the exterior unit normal $\mathbf{n}$ to $\partial\Omega$ is written $(\mathbf{n}_\perp, n_3)$. The boundary of Ω is

$$\partial\Omega = (\partial\omega \times \overline{I}) \cup (\overline{\omega} \times \partial I).$$

On $\omega \times \partial I$, $\mathbf{n}_\perp = 0$ and $n_3 = \pm 1$. On $\partial\omega \times I$, $\mathbf{n}_\perp$ is the exterior unit normal to $\partial\omega$, $n_3 = 0$, and the tangential component of $\mathbf{u}_\perp$ is $\mathbf{u}_\perp \times \mathbf{n}_\perp = u_1 n_2 - u_2 n_1$. The *electric boundary conditions* $\mathbf{u} \times \mathbf{n} = 0$ *on* $\partial\Omega$ are equivalent to

$$\begin{aligned} &\mathbf{u}_\perp \times \mathbf{n}_\perp = 0 \quad \text{and} \quad u_3 = 0 \quad \text{on} \quad \partial\omega \times I, \\ &\mathbf{u}_\perp = 0 \quad \text{on} \quad \omega \times \partial I. \end{aligned} \tag{2.3}$$

The gradient and the Laplacian in the transverse plane containing ω are denoted by $\nabla_\perp$ and $\Delta_\perp$:

$$\nabla_\perp v = \begin{pmatrix} \partial_1 v \\ \partial_2 v \end{pmatrix} \quad \text{and} \quad \Delta_\perp v = \partial_1^2 v + \partial_2^2 v.$$

The vector and scalar curls in 2D are given by

$$\mathbf{curl}_\perp\, v = \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix} \quad \text{and} \quad curl_\perp\, \mathbf{v} = \partial_1 v_2 - \partial_2 v_1.$$

We have the formula

$$\operatorname{curl} \mathbf{u} = \begin{pmatrix} \mathbf{curl}_\perp\, u_3 \\ curl_\perp\, \mathbf{u}_\perp \end{pmatrix} + \partial_3 \begin{pmatrix} -u_2 \\ u_1 \\ 0 \end{pmatrix}. \tag{2.4}$$

2.3 The M, N ansatz and the TE or TM polarizations

The interior partial differential equation satisfied by eigenpairs is the system:

$$\operatorname{curl}\operatorname{curl} \mathbf{u} = k^2 \mathbf{u} \quad \text{and} \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega. \tag{2.5}$$

There is a well-known ansatz to solve these equations, called vector wave functions **M** and **N**. They depend on the choice of a unit *piloting vector* $\hat{\mathbf{c}}$, and then **M** and **N** are generated by scalar potentials $q = q(x)$ according to

$$\mathbf{M}[q] = \operatorname{curl}(q\,\hat{\mathbf{c}}) \quad \text{and} \quad \mathbf{N}[q] = \operatorname{curl} \mathbf{M}[q] = \operatorname{curl}\operatorname{curl}(q\,\hat{\mathbf{c}}). \tag{2.6}$$

In a slightly modified form where one takes $\hat{\mathbf{c}} = \frac{x}{|x|}$, the ansatz **M** and **N** are the corner stone for the construction of spherical wave functions; cf. Section 8.

For our study, we choose

$$\hat{\mathbf{c}} = \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.7}$$

Direct calculations yield the following.

Lemma 2.2. *Let* $q \in H^1(\Omega)$ *and set* $\mathbf{M}[q] = \operatorname{curl}(q\,\mathbf{e}_3)$. *Then*

$$\mathbf{M}[q] = \begin{pmatrix} \mathbf{curl}_\perp q \\ 0 \end{pmatrix} \quad \text{and} \quad \operatorname{curl} \mathbf{M}[q] = \nabla(\partial_3 q) - \Delta q\,\mathbf{e}_3. \tag{2.8}$$

With $\mathbf{N}[q] = \operatorname{curl} \mathbf{M}[q]$, *we have*

$$\mathbf{N}[q] = \nabla(\partial_3 q) - \Delta q\,\mathbf{e}_3 \quad \text{and} \quad \operatorname{curl} \mathbf{N}[q] = -\mathbf{M}[\Delta q]. \tag{2.9}$$

The form of **M** and curl **N** with their third component zero explains why **M**, when describing an electric field, represents the TE (transverse electric) polarization, and **N**, the TM (transverse magnetic) polarization. For the description of a magnetic field, the converse happens: **M** is TM and **N** is TE.

As a consequence, we find that

$$\begin{aligned} (\operatorname{curl}\operatorname{curl} - k^2)\mathbf{M}[q] &= -\mathbf{M}[\Delta q + k^2 q], \\ (\operatorname{curl}\operatorname{curl} - k^2)\mathbf{N}[q] &= -\mathbf{N}[\Delta q + k^2 q]. \end{aligned} \tag{2.10}$$

Thus, looking for solutions of (2.5) amounts to considering $\mathbf{M}[q]$ and $\mathbf{N}[q]$ with q solution of the Helmholtz equation $\Delta q + \kappa^2 q = 0$.

3 Electric eigenmodes in a product domain

In this section, we look for solutions $(k^2, \mathbf{E})$ of the electric problem (2.1) with the gauge constraint $\operatorname{div} \mathbf{E} = 0$. For this, we use the **M**, **N** ansatz, we find sufficient conditions on the potentials q, construct families of eigenpairs, and prove that this system is complete.

3.1 TE modes

Let $\mathbf{E} = \mathbf{M}[q]$ be a TE mode. By construction, $\operatorname{div} \mathbf{E} = 0$. By (2.10), q has to satisfy

$$\Delta q + \kappa^2 q = 0. \tag{3.1}$$

It remains to verify the electric boundary conditions $\mathbf{E} \times \mathbf{n} = 0$ on $\partial\Omega$. Combining (2.3) and (2.8), we find

$$\begin{aligned} \mathbf{curl}_\perp q \times \mathbf{n}_\perp = 0 \quad &\text{on } \partial\omega \times I, \\ \mathbf{curl}_\perp q = 0 \quad &\text{on } \omega \times \partial I, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \partial_n q = 0 \quad &\text{on } \partial\omega \times I, \\ \nabla_\perp q = 0 \quad &\text{on } \omega \times \partial I. \end{aligned} \tag{3.2}$$

Sufficient conditions for this are Dirichlet conditions on $\omega \times \partial I$ combined with Neumann conditions on $\partial\omega \times I$. This is a tensor product of a Neumann problem on ω and a Dirichlet problem on I. Along the same principle than for pure Dirichlet problem (cf. Theorem 1.1), we find a spectral basis for q in the form

$$q_{jm} = v_j^{\mathsf{neu}} \otimes w_m^{\mathsf{dir}}, \quad k^2 = \lambda_j^{\mathsf{neu}} + \mu_m^{\mathsf{dir}}, \quad j \geq 1,\ m \geq 1. \tag{3.3}$$

Here, $j = 0$ (corresponding to $v_0^{\mathsf{neu}} = 1$) is discarded because functions q independent of $x_\perp$ give $\mathbf{M}[q] = 0$.

Thus we have found the following families of TE modes.

Lemma 3.1. *For all $j \geq 1$, $m \geq 1$, the field $\mathbf{E}_{jm}^{\mathsf{TE}} := \mathbf{M}[v_j^{\mathsf{neu}} \otimes w_m^{\mathsf{dir}}]$, i. e.,*

$$\mathbf{E}_{jm}^{\mathsf{TE}}(x_\perp, x_3) = \begin{pmatrix} \mathbf{curl}_\perp\, v_j^{\mathsf{neu}}(x_\perp) \\ 0 \end{pmatrix} w_m^{\mathsf{dir}}(x_3), \tag{3.4}$$

is a TE mode for problem (2.1) *associated with the eigenvalue $\Lambda_{jm}^{\mathsf{TE}} = \lambda_j^{\mathsf{neu}} + \mu_m^{\mathsf{dir}}$.*

3.2 TM modes

Let $\mathbf{E} = \mathbf{N}[q]$ be a TM mode. Again, $\operatorname{div} \mathbf{E} = 0$, q has to satisfy (3.1), and it remains to verify the electric boundary conditions $\mathbf{E} \times \mathbf{n} = 0$ on $\partial\Omega$: Using (2.8), we find that

$$\mathbf{E}_\perp = \nabla_\perp(\partial_3 q) \quad \text{and} \quad E_3 = -\Delta_\perp q$$

Hence, with (2.3)

$$\begin{aligned} &\nabla_\perp(\partial_3 q) \times \mathbf{n}_\perp = 0 \quad \text{and} \quad \Delta_\perp q = 0 \quad \text{on} \quad \partial\omega \times I, \\ &\nabla_\perp(\partial_3 q) = 0 \quad \text{on} \quad \omega \times \partial I. \end{aligned}$$

We obtain sufficient conditions through the separation of variable ansatz

$$q(x) = v(x_\perp)\, w(x_3)$$

with

$$-\Delta_\perp v = \lambda v \ \text{ in } \omega \quad \text{and} \quad -\partial_3^2 w = \mu w \ \text{ in } I \quad \text{with } \lambda + \mu = k^2 = \Lambda, \tag{3.5}$$

and the boundary conditions become

$$\begin{cases} (\mathbf{n}_\perp \times \nabla_\perp) v(x_\perp)\, \partial_3 w(x_3) = 0 & \forall x_\perp \in \partial\omega,\ \forall x_3 \in I, \\ \Delta_\perp v(x_\perp)\, w(x_3) = 0 & \forall x_\perp \in \partial\omega,\ \forall x_3 \in I, \\ \nabla_\perp v(x_\perp) \partial_3 w(x_3) = 0 & \forall x_\perp \in \omega,\ \forall x_3 \in \partial I, \end{cases}$$

which yields, with $\partial_d\omega$, $d = 1, \ldots, D$, the connected components of $\partial\omega$,

$$\begin{cases} v = \text{const.} \quad \text{on each } \partial_d\omega & \text{or} \quad \partial_3 w \equiv 0 \quad \text{in } I, \\ \Delta_\perp v = 0 \quad \text{on } \partial\omega & \text{or} \quad w \equiv 0 \quad \text{in } I, \\ \nabla_\perp v \equiv 0 \quad \text{in } \omega & \text{or} \quad \partial_3 w = 0 \quad \text{on } \partial I. \end{cases} \tag{3.6}$$

The conditions $\nabla_\perp v \equiv 0$ and $w \equiv 0$ have to be discarded since they imply $\mathbf{E} \equiv 0$. Therefore we should have $\partial_3 w = 0$ on ∂I and $\Delta_\perp v = 0$ on $\partial\omega$. The latter condition implies that $v = 0$ on $\partial\omega$ in the case when $\lambda \neq 0$. When $\lambda = 0$, the condition $v = \text{const.}$ on each $\partial_d\omega$ is sufficient. Thus we have shown that (3.5)–(3.6) can be summarized as follows: Either

$$\begin{cases} -\Delta_\perp v = \lambda v \quad \text{in } \omega \quad \text{and} \quad v = 0 \quad \text{on } \partial\omega \\ -\partial_3^2 w = \mu w \quad \text{in } I \quad \text{and} \quad \partial_3 w = 0 \quad \text{on } \partial I \end{cases} \quad \text{with } \lambda \neq 0,\ \lambda + \mu = \Lambda, \tag{3.7}$$

or

$$\begin{cases} -\Delta_\perp v = 0 \quad \text{in } \omega \quad \text{and} \quad v = \text{const} \quad \text{on each } \partial_d\omega \\ -\partial_3^2 w = \mu w \quad \text{in } I \quad \text{and} \quad \partial_3 w = 0 \quad \text{on } \partial I \end{cases} \quad \text{with } \mu = \Lambda. \tag{3.8}$$

Hence we have found the following two families of TM modes. First, we have the standard one.

Lemma 3.2. *For all $j \geq 1$, $m \geq 0$, the field $\mathbf{E}^{\mathsf{TM}}_{jm} := \mathbf{N}[v_j^{\mathsf{dir}} \otimes w_m^{\mathsf{neu}}]$, i. e.,*

$$\mathbf{E}^{\mathsf{TM}}_{jm}(x_\perp, x_3) = \begin{pmatrix} \nabla_\perp v_j^{\mathsf{dir}}(x_\perp) \\ 0 \end{pmatrix} \partial_3 w_m^{\mathsf{neu}}(x_3) - \begin{pmatrix} 0 \\ \Delta_\perp v_j^{\mathsf{dir}}(x_\perp) \end{pmatrix} w_m^{\mathsf{neu}}(x_3), \tag{3.9}$$

is a TM mode for problem (2.1) *associated with the eigenvalue $\Lambda^{\mathsf{TM}}_{jm} = \lambda_j^{\mathsf{dir}} + \mu_m^{\mathsf{neu}}$.*

The second family appears if ω has a nontrivial topology (i. e., if $D \geq 2$), and shares the features of TE and TM polarization (vanishing third component of the electric and magnetic fields).

Lemma 3.3. *There exist D linearly independent harmonic potentials v_d^{top} that have constant traces on each connected component $\partial_d\omega$ of $\partial\omega$. They can be chosen such that v_D^{top} is constant in ω. If $\partial\omega$ has more than one connected component, then the v_d^{top}, $d = 1, \ldots, D-1$, have linearly independent gradients, and they generate the family of TEM modes defined for all $d = 1, \ldots, D-1$ and $m \geq 1$ as the fields $\mathbf{E}^{\mathsf{TEM}}_{dm} := \mathbf{N}[v_d^{\mathsf{top}} \otimes w_m^{\mathsf{neu}}]$ which can also be written as*

$$\mathbf{E}^{\mathsf{TEM}}_{dm}(x_\perp, x_3) = \begin{pmatrix} \nabla_\perp v_d^{\mathsf{top}}(x_\perp) \\ 0 \end{pmatrix} w_m^{\mathsf{dir}}(x_3), \tag{3.10}$$

and is associated with the eigenvalue $\Lambda^{\mathsf{TEM}}_{dm} = \mu_m^{\mathsf{dir}}$.

Note that to obtain (3.10) we have used that the derivatives $\partial_3 w_k^{\mathsf{neu}}$ for $k \geq 1$ are an eigenvector basis for the Dirichlet problem on the interval I.

Remark 3.4. Let us borrow the following objects from [1]: Let ω° be $\omega \setminus \Sigma$, where $\Sigma = \bigcup_{d=1}^{D-1} \Sigma_d$ is a minimal set of cuts so that ω° is simply connected. Then we can define the space $\Theta(\omega)$ as

$$\Theta(\omega) = \{\varphi \in H^1(\omega^\circ) \mid [\varphi]_{\Sigma_d} = \mathrm{const}(d),\ d = 1, \dots, D-1\}.$$

For $\varphi \in \Theta(\omega)$, its extended $\mathbf{curl}_\perp$ denoted by $\widetilde{\mathbf{curl}}_\perp\, \varphi$ is defined as its $\mathbf{curl}_\perp$ in ω°, considered as an element of $L^2(\omega)$. Then there exist "conjugate" potentials $\tilde{v}_d^{\mathsf{top}} \in \Theta(\omega)$ such that for any $d \leq D-1$, there holds

$$\widetilde{\mathbf{curl}}_\perp\, \tilde{v}_d^{\mathsf{top}} = \nabla_\perp v_d^{\mathsf{top}}. \tag{3.11}$$

Therefore, for all $m \geq 1$, the mode $\mathbf{E}_{dm}^{\mathsf{TEM}}$ is also an extended TE mode. This is why it is called a TEM mode.

3.3 Completeness

The aim of this section is to prove the following.

Lemma 3.5. *Let* $\mathbf{u} \in \mathbf{X}_{\mathsf{N}}(\Omega)$ *such that* $\operatorname{div} \mathbf{u} = 0$. *We assume that for all integers* $j \geq 1$ *and* $d \in [1, D-1]$:

$$\langle \mathbf{u}, \mathbf{E}_{jm}^{\mathsf{TE}} \rangle = 0\ (\forall m \geq 1), \quad \langle \mathbf{u}, \mathbf{E}_{jm}^{\mathsf{TM}} \rangle = 0\ (\forall m \geq 0) \quad \textit{and} \quad \langle \mathbf{u}, \mathbf{E}_{dm}^{\mathsf{TEM}} \rangle = 0\ (\forall m \geq 1).$$

Here, $\langle \cdot, \cdot \rangle$ *is the* L^2 *scalar product on* Ω. *Then* $\mathbf{u} = 0$.

Proof. We first draw consequences from the orthogonality properties against the TM modes: We fix j and m and set $v = v_j^{\mathsf{dir}}$, $w = w_m^{\mathsf{neu}}$ and integrate by parts:

$$\begin{aligned}
0 &= \int_I \int_\omega \mathbf{u}_\perp(x_\perp, x_3)\, \nabla_\perp v(x_\perp) \partial_3 w(x_3) - u_3(x_\perp, x_3)\, \Delta_\perp v(x_\perp) w(x_3)\ \mathrm{d}x_\perp \mathrm{d}x_3 \\
&= \int_I \int_\omega -\operatorname{div}_\perp \mathbf{u}_\perp(x_\perp, x_3)\, v(x_\perp) \partial_3 w(x_3) - u_3(x_\perp, x_3)\, \Delta_\perp v(x_\perp) w(x_3)\ \mathrm{d}x_\perp \mathrm{d}x_3 \\
&= \int_I \int_\omega \partial_3 u_3(x_\perp, x_3)\, v(x_\perp) \partial_3 w(x_3) - u_3(x_\perp, x_3)\, \Delta_\perp v(x_\perp) w(x_3)\ \mathrm{d}x_\perp \mathrm{d}x_3 \\
&= \int_I \int_\omega -u_3(x_\perp, x_3)\, v(x_\perp) \partial_3^2 w(x_3) - u_3(x_\perp, x_3)\, \Delta_\perp v(x_\perp) w(x_3)\ \mathrm{d}x_\perp \mathrm{d}x_3.
\end{aligned}$$

Here, we have used that $\operatorname{div} \mathbf{u} = 0$, replacing $\operatorname{div}_\perp \mathbf{u}_\perp$ by $-\partial_3 u_3$. Coming back to the properties of $v = v_j^{\mathsf{dir}}$ and $w = w_m^{\mathsf{neu}}$, we find for all $j \geq 1$ and $m \geq 0$:

$$\int_I \int_\omega u_3(x_\perp, x_3)\, (\lambda_j^{\mathsf{dir}} + \mu_m^{\mathsf{neu}}) v_j^{\mathsf{dir}}(x_\perp) w_m^{\mathsf{neu}}(x_3)\ \mathrm{d}x_\perp \mathrm{d}x_3 = 0.$$

Since $\lambda_j^{\mathrm{dir}} + \mu_m^{\mathrm{neu}}$ is never 0, we deduce that for all $j \geq 1$ and $m \geq 0$:

$$\int_I \int_\omega u_3(x_\perp, x_3)\, v_j^{\mathrm{dir}}(x_\perp) w_m^{\mathrm{neu}}(x_3)\, \mathrm{d}x_\perp \mathrm{d}x_3 = 0.$$

The set $v_j^{\mathrm{dir}}(x_\perp) w_m^{\mathrm{neu}}(x_3)$ being a complete basis in $L^2(\Omega)$, we deduce that $u_3 = 0$.

Next, we use the orthogonality against the TE modes: for all $j \geq 1$ and $m \geq 1$, there holds:

$$\int_I w_m^{\mathrm{dir}}(x_3) \int_\omega \mathbf{u}_\perp(x_\perp, x_3) \cdot \mathbf{curl}_\perp\, v_j^{\mathrm{neu}}(x_\perp)\, \mathrm{d}x_\perp \mathrm{d}x_3 = 0.$$

Therefore, for all $j \geq 1$:

$$\int_\omega \mathbf{u}_\perp(x_\perp, x_3) \cdot \mathbf{curl}_\perp\, v_j^{\mathrm{neu}}(x_\perp)\, \mathrm{d}x_\perp = 0, \quad \text{for a. e. } x_3 \in I.$$

We deduce that $curl_\perp\, \mathbf{u}_\perp(\cdot, x_3)$ is orthogonal to all v_j^{neu} for $j \geq 1$, which means that $curl_\perp\, \mathbf{u}_\perp(\cdot, x_3)$ is constant with respect to $x_\perp$. There exists a function $z = z(x_3)$ such that

$$(*) \qquad curl_\perp\, \mathbf{u}_\perp(x_\perp, x_3) = z(x_3).$$

Since $\operatorname{div} \mathbf{u} = 0$ and $u_3 = 0$, we have $\operatorname{div}_\perp \mathbf{u}_\perp = 0$, which implies that locally $\mathbf{u}_\perp$ is a $\mathbf{curl}_\perp$ of a scalar potential and that

$$\int_{\partial\omega} \mathbf{u}_\perp \cdot \mathbf{n}_\perp\, \mathrm{d}\sigma = 0.$$

Additionally, the orthogonality relations against the TEM modes yield for all $m \geq 1$ and $d \leq D - 1$:

$$\int_I w_m^{\mathrm{dir}}(x_3) \int_\omega \mathbf{u}_\perp(x_\perp, x_3) \cdot \nabla_\perp v_d^{\mathrm{top}}(x_\perp)\, \mathrm{d}x_\perp \mathrm{d}x_3 = 0.$$

We deduce that

$$\int_\omega \mathbf{u}_\perp(x_\perp, x_3) \cdot \nabla_\perp v_d^{\mathrm{top}}(x_\perp)\, \mathrm{d}x_\perp = 0, \quad \text{for a. e. } x_3 \in I,$$

from which we find that (we recall that $\partial_d\omega$ are the connected components of $\partial\omega$)

$$\int_{\partial_d\omega} \mathbf{u}_\perp \cdot \mathbf{n}_\perp\, \mathrm{d}\sigma = 0, \quad d = 1, \dots, D.$$

These are the flux conditions that provide the existence of a global scalar potential $y \in L^2(I, H^1(\omega))$ such that

$$\mathbf{u}_\perp(x_\perp, x_3) = \mathbf{curl}_\perp\, y(x_\perp, x_3).$$

As $\mathbf{u}_\perp(\cdot, x_3)$ satisfies the tangential boundary condition on $\partial\omega$ for a. e. $x_3 \in I$, then $y(\cdot, x_3)$ satisfies in turn the Neumann boundary condition on $\partial\omega$ for a. e. $x_3 \in I$. With $(*)$, we find

$$-\Delta_\perp y(x_\perp, x_3) = z(x_3).$$

Since y satisfies the homogeneous Neumann condition with respect to $x_\perp$, this implies that $z(x_3) = 0$ for all x_3. Finally we have obtained that $\mathbf{u}_\perp = 0$. □

3.4 Eigenmodes

Summarizing, we have proved the following.

Theorem 3.6. *Let $\Omega = \omega \times I$. The eigenpairs with zero divergence of the electric Maxwell operator* (2.1) *can be organized in the three families:*

(i) $\mathbf{E}^{\mathsf{TE}}_{jm} = \begin{pmatrix} \mathbf{curl}_\perp v_j^{\mathsf{neu}}(x_\perp) \\ 0 \end{pmatrix} w_m^{\mathsf{dir}}(x_3)$ *with* $\Lambda^{\mathsf{TE}}_{jm} = \lambda_j^{\mathsf{neu}} + \mu_m^{\mathsf{dir}}$, $j \geq 1$, $m \geq 1$;

(ii) $\mathbf{E}^{\mathsf{TM}}_{jm} = \begin{pmatrix} \nabla_\perp v_j^{\mathsf{dir}}(x_\perp) \\ 0 \end{pmatrix} \partial_3 w_m^{\mathsf{neu}}(x_3) - \begin{pmatrix} 0 \\ \Delta_\perp v_j^{\mathsf{dir}}(x_\perp) \end{pmatrix} w_m^{\mathsf{neu}}(x_3)$

with $\Lambda^{\mathsf{TM}}_{jm} = \lambda_j^{\mathsf{dir}} + \mu_m^{\mathsf{neu}}$, $j \geq 1$, $m \geq 0$;

(iii) *and, if ω is not simply connected (i. e., $D \geq 2$)*

$\mathbf{E}^{\mathsf{TEM}}_{dm} = \begin{pmatrix} \nabla_\perp v_d^{\mathsf{top}}(x_\perp) \\ 0 \end{pmatrix} w_m^{\mathsf{dir}}(x_3)$ *with* $\Lambda^{\mathsf{TEM}}_{dm} = \mu_m^{\mathsf{dir}}$, $1 \leq d \leq D-1$, $m \geq 1$.

See Notation 1.3, *Lemmas* 3.2 *and* 3.3 *for the notation of the 2D and 1D quantities. All the associated eigenvalues* $\Lambda^{\mathsf{TE}}_{jm}$, $\Lambda^{\mathsf{TM}}_{jm}$ *and* $\Lambda^{\mathsf{TEM}}_{dm}$ *are nonzero.*

4 Magnetic eigenmodes in a product domain

Since the magnetic field $\mathbf{H}$ associated with the electric field $\mathbf{E}$ is given by

$$\mathbf{H} = \frac{1}{ik}\,\operatorname{curl}\mathbf{E}, \quad \text{for } k = \pm\sqrt{\Lambda}$$

for any nonzero eigenvalue Λ, we deduce the following.

Corollary 4.1. *Under the conditions of Theorem* 3.6, *we set $k = \pm\sqrt{\Lambda}$. The associated magnetic fields are given by*

$$\mathbf{H}^{\mathsf{TE}}_{jm} = \frac{1}{ik^{\mathsf{TE}}_{jm}} \left\{ \begin{pmatrix} \nabla_\perp v_j^{\mathsf{neu}}(x_\perp) \\ 0 \end{pmatrix} \partial_3 w_m^{\mathsf{dir}}(x_3) - \begin{pmatrix} 0 \\ \Delta_\perp v_j^{\mathsf{neu}}(x_\perp) \end{pmatrix} w_m^{\mathsf{dir}}(x_3) \right\} \quad j, m \geq 1,$$

$$\mathbf{H}^{\mathsf{TM}}_{jm} = -ik^{\mathsf{TM}}_{jm} \begin{pmatrix} \mathbf{curl}_\perp v_j^{\mathsf{dir}}(x_\perp) \\ 0 \end{pmatrix} w_m^{\mathsf{neu}}(x_3) \quad j \geq 1,\ m \geq 0,$$

$$\mathbf{H}_{dm}^{\mathsf{TEM}} = \frac{i}{k_{dm}^{\mathsf{TEM}}} \begin{pmatrix} \mathbf{curl}_\perp v_d^{\mathsf{top}}(x_\perp) \\ 0 \end{pmatrix} \partial_3 w_m^{\mathsf{dir}}(x_3) \quad 1 \le d \le D-1,\ m \ge 1,$$

and the triples $(k, \mathbf{E}, \mathbf{H})$ *are Maxwell eigenmodes.*

Remark 4.2.

(i) The *electric* fields in the pairs $(\mathbf{E}^{\mathsf{TE}}, \mathbf{H}^{\mathsf{TE}})$ are transverse to the axis x_3, while in the pairs $(\mathbf{E}^{\mathsf{TM}}, \mathbf{H}^{\mathsf{TM}})$ the *magnetic* fields are transverse to the axis x_3, which justifies the labels of the polarizations.

(ii) We notice that for all $m \ge 1$, $\mathbf{H}_{dm}^{\mathsf{TEM}}$ can also be written as

$$\mathbf{H}_{dm}^{\mathsf{TEM}} = i \begin{pmatrix} \mathbf{curl}_\perp v_d^{\mathsf{top}}(x_\perp) \\ 0 \end{pmatrix} w_m^{\mathsf{neu}}(x_3).$$

The expression above also makes sense for $m = 0$. The associated eigenvalue is 0 and the corresponding electric field is 0. These magnetostatic Maxwell eigenmodes $(0, \mathbf{0}, \mathbf{H}_{d0}^{\mathsf{TEM}})$ are those produced by the 3D topological nontriviality of Ω.

Remark 4.3. If ω contains holes, i. e., if TEM modes are present, they often contribute the smallest positive eigenvalues. Let us make formulas for eigenvalues more explicit: Let ℓ be the length of the interval I and let us assume that ω has *one hole*. Besides the magnetostatic zero eigenvalue, we find

$$\Lambda_{jm}^{\mathsf{TE}} = \lambda_j^{\mathsf{neu}} + \left(\frac{m\pi}{\ell}\right)^2 \ (\forall j, m \ge 1), \quad \Lambda_{jm}^{\mathsf{TM}} = \lambda_j^{\mathsf{dir}} + \left(\frac{m\pi}{\ell}\right)^2 \ (\forall j \ge 1, m \ge 0),$$

and

$$\Lambda_m^{\mathsf{TEM}} = \left(\frac{m\pi}{\ell}\right)^2 \ (\forall m \ge 1).$$

Then the smallest positive eigenvalue is either $\Lambda_{1,0}^{\mathsf{TM}}$ or Λ_1^{TEM}. If ω is fixed and ℓ large enough, Λ_1^{TEM} is smaller than $\Lambda_{1,0}^{\mathsf{TM}}$.

We summarize the results of Sections 3 and 4 in Table 6.1.

Table 6.1: Synthetic description of Maxwell eigenmodes, using **M** and **N** (2.6).

Polarization	k^2	**E**	**H**
TE	$\lambda_j^{\mathsf{neu}} + (\frac{m\pi}{\ell})^2$	$\mathbf{M}[v_j^{\mathsf{neu}} \otimes \sin(\frac{m\pi}{\ell}\cdot)]$	$\frac{1}{ik}\,\mathbf{N}[v_j^{\mathsf{neu}} \otimes \sin(\frac{m\pi}{\ell}\cdot)]$
TM	$\lambda_j^{\mathsf{dir}} + (\frac{m\pi}{\ell})^2$	$\mathbf{N}[v_j^{\mathsf{dir}} \otimes \cos(\frac{m\pi}{\ell}\cdot)]$	$ik\,\mathbf{M}[v_j^{\mathsf{dir}} \otimes \cos(\frac{m\pi}{\ell}\cdot)]$
TEM	$(\frac{m\pi}{\ell})^2$	$\mathbf{N}[v_d^{\mathsf{top}} \otimes \cos(\frac{m\pi}{\ell}\cdot)]$	$ik\,\mathbf{M}[v_d^{\mathsf{top}} \otimes \cos(\frac{m\pi}{\ell}\cdot)]$
Magnetostatic	0	$\mathbf{0}$	$\mathbf{M}[v_d^{\mathsf{top}} \otimes 1]$

5 Mixed perfectly conducting or insulating conditions

Consider now the situation where a part $\partial\Omega_{\mathrm{cd}}$ of the boundary of Ω represents *perfectly conducting walls* whereas another part $\partial\Omega_{\mathrm{ins}}$ represents *perfectly insulating* walls, with

$$\partial\Omega = \partial\Omega_{\mathrm{cd}} \cup \partial\Omega_{\mathrm{ins}}, \quad \partial\Omega_{\mathrm{cd}} \cap \partial\Omega_{\mathrm{ins}} = \emptyset. \tag{5.1}$$

Boundary conditions are then

$$\begin{cases} \mathbf{E}\times\mathbf{n} = 0 \quad \text{and} \quad \mathbf{H}\cdot\mathbf{n} = 0, & \text{on } \partial\Omega_{\mathrm{cd}}, \quad \text{(perfect conductor b. c.)} \\ \mathbf{E}\cdot\mathbf{n} = 0 \quad \text{and} \quad \mathbf{H}\times\mathbf{n} = 0, & \text{on } \partial\Omega_{\mathrm{ins}}, \quad \text{(perfect insulator b. c.)} \end{cases}$$

Similar results as above hold for *mixed boundary conditions* when the perfectly conducting or insulating parts $\partial\Omega_{\mathrm{cd}}$ and $\partial\Omega_{\mathrm{ins}}$ are chosen to be either $\partial\omega\times I$ or $\omega\times\partial I$. Let us give two examples.

Example 5.1. Let us consider the case when

$$\partial\Omega_{\mathrm{cd}} = \partial\omega\times I \quad \text{and} \quad \partial\Omega_{\mathrm{ins}} = \omega\times\partial I.$$

Then the essential boundary condition for the electric field $\mathbf{E}$ on $\omega\times\partial I$ is $\mathbf{E}_3 = 0$ and the natural boundary condition is $\operatorname{curl}\mathbf{E}\times\mathbf{n} = 0$, reducing to $\partial_3\mathbf{E}_\perp = 0$. Thus we find the three families of electric eigenfunctions:

$$\mathbf{E}^{\mathrm{TE}}_{jm} = \begin{pmatrix} \mathbf{curl}_\perp v^{\mathrm{neu}}_j(x_\perp) \\ 0 \end{pmatrix} w^{\mathrm{neu}}_m(x_3) \quad \text{with } j\ge 1,\ m\ge 0,$$

$$\mathbf{E}^{\mathrm{TM}}_{jm} = \begin{pmatrix} \nabla_\perp v^{\mathrm{dir}}_j(x_\perp) \\ 0 \end{pmatrix} \partial_3 w^{\mathrm{dir}}_m(x_3) - \begin{pmatrix} 0 \\ \Delta_\perp v^{\mathrm{dir}}_j(x_\perp) \end{pmatrix} w^{\mathrm{dir}}_m(x_3), \quad \text{with } j\ge 1,\ m\ge 1,$$

$$\mathbf{E}^{\mathrm{TEM}}_{dm} = \begin{pmatrix} \nabla_\perp v^{\mathrm{top}}_d(x_\perp) \\ 0 \end{pmatrix} w^{\mathrm{neu}}_m(x_3) \quad \text{with } 1\le d\le D-1,\ m\ge 0,$$

associated with the eigenvalues $\Lambda^{\mathrm{TE}}_{jm} = \lambda^{\mathrm{neu}}_j + \mu^{\mathrm{neu}}_m$, $\Lambda^{\mathrm{TM}}_{jm} = \lambda^{\mathrm{dir}}_j + \mu^{\mathrm{dir}}_m$, and $\Lambda^{\mathrm{TEM}}_{dm} = \mu^{\mathrm{neu}}_m$.

Example 5.2. We set $I = (0,\ell)$. Let us consider the case when

$$\partial\Omega_{\mathrm{cd}} = (\partial\omega\times I)\cup(\omega\times\{0\}) \quad \text{and} \quad \partial\Omega_{\mathrm{ins}} = \omega\times\{\ell\}.$$

The axial generators w_m can be described thanks to the eigenvectors w^{mix}_m, $m\ge 1$, of the *mixed* problem in ω:

$$-\partial_3^2 w = \mu w, \quad w(0) = 0, \quad \partial_3 w(\ell) = 0.$$

We find

$$\mathbf{E}^{\mathrm{TE}}_{jm} = \begin{pmatrix} \mathbf{curl}_\perp\, v_j^{\mathrm{neu}}(x_\perp) \\ 0 \end{pmatrix} w_m^{\mathrm{mix}}(x_3) \quad \text{with } j \ge 1,\ m \ge 1,$$

$$\mathbf{E}^{\mathrm{TM}}_{jm} = \begin{pmatrix} \nabla_\perp v_j^{\mathrm{dir}}(x_\perp) \\ 0 \end{pmatrix} \partial_3^2 w_m^{\mathrm{mix}}(x_3) - \begin{pmatrix} 0 \\ \Delta_\perp v_j^{\mathrm{dir}}(x_\perp) \end{pmatrix} \partial_3 w_m^{\mathrm{mix}}(x_3), \quad \text{with } j \ge 1,\ m \ge 1,$$

$$\mathbf{E}^{\mathrm{TEM}}_{dm} = \begin{pmatrix} \nabla_\perp v_d^{\mathrm{top}}(x_\perp) \\ 0 \end{pmatrix} w_m^{\mathrm{mix}}(x_3) \quad \text{with } 1 \le d \le D-1,\ m \ge 1.$$

If ω contains holes, TEM modes are present and contribute the smallest positive eigenvalue $(\frac{\pi}{2\ell})^2$.

6 Application 1: Maxwell eigenvalues of cuboids

6.1 Cube

Let Ω be the cube $(0,\pi)^3$. We can apply Theorem 3.6 with $\omega = (0,\pi)^2$ and $I = (0,\pi)$. Since ω is simply connected, we have TE and TM modes only. Therefore, the normalized Maxwell eigenvalues are

$$\lambda_j^{\mathrm{neu}} + \mu_m^{\mathrm{dir}}, \quad j \ge 1,\ m \ge 1 \quad \text{and} \quad \lambda_j^{\mathrm{dir}} + \mu_m^{\mathrm{neu}}, \quad j \ge 1,\ m \ge 0.$$

We have

$$\mu_m^{\mathrm{dir}} = m^2, \quad m \ge 1 \quad \text{and} \quad \mu_m^{\mathrm{neu}} = m^2, \quad m \ge 0.$$

The Dirichlet eigenvalues on ω are

$$k_1^2 + k_2^2, \quad k_1, k_2 \ge 1.$$

The nonzero Neumann eigenvalues are

$$k_1^2 + k_2^2, \quad k_1, k_2 \ge 0, \quad k_1 \text{ or } k_2 \ne 0.$$

Therefore, the TE eigenvalues are

$$k_1^2 + k_2^2 + k_3^2, \quad k_1, k_2 \ge 0, \quad k_1 \text{ or } k_2 \ne 0, \quad k_3 \ge 1.$$

The TM eigenvalues are

$$k_1^2 + k_2^2 + k_3^2, \quad k_1, k_2 \ge 1, \quad k_3 \ge 0.$$

Therefore, we have once

$$k_1^2 + k_2^2 + k_3^2, \quad k_1, k_2, k_3 \ge 0 \quad \text{with exactly one index } \nu \in \{1,2,3\} \text{ such that } \quad k_\nu = 0,$$

and twice

$$k_1^2 + k_2^2 + k_3^2, \quad k_1, k_2, k_3 \geq 1.$$

The first eigenvalues are

$$2 \text{ (mult. 3)}, \quad 3 \text{ (mult. 2)}, \quad 5 \text{ (mult. 6)}, \quad 6 \text{ (mult. 6)}, \quad 8 \text{ (mult. 3)}, \dots$$

A larger multiplicity of 12 is attained for example for $14 = 1 + 4 + 9$. But 12 is not the maximal multiplicity (e. g., the multiplicity of $26 = 25 + 1 + 0 = 16 + 9 + 1$ is 18).

The Dirichlet eigenvectors on $(0, \pi)$ are $\zeta \mapsto \sin k\zeta$, $k \geq 1$, and the Neumann eigenvectors are $\cos k\zeta$, $k \geq 0$. The components of the electric eigenvectors in the cube are (sums of) products of two sin terms by one cos term.

6.2 Cuboids

For a *rectangular parallelepiped,*

$$\Omega = (0, \ell_1) \times (0, \ell_2) \times (0, \ell_3),$$

we find the eigenvalues: Once

$$\left(\frac{k_1\pi}{\ell_1}\right)^2 + \left(\frac{k_2\pi}{\ell_2}\right)^2 + \left(\frac{k_3\pi}{\ell_3}\right)^2,$$
$$\forall k_1, k_2, k_3 \geq 0 \quad \text{with exactly one index } \nu \in \{1, 2, 3\} \text{ such that} \quad k_\nu = 0,$$

and twice

$$\left(\frac{k_1\pi}{\ell_1}\right)^2 + \left(\frac{k_2\pi}{\ell_2}\right)^2 + \left(\frac{k_3\pi}{\ell_3}\right)^2, \quad \forall k_1, k_2, k_3 \geq 1.$$

7 Application 2: Maxwell eigenvalues in axisymmetric product domains

We assume now, besides the assumption that $\Omega = \omega \times I$, that the domain Ω is axisymmetric. In this case, the separation of variables method can be used once more, giving explicit formulas for the Laplace eigenvectors and eigenfunctions, and hence more explicit formulas for the Maxwell eigenmodes. Now Ω axisymmetric implies that ω is an axisymmetric domain in dimension 2. Hence ω is either a disc or an annulus (i. e., a disc with a concentric hole). We investigate both situations.

7.1 Axisymmetric domains

Let R be the external radius of ω and r_0 be its internal radius, with the convention that $r_0 = 0$ corresponds to the case when ω is a disc. Let us denote by $\mathbb{T}$ the one-dimensional torus

$$\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}).$$

We use *cylindrical coordinates* $(r, \varphi, x_3) \in (r_0, R) \times \mathbb{T} \times I$. Setting

$$\breve{u}(r, \varphi, x_3) = u(x),$$

we introduce *cylindrical components* (u_r, u_φ, u_3) of the field $\mathbf{u} = (u_1, u_2, u_3)$,

$$u_r = \breve{u}_1 \cos\varphi + \breve{u}_2 \sin\varphi \quad \text{and} \quad u_\varphi = -\breve{u}_1 \sin\varphi + \breve{u}_2 \cos\varphi.$$

In particular, for a scalar function q, the radial and angular components of $\nabla_\perp q$ are $\partial_r q$ and $\frac{1}{r}\partial_\varphi q$, and those of $\mathbf{curl}_\perp\, q$ are $\frac{1}{r}\partial_\varphi q$ and $-\partial_r q$. With this, we find the representation in cylindrical coordinates of the ansatz $\mathbf{M}[q]$ and $\mathbf{N}[q]$ when q has the tensor form $v \otimes w$:

$$\begin{cases} \mathbf{M}_r[v \otimes w] = \dfrac{1}{r}\partial_\varphi v(r, \varphi)\, w(x_3), \\ \mathbf{M}_\varphi[v \otimes w] = -\partial_r v(r, \varphi)\, w(x_3), \\ \mathbf{M}_3[v \otimes w] = 0, \end{cases} \tag{7.1}$$

and

$$\begin{cases} \mathbf{N}_r[v \otimes w] = \partial_r v(r, \varphi)\, \partial_3 w(x_3), \\ \mathbf{N}_\varphi[v \otimes w] = \dfrac{1}{r}\partial_\varphi v(r, \varphi)\, \partial_3 w(x_3), \\ \mathbf{N}_3[v \otimes w] = -\dfrac{1}{r^2}((r\partial_r)^2 + \partial_\varphi^2) v(r, \varphi)\, w(x_3). \end{cases} \tag{7.2}$$

To describe the Maxwell eigenmodes in the axisymmetric case, we use Table 6.1 and make explicit the Dirichlet and Neumann eigenvectors v^{dir} and v^{neu} on ω, and also v^{top} when there is a hole ($r_0 > 0$).

It is a classical technique to use the invariance under rotation of the Laplace operator $\Delta_\perp$ for diagonalizing it by Fourier series with respect to $\varphi \in \mathbb{T}$. This leads to the following representations:

$$v^{\mathrm{dir}} = h^{\mathrm{dir}}_{np}(r)\, e^{in\varphi} \quad \text{and} \quad v^{\mathrm{neu}} = h^{\mathrm{neu}}_{np}(r)\, e^{in\varphi}, \quad n \in \mathbb{Z}, \quad p \geq 1, \tag{7.3}$$

where for each $n \in \mathbb{Z}$, the functions $(h^{\mathrm{dir}}_{np})_p$ and $(h^{\mathrm{neu}}_{np})_p$ are bases of eigenfunctions for the operator

$$h \longmapsto \left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{n^2}{r^2}\right)h, \quad r \in (r_0, R) \tag{7.4}$$

with appropriate boundary conditions.

7.2 The cylinder (ω is a disc)

For $h^{\rm dir}$, the boundary condition at R is $h^{\rm dir}(R) = 0$, for $h^{\rm neu}$ this is $\partial_r h^{\rm neu}(R) = 0$. At the other end, $r = 0$ of the interval $(0, R)$, the boundary conditions are driven by integrability properties (cf. [3]): For $h^{\rm dir}$ and $h^{\rm neu}$, they are

$$\partial_r h(0) = 0 \quad \text{if } n = 0, \quad \text{and} \quad h(0) = 0 \quad \text{if } n \neq 0. \tag{7.5}$$

As a consequence, both $h^{\rm dir}$ and $h^{\rm neu}$ are given by the Bessel functions of the first kind J_n that satisfy (7.5) and the equation $(-\partial_r^2 - \frac{1}{r}\partial_r + n^2)J_n = J_n$; cf (7.4). One finds the following.

Lemma 7.1 ([6]).

(i) *Let $(z_{np})_{p\geq 1}$ be the increasing sequence of the positive zeros of J_n. Then a spectral sequence for the Dirichlet problem for $-\Delta_\perp$ on ω is*

$$\lambda_{np}^{\rm dir} = \left(\frac{z_{np}}{R}\right)^2 \quad \text{and} \quad v_{np}^{\rm dir} = J_n\left(\frac{z_{np}\, r}{R}\right) e^{in\varphi}, \quad n \in \mathbb{Z},\ p \geq 1 \tag{7.6}$$

(ii) *Let $(z'_{np})_{p\geq 1}$ be the increasing sequence of the positive zeros of J'_n. Then a spectral sequence for the Neumann problem for $-\Delta_\perp$ on ω is, in addition to the constant eigenfunction,*

$$\lambda_{np}^{\rm neu} = \left(\frac{z'_{np}}{R}\right)^2 \quad \text{and} \quad v_{np}^{\rm neu} = J_n\left(\frac{z'_{np}\, r}{R}\right) e^{in\varphi}, \quad n \in \mathbb{Z},\ p \geq 1 \tag{7.7}$$

We summarize results in Table 6.2.

Table 6.2: Maxwell eigenmodes in a cylinder of radius R and length ℓ, using **M** (7.1)–**N** (7.2), and $v_{np}^{\rm dir}$ (7.6)–$v_{np}^{\rm neu}$ (7.7).

Polarization	k^2	E	H
TE	$(\frac{z'_{np}}{R})^2 + (\frac{m\pi}{\ell})^2$	$\mathbf{M}[v_{np}^{\rm neu} \otimes \sin(\frac{m\pi}{\ell}\cdot)]$	$\frac{1}{ik}\,\mathbf{N}[v_{np}^{\rm neu} \otimes \sin(\frac{m\pi}{\ell}\cdot)]$
TM	$(\frac{z_{np}}{R})^2 + (\frac{m\pi}{\ell})^2$	$\mathbf{N}[v_{np}^{\rm dir} \otimes \cos(\frac{m\pi}{\ell}\cdot)]$	$ik\,\mathbf{M}[v_{np}^{\rm dir} \otimes \cos(\frac{m\pi}{\ell}\cdot)]$

Table 6.3: The first three zeros of $J_0, J_1, J_2, J'_0, J'_1, J'_2$.

$z_{0,j}$	$z_{1,j}$	$z_{2,j}$	$z'_{0,j}$	$z'_{1,j}$	$z'_{2,j}$
2.4048	3.8317	5.1356	3.8317	1.8412	3.0542
5.5201	7.0156	8.4172	7.0156	5.3314	6.7061
8.6537	10.173	11.620	10.173	8.5363	9.9695

We give in Table 6.3 values for the first three zeros $z_{n,j}$ and $z'_{n,j}$ for $n = 0, 1, 2$. We use the relation $J_{\nu-1} - J_{\nu+1} = 2J'_\nu$ to compute $z'_{n,j}$. Since $J_{-1} = -J_1$, we note that there holds

$$z'_{0,j} = z_{1,j}, \quad \forall j \geq 1.$$

7.3 The coaxial cylindrical hole (ω is an annulus)

In this case again, there exist explicit formulas for the Laplace eigenvectors and eigenfunctions. This is classical knowledge; see, e. g., [7, 8]. The boundary conditions on h^{dir} and h^{neu} are now the standard ones at r_0 and R. We have to find the associated eigenpairs of the operator (7.4) for any $n \in \mathbb{N}$. We find that the radial eigenvectors h^{dir} and h^{neu} are linear combinations of the Bessel functions J_n and Y_n of first and second kind:

$$h^{\mathrm{dir}}_{np}(r) = \alpha_{np} J_n(k_{np}\, r) + \beta_{np} Y_n(k_{np}\, r)$$

with eigenvalues $\lambda^{\mathrm{dir}}_{np} = (k_{np})^2$, where k_{np} are the positive zeros of the determinant function

$$k \longmapsto J_n(kr_0)\, Y_n(kR) - Y_n(kr_0)\, J_n(kR).$$

Analogous formulas exist for h^{neu}.

Since ω has one hole, the number L of the connected components of its boundary is 2. There exists a nonconstant harmonic potential v^{top} that takes two distinct constant values on the two connected components of $\partial\omega$. This generator v^{top} can be defined as

$$v^{\mathrm{top}}(x_\perp) = \log r.$$

In connection with Remark 3.4, we note that the conjugate potential $\tilde{v}^{\mathrm{top}}$ is the function $x_\perp \mapsto \varphi$. In cylindrical components, there holds

$$\begin{pmatrix} \widetilde{\mathbf{curl}}_\perp\, \tilde{v}^{\mathrm{top}} \\ 0 \end{pmatrix} = \begin{pmatrix} \nabla_\perp v^{\mathrm{top}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathbf{curl}_\perp\, v^{\mathrm{top}} \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ \frac{1}{r} \\ 0 \end{pmatrix}.$$

We summarize the results concerning TEM modes for $\Omega = \omega \times I$ with the annulus ω.

Corollary 7.2. *Let ℓ be the length of the cylinder Ω with coaxial hole. Its family of TEM modes is axisymmetric and has the form $(\frac{m\pi}{\ell}, \mathbf{E}^{\mathrm{TEM}}, \mathbf{H}^{\mathrm{TEM}})$ with:*

(a) *for $m \geq 1$,*

$$\begin{cases} E^{\mathrm{TEM}}_r = \dfrac{1}{r} \sin\Big(\dfrac{m\pi}{\ell} x_3\Big), \\ E^{\mathrm{TEM}}_\varphi = 0, \\ E^{\mathrm{TEM}}_3 = 0, \end{cases} \quad \text{and} \quad \begin{cases} H^{\mathrm{TEM}}_r = 0, \\ H^{\mathrm{TEM}}_\varphi = -i\dfrac{m\pi}{\ell}\dfrac{1}{r} \cos\Big(\dfrac{m\pi x_3}{\ell}\Big), \\ H^{\mathrm{TEM}}_3 = 0 \end{cases} \tag{7.8}$$

(b) *for $m = 0$, $\mathbf{E} = \mathbf{0}$ and $\mathbf{H} = (0\ 1\ 0)^\top$.*

Remark 7.3. As r_0 tends to 0, the Dirichlet and Neumann eigenmodes of the annulus tend to the Dirichlet and Neumann eigenvalues of the disc of same radius R. Hence the TE and TM modes of the cylinder with hole tend to the TE and TM modes of the cylinder without hole. In contrast, the TEM modes do not depend on r_0 as long as $r_0 \neq 0$, but disappear at the limit when $r_0 = 0$. This fact has a practical importance when thin conductor wires are present.

8 Maxwell eigenmodes in a ball

For the sake of comparison, we revisit known results about Maxwell eigenmodes in a ball; see [9, Chapter 10]. Let $\Omega \subset \mathbb{R}^3$ be the ball of center 0 and radius R. Here, we use spherical coordinates $(\theta, \varphi, \rho) \in [0, \pi] \times \mathbb{T} \times [0, R]$, associate with the orthonormal basis

$$(\hat{\theta}, \hat{\varphi}, \hat{\rho}).$$

Formulas for Maxwell eigenmodes are based on Debye potentials. This is the **M**, **N** ansatz, in a form slightly different from (2.6): The piloting vector is replaced by the unit field

$$\hat{\boldsymbol{x}} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \quad \text{i. e.} \quad \hat{\boldsymbol{x}} = \hat{\rho}.$$

The **M**, **N** ansatz takes the form

$$\mathbf{M}[q] = \operatorname{curl}(q\,\hat{\boldsymbol{x}}) \quad \text{and} \quad \mathbf{N}[q] = \operatorname{curl} \mathbf{M}[q] = \operatorname{curl}\operatorname{curl}(q\,\hat{\boldsymbol{x}}). \tag{8.1}$$

Using for instance identities (cf. [5, Section 6.2]),

$$\operatorname{curl}(p\boldsymbol{x}) = \nabla p \times \boldsymbol{x} \quad \text{and} \quad \operatorname{curl}(\boldsymbol{a} \times \boldsymbol{x}) = (\rho\partial_\rho + 2)\boldsymbol{a} - \boldsymbol{x} \operatorname{div} \boldsymbol{a}$$

we find the following formulas where we express vectors in spherical components on the basis $(\hat{\theta}, \hat{\varphi}, \hat{\rho})$:

$$\mathbf{M}[q] = \nabla\left(\frac{q}{\rho}\right) \times \boldsymbol{x} = \nabla q \times \hat{\boldsymbol{x}} = \begin{pmatrix} \frac{1}{\rho}\partial_\theta q \\ -\frac{1}{\rho \sin\theta}\partial_\varphi q \\ 0 \end{pmatrix} \tag{8.2}$$

and

$$\mathbf{N}[q] = \operatorname{curl} \mathbf{M}[q] = \nabla(\partial_\rho q) - \hat{\boldsymbol{x}}\,\rho\,\Delta\left(\frac{q}{\rho}\right). \tag{8.3}$$

Therefore,

$$\operatorname{curl} \mathbf{N}[q] = \operatorname{curl}\operatorname{curl} \mathbf{M}[q] = -\mathbf{M}\left[\rho\,\Delta\left(\frac{q}{\rho}\right)\right]. \tag{8.4}$$

Introduce the operator

$$\mathfrak{L} : q \longmapsto \mathfrak{L}q = -\rho\,\Delta\Big(\frac{q}{\rho}\Big).$$

Then the equations curl curl $\mathbf{u} - k^2\mathbf{u} = 0$ for $\mathbf{M}[q]$ and $\mathbf{N}[q]$ are equivalent to

$$\mathbf{M}[(\mathfrak{L} - k^2)q] = 0 \quad \text{and} \quad \mathbf{N}[(\mathfrak{L} - k^2)q] = 0.$$

Thus we are interested in scalar solutions q of the equation

$$\mathfrak{L}q = k^2 q \quad \text{in } [0,\pi] \times \mathbb{T} \times [0,R]. \tag{8.5}$$

We note that

$$\mathfrak{L} = -\partial_\rho^2 - \frac{1}{\rho^2}\,\Delta_{\mathbb{S}^2} \tag{8.6}$$

with the Laplace–Beltrami operator $\Delta_{\mathbb{S}^2}$ on the unit sphere $\mathbb{S}^2$,

$$\Delta_{\mathbb{S}^2} = \frac{1}{\sin\theta}\,\partial_\theta \sin\theta\partial_\theta + \frac{1}{\sin^2\theta}\,\partial_\varphi^2 .$$

The equation (8.5) is satisfied by all functions in tensor form

$$q(\theta,\varphi,\rho) = Y_n^m(\theta,\varphi)\,h(k\rho)\,,$$

where Y_n^m are the spherical harmonics and h is a linear combination of the Riccati–Bessel functions ψ_n and χ_n (sometimes written as S_n and C_n). Following Debye's notation, we use the definition

$$\psi_n(x) = xj_n(x) = \sqrt{\frac{\pi x}{2}}J_{n+\frac{1}{2}}(x) \quad \text{and} \quad \chi_n(x) = -xy_n(x) = -\sqrt{\frac{\pi x}{2}}Y_{n+\frac{1}{2}}(x),$$

where J_ν, Y_ν are the Bessel functions and j_n, y_n the spherical Bessel functions of first, second kind, respectively. Because of integrability conditions in 0, χ_n has to be discarded. It remains to look for potentials q of type $Y_n^m \otimes \psi_n(k\,\cdot)$ so that either $\mathbf{M}[q]$ or $\mathbf{N}[q]$ satisfy the electric boundary condition on the boundary of the ball, i. e.,

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{if } \rho = R.$$

Using formulas (8.2) and (8.3), we find that this boundary condition is satisfied by $\mathbf{M}[Y_n^m \otimes \psi_n(k\,\cdot)]$ if $\psi_n(kR) = 0$ and by $\mathbf{N}[Y_n^m \otimes \psi_n(k\,\cdot)]$ if $\psi_n'(kR) = 0$. The remarkable fact is that the related potentials are then eigenvectors of the operator $\mathfrak{L}$ with the eigenvalue k^2 for Dirichlet or Neumann conditions. Note that the operator $\mathfrak{L}$ is associated with the coercive bilinear form

$$a(q,\tilde{q}) = \int_0^R \Big[\int_{\mathbb{S}^2} \Big(\partial_\rho q\,\partial_\rho \tilde{q} + \frac{1}{\rho^2}\,\partial_\theta q\,\partial_\theta \tilde{q} + \frac{1}{\rho^2 \sin^2\theta}\,\partial_\varphi q\,\partial_\varphi \tilde{q}\Big)\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\Big]\mathrm{d}\rho$$

on the space

$$V = \left\{ q \in L^2(\mathbb{S}^2 \times (0,R)),\ \partial_\rho q,\ \frac{1}{\rho}\nabla_\perp q \in L^2(\mathbb{S}^2 \times (0,R)) \right\}.$$

Completed with either Dirichlet or Neumann boundary conditions on $\rho = R$, $\mathfrak{L}$ is self-adjoint. We have obtained the following.

Theorem 8.1.

(i) *The Dirichlet eigenpairs of* $\mathfrak{L}$ *have the form* $(k_{np}^2, q_{nmp}^{\mathrm{dir}})$, $n \geq 0$, $|m| \leq n$, $p \geq 1$ *with* $(k_{np})_{p\geq 1}$ *the enumeration of the positive zeros of the function* $k \mapsto \psi_n(kR)$ *and* $q_{nmp} = Y_n^m \otimes \psi_n(k_{np}\,\cdot\,)$. *All triples* $(k_{np}, \mathbf{M}[q_{nmp}^{\mathrm{dir}}], \frac{1}{ik}\mathbf{N}[q_{nmp}^{\mathrm{dir}}])$ *are Maxwell eigenmodes on the ball of radius* R.

(ii) *The nonconstant Neumann eigenpairs of* $\mathfrak{L}$ *have the form* $((k'_{np})^2, q_{nmp}^{\mathrm{neu}})$, $n \geq 0$, $|m| \leq n$, $p \geq 1$ *with* $(k'_{np})_{p\geq 1}$ *the enumeration of the positive zeros of the function* $k \mapsto \psi'_n(kR)$ *and* $q_{nmp} = Y_n^m \otimes \psi_n(k'_{np}\,\cdot\,)$. *All triples* $(k'_{np}, \mathbf{N}[q_{nmp}^{\mathrm{neu}}], ik\mathbf{M}[q_{nmp}^{\mathrm{neu}}])$ *are Maxwell eigenmodes on the ball of radius* R.

Remark 8.2. In the literature, the **M** ansatz is frequently written in a slightly different way which we distinguish with an asterisk:

$$\mathbf{M}^\star[q^\star] = \operatorname{curl}(q^\star \boldsymbol{x})$$

instead of $\mathbf{M}[q] = \operatorname{curl}(q\hat{\boldsymbol{x}})$. As usual, $\mathbf{N}^\star = \operatorname{curl}\mathbf{M}^\star$. The outcome for the Maxwell eigenmodes is the same of course. Nevertheless, the interpretation of the potentials is different. We have

$$q^\star = \frac{q}{\rho}.$$

1. Concerning Dirichlet modes, the functions $q_{nmp}^\star$ defined as $q_{nmp}^{\mathrm{dir}}/\rho$ are the eigenfunctions of the Dirichlet problem for the standard positive Laplace operator $-\Delta$ on the ball. In other words, the eigenvalues k_{np}^2 are also the standard Laplace eigenvalues.
2. But, when Neumann modes are concerned, the functions $q_{nmp}^\star$ defined as $q_{nmp}^{\mathrm{neu}}/\rho$ are not Neumann eigenfunctions for $-\Delta$.

Remark 8.3. The tensor product potentials $Y_n^m \otimes h(k\,\cdot\,)$ with h being any of the Riccati–Bessel functions have been used more than a century ago to describe scattering of plane waves by a dielectric sphere (Mie series). Scattering resonances (with negative imaginary part) have also been investigated at that time. More recently, whispering gallery modes have been analytically calculated by a similar method [2]. All of these problems are transmission problems between the ball and its exterior. Inside the ball h has the form $\psi_n(\mathrm{n_{opt}}k\,\cdot\,)$ where $\mathrm{n_{opt}}$ is the refractive (or optical) index of the ball. Outside the ball, h is either $\zeta_n^{(1)}(k\,\cdot\,)$ for scattering, or $\chi_n(k\,\cdot\,)$ for whispering gallery modes.

We end this section by a completeness result that can be seen as a consequence of Theorem 8.1.

Corollary 8.4. *The union of the two families*

$$\Big(k_{np}, \mathbf{M}[q^{\mathsf{dir}}_{nmp}], \frac{1}{ik}\mathbf{N}[q^{\mathsf{dir}}_{nmp}]\Big)_{nmp} \quad \textit{and} \quad (k'_{np}, \mathbf{N}[q^{\mathsf{neu}}_{nmp}], ik\mathbf{M}[q^{\mathsf{neu}}_{nmp}])_{nmp}$$

described in Theorem 8.1 *form a complete set of Maxwell eigenmodes.*

Proof. Let $\mathbf{u} \in \mathbf{X}_{\mathsf{N}}(\Omega)$ such that $\operatorname{div}\mathbf{u} = 0$. We assume that $\mathbf{u}$ is orthogonal to all electric eigenvectors $\mathbf{M}[q^{\mathsf{dir}}_{nmp}]$ and $\mathbf{N}[q^{\mathsf{neu}}_{nmp}]$. We prove that $\mathbf{u} = 0$ by contradiction. Assuming that $\mathbf{u} \neq 0$ and relying on the fact that the Maxwell problem possesses an orthonormal basis of eigenfunctions, we may suppose that $\mathbf{u}$ is an eigenvector itself, associated with an eigenvalue k^2. Since the ball Ω is topologically trivial, the condition $\operatorname{div}\mathbf{u} = 0$ implies that $k \neq 0$, whence $\mathbf{u} = \frac{1}{k^2}\operatorname{curl}\operatorname{curl}\mathbf{u}$. The orthogonality of $\mathbf{u}$ against all eigenvectors $\mathbf{M}[q^{\mathsf{dir}}_{nmp}]$ implies through integration by parts that $\operatorname{curl}\mathbf{u}$ is orthogonal to all $q^{\mathsf{dir}}_{nmp}\,\hat{\boldsymbol{x}}$, hence $\operatorname{curl}\mathbf{u}$ has a zero radial component. In a similar way, the orthogonality of against all eigenvectors $\mathbf{N}[q^{\mathsf{neu}}_{nmp}]$ implies that $\operatorname{curl}\operatorname{curl}\mathbf{u}$, hence $\mathbf{u}$, has a zero radial component. Finally, the implication

$$\mathbf{u}\cdot\hat{\boldsymbol{x}} = 0, \quad \operatorname{curl}\mathbf{u}\cdot\hat{\boldsymbol{x}} = 0, \quad \text{and} \quad \operatorname{div}\mathbf{u} = 0 \quad \Longrightarrow \quad \mathbf{u} = 0$$

can be found in [11] and leads to a contradiction, which proves the completeness. □

9 Extension to nonconstant electric permittivity

Let us consider the original Maxwell system (A.4). We still assume that the magnetic permeability μ is equal to μ_0 in the whole domain Ω. But we allow now that the electric permittivity ε may vary in Ω. We set

$$\varepsilon = \varepsilon_{\mathsf{rel}}\varepsilon_0, \quad \varepsilon_{\mathsf{rel}} \geq 1.$$

We consider domains Ω in the product form $\omega \times I$. We assume that

$$\varepsilon_{\mathsf{rel}}(x) = \varepsilon_{\mathsf{rel}}(x_\perp), \quad \varepsilon_{\mathsf{rel}} \in L^\infty(\omega), \tag{9.1}$$

like in wave guides or optic fibers. The Maxwell system takes now the form (A.6) instead of (1.1). Then the classification of eigenvectors into TE, TM, and TEM does not hold any more (at least not in the form given by Theorem 3.6 and Corollary 4.1). Nevertheless, the splitting of the spectrum according to frequencies with respect to the axial variable x_3 remains possible, as well as a tensor product form. We are going to investigate the magnetic field $\mathbf{H}$, taking advantage of its local regularity even if $\varepsilon_{\mathrm{rel}}$ is not continuous. The magnetic variational formulation becomes, instead of (2.2):

Find the eigenpairs $(\Lambda = \kappa^2, \mathbf{u})$ *with* $\mathbf{u} \neq 0$ *in* $\mathbf{X}_\mathsf{T}(\Omega)$ *and* $\operatorname{div} \mathbf{u} = 0$ *such that*

$$\int_\Omega \frac{1}{\varepsilon_{\mathrm{rel}}} \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{u}' + s \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, \mathrm{d}\mathbf{x} = \Lambda \int_\Omega \mathbf{u} \cdot \mathbf{u}' \, \mathrm{d}\mathbf{x}, \quad \forall \mathbf{u}' \in \mathbf{X}_\mathsf{T}(\Omega). \tag{9.2}$$

Here, s is nonnegative. The choice $s > 0$ corresponds to an elliptic regularization of the system. To simplify notation, let us assume that

$$\boxed{\mathrm{I} = (0, \pi)} \tag{9.3}$$

In the constant material case, considering the Maxwell eigenmodes from the magnetic point of view, we note that the magnetic part of eigenmodes given in Corollary 4.1 have the following form:

$$\mathbf{H}^{\mathrm{TE}} = \begin{pmatrix} k \nabla_\perp v(x_\perp) \cos(m x_3) \\ -\Delta_\perp v(x_\perp) \sin(m x_3) \end{pmatrix} \quad \text{and} \quad \mathbf{H}^{\mathrm{TM}} = \begin{pmatrix} \mathbf{curl}_\perp v(x_\perp) \cos(m x_3) \\ 0 \end{pmatrix} \tag{9.4}$$

We are going to prove that we still have a similar structure with respect to the axial variable x_3.

Theorem 9.1. *With the assumptions* (9.1) *and* (9.3), *the magnetic eigenmodes solution of* (9.2) *can be organized in a sequence of independent families* $\mathfrak{H}_m$ *with index* $m \in \mathbb{N}$ *in which each eigenvector has the tensor product form*

$$\mathbf{H} = \begin{pmatrix} \mathbf{v}_\perp(x_\perp) \cos(m x_3) \\ v_3(x_\perp) \sin(m x_3) \end{pmatrix}. \tag{9.5}$$

For any $m \in \mathbb{N}$, *let* Λ_j^m *and* $\mathbf{v}_j^m := (\mathbf{v}_{\perp,j}^m, v_{3,j}^m)$ *be the eigenpairs of the problem:*

Find $\Lambda \in \mathbb{R}$, $\mathbf{v} = (\mathbf{v}_\perp, v_3) \neq 0$ *in* $\mathbf{X}_\mathsf{T}(\omega) \times H^1(\omega)$ *with* $\operatorname{div}_\perp \mathbf{v}_\perp + m v_3 = 0$ *such that*

$$\begin{aligned} \int_\omega \frac{1}{\varepsilon_{\mathrm{rel}}} \{ curl_\perp \mathbf{v}_\perp \; curl_\perp \mathbf{v}'_\perp + (\nabla_\perp v_3 + m \mathbf{v}_\perp) \cdot (\nabla_\perp v'_3 + m \mathbf{v}'_\perp) \} \, \mathrm{d}\mathbf{x} \\ = \Lambda \int_\omega \mathbf{v} \cdot \mathbf{v}' \, \mathrm{d}\mathbf{x}, \quad \forall \mathbf{v}' \in \mathbf{X}_\mathsf{T}(\omega) \times H^1(\omega). \end{aligned} \tag{9.6}$$

Denote by $\mathbf{H}_j^m$ *the vector of form* (9.5) *with* $\mathbf{v} = \mathbf{v}_j^m$. *Then the eigenpairs* $(\Lambda_j^m, \mathbf{H}_j^m)_{j \geq 1}$ *span the family* $\mathfrak{H}_m$.

Proof. Solutions of (9.2) satisfy on $\omega \times \{0\}$ the essential boundary condition $u_3 = 0$, and the natural boundary condition $\frac{1}{\varepsilon_{\mathrm{rel}}} \operatorname{curl} \mathbf{u} \times \mathbf{e}_3 = 0$. Since $u_3 = 0$ on $\omega \times \{0\}$, $\partial_1 u_3$ and $\partial_2 u_3$ are also 0 on $\omega \times \{0\}$, and the natural boundary condition implies that $\partial_3 u_1 = \partial_3 u_2 = 0$ on $\omega \times \{0\}$. Therefore, defining the extension

$$\tilde{\mathbf{u}}_\perp(x_\perp, -x_3) = \mathbf{u}_\perp(x_\perp, x_3) \quad \text{and} \quad \tilde{u}_3(x_\perp, -x_3) = -u_3(x_\perp, x_3), \quad \forall x_3 \in (0, \pi)$$

we obtain an element $\tilde{\mathbf{u}} \in \mathbf{X}_T(\omega \times (-\pi,\pi))$ which satisfies $\operatorname{div}\tilde{\mathbf{u}} = 0$ and is a solution of (9.2) on the extended domain $\omega \times (-\pi,\pi)$. Moreover, $\mathbf{u}(x_\perp,-\pi) = \mathbf{u}(x_\perp,\pi)$ and $\partial_3\mathbf{u}(x_\perp,-\pi) = \partial_3\mathbf{u}(x_\perp,\pi)$ for all $x_\perp \in \omega$. We deduce that $\tilde{\mathbf{u}}$ is a solution of (9.2) on the domain $\mathbf{X}_T(\omega \times \mathbb{T})$ where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Since the coefficient $\varepsilon_{\mathrm{rel}}$ does not depend on x_3, the underlying Maxwell operator commutes with ∂_3. Therefore, the spectrum of problem (9.2) can be decomposed according to the eigenvectors of ∂_3 on $\mathbb{T}$, which are the functions $x_3 \mapsto e^{imx_3}$, $m \in \mathbb{Z}$.

For any positive integer m, we notice that if $(\mathbf{v}_\perp(x_\perp), v_3(x_\perp))e^{imx_3}$ is a solution of (9.2) on the domain $\mathbf{X}_T(\omega\times\mathbb{T})$, then $(\mathbf{v}_\perp(x_\perp), -v_3(x_\perp))e^{-imx_3}$ is also a solution of the same problem. Therefore, their sum is also a solution of the same problem. Moreover, this sum has the form (9.5) and satisfies the boundary conditions (perfectly conducting walls)[1] of the space $\mathbf{X}_T(\Omega)$. Conversely, this sum is, up to a multiplicative constant, the only linear combination of $(\mathbf{v}_\perp(x_\perp), v_3(x_\perp))e^{imx_3}$ and $(\mathbf{v}_\perp(x_\perp), -v_3(x_\perp))e^{-imx_3}$ which satisfies the boundary conditions of the space $\mathbf{X}_T(\Omega)$.

Calculating

$$\int_\Omega \frac{1}{\varepsilon_{\mathrm{rel}}} \operatorname{curl}\mathbf{u}\ \operatorname{curl}\mathbf{u}'\ \mathrm{d}\mathbf{x}$$

for

$$\mathbf{u} = \begin{pmatrix} \mathbf{v}(x_\perp)\cos(mx_3) \\ v_3(x_\perp)\sin(mx_3) \end{pmatrix} \quad \text{and} \quad \mathbf{u}' = \begin{pmatrix} \mathbf{v}'(x_\perp)\cos(mx_3) \\ v_3'(x_\perp)\sin(mx_3) \end{pmatrix},$$

we find

$$\int_\omega \frac{1}{\varepsilon_{\mathrm{rel}}} \{ curl_\perp \mathbf{v}_\perp\ curl_\perp \mathbf{v}'_\perp + (\mathbf{curl}_\perp v_3 + m\mathbf{v}_\perp \times \mathbf{e}_3)\cdot(\mathbf{curl}_\perp v_3' + m\mathbf{v}'_\perp \times \mathbf{e}_3)\}\,\mathrm{d}\mathbf{x}$$

which coincides with the bilinear form in problem (9.6). □

Remark 9.2. The bilinear form of problem (9.6) can be regularized by

$$\int_\omega \frac{1}{\varepsilon_{\mathrm{rel}}} \{(\operatorname{div}_\perp \mathbf{v}_\perp + mv_3)(\operatorname{div}_\perp \mathbf{v}'_\perp + mv_3')\}\,\mathrm{d}\mathbf{x}.$$

We can check that if $\varepsilon_{\mathrm{rel}}$ is constant, the resulting bilinear form is equal to

$$\frac{1}{\varepsilon_{\mathrm{rel}}} \int_\omega curl_\perp \mathbf{v}_\perp\ curl_\perp \mathbf{v}'_\perp + \nabla_\perp v_3 \cdot \nabla_\perp v_3' + \operatorname{div}_\perp \mathbf{v}_\perp\ \operatorname{div}_\perp \mathbf{v}'_\perp + m^2(\mathbf{v}_\perp\cdot\mathbf{v}'_\perp + v_3 v_3')\,\mathrm{d}\mathbf{x}.$$

---

1 Considering the *difference* instead the sum, we would find the perfectly insulating boundary conditions on $\omega \times \partial I$.

Remark 9.3. For $m = 0$, problem (9.6) reduces to two uncoupled problems: The magnetic 2D Maxwell eigenvalue problem in ω for $\mathbf{v}_\perp$ and the Neumann eigenvalue problem for $-\Delta_\perp$ in ω for v_3. This last problem does not yield any nontrivial solution of (9.6) since for $m = 0$, the third component in the Ansatz (9.5) is zero. Moreover, we can show that the solutions of the magnetic 2D Maxwell eigenvalue problem in ω are the pairs $(\mathbf{curl}_\perp v_j^{\mathrm{dir}}, \lambda_j^{\mathrm{dir}})$, $j \geq 1$, with the eigenpairs $(v_j^{\mathrm{dir}}, \lambda_j^{\mathrm{dir}})$ of the problem

$$- \Delta_\perp v = \lambda \varepsilon v \quad \text{in } \omega,\ v \in H_0^1(\omega). \tag{9.7}$$

Thus we have found for $m = 0$ the family of TM modes:

$$\mathbf{H}_j^{\mathrm{TM}} = \begin{pmatrix} \mathbf{curl}_\perp v_j^{\mathrm{dir}}(x_\perp) \\ 0 \end{pmatrix} \qquad j \geq 1.$$

Appendix A. Normalizing Maxwell equations

Let ε and μ are the electric permittivity and the magnetic permeability of the material inside Ω. We assume that the boundary of Ω represents *perfectly conducting* or *perfectly insulating* walls:

$$\partial\Omega = \partial\Omega_{\mathrm{cd}} \cup \partial\Omega_{\mathrm{ins}}, \quad \partial\Omega_{\mathrm{cd}} \cap \partial\Omega_{\mathrm{ins}} = \emptyset, \tag{A.1}$$

where $\partial_{\mathrm{cd}}\Omega$ is the perfectly conducting part and $\partial_{\mathrm{ins}}\Omega$ the perfectly insulating part.

The cavity resonator problem is to find the frequencies $\varpi \in \mathbb{R}_+$ and the nonzero electromagnetic fields $(\hat{\mathbf{E}}, \hat{\mathbf{H}}) \in L^2(\Omega)^6$ such that

$$\begin{cases} \operatorname{curl} \hat{\mathbf{E}} - i\varpi\mu\hat{\mathbf{H}} = 0 & \text{in } \Omega, \quad \text{(Faraday law)} \\ \operatorname{curl} \hat{\mathbf{H}} + i\varpi\varepsilon\hat{\mathbf{E}} = 0 & \text{in } \Omega, \quad \text{(Ampère law)} \\ \operatorname{div} \varepsilon\hat{\mathbf{E}} = 0 \quad \text{and} \quad \operatorname{div} \mu\hat{\mathbf{H}} = 0 & \text{in } \Omega, \quad \text{(gauge conditions).} \end{cases} \tag{A.2a}$$

with boundary conditions

$$\begin{cases} \hat{\mathbf{E}} \times \mathbf{n} = 0 \quad \text{and} \quad \hat{\mathbf{H}} \cdot \mathbf{n} = 0, & \text{on } \partial\Omega_{\mathrm{cd}}, \quad \text{(perfect conductor b. c.)} \\ \hat{\mathbf{E}} \cdot \mathbf{n} = 0 \quad \text{and} \quad \hat{\mathbf{H}} \times \mathbf{n} = 0, & \text{on } \partial\Omega_{\mathrm{ins}}, \quad \text{(perfect insulator b. c.)} \end{cases} \tag{A.2b}$$

In this paper, we consider the nonmagnetic case, i. e., when $\mu \equiv \mu_0$ in Ω. We can set

$$\varepsilon = \mathsf{n}_{\mathrm{opt}}^2 \varepsilon_0 = \varepsilon_{\mathrm{rel}}\, \varepsilon_0 \tag{A.3}$$

where $\mathsf{n}_{\mathrm{opt}}$ is the refractive index of the material and $\varepsilon_{\mathrm{rel}}$ the relative permittivity. Then (A.2a) reduces to

$$\begin{cases} \operatorname{curl} \hat{\mathbf{E}} - i\varpi\mu_0\hat{\mathbf{H}} = 0 & \text{in } \Omega, \quad \text{(Faraday law)} \\ \operatorname{curl} \hat{\mathbf{H}} + i\varpi\varepsilon_{\mathrm{rel}}\, \varepsilon_0\hat{\mathbf{E}} = 0 & \text{in } \Omega, \quad \text{(Ampère law)} \\ \operatorname{div} \varepsilon_{\mathrm{rel}}\, \varepsilon_0\hat{\mathbf{E}} = 0 \quad \text{and} \quad \operatorname{div} \mu_0\hat{\mathbf{H}} = 0 & \text{in } \Omega, \quad \text{(gauge conditions).} \end{cases} \tag{A.4}$$

With the normalization,

$$k = \varpi\sqrt{\varepsilon_0\mu_0} \quad \text{(wave number)}, \quad \mathbf{E} = \sqrt{\varepsilon_0}\,\hat{\mathbf{E}} \quad \text{and} \quad \mathbf{H} = \sqrt{\mu_0}\,\hat{\mathbf{H}}, \tag{A.5}$$

system (A.4) is transformed into

$$\begin{cases} \operatorname{curl}\mathbf{E} - ik\mathbf{H} = 0 & \text{in } \Omega, \\ \operatorname{curl}\mathbf{H} + ik\varepsilon_{\mathrm{rel}}\mathbf{E} = 0 & \text{in } \Omega, \\ \operatorname{div}\varepsilon_{\mathrm{rel}}\mathbf{E} = 0 \quad \text{and} \quad \operatorname{div}\mathbf{H} = 0 & \text{in } \Omega. \end{cases} \tag{A.6}$$

Bibliography

[1] C. Amrouche, Ch. Bernardi, M. Dauge and V. Girault, Vector potentials in three-dimensional nonsmooth domains, *Math. Methods Appl. Sci.*, **21** (1998), 823–864.

[2] S. Balac and P. Féron, *Whispering Gallery Modes Volume Computation in Optical Micro-Spheres*, FOTON, UMR CNRS 6082, Research report, December 2014.

[3] Ch. Bernardi, M. Dauge and Y. Maday, *Spectral Methods for Axisymmetric Domains*, Series in Applied Mathematics (Paris), vol. 3, Gauthier-Villars, Éditions Scientifiques et Médicales Elsevier, Paris, 1999.

[4] M. Costabel and M. Dauge, Maxwell and Lamé eigenvalues on polyhedra, *Math. Methods Appl. Sci.*, **22** (1999), 243–258.

[5] M. Costabel and M. Dauge, Singularities of electromagnetic fields in polyhedral domains, *Arch. Ration. Mech. Anal.*, **151** (2000), 221–276.

[6] R. Courant and D. Hilbert, *Methods of Mathematical Physics. Vol. I*, Interscience Publishers, Inc., New York, N.Y., 1953.

[7] H. P. W. Gottlieb, Harmonic properties of the annular membrane, *J. Acoust. Soc. Am.*, **66** (1979), 647–650.

[8] H. P. W. Gottlieb, Eigenvalues of the Laplacian with Neumann boundary conditions, *J. Aust. Math. Soc. Ser. B*, **26** (1985), 293–309.

[9] G. W. Hanson and A. B. Yakovlev, *Operator Theory for Electromagnetics: An Introduction*, Springer-Verlag, New York, 2002.

[10] J. D. Jackson, *Classical Electrodynamics*, 3rd edition, Wiley, New York, 1998.

[11] J. R. Schulenberger, The Debye potential: a scalar factorization for Maxwell's equations, *J. Math. Anal. Appl.*, **63** (1978), 502–520.

Ralf Hiptmair and Clemens Pechstein

7 Discrete regular decompositions of tetrahedral discrete 1-forms

Abstract: For a piecewise polynomial finite element space $\mathcal{W}^1_{p,\Gamma_D}(\mathcal{T}) \subset \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ built on a mesh $\mathcal{T}$ of a Lipschitz domain $\Omega \subset \mathbb{R}^3$ and with vanishing tangential trace on $\Gamma_D \subset \partial\Omega$, a discrete regular decomposition is a *stable* splitting of elements of $\mathcal{W}^1_{p,\Gamma_D}(\mathcal{T})$ into (i) piecewise polynomial continuous vector fields on Ω, vanishing on Γ_D, (ii) gradients of piecewise polynomial continuous scalar finite element functions, and (iii) a "small" remainder. Such decompositions have turned out to be a key tool in the numerical analysis of "edge" finite element methods for variational problems in $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ that commonly occur in computational electromagnetics.

We show the existence of such decompositions for Nédélec's tetrahedral edge element spaces of any polynomial degree with stability depending only on Ω, Γ_D, and the shape regularity of the mesh. Our decompositions also respect homogeneous boundary conditions on a part of the boundary of Ω. Key tools for our construction are continuous regular decompositions, boundary-aware local co-chain projections, projection-based interpolation, and quasi-interpolation with low regularity requirements.

Keywords: Regular decomposition, edge elements, *hp*-FEM, polynomial extension, projection-based interpolation, quasi-interpolation

MSC 2010: 65N30

1 Introduction

We study an important aspect of the theory of finite element subspaces of $\boldsymbol{H}(\mathbf{curl}, \Omega)$, $\Omega \subset \mathbb{R}^3$ a bounded domain whose properties will be specified below. We restrict ourselves to spaces introduced as spaces of discrete 1-forms on simplicial meshes in finite element exterior calculus (FEEC). They are also known as edge elements and their pivotal role in the Galerkin discretization of electromagnetic boundary value problem is no longer a moot point.

The starting point are stable decompositions of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ into vector fields with components in $H^1(\Omega)$ and gradients, which have been developed as powerful tools in the theory of function spaces [8, 11, 17, 18]. We refer to them as *regular decomposi-*

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tions. In Section 2, we are going to present a particular instance. It later turned out that discrete counterparts of regular decompositions of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ are similarly useful in the numerical analysis of edge element schemes. We are going to survey a few applications and give references in Section 1.5.

Section 3 will be devoted to proving a discrete regular decomposition theorem for lowest order tetrahedral edge elements, also known as Whitney-1-forms. Compared to what was known previously, we establish enhanced stability properties also in $L^2(\Omega)$. We owe these stronger results to the use of so-called local commuting co-chain projections pioneered by Falk and Winther [27, 28]. A tailored version of those will be introduced and examined in Section 3.2.

Subsequently, in Section 4, we tackle tetrahedral discrete 1-forms of higher (uniform) polynomial degree p. For them, we can establish p-uniformly stable discrete regular decompositions, with weaker stability properties than those achievable for Whitney 1-forms, though. The key tool are commuting local projection based interpolation operators presented in Section 4.1 combined with a p-stable quasi-interpolation borrowed from [47].

The focus of this work is on numerical analysis techniques required to establish existence and properties of discrete regular decompositions. In detail, we gather, review, assemble, and, sometimes, extend theoretical results from the finite element literature, with the intention of conveying the guiding ideas and tricks underlying the proofs. The actual use of regular decompositions will be addressed only briefly in Section 1.5.

1.1 Geometric setting

Since subtle geometric arguments will play a major role for parts of the theory, we have to give a precise characterization of the geometric setting: We let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected *Lipschitz polyhedron*. Its boundary $\Gamma := \partial\Omega$, is partitioned according to $\Gamma = \Gamma_D \cup \Sigma \cup \Gamma_N$, with relatively open sets Γ_D and Γ_N. We assume that this provides a *piecewise C^1 dissection* of Γ in the sense of [31, Definition 2.2]. Sloppily speaking, this means that Σ is the union of closed curves that are piecewise C^1. This is the proper setting for the continuous regular decomposition studied in Section 2. When we look at discrete regular decompositions we further restrict the shape of Ω and Γ_D: In this case, we demand that Σ consists of disjoint closed polygons.

We triangulate Ω with a simplicial mesh $\mathcal{T}$, which will be identified with its set of tetrahedral elements: $\mathcal{T} = \{T\}$. We assume that the partitioning of the boundary Γ is resolved by the mesh. We endow edges and faces of $\mathcal{T}$ with intrinsic orientations; see Section 3.2.1.

Assumption 1.1. Both $\overline{\Gamma}_D$ and $\overline{\Gamma}_N$ are unions of closed faces of elements of $\mathcal{T}$.

We write h_T for the local mesh size, that is, the diameter of $T \in \mathcal{T}$, and r_T for the radius of the largest ball contained in T. These numbers enter the global *shape regularity measure* $\rho(\mathcal{T})$ of the mesh defined as [15], [49, Section II.4],

$$\rho(\mathcal{T}) := \max\{{}^{h_T}/_{r_T},\ T \in \mathcal{T}\}\ . \tag{1.1}$$

The symbol h will also denote a function $h \in L^\infty(\Omega)$ with $h(\mathbf{x}) := h_T$ for $\mathbf{x} \in T$, $T \in \mathcal{T}$.

1.2 Notation and function spaces

We adhere to the de-facto standard notation for function spaces in the numerical analysis literature [36, Section 2.4]. In particular, we write $H^s(D)$, $s \in \mathbb{R}$, for the Sobolev (Hilbert) space of order s on the domain D, see [50, Chapter 3]. It is endowed with the usual norm $\|\cdot\|_{s,D}$, and the semi-norm $|\cdot|_{s,D}$. We write $H^s_\Sigma(D)$, $s > \frac{1}{2}$, for the subspace with zero boundary conditions imposed on $\Sigma \subset \partial D$. Bold typeface distinguishes (spaces of) vector valued functions, e. g., $\boldsymbol{H}^s_\Sigma(D)$. The notation $\boldsymbol{H}_\Sigma(\mathbf{curl}, D)$ and $\boldsymbol{H}_\Sigma(\mathrm{div}, D)$ stand for spaces of vector fields with rotation and divergence, respectively, in $\boldsymbol{L}^2(D)$, and zero tangential/normal trace on $\Sigma \subset \partial D$. The associated norms read $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl},D)}$ and $\|\cdot\|_{\boldsymbol{H}(\mathrm{div},D)}$.

1.3 Tetrahedral discrete differential forms

Discrete differential forms provide finite element spaces of differential forms. They are studied in the new field of Finite Element Exterior Calculus (FEEC) using tools from the calculus of differential forms [34, 4, 5]. In this article, we stick to the classical calculus of vector analysis, because all developments are set in 3D Euclidean space. Yet, the differential forms background has inspired our notation: integer superscripts label spaces and operators related to differential forms of a particular degree.

We restrict ourselves to the so-called first family of simplicial discrete differential forms. It comprises the following $\mathcal{T}$-piecewise polynomial finite element spaces.

① Discrete 0-forms, continuous Lagrangian finite elements:

$$\mathcal{W}^0_{p,\Gamma_D}(\mathcal{T}) := \{v \in H^1_{\Gamma_D}(\Omega),\ v_{|T} \in \mathcal{W}^0_p(T) \quad \forall T \in \mathcal{T}\}\ ,$$
$$\mathcal{W}^0_p(T) := \mathcal{P}_{p+1}(\mathbb{R}^3)\ ,$$

② Discrete 1-forms, Nedéléc's first family of **curl**-conforming elements ("edge elements"):

$$\boldsymbol{\mathcal{W}}^1_{p,\Gamma_D}(\mathcal{T}) := \{\mathbf{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega),\ \mathbf{v}_{|T} \in \boldsymbol{\mathcal{W}}^1_p(T) \quad \forall T \in \mathcal{T}\}\ ,$$
$$\boldsymbol{\mathcal{W}}^1_p(T) := \{\boldsymbol{x} \mapsto \mathbf{p}(\boldsymbol{x}) + \mathbf{q}(\boldsymbol{x}) \times \boldsymbol{x},\ \mathbf{p}, \mathbf{q} \in \boldsymbol{\mathcal{P}}_p(\mathbb{R}^3)\}\ ,$$

③ Discrete 2-forms, div-conforming Raviart–Thomas finite elements (“face elements”):

$$\begin{aligned}\boldsymbol{\mathcal{W}}^2_{p,\Gamma_D}(\mathcal{T}) &:= \{\mathbf{v} \in \boldsymbol{H}_{\Gamma_D}(\mathrm{div},\Omega),\ \mathbf{v}_{|T} \in \boldsymbol{\mathcal{W}}^2_p(T) \quad \forall T \in \mathcal{T}\}\,,\\ \boldsymbol{\mathcal{W}}^2_p(T) &:= \{\boldsymbol{x} \mapsto \mathbf{p}(\boldsymbol{x}) + q(\boldsymbol{x})\boldsymbol{x},\ \mathbf{p} \in \boldsymbol{\mathcal{P}}_p(\mathbb{R}^3),\ q \in \mathcal{P}_p(\mathbb{R}^3)\}\,,\end{aligned}$$

④ Discrete 3-forms, discontinuous piecewise polynomials:

$$\begin{aligned}\mathcal{W}^3_p(\mathcal{T}) &:= \{v \in L^2(\Omega),\ v_{|T} \in \mathcal{W}^3_p(T) \quad \forall T \in \mathcal{T}\}\,,\\ \mathcal{W}^3_p(T) &:= \mathcal{P}_p(\mathbb{R}^3)\,.\end{aligned}$$

Here, $p \in \mathbb{N}$ stands for the polynomial degree and $\mathcal{P}_p(\mathbb{R}^3)/\boldsymbol{\mathcal{P}}_p(\mathbb{R}^3)$ for the spaces of polynomials/polynomials vector fields of degree $\le p$ in three variables. Dropping the Γ_D subscript indicates that no boundary conditions are enforced. Notice that our notation above differ from what is adopted in the seminal work [4] on FEEC, where the authors write $\mathcal{P}^-_p\Lambda^\ell(\mathcal{T})$ instead of $\mathcal{W}^\ell_p(\mathcal{T})$.

First-order differential operators related to the exterior derivative connect these spaces to a discrete de Rham complex:

$$\mathcal{K}_{\Gamma_D}(\Omega) \xrightarrow{\mathrm{Id}} \mathcal{W}^0_{p,\Gamma_D}(\mathcal{T}) \xrightarrow{\mathbf{grad}} \boldsymbol{\mathcal{W}}^1_{p,\Gamma_D}(\mathcal{T}) \xrightarrow{\mathbf{curl}} \boldsymbol{\mathcal{W}}^2_{p,\Gamma_D}(\mathcal{T}) \xrightarrow{\mathrm{div}} \mathcal{W}^3_p(\mathcal{T}) \xrightarrow{0} \{0\}. \tag{1.2}$$

Here, the space of constants is given by

$$\mathcal{K}_{\Gamma_D}(\Omega) := \{v \in H^1_{\Gamma_D}(\Omega)\colon v_{|\Omega} = \mathrm{const}\} = \begin{cases}\mathrm{span}\{1\} & \text{if } \Gamma_D = \emptyset,\\ \{0\} & \text{otherwise.}\end{cases} \tag{1.3}$$

In the complex (1.2), the range of an operator is contained in the kernel of the subsequent operator.

In the lowest-order case ($p = 0$), the elements of $\mathcal{W}^\ell_{0,\Gamma_D}(\mathcal{T})$ are called Whitney forms. In the sections devoted to these spaces, we are going to replace the subscript $p = 0$ with h and write $\mathcal{W}^\ell_{h,\Gamma_D}(\mathcal{T}) := \mathcal{W}^\ell_{0,\Gamma_D}(\mathcal{T})$.

Finally, we need spaces of vectorial continuous Lagrangian finite element functions,

$$\boldsymbol{\mathcal{V}}^0_{p,\Gamma_D}(\mathcal{T}) := [\mathcal{W}^0_{p,\Gamma_D}(\mathcal{T})]^3, \qquad \boldsymbol{\mathcal{V}}^0_{h,\Gamma_D}(\mathcal{T}) := [\mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})]^3\,. \tag{1.4}$$

1.4 Main results

Our main theorem about the discrete regular decomposition of the spaces of Whitney 1-forms (“edge elements”) involves a *local* projection operator $\boldsymbol{R}^1_D\colon \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega) \to \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$ that respects the homogeneous boundary conditions. This operator and a related one will be constructed in Section 3.2.6 below, together with several stability estimates.

Theorem 1.2 (Stable discrete regular decomposition for Whitney-1-forms in 3D). *For every discrete 1-form of the lowest-order first family* $\mathbf{v}_h \in \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$, *there exists a continuous and piecewise linear vector field* $\mathbf{z}_h \in \boldsymbol{\mathcal{V}}^0_{h,\Gamma_D}(\mathcal{T}) = [\mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})]^3$, *a continuous and piecewise linear scalar function* $\varphi_h \in \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$, *and a remainder* $\tilde{\mathbf{v}}_h \in \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$, *all depending linearly on* $\mathbf{v}_h$, *providing the* discrete regular decomposition

$$\mathbf{v}_h = \boldsymbol{R}^1_D \mathbf{z}_h + \tilde{\mathbf{v}}_h + \mathbf{grad}\,\varphi_h \ ,$$

and satisfying the norm estimates

$$\|\mathbf{z}_h\|_{0,\Omega} \le C\,\|\mathbf{v}_h\|_{0,\Omega} \ , \qquad |\mathbf{z}_h|_{1,\Omega} \le C\left(\frac{1}{d}\,\|\mathbf{v}_h\|_{0,\Omega} + \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega}\right), \tag{1.5}$$

$$|\varphi_h|_{1,\Omega} \le C\,\|\mathbf{v}_h\|_{0,\Omega} \ , \tag{1.6}$$

$$\|\tilde{\mathbf{v}}_h\|_{0,\Omega} \le C\,\|\mathbf{v}_h\|_{0,\Omega} \ , \quad \left\|h^{-1}\tilde{\mathbf{v}}_h\right\|_{0,\Omega} \le C\left(\frac{1}{d}\,\|\mathbf{v}_h\|_{0,\Omega} + \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega}\right), \tag{1.7}$$

with $d = \operatorname{diam}(\Omega)$ *and constants* $C > 0$ depending only *on the shape of* Ω, Γ_D, *and the shape regularity measure* $\rho(\mathcal{T})$.

Similar but weaker results are stated in [39, Lemma 5.1] and [41, Lemma 5.1]. These estimates did not bound the $\boldsymbol{L}^2(\Omega)$-norm of $\mathbf{z}_h$ by the $\boldsymbol{L}^2(\Omega)$-norm of $\mathbf{v}_h$. The proof of Theorem 1.2 is given in Section 3 and it will demonstrate the substantial additional effort required to establish stability in $\boldsymbol{L}^2(\Omega)$.

The next result presents a "p-version" counterpart of Theorem 1.2, because it targets spaces of discrete 1-forms with *arbitrary* polynomial degree p with a focus on p-uniform stability estimates.

Theorem 1.3 (Discrete regular decomposition for discrete 1-forms). *For every discrete 1-form of the first family* $\mathbf{v}_p \in \boldsymbol{\mathcal{W}}^1_{p,\Gamma_D}(\mathcal{T})$, $p \in \mathbb{N}_0$, *there exists a continuous vector field* $\mathbf{z}_p \in \boldsymbol{\mathcal{V}}^0_{p,\Gamma_D}(\mathcal{T}) \subset \boldsymbol{H}^1_{\Gamma_D}(\Omega)$, $\mathcal{T}$*-piecewise polynomial of degree* $\le p+1$, *a continuous,* $\mathcal{T}$*-piecewise polynomial scalar function* $\varphi_p \in \mathcal{W}^0_{p,\Gamma_D}(\mathcal{T})$, *and a remainder* $\tilde{\mathbf{v}}_p \in \boldsymbol{\mathcal{W}}^1_{p,\Gamma_D}(\mathcal{T})$,

(I) *all depending linearly on* $\mathbf{v}_p$*;*

(II) *satisfying the norm estimates*

$$\left\|\mathbf{z}_p\right\|_{0,\Omega} \le C\left\|\mathbf{v}_p\right\|_{0,\Omega} \ , \quad \left|\mathbf{z}_p\right|_{1,\Omega} \le C\left(\frac{1}{d}\,\|\mathbf{v}_h\|_{0,\Omega} + \left\|\mathbf{curl}\,\mathbf{v}_p\right\|_{0,\Omega}\right), \tag{1.8}$$

$$\left|\varphi_p\right|_{1,\Omega} \le C\left(\left\|\mathbf{v}_p\right\|_{0,\Omega} + \max_{T\in\mathcal{T}}\left\{(1+\log(p+1))^{3/2}\frac{h_T}{p}\right\}\left\|\mathbf{curl}\,\mathbf{v}_p\right\|_{0,\Omega}\right), \tag{1.9}$$

$$\left(\sum_{T\in\mathcal{T}}\left\|\frac{p+1}{h_T}\tilde{\mathbf{v}}_p\right\|^2_{0,T}\right)^{1/2} \le C(1+\log(p+1))^{3/2}\left(\frac{1}{d}\,\left\|\mathbf{v}_p\right\|_{0,\Omega} + \left\|\mathbf{curl}\,\mathbf{v}_p\right\|_{0,\Omega}\right), \tag{1.10}$$

with $d := \operatorname{diam}(\Omega)$ *and constants* $C > 0$ depending only *on the shape of* Ω, Γ_D, *and the shape regularity measure* $\rho(\mathcal{T})$*;*

(III) *and providing the* discrete regular decomposition

$$\mathbf{v}_p = \Pi_p^1 \mathbf{z}_p + \tilde{\mathbf{v}}_p + \mathbf{grad}\,\varphi_p \ ,$$

where $\Pi_p^1 : \mathcal{V}^0_{p,\Gamma_D}(\mathcal{T}) \to \mathcal{W}^1_{p,\Gamma_D}(\mathcal{T})$ *is a strictly local linear interpolation operator.*

This result has no precursor in the literature. Its obvious shortcoming is the restriction to a uniform polynomial degree p. More desirable would be a version admitting variable polynomial degree, and thus, encompassing finite element spaces created by *hp*-refinement; see [2]. However, there is a single technical obstacle that has prevented us from admitting variable p; refer to Remark 4.17.

Another class of results on discrete regular decompositions beyond the scope of the above two theorems addresses stability estimates with non-constant positive weight functions entering the norms. Currently (2017), this is an area of active research and first results for piecewise constant weight functions are reported in [46, 44, 45].

1.5 Applications

The discrete regular decompositions of the kind provided by Theorem 1.2 have turned out to be a powerful tool for the numerical analysis of various aspects of edge finite element methods. We emphasize their role as *theoretical tool*, because there is not a single algorithm, which relies on the actual computation of the finite element functions comprising a discrete regular decomposition. The following, probably incomplete, list mentions a few pieces of research in numerical analysis, where h-version discrete regular decompositions played a pivotal role:

- Analysis of geometric multigrid methods for $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$-elliptic variational problems discretized by means of edge elements [41, 35, 62]: Here, discrete regular decompositions allow to harness results on the stability of multilevel nodal decompositions of $\mathcal{V}^0_1(\mathcal{T})$.
- Convergence theory of domain decomposition methods for discrete $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$-elliptic variational problems [55, 25, 24, 46, 42, 43, 45]: In the same vein as multigrid theory, these approaches manage to exploit results for Lagrangian finite elements and $H^1(\Omega)$-elliptic variational problems.
- Foundation of nodal auxiliary space preconditioners [40, 39, 48]: the stable discrete regular decomposition directly spawns a subspace correction method for discrete $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$-elliptic variational problems whose key step amounts to the solution of scalar elliptic boundary value problems.
- Analysis of geometric auxiliary space methods for edge elements [38].
- Reliability estimates for residual based local error estimators for $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$-elliptic variational problems [23, 13, 58].

2 Continuous regular decomposition

It goes without saying that all results about discrete regular decompositions have their roots in stability properties of continuous regular decompositions of the function space $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$. Now we state and prove the corresponding key estimates. For ease of presentation, we set $\operatorname{diam}(\Omega) = 1$ throughout the remainder of this manuscript. Simple scaling arguments will then produce the more general estimates of Theorem 1.2 and Theorem 1.3.

The following result can essentially be found in [41, 32], except that we also assert extra L^2-stability. Note that there are neither restrictions on the topology of Ω nor on the connectedness of the Dirichlet boundary Γ_D. A more general version of the theory can be found in [56].

Theorem 2.1 (Boundary aware regular decomposition). *For each* $\mathbf{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ *there exists a vector field* $\mathbf{z} \in \boldsymbol{H}^1_{\Gamma_D}(\Omega)$ *and a scalar function* $\varphi \in H^1_{\Gamma_D}(\Omega)$ *depending linearly on* $\mathbf{v}$ *such that*

$$\mathbf{v} = \mathbf{z} + \mathbf{grad}\,\varphi,$$

and

$$\|\mathbf{z}\|_{0,\Omega} \leq C\,\|\mathbf{v}\|_{0,\Omega}\,, \qquad |\mathbf{z}|_{1,\Omega} \leq C\,\|\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\,, \tag{2.1}$$

$$\|\varphi\|_{1,\Omega} \leq C\,\|\mathbf{v}\|_{0,\Omega}\,, \tag{2.2}$$

with constants independent of $\mathbf{v}$.

For the proof, we need a few auxiliary results that will be provided in the next three sections.

2.1 Collars and bulges

Under the assumptions on Ω made in Section 1.1, [31, Lemma 4.4] guarantees the existence of an open Lipschitz neighborhood Ω_Γ ("Lipschitz collar") of $\Gamma := \partial\Omega$ and of a smooth vector field $\tilde{\boldsymbol{n}} \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ with $\|\tilde{\boldsymbol{n}}\| \equiv 1$ on Ω_Γ that is *transversal* to Γ:

$$\exists\kappa > 0: \quad \tilde{\boldsymbol{n}}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) \geq \kappa \quad \text{for almost all } \boldsymbol{x} \in \Gamma\,. \tag{2.3}$$

Extrusion of Γ_D by the local flow induced by $\tilde{\boldsymbol{n}}$ spawns the "bulge" $\Upsilon_D \subset \Omega_\Gamma \setminus \Omega$; see Figure 7.1. We recall the properties of bulge domains from [31, Section 2].

Theorem 2.2 (Bulge-augmented domain). *There exists a* Lipschitz domain $\Upsilon_D \subset \mathbb{R}^3 \setminus \overline{\Omega}$, *such that* $\overline{\Upsilon}_D \cap \overline{\Omega} = \Gamma_D$, $\Omega^e := \Upsilon_D \cup \Gamma_D \cup \Omega$ *is Lipschitz,* $\operatorname{diam}(\Omega^e) \leq 2$, *and* $\overline{\Upsilon}_D \subset \Omega_\Gamma$.

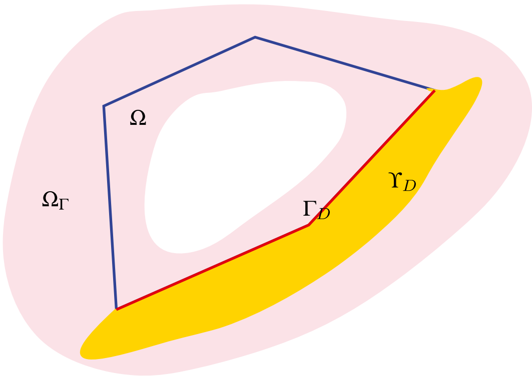

Figure 7.1: Collar domain Ω_Γ (pink) and bulge domain Υ_D (gold).

Remark 2.3. If Γ_D has several components Γ_k, $k = 1, \ldots, N$, then each of them gives rise to a separate bulge Υ_k with $\overline{\Upsilon}_k \cap \overline{\Omega} = \Gamma_k$, and the individual bulges have positive distance from each other. This is a consequence of our assumptions on Γ and has to be kept in mind though we are not going to mention this fact explicitly in the sequel.

2.2 Extension operators

Lemma 2.4 ([60]). *Let $\mathcal{D}$ be a bounded Lipschitz domain with* $\operatorname{diam}(\mathcal{D}) = 1$. *Then there exists a bounded linear extension operator $E_{\mathcal{D}}: L^2(\mathcal{D}) \to L^2(\mathbb{R}^3)$ such that for $k \in \mathbb{N}_0$,*

$$\|E_{\mathcal{D}} v\|_{k,\mathcal{D}} \leq C\|v\|_{k,\mathcal{D}} \qquad \forall v \in H^k(\mathcal{D}), \tag{2.4}$$

with C depending only on $\mathcal{D}$ and k. Moreover, $E_{\mathcal{D}} v$ has compact support in $\mathbb{R}^3$.

We apply this fundamental result to the bulge domain Υ_D introduced in Section 2.1.

Corollary 2.5. *There exists an extension operator $E^{(2)}_{\Upsilon_D}: L^2(\Upsilon_D) \to L^2(\mathbb{R}^3)$ such that for $k \in \mathbb{N}_0$,*

$$\|E^{(2)}_{\Upsilon_D} v\|_{k,\mathbb{R}^3} \leq C\|v\|_{k,\Upsilon_D} \qquad \forall v \in H^k(\Upsilon_D), \tag{2.5}$$

where the constant C depends on Ω, Υ_D, and k.

Lemma 2.6. *For a Lipschitz domain $\mathcal{D}$ with* $\operatorname{diam}(\mathcal{D}) = 1$ *there exists a bounded linear extension operator $E^{\mathbf{curl}}_{\mathcal{D}}: \boldsymbol{L}^2(\mathcal{D}) \to \boldsymbol{L}^2(\mathbb{R}^3)$ such that, with constants depending only on $\mathcal{D}$,*

$$\begin{aligned} \|E^{\mathbf{curl}}_{\mathcal{D}} \mathbf{v}\|_{0,\mathbb{R}^3} &\leq C\|\mathbf{v}\|_{0,\mathcal{D}} && \forall \mathbf{v} \in \boldsymbol{L}^2(\mathcal{D}), \\ \|E^{\mathbf{curl}}_{\mathcal{D}} \mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)} &\leq C\|\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\mathcal{D})} && \forall \mathbf{v} \in \boldsymbol{H}(\mathbf{curl}, \mathcal{D}). \end{aligned}$$

Moreover, $E^{\mathbf{curl}}_{\mathcal{D}} \mathbf{v}$ has compact support in $\mathbb{R}^3$.

Proof. Since $\mathcal{D}$ is (strong) Lipschitz, it is also *weak* Lipschitz, and so the Lipschitz collar is locally the image of the unit cube under a bi-Lipschitz mapping such that the exterior is mapped to the upper half-space [49, Section VII.1]. On the cube, we define the extension of $\mathbf{w}(x_1, x_2, x_3)$ as $\operatorname{diag}(1, 1, -1)\mathbf{w}(x_1, x_2, -x_3)$. Mapping back to the collar and using a partition of unity, one obtains the desired result, since the bi-Lipschitz mapping preserves the **curl**-operator. □

We note that a similar result with higher order **curl**-derivatives (but not with the pure L^2-stability) has been shown in [37].

2.3 A Fourier-based projection

The next lemma builds on similar results from [3, Lemma 3.5], [36, Lemma 2.5], and [37, Lemma 5.1].

Lemma 2.7. *There exists a bounded linear operator* $\mathsf{L}_{\mathbf{curl}}: \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \to \boldsymbol{H}^1(\mathbb{R}^3)$ *such that for all* $\mathbf{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$

(L_1) $\mathbf{curl}\, \mathsf{L}_{\mathbf{curl}}\mathbf{v} = \mathbf{curl}\ \mathbf{v}$;

(L_2) $\operatorname{div} \mathsf{L}_{\mathbf{curl}}\mathbf{v} = 0$;

(L_3) $\|\mathsf{L}_{\mathbf{curl}}\mathbf{v}\|_{0,\mathbb{R}^3} \leq \|\mathbf{v}\|_{0,\mathbb{R}^3}$ *and* $\|(I - \mathsf{L}_{\mathbf{curl}})\mathbf{v}\|_{0,\mathbb{R}^3} \leq \|\mathbf{v}\|_{0,\mathbb{R}^3}$;

(L_4) $\|\nabla \mathsf{L}_{\mathbf{curl}}\mathbf{v}\|_{0,\mathbb{R}^3} \leq \|\mathbf{curl}\,\mathbf{v}\|_{0,\mathbb{R}^3}$;

(L_5) $\mathsf{L}^2_{\mathbf{curl}}\mathbf{v} = \mathsf{L}_{\mathbf{curl}}\mathbf{v}$, *i. e.,* $\mathsf{L}_{\mathbf{curl}}$ *is a projection.*

In the statement (L_4), ∇ applied to a vector field yields the Jacobian.

Proof. The proof is classical; see, e. g., [29, Chapter I, Theorem 3.4] and [55, Lemma 2.1]. Let $\widehat{\mathbf{v}}(\boldsymbol{\xi}) := (\mathcal{F}\mathbf{v})(\boldsymbol{\xi}) := \int_{\mathbb{R}^3} e^{-2\pi \mathrm{i}\mathbf{x}\cdot\boldsymbol{\xi}}\mathbf{v}(\mathbf{x})\, d\mathbf{x}$ denote the (component-wise) Fourier transform of $\mathbf{v} \in \boldsymbol{L}^2(\mathbb{R}^3)$. Recall that $\partial_k \mathbf{v}$, $\mathbf{curl}\,\mathbf{v}$, $\operatorname{div}\ \mathbf{v}$ correspond to $2\pi\mathrm{i}\boldsymbol{\xi}_k\widehat{\mathbf{v}}$, $2\pi\mathrm{i}\boldsymbol{\xi} \times \widehat{\mathbf{v}}$, and $2\pi\mathrm{i}\boldsymbol{\xi} \cdot \widehat{\mathbf{v}}$, respectively. We set

$$\mathsf{L}_{\mathbf{curl}}\mathbf{v} := \mathcal{F}^{-1}\widehat{\mathbf{w}}, \qquad \text{with } \widehat{\mathbf{w}}(\boldsymbol{\xi}) := -|\boldsymbol{\xi}|^{-2}(\boldsymbol{\xi} \times \boldsymbol{\xi} \times \widehat{\mathbf{v}}(\boldsymbol{\xi})).$$

Elementary properties of $\widehat{\mathbf{w}} \in \boldsymbol{L}^2(\mathbb{R}^3)$ yield most of the assertions: (L_1) from $2\pi\boldsymbol{\xi} \times \widehat{\mathbf{w}} = 2\pi\boldsymbol{\xi} \times \widehat{\mathbf{v}}$. ($L_2$) from $2\pi\boldsymbol{\xi} \cdot \widehat{\mathbf{w}} = 0$. ($L_3$) from $|\widehat{\mathbf{w}}| \leq |\widehat{\mathbf{v}}|$, because due to Plancherel's theorem,

$$\left.\begin{aligned} \|\mathsf{L}_{\mathbf{curl}}\mathbf{v}\|_{0,\mathbb{R}^3} &= \|\widehat{\mathbf{w}}\|_{0,\mathbb{R}^3} \\ \|(I - \mathsf{L}_{\mathbf{curl}})\mathbf{v}\|_{0,\mathbb{R}^3} &= \|\underbrace{\widehat{\mathbf{v}} + |\boldsymbol{\xi}|^{-2}\boldsymbol{\xi} \times \boldsymbol{\xi} \times \widehat{\mathbf{v}}}_{=|\boldsymbol{\xi}|^{-2}(\widehat{\mathbf{v}}\cdot\boldsymbol{\xi})\boldsymbol{\xi}}\|_{0,\mathbb{R}^3} \end{aligned}\right\} \leq \|\widehat{\mathbf{v}}\|_{0,\mathbb{R}^3} = \|\mathbf{v}\|_{0,\mathbb{R}^3}.$$

(L_4) is obtained as follows:

$$\|\nabla \mathsf{L}_{\mathbf{curl}}\mathbf{v}\|^2_{0,\mathbb{R}^3} = \sum_{k=1}^{3} \|2\pi\mathrm{i}\xi_k\widehat{\mathbf{w}}\|^2_{0,\mathbb{R}^3} \leq \sum_{k=1}^{3} \left\| \frac{|\xi_k|^2}{|\boldsymbol{\xi}|^2} 2\pi\mathrm{i}\boldsymbol{\xi} \times \widehat{\mathbf{v}} \right\|^2_{0,\mathbb{R}^3} \leq \|\mathbf{curl}\,\mathbf{v}\|^2_{0,\mathbb{R}^3}.$$

The last estimate shows that indeed $\mathsf{L}_{\mathbf{curl}}\mathbf{v} \in \boldsymbol{H}^1(\mathbb{R}^3)$. ($L_5$) is checked easily. □

2.4 Proof of Theorem 2.1

We follow the proof as in [41, Theorem 5.9] and establish the L^2-stability using the ideas from [55, Lemma 2.2]. Let $\mathbf{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ be arbitrary but fixed.

Step 1: We extend $\mathbf{v}$ by zero to a function in $\boldsymbol{H}(\mathbf{curl}, \Omega^e)$, where Ω^e is the extended domain from Section 2.1 and then to $\tilde{\mathbf{v}} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$ using $E_{\Omega^e}^{\mathbf{curl}}$. We observe $\tilde{\mathbf{v}}_{|\Upsilon_D} \equiv 0$ and that Lemma 2.6 implies

$$\|\tilde{\mathbf{v}}\|_{0,\mathbb{R}^3} \leq C\|\mathbf{v}\|_{0,\Omega}, \quad \|\mathbf{curl}\,\tilde{\mathbf{v}}\|_{0,\mathbb{R}^3} \leq C\,\|\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\,. \tag{2.6}$$

Step 2: Let $B \supseteq \Omega^e$ be a ball such that $1 \leq \operatorname{diam}(B) \leq 2$ and define

$$\mathbf{w} := (\mathsf{L}_{\mathbf{curl}}\tilde{\mathbf{v}})_{|B}.$$

Due to (L_1) of Lemma 2.7, $\mathbf{curl}\,\mathbf{w} = \mathbf{curl}\,\tilde{\mathbf{v}}$ in B. Since B is simply connected, there exists a scalar potential $\psi \in H^1(B)$ with zero average $\int_B \psi\, d\mathbf{x} = 0$ such that

$$\tilde{\mathbf{v}} = \mathbf{w} + \mathbf{grad}\,\psi.$$

Lemma 2.7 together with (2.6) implies

$$\begin{aligned}
\|\mathbf{w}\|_{0,B} &= \|\mathsf{L}_{\mathbf{curl}}\tilde{\mathbf{v}}\|_{0,B} \leq \|\tilde{\mathbf{v}}\|_{0,\mathbb{R}^3} \leq C\|\mathbf{v}\|_{0,\Omega}\,,\\
\|\mathbf{grad}\,\psi\|_{0,B} &= \|(I - \mathsf{L}_{\mathbf{curl}})\tilde{\mathbf{v}}\|_{0,B} \leq \|\tilde{\mathbf{v}}\|_{0,\mathbb{R}^3} \leq C\|\mathbf{v}\|_{0,\Omega}\,,\\
\|\nabla\mathbf{w}\|_{0,B} &\leq \|\mathbf{curl}\,\tilde{\mathbf{v}}\|_{0,\mathbb{R}^3} \leq C\|\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\,,\\
\|\psi\|_{0,B} &\leq C\|\mathbf{grad}\,\psi\|_{0,B} \leq C\|\mathbf{v}\|_{0,\Omega}\,,
\end{aligned} \tag{2.7}$$

where in the last estimate we have used Poincaré's inequality on the convex ball B [6].

Step 3: Since

$$0 = \mathbf{w} + \mathbf{grad}\,\psi \quad \text{in } \Upsilon_D\,,$$

we conclude that $\psi_{|\Upsilon_D} \in H^2(\Upsilon_D)$. We define $\tilde{\psi} := (E_{\Upsilon_D}^{(2)}\psi)_{|B} \in H^2(B)$. From Corollary 2.5, we obtain

$$\begin{aligned}
\|\tilde{\psi}\|_{0,B} &\leq C\|\psi\|_{0,\Upsilon_D} \leq C\|\mathbf{v}\|_{0,\Omega}\,,\\
\|\mathbf{grad}\,\tilde{\psi}\|_{0,B} &\leq C\|\psi\|_{1,\Upsilon_D} \leq C\|\mathbf{grad}\,\psi\|_{0,B} \leq C\|\mathbf{v}\|_{0,\Omega}\,,\\
\|\nabla\,\mathbf{grad}\,\tilde{\psi}\|_{0,B} &\leq C\big(\|\underbrace{\nabla\,\mathbf{grad}\,\psi}_{=-\nabla\mathbf{w}}\|_{0,\Upsilon_D}^2 + \|\psi\|_{1,\Upsilon_D}^2\big)^{1/2} \leq C\|\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\,,
\end{aligned} \tag{2.8}$$

where $\nabla\,\mathbf{grad}$ indicates the Hessian.

Step 4: In B, it holds that

$$\tilde{\mathbf{v}} = \mathbf{w} + \mathbf{grad}\,\psi = \underbrace{\mathbf{w} + \mathbf{grad}\,\tilde{\psi}}_{=:\mathbf{z}\in\boldsymbol{H}^1} + \mathbf{grad}(\underbrace{\psi - \tilde{\psi}}_{=:\varphi\in H^1}).$$

It is easy to see that $\varphi = 0$ in Υ_D and so $\varphi \in H^1_{\Gamma_D}(\Omega)$. Correspondingly, $\mathbf{grad}\,\varphi = 0$ and $\tilde{\mathbf{v}} = 0$ in Υ_D, and so $\mathbf{z} \in \boldsymbol{H}^1_{\Gamma_D}(\Omega)$. Combining (2.7) and (2.8) yields the desired estimates for $\mathbf{z}$ and p.

3 Discrete regular decomposition: lowest-order case

Now we tackle the proof of Theorem 1.2. We employ an extended version of the local projectors invented by Falk and Winther in [27]; see also [28]. Our extension is aimed at enforcing compliance with the boundary conditions on Γ_D and the sophisticated technical details will be elaborated in Section 3.2. With this tool at our disposal, the proof of Theorem 1.2 can be done in a few simple steps as we are going to demonstrate in Section 3.3.

3.1 Outline: proof by projection

As a little preview, we give a sketch of the proof, in order to motivate the need for the operators we construct step by step in the sequel. Let $\boldsymbol{R}_D^1 : \boldsymbol{L}^2(\Omega) \to \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$ be a projector onto the finite element space. Given $\mathbf{v}_h \in \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$, we apply the projector property as well as the *continuous* regular decomposition from Theorem 2.1:

$$\mathbf{v}_h = \boldsymbol{R}_D^1 \mathbf{v}_h = \boldsymbol{R}_D^1 \mathbf{z} + \boldsymbol{R}_D^1 \,\mathbf{grad}\, \varphi. \tag{3.1}$$

In the following, we also construct a companion operator $R_D^0 : H^1(\Omega) \to \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$ which *commutes* with $\boldsymbol{R}_D^1$ under the gradient operator, such that we obtain

$$\mathbf{v}_h = \boldsymbol{R}_D^1 \mathbf{z} + \mathbf{grad}\, \underbrace{R_D^0 \varphi}_{=:\varphi_h}. \tag{3.2}$$

As regards Theorem 1.2, it will be essential that R_D^0 and $\boldsymbol{R}_D^1$ respect the homogeneous Dirichlet boundary conditions on Γ_D.

The function $\mathbf{z}$ above is in $\mathbf{H}^1(\Omega)$ but not in the finite element space. This is why we have to introduce a third term into the splitting by means of a *Clèment-type operator* $\boldsymbol{M}_D^0 : \boldsymbol{L}^2(\Omega) \to \boldsymbol{\mathcal{V}}^0_{h,\Gamma_D}(\mathcal{T}) = (\mathcal{W}^0_{h,\Gamma_D}(\mathcal{T}))^3$ (defined component-wise), that, again, respects homogeneous boundary conditions. We then obtain

$$\mathbf{v}_h = \boldsymbol{R}_D^1 \underbrace{\boldsymbol{M}_D^0 \mathbf{z}}_{=:\mathbf{z}_h} + \underbrace{\boldsymbol{R}_D^1 (I - \mathbf{M}_D^0)\mathbf{z}}_{=:\tilde{\mathbf{v}}_h} + \mathbf{grad}\, \underbrace{R_D^0 \varphi}_{=:\varphi_h}. \tag{3.3}$$

The norm estimates from Theorem 1.2 require a series of stability properties of the operators R_D^0, $\boldsymbol{R}_D^1$, and $\boldsymbol{M}_D^0$, in particular L^2-stability.

Instead of the operator $\mathbf{R}_D^1$ above, a simple *interpolation* operator Π_h^1 mapping to the Nèdèlec space $\mathcal{W}_h^1(\mathcal{T})$ can be used. Clearly, $\mathbf{\Pi}_h^1$ is a projector too, but its domain of definition is a genuine subspace of $\boldsymbol{H}(\mathbf{curl}, \Omega)$, and its stability properties are fairly different from those of $\mathbf{R}_D^1$. Nevertheless, when applying the projection property and the continuous regular decomposition $\mathbf{v}_h = \mathbf{z} + \mathbf{grad}\, \varphi$, one obtains

$$\mathbf{v}_h = \mathbf{\Pi}_h^1 \mathbf{z} + \mathbf{\Pi}_h^1 \,\mathbf{grad}\, \varphi = \mathbf{\Pi}_h^1 \mathbf{z} + \mathbf{grad}\, \Pi_h^0 \varphi, \tag{3.4}$$

using the commuting property with the nodal interpolation operator Π_h^0. However, one must make sure that all the terms are well-defined. Firstly, since $\mathbf{curl}\,\mathbf{z} = \mathbf{curl}\,\mathbf{v}_h$ and $\mathbf{curl}\,\mathbf{grad}\,\varphi = 0$, the terms $\mathbf{\Pi}_h^1\mathbf{z}$ and $\mathbf{\Pi}_h^1\,\mathbf{grad}\,\varphi$ are well-defined; cf. [36, Lemma 4.6]. Secondly, since $\mathbf{grad}\,\varphi = \mathbf{v}_h - \mathbf{z}$ is in H^1, piecewise on each element of the mesh, we can conclude that φ is piecewise in H^2. Therefore, its point evaluation at the nodes of the mesh is well-defined and so is $\Pi_h^0\varphi$. Finally, using the Clément-type quasi-interpolation operator $\mathbf{M}_D^0$, we obtain

$$\mathbf{v}_h = \mathbf{\Pi}_h^1 \underbrace{\mathbf{M}_D^0\mathbf{z}}_{=:\mathbf{z}_h} + \underbrace{\mathbf{\Pi}_h^1(\mathbf{z} - \Pi_h^1\mathbf{z})}_{=:\tilde{\mathbf{v}}_h} + \mathbf{grad}\,\underbrace{\Pi_h^0\varphi}_{=:\varphi_h}\,. \tag{3.5}$$

The above alternative strategy will be used in Section 4 to tackle the proof of Theorem 1.3 for the p-version. However, due to poorer stability properties $\mathbf{\Pi}_h^1$, the resulting stability estimates will be weaker. In particular, we cannot prove pure L^2 estimates.

Remark 3.1. The first such three-term splitting in the literature can be found in [38]. Shortly later, improved versions were given in [39, Lemma 5.1] and [41, Lemma 5.1]. The arguments therein are slightly different. Instead of using that φ is piecewise in H^2, it is shown that $\mathbf{curl}(\mathbf{z} - \Pi_h^1\mathbf{z}) = 0$. From the additional property that the integral of $\mathbf{z} - \Pi_h^1\mathbf{z}$ over each edge of the mesh vanishes, one can conclude that there exists $q \in H^1(\Omega)$ with $\mathbf{z} - \Pi_h^1\mathbf{z} = \mathbf{grad}\,q$, cf. [41, Lemma 2.3]. Summarizing,

$$\mathbf{v}_h = \Pi_h^1\mathbf{z} + \underbrace{(\mathbf{z} - \Pi_h^1\mathbf{z})}_{=\mathbf{grad}\,q} + \mathbf{grad}\,\varphi = \Pi_h^1\mathbf{z} + \mathbf{grad}(\varphi + q).$$

Since $\mathbf{grad}(\varphi + q) \in \mathcal{W}_{p,\Gamma_D}^1(\mathcal{T})$, we conclude from the discrete de Rham complex that $\varphi + q \in \mathcal{W}_{p,\Gamma_D}^0(\mathcal{T})$ and we can set $\varphi_h = \varphi + q$. Indeed, comparing with the splitting (3.4), we see that

$$\mathbf{grad}(\varphi + q) = \mathbf{grad}\,\varphi + \mathbf{z} - \Pi_h^1\mathbf{z} = \mathbf{v}_h - \Pi_h^1 z \overset{(3.4)}{=} \mathbf{grad}\,\varphi_h\,.$$

3.2 Local bounded boundary-aware co-chain projections

In this section, we construct two sets of operators parallel to developments in [27], from where we have also borrowed a good deal of the notation. The first one are modified Clément-type operators $M_D^0 : L^2(\Omega) \to \mathcal{W}_{h,\Gamma_D}^0(\mathcal{T})$ and $\boldsymbol{M}_D^1 : \boldsymbol{L}^2(\Omega) \to \boldsymbol{\mathcal{W}}_{h,\Gamma_D}^1(\mathcal{T})$ that commute with the gradient on $H_{\Gamma_D}^1(\Omega)$:

$$\begin{array}{ccc} H_{\Gamma_D}^1(\Omega) & \overset{\mathbf{grad}}{\longrightarrow} & \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega) \\ \downarrow M_D^0 & & \downarrow \boldsymbol{M}_D^1 \\ \mathcal{W}_{h,\Gamma_D}^0(\mathcal{T}) & \overset{\mathbf{grad}}{\longrightarrow} & \boldsymbol{\mathcal{W}}_{h,\Gamma_D}^1(\mathcal{T}) \end{array} \tag{3.6}$$

The operators feature also some of the local stability and approximation properties of the classical Clément quasi-interpolant [16]; see below. The second class of operators are so-called bounded co-chain projections, originally introduced by Falk and Winther [27]. The operators are defined on the spaces of the de Rham complex, they are projections onto spaces of discrete differential forms, commute with the exterior derivative, and are locally defined. Here, we modify two of these operators, in the sequel called R_D^0 and $\boldsymbol{R}_D^1$ such that they additionally respect homogeneous boundary conditions. We have the commuting diagram

$$\begin{array}{ccc} H^1_{\Gamma_D}(\Omega) & \overset{\textbf{grad}}{\longrightarrow} & \boldsymbol{H}_{\Gamma_D}(\textbf{curl},\Omega) \\ \downarrow R_D^0 & & \downarrow \boldsymbol{R}_D^1 \\ \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T}) & \overset{\textbf{grad}}{\longrightarrow} & \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T}), \end{array} \tag{3.7}$$

where opposed to (3.6), the operators are projectors.

3.2.1 Notation and assumptions

We need a little more notation for the subsequent construction. Let $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{F}$ denote the set of vertices, edges, and faces (resp.) of the mesh $\mathcal{T}$. We also introduce the sets $\mathcal{V}_f := \{\mathsf{v} \in \mathcal{V}: \mathsf{v} \notin \overline{\Gamma_D}\}$, $\mathcal{E}_f := \{\mathsf{e} \in \mathcal{E}: \mathsf{e} \not\subset \overline{\Gamma_D}\}$, and $\mathcal{F}_f := \{\mathsf{f} \in \mathcal{F}: \mathsf{f} \not\subset \overline{\Gamma_D}\}$ of "free" vertices, edges, and faces, respectively. Let φ_v denote the nodal vertex basis function fulfilling $\varphi_\mathsf{v}(\mathsf{v}') = \delta_{\mathsf{v}\mathsf{v}'}$ for $\mathsf{v}, \mathsf{v}' \in \mathcal{V}$. Edges and faces have to be oriented: For an edge $\mathsf{e} = [\mathsf{e}_1, \mathsf{e}_2]$ with endpoints e_1, $\mathsf{e}_2 \in \mathcal{V}$, the orientation is given by the unit tangent $\boldsymbol{\tau}_\mathsf{e} := (\mathsf{e}_2-\mathsf{e}_1)/|\mathsf{e}_2-\mathsf{e}_1|$. The orientation of a face $\mathsf{f} \in \mathcal{F}$ is provided by the unit normal $\mathbf{n}_\mathsf{f}$. By $\boldsymbol{\psi}_\mathsf{e} \in \boldsymbol{\mathcal{W}}^1_h(\mathcal{T})$ and $\boldsymbol{\zeta}_\mathsf{f} \in \mathcal{W}^2_h(\mathcal{T})$ we denote the Nédélec edge and face basis functions, fulfilling $\int_{\mathsf{e}'} \boldsymbol{\psi}_\mathsf{e} \cdot \boldsymbol{\tau}_\mathsf{e}\, ds = \delta_{\mathsf{e}\mathsf{e}'}$ for $\mathsf{e}, \mathsf{e}' \in \mathcal{E}$ and $\int_{\mathsf{f}'} \boldsymbol{\zeta}_\mathsf{f} \cdot \mathbf{n}_{\mathsf{f}'}\, ds = \delta_{\mathsf{f}\mathsf{f}'}$ for $\mathsf{f}, \mathsf{f}' \in \mathcal{F}$. We find that

$$\begin{aligned} \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T}) &= \operatorname{span}\{\varphi_\mathsf{v}\}_{\mathsf{v}\in\mathcal{V}_f}, \\ \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T}) &= \operatorname{span}\{\boldsymbol{\psi}_\mathsf{e}\}_{\mathsf{e}\in\mathcal{E}_f}, \\ \boldsymbol{\mathcal{W}}^2_{h,\Gamma_D}(\mathcal{T}) &= \operatorname{span}\{\boldsymbol{\zeta}_\mathsf{f}\}_{\mathsf{f}\in\mathcal{F}_f}. \end{aligned}$$

Finally, for a vertex $\mathsf{v} \in \mathcal{V}$, its node patch ω_v is defined by

$$\omega_\mathsf{v} := \bigcup_{T\in\mathcal{T}:\mathsf{v}\in\overline{T}} \overline{T}.$$

For an edge $\mathsf{e} = [\mathsf{e}_1, \mathsf{e}_2] \in \mathcal{E}$ and a triangular face $\mathsf{f} = [\mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_2] \in \mathcal{F}$, the corresponding patches are given by

$$\omega_\mathsf{e} = \omega_{\mathsf{e}_1} \cup \omega_{\mathsf{e}_2}, \qquad \omega_\mathsf{f} = \omega_{\mathsf{f}_1} \cup \omega_{\mathsf{f}_2} \cup \omega_{\mathsf{f}_3}.$$

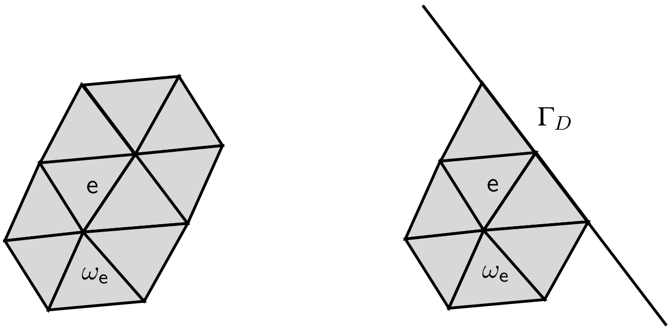

Figure 7.2: Sketch of edge patches.

See Figure 7.2 for a sketch of two edge patches. Finally, the element patch corresponding to an element $T \in \mathcal{T}$ is given by

$$\omega_T = \bigcup_{\mathsf{v} \in \mathcal{V} \cap \overline{T}} \omega_{\mathsf{v}} .$$

For a patch $\omega \subset \Omega$ of elements of $\mathcal{T}$, we will frequently use the space $H^1_{\Gamma_D}(\omega) := \{u \in H^1(\omega) \colon u_{|\Gamma_D} = 0\}$. If $\text{meas}_2(\partial\omega \cap \Gamma_D) = 0$ the functions in this space fulfill no boundary condition.

The following, technical assumption is fulfilled for standard meshes.

Assumption 3.2. For each vertex $\mathsf{v} \in \mathcal{V}$, edge $\mathsf{e} \in \mathcal{E}$, and face $\mathsf{f} \in \mathcal{F}$, the vertex patch ω_{v}, edge patch ω_{e}, and face patch ω_{f}, respectively, is simply connected and has a simply connected boundary.

The following results will be helpful in the development of our theory later on.

Lemma 3.3. *Let* $\mathsf{e} = [\mathsf{e}_1, \mathsf{e}_2] \in \mathcal{E}_f$ *with* $\mathsf{e}_1 \in \overline{\Gamma_D}$ *(or* $\mathsf{e}_2 \in \overline{\Gamma_D}$*). Then there exists a face* $\mathsf{f} \subset \partial\omega_{\mathsf{e}} \cap \Gamma_D$ *with* $\mathsf{e}_1 \in \bar{\mathsf{f}}$ *(or* $\mathsf{e}_2 \in \bar{\mathsf{f}}$*, resp.).*

Proof. Suppose that $\mathsf{e}_1 \in \overline{\Gamma_D}$. Then there exists a face $\mathsf{f} \subset \Gamma_D$ with $\mathsf{e}_1 \subset \bar{\mathsf{f}}$. Since $\omega_{\mathsf{e}} \supset \omega_{\mathsf{e}_1}$, there is an element $T \subset \omega_{\mathsf{e}}$ such that f is a face of T, and moreover, $\mathsf{f} \subset \partial\omega_{\mathsf{e}}$. □

We will use a couple of times that

$$\begin{aligned}
\text{diam}(\omega_{\mathsf{v}}) &\le C h_T , & \text{diam}(\omega_{\mathsf{v}})^{-1} &\le C h_T^{-1} & \forall \mathsf{v} &\in \mathcal{V} \cap \overline{T}, \\
\text{diam}(\omega_{\mathsf{e}}) &\le C h_T , & \text{diam}(\omega_{\mathsf{e}})^{-1} &\le C h_T^{-1} & \forall \mathsf{e} &\in \mathcal{E} \cap \overline{T}, \\
h_{\mathsf{e}} := \text{diam}(\mathsf{e}) &\le C h_T , & h_{\mathsf{e}}^{-1} &\le C h_T^{-1} & \forall \mathsf{e} &\in \mathcal{E} \cap \overline{T},
\end{aligned}$$

with a (generic) constant C only depending on the shape regularity of $\mathcal{T}$. Furthermore, we need the following discrete estimates:

Lemma 3.4. *For any element $T \in \mathcal{T}$ and any vertex* $\mathsf{v} \subset \overline{T}$,

$$|u_h(\mathsf{v})| \le C h_T^{-3/2} \|u_h\|_{0,T} \qquad \forall u_h \in \mathcal{W}_h^0(T),$$
$$\|\mathbf{grad}\, \varphi_{\mathsf{v}}\|_{0,T} \le C h_T^{1/2} .$$

Moreover, for every edge $\mathsf{e} \subset \overline{T}$,

$$\left| \int_{\mathsf{e}} \mathbf{w}_h \cdot \tau_{\mathsf{e}}\, ds \right| \le C h_T^{-1/2} \|\mathbf{w}_h\|_{0,T} \qquad \forall \mathbf{w}_h \in \boldsymbol{\mathcal{W}}_h^1(T),$$
$$\|\boldsymbol{\psi}_{\mathsf{e}}\|_{0,T} \le C h_T^{1/2} .$$

Proof. The proof is carried out using standard techniques from finite elements, transformation to the reference element, and an eigenvalue analysis of the reference element mass matrix. □

3.2.2 Locally exact sequences and Poincaré–Friedrichs-type inequalities

Let ω be a patch of elements which is simply connected with simply connected boundary, and let $\gamma \subset \partial\omega$ be a simply connected surface that is a union of faces of elements; the cases $\gamma = \emptyset$ and $\gamma = \partial\omega$ are admitted. Then the local sequence

$$\mathcal{K}_\gamma(\omega) \xrightarrow{\text{id}} \mathcal{W}_{h,\gamma}^0(\omega) \xrightarrow{\mathbf{grad}} \boldsymbol{\mathcal{W}}_{h,\gamma}^1(\omega) \xrightarrow{\mathbf{curl}} \boldsymbol{\mathcal{W}}_{h,\gamma}^2(\omega) \xrightarrow{\text{div}} \mathcal{W}_{h,\gamma}^3(\omega) \xrightarrow{0} \{0\} \tag{3.8}$$

is *exact*, i. e., the range of an operator is equal to the kernel of the subsequent operator [4, 5]. Above, $\mathcal{K}_\gamma(\omega)$ is the space of constants if $\gamma = \emptyset$ and $\mathcal{K}_\gamma(\omega) = \{0\}$ otherwise, and

$$\mathcal{W}_{h,\gamma}^3(\omega) = \begin{cases} \{v \in \mathcal{W}_h^3(\omega) \colon \int_\omega v\, dx = 0\} & \text{if } \gamma = \partial\omega \\ \mathcal{W}_h^3(\omega) & \text{otherwise.} \end{cases}$$

We have the classical Poincaré inequality

$$\|u - \overline{u}^\omega\|_{0,\omega} \le C h_\omega \|\mathbf{grad}\, u\|_{0,\omega} \qquad \forall u \in H^1(\omega), \tag{3.9}$$

where $\overline{u}^\omega := |\omega|^{-1} \int_\omega u\, dx$ and $h_\omega := \text{diam}(\omega)$, and the Friedrichs inequality

$$\|u\|_{0,\omega} \le C h_\omega \|\mathbf{grad}\, u\|_{0,\omega} \qquad \forall u \in H_\gamma^1(\omega), \qquad \text{if meas}_2(\gamma) > 0. \tag{3.10}$$

Above, the constants C depend only on the shape regularity of the mesh $\mathcal{T}$; for a proof see, e. g., [57]. We can write these inequalities in a more abstract way by introducing the $L^2(\omega)$-orthogonal projector $\Pi_{\omega,\gamma}^0 \colon H_\gamma^1(\omega) \to \mathcal{K}_\gamma(\omega) = \ker(\mathbf{grad}_{|H_\gamma^1(\omega)})$:

$$\|u - \Pi_{\omega,\gamma}^0 u\|_{0,\omega} \le C h_\omega \|\mathbf{grad}\, u\|_{0,\omega} \quad \forall u \in H_\gamma^1(\omega) . \tag{3.11}$$

For the other spaces in (3.8), let

$$\mathbf{\Pi}^1_{h,\omega,y}: \boldsymbol{H}(\mathbf{curl},\omega) \to \mathbf{grad}\,\mathcal{W}^0_{h,y}(\omega), \quad \mathbf{\Pi}^2_{h,\omega,y}: \boldsymbol{H}(\mathrm{div},\omega) \to \mathbf{curl}\,\boldsymbol{\mathcal{W}}^1_{h,y}(\omega)$$

denote the $\boldsymbol{L}^2(\omega)$-orthogonal projectors onto $\mathbf{grad}\,\mathcal{W}^0_{h,y}(\omega) = \ker(\mathbf{curl}_{|\boldsymbol{\mathcal{W}}^1_{h,y}(\omega)})$ and $\mathbf{curl}\,\boldsymbol{\mathcal{W}}^1_{h,y}(\omega) = \ker(\mathrm{div}_{|\boldsymbol{\mathcal{W}}^2_{h,y}(\omega)})$, respectively. Then the following discrete Poincaré–Friedrichs-type inequalities hold:

$$\|\mathbf{w} - \mathbf{\Pi}^1_{h,\omega,y}\mathbf{w}\|_{0,\omega} \le C h_\omega \|\mathbf{curl}\ \mathbf{w}\|_{0,\omega} \qquad \forall \mathbf{w} \in \boldsymbol{\mathcal{W}}^1_{h,y}(\omega), \tag{3.12}$$

$$\|\mathbf{q} - \mathbf{\Pi}^2_{h,\omega,y}\mathbf{q}\|_{0,\omega} \le C h_\omega \|\operatorname{div}\ \mathbf{q}\|_{0,\omega} \qquad \forall \mathbf{q} \in \boldsymbol{\mathcal{W}}^2_{h,y}(\omega), \tag{3.13}$$

where the constant C depends only on the shape regularity of $\mathcal{T}$. These important results can be shown by transformation to a few number of reference patches. From the L^2-projection property, we obtain that

$$\|\mathbf{\Pi}^1_{h,\omega,y}\mathbf{w}\|_{0,\omega} \le \|\mathbf{w}\|_{0,\omega} \qquad \forall \mathbf{w} \in \boldsymbol{H}(\mathbf{curl},\omega), \tag{3.14}$$

$$\|\mathbf{\Pi}^2_{h,\omega,y}\mathbf{q}\|_{0,\omega} \le \|\mathbf{q}\|_{0,\omega} \qquad \forall \mathbf{q} \in \boldsymbol{H}(\mathrm{div},\omega). \tag{3.15}$$

3.2.3 Modified Clément operators

We define $M^0_D: L^2(\Omega) \to \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$ by

$$M^0_D u := \sum_{\mathsf{v}\in\mathcal{V}_f} \overline{u}^{\omega_\mathsf{v}}\, \varphi_\mathsf{v}\,. \tag{3.16}$$

Recall again that $\overline{u}^{\omega_\mathsf{v}} := \frac{1}{|\omega_\mathsf{v}|}\int_{\omega_\mathsf{v}} u\,dx$ is the mean value of u over ω_v. As a simple but useful property,

$$(M^0_D u)(\mathsf{v}) = \begin{cases} \overline{u}^{\omega_\mathsf{v}} & \text{if } \mathsf{v} \in \mathcal{V}_f \\ 0 & \text{otherwise,} \end{cases} \tag{3.17}$$

i. e., the operator respects the homogeneous boundary conditions. Next, we define $\boldsymbol{M}^1_D: \boldsymbol{L}^2(\Omega) \to \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$ by

$$\boldsymbol{M}^1_D\mathbf{w} := \sum_{\mathsf{e}\in\mathcal{E}_f} \int_{\omega_\mathsf{e}} \mathbf{w}\cdot\mathbf{z}^1_\mathsf{e}\,dx\,\boldsymbol{\psi}_\mathsf{e}\,, \tag{3.18}$$

where the weight function $\mathbf{z}^1_\mathsf{e} \in \boldsymbol{H}(\mathrm{div},\omega_\mathsf{e})$ is yet to be constructed. Beforehand, we define for $\mathsf{e} = [\mathsf{e}_1,\mathsf{e}_2]$ the piecewise constant function

$$y^0_\mathsf{e} := \sum_{\mathsf{v}\in\overline{\mathsf{e}}\cap\mathcal{V}_f} \sigma^\mathsf{v}_\mathsf{e}\frac{1}{|\omega_\mathsf{v}|}\chi_{\omega_\mathsf{v}} = \begin{cases} \frac{1}{|\omega_{\mathsf{e}_2}|}\chi_{\omega_{\mathsf{e}_2}} - \frac{1}{|\omega_{\mathsf{e}_1}|}\chi_{\omega_{\mathsf{e}_1}} & \text{if } \mathsf{e}_1 \notin \overline{\Gamma_D}, \mathsf{e}_2 \notin \overline{\Gamma_D}\,, \\ \frac{1}{|\omega_{\mathsf{e}_2}|}\chi_{\omega_{\mathsf{e}_2}} & \text{if } \mathsf{e}_1 \in \overline{\Gamma_D}\,, \\ -\frac{1}{|\omega_{\mathsf{e}_1}|}\chi_{\omega_{\mathsf{e}_1}} & \text{if } \mathsf{e}_2 \in \overline{\Gamma_D}\,. \end{cases} \tag{3.19}$$

Above, $\chi_{\omega_{\mathsf{v}}}$ is the characteristic function, $\sigma_{\mathsf{e}}^{\mathsf{v}} = -1$ if v is the starting point of e and $\sigma_{\mathsf{e}}^{\mathsf{v}} = +1$ if v is its endpoint (i. e., $\sigma_{\mathsf{e}}^{\mathsf{v}}$ is an entry of the edge-vertex incidence matrix). It is seen easily that $y_{\mathsf{e}}^0 \in \mathcal{W}_h^3(\omega_{\mathsf{e}})$ and that

$$\mathsf{e}_1 \notin \overline{\Gamma_D} \text{ and } \mathsf{e}_2 \notin \overline{\Gamma_D} \quad \Longrightarrow \quad \int_{\omega_{\mathsf{e}}} y_{\mathsf{e}}^0\, dx = 0. \tag{3.20}$$

We require that

$$-\operatorname{div} \mathbf{z}_{\mathsf{e}}^1 = y_{\mathsf{e}}^0 \qquad \text{in } \omega_{\mathsf{e}}, \tag{3.21}$$

$$\mathbf{z}_{\mathsf{e}}^1 \cdot \mathbf{n} = 0 \qquad \text{on } \partial\omega_{\mathsf{e}} \setminus \gamma_{\mathsf{e}}, \tag{3.22}$$

where γ_{e} is constructed as follows:

(i) If $\mathsf{e}_1 \notin \overline{\Gamma_D}$ and $\mathsf{e}_2 \notin \overline{\Gamma_D}$, we set $\gamma_{\mathsf{e}} := \emptyset$.

(ii) If one of the endpoints of e, say e_1, lies on $\overline{\Gamma_D}$, then we set $\gamma_{\mathsf{e}} := \mathsf{f}$, where f is the triangular face from Lemma 3.3 such that $\mathsf{f} \subset \partial\omega_{\mathsf{e}} \cap \Gamma_D$ and $\mathsf{e}_1 \subset \overline{\mathsf{f}}$. See Figure 7.3 for an illustration.

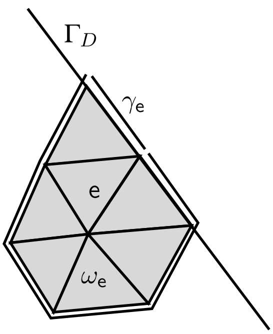

Figure 7.3: Sketch of an edge patch ω_{e} and the surface γ_{e} for the case that one of the endpoints of the edge lies on the Dirichlet boundary Γ_D. The weight function $\mathbf{z}_{\mathsf{e}}^1$ has vanishing normal component on γ_{e}^c (dotted line).

From the construction of γ_{e} and from (3.20) we can conclude that

$$y_{\mathsf{e}}^0 \in \mathcal{W}_{h,\gamma_{\mathsf{e}}^c}^3(\omega_{\mathsf{e}}), \tag{3.23}$$

where $\gamma_{\mathsf{e}}^c := \partial\omega_{\mathsf{e}} \setminus \gamma_{\mathsf{e}}$. In particular, for the case $\gamma_{\mathsf{e}} = \emptyset$, (3.20) serves as a compatibility condition for (3.21)–(3.22) due to Gauss' theorem. In order to fix $\mathbf{z}_{\mathsf{e}}^1$ uniquely, we require two additional properties:

$$\mathbf{z}_{\mathsf{e}}^1 \in \boldsymbol{\mathcal{W}}_{h,\gamma_{\mathsf{e}}^c}^2(\omega_{\mathsf{e}}), \tag{3.24}$$

$$\int_{\omega_e} \mathbf{z}_e^1 \cdot \mathbf{curl}\ \mathbf{w}_h\, dx = 0 \qquad \forall \mathbf{w}_h \in \mathcal{W}^1_{h,\gamma_e^c}(\omega_e). \tag{3.25}$$

Recall that due to Assumption 3.2, ω_e is simply connected with simply connected boundary. Therefore, since γ_e is either empty or a triangular face, the complementary surface γ_e^c is simply connected. Therefore, the sequence (3.8) with $\omega \mapsto \omega_e$ and $\gamma \mapsto \gamma_e^c$ is exact, and it follows that the weight function $\mathbf{z}_e^1$ indeed exists and is unique.

From (3.21)–(3.22), we can conclude that

$$\int_{\omega_e} \mathbf{grad}\, q \cdot \mathbf{z}_e^1\, dx = \int_{\omega_e} q\, y_e^0\, dx \qquad \forall q \in \begin{cases} H^1(\omega_e) & \text{if } e_1 \notin \overline{\Gamma_D} \text{ and } e_2 \notin \overline{\Gamma_D}, \\ H^1_{\gamma_e}(\omega_e) & \text{if } e_1 \in \overline{\Gamma_D} \text{ or } e_2 \in \overline{\Gamma_D}. \end{cases} \tag{3.26}$$

Lemma 3.5. *For all $u \in H^1_{\Gamma_D}(\Omega)$, we have the commuting property:*

$$\boldsymbol{M}_D^1\, \mathbf{grad}\, u = \mathbf{grad}\, M_D^0 u.$$

Moreover, for an edge $e \in \mathcal{E}_f$ with $e_1 \in \overline{\Gamma_D}$ or $e_2 \in \overline{\Gamma_D}$ and for $u_h \in \mathcal{W}^0_{h,\gamma_e}(\omega_e)$,

$$\int_e (\boldsymbol{M}_D^1\, \mathbf{grad}\, u_h) \cdot \boldsymbol{\tau}_e\, ds = \int_e (\mathbf{grad}\, M_D^0 u_h) \cdot \boldsymbol{\tau}_e\, ds,$$

where the two expressions are well-defined.

Proof. For the first part of the proof, we just consider $u \in H^1(\Omega)$. By construction, both $\boldsymbol{M}_D^1\, \mathbf{grad}\, u$ and $\mathbf{grad}\, M^0 u$ belong to $\mathcal{W}^1_{h,\Gamma_D}(\mathcal{T})$, even for a non-trivial topology of Ω, Γ_D. Therefore, in order to show the first identity, it suffices to check all the edge integrals on $e = [e_1, e_2] \in \mathcal{E}_f$:

$$\begin{aligned} &\int_e (\mathbf{grad}\, M^0 u) \cdot \boldsymbol{\tau}_e\, ds \\ &\quad = \sum_{v \in \mathcal{V}^f} \overline{u}^{\omega_v} \int_e \mathbf{grad}\, \varphi_v \cdot \boldsymbol{\tau}_e\, ds = \begin{cases} \overline{u}^{\omega_{e_2}} - \overline{u}^{\omega_{e_1}} & \text{if } e_1 \notin \overline{\Gamma_D}, e_2 \notin \overline{\Gamma_D}, \\ \overline{u}^{\omega_{e_2}} & \text{if } e_1 \in \overline{\Gamma_D}, \\ -\overline{u}^{\omega_{e_1}} & \text{if } e_2 \in \overline{\Gamma_D}. \end{cases} \end{aligned}$$

Since

$$\overline{u}^{\omega_{e_i}} = \int_{\omega_e} u\, \chi_{\omega_{e_i}}\, dx,$$

we can conclude from (3.19) that

$$\int_e (\mathbf{grad}\, M_D^0 u) \cdot \boldsymbol{\tau}_e\, ds = \int_{\omega_e} u\, y_e^0\, dx \qquad \forall u \in H^1(\Omega). \tag{3.27}$$

We now show the first identity and assume that $u \in H^1_{\Gamma_D}(\Omega)$. Consequently, $u_{|\omega_e} \in H^1_{\Gamma_D}(\omega_e)$, in particular $u_{|\omega_e} \in H^1_{\gamma_e}(\omega_e)$, and so (3.26) and the definition (3.18) of $\boldsymbol{M}^1_D$ imply

$$\begin{aligned}\int_e (\mathbf{grad}\, M^0_D u)\cdot \boldsymbol{\tau}_e\, ds &= \int_{\omega_e} \mathbf{grad}\, u \cdot \mathbf{z}^1_e\, dx \\ &= \int_e \int_{\omega_e} \mathbf{grad}\, u \cdot \mathbf{z}^1_e\, dx\, \boldsymbol{\psi}_e \cdot \boldsymbol{\tau}_e\, ds = \int_e (\boldsymbol{M}^1_D\, \mathbf{grad}\, u)\cdot \tau_e\, ds.\end{aligned}$$

The second identity follows by the same arguments and the locality of M^0_D, $\boldsymbol{M}^1_D$. □

Lemma 3.6. *For all $u \in L^2(\Omega)$ and $T \in \mathcal{T}$,*

$$\|M^0_D u\|_{0,T} \le C\|u\|_{0,\omega_T}\,.$$

Proof. From the definition of M^0_D, we derive

$$\|M^0_D u\|_{0,T} \le \sum_{v\in\mathcal{V}_f\cap\overline{T}} |\overline{u}^{\omega_v}|\, \|\varphi_v\|_{0,T}\,.$$

Cauchy's inequality yields $|\overline{u}^{\omega_v}| \le |\omega_v|^{-1/2}\|u\|_{0,\omega_v}$ and standard FE arguments show that $|\omega_v| \ge c h_T^3$ and $\|\varphi_v\|_{0,\omega_T} \le C h_T^{3/2}$. □

For the approximation property of M^0_D, we need another construction. For elements T where

$$\partial\omega_T \cap \Gamma_D \ne \emptyset \quad \text{but} \quad \mathrm{meas}_2(\partial\omega_T \cap \Gamma_D) = 0,$$

we define a *slightly enlarged* element patch $\widetilde{\omega}_T \supset \omega_T$ such that

$$\mathrm{meas}_2(\partial\widetilde{\omega}_T \cap \Gamma_D) > 0 \qquad \text{and} \qquad \mathrm{diam}(\widetilde{\omega}_T) \le C h_T\,, \tag{3.28}$$

with a uniform constant C depending only on the shape regularity of $\mathcal{T}$; see Figure 7.4 for an illustration. For all other elements, we simply set $\widetilde{\omega}_T = \omega_T$.

Lemma 3.7. *For all $u \in H^1_{\Gamma_D}(\Omega)$ and $T \in \mathcal{T}$,*

$$\left\|u - M^0_D u\right\|_{0,T} \le C h_T \|\mathbf{grad}\, u\|_{0,\widetilde{\omega}_T}$$

Proof. Let T be such that $\mathrm{meas}_2(\partial\widetilde{\omega}_T \cap \Gamma_D) = 0$, which implies that $\partial\omega_T \cap \Gamma_D = \emptyset$, and so all vertices on $\overline{\omega}_T$ are in $\mathcal{V}_f$. Due to the partition of unity property of the vertex basis functions,

$$(M^0_D c)_{|\omega_T} = c \qquad \text{for any constant } c.$$

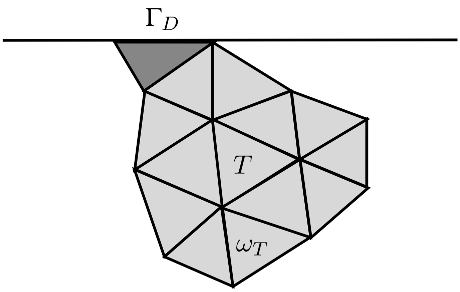

Figure 7.4: Sketch of construction of enlarged element patch $\widetilde{\omega}_T$. Light grey area: original patch ω_T. Dark grey area: element that is added in order to obtain $\widetilde{\omega}_T$.

Hence,

$$u - M_D^0 u = u - \overline{u}^{\omega_e} - M_D^0(u - \overline{u}^{\omega_e}).$$

From the triangle inequality and the L^2-estimate from Lemma 3.6, we obtain

$$\|u - M_D^0 u\|_{0,T} \leq C\|u - \overline{u}^{\omega_e}\|_{0,\omega_T} \leq Ch_T\|\mathbf{grad}\, u\|_{0,\omega_T},$$

where in the last step, we have used Poincaré's inequality (3.9). Finally, let T be such that $\mathrm{meas}_2(\partial\widetilde{\omega}_T \cap \Gamma_D) > 0$. We apply Lemma 3.6 directly, leading to

$$\|u - M_D^0 u\|_{0,T} \leq C\|u\|_{0,\omega_T} \leq C\|u\|_{0,\widetilde{\omega}_T}.$$

Since u vanishes on $\partial\widetilde{\omega}_T \cap \Gamma_D$ by assumption, Friedrichs' inequality (3.10) yields the desired bound. □

The stability of M_D^0 in the H^1-semi norm will be a consequence of Lemma 3.9 below. The L^2-stability of $\boldsymbol{M}_D^1$ involves the particular choice of the weight function $\mathbf{z}_e^1$ and needs the following auxiliary estimate.

Lemma 3.8. *Let the weight function* $\mathbf{z}_e^1$ *be defined by* (3.21), (3.22), (3.24), *and* (3.25). *Then*

$$\|\mathbf{z}_e^1\|_{0,\omega_e} \leq Ch_e^{-1/2}.$$

Proof. The orthogonality condition (3.25) implies that $\mathbf{\Pi}^2_{h,\omega_e,\gamma_e}\mathbf{z}_e^1 = 0$, and so the discrete Poincaré–Friedrichs-type inequality (3.13) implies

$$\|\mathbf{z}_e^1\|_{0,\omega_e} \leq Ch_e\|\mathrm{div}\ \mathbf{z}_e^1\|_{0,\omega_e} = Ch_e\|y_e^0\|_{0,\omega_e},$$

where we have used (3.21). From the definition (3.19) of y_e^0, we see that $\|y_e^0\|_{0,\omega_e} \leq Ch_e^{3/2}h_e^{-3} = Ch_e^{-3/2}$. □

Lemma 3.9. *For all* $\mathbf{w} \in \boldsymbol{L}^2(\Omega)$ *and* $T \in \mathcal{T}$,

$$\|\boldsymbol{M}_D^1 \mathbf{w}\|_{0,T} \leq C\|\mathbf{w}\|_{0,\omega_T}\,.$$

Proof.

$$\|\boldsymbol{M}_D^1 \mathbf{w}\|_{0,T} \leq \sum_{\mathrm{e}\in\mathcal{E}_f\cap\overline{T}} \left|\int_{\omega_\mathrm{e}} \mathbf{w}\cdot\mathbf{z}_\mathrm{e}^1\,dx\right| \|\boldsymbol{\psi}_\mathrm{e}\|_{0,\omega_\mathrm{e}} \leq \sum_{\mathrm{e}\in\mathcal{E}_f\cap\overline{T}} \|\mathbf{w}\|_{0,\omega_\mathrm{e}} \|\mathbf{z}_\mathrm{e}^1\|_{0,\omega_\mathrm{e}} \|\boldsymbol{\psi}_\mathrm{e}\|_{0,\omega_\mathrm{e}}\,.$$

From Lemma 3.4, $\|\boldsymbol{\psi}_\mathrm{e}\|_{0,\omega_\mathrm{e}} \leq C h_\mathrm{e}^{-1} h_\mathrm{e}^{3/2}$. The proof is concluded by applying Lemma 3.8. □

Corollary 3.10. *For all* $u \in H^1_{\Gamma_D}(\Omega)$ *and* $T \in \mathcal{T}$,

$$\|\mathbf{grad}\, M_D^0 u\|_{0,T} \leq C\|\mathbf{grad}\, u\|_{0,\omega_T}\,.$$

Proof. Due to Lemma 3.5, $\mathbf{grad}\, M_D^0 u = \boldsymbol{M}_D^1\, \mathbf{grad}\, u$ for all $u \in H^1_{\Gamma_D}(\Omega)$, so the statement follows from Lemma 3.9. □

3.2.4 Auxiliary projectors on local patches

Let ω be a simply connected patch of a few elements with simply connected boundary and $\gamma \subset \partial\omega$ a simply connected union of faces such that the exact sequence property (3.8) holds; the cases $\gamma = \emptyset$, $\gamma = \partial\omega$ are admitted. We define $Q^0_{\omega,\gamma}: H^1(\omega) \to \mathcal{W}^0_{h,\gamma}(\omega)$ by

$$\int_\omega Q^0_{\omega,\gamma} u\,dx = \int_\omega u\,dx \qquad \text{if } \gamma = \emptyset, \tag{3.29}$$

$$\int_\omega \mathbf{grad}(Q^0_{\omega,\gamma} u)\cdot \mathbf{grad}\, p_h\,dx = \int_\omega \mathbf{grad}\, u \cdot \mathbf{grad}\, p_h\,dx \qquad \forall p_h \in \mathcal{W}^0_{h,\gamma}(\omega), \tag{3.30}$$

and $\boldsymbol{Q}^1_{\omega,\gamma}: \boldsymbol{H}(\mathbf{curl},\omega) \to \boldsymbol{\mathcal{W}}^1_{h,\gamma}(\omega)$ by

$$\int_\omega \boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w}\cdot\mathbf{grad}\, p_h\,dx = \int_\omega \mathbf{w}\cdot\mathbf{grad}\, p_h\,dx \qquad \forall p_h \in \mathcal{W}^0_{h,\gamma}(\omega), \tag{3.31}$$

$$\int_\omega \mathbf{curl}(\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w})\cdot\mathbf{curl}\ \mathbf{q}_h\,dx = \int_\omega \mathbf{curl}\ \mathbf{w}\cdot\mathbf{curl}\ \mathbf{q}_h\,dx \qquad \forall \mathbf{q}_h \in \boldsymbol{\mathcal{W}}^1_{h,\gamma}(\omega). \tag{3.32}$$

Obviously,

$$\boldsymbol{Q}^1_{\omega,\gamma}\,\mathbf{grad}\, u = \mathbf{grad}\, Q^0_{\omega,\gamma} u \qquad \forall u \in H^1(\omega), \tag{3.33}$$

$$Q^0_{\omega,\gamma} u_h = u_h \qquad \forall u_h \in \mathcal{W}^0_{h,\gamma}(\omega), \tag{3.34}$$

$$\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w}_h = \mathbf{w}_h \qquad \forall \mathbf{w}_h \in \boldsymbol{\mathcal{W}}^1_{h,\gamma}(\omega). \tag{3.35}$$

Finally, we define a *lifting operator* $Q^1_{\omega,\gamma,-}: \boldsymbol{H}(\mathbf{curl},\omega) \to \mathcal{W}^0_{h,\gamma}(\omega)$ by

$$\int_\omega Q^1_{\omega,\gamma,-}\mathbf{w}\,dx = 0 \qquad \text{if } \gamma = \emptyset, \tag{3.36}$$

$$\int_\omega \mathbf{grad}(Q^1_{\omega,\gamma,-}\mathbf{w}) \cdot \mathbf{grad}\, p_h\,dx = \int_\omega \mathbf{w} \cdot \mathbf{grad}\, p_h\,dx \qquad \forall p_h \in \mathcal{W}^0_{h,\gamma}(\omega). \tag{3.37}$$

Summarizing, we have

$$Q^0_{\omega,\gamma}u = \begin{cases} \overline{u}^\omega + Q^1_{\omega,\gamma,-}\,\mathbf{grad}\,u & \text{if } \gamma = \emptyset, \\ Q^1_{\omega,\gamma,-}\,\mathbf{grad}\,u & \text{otherwise.} \end{cases} \tag{3.38}$$

Lemma 3.11. *For* $u \in H^1(\omega)$,

$$\begin{aligned} \|\mathbf{grad}\, Q^0_{\omega,\gamma}u\|_{0,\omega} &\le \|\mathbf{grad}\,u\|_{0,\omega} \\ \|Q^0_{\omega,\gamma}u\|_{0,\omega} &\le \|u\|_{0,\omega} + C\,\mathrm{diam}(\omega)\|\mathbf{grad}\,u\|_{0,\omega}\,. \end{aligned}$$

Proof. The first estimate follows immediately from (3.30) by setting $p_h = Q^0_{\omega,\gamma}u$ and applying Cauchy's inequality. For the second estimate, we treat two cases:

– If $\mathrm{meas}_2(\gamma) = 0$ then the mean value property (3.29) implies

$$Q^0_{\omega,\gamma}u = Q^0_{\omega,\gamma}(u - \overline{u}^\omega) + \overline{u}^\omega$$

and the first term has vanishing mean over ω. From the triangle inequality, Cauchy–Schwarz, and Poincaré's inequality (3.9), we obtain

$$\begin{aligned} \|Q^0_{\omega,\gamma}u\|_{0,\omega} &\le \|Q^0_{\omega,\gamma}(u - \overline{u}^\omega)\|_{0,\omega} + \|\overline{u}^\omega\|_{0,\omega} \\ &\le C\,\mathrm{diam}(\omega)\|\mathbf{grad}\,Q^0_{\omega,\gamma}u\|_{0,\omega} + \|u\|_{0,\omega}\,. \end{aligned}$$

– If $\mathrm{meas}_2(\gamma) > 0$, we obtain from Friedrichs' inequality (3.10) that

$$\|Q^0_{\omega,\gamma}u\|_{0,\omega} \le C\,\mathrm{diam}(\omega)\|\mathbf{grad}\,Q^0_{\omega,\gamma}u\|_{0,\omega}\,.$$

In both cases, employing the first estimate concludes the proof. □

Lemma 3.12. *For* $\mathbf{w} \in \boldsymbol{H}(\mathbf{curl},\omega)$,

$$\begin{aligned} \|\mathbf{curl}\;\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w}\|_{0,\omega} &\le \|\mathbf{curl}\;\mathbf{w}\|_{0,\omega} \\ \|\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w}\|_{0,\omega} &\le \|\mathbf{w}\|_{0,\omega} + C\,\mathrm{diam}(\omega)\|\mathbf{curl}\;\mathbf{w}\|_{0,\omega}\,. \end{aligned}$$

Proof. The first estimate follows immediately from (3.32) by setting $\mathbf{q}_h = \mathbf{Q}^1_{\omega,\gamma}\mathbf{w}$ and applying Cauchy's inequality. For the second estimate, recall the projection operator $\mathbf{\Pi}^1_{h,\omega,\gamma}: \boldsymbol{H}(\mathbf{curl},\omega) \to \mathbf{grad}\,\mathcal{W}^0_{h,\gamma}(\omega)$, from Section 3.2.2, which has the property

$$\int_\omega \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w} \cdot \mathbf{grad}\,p_h\,dx = \int_\omega \mathbf{w} \cdot \mathbf{grad}\,p_h\,dx \qquad \forall p_h \in \mathcal{W}^0_{h,\gamma}(\omega). \tag{3.39}$$

Since $\mathbf{\Pi}^1_{h,\omega,\gamma}$, $\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{\Pi}^1_{h,\omega,\gamma}$, and $\mathbf{\Pi}^1_{h,\omega,\gamma}\boldsymbol{Q}^1_{\omega,\gamma}$ have the same range, we can conclude from (3.31) and (3.39) that

$$\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w} = \mathbf{\Pi}^1_{h,\omega,\gamma}\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w} = \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w}.$$

Therefore,

$$\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w} = \boldsymbol{Q}^1_{\omega,\gamma}(\mathbf{w} - \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w}) + \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w}$$

and

$$\mathbf{\Pi}^1_{h,\omega,\gamma}\boldsymbol{Q}^1_{\omega,\gamma}(\mathbf{w} - \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w}) = 0.$$

Hence, the discrete Poincaré–Friedrichs-type inequality (3.12) together with the L^2-stability (3.14) of $\mathbf{\Pi}^1_{h,\omega,\gamma}$ yields

$$\begin{aligned}\|\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w}\|_{0,\omega} &\le \|\boldsymbol{Q}^1_{\omega,\gamma}(\mathbf{w} - \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w})\|_{0,\omega} + \|\mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w}\|_{0,\omega} \\ &\le C\,\mathrm{diam}(\omega)\|\underbrace{\mathbf{curl}\ \boldsymbol{Q}^1_{\omega,\gamma}(\mathbf{w} - \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w})}_{\mathbf{curl}\ \boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w}}\|_{0,\omega} + \|\mathbf{w}\|_{0,\omega}\,.\end{aligned}$$

Employing the first estimate once again concludes the proof. □

Finally, we need stability estimates for the lifting operator $Q^1_{\omega,\gamma,-}$:

Lemma 3.13. *For any* $\mathbf{w} \in \boldsymbol{H}(\mathbf{curl}, \omega)$,

$$\begin{aligned}\|\mathbf{grad}\, Q^1_{\omega,\gamma,-}\mathbf{w}\|_{0,\omega} &\le \|\mathbf{w}\|_{0,\omega}\,, \\ \|Q^1_{\omega,\gamma,-}\mathbf{w}\|_{0,\omega} &\le C\,\mathrm{diam}(\omega)\|\mathbf{w}\|_{0,\omega}\,.\end{aligned}$$

Proof. Choosing $p_h := Q^1_{\omega,\gamma,-}\mathbf{w}$ in (3.37) applying Cauchy–Schwarz, we find that

$$\|\mathbf{grad}\, Q^1_{\omega,\gamma,-}\mathbf{w}\|^2_{0,\omega} = \int_\omega \mathbf{w}\cdot\mathbf{grad}(Q^1_{\omega,\gamma,-}\mathbf{w})\,dx \le \|\mathbf{w}\|_{0,\omega}\,\|Q^1_{\omega,\gamma,-}\mathbf{w}\|_{0,\omega}\,,$$

which implies the first inequality. For $\gamma = \emptyset$, the second inequality follows from the first one by Poincaré's inequality (3.9) because $Q^1_{\omega,\emptyset,-}\mathbf{w}$ has vanishing mean over ω. If $\mathrm{meas}_2(\gamma) > 0$, then we can use Friedrichs' inequality (3.10) to obtain the same result. □

3.2.5 The auxiliary operators S^0_D and S^1_D

For $\mathsf{v} \in \mathcal{V}_f$, we set

$$Q^0_{\mathsf{v}} := Q^0_{\omega_{\mathsf{v}},\emptyset}\,, \qquad \boldsymbol{Q}^1_{\mathsf{v}} := \boldsymbol{Q}^1_{\omega_{\mathsf{v}},\emptyset}\,, \qquad Q^1_{\mathsf{v},-} := Q^1_{\omega_{\mathsf{v}},\emptyset,-}\,. \tag{3.40}$$

We define $S_D^0\colon H^1(\Omega) \to \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$ by

$$S_D^0 u := M_D^0 u, \tag{3.41}$$

and $\boldsymbol{S}_D^1\colon \boldsymbol{H}(\mathbf{curl},\Omega) \to \mathcal{W}^1_{h,\Gamma_D}(\mathcal{T})$ by

$$\boldsymbol{S}_D^1 \mathbf{w} := \boldsymbol{M}_D^1 \mathbf{w} + \sum_{\mathsf{v}\in\mathcal{V}_f} (Q^1_{\mathsf{v},-}\mathbf{w})(\mathsf{v})\ \mathbf{grad}\,\varphi_{\mathsf{v}}\,. \tag{3.42}$$

Remark 3.14. Following the original paper by Falk and Winther [27], the operator $\boldsymbol{S}_D^1$ should be defined by

$$\boldsymbol{S}_D^1 \mathbf{w} := \boldsymbol{M}_D^1 \mathbf{w} + \sum_{\mathsf{v}\in\mathcal{V}_f} [(I - S_D^0)Q^1_{\mathsf{v},-}\mathbf{w}](\mathsf{v})\ \mathbf{grad}\,\varphi_{\mathsf{v}} \tag{3.43}$$

and one needs to argue firstly that the expression $[(I - S_D^0)Q^1_{\mathsf{v},-}\mathbf{w}](\mathsf{v})$ is well-defined. Indeed, for $\mathsf{v} \in \mathcal{V}_f$,

$$[(I - \underbrace{S_D^0}_{M_D^0})Q^1_{\mathsf{v},-}\mathbf{w}](\mathsf{v}) \;=\; (Q^1_{\omega_{\mathsf{v}},\emptyset,-}\mathbf{w})(\mathsf{v}) - \underbrace{\overline{Q^1_{\omega_{\mathsf{v}},\emptyset,-}\mathbf{w}}^{\omega_{\mathsf{v}}}}_{=0},$$

which is also the reason for the simplified definition (3.42) compared to (3.43).

Unlike M_D^0, $\boldsymbol{M}_D^1$, the operators S_D^0 and $\boldsymbol{S}_D^1$ do not commute and they are not projections either. The key property of $\boldsymbol{S}_D^1$ is the following one.

Lemma 3.15. *For all* $\mathsf{e} = [\mathsf{e}_1, \mathsf{e}_2] \in \mathcal{E}$ *and for all* $u_h \in \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$,

$$\int_{\mathsf{e}} (\boldsymbol{S}_D^1\,\mathbf{grad}\,u_h)\cdot\boldsymbol{\tau}_{\mathsf{e}}\,ds = \int_{\mathsf{e}} \mathbf{grad}\,u_h\cdot\boldsymbol{\tau}_{\mathsf{e}}\,ds.$$

The same identity holds for a particular edge e *if* u_h *is only given in* $\mathcal{W}^0_{h,\gamma_{\mathsf{e}}}(\omega_{\mathsf{e}})$.

Proof. For edges e on the Dirichlet boundary Γ_D, both integrals evaluate to zero. Let us therefore consider $\mathsf{e} \in \mathcal{E}_f$ and $u_h \in \mathcal{W}^0_h(\omega_{\mathsf{e}})$. We will specify boundary conditions for u_h later on. Insertion of the definition of $\boldsymbol{S}_D^1$ into the left-hand side yields

$$\begin{aligned}
&\int_{\mathsf{e}} (\boldsymbol{S}_D^1\,\mathbf{grad}\,u_h)\cdot\boldsymbol{\tau}_{\mathsf{e}}\,ds \\
&\quad= \int_{\mathsf{e}} (\boldsymbol{M}_D^1\,\mathbf{grad}\,u_h)\cdot\boldsymbol{\tau}_{\mathsf{e}}\,ds + \int_{\mathsf{e}} \sum_{\mathsf{v}\in\mathcal{V}_f} (Q^1_{\mathsf{v},-}\,\mathbf{grad}\,u_h)(\mathsf{v})\,\mathbf{grad}\,\varphi_{\mathsf{v}}\cdot\boldsymbol{\tau}_{\mathsf{e}}\,ds \\
&\quad= \int_{\mathsf{e}} (\boldsymbol{M}_D^1\,\mathbf{grad}\,u_h)\cdot\boldsymbol{\tau}_{\mathsf{e}}\,ds + \sum_{\mathsf{v}\in\bar{\mathsf{e}}\cap\mathcal{V}_f} \sigma^{\mathsf{e}}_{\mathsf{v}}(Q^1_{\mathsf{v},-}\,\mathbf{grad}\,u_h)(\mathsf{v}),
\end{aligned}$$

where $\sigma^{\mathrm{e}}_{\mathrm{e}_2} = +1$ and $\sigma^{\mathrm{e}}_{\mathrm{e}_1} = -1$. Apparently, these expressions are well-defined although u_h is only given in $\mathcal{W}^0_h(\omega_{\mathrm{e}})$. Identity (3.38) and the projection property (3.34) of Q^0_{v} yield

$$Q^1_{\mathrm{v},-}\,\mathbf{grad}\,u_h = Q^0_{\mathrm{v}} u_h - \overline{u_h}^{\omega_{\mathrm{v}}} = u_h - \overline{u_h}^{\omega_{\mathrm{v}}}\,.$$

Therefore,

$$(Q^1_{\mathrm{v},-}\,\mathbf{grad}\,u_h)(\mathrm{v}) = u_h(\mathrm{v}) - \overline{u_h}^{\omega_{\mathrm{v}}}\,.$$

Substitution in the earlier formula yields (still for arbitrary $u_h \in \mathcal{W}^0_h(\omega_{\mathrm{e}})$)

$$\begin{aligned}
&\int_{\mathrm{e}} (\boldsymbol{S}^1_D\,\mathbf{grad}\,u_h)\cdot\boldsymbol{\tau}_{\mathrm{e}}\,ds \\
&\quad= \int_{\mathrm{e}} (\boldsymbol{M}^1_D\,\mathbf{grad}\,u_h)\cdot\boldsymbol{\tau}_{\mathrm{e}}\,ds + \sum_{\mathrm{v}\in\bar{\mathrm{e}}\cap\mathcal{V}_f} \sigma^{\mathrm{e}}_{\mathrm{v}}\big(u_h(\mathrm{v}) - \overline{u_h}^{\omega_{\mathrm{v}}}\big) \\
&\quad= \underbrace{\int_{\mathrm{e}} (\boldsymbol{M}^1_D\,\mathbf{grad}\,u_h)\cdot\boldsymbol{\tau}_{\mathrm{e}}\,ds}_{\text{(I)}} - \underbrace{\sum_{\mathrm{v}\in\bar{\mathrm{e}}\cap\mathcal{V}_f} \sigma^{\mathrm{e}}_{\mathrm{v}}\overline{u_h}^{\omega_{\mathrm{v}}}}_{\text{(II)}} + \underbrace{\sum_{\mathrm{v}\in\bar{\mathrm{e}}\cap\mathcal{V}_f} \sigma^{\mathrm{e}}_{\mathrm{v}} u_h(\mathrm{v})}_{=\int_{\mathrm{e}} \mathbf{grad}\,u_h\cdot\boldsymbol{\tau}_{\mathrm{e}}\,ds}\,.
\end{aligned}$$

For the remainder of the proof, we treat two cases:

– If $u_h \in \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$, then we obtain from the first commuting property of Lemma 3.5 and Identity (3.27) in its proof that

$$\text{(I)} = \int_{\mathrm{e}} (\boldsymbol{M}^1_D\,\mathbf{grad}\,u_h)\cdot\boldsymbol{\tau}_{\mathrm{e}}\,ds = \int_{\mathrm{e}} \mathbf{grad}(M^0_D u_h)\cdot\boldsymbol{\tau}_{\mathrm{e}}\,ds = \int_{\omega_{\mathrm{e}}} u_h\, y^0_{\mathrm{e}}\,dx = \text{(II)}.$$

– If $u_h \in \mathcal{W}^0_{h,y_{\mathrm{e}}}(\omega_{\mathrm{e}})$, then the second commuting property of Lemma 3.5 and Identity (3.27) in its proof imply the same formula. □

Next, we provide a stability estimate for $\boldsymbol{S}^1_D$.

Lemma 3.16. *For all* $\mathbf{w} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)$ *and* $T \in \mathcal{T}$,

$$\|\boldsymbol{S}^1_D\mathbf{w}\|_{0,T} \le C\|\mathbf{w}\|_{0,\omega_T}$$

Proof. The definition of $\boldsymbol{S}^1_D$ and the triangle inequality imply

$$\|\boldsymbol{S}^1_D\mathbf{w}\|_{0,T} \le \|\boldsymbol{M}^1_D\mathbf{w}\|_{0,T} + \sum_{\mathrm{v}\in\mathcal{V}_f\cap\overline{T}} \big|(Q^1_{\mathrm{v},-}\mathbf{w})(\mathrm{v})\big|\,\|\mathbf{grad}\,\varphi_{\mathrm{v}}\|_{0,T}\,.$$

The first term can be estimated from above by $C\|\mathbf{w}\|_{0,\omega_T}$; cf. Lemma 3.9. Using Lemma 3.4, we can now estimate the second term:

$$\sum_{\mathrm{v}\in\mathcal{V}_f\cap\overline{T}} \big|(Q^1_{\mathrm{v},-}\mathbf{w})(\mathrm{v})\big|\,\|\mathbf{grad}\,\varphi_{\mathrm{v}}\|_{0,T} \le C \sum_{\mathrm{v}\in\mathcal{V}_f\cap\overline{T}} h_T^{-1}\|Q^1_{\mathrm{v},-}\mathbf{w}\|_{0,T}\,.$$

Recall that $\overline{Q^1_{\mathsf{v},-}\mathbf{w}}^{\omega_\mathsf{v}} = 0$, so Poincaré's inequality (3.9) implies

$$h_T^{-1}\|Q^1_{\mathsf{v},-}\mathbf{w}\|_{0,T} \leq C\|\mathbf{grad}\,Q^1_{\mathsf{v},-}\mathbf{w}\|_{0,\omega_\mathsf{v}} \leq C\|\mathbf{w}\|_{0,\omega_\mathsf{v}}\,,$$

where in the last step, we have used Lemma 3.13. Combination of the above yields

$$\sum_{\mathsf{v}\in\mathcal{V}_f\cap\overline{T}} \big|(Q^1_{\mathsf{v},-}\mathbf{w})(\mathsf{v})\big|\,\|\mathbf{grad}\,\varphi_\mathsf{v}\|_{0,T} \leq C\|\mathbf{w}\|_{0,\omega_T}\,.$$

Combination of the estimates for the first and second term concludes the proof. □

In addition to the previous lemma, we need another local estimate for $\boldsymbol{S}^1_D$:

Lemma 3.17. *For all* $\mathbf{w} \in \boldsymbol{H}(\mathbf{curl}, \omega_\mathsf{e})$,

$$\Big|\int_\mathsf{e}(\boldsymbol{S}^1_D\mathbf{w})\cdot\tau_\mathsf{e}\,ds\Big| \leq Ch_\mathsf{e}^{-1/2}\|\mathbf{w}\|_{0,\omega_\mathsf{e}}\,.$$

Proof. From the definition of $\boldsymbol{S}^1_D$, we see that

$$\begin{aligned}&\Big|\int_\mathsf{e}(\boldsymbol{S}^1_D\mathbf{w})\cdot\tau_\mathsf{e}\,ds\Big| \\ &\quad\leq \Big|\int_\mathsf{e}(\boldsymbol{M}^1_D\mathbf{w})\cdot\tau_\mathsf{e}\,ds\Big| + \sum_{\mathsf{v}\in\mathcal{V}_f\cap\overline{\mathsf{e}}}\big|(Q^1_{\mathsf{v},-}\mathbf{w})(\mathsf{v})\big|\,\underbrace{\Big|\int_\mathsf{e}\mathbf{grad}\,\varphi_\mathsf{v}\cdot\tau_\mathsf{e}\,ds\Big|}_{=\pm1}.\end{aligned}$$

From the definition of $\boldsymbol{M}^1_D$, we easily conclude from Lemma 3.8 that

$$\Big|\int_\mathsf{e}(\boldsymbol{M}^1_D\mathbf{w})\cdot\tau_\mathsf{e}\,ds\Big| \leq \Big|\int_{\omega_\mathsf{e}}\mathbf{w}\cdot\mathbf{z}^1_\mathsf{e}\,dx\Big| \leq Ch_\mathsf{e}^{-1/2}\|\mathbf{w}\|_{0,\omega_\mathsf{e}}\,.$$

Due to Lemma 3.4 and Lemma 3.13,

$$\big|(Q^1_{\mathsf{v},-}\mathbf{w})(\mathsf{v})\big| \leq Ch_T^{-3/2}\|Q^1_{\mathsf{v},-}\mathbf{w}\|_{0,T} \leq Ch_T^{-1/2}\|\mathbf{w}\|_{0,\omega_\mathsf{v}}\,.$$

Summation over the above estimates yields the desired result. □

3.2.6 The bounded co-chain projectors

Recall that we defined, for $\mathsf{v} \in \mathcal{V}_f$,

$$Q^0_\mathsf{v} := Q^0_{\omega_\mathsf{v},\emptyset}\,,\qquad \boldsymbol{Q}^1_\mathsf{v} := \boldsymbol{Q}^1_{\omega_\mathsf{v},\emptyset}\,,\qquad Q^0_{\mathsf{v},-} := Q^0_{\omega_\mathsf{v},\emptyset,-}\,.$$

In addition, for $\mathsf{e} \in \mathcal{E}_f$, we set

$$Q^0_\mathsf{e} := Q^0_{\omega_\mathsf{e},\gamma_\mathsf{e}}\,,\qquad \boldsymbol{Q}^1_\mathsf{e} := \boldsymbol{Q}^1_{\omega_\mathsf{e},\gamma_\mathsf{e}}\,,\qquad Q^1_{\mathsf{e},-} := Q^1_{\omega_\mathsf{e},\gamma_\mathsf{e},-}\,, \tag{3.44}$$

where γ_{e} is constructed as in Sect. 3.2.3 when specifying the weight function $\mathbf{z}^1_{\mathsf{e}}$. Recall that $\gamma_{\mathsf{e}} = \emptyset$ for the case that $\mathsf{e}_1 \notin \overline{\Gamma_D}$ and $\mathsf{e}_2 \notin \overline{\Gamma_D}$.

Based on these operators, we define $R^0_D \colon H^1(\Omega) \to \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$ by

$$R^0_D u = S^0_D u + \sum_{\mathsf{v}\in\mathcal{V}_f} [(I - S^0_D)Q^0_{\mathsf{v}} u](\mathsf{v})\, \varphi_{\mathsf{v}}$$

and $\boldsymbol{R}^1_D \colon \boldsymbol{H}(\mathbf{curl}, \Omega) \to \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$ by

$$\boldsymbol{R}^1_D \mathbf{w} = \boldsymbol{S}^1_D \mathbf{w} + \sum_{\mathsf{e}\in\mathcal{E}_f} \int_{\mathsf{e}} [(I - \boldsymbol{S}^1_D)\boldsymbol{Q}^1_{\mathsf{e}} \mathbf{w}] \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds\, \boldsymbol{\psi}_{\mathsf{e}}\,.$$

Before we continue, we have to argue that the two operators are well-defined. For R^0_D, observe that

$$[(I - S^0_D)Q^0_{\mathsf{v}} u](\mathsf{v}) = [(I - M^0_D)Q^0_{\mathsf{v}} u](\mathsf{v}).$$

Since for any $p \in H^1(\Omega)$, the value $(M^0_D p)(\mathsf{v})$ depends only on $p_{|\omega_{\mathsf{v}}}$, the expression above is valid. For $\boldsymbol{R}^1_D$, recall that for $\bar{\mathbf{w}} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$,

$$\boldsymbol{S}^1_D \bar{\mathbf{w}} = \boldsymbol{M}^1_D \bar{\mathbf{w}} + \sum_{\mathsf{v}\in\mathcal{V}_f} (Q^1_{\mathsf{v},-} \bar{\mathbf{w}})(\mathsf{v})\, \mathbf{grad}\, \varphi_{\mathsf{v}}\,.$$

From the definition of $\boldsymbol{M}^1_D$, we see that $\int_{\mathsf{e}} (\boldsymbol{M}^1_D \bar{\mathbf{w}}) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds$ depends only on $\bar{\mathbf{w}}_{|\omega_{\mathsf{e}}}$. Since $(Q^1_{\mathsf{v},-} \bar{\mathbf{w}})(\mathsf{v})$ depends only on $\bar{\mathbf{w}}_{|\omega_{\mathsf{v}}}$, we can conclude altogether that $\int_{\mathsf{e}} (\boldsymbol{S}^1_D \bar{\mathbf{w}}) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds$ only depends on $\bar{\mathbf{w}}_{|\omega_{\mathsf{e}}}$. Setting (formally) $\bar{\mathbf{w}} = \boldsymbol{Q}^1_{\mathsf{e}} \mathbf{w}$ shows that $\boldsymbol{R}^1_D$ is well-defined.

As a next step, we show the projection property of R^0_D and $\boldsymbol{R}^1_D$.

Lemma 3.18. *For all $u_h \in \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$,*

$$R^0_D u_h = u_h\,.$$

Proof. Since both expressions are in $\mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$, it suffices to check the values at each free vertex $\mathsf{v} \in \mathcal{V}_f$:

$$(R^0_D u_h)(\mathsf{v}) = (S^0_D u_h)(\mathsf{v}) + [(I - S^0_D)\, \underbrace{Q^0_{\mathsf{v}} u_h}_{=u_{h|\omega_{\mathsf{v}}}}](\mathsf{v}) = u_h(\mathsf{v}),$$

where we have used (3.34). □

Lemma 3.19. *For all $\mathbf{w}_h \in \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$,*

$$\boldsymbol{R}^1_D \mathbf{w}_h = \mathbf{w}_h\,.$$

Proof. Since both expressions are in $\mathcal{W}^1_{h,\Gamma_D}(\mathcal{T})$, it suffices to check the integrals over each free edge $\mathsf{e} \in \mathcal{E}_f$:

$$\int_{\mathsf{e}} (\boldsymbol{R}^1_D \mathbf{w}_h) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds = \int_{\mathsf{e}} (\boldsymbol{S}^1_D \mathbf{w}_h) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds + \int_{\mathsf{e}} [(I - \boldsymbol{S}^1_D)\underbrace{\boldsymbol{Q}^1_{\mathsf{e}} \mathbf{w}_h}_{=\mathbf{w}_{h|\omega_{\mathsf{e}}}}] \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds = \int_{\mathsf{e}} \mathbf{w}_h \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds,$$

where we have used (3.35). □

The following lemma shows the commuting property of R^0_D, $\boldsymbol{R}^1_D$.

Lemma 3.20. *For all $u \in H^1_{\Gamma_D}(\Omega)$,*

$$\boldsymbol{R}^1_D \,\mathbf{grad}\, u = \mathbf{grad}\, R^0_D u.$$

Proof. Let $u \in H^1_{\Gamma_D}(\Omega)$. Firstly, using the definition of R^0_D, $S^0_D = M^0_D$, and Lemma 3.5 we obtain

$$\mathbf{grad}\, R^0_D u \;=\; \underbrace{\mathbf{grad}\, M^0_D u}_{=\boldsymbol{M}^1_D \,\mathbf{grad}\, u} + \sum_{\mathsf{v}\in\mathcal{V}_f} \underbrace{[(I - M^0_D)Q^0_{\mathsf{v}} u](\mathsf{v})}_{=[(I-M^0_D)Q^1_{\mathsf{v},-}\,\mathbf{grad}\, u](\mathsf{v})} \,\mathbf{grad}\, \varphi_{\mathsf{v}} \;=\; \boldsymbol{S}^1_D \,\mathbf{grad}\, u,$$

where in the last steps we have used that M^0_D preserves constants on each of the patches ω_{v}, $\mathsf{v} \in \mathcal{V}_f$ as well as representation (3.43) of $\boldsymbol{S}^1_D$. Secondly, by the commuting property (3.33) of the operators Q^0_{e}, $\boldsymbol{Q}^1_{\mathsf{e}}$,

$$\boldsymbol{R}^1_D \,\mathbf{grad}\, u - \boldsymbol{S}^1_D \,\mathbf{grad}\, u \;=\; \sum_{\mathsf{e}\in\mathcal{E}_f} \int_{\mathsf{e}} [(I - \boldsymbol{S}^1_D)\underbrace{\boldsymbol{Q}^1_{\mathsf{e}} \,\mathbf{grad}\, u}_{\mathbf{grad}\, Q^0_{\mathsf{e}} u}] \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds\, \boldsymbol{\psi}_{\mathsf{e}}\,.$$

Recall, for any $\mathsf{e} \in \mathcal{E}_f$, that $Q^0_{\mathsf{e}} u \in \mathcal{W}^0_{h,\gamma_{\mathsf{e}}}(\omega_{\mathsf{e}})$, see (3.44). Therefore, we can apply Lemma 3.15 and obtain that

$$\boldsymbol{R}^1_D \,\mathbf{grad}\, u - \boldsymbol{S}^1_D \,\mathbf{grad}\, u = 0.$$

To summarize,

$$\mathbf{grad}\, R^0_D u = \boldsymbol{S}^1_D \,\mathbf{grad}\, u = \boldsymbol{R}^1_D \,\mathbf{grad}\, u.$$ □

In the following, we show stability estimates for R^0_D, $\boldsymbol{R}^1_D$.

Lemma 3.21. *For all $u \in H^1(\Omega)$ and $T \in \mathcal{T}$,*

$$\|R^0_D u\|_{0,T} \le C(\|u\|_{0,\omega_T} + h_T \|\,\mathbf{grad}\, u\|_{0,\omega_T}).$$

Proof. Following the definition of R^0_D, we obtain from the triangle inequality that

$$\|R^0_D u\|_{0,T} \le \|M^0_D u\|_{0,T} + \sum_{\mathsf{v}\in\mathcal{V}_f\cap\overline{T}} \left|[(I - M^0_D)Q^0_{\mathsf{v}} u](\mathsf{v})\right| \|\varphi_{\mathsf{v}}\|_{0,T}\,.$$

The first term is bounded by $C\|u\|_{0,\omega_T}$; cf. Lemma 3.6. We bound the second term step by step. Let $\mathsf{v} \in \mathcal{V}_f \cap \overline{T}$. Using the definitions of M_D^0 and Q_v^0, we find that

$$(M_D^0 Q_\mathsf{v}^0 u)(\mathsf{v}) = \overline{Q_\mathsf{v}^0 u}^{\omega_\mathsf{v}} = \overline{u}^{\omega_\mathsf{v}},$$

and so, together with Lemma 3.4, we obtain

$$\begin{aligned}\big|[(I - M_D^0)Q_\mathsf{v}^0 u](\mathsf{v})\big| &\le \big|(Q_\mathsf{v}^0 u)(\mathsf{v})\big| + \big|(M_D^0 Q_\mathsf{v}^0 u)(\mathsf{v})\big| \\ &\le C h_T^{-3/2}\|Q_\mathsf{v}^0 u\|_{0,T} + |\overline{u}^{\omega_\mathsf{v}}|.\end{aligned}$$

Due to Lemma 3.11,

$$\|Q_\mathsf{v}^0 u\|_{0,T} \le \|u\|_{0,\omega_\mathsf{v}} + C h_T \|\mathbf{grad}\, u\|_{0,\omega_\mathsf{v}},$$

and with the Cauchy–Schwarz inequality,

$$|\overline{u}^{\omega_\mathsf{v}}| = \frac{1}{|\omega_\mathsf{v}|}\left|\int_{\omega_\mathsf{v}} u\,dx\right| \le \frac{1}{|\omega_\mathsf{v}|^{1/2}}\|u\|_{\omega_\mathsf{v}} \le C h_T^{-3/2}\|u\|_{\omega_\mathsf{v}}.$$

Combining all the estimate from above, we can conclude that

$$\big|[(I - M_D^0)Q_\mathsf{v}^0 u](\mathsf{v})\big| \le C h_T^{-3/2}(\|u\|_{0,\omega_\mathsf{v}} + h_T\|\mathbf{grad}\, u\|_{0,\omega_\mathsf{v}}).$$

Since $\|\varphi_\mathsf{v}\|_{0,T} \le C h_T^{3/2}$, we obtain the following bound for the second term:

$$\sum_{\mathsf{v}\in\mathcal{V}_f\cap\overline{T}} \big|[(I - M_D^0)Q_\mathsf{v}^0 u](\mathsf{v})\big|\,\|\varphi_\mathsf{v}\|_{0,T} \le C(\|u\|_{0,\omega_T} + h_T\|\mathbf{grad}\, u\|_{0,\omega_T}),$$

which concludes the proof. □

Lemma 3.22. *For all* $\mathbf{w} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ *and* $T \in \mathcal{T}$,

$$\|\boldsymbol{R}_D^1\mathbf{w}\|_{0,T} \le C(\|\mathbf{w}\|_{0,\omega_T} + h_T\|\mathbf{curl}\ \mathbf{w}\|_{0,\omega_T}).$$

Proof. Following the definition of $\boldsymbol{R}_D^1$, we find that

$$\|\boldsymbol{R}_D^1\mathbf{w}\|_{0,T} \le \|\boldsymbol{S}_D^1\mathbf{w}\|_{0,T} + \sum_{\mathsf{e}\in\mathcal{E}_f\cap\overline{T}}\left|\int_\mathsf{e}[(I - \boldsymbol{S}_D^1)\boldsymbol{Q}_\mathsf{e}^1\mathbf{w}]\cdot\tau_\mathsf{e}\,ds\right|\|\boldsymbol{\psi}_\mathsf{e}\|_{0,T}$$

The first term can be bounded by $C\|\mathbf{w}\|_{0,\omega_T}$; cf. Lemma 3.16. The second term is bounded step by step. Let $\mathsf{e} \in \mathcal{E}_f \cap \overline{T}$. Then due to Lemma 3.4, Lemma 3.17, and Lemma 3.12,

$$\left|\int_\mathsf{e}[(I - \boldsymbol{S}_D^1)\boldsymbol{Q}_\mathsf{e}^1\mathbf{w}]\cdot\tau_\mathsf{e}\,ds\right| \le \left|\int_\mathsf{e}(\boldsymbol{Q}_\mathsf{e}^1\mathbf{w})\cdot\tau_\mathsf{e}\,ds\right| + \left|\int_\mathsf{e}(\boldsymbol{S}_D^1\boldsymbol{Q}_\mathsf{e}^1\mathbf{w})\cdot\tau_\mathsf{e}\,ds\right|$$

$$\leq Ch_T^{-1/2}\|\mathbf{Q}_e^1\mathbf{w}\|_{0,T} + Ch_T^{-1/2}\|\mathbf{Q}_e^1\mathbf{w}\|_{0,T}$$
$$\leq Ch_T^{-1/2}(\|\mathbf{w}\|_{0,\omega_e} + h_T\|\mathbf{curl}\ \mathbf{w}\|_{0,\omega_e}).$$

Since $\|\boldsymbol{\psi}_e\| \leq Ch_T^{1/2}$ (Lemma 3.4), summation over the free edges of T and incorporating the estimate for $\boldsymbol{S}_D^1\mathbf{w}$ yields

$$\|\boldsymbol{R}_D^1\mathbf{w}\|_{0,T} \leq C\|\mathbf{w}\|_{0,\omega_T} + \sum_{e\in\mathcal{E}_f\cap\overline{T}} C(\|\mathbf{w}\|_{0,\omega_e} + h_T\|\mathbf{curl}\ \mathbf{w}\|_{0,\omega_e})$$
$$\leq C(\|\mathbf{w}\|_{0,\omega_T} + h_T\|\mathbf{curl}\ \mathbf{w}\|_{0,\omega_T}),$$

which concludes the proof. □

Corollary 3.23. *For all $u \in H^1_{\Gamma_D}(\Omega)$ and $T \in \mathcal{T}$,*

$$|R_D^0 u|_{1,T} \leq C|u|_{1,\omega_T}.$$

Proof. The statement follows immediately from Lemma 3.20 and Lemma 3.22. □

3.3 Proof of Theorem 1.2

Throughout the proof, we assume that $\operatorname{diam}(\Omega) = 1$, because the general case then follows by a simple scaling argument. Given $\mathbf{v}_h \in \mathcal{W}^1_{h,\Gamma_D}(\mathcal{T})$, we apply the continuous regular decomposition from Theorem 2.1, so

$$\mathbf{v}_h = \mathbf{z} + \mathbf{grad}\,\varphi$$

with $\mathbf{z} \in \boldsymbol{H}^1_{\Gamma_D}(\Omega)$, $\varphi \in H^1_{\Gamma_D}(\Omega)$ depending linearly on $\mathbf{v}_h$, and

$$\|\varphi\|_{1,\Omega} \leq C\|\mathbf{v}_h\|_{0,\Omega}, \tag{3.45}$$
$$\|\mathbf{z}\|_{0,\Omega} \leq C\|\mathbf{v}_h\|_{0,\Omega}, \tag{3.46}$$
$$\|\mathbf{z}\|_{1,\Omega} \leq C\|\mathbf{v}_h\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}. \tag{3.47}$$

Recall the projection operators R_D^0 and $\boldsymbol{R}_1^D$ from Section 3.2.6 and the modified Clément operator M_D^0 from Section 3.2.3. Let $\boldsymbol{M}_D^0 : \boldsymbol{L}^2(\Omega) \to \boldsymbol{\mathcal{V}}^0_{h,\Gamma_D}(\mathcal{T}) = (\mathcal{W}^0_{h,\Gamma_D}(\mathcal{T}))^3$ denote the corresponding vector-valued operator (defined component-wise). Due to the projection property Lemma 3.19, $\boldsymbol{R}_D^1\mathbf{v}_h = \mathbf{v}_h$, and so

$$\mathbf{v}_h = \boldsymbol{R}_D^1\mathbf{z} + \underbrace{\boldsymbol{R}_D^1\,\mathbf{grad}\,\varphi}_{=\mathbf{grad}\,R_D^0\varphi} = \boldsymbol{R}_D^1\underbrace{\boldsymbol{M}_D^0\mathbf{z}}_{=:\mathbf{z}_h} + \underbrace{\boldsymbol{R}_D^1(I-\boldsymbol{M}_D^0)\mathbf{z}}_{=:\tilde{\mathbf{v}}_h} + \mathbf{grad}\underbrace{R_D^0\varphi}_{=:\varphi_h}. \tag{3.48}$$

From Lemma 3.6, Lemma 3.7, and Corollary 3.10, we obtain

$$\|\boldsymbol{M}_D^0\mathbf{z}\|_{0,T} \leq C\|\mathbf{z}\|_{0,\omega_T}, \tag{3.49}$$

$$|\boldsymbol{M}_D^0 \mathbf{z}|_{1,T} \le C|\mathbf{z}|_{1,\omega_T}\,, \tag{3.50}$$

$$\|(I - \boldsymbol{M}_D^0)\mathbf{z}\|_{0,T} \le C h_T\, |\mathbf{z}|_{1,\widetilde{\omega}_T}\,, \tag{3.51}$$

where $\widetilde{\omega}_T$ is the possibly enlarged element patch; see (3.28).

Due to the mapping properties of R_D^0 and $\boldsymbol{R}_D^1$, we obtain that

$$\mathbf{v}_h = \boldsymbol{R}_D^1 \mathbf{z}_h + \widetilde{\mathbf{v}}_h + \mathbf{grad}\,\varphi_h$$

with

$$\mathbf{z}_h \in \mathcal{V}^0_{h,\Gamma_D}(\mathcal{T}), \qquad \widetilde{\mathbf{v}}_h \in \mathcal{W}^1_{h,\Gamma_D}(\mathcal{T}), \qquad \varphi_h \in \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T}).$$

Combining (3.46), (3.47), (3.49), and (3.50) imply the following estimates for $\mathbf{z}_h$:

$$\begin{aligned}
\|\mathbf{z}_h\|_{0,\Omega} &= \|\boldsymbol{M}_D^0 \mathbf{z}\|_{0,\Omega} \le C\|\mathbf{z}\|_{0,\Omega} \le C\|\mathbf{v}_h\|_{0,\Omega}\,,\\
|\mathbf{z}_h|_{1,\Omega} &= |\boldsymbol{M}_D^0 \mathbf{z}|_{1,\Omega} \le C|\mathbf{z}|_{1,\Omega} \le C\|\mathbf{v}_h\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\,.
\end{aligned}$$

From Lemma 3.22 and an inverse inequality, we conclude

$$\begin{aligned}
\|\boldsymbol{R}_D^1 \mathbf{z}_h\|^2_{0,\Omega} &= \sum_{T\in\mathcal{T}} \|\boldsymbol{R}_D^1 \mathbf{z}_h\|^2_{0,T}\\
&\le C \sum_{T\in\mathcal{T}} \big(\|\mathbf{z}_h\|^2_{0,\omega_T} + h_T^2 \underbrace{\|\mathbf{curl}\,\mathbf{z}_h\|^2_{0,\omega_T}}_{\le |\mathbf{z}_h|^2_{1,\omega_T}}\big) \;\le\; C\|\mathbf{z}_h\|^2_{0,\Omega}\,.
\end{aligned}$$

Our next term to be considered is $\widetilde{\mathbf{v}}_h$. Lemma 3.22, (3.51), and (3.50) yield

$$\begin{aligned}
\|h^{-1}\widetilde{\mathbf{v}}_h\|^2_{0,\Omega} &= \sum_{T\in\mathcal{T}} h_T^{-2} \|\boldsymbol{R}_D^1 (I - \boldsymbol{M}_D^0)\mathbf{z}\|^2_{0,T}\\
&\le C \sum_{T\in\mathcal{T}} h_T^{-2} \big(\underbrace{\|(I - \boldsymbol{M}_D^0)\mathbf{z}\|^2_{0,\omega_T}}_{\le C h_T^2 |\mathbf{z}|^2_{1,\widetilde{\omega}_T}} + h_T^2 \underbrace{|(I - \boldsymbol{M}_D^0)\mathbf{z}|^2_{1,\omega_T}}_{\le C|\mathbf{z}|^2_{1,\omega_T}}\big)\\
&\le C|\mathbf{z}|^2_{1,\Omega} \;\le\; C\|\mathbf{v}_h\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\,.
\end{aligned}$$

For the same vector field without the scaling factor, we obtain from Lemma 3.22

$$\begin{aligned}
\|\widetilde{\mathbf{v}}_h\|_{0,\Omega} &\le C \sum_{T\in\mathcal{T}} \big(\|(I - \boldsymbol{M}_D^0)\mathbf{z}\|_{0,\omega_T} + h_T \|\mathbf{curl}\,(I - \boldsymbol{M}_D^0 \mathbf{z})\|_{0,\omega_T}\big)\\
&\le C \sum_{T\in\mathcal{T}} \big(\|\mathbf{z}\|_{0,\omega_T} + \|\boldsymbol{M}_D^0 \mathbf{z}\|_{0,\omega_T} + h_T \|\mathbf{curl}\;\mathbf{z}\|_{0,\omega_T} + h_T \|\mathbf{curl}\;\boldsymbol{M}_D^0 \mathbf{z}\|_{0,\omega_T}\big).
\end{aligned}$$

Since $\mathbf{curl}\,\mathbf{z} = \mathbf{curl}\,\mathbf{v}_h$, local inverse inequalities imply

$$\begin{aligned}
h_T \|\mathbf{curl}\,\mathbf{z}\|_{0,\omega_T} &\le C\|\mathbf{v}_h\|_{0,\omega_T}\,,\\
h_T \|\mathbf{curl}\,\boldsymbol{M}_D^0 \mathbf{z}\|_{0,\omega_T} &\le C\|\boldsymbol{M}_D^0 \mathbf{z}\|_{0,\omega_T}\,.
\end{aligned}$$

Together with (3.49) and (3.46), we find that

$$\|\tilde{\mathbf{v}}_h\|_{0,T} \le C\big(\|\mathbf{z}\|_{0,\Omega} + \|\mathbf{v}_h\|_{0,\Omega}\big) \le C\|\mathbf{v}_h\|_{0,\Omega}\,.$$

Finally, we consider the scalar potential. From Corollary 3.23 and (3.45), we obtain

$$|\varphi_h|_{1,\Omega} = \big|R_D^0\varphi\big|_{1,\Omega} \le C|\varphi|_{1,\Omega} \le C\|\mathbf{v}_h\|_{0,\Omega}\,.$$

For an estimate in the L^2-norm, we use the (global) Friedrichs (for $\Gamma_D \neq \emptyset$) or Poincaré inequality

$$\|\varphi_h\|_{0,\Omega} \le C|\varphi_h|_{1,\Omega} \le C\|\mathbf{v}_h\|_{0,\Omega}\,,$$

(recall that $\operatorname{diam}(\Omega) = 1$). This implies an overall estimate in the full H^1-norm and concludes the proof of Theorem 1.2.

4 Discrete regular decomposition: *p*-version

Now we aim to establish existence and stability of discrete regular decompositions of the finite element space $\mathcal{W}^1_{\Gamma_D}(\mathcal{T}) \subset \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ for arbitrary polynomial degree $p \in \mathbb{N}_0$. The final result has already been stated in Theorem 1.3. The key objective is to ensure that stability holds *uniformly in p*, in addition to independence of the local mesh width of $\mathcal{T}$, of course. Thus, in this section, we use the symbols $\lesssim$, $\gtrsim$, and $\approx$ to express one- and two-sided inequalities up to constants that may depend only on Ω, Γ_D, and the shape regularity measure $\rho(\mathcal{T})$ of the mesh as defined in (1.1); the constants must not depend on p!

The proof of Theorem 1.3 given in this section runs structurally parallel to that of Theorem 1.2 as presented in Section 3.3. There are substantial differences in the two main ingredients, the commuting projector and quasi-interpolation operator:

(I) For want of p-stable local commuting co-chain projections generalizing the construction of Section 3.2, we have to resort to an alternative tool: commuting projection-based interpolation operators, whose details will be explained in Section 4.1.

(II) The modified Clement operator $\mathbf{M}_D^0$ will be replaced with smoothed interpolation, which will be elaborated in Section 4.2.

4.1 Projection-based interpolation

Projection based interpolation supplies perfectly local projectors onto the local spaces of discrete differential form that commute with the differential operators **grad**, **curl**, div, respectively. Locality also extends to the values on the facets (vertices, edges, faces) of tetrahedra, which makes it possible to assemble the local operators into projectors onto $\mathcal{W}^l_{\Gamma_D}(\mathcal{T})$.

The design of these operators is an intricate multi-stage procedure and we follow [36, Section 3.5]. Their main algebraic properties are stated in Lemmata 4.5, 4.6, and 4.7. Even more demanding is the proof of p-uniform approximation properties, which was accomplished in [20]. We recall the result only for 0-forms, that is, scalar functions, in Theorem 4.10, since it will be instrumental for getting the special interpolation error estimate of Lemma 4.16. Its proof will also hinge on a special stable lifting operator from [19] that we recall in the next section.

All considerations in this section are purely local. Therefore, in the beginning we single out an arbitrary tetrahedron $T \in \mathcal{T}$. All constants in estimates may only depend on its shape regularity measure $\rho(T) := {}^{h_T}/_{r_T}$.

4.1.1 Tool: smoothed Poincaré lifting

Let $D \subset \mathbb{R}^3$ stand for a bounded domain that is star-shaped with respect to a subdomain $B \subset D$, that is,

$$\forall \boldsymbol{a} \in B, \boldsymbol{x} \in D : \quad \{t\boldsymbol{a} + (1-t)\boldsymbol{x},\ 0 < t < 1\} \subset D\,. \tag{4.1}$$

Definition 4.1. The *Poincaré lifting* $\mathsf{R}_{\boldsymbol{a}} : \boldsymbol{C}^0(\overline{\Omega}) \mapsto \boldsymbol{C}^0(\overline{\Omega})$, $\boldsymbol{a} \in B$, is defined as

$$\mathsf{R}_{\boldsymbol{a}}(\mathbf{u})(\boldsymbol{x}) := \int_0^1 t\mathbf{u}\big(\boldsymbol{x} + t(\boldsymbol{x} - \boldsymbol{a})\big)\,\mathrm{d}t \times (\boldsymbol{x} - \boldsymbol{a})\,, \quad \boldsymbol{x} \in D\,, \tag{4.2}$$

where $\times$ designates the cross product of two vectors in $\mathbb{R}^3$.

This is a special case of the generalized path integral formula for differential forms, which is instrumental in proving the exactness of closed forms on star-shaped domains, the so-called "Poincaré lemma"; see [12, Section 2.13].

The linear mapping $\mathsf{R}_{\boldsymbol{a}}$ provides a right inverse of the **curl**-operator on divergence-free vector fields, see [30, Proposition 2.1] for the simple proof, and [12, Section 2.13] for a general proof based on differential forms.

Lemma 4.2. *If* $\operatorname{div}\mathbf{u} = 0$, *then for any* $\boldsymbol{a} \in B$, $\mathbf{curl}\,\mathsf{R}_{\boldsymbol{a}}\mathbf{u} = \mathbf{u}$ *for all* $\mathbf{u} \in \boldsymbol{C}^1(\overline{D})$.

Unfortunately, the mapping $\mathsf{R}_{\boldsymbol{a}}$ cannot be extended to a continuous mapping $\boldsymbol{L}^2(D) \mapsto \boldsymbol{H}^1(D)$, *cf.* [30, Theorem 2.1]. As discovered in the breakthrough paper [19] based on earlier work of Bogovskiĭ [10], it takes a smoothed version to accomplish this: we introduce the *smoothed Poincaré lifting*[1]

$$\mathsf{R}(\mathbf{u}) := \int_B \Phi(\boldsymbol{a})\mathsf{R}_{\boldsymbol{a}}(\mathbf{u})\,\mathrm{d}\boldsymbol{a}\,, \tag{4.3}$$

1 The dependence of R on Φ is dropped from the notation.

where

$$\Phi \in C^\infty(\mathbb{R}^3)\,, \quad \operatorname{supp}\Phi \subset B\,, \quad \int_B \Phi(\boldsymbol{a})\,\mathrm{d}\boldsymbol{a} = 1\,. \tag{4.4}$$

The substitution

$$\boldsymbol{y} := \boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a}), \qquad \tau := \frac{1}{1-t}\,, \tag{4.5}$$

transforms the integral (4.4) into

$$\begin{aligned} \mathsf{R}(\mathbf{u})(\boldsymbol{x}) &= \int_{\mathbb{R}^3}\int_1^\infty \tau(1-\tau)\mathbf{u}(\boldsymbol{y}) \times (\boldsymbol{x}-\boldsymbol{y})\Phi(\boldsymbol{y}+\tau(\boldsymbol{y}-\boldsymbol{x}))\,\mathrm{d}\tau\mathrm{d}\boldsymbol{y} \\ &= \int_{\mathbb{R}^3} \mathbf{k}(\boldsymbol{x},\boldsymbol{y}-\boldsymbol{x}) \times \mathbf{u}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,, \end{aligned} \tag{4.6}$$

that is, R is a convolution-type integral operator with kernel

$$\begin{aligned} \mathbf{k}(\boldsymbol{x},\boldsymbol{z}) &= \int_1^\infty \tau(1+\tau)\Phi(\boldsymbol{x}+\tau\boldsymbol{z})\boldsymbol{z}\,\mathrm{d}\tau \\ &= \frac{\boldsymbol{z}}{|\boldsymbol{z}|^2}\int_1^\infty \zeta\Phi\Big(\boldsymbol{x}+\zeta\frac{\boldsymbol{z}}{|\boldsymbol{z}|}\Big)\mathrm{d}\zeta + \frac{\boldsymbol{z}}{|\boldsymbol{z}|^3}\int_1^\infty \zeta^2\Phi\Big(\boldsymbol{x}+\zeta\frac{\boldsymbol{z}}{|\boldsymbol{z}|}\Big)\mathrm{d}\zeta\,. \end{aligned} \tag{4.7}$$

The kernel can be bounded by $|\mathbf{k}(\boldsymbol{x},\boldsymbol{z})| \le K(\boldsymbol{x})|\boldsymbol{z}|^{-2}$, where $K \in C^\infty(\mathbb{R}^3)$ depends only on Φ and is locally uniformly bounded. As a consequence, (4.6) exists as an improper integral.

The intricate but elementary analysis of [19, Section 3.3] further shows, that $\mathbf{k}$ belongs to the Hörmander symbol class $S^{-1}_{1,0}(\mathbb{R}^3)$; see [61, Chapter 7]. Invoking the theory of pseudo-differential operators [61, Proposition 5.5], we obtain the following continuity result, which is a special case of [19, Corollary 3.4].

Theorem 4.3. *The mapping* R *can be extended to a* continuous *linear operator* $\boldsymbol{L}^2(D) \mapsto \boldsymbol{H}^1(D)$*, which is still denoted by* R*. It satisfies*

$$\mathbf{curl}\,\mathsf{R}\mathbf{u} = \mathbf{u} \quad \forall \mathbf{u} \in \boldsymbol{H}(\mathrm{div}, D),\ \mathrm{div}\,\mathbf{u} = 0\,. \tag{4.8}$$

The smoothed Poincaré lifting shares this continuity property with many other mappings; see [36, Section 2.4]. Yet, it enjoys another essential feature, which is immediate from its definition (4.2): R maps polynomials of degree p to other polynomials of degree $\le p+1$. The next section will highlight the significance of this observation.

4.1.2 $\mathcal{W}^1_p(\mathcal{T})$: a local view

According to [34, Section 3], for any $T \in \mathcal{T}$, $\boldsymbol{a} \in T$, we can obtain the local space of discrete 1-forms of the first family as

$$\mathcal{W}^1_p(T) = \boldsymbol{\mathcal{P}}_p(\mathbb{R}^3) + \mathsf{R}_{\boldsymbol{a}}(\{\mathbf{q} \in \boldsymbol{\mathcal{P}}_p(\mathbb{R}^3),\ \mathrm{div}\,\mathbf{q} = 0\})\,. \tag{4.9}$$

Independence of $\boldsymbol{a}$ is discussed in [34, Section 3]. The representation (4.9) can be established by dimensional arguments: from the formula (4.2) for the Poincaré lifting we immediately see that $\boldsymbol{\mathcal{P}}_p(\mathbb{R}^3) + \mathsf{R}_{\boldsymbol{a}}(\boldsymbol{\mathcal{P}}_p(\mathbb{R}^3)) \subset \mathcal{W}^1_p(T)$. In addition, from [54, Lemma 4] and [34, Theorem 6, case $l = 1$, $n = 3$] we learn that the dimensions of both spaces agree and are equal to

$$\dim \mathcal{W}^1_p(T) = \tfrac{1}{2}(1+p)(3+p)(4+p) . \tag{4.10}$$

As a consequence, the two finite dimensional spaces must agree.

For the remainder of this section, which focuses on local spaces, we single out a tetrahedron $T \in \mathcal{T}$. On T we can introduce a smoothed Poincaré lifting R_T according to (4.3) with $B = T$ and a suitable $\Phi \in C^\infty_0(T)$ complying with (4.4). An immediate consequence of (4.9) is that

$$\mathsf{R}_T(\{\mathbf{v} \in \boldsymbol{\mathcal{P}}_p(\mathbb{R}^3) : \operatorname{div} \mathbf{v} = 0\}) \subset \mathcal{W}^1_p(T) . \tag{4.11}$$

We introduce the notation $\mathcal{F}_m(T)$ for the set of all m-dimensional facets of T, $m = 0, 1, 2, 3$. Hence, $\mathcal{F}_0(T)$ contains the vertices of T, $\mathcal{F}_1(T)$ the edges, $\mathcal{F}_2(T)$ the faces, and $\mathcal{F}_3(T) = \{T\}$. Moreover, for some $F \in \mathcal{F}_m(T)$, $m = 1, 2, 3$, $\mathcal{P}_p(F)$ denotes the space of m-variate polynomials of total degree $\leq p$ in a local coordinate system of the facet F, and $\boldsymbol{\mathcal{P}}_p(F)$ will designate corresponding tangential polynomial vector fields. Further, we write

$$\mathcal{W}^1_p(e) = \mathcal{W}^1_p(T) \cdot \boldsymbol{t}_e , \quad \boldsymbol{t}_e \text{ the unit tangent vector of } e,\ e \in \mathcal{F}_1(T) , \tag{4.12}$$

$$\mathcal{W}^1_p(f) = \mathcal{W}^1_p(T) \times \boldsymbol{n}_f , \quad \boldsymbol{n}_f \text{ the unit normal vector of } f,\ f \in \mathcal{F}_2(T) , \tag{4.13}$$

for the tangential traces of local edge element vector fields onto edges and faces. Simple vector analytic manipulations permit us to deduce from (4.9) that

$$\mathcal{W}^1_p(e) = \mathcal{P}_p(e) , \quad e \in \mathcal{F}_1(T) , \tag{4.14}$$

$$\mathcal{W}^1_p(f) = \boldsymbol{\mathcal{P}}_p(f) + \mathsf{R}^{2D}_{\boldsymbol{a}}(\mathcal{P}_p(f)) , \quad \boldsymbol{a} \in f , \quad f \in \mathcal{F}_2(T) , \tag{4.15}$$

where the projection $\mathsf{R}^{2D}_{\boldsymbol{a}}$ of the Poincaré lifting in the plane reads

$$\mathsf{R}^{2D}_{\boldsymbol{a}}(u)(\boldsymbol{x}) := \int_0^1 t u(\boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a})](\boldsymbol{x} - \boldsymbol{a})\, \mathrm{d}t , \quad \boldsymbol{a} \in \mathbb{R}^2 . \tag{4.16}$$

It satisfies $\operatorname{div}_\Gamma \mathsf{R}^{2D}_{\boldsymbol{a}}(u) = u$ for all $u \in C^\infty(\mathbb{R}^2)$. We point out that, along with (4.9), the formulas (4.14) and (4.15) are special versions of the general representation formula for discrete 1-forms; see [34, Formula (16)] and [4, Section 3.2]. Special facet tangential trace spaces with "zero boundary conditions" will also be needed:

$$\mathring{\mathcal{W}}^1_p(e) := \left\{ u \in \mathcal{W}^1_p(e) : \int_e u \, \mathrm{d}l = 0 \right\} , \quad e \in \mathcal{F}_1(T) , \tag{4.17}$$

$$\mathring{\mathcal{W}}_p^1(f) := \{\mathbf{u} \in \mathcal{W}_p^1(f) : \ \mathbf{u} \cdot \boldsymbol{n}_{e,f} \equiv 0 \ \forall e \in \mathcal{F}_1(T), \ e \subset \partial f\} , \quad f \in \mathcal{F}_2(T) , \tag{4.18}$$
$$\mathring{\mathcal{W}}_p^1(T) := \{\mathbf{u} \in \mathcal{W}_p^1(T) : \ \mathbf{u} \times \boldsymbol{n}_f \equiv 0 \ \forall f \in \mathcal{F}_2(T)\} . \tag{4.19}$$

Here, $\boldsymbol{n}_f$ represents an exterior face unit normal of T, $\boldsymbol{n}_{e,f}$ the in-plane normal of a face w. r. t. an edge $e \subset \partial f$.

According to [54, Section 1.2], [34, Section 4], and [4, Section 4.3], the local degrees of freedom for $\mathcal{W}_p^1(T)$ are given by the first $p-2$ vectorial moments on the cells of $\mathcal{T}$, the first $p-1$ vectorial moments of the tangential components on the faces of $\mathcal{T}$ and the first p tangential moments along the edges of T; see (4.21) for concrete formulas. Then the set $\mathrm{dof}_p^1(T)$ of local degrees of freedom can be partitioned as [49, Chapter 3], [4, Section 4.5],

$$\mathrm{dof}_p^1(T) = \bigcup_{e \in \mathcal{F}_1(T)} \mathrm{ldf}_p^1(e) \ \cup \bigcup_{f \in \mathcal{F}_2(T)} \mathrm{ldf}_p^1(f) \ \cup \ \mathrm{ldf}_p^1(T) , \tag{4.20}$$

where the functionals in $\mathrm{ldf}_p^1(e)$, $\mathrm{ldf}_p^1(f)$, and $\mathrm{ldf}_p^1(T)$ are supported on an edge, face, and T, respectively, and read

$$\begin{aligned}
\kappa \in \mathrm{ldf}_p^1(e) \quad &\Rightarrow \quad \kappa(\mathbf{u}) = \textstyle\int_e q\boldsymbol{\xi} \cdot \boldsymbol{t}_e \,\mathrm{d}l &&\text{for } e \in \mathcal{F}_1(T), \text{ suitable } q \in \mathcal{P}_p(e) ,\\
\kappa \in \mathrm{ldf}_p^1(f) \quad &\Rightarrow \quad \kappa(\mathbf{u}) = \textstyle\int_f \mathbf{q} \cdot (\boldsymbol{\xi} \times \mathbf{n}) \,\mathrm{d}S &&\text{for } f \in \mathcal{F}_2(T), \text{ suitable } \mathbf{q} \in \boldsymbol{\mathcal{P}}_{p-1}(f) ,\\
\kappa \in \mathrm{ldf}_p^1(T) \quad &\Rightarrow \quad \kappa(\mathbf{u}) := \textstyle\int_T \mathbf{q} \cdot \boldsymbol{\xi} \,\mathrm{d}\boldsymbol{x} &&\text{for certain } \mathbf{q} \in \boldsymbol{\mathcal{P}}_{p-2}(T) .
\end{aligned} \tag{4.21}$$

These functionals are unisolvent on $\mathcal{W}_p^1(T)$ and locally fix the tangential trace of $\mathbf{u} \in \mathcal{W}_p^1(T)$. There is a splitting of $\mathcal{W}_p^1(T)$ dual to (4.20): Defining

$$\mathcal{Y}_p^1(F) := \{\mathbf{v} \in \mathcal{W}^1(T) : \ \kappa(\mathbf{v}) = 0 \ \forall \kappa \in \mathrm{dof}_p^1(T) \setminus \mathrm{ldf}_p^1(F)\} \tag{4.22}$$

for $F \in \mathcal{F}_m(T)$, $m = 1, 2, 3$, we find the direct sum decomposition

$$\mathcal{W}_p^1(T) = \sum_{m=1}^{3} \sum_{F \in \mathcal{F}_m(T)} \mathcal{Y}_p^1(F) . \tag{4.23}$$

In addition, note that the tangential trace of $\mathbf{u} \in \mathcal{Y}_p^1(F)$ vanishes on all facets $\neq F$, whose dimension is smaller or equal the dimension of F. By the unisolvence of $\mathrm{dof}_p^1(T)$, there are bijective linear *extension operators*

$$\mathsf{E}_{e,p}^1 : \mathcal{W}_p^1(e) \mapsto \mathcal{Y}_p^1(e) , \quad e \in \mathcal{F}_1(T) , \tag{4.24}$$
$$\mathsf{E}_{f,p}^1 : \mathring{\mathcal{W}}_p^1(f) \mapsto \mathcal{Y}_p^1(f) , \quad f \in \mathcal{F}_2(T) . \tag{4.25}$$

Similar relationships hold for discrete 2-forms, for which we have the following alternative representation of the local space [34, formula (16) for $l = 2$, $n = 3$]:

$$\mathcal{W}_p^2(T) = \boldsymbol{\mathcal{P}}_p(T) + \mathsf{D}_{\boldsymbol{a}}(\mathcal{P}_p(T)) , \tag{4.26}$$

where the appropriate version of the Poincaré lifting reads

$$(\mathsf{D}_{\boldsymbol{a}}u)(\boldsymbol{x}) := \int_0^1 t^2 u(\boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a}))(\boldsymbol{x} - \boldsymbol{a})\,\mathrm{d}t\,, \quad \boldsymbol{a} \in T\,. \tag{4.27}$$

Like (4.9) this is a special incarnation of the general formula (16) in [34]. Again, dimensional arguments based on [54, Section 1.3] and [34, Theorem 6] confirm the representation (4.27). We remark that $\operatorname{div}\mathsf{D}_{\boldsymbol{a}}u = u$, see [30, Proposition 1.2].

The normal trace space of $\mathcal{W}_p^2(T)$ onto a face is

$$\mathcal{W}_p^2(f) := \mathcal{W}_p^2(T)\cdot\boldsymbol{n}_f = \mathcal{P}_p(f)\,, \quad f \in \mathcal{F}_2(T)\,, \tag{4.28}$$

and as relevant space "with zero trace" we are going to need

$$\mathring{\mathcal{W}}_p^2(f) := \left\{u \in \mathcal{W}_p^2(f) : \int_f u\,\mathrm{d}S = 0\right\}, \quad f \in \mathcal{F}_2(T)\,, \tag{4.29}$$

$$\mathring{\mathcal{W}}_p^2(T) := \{\mathbf{u} \in \mathcal{W}_p^2(T) :\ \mathbf{u}\cdot\boldsymbol{n}_{\partial T} = 0\}\,. \tag{4.30}$$

The connection between the local spaces $\mathcal{W}_p^1(T)$, $\mathcal{W}_p^2(T)$ and full polynomial spaces is established through a local exact sequence [34, Section 5]. To elucidate the relationship between differential operators and various traces onto faces and edges, we also include those in the statement of the following theorem. There $\boldsymbol{n}_f$ stands for an exterior face unit normal of T, $\boldsymbol{n}_{e,f}$ for the in-plane normal of a face w. r. t. an edge $e \subset \partial f$, and $\frac{d}{ds}$ is the differentiation w. r. t. arc length on an edge.

Theorem 4.4 (Local exact sequences). *For $f \in \mathcal{F}_2(T)$, $e \in \mathcal{F}_1(T)$, $e \subset \partial f$, all the sequences in*

$$\begin{array}{ccccccccccc}
\text{const} & \xrightarrow{\mathsf{Id}} & \mathcal{P}_{p+1}(T) & \xrightarrow{\mathbf{grad}} & \mathcal{W}_p^1(T) & \xrightarrow{\mathbf{curl}} & \mathcal{W}_p^2(T) & \xrightarrow{\operatorname{div}} & \mathcal{P}_p(T) & \xrightarrow{0} & \{0\} \\
 & & \downarrow{\scriptstyle \cdot|_f} & & \downarrow{\scriptstyle .\times\boldsymbol{n}_{f|f}} & & \downarrow{\scriptstyle \cdot\cdot\boldsymbol{n}_{f|f}} & & & & \\
\text{const} & \xrightarrow{\mathsf{Id}} & \mathcal{P}_{p+1}(f) & \xrightarrow{\mathbf{curl}_\Gamma} & \mathcal{W}_p^1(f) & \xrightarrow{\operatorname{div}_\Gamma} & \mathcal{P}_p(f) & \xrightarrow{0} & \{0\} & & \\
 & & \downarrow{\scriptstyle \cdot|_e} & & \downarrow{\scriptstyle \cdot\cdot\boldsymbol{n}_{e,f|e}} & & & & & & \\
\text{const} & \xrightarrow{\mathsf{Id}} & \mathcal{P}_{p+1}(e) & \xrightarrow{\frac{d}{ds}} & \mathcal{P}_p(e) & \xrightarrow{0} & \{0\} & & & &
\end{array}$$

are exact and the diagram commutes.

4.1.3 Projections, liftings, and extensions

Following the developments of [36, Section 3.5], projection based interpolation requires building blocks in the form of local *orthogonal* projections P_*^l and liftings L_*^l.[2]

[2] The parameter l in the notation for the extension operators E_*^l, the projections P_*^l, and the liftings L_*^l refers to the degree of the discrete differential form they operate on. This is explained in more detail in [36, Section 3.5].

Some operators will depend on a regularity parameter $0 < \epsilon < \frac{1}{2}$, which is considered fixed below and will be specified in Section 4.1.5. To begin with, we define for every $e \in \mathcal{F}_1(T)$

$$\mathsf{P}^1_{e,p} : H^{-1+\epsilon}(e) \mapsto \frac{d}{ds}\mathring{\mathcal{P}}_{p+1}(e) = \mathring{\mathcal{W}}^1_p(e) \tag{4.31}$$

as the $H^{-1+\epsilon}(e)$-orthogonal projection. Here, $\mathring{\mathcal{P}}_p(F)$ denotes the space of degree p polynomials on a facet F that vanish on ∂F.

Similarly, for every face $f \in \mathcal{F}_2(T)$ introduce

$$\mathsf{P}^1_{f,p} : \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(f) \mapsto \mathbf{curl}_\Gamma \mathring{\mathcal{P}}_{p+1}(f) = \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(f) : \ \mathrm{div}_\Gamma \mathbf{v} = 0\}\ , \tag{4.32}$$

$$\mathsf{P}^2_{f,p} : \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(f) \mapsto \mathrm{div}_\Gamma \mathring{\mathcal{W}}^1_p(f) = \mathring{\mathcal{W}}^2_p(f)\ , \tag{4.33}$$

as the corresponding $\boldsymbol{H}^{-\frac{1}{2}+\epsilon}(f)$-orthogonal projections. Eventually, let

$$\mathsf{P}^1_{T,p} : \boldsymbol{L}^2(T) \mapsto \mathbf{grad}\, \mathring{\mathcal{P}}_{p+1}(T) = \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(T) : \ \mathbf{curl}\, \mathbf{v} = 0\}\ , \tag{4.34}$$

$$\mathsf{P}^2_{T,p} : \boldsymbol{L}^2(T) \mapsto \mathbf{curl}\, \mathring{\mathcal{W}}^1_p(T) = \{\mathbf{v} \in \mathring{\mathcal{W}}^2_p(T) : \ \mathrm{div}\, \mathbf{v} = 0\}\ , \tag{4.35}$$

$$\mathsf{P}^3_{T,p} : L^2(T) \mapsto \mathrm{div}\, \mathring{\mathcal{W}}^2_p(T) = \left\{ v \in \mathcal{P}_p(T) : \ \int_T v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0 \right\}\ , \tag{4.36}$$

stand for the respective $L^2(T)$-orthogonal projections. Local exact sequences have tacitly been used in these statements; see (4.46) below.

The lifting operators

$$\mathsf{L}^1_{e,p} : \mathring{\mathcal{W}}^1_p(e) \mapsto \mathring{\mathcal{P}}_{p+1}(e)\ , \quad e \in \mathcal{F}_1(T)\ , \tag{4.37}$$

$$\mathsf{L}^1_{f,p} : \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(f) : \ \mathrm{div}_\Gamma \mathbf{v} = 0\} \mapsto \mathring{\mathcal{P}}_{p+1}(f)\ , \quad f \in \mathcal{F}_2(T)\ , \tag{4.38}$$

$$\mathsf{L}^1_{T,p} : \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(T) : \ \mathbf{curl}\, \mathbf{v} = 0\} \mapsto \mathring{\mathcal{P}}_{p+1}(T)\ , \tag{4.39}$$

are uniquely defined by requiring

$$\frac{d}{ds}\mathsf{L}^1_{e,p} u = u \quad \forall u \in \mathring{\mathcal{W}}^1_p(e)\ , \tag{4.40}$$

$$\mathbf{curl}_\Gamma \mathsf{L}^1_{f,p} \mathbf{u} = \mathbf{u} \quad \forall \mathbf{u} \in \{\mathring{\mathcal{W}}^1_p(f) : \ \mathrm{div}_\Gamma \mathbf{v} = 0\}\ , \tag{4.41}$$

$$\mathbf{grad}\, \mathsf{L}^1_{T,p} \mathbf{u} = \mathbf{u} \quad \forall \mathbf{u} \in \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(T) : \ \mathbf{curl}\, \mathbf{v} = 0\}\ . \tag{4.42}$$

Another class of liftings provides right inverses for **curl** and div_Γ: Pick a face $f \in \mathcal{F}_2(T)$, and, without loss of generality, assume the vertex opposite to the edge $\tilde{e}$ to coincide with 0. Then define

$$\mathsf{L}^2_{f,p} : \begin{cases} \mathrm{div}_\Gamma \mathring{\mathcal{W}}^1_p(f) & \mapsto \quad \mathring{\mathcal{W}}^1_p(f) \\ \quad u & \mapsto \quad \mathsf{R}^{2D}_0 u - \mathbf{curl}_\Gamma \mathsf{E}^0_{\tilde{e},p} \mathsf{L}^1_{\tilde{e},p}(\mathsf{R}^{2D}_0 u \cdot \boldsymbol{n}_{\tilde{e},f})\ . \end{cases} \tag{4.43}$$

This is a valid definition, since, by virtue of definition (4.16), the normal components of $\mathsf{R}_0^{2D}u$ will vanish on $\partial f \setminus \widetilde{e}$. Moreover, $\mathrm{div}_\Gamma \mathsf{R}_0^{2D}u = u$ ensures that the normal component of $\mathsf{R}_0^{2D}u$ has zero average on $\widetilde{e}$. We infer

$$(\mathbf{curl}_\Gamma \mathsf{E}^0_{\widetilde{e},p}\mathsf{L}^1_{\widetilde{e},p}((\mathsf{R}_0^{2D}\mathbf{u}\cdot \boldsymbol{n}_{\widetilde{e},f})_{|\widetilde{e}})\cdot \boldsymbol{n}_{\widetilde{e},f})_{|\widetilde{e}} = \frac{d}{ds}\mathsf{L}^1_{\widetilde{e},p}((\mathsf{R}_0^{2D}\mathbf{u})\cdot \boldsymbol{n}_{\widetilde{e},f})_{|\widetilde{e}} = \mathsf{R}_0^{2D}\mathbf{u}\cdot \boldsymbol{n}_{\widetilde{e},f} \quad \text{on } \widetilde{e}\,,$$

and see that the zero trace condition on ∂f is satisfied. The same idea underlies the definition of

$$\mathsf{L}^2_{T,p} : \begin{cases} \mathbf{curl}\,\mathring{\mathcal{W}}^1_p(T) & \mapsto \quad \mathring{\mathcal{W}}^1_p(T) \\ \qquad \mathbf{u} & \mapsto \quad \mathsf{R}_0\mathbf{u} - \mathbf{grad}\,\mathsf{E}^0_{\widetilde{f},p}\mathsf{L}^1_{\widetilde{f},p}(((\mathsf{R}_0\mathbf{u})\times \boldsymbol{n}_{\widetilde{f}})_{|\widetilde{f}})\,, \end{cases} \tag{4.44}$$

where $\widetilde{f}$ is the face opposite to vertex 0, and the definition of

$$\mathsf{L}^3_{T,p} : \begin{cases} \mathrm{div}\,\mathring{\mathcal{W}}^2_p(T) & \mapsto \quad \mathring{\mathcal{W}}^2_p(T) \\ \qquad u & \mapsto \quad \mathsf{D}_0 u - \mathbf{curl}\,\mathsf{E}^1_{\widetilde{f},p}\mathsf{L}^2_{\widetilde{f},p}((\mathsf{D}_0 u\cdot \boldsymbol{n}_{\widetilde{f}})_{|\widetilde{f}})\,. \end{cases} \tag{4.45}$$

The relationships between the various facet function spaces with vanishing traces can be summarized in the following exact sequences:

$$\begin{array}{lllllllllll}
\{0\} & \xrightarrow{\mathsf{Id}} & \mathring{\mathcal{P}}_{p+1}(T) & \xrightarrow{\mathbf{grad}} & \mathring{\mathcal{W}}^1_p(T) & \xrightarrow{\mathbf{curl}} & \mathring{\mathcal{W}}^2_p(T) & \xrightarrow{\mathrm{div}} & \overline{\mathcal{P}}_p(T) & \xrightarrow{0} & \{0\}, \\
\{0\} & \xrightarrow{\mathsf{Id}} & \mathring{\mathcal{P}}_{p+1}(f) & \xrightarrow{\mathbf{curl}_\Gamma} & \mathring{\mathcal{W}}^1_p(f) & \xrightarrow{\mathrm{div}_\Gamma} & \overline{\mathcal{P}}_p(f) & \xrightarrow{0} & \{0\}, \\
\{0\} & \xrightarrow{\mathsf{Id}} & \mathring{\mathcal{P}}_{p+1}(e) & \xrightarrow{\frac{d}{ds}} & \overline{\mathcal{P}}_p(e) & \xrightarrow{0} & \{0\}\,,
\end{array} \tag{4.46}$$

where $\overline{\mathcal{P}}_p(F)$ designates degree p polynomial spaces on F with vanishing mean. These relationships and the lifting mappings $\mathsf{L}^l_{*,p}$ are studied in [36, Section 3.4].

Finally, we need polynomial extension operators

$$\mathsf{E}^0_{e,p} : \mathring{\mathcal{P}}_{p+1}(e) \mapsto \mathcal{P}_{p+1}(T)\,, \tag{4.47}$$
$$\mathsf{E}^0_{f,p} : \mathring{\mathcal{P}}_{p+1}(f) \mapsto \mathcal{P}_{p+1}(T) \tag{4.48}$$

that satisfy

$$\mathsf{E}^0_{e,p}u_{|e'} = 0 \quad \forall e' \in \mathcal{F}_1(T)\setminus\{e\}\,, \tag{4.49}$$
$$\mathsf{E}^0_{f,p}u_{|f'} = 0 \quad \forall f' \in \mathcal{F}_2(T)\setminus\{f\}\,. \tag{4.50}$$

Such extension operators can be constructed relying on a representation of a polynomial on F, $F \in \mathcal{F}_m(T)$, $m = 1, 2$, as a homogeneous polynomial in the barycentric coordinates of F; see [36, Lemma 3.4] of [49, Section IV.3]. As an alternative, one may use the polynomial preserving extension operators proposed in [53, 21] and [1]. We stress that continuity properties of these extensions do not matter for our purpose.

4.1.4 Interpolation operators

Now we are in a position to define the projection based interpolation operators locally on a generic tetrahedron T with vertices $\boldsymbol{a}_i$, $i = 1, 2, 3, 4$.

First, we devise a suitable projection (depending on the regularity parameter $0 < \epsilon < \frac{1}{2}$, which is usually suppressed to keep notation manageable)

$$\Pi^0_{T,p}(= \Pi^0_{T,p}(\epsilon)) \ : \ C^\infty(\overline{T}) \mapsto \mathcal{P}_{p+1}(T) \tag{4.51}$$

for degree p Lagrangian $H^1(\Omega)$-conforming finite elements. For $u \in C^\infty(\overline{T})$ define (λ_i is the barycentric coordinate function belonging to vertex $\boldsymbol{a}_i$ of T)

$$u^{(0)} := u - \underbrace{\sum_{i=1}^{4} u(\boldsymbol{a}_i)\lambda_i}_{:=w^{(0)}} , \tag{4.52}$$

$$u^{(1)} := u^{(0)} - \underbrace{\sum_{e \in \mathcal{F}_1(T)} \mathsf{E}^0_{e,p}\mathsf{L}^1_{e,p}\mathsf{P}^1_{e,p}\frac{d}{ds}u^{(0)}{}_{|e}}_{:=w^{(1)}} , \tag{4.53}$$

$$u^{(2)} := u^{(1)} - \underbrace{\sum_{f \in \mathcal{F}_1(T)} \mathsf{E}^0_{f,p}\mathsf{L}^1_{f,p}\mathsf{P}^1_{f,p}\mathbf{curl}_\Gamma(u^{(1)}{}_{|f})}_{:=w^{(2)}} , \tag{4.54}$$

$$\Pi^0_{T,p}u := \mathsf{L}^1_{T,p}\mathsf{P}^1_{T,p}\,\mathbf{grad}\,u^{(2)} + w^{(2)} + w^{(1)} + w^{(0)} . \tag{4.55}$$

Observe that $w^{(i)}{}_{|F} = 0$ for all $F \in \mathcal{F}_m(T)$, $0 \leq m < i \leq 3$. We point out that $w^{(0)}$ is the standard linear interpolant of u.

Lemma 4.5. *The linear mapping $\Pi^0_{T,p}$, $p \in \mathbb{N}_0$, is a projection onto $\mathcal{P}_{p+1}(T)$.*

Proof. Assume $u \in \mathcal{P}_{p+1}(T)$, which will carry over to all intermediate functions. Since $u^{(0)}(\boldsymbol{a}_i) = 0$, $i = 1, \dots, 4$, we conclude from the projection property of $\mathsf{P}^1_{e,p}$ that $\mathsf{L}^1_e\mathsf{P}^1_e\frac{d}{ds}u^{(0)}{}_{|e} = u^{(0)}{}_{|e}$ for any edge $e \in \mathcal{F}_1(T)$. As a consequence,

$$u^{(1)} = u^{(0)} - \sum_{e \in \mathcal{F}_1(T)} \mathsf{E}^0_{e,p}u^{(0)}{}_{|e} \quad \Rightarrow \quad u^{(1)}_{|e} = 0 \quad \forall e \in \mathcal{F}_1(T) . \tag{4.56}$$

We infer $\mathsf{L}^1_{f,p}\mathsf{P}^1_f\mathbf{curl}_\Gamma(u^{(1)}{}_{|f}) = u^{(1)}{}_{|f}$ on each face $f \in \mathcal{F}_2(T)$, which implies

$$u^{(2)} = u^{(1)} - \sum_{f \in \mathcal{F}_1(T)} \mathsf{E}^0_{f,p}(u^{(1)}{}_{|f}) \quad \Rightarrow \quad u^{(2)}{}_{|f} = 0 \quad \forall f \in \mathcal{F}_2(T) . \tag{4.57}$$

This means that $\mathsf{L}^1_{T,p}\mathsf{P}^1_{T,p}\,\mathbf{grad}\,u^{(2)} = u^{(2)}$ and a telescopic sum argument completes the proof. □

A similar stage by stage construction applies to edge elements and gives a projection

$$\Pi^1_{T,p}(=\Pi^1_{T,p}(\epsilon)) \ : \ \boldsymbol{C}^\infty(\overline{T}) \mapsto \mathcal{W}^1(T) \ : \tag{4.58}$$

for a directed edge $e := [\mathbf{a}_i, \mathbf{a}_j]$ we introduce the Whitney-1-form basis function

$$\mathbf{b}_e = \lambda_i \,\mathbf{grad}\,\lambda_j - \lambda_j \,\mathbf{grad}\,\lambda_i \ . \tag{4.59}$$

These functions span $\mathcal{W}^1_0(T)$. Next, for $\mathbf{u} \in \boldsymbol{C}^\infty(\overline{T})$ define

$$\mathbf{u}^{(0)} := \mathbf{u} - \underbrace{\left(\sum_{e\in\mathcal{F}_1(T)} \int_e \mathbf{u}\cdot d\vec{s}\right)\mathbf{b}_e}_{:=\mathbf{w}^{(0)}} \ , \tag{4.60}$$

$$\mathbf{u}^{(1)} := \mathbf{u}^{(0)} - \underbrace{\sum_{e\in\mathcal{F}_1(T)} \mathbf{grad}\,\mathsf{E}^0_{e,p}\mathsf{L}^1_{e,p}\mathsf{P}^1_{e,p}((\mathbf{u}^{(0)}\cdot \boldsymbol{t}_e)_{|e})}_{:=\mathbf{w}^{(1)}} \ , \tag{4.61}$$

$$\mathbf{u}^{(2)} := \mathbf{u}^{(1)} - \underbrace{\sum_{f\in\mathcal{F}_2(T)} \mathsf{E}^1_{f,p}\mathsf{L}^2_{f,p}\mathsf{P}^2_{f,p}\mathrm{div}_\Gamma((\mathbf{u}^{(1)}\times \boldsymbol{n}_f)_{|f})}_{:=\mathbf{w}^{(2)}} \ , \tag{4.62}$$

$$\mathbf{u}^{(3)} := \mathbf{u}^{(2)} - \underbrace{\sum_{f\in\mathcal{F}_2(T)} \mathbf{grad}\,\mathsf{E}^0_{f,p}\mathsf{L}^1_{f,p}\mathsf{P}^1_{f,p}((\mathbf{u}^{(2)}\times \boldsymbol{n}_f)_{|f})}_{:=\mathbf{w}^{(3)}} \ , \tag{4.63}$$

$$\mathbf{u}^{(4)} := \mathbf{u}^{(3)} - \underbrace{\mathsf{L}^2_{T,p}\mathsf{P}^2_{T,p}\,\mathbf{curl}\,\mathbf{u}^{(3)}}_{:=\mathbf{w}^{(4)}} \ , \tag{4.64}$$

$$\Pi^1_{T,p}\mathbf{u} := \mathbf{grad}\,\mathsf{L}^1_{T,p}\mathsf{P}^1_{T,p}\mathbf{u}^{(4)} + \mathbf{w}^{(4)} + \mathbf{w}^{(3)} + \mathbf{w}^{(2)} + \mathbf{w}^{(1)} + \mathbf{w}^{(0)} \ . \tag{4.65}$$

The contribution $\mathbf{w}^{(0)}$ is the standard interpolant $\Pi^1_{T,0}$ of $\mathbf{u}$ onto the local space of Whitney-1-forms (lowest order edge elements). The extension operators were chosen in a way that guarantees that $\mathbf{w}^{(2)}\cdot \boldsymbol{t}_e = 0$ and $\mathbf{w}^{(3)}\cdot \boldsymbol{t}_e = 0$ for all $e \in \mathcal{F}_1(T)$.

Lemma 4.6. *The linear mapping $\Pi^1_{T,p}$, $p \in \mathbb{N}_0$, is a projection onto $\mathcal{W}^1_p(T)$ and satisfies the* commuting diagram property

$$\Pi^1_{T,p} \circ \mathbf{grad} = \mathbf{grad} \circ \Pi^0_{T,p} \quad \textit{on } C^\infty(\overline{T}) \ . \tag{4.66}$$

Proof. The proof of the projection property runs parallel to that of Lemma 4.5. Assuming $\mathbf{u} \in \mathcal{W}^1_p(T)$, it is obvious that the same will hold for all $\mathbf{u}^{(i)}$ and $\mathbf{w}^{(i)}$ from (4.60)–(4.65). In order to confirm that all projections can be discarded, we have to check that their arguments satisfy conditions of zero trace on the facet boundaries and, in some cases, belong to the kernel of differential operators.

First, recalling the properties of the interpolation operator Π^1_0 for Whitney-1-forms, we find $(\mathbf{u}^{(0)}\cdot \boldsymbol{t}_e)_{|e} \in \mathring{\mathcal{W}}^1_p(e)$. This implies

$$\mathbf{grad}\,\mathsf{E}^0_{e,p}\mathsf{L}^1_{e,p}\mathsf{P}^1_{e,p}((\mathbf{u}^{(0)}\cdot \boldsymbol{t}_e)_{|e}) = (\mathbf{u}^{(0)}\cdot \boldsymbol{t}_e)_{|e} \quad \forall e \in \mathcal{F}_1(T) \ , \tag{4.67}$$

and

$$(\mathbf{u}^{(1)} \cdot \boldsymbol{t}_e)_{|e} \equiv 0 \quad \forall e \in \mathcal{F}_1(T)\,. \tag{4.68}$$

We see that $(\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f} \in \mathring{\mathcal{W}}_p^1(f)$ for any $f \in \mathcal{F}_2(T)$, so that

$$\mathsf{P}^2_{f,p}\mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) = \mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) \tag{4.69}$$

$$\Rightarrow \quad \mathrm{div}_\Gamma \mathsf{L}^2_{f,p}\mathsf{P}^2_{f,p}\mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) = \mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) \tag{4.70}$$

$$\Rightarrow \quad \mathrm{div}_\Gamma((\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f}) = 0 \quad \forall f \in \mathcal{F}_2(T)\,, \quad (\mathbf{u}^{(2)} \cdot \boldsymbol{t}_e)_{|e} \equiv 0 \quad \forall e \in \mathcal{F}_1(T) \tag{4.71}$$

$$\Rightarrow \quad \mathsf{P}^1_{f,p}((\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f}) = (\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f} \quad \forall f \in \mathcal{F}_2(T) \tag{4.72}$$

$$\Rightarrow \quad \mathbf{grad}\, \mathsf{E}^0_{f,p}\mathsf{L}^1_{f,p}\mathsf{P}^1_{f,p}((\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f}) \times \boldsymbol{n}_f = (\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f} \quad \forall f \in \mathcal{F}_2(T) \tag{4.73}$$

$$\Rightarrow \quad (\mathbf{u}^{(3)} \times \boldsymbol{n}_f)_{|f} = 0 \quad \forall f \in \mathcal{F}_2(T) \tag{4.74}$$

$$\Rightarrow \quad \mathsf{P}^2_{T,p}\, \mathbf{curl}\, \mathbf{u}^{(3)} = \mathbf{curl}\, \mathbf{u}^{(3)} \tag{4.75}$$

$$\Rightarrow \quad \mathbf{curl}\, \mathsf{L}^2_{T,p}\mathsf{P}^2_{T,p}\, \mathbf{curl}\, \mathbf{u}^{(3)} = \mathbf{curl}\, \mathbf{u}^{(3)} \tag{4.76}$$

$$\Rightarrow \quad \mathbf{curl}\, \mathbf{u}^{(4)} = 0 \quad \Rightarrow \quad \mathsf{P}^1_T \mathbf{u}^{(4)} = \mathbf{u}^{(4)} \tag{4.77}$$

$$\Rightarrow \quad \mathbf{grad}\, \mathsf{L}^1_T\mathsf{P}^1_T\mathbf{u}^{(4)} = \mathbf{u}^{(4)}\,, \tag{4.78}$$

which confirms the projector property.

Now assume $\mathbf{u} = \mathbf{grad}\, u$ for some $u \in C^\infty(\overline{T})$. The commuting diagram property will follow, if we manage to show $\mathbf{grad}\, u^{(0)} = \mathbf{u}^{(0)}$, $\mathbf{grad}\, u^{(1)} = \mathbf{u}^{(1)}$, $\mathbf{grad}\, u^{(2)} = \mathbf{u}^{(3)}$, etc., for the intermediate functions in (4.52)–(4.55) and (4.60)–(4.65), respectively.

By the commuting diagram property for the standard local interpolation operators onto the spaces of Whitney-0-forms (linear polynomials) and Whitney-1-forms, we conclude

$$\mathbf{grad}\, u^{(0)} = \mathbf{u}^{(0)} \quad \Rightarrow \quad \frac{d}{ds}u^{(0)}{}_{|e} = (\mathbf{u}^{(0)} \cdot \boldsymbol{t}_e)_{|e} \quad \forall e \in \mathcal{F}_1(T) \tag{4.79}$$

$$\Rightarrow \quad \mathbf{u}^{(1)} = \mathbf{grad}\, u^{(1)} \quad \Rightarrow \quad \mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) = 0 \quad \forall f \in \mathcal{F}_2(T) \tag{4.80}$$

$$\Rightarrow \quad \mathbf{u}^{(2)} = \mathbf{u}^{(1)} \tag{4.81}$$

$$\Rightarrow \quad (\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f} = \mathbf{curl}_\Gamma u^{(1)}{}_f \quad \forall f \in \mathcal{F}_2(T) \quad \Rightarrow \quad \mathbf{u}^{(3)} = \mathbf{grad}\, u^{(2)} \tag{4.82}$$

$$\Rightarrow \quad \mathbf{u}^{(4)} = \mathbf{u}^{(3)}\,. \tag{4.83}$$

Of course, analogous relationships for the functions $w^{(i)}$ and $\mathbf{w}^{(i)}$ hold, which yields $\Pi^1_{T,p}\mathbf{u} = \mathbf{grad}\, \Pi^0_{T,p}u$. □

Following [36, Section 3.5], a projection based interpolation onto $\mathcal{W}^2_p(T)$, the operator $\Pi^2_{T,p}(= \Pi^2_{T,p}(\epsilon)) : \boldsymbol{C}^\infty(\overline{T}) \mapsto \mathcal{W}^2_p(T)$, involves the stages

$$\mathbf{u}^{(0)} := \mathbf{u} - \underbrace{\left(\sum_{f \in \mathcal{F}_2(T)} \int_f \mathbf{u} \cdot \boldsymbol{n}_f \,\mathrm{d}S\right) \mathbf{b}_f}_{:=\mathbf{w}^{(0)}}\,, \tag{4.84}$$

$$\mathbf{u}^{(1)} := \mathbf{u}^{(0)} - \underbrace{\sum_{f\in\mathcal{F}_2(T)} \mathbf{curl}\, \mathsf{E}^1_{f,p}\mathsf{L}^2_{f,p}\mathsf{P}^2_{f,p}((\mathbf{u}^{(0)}\cdot \boldsymbol{n}_f)_{|f})}_{:=\mathbf{w}^{(1)}} \tag{4.85}$$

$$\mathbf{u}^{(2)} := \mathbf{u}^{(1)} - \underbrace{\mathsf{L}^3_{T,p}\mathsf{P}^3_{T,p}\,\mathrm{div}\,\mathbf{u}^{(1)}}_{:=\mathbf{w}^{(2)}} \tag{4.86}$$

$$\Pi^2_{T,p}\mathbf{u} := \mathbf{curl}\,\mathsf{L}^2_{T,p}\mathsf{P}_{T,p}\mathbf{u}^{(2)} + \mathbf{w}^{(0)} + \mathbf{w}^{(1)} + \mathbf{w}^{(2)} . \tag{4.87}$$

Here, $\mathbf{b}_f$ refers to the local basis functions for Whitney-2-forms [36, Section 3.2]:

$$\mathbf{b}_f = \lambda_i\,\mathbf{grad}\,\lambda_j \times \mathbf{grad}\,\lambda_k + \lambda_j\,\mathbf{grad}\,\lambda_k \times \lambda_i + \lambda_k\,\mathbf{grad}\,\lambda_i \times \lambda_j . \tag{4.88}$$

Analogous to Lemma 4.6, one proves the following result.

Lemma 4.7. *The linear operator $\Pi^2_{T,p}$, $p \in \mathbb{N}_0$, is a projection onto $\mathcal{W}^2_p(T)$ and satisfies the* commuting diagram property

$$\Pi^2_{T,p} \circ \mathbf{curl} = \mathbf{curl} \circ \Pi^1_{T,p} \quad \text{on } \boldsymbol{C}^\infty(\overline{T}) . \tag{4.89}$$

The next lemma makes it possible to patch together the local projection based interpolation operator to obtain global interpolation operators

$$\Pi^l_p : \boldsymbol{C}^\infty(\overline{\Omega}) \mapsto \mathcal{W}^l_p(\mathcal{T}) , \quad l = 1, 2 . \tag{4.90}$$

Lemma 4.8. *For any $F \in \mathcal{F}_m(T)$, $m = 0, 1, 2$, and $u \in C^\infty(\overline{T})$ the restriction $\Pi^0_{T,p}u_{|F}$ depends only on $u_{|F}$.*

For any $F \in \mathcal{F}_m(T)$, $m = 1, 2$, and $\mathbf{u} \in \boldsymbol{C}^\infty(\overline{T})$ the tangential trace of $\Pi^1_{T,p}\mathbf{u}$ onto F depends only on the tangential trace of $\mathbf{u}$ on F.

For any face $f \in \mathcal{F}_2(T)$ and $\mathbf{u} \in \boldsymbol{C}^\infty(\overline{T})$ the normal trace of $\Pi^2_{T,p}\mathbf{u}$ onto f depends only on the normal component of $\mathbf{u}$ on f.

Proof. The assertion is immediate from the construction, in particular, the properties of the extension operators used therein. □

It goes without saying that density arguments permit us to extend Π^l_p, $l = 0, 1, 2$, to Sobolev spaces, as long as they are continuous in the respective norms. (Repeated) application of trace theorems [33, Section 1.5] reveals that it is possible to obtain continuous projectors

$$\Pi^0_p : H^{1+s}(\Omega) \mapsto \mathcal{W}^0_p(\mathcal{T}) , \tag{4.91}$$

$$\Pi^1_p : \boldsymbol{H}^{\frac{1}{2}+s}(\Omega) \mapsto \boldsymbol{\mathcal{W}}^1_p(\mathcal{T}) , \tag{4.92}$$

$$\Pi^2_p : \boldsymbol{H}^s(\Omega) \mapsto \boldsymbol{\mathcal{W}}^2_p(\mathcal{T}) , \tag{4.93}$$

for any $s > \frac{1}{2}$. In addition, by virtue of Lemma 4.8 and the resolution of Γ_D by $\mathcal{T}$, zero pointwise/tangential/normal trace on Γ_D of the argument function will be preserved by Π^l_p, $l = 0, 1, 2$, for instance,

$$\Pi^1_p\Big(\boldsymbol{H}^{\frac{1}{2}+s}(\Omega) \cap \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)\Big) = \boldsymbol{\mathcal{W}}^1_{p,\Gamma_D}(\mathcal{T}) \cap \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega) . \tag{4.94}$$

4.1.5 Local interpolation error estimates

Closely following [20, Section 6], we first examine the interpolation error for $\Pi^0_{T,p}$. Please notice that $\Pi^0_{T,p}$ still depends on the fixed regularity parameter $0 < \epsilon < \frac{1}{2}$. The argument function of $\Pi^0_{T,p}$ is assumed to lie in $H^2(T)$. The continuous embedding $H^2(T) \hookrightarrow C^0(\overline{T})$ plus trace theorems for Sobolev spaces render all operators well-defined in this case.

We start with an observation related to the local best approximation properties of the projection based interpolant.

Lemma 4.9. *For any $u \in H^2(T)$ holds true*

$$(\mathbf{grad}(u - \Pi^0_{T,p}u), \mathbf{grad}\, v)_{L^2(T)} = 0 \quad \forall v \in \mathring{\mathcal{P}}_{p+1}(T)\,, \tag{4.95}$$

$$(\mathbf{curl}_\Gamma (u - \Pi^0_{T,p}u)_{|f}, \mathbf{curl}_\Gamma v)_{H^{-\frac{1}{2}+\epsilon}(f)} = 0 \quad \forall v \in \mathring{\mathcal{P}}_{p+1}(f),\ f \in \mathcal{F}_2(T)\,, \tag{4.96}$$

$$\left(\frac{d}{ds}(u - \Pi^0_{T,p}u)_{|e}, \frac{d}{ds}v\right)_{H^{-1+\epsilon}(e)} = 0 \quad \forall v \in \mathring{\mathcal{P}}_{p+1}(e),\ e \in \mathcal{F}_1(T)\,. \tag{4.97}$$

Proof. We use the notation of (4.52)–(4.55). Setting $w := w^{(0)} + w^{(1)} + w^{(2)}$, we find

$$\Pi^0_{T,p}u = \mathsf{L}^1_{T,p}\mathsf{P}^1_{T,p}\,\mathbf{grad}(u - w) + w\,, \tag{4.98}$$

which implies, because $\mathsf{L}^1_{T,p}$ is a right inverse of **grad**,

$$\mathbf{grad}\,\Pi^0_{T,p}u = \mathsf{P}^1_{T,p}\,\mathbf{grad}\,u + (\mathsf{Id} - \mathsf{P}^1_{T,p})\,\mathbf{grad}\,w\,. \tag{4.99}$$

This means that $\mathbf{grad}\,u - \mathbf{grad}\,\Pi^0_{T,p}u$ belongs to the range of $\mathsf{Id} - \mathsf{P}^1_{T,p}$ and (4.95) follows from (4.34) and the properties of orthogonal projections. Similar manipulations establish (4.96):

$$\begin{aligned}\mathbf{curl}_\Gamma \Pi^0_{T,p}u_{|f} &= \mathbf{curl}_\Gamma w_{|f}\\ &= \underbrace{\mathbf{curl}_\Gamma \mathsf{L}^1_{f,p}}_{=\mathsf{Id}}\,\mathsf{P}^1_{f,p}\mathbf{curl}_\Gamma u^{(1)} + \mathbf{curl}_\Gamma (w^{(0)} + w^{(1)})_{|f}\\ &= \mathsf{P}^1_{f,p}\mathbf{curl}_\Gamma u_{|f} + (\mathsf{Id} - \mathsf{P}^1_{f,p})\mathbf{curl}_\Gamma (w^{(0)} + w^{(1)}) \quad \forall f \in \mathcal{F}_2(T)\,.\end{aligned}$$

The same arguments as above verify (4.97). □

From this, we can conclude the result of [20, Section 6, Corollary 1]. To state it, we now assume a dependence

$$0 < \epsilon = \epsilon(p) := \frac{1}{10\log(p+2)} < \frac{1}{4}\,, \quad p \in \mathbb{N}\,, \tag{4.100}$$

of the parameter ϵ in the definition of the local projection based interpolation operators. Below, all parameters ϵ are linked to p via (4.100). Please note that we retain the notation $(\Pi^l_{T,p})_{p\in\mathbb{N}}$, $l = 0, 1, 2$, for these new families of operators.

Theorem 4.10 (Spectral interpolation error estimate for $\Pi^0_{T,p}$). *With a constant merely depending on the shape-regularity of T*

$$\left|(\mathsf{Id} - \Pi^0_{T,p})v\right|_{1,T} \lesssim (1 + \log^{3/2}(p+1))\frac{h_T}{p+1}|v|_{2,T} \quad \forall v \in H^2(T)\,. \tag{4.101}$$

Stable polynomial extensions are instrumental for the proof, which will be postponed until page 245. First, we recall the results of [53, Theorem 1] and [1, Theorem 1].

Theorem 4.11 (Stable polynomial extension for tetrahedra). *For a tetrahedron T, there is linear operator* $\mathsf{S}_T : H^{\frac{1}{2}}(\partial T) \mapsto H^1(T)$ *such that*

$$\mathsf{S}_T u_{|\partial T} = u \quad \forall u \in H^{\frac{1}{2}}(\partial T)\,, \tag{4.102}$$

$$\left|\mathsf{S}_T u\right|_{1,T} \lesssim |u|_{\frac{1}{2},\partial T} \quad \forall u \in H^{\frac{1}{2}}(\partial T)\,, \tag{4.103}$$

$$\mathsf{S}_T w \in \mathcal{P}_{p+1}(T) \quad \forall w \in \mathcal{P}_{p+1}(T)_{|\partial T}\,. \tag{4.104}$$

Theorem 4.12 (Stable polynomial extension for triangles). *Given a triangle F, there is a linear mapping* $\mathsf{S}_F : L^2(\partial F) \mapsto H^{\frac{1}{2}}(F)$ *such that*

$$\left|\mathsf{S}_F u\right|_{\frac{1}{2},F} \lesssim \|u\|_{0,\partial F} \quad \forall u \in L^2(\partial F)\,, \tag{4.105}$$

$$\left|\mathsf{S}_F u\right|_{1,F} \lesssim |u|_{\frac{1}{2},\partial F} \quad \forall u \in H^{\frac{1}{2}}(\partial F)\,, \tag{4.106}$$

$$\mathsf{S}_F w \in \mathcal{P}_{p+1}(F) \quad \forall w \in \mathcal{P}_{p+1}(F)_{|\partial F}\,, \tag{4.107}$$

where the constants depend only on the shape regularity measure of T.

By interpolation in Sobolev scale from the last theorem, we can conclude

$$\left|\mathsf{S}_F u\right|_{s,F} \lesssim |u|_{s-\frac{1}{2},\partial F} \quad \forall u \in H^{s-\frac{1}{2}}(\partial F), \quad \frac{1}{2} \le s \le 1\,. \tag{4.108}$$

We also need to deal with the awkward property of the $H^{\frac{1}{2}}(\partial T)$-norm that it cannot be split into face contributions. To that end, we resort to a result from [50, Proof of Lemma 3.31]; see also [20, Lemma 13].

Lemma 4.13 (Splitting of $H^{\frac{1}{2}}(\partial T)$-norm). *With a constants depending only on the shape regularity of the tetrahedron T, there holds*

$$|u|_{s,\partial T} \lesssim \frac{1}{s-\frac{1}{2}} \sum_{f \in \mathcal{F}_2(T)} |u|_{s,f} \quad \forall u \in H^{\frac{1}{2}+s}(\partial T), \quad \frac{1}{2} < s \le 1\,. \tag{4.109}$$

Another natural ingredient for the proof are polynomial best approximation estimates; see [59] or [53, Section 3].

Lemma 4.14. *Let* $0 \le r \le 1$, $0 \le s \le 2$, *and F be either a tetrahedron or a triangle. Then*

$$\inf_{v_p \in \mathcal{P}_{p+1}(F)} \left|u - v_p\right|_{r,F} \lesssim \left(\frac{h_F}{p}\right)^{s+1-r} |u|_{s+1,F} \quad \forall u \in H^{s+1}(F)\,. \tag{4.110}$$

Define a semi-norm projection $\mathsf{Q}_{T,p} : H^1(T) \mapsto \mathcal{P}_{p+1}(T)$ on the tetrahedron T by

$$\begin{aligned}\int_T \mathbf{grad}(u - \mathsf{Q}_{T,p}u) \cdot \mathbf{grad}\, v_p \,\mathrm{d}\boldsymbol{x} &= 0 \quad \forall v_p \in \mathcal{P}_{p+1}(T)\,, \\ \int_T u - \mathsf{Q}_{T,p}u \,\mathrm{d}\boldsymbol{x} &= 0\,,\end{aligned} \tag{4.111}$$

and, for $\frac{1}{2} \le s \le 1$, semi-norm projections $\mathsf{Q}_{f,p} : H^{s-\frac{1}{2}}(f) \mapsto \mathcal{P}_{p+1}(f), f \in \mathcal{F}_2(T)$, by

$$\begin{aligned}\left(\mathbf{curl}_\Gamma(u - \mathsf{Q}_{f,p}u), \mathbf{curl}_\Gamma v_p\right)_{H^{s-\frac{1}{2}}(f)} &= 0 \quad \forall v_p \in \mathcal{P}_{p+1}(T)\,, \\ \int_f u - \mathsf{Q}_{f,p}u \,\mathrm{d}\boldsymbol{x} &= 0\,.\end{aligned} \tag{4.112}$$

These definitions involve best approximation properties of $\mathsf{Q}_{T,p}u$ and $\mathsf{Q}_{f,p}u$. Thus, we learn from Lemma 4.14 that with constants independent of $0 < \epsilon < \frac{1}{2} < s \le 1$,

$$\left|u - \mathsf{Q}_{T,p}u\right|_{1,T} \lesssim \left(\frac{h_T}{p+1}\right)^s |u|_{1+s,T} \quad \forall u \in H^s(T)\,, \tag{4.113}$$

$$\left|u - \mathsf{Q}_{f,p}u\right|_{\frac{1}{2}+\epsilon,f} \lesssim \left(\frac{h_T}{p+1}\right)^{s-\epsilon} |u|_{\frac{1}{2}+s,T} \quad \forall u \in H^{\frac{1}{2}+s}(f)\,. \tag{4.114}$$

The latter estimate follows from the fact that $|\cdot|_{\frac{1}{2}+\epsilon,f}$ and $\|\mathbf{curl}_\Gamma\cdot\|_{-\frac{1}{2}+\epsilon,f}$ are equivalent semi-norms, uniformly in ϵ.

We also need error estimates for the $L^2(e)$-orthogonal projections,

$$\mathsf{Q}^*_{e,p} : L^2(e) \mapsto \mathring{\mathcal{P}}_{p+1}(e)\,, \quad e \in \mathcal{F}_1(T)\,. \tag{4.115}$$

Lemma 4.15 (see [20, Lemma 18]). *With a constant independent of p,* $0 \le \epsilon \le \frac{1}{2}$*, and* $2\epsilon \le r \le 1 + \epsilon$*,*

$$\left|u - \mathsf{Q}^*_{e,p}u\right|_{\epsilon,e} \lesssim \left(\frac{h_e}{p+1}\right)^{r-2\epsilon} |u|_{r,e} \quad \forall u \in H^r(e) \cap H^1_0(e)\,.$$

Proof. By scaling arguments, we may assume $h_e = 1$. Write $\mathsf{I}_{e,p} : H^1_0(e) \mapsto \mathring{\mathcal{P}}_{p+1}$ for the interpolation operator

$$(\mathsf{I}_{e,p}u)(\xi) = u(0) + \int_0^\xi \left(\mathsf{Q}_{e,p}\frac{du}{d\xi}\right)(\tau)\,\mathrm{d}\tau\,, \quad 0 \le \xi \le |e|\,,$$

where ξ is the arc length parameter for the edge e and $\mathsf{Q}_{e,p} : L^2(\Omega) \mapsto \mathcal{P}_p(e)$ is the $L^2(e)$-orthogonal projection. From [59, Section 3.3.1, Theorem 3.17], we learn that

$$\left|u - \mathsf{I}_{e,p}u\right|_{1,e} \lesssim (p+1)^{-1}|u|_{2,e} \quad \forall u \in H^2(e)\,, \tag{4.116}$$

$$\left\|u - \mathsf{I}_{e,p}u\right\|_{0,e} \lesssim (p+1)^{-m}|u|_{m,e} \quad \forall u \in H^m(e)\,, \quad m = 1, 2\,. \tag{4.117}$$

As $\mathsf{I}_{e,p}u \in \mathring{\mathcal{P}}_{p+1}(e)$ for $u \in H_0^1(e)$, this permits us to conclude

$$\|u - \mathsf{Q}_{e,p}^* u\|_{0,e} \le \|u - \mathsf{I}_{e,p}u\|_{0,e} \lesssim (p+1)^{-1}\|u\|_{1,e}\ , \tag{4.118}$$

which yields, by interpolation between $H^1(e)$ and $L^2(e)$,

$$\|u - \mathsf{Q}_{e,p}^* u\|_{0,e} \lesssim (p+1)^{-q}\|u\|_{q,e}\ , \quad 0 \le q \le 1\ , \tag{4.119}$$

where the constant is independent of q. On the other hand, using the inverse inequality [7, Lemma 1]

$$\|u\|_{1,e} \lesssim (p+1)^2\|u\|_{0,e} \quad \forall u \in \mathcal{P}_{p+1}(e) \tag{4.120}$$

and (4.116), (4.117), we find the estimate

$$\begin{aligned} |u - \mathsf{Q}_{e,p}^* u|_{1,e} &\le |u - \mathsf{I}_{e,p}u|_{1,e} + |\mathsf{Q}_{e,p}^* u - \mathsf{I}_{e,p}u|_{1,e} \\ &\le |u - \mathsf{I}_{e,p}u|_{1,e} + (p+1)^2\|\mathsf{Q}_{e,p}^* u - \mathsf{I}_{e,p}u\|_{0,e} \\ &\lesssim |u - \mathsf{I}_{e,p}u|_{1,e} + (p+1)^2\|u - \mathsf{I}_{e,p}u\|_{0,e} \lesssim \|u\|_{2,e}\ . \end{aligned} \tag{4.121}$$

Interpolation between (4.119) with $q = \frac{r-2\epsilon}{1-\epsilon}$ and (4.121) completes the proof. □

Proof of Theorem 4.10, cf. [20, *Section 6*]. Orthogonality (4.95) of Lemma 4.9 combined with the definition of $\mathsf{Q}_{T,p}$ involves

$$\int_T \mathbf{grad}((\Pi_{T,p}^0 - \mathsf{Q}_{T,p})u) \cdot \mathbf{grad}\, v_p \,\mathrm{d}\boldsymbol{x} = 0 \quad \forall v_p \in \mathring{\mathcal{P}}_{p+1}(T)\ . \tag{4.122}$$

Hence, $(\Pi_{T,p}^0 - \mathsf{Q}_{T,p})u$ turns out to be the $|\cdot|_{1,T}$-minimal degree $p+1$ polynomial extension of $(\Pi_{T,p}^0 - \mathsf{Q}_{T,p})u_{|\partial T}$, which, thanks to Theorem 4.11, implies

$$\begin{aligned} |(\Pi_{T,p}^0 - \mathsf{Q}_{T,p})u|_{1,T} &\le |\mathsf{S}_T((\Pi_{T,p}^0 u - \mathsf{Q}_{T,p}u)_{|\partial T})|_{1,T} \\ &\lesssim |(\Pi_{T,p}^0 u - \mathsf{Q}_{T,p}u)_{|\partial T}|_{\frac12,\partial T}\ . \end{aligned} \tag{4.123}$$

Thus, by the continuity of the trace operator $H^1(T) \mapsto H^{\frac12}(\partial T)$,

$$\begin{aligned} |u - \Pi_{T,p}^0 u|_{1,T} &\lesssim |u - \mathsf{Q}_{T,p}u|_{1,T} + |(u - \Pi_{T,p}^0 u)_{|\partial T}|_{\frac12,\partial T} + |(u - \mathsf{Q}_{T,p}u)_{|\partial T}|_{\frac12,\partial T} \\ &\lesssim \left(|u - \mathsf{Q}_{T,p}u|_{1,T} + |(u - \Pi_{T,p}^0 u)_{|\partial T}|_{\frac12,\partial T}\right)\ . \end{aligned} \tag{4.124}$$

To estimate $|(u - \Pi_{T,p}^0 u)_{|\partial T}|_{\frac12,\partial T}$, we appeal to Lemma 4.13 and get

$$\begin{aligned} |(u - \Pi_{T,p}^0 u)_{|\partial T}|_{\frac12,\partial T} &\le |(u - \Pi_{T,p}^0 u)_{|\partial T}|_{\frac12+\epsilon,\partial T} \\ &\lesssim \frac{1}{\epsilon} \sum_{f \in \mathcal{F}_2(T)} |(u - \Pi_{T,p}^0 u)_{|f}|_{\frac12+\epsilon,f}\ . \end{aligned} \tag{4.125}$$

Next, we use (4.96) from Lemma 4.9 together with (4.112), which confirms that $(\Pi^0_{T,p}u)_{|f} - \mathsf{Q}_{f,p}u$ is the minimum $|\cdot|_{\frac{1}{2}+\epsilon,f}$-semi-norm polynomial extension of $(\Pi^0_{T,p}u)_{|\partial f} - \mathsf{Q}_{f,p}(u)_{|\partial f}$. Hence, based on arguments parallel to the derivation of (4.124), this time using Theorem 4.12, we can bound

$$\left|(u - \Pi^0_{T,p}u)_{|f}\right|_{\frac{1}{2}+\epsilon,f} \lesssim \left|u_{|f} - \mathsf{Q}_{f,p}u\right|_{\frac{1}{2}+\epsilon,f} + \left|(\Pi^0_{T,p}u - \mathsf{Q}_{f,p}u)_{|\partial f}\right|_{\epsilon,\partial f}, \tag{4.126}$$

where the (ϵ-independent!) continuity constant of the trace mapping S_f enters the constant. Also recall the continuity of the trace mapping $H^{\frac{1}{2}+\epsilon}(f) \mapsto H^\epsilon(\partial f)$ [50, Proof of Lemma 3.35]: with a constant independent of ϵ,

$$\left\|u_{|\partial f}\right\|_{\epsilon,\partial f} \lesssim \frac{1}{\sqrt{\epsilon}}\|u\|_{\frac{1}{2}+\epsilon,f} \quad \forall u \in H^{\frac{1}{2}+\epsilon}(f). \tag{4.127}$$

Use this to continue the estimate (4.126)

$$\left|(u - \Pi^0_{T,p}u)_{|f}\right|_{\frac{1}{2}+\epsilon,f} \lesssim \frac{1}{\sqrt{\epsilon}}\left|u_{|f} - \mathsf{Q}_{f,p}u\right|_{\frac{1}{2}+\epsilon,f} + \left|(u - \Pi^0_{T,p}u)_{|\partial f}\right|_{\epsilon,\partial f}. \tag{4.128}$$

As $\epsilon < \frac{1}{2}$, we can localize the norm $\left|(u - \Pi^0_{T,p}u)_{|\partial f}\right|_{\epsilon,\partial f}$ to the edges of f, similarly to Lemma 4.13,

$$\left|(u - \Pi^0_{T,p}u)_{|\partial f}\right|_{\epsilon,\partial f} \lesssim \frac{1}{\frac{1}{2} - \epsilon} \sum_{e \in \mathcal{F}_1(T), e \subset \partial f} \left|(u - \Pi^0_{T,p}u)_{|e}\right|_{\epsilon,e}. \tag{4.129}$$

Recall the ϵ-uniform equivalence of the norms $|\cdot|_{\epsilon,e}$ and $\|\frac{d}{ds}\cdot\|_{-1+\epsilon,e}$. Hence, owing to (4.97), we have from Lemma 4.15 with $r = 1$:

$$\begin{aligned}\left|(u - \Pi^0_{T,p}u)_{|e}\right|_{\epsilon,e} &\lesssim \inf_{v_p \in \mathring{\mathcal{P}}_{p+1}} \left|(u - \Pi^0_{T,0}u)_{|e} - v_p\right|_{\epsilon,e} \\ &\lesssim \left|(u - \Pi^0_{T,0}u)_{|e} - \mathsf{Q}^*_{e,p}((u - \Pi^0_{T,0}u)_{|e})\right|_{\epsilon,e} \\ &\lesssim \left(\frac{h_T}{p+1}\right)^{1-2\epsilon} \left|(u - \Pi^0_{T,0}u)_{|e}\right|_{s,e}.\end{aligned} \tag{4.130}$$

Moreover, $H^2(T)$ is continuously embedded into $C^0(\overline{T})$. Consequently, applying trace theorems twice and appealing to the equivalence of all norms on the finite dimensional space $\mathcal{P}_1(T)$,

$$\left|(u - \Pi^0_{T,0}u)_{|e}\right|_{s,e} \le \left|u_{|e}\right|_{s,e} + \left|(\Pi^0_{T,0}u)_{|e}\right|_{s,e} \lesssim |u|_{1+s,T}, \tag{4.131}$$

where the constant may depend on s. Combining the estimates (4.124), (4.125), (4.128), and (4.129), (4.130) with (4.131), we find

$$\begin{aligned}\left|u - \Pi^0_{T,p}u\right|_{1,T} &\lesssim \left|u - \mathsf{Q}_{T,p}u\right|_{1,T} + \frac{1}{\epsilon^{3/2}} \sum_{f \in \mathcal{F}_2(T)} \left|u_{|f} - \mathsf{Q}_{f,p}(u_{|f})\right|_{\frac{1}{2}+\epsilon,f} \\ &\quad + \left(\frac{h_T}{p+1}\right)^{s-2\epsilon} \frac{1}{\epsilon(\frac{1}{2} - \epsilon)} \sum_{e \in \mathcal{F}_1(T)} |u|_{2,T}.\end{aligned} \tag{4.132}$$

Finally, we plug in the projection error estimates (4.113), (4.114), and arrive at

$$\left|u - \Pi^0_{T,p}(\epsilon)u\right|_{1,T} \lesssim \frac{h_T}{p+1}|u|_{2,T} + \left(\frac{h_T}{p+1}\right)^{1-+\epsilon} \frac{1}{\epsilon^{3/2}} \sum_{f\in\mathcal{F}_2(T)} |u|_{3/2,f} + \left(\frac{h_T}{p+1}\right)^{1-2\epsilon} \frac{1}{\epsilon(\frac{1}{2}-\epsilon)} \sum_{e\in\mathcal{F}_1(T)} |u|_{1,e} \,, \tag{4.133}$$

with constants also independent of ϵ. The choice (4.100) of ϵ together with an application of trace theorems then completes the proof. □

The next lemma plays the role of [9, Lemma 9] and makes it possible to adapt the approach of [9, Section 4.4] to 3D edge elements.

Lemma 4.16. *If* $\mathbf{u} \in \boldsymbol{H}^1(T) \cap \boldsymbol{H}(\mathbf{curl}, T)$ *possesses a polynomial* **curl** *in the sense that* $\mathbf{curl}\,\mathbf{u} \in \mathcal{P}_p(T)$, *then*

$$\left\|(\mathsf{Id} - \Pi^1_p)\mathbf{u}\right\|_{0,\Omega} \lesssim (1 + \log^{3/2}(p+1))\frac{h_T}{p}\,|\mathbf{u}|_{1,T} \,. \tag{4.134}$$

Proof. Pick any $\mathbf{u}$ complying with the assumptions of the lemma and split

$$\mathbf{u} = (\mathbf{u} - \mathsf{R}_T\,\mathbf{curl}\,\mathbf{u}) + \mathsf{R}_T\,\mathbf{curl}\,\mathbf{u} \,. \tag{4.135}$$

Note that the properties of the smoothed Poincaré lifting R_T stated in Theorem 4.3 imply:

(i) $\mathbf{curl}(\mathbf{u} - \mathsf{R}_T\,\mathbf{curl}\,\mathbf{u}) = 0$ on T, as a consequence of (4.8), and
(ii) $\mathsf{R}_T\,\mathbf{curl}\,\mathbf{u} \in \boldsymbol{H}^1(T)$ and the bound

$$\left\|\mathsf{R}_T\,\mathbf{curl}\,\mathbf{u}\right\|_{1,T} \lesssim \left\|\mathbf{curl}\,\mathbf{u}\right\|_{0,\Omega} \,, \tag{4.136}$$

where here and below no constant may depend on $\mathbf{u}$ or p.

Hence, as $\mathbf{u} \in \boldsymbol{H}^1(T)$, there exists $v \in H^2(T)$ such that

$$\mathbf{u} = \mathbf{grad}\,v + \mathsf{R}_T\,\mathbf{curl}\,\mathbf{u} \,. \tag{4.137}$$

The continuity of R_T reveals that

$$|v|_{2,T} \le \|\mathbf{u}\|_{1,T} + \left|\mathsf{R}_T\,\mathbf{curl}\,\mathbf{u}\right|_{1,T} \lesssim \|\mathbf{u}\|_{1,T} + \|\mathbf{curl}\,\mathbf{u}\|_{0,T} \,. \tag{4.138}$$

By the assumptions of the lemma and (4.11), we know that

$$\mathsf{R}_T\,\mathbf{curl}\,\mathbf{u} \in \mathcal{W}^1_p(T) \,. \tag{4.139}$$

By the commuting diagram property from Lemma 4.6 and the projector property of $\Pi^1_{T,p}$, the task is reduced to an interpolation estimate for $\Pi^0_{T,p}$:

$$(\mathsf{Id} - \Pi^1_{T,p})\mathbf{u} \overset{(4.137)}{=} \mathbf{grad}(\mathsf{Id} - \Pi^0_{T,p})v + \underbrace{(\mathsf{Id} - \Pi^1_{T,p})\mathsf{R}_T\,\mathbf{curl}\,\mathbf{u}}_{=0} . \tag{4.140}$$

As a consequence, invoking Theorem 4.10,

$$\begin{aligned}\left\|(\mathsf{Id} - \Pi^1_{T,p})\mathbf{u}\right\|_{0,T} &\overset{(4.140)}{=} \left|(\mathsf{Id} - \Pi^0_{T,p})v\right|_{1,T} \lesssim (1 + \log^{2/3}(p+1))\frac{h_T}{p}\,|v|_{2,T} \\ &\overset{(4.138)}{\lesssim} (1 + \log^{3/2}(p+1))\frac{h_T}{p}\left(\|\mathbf{u}\|_{1,T} + \|\mathbf{curl}\,\mathbf{u}\|_{0,T}\right) ,\end{aligned} \tag{4.141}$$

which gives the assertion of the lemma. □

Remark 4.17. In principle, the very construction of projection based interpolation operators well fits spaces of discrete differential forms with variable polynomial degree ("*hp*-spaces) as long as the so-called minimum rule for the degrees; see [49, Remark IV.3.2] or [22], is fulfilled. Unfortunately, it is not clear how to adapt the splitting (4.135) to the *hp* setting and our proof of the key Lemma 4.16 cannot be extended.

4.2 Boundary-aware *p*-stable quasi-interpolation for Lagrangian finite elements

In this section, we sketch the construction of a local quasi-interpolation operator into $\mathcal{W}^0_{\Gamma_D}(\mathcal{T})$ following the policy of smoothing projections by local regularization that as developed in [14, 26], [49, Chapter VII] and [47]. The latter fundamental work is our main source and [47, Corollary 3.7] already asserts the existence of suitable quasi-interpolation operator in the case $\Gamma_D = \partial\Omega$. We extend this to zero boundary conditions on parts of $\partial\Omega$, borrowing a distortion technique from [49, Section VII.2]. We point out that [51, Theorem 3.3] provides exactly the kind of quasi-interpolation we need, unfortunately only in two dimensions. The extension to 3D looks formidably technical.

According to [14, Section 4.1], the flow induced by the vector field $\tilde{\boldsymbol{n}}$ introduced in Section 2.1 can be used to define a "reflection at the boundary Γ", a map $\mathsf{R}_\Gamma : \Omega_\Gamma \to \Omega_\Gamma$ satisfying

(R_1) $\mathsf{R}_\Gamma(\Omega \cap \Omega_\Gamma) = (\mathbb{R}^3 \setminus \overline{\Omega}) \cap \Omega_\Gamma$;

(R_2) $\mathsf{R}_\Gamma(\boldsymbol{x}) = \boldsymbol{x} \quad \forall \boldsymbol{x} \in \Gamma$;

(R_3) R_Γ is bi-Lipschitz with Lipschitz constants depending only on Γ.

We introduce the p-scaled mesh width function $\varepsilon_h \in L^\infty(\Omega)$, $\varepsilon_h(\boldsymbol{x}) = {}^{h_T}/_{p+1}$ on $T \in \mathcal{T}$: $\varepsilon_h := {}^{h}/_{p+1}$. We can extend it to a function $\varepsilon_h \in L^\infty(\Omega)$ on the expanded domain $\widetilde{\Omega} := \Omega \cup \Omega_\Gamma$ by reflection:

$$\varepsilon_h(\boldsymbol{x}) := \varepsilon_h\big(\mathsf{R}_\Gamma^{-1}(\boldsymbol{x})\big) \quad \text{for almost all } \boldsymbol{x} \in \Omega_\Gamma \setminus \Omega .$$

From [47, Lemma 3.1] or [49, Lemma VII.8.2], we learn that convolution of ε_h with a simple mollifier yields a smoothed extended mesh width function with bounded derivatives.

Lemma 4.18 (Smooth extended mesh width function). *The exists a smooth function $\varepsilon \in C^\infty(\widetilde{\Omega})$ such that*
(E$_1$) *$\varepsilon \approx \varepsilon_h$ almost everywhere in $\widetilde{\Omega}$;*
(E$_2$) *$|D^{\mathbf{r}}\varepsilon| \lesssim |\varepsilon|^{1-|\mathbf{r}|}$ for all $\mathbf{r} \in \mathbb{N}_0^3$ pointwise in $\widetilde{\Omega}$.*

Thus, ε qualifies as an *admissible length scale function* in the parlance of [47, Def. 2.1]. In particular, ε is uniformly positive and Lipschitz continuous; we write $L_\varepsilon > 0$ for its Lipschitz constant that depends on Ω and $\rho(\mathcal{T})$ alone.

To handle zero boundary conditions on Γ_D, we take the cue from [49, Section VII.2] and consider a blow-up map for the bulge domain Υ_D introduced in Section 2.1, Theorem 2.2.

Lemma 4.19 (Shrinkage mapping for bulge domain [49, Theorem VII.2.1]). *We can find constants $\delta_D > 0$ and $L_D > 0$ depending only on $\widetilde{\Omega}$ and Υ_D such that for* any *function $\xi : \widetilde{\Omega} \to \mathbb{R}^+$ with*
– $|\xi(\boldsymbol{x}) - \xi(\boldsymbol{y})| \leq \delta_D \|\boldsymbol{x} - \boldsymbol{y}\|$ *, for all $\boldsymbol{x}, \boldsymbol{y} \in \widetilde{\Omega}$;*
– $|\xi(\boldsymbol{x})| \leq \delta_D$ *for all $\boldsymbol{x} \in \widetilde{\Omega}$,*

there exists a bi-Lipschitz *mapping $\mathsf{T}_\xi : \widetilde{\Omega} \to \widetilde{\Omega}$ with*[3]
(T$_1$) $\left\|\mathsf{T}_\xi(\boldsymbol{x}) - \mathsf{T}_\xi(\boldsymbol{y})\right\| \leq L_D(1 + \delta_D)\|\boldsymbol{x} - \boldsymbol{y}\|$ *for all $\boldsymbol{x}, \boldsymbol{y} \in \widetilde{\Omega}$;*
(T$_2$) $\left\|\mathsf{T}_\xi^{-1}(\boldsymbol{x}) - \mathsf{T}_\xi^{-1}(\boldsymbol{y})\right\| \leq L_D(1 + \delta_D)\|\boldsymbol{x} - \boldsymbol{y}\|$ *for all $\boldsymbol{x}, \boldsymbol{y} \in \widetilde{\Omega}$;*
(T$_3$) $\left\|\mathsf{T}_\xi(\boldsymbol{x}) - \boldsymbol{x}\right\| \leq L_D\xi(\boldsymbol{x})$ *, $\boldsymbol{x} \in \widetilde{\Omega}$;*
(T$_4$) *$\mathsf{T}_\xi(\boldsymbol{x}) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \widetilde{\Omega}$ with* $\operatorname{dist}(\boldsymbol{x}, \partial\Upsilon_D) \geq L_D\xi(\boldsymbol{x})$*;*
(T$_5$) *for all $\boldsymbol{x} \in \overline{\Upsilon_D}$ there holds $\mathsf{T}_\xi(B_{\xi(x)/L_D}(\boldsymbol{x}) \cap \widetilde{\Omega}) \subset \Upsilon_D$;*
(T$_6$) $\det D\mathsf{T}_\xi(\boldsymbol{x}) \approx 1$ *for all $\boldsymbol{x} \in \widetilde{\Omega}$.*

Casually speaking, by (T$_5$) T_ξ is a mapping that pulls a neighborhood of Υ_D into Υ_D. The property (T$_3$) ensures that the amount of local distortion effected by T_ξ can be controlled by ξ. The next result is borrowed from [47, Lemma 5.1 and 5.7] and paves the way for localization arguments.

Lemma 4.20 (Finite cover). *We can find "small constants"*

$$\alpha, \beta > 0\,, \quad \alpha < \beta\,, \quad \beta < \min\left\{1, \frac{\operatorname{dist}(\Omega^e, \partial\widetilde{\Omega})}{\|\varepsilon\|_{\infty,\widetilde{\Omega}}}, \frac{1}{2L_\varepsilon}\right\}, \tag{4.142}$$

and a finite set of points $\mathcal{Z} \subset \widetilde{\Omega}$ such that

3 The symbol $B_r(\boldsymbol{z})$ designates the open ball around $\boldsymbol{z} \in \mathbb{R}^3$ with radius $r > 0$.

(C_1) $\widetilde{\Omega} \subset \bigcup\{B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z}),\, \boldsymbol{z} \in \mathcal{Z}\}$ (*covering property*)
(C_2) $\operatorname{card}\{\boldsymbol{z} \in \mathcal{Z} : \boldsymbol{x} \in B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})\} \lesssim 1$ *for all* $\boldsymbol{x} \in \widetilde{\Omega}$ (*uniform finite overlap*).

From now, we fix α, β according to Lemma 4.20. From the covering and finite overlap property, we conclude for any $m \in \mathbb{N}_0$,

$$\sum_{\boldsymbol{z}\in\mathcal{Z}} \|v\|^2_{m,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \approx \sum_{\boldsymbol{z}\in\mathcal{Z}} \|v\|^2_{m,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \approx \|v\|^2_{m,\widetilde{\Omega}} \quad \forall v \in H^m(\widetilde{\Omega}) \,. \tag{4.143}$$

In addition, by the triangle inequality the bound on β ensures that for any $\boldsymbol{z} \in \Omega^e$

$$B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z}) \subset \widetilde{\Omega} \quad \text{and} \quad \tfrac{1}{2}\varepsilon(\boldsymbol{z}) \le \varepsilon(\boldsymbol{x}) \le \tfrac{3}{2}\varepsilon(\boldsymbol{z}) \quad \forall \boldsymbol{x} \in B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z}) \,. \tag{4.144}$$

Next, set $\tau := \frac{1}{2}(\alpha + \beta)$ and choose a small number $\delta > 0$ satisfying the following inequalities:
(δ_1) $2L_D^2\delta \le \beta - \tau$, with L_D from Lemma 4.19;
(δ_2) $\delta L_\varepsilon \le 1$ for the Lipschitz constant L_ε of ε;
(δ_3) $\delta L_D L_\varepsilon < \delta_D$, and $\delta L_D \|\varepsilon\|_{\infty,\widetilde{\Omega}} \le \delta_D$;
(δ_4) $2\delta + \alpha < \tau$ and $2\delta + \tau < \beta$.

Now, recall Lemma 4.19 and define a concrete distortion map T_ε by setting $\mathsf{T}_\varepsilon := \mathsf{T}_\xi$ with the particular control function $\xi(\boldsymbol{x}) := L_D\delta\varepsilon(\boldsymbol{x})$, $\boldsymbol{x} \in \widetilde{\Omega}$. Owing to ($\delta_3$), this choice of $\xi : \widetilde{\Omega} \to \mathbb{R}^+$ satisfies the assumptions of Lemma 4.19. Thanks to Lemma 4.19, (T_5) we infer

$$\mathsf{T}_\varepsilon(B_{\delta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})) \subset \Upsilon_D \quad \forall \boldsymbol{z} \in \Gamma_D \,. \tag{4.145}$$

As a consequence of (4.144), (δ_1), and Theorem 4.19, (T_3) we note

$$\forall \boldsymbol{z} \in \widetilde{\Omega} : \quad \begin{aligned} &\mathsf{T}_\varepsilon(B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z}) \cap \widetilde{\Omega}) \subset B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z}) \,, \\ &\mathsf{T}_\varepsilon(B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z}) \cap \widetilde{\Omega}) \subset B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z}) \,. \end{aligned} \tag{4.146}$$

We now study the pullback of functions under the distortion $\mathsf{T}_\varepsilon : \widetilde{\Omega} \to \widetilde{\Omega}$,

$$(\mathsf{T}_\varepsilon^* v)(\boldsymbol{x}) := v(\mathsf{T}_\varepsilon(\boldsymbol{x})) \quad \boldsymbol{x} \in \widetilde{\Omega} \quad \text{for} \quad v : \widetilde{\Omega} \to \mathbb{R} \,. \tag{4.147}$$

Lemma 4.21 (Estimates for pullback). *With constants depending only on Ω and the Lipschitz constant L_ε of ε the following estimates hold true:*
(PB_1) $\|\mathsf{T}_\varepsilon^* v\|_{0,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})\cap\widetilde{\Omega}} \approx \|v\|_{0,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})\cap\widetilde{\Omega}}$ *for all* $\boldsymbol{z} \in \widetilde{\Omega}$, $v \in L^2(\widetilde{\Omega})$,
(PB_2) $|\mathsf{T}_\varepsilon^* v|_{1,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})\cap\widetilde{\Omega}} \lesssim |v|_{1,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})\cap\widetilde{\Omega}}$ *for all* $\boldsymbol{z} \in \widetilde{\Omega}$, $v \in H^1(\widetilde{\Omega})$,
(PB_3) $\|(\mathsf{Id} - \mathsf{T}_\varepsilon^*)v\|_{0,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})\, |v|_{1,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}$ *for all* $\boldsymbol{z} \in \Omega^e$ *and* $v \in H^1(\widetilde{\Omega})$.

Proof. The assertions (PB_1) and (PB_2) follow from (4.146), $\|D\mathsf{T}_\varepsilon\|_{\infty,\widetilde{\Omega}}, \|D\mathsf{T}_\varepsilon^{-1}\|_{\infty,\widetilde{\Omega}} \lesssim 1$, Theorem 4.19, ($T_6$), the chain rule and the transformation formula for integrals.

To show (PB_3), we resort to convolution with a mollifier $\rho \in C^\infty(\mathbb{R}^3)$ that satisfies $\rho \geq 0$, $\operatorname{supp}(\rho) \subset B_1(0)$, and $\int_{\mathbb{R}^3} \rho(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$. Writing $\rho_\nu(\boldsymbol{x}) := \nu^{-3}\rho(\boldsymbol{x}/\nu)$, $\nu > 0$, we define for some function $\xi : \widetilde{\Omega} \to \mathbb{R}^+$,

$$(\mathsf{M}_\xi v)(\boldsymbol{x}) := \int_{\mathbb{R}^3} v(\boldsymbol{x}-\boldsymbol{y})\rho_{\xi(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{y}\,, \quad v \in L^1(\mathbb{R}^3)\,. \tag{4.148}$$

Since $\|\rho_\nu\|^2_{0,\mathbb{R}^3} = \nu^{-3}\,\|\rho\|^2_{0,\mathbb{R}^3}$, the Cauchy–Schwarz inequality yields

$$|(\mathsf{M}_\xi v)(\boldsymbol{x})| \leq \left\|\rho_{\xi(\boldsymbol{x})}\right\|_{0,\mathbb{R}^3} \|v\|_{0,B_{\xi(\boldsymbol{x})}(\boldsymbol{x})} \lesssim \xi(\boldsymbol{x})^{-3/2}\,\|v\|_{0,B_{\xi(\boldsymbol{x})}(\boldsymbol{x})}\ . \tag{4.149}$$

From now on, we set $\xi(\boldsymbol{x}) := L_D\delta\varepsilon(\boldsymbol{x})$ and, by (4.144), (4.149), and (δ_1), conclude for every $\boldsymbol{z} \in \Omega^e$

$$\left\|\mathsf{M}_\xi(\boldsymbol{x})\right\|^2_{0,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \leq (\tau\varepsilon(\boldsymbol{z}))^3 \max_{\boldsymbol{z}\in B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} L_D\delta\varepsilon(\boldsymbol{x})^{-3} \lesssim \|v\|^2_{0,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}\ . \tag{4.150}$$

The properties of ρ ensure that M_ξ preserves constants, so that we obtain by a scaling argument and the Bramble–Hilbert lemma [47, Lemma 4.3]:

$$\begin{aligned}\left\|v - \mathsf{M}_\xi v\right\|_{0,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} &= \inf_{c\in\mathbb{R}} \left\|(v-c) - \mathsf{M}_\xi(v-c)\right\|_{0,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \\ &\lesssim \inf_{c\in\mathbb{R}} \|v-c\|_{0,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \beta\varepsilon(\boldsymbol{z})\,|v|_{1,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}\ ,\end{aligned} \tag{4.151}$$

for any $v \in H^1(\widetilde{\Omega})$. Fixing $v \in H^1(\widetilde{\Omega})$ and $\boldsymbol{z} \in \Omega^e$ we continue with the triangle inequality:

$$\begin{aligned}\left\|v - \mathsf{T}^*_\varepsilon v\right\|_{0,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} &\leq \left\|v - \mathsf{M}_\xi v\right\|_{0,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \\ &\quad + \left\|(\mathsf{Id} - \mathsf{T}^*_\varepsilon)\mathsf{M}_\xi v\right\|_{0,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} + \left\|\mathsf{T}^*_\varepsilon(\mathsf{M}_\xi - \mathsf{Id})v\right\|_{0,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}\ .\end{aligned} \tag{4.152}$$

By means of (4.151) and Theorem 4.21, (PB_1) the first and last term can be estimated by $\lesssim \varepsilon(\boldsymbol{z})\,|v|_{1,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}$. Concerning the middle term, we appeal to the mean value theorem applied to $w := \mathsf{M}_\xi v$ and, by Theorem 4.19, (T_3), (4.146), get for $\boldsymbol{x} \in B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})$

$$\left|w(\boldsymbol{x}) - w(\mathsf{T}_\varepsilon(\boldsymbol{x}))\right| \leq \|\mathbf{grad}\,w\|_{\infty,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}\ \|\boldsymbol{x} - \mathsf{T}_\varepsilon(\boldsymbol{x})\| \lesssim \left\|\mathbf{grad}\,\mathsf{M}_\xi v\right\|_{\infty,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}\ .$$

Since **grad** commutes with convolution, the maximum norm of **grad** $\mathsf{M}_\xi v$ can be estimated as in (4.149) above:

$$\left\|\mathbf{grad}\,\mathsf{M}_\xi v\right\|_{\infty,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})^{-3/2}\,\|\mathbf{grad}\,v\|_{0,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}\ .$$

Ultimately, this yields

$$\left\|(\mathsf{Id} - \mathsf{T}^*_\varepsilon)\mathsf{M}_\xi v\right\|^2_{0,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})^3\,\left\|\mathbf{grad}\,\mathsf{M}_\xi v\right\|^2_{\infty,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \|\mathbf{grad}\,v\|^2_{0,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}\ ,$$

and the assertion (PB_3) when plugged into (4.152). □

Following [47, Section 5.2], we now outline the key idea of regularization by mollification. We employ a mollifier of order $6 =: k_{\max} \in \mathbb{N}_0$, that is, a function $\rho \in C^\infty(\mathbb{R}^3)$ with $\mathrm{supp}(\rho) \subset B_1(0)$, and [47, equation (4.1)],

$$\int_{\mathbb{R}^3} \boldsymbol{y}^{\mathbf{r}} \rho(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = \begin{cases} 1 & \text{if } \mathbf{r} = 0\,, \\ 0 & \text{else,} \end{cases} \tag{4.153}$$

for every multi-index $\mathbf{r} \in \mathbb{N}_0^3$ with $|\mathbf{r}| \le k_{\max}$. This property leads to the preservation of polynomials of degree up to $k_{\max}$ under convolution with ρ_ν. Analogously to (4.148), we define the mollification

$$(\mathsf{E}v)(\boldsymbol{x}) := \int_{\mathbb{R}^3} v(\boldsymbol{y}) \rho_{\delta\varepsilon(\boldsymbol{x})}(\boldsymbol{x} - \boldsymbol{y})\, \mathrm{d}\boldsymbol{y}\,, \quad \boldsymbol{x} \in \Omega^e\,, \quad v \in L^1(\widetilde{\Omega})\,. \tag{4.154}$$

From [47, Lemma 5.3], we learn that for every $\boldsymbol{z} \in \Omega^e$ and integers $0 \le m \le \ell$, with $\ell, m \le k_{\max} + 1$,

$$|\mathsf{E}v|_{\ell, B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})^{m-\ell}\, |v|_{m, B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \quad \forall v \in H^\ell(\widetilde{\Omega})\,, \tag{4.155}$$

$$|(\mathsf{Id} - \mathsf{E})v|_{m, B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})^{\ell-m}\, |v|_{\ell, B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \quad \forall v \in H^\ell(\widetilde{\Omega})\,. \tag{4.156}$$

The composition of mollification and distortion pullback yields the regularizing operator

$$\mathsf{J} := \mathsf{E} \circ \mathsf{T}_\varepsilon^* : L^1(\widetilde{\Omega}) \to C^\infty(\Omega^e)\,. \tag{4.157}$$

In light of (4.145), it is immediate that

$$\boxed{v_{|\Upsilon_D} = 0 \quad \Rightarrow \quad \mathsf{J}v_{|\Gamma_D} = 0}\,. \tag{4.158}$$

Using (4.155) for $m = \ell = 0$, (4.156) for $m = 0, \ell = 1$, and Theorem 4.21, (PB_3), for any $\boldsymbol{z} \in \Omega^e$ we find the bound

$$\begin{aligned} \|(\mathsf{Id} - \mathsf{J})v\|_{0, B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} &\le \|(\mathsf{Id} - \mathsf{E})v\|_{0, B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} + \left\|\mathsf{E}(\mathsf{Id} - \mathsf{T}_\varepsilon^*)v\right\|_{0, B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \\ &\lesssim \varepsilon(\boldsymbol{z})\, |v|_{1, B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} + \left\|(\mathsf{Id} - \mathsf{T}_\varepsilon^*)v\right\|_{0, B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})\, |v|_{1, B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}\,. \end{aligned} \tag{4.159}$$

By means of (4.155) for $m = 0, 1$ we get for any $1 \le \ell \le k_{max} + 1$,

$$|\mathsf{J}v|_{\ell, B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})^{-\ell} \left\|\mathsf{T}_\varepsilon^* v\right\|_{0, B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}\,, \quad \forall v \in H^1(\widetilde{\Omega})\,, \tag{4.160}$$

$$|\mathsf{J}v|_{\ell, B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})^{1-\ell} \left|\mathsf{T}_\varepsilon^* v\right|_{1, B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}\,, \quad \forall v \in H^1(\widetilde{\Omega})\,. \tag{4.161}$$

Further, (4.155) for $m = \ell = 1$ and Theorem 4.21, (PB_3) lead to

$$|\mathsf{J}v|_{1, B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim |v|_{1, B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \quad \forall v \in H^1(\widetilde{\Omega})\,. \tag{4.162}$$

The final step is inspired by [47, Section 3.1]. To build the desired quasi-interpolation operator, we apply the perfectly local projection-based interpolation operators $\mathsf{I}_p : H^6(\Omega) \to \mathcal{W}_p^0(\mathcal{T})$ from [52, Corollary 7.4] to the regularized function:

$$\boxed{\mathsf{Q}_p := \mathsf{I}_p \circ \mathsf{J} : L^1(\widetilde{\Omega}) \to \mathcal{W}_p^0(\mathcal{T})} \, . \tag{4.163}$$

We recall properties of I_p from [52, Section 7]. Firstly, it enjoys locality in the sense that

- $(\mathsf{I}_p v)(\boldsymbol{a}) = v(\boldsymbol{a})$ for every vertex $\boldsymbol{a}$ of the mesh $\mathcal{T}$;
- $(\mathsf{I}_p v)_{|e}$ is uniquely determined by $v_{|e}$ for every edge e;
- $(\mathsf{I}_p v)_{|F}$ depends only on $v_{|F}$ for every face F;
- and $(\mathsf{I}_p v)_{|T}$ exclusively relies on $v_{|T}$ for all tetrahedra $T \in \mathcal{T}$.

Obviously, if $v_{|F} = 0$, then $(\mathsf{I}_p v)_{|F} = 0$. As $\mathcal{T}$ was supposed to resolve Γ_D, applying I_p to a smooth function vanishing on Γ_D will result in an interpolant with the same property. This accounts for the range of Q_p stated in (4.163).

The locality of I_p comes at the price of poor stability. In [52, Corollary 7.4], the authors showed p-uniform local continuity of

$$\mathsf{I}_{p\,|T} : H^6(T) \to \mathcal{P}_p(\mathbb{R}^3) \, , \tag{4.164}$$

and an estimate of the form

$$h_T/p \left|(\mathsf{Id} - \mathsf{I}_p)v\right|_{1,T} + \left\|(\mathsf{Id} - \mathsf{I}_p)v\right\|_{0,T} \lesssim (h_T/p)^6 \, \|v\|_{6,T} \qquad \forall v \in H^6(T) \, , \tag{4.165}$$

where the constants depends merely on the shape regularity measure of the tetrahedron $T \in \mathcal{T}$. Since $\mathsf{J}v \in C^\infty(\Omega^e)$, the tight smoothness requirements of I_p can be accommodated. This is the main rationale behind using the regularizer J.

H^1-Stability of Q_p is straightforward from (4.165), (4.161), and the finite overlap property from Lemma 4.20. To begin with, we get

$$\begin{aligned} \left\|\mathsf{Q}_p u\right\|_{0,T} &= \left\|\mathsf{I}_p(\mathsf{J}u)\right\|_{0,T} \lesssim \|\mathsf{J}u\|_{0,T} + (h_Y/p)^6 \, \|\mathsf{J}u\|_{6,T} \lesssim \|u\|_{0,U_T} \, , \\ \left|\mathsf{Q}_p u\right|_{1,T} &= \left|\mathsf{I}_p(\mathsf{J}u)\right|_{1,T} \lesssim |\mathsf{J}u|_{1,T} + (h_Y/p)^5 \, \|\mathsf{J}u\|_{6,T} \lesssim |u|_{1,U_T} \, , \end{aligned} \tag{4.166}$$

where $U_T := \bigcup\{B_{\beta\varepsilon(\boldsymbol{x})}(\boldsymbol{x}), \, \boldsymbol{x} \in T\}$ is a local neighborhood of T. Local approximation estimates can be deduced from (4.159), (4.161), and (4.165):

$$\begin{aligned} \left\|(\mathsf{Id} - \mathsf{Q}_p)u\right\|_{0,T} &\le \|(\mathsf{Id} - \mathsf{J})u\|_{0,T} + \left\|(\mathsf{Id} - \mathsf{I}_p)\mathsf{J}u\right\|_{0,T} \\ &\lesssim h_T/p \, |u|_{1,U_T} + (h_T/p)^6 \, \|\mathsf{J}u\|_{6,T} \lesssim h_T/p \, |u|_{1,U_T} \, . \end{aligned} \tag{4.167}$$

Squaring and adding both (4.166) and (4.167) establishes global stability and approximation properties of our quasi-interpolation Q_p.

Theorem 4.22 (Quasi-Interpolation operator). *The operators* $\mathsf{Q}_p : L^1(\widetilde{\Omega}) \to \mathcal{W}_p^0(\mathcal{T}) \subset H^1(\Omega)$ *satisfy:*

(Q_1) $\|\mathsf{Q}_p u\|_{0,\Omega} \lesssim \|u\|_{0,\widetilde{\Omega}}$ *for all* $u \in L^2(\widetilde{\Omega})$*;*
(Q_2) $|\mathsf{Q}_p u|_{1,\Omega} \lesssim |u|_{1,\widetilde{\Omega}}$ *for all* $u \in H^1(\widetilde{\Omega})$*;*
(Q_3) $\|\varepsilon^{-1}(\mathsf{Id} - \mathsf{Q}_p)u\|_{0,\Omega} \lesssim |u|_{1,\widetilde{\Omega}}$ *for all* $u \in H^1(\widetilde{\Omega})$*,*

with constants depending only on Ω*,* Γ_D*, and the shape regularity measure* $\rho(\mathcal{T})$*.*

Further, $\mathsf{Q}_p v \in \mathcal{W}^0_{p,\Gamma_D}(\mathcal{T}) \subset H^1_{\Gamma_D}(\Omega)$*, if* $v_{|\Upsilon_D} = 0$*.*

The last assertion of the theorem follows from (4.158) and the locality of I_p discussed above.

4.3 Proof of Theorem 1.3

With local commuting projectors Π^1_p from Section 4.1 and stable quasi-interpolation operator Q_p from Section 4.2 at our disposal, the construction and analysis of p-uniformly stable discrete regular decompositions of $\mathcal{W}^1_{\Gamma_D}(\mathcal{T})$ runs rather parallel to the lowest-order case presented in Section 3.3.

We fix $\mathbf{v}_p \in \mathcal{W}^1_{p,\Gamma_D}(\mathcal{T}) \subset \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ and consider its regular decomposition supplied by Theorem 2.1 and its proof:

$$\mathbf{v}_p = \mathbf{z}_{|\Omega} + \mathbf{grad}\,\varphi\,, \quad \mathbf{z} \in \boldsymbol{H}^1(\mathbb{R}^3),\ \mathbf{z}_{|\Upsilon_D} \equiv 0\,, \quad \varphi \in H^1_{\Gamma_D}(\Omega)\,, \tag{4.168}$$

with norm bounds

$$\|\mathbf{z}\|_{0,\mathbb{R}^3} \lesssim \|\mathbf{v}_p\|_{0,\Omega}\,, \quad |\mathbf{z}|_{1,\mathbb{R}^3} \lesssim \|\mathbf{v}_p\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\,, \quad \|\varphi\|_{1,\Omega} \lesssim \|\mathbf{v}_p\|_{0,\Omega}\,. \tag{4.169}$$

None of the constants depends on $\mathbf{v}_p$. Since $\mathbf{curl}\,\mathbf{z} = \mathbf{curl}\,\mathbf{v}_p$, that is, $\mathbf{z}$ has a piecewise polynomial **curl**, Lemma 4.16 ensures that $\Pi^1_p\mathbf{z}$ is well-defined. In addition, for every $T \in \mathcal{T}$ we have $\mathbf{grad}\,\varphi_{|T} = \mathbf{v}_{p\,|T} - \mathbf{z}_{|T} \in \boldsymbol{H}^1(T)$, which implies $\varphi_{|T} \in H^2(T)$. Hence, φ possesses enough local regularity to render also $\Pi^0_p\varphi$ well-defined. This permits us to rely on the commuting diagram property of Lemma 4.6 when letting Π^1_p act on $\mathbf{v}_p$:

$$\mathbf{v}_p = \Pi^1_p\mathbf{v}_p = \Pi^1_p\mathbf{z} + \mathbf{grad}\,\Pi^0_p\varphi\,.$$

In order to obtain a contribution in $\boldsymbol{H}^1_{\Gamma_D}(\Omega)$, we insert a boundary-aware quasi-interpolant to generate the regular part $\mathbf{z}_p$ of the decomposition (III):

$$\mathbf{v}_p = \Pi^1_p \underbrace{\mathsf{Q}_p\mathbf{z}}_{=:\mathbf{z}_p} + \Pi^1_p(\mathsf{Id} - \mathsf{Q}_p)\mathbf{z} + \mathbf{grad}\,\underbrace{\Pi^0_p\varphi}_{=:\varphi_p}\,, \qquad \begin{array}{l}\mathbf{z}_p \in \boldsymbol{\mathcal{V}}^0_{p,\Gamma_D}(\mathcal{T})\,,\\ \varphi_p \in \mathcal{W}^0_{\Gamma_D}(\mathcal{T})\,.\end{array} \tag{4.170}$$

Writing $\widetilde{\mathbf{v}}_p := \Pi^1_p(\mathsf{Id} - \mathsf{Q}_p)\mathbf{z} \in \mathcal{W}^1_{\Gamma_D}(\mathcal{T})$, we have split $\mathbf{v}_p \in \mathcal{W}^1_{\Gamma_D}(\mathcal{T})$ as

$$\mathbf{v}_p = \Pi^1_p\mathbf{z}_p + \widetilde{\mathbf{v}}_p + \mathbf{grad}\,\varphi_p\,. \tag{III}$$

Next, we investigate the stability of this splitting, bounding norms of its terms by norms of $\mathbf{v}_p$.

① Estimating norms of $\mathbf{z}_p = \mathsf{Q}_p\mathbf{z}$ based on Theorem 4.22 is straightforward

$$\text{Theorem 4.22, } (\mathsf{Q}_1) \quad \Rightarrow \quad \|\mathbf{z}_p\|_{0,\Omega} \lesssim \|\mathbf{z}\|_{0,\Omega} \lesssim \|\mathbf{v}_p\|_{0,\Omega}\ , \tag{4.171}$$

$$\text{Theorem 4.22, } (\mathsf{Q}_2) \quad \Rightarrow \quad |\mathbf{z}_p|_{1,\Omega} \lesssim |\mathbf{z}|_{1,\Omega} \lesssim \|\mathbf{v}_p\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\ . \tag{4.172}$$

② Interpolation error estimates from Lemma 4.16 for Π_p^1 and Theorem 4.22, (Q_3) give bounds for $\tilde{\mathbf{v}}_p$, local ones first: for any tetrahedron $T \in \mathcal{T}$

$$\begin{aligned} \|\tilde{\mathbf{v}}_p\|_{0,T} &\lesssim \left\|(\mathsf{Id} - \Pi_p^1)(\mathsf{Id} - \mathsf{Q}_p)\mathbf{z}\right\|_{0,T} + \left\|(\mathsf{Id} - \mathsf{Q}_p)\mathbf{z}\right\|_{0,T} \\ &\lesssim (1 + \log(p+1))^{3/2} \frac{h_T}{p+1} \left|(\mathsf{Id} - \mathsf{Q}_p)\mathbf{z}\right|_{1,T} + \frac{h_T}{p+1} |\mathbf{z}|_{1,T} \\ &\lesssim (1 + \log(p+1))^{3/2} \frac{h_T}{p+1} \left|(\mathsf{Id} - \mathsf{Q}_p)\mathbf{z}\right|_{1,T}\ , \end{aligned} \tag{4.173}$$

which implies after squaring and summing that

$$\left(\sum_{T\in\mathcal{T}} \left\| \frac{p+1}{h_T} \tilde{\mathbf{v}}_p \right\|_{0,T}^2 \right)^{1/2} \lesssim (1 + \log(p+1))^{3/2} \|\mathbf{v}_p\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\ . \tag{4.174}$$

③ Norm estimates for φ_p rely on those for $\mathbf{z}_p$ and the local interpolation error estimate of Lemma 4.16:

$$\begin{aligned} |\varphi_p|_{1,T} &\le \|\mathbf{v}_p\|_{0,T} + \left\|\Pi_p^1 \mathbf{z}\right\|_{0,T} \\ &\lesssim \|\mathbf{v}_p\|_{0,T} + \|\mathbf{z}\|_{0,T} + (1 + \log(p+1))^{3/2} \frac{h_T}{p} |\mathbf{z}|_{1,T}\ . \end{aligned} \tag{4.175}$$

As a consequence of (4.169), we end up with

$$|\varphi_p|_{1,\Omega} \lesssim \|\mathbf{v}_p\|_{0,\Omega} + \max_{T\in\mathcal{T}} \left\{ (1 + \log(p+1))^{3/2} \frac{h_T}{p+1} \right\} \|\mathbf{curl}\,\mathbf{v}_p\|_{0,\Omega}\ . \tag{4.176}$$

Thus we are done, because Theorem 1.3 merely collects the estimates (4.171), (4.172), (4.174), and (4.176).

Bibliography

[1] M. Ainsworth and L. Demkowicz, Explicit polynomial preserving trace liftings on a triangle, *Math. Nachr.*, **282** (2009), 640–658.

[2] M. Ainsworth and K. Pinchedez, HP-Approximation theory for BDM/RT finite elements and applications, *SIAM J. Numer. Anal.*, **40** (2002), 2047–2068.

[3] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, Vector potentials in three-dimensional non-smooth domains, *Math. Methods Appl. Sci.*, **21** (1998), 823–864.
[4] D. N. Arnold, R. S. Falk, and R. Winther, Finite element exterior calculus, homological techniques, and applications, *Acta Numer.*, **15** (2006), 1–155.
[5] D. N. Arnold, R. S. Falk, and R. Winther, Finite element exterior calculus: from Hodge theory to numerical stability, *Bull. Am. Math. Soc.*, **47** (2010), 281–354.
[6] M. Bebendorf, A note on the Poincaré inequality for convex domains, *Z. Anal. Anwend.*, **22** (2003), 751–756.
[7] C. Bernardi and Y. Maday, Spectral methods, in *Handbook of Numerical Analysis, vol. V* (P. G. Ciarlet and J.-L. Lions, eds.), pp. 209–485, North-Holland, Amsterdam, 1997.
[8] M. S. Birman and M. Z. Solomyak, L_2-theory of the Maxwell operator in arbitrary domains, *Russ. Math. Surv.*, **42** (1987), 75–96.
[9] D. Boffi, M. Costabel, M. Dauge, and L. Demkowicz, Discrete compactness for the *hp* version of rectangular edge finite elements, *SIAM J. Numer. Anal.*, **44** (2006), 979–1004.
[10] M. E. Bogovskii, Solution of the first boundary value problem for an equation of continuity of an incompressible medium, *Dokl. Akad. Nauk SSSR*, **248** (1979), 1037–1040.
[11] A. Buffa, M. Costabel, and D. Sheen, On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains, *J. Math. Anal. Appl.*, **276** (2002), 845–867.
[12] H. Cartan, *Formes Différentielles*, Hermann, Paris, 1967.
[13] J. Chen, Y. Xu, and J. Zou, An adaptive edge element method and its convergence for a saddle-point problem from magnetostatics, *Numer. Methods Partial Differ. Equ.*, **28** (2011), 1643–1666.
[14] S. H. Christiansen and R. Winther, Smoothed projections in finite element exterior calculus, *Math. Comput.*, **77** (2008), 813–829.
[15] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Studies in Mathematics and its Applications, vol. 4, North-Holland, Amsterdam, 1978.
[16] P. Clément, Approximation by finite element functions using local regularization, *RAIRO Anal. Numér.*, **9** (1975), 77–84.
[17] M. Costabel, A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains, *Math. Methods Appl. Sci.*, **12** (1990), 365–368.
[18] M. Costabel, M. Dauge, and S. Nicaise, Singularities of Maxwell interface problems, *Modél. Math. Anal. Numér.*, **33** (1999), 627–649.
[19] M. Costabel and A. McIntosh, On Bogovskii and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains, *Math. Z.*, **265** (2010), 297–320.
[20] L. Demkowicz and A. Buffa, H^1, $\mathbf{H}(\mathbf{curl})$, and $\mathbf{H}(\mathrm{div})$-conforming projection-based interpolation int three dimensions. Quasi-optimal p-interpolation estimates, *Comput. Methods Appl. Mech. Eng.*, **194** (2005), 267–296.
[21] L. Demkowicz, J. Gopalakrishnan, and J. Schöberl, Polynomial extension operators. Part I, *SIAM J. Numer. Anal.*, **46** (2008), 3006–3031.
[22] L. Demkowicz, P. Monk, L. Vardapetyan, and W. Rachowicz, de Rham diagram for *hp* finite element spaces, *Comput. Math. Appl.*, **39** (2000), 29–38.
[23] A. Demlow and A. N. Hirani, A posteriori error estimates for finite element exterior calculus: the de Rham complex, *Found. Comput. Math.*, **14** (2014), 1337–1371.
[24] C. R. Dohrmann and O. B. Widlund, An iterative substructuring algorithm for two-dimensional problems in H(curl), *SIAM J. Numer. Anal.*, **50** (2011), 1004–1028.
[25] C. R. Dohrmann and O. B. Widlund, Some recent tools and a BDDC algorithm for 3D problems in H(curl), in *Domain Decomposition Methods in Science and Engineering XX* (R. Bank, M. Holst, O. Widlund and J. Xu, eds.), pp. 15–25, Lecture Notes in Computational Science and Engineering, vol. 91, Springer, Berlin, Heidelberg, 2013.

[26] A. Ern and J.-L. Guermond, Finite element quasi-interpolation and best approximation, *Modél. Math. Anal. Numér.*, **51** (2017), 1367–1385.
[27] R. S. Falk and R. Winther, Local bounded cochain projections, *Math. Comput.*, **83** (2014), 2631–2656.
[28] R. S. Falk and R. Winther, Double complexes and local cochain projections, *Numer. Methods Partial Differ. Equ.*, **31** (2015), 541–551.
[29] V. Girault and P. A. Raviart, *Finite Element Methods for Navier–Stokes Equations*, Springer, New York, 1986.
[30] J. Gopalakrishnan and L. Demkowicz, Quasioptimality of some spectral mixed method, *J. Comput. Appl. Math.*, **167** (2004), 163–182.
[31] J. Gopalakrishnan and W. Qiu, Partial expansion of a Lipschitz domain and some applications, *Front. Math. China*, **7** (2012), 249–272.
[32] J. Gopalakrishnan and W. Qiu, An analysis of the practical DPG method, *Math. Comput.*, **83** (2014), 537–552.
[33] P. Grisvard, *Singularities in Boundary Value Problems*, Recherches en Mathématiques Appliquées [Research in Applied Mathematics], vol. 22, Masson/Springer-Verlag, Paris/Berlin, 1992.
[34] R. Hiptmair, Canonical construction of finite elements, *Math. Comput.*, **68** (1999), 1325–1346.
[35] R. Hiptmair, *Multigrid for Eddy Current Computation*, Sonderforschungsbereich 382, Universität Tübingen, Report no. 154, Tübingen, Germany, April 2000.
[36] R. Hiptmair, Finite elements in computational electromagnetism, *Acta Numer.*, **11** (2002), 237–339.
[37] R. Hiptmair, J. Li, and J. Zou, Universal extension for Sobolev spaces of differential forms and applications, *J. Funct. Anal.*, **263** (2012), 364–382.
[38] R. Hiptmair, G. Widmer, and J. Zou, Auxiliary space preconditioning in $\mathbf{H}_0(\mathbf{curl}, \Omega)$, *Numer. Math.*, **103** (2006), 435–459.
[39] R. Hiptmair and J. Xu, Nodal auxiliary space preconditioning in H(curl) and H(div) spaces, *SIAM J. Numer. Anal.*, **45** (2007), 2483–2509.
[40] R. Hiptmair and J. Xu, Auxiliary space preconditioning for edge elements, *IEEE Trans. Magn.*, **44** (2008), 938–941.
[41] R. Hiptmair and W.-Y. Zheng, Local multigrid in $\mathbf{H}(\mathbf{curl})$, *J. Comput. Math.*, **27** (2009), 573–603, Extended version published in arXiv:0901.0764 [math.NA].
[42] Q. Hu and J. Zou, A non-overlapping domain decomposition method for Maxwell's equation in three dimensions, *SIAM J. Numer. Anal.*, **41** (2003), 1682–1708.
[43] Q. Hu and J. Zou, Substructuring preconditioners for saddle-point problems arising from Maxwell's equations in three dimensions, *Math. Comput.*, **73** (2003), 35–61.
[44] Q.-Y. Hu, Convergence of HX preconditioner for Maxwell's equations with jump coefficients (i): Various extensions of the regular Helmholtz decomposition, arXiv:1708.05850, 2017.
[45] Q.-Y. Hu, Convergence of HX preconditioner for Maxwell's equations with jump coefficients (ii): The main results, arXiv:1708.05853, 2017.
[46] Q.-Y. Hu, S. Shu, and J. Zou, A discrete weighted Helmholtz decomposition and its application, *Numer. Math.*, **125** (2013), 153–189.
[47] M. Karkulik and J. M. Melenk, Local high-order regularization and applications to *hp*-methods, *Comput. Math. Appl.*, **70** (2015), 1606–1639.
[48] T. V. Kolev and P. S. Vassilevski, Parallel auxiliary space AMG for $\mathbf{H}(\mathbf{curl})$ problems, *J. Comput. Math.*, **27** (2009), 604–623.
[49] M. W. Licht, *On the A Priori and A Posteriori Error Analysis in Finite Element Exterior Calculus*, Dissertation, Department of Mathematics, University of Oslo, Norway, 2017.

[50] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, UK, 2000.
[51] J. M. Melenk, *hp*-interpolation of nonsmooth functions and an application to *hp*-a posteriori error estimation, *SIAM J. Numer. Anal.*, **43** (2005), 127–155 (electronic).
[52] J. M. Melenk, A. Parsania, and S. Sauter, General DG-methods for highly indefinite Helmholtz problems, *J. Sci. Comput.*, **57** (2013), 536–581.
[53] R. Muñoz-Sola, Polynomial liftings on a tetrahedron and applications to the *hp*-version of the finite element method in three dimensions, *SIAM J. Numer. Anal.*, **34** (1997), 282–314.
[54] J. C. Nédélec, Mixed finite elements in $\mathbb{R}^3$, *Numer. Math.*, **35** (1980), 315–341.
[55] J. E. Pasciak and J. Zhao, Overlapping Schwarz methods in H(curl) on polyhedral domains, *Numer. Math.*, **10** (2002), 211–234.
[56] R. Hiptmair and C. Pechstein, *Regular decompositions of vector fields – Continuous, discrete, and structure-preserving*, SAM-Report No. 2019-18, ETH-Zürich, Rämistrasse 101, CH-8092 Zürich, Switzerland, April 2019.
[57] C. Pechstein and R. Scheichl, Weighted Poincaré inequalities, *IMA J. Numer. Anal.*, **33** (2013), 652–686.
[58] J. Schöberl, A posteriori error estimates for Maxwell equations, *Math. Comput.*, **77** (2008), 633,649.
[59] C. Schwab, *p- and hp-Finite Element Methods. Theory and Applications in Solid and Fluid Mechanics*, Numerical Mathematics and Scientific Computation, Clarendon Press, Oxford, 1998.
[60] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Math Series, vol. 30, Princeton University Press, Princeton, New Jersey, 1970.
[61] M. E. Taylor, *Partial Differential Equations II. Qualitative Studies of Linear Equations*, Applied Mathematical Sciences, vol. 116, Springer, New York, 1996.
[62] J.-C. Xu, L. Chen, and R. H. Nochetto, Optimal multilevel methods for H(grad), H(curl), and H(div) systems on graded and unstructured grids, in *Multiscale, Non-Linear and Adaptive Approximation* (R. A. DeVore and A. Kunoth, eds.), pp. 599–659, Springer, Berlin, 2009.

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8 Some old and some new results in inverse obstacle scattering

Abstract: We will survey on uniqueness, that is, identifiability and on reconstruction issues for inverse obstacle scattering for time-harmonic acoustic and electromagnetic waves. In the first part, we begin by presenting two classical uniqueness proofs and after that proceed with two recent uniqueness results for inverse obstacle scattering subject to a generalized impedance boundary condition. Then we proceed with an iterative reconstruction algorithm via nonlinear boundary integral equations for the case of the generalized impedance boundary condition. In the final part, we present new integral equation formulations for transmission eigenvalues that play an important role through their connections with the linear sampling method and the factorization method for inverse scattering problems for penetrable objects.

Keywords: Uniqueness, generalized impedance boundary condition, transmission eigenvalues, boundary integral equations

MSC 2010: 35P25, 35P30, 35R30, 45A05

1 Uniqueness in inverse obstacle scattering

Scattering theory is concerned with the effects that obstacles and inhomogeneities have on the propagation of waves and in particular time-harmonic waves. For simplicity, we focus our attention on acoustic waves and only give passing references to electromagnetic waves. Throughout the paper, we will consider scattering objects within a homogeneous background that are described by a bounded domain $D \subset \mathbb{R}^m$ for $m = 2, 3$ with a connected C^2 smooth boundary ∂D and can be either impenetrable or penetrable. We note that the smoothness assumption, in principle, can be weakened and Lipschitz boundaries can also be allowed.

Given as incident field a plane wave $u^i(x) = e^{ikx\cdot d}$ propagating in the direction $d \in \mathbb{S}^{m-1} := \{x \in \mathbb{R}^m : |x| = 1\}$, the simplest obstacle scattering problem is to find the total field $u \in H^1_{\text{loc}}(\mathbb{R}^m \setminus \bar{D})$ as superposition $u = u^i + u^s$ of the incident field and the scattered field u^s such that the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^m \setminus \bar{D} \tag{1.1}$$

with positive wave number k and the boundary condition

$$u = 0 \quad \text{on } \partial D \tag{1.2}$$

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are satisfied together with the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{m-1}{2}}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |x|, \tag{1.3}$$

uniformly for all directions. The homogeneous Dirichlet boundary condition (1.2) corresponds to a *sound-soft* obstacle. Boundary conditions other than (1.2) to be considered are the homogeneous Neumann or *sound-hard* boundary condition or the impedance boundary condition, also known as Leontovich boundary condition,

$$\frac{\partial u}{\partial \nu} + ik\lambda u = 0 \quad \text{on } \partial D \tag{1.4}$$

where ν is the unit outward normal to ∂D and λ is a given continuous complex valued function with nonnegative real part. In addition to plane waves, other incident fields can be considered.

The radiation condition (1.3) was introduced by Sommerfeld in 1912 to characterize an outward energy flux. It is equivalent to the asymptotic behavior

$$u^s(x) = \frac{e^{ik|x|}}{|x|^{\frac{m-1}{2}}}\left[u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right], \quad |x| \to \infty,$$

uniformly for all directions $\hat{x} = x/|x|$ and where u_∞ is defined on $\mathbb{S}^{m-1}$ and is called the *far field pattern* of u^s. Solutions to the Helmholtz equation satisfying (1.3) are called radiating. For plane wave incidence, we will indicate the dependence of the far field pattern on the incident direction d by writing $u_\infty(\hat{x}, d)$.

Uniqueness of a solution to the obstacle scattering problem is a consequence of the following fundamental lemma which is due to Rellich (1943) and Vekua (1943) and is known as Rellich's lemma. This lemma later on also plays an essential role in connection with uniqueness for the inverse scattering problems. For a proof, we refer to [16]. Existence of a solution was first established by Vekua, Weyl and Müller in the 1950s by a boundary integral equation approach.

Lemma 1.1. *Any radiating solution $u^s \in H^1_{\text{loc}}(\mathbb{R}^m \setminus \bar{D})$ to the Helmholtz equation with far field pattern $u_\infty = 0$ vanishes identically in $\mathbb{R}^m \setminus \bar{D}$.*

Given the incident field $u^i(x) = e^{ikx\cdot d}$, the basic *inverse obstacle scattering problem* is to determine D from a knowledge of the far field pattern $u_\infty(\hat{x}, d)$ for all observation directions $\hat{x} \in \mathbb{S}^{m-1}$ and one or a few incident directions $d \in \mathbb{S}^{m-1}$ and a fixed wave number k. This inverse problem serves as a model problem for analyzing inverse scattering techniques in nondestructive evaluation such as radar, sonar, ultrasound imaging, seismic imaging, etc. However, we note that in practical applications the inverse scattering problem will never occur in the above idealized form. In particular, the far field pattern or some other measured quantity of the scattered wave will be available only for observation directions within a limited aperture either in the near or in the far field region.

We begin by noting that the inverse obstacle scattering problem is non-linear in the sense that the scattered wave depends nonlinearly on the scatterer D. More importantly, it is ill-posed since the determination of D does not depend continuously on the far field pattern in any reasonable norm.

We illustrate the nonlinearity and ill-posedness of the inverse obstacle scattering problem by looking at a simple example. For this, we consider as incident field the entire solution v^i to the Helmholtz equation given by

$$v^i(x) = \frac{\sin k|x|}{|x|}, \quad x \in \mathbb{R}^3. \tag{1.5}$$

Because of

$$\frac{\sin k|x|}{|x|} = \frac{k}{4\pi} \int_{\mathbb{S}^2} e^{ikx\cdot d}\, ds(d), \quad x \in \mathbb{R}^3,$$

the field v^i is a superposition of plane waves. For D, a sound-soft ball of radius R centered at the origin the scattered wave is given by

$$v^s(x) = -\frac{\sin kR}{e^{ikR}} \frac{e^{ik|x|}}{|x|}, \quad |x| \geq R. \tag{1.6}$$

This leads to the total wave

$$v(x) = \frac{1}{|x|e^{ikR}} \sin k(|x| - R), \quad |x| \geq R, \tag{1.7}$$

and the constant far field pattern

$$v_\infty(\hat{x}) = -\frac{\sin kR}{e^{ikR}}, \quad \hat{x} \in \mathbb{S}^2. \tag{1.8}$$

Therefore, assuming the a priori information that the scatterer is a ball centered at the origin, (1.8) provides a nonlinear equation for determining the radius R.

Concerning the ill-posedness, we consider a perturbed far field pattern

$$v^\delta_\infty(\hat{x}) = -\frac{\sin kR}{e^{ikR}} + \delta Y_n(\hat{x})$$

with some $\delta \in \mathbb{R}$ and a spherical harmonic Y_n of degree n. Then, in view of the asymptotic behavior of the spherical Hankel functions for large argument, the corresponding total field is given in terms of an outgoing spherical wave function

$$v^\delta(x) = \frac{\sin k(|x| - R)}{e^{ikR}|x|} + \delta k i^{n+1} h^{(1)}_n(k|x|) Y_n\left(\frac{x}{|x|}\right)$$

with the spherical Hankel function $h^{(1)}_n$ of order n and of the first kind (see in [16, Theorem 2.16]). This implies

$$v^\delta(x) = \delta k i^{n+1} h^{(1)}_n(kR) Y_n\left(\frac{x}{|x|}\right), \quad |x| = R,$$

and consequently, by the asymptotics of the spherical Hankel functions for large order, it follows that

$$|v^\delta(x)| \approx \delta k \left(\frac{2n}{ekR}\right)^n Y_n\left(\frac{x}{|x|}\right), \quad |x| = R.$$

This illustrates that small changes in the data v_∞ can cause large errors in the solution of the inverse problem, or a solution even may not exist anymore since v^δ may fail to have a closed surface as zero level surface.

From a functional analytic point of view, the ill-posedness is a consequence of the compactness property of the mapping $\partial D \mapsto u_\infty$ (see [16, Theorem 5.7]).

The following classical uniqueness result is due to Schiffer.

Theorem 1.2. *Assume that D_1 and D_2 are two sound-soft scatterers such that their far field patterns coincide for all $\hat{x}, d \in \mathbb{S}^{m-1}$ and one fixed wave number k. Then $D_1 = D_2$.*

Proof. Assume that $D_1 \neq D_2$. By Rellich's lemma, for each incident wave u^i the scattered waves u_1^s and u_2^s for the obstacles D_1 and D_2 coincide in the unbounded component G of the complement of $D_1 \cup D_2$. Without loss of generality, one can assume that $D^* := (\mathbb{R}^m \setminus G) \setminus \bar{D}_2$ is nonempty. Then u_2^s is defined in D^*, and the total wave $u = u^i + u_2^s$ satisfies the Helmholtz equation in D^* and the homogeneous boundary condition $u = 0$ on ∂D^*. Hence, u is a Dirichlet eigenfunction of $-\Delta$ in the domain D^* with eigenvalue k^2. The proof is now completed by showing that the total fields for distinct incident plane waves are linearly independent, since this contradicts the fact that for a fixed eigenvalue the Dirichlet eigenspace of $-\Delta$ in $H_0^1(D^*)$ has finite dimension. □

Schiffer's uniqueness result was obtained around 1960 and appeared as a private communication in the monograph by Lax and Philipps [33]. This is notable since nowadays in a time of permanent evaluation and competition for grants nobody would want to give away such a valuable result as a private communication. Noting that the proof presented in [33] contains a slight technical fault since the fact that the complement of $D_1 \cup D_2$ might be disconnected was overlooked, it is comforting to observe that even eminent authors can have errors in their books.

Using the strong monotonicity property of the Dirichlet eigenvalues of $-\Delta$, extending Schiffer's ideas in 1983 Colton and Sleeman [17] showed that a sound-soft scatterer is uniquely determined by the far field pattern for one incident wave under the a priori assumption that it is contained in a ball of radius R such that $kR < c_{m,0}$. Here, $c_{2,0}$ and $c_{3,0} = \pi$ are the smallest zeros of the Bessel function J_0 and the spherical Bessel function j_0, respectively, representing the smallest eigenvalue for the unit ball which is a simple eigenvalue. Hence, exploiting the fact that the wave functions are complex valued with linearly independent real and imaginary parts, in 2005 Gintides [21] improved this bound to $kR < c_{m,1}$ in terms of the smallest positive zeros $c_{2,1}$ and $c_{3,1} = 4.49\ldots$ of the Bessel function J_1 and the spherical Bessel function j_1, respectively. For other than the Dirichlet boundary condition, there is no analogue to the results in [17, 21] since there is no monotonicity property for the eigenvalues of $-\Delta$ for other boundary conditions.

Although there is widespread belief that the far field pattern for one single incident direction and one single wave number determines the scatterer without any additional a priori information, establishing this result still remains a challenging open problem. To illustrate the difficulty of a proof, we consider scattering of the entire solution v^i given by (1.5) from a sound-soft ball D of radius R centered at the origin. Then from (1.7) we observe that the total field v vanishes on the spheres with radius $R_n := R + n\pi/k$ centered at the origin for all integers n for which $R_n > 0$. This indicates that proving uniqueness for the inverse obstacle scattering problem with one single incident plane wave needs to incorporate special features of the incident field.

Starting in 2003 in a series of papers by Alessandrini, Cheng, Liu, Rondi and Yamamoto [1, 13, 34, 35], it was established that one incident plane wave is sufficient to uniquely determine a sound-soft polyhedron. Assuming that there exist two polyhedral scatterers producing the same far field pattern for one incident plane wave, the main idea of their proofs is to use the reflexion principle to construct a zero field line extending to infinity. However, in view of the fact that the scattered wave tends to zero uniformly at infinity, this contradicts the property that the incident plane wave has modulus one everywhere. These results for the polyhedron have analogs for other boundary conditions and also for electromagnetic waves.

The finiteness of the dimension of the eigenspaces for eigenvalues of $-\Delta$ for the Neumann or impedance boundary condition requires the boundary of the intersection D^* from the proof of Theorem 1.2 to be sufficiently smooth which, in general, is not the case. Therefore, there does not exist an immediate extension of Schiffer's approach to other boundary conditions.

Assuming that two different scatterers have the same far field patterns for all incident directions, in 1990 Isakov [23] obtained a contradiction by considering a sequence of solutions with a singularity moving towards a boundary point of one scatterer that is not contained in the other scatterer. He used weak solutions and the proofs are technically involved. During a hike in the Dolomites, on a long downhill walk in 1993 Kirsch and Kress [29] realized that these proofs can be simplified by using classical solutions rather than weak solutions and by obtaining the contradiction by considering point wise limits of the singular solutions rather than limits of L^2 norms. For boundary conditions of the form $Bu = 0$ on ∂D, where $Bu = u$ for a sound-soft scatterer and $Bu = \partial u/\partial\nu + ik\lambda u$ for the impedance boundary condition one can state the following theorem. For its proof and for later use throughout the remainder of the paper, we introduce the notation

$$\Phi_k(x,y) := \begin{cases} \dfrac{1}{4\pi}\dfrac{e^{ik|x-y|}}{|x-y|}\,, & m = 3, \\ \dfrac{i}{4}H_0^{(1)}(k|x-y|)\,, & m = 2, \end{cases} \tag{1.9}$$

for the fundamental solution of the Helmholtz equation, where $H_0^{(1)}$ denotes the Hankel function of order zero of the first kind.

Theorem 1.3. *Let two scatterers D_1 and D_2 with boundary conditions B_1 and B_2 have the same far field patterns for all $\hat{x}, d \in \mathbb{S}^{m-1}$ and one fixed wave number k. Then $D_1 = D_2$ and $B_1 = B_2$.*

Proof. In addition to scattering of plane waves, we also consider scattering of point sources $\Phi_k(\cdot, z)$ with source location z in $\mathbb{R}^m \setminus \bar{D}$. We will make use of the mixed reciprocity relation

$$w^s_\infty(-d, z) = \gamma_m u^s(z, d), \quad z \in \mathbb{R}^m \setminus \bar{D},\ d \in \mathbb{S}^2, \tag{1.10}$$

which, for scattering of a point source located in z, connects the far field pattern w_∞ of the scattered wave in observation direction $-d$ with the scattered wave u^s for plane wave incidence in direction d evaluated at z and where

$$\gamma_2 = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} \quad \text{and} \quad \gamma_3 = \frac{1}{4\pi} \tag{1.11}$$

(see [16, Theorem 3.16], [18]). Using Rellich's lemma and (1.10) from the assumption of the theorem one can deduce that $w^s_1(x, z) = w^s_2(x, z)$ for all $x, z \in G$. Here, we assume again that $D_1 \neq D_2$ and that G is defined as in the proof of Theorem 1.2 and w_1 and w_2 are the scattered waves for point source incidence for the obstacles D_1 and D_2, respectively. Now a contradiction can be obtained choosing $x \in \partial G$ such that $x \in \partial D_1$ and $x \notin \partial D_2$ and a sequence $z_n \in G$ such that $z_n \to x$ as $n \to \infty$. Hence $D_1 = D_2$ and then $B_1 = B_2$ follows from $u_1 = u_2$. □

The idea of the proof for Theorem 1.3 has been applied to a number of other boundary conditions such as for example a generalized impedance boundary condition by Bourgeois, Chaulet and Haddar [4] and other differential equations such as the Maxwell equations for electromagnetic waves. The generalized impedance boundary condition will be the subject of the next section.

2 Generalized impedance boundary condition

Given the plane wave $u^i(x) = e^{ikx\cdot d}$ as incident field, the obstacle scattering problem with the generalized impedance boundary condition (GIBC) consists in finding the total field $u \in H^2_{\text{loc}}(\mathbb{R}^m \setminus \bar{D})$ as superposition $u = u^i + u^s$ of the incident field and the scattered field u^s such that u satisfies the Helmholtz equation (1.1) and the boundary condition

$$\frac{\partial u}{\partial \nu} + ik(\lambda u - \operatorname{Div} \mu \operatorname{Grad} u) = 0 \quad \text{on } \partial D \tag{2.1}$$

together with the Sommerfeld radiation condition (1.3). Here, Grad and Div denote the surface gradient and surface divergence on ∂D and $\mu \in C^2(\partial D)$ and $\lambda \in C^1(\partial D)$ are given complex valued functions with nonnegative real parts. In the two-dimensional case both Grad and Div correspond to the tangential derivative d/ds. Recall that the unit normal vector ν to ∂D is directed towards the exterior of D. The boundary condition (2.1) requires $u \in H^2_{\mathrm{loc}}(\mathbb{R}^m \setminus \bar{D})$ and, in view of $u|_{\partial D} \in H^{\frac{3}{2}}(\partial D)$ has to be understood in the weak sense, that is,

$$\int_{\partial D} \left(\eta \frac{\partial u}{\partial \nu} + ik\lambda\eta u + ik\mu \operatorname{Grad} \eta \cdot \operatorname{Grad} u \right) ds = 0 \tag{2.2}$$

for all $\eta \in H^{\frac{3}{2}}(\partial D)$.

We note that the classical Leontovich boundary condition (1.4) is contained in (2.1) as the special case where $\mu = 0$. As compared with the Leontovich condition, the wider class of impedance conditions (2.1) provides more accurate models, for example, for imperfectly conducting obstacles (see [20, 22, 39]). For further interpretation of the generalized impedance boundary condition, we refer to [3, 4, 5] where the direct and the inverse scattering problem are analyzed by variational methods. Here, we will base our analysis on boundary integral equations.

2.1 The direct problem

Extending the analysis in [32] from the two-dimensional to the three-dimensional case, we briefly sketch an existence analysis via boundary integral equations.

Theorem 2.1. *Any solution $u \in H^2_{\mathrm{loc}}(\mathbb{R}^m \setminus \bar{D})$ to* (1.1) *and* (2.1) *satisfying the Sommerfeld radiation condition vanishes identically.*

Proof. Inserting $\eta = \bar{u}|_{\partial D}$ in the weak form (2.2) of the boundary condition we obtain that

$$\int_{\partial D} \bar{u} \frac{\partial u}{\partial \nu}\, ds = -ik \int_{\partial D} \{\lambda |u|^2 + \mu |\operatorname{Grad} u|^2\} ds.$$

Hence in view of our assumption $\operatorname{Re}\lambda \geq 0$ and $\operatorname{Re}\mu \geq 0$ we can conclude that

$$\operatorname{Im} \int_{\partial D} \bar{u} \frac{\partial u}{\partial \nu}\, ds \leq 0$$

and from this and the radiation condition the statement of the theorem follows from Theorem 2.13 in [16]. □

Corollary 2.2. *The obstacle scattering problem with generalized impedance boundary condition has at most one solution.*

For the existence analysis, following [16, Section 3.1] we introduce the classical boundary integral operators in scattering theory given by the single- and double-layer operators

$$(S_k\varphi)(x) := 2\int_{\partial D} \Phi_k(x,y)\varphi(y)\,ds(y) \tag{2.3}$$

and

$$(K_k\varphi)(x) := 2\int_{\partial D} \frac{\partial\Phi_k(x,y)}{\partial\nu(y)}\varphi(y)\,ds(y) \tag{2.4}$$

and the corresponding normal derivative operators

$$(K_k'\varphi)(x) := 2\int_{\partial D} \frac{\partial\Phi_k(x,y)}{\partial\nu(x)}\varphi(y)\,ds(y) \tag{2.5}$$

and

$$(T_k\varphi)(x) := 2\frac{\partial}{\partial\nu(x)}\int_{\partial D} \frac{\partial\Phi_k(x,y)}{\partial\nu(y)}\varphi(y)\,ds(y) \tag{2.6}$$

for $x \in \partial D$. We note that for $\partial D \in C^{4,\alpha}$ the operators $S_k : H^{\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$, and $K_k' : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ are bounded (see [26, 36]). (The subscript k for the operators will be needed in the next section.)

We seek the solution in the form of a single-layer potential for the scattered wave

$$u^s(x) = \int_{\partial D} \Phi_k(x,y)\varphi(y)\,ds(y), \quad x \in \mathbb{R}^m \setminus \overline{D}, \tag{2.7}$$

with density $\varphi \in H^{\frac{1}{2}}(\partial D)$ and note that the regularity $\varphi \in H^{\frac{1}{2}}(\partial D)$ guarantees that $u \in H^2_{\mathrm{loc}}(\mathbb{R}^m \setminus \overline{D})$ (see [36]). From the jump relations for single-layer potentials (see [16, Theorem 3.1]) we observe that the boundary condition (2.1) is satisfied provided φ solves the integro-differential equation

$$\varphi - K_k'\varphi - ik(\lambda - \operatorname{Div}\mu\operatorname{Grad})S_k\varphi = g \tag{2.8}$$

where we set

$$g := 2\left.\frac{\partial u^i}{\partial\nu}\right|_{\partial D} + 2ik(\lambda - \operatorname{Div}\mu\operatorname{Grad})u^i|_{\partial D} \tag{2.9}$$

in terms of the incident wave u^i. After defining a bounded linear operator $A_k : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ by

$$A_k\varphi := \varphi - K_k'\varphi - ik(\lambda - \operatorname{Div}\mu\operatorname{Grad})S_k\varphi \tag{2.10}$$

we summarize the above into the following theorem.

Theorem 2.3. *The single-layer potential* (2.7) *solves the scattering problem* (1.1), (2.1) *and* (1.3) *provided the density* φ *satisfies the equation*

$$A_k\varphi = g. \tag{2.11}$$

Lemma 2.4. *The modified Laplace–Beltrami operator given by*

$$L\varphi := -\operatorname{Div}\operatorname{Grad}\varphi + \varphi \tag{2.12}$$

is an isomorphism from $H^{\frac{3}{2}}(\partial D)$ *onto* $H^{-\frac{1}{2}}(\partial D)$.

Proof. In view of the Gauss surface divergence theorem, the surface divergence of a vector field $w \in L^2(\partial D)$ is given by the duality pairing

$$(\operatorname{Div} w, \psi) = -(w, \operatorname{Grad}\psi), \quad \psi \in H^1(\partial D).$$

This in turn implies

$$(L\varphi, \psi) = (\operatorname{Grad}\varphi, \operatorname{Grad}\psi) + (\varphi, \psi)$$

for $\varphi, \psi \in H^1(\partial D)$ and consequently

$$\|L\varphi\|_{H^{-1}(\partial D)} = \sup_{\|\psi\|_{H^1(\partial D)}=1} |(L\varphi, \psi)| \leq C_1\|\varphi\|_{H_1(\partial D)} \tag{2.13}$$

and

$$|(L\varphi, \varphi)| \geq C_2\|\varphi\|^2_{H_1(\partial D)} \tag{2.14}$$

for all $\varphi \in H^1(\partial D)$ and some positive constants C_1 and C_2. From (2.13), we have that $L : H^1(\partial D) \to H^{-1}(\partial D)$ is bounded and from (2.14) we can conclude that it is injective and has closed range. Assuming that it is not surjective implies the existence of some $\chi \neq 0$ in the dual space $(H^{-1}(\partial D))^* = H^1(\partial D)$ that vanishes on $L(H^1(\partial D))$, that is,

$$(L\varphi, \chi) = 0$$

for all $\varphi \in H^1(\partial D)$. Choosing $\varphi = \chi$ yields $(L\chi, \chi) = 0$ and from (2.14), we obtain the contradiction $\chi = 0$. Hence $L : H^1(\partial D) \to H^{-1}(\partial D)$ is bijective, and consequently by Banach's open mapping theorem it is an isomorphism.

Clearly, the operator $L : H^2(\partial D) \to L^2(\partial D)$ is bounded and proceeding as in the proof of Theorem 1.3 in [41] using elliptic regularity analysis it can be shown that its inverse is also bounded (see also Lemma 3.2 below). Now the statement of the lemma follows by Sobolev space interpolation. □

Lemma 2.5. *The operator*

$$A_k + ik\mu LS_k : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$$

is compact.

Proof. The boundedness of the operators $S_k : H^{\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$ and $K_k' : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ mentioned above and our assumption $\lambda \in C^1(\partial D)$ imply that all terms in the sum (2.10) defining the operator A_k are bounded from $H^{\frac{1}{2}}(\partial D)$ into $H^{\frac{1}{2}}(\partial D)$ except the term

$$\varphi \mapsto ik \operatorname{Div} \mu \operatorname{Grad} S_k \varphi.$$

Therefore, after splitting

$$\operatorname{Div} \mu \operatorname{Grad} S_k \varphi = \mu \operatorname{Div} \operatorname{Grad} S_k \varphi + \operatorname{Grad} \mu \cdot \operatorname{Grad} S_k \varphi$$

we observe that the operator $A_k + ik\mu L S_k : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is bounded since we assumed $\mu \in C^2(\partial D)$. Hence the statement of the lemma follows from the compact embedding of $H^{\frac{1}{2}}(\partial D)$ into $H^{-\frac{1}{2}}(\partial D)$. □

Theorem 2.6. *Assume that* $|\mu| > 0$ *and that* k^2 *is not a Dirichlet eigenvalue for* $-\Delta$ *in* D*. Then for each* $g \in H^{-\frac{1}{2}}(\partial D)$ *the equation* (2.11) *has a unique solution* $\varphi \in H^{\frac{1}{2}}(\partial D)$ *and this solution depends continuously on* g*.*

Proof. Since under our assumption on k the operator $S_k : H^{\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$ is an isomorphism, by Lemma 2.4 and our assumptions on μ the operator $ik\mu L S_k : H^{\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ also is an isomorphism. Therefore, in view of Lemma 2.5, by the Riesz theory it suffices to show that the operator A_k is injective. Assume that $\varphi \in H^{\frac{1}{2}}(\partial D)$ satisfies $A_k \varphi = 0$. Then, by Theorem 2.3 the single-layer potential u defined by (2.7) solves the scattering problem for the incident wave $u^i = 0$. Hence, by the uniqueness Theorem 2.1 we have $u = 0$ in $\mathbb{R}^m \setminus \overline{D}$. Taking the boundary trace of u, it follows that $S_k \varphi = 0$, and consequently $\varphi = 0$. □

To remedy the failure of the single-layer potential approach at the interior Dirichlet eigenvalues, as in the case of the classical impedance condition, we modify it into the form of a combined single- and double-layer potential for the scattered wave

$$u^s(x) = \int_{\partial D} \left\{ \Phi_k(x, y) + i \frac{\partial \Phi_k(x, y)}{\partial \nu(y)} \right\} \varphi(y)\, ds(y), \quad x \in \mathbb{R}^m \setminus \overline{D}, \tag{2.15}$$

with density $\varphi \in H^{\frac{3}{2}}(\partial D)$. The boundary condition (2.1) is satisfied provided φ solves the integro-differential equation

$$\varphi - K_k' \varphi - i T_k \varphi - ik(\lambda - \operatorname{Div} \mu \operatorname{Grad})(S_k \varphi + i\varphi + i K_k \varphi) = g \tag{2.16}$$

with g given by (2.9). Then with the same ideas as applied in the analysis of the integro-differential equation (2.8) the following existence result can be established. For the two-dimensional case, we refer to [32].

Theorem 2.7. *Under the assumption* $|\mu| > 0$*, the direct scattering problem with generalized impedance boundary condition has a unique solution.*

For the numerical solution in two dimensions, collocation methods based on numerical quadratures using trigonometric polynomial approximations are the most efficient methods for solving boundary integral equations for scattering problems in planar domains with smooth boundaries (see [16, Section 3.5]). Here, additionally an approximation is required for the operator $\varphi \mapsto d/ds\,\mu\,d\varphi/ds$ as the new feature in the integro-differential equations for the generalized impedance boundary condition. For this, we recommend trigonometric differentiation. It can be shown that this approach leads to spectral convergence for infinitely smooth boundaries and impedance coefficients. Details on this, including numerical examples, are presented in [32].

In three dimensions for smooth boundaries that are homeomorphic to the unit sphere numerical methods with spectral convergence for the boundary integral equations for scattering problems can be obtained via approximations by spherical harmonics by means of a hyperinterpolation operator on the unit sphere (see [16, Section 3.6]). This operator, in principle, can also be employed to approximate the surface gradient and the surface divergence. However, a numerical implementation of this idea at the time of this writing has not yet been done.

2.2 The inverse problem

The most general inverse scattering problem is the *inverse shape and impedance problem* to determine ∂D, μ and λ from a knowledge of a number of far field patterns u_∞ of solutions u to (1.1), (2.1) and (1.3). Here, we will be only concerned with two less general cases, namely the *inverse shape problem* and the *inverse impedance problem*. The inverse shape problem consists in determining ∂D knowing the impedance coefficients μ and λ. With the roles reversed, the inverse impedance problem requires to determine the impedance functions μ and λ for a known shape ∂D.

We briefly discuss the uniqueness issue and begin with the inverse impedance problem. In two dimensions, Cakoni and Kress [11] have shown that for a given shape ∂D three far field patterns corresponding to the scattering of three plane waves with different incident directions uniquely determine the impedance functions μ and λ. For two cylindrical wave functions as incident fields in [32], a counterexample is given where different impedance coefficients lead to the same two far field patterns. The uniqueness proof is sort of constructive and can be employed for an algorithm for the solution of the inverse impedance problem. For details and numerical reconstructions, we refer to [32].

The following uniqueness result for the full inverse shape and impedance problem (in two and three dimensions) was obtained by Bourgeois, Chaulet and Haddar [4] by using the method presented in Theorem 1.3.

Theorem 2.8. *Let two scatterers D_1 and D_2 with impedance functions λ_1, μ_1 and λ_2, μ_2 have the same far field patterns for all $\hat{x}, d \in \mathbb{S}^{m-1}$ and one fixed wave number. Then $D_1 = D_2$, $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$.*

We now outline an iterative algorithm for approximately solving the inverse shape problem which extends the method proposed by Johansson and Sleeman [24] for sound-soft or perfectly conducting obstacles. For this, we introduce the operator

$$S_\infty : H^{\frac{1}{2}}(\partial D) \to L^2(\mathbb{S}^{m-1})$$

by

$$(S_\infty \varphi)(\hat{x}) := \gamma_m \int_{\partial D} e^{-ik\hat{x}\cdot y} \varphi(y)\, ds(y), \quad \hat{x} \in \mathbb{S}^{m-1}, \tag{2.17}$$

where γ_m is given by (1.11). Then, in view of the asymptotic for the Hankel functions, the far field pattern for the solution to the scattering problem (1.1), (2.1) and (1.3) is given by

$$u_\infty = S_\infty \varphi \tag{2.18}$$

in terms of the solution φ to (2.8). Hence we can state the following theorem as theoretical basis of the inverse algorithm.

Theorem 2.9. *For a given incident field u^i and a given far field pattern u_∞, assume that ∂D and the density φ satisfy the system*

$$\varphi - K_k' \varphi - ik(\lambda - \operatorname{Div} \mu \operatorname{Grad}) S_k \varphi = g \tag{2.19}$$

and

$$S_\infty \varphi = u_\infty \tag{2.20}$$

where g is given in terms of the incident field by (2.9). *Then ∂D solves the inverse shape problem.* (*Note that the operators S_k, K_k' and S_∞ and the right hand side g depend on ∂D.*)

The operator S_∞ is compact with exponentially decreasing singular values and therefore the linear equation (2.20) is severely ill-posed reflecting the ill-posedness of the inverse shape problem. We denote this equation as the *data equation*. Note that the system (2.19)–(2.20) is linear with respect to the density φ and nonlinear with respect to the boundary ∂D. This opens up a variety of approaches to solve (2.19)–(2.20) by linearization and iteration. Here, we are going to proceed as follows. Given an approximation for the unknown ∂D, we solve the equation (2.19) that we denote as the *field equation* for the unknown density φ, that is, we solve the forward problem for the approximate boundary. Then, keeping φ fixed we linearize the data equation (2.20) with respect to the boundary to update the approximation.

To describe this in more detail, for simplicity, we assume ∂D to be star-like with respect to the origin, i. e., ∂D is represented in the parametric form

$$\partial D := \{r(z)z : z \in \mathbb{S}^{m-1}\} \tag{2.21}$$

with a positive function $r \in C^2(\mathbb{S}^{m-1})$. Then, indicating its dependence on the boundary ∂D, the parametrized form

$$\widetilde{S}_\infty : H^{\frac{1}{2}}(\mathbb{S}^{m-1}) \times C^2_+(\mathbb{S}^{m-1}) \to L^2(\mathbb{S}^{m-1})$$

of the operator S_∞ is given by

$$(\widetilde{S}_\infty(\psi, r))(\hat{x}) = \gamma_m \int_{\mathbb{S}^{m-1}} e^{-ikr(\hat{y})\hat{x}\cdot\hat{y}} J_r(\hat{y})\psi(\hat{y})\, ds(\hat{y}), \quad \hat{x} \in \mathbb{S}^{m-1}. \tag{2.22}$$

Here, J_r is the Jacobian of the mapping (2.21) given by $J_r = \sqrt{r^2 + [dr/ds]^2}$ if $m = 2$ and $J_r = r\sqrt{r^2 + |\operatorname{Grad} r|^2}$ if $m = 3$. For notational convenience, we introduce the mapping p taking the scalar function r onto the vector function $(p(r))(z) := r(z)z$ for $z \in \mathbb{S}^{m-1}$. Then the parameterized form of (2.20) is given by

$$\widetilde{S}_\infty(\psi, r) = u_\infty \tag{2.23}$$

where $\psi = \varphi \circ p(r)$. Its linearization with respect to r in direction q becomes

$$\widetilde{S}_\infty(\psi, r) + \widetilde{S}'_\infty(\psi, r; q) = u_\infty \tag{2.24}$$

and is an ill-posed linear equation for the perturbation q to obtain the update $r + q$. Here, the Fréchet derivative $\widetilde{S}'_\infty$ of the operator $\widetilde{S}_\infty$ with respect to the boundary r in the direction q is given by

$$\widetilde{S}'_\infty(\psi, r; q)(\hat{x}) := \gamma_m \int_{S^{m-1}} e^{-ikr(\hat{y})\hat{x}\cdot\hat{y}} [-ikq(\hat{y})\hat{x}\cdot\hat{y} J_r(\hat{y}) + (J'_r q)(\hat{y})]\psi(\hat{y})\, ds(\hat{y})$$

for $\hat{x} \in \mathbb{S}^{m-1}$ where $J'_r q$ denotes the Fréchet derivative of J_r in direction q. We have

$$J'_r q = \frac{rq + r'q'}{\sqrt{r^2 + [dr/ds]^2}}$$

if $m = 2$ and

$$J'_r q = q\sqrt{r^2 + |\operatorname{Grad} r|^2} + r\frac{rq + \operatorname{Grad} r \cdot \operatorname{Grad} q}{\sqrt{r^2 + |\operatorname{Grad} r|^2}}$$

if $m = 3$.

Now, given an approximation for ∂D with parameterization r, each iteration step of the proposed inverse algorithm consists of two parts:

1. We solve the parameterized well-posed field equation (2.19) for ψ. In two dimensions, this can be done through the numerical method described at the end of the previous subsection.
2. Then we solve the ill-posed linearized equation (2.24) for q and obtain an updated approximation for ∂D with the parameterization $r + q$. Since the kernels of the integral operators in (2.24) are smooth, for its numerical approximation the composite trapezoidal rule in two dimensions or the Gauss trapezoidal rule in three dimensions can be employed. Because of the ill-posedness, the solution of (2.24) requires stabilization, for example, by Tikhonov regularization.

This algorithm has a straightforward extension for the case of more than one incident wave. Assume that $u_1^i, \dots, u_N^i$ are N incident waves with different incident directions and $u_{\infty,1}, \dots, u_{\infty,N}$ the corresponding far field patterns for scattering from ∂D. Given an approximation r for the boundary, we first solve the field equations (2.19) for the N different incident fields to obtain N densities $\psi_1, \dots, \psi_N$. Then we solve the linearized equations

$$\widetilde{S}_\infty(\psi_n, r) + \widetilde{S}'_\infty(\psi_n, r; q) = u_{\infty,n}, \quad n = 1, \dots, N, \tag{2.25}$$

for the update $r + q$ by interpreting them as one ill-posed equation with an operator from $L^2(\mathbb{S}^{m-1})$ into $(L^2(\mathbb{S}^{m-1}))^N$ and applying Tikhonov regularization.

For more details on the numerical implementation and numerical examples in two dimensions, we refer to [32]. Numerical examples in three dimensions are not available for the time being. Further research is required for the solution of the full inverse problem by simultaneous linearization of both equations (2.19) and (2.20) with respect to the shape ∂D, the impedance functions λ and μ and the density φ analogous to [9].

3 Transmission eigenvalues

Roughly speaking, for the solution of inverse scattering problems one can distinguish between two main groups of methods, namely iterative methods and sampling methods. Iterative methods reformulate the inverse problem as a nonlinear ill-posed operator equation and solve it by iteration schemes such as regularized Newton methods, Landweber iterations or conjugate gradient methods. Sampling methods develop criteria in terms of the behavior of appropriately chosen ill-posed linear integral equations that decide on whether a point lies inside or outside the scatterer. In the previous section, we met the approach by Johansson and Sleeman as an example for an iteration method and two of the prominent examples for a sampling method are the linear sampling method and the factorization method. They are based on the far field operator

$$F : L^2(\mathbb{S}^{m-1}) \to L^2(\mathbb{S}^{m-1})$$

defined by

$$Fg(\hat{x}) := \int_{\mathbb{S}^{m-1}} u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^{m-1},$$

that is, the integral operator with the far field pattern as the kernel. Further for the far field $w_\infty(\cdot, z)$ of the point source $\gamma_m^{-1}\Phi(\cdot, z)$ located at z (where γ_m is given by (1.11)) we note that

$$w_\infty(\hat{x}, z) = e^{-ik\hat{x}\cdot z}, \quad \hat{x} \in \mathbb{S}^{m-1},\ z \in \mathbb{R}^m.$$

Now, the following theorem due to Kirsch [27] provides a short and concise description of the factorization method. Its proof relies on deep functional analytic tools, together with a factorization of the far field operator that explains the name of the method.

Theorem 3.1. *Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ for D. Then the equation*

$$(F^* F)^{1/4} g(\cdot, z) = w_\infty(\cdot, z),$$

where F^* is the adjoint operator of F, is solvable in $L^2(\mathbb{S}^{m-1})$ if and only if $z \in D$.

Picard's theorem (see [16, Theorem 4.8]) on the solution of equations of the first kind with compact operators can be employed for the numerical implementation of this criterion.

The linear sampling method introduced by Colton and Kirsch [14] is based on the far field equation

$$Fg(\cdot, z) = w_\infty(\cdot, z)$$

and decides on the behavior of its Tikhonov solution whether z belongs to the scatterer D. We refrain from the concise formulation since it is more involved as compared with Theorem 3.1 and only note that the linear sampling method also requires that k^2 is not a Dirichlet eigenvalue of $-\Delta$ for D.

An important feature of the factorization method and the linear sampling method is that they both work independently on the nature of the scatterer and also for scattering from inhomogeneous media.

Deviating for a couple of paragraphs from the theme of obstacle scattering, we consider the case of an isotropic medium with refractive index n. We assume that n is real valued and nonnegative and that the contrast $m := 1 - n$ has support given by our obstacle domain $\bar{D}$ and is continuous in $\bar{D}$. Then, for an incident plane wave $u^i(x) = e^{ikx\cdot d}$, the simplest inhomogeneous medium scattering problem is to find the total field $u \in H^1_{\mathrm{loc}}(\mathbb{R}^m)$ such that $u = u^i + u^s$ satisfies

$$\Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^m \tag{3.1}$$

and u^s satisfies the Sommerfeld radiation condition (1.3).

As shown by Kirsch [28], Theorem 3.1 remains valid for the inhomogeneous medium scattering problem, and also the linear sampling method has been extended to this case. Only the assumption on the wave number has to be modified, for the medium problem k is required not to be an interior transmission eigenvalue. A complex number k is called a transmission eigenvalue it there exist nontrivial functions $v, w \in L^2(D)$ with $\Delta v, \Delta w \in L^2(D)$ and $w - v \in H^2(D)$ such that

$$\Delta v + k^2 v = 0, \quad \Delta w + k^2 n w = 0 \quad \text{in } D \tag{3.2}$$

and

$$v = w, \quad \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \quad \text{on } \partial D \tag{3.3}$$

In view of the transmission condition (3.3), the space $H^2(D)$ of functions u with vanishing trace $u|_{\partial D}$ and normal trace $\partial_\nu u|_{\partial D}$ is the natural solution space for the difference $v - w$. Then from

$$\Delta v - \Delta w = k^2(w - v) - k^2 m w$$

we observe that we must demand $v, w \in L^2(D)$ and $\Delta v, \Delta w \in L^2(D)$.

This eigenvalue problem was first introduced by Kirsch [25] in 1986 in connection with the denseness and injectivity of the far field operator. The transmission eigenvalues can be seen as the extension of the idea of resonant frequencies for impenetrable obstacles to the case of penetrable media and related to nonscattering frequencies. As shown in [8], if k is a real transmission eigenvalue and v can be extended outside D as a solution to the Helmholtz equation, then if the extended field is used as incident field the corresponding scattered wave is identically zero, i. e., this field does not scatter at the wave number k. The transmission eigenvalue problem is a nonselfadjoint eigenvalue problem that is not covered by the standard theory of eigenvalue problems for elliptic equations. With respect to the factorization method and the linear sampling method, for a long time transmission eigenvalues were viewed as something to avoid, and only in 2008, Päivärinta and Sylvester [37] proved the existence of real transmission eigenvalues. Discreteness of the set of transmission eigenvalues was shown much earlier by Colton, Kirsch and Päivärinta [15] and Rynne and Sleeman [40]. More recently, it has been indicated that monotonicity properties of transmission eigenvalues in terms of the refractive index [6, 7] might open the possibility to use transmission eigenvalues as target signature for inverse media problems.

Here, following the recent work of Cakoni and Kress [12], we want to illustrate how boundary integral equations can be used to characterize and compute transmission eigenvalues in the case were n is constant in D. The main idea is to derive an integral equation from a characterization of the transmission eigenvalues in terms of the Robin-to-Neumann operator as defined by

$$N_k : f \mapsto \frac{\partial u}{\partial \nu} \tag{3.4}$$

where $u \in H^1(D)$ is the unique solution to

$$\Delta u + k^2 u = 0 \quad \text{in } D \tag{3.5}$$

satisfying the nonlocal impedance boundary condition

$$u + i\eta P^3 \frac{\partial u}{\partial \nu} = f \quad \text{on } \partial D \tag{3.6}$$

for $f \in H^{\frac{1}{2}}(\partial D)$. Here, η is a positive constant and P is a positive definite pseudo-differential operator of order -1. For example, we may choose $P = S_0$ where S_0 is the single-layer boundary integral operator (2.3) for the Laplace case $k = 0$ which needs to be modified in the two-dimensional case as in Theorem 7.41 in [31]. Our approach differs slightly from that in [12] through the use of the nonlocal impedance boundary condition rather than the classical Leontovich impedance condition (1.4). Using the smoothing operator P slightly simplifies the analysis.

For any solution of (3.5) and (3.6) for $f = 0$ from Green's integral theorem we have that

$$\int_D [|\operatorname{grad} u|^2 - k^2|u|^2]\,dx = i\eta \int_{\partial D} \left| P^{\frac{3}{2}} \frac{\partial u}{\partial \nu} \right|^2 ds$$

which implies uniqueness of the solution for all k with $\operatorname{Re} k > 0$ and $\operatorname{Im} k \geq 0$. Existence of a solution can be shown analogous to Theorem 2.6. The single-layer potential with density $\varphi \in H^{-\frac{1}{2}}(\partial D)$ solves (3.5) and (3.6) provided φ satisfies the equation

$$A_k \varphi = f \tag{3.7}$$

where we redefined

$$A_k := S_k + i\eta P^3 (I + K_k'). \tag{3.8}$$

From uniqueness both for the interior impedance problem (3.5) and (3.6) in D and for the exterior Dirichlet problem in $\mathbb{R}^m \setminus \bar{D}$ together with the jump relations for the single-layer potential, it can be checked that A_k has a trivial kernel in $H^{-\frac{1}{2}}(\partial D)$ for all k with positive real part and nonnegative imaginary part. After picking a wave number k_0 such that k_0^2 is not a Dirichlet eigenvalue for $-\Delta$ in D, we write $A_k = S_{k_0} + B_k$ where

$$B_k := S_k - S_{k_0} + i\eta P^3 (I + K_k').$$

Then $S_{k_0} : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is an isomorphism and $B_k : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is compact since the difference $S_k - S_{k_0}$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^{\frac{3}{2}}(\partial D)$ (see [16, Lemma 5.37]) and $P^3(I + K_k')$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^{\frac{5}{2}}(\partial D)$ because of our assumption on P. Therefore, by the Riesz theory $A_k : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is an isomorphism and we can write

$$N_k = (I + K_k')A_k^{-1}. \tag{3.9}$$

Now, setting

$$k_n := k\sqrt{n},$$

we have that k is a transmission eigenvalue if and only if the kernel of the operator

$$M(k;\eta) := N_k - N_{k_n} \tag{3.10}$$

in nontrivial.

We need to adjust the spaces in which we have to investigate the kernel of $M(k;\eta)$ since we must search for the eigenfunctions v, w in $L^2(D)$. This implies that their trace and their normal derivative on the boundary belong to $H^{-\frac{1}{2}}(\partial D)$ and $H^{-\frac{3}{2}}(\partial D)$, respectively. Indeed if $u \in L^2_\Delta(D) := \{u \in L^2(D) : \Delta u \in L^2(D)\}$ then its trace $u \in H^{-\frac{1}{2}}(\partial D)$ is defined by duality using the identity

$$\langle u, \tau\rangle_{H^{-\frac{1}{2}}(\partial D), H^{\frac{1}{2}}(\partial D)} = \int_D (u\Delta w - w\Delta u)dx$$

where $w \in H^2(D)$ is such that $w = 0$ and $\partial_\nu w = \tau$ on ∂D. Similarly, the trace of $\partial_\nu u \in H^{-\frac{3}{2}}(\partial D)$ is defined by duality using the identity

$$\left\langle \frac{\partial u}{\partial \nu}, \tau \right\rangle_{H^{-\frac{3}{2}}(\partial D), H^{\frac{3}{2}}(\partial D)} = -\int_D (u\Delta w - w\Delta u)dx$$

where $w \in H^2(D)$ is such that $w = \tau$ and $\partial_\nu w = 0$ on ∂D.

Therefore, when we represent v and w by single-layer potentials we must work with densities in $H^{-\frac{3}{2}}(\partial D)$. For convenience, we introduce

$$(\mathcal{S}_k\varphi)(x) := 2\int_{\partial D} \varphi(y)\Phi_k(x,y)\, ds(y), \quad x \in D.$$

Obviously, $\mathcal{S}_k\varphi$ satisfies the Helmholtz equation, hence we can conclude that $\mathcal{S}_k : H^{-\frac{3}{2}}(\partial D) \to L^2_\Delta(D)$ is bounded. Further, by a duality argument it is possible to extend the jump relations for single-layer potentials across ∂D to the case of densities in $H^{-\frac{3}{2}}(\partial D)$. The standard theory of single-layer potentials implies that both operators $S_{k_0} : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ and $S_{k_0} : H^{\frac{1}{2}}(\partial D) \to H^{\frac{3}{2}}(\partial D)$ are isomorphisms under our assumption on k_0^2 not to be a Dirichlet eigenvalue. From this, again by duality it follows that $S_{k_0} : H^{-\frac{3}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is an isomorphism. Consequently, from the above we have that A_k also is an isomorphism from $H^{-\frac{3}{2}}(\partial D)$ onto $H^{-\frac{1}{2}}(\partial D)$.

We note that the above statements remain valid in the case when $k = i$ and $\eta = i$ because of the uniqueness for the Robin problem $\Delta u - u = 0$ in D with $u + P^3\partial_\nu u = 0$ on ∂D.

To analyze the kernel of $M(k)$, we now want to show that

$$M(k;\eta) = (I + K_k')A_k^{-1} - (I + K_{k_n}')A_{k_n}^{-1} : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$$

is a Fredholm operator of index zero and to this end we begin with a regularity result.

Lemma 3.2. *Let $F \in H^m(D)$ and $g \in H^{m+\frac{3}{2}}(\partial D)$. Then the unique solution $v \in L^2(D)$ of $\Delta v = F$ in D and $v = g$ on ∂D belongs to $H^{m+2}(D)$ and the mapping taking (F, g) into v is bounded from $H^m(D) \times H^{m+\frac{3}{2}}(\partial D)$ into $H^{m+2}(D)$ for $m = 0, 1, \dots$*

Proof. We make use of a regularity theorem on the Poisson equation which guarantees that the unique solution $v \in H_0^1(D)$ of $\Delta v = F$ for $F \in H^m(D)$ belongs to $H^{m+2}(D)$ and that the linear mapping taking F into v is bounded from $H^m(D)$ into $H^{m+2}(D)$ for $m = 0, 1, \dots$ (see Theorem 1.3 in [41, p. 305]).

First, we show that this property can be extended to solutions $v \in L^2(D)$ that vanish on ∂D in the sense of the $H^{-\frac{1}{2}}(\partial D)$ trace. For this, we observe from the definition of the $H^{-\frac{1}{2}}(\partial D)$ trace that for any harmonic function $v \in L^2(D)$ vanishing on the boundary ∂D we have that $\int_D v\Delta w dx = 0$ for all $w \in H^2(D)$ with $w = 0$ on ∂D. Inserting the solution $w \in H_0^1(D)$ of $\Delta w = \bar{v}$ which automatically belongs to $H^2(D)$ by the above theorem yields $v = 0$ in D. For a solution $v \in L^2(D)$ of $\Delta v = F$ for $F \in L^2(D)$ with vanishing $H^{-\frac{1}{2}}(\partial D)$ trace on ∂D, we denote by $\tilde{v}$ the solution of $\Delta\tilde{v} = F$ in $H_0^1(D)$ and apply the uniqueness result for the difference $v - \tilde{v}$ to obtain that $v = \tilde{v} \in H_0^1(D)$.

The statement of the lemma now follows from the observation that the unique solution $w \in H^1(D)$ of the Laplace equation $\Delta w = 0$ with boundary condition $w = g$ on ∂D is in $H^{m+2}(D)$ and that the mapping taking g into w is bounded from $H^{m+\frac{3}{2}}(\partial D)$ into $H^{m+2}(D)$ as can be observed from the single-layer boundary integral equation approach. □

Lemma 3.3. *The linear operators*

$$\varphi \mapsto \mathcal{S}_k A_k^{-1}\varphi - \mathcal{S}_{k_n} A_{k_n}^{-1}\varphi \tag{3.11}$$

from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$ and $M(k;\eta) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ are bounded.

Proof. By definition, $M(k)\varphi$ is the normal derivative trace on the boundary ∂D of

$$u := \mathcal{S}_k A_k^{-1}\varphi - \mathcal{S}_{k_n} A_{k_n}^{-1}\varphi, \quad \varphi \in H^{-\frac{1}{2}}(\partial D).$$

Then,

$$\Delta u = -k^2\mathcal{S}_k A_k^{-1}\varphi + k_n^2\mathcal{S}_{k_n} A_{k_n}^{-1}\varphi$$

is in $L^2(D)$ and the mapping $\varphi \to \Delta u$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $L^2(D)$. Furthermore, from

$$[\mathcal{S}_k A_k^{-1}\varphi]|_{\partial D} + i\eta P^3 \frac{\partial}{\partial \nu}\mathcal{S}_k A_k^{-1}\varphi = [S_k + i\eta P^3(I + K_k')]A_k^{-1}\varphi = \varphi$$

we have that

$$u = g \quad \text{on } \partial D$$

with

$$g = -i\eta P^3 \frac{\partial u}{\partial \nu}.$$

Since $\varphi \mapsto u$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $L^2(D)$, we have that $\varphi \mapsto \partial_\nu u$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ to $H^{-\frac{3}{2}}(\partial D)$ and our assumption on the operator P finally ensures that the mapping $\varphi \to g$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^{\frac{3}{2}}(\partial D)$.

From this, Lemma 3.2 for $m = 0$ implies the first statement and the second follows by taking the normal trace. □

Theorem 3.4. *Let $\kappa > 0$ and $\kappa_n := \kappa\sqrt{n}$. Then*

$$(\kappa^2 - \kappa_n^2)M(i\kappa; i) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$$

is coercive.

Proof. For $u, v \in H^2(D)$, we can transform

$$\begin{aligned}
&\int_D v(\Delta - \kappa^2)(\Delta - \kappa_n^2)u\,dx \\
&\quad - \int_D [\Delta u \Delta v + (\kappa^2 + \kappa_n^2)\operatorname{grad} u \cdot \operatorname{grad} v + \kappa^2\kappa_n^2 uv]dx \\
&\quad = \int_D (v\Delta\Delta u - \Delta v \Delta u)\,dx - (\kappa^2 + \kappa_n^2)\int_D (v\Delta u + \operatorname{grad} u \cdot \operatorname{grad} v)\,dx
\end{aligned}$$

From this, by Green's theorem we obtain

$$\begin{aligned}
&\int_D v(\Delta - \kappa^2)(\Delta - \kappa_n^2)u\,dx \\
&\quad - \int_D [\Delta u \Delta v + (\kappa^2 + \kappa_n^2)\operatorname{grad} u \cdot \operatorname{grad} v + \kappa^2\kappa_n^2 uv]dx \\
&\quad = \int_{\partial D} \left(v\frac{\partial \Delta u}{\partial \nu} - \Delta u \frac{\partial v}{\partial \nu}\right) ds - (\kappa^2 + \kappa_n^2)\int_{\partial D} v\frac{\partial u}{\partial \nu}\,ds.
\end{aligned} \tag{3.12}$$

For $v = \bar{u}$, the second domain integral is equivalent to the $\|\cdot\|_{H^2}$ norm as can be seen with the aid of Green's representation formula, that is,

$$\int_D [|\Delta u|^2 + (\kappa^2 + \kappa_n^2)|\operatorname{grad} u|^2 + \kappa^2\kappa_n^2|u|^2]\, dx \geq c\|u\|^2_{H^2(D)} \tag{3.13}$$

for all $u \in H^2(D)$ and some constant $c > 0$.

Now, for $\varphi \in H^{-\frac{1}{2}}(\partial D)$ as above we define

$$u := \mathcal{S}_{i\kappa}A^{-1}_{i\kappa}\varphi - \mathcal{S}_{i\kappa_n}A^{-1}_{i\kappa_n}\varphi$$

which belongs to $H^2(D)$ by Lemma 3.3. Then

$$(\Delta - \kappa^2)(\Delta - \kappa_n^2)u = 0 \tag{3.14}$$

and

$$\Delta u = \kappa^2\mathcal{S}_{i\kappa}A^{-1}_{i\kappa}\varphi - \kappa_n^2\mathcal{S}_{i\kappa_n}A^{-1}_{i\kappa_n}\varphi.$$

From this, as in the proof of Lemma 3.3, we obtain the boundary conditions

$$u + P^3\frac{\partial u}{\partial \nu} = 0 \quad \text{and} \quad \Delta u + P^3\frac{\partial \Delta u}{\partial \nu} = (\kappa^2 - \kappa_n^2)\varphi \quad \text{on } \partial D. \tag{3.15}$$

We set $v = \bar{u}$ in (3.12) and use (3.15) and the self-adjointness of P to find that

$$\begin{aligned}&\int_D [|\Delta u|^2 + (\kappa^2 + \kappa_n^2)|\operatorname{grad} u|^2 + \kappa^2\kappa_n^2|u|^2]dx\\ &\quad = (\kappa^2 - \kappa_n^2)\int_{\partial D}\varphi\frac{\partial \bar{u}}{\partial \nu}\, ds - (\kappa^2 + \kappa_n^2)\int_{\partial D} P^{\frac{3}{2}}\frac{\partial u}{\partial \nu}P^{\frac{3}{2}}\frac{\partial \bar{u}}{\partial \nu}\, ds.\end{aligned}$$

Inserting $\partial_\nu u = M(i\kappa, i)\varphi$ and using the positive definiteness of P and (3.13), we get the coercivity estimate

$$(\kappa^2 - \kappa_n^2)\int_{\partial D}\varphi\overline{M(i\kappa)\varphi}\, ds \geq \tilde{C}\|u\|^2_{H^2(D)} \geq C\|\varphi\|^2_{H^{-\frac{1}{2}}(\partial D)} \tag{3.16}$$

for $\varphi \in H^{-\frac{1}{2}}(\partial D)$ and some constants $\tilde{C}, C > 0$, where for the latter inequality we used (3.15) and the definition of the trace of φ by duality. □

Theorem 3.5. *The operator*

$$M(k;\eta) + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2}M(i|k|;i) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$$

is compact.

Proof. For $\varphi \in H^{-\frac{1}{2}}(\partial D)$, we define

$$u := \mathcal{S}_k A_k^{-1}\varphi - \mathcal{S}_{k_n} A_{k_n}^{-1}\varphi \quad \text{and} \quad u_i := \mathcal{S}_{i|k|} A_{i|k|}^{-1}\varphi - \mathcal{S}_{i|k_n|} A_{i|k_n|}^{-1}\varphi$$

and let

$$U := u + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} u_i. \tag{3.17}$$

Then, by Lemma 3.3 we have $u, U \in H^2(D)$ with the mappings $\varphi \mapsto U$ and $\varphi \mapsto u$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$. Further, U satisfies the boundary conditions (see (3.15))

$$U = -P^3 \frac{\partial U}{\partial \nu} + (1 - i\eta) P^3 \frac{\partial u}{\partial \nu} \tag{3.18}$$

and

$$\Delta U = -P^3 \frac{\partial \Delta U}{\partial \nu} + (1 - i\eta) P^3 \frac{\partial \Delta u}{\partial \nu} \tag{3.19}$$

on ∂D. (We note that the coefficient in the definition of U in (3.17) is chosen such that we obtain (3.19).) By Lemma 3.3, the mappings $\varphi \mapsto U$ and $\varphi \mapsto u$ are bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$. Therefore, in view of our assumption on P, the right-hand side g_1 of (3.18) is in $H^{\frac{7}{2}}(\partial D)$ with the mapping $\varphi \mapsto g_1$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^{\frac{7}{2}}(\partial D)$. The right-hand side g_2 of (3.19) is in $H^{\frac{3}{2}}(\partial D)$ with the mapping $\varphi \mapsto g_1$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^{\frac{3}{2}}(\partial D)$.

Furthermore, it is straightforward to check that

$$\Delta\Delta U = F(u, u_i) \tag{3.20}$$

where

$$\begin{aligned} F(u, u_i) := & -k^2 k_n^2 u - (k^2 + k_n^2)\Delta u \\ & - \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2}\left[|k|^2 |k_n|^2 u_i - (|k|^2 + |k_n|^2)\Delta u_i\right] \end{aligned} \tag{3.21}$$

belongs to $L^2(D)$ with the mapping $\varphi \to F$ bounded from $H^{-\frac{1}{2}}(\partial D)$ to $L^2(D)$.

Now, we can use Lemma 3.2 again. Applying it first for ΔU we obtain that $\Delta U \in H^2(D)$ with the mapping $\varphi \mapsto \Delta U$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^2(D)$. Applying the lemma then for U shows that $U \in H^4(D)$ with the mapping $\varphi \mapsto U$ bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^4(D)$. Therefore, the mapping $\varphi \mapsto \partial_\nu U$ is bounded from $H^{-\frac{1}{2}}(\partial D)$ into $H^{\frac{5}{2}}(\partial D)$. Now, in view of

$$\frac{\partial U}{\partial \nu} = M(k;\eta) + \frac{k^2 - k_n^2}{|k|^2 - |k_n|^2} M(i|k|; i)$$

the statement of the theorem follows from the compact embedding of $H^{\frac{5}{2}}(\partial D)$ into $H^{\frac{1}{2}}(\partial D)$. □

Noting that $M(k;\eta)$ is analytic in k since the kernels of S_k and K_k' are analytic in k, now Theorems 3.4 and 3.5 imply the following final result. From this, in particular, we can reestablish the discreteness of the set of transmission eigenvalues for the special case of a constant refractive index and the finite multiplicity of the transmission eigenvalues.

Theorem 3.6. *$M(k;\eta) : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is a Fredholm operator with index zero and analytic in $\{k \in \mathbb{C} : \operatorname{Re} k > 0$ and $\operatorname{Im} k \geq 0\}$.*

Cakoni and Kress [12] also used their boundary integral formulations for actual computations of transmission eigenvalues with the aid of the attractive new algorithm for solving nonlinear eigenvalue problems for large sized matrices A that are analytic with respect to the eigenvalue parameter as proposed by Beyn [2]. So far, in the literature, the majority of numerical methods were based on finite element methods applied after a transformation of the homogeneous interior transmission problem to a generalized eigenvalue problem for a fourth-order partial differential equation. Boundary integral equations had been employed for the computation of transmission eigenvalues only by Cossonnière [18] and Kleefeld [30] using a two-by-two system of boundary integral equations proposed by Cossonnière and Haddar [19]. Comparing the computational costs for Beyn's algorithm as applied to Cossonnière and Haddar's two-by-two system, it can be shown that the approach presented here reduces the costs in the application of Beyn's algorithm by a about 50 percent. For details of the implementation and numerical results, we refer to [12] and for a very recent extension of this approach to the Maxwell equations including numerical results for transmission eigenvalues we refer to [10].

Bibliography

[1] G. Alessandrini and L. Rondi, Determining a sound–soft polyhedral scatterer by a single far–field measurement, *Proc. Am. Math. Soc.*, **133** (2005), 1685–1691.

[2] W.-J. Beyn, An integral method for solving non-linear eigenvalue problems, *Linear Algebra Appl.*, **436** (2012), 3839–3863.

[3] L. Bourgeois, N. Chaulet, and H. Haddar, Stable reconstruction of generalized impedance boundary conditions, *Inverse Probl.*, **27** (2011), 095002.

[4] L. Bourgeois, N. Chaulet, and H. Haddar, On simultaneous identification of a scatterer and its generalized impedance boundary condition, *SIAM J. Sci. Comput.*, **34** (2012), A1824–A1848.

[5] L. Bourgeois and H. Haddar, Identification of generalized impedance boundary conditions in inverse scattering problems, *Inverse Probl. Imaging*, **4** (2010), 19–38.

[6] F. Cakoni and D. Colton, *A Qualitative Approach to Inverse Scattering Theory*, Springer, Berlin, 2014.

[7] F. Cakoni, D. Gintides, and H. Haddar, The existence of an infinite discrete set of transmission eigenvalues, *SIAM J. Math. Anal.*, **42** (2010), 237–255.

[8] F. Cakoni and H. Haddar, Transmission eigenvalues in inverse scattering theory. *Inverse Problems and Applications, Inside Out* **60**, MSRI Publications 2012.

[9] F. Cakoni, Y. Hu, and R. Kress, Simultaneous reconstruction of shape and generalized impedance functions in electrostatic imaging, *Inverse Probl.*, **30** (2014), 105009.

[10] F. Cakoni, O. Ivanyshyn, R. Kress, and F. Le Louër, A boundary integral equation for the transmission eigenvalue problem for Maxwell's equation, *Math. Methods Appl. Sci.* (2018), DOI: 10.1002/mma.4664.

[11] F. Cakoni and R. Kress, Integral equation methods for the inverse obstacle problem with generalized impedance boundary condition, *Inverse Probl.*, **29** (2013), 015005.

[12] F. Cakoni and R. Kress, A boundary integral equation method for the transmission eigenvalue problem, *Appl. Anal.*, **96** (2017), 23–38.

[13] J. Cheng and M. Yamamoto, Uniqueness in an inverse scattering problem within non-trapping polygonal obstacles with at most two incoming waves, *Inverse Probl.*, **19** (2003), 1361–1384.

[14] D. Colton and A. Kirsch, A simple method for solving inverse scattering problems in the resonance region, *Inverse Probl.*, **12** (1996), 383–393.

[15] D. Colton, A. Kirsch, and L. Päivärinta, Far field patterns for acoustic waves in an inhomogeneous medium, *SIAM J. Math. Anal.*, **20** (1989), 1472–1483.

[16] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 3rd edition, Springer, New York, 2013.

[17] D. Colton and B. D. Sleeman, Uniqueness theorems for the inverse problem of acoustic scattering, *IMA J. Appl. Math.*, **31** (1983), 253–259.

[18] A. Cossonnière, *Valeurs propres de transmission et leur utilisation dans l'identification d'inclusions à partir de mesures é lectromagnétiques*, PhD Thesis, Université Toulouse, 2011.

[19] A. Cossonnière and H. Haddar, Surface integral formulation of the interior transmission problem, *J. Integral Equ. Appl.*, **25** (2013), 341–376.

[20] M. Duruflé, H. Haddar, and P. Joly, High order generalized impedance boundary conditions in electromagnetic scattering problems, *C. R. Phys.*, **7** (2006), 533–542.

[21] D. Gintides, Local uniqueness for the inverse scattering problem in acoustics via the Faber–Krahn inequality, *Inverse Probl.*, **21** (2005), 1195–1205.

[22] H. Haddar, P. Joly, and H. M. Nguyen, Generalized impedance boundary conditions for scattering by strongly absorbing obstacles: the scalar case, *Math. Models Methods Appl. Sci.*, **15** (2005), 1273–1300.

[23] V. Isakov, On uniqueness in the inverse transmission scattering problem, *Commun. Partial Differ. Equ.*, **15** (1990), 1565–1587.

[24] T. Johansson and B. Sleeman, Reconstruction of an acoustically sound-soft obstacle from one incident field and the far field pattern, *IMA J. Appl. Math.*, **72** (2007), 96–112.

[25] A. Kirsch, The denseness of the far field patterns for the transmission problem, *IMA J. Appl. Math.*, **37** (1986), 213–225.

[26] A. Kirsch, Surface gradient and continuity properties of some integral operators in classical scattering theory, *Math. Methods Appl. Sci.*, **11** (1989), 789–804.

[27] A. Kirsch, Characterization of the shape of the scattering obstacle by the spectral data of the far field operator, *Inverse Probl.*, **14** (1998), 1489–1512.

[28] A. Kirsch, Factorization of the far field operator for the inhomogeneous medium case and an application to inverse scattering theory, *Inverse Probl.*, **15** (1999), 413–429.

[29] A. Kirsch and R. Kress, Uniqueness in inverse obstacle scattering, *Inverse Probl.*, **9** (1993), 285–299.

[30] A. Kleefeld, A numerical method to compute interior transmission eigenvalues, *Inverse Probl.*, **29** (2013), 104012.

[31] R. Kress, *Integral Equations*, 3rd edition, Springer, New York, 2014.

[32] R. Kress, Integral equation methods in inverse obstacle scattering with a generalized impedance boundary condition, in J. Dick, F. Y. Kuo, and H. Wozniakowski (eds.), *Contemporary*

Computational Mathematics – A Celebration of the 80th Birthday of Ian Sloan, Springer-Verlag, 2018.
[33] P. D. Lax and R. S. Phillips, *Scattering Theory*, Academic Press, New York, 1967.
[34] H. Liu, M. Yamamoto, and J. Zou, Reflection principle for the Maxwell equations and its application to inverse electromagnetic scattering, *Inverse Probl.*, **23** (2007), 2357–2366.
[35] H. Liu and J. Zou, Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers, *Inverse Probl.*, **22** (2006), 515–524.
[36] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.
[37] L. Päivärinta and J. Sylvester, Transmission eigenvalues, *SIAM J. Math. Anal.*, **40** (2008), 738–758.
[38] R. Potthast, *Point-Sources and Multipoles in Inverse Scattering Theory*, Chapman & Hall, London, 2001.
[39] T. B. A. Senior and J. L. Volakis, *Approximate Boundary Conditions in Electromagnetics*, IEEE Electromagnetic Waves Series, vol. 41, The Institution of Electrical Engineers, London, 1995.
[40] B. P. Rynne and B. D. Sleeman, The interior transmission problem and inverse scattering from inhomogeneous media, *SIAM J. Math. Anal.*, **22** (1991), 1755–1762.
[41] M. E. Taylor, *Partial Differential Equations*, Springer, New York, 1996.

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9 The time-harmonic Maxwell equations with impedance boundary conditions in polyhedral domains

Abstract: In this paper, we first develop a variational formulation of the time-harmonic Maxwell equations with impedance boundary conditions in polyhedral domains similar to the one for domains with smooth boundary proposed in Section 4.5.d of Costabel et al., *Corner Singularities and Analytic Regularity for Linear Elliptic Systems. Part I: Smooth Domains*, 2010. It turns out that the variational space is embedded in H^1 as soon as the domain is convex. In such a case, the existence of a weak solution follows by a compact perturbation argument. As the associated boundary value problem is an elliptic system, standard shift theorem from Dauge, *Elliptic Boundary Value Problems on Corner Domains – Smoothness and Asymptotics of Solutions*, Springer, 1988 can be applied if the corner and edge singularities are explicitly known. We therefore describe such singularities, by adapting the general strategy from Costabel and Dauge, *Arch. Ration. Mech. Anal.*, **151** (2000), 221–276. Finally in order to perform a wavenumber explicit error analysis of our problem, a stability estimate is mandatory (see Melenk and Sauter, *Math. Comput.*, **79** (2010), 1871–1914 and Melenk and Sauter, *SIAM J. Numer. Anal.*, **49** (2011), 1210–1243 for the Helmholtz equation). We then prove such an estimate for some particular configurations. We end up with the study of a Galerkin (*h*-version) finite element method using Lagrange elements and give wave number explicit error bounds in the asymptotic ranges. Some numerical tests that illustrate our theoretical results are also presented.

Keywords: Maxwell equations, absorbing boundary conditions, non-smooth domains, finite elements

MSC 2010: 35J57, 35B65, 65N12, 65N30

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1 Introduction

In this paper, we are interested in the time-harmonic Maxwell equations for electromagnetic waves in a bounded, simply connected polyhedral domain Ω of $\mathbb{R}^3$ with a Lipschitz boundary (simply called polyhedron later on) filled by an isotropic homogeneous material with an absorbing boundary condition (also called Leontovich condition) that takes the form

$$\begin{cases} \operatorname{curl}\mathbf{E} - ik\mathbf{H} = \mathbf{0} \quad \text{and} \quad \operatorname{curl}\mathbf{H} + ik\mathbf{E} = \mathbf{J} \quad \text{in } \Omega, \\ \mathbf{H}\times\mathbf{n} - \lambda_{\mathrm{imp}}\mathbf{E}_t = \mathbf{0} \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, $\mathbf{E}$ is the electric part and $\mathbf{H}$ is the magnetic part of the electromagnetic field, and the constant k corresponds to the wave number or frequency and is, for the moment, supposed to be non-negative. The right-hand side $\mathbf{J}$ is the current density which – in the absence of free electric charges – is divergence-free, namely

$$\operatorname{div}\mathbf{J} = 0 \quad \text{in } \Omega.$$

As usual, $\mathbf{n}$ is the unit vector normal to $\partial\Omega$ pointing outside Ω and $\mathbf{E}_t = \mathbf{E} - (\mathbf{E}\cdot\mathbf{n})\mathbf{n}$ is the tangential component of $\mathbf{E}$. The impedance λ_{imp} is a smooth function[1] defined on $\partial\Omega$ satisfying

$$\lambda_{\mathrm{imp}} : \partial\Omega \to \mathbb{R}, \quad \text{such that} \quad \forall x \in \partial\Omega, \quad \lambda_{\mathrm{imp}}(x) > 0; \tag{1.2}$$

see, for instance, [35, 34]. The case $\lambda_{\mathrm{imp}} \equiv 1$ is also called the Silver–Müller boundary condition [3].

In practice, absorbing boundary conditions are used to reduce an unbounded domain of calculations into a bounded one; see [35, 34].

As variational formulation, a first attempt is to eliminate $\mathbf{H}$ by the relation $\mathbf{H} = \frac{1}{ik}\operatorname{curl}\mathbf{E}$, that transforms the impedance condition in the form

$$(\operatorname{curl}\mathbf{E})\times\mathbf{n} - ik\lambda_{\mathrm{imp}}\mathbf{E}_t = \mathbf{0} \quad \text{on } \partial\Omega.$$

Unfortunately, such a boundary condition has no meaning in $\mathbf{H}(\operatorname{curl},\Omega)$, hence a solution is to introduce the subspace

$$\mathbf{H}_{\mathrm{imp}}(\Omega) = \{\mathbf{u}\in\mathbf{H}(\operatorname{curl};\Omega) : \gamma_0\mathbf{u}_t \in \mathbf{L}^2(\partial\Omega)\}.$$

Then eliminating $\mathbf{H}$ in the second identity of (1.1), and multiplying by a test function, we arrive at

$$\int_\Omega (\operatorname{curl}\mathbf{E}\cdot\operatorname{curl}\bar{\mathbf{E}}' - k^2\mathbf{E}\cdot\bar{\mathbf{E}}')\,dx - ik\int_{\partial\Omega}\lambda_{\mathrm{imp}}\mathbf{E}_t\cdot\bar{\mathbf{E}}'_t\,d\sigma = ik\int_\Omega \mathbf{J}\cdot\bar{\mathbf{E}}'\,dx, \quad \forall\mathbf{E}'\in\mathbf{H}_{\mathrm{imp}}(\Omega). \tag{1.3}$$

1 $\lambda_{\mathrm{imp}} \in C^{0,1}(\partial\Omega)$ is sufficient.

Error analyses of (1.3) using Nédélec elements are available in [34, 19], but no explicit dependence with respect to k is proved. Moreover, there is no hope to get easily regularity results of the solution by applying the theory of elliptic boundary value problems to the system associated with (1.3) because it is not elliptic (see [14, Section 4.5.d]).

A second attempt, proposed in [14, Section 4.5.d] for smooth boundaries and inspired from [35, Section 5.4.3], is to keep the full electromagnetic field and use the variational space

$$\mathbf{V} = \{(\mathbf{E}, \mathbf{H}) \in (\mathbf{H}(\mathrm{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega))^2 : \ \mathbf{H} \times \mathbf{n} = \lambda_{\mathrm{imp}} \mathbf{E}_t \ \text{ on } \partial\Omega\}, \tag{1.4}$$

considering the impedance condition in (1.1) as an essential boundary condition. Hence the proposed variational formulation is: Find $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ such that

$$\mathbf{a}_{k,s}((\mathbf{E}, \mathbf{H}), (\mathbf{E}', \mathbf{H}')) = \int_\Omega (ik\mathbf{J} \cdot \bar{\mathbf{E}}' + \mathbf{J} \cdot \mathrm{curl}\, \bar{\mathbf{H}}')\, dx, \quad \forall (\mathbf{E}', \mathbf{H}') \in \mathbf{V}, \tag{1.5}$$

with the choice

$$\mathbf{a}_{k,s}((\mathbf{E}, \mathbf{H}), (\mathbf{E}', \mathbf{H}')) = a_{k,s}(\mathbf{E}, \mathbf{E}') + a_{k,s}(\mathbf{H}, \mathbf{H}') - ik \int_{\partial\Omega} \left(\lambda_{\mathrm{imp}} \mathbf{E}_t \cdot \bar{\mathbf{E}}_t' + \frac{1}{\lambda_{\mathrm{imp}}} \mathbf{H}_t \cdot \bar{\mathbf{H}}_t' \right) d\sigma,$$

with a positive real parameter s that may depend on k but is assumed to be in a fixed interval $[s_0, s_1]$ with $0 < s_0 \leq s_1 < \infty$ independent of k (see Section 5 below for more details) and

$$a_{k,s}(\mathbf{u}, \mathbf{v}) = \int_\Omega (\mathrm{curl}\, \mathbf{u} \cdot \mathrm{curl}\, \bar{\mathbf{v}} + s\, \mathrm{div}\, \mathbf{u}\, \mathrm{div}\, \bar{\mathbf{v}} - k^2 \mathbf{u} \cdot \bar{\mathbf{v}})\, dx.$$

The natural norm $\|\cdot\|_k$ of V associated with problem (1.5) is defined by

$$\begin{aligned}\|(\mathbf{E}, \mathbf{H})\|_k^2 &= \|\mathrm{curl}\, \mathbf{E}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathrm{div}\, \mathbf{E}\|_{L^2(\Omega)}^2 + k^2 \|\mathbf{E}\|_{\mathbf{L}^2(\Omega)}^2 \\ &\quad + \|\mathrm{curl}\, \mathbf{H}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathrm{div}\, \mathbf{H}\|_{L^2(\Omega)}^2 + k^2 \|\mathbf{H}\|_{\mathbf{L}^2(\Omega)}^2.\end{aligned}$$

This new formulation (1.5) has the advantage that its associated boundary value problem is an elliptic system (see [14, Section 4.5.d]), hence standard shift regularity results can be used. Nevertheless, this problem is still difficult to solve numerically as the wave number k is large, because oscillatory solutions exist and because of the so-called *pollution effect* [26, 27]: when the number of wavelengths inside the propagation domain is important, the numerical solution is only meaningful under restrictive conditions on the mesh size. This effect is manifested by a gap between the error of the best approximation the finite element scheme and the error of the numerical solution that is actually produced. This gap becomes more important as the frequency increases, unless additional discretization points per wavelength or higher order elements are employed. This problem, typical for wave-type equations, is also related to a lack of stability of the finite element scheme, since the associated sesquilinear forms are not coercive. Consequently, the quasi-optimality of the finite element solution in the energy norm is not guaranteed for arbitrary meshes, but is achieved only in an asymptotic range, i. e., for small enough mesh sizes, that depends on the frequency and the discretization order.

The behaviour of the asymptotic range with respect to the frequency, the mesh size, and the discretization order is the key to understand the efficiency of a finite element method. For the Helmholtz equation in domain with analytic boundaries, the asymptotic range for *hp*-finite element methods has been characterized in a sequence of papers by J. M. Melenk and collaborators [17, 32, 33]. For less regular boundaries, similar asymptotic ranges can be achieved using an expansion of the solution in powers of k [10].

The goal of the present paper is therefore to perform a similar analysis for the second variational problem of the time-harmonic Maxwell equations with impedance boundary conditions set on polyhedral domains. In such a situation, several difficulties appear: The first one is to show the well-posedness of the problem that requires to show that the variational space $\mathbf{V}$ is compactly embedded into $L^2(\Omega)^6$. In the smooth case (see [3, 14]), this is based on the hidden regularity of $\mathbf{V}$, namely on the embedding of $\mathbf{V}$ into $H^1(\Omega)^6$, hence we show that a similar embedding is valid for the largest possible class of polyhedra, namely this embedding holds if and only if Ω is convex. Secondly, error estimates are usually based on regularity results of the solution of the analyzed problem. Since our domain is not smooth, we then need to determine the corner and edge singularities of our system. This is here done by adapting the techniques from [16, 13]. The third obstacle is to prove the stability estimate for problem (1.5) and its adjoint one. For problem (1.3), the difficulty comes from the lack of stability estimate of the adjoint problem with a non-divergence-free right-hand side; but here by an appropriate choice of the parameter s, this difficulty can be avoided, at least for some particular domains. With these key results in hand, we are finally able to study a Galerkin h-finite element method using Lagrange elements and to give wave number explicit error bounds in an asymptotic range, characterized by the stability estimate and the minimal regularity of the solution of the adjoint problem. Since this minimal regularity could be quite poor, this asymptotic range could be quite strong for quasi-uniform meshes, hence in the absence of edge singularities, we improve it by using adapted meshes, namely meshes refined near the corners of the domain.

Our paper is organized as follows: The hidden regularity of the variational space is proved in Section 2. In Section 3, the well-posedness of our variational problem is proved and some useful properties are given. In Section 4, we describe the edge and corner singularities of our problem. The next Section 5 is devoted to the proof of the stability estimate. Finally, in Section 6 some h-finite element approximations are studied and some numerical tests that confirm our theoretical analysis are presented.

Let us finish this section with some notation used in the remainder of the paper. For a bounded domain D, the usual norm and semi-norm of $H^t(D)$ ($t \geq 0$) are denoted by $\|\cdot\|_{t,D}$ and $|\cdot|_{t,D}$, respectively. For $t = 0$, we will drop the index t. For shortness, we further write $\mathbf{H}^t(D) = H^t(D)^3$. Here and below, γ_0 is a generic notation for the trace operator from $H^t(\mathcal{O})$ to $H^{t-\frac{1}{2}}(\partial\mathcal{O})$, for all $t > \frac{1}{2}$. Furthermore, the notation $A \lesssim B$ (resp., $A \gtrsim B$) means the existence of a positive constant C_1 (resp. C_2), which is independent of A, B, the wave number k, the parameter s and any mesh size h such that $A \leq C_1 B$ (resp., $A \geq C_2 B$). The notation $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$ hold simultaneously.

2 Hidden regularity of the variational space

If $\partial\Omega$ is of class $\mathcal{C}^2$, it is well known that the continuous embedding

$$\mathbf{V} \hookrightarrow \left(\mathbf{H}^1(\Omega)\right)^2 \tag{2.1}$$

holds, which means that $\mathbf{V} \subset (\mathbf{H}^1(\Omega))^2$ with the estimate

$$\begin{aligned} \|(\mathbf{E}, \mathbf{H})\|_{\mathbf{H}^1(\Omega)^2} \lesssim{} & \|\operatorname{curl} \mathbf{E}\|_{\mathbf{L}^2(\Omega)} + \|\operatorname{div} \mathbf{E}\|_{L^2(\Omega)} + \|\mathbf{E}\|_{\mathbf{L}^2(\Omega)} \\ & + \|\operatorname{curl} \mathbf{H}\|_{\mathbf{L}^2(\Omega)} + \|\operatorname{div} \mathbf{H}\|_{L^2(\Omega)} + \|\mathbf{H}\|_{\mathbf{L}^2(\Omega)}, \quad \forall (\mathbf{E}, \mathbf{H}) \in \mathbf{V}. \end{aligned} \tag{2.2}$$

A proof of this result is available in [3] for a smooth boundary and in Lemma 4.5.5 of [14] for a $\mathcal{C}^2$ boundary. In both cases, the three main steps of the proof are:

1. The continuity of the trace operator

   $$\mathbf{H}(\operatorname{curl}, \Omega) \to \mathbf{H}^{-1/2}(\operatorname{div}; \partial\Omega) : \mathbf{U} \to \mathbf{U} \times \mathbf{n},$$

   proved in [38] (see also [35, Theorem 5.4.2]).
2. The elliptic regularity of the Laplace–Beltrami operator $\Delta_{\mathrm{LB}} = \operatorname{div}_t \nabla_t$ on a smooth manifold without boundary that implies that $\Delta_{\mathrm{LB}} - I$ is an isomorphism from $H^{\frac{3}{2}}(\Gamma)$ into $H^{-\frac{1}{2}}(\Gamma)$; see, for instance, [29].
3. The operator

   $$H^2(\Omega) \to L^2(\Omega) \times H^{\frac{3}{2}}(\Gamma) : u \to (-\Delta u, \gamma_0 u),$$

   is an isomorphism; see again [29].

If we want to extend this result to polyhedra, we then need to check if the three main points before are available. This is indeed the case, since point 1 can be found in [6], point 2 is proved in [8, Theorem 8] under a geometrical assumption (see (2.3) below), while point 3 is a consequence of [16].

To be more precise, let us first introduce the following notation (see [6] or [36, Chapter 2]): as Ω is a polyhedron, its boundary Γ is a finite union of (open and disjoint) faces Γ_j, $j = 1, \ldots, N$ such that $\Gamma = \bigcup_{j=1}^{N} \bar{\Gamma}_j$. As usual, $\mathbf{n}$ is the unit outward normal vector to Ω and we will set $\mathbf{n}_i = \mathbf{n}_{|\Gamma_i}$ its restriction to Γ_i. When Γ_i and Γ_j are two adjacent faces, we denote by e_{ij} their common (open) edge and by τ_{ij} a unit vector parallel to e_{ij}. By convention, we assume that $\tau_{ij} = \tau_{ji}$. We further set $\mathbf{n}_{ij} = \tau_{ij} \times \mathbf{n}_i$. Note that the pair $(\mathbf{n}_{ij}, \tau_{ij})$ is an orthonormal basis of the plane generated by Γ_i and consequently $\mathbf{n}_{ij}$ is a normal vector to Γ_i along e_{ij}. For shortness, we introduce the set

$$\mathcal{E} = \{(i,j) : i < j \text{ and such that } \bar{\Gamma}_i \cap \bar{\Gamma}_j = \bar{e}_{ij}\}.$$

We denote by $\mathcal{C}$ the set of vertices of Γ (that are the vertices of Ω). Furthermore, for any $c \in \mathcal{C}$, we denote by G_c the intersection between the infinite three-dimensional cone Ξ_c that coincides with Ω in a neighbourhood of c and the unit sphere centred at c and by ω_c the length of (in radians) of the boundary of G_c.

We first introduce the set

$$\mathbf{L}^2_t(\Gamma) = \{\mathbf{w} \in \mathbf{L}^2(\Gamma) : \mathbf{w} \cdot \mathbf{n} = \mathbf{0} \text{ on } \Gamma\}.$$

For a function $v \in L^2(\Gamma)$, we denote by v_j its restriction to Γ_j. As Γ is Lipschitz, we can define $H^1(\Gamma)$ via local charts, but we can notice that

$$\begin{aligned} H^1(\Gamma) = \{u \in L^2(\Gamma) : u_j \in H^1(\Gamma_j), \quad &\forall j = 1, \ldots, N \text{ satisfying} \\ &\gamma_0 u_i = \gamma_0 u_j \text{ on } e_{ij}, \quad \forall (i,j) \in \mathcal{E}\}. \end{aligned}$$

As Γ is only Lipschitz, we cannot directly define $H^t(\Gamma)$ for $t > 1$, but following [6] (or [8]), we define

$$H^{\frac{3}{2}}(\Gamma) = \{\gamma_0 u : u \in H^2(\Omega)\},$$

with

$$\|w\|_{\frac{3}{2},\Gamma} = \inf_{u \in H^2(\Omega) : \gamma_0 u = w} \|u\|_{2,\Omega}.$$

Let us notice that according to Theorem 3.4 of [6], we have

$$H^{\frac{3}{2}}(\Gamma) = \{w \in H^1(\Gamma) : \nabla_t w \in \mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma)\},$$

with

$$\|w\|_{\frac{3}{2},\Gamma} \sim \|w\|_{1,\Gamma} + \|\nabla_t w\|_{\|,\frac{1}{2},\Gamma}, \quad \forall w \in H^{\frac{3}{2}}(\Gamma),$$

where $\nabla_t u$ is the tangential gradient of u and $\mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma)$ is defined by

$$\mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma) = \{\mathbf{u} \in \mathbf{L}^2_t(\Gamma) : \mathbf{u}_i \in (H^{\frac{1}{2}}(\Gamma_i))^3, \quad \forall i = 1, \ldots, N, \text{ and } \mathcal{N}_{ij}^{\|}(\mathbf{u}) < \infty, \quad \forall (i,j) \in \mathcal{E}\},$$

where

$$\mathcal{N}_{ij}^{\|}(\mathbf{u}) = \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{u}_i(x)\cdot\tau_{ij} - \mathbf{u}_j(y)\cdot\tau_{ij}|^2}{|x-y|^3}\, d\sigma(x)d\sigma(y),$$

and finally

$$\|\mathbf{u}\|^2_{\|,\frac{1}{2},\Gamma} = \sum_{i=1}^{N} \|\mathbf{u}_i\|^2_{\frac{1}{2},\Gamma_i} + \sum_{(i,j)\in\mathcal{E}} \mathcal{N}_{ij}^{\|}(\mathbf{u}), \quad \forall \mathbf{u} \in \mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma).$$

For further uses, we also introduce

$$\mathbf{H}_{\perp}^{\frac{1}{2}}(\Gamma) = \{\mathbf{u} \in \mathbf{L}_t^2(\Gamma) : \mathbf{u}_i \in (H^{\frac{1}{2}}(\Gamma_i))^3,\ \forall i = 1,\dots,N, \text{ and } \mathcal{N}_{ij}^{\perp}(\mathbf{u}) < \infty,\ \forall (i,j) \in \mathcal{E}\},$$

where

$$\mathcal{N}_{ij}^{\perp}(\mathbf{u}) = \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{u}_i(x)\cdot\mathbf{n}_{ij} - \mathbf{u}_j(y)\cdot\mathbf{n}_{ji}|^2}{|x-y|^3}\, d\sigma(x)d\sigma(y),$$

and finally

$$\|\mathbf{u}\|^2_{\perp,\frac{1}{2},\Gamma} = \sum_{i=1}^{N} \|\mathbf{u}_i\|^2_{\frac{1}{2},\Gamma_i} + \sum_{(i,j)\in\mathcal{E}} \mathcal{N}_{ij}^{\perp}(\mathbf{u}), \quad \forall \mathbf{u} \in \mathbf{H}_{\perp}^{\frac{1}{2}}(\Gamma).$$

Let us also define (cf. [6]) $\mathbf{H}_{\|}^{-\frac{1}{2}}(\Gamma)$ as the dual of $\mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma)$ (with pivot space $\mathbf{L}_t^2(\Gamma)$) and introduce the tangential divergence $\operatorname{div}_t : \mathbf{H}_{\|}^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{3}{2}}(\Gamma)$ as the adjoint of $-\nabla_t$, namely

$$\langle \operatorname{div}_t \mathbf{u}, \varphi\rangle_{H^{-\frac{3}{2}}(\Gamma)-H^{\frac{3}{2}}(\Gamma)} = -\langle \mathbf{u}, \nabla_t\varphi\rangle_{\mathbf{H}_{\|}^{-\frac{1}{2}}(\Gamma)-\mathbf{H}_{\|}^{\frac{1}{2}}(\Gamma)}, \quad \forall \mathbf{u} \in \mathbf{H}_{\|}^{-\frac{1}{2}}(\Gamma), \varphi \in H^{\frac{3}{2}}(\Gamma).$$

Finally, let us define

$$\mathbf{H}_{\|}^{-1/2}(\operatorname{div};\Gamma) = \{\mathbf{w} \in \mathbf{H}_{\|}^{-1/2}(\Gamma) : \operatorname{div}_t \mathbf{w} \in H^{-1/2}(\Gamma)\},$$

and recall the next result proved in [6, Theorem 3.9]:

Theorem 2.1. *[[9, Theorem 4.1]] The trace mapping*

$$\mathbf{H}(\operatorname{curl},\Omega) \to \mathbf{H}_{\|}^{-1/2}(\operatorname{div};\Gamma) : \mathbf{U} \to \mathbf{U}\times\mathbf{n},$$

is linear, continuous and surjective.

Theorem 2.2. *If Ω is a polyhedron satisfying*

$$\omega_c < 4\pi, \quad \forall c \in \mathcal{C}, \tag{2.3}$$

then for any $h \in H^{-\frac{1}{2}}(\Gamma)$*, there exists a unique* $u \in H^{\frac{3}{2}}(\Gamma)$ *such that*

$$u - \operatorname{div}_t \nabla_t u = h \text{ in } H^{-\frac{1}{2}}(\Gamma), \tag{2.4}$$

with

$$\|u\|_{\frac{3}{2},\Gamma} \lesssim \|h\|_{-\frac{1}{2},\Gamma}. \tag{2.5}$$

Proof. Fix $h \in H^{-\frac{1}{2}}(\Gamma)$. Then there exists a unique solution $u \in H^1(\Gamma)$ of

$$\int_\Gamma (\nabla_t u \cdot \nabla_t \bar{v} + u\bar{v})\, d\sigma(x) = \langle h, v\rangle, \quad \forall v \in H^1(\Gamma).$$

This solution clearly satisfies (2.4). Furthermore, owing to our assumption (2.3), Theorem 8 from [8] (with $t = \frac{1}{2}$, valid since $\frac{2\pi}{\omega_c} > \frac{1}{2}$ for all corners c) guarantees that $u \in H^{\frac{3}{2}}(\Gamma)$ since $h - u$ belongs to $H^{-\frac{1}{2}}(\Gamma)$.

To obtain the estimate (2.5), we take advantage of the closed graph theorem. Indeed introduce the mapping

$$T : \{v \in H^{\frac{3}{2}}(\Gamma) : \operatorname{div}_t \nabla_t v \in H^{-\frac{1}{2}}(\Gamma)\} \to H^{-\frac{1}{2}}(\Gamma) : u \to u - \operatorname{div}_t \nabla_t u,$$

that is well-defined and continuous. Since the above arguments show that it is bijective, its inverse is also continuous, which yields

$$\|u\|_{\frac{3}{2},\Gamma} \lesssim \|u - \operatorname{div}_t \nabla_t u\|_{-\frac{1}{2},\Gamma},$$

and is exactly (2.5). □

Remark 2.3. Any convex polyhedron satisfies (2.3), since by [43, problem 1.10.1], one always have $\omega_c < 2\pi$, for all $c \in \mathcal{C}$. But the class of polyhedra satisfying (2.3) is quite larger since the Fichera corner and any prism $D \times I$, where D is any polygon with a Lipschitz boundary and I is an interval that satisfies (2.3).

Theorem 2.4. *If* Ω *is a convex polyhedron, then the continuous embedding* (2.1) *remains valid.*

Proof. The proof follows the one of Lemma 4.5.5 of [14] with the necessary adaptation. Let $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$. Let us prove that $\mathbf{E} \in \mathbf{H}^1(\Omega)$. The proof for $\mathbf{H}$ is similar.

By Theorems 2.17 and 3.12 of [1], there exists a vector potential $\mathbf{w} \in \mathbf{H}_T(\Omega) = \{\mathbf{w} \in \mathbf{H}^1(\Omega)^3 : \mathbf{w} \cdot \mathbf{n} = \mathbf{0} \text{ on } \Gamma\}$ such that $\operatorname{div} \mathbf{w} = \mathbf{0}$ and

$$\operatorname{curl} \mathbf{w} = \operatorname{curl} \mathbf{E} \quad \text{in } \Omega,$$

and satisfying

$$\|\mathbf{w}\|_{1,\Omega} \lesssim \|\operatorname{curl} \mathbf{E}\|_\Omega. \tag{2.6}$$

Thus, there exists a potential $\varphi \in H^1(\Omega)$ such that

$$\nabla\varphi = \mathbf{E} - \mathbf{w}, \tag{2.7}$$

with (by assuming that $\int_\Omega \varphi\, dx = 0$)

$$\|\varphi\|_{1,\Omega} \lesssim \|\mathbf{E}\|_\Omega + \|\mathbf{w}\|_\Omega \lesssim \|\mathbf{E}\|_{H(\mathrm{curl},\Omega)}.$$

Therefore, as a consequence of $\operatorname{div}\mathbf{E} \in L^2(\Omega)$ we find that

$$\operatorname{div}\nabla\varphi \in L^2(\Omega), \tag{2.8}$$

with

$$\|\operatorname{div}\nabla\varphi\|_\Omega \lesssim \|\operatorname{div}\mathbf{E}\|_\Omega. \tag{2.9}$$

By (2.7), the trace $\mathbf{E}_t$ coincides with $\mathbf{w}_t + \nabla_t\varphi$, i. e.,

$$\mathbf{E}_t = \mathbf{w}_t + \nabla_t\varphi \quad \text{on } \Gamma.$$

As $\mathbf{H}$ belongs to $\mathbf{H}(\mathrm{curl},\Omega)$, by Theorem 2.1 its trace $\mathbf{H}\times\mathbf{n}$ belongs to $\mathbf{H}_\parallel^{-1/2}(\mathrm{div};\Gamma)$. By the impedance condition $\mathbf{H}\times\mathbf{n} = \lambda_{\mathrm{imp}}\mathbf{E}_t$, we deduce that $\lambda_{\mathrm{imp}}\mathbf{E}_t$ also belongs to $\mathbf{H}_\parallel^{-1/2}(\mathrm{div};\Gamma)$ with

$$\|\lambda_{\mathrm{imp}}\mathbf{E}_t\|_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div};\Gamma)} \lesssim \|H\|_{\mathbf{H}(\mathrm{curl},\Omega)}. \tag{2.10}$$

Likewise, as $\mathbf{w}\cdot\mathbf{n} = 0$ and $\mathbf{w} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$, let us show that $\mathbf{w}_t$ also belongs $\mathbf{H}_\parallel^{-1/2}(\mathrm{div};\Gamma)$ with

$$\|\mathbf{w}_t\|_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div};\Gamma)} \lesssim \|\operatorname{curl}\mathbf{E}\|_\Omega. \tag{2.11}$$

Indeed the above properties imply that

$$\mathbf{w}_t = \mathbf{w} \in \mathbf{H}_\perp^{1/2}(\Gamma). \tag{2.12}$$

Namely to show that property we simply need to show that for any $(i,j) \in \mathcal{E}$, one has

$$\int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{w}_i(x)\cdot\mathbf{n}_{ij} - \mathbf{w}_j(y)\cdot\mathbf{n}_{ji}|^2}{|x-y|^3}\, d\sigma(x)d\sigma(y) \lesssim \|\mathbf{w}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}. \tag{2.13}$$

But for such a pair, $\mathbf{n}_{ij}$ is a linear combination of $\mathbf{n}_i$ and $\mathbf{n}_j$, and consequently,

$$\int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{w}_i(x)|\cdot\mathbf{n}_{ij}^2}{|x-y|^3}\, d\sigma(x)d\sigma(y) \lesssim \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{w}_i(x)\cdot\mathbf{n}_j|^2}{|x-y|^3}\, d\sigma(x)d\sigma(y)$$

$$= \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{w}_i(x)\cdot\mathbf{n}_j - \mathbf{w}_j(y)\cdot\mathbf{n}_j|^2}{|x-y|^3}\, d\sigma(x)d\sigma(y)$$

since $\mathbf{w}_i \cdot \mathbf{n}_i = 0$ on Γ_i and $\mathbf{w}_j \cdot \mathbf{n}_j = 0$ on Γ_j. This shows that

$$\int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{w}_i(x)\cdot\mathbf{n}_{ij}|^2}{|x-y|^3}\, d\sigma(x)d\sigma(y) \lesssim \int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{w}_i(x)-\mathbf{w}_j(y)|^2}{|x-y|^3}\, d\sigma(x)d\sigma(y) \lesssim \|\mathbf{w}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)},$$

as well as (by exchanging the role of Γ_i and Γ_j)

$$\int_{\Gamma_i}\int_{\Gamma_j} \frac{|\mathbf{w}_j\cdot\mathbf{n}_{ji}(y)|^2}{|x-y|^3}\, d\sigma(x)d\sigma(y) \lesssim \|\mathbf{w}\|^2_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}.$$

Hence (2.13) holds. As mentioned in [7, p. 39], Theorem 2.1, a density argument and a duality argument lead to the continuity of div_t from $\mathbf{H}^{\frac{1}{2}}_{\perp}(\Gamma)$ to $H^{-\frac{1}{2}}(\Gamma)$, and by (2.12) we deduce that

$$\operatorname{div}_t \mathbf{w}_t = \operatorname{div}_t \mathbf{w} \in H^{-\frac{1}{2}}(\Gamma).$$

Altogether, we finally obtain that $\lambda_{\text{imp}}\nabla_t\varphi$ belongs to $\mathbf{H}^{-1/2}_{\parallel}(\operatorname{div};\Gamma)$ and since λ_{imp} is smooth and never 0 on Γ, we conclude that

$$\operatorname{div}_t \nabla_t\varphi \in H^{-\frac{1}{2}}(\Gamma),$$

and since φ is in $H^{-\frac{1}{2}}(\Gamma)$,

$$\varphi - \operatorname{div}_t \nabla_t\varphi \in H^{-\frac{1}{2}}(\Gamma),$$

with

$$\|\varphi - \operatorname{div}_t \nabla_t\varphi\|_{-\frac{1}{2},\Gamma} \lesssim \|H\|_{\mathbf{H}(\text{curl};\Omega)} + \|\mathbf{E}\|_{\mathbf{H}(\text{curl};\Omega)}. \tag{2.14}$$

By Theorem 2.2, we deduce that

$$\varphi_{|\Gamma} \in H^{\frac{3}{2}}(\Gamma), \tag{2.15}$$

with

$$\|\varphi\|_{\frac{3}{2},\Gamma} \lesssim \|H\|_{\mathbf{H}(\text{curl};\Omega)} + \|\mathbf{E}\|_{\mathbf{H}(\text{curl};\Omega)}. \tag{2.16}$$

Now, using the elliptic regularity for φ solution of the Dirichlet problem (2.8)–(2.15) in Ω (see [16, Corollary 18.19]), we find $\varphi \in H^2(\Omega)$ with

$$\begin{aligned} \|\varphi\|_{2,\Omega} &\lesssim \|\operatorname{div}\nabla\varphi\|_{\Omega} + \|\varphi\|_{\frac{3}{2},\Gamma} \\ &\lesssim \|\mathbf{H}\|_{\mathbf{H}(\text{curl};\Omega)} + \|\mathbf{E}\|_{\mathbf{H}(\text{curl};\Omega)} + \|\operatorname{div}\mathbf{E}\|_{\Omega}. \end{aligned} \tag{2.17}$$

Coming back to (2.7), we have obtained that $\mathbf{E} \in \mathbf{H}^1(\Omega)$ with

$$\|\mathbf{E}\|_{1,\Omega} \leq \|\mathbf{w}\|_{1,\Omega} + \|\nabla\varphi\|_{1,\Omega}.$$

Hence taking into account (2.6) and (2.17), we arrive at the estimate

$$\|\mathbf{E}\|_{1,\Omega} \lesssim \|\mathbf{H}\|_{\mathbf{H}(\mathrm{curl};\Omega)} + \|\mathbf{E}\|_{\mathbf{H}(\mathrm{curl};\Omega)} + \|\operatorname{div}\mathbf{E}\|_{\Omega}.$$

As said before, exchanging the role of $\mathbf{E}$ and $\mathbf{H}$ we can show that $\mathbf{H} \in \mathbf{H}^1(\Omega)$ with

$$\|\mathbf{H}\|_{1,\Omega} \lesssim \|\mathbf{H}\|_{\mathbf{H}(\mathrm{curl};\Omega)} + \|\mathbf{E}\|_{\mathbf{H}(\mathrm{curl};\Omega)} + \|\operatorname{div}\mathbf{H}\|_{\Omega}.$$

The proof is then completed. □

It turns out that the convexity condition is a necessary and sufficient condition that guarantees the continuous embedding (2.1), namely we have the following.

Corollary 2.5. *If Ω is a polyhedron. Then Ω is convex if and only if the continuous embedding* (2.1) *is valid.*

Proof. It suffices to prove that the convexity condition is a necessary condition. For that purpose, we use a contradiction argument. Assume that Ω is not convex, then by [16] (see also [13, Section 1]), there exists a (singular) function $\varphi \in H_0^1(\Omega) \setminus H^2(\Omega)$ such that

$$\Delta\varphi \in L^2(\Omega).$$

In that way, the pair $(\nabla\varphi, \nabla\varphi)$ belongs to $\mathbf{V}$, but that cannot be in $\mathbf{H}^1(\Omega)^2$ since $\varphi \notin H^2(\Omega)$. This proves that (2.1) is not valid. □

3 Well-posedness

Let us start with a coerciveness result for the sesquilinear form a.

Theorem 3.1. *If Ω is a convex polyhedron, then the sesquilinear form $\mathbf{a}_{k,s}(\cdot,\cdot)$ is weakly coercive on $\mathbf{V}$, in the sense that there exists $c > 0$ independent of k and s such that*

$$\Re\mathbf{a}_{k,s}\big((\mathbf{E},\mathbf{H}),(\mathbf{E},\mathbf{H})\big) \geq c\big(\|\mathbf{E}\|_{1,\Omega}^2 + \|\mathbf{H}\|_{1,\Omega}^2\big) - \big(k^2+1\big)\big(\|\mathbf{E}\|_{\Omega}^2 + \|\mathbf{H}\|_{\Omega}^2\big), \quad \forall(\mathbf{E},\mathbf{H}) \in \mathbf{V}. \quad (3.1)$$

Proof. Direct consequence of Theorem 2.4, recalling our assumption on λ_{imp} to be real valued. □

Remark 3.2. Under the assumptions of the previous Theorem, for $k \geq 1$, we have

$$\|(\mathbf{E},\mathbf{H})\|_k \gtrsim \|(\mathbf{E},\mathbf{H})\|_{\mathbf{H}^1(\Omega)^2}.$$

The existence of a weak solution to (1.5) for $k > 0$ directly follows from this coerciveness and the next uniqueness result for problem (1.1).

Lemma 3.3. *Let* $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ *be a solution of*

$$\begin{cases} \operatorname{curl} \mathbf{E} - ik\mathbf{H} = \mathbf{0} \quad \text{and} \quad \operatorname{curl} \mathbf{H} + ik\mathbf{E} = \mathbf{0} \quad \text{in } \Omega, \\ \mathbf{H} \times \mathbf{n} - \lambda_{\text{imp}} \mathbf{E}_t = \mathbf{0} \quad \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Assume that $\mathbf{E}$ *and* $\mathbf{H}$ *are divergence-free. Then* $(\mathbf{E}, \mathbf{H}) = (\mathbf{0}, \mathbf{0})$.

Proof. By Green's formula (see [20, Theorem I.2.11]), we have

$$\begin{aligned} \int_\Omega (|\operatorname{curl} \mathbf{E}|^2 + |\operatorname{curl} \mathbf{H}|^2)\, dx &= ik \int_\Omega (\operatorname{curl} \mathbf{H} \cdot \bar{\mathbf{E}} - \operatorname{curl} \mathbf{E} \cdot \bar{\mathbf{H}})\, dx \\ &= ik \int_\Omega (\mathbf{H} \cdot \operatorname{curl} \bar{\mathbf{E}} - \operatorname{curl} \mathbf{E} \cdot \bar{\mathbf{H}})\, dx - ik \int_{\partial\Omega} (\mathbf{H} \times \mathbf{n} \cdot \bar{\mathbf{E}})\, d\sigma(x). \end{aligned}$$

Hence using the impedance boundary condition in (3.2), we find that

$$\int_\Omega (|\operatorname{curl} \mathbf{E}|^2 + |\operatorname{curl} \mathbf{H}|^2)\, dx = ik \int_\Omega (\mathbf{H} \cdot \operatorname{curl} \bar{\mathbf{E}} - \operatorname{curl} \mathbf{E} \cdot \bar{\mathbf{H}})\, dx - ik \int_{\partial\Omega} \lambda_{\text{imp}} |\mathbf{E}_t|^2\, d\sigma(x).$$

Taking the imaginary part of this identity, we find that

$$k \int_{\partial\Omega} \lambda_{\text{imp}} |\mathbf{E}_t|^2\, d\sigma(x) = 0.$$

Hence if $k > 0$, we deduce that

$$\mathbf{E}_t = \mathbf{0} \quad \text{on } \partial\Omega,$$

as λ_{imp} is positive on $\partial\Omega$. Again by the impedance boundary condition, $\mathbf{H}$ also satisfies

$$\mathbf{H} \times \mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega.$$

This means that we can extend $\mathbf{E}$ and $\mathbf{H}$ by zero outside Ω and that these extensions belong to $H(\operatorname{curl}, \mathbb{R}^3)$. Owing to Theorem 4.13 of [34], we conclude that $(\mathbf{E}, \mathbf{H}) = (\mathbf{0}, \mathbf{0})$.

For $k = 0$, we notice that (3.2) implies that $\mathbf{E}$ and $\mathbf{H}$ are curl-free, hence as Ω is supposed to be simply connected, by Theorem I.2.6 of [20], there exist $\Phi_E, \Phi_H \in H^1(\Omega)$ such that

$$\mathbf{E} = \nabla\Phi_E, \quad \mathbf{H} = \nabla\Phi_H.$$

Due to the H^1 regularity of $\mathbf{E}$ and $\mathbf{H}$, Φ_E and Φ_H both belong to $H^2(\Omega)$. Now using the impedance boundary condition, we have

$$\operatorname{div}_t(\lambda_{\text{imp}} \nabla_t \Phi_E) = \operatorname{div}_t(\nabla\Phi_H \times \mathbf{n}) \quad \text{on } \partial\Omega,$$

and by the standard property

$$\operatorname{div}_t(\mathbf{v}\times\mathbf{n}) = \operatorname{curl}\mathbf{v}\cdot\mathbf{n},$$

valid for all $\mathbf{v}\in H(\operatorname{curl},\Omega)$ (see [6, p. 23]), we deduce that

$$\operatorname{div}_t(\lambda_{\mathrm{imp}}\nabla_t\Phi_E) = 0 \quad \text{on } \partial\Omega.$$

By its definition (see [6, Definition 3.3]), this property implies that

$$\int_{\partial\Omega} |\lambda_{\mathrm{imp}}\nabla_t\Phi_E|^2\, d\sigma(x) = 0.$$

Consequently, Φ_E is constant on the whole boundary. As $\mathbf{E}$ is divergence-free, Φ_E is harmonic in Ω and consequently it is constant on the whole Ω, which guarantees that $\mathbf{E}=\mathbf{0}$. With this property and recalling the impedance boundary condition, we deduce that $\nabla_t\Phi_H = \mathbf{0}$ on the whole boundary. As $\mathbf{H}$ is also divergence-free, Φ_H is harmonic in Ω and we conclude that $\mathbf{H}=\mathbf{0}$. □

Our next goal is to prove an existence and uniqueness result to problem (1.5), that can be formulated in the more general form

$$\mathbf{a}_{k,s}((\mathbf{E},\mathbf{H});(\mathbf{E}',\mathbf{H}')) = \langle \mathbf{F};(\mathbf{E}',\mathbf{H}')\rangle, \quad \forall(\mathbf{E}',\mathbf{H}')\in\mathbf{V}, \tag{3.3}$$

with $\mathbf{F}\in\mathbf{V}'$. First, we need to show extra regularities of the divergence of any solution $(\mathbf{E},\mathbf{H})$ of this problem under the assumption that $\mathbf{F}$ belongs to $\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$ in the sense that

$$\langle \mathbf{F};(\mathbf{E}',\mathbf{H}')\rangle = \int_\Omega (\mathbf{f}_1\cdot\bar{\mathbf{E}}' + \mathbf{f}_2\cdot\bar{\mathbf{H}}')\,dx, \tag{3.4}$$

with $\mathbf{f}_1,\mathbf{f}_2\in\mathbf{L}^2(\Omega)$.

Lemma 3.4. *If the impedance function λ_{imp} satisfies* (1.2) *and $-k^2/s$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary conditions in Ω, then for all $\mathbf{f}_1,\mathbf{f}_2\in\mathbf{L}^2(\Omega)$, any solution $(\mathbf{E},\mathbf{H})\in\mathbf{V}$ to the problem*

$$\mathbf{a}_{k,s}((\mathbf{E},\mathbf{H});(\mathbf{E}',\mathbf{H}')) = \int_\Omega (\mathbf{f}_1\cdot\bar{\mathbf{E}}' + \mathbf{f}_2\cdot\bar{\mathbf{H}}')\,dx, \quad \forall(\mathbf{E}',\mathbf{H}')\in\mathbf{V}, \tag{3.5}$$

satisfies

$$\operatorname{div}\mathbf{E},\operatorname{div}\mathbf{H}\in H_0^1(\Omega),$$

with

$$\operatorname{div}\mathbf{E} = -(s\Delta+k^2)^{-1}\operatorname{div}\mathbf{f}_1, \quad \operatorname{div}\mathbf{H} = -(s\Delta+k^2)^{-1}\operatorname{div}\mathbf{f}_2.$$

Proof. We basically follow the proof of Lemma 4.5.8 of [14] with a slight adaptation due to the change of right-hand side in (3.5) with respect to [14]. In (3.5), we first take test functions in the form $(\nabla\varphi, \mathbf{0})$ with an arbitrary $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$. This directly implies that $(\nabla\varphi, \mathbf{0})$ belongs to $\mathbf{V}$ and, therefore, we get

$$s\int_\Omega \operatorname{div}\mathbf{E}\ \operatorname{div}\nabla\bar{\varphi}\,dx - k^2\int_\Omega \mathbf{E}\cdot\nabla\bar{\varphi}\,dx = \int_\Omega \mathbf{f}_1\cdot\nabla\bar{\varphi}\,dx.$$

Consequently, one deduces that

$$\int_\Omega \operatorname{div}\mathbf{E}\ (s\Delta + k^2)\varphi\,dx = -\langle \operatorname{div}\mathbf{f}_1; \varphi\rangle, \quad \forall\varphi \in H^2(\Omega)\cap H_0^1(\Omega). \tag{3.6}$$

On the other hand, as $-k^2/s$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary conditions in $H^2(\Omega)$, there exists a unique solution $q \in H_0^1(\Omega)$ to

$$(s\Delta + k^2)q = -\operatorname{div}\mathbf{f}_1.$$

Taking the duality with $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, after an integration by parts, we obtain equivalently that

$$\int_\Omega q\ (s\Delta + k^2)\varphi\,dx = -\langle \operatorname{div}\mathbf{f}_1; \varphi\rangle, \quad \forall\varphi \in H^2(\Omega)\cap H_0^1(\Omega).$$

Comparing this identity with (3.6), we find that

$$\int_\Omega (\operatorname{div}\mathbf{E} - q)\ (s\Delta + k^2)\varphi\,dx = 0, \quad \forall\varphi \in H^2(\Omega)\cap H_0^1(\Omega),$$

and since the range of $(s\Delta + \omega^2)$ is the whole $L^2(\Omega)$, one gets that $\operatorname{div}\mathbf{E} = q$, as announced.

The result for $\mathbf{H}$ follows in the same way by choosing test functions in the form $(\mathbf{0}, \nabla\bar{\varphi})$. □

We are now ready to prove an existence and uniqueness result to (3.3).

Theorem 3.5. *If Ω is a convex polyhedron, the impedance function λ_{imp} satisfies* (1.2) *and $-k^2/s$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary conditions in Ω, then for any $\mathbf{F} \in \mathbf{V}'$, the problem* (3.3) *has a unique solution* $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$.

Proof. We associate to problem (3.3) the continuous operator $A_{k,s}$ from $\mathbf{V}$ into its dual by

$$(A_{k,s}\mathbf{u})(\mathbf{v}) = \mathbf{a}_{k,s}(\mathbf{u}, \mathbf{v}), \quad \forall\mathbf{u}, \mathbf{v} \in \mathbf{V}.$$

Now according to Theorem 3.1, the sesquilinear form

$$\mathbf{a}_{k,s}((\mathbf{E}, \mathbf{H}), (\mathbf{E}, \mathbf{H})) + (k^2 + 1)(\|\mathbf{E}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{H}\|^2_{\mathbf{L}^2(\Omega)}),$$

is strongly coercive in $\mathbf{V}$ and by Lax-Milgram lemma, the operator $A_{k,s} + (k^2 + 1)\mathbb{I}$ is an isomorphism $\mathbf{V}$ into its dual. As $\mathbf{V}$ is compactly embedded into $\mathbf{L}^2(\Omega)^6$, the operator $A_{k,s}$ is a Fredholm operator of index zero. Hence uniqueness implies existence and uniqueness.

So let us fix $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ be a solution of (3.3) with $\mathbf{F} = \mathbf{0}$. Then by Lemma 4.5.9 of [14] (valid due to Lemma 3.3), we find that $(\mathbf{E}, \mathbf{H})$ is solution of the original problem (1.1) with $\mathbf{J} = \mathbf{0}$, namely (3.2). We further notice that Lemma 3.4 guarantees that $\mathbf{E}$ and $\mathbf{H}$ are divergence-free (only useful for $k = 0$). As Lemma 3.3 yields that $(\mathbf{E}, \mathbf{H}) = (\mathbf{0}, \mathbf{0})$, we conclude an existence and uniqueness result. □

As already mentioned, for the particular choice

$$\langle \mathbf{F}; (\mathbf{E}', \mathbf{H}') \rangle = \int_\Omega (i\omega \mathbf{J} \cdot \bar{\mathbf{E}}' + \mathbf{J} \cdot \operatorname{curl} \bar{\mathbf{H}}')\, dx,$$

with $\mathbf{J} \in \mathbf{L}^2(\Omega)$, problem (3.3) reduces to (1.5). Hence under the assumptions of Theorem 3.5 and if $\mathbf{J} \in \mathbf{L}^2(\Omega)$, this last problem has a unique solution $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$, that owing to Lemma 4.5.9 of [14] is moreover solution of the original problem (1.1) under the additional assumption that $\mathbf{J} \in \mathbf{H}(\operatorname{div}; \Omega)$.

Now under the assumptions of Theorem 3.5, given two functions $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, we denote by $(\mathbf{E}, \mathbf{H}) = \mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$, the unique solution of (3.3) with $\mathbf{F}$ given by (3.4) or equivalently solution of (3.5). Note that the general considerations from [14, Section 4.5.d] implies that $(\mathbf{E}, \mathbf{H})$ is actually the solution of the boundary value elliptic system

$$\begin{cases} \left.\begin{aligned} L_{k,s}(\mathbf{E}) = \mathbf{f}_1 \\ L_{k,s}(\mathbf{H}) = \mathbf{f}_2 \end{aligned}\right\} \quad \text{in } \Omega \\ \left.\begin{aligned} \operatorname{div} \mathbf{E} = 0 \\ \operatorname{div} \mathbf{H} = 0 \\ T(\mathbf{E}, \mathbf{H}) = 0 \\ B_k(\mathbf{E}, \mathbf{H}) = 0 \end{aligned}\right\} \quad \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

where

$$\begin{aligned} L_{k,s}(\mathbf{u}) &= \operatorname{curl} \operatorname{curl} \mathbf{u} - s\nabla \operatorname{div} \mathbf{u} - k^2 \mathbf{u}, \\ T(\mathbf{E}, \mathbf{H}) &= \mathbf{H} \times \mathbf{n} - \lambda_{\mathrm{imp}} \mathbf{E}_t, \\ B_k(\mathbf{E}, \mathbf{H}) &= (\operatorname{curl} \mathbf{H}) \times \mathbf{n} + \frac{1}{\lambda_{\mathrm{imp}}} (\operatorname{curl} \mathbf{E})_t - \frac{ik}{\lambda_{\mathrm{imp}}} \mathbf{H}_t + ik\mathbf{E} \times \mathbf{n}. \end{aligned}$$

Remark 3.6. As suggested by its definition, under the assumptions of Theorem 3.5, $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ depends on s, but if the data $\mathbf{f}_1$ and $\mathbf{f}_2$ are divergence-free, then as Lemma 3.4 guarantees that each component of $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ is divergence-free, we deduce that

$$\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) = \mathbb{S}_{k,s'}(\mathbf{f}_1, \mathbf{f}_2),$$

for all $s' > 0$ such that $-k^2/s'$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary conditions in Ω. In other words, in that case $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ does not depend on s and hence the parameter s can be chosen independent of k. This is of particular interest for practical applications (see problem (1.5)), since the data $\mathbf{f}_1$ and $\mathbf{f}_2$ are divergence-free. The interest of considering non-divergence-free right-hand side will appear in the error analysis of our numerical schemes; see Remark 6.6.

Let us end up this section with an extra regularity result of the curl of each component of $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ if $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$ are divergence-free.

Lemma 3.7. *Under the assumptions of Theorem* 3.5, *let* $(\mathbf{E}, \mathbf{H}) = \mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$, *with* $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$ *such that*

$$\operatorname{div} \mathbf{f}_1 = \operatorname{div} \mathbf{f}_1 = 0.$$

Then $(\mathbf{U}, \mathbf{W}) = (\operatorname{curl} \mathbf{E} - ik\mathbf{H}, \operatorname{curl} \mathbf{H} + ik\mathbf{E})$ *belongs to* $\mathbf{V}$ *and satisfies the Maxwell system*

$$\operatorname{curl} \mathbf{U} + ik\mathbf{W} = \mathbf{f}_1 \quad \text{and} \quad \operatorname{curl} \mathbf{W} - ik\mathbf{U} = \mathbf{f}_2 \quad \text{in } \Omega. \tag{3.8}$$

Proof. According to Lemma 3.4, **E** and **H** are divergence-free, hence **U** and **W** as well. Hence the identities (3.8) directly follows from the two first identities of (3.7). This directly furnishes the regularities

$$\operatorname{curl} \mathbf{U}, \operatorname{curl} \mathbf{W} \in \mathbf{L}^2(\Omega).$$

Finally, the boundary conditions

$$\mathbf{W} \times \mathbf{n} - \lambda_{\text{imp}} \mathbf{U}_t = \mathbf{0} \quad \text{on } \partial\Omega,$$

directly follows from the last boundary conditions in (3.7). □

4 Corner/edge singularities

Here, for the sake of simplicity we assume that $\lambda_{\text{imp}} = 1$ and want to describe the regularity/singularity of $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ with $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{H}^t(\Omega)$, for $t \geq 0$. As said before, as the system (3.7) is an elliptic system, the shift property will be valid far from the corners and edges of Ω, in other words, $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ belongs to $\mathbf{H}^{t+2}(\Omega \setminus \mathcal{V}) \times \mathbf{H}^{t+2}(\Omega \setminus \mathcal{V})$, for any neighborhood $\mathcal{V}$ of the corners and edges.

We therefore need to determine the corner and edge singularities of system (3.7).

4.1 Corner singularities

For c be a corner of Ω, we recall that Ξ_c is the three-dimensional cone that coincides with Ω in a neighbourhood of c and that G_c is its section with the unit sphere. For shortness, if no confusion is possible, we will drop the index c. As usual denote by (r, ϑ) the spherical coordinates centred at c. The standard antsatz [16, 21, 28] is to look for the corner singularities $(\mathbf{E}, \mathbf{H})$ of problem (3.7) in the form

$$(\mathbf{E}, \mathbf{H}) = r^{\lambda}(\mathbf{U}(\vartheta), \mathbf{V}(\vartheta)), \tag{4.1}$$

with $\lambda \in \mathbb{C}$ such that $\Re\lambda > -\frac{1}{2}$ and $\mathbf{U}, \mathbf{V} \in \mathbf{H}^1(G)$ that is solution of (as our system is invariant by translation)

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - s\nabla\operatorname{div}\mathbf{E} = \mathbf{0} & \text{in } \Xi, \\ \operatorname{curl}\operatorname{curl}\mathbf{H} - s\nabla\operatorname{div}\mathbf{H} = \mathbf{0} & \text{in } \Xi, \\ \operatorname{div}\mathbf{E} = \operatorname{div}\mathbf{H} = 0 & \text{on } \partial\Xi, \\ \mathbf{H}\times n - \mathbf{E}_t = (\operatorname{curl}\mathbf{H})\times n + (\operatorname{curl}\mathbf{E})_t = \mathbf{0} & \text{on } \partial\Xi. \end{cases} \tag{4.2}$$

Remark 4.1. For the sake of simplicity, we consider here the spectral condition that is stronger than the notion of injectivity modulo the polynomials (from [16]) that consists in replacing the right-hand side in the two first identities of (4.2) by a polynomial of degree $\lambda - 2$. As a consequence, we eventually add some integer ≥ 2 in the set of corner singular exponent, that at least do not affect the regularity results up to $\frac{7}{2}$.

Inspired from [13], we introduce the auxiliary variables

$$q_E = \operatorname{div}\mathbf{E}, \quad q_H = \operatorname{div}\mathbf{H}, \quad \boldsymbol{\psi}_E = \operatorname{curl}\mathbf{E}, \quad \boldsymbol{\psi}_H = \operatorname{curl}\mathbf{H},$$

and re-write the above system in the equivalent form

$$\begin{cases} \Delta q_E = 0 & \text{in } \Xi, \\ q_E = 0 & \text{on } \partial\Xi, \end{cases} \qquad \begin{cases} \Delta q_H = 0 & \text{in } \Xi, \\ q_H = 0 & \text{on } \partial\Xi, \end{cases} \tag{4.3a}$$

$$\begin{cases} \operatorname{curl}\boldsymbol{\psi}_E = s\nabla q_E & \text{in } \Xi, \\ \operatorname{curl}\boldsymbol{\psi}_H = s\nabla q_H & \text{in } \Xi, \\ \operatorname{div}\boldsymbol{\psi}_E = \operatorname{div}\boldsymbol{\psi}_H = 0 & \text{on } \partial\Xi, \\ \boldsymbol{\psi}_H \times \mathbf{n} = -(\boldsymbol{\psi}_E)_t & \text{on } \partial\Xi, \end{cases} \tag{4.3b}$$

$$\begin{cases} \operatorname{curl}\mathbf{E} = \boldsymbol{\psi}_E, \quad \operatorname{div}\mathbf{E} = q_E & \text{in } \Xi, \\ \operatorname{curl}\mathbf{H} = \boldsymbol{\psi}_H, \quad \operatorname{div}\mathbf{H} = q_H & \text{in } \Xi, \\ \mathbf{H}\times n = \mathbf{E}_t & \text{on } \partial\Xi. \end{cases} \tag{4.3c}$$

Then three types of singularities appear:

Type 1: $(q_E, q_H) = (0, 0)$, $(\boldsymbol{\psi}_E, \boldsymbol{\psi}_H) = (\mathbf{0}, \mathbf{0})$ and $(\mathbf{E}, \mathbf{H})$ general non-zero solution of (4.3c).

Type 2: $(q_E, q_H) = (0, 0)$, $(\boldsymbol{\psi}_E, \boldsymbol{\psi}_H)$ general non-zero solution of (4.3b) and $(\mathbf{E}, \mathbf{H})$ particular solution of (4.3c).

Type 3: (q_E, q_H) general non-zero solution of (4.3a), $(\boldsymbol{\psi}_E, \boldsymbol{\psi}_H)$ particular solution of (4.3b) and $(\mathbf{E}, \mathbf{H})$ particular solution of (4.3c).

These singularities are different from those from [13] essentially due to the boundary conditions

$$\mathbf{H} \times n - \mathbf{E}_t = (\operatorname{curl} \mathbf{H}) \times n + (\operatorname{curl} \mathbf{E})_t = \mathbf{0} \quad \text{on } \partial\Xi.$$

Some singularities from [13] will be also singularities of our problem but not the converse; see below. To describe them, we recall the corner singularities of the Laplace operator with Dirichlet boundary conditions in Ξ; see [21, 16, 13] for instance. We first denote by L_G^{Dir} the positive Laplace–Beltrami operator with Dirichlet boundary conditions on G. Recall that L_G^{Dir} is a self-adjoint operators with a compact resolvent in $L^2(G)$, hence we denote its spectrum by $\sigma(L_G^{\text{Dir}})$. Then we make the following definition.

Definition 4.2. The set $\Lambda_{\text{Dir}}(\Gamma)$ of corner singular exponents of the Laplace operator with Dirichlet boundary conditions in Ξ is defined as the set of $\lambda \in \mathbb{C}$ such that there exists a non-trivial solution $\varphi \in H_0^1(G)$ of

$$\Delta(r^\lambda \varphi(\vartheta)) = 0. \tag{4.4}$$

We denote by Z_{Dir}^λ the set of such solutions.

Due to the relation

$$r^2\Delta = (r\partial_r)^2 + (r\partial_r) + \Delta_G,$$

for any $\lambda \in \mathbb{C}$ and $\varphi \in H^1(G)$, we have

$$\Delta(r^\lambda \varphi) = r^{\lambda-2}\mathcal{L}(\lambda)\varphi, \tag{4.5}$$

where

$$\mathcal{L}(\lambda)\varphi = \Delta_G\varphi + \lambda(\lambda+1)\varphi, \tag{4.6}$$

with Δ_G the Laplace–Beltrami operator on G. Consequently, the set $\Lambda_{\text{Dir}}(\Gamma)$ is related to the spectrum $\sigma(L_G^{\text{Dir}})$ of L_G^{Dir} as follows (see [13, Lemma 2.4]):

$$\Lambda_{\text{Dir}}(\Gamma) = \left\{-\frac{1}{2} \pm \sqrt{\mu + \frac{1}{4}} : \mu \in \sigma(L_G^{\text{Dir}})\right\}.$$

For $\lambda \in \Lambda_{\text{Dir}}(\Gamma)$, the elements of Z_{Dir}^λ are related to the set $V_{\text{Dir}}(\lambda)$ of eigenvectors of L_G^{Dir} associated with $\mu = \lambda(\lambda+1)$ via the relation

$$Z_{\text{Dir}}^\lambda = \{r^\lambda \varphi : \varphi \in V_{\text{Dir}}(\lambda)\}.$$

Recalling from the previous section that ω_c is the length of the network $\mathcal{R}_c$, we finally set

$$\Upsilon_c = \left\{\frac{2k\pi}{\omega_c} : k \in \mathbb{Z}\right\},$$

as well as

$$\Upsilon_c^* = \left\{ \frac{2k\pi}{\omega_c} : k \in \mathbb{Z} \setminus \{0\} \right\}.$$

We are ready to consider our different types of singularities. We start with singularities of type 1.

Lemma 4.3. *Let $\lambda \in \mathbb{C}$ be different from -1. Then $(\mathbf{E}, \mathbf{H})$ in the form (4.1) is a singularity of type 1 if and only if $\lambda + 1 \in \Lambda_{\mathrm{Dir}}(\Gamma) \cup \Upsilon_c^*$.*

Proof. $(\mathbf{E}, \mathbf{H})$ in the form (4.1) is a singularity of type 1 if and only if it satisfies

$$\begin{cases} \operatorname{curl} \mathbf{E} = \mathbf{0}, & \operatorname{div} \mathbf{E} = 0 & \text{in } \Xi, \\ \operatorname{curl} \mathbf{H} = \mathbf{0}, & \operatorname{div} \mathbf{H} = 0 & \text{in } \Xi, \\ \mathbf{H} \times n = \mathbf{E}_t & & \text{on } \partial\Xi. \end{cases} \tag{4.7}$$

(i) Since a singularity of type 1 from [13] is a vector field $\mathbf{E}_{\mathrm{CD}}$ that satisfies

$$\begin{cases} \operatorname{curl} \mathbf{E}_{\mathrm{CD}} = \mathbf{0}, & \operatorname{div} \mathbf{E}_{\mathrm{CD}} = 0 & \text{in } \Xi, \\ \mathbf{E}_{\mathrm{CD}} \times \mathbf{n} = \mathbf{0} & & \text{on } \partial\Xi, \end{cases}$$

by Lemma 6.4 of [13], we deduce that any $\lambda \in \mathbb{C}$ such that $\lambda + 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$ induces a singularity of type 1 for our problem (pairs like $(\mathbf{E}_{\mathrm{CD}}, \mathbf{0})$ for instance).

(ii) We now show that other singular exponents appear. As $\lambda \neq -1$, by Lemma 6.1 of [13], the scalar fields

$$\Phi_E = \frac{1}{\lambda + 1} \mathbf{E} \cdot \mathbf{x}, \quad \Phi_H = \frac{1}{\lambda + 1} \mathbf{H} \cdot \mathbf{x},$$

are scalar potentials of $\mathbf{E}$ and $\mathbf{H}$, namely

$$\mathbf{E} = \nabla \Phi_E, \quad \mathbf{H} = \nabla \Phi_H \quad \text{in } \Xi. \tag{4.8}$$

Consequently, by the divergence-free property of $\mathbf{E}$ and $\mathbf{H}$, we deduce that

$$\Delta \Phi_E = \Delta \Phi_H = 0 \quad \text{in } \Xi. \tag{4.9}$$

Hence if we set

$$u_E(\vartheta) = \frac{1}{\lambda + 1} \mathbf{E}(\vartheta) \cdot \vartheta, u_H(\vartheta) = \frac{1}{\lambda + 1} \mathbf{H}(\vartheta) \cdot \vartheta,$$

we have

$$\Phi_E = r^{\lambda+1} u_E(\vartheta), \quad \Phi_H = r^{\lambda+1} u_H(\vartheta), \tag{4.10}$$

and by the identity (4.5), we get

$$\mathcal{L}(\lambda + 1) u_E = \mathcal{L}(\lambda + 1) u_H = 0 \quad \text{in } G. \tag{4.11}$$

Now we come back to the boundary condition in (4.7) that can be written in polar coordinates (r,θ) in the form

$$\begin{cases} \partial_r\phi_H = -\frac{1}{r}\partial_\theta\phi_E, \\ \frac{1}{r}\partial_\theta\phi_H = \partial_r\phi_E. \end{cases}$$

Due to (4.10), in term of u_E and u_H, this is equivalent to

$$\begin{cases} u_H = -\frac{1}{\lambda+1}\partial_\theta u_E, \\ \partial_\theta u_H = (\lambda+1)u_E. \end{cases}$$

These two identities imply that u_H is known if u_E is (or the converse) and then u_E has to satisfy

$$\partial_\theta^2 u_E + (\lambda+1)^2 u_E = 0 \text{ on } \mathcal{R}_c. \tag{4.12}$$

In other words, u_E is an eigenvector of the positive Laplace operator on $\mathcal{R}_c$ of eigenvalue $(\lambda+1)^2$. As the set of such eigenvalue is precisely made of μ^2, with $\mu \in \Upsilon_c$, two alternatives occur:

a. $\lambda+1$ does not belong to Υ_c, hence in that case $u_E = u_H = 0$ and, therefore,

$$\Phi_E = \Phi_H = 0 \text{ on } \partial\Xi,$$

and we conclude as in Lemma 6.4 of [13] that $\lambda+1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$.

b. $\lambda+1$ belongs to Υ_c, hence a non-trivial solution u_E of (4.12) exists (it is a multiple of an associated eigenvector) and then $u_H = -\frac{1}{\lambda+1}\partial_\theta u_E$. This means that the trace of u_E and u_H are prescribed on ∂G (i. e., $\mathcal{R}_c$), and call them φ_E and φ_H. Recalling (4.11), this means that u_E and u_H are respective solution of the following boundary value problems on G:

$$\begin{cases} \mathcal{L}(\lambda+1)u_E = 0 & \text{in } G, \\ u_E = \varphi_E & \text{on } \partial G. \end{cases} \qquad \begin{cases} \mathcal{L}(\lambda+1)u_H = 0 & \text{in } G, \\ u_H = \varphi_H & \text{on } \partial G. \end{cases}$$

For both problems, either $\lambda+1 \notin \Lambda_{\mathrm{Dir}}(\Gamma)$ and a solution exists, or $\lambda+1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$ and no matter that a solution exists or not, because, by point i), this case already gives rise to a singular exponent. □

We go on with singularities of type 2.

Lemma 4.4. *Let* $\lambda \in \mathbb{C}$. *If* $(\mathbf{E},\mathbf{H})$ *in the form* (4.1) *is a singularity of type 2, then* $\lambda \in \Lambda_{\mathrm{Dir}}(\Gamma) \cup \Upsilon_c^*$.

Proof. If $(\mathbf{E},\mathbf{H})$ in the form (4.1) is a singularity of type 2, then (see (4.3b)) $(\boldsymbol{\psi}_E, \boldsymbol{\psi}_H)$ satisfies

$$\begin{cases} \operatorname{curl} \boldsymbol{\psi}_E = \mathbf{0} & \text{in } \Xi, \\ \operatorname{curl} \boldsymbol{\psi}_H = \mathbf{0} & \text{in } \Xi, \\ \operatorname{div} \boldsymbol{\psi}_E = \operatorname{div} \boldsymbol{\psi}_H = 0 & \text{on } \partial\Xi, \\ \boldsymbol{\psi}_H \times \mathbf{n} = -(\boldsymbol{\psi}_E)_t & \text{on } \partial\Xi. \end{cases}$$

If we compare this system with (4.7), we deduce equivalently that λ belongs to $\Lambda_{\mathrm{Dir}}(\Gamma) \cup \Upsilon_c^*$, recalling that $(\boldsymbol{\psi}_E, \boldsymbol{\psi}_H)$ behaves like $r^{\lambda-1}$. Hence we have found that $\lambda \in \Lambda_{\mathrm{Dir}}(\Gamma) \cup \Upsilon_c^*$ is a necessary condition. □

We end up with singularities of type 3.

Lemma 4.5. *Let* $\lambda \in \mathbb{C}$. *If* $(\mathbf{E}, \mathbf{H})$ *in the form* (4.1) *is a singularity of type 3, then* $\lambda - 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$.

Proof. If $(\mathbf{E}, \mathbf{H})$ in the form (4.1) is a singularity of type 3, then (q_E, q_H) is a solution of (4.3a), which means equivalently that $\lambda - 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$ is a necessary condition. □

Among the corner singular exponents exhibited in the previous lemmas, according to Lemma 3.4, we have to remove the ones for which

$$\operatorname{div} \mathbf{E} \notin H^1_{\mathrm{loc}}(\Xi) \quad \text{or} \quad \operatorname{div} \mathbf{H} \notin H^1_{\mathrm{loc}}(\Xi).$$

No more constraint appears for singularities of type 1 or 2 since $\mathbf{E}$ and $\mathbf{H}$ are divergence-free. On the contrary for singularities of type 3 as $\operatorname{div} \mathbf{E} = q_E$ (resp., $\operatorname{div} \mathbf{H} = q_H$), we get the restriction

$$\lambda - 1 > -\frac{1}{2}.$$

As Lemma 4.5 also says that $\lambda - 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$ and as the set $\Lambda_{\mathrm{Dir}}(\Gamma) \cap [-1, 0]$ is always empty, we get the final constraint

$$\lambda - 1 > 0.$$

In summary, if we denote by Λ_c the set of corner singular exponents of the variational problem (3.7) (in $\mathbf{H}^1$), we have shown that

$$\Lambda_{c,1} \subset \Lambda_c \subset \Lambda_{c,1} \cup \Lambda_{c,2} \cup \Lambda_{c,3}, \tag{4.13}$$

where we have set

$$\Lambda_{c,1} = \left\{\lambda \in \mathbb{R} : \lambda > -\frac{1}{2} \text{ and } \lambda + 1 \in \Lambda_{\mathrm{Dir}}(\Gamma) \cup \Upsilon_c^*\right\}$$
$$\Lambda_{c,2} = \left\{\lambda \in \mathbb{R} : \lambda > -\frac{1}{2} \text{ and } \lambda \in \Lambda_{\mathrm{Dir}}(\Gamma) \cup \Upsilon_c^*\right\},$$
$$\Lambda_{c,3} = \{\lambda \in \mathbb{R} : \lambda > 1 \text{ and } \lambda - 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)\}.$$

Note that in the particular case of a cuboid, for all corners we have $\omega_c = \frac{3\pi}{2}$, while Proposition 18.8 of [16] yields

$$\Lambda_{\mathrm{Dir}}(\Gamma) = \{3 + 2d : d \in \mathbb{N}\} \cup \{-(4 + 2d) : d \in \mathbb{N}\}.$$

Consequently, one easily checks that

$$\Lambda_{c,1} = \{2 + 2d : d \in \mathbb{N}\} \cup \left\{\frac{4k}{3} - 1 : k \in \mathbb{N}^*\right\},$$
$$\Lambda_{c,2} = \{3 + 2d : d \in \mathbb{N}^*\} \cup \left\{\frac{4k}{3} : k \in \mathbb{N}^*\right\},$$
$$\Lambda_{c,3} = \{4 + 2d : d \in \mathbb{N}\}.$$

Hence the smallest corner singular exponent is equal to $\frac{1}{3}$.

Similarly, with the help of Lemma 18.7 of [16], the sets $\Lambda_{c,i}$, $i = 1, 2, 3$ can be characterized for any prism $D \times I$, where D is any polygon with a Lipschitz boundary and I is an interval.

4.2 Edge singularities

Our goal is to describe the edge singularities of problem (3.7). Let us then fix an edge e of Ω, then near an interior point of e, as our system (3.7) is invariant by translation and rotation (using a Piola transformation, that in this case corresponds to the covariant transformation), we may suppose that Ω behaves like $W_e = C_e \times \mathbb{R}$ where C_e is a two-dimensional cone centred at $(0, 0)$ of opening $\omega_e \in (0, 2\pi)$, with $\omega_e \neq \pi$. Here, for the sake of generality, we do not assume that $\omega_e < \pi$. Below we will also use the polar coordinates (r, θ) in C_e centred at $(0, 0)$. Let us recall that the set $\Lambda_{\mathrm{Dir}}(C_e)$ of singular exponents of the Laplace operator with Dirichlet boundary conditions in C_e is defined by

$$\Lambda_{\mathrm{Dir}}(C_e) = \left\{\frac{k\pi}{\omega_e} : k \in \mathbb{Z} \setminus \{0\}\right\}.$$

Similarly, we recall that the set of singular exponents of the Laplace operator with Neumann boundary conditions in C_e is defined by

$$\Lambda_{\mathrm{Neu}}(C) = \left\{\frac{k\pi}{\omega_e} : k \in \mathbb{Z}\right\}.$$

For convenience, when no confusion is possible, we will drop the index e. As usual, for $\lambda \in \mathbb{C}$, the edge singularities are obtained by looking for a non-polynomial solution $(\mathbf{E}, \mathbf{H})$ (independent of the x_3 variable) in the form of

$$(\mathbf{E}, \mathbf{H}) = r^\lambda \sum_{q=0}^{Q} (\ln r)^q (\mathbf{U}_q(\vartheta), \mathbf{V}_q(\vartheta)), \tag{4.14}$$

of

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - s\nabla\operatorname{div}\mathbf{E} = \mathbf{F}_E & \text{in } W, \\ \operatorname{curl}\operatorname{curl}\mathbf{H} - s\nabla\operatorname{div}\mathbf{H} = \mathbf{F}_H & \text{in } W, \\ \operatorname{div}\mathbf{E} = \operatorname{div}\mathbf{H} = 0 & \text{on } \partial W, \\ \mathbf{H}\times n - \mathbf{E}_t = (\operatorname{curl}\mathbf{H})\times n + (\operatorname{curl}\mathbf{E})_t = \mathbf{0} & \text{on } \partial W, \end{cases} \tag{4.15}$$

$\mathbf{F}_E$, $\mathbf{F}_H$ being a polynomial in the x_1, x_2 variables. In that way, we see that the pair $\mathsf{E} = (E_1, E_2)$ made of the two first components of $\mathbf{E}$ and the third component $h := H_3$ of $\mathbf{H}$ satisfy

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathsf{E} - s\nabla\operatorname{div}\mathsf{E} = \mathbf{F}_E & \text{in } C, \\ \Delta h = g & \text{in } C, \\ \operatorname{div}\mathsf{E} = 0 & \text{on } \partial C, \\ h + \mathsf{E}_t = \partial_n h - \operatorname{curl}\mathsf{E} = 0 & \text{on } \partial C, \end{cases} \tag{4.16}$$

F, g being a polynomial (in the x_1, x_2 variables) and as usual

$$\operatorname{curl}\mathsf{E} = \partial_1 E_2 - \partial_2 E_1,$$

and

$$\mathsf{E}_t = n_1 E_2 - n_2 E_1 \quad \text{on } \partial C,$$

if $\mathbf{n} = (n_1, n_2)$ on ∂C, further for a scalar field φ we have

$$\operatorname{curl}\varphi = \begin{pmatrix} \partial_2\varphi \\ -\partial_1\varphi \end{pmatrix}.$$

The pair (H_1, H_2) made of the two first components of $\mathbf{H}$ and $-E_3$, where E_3 is the third component of $\mathbf{E}$ that satisfy the same system, hence we only need to characterize the singularities of (4.16).

Inspired from [13], the singularities of system (4.16) are obtained by introducing the scalar variables $q = \operatorname{div}\mathsf{E}$ and $\psi = \operatorname{curl}\mathsf{E}$. In this way, if $\lambda \notin \mathbb{N}_2 := \{n \in \mathbb{N} : n \geq 2\}$ (or equivalently λ is not an integer or is an integer ≤ 1), we find the equivalent system

$$\begin{cases} \Delta q = 0 & \text{in } C, \\ q = 0 & \text{on } \partial C, \end{cases} \tag{4.17a}$$

$$\begin{cases} \operatorname{curl}\boldsymbol{\psi} = s\nabla q & \text{in } C, \\ \Delta h = 0 & \text{in } C, \\ \partial_n h - \boldsymbol{\psi} = 0 & \text{on } \partial C, \end{cases} \tag{4.17b}$$

$$\begin{cases} \operatorname{curl}\mathsf{E} = \boldsymbol{\psi}, \quad \operatorname{div}\mathsf{E} = q & \text{in } C, \\ \mathsf{E}_t = -h & \text{on } \partial C. \end{cases} \tag{4.17c}$$

As before three types of singularities appear:

Type 1: $q = 0$, $\boldsymbol{\psi} = 0$ and E general non-zero solution of (4.17c).
Type 2: $q = 0$, $\boldsymbol{\psi}$ general non-zero solution of (4.17b) and E particular solution of (4.17c).
Type 3: q general non-zero solution of (4.17a), $\boldsymbol{\psi}$ particular solution of (4.17b) and E particular solution of (4.17c).

The singularities of type 1 were treated in [13, Section 5c], where it is shown that $\lambda \notin \mathbb{N}_2$ is such that $\lambda + 1 \in \Lambda_{\mathrm{Dir}}(C) \setminus \{2\}$.

Let us now look at singularities of type 2.

Lemma 4.6. *Let $\lambda \notin \mathbb{N}_2$ be such that $\Re\lambda > 0$. Then λ is a singularity of type 2 if and only if $\lambda \in \Lambda_{\mathrm{Neu}}(C)$.*

Proof. If (E, h) in the form

$$\mathsf{E} = r^\lambda \sum_{q=0}^{Q} (\ln r)^q \mathsf{U}(\vartheta), \quad h = r^\lambda \sum_{q=0}^{Q} (\ln r)^q v_q(\vartheta), \tag{4.18}$$

is a singularity of type 2, then $\psi = \operatorname{curl} \mathsf{E}$ satisfies (see (4.17b))

$$\begin{cases} \operatorname{curl} \boldsymbol{\psi} = 0 & \text{in } C, \\ \Delta h = 0 & \text{in } C, \\ \partial_n h - \boldsymbol{\psi} = 0 & \text{on } \partial C. \end{cases}$$

In this case, ψ is constant in the whole C. Hence we distinguish the case $\lambda = 1$ or not:

1. If $\lambda \neq 1$, then $\psi = 0$ and consequently h satisfies

$$\begin{cases} \Delta h = 0 & \text{in } C, \\ \partial_n h = 0 & \text{on } \partial C, \end{cases} \tag{4.19}$$

   which means that λ belongs to $\Lambda_{\mathrm{Neu}}(C)$ and h is in the form

$$h = r^\lambda \cos(\lambda\theta).$$

2. If $\lambda = 1$, then there exists a constant c such that $\psi = c$, and consequently h satisfies

$$\begin{cases} \Delta h = 0 & \text{in } C, \\ \partial_n h = c & \text{on } \partial C. \end{cases} \tag{4.20}$$

   For two parameters c_1 and c_2, denote by

$$h_0 = c_1 x_1 + c_2 x_2 = r(c_1 \cos\theta + c_2 \sin\theta).$$

   Clearly, h_0 is harmonic and satisfies

$$\partial_n h_0(\theta = 0) = -c_2,$$

$$\partial_n h_0(\theta = \omega) = -c_1 \sin\omega + c_2 \cos\omega,$$

hence it fulfils (4.20) if and only if (c_1, c_2) satisfies the 2×2 linear system

$$c_2 = -c, -c_1 \sin\omega + c_2 \cos\omega = c.$$

Since $\sin\omega$ is different from zero, such a solution exists and therefore $d = h - h_0$ satisfies (4.19). This would mean that 1 belongs to $\Lambda_{\mathrm{Neu}}(C)$, which is not possible.

Once ψ and h are found, we look for a particular solution E of (4.17c) with $q = 0$. From its curl-free property, we look for E in the form

$$E = \nabla\Phi,$$

with

$$\Phi = r^{\lambda+1}\varphi(\theta),$$

where φ has to satisfy

$$\begin{cases} \varphi'' + (\lambda+1)^2\varphi = 0 & \text{in } (0,\omega), \\ (\lambda+1)\varphi(0) = -1, \quad (\lambda+1)\varphi(\omega) = -\cos(\lambda\omega). \end{cases}$$

As $\lambda + 1$ does not belong to $\Lambda_{\mathrm{Dir}}(C)$ and is different from zero, such a solution φ always exists. □

Lemma 4.7. *Let $\lambda \notin \mathbb{N}_2$ be such that $\Re\lambda > 0$. Then λ is a singularity of type 3 if and only if $\lambda - 1 \in \Lambda_{\mathrm{Dir}}(C)$.*

Proof. If (E, h) in the form (4.18) is a singularity of type 3, then $q = \operatorname{div} \mathsf{E}$ satisfies (4.17a) and consequently $\lambda - 1$ belongs to $\Lambda_{\mathrm{Dir}}(C)$ and q is equal to

$$q = r^{\lambda-1}\sin((\lambda-1)\theta),$$

up to a non-zero multiplicative factor (that we then fix to be 1).

Now we look for (ψ, h) a particular solution of (4.17b). As simple calculations yield

$$\operatorname{curl}(r^{\lambda-1}\cos((\lambda-1)\theta)) = -\nabla r^{\lambda-1}\sin((\lambda-1)\theta),$$

we deduce that

$$\psi = -sr^{\lambda-1}\cos((\lambda-1)\theta) + k,$$

for some constant k, that we can fix to be zero since we look for particular solutions. Hence it remains to find h solution of

$$\begin{cases} \Delta h = 0 & \text{in } C, \\ \partial_n h = -sr^{\lambda-1}\cos((\lambda-1)\theta) & \text{on } \partial C. \end{cases}$$

Such a h exists in the form

$$h = r^{\lambda}\eta(\theta),$$

since the previous problem is equivalent to

$$\begin{cases} \eta'' + \lambda^2\eta = 0 & \text{in } (0,\omega), \\ \eta'(0) = s, \quad \eta'(\omega) = \pm s(-1)^k, \end{cases}$$

when $\lambda = \frac{k\pi}{\omega}$ and this system has a unique solution since $\lambda \notin \Lambda_{\mathrm{Neu}}(C)$.

Now we look for E a particular solution of (4.17c) with the functions q, ψ and h found before, which then takes the form

$$\begin{cases} \operatorname{curl} \mathsf{E} = -sr^{\lambda-1}\cos((\lambda-1)\theta) & \text{in } C, \\ \operatorname{div} \mathsf{E} = r^{\lambda-1}\sin((\lambda-1)\theta) & \text{in } C, \\ \mathsf{E}_t = -r^{\lambda}\eta(\theta) & \text{on } \partial C. \end{cases}$$

Hence we look for E in the form

$$\mathsf{E} = -\frac{s}{4\lambda}\operatorname{curl}(r^{\lambda+1}\cos((\lambda-1)\theta)) + \nabla\Phi.$$

As simple calculations yield

$$\operatorname{curl}\operatorname{curl}(r^{\lambda+1}\cos((\lambda-1)\theta)) = 4\lambda\cos((\lambda-1)\theta),$$

we deduce that the previous system in E is equivalent to

$$\begin{cases} \Delta\Phi = r^{\lambda-1}\sin((\lambda-1)\theta) & \text{in } C, \\ \partial_r\Phi(r,0) = c_0 r^{\lambda}, \quad \partial_r\Phi(r,\omega) = c_\omega r^{\lambda}, \end{cases} \tag{4.21}$$

for two constants c_0 and c_ω. If $\lambda+1 \notin \Lambda_{\mathrm{Dir}}(C)$, then a solution Φ of this problem always exists in the form

$$r^{\lambda+1}\varphi(\theta),$$

since it is then equivalent to

$$\begin{cases} \varphi'' + (\lambda+1)^2\varphi = \sin((\lambda-1)\theta) & \text{in } C, \\ \varphi(0) = \dfrac{c_0}{\lambda+1}, \quad \partial_r\Phi(r,\omega) = \dfrac{c_\omega}{\lambda+1}. \end{cases}$$

On the contrary if $\lambda+1 \in \Lambda_{\mathrm{Dir}}(C)$ (that only occurs when $\omega = \frac{3\pi}{2}$), then we look for Φ in the form

$$r^{\lambda+1}(\varphi_0(\theta) + \log r\varphi_1(\theta)). \tag{4.22}$$

Since, in this particular choice, problem (4.21) is equivalent to

$$\begin{cases} \Delta\Phi = r^{\lambda-1}\sin((\lambda-1)\theta) & \text{in } C, \\ \Phi(r,0) = \dfrac{c_0}{\lambda+1}r^{\lambda+1}, \quad \Phi(r,\omega) = \dfrac{c_\omega}{\lambda+1}r^{\lambda+1}, \end{cases}$$

by Theorem 4.22 of [36], we deduce that a solution Φ in the form (4.22) exists.

In both cases, a solution Φ exists, hence the existence of E. □

As before among the edge singular exponents, we have to remove the ones for which

$$\operatorname{div}\mathbf{E} \notin H^1_{\text{loc}}(W) \quad \text{or} \quad \operatorname{div}\mathbf{H} \notin H^1_{\text{loc}}(W).$$

No more constraint appears for singularities of type 1 or 2 since **E** and **H** are divergence-free. On the contrary for singularities of type 3, we get the restriction

$$\lambda > 1.$$

In summary, if we denote by Λ_e the set of edge singular exponents $\notin \mathbb{N}_2$ of the variational problem (3.7) (in $\mathbf{H}^1$, i. e., with $\Re\lambda > 0$), we have shown that

$$\Lambda_e = \Lambda_{e,1} \cup \Lambda_{e,2} \cup \Lambda_{e,3}, \tag{4.23}$$

where we have set

$$\begin{aligned} \Lambda_{e,1} &= \{\lambda \in \mathbb{R} : \lambda > 0 \text{ and } \lambda + 1 \in \Lambda_{\text{Dir}}(C) \setminus \{2\}\} \\ \Lambda_{e,2} &= \{\lambda \in \mathbb{R} : \lambda > 0 \text{ and } \lambda \in \Lambda_{\text{Neu}}(C)\}, \\ \Lambda_{c,3} &= \{\lambda \in \mathbb{R} : \lambda > 1 \text{ and } \lambda - 1 \in \Lambda_{\text{Dir}}(C)\}. \end{aligned}$$

Note that in the particular case of a cuboid, for all edges we have $\omega_e = \frac{\pi}{2}$, and consequently $\Lambda_e = \emptyset$ (recalling that the natural number in $\mathbb{N}_2$ are excluded from this set). Since one can show that $\lambda = 2$ is a singular exponent, the maximal regularity along the edge is $H^{3-\varepsilon}$, for any $\varepsilon > 0$.

In conclusion, for any convex polyhedral domain, there exists $t_\Omega \in (1,2]$ such that for any $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, $\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)$ belongs to $\mathbf{H}^t(\Omega)^2$, for all $t < t_\Omega$. For instance for a cuboid, we have $t_\Omega = \frac{11}{6}$.

5 Wavenumber explicit stability analysis

The basic block for a wavenumber explicit error analysis of problem (3.7) (or (3.5)) is a so-called stability estimate at the energy level; for the Helmholtz equation, see [15, 17, 22], while for problem (1.3), see [23]. Hence we make the following definition.

Definition 5.1. We will say that system (3.7) satisfies the k-stability property with exponent $\alpha \geq 0$ (independent of k and s) if there exists $k_0 > 0$ such that for all $k \geq k_0$ and all $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, the solution $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ of (3.5) satisfies

$$\|(\mathbf{E}, \mathbf{H})\|_k \lesssim k^\alpha(\|\mathbf{f}_1\|_{0,\Omega} + \|\mathbf{f}_2\|_{0,\Omega}). \tag{5.1}$$

Before going on, let us show that this property is valid for some particular domains, in particular it will be valid for rectangular cuboids of rational lengths, some tetrahedra and some prisms. To prove such a result, we first start with a similar property with divergence-free data. In this case, our proof is a simple consequence of a result obtained in [37] for the time-dependent Maxwell system with impedance boundary conditions combined with the next result of functional analysis [40, 25].

Lemma 5.2. *A C_0 semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ of contractions on a Hilbert space H is exponentially stable, i. e., satisfies*

$$\|e^{t\mathcal{A}} U_0\|_H \leq Ce^{-\omega t}\|U_0\|_H, \quad \forall U_0 \in H, \quad \forall t \geq 0,$$

for some positive constants C and ω if and only if

$$\rho(\mathcal{A}) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R}, \tag{5.2}$$

and

$$\sup_{\beta\in\mathbb{R}} \|(i\beta\mathbb{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} < \infty, \tag{5.3}$$

where $\rho(\mathcal{A})$ denotes the resolvent set of the operator $\mathcal{A}$.

Theorem 5.3. *In addition to the assumptions of Theorem 3.5, assume that Ω is star-shaped with respect to a point. Then for all $k \geq 0$ and all $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$ such that $\operatorname{div} \mathbf{f}_1 = \operatorname{div} \mathbf{f}_2 = 0$, the solution $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ of (3.5) satisfies (5.1) with $\alpha = 1$.*

Proof. As the data are divergence-free, by Lemma 3.7, the auxiliary unknown $(\mathbf{U}, \mathbf{W}) = (\operatorname{curl} \mathbf{E} - ik\mathbf{H}, \operatorname{curl} \mathbf{H} + ik\mathbf{E})$ belongs to $\mathbf{V}$, is divergence-free and satisfies the Maxwell system (3.8).

Now we notice that Theorem 4.1 of [37] (valid for star-shaped domain with a Lipschitz boundary) shows that the time-dependent Maxwell system

$$\begin{cases} \partial_t \mathbf{E} + \operatorname{curl} \mathbf{H} = \mathbf{0} \quad \text{and} \quad \partial_t \mathbf{H} - \operatorname{curl} \mathbf{E} = \mathbf{0} & \text{in } \Omega, \\ \mathbf{H} \times \mathbf{n} - \lambda_{\text{imp}} \mathbf{E_t} = \mathbf{0} & \text{on } \Xi, \end{cases}$$

is exponentially stable in $\mathcal{H} = \{(\mathbf{E}, \mathbf{H}) \in \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) : \operatorname{div} \mathbf{E} = \operatorname{div} \mathbf{H} = 0\}$. This equivalently means that the operator $\mathcal{A}$ defined by

$$\mathcal{A}(\mathbf{E}, \mathbf{H}) = (-\operatorname{curl} \mathbf{H}, \operatorname{curl} \mathbf{E}), \quad \forall (\mathbf{E}, \mathbf{H}) \in D(\mathcal{A}),$$

with domain

$$D(\mathcal{A}) = \{(\mathbf{E}, \mathbf{H}) \in \mathbf{V} : \operatorname{div} \mathbf{E} = \operatorname{div} \mathbf{H} = 0\},$$

generates an exponentially stable C_0 semigroup in $\mathcal{H}$. Hence by Lemma 5.2, we deduce that its resolvent is bounded on the imaginary axis. This precisely implies that

$$\|\mathbf{U}\|_\Omega + \|\mathbf{W}\|_\Omega \lesssim \|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega, \tag{5.4}$$

for all $k \geq 0$. But coming back to the definition of $\mathbf{U}$ and $\mathbf{W}$, we can look at $(\mathbf{E}, \mathbf{H})$ as a solution in $D(\mathcal{A})$ of the Maxwell system

$$\operatorname{curl} \mathbf{E} - ik\mathbf{H} = \mathbf{U}, \quad \operatorname{curl} \mathbf{H} + ik\mathbf{E} = \mathbf{W}.$$

Hence the previous arguments show that

$$\|\mathbf{E}\|_\Omega + \|\mathbf{H}\|_\Omega \lesssim \|\mathbf{U}\|_\Omega + \|\mathbf{W}\|_\Omega.$$

By the estimate (5.4), we deduce that

$$\|\mathbf{E}\|_\Omega + \|\mathbf{H}\|_\Omega \lesssim \|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega. \tag{5.5}$$

Finally, as

$$\|(\mathbf{E}, \mathbf{H})\|_k \sim \|\operatorname{curl} \mathbf{E}\|_\Omega + \| \operatorname{curl} \mathbf{H}\|_\Omega + k(\|\mathbf{E}\|_\Omega + \|\mathbf{H}\|_\Omega),$$

by the triangular inequality, we get that

$$\begin{aligned} \|(\mathbf{E}, \mathbf{H})\|_k &\lesssim \|\operatorname{curl} \mathbf{E} - ik\mathbf{H}\|_\Omega + \|\operatorname{curl} \mathbf{H} + ik\mathbf{E}\|_\Omega + k(\|\mathbf{E}\|_\Omega + \|\mathbf{H}\|_\Omega) \\ &\lesssim \|\mathbf{U}\|_\Omega + \|\mathbf{V}\|_\Omega + k(\|\mathbf{E}\|_\Omega + \|\mathbf{H}\|_\Omega). \end{aligned}$$

By the estimates (5.4) and (5.5), we conclude that

$$\|(\mathbf{E}, \mathbf{H})\|_k \lesssim k(\|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega),$$

as announced. □

Now we leave out the divergence-free constraint on the data. Before let us denote by $\{\lambda_n\}_{n\in\mathbb{N}^*}$, the set of eigenvalues enumerated in increasing order (and not repeated according to their multiplicity) of the positive Laplace operator $-\Delta$ with Dirichlet boundary conditions in Ω. For each $n \in \mathbb{N}^*$, we also denote by $\varphi_{n,\ell}$, $\ell = 1, \dots, m(n)$, the orthonormal eigenvectors associated with λ_n. For all $k > 0$ and each $s \in [1, 2]$, let us define the unique integer $n(k, s)$ such that

$$\lambda_{n(k,s)} \leq \frac{k^2}{s} < \lambda_{n(k,s)+1}, \tag{5.6}$$

and denote by

$$g_{n(k,s)} = \lambda_{n(k,s)+1} - \lambda_{n(k,s)},$$

the gap between these consecutive eigenvalues. Now we show that if $g_{n(k,s)}$ satisfies some uniform lower bound, then the k-stability property holds.

Lemma 5.4. *In addition to the assumptions of Theorem* 5.3, *assume that there exists a non-negative real number β and two positive real number γ_0 and k_1 such that*

$$\forall k \geq k_1 \; \exists s \in [1, 2] : g_{n(k,s)} \geq \gamma_0 k^{-2\beta}. \tag{5.7}$$

Then there exist two positive real numbers s_0, s_1 such that $s_0 < s_1$ (depending on β, γ_0 and k_1) and for an appropriate choice of $s \in [s_0, s_1]$ (but such that $-k^2/s$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary conditions in Ω), the k-stability property with exponent $\alpha = 2\beta + 1$ holds.

Proof. The first step is to reduce the problem to divergence-free right-hand sides. For that purpose, for $i = 1$ or 2, we consider $u_i, \varphi_i \in H_0^1(\Omega)$ variational solutions of

$$\begin{aligned} \Delta u_i &= \operatorname{div} \mathbf{f}_i \quad \text{in } \Omega, \\ \left(\Delta\varphi_i + \frac{k^2}{s}\varphi_i\right) &= -s^{-1} u_i \quad \text{in } \Omega. \end{aligned}$$

Then simple calculations show that $(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) = (\mathbf{E} - \nabla\varphi_1, \mathbf{H} - \nabla\varphi_2)$ belongs to $\mathbf{V}$ and is solution of (3.7) with divergence-free right-hand side, namely

$$\begin{cases} \left.\begin{aligned} L_{k,s}(\tilde{\mathbf{E}}) = \tilde{\mathbf{f}}_1 = \mathbf{f}_1 - \nabla u_1, \\ L_{k,s}(\tilde{\mathbf{H}}) = \tilde{\mathbf{f}}_2 = \mathbf{f}_2 - \nabla u_2, \end{aligned}\right\} \quad \text{in } \Omega, \\ \left.\begin{aligned} \operatorname{div} \tilde{\mathbf{E}} = 0 \\ \operatorname{div} \tilde{\mathbf{H}} = 0 \\ T(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) = 0 \\ B(\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) = 0 \end{aligned}\right\} \quad \text{on } \partial\Omega. \end{cases} \tag{5.8}$$

In a first step, we estimate the H^1-norm of φ_i. Since we assume that $\frac{k^2}{s}$ does not encounter the spectrum of the Laplace operator, by the spectral theorem, we can write

$$\varphi_i = -s^{-1} \sum_{n\in\mathbb{N}^*} \left(\frac{k^2}{s} - \lambda_n\right)^{-1} \sum_{\ell=1}^{m(n)} (u_i, \varphi_{n,\ell})_\Omega \varphi_{n,\ell}.$$

Consequently, we have

$$\|\varphi_i\|_{1,\Omega}^2 \sim s^{-2} \sum_{n\in\mathbb{N}^*} \left(\frac{k^2}{s} - \lambda_n\right)^{-2} \sum_{\ell=1}^{m(n)} |(u_i, \varphi_{n,\ell})_\Omega|^2 \lambda_n. \tag{5.9}$$

Hence our goal is to choose s in an interval $[s_0, s_1]$ with s_0 and s_1 independent of k satisfying $0 < s_0 \leq s_1 < \infty$ and such that

$$\left|\frac{k^2}{s} - \lambda_n\right| \gtrsim k^{-2\beta}, \quad \forall n \in \mathbb{N}^*, \ k \geq k_0, \tag{5.10}$$

with k_0 large enough. Indeed if this estimate is valid, then (5.9) can be transformed into

$$\|\varphi_i\|_{1,\Omega}^2 \lesssim k^{4\beta} \sum_{n\in\mathbb{N}^*} \sum_{\ell=1}^{m(n)} |(u_i, \varphi_{n,\ell})_\Omega|^2 \lambda_n$$

and, therefore,

$$\|\varphi_i\|_{1,\Omega} \lesssim k^{2\beta} \|u_i\|_{1,\Omega}.$$

As clearly

$$\|u_i\|_{1,\Omega} \lesssim \|\mathbf{f}_i\|_\Omega, \tag{5.11}$$

we conclude that

$$\|\varphi_i\|_{1,\Omega} \lesssim k^{2\beta} \|\mathbf{f}_i\|_\Omega. \tag{5.12}$$

As

$$\|(\nabla\varphi_1, \nabla\varphi_2)\|_k \sim \sqrt{s}(\|\Delta\varphi_1\|_\Omega + \|\Delta\varphi_2\|_\Omega) + k(\|\varphi_1\|_{1,\Omega} + \|\varphi_2\|_{1,\Omega}), \tag{5.13}$$

we need to estimate the L^2-norm of $\Delta\varphi_1$. But from its definition, we have

$$\Delta\varphi_i + \frac{k^2}{s}\varphi_i = -s^{-1}u_i,$$

and taking the L^2-inner product with φ_i, we get

$$(\Delta\varphi_i, \varphi_i)_\Omega + \frac{k^2}{s}\|\varphi_i\|_\Omega^2 = -s^{-1}(u_i, \varphi_i)_\Omega.$$

Using Cauchy–Schwarz's inequality, we get

$$\frac{k^2}{s}\|\varphi_i\|_\Omega^2 \leq s^{-1}\|u_i\|_\Omega \|\varphi_i\|_\Omega + |\varphi_i|_{1,\Omega}^2.$$

With the help of (5.11) and (5.12), we obtain

$$k^2\|\varphi_i\|_\Omega^2 \lesssim \|\mathbf{f}_i\|_\Omega \|\varphi_i\|_\Omega + k^{4\beta}\|\mathbf{f}_i\|_\Omega^2.$$

Hence by Young's inequality, we get

$$k^2\|\varphi_i\|_\Omega^2 \lesssim k^{4\beta}\|\mathbf{f}_i\|_\Omega^2,$$

which proves that

$$\|\varphi_i\|_\Omega \lesssim k^{2\beta-1}\|\mathbf{f}_i\|_\Omega. \tag{5.14}$$

This directly implies that

$$\|\Delta\varphi_i\|_\Omega \leq \frac{k^2}{s}\|\varphi_i\|_\Omega + s^{-1}\|u_i\|_\Omega \lesssim k^{2\beta+1}\|\mathbf{f}_i\|_\Omega.$$

Using this estimate and (5.12) in (5.13) leads to

$$\|(\nabla\varphi_1, \nabla\varphi_2)\|_k \lesssim k^{2\beta+1}(\|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega). \tag{5.15}$$

At this stage, we use Theorem 5.3 that yields

$$\|(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})\|_k \lesssim k(\|\tilde{\mathbf{f}}_1\|_\Omega + \|\tilde{\mathbf{f}}_2\|_\Omega).$$

Hence by the definition of $\tilde{\mathbf{f}}_i$ and (5.11), we deduce that

$$\|(\tilde{\mathbf{E}}, \tilde{\mathbf{H}})\|_k \lesssim k(\|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega).$$

As $(\mathbf{E}, \mathbf{H}) = (\tilde{\mathbf{E}}, \tilde{\mathbf{H}}) + (\nabla\varphi_1, \nabla\varphi_2)$, the combination of this last estimate with (5.15) leads to

$$\|(\mathbf{E}, \mathbf{H})\|_k \lesssim k^{2\beta+1}(\|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega), \tag{5.16}$$

which proves the stability estimate with $\alpha = 2\beta + 1$.

It remains to prove that (5.10) holds for an appropriate choice of s. This is done with the help of our assumption (5.7), by an eventual slight modification of s from this assumption. To be more precise, for all $k \geq k_1$, we fix one $s \in [1, 2]$ such that (5.7) holds and denote it by $s(k)$. We now distinguish between three cases:

a) If $\lambda_{n(k,s(k))} \leq \frac{k^2}{s(k)} \leq \lambda_{n(k,s(k))} + \frac{\gamma_0}{3k^{2\beta}}$, then we fix s such that

$$\frac{k^2}{s} = \lambda_{n(k,s(k))} + \frac{\gamma_0}{3k^{2\beta}}. \tag{5.17}$$

With this choice, we clearly have

$$\frac{k^2}{s} - \lambda_{n(k,s(k))} = \frac{\gamma_0}{3k^{2\beta}},$$

while

$$\lambda_{n(k,s(k))+1} - \frac{k^2}{s} = \lambda_{n(k,s(k))+1} - \lambda_{n(k,s(k))} - \frac{\gamma_0}{3k^{2\beta}} \geq \frac{2\gamma_0}{3k^{2\beta}},$$

which proves that (5.10) holds. Let us now show that s remains in a (uniformly) bounded interval. Indeed (5.17) is equivalent to

$$s = \frac{k^2}{\lambda_{n(k,s(k))} + \frac{\gamma_0}{3k^{2\beta}}}.$$

As by assumption $k^2 \leq s(k)(\lambda_{n(k,s(k))} + \frac{\gamma_0}{3k^{2\beta}})$, we directly deduce that $s \leq s(k) \leq 2$. Conversely, from (5.6), we deduce that

$$\begin{aligned}\frac{k^2}{\lambda_{n(k,s(k))} + \frac{\gamma_0}{3k^{2\beta}}} &\geq \frac{k^2}{\frac{k^2}{s(k)} + \frac{\gamma_0}{3k^{2\beta}}} \\ &\geq \frac{s(k)}{1 + \frac{\gamma_0 s(k)}{3k^{2(\beta+1)}}} \\ &\geq \frac{1}{1 + \frac{2\gamma_0}{3k_1^{2(\beta+1)}}}.\end{aligned}$$

b) If $\lambda_{n(k,s(k))+1} - \frac{\gamma_0}{3k^{2\beta}} \leq \frac{k^2}{s(k)} \leq \lambda_{n(k,s(k))+1}$, then we fix s such that

$$\frac{k^2}{s} = \lambda_{n(k,s(k))+1} - \frac{\gamma_0}{3k^{2\beta}}.$$

We check exactly as in the first case that (5.10) holds. Furthermore, by assumption $s \geq 1$, while for the lower bound we see that

$$s = \frac{k^2}{\lambda_{n(k,s(k))+1} - \frac{\gamma_0}{3k^{2\beta}}} \leq \frac{k^2}{\frac{k^2}{s(k)} - \frac{\gamma_0}{3k^{2\beta}}} \leq \frac{s(k)}{1 - \frac{s(k)\gamma_0}{3k^{2\beta}}} \leq \frac{2}{1 - \frac{2\gamma_0}{3k^{2\beta}}}.$$

Hence $s \leq 3$ for $k \geq k_0$ with k_0 large enough.

c) If $\lambda_{n(k,s(k))} + \frac{\gamma_0}{3k^{2\beta}} < \frac{k^2}{s(k)} < \lambda_{n(k,s(k))+1} - \frac{\gamma_0}{3k^{2\beta}}$, then we fix $s = s(k)$. In such a case, we directly see that (5.10) holds since

$$k^2 - \lambda_{n(k,s(k))} \geq \frac{\gamma_0}{3k^{2\beta}}, \text{ and } \lambda_{n(k,s(k))+1} - k^2 \geq \frac{\gamma_0}{3k^{2\beta}}.$$

The proof is then complete. □

Remark 5.5. The parameter s fixed in the previous lemma clearly depends on k. Furthermore, if β is positive, the quantity $\frac{k^2}{s}$ approaches the spectrum of $-\Delta$, and hence the norm of the resolvent operator $\Delta + \frac{k^2}{s}$ blows up, but the estimate (5.16) controls this blow up since it yields

$$\|\operatorname{div} \mathbf{E}\|_\Omega + \|\operatorname{div} \mathbf{H}\|_\Omega \lesssim k^{2\beta+1}(\|\mathbf{f}_1\|_\Omega + \|\mathbf{f}_2\|_\Omega).$$

Let us now show that (5.7) always holds with $\beta = \frac{1}{2}$.

Lemma 5.6. *For all bounded domain* Ω (*of* $\mathbb{R}^3$), *the assumption* (5.7) *holds with* $\beta = \frac{1}{2}$.

Proof. Assume that (5.7) does not hold with $\beta = \frac{1}{2}$, in other words

$$\forall \gamma_0 > 0,\ k_1 > 0\ \exists k \geq k_1\ \forall s \in [1, 2] : g_{n(k,s)} < \gamma_0 k^{-1}. \tag{5.18}$$

We first fix γ_0 such that

$$\gamma_0 < \frac{1}{48\sqrt{2}c|\Omega|}, \tag{5.19}$$

where $|\Omega|$ is the measure of Ω and $c = \frac{1}{6\pi^2}$ is the universal constant such that Weyl's formula

$$\lim_{t\to\infty} \frac{N(t)}{c|\Omega|t^{\frac{3}{2}}} = 1, \tag{5.20}$$

holds, where $N(t)$ is the eigenvalue counting function of the positive Laplace operator $-\Delta$ with Dirichlet boundary conditions in Ω, i. e., the number of its eigenvalues, which are less than t. Then we fix k_1 large enough, namely $k_1^3 \geq 12\gamma_0$. Then for all $k \geq k_1$, we define the real numbers

$$s_i = 1 + \frac{3\gamma_0 i}{k^3}, \quad \forall i = 1, \ldots, N_k,$$

where $N_k = \lfloor \frac{k^3}{6\gamma_0} \rfloor - 1$ (where $\lfloor x \rfloor$ is the integral part of any real number x, namely the unique integer such that $x \leq \lfloor x \rfloor < x + 1$). By our assumption N_k is larger than 1 and for k large it behaves like k^3. It is easy to see that all s_i belongs to $[1, \frac{3}{2}]$. Now we look at the intervals

$$I_i = \left[\frac{k^2}{s_i} - \frac{\gamma_0}{2k}, \frac{k^2}{s_i} + \frac{\gamma_0}{2k}\right], \quad \forall i = 1, \ldots, N_k,$$

and show that they are disjoint, i. e.,

$$I_i \cap I_j = \emptyset, \quad \forall i \neq j, \tag{5.21}$$

and included into the closed interval $[\frac{k^2}{2}, 2k^2]$:

$$I_i \subset \left[\frac{k^2}{2}, 2k^2\right], \quad \forall i = 1, \ldots, N_k. \tag{5.22}$$

Indeed for the second assertion it suffices to show that

$$\frac{k^2}{s_i} - \frac{\gamma_0}{2k} \geq \frac{k^2}{2}, \tag{5.23}$$

and that

$$\frac{k^2}{s_i} + \frac{\gamma_0}{2k} \geq 2k^2. \tag{5.24}$$

This second estimate holds if and only if

$$\frac{k^2}{s_1} + \frac{\gamma_0}{2k} \geq 2k^2,$$

or equivalently

$$\frac{1}{s_1} \leq 2 - \frac{\gamma_0}{2k^3}.$$

Since $s_1 = 1 + \frac{3\gamma_0}{k^3}$, this holds if and only if

$$\left(2 - \frac{\gamma_0}{2k^3}\right)\left(1 + \frac{3\gamma_0}{k^3}\right) \geq 1,$$

which means that $\frac{\gamma_0}{k^3}$ has to satisfy

$$\frac{11 - \sqrt{145}}{6} \leq \frac{\gamma_0}{k^3} \leq \frac{11 + \sqrt{145}}{6},$$

that is valid owing to our assumption on k_1 (and the fact that $k \geq k_1$).

In the same spirit, the estimate (5.23) holds if and only if

$$s_{N_k} \leq \frac{2}{1 + \frac{\gamma_0}{k^3}},$$

which holds because our assumption on k_1 implies that

$$\frac{3}{2} \leq \frac{2}{1 + \frac{2\gamma_0}{k^3}}.$$

Now to prove (5.21), it suffices to show that

$$I_i \cap I_{i+1} = \emptyset, \quad \forall i = 1, \ldots, N_k - 1,$$

or

$$\frac{k^2}{s_{i+1}} + \frac{\gamma_0}{2k} < \frac{k^2}{s_i} - \frac{\gamma_0}{2k}, \quad \forall i = 1, \ldots, N_k - 1.$$

By the definition of the s_i, this holds if and only if

$$s_i s_{i+1} < 3.$$

Since $s_i s_{i+1} \leq \frac{9}{4}$, we deduce that (5.21) is valid.

Since the length of I_i is exactly equal to $\frac{\gamma_0}{k}$ and due to our assumption (5.18), $\lambda_{n(k,s_i)}$ or $\lambda_{n(k,s_i)+1}$ belongs to I_i. Due to (5.21) and (5.22), for all $k \geq k_1$, we have found N_k distinct eigenvalues in the interval $[\frac{k^2}{2}, 2k^2]$. This implies that

$$N(2k^2) \geq N_k \geq \frac{k^3}{6\gamma_0} - 1 \geq \frac{k^3}{12\gamma_0}, \quad \forall k \geq k_1.$$

But Weyl's formula (5.20) implies that there exists $k_2 > 0$ large enough such that

$$N(2k^2) \leq 2c|\Omega|(2k^2)^{\frac{3}{2}}, \quad \forall k \geq k_2.$$

These two estimates yield

$$\gamma_0 \geq \frac{1}{48\sqrt{2}c|\Omega|},$$

which contradicts (5.19). □

We now notice that (5.7) may hold for $\beta \leq \frac{1}{2}$, in particular it holds with $\beta = 0$ once the next gap condition

$$\exists g_0 > 0 : \lambda_{n+1} - \lambda_n \geq g_0, \quad \forall n \in \mathbb{N}^*, \tag{5.25}$$

holds.

Lemma 5.7. *Assume that* (5.25) *holds, then the assumption* (5.7) *is valid with* $\beta = 0$ *and* $\gamma_0 = g_0$.

Proof. If $\frac{k^2}{2}$ is different from $\lambda_{n(k,2)}$, then we take $s = 2$ and find

$$g_{n(k,2)} \geq g_0,$$

hence the result. On the contrary if $\frac{k^2}{2} = \lambda_{n(k,2)}$, then we choose $s = 2 - \varepsilon$ with $\varepsilon \in (0, 1)$ small enough such that

$$\frac{k^2}{2 - \varepsilon} < \lambda_{n(k,2)+1}.$$

Since $k^2 = 2\lambda_{n(k,2)}$, this means that we additionally require that

$$\varepsilon < 2\left(1 - \frac{\lambda_{n(k,2)}}{\lambda_{n(k,2)+1}}\right),$$

which is always possible since this right-hand side is positive. With this choice, we have that $n(k, s) = n(k, 2)$ and we conclude that $g_{n(k,s)} \geq g_0$. □

Corollary 5.8. *Assume that* $\Omega = (0, \sqrt{a_1})\times(0, \sqrt{a_2})\times(0, \sqrt{a_3})$, *with positive real numbers* a_i, $i = 1, 2, 3$ *such that* $\frac{a_i}{a_1}$ *is a rational number,* $i = 2, 3$. *Then the gap condition* (5.25) *holds with* $\beta = 0$, *and hence for an appropriate choice of* s, *the* k*-stability property with exponent* $\alpha = 1$ *holds.*

Proof. For such a cuboid, it is well known that the spectrum of the Laplace operator $-\Delta$ with Dirichlet boundary condition is given by

$$\pi^2\left(\frac{k_1^2}{a_1} + \frac{k_2^2}{a_2} + \frac{k_3^2}{a_3}\right),$$

for any $k_i \in \mathbb{N}^*$, $i = 1, 2, 3$. Hence writing $\frac{a_i}{a_1} = \frac{n_i}{d}$, with $n_i, d \in \mathbb{N}^*$, the spectrum is equivalently characterized by the set of

$$\frac{\pi^2}{a_1 n_2 n_3}(k_1^2 n_2 n_3 + k_2^2 n_1 n_3 + k_3^2 n_1 n_2),$$

for any $k_i \in \mathbb{N}^*$, $i = 1, 2, 3$. Since, in our situation, $k_1^2 n_2 n_3 + k_2^2 n_1 n_3 + k_3^2 n_1 n_2$ is a natural number, the spectrum is a subset of

$$g_0 \mathbb{N}^*,$$

where $g_0 = \frac{\pi^2}{a_1 n_2 n_3}$. Hence the distance between two consecutive different eigenvalues is at most larger than g_0. □

Remark 5.9. If the cuboid $\Omega = (0, \sqrt{a_1})\times(0, \sqrt{a_2})\times(0, \sqrt{a_3})$, with positive real numbers a_i, $i = 1, 2, 3$ such that $\frac{a_2}{a_1} = \frac{a_3}{a_1}$ is an irrational number badly approximable. Then by the same arguments than before and the use of Proposition 2.1 of [5], the gap condition (5.25) holds with $\beta = 1$, and hence for an appropriate choice of s, the k-stability property with exponent $\alpha = 3$ holds.

Corollary 5.10. *Assume that* Ω *is a prism in the form* $\Omega = T_a \times (0, \sqrt{h})$, *with positive real numbers* a *and* h *such that* $\frac{h}{a}$ *is a rational number and* T_a *is an equilateral triangle of side of length* $\sqrt{a}$. *Then the gap condition* (5.25) *holds with* $\beta = 0$, *and hence for an appropriate choice of* s, *the* k*-stability property with exponent* $\alpha = 1$ *holds.*

Proof. For such a prism, using a separation of variables, a scaling argument and Theorem 1 of [39] (see also Theorem 3.2 of [24], case of type A_2), we deduce that the spectrum of the Laplace operator $-\Delta$ with Dirichlet boundary condition is given by

$$\frac{16\pi^2}{27a}(k_1^2 + k_2^2 + k_1 k_2) + \frac{k_3^2 \pi^2}{h},$$

for any $k_3 \in \mathbb{N}^*$ and $k_1 \in \mathbb{Z}^*$, $k_2 \in \mathbb{Z}$ such that $k_1 + k_2 \neq 0$. Hence writing $\frac{h}{a} = \frac{n}{d}$ with $n, d \in \mathbb{N}^*$, the eigenvalues can be written as

$$\frac{\pi^2}{27an}((k_1^2 + k_2 + k_1 k_2)n + 27dk_3^2),$$

for the previous parameters k_i. As in the previous corollary, this means that the distance between two consecutive different eigenvalues is at most larger than $g_0 = \frac{\pi^2}{27an}$. □

Remark 5.11. By Theorem 3.2 of [24] (case of type C_2 or D_2, see also [4, Proposition 9]), Corollary 5.10 remains valid is T_a is an isosceles right triangle with two sides of length $\sqrt{a}$, with a positive number a.

Corollary 5.12. *Assume that* Ω *is a tetrahedron with vertices* $(0,0,0)$, $(\sqrt{a},0,0)$, $(\sqrt{a}/2,\sqrt{a}/2,-\sqrt{a}/2)$, $(\sqrt{a}/2,\sqrt{a}/2,\sqrt{a}/2)$, *with a positive number a. Then the gap condition* (5.25) *holds with* $\beta = 0$, *and hence for appropriate choice of s, the k-stability property with exponent* $\alpha = 1$ *holds.*

Proof. For such a tetrahedron, by a scaling argument and Theorem 3.2 of [24] (case of type $A_3 = D_3$, see also [4, Proposition 9]) we deduce that the spectrum of the Laplace operator $-\Delta$ with Dirichlet boundary condition is given by

$$\frac{4\pi^2}{a}\left(k_1^2 + \frac{3}{4}(k_2^2 + k_3^2) + k_1k_2 + k_1k_3 + \frac{1}{2}k_2k_3\right),$$

for any $k_i \in \mathbb{N}^*$, $i = 1,2,3$. This means that the distance between two consecutive different eigenvalues is at most larger than $g_0 = \frac{\pi^2}{a}$. □

Remark 5.13. By Theorem 3.2 of [24] (see also [4, Proposition 9]), Corollary 5.12 remains valid for a tetrahedron T_a with vertices $(0,0,0)$, $(\sqrt{a},0,0)$, $(\sqrt{a}/2,\sqrt{a}/2,0)$, $(\sqrt{a}/2,\sqrt{a}/2,\sqrt{a}/2)$ (case of type B_3) and for a tetrahedron T_a with vertices $(0,0,0)$, $(\sqrt{a}/2,0,0)$, $(\sqrt{a}/2,\sqrt{a}/2,0)$, $(\sqrt{a}/2,\sqrt{a}/2,\sqrt{a}/2)$ (case of type C_3), with a positive number a.

6 *h*-finite element approximations

For the sake of simplicity, we here perform some error analyses when $\lambda_{\text{imp}} = 1$, but for convex polyhedral domains and for which the stability estimate is valid. Before stating some convergence results for different finite element approximations, we state some regularity results and a priori bounds.

6.1 Some regularity results and a priori bounds

Theorem 6.1. *Assume that* $\lambda_{\text{imp}} = 1$, *and that* Ω *is a convex polyhedron and that the k-stability property with exponent* α *holds. Then for any* $\mathbf{f}_1,\mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, $\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)$ *belongs to* $\mathbf{H}^t(\Omega)^2$, *for all* $t < t_\Omega$ *with*

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)\|_{t,\Omega} \lesssim (1 + k^{1+\alpha})\|(\mathbf{f}_1,\mathbf{f}_2)\|_\Omega. \tag{6.1}$$

Proof. Since the regularity of $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ was already stated in Section 4, we only concentrate on the estimate (6.1). It indeed holds by looking at $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ as solution of (3.3) with $k = 0$ and a right-hand side defined by

$$\langle F, (\mathbf{E}', \mathbf{H}')\rangle = \int_{\Omega} ((\mathbf{f}_1 + k^2\mathbf{E}) \cdot \bar{\mathbf{E}}' + (\mathbf{f}_2 + k^2\mathbf{H}) \cdot \bar{\mathbf{H}}')\, dx + ik \int_{\partial\Omega} (\mathbf{E}_t \cdot \bar{\mathbf{E}}'_t + \mathbf{H}_t \cdot \bar{\mathbf{H}}'_t)\, d\sigma.$$

By elliptic regularity and the stability estimate (5.1), we obtain

$$\begin{aligned}\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)\|_{t,\Omega} &\lesssim \|(\mathbf{f}_1, \mathbf{f}_2)\|_{\Omega} + k^2\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)\|_{\Omega} + k\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)\|_{\frac{1}{2},\partial\Omega}\\ &\lesssim (1 + k^{1+\alpha})\|(\mathbf{f}_1, \mathbf{f}_2)\|_{\Omega},\end{aligned}$$

which proves (6.1). □

Now we show similar results in weighted Sobolev spaces (in the absence of edge singularities), namely for all $\ell \in \mathbb{N}$, $\ell \geq 2$, and all non-negative real numbers ν, if $r(\mathbf{x})$ is the distance from $\mathbf{x}$ to the corners of Ω, then we introduce the weighted space

$$H^{\ell,\nu}(\Omega) := \{v \in H^1(\Omega) : r^{\alpha} D^{\beta} v \in L^2(\Omega),\ \forall \beta \in \mathbb{N}^3 : 2 \leq |\beta| \leq \ell\},$$

which is a Hilbert space with its natural norm $\|\cdot\|_{\ell,\nu;\Omega}$.

Theorem 6.2. *In addition to the assumptions of Theorem* 6.1, *assume that* $\omega_e \leq \frac{\pi}{2}$, *for all edge e of* Ω *and that* $\lambda \neq \frac{1}{2}$, *for all* $\lambda \in \Lambda_c$ *and all corners c of* Ω. *Then for any* $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ *can be decomposed as follows:*

$$\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{E}_R, \mathbf{H}_R) + \sum_{c \in \mathcal{C}} \sum_{\lambda \in \Lambda_c \cap (-\frac{1}{2}, \frac{1}{2})} \kappa_{c,\lambda} r_c^{\lambda} (\varphi_{E,c,\lambda}(\vartheta_c), \varphi_{H,c,\lambda}(\vartheta_c)), \tag{6.2}$$

with $(\mathbf{E}_R, \mathbf{H}_R) \in \mathbf{H}^2(\Omega)^2$, $\mathcal{C}$ *is the set of corners of* Ω, (r_c, ϑ_c) *are the spherical coordinates centred at c*, $\kappa_{c,\lambda}$ *is a constant and* $\varphi_{E,c,\lambda}$, $\varphi_{H,c,\lambda}$ *belongs to* $\mathbf{H}^2(G_c)$. *Furthermore, we will have*

$$\|(\mathbf{E}_R, \mathbf{H}_R)\|_{2,\Omega} + \sum_{c \in \mathcal{C}} \sum_{\lambda \in \Lambda_c : 0 < \lambda < \frac{1}{2}} |\kappa_{c,\lambda}| \lesssim (1 + k^{1+\alpha})\|(\mathbf{f}_1, \mathbf{f}_2)\|_{\Omega}. \tag{6.3}$$

In particular it holds $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) \in H^{2,\nu}(\Omega)^6$, *for all* $\nu > 2 - t_\Omega$ *with*

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)\|_{2,\nu;\Omega} \lesssim (1 + k^{1+\alpha})\|(\mathbf{f}_1, \mathbf{f}_2)\|_{\Omega}. \tag{6.4}$$

Proof. Since there is no edge singular exponent in the interval $[0, 1]$, the results of Section 4 and of Section 8.2 of [28] (global regularity results in weighted Sobolev spaces for elliptic systems on domains with point singularities) allow to show that the splitting (6.2) and the estimate (6.3) hold. The regularity $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) \in H^{2,\nu}(\Omega)^6$, for all $\nu > 2 - t_\Omega$ and the estimate (6.4) directly follow from the fact that $r_c^{\lambda}(\varphi_{E,c,\lambda}(\vartheta_c), \varphi_{H,c,\lambda}(\vartheta_c))$ belongs to $H^{2,\nu}(\Omega)^6$, for all $\nu > 2 - t_\Omega$. □

Finally, still in the absence of edge singularities, we want to improve the previous result for a regular part almost in H^3, namely we prove the next result.

Theorem 6.3. *Under the assumptions of Theorem 6.2, for any $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ can be decomposed as follows:*

$$\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) = \mathbb{S}_{0,s}(\mathbf{f}_1, \mathbf{f}_2) + (\mathbf{R}_{E,\mathrm{reg}}, \mathbf{R}_{H,\mathrm{reg}}) + (\mathbf{R}_{E,\mathrm{sing}}, \mathbf{R}_{H,\mathrm{sing}}), \tag{6.5}$$

with $\mathbb{S}_{0,s}(\mathbf{f}_1, \mathbf{f}_2) \in H^{2,\nu}(\Omega)^6$, for any $\nu > 2 - t_\Omega$, satisfying

$$\|\mathbb{S}_{0,s}(\mathbf{f}_1, \mathbf{f}_2)\|_{2,\nu;\Omega} \lesssim \|(\mathbf{f}_1, \mathbf{f}_2)\|_\Omega, \tag{6.6}$$

$(\mathbf{R}_{E,\mathrm{reg}}, \mathbf{R}_{H,\mathrm{reg}}) \in \mathbf{H}^{3-\varepsilon}(\Omega)^2$ and $(\mathbf{R}_{E,\mathrm{sing}}, \mathbf{R}_{H,\mathrm{sing}}) \in H^{3,\nu_0}(\Omega)^6$ (for shortness their dependence in s is skipped), for any $\varepsilon > 0$ and any $\nu_0 > 3 - t_\Omega$, such that

$$\|(\mathbf{R}_{E,\mathrm{reg}}, \mathbf{R}_{H,\mathrm{reg}})\|_{3-\varepsilon,\Omega} + \|(\mathbf{R}_{E,\mathrm{sing}}, \mathbf{R}_{H,\mathrm{sing}})\|_{3,\nu_0;\Omega} \lesssim (1 + k^{2+\alpha})\|(\mathbf{f}_1, \mathbf{f}_2)\|_\Omega. \tag{6.7}$$

Proof. In a first step, we split up $(\mathbf{E}, \mathbf{H}) := \mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ (see [10] for a similar approach in domains with a smooth boundary) as follows:

$$\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) = \mathbb{S}_{0,s}(\mathbf{f}_1, \mathbf{f}_2) + (\mathbf{R}_E, \mathbf{R}_H), \tag{6.8}$$

where the remainder $(\mathbf{R}_E, \mathbf{R}_H) \in \mathbf{V}$ (for shortness it dependence in s is skipped) satisfies

$$\begin{aligned}\mathbf{a}_{0,s}((\mathbf{R}_E, \mathbf{R}_H), (\mathbf{E}', \mathbf{H}')) = k^2 &\int_\Omega (\mathbf{E} \cdot \bar{\mathbf{E}}' + \mathbf{H} \cdot \bar{\mathbf{H}}')\, dx \\ &- ik \int_{\partial\Omega} (\mathbf{E}_t \cdot \bar{\mathbf{E}}_t' + \mathbf{H}_t \cdot \bar{\mathbf{H}}_t')\, d\sigma, \quad \forall (\mathbf{E}', \mathbf{H}') \in \mathbf{V}.\end{aligned} \tag{6.9}$$

By Theorem 3.5, the existence and uniqueness of $\mathbb{S}_{0,s}(\mathbf{f}_1, \mathbf{f}_2)$ and of $(\mathbf{R}_E, \mathbf{R}_H)$ are guaranteed. Moreover from the estimate (6.4) (with $k = 0$), we see that $\mathbb{S}_{0,s}(\mathbf{f}_1, \mathbf{f}_2)$ belongs to $H^{2,\nu}(\Omega)^6$, for any $\nu > 2 - t_\Omega$ and that the estimate (6.6) holds. A similar result is valid for $(\mathbf{R}_E, \mathbf{R}_H)$, but we are interested in an improved regularity. More precisely, we want to show that

$$(\mathbf{R}_E, \mathbf{R}_H) = (\mathbf{R}_{E,\mathrm{reg}}, \mathbf{R}_{H,\mathrm{reg}}) + (\mathbf{R}_{E,\mathrm{sing}}, \mathbf{R}_{H,\mathrm{sing}}), \tag{6.10}$$

with $(\mathbf{R}_{E,\mathrm{reg}}, \mathbf{R}_{H,\mathrm{reg}})$ and $(\mathbf{R}_{E,\mathrm{sing}}, \mathbf{R}_{H,\mathrm{sing}})$ as stated in the Theorem. Indeed we first notice that the volumic term in the right-hand side of (6.9) has the appropriate regularity to obtain a decomposition of $(\mathbf{R}_E, \mathbf{R}_H)$ into a regular part in $\mathbf{H}^{3-\varepsilon}(\Omega)^2$ and a singular (corner) part. Unfortunately, this is not the case for the boundary term, because $(\mathbf{E}, \mathbf{H})$ is not in $\mathbf{H}^2(\Omega)^2$, but due to its splitting (6.2), we can use a lifting of the singular part.

More precisely by using Lemma 6.1.13 of [28], for all corners c, and all $\lambda \in \bigcap(-\frac{1}{2}, \frac{1}{2})$, there exists a field $(\mathbf{E}_{c,\lambda}, \mathbf{H}_{c,\lambda})$ in the form

$$(\mathbf{E}_{c,\lambda}, \mathbf{H}_{c,\lambda}) = r_c^{1+\lambda} \sum_{\ell=0}^{\kappa(\lambda)} \varphi_{c,\lambda,\ell}(\vartheta_c)(\ln r_c)^\ell,$$

with $\kappa(\lambda) \in \mathbb{N}$ and $\varphi_{c,\lambda,\ell} \in \mathbf{H}^{3-\varepsilon}(G_c)$ such that

$$\begin{cases} \left.\begin{array}{r} L_{k,s}(\mathbf{E}_{c,\lambda}) = \mathbf{0} \\ L_{k,s}(\mathbf{H}_{c,\lambda}) = \mathbf{0} \end{array}\right\} \quad \text{in } \Xi_c \\ \left.\begin{array}{r} \operatorname{div} \mathbf{E}_{c,\lambda} = 0 \\ \operatorname{div} \mathbf{H}_{c,\lambda} = 0 \\ T(\mathbf{E}_{c,\lambda}, \mathbf{H}_{c,\lambda}) = 0 \\ B_0(\mathbf{E}_{c,\lambda}, \mathbf{H}_{c,\lambda}) = 2\varphi_{E,c,\lambda,t} \end{array}\right\} \quad \text{on } \partial\Xi_c. \end{cases}$$

Hence for any corner c by fixing a smooth cut-off function η_c equal to 1 near c and equal to 0 near the other corners, we introduce

$$(\tilde{\mathbf{R}}_E, \tilde{\mathbf{R}}_H) = (\mathbf{R}_E, \mathbf{R}_H) - ik \sum_{c \in \mathcal{C}} \sum_{\lambda \in \Lambda_c \cap (-\frac{1}{2}, \frac{1}{2})} \kappa_{c,\lambda} \eta_c (\mathbf{E}_{c,\lambda}, \mathbf{H}_{c,\lambda}), \tag{6.11}$$

that still belongs to $\mathbf{V}$ and is solution of

$$\begin{aligned} \mathbf{a}_{0,s}((\tilde{\mathbf{R}}_E, \tilde{\mathbf{R}}_H), (\mathbf{E}', \mathbf{H}')) = k^2 \int_\Omega (\mathbf{E} \cdot \bar{\mathbf{E}}' + \mathbf{H} \cdot \bar{\mathbf{H}}') \, dx \\ - ikF(\mathbf{E}', \mathbf{H}'), \quad \forall (\mathbf{E}', \mathbf{H}') \in \mathbf{V}, \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} F(\mathbf{E}', \mathbf{H}') &= \int_{\partial\Omega} (\mathbf{E}_t \cdot \bar{\mathbf{E}}_t' + \mathbf{H}_t \cdot \bar{\mathbf{H}}_t') \, d\sigma - \sum_{c \in \mathcal{C}} \sum_{\lambda \in \Lambda_c \cap (-\frac{1}{2}, \frac{1}{2})} \kappa_{c,\lambda} \mathbf{a}_{0,s}(\eta_c(\mathbf{E}_c, \mathbf{H}_c), (\mathbf{E}', \mathbf{H}')) \\ &= \int_{\partial\Omega} (\mathbf{E}_{R,t} \cdot \bar{\mathbf{E}}_t' + \mathbf{H}_{R,t} \cdot \bar{\mathbf{H}}_t') \, d\sigma \\ &\quad + \sum_{c \in \mathcal{C}} \sum_{\lambda \in \Lambda_c \cap (-\frac{1}{2}, \frac{1}{2})} \kappa_{c,\lambda} \int_{\partial\Omega} r_c^\lambda (1 - \eta_c)(\varphi_{E,c,\lambda,t} \cdot \bar{\mathbf{E}}_t' + \varphi_{H,c,\lambda,t} \cdot \bar{\mathbf{H}}_t') \, d\sigma \\ &\quad - \sum_{c \in \mathcal{C}} \sum_{\lambda \in \Lambda_c \cap (-\frac{1}{2}, \frac{1}{2})} \kappa_{c,\lambda} \int_\Omega (L_{k,s}(\eta_c \mathbf{E}_{c,\lambda}) \cdot \bar{\mathbf{E}}' + L_{k,s}(\eta_c \mathbf{H}_{c,\lambda}) \cdot \bar{\mathbf{H}}') \, dx. \end{aligned}$$

Since $(1 - \eta_c)\varphi_{E,c,\lambda,t}$, $(1 - \eta_c)\varphi_{H,c,\lambda,t}$, $L_{k,s}(\eta_c \mathbf{E}_{c,\lambda})$, $L_{k,s}(\eta_c \mathbf{H}_{c,\lambda})$ are sufficiently regular, by the shift theorem, we deduce that $(\tilde{\mathbf{R}}_E, \tilde{\mathbf{R}}_H)$ admits a decomposition into a regular part in $\mathbf{H}^{3-\varepsilon}(\Omega)^2$ for any $\varepsilon > 0$ and a singular part that corresponds to corner singularities, namely

$$(\tilde{\mathbf{R}}_E, \tilde{\mathbf{R}}_H) = (\mathbf{R}_{E,\mathrm{reg}}, \mathbf{R}_{H,\mathrm{reg}}) + \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{3}{2}-\varepsilon)} \kappa'_{\lambda,c}\mathbf{S}_c^\lambda, \tag{6.13}$$

where $(\mathbf{R}_{E,\mathrm{reg}}, \mathbf{R}_{H,\mathrm{reg}}) \in \mathbf{H}^{3-\varepsilon}(\Omega)^2$, $\mathbf{S}_c^\lambda$ is the singular function associated with λ, and $\kappa'_{\lambda,c} \in \mathbb{C}$. Furthermore, we have the estimate

$$\begin{aligned}\|(\mathbf{R}_{E,\mathrm{reg}}, \mathbf{R}_{H,\mathrm{reg}})\|_{3-\varepsilon,\Omega} + \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{3}{2}-\varepsilon)} |\kappa'_{\lambda,c}| &\lesssim k^2\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)\|_\Omega \\ &+ k\|(\mathbf{E}_R,\mathbf{H}_R)\|_{2,\Omega} + k\sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{1}{2})} |\kappa_{c,\lambda}|.\end{aligned}$$

Hence by the stability estimate (5.1) and the estimate (6.3), we get

$$\|(\mathbf{R}_{E,\mathrm{reg}}, \mathbf{R}_{H,\mathrm{reg}})\|_{3-\varepsilon,\Omega} + \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{3}{2}-\varepsilon)} |\kappa'_{\lambda,c}| \lesssim (1+k^{2+\alpha})\|(\mathbf{f}_1,\mathbf{f}_2)\|_{0,\Omega}. \tag{6.14}$$

Coming back to the definition (6.11) of $(\tilde{\mathbf{R}}_E, \tilde{\mathbf{R}}_H)$ and using its splitting (6.13), we find the decomposition (6.10) of $(\mathbf{R}_E, \mathbf{R}_H)$ with

$$(\mathbf{R}_{E,\mathrm{sing}}, \mathbf{R}_{H,\mathrm{sing}}) = ik\sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{1}{2})} \kappa_{c,\lambda}\eta_c(\mathbf{E}_{c,\lambda},\mathbf{H}_{c,\lambda}) + \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{3}{2}-\varepsilon)} \kappa'_{\lambda,c}\mathbf{S}_c^\lambda,$$

that clearly belongs to $H^{3,\nu_0}(\Omega)^6$ for any $\nu_0 > 3 - t_\Omega$, with the estimate

$$\|(\mathbf{R}_{E,\mathrm{sing}}, \mathbf{R}_{H,\mathrm{sing}})\|_{3,nu_0;\Omega} \lesssim k \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{1}{2})} |\kappa_{c,\lambda}| + \sum_{c\in\mathcal{C}} \sum_{\lambda\in\Lambda_c\cap(-\frac{1}{2},\frac{3}{2}-\varepsilon)} |\kappa'_{\lambda,c}|.$$

Using the estimates (6.3) and (6.14), we conclude that (6.7) is valid. □

Obviously, the same regularity results are valid for the solution $(\mathbf{E}^*, \mathbf{H}^*) = \mathbb{S}^*_{k,s}(\mathbf{F},\mathbf{G})$ of the adjoint problem

$$\mathbf{a}_{k,s}((\mathbf{E}',\mathbf{H}'),(\mathbf{E}^*,\mathbf{H}^*)) = \int_\Omega (\bar{\mathbf{F}}\cdot\mathbf{E}' + \bar{\mathbf{G}}\cdot\mathbf{H}'), \quad \forall(\mathbf{E}',\mathbf{H}') \in \mathbf{V}. \tag{6.15}$$

Indeed as

$$\mathbf{a}_{k,s}((\mathbf{E}',\mathbf{H}'),(\mathbf{E}^*,\mathbf{H}^*)) = a_{k,s}(\bar{\mathbf{E}}^*,\bar{\mathbf{E}}') + a_{k,s}(\bar{\mathbf{H}}^*,\bar{\mathbf{H}}') + ik\int_{\partial\Omega}(\bar{\mathbf{E}}^*_t\cdot\mathbf{E}'_t + \bar{\mathbf{H}}^*_t\cdot\mathbf{H}'_t)\,d\sigma,$$

we deduce that

$$(\bar{\mathbf{E}}^*, \bar{\mathbf{H}}^*) = \mathbb{S}_{k,s}(\bar{\mathbf{F}},\bar{\mathbf{G}}).$$

6.2 Wavenumber explicit error analyses

With the above regularity results from Theorems 6.1 or 6.2 in hands, we can perform some error analyses following a standard approach (see [31, Chapter 8] and [32, Section 4]), the differences with these references are the loss of regularity and/or the use of refined meshes. The situation from Theorem 6.3 is different and uses similar ideas than in [10].

6.2.1 $\mathbb{P}_1$-elements with regular meshes

We start with the simplest case where we approximate $\mathbf{V}$ by a subspace made of piecewise polynomials of degree 1 on a regular (in the Ciarlet sense) mesh $\mathcal{T}_h$ of Ω made of tetrahedra, namely we take

$$\mathbf{V}_h := \mathbf{V} \cap \mathbb{P}_{1,h},$$

where

$$\mathbb{P}_{1,h} := \{(\mathbf{E}_h, \mathbf{H}_h) \in \mathbf{L}^2(\Omega)^2 : \mathbf{E}_{h|T}, \mathbf{H}_{h|T} \in (\mathbb{P}_1(T))^3, \ \forall T \in \mathcal{T}_h\}.$$

At this stage, a finite element approximation of $(\mathbf{E}, \mathbf{H}) = \mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) \in \mathbf{V}$ with $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$ consists in looking for $(\mathbf{E}_h, \mathbf{H}_h) = \mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2) \in \mathbf{V}_h$ solution of

$$\mathbf{a}_{k,s}((\mathbf{E}_h, \mathbf{H}_h); (\mathbf{E}', \mathbf{H}')) = \int_\Omega (\mathbf{f}_1 \cdot \bar{\mathbf{E}}'_h + \mathbf{f}_1 \cdot \bar{\mathbf{H}}'_h), \quad \forall (\mathbf{E}'_h, \mathbf{H}'_h) \in \mathbf{V}_h. \tag{6.16}$$

To analyse the existence of such a solution $\mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2)$ and the error between this approximated solution and $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$, according to a general principle (see, for instance, [32, 33] for the Helmholtz equation), we introduce the adjoint approximability

$$\eta(\mathbf{V}_h) = \sup_{(\mathbf{F},\mathbf{G}) \in \mathbf{L}^2(\Omega)^2 \setminus \{(\mathbf{0},\mathbf{0})\}} \inf_{(\mathbf{U}_h,\mathbf{V}_h) \in \mathbf{V}_h} \frac{\|\mathbb{S}^*_{k,s}(\mathbf{F}, \mathbf{G}) - (\mathbf{U}_h, \mathbf{V}_h)\|_k}{\|(\mathbf{F}, \mathbf{G})\|_\Omega}.$$

By Theorem 4.2 of [32] (that directly extends to our setting), the existence and uniqueness of a solution to (6.16) is guaranteed if $k\eta(\mathbf{V}_h)$ is small enough (stated precisely below).

To show such a result, we will use the standard Lagrange interpolant. Namely, for any $(\mathbf{E}, \mathbf{H}) \in \mathbf{H}^t(\Omega)^2$, with $t > \frac{3}{2}$, by the Sobolev embedding theorem, its Lagrange interpolant $I_h(\mathbf{E}, \mathbf{H})$ (defined as the unique element of $\mathbb{P}_{1,h}$ that coincides with $(\mathbf{E}, \mathbf{H})$ at the nodes of the triangulation) has a meaning. If furthermore $(\mathbf{E}, \mathbf{H})$ belongs to $\mathbf{V}$, then $I_h(\mathbf{E}, \mathbf{H})$ will be also in $\mathbf{V}$, hence in $\mathbf{V}_h$, since the normal vector is constant along the faces of Ω.

Recall that for any $t > \frac{3}{2}$, we also have the error estimate

$$\|(\mathbf{E}, \mathbf{H}) - I_h(\mathbf{E}, \mathbf{H})\|_{\ell,\Omega} \lesssim h^{t-\ell} \|(\mathbf{E}, \mathbf{H})\|_{t,\Omega}, \tag{6.17}$$

for $\ell = 0$ or 1, see [12, Theorem 3.2.1] in the case $t \in \mathbb{N}$ and easily extended to non-integer t.

These estimates directly allow to bound $\eta(\mathbf{V}_h)$.

Lemma 6.4. *In addition to the assumptions of Theorem* 6.1, *assume that* $t_\Omega > \frac{3}{2}$. *Then for all* $t \in (\frac{3}{2}, t_\Omega)$ *and all* $k \geq k_0$, *we have*

$$\eta(\mathbf{V}_h) \lesssim k^{1+\alpha} h^{t-1}(1 + kh). \tag{6.18}$$

Proof. Fix an arbitrary datum $(\mathbf{F}, \mathbf{G}) \in \mathbf{L}^2(\Omega)^2$ and denote $(\mathbf{E}^*, \mathbf{H}^*) = \mathbb{S}^*_{k,s}(\mathbf{F}, \mathbf{G})$. Then owing to (6.17), we have

$$\begin{aligned}\|(\mathbf{E}^*, \mathbf{H}^*) - I_h(\mathbf{E}^*, \mathbf{H}^*)\|_k &\lesssim k\|(\mathbf{E}^*, \mathbf{H}^*) - I_h(\mathbf{E}^*, \mathbf{H}^*)\|_{0,\Omega} + \|(\mathbf{E}^*, \mathbf{H}^*) - I_h(\mathbf{E}^*, \mathbf{H}^*)\|_{1,\Omega} \\ &\lesssim (kh^t + h^{t-1})\|(\mathbf{E}^*, \mathbf{H}^*)\|_{t,\Omega}.\end{aligned}$$

The estimate (6.1) allows to obtain the result. □

Corollary 6.5. *Under the assumptions of Lemma* 6.4, *for any fixed* $t \in (\frac{3}{2}, t_\Omega)$, *there exists* $C > 0$ (*small enough and depending only on* Ω *and* t) *such that if*

$$k^{\frac{2+\alpha}{t-1}} h \leq C, \tag{6.19}$$

then for all $k \geq k_0$ *and all* $\mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, *problem* (6.16) *has a unique solution* $\mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2)$ *and the following error estimate holds:*

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2)\|_k \lesssim k^{1+\alpha} h^{t-1}. \tag{6.20}$$

Proof. We first notice that the assumption (6.19) is equivalent to

$$k^{2+\alpha} h^{t-1} \leq C^{t-1}$$

and also implies that

$$kh \leq C,$$

since $t \leq 2$. As (6.18) means that there exists $C_0 > 0$ (independent of k, s, and h) such that

$$k\eta(\mathbf{V}_h) \leq C_0 k^{2+\alpha} h^{t-1}(1 + kh),$$

we deduce that

$$k\eta(\mathbf{V}_h) \leq C_0 k^{2+\alpha} h^{t-1}(1 + kh) \leq C_0 C^{t-1}(1 + C).$$

As mentioned before, the existence of $\mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2)$ then follows from Theorem 4.2 of [32] if

$$C_0 C^{t-1}(1 + C) \leq \frac{1}{4C_c},$$

where C_c is the continuity constant of $\mathbf{a}_{k,s}$ (that here is equal to $\max\{1, s_1\}$).

Now, we use the arguments from Theorem 4.2 of [32]. Namely, we notice that

$$\Re \mathbf{a}_{k,s}\big((\mathbf{U},\mathbf{W}),(\mathbf{U},\mathbf{W})\big) \geq \min\{1,s_0\}\|(\mathbf{U},\mathbf{W})\|_k^2 - 2k^2\big(\|\mathbf{U}\|_\Omega^2 + \|\mathbf{W}\|_\Omega^2\big),$$

where for shortness we write $(\mathbf{U},\mathbf{W}) = \mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)$. Therefore, by (6.15), one has

$$\begin{aligned}\Re \mathbf{a}_{k,s}\big((\mathbf{U},\mathbf{W}),(\mathbf{U},\mathbf{W}) + 2k^2\mathbb{S}_{k,s}^*(\mathbf{U},\mathbf{W})\big) &= \Re \mathbf{a}_{k,s}\big((\mathbf{U},\mathbf{W}),(\mathbf{U},\mathbf{W})\big)\\ &\quad + 2k^2\Re \mathbf{a}_{k,s}\big((\mathbf{U},\mathbf{W}),\mathbb{S}_{k,s}^*(\mathbf{U},\mathbf{W})\big)\\ &= \Re \mathbf{a}_{k,s}\big((\mathbf{U},\mathbf{W}),(\mathbf{U},\mathbf{W})\big) + 2k^2\big(\|\mathbf{U}\|_\Omega^2 + \|\mathbf{W}\|_\Omega^2\big),\end{aligned}$$

and by the previous estimate we deduce that

$$\min\{1,s_0\}\|(\mathbf{U},\mathbf{W})\|_k^2 \leq \Re \mathbf{a}_{k,s}\big((\mathbf{U},\mathbf{W}),(\mathbf{U},\mathbf{W}) + 2k^2\mathbb{S}_{k,s}^*(\mathbf{U},\mathbf{W})\big).$$

By Galerkin orthogonality, we can transform the right-hand side of this estimate as follows:

$$\begin{aligned}\Re \mathbf{a}_{k,s}\big((\mathbf{U},\mathbf{W}),(\mathbf{U},\mathbf{W}) + 2k^2\mathbb{S}_{k,s}^*(\mathbf{U},\mathbf{W})\big) &= \Re \mathbf{a}_{k,s}\big((\mathbf{U},\mathbf{W}),\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - (\mathbf{Y}_h,\mathbf{Z}_h)\big)\\ &\quad + 2k^2\Re \mathbf{a}_{k,s}\big((\mathbf{U},\mathbf{W}),\mathbb{S}_{k,s}^*(\mathbf{U},\mathbf{W}) - (\mathbf{U}_h,\mathbf{W}_h)\big),\end{aligned}$$

for any $(\mathbf{U}_h,\mathbf{W}_h),(\mathbf{Y}_h,\mathbf{Z}_h) \in \mathbf{V}_h$. By the continuity of the sesquilinear form a with respect to the norm $\|\cdot\|_k$, the previous estimate and identity yield

$$\|(\mathbf{U},\mathbf{W})\|_k^2 \lesssim \|(\mathbf{U},\mathbf{W})\|_k\big(\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - (\mathbf{Y}_h,\mathbf{Z}_h)\|_k + k^2\|\mathbb{S}_{k,s}^*(\mathbf{U},\mathbf{W}) - (\mathbf{U}_h,\mathbf{W}_h)\|_k\big).$$

As $(\mathbf{U}_h,\mathbf{W}_h)$ and $(\mathbf{Y}_h,\mathbf{Z}_h)$ are arbitrary in $\mathbf{V}_h$, by taking the infimum, we deduce that

$$\begin{aligned}\|(\mathbf{U},\mathbf{W})\|_k &\lesssim \inf_{(\mathbf{Y}_h,\mathbf{Z}_h)\in\mathbf{V}_h} \|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - (\mathbf{Y}_h,\mathbf{Z}_h)\|_k + k^2\eta(\mathbf{V}_h)\|(\mathbf{U},\mathbf{W})\|_\Omega\\ &\lesssim \inf_{(\mathbf{Y}_h,\mathbf{Z}_h)\in\mathbf{V}_h} \|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - (\mathbf{Y}_h,\mathbf{Z}_h)\|_k + k\eta(\mathbf{V}_h)\|(\mathbf{U},\mathbf{W})\|_k.\end{aligned}$$

Hence for $k\eta(\mathbf{V}_h)$ small enough, we deduce that

$$\|(\mathbf{U},\mathbf{W})\|_k \lesssim \inf_{(\mathbf{Y}_h,\mathbf{Z}_h)\in\mathbf{V}_h} \|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - (\mathbf{Y}_h,\mathbf{Z}_h)\|_k. \tag{6.21}$$

By the estimates (6.1) and (6.17), we conclude that

$$\|(\mathbf{U},\mathbf{W})\|_k \lesssim (kh^t + h^{t-1})k^{1+\alpha} = k^{1+\alpha}h^{t-1}(1+kh) \lesssim k^{1+\alpha}h^{t-1}. \qquad \square$$

Remark 6.6. The interest of considering non-divergence-free right-hand side in problem (3.5) appears in the definition of $\eta(\mathbf{V}_h)$ (and its estimate) and in the above proof. In both cases, the problem comes from the fact that even for divergence-free fields $\mathbf{f}_1$, $\mathbf{f}_2$, each component of $\mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)$ is not divergence-free. As a consequence,

$\mathbb{S}^*_{k,s}(\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)-\mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2))$ depends on s, but this plays no role in the estimate (6.20), except that s has to be fixed so that the stability estimate holds. Consequently, at least theoretically $\mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)$ has to be computed with such an s, even if $\mathbb{S}_{k,h}(\mathbf{f}_1,\mathbf{f}_2)$ is independent of s in case of divergence-free fields $\mathbf{f}_1$, $\mathbf{f}_2$, while practically (see below) it is fixed by comparing k^2 with the spectrum of the Laplace operator $-\Delta$ with Dirichlet boundary condition in Ω (or an approximation of it).

Remark 6.7. For the unit cuboid, as $\alpha = 1$ (see Corollary 5.8) and t can be as close as we want to $\frac{11}{6}$, the condition (6.19) is mostly $k^{\frac{18}{5}}h$ small enough.

Remark 6.8. Let us notice that the estimate (6.21) is valid under the above assumptions, but if $\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)$ belongs to $\mathbf{H}^{p+1}(\Omega)^2$ and polynomials of degree p will be used to define $\mathbf{V}_h$, then the rate of convergence in h in the estimate (6.20) will be improved, passing from h^{t-1} to h^p.

6.2.2 $\mathbb{P}_1$-elements with refined meshes

Here, we assume that the assumptions of Theorem 6.2 hold and want to take advantage of the regularity of $\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)$ in $H^{2,\nu}(\Omega)^6$, for any $\nu > 2-t_\Omega$ (see estimate (6.4)). More precisely following the arguments from [30, Theorem 3.3] (see also [2]) using a family of refined meshes $\mathcal{T}_h$ satisfying the refined rules

$$h_T \lesssim h \inf_{\mathbf{x}\in T} r(\mathbf{x})^\nu \quad \text{if } T \text{ is far away from the corners of } \Omega, \tag{6.22}$$

$$h_T \lesssim h^{\frac{1}{1-\nu}} \quad \text{if } T \text{ has a corners of } \Omega \text{ as vertex}, \tag{6.23}$$

with a fixed but arbitrary $\nu \in (2-t_\Omega, 1)$ (as close as we want from $2-t_\Omega$), we have that

$$\|(\mathbf{E},\mathbf{H}) - I_h(\mathbf{E},\mathbf{H})\|_{\ell,\Omega} \lesssim h^{2-\ell}\|(\mathbf{E},\mathbf{H})\|_{2,\nu;\Omega},$$

for $\ell = 0$ or 1. Consequently, as in the previous subsection, for $\mathbf{V}_h$ build on such meshes, there exists a positive constant C (independent of k, s and h) such that if

$$k^{2+\alpha}h \le C,$$

then for all $k \ge k_0$ and all $\mathbf{f}_1,\mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, problem (6.16) has a unique solution $\mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)$ and the following error estimate holds:

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)\|_k \lesssim k^{1+\alpha}h. \tag{6.24}$$

6.2.3 $\mathbb{P}_2$-elements with refined meshes

Under the assumptions of Theorem 6.2, we can improve the previous orders of convergence and reduce the constraint between k and h. For those purposes, we use the splitting (6.8) of $\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$ and the estimates (6.6) and (6.7) (recalling (6.10)). Then as in the previous subsection, we need to use a family of refined meshes $\mathcal{T}_h$ satisfying the refined rules:

$$h_T \lesssim h \inf_{\mathbf{x}\in T} r(\mathbf{x})^{\frac{\nu_0}{2}} \quad \text{if } T \text{ is far away from the corners of } \Omega, \tag{6.25}$$

$$h_T \lesssim h^{\frac{2}{2-\nu_0}} \quad \text{if } T \text{ has a corners of } \Omega \text{ as vertex}, \tag{6.26}$$

with a fixed but arbitrary $\nu_0 \in (3 - t_\Omega, 2)$. In such a situation, again by (6.17) and by [30, Theorem 3.3], we have

$$\|(\mathbf{R}_{E,\text{reg}}, \mathbf{R}_{H,\text{reg}}) - I_h(\mathbf{R}_{E,\text{reg}}, \mathbf{R}_{H,\text{reg}})\|_{\ell,\Omega} \lesssim h^{3-\varepsilon-\ell}\|(\mathbf{R}_{E,\text{reg}}, \mathbf{R}_{H,\text{reg}})\|_{3-\varepsilon,\Omega}, \tag{6.27}$$

$$\|(\mathbf{R}_{E,\text{sing}}, \mathbf{R}_{H,\text{sing}}) - I_h(\mathbf{R}_{E,\text{sing}}, \mathbf{R}_{H,\text{sing}})\|_{\ell,\Omega} \lesssim h^{3-\ell}\|(\mathbf{R}_{E,\text{sing}}, \mathbf{R}_{H,\text{sing}})\|_{3,\nu_0;\Omega}, \tag{6.28}$$

for $\ell = 0$ or 1.

Let us now show that (6.25) (resp., (6.26)) guarantees that (6.22) (resp., (6.23)) holds with $\nu = \nu_0 - 1$. In the first case, we simply notice that

$$r(\mathbf{x})^{\frac{\nu_0}{2}} = r(\mathbf{x})^{\frac{\nu+1}{2}}$$

and, therefore,

$$r(\mathbf{x})^{\frac{\nu+1}{2}} \lesssim r(\mathbf{x})^{\nu}$$

if and only if

$$r(\mathbf{x})^{\nu+1} \lesssim r(\mathbf{x})^{2\nu}.$$

This last estimate is valid for any $x \in T$ because ν belongs to $(0, 1)$ and $r(\mathbf{x})$ is bounded. The second implication is a simple consequence of the fact that

$$h^{\frac{2}{2-\nu_0}} = h^{\frac{2}{1-\nu}} \lesssim h^{\frac{1}{1-\nu}}.$$

Since our family of meshes then satisfies (6.22) and (6.23) with $\nu = \nu_0 - 1 > 2 - t_\Omega$, we deduce that

$$\|\mathbb{S}_0(\mathbf{f}_1, \mathbf{f}_2) - I_h\mathbb{S}_0(\mathbf{f}_1, \mathbf{f}_2)\|_{\ell,\Omega} \lesssim h^{2-\ell}\|\mathbb{S}_0(\mathbf{f}_1, \mathbf{f}_2)\|_{2,\nu;\Omega}, \tag{6.29}$$

for $\ell = 0$ or 1. With such estimates in hand, we can estimate the adjoint approximability.

Lemma 6.9. *For $\mathbf{V}_h$ build on meshes satisfying* (6.25) *and* (6.26), *we have*

$$\eta(\mathbf{V}_h) \lesssim (1 + kh)(h + k^3 h^{2-\varepsilon}). \tag{6.30}$$

Proof. Fix an arbitrary datum $(\mathbf{F},\mathbf{G}) \in \mathbf{L}^2(\Omega)^2$, we denote $(\mathbf{E}^*,\mathbf{H}^*) = \mathbb{S}^*_{k,s}(\mathbf{F},\mathbf{G})$. Then we use its splitting

$$(\mathbf{E}^*,\mathbf{H}^*) = \mathbb{S}^*_0(\mathbf{F},\mathbf{G}) + (\mathbf{R}^*_{E,\mathrm{reg}},\mathbf{R}^*_{H,\mathrm{reg}}) + (\mathbf{R}^*_{E,\mathrm{sing}},\mathbf{R}^*_{H,\mathrm{sing}}).$$

Owing to (6.27), (6.28) and (6.29), we have

$$\begin{aligned}\|(\mathbf{E}^*,\mathbf{H}^*) - I_h(\mathbf{E}^*,\mathbf{H}^*)\|_k \lesssim{}& (1+kh)h\|\mathbb{S}_0(\mathbf{f}_1,\mathbf{f}_2)\|_{2,\nu;\Omega}\\ &+ (1+kh)h^{2-\varepsilon}\|(\mathbf{R}^*_{E,\mathrm{reg}},\mathbf{R}^*_{H,\mathrm{reg}})\|_{3-\varepsilon,\Omega}\\ &+ (1+kh)h^2\|(\mathbf{R}_{E,\mathrm{sing}},\mathbf{R}_{H,\mathrm{sing}})\|_{3,\nu_0;\Omega}.\end{aligned}$$

The estimates (6.6) and (6.7) allow to obtain the result. □

Consequently, as in the previous subsection, for $\mathbf{V}_h$ build on such meshes, there exists a positive constant C (independent of k, s and h) such that if

$$k^4h^{2-\varepsilon} \le C,$$

then for all $k \ge k_0$ and all $\mathbf{f}_1,\mathbf{f}_2 \in \mathbf{L}^2(\Omega)$, problem (6.16) has a unique solution $\mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)$ with the error estimate

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)\|_k \lesssim k^3h^{2-\varepsilon}.$$

Remark 6.10. Note that the impedance boundary conditions are imposed as essential boundary conditions. As we are dealing with polyhedral domains, Lagrange elements can be used to construct conforming subspaces $\mathbf{V}_h$. The extension to curved domains seems to be difficult, but a penalisation technique can be used [42].

6.3 Some numerical tests

For the sake of simplicity, we restrict ourselves to the TE/TH polarization of the problem (3.7). In other words, we take

$$\Omega = D \times \mathbb{R},$$

where D is a two-dimensional polygon and assume that the solution of our problem is independent of the third variable. In such a case, the original problem splits up into a TE polarization problem in (E_1,E_2,H_3) in D, and a TH polarization one in (H_1,H_2,E_3) in D, whose variational formulations are fully similar to (3.3). Furthermore, the singularities of such problems correspond to the edge singularities of the original one.

We first use a toy experiment in the unit square $D = (0,1)^2$ to illustrate our results. In such a case, as exact solution, we take

$$E_1(x_1,x_2) = -\ell\pi\cos(\ell\pi x_1)\sin(\ell\pi x_2),$$

$$E_2(x_1, x_2) = \ell\pi \sin(\ell\pi x_1)\cos(\ell\pi x_2),$$
$$H_3(x_1, x_2) = \sin(\ell\pi x_1)\sin(\ell\pi x_2),$$

where $\ell \in \mathbb{N}^*$. With such a choice, we notice that (E_1, E_2) is divergence-free, that

$$\Delta E_1 + k^2 E_1 = \Delta E_1 + k^2 E_2 = \Delta H_3 + k^2 H_3 = 0,$$

with $k^2 = 2\ell^2\pi^2$ and that they satisfy the impedance boundary condition. We then compute the right-hand side of (3.3) accordingly (where only a boundary term occurs). In our numerical experiments, we have chosen either $\ell = 2, 5, 8, 10, 15$ or 29 and $s = 14.3$. This choice of s is made because it yields satisfactory numerical results, but it is also in accordance with the condition that $-\frac{k^2}{s}$ is different from the eigenvalues of the Laplace operator Δ with Dirichlet boundary conditions in D, which in this case means that

$$\frac{k^2}{s} \neq (\ell_1^2 + \ell_2^2)\pi^2, \tag{6.31}$$

for all positive integers ℓ_1, ℓ_2. Indeed, in the first case $\ell = 2$, the ratio $\frac{k^2}{s}$ is smaller than the smallest eigenvalue $2\pi^2$, while in the other cases, it is strictly between two eigenvalues.

In Figures 9.1–9.3, we have depicted the different orders of convergence for different values of h, k and $p = 1, 2$, and 4. From these figures, we see that if polynomials of order p are used, then in the asymptotic regime, the convergence rate is p for h small enough as theoretically expected, since the solution is smooth (see Remark 6.8).

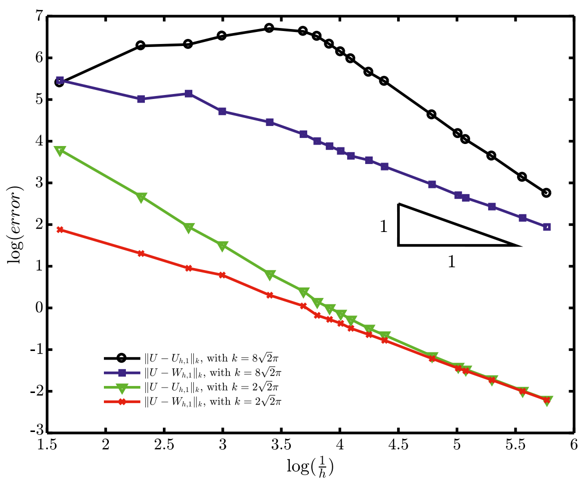


Figure 9.1: Rates of convergence for $p = 1$, $k = 2\sqrt{2}\pi$ or $8\sqrt{2}\pi$ ($U = \mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$, $U_{h,p} = \mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2)$, $W_{h,p} = \mathbb{P}_h\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)$).

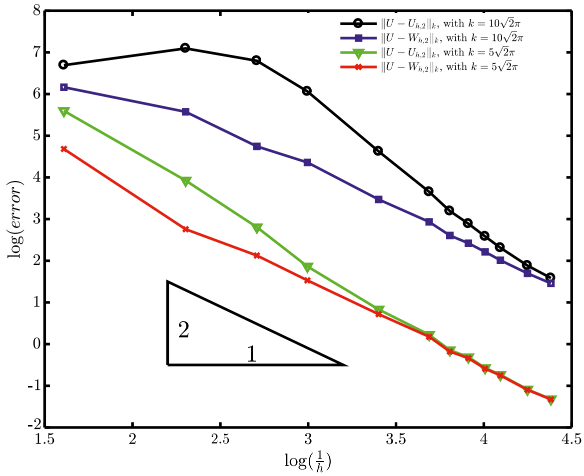


Figure 9.2: Rates of convergence for $p = 2$, $k = 5\sqrt{2}\pi$ or $10\sqrt{2}\pi$.

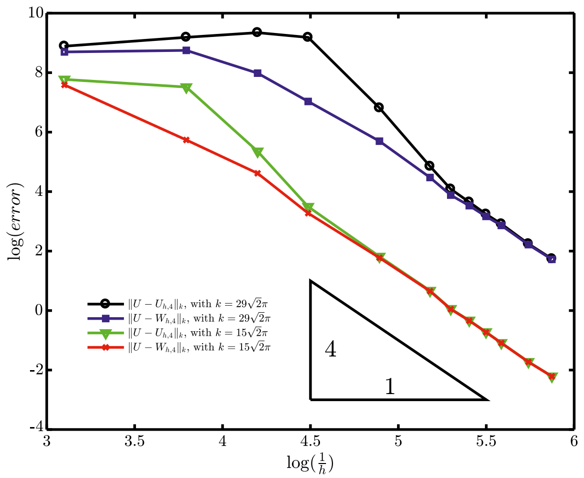


Figure 9.3: Rates of convergence for $p = 4$, $k = 15\sqrt{2}\pi$ or $29\sqrt{2}\pi$.

The second main result from Sections 6.2.2 and 6.2.3 states that if $k^{p+2}h^p \lesssim 1$ with $p = 1$ or 2 (up to ε for $p = 2$), then

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1, \mathbf{f}_2)\|_k \lesssim \|\mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2) - \mathbb{P}_h \mathbb{S}_{k,s}(\mathbf{f}_1, \mathbf{f}_2)\|_k, \tag{6.32}$$

where $\mathbb{P}_h$ is the orthogonal projection on $\mathbf{V}_h$ for the inner product associated with the norm $\|\cdot\|_k$, namely for $(\mathbf{U}, \mathbf{V}) \in \mathbf{V}$, $\mathbb{P}_h(\mathbf{U}, \mathbf{V})$ is the unique solution of

$$(\mathbb{P}_h(\mathbf{U}, \mathbf{V}), (\mathbf{U}'_h, \mathbf{V}'_h))_k = ((\mathbf{U}, \mathbf{V}), (\mathbf{U}'_h, \mathbf{V}'_h))_k, \quad \forall (\mathbf{U}'_h, \mathbf{V}'_h) \in \mathbf{V}_h,$$

where

$$((\mathbf{U},\mathbf{V}),(\mathbf{U}',\mathbf{V}'))_k = \int_\Omega (\operatorname{curl}\mathbf{U}\cdot\operatorname{curl}\bar{\mathbf{U}}' + s\operatorname{div}\mathbf{U}\operatorname{div}\bar{\mathbf{U}}' + k^2\mathbf{U}\cdot\bar{\mathbf{U}}')\,dx + \int_\Omega (\operatorname{curl}\mathbf{V}\cdot\operatorname{curl}\bar{\mathbf{V}}' + s\operatorname{div}\mathbf{V}\operatorname{div}\bar{\mathbf{V}}' + k^2\mathbf{V}\cdot\bar{\mathbf{V}}')\,dx.$$

In order to see if this bound is sharp or not, we compute $\mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)$ and $\mathbb{P}_h\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)$ for different values of h, p and k. For each k and p, we denote by $h^\star(k)$ the greatest value h_0 such that

$$\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)\|_k \le 2\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{P}_h\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)\|_k, \quad \forall h \le h_0. \tag{6.33}$$

The value of $h^\star(k)$ for a given k is obtained by inspecting the ratio

$$\frac{\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{S}_{k,s,h}(\mathbf{f}_1,\mathbf{f}_2)\|_k}{\|\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2) - \mathbb{P}_h\mathbb{S}_{k,s}(\mathbf{f}_1,\mathbf{f}_2)\|_k}.$$

Condition (6.33) states that the finite element solution must be quasi optimal in the $\|\cdot\|_k$ norm, uniformly in k (with the arbitrary constant 2).

The graph of $h^\star(k)$ is represented in Figure 9.4(a), 9.4(b) and 9.4(c) for $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_4$ elements, respectively. We observe that in both cases $h^\star(k) \sim k^{-1-1/p}$, which is better than the condition $k^{p+2}h^p \lesssim 1$ that would furnish $h^\star(k) \sim k^{-1-2/p}$. Indeed, it means that quasi-optimality in the sense of (6.33) is achieved under the condition that $h \le h^\star(k) \sim k^{-1-1/p}$, which is equivalent to $k^{p+1}h^p \le k^{p+1}[h^\star(k)]^p \lesssim 1$, that is better than $k^{p+2}h^p \lesssim 1$. We thus conclude that our stability condition seems to be not sharp and can probably be improved. Note that our experiments indicate that this stability condition remains valid for values of p larger than the theoretical one, that is, here equal to 2.

As a second example, we take on the square $(-1,1)^2$ the exact solution given by

$$E_1(x_1,x_2) = x_2 e^{ikx_1},$$
$$E_2(x_1,x_2) = -x_1 e^{ikx_1},$$
$$H_3(x_1,x_2) = \lambda_{\mathrm{imp}} e^{ikx_1},$$

that satisfies the homogeneous impedance boundary condition

$$H_3 - \lambda_{\mathrm{imp}}\mathbf{E_t} = 0 \quad \text{on } \partial D.$$

We have computed the numerical approximation of this solution for $k = 30$, the choice $s = 14.3$ (again with this choice, $\frac{k^2}{s}$ is smaller than the smallest eigenvalue $2\pi^2$), and for different values of λ_{imp}, namely we have chosen $\lambda_{\mathrm{imp}} = 1, 10, 50$ and 100. In Figure 9.5, we have depicted the different orders of convergence for $p = 1, 2$ and 4 and different values of h. Again since the solution is regular, the rate of convergence p is

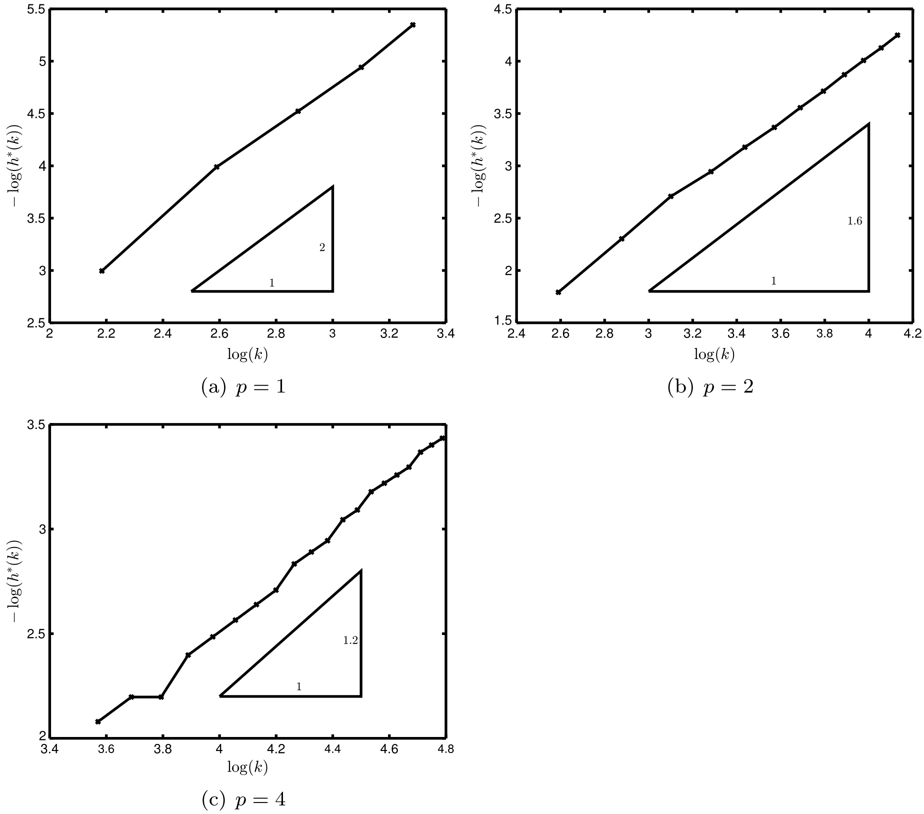


Figure 9.4: Asymptotic range of $h^*(k)$ for $p = 1, 2, 4$.

observed in the asymptotic regime and seems not to be affected by the variation of λ_{imp}.

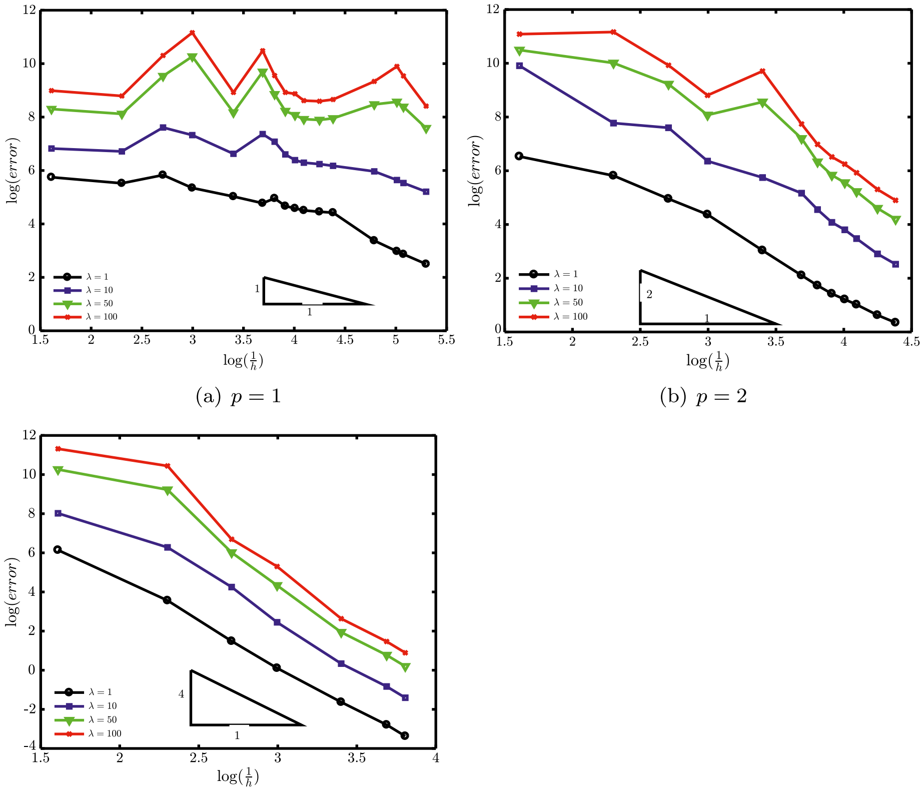
Finally, we have tested the case when a corner singularity appears. Namely, on the L-shaped domain $L = (-1,1)^2 \setminus ((0,1) \times (-1,0))$, we take as exact solution (written in polar coordinates (r,θ) centred at $(0,0)$)

$$\mathbf{E}(r,\theta) = \nabla\left(r^{\frac{4}{3}} \sin\left(\frac{4\theta}{3}\right) e^{ikr}\right),$$
$$H_3(r,\theta) = 0.$$

This solution exhibits the typical edge singularity of our Maxwell system described in Section 4.2.

This solution does not satisfies the homogeneous impedance boundary condition (with $\lambda_{\text{imp}} = 1$), hence we have imposed to our numerical solutions $(\mathbf{E}_h, H_{3h})$ to satisfy

$$H_{3h}(v) - \mathbf{E}_{h,\mathbf{t}}(v) = -\mathbf{E}_{\mathbf{t}}(v),$$



(a) $p = 1$ (b) $p = 2$

(c) $p = 4$

Figure 9.5: Rates of convergence for $\lambda_{\text{imp}} = 1, 10, 50, 100$ with $p = 1, 2, 4$.

at all nodes of the boundary of L. The convergence rates for $k = 1, 50$ and 100 are presented in Figures 9.6 and 9.7 for different values of h and p. There we observe, in the asymptotic regime, that for $k = 1$, the use of quasi-uniform meshes affects the rate of convergence since for $p = 1$ it is equal to $\frac{1}{3}$, while the use of refined meshes restores the optimal rate of convergence 1 (as theoretically expected). On the contrary for $k = 50$ or 100, we see, again in the asymptotic range, that the rate of convergence is p. This observation is in accordance with a recent result proved in [11] for Helmholtz problems in polygonal domains, which shows that in high frequency the dominant part of the solution is the regular part of the solution (which in our case is zero). Note that we have also chosen $s = 14.3$. Indeed for $k = 1$, the spectral condition on $\frac{k^2}{s}$ holds since the smallest eigenvalue of the Laplace operator with Dirichlet boundary conditions in L is approximatively equal to 9.6387; see [18, 41]. We are not able to check if the spectral condition is valid for $k = 50$ or 100 since the approximated values of the eigenvalues of the Laplace operator with Dirichlet boundary conditions in L seem to be only available up to 97 (see [41, Table 1]), but since our numerical results are satisfactory, we suppose that it is satisfied.

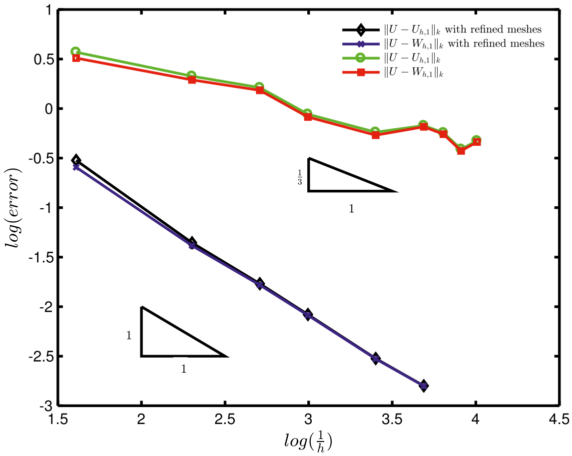
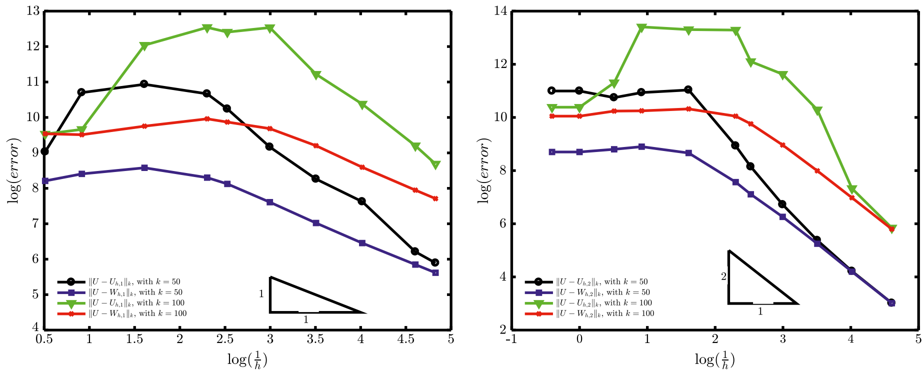


Figure 9.6: Rates of convergence for the singular solution in the *L*-shaped domain for $k = 1$ with uniform and refined meshes for $p = 1$.



(a) $p = 1$ (b) $p = 2$

Figure 9.7: Rates of convergence for the singular solution in the *L*-shaped domain for $k = 50$ or 100 with $p = 1$ (left) and $p = 2$ (right).

Note that our numerical tests are performed with the help of XLife++, a FEM library developed in C++ by P. O. E. M. S. (Ensta) and I. R. M. A. R. (Rennes) laboratories.

Bibliography

[1] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault, Vector potentials in three-dimensional nonsmooth domains, *Math. Methods Appl. Sci.*, **21** (1998), 823–864.

[2] T. Apel, A. Sändig, and J. Whiteman, Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non-smooth domains, *Math. Methods Appl. Sci.*, **19** (1996), 63–85.

[3] H. Barucq and B. Hanouzet, Etude asymptotique du système de Maxwell avec la condition aux limites absorbante de Silver-Müller II, *C. R. Acad. Sci. Paris Sér. I*, **316** (1993), 1019–1024.
[4] P. H. Bérard, Spectres et groupes cristallographiques. I. Domaines euclidiens, *Invent. Math.*, **58** (1980), 179–199.
[5] V. Blomer, J. Bourgain, M. Radziwiłł, and Z. Rudnick, Small gaps in the spectrum of the rectangular billard, *Ann. Sci. Éc. Norm. Supér. (4)*, **50** (2017), 1283–1300.
[6] A. Buffa and P. Ciarlet Jr., On traces for functional spaces related to Maxwell's equations. I. An integration by parts formula in Lipschitz polyhedra, *Math. Methods Appl. Sci.*, **24**(1) (2001), 9–30.
[7] A. Buffa and P. Ciarlet Jr., On traces for functional spaces related to Maxwell's equations. II. Hodge decompositions on the boundary of Lipschitz polyhedra and applications, *Math. Methods Appl. Sci.*, **24**(1) (2001), 31–48.
[8] A. Buffa, M. Costabel, and C. Schwab, Boundary element methods for Maxwell's equations on non-smooth domains, *Numer. Math.*, **92**(4) (2002), 679–710.
[9] A. Buffa, M. Costabel, and D. Sheen, On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains, *J. Math. Anal. Appl.*, **276**(2) (2002), 845–867.
[10] T. Chaumont-Frelet and S. Nicaise, *Wavenumber Explicit Convergence Analysis for Finite Element Discretizations of General Wave Propagation Problems*, Technical report, Lamav, Univ. Valenciennes, 2017.
[11] T. Chaumont-Frelet and S. Nicaise, High frequency behaviour of corner singularities in Helmholtz problems, *ESAIM Math. Model. Numer. Anal.*, **52**(5) (2018), 1803–1845.
[12] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
[13] M. Costabel and M. Dauge, Singularities of electromagnetic fields in polyhedral domains, *Arch. Ration. Mech. Anal.*, **151** (2000), 221–276.
[14] M. Costabel, M. Dauge, and S. Nicaise, *Corner Singularities and Analytic Regularity for Linear Elliptic Systems. Part I: Smooth Domains*, http://hal.archives-ouvertes.fr/hal-00453934/en/ Online version of Chapters 1 to 5 (2010).
[15] P. Cummings and X. Feng, Sharp regularity coefficient estimates for complex-valued acoustic and elastic Helmholtz equations, *Math. Models Methods Appl. Sci.*, **16**(1) (2006), 139–160.
[16] M. Dauge, *Elliptic Boundary Value Problems on Corner Domains – Smoothness and Asymptotics of Solutions*, Lecture Notes in Mathematics, vol. 1341, Springer, Berlin, 1988.
[17] S. Esterhazy and J. Melenk, On stability of discretizations of the Helmholtz equation, in *Lecture Notes in Computational Science and Engineering*, pp. 285–324, Springer-Verlag Berlin Heidelberg, 2012.
[18] L. Fox, P. Henrici, and C. Moler, Approximations and bounds for eigenvalues of elliptic operators, *SIAM J. Numer. Anal.*, **4** (1967), 89–102.
[19] G. N. Gatica and S. Meddahi, Finite element analysis of a time harmonic Maxwell problem with an impedance boundary condition, *IMA J. Numer. Anal.*, **32**(2) (2012), 534–552.
[20] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier–Stokes Equations, Theory and Algorithms*, Springer Series in Computational Mathematics, vol. 5, Springer, Berlin, 1986.
[21] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, London, Melbourne, 1985.
[22] U. Hetmaniuk, Stability estimates for a class of Helmholtz problems, *Commun. Math. Sci.*, **5**(3) (2007), 665–678.
[23] R. Hiptmair, A. Moiola, and I. Perugia, Stability results for the time-harmonic Maxwell equations with impedance boundary conditions, *Math. Models Methods Appl. Sci.*, **21**(11) (2011), 2263–2287.
[24] Y. Hoshikawa and H. Urakawa, Affine Weyl groups and the boundary value eigenvalue problems of the Laplacian, *Interdiscip. Inf. Sci.*, **16**(1) (2010), 93–109.

[25] F. L. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. Differ. Equ.*, **1**(1) (1985), 43–56.
[26] F. Ihlenburg and I. Babuška, Finite element solution of the Helmholtz equation with high wave number part I: the h-version of the fem, *Comput. Math. Appl.*, **30**(9) (1995), 9–37.
[27] F. Ihlenburg and I. Babuška, Finite element solution of the Helmholtz equation with high wave number part II: the h-p version of the fem, *SIAM J. Numer. Anal.*, **34**(1) (1997), 315–358.
[28] V. A. Kozlov, V. G. Maz'ya, and J. Rossmann, *Elliptic Boundary Value Problems in Domains with Point Singularities*, Mathematical Surveys and Monographs, vol. 52, American Mathematical Society, Providence, RI, 1997.
[29] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications. Vol. 1*, Travaux et Recherches Mathématiques, vol. 17, Dunod, Paris, 1968.
[30] J. M.-S. Lubuma and S. Nicaise, Dirichlet problems in polyhedral domains II: approximation by FEM and BEM, *J. Comput. Appl. Math.*, **61** (1995), 13–27.
[31] J. Melenk, *On Generalized Finite Element Methods*, PhD thesis, University of Maryland, 1995.
[32] J. Melenk and S. Sauter, Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions, *Math. Comput.*, **79**(272) (2010), 1871–1914.
[33] J. Melenk and S. Sauter, Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation, *SIAM J. Numer. Anal.*, **49**(3) (2011), 1210–1243.
[34] P. Monk, *Finite Element Methods for Maxwell's Equations*, Numerical Mathematics and Scientific Computation, Oxford University Press, 2003.
[35] J.-C. Nédélec, *Acoustic and Electromagnetic Equations*, Applied Mathematical Sciences, vol. 144, Springer-Verlag, New York, 2001.
[36] S. Nicaise, *Polygonal Interface Problems*, Methoden und Verfahren der mathematischen Physik, vol. 39, Peter Lang GmbH, Europäischer Verlag der Wissenschaften, Frankfurt/M., 1993.
[37] S. Nicaise and C. Pignotti, Boundary stabilization of Maxwell's equations with space-time variable coefficients, *ESAIM Control Optim. Calc. Var.*, **9** (2003), 563–578, electronic.
[38] L. Paquet, Problèmes mixtes pour le système de Maxwell, *Ann. Fac. Sci. Toulouse, Math.*, **4** (1982), 103–141.
[39] M. A. Pinsky, The eigenvalues of an equilateral triangle, *SIAM J. Math. Anal.*, **11**(5) (1980), 819–827.
[40] J. Prüss, On the spectrum of C_0-semigroups, *Trans. Am. Math. Soc.*, **284**(2) (1984), 847–857.
[41] G. Still, Approximation theory methods for solving elliptic eigenvalue problems, *Z. Angew. Math. Mech.*, **83**(7) (2003), 468–478.
[42] J. Tomezyk, *Méthodes d'éléments finis pour des problèmes aux limites non définis positifs*, PhD thesis, University of Valenciennes, 2019, in preparation.
[43] V. A. Toponogov, *Differential Geometry of Curves and Surfaces*, Birkhäuser Boston, Inc., Boston, MA, 2006. A concise guide, with the editorial assistance of Vladimir Y. Rovenski.

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10 Time-harmonic electro-magnetic scattering in exterior weak Lipschitz domains with mixed boundary conditions

Abstract: This paper treats the time-harmonic electro-magnetic scattering or radiation problem governed by Maxwell's equations, i. e.,

$$\begin{aligned} -\operatorname{rot} H + i\omega\varepsilon E &= F \quad \text{in } \Omega, & E \times \nu &= 0 \quad \text{on } \Gamma_1, \\ \operatorname{rot} E + i\omega\mu H &= G \quad \text{in } \Omega, & H \times \nu &= 0 \quad \text{on } \Gamma_2, \end{aligned}$$

where $\omega \in \mathbb{C} \setminus (0)$ and $\Omega \subset \mathbb{R}^3$ is an exterior weak Lipschitz domain with boundary Γ divided into two disjoint parts Γ_1 and Γ_2. We will present a solution theory using the framework of polynomially weighted Sobolev spaces for the rotation and divergence. For the physically interesting case $\omega \in \mathbb{R} \setminus (0)$, we will show a Fredholm alternative type result to hold using the principle of limiting absorption introduced by Eidus in the 1960s. The necessary a priori estimate and polynomial decay of eigenfunctions for the Maxwell equations will be obtained by transferring well-known results for the Helmholtz equation using a suitable decomposition of the fields E and H. The crucial point for existence is a local version of Weck's selection theorem, also called Maxwell compactness property.

Keywords: Maxwell equations, radiating solutions, exterior boundary value problems, polynomial decay, mixed boundary conditions, weighted Sobolev spaces, Hodge–Helmholtz decompositions

MSC 2010: 35Q60, 78A25, 78A30

1 Introduction

The equations that describe the behavior of electro-magnetic vector fields in some space-time domain $I \times \Omega \subset \mathbb{R} \times \mathbb{R}^3$, first completely formulated by J. C. Maxwell in 1864, are

$$\begin{aligned} -\operatorname{rot} H + \partial_t D &= J, & \operatorname{rot} E + \partial_t B &= 0, & &\text{in } I \times \Omega, \\ \operatorname{div} D &= \rho, & \operatorname{div} B &= 0, & &\text{in } I \times \Omega, \end{aligned}$$

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where E, H are the electric, respectively, magnetic field, D, B represent the displacement current and magnetic induction and J, ρ describe the current density, respectively, the charge density. Excluding, e. g., ferromagnetic, respectively, ferroelectric materials, the parameters linking E and H with D and B are often assumed to be of the linear form $D = \varepsilon E$ and $B = \mu H$, where ε and μ are matrix-valued functions describing the permittivity and permeability of the medium filling Ω. Here, we are especially interested in the case of an exterior domain $\Omega \subset \mathbb{R}^3$, i. e., a connected open subset with compact complement. Applying the divergence to the first two equations, we see that the latter two equations are implicitly included in the first two and may be omitted. Hence, neglecting the static case, Maxwell's equations reduce to

$$-\operatorname{rot} H + \partial_t(\varepsilon E) = F, \quad \operatorname{rot} E + \partial_t(\mu H) = G, \quad \text{in } I \times \Omega,$$

with arbitrary right-hand sides F, G. Among the wide range of phenomena described by these equations one important case is the discussion of *"time-harmonic"* electromagnetic fields where all fields vary sinusoidally in time with frequency $\omega \in \mathbb{C} \setminus (0)$, i. e.,

$$E(t,x) = e^{i\omega t}E(x), \quad H(t,x) = e^{i\omega t}H(x), \quad G(t,x) = e^{i\omega t}G(x), \quad F(t,x) = e^{i\omega t}F(x)\,.$$

Substituting this ansatz into the equations (or using Fourier transformation in time) and assuming that ε and μ are time-independent we are lead to what is called *"time-harmonic Maxwell's equations"*:

$$\operatorname{rot} E + i\omega\mu H = G, \quad -\operatorname{rot} H + i\omega\varepsilon E = F, \quad \text{in } \Omega\,. \tag{1.1}$$

This system equipped with suitable boundary conditions describes, e. g., the scattering of time-harmonic electro-magnetic waves which is of high interest in many applications like geophysics, medicine, electrical engineering, biology and many others.

First existence results concerning boundary value problems for the time-harmonic Maxwell system in bounded and exterior domains have been given by Müller [13, 12]. He studied isotropic and homogeneous media and used integral equation methods. Using alternating differential forms, Weyl [29] investigated these equations on Riemannian manifolds of arbitrary dimension, while Werner [28] was able to transfer Müller's results to the case of inhomogeneous but isotropic media. However, for general inhomogeneous anisotropic media and arbitrary exterior domains, boundary integral methods are less useful since they heavily depend on the explicit knowledge of the fundamental solution and strong assumptions on boundary regularity. That is why Hilbert space methods are a promising alternative. Unfortunately, Maxwell's equations are nonelliptic, hence it is in general not possible to estimate all first derivatives of a solution. In [9], Leis could overcome this problem by transforming the boundary value problem for Maxwell's system into a boundary value problem for the Helmholtz equation, assuming that the medium filling Ω, is inhomogeneous and anisotropic within a bounded subset of Ω. Nevertheless, he still needed boundary regularity to gain equivalence of both problems. But also for nonsmooth boundaries Hilbert space methods are expedient. In fact, as shown by Leis [10], it is sufficient that Ω satisfies a certain selection theorem, later called *Weck's selection theorem* or *Maxwell compactness property*, which holds for a class of boundaries much larger than those accessible by the detour over H^1 (cf. Weck [24], Costabel [2] and Picard, Weck, Witsch [20]). See [11] for a detailed monograph and [1] for the most recent result and an overview. The most recent result regarding a solution theory is due to Pauly [16] (see also [14]) and in its structure comparable to the results of Picard [18] and Picard, Weck and Witsch [20]. While all these results above have been obtained for full boundary conditions, in the present paper we study the case of mixed boundary conditions. More precisely, we are interested in solving the system (1.1) for $\omega \in \mathbb{C} \setminus (0)$ in an exterior domain $\Omega \subset \mathbb{R}^3$, where we assume that $\Gamma := \partial\Omega$ is decomposed into two relatively open subsets Γ_1 and its complement $\Gamma_2 := \Gamma \setminus \overline{\Gamma}_1$ and impose homogeneous boundary conditions, which in classical terms can be written as

$$\nu \times E = 0 \text{ on } \Gamma_1, \qquad \nu \times H = 0 \text{ on } \Gamma_2, \qquad (\nu : \text{ outward unit normal}). \tag{1.2}$$

Conveniently, we can apply the same methods as in [15] (see also Picard, Weck and Witsch [20], Weck and Witsch [27, 25]) to construct a solution. Indeed, most of the proofs carry over practically verbatim. For $\omega \in \mathbb{C} \setminus \mathbb{R}$, the solution theory is obtained by standard Hilbert space methods as ω belongs to the resolvent set of the Maxwell operator. In the case of $\omega \in \mathbb{R} \setminus (0)$, i. e., ω is in the continuous spectrum of the Maxwell operator, we use the limiting absorption principle introduced by Eidus [4] and approximate solutions to $\omega \in \mathbb{R} \setminus (0)$ by solutions corresponding to $\omega \in \mathbb{C} \setminus \mathbb{R}$. This will be sufficient to show a generalized Fredholm alternative (cf. our main result, Theorem 3.10) to hold. The essential ingredients needed for the limit process are

- the polynomial decay of eigensolutions;
- an a priori estimate for solutions corresponding to nonreal frequencies;
- a Helmholtz-type decomposition;
- and *Weck's local selection theorem* (*WLST*), that is,

$$\mathbf{R}_{\Gamma_1}(\Omega) \cap \varepsilon^{-1}\mathbf{D}_{\Gamma_2}(\Omega) \longrightarrow \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega}) \text{ is compact.}$$

While the first two are obtained by transferring well-known results for the scalar Helmholtz equation to the time-harmonic Maxwell equations using a suitable decomposition of the fields E and H, Lemma 4.1, the last one is an assumption on the quality of the boundary. As we will see, WLST is an immediate consequence of *Weck's selection theorem* (*WST*), i. e.,

$$\mathbf{R}_{\Gamma_1}(\Theta) \cap \varepsilon^{-1}\mathbf{D}_{\Gamma_2}(\Theta) \longrightarrow \mathsf{L}^2(\Theta) \text{ is compact,}$$

which holds in bounded weak Lipschitz domains $\Theta \subset \mathbb{R}^3$, but fails in unbounded such as exterior domains (cf. Bauer, Pauly, Schomburg [1] and the references therein). For strong Lipschitz-domains, see Jochmann [7] and Fernandes, Gilardis [5].

2 Preliminaries and notation

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the usual sets of integers, natural, real and complex numbers, respectively. Furthermore, let i be the imaginary unit, $\operatorname{Re} z$, $\operatorname{Im} z$ and $\overline{z}$ real part, imaginary part and complex conjugate of $z \in \mathbb{C}$, as well as

$$\mathbb{R}_+ := \{s \in \mathbb{R} \,|\, s > 0\}, \quad \mathbb{C}_+ := \{z \in \mathbb{C} \,|\, \operatorname{Im} z \geq 0\}, \quad \mathbb{I} := \left\{(2m+1)/2 \,\middle|\, m \in \mathbb{Z} \setminus (0)\right\}.$$

For $x \in \mathbb{R}^n$ with $x \neq 0$ we set $r(x) := |x|$ and $\xi(x) := x/|x|$ ($|\cdot|$:Euclidean norm in $\mathbb{R}^n$). Moreover, $\mathrm{U}(\tilde{r})$, respectively, $\mathrm{B}(\tilde{r})$ indicate the open, respectively, closed ball of radius $\tilde{r}$ in $\mathbb{R}^n$ centered in the origin and we define

$$\mathrm{S}(\tilde{r}) := \mathrm{B}(\tilde{r}) \setminus \mathrm{U}(\tilde{r}), \quad \breve{\mathrm{U}}(\tilde{r}) := \mathbb{R}^3 \setminus \mathrm{B}(\tilde{r}), \quad \mathrm{G}(\tilde{r}, \hat{r}) := \breve{\mathrm{U}}(\tilde{r}) \cap \mathrm{U}(\hat{r})$$

with $\hat{r} > \tilde{r}$. If $f : \mathrm{X} \longrightarrow \mathrm{Y}$ is a function mapping X to Y the restriction of f to a subset $\mathrm{U} \subset \mathrm{X}$ will be marked with $f|_U$ and $\mathcal{D}(f)$, $\mathcal{N}(f)$, $\mathcal{R}(f)$, and $\operatorname{supp} f$ denote domain of definition, kernel, range, and support of f, respectively. For Banach or Hilbert spaces X and Y we denote by $\mathrm{L}(\mathsf{X}, \mathsf{Y})$ and $\mathrm{B}(\mathsf{X}, \mathsf{Y})$ the sets of linear respectively bounded linear operators mapping X to Y. For X, Y subspaces of a normed vector space V, $\mathrm{X} + \mathrm{Y}$, $\mathrm{X} \dotplus \mathrm{Y}$, and $\mathrm{X} \oplus \mathrm{Y}$ indicate the sum, the direct sum, and the orthogonal sum of X and Y, where in the last case we presume the existence of a scalar product $\langle \cdot, \cdot \rangle_{\mathrm{V}}$ on V. Moreover, $\langle \cdot, \cdot \rangle_{\mathrm{X}\times\mathrm{Y}}$, respectively, $\|\cdot\|_{\mathrm{X}\times\mathrm{Y}}$ denote the natural scalar product resp. induced norm on $\mathrm{X} \times \mathrm{Y}$. If $\mathrm{X} = \mathrm{Y}$, we often simply use the index X instead of $\mathrm{X} \times \mathrm{X}$.

2.1 General assumptions and weighted Sobolev spaces

Unless stated otherwise, from now on and throughout this paper, it is assumed that $\Omega \subset \mathbb{R}^3$ is an exterior weak Lipschitz domain with weak Lipschitz interface in the sense of [1, Definition 2.3, Definition 2.5], which in principle means that $\Gamma = \partial\Omega$ is a Lipschitz-manifold and Γ_1 respectively Γ_2 are Lipschitz-submanifolds of Γ. For later purposes, we fix $r_0 > 0$ such that $\mathbb{R}^3 \setminus \Omega \Subset \mathrm{U}(r_0)$ and define for arbitrary $\tilde{r} \geq r_0$,

$$\Omega(\tilde{r}) := \Omega \cap \mathrm{U}(\tilde{r}).$$

With $r_k := 2^k r_0$, $k \in \mathbb{N}$ and $\tilde{\eta} \in \mathsf{C}^\infty(\mathbb{R})$ such that

$$0 \leq \tilde{\eta} \leq 1, \quad \operatorname{supp} \tilde{\eta} \subset (-\infty, 2-\delta), \quad \tilde{\eta}|_{(-\infty,1+\delta)} = 1, \tag{2.1}$$

for some $0 < \delta < 1$, we define functions $\eta, \check{\eta}, \eta_k, \check{\eta}_k \in \mathsf{C}^\infty(\mathbb{R}^3)$ by

$$\eta(x) := \tilde{\eta}(r(x)/r_0), \quad \check{\eta}(x) := 1 - \eta(x), \quad \eta_k(x) := \tilde{\eta}\big(r(x)/r_k\big), \quad \text{respectively} \quad \check{\eta}_k(x) := 1 - \eta_k(x),$$

meaning

$$\begin{aligned} &\operatorname{supp}\eta \subset \mathrm{B}(r_1) \quad \text{with } \eta = 1 \text{ on } \mathrm{U}(r_0), && &&\operatorname{supp}\eta_k \subset \mathrm{U}(r_{k+1}) \quad \text{with} \quad \eta_k = 1 \text{ on } \mathrm{U}(r_k),\\ &&&\text{respectively}&&\\ &\operatorname{supp}\check{\eta} \subset \check{\mathrm{U}}(r_0) \quad \text{with } \check{\eta} = 1 \text{ on } \check{\mathrm{U}}(r_1), && &&\operatorname{supp}\check{\eta}_k \subset \check{\mathrm{U}}(r_k) \quad \text{with} \quad \check{\eta}_k = 1 \text{ on } \check{\mathrm{U}}(r_{k+1}).\end{aligned}$$

These functions will later be utilized for particular cut-off procedures.

Next, we introduce our notation for Lebesgue and Sobolev spaces needed in the following discussion. Note that we will not indicate whether the elements of these spaces are scalar functions or vector fields. This will be always clear from the context. The example[1]

$$\begin{aligned} E &:= \nabla \ln(r) \in \mathsf{H}^1_{\mathrm{loc}}(\check{\mathrm{U}}(1)), & \operatorname{rot} E &= 0 \in \mathsf{L}^2(\check{\mathrm{U}}(1)),\\ \nu \times E|_{\mathrm{S}(1)} &= 0, & \operatorname{div} E &= r^{-2} \in \mathsf{L}^2(\check{\mathrm{U}}(1))\end{aligned}$$

shows that a standard L^2-setting is not appropriate for exterior domains. Even for square-integrable right-hand sides, we cannot expect to find square-integrable solutions. Indeed, it turns out that we have to work in weighted Lebesgue and Sobolev spaces to develop a solution theory. With $\rho := (1 + r^2)^{1/2}$, we introduce for an arbitrary domain $\Omega \subset \mathbb{R}^3$, $t \in \mathbb{R}$, and $m \in \mathbb{N}$

$$\begin{aligned} \mathsf{L}^2_{\mathrm{t}}(\Omega) &:= \{w \in \mathsf{L}^2_{\mathrm{loc}}(\Omega) \mid \rho^t w \in \mathsf{L}^2(\Omega)\},\\ \mathbf{H}^{\mathrm{m}}_{\mathrm{t}}(\Omega) &:= \{w \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \mid \forall\, |\alpha| \le m: \ \partial^\alpha w \in \mathsf{L}^2_{\mathrm{t}}(\Omega)\},\\ \mathsf{H}^{\mathrm{m}}_{\mathrm{t}}(\Omega) &:= \{w \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \mid \forall\, |\alpha| \le m: \ \partial^\alpha w \in \mathsf{L}^2_{\mathrm{t}+|\alpha|}(\Omega)\},\end{aligned}$$

$$\begin{aligned} \mathbf{R}_{\mathrm{t}}(\Omega) &:= \{E \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \mid \operatorname{rot} E \in \mathsf{L}^2_{\mathrm{t}}(\Omega)\}, & \mathsf{R}_{\mathrm{t}}(\Omega) &:= \{E \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \mid \operatorname{rot} E \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega)\},\\ \mathbf{D}_{\mathrm{t}}(\Omega) &:= \{H \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \mid \operatorname{div} H \in \mathsf{L}^2_{\mathrm{t}}(\Omega)\}, & \mathsf{D}_{\mathrm{t}}(\Omega) &:= \{H \in \mathsf{L}^2_{\mathrm{t}}(\Omega) \mid \operatorname{div} H \in \mathsf{L}^2_{\mathrm{t}+1}(\Omega)\},\end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ is a multi-index and $\partial^\alpha w := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} w$, $\operatorname{rot} E$, and $\operatorname{div} H$ are the usual distributional or weak derivatives. Equipped with the induced norms,

$$\begin{aligned} \|w\|^2_{\mathsf{L}^2_{\mathrm{t}}(\Omega)} &:= \left\|\rho^t w\right\|^2_{\mathsf{L}^2(\Omega)},\\ \|w\|^2_{\mathbf{H}^{\mathrm{m}}_{\mathrm{t}}(\Omega)} &:= \sum_{|\alpha| \le m} \left\|\partial^\alpha w\right\|^2_{\mathsf{L}^2_{\mathrm{t}}(\Omega)},\\ \|w\|^2_{\mathsf{H}^{\mathrm{m}}_{\mathrm{t}}(\Omega)} &:= \sum_{|\alpha| \le m} \left\|\partial^\alpha w\right\|^2_{\mathsf{L}^2_{\mathrm{t}+|\alpha|}(\Omega)},\end{aligned}$$

1 Although the right-hand sides 0 and r^{-2} are $\mathsf{L}^2(\check{\mathrm{U}}(1))$-functions, we have $E = \xi/r \notin \mathsf{L}^2(\check{\mathrm{U}}(1))$, but $E \in \mathsf{L}^2_{-1}(\check{\mathrm{U}}(1))$.

$$\|E\|^2_{\mathbf{R}_t(\Omega)} := \|E\|^2_{\mathsf{L}^2_t(\Omega)} + \|\operatorname{rot} E\|^2_{\mathsf{L}^2_t(\Omega)}, \qquad \|E\|^2_{\mathsf{R}_t(\Omega)} := \|E\|^2_{\mathsf{L}^2_t(\Omega)} + \|\operatorname{rot} E\|^2_{\mathsf{L}^2_{t+1}(\Omega)},$$
$$\|H\|^2_{\mathbf{D}_t(\Omega)} := \|H\|^2_{\mathsf{L}^2_t(\Omega)} + \|\operatorname{div} H\|^2_{\mathsf{L}^2_t(\Omega)}, \qquad \|H\|^2_{\mathsf{D}_t(\Omega)} := \|H\|^2_{\mathsf{L}^2_t(\Omega)} + \|\operatorname{div} H\|^2_{\mathsf{L}^2_{t+1}(\Omega)},$$

they become Hilbert spaces. As usual, the subscript "loc" respectively "vox" indicates local square-integrability respectively bounded support. Please note, that the bold spaces with weight $t = 0$ correspond to the classical Lebesque and Sobolev spaces and for bounded domains "nonweighted" and weighted spaces even coincide:

$$\Omega \subset \mathbb{R}^3 \text{ bounded} \implies \forall\, t \in \mathbb{R}: \begin{cases} \mathbf{H}^1_t(\Omega) = \mathsf{H}^1_t(\Omega) = \mathsf{H}^1_0(\Omega) = \mathsf{H}^1(\Omega) \\ \mathbf{R}_t(\Omega) = \mathsf{R}_t(\Omega) = \mathsf{R}_0(\Omega) = \mathsf{H}(\operatorname{rot}, \Omega) \\ \mathbf{D}_t(\Omega) = \mathsf{D}_t(\Omega) = \mathsf{D}_0(\Omega) = \mathsf{H}(\operatorname{div}, \Omega) \end{cases}$$

Besides the usual set $\mathring{\mathsf{C}}^\infty(\Omega)$ of test fields (resp., test functions), we introduce

$$\mathsf{C}^\infty_{\Gamma_i}(\Omega) := \left\{ \varphi|_\Omega \;\middle|\; \varphi \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^3) \text{ and } \operatorname{dist}(\operatorname{supp}\varphi, \Gamma_i) > 0 \right\}, \qquad i = 1, 2$$

to formulate boundary conditions in the weak sense:

$$\begin{aligned} \mathbf{H}^m_{t,\Gamma_i}(\Omega) &:= \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathbf{H}^m_t(\Omega)}}, \quad \mathbf{R}_{t,\Gamma_i}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathbf{R}_t(\Omega)}}, \quad \mathbf{D}_{t,\Gamma_i}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathbf{D}_t(\Omega)}}, \\ \mathsf{H}^m_{t,\Gamma_i}(\Omega) &:= \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathsf{H}^m_t(\Omega)}}, \quad \mathsf{R}_{t,\Gamma_i}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathsf{R}_t(\Omega)}}, \quad \mathsf{D}_{t,\Gamma_i}(\Omega) := \overline{\mathsf{C}^\infty_{\Gamma_i}(\Omega)}^{\|\cdot\|_{\mathsf{D}_t(\Omega)}}. \end{aligned} \tag{2.2}$$

These spaces indeed generalize vanishing scalar, tangential and normal Dirichlet boundary conditions even and in particular to boundaries for which the notion of a normal vector may not make any sense. Moreover, 0 at the lower left corner denotes vanishing rotation respectively divergence, e. g.,

$${}_0\mathsf{R}_t(\Omega) := \left\{ E \in \mathsf{R}_t(\Omega) \,\middle|\, \operatorname{rot} E = 0 \right\}, \quad {}_0\mathsf{D}_{t,\Gamma_1}(\Omega) := \left\{ H \in \mathsf{D}_{t,\Gamma_1}(\Omega) \,\middle|\, \operatorname{div} H = 0 \right\}, \quad \dots,$$

and if $t = 0$ in any of the definitions given above, we will skip the weight, e. g.,

$$\mathbf{H}^m(\Omega) = \mathbf{H}^m_0(\Omega), \quad \mathsf{R}_{\Gamma_1}(\Omega) = \mathsf{R}_{0,\Gamma_1}(\Omega), \quad \mathbf{D}_{\Gamma_1}(\Omega) = \mathbf{D}_{0,\Gamma_1}(\Omega), \quad \dots .$$

Finally we set

$$\mathsf{X}_{<s} := \bigcap_{t<s} \mathsf{X}_t \quad \text{and} \quad \mathsf{X}_{>s} := \bigcup_{t>s} \mathsf{X}_t \quad (s \in \mathbb{R}),$$

for X_t being any of the spaces above. If $\Omega = \mathbb{R}^3$ we omit the space reference, e.g.,

$$\mathbf{H}^m_t := \mathbf{H}^m_t(\mathbb{R}^3), \quad \mathsf{R}_{t,\Gamma_1} := \mathsf{R}_{t,\Gamma_1}(\mathbb{R}^3), \quad \mathbf{D}_t := \mathbf{D}_t(\mathbb{R}^3), \quad \mathsf{H}^m_{t,\Gamma_2} := \mathsf{H}^m_{t,\Gamma_2}(\mathbb{R}^3), \quad \dots \quad .$$

The material parameters ε and μ are assumed to be κ-admissible in the following sense.

Definition 2.1. Let $\kappa \geq 0$. We call a transformation γ κ-admissible, if
- $\gamma : \Omega \longrightarrow \mathbb{R}^{3\times 3}$ is an L^∞-matrix field,
- γ is symmetric, i. e.,

$$\forall\, E, H \in \mathsf{L}^2(\Omega): \quad \langle\, E\,, \gamma H\,\rangle_{\mathsf{L}^2(\Omega)} = \langle\, \gamma E\,, H\,\rangle_{\mathsf{L}^2(\Omega)}\,,$$

- γ is uniformly positive definite, i. e.,

$$\exists\, c > 0\ \forall\, E \in \mathsf{L}^2(\Omega): \quad \langle\, E\,, \gamma E\,\rangle_{\mathsf{L}^2(\Omega)} \geq c \cdot \|E\|^2_{\mathsf{L}^2(\Omega)}\,,$$

- γ is asymptotically a multiple of the identity, i. e.,

$$\gamma = \gamma_0 \cdot \mathbb{1} + \hat{\gamma} \quad \text{with} \quad \gamma_0 \in \mathbb{R}_+ \quad \text{and} \quad \hat{\gamma} = \mathcal{O}(r^{-\kappa}) \quad \text{as} \quad r \longrightarrow \infty\,.$$

Then ε, μ are pointwise invertible and ε^{-1}, μ^{-1} defined by

$$\varepsilon^{-1}(x) := \big(\varepsilon(x)\big)^{-1} \quad \text{and} \quad \mu^{-1}(x) := \big(\mu(x)\big)^{-1}, \quad x \in \Omega\,,$$

are also κ-admissible. Moreover,

$$\langle\,\cdot\,,\cdot\,\rangle_\varepsilon := \langle\, \varepsilon\,\cdot\,,\cdot\,\rangle_{\mathsf{L}^2(\Omega)} \quad \text{and} \quad \langle\,\cdot\,,\cdot\,\rangle_\mu := \langle\, \mu\,\cdot\,,\cdot\,\rangle_{\mathsf{L}^2(\Omega)}$$

define scalar products on $\mathsf{L}^2(\Omega)$ inducing norms equivalent to the standard ones. Consequently,

$$\mathsf{L}^2_\varepsilon(\Omega) := \big(\mathsf{L}^2(\Omega), \langle\,\cdot\,,\cdot\,\rangle_\varepsilon\big), \quad \mathsf{L}^2_\mu(\Omega) := \big(\mathsf{L}^2(\Omega), \langle\,\cdot\,,\cdot\,\rangle_\mu\big), \quad \text{and} \quad \mathsf{L}^2_\Lambda(\Omega) := \mathsf{L}^2_\varepsilon(\Omega) \times \mathsf{L}^2_\mu(\Omega)$$

are Hilbert spaces and we denote by

$$\|\cdot\|_\varepsilon\,,\ \|\cdot\|_\mu\ \ \|\cdot\|_\Lambda\,, \quad \oplus_\varepsilon,\ \oplus_\mu,\ \oplus_\Lambda, \quad \text{and} \quad \perp_\varepsilon,\ \perp_\mu,\ \perp_\Lambda$$

the norm, the orthogonal sum and the orthogonal complement in these spaces. For further simplification and to shorten notation, we also introduce for $\varepsilon = \varepsilon_0 \cdot \mathbb{1} + \hat{\varepsilon}$ and $\mu = \mu_0 \cdot \mathbb{1} + \hat{\mu}$ (recalling $\xi(x) = x/r(x)$) the formal matrix operators

$$\Lambda := \begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}, \qquad \Lambda^{-1} := \begin{bmatrix} \varepsilon^{-1} & 0 \\ 0 & \mu^{-1} \end{bmatrix}, \qquad \hat{\Lambda} := \begin{bmatrix} \hat{\varepsilon} & 0 \\ 0 & \hat{\mu} \end{bmatrix},$$

$$\Lambda\,(E,H) = (\varepsilon E, \mu H), \qquad \Lambda^{-1}\,(E,H) = (\varepsilon^{-1} E, \mu^{-1} H), \qquad \hat{\Lambda}\,(E,H) = (\hat{\varepsilon} E, \hat{\mu} H),$$

$$\Lambda_0 := \begin{bmatrix} \varepsilon_0 & 0 \\ 0 & \mu_0 \end{bmatrix}, \qquad \widetilde{\Lambda}_0 := \begin{bmatrix} \mu_0 & 0 \\ 0 & \varepsilon_0 \end{bmatrix}, \qquad \Xi := \begin{bmatrix} 0 & -\xi\times \\ \xi\times & 0 \end{bmatrix},$$

$$\Lambda_0\,(E,H) = (\varepsilon_0 E, \mu_0 H), \qquad \widetilde{\Lambda}_0\,(E,H) = (\mu_0 E, \varepsilon_0 H), \qquad \Xi\,(E,H) = (-\xi \times H, \xi \times E),$$

$$\operatorname{Rot} := \begin{bmatrix} 0 & -\operatorname{rot} \\ \operatorname{rot} & 0 \end{bmatrix}, \qquad \mathrm{M} := i\Lambda^{-1}\operatorname{Rot} = \begin{bmatrix} 0 & -i\varepsilon^{-1}\operatorname{rot} \\ i\mu^{-1}\operatorname{rot} & 0 \end{bmatrix},$$

$$\operatorname{Rot}(E,H) = (-\operatorname{rot} H, \operatorname{rot} E), \qquad \mathrm{M}(E,H) = (-i\varepsilon^{-1}\operatorname{rot} H, i\mu^{-1}\operatorname{rot} E).$$

We end this section with a lemma, showing that the spaces defined in (2.2) indeed generalize vanishing scalar, tangential and normal boundary conditions.

Lemma 2.2. *For $t \in \mathbb{R}$ and $i \in (1,2)$, the following inclusions hold:*

(a) $\mathsf{H}^m_{t,\Gamma_i}(\Omega) \subset \mathbf{H}^m_{t,\Gamma_i}(\Omega)$, $\mathsf{R}_{t,\Gamma_i}(\Omega) \subset \mathbf{R}_{t,\Gamma_i}(\Omega)$, $\mathsf{D}_{t,\Gamma_i}(\Omega) \subset \mathbf{D}_{t,\Gamma_i}(\Omega)$

(b) $\nabla \mathbf{H}^1_{t,\Gamma_i}(\Omega) \subset {}_0\mathsf{R}_{t,\Gamma_i}(\Omega)$, $\nabla \mathsf{H}^1_{t,\Gamma_i}(\Omega) \subset {}_0\mathsf{R}_{t+1,\Gamma_i}(\Omega)$

(c) $\operatorname{rot} \mathbf{R}_{t,\Gamma_i}(\Omega) \subset {}_0\mathsf{D}_{t,\Gamma_i}(\Omega)$, $\operatorname{rot} \mathsf{R}_{t,\Gamma_i}(\Omega) \subset {}_0\mathsf{D}_{t+1,\Gamma_i}(\Omega)$

Additionally, we have for $i,j \in (1,2)$, $i \neq j$:

$$\mathbf{H}^1_{t,\Gamma_i}(\Omega) = \boldsymbol{\mathcal{H}}^1_{t,\Gamma_i}(\Omega) := \left\{ w \in \mathbf{H}^1_t(\Omega) \;\middle|\; \forall\, \Phi \in \mathsf{C}^\infty_{\Gamma_j}(\Omega) : \; \langle w, \operatorname{div}\Phi \rangle_{\mathsf{L}^2(\Omega)} = -\langle \nabla w, \Phi \rangle_{\mathsf{L}^2(\Omega)} \right\},$$
$$\mathbf{R}_{t,\Gamma_i}(\Omega) = \boldsymbol{\mathcal{R}}_{t,\Gamma_i}(\Omega) := \left\{ E \in \mathbf{R}_t(\Omega) \;\middle|\; \forall\, \Phi \in \mathsf{C}^\infty_{\Gamma_j}(\Omega) : \; \langle E, \operatorname{rot}\Phi \rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{rot} E, \Phi \rangle_{\mathsf{L}^2(\Omega)} \right\},$$
$$\mathbf{D}_{t,\Gamma_i}(\Omega) = \boldsymbol{\mathcal{D}}_{t,\Gamma_i}(\Omega) := \left\{ H \in \mathbf{D}_t(\Omega) \;\middle|\; \forall\, \phi \in \mathsf{C}^\infty_{\Gamma_j}(\Omega) : \; \langle H, \nabla\phi \rangle_{\mathsf{L}^2(\Omega)} = -\langle \operatorname{div} H, \phi \rangle_{\mathsf{L}^2(\Omega)} \right\},$$

and

$$\mathsf{H}^1_{t,\Gamma_i}(\Omega) = \mathcal{H}^1_{t,\Gamma_i}(\Omega) := \left\{ w \in \mathsf{H}^1_t(\Omega) \;\middle|\; \forall\, \Phi \in \mathsf{C}^\infty_{\Gamma_j}(\Omega) : \; \langle w, \operatorname{div}\Phi \rangle_{\mathsf{L}^2(\Omega)} = -\langle \nabla w, \Phi \rangle_{\mathsf{L}^2(\Omega)} \right\},$$
$$\mathsf{R}_{t,\Gamma_i}(\Omega) = \mathcal{R}_{t,\Gamma_i}(\Omega) := \left\{ E \in \mathsf{R}_t(\Omega) \;\middle|\; \forall\, \Phi \in \mathsf{C}^\infty_{\Gamma_j}(\Omega) : \; \langle E, \operatorname{rot}\Phi \rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{rot} E, \Phi \rangle_{\mathsf{L}^2(\Omega)} \right\},$$
$$\mathsf{D}_{t,\Gamma_i}(\Omega) = \mathcal{D}_{t,\Gamma_i}(\Omega) := \left\{ H \in \mathsf{D}_t(\Omega) \;\middle|\; \forall\, \phi \in \mathsf{C}^\infty_{\Gamma_j}(\Omega) : \; \langle H, \nabla\phi \rangle_{\mathsf{L}^2(\Omega)} = -\langle \operatorname{div} H, \phi \rangle_{\mathsf{L}^2(\Omega)} \right\},$$

where (by continuity of the L^2-scalar product) we may also replace $\mathsf{C}^\infty_{\Gamma_j}(\Omega)$ by

$$\mathbf{H}^1_{s,\Gamma_j}(\Omega),\; \mathbf{R}_{s,\Gamma_j}(\Omega),\; \mathbf{D}_{s,\Gamma_j}(\Omega) \quad \textit{resp.} \quad \mathsf{H}^1_{s,\Gamma_j}(\Omega),\; \mathsf{R}_{s,\Gamma_j}(\Omega),\; \mathsf{D}_{s,\Gamma_j}(\Omega),$$

with $s + t \geq 0$ resp. $s + t \geq -1$.

Proof. As representatives of the arguments, we show

$$\text{(i)}\; \operatorname{rot} \mathbf{R}_{t,\Gamma_2}(\Omega) \subset {}_0\mathsf{D}_{t,\Gamma_2}(\Omega) \quad \text{and} \quad \text{(ii)}\; \mathbf{R}_{t,\Gamma_1}(\Omega) = \boldsymbol{\mathcal{R}}_{t,\Gamma_1}(\Omega).$$

For $E \in \operatorname{rot} \mathbf{R}_{t,\Gamma_2}(\Omega)$, there exists a sequence $(\mathcal{E}_n)_{n\in\mathbb{N}} \subset \mathsf{C}^\infty_{\Gamma_2}(\Omega)$ such that $\operatorname{rot} \mathcal{E}_n \longrightarrow E$ in $\mathsf{L}^2_t(\Omega)$. Then

$$\forall\,\phi \in \mathring{\mathsf{C}}^\infty(\Omega): \quad \langle E, \nabla\phi \rangle_{\mathsf{L}^2(\Omega)} = \lim_{n\to\infty} \langle \operatorname{rot} \mathcal{E}_n, \nabla\phi \rangle_{\mathsf{L}^2(\Omega)}$$
$$= -\lim_{n\to\infty} \langle \operatorname{div}\big(\operatorname{rot} \mathcal{E}_n\big), \phi \rangle_{\mathsf{L}^2(\Omega)} = 0\,,$$

hence E has vanishing divergence and $(E_n)_{n\in\mathbb{N}}$ defined by $E_n := \operatorname{rot} \mathcal{E}_n$ satisfies

$$(E_n)_{n\in\mathbb{N}} \subset \mathsf{C}^\infty_{\Gamma_2}(\Omega), \quad E_n \xrightarrow{\mathsf{L}^2_t(\Omega)} E \quad \text{and} \quad \operatorname{div} E_n = \operatorname{div}\big(\operatorname{rot} \mathcal{E}_n\big) = 0 \xrightarrow{\mathsf{L}^2_{t+1}(\Omega)} 0\,.$$

Thus $E \in {}_0\mathsf{D}_{t,\Gamma_2}(\Omega)$, showing (i). Let us show (ii). We have $\mathbf{R}_{t,\Gamma_1}(\Omega) \subset \mathcal{R}_{t,\Gamma_1}(\Omega)$. For the other direction, let $E \in \mathcal{R}_{t,\Gamma_1}(\Omega)$ and $\delta > 0$. Using the cut-off function from above we define $(E_k)_{k\in\mathbb{N}}$ by $E_k := \eta_k E$. Then $E_k \in \mathcal{R}_{\tilde\Gamma_1}(\Omega(2r_k))$, $\tilde\Gamma_1 := \Gamma_1 \cup \mathsf{S}(2r_k)$, since for $\Phi \in \mathsf{C}^\infty_{\Gamma_2}(\Omega(2r_k))$ it holds by $\eta_k\Phi \in \mathsf{C}^\infty_{\Gamma_2}(\Omega)$

$$\langle E_k, \operatorname{rot}\Phi \rangle_{\mathsf{L}^2(\Omega(2r_k))} = \langle \eta_k E, \operatorname{rot}\Phi \rangle_{\mathsf{L}^2(\Omega)}$$
$$= \langle E, \operatorname{rot}(\eta_k\Phi) \rangle_{\mathsf{L}^2(\Omega)} - \langle E, \nabla\eta_k \times \Phi \rangle_{\mathsf{L}^2(\Omega)}$$
$$= \langle \eta_k \operatorname{rot} E + \nabla\eta_k \times E, \Phi \rangle_{\mathsf{L}^2(\Omega(2r_k))} = \langle \operatorname{rot} E_k, \Phi \rangle_{\mathsf{L}^2(\Omega(2r_k))}\,.$$

By means of monotone convergence, we have[2]

$$\|E - E_k\|_{\mathbf{R}_t(\Omega)} = \|\check\eta_k E\|_{\mathbf{R}_t(\Omega)} \le c \cdot \Big(\|E\|_{\mathbf{R}_t(\check{\mathsf{U}}(r_k))} + \frac{1}{2^k} \cdot \|E\|_{\mathsf{L}^2_t(\Omega)} \Big) \to 0\,,$$

hence we can choose $\hat k > 0$ such that $\|E - E_{\hat k}\|_{\mathbf{R}_t(\Omega)} < \delta/2$. As $\Omega(2r_{\hat k}) = \Omega \cap \mathsf{U}(2r_{\hat k})$ is a bounded weak Lipschitz domain, we obtain $\mathcal{R}_{\tilde\Gamma_1}(\Omega(2r_{\hat k})) = \mathbf{R}_{\tilde\Gamma_1}(\Omega(2r_{\hat k}))$ by [1, Section 3.3], yielding the existence of some $\Psi \in \mathsf{C}^\infty_{\tilde\Gamma_1}(\Omega(2r_{\hat k}))$ such that

$$\|E_{\hat k} - \Psi\|_{\mathbf{R}_t(\Omega(2r_{\hat k}))} \le c \cdot \|E_{\hat k} - \Psi\|_{\mathbf{R}(\Omega(2r_{\hat k}))} < \delta/2\,.$$

Extending Ψ by zero to Ω, we obtain (by abuse of notation) $\Psi \in \mathsf{C}^\infty_{\Gamma_1}(\Omega)$ with

$$\|E - \Psi\|_{\mathbf{R}_t(\Omega)} \le \|E - E_{\hat k}\|_{\mathbf{R}_t(\Omega)} + \|E_{\hat k} - \Psi\|_{\mathbf{R}_t(\Omega(2r_{\hat k}))} < \frac{\delta}{2} + \frac{\delta}{2} = \delta\,,$$

which completes the proof. □

2.2 Some functional analysis

Let H_1 and H_2 be Hilbert spaces and let $\mathrm{A} : \mathcal{D}(\mathrm{A}) \subset \mathsf{H}_1 \longrightarrow \mathsf{H}_2$ be a linear, densely defined, and closed linear operator with the adjoint $\mathrm{A}^* : \mathcal{D}(\mathrm{A}^*) \subset \mathsf{H}_2 \longrightarrow \mathsf{H}_1$, which is then linear, densely defined, and closed as well. Note that A^* is characterized by

$$\langle \mathrm{A}\,x, y \rangle_{\mathsf{H}_2} = \langle x, \mathrm{A}^*\,y \rangle_{\mathsf{H}_1} \quad \forall\, x \in \mathcal{D}(\mathrm{A}),\ y \in \mathcal{D}(\mathrm{A}^*)\,.$$

2 Here and hereafter, $c > 0$ denotes some generic constant.

By the projection theorem, we have the following Helmholtz-type decompositions:

$$\mathsf{H}_1 = \overline{\mathcal{R}(\mathrm{A}^*)} \oplus \mathcal{N}(\mathrm{A}), \quad \text{and} \quad \mathsf{H}_2 = \overline{\mathcal{R}(\mathrm{A})} \oplus \mathcal{N}(\mathrm{A}^*),$$

which propose the corresponding reduced operators $\mathcal{A} := \mathrm{A}|_{\mathcal{N}(\mathrm{A})^\perp}$, $\mathcal{A}^* := \mathrm{A}^*|_{\mathcal{N}(\mathrm{A}^*)^\perp}$, i. e.,

$$\begin{aligned} \mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \overline{\mathcal{R}(\mathrm{A}^*)} &\longrightarrow \overline{\mathcal{R}(\mathrm{A})}, \\ \mathcal{D}(\mathcal{A}) &= \mathcal{D}(\mathrm{A}) \cap \overline{\mathcal{R}(\mathrm{A}^*)}, \end{aligned} \quad \text{resp.} \quad \begin{aligned} \mathcal{A}^* : \mathcal{D}(\mathcal{A}^*) \subset \overline{\mathcal{R}(\mathrm{A})} &\longrightarrow \overline{\mathcal{R}(\mathrm{A}^*)}, \\ \mathcal{D}(\mathcal{A}^*) &= \mathcal{D}(\mathrm{A}^*) \cap \overline{\mathcal{R}(\mathrm{A})}. \end{aligned}$$

These operators are also closed, densely defined and indeed adjoint to each other. Moreover, by definition $\mathcal{A}$ and $\mathcal{A}^*$ are injective and, therefore, the inverse operators

$$\mathcal{A}^{-1} : \mathcal{R}(\mathrm{A}) \longrightarrow \mathcal{D}(\mathcal{A}) \quad \text{and} \quad (\mathcal{A}^*)^{-1} : \mathcal{R}(\mathrm{A}^*) \longrightarrow \mathcal{D}(\mathcal{A}^*)$$

exist. The pair $(\mathcal{A}, \mathcal{A}^*)$ satisfies the following result of the so-called *Functional Analysis Toolbox* (see, e. g., [17, Section 2]), from which we will derive some Poincaré-type estimates for the time-harmonic Maxwell operator $(\mathcal{M} - \omega)$ (cf. Remark 3.11 and Remark 3.7).

Lemma 2.3. *The following assertions are equivalent:*
(1) $\exists\, c_\mathrm{A} \in (0, \infty) \ \ \forall\, x \in \mathcal{D}(\mathcal{A})$: $\|x\|_{\mathsf{H}_1} \le c_\mathrm{A} \|\mathrm{A}\, x\|_{\mathsf{H}_2}$.
(1*) $\exists\, c_{\mathrm{A}^*} \in (0, \infty) \ \ \forall\, y \in \mathcal{D}(\mathcal{A}^*)$: $\|y\|_{\mathsf{H}_2} \le c_{\mathrm{A}^*} \|\mathrm{A}^*\, y\|_{\mathsf{H}_1}$.
(2) $\mathcal{R}(\mathrm{A}) = \mathcal{R}(\mathcal{A})$ *is closed in* H_2.
(2*) $\mathcal{R}(\mathrm{A}^*) = \mathcal{R}(\mathcal{A}^*)$ *ist closed in* H_1.
(3) $\mathcal{A}^{-1} : \mathcal{R}(\mathrm{A}) \longrightarrow \mathcal{D}(\mathcal{A})$ *is continuous.*
(3*) $(\mathcal{A}^*)^{-1} : \mathcal{R}(\mathrm{A}^*) \longrightarrow \mathcal{D}(\mathcal{A}^*)$ *is continuous.*

Note that for the "best" constants c_A and c_{A^*} it holds

$$\left\|\mathcal{A}^{-1}\right\|_{\mathcal{R}(\mathrm{A}),\mathcal{R}(\mathrm{A}^*)} = c_\mathrm{A} = c_{\mathrm{A}^*} = \left\|(\mathcal{A}^*)^{-1}\right\|_{\mathcal{R}(\mathrm{A}^*),\mathcal{R}(\mathrm{A})}.$$

3 Solution theory for time-harmonic Maxwell equations

As mentioned above, we shall treat the time-harmonic Maxwell equations with mixed boundary conditions

$$\begin{aligned} -\operatorname{rot} H + i\omega\varepsilon E &= F \text{ in } \Omega, & E \times \nu &= 0 \quad \text{on } \Gamma_1, \\ \operatorname{rot} E + i\omega\mu H &= G \text{ in } \Omega, & H \times \nu &= 0 \quad \text{on } \Gamma_2, \end{aligned} \tag{3.1}$$

in an exterior weak Lipschitz domain $\Omega \subset \mathbb{R}^3$ and for frequencies $\omega \in \mathbb{C}\setminus(0)$. Moreover, we suppose that the material parameters ε and μ are κ-admissible with $\kappa \geq 0$. Using the abbreviations from above and rewriting

$$u := (E, H), \qquad f := i\Lambda^{-1}(-F, G),$$

the weak formulation of these boundary value problem reads:

$$\textit{For } f \in \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega}) \textit{ find } u \in \mathsf{R}_{\mathrm{loc},\Gamma_1}(\overline{\Omega}) \times \mathsf{R}_{\mathrm{loc},\Gamma_2}(\overline{\Omega}) \textit{ such that } (\mathrm{M} - \omega)\, u = f. \tag{3.2}$$

We shall solve this problem using polynomially weighted Hilbert spaces. In doing so, we avoid additional assumptions on boundary regularity for Ω, since only a compactness result comparable to Rellich's selection theorem is needed. More precisely, we will show that Ω satisfies *"Weck's* (*local*) *selection theorem"*, also called "(*local*) *Maxwell compactness property"*, which in fact is also an assumption on the quality of the boundary and in some sense supersedes assumptions on boundary regularity.

Definition 3.1. Let γ be κ-admissible with $\kappa \geq 0$ and let $\Omega \subset \mathbb{R}^3$ be open. Ω satisfies "Weck's local selection theorem" (WLST) (or has the "local Maxwell compactness property"), if the embedding

$$\mathbf{R}_{\Gamma_1}(\Omega) \cap \gamma^{-1}\mathbf{D}_{\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega}) \tag{3.3}$$

is compact. Ω satisfies "Weck's selection theorem" (WST) (or has the "Maxwell compactness property") if the embedding

$$\mathbf{R}_{\Gamma_1}(\Omega) \cap \gamma^{-1}\mathbf{D}_{\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \tag{3.4}$$

is compact.

Remark 3.2. Note that Weck's (local) selection theorem is essentially independent of γ meaning that a domain $\Omega \subset \mathbb{R}^3$ satisfies WST respectively WLST, if and only if the imbedding

$$\mathbf{R}_{\Gamma_1}(\Omega) \cap \mathbf{D}_{\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega) \quad \text{resp.} \quad \mathbf{R}_{\Gamma_1}(\Omega) \cap \mathbf{D}_{\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$$

is compact. The proof is practically identical with the one of [19, Lemma 2] (see also [24, 22]).

Lemma 3.3. *Let γ be κ-admissible with $\kappa \geq 0$ and let $\Omega \subset \mathbb{R}^3$ be an exterior domain. Then the following statements are equivalent:*

(a) *Ω satisfies WLST.*

(b) *For all $\tilde{r} > r_{0,}$ the imbedding*

$$\mathbf{R}_{\tilde{\Gamma}_1}(\Omega(\tilde{r})) \cap \gamma^{-1}\mathbf{D}_{\Gamma_2}(\Omega(\tilde{r})) \hookrightarrow \mathsf{L}^2(\Omega(\tilde{r}))$$

with $\tilde{\Gamma}_1 := \Gamma_1 \cup \mathrm{S}(\tilde{r})$ is compact, i.e., $\Omega(\tilde{r})$ satisfies WST.

(c) *For all* $\tilde{r} > r_0$, *the imbedding*

$$\mathsf{R}_{\Gamma_1}(\Omega(\tilde{r})) \cap \gamma^{-1}\mathsf{D}_{\tilde{\Gamma}_2}(\Omega(\tilde{r})) \hookrightarrow \mathsf{L}^2(\Omega(\tilde{r}))$$

with $\tilde{\Gamma}_2 := \Gamma_2 \cup \mathrm{S}(\tilde{r})$ *is compact, i.e.,* $\Omega(\tilde{r})$ *satisfies WST.*

(d) *For all* $s, t \in \mathbb{R}$ *with* $t < s$, *the imbedding*

$$\mathsf{R}_{s,\Gamma_1}(\Omega) \cap \gamma^{-1}\mathsf{D}_{s,\Gamma_2}(\Omega) \hookrightarrow \mathsf{L}^2_t(\Omega)$$

is compact.

Proof. (a)⇒(b): Let $\tilde{r} > r_0$. By Remark 3.2, it is sufficient to show the compactness of

$$\mathsf{R}_{\tilde{\Gamma}_1}(\Omega(\tilde{r})) \cap \mathsf{D}_{\Gamma_2}(\Omega(\tilde{r})) \hookrightarrow \mathsf{L}^2(\Omega(\tilde{r}))\,.$$

Therefore, let $(E_n)_{n\in\mathbb{N}} \subset \mathsf{R}_{\tilde{\Gamma}_1}(\Omega(\tilde{r})) \cap \mathsf{D}_{\Gamma_2}(\Omega(\tilde{r}))$ be bounded, choose $r_0 < \hat{r} < \tilde{r}$ and a cut-off function $\chi \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^3)$ with $\operatorname{supp}\chi \subset \mathrm{U}(\tilde{r})$ and $\chi|_{\mathrm{B}(\hat{r})} = 1$. Then, for every $n \in \mathbb{N}$ we have

$$E_n = \check{E}_n + \hat{E}_n := \chi E_n + (1-\chi)E_n\,, \quad \operatorname{supp}\check{E}_n \subset \Omega(\tilde{r})\,, \quad \operatorname{supp}\hat{E}_n \subset \mathrm{G}(\hat{r},\tilde{r})\,,$$

splitting $(E_n)_{n\in\mathbb{N}}$ into $(\check{E}_n)_{n\in\mathbb{N}}$ and $(\hat{E}_n)_{n\in\mathbb{N}}$. Extending $\check{E}_n$ respectively $\hat{E}_n$ by zero, we obtain (by abuse of notation) sequences

$$(\check{E}_n)_{n\in\mathbb{N}} \subset \mathsf{R}_{\Gamma_1}(\Omega) \cap \mathsf{D}_{\Gamma_2}(\Omega) \quad \text{and} \quad (\hat{E}_n)_{n\in\mathbb{N}} \subset \mathsf{R}_{\mathrm{S}(\tilde{r})}(\mathrm{U}(\tilde{r})) \cap \mathsf{D}(\mathrm{U}(\tilde{r}))$$

which are bounded in the respective spaces. Thus, using Weck's local selection theorem and Remark 3.2, we can choose a subsequence $(\check{E}_{\pi(n)})_{n\in\mathbb{N}}$ of $(\check{E}_n)_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$. The corresponding subsequence $(\hat{E}_{\pi(n)})_{n\in\mathbb{N}}$ is of course also bounded in $\mathsf{R}_{\mathrm{S}(\tilde{r})}(\mathrm{U}(\tilde{r})) \cap \mathsf{D}(\mathrm{U}(\tilde{r}))$ and by [23, Theorem 2.2], even in $\mathsf{H}^1(\mathrm{U}(\tilde{r}))$, hence (Rellich's selection theorem) has a subsequence $(\hat{E}_{\tilde{\pi}(n)})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2(\mathrm{U}(\tilde{r}))$. Thus

$$\begin{aligned}
&\|E_{\tilde{\pi}(n)} - E_{\tilde{\pi}(m)}\|_{\mathsf{L}^2(\Omega(\tilde{r}))} \\
&\quad \le c \cdot \Big(\|\chi(E_{\tilde{\pi}(n)} - E_{\tilde{\pi}(m)})\|_{\mathsf{L}^2(\Omega(\tilde{r}))} + \|(1-\chi)(E_{\tilde{\pi}(n)} - E_{\tilde{\pi}(m)})\|_{\mathsf{L}^2(\Omega(\tilde{r}))} \Big) \\
&\quad \le c \cdot \Big(\|\check{E}_{\tilde{\pi}(n)} - \check{E}_{\tilde{\pi}(m)}\|_{\mathsf{L}^2(\Omega(\tilde{r}))} + \|\hat{E}_{\tilde{\pi}(n)} - \hat{E}_{\tilde{\pi}(m)}\|_{\mathsf{L}^2(\mathrm{U}(\tilde{r}))} \Big) \xrightarrow{m,n\to\infty} 0\,,
\end{aligned}$$

meaning that $(E_{\tilde{\pi}(n)})_{n\in\mathbb{N}} \subset (E_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathsf{L}^2(\Omega(\tilde{r}))$.

(b)⇒(d): Let $s, t \in \mathbb{R}$ with $s > t$ and let $(E_n)_{n\in\mathbb{N}} \subset \mathsf{R}_{s,\Gamma_1}(\Omega) \cap \gamma^{-1}\mathsf{D}_{s,\Gamma_2}(\Omega)$ be bounded. Then there exists a subsequence $(E_{\pi(n)})_{n\in\mathbb{N}} \subset (E_n)_{n\in\mathbb{N}}$ which converges weakly in $\mathsf{R}_{s,\Gamma_1}(\Omega) \cap \gamma^{-1}\mathsf{D}_{s,\Gamma_2}(\Omega)$ to some vector field $E \in \mathsf{R}_{s,\Gamma_1}(\Omega) \cap \gamma^{-1}\mathsf{D}_{s,\Gamma_2}(\Omega)$. We now construct a subsequence $(E_{\tilde{\pi}(n)})_{n\in\mathbb{N}}$ of $(E_{\pi(n)})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$ to the same limit E. For this, observe that

$$(E_{\pi(n),1})_{n\in\mathbb{N}} \quad \text{with} \quad E_{\pi(n),1} := \eta_1 E_{\pi(n)}$$

is bounded in $\mathsf{R}_{\tilde{\Gamma}_1}(\Omega(r_2)) \cap \gamma^{-1}\mathsf{D}_{\Gamma_2}(\Omega(r_2))$, $\tilde{\Gamma}_1 := \Gamma_1 \cup \mathrm{S}(r_2)$ such that by assumption there exists a subsequence $(E_{\pi_1(n),1})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2(\Omega(r_2))$. Then $(E_{\pi_1(n)})_{n\in\mathbb{N}} \subset (E_{\pi(n)})_{n\in\mathbb{N}}$ is converging in $\mathsf{L}^2(\Omega(r_1))$, and, as $(E_{\pi_1(n)})_{n\in\mathbb{N}}$ is also weakly convergent in $\mathsf{L}^2(\Omega(r_1))$, we have

$$E_{\pi_1(n)} \longrightarrow E \quad \text{in } \mathsf{L}^2(\Omega(r_1))\,.$$

Multiplying $(E_{\pi_1(n)})_{n\in\mathbb{N}}$ with η_2, we obtain a sequence $(E_{\pi_1(n),2})_{n\in\mathbb{N}}$, $E_{\pi_1(n),2} := \eta_2 E_{\pi_1(n)}$ bounded in $\mathsf{R}_{\tilde{\Gamma}_1}(\Omega(r_3))\cap\gamma^{-1}\mathsf{D}_{\Gamma_2}(\Omega(r_3))$, $\tilde{\Gamma}_1 := \Gamma_1 \cup \mathrm{S}(r_3)$, and, as before, we construct a subsequence $(E_{\pi_2(n),2})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2(\Omega(r_3))$, giving again a converging subsequence $(E_{\pi_2(n)})_{n\in\mathbb{N}} \subset (E_{\pi_1(n)})_{n\in\mathbb{N}}$ with

$$E_{\pi_2(n)} \longrightarrow E \quad \text{in } \mathsf{L}^2(\Omega(r_2))\,.$$

Continuing like this, we successively construct converging subsequences $(E_{\pi_k(n)})_{n\in\mathbb{N}}$ with $E_{\pi_k(n)} \longrightarrow E$ in $\mathsf{L}^2(\Omega(r_k))$ and switching to the diagonal sequence we indeed end up with a sequence $(E_{\tilde{\pi}(n)})_{n\in\mathbb{N}}$, $\tilde{\pi}(n) := \pi_n(n)$, with $E_{\tilde{\pi}(n)} \longrightarrow E$ in $\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$. Now Lemma A.1 implies for arbitrary $\theta > 0$

$$\left\|E_{\tilde{\pi}(n)} - E\right\|_{\mathsf{L}^2_t(\Omega)} \le c \cdot \left\|E_{\tilde{\pi}(n)} - E\right\|_{\mathsf{L}^2(\Omega(\delta))} + \theta\,,$$

with $c, \delta \in (0,\infty)$ independent of $E_{\tilde{\pi}(n)}$. Hence

$$\limsup_{n\to\infty} \left\|E_{\tilde{\pi}(n)} - E\right\|_{\mathsf{L}^2_t(\Omega)} \le \theta\,,$$

and we obtain $E_{\tilde{\pi}(n)} \longrightarrow E$ in $\mathsf{L}^2_t(\Omega)$.

(d)⇒(a): For $(E_n)_{n\in\mathbb{N}}$ bounded in $\mathsf{R}_{\Gamma_1}(\Omega) \cap \gamma^{-1}\mathsf{D}_{\Gamma_2}(\Omega)$, assertion (c) implies the existence of a subsequence $(E_{\pi(n)})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2_{-1}(\Omega)$ to some $E \in \mathsf{L}^2_{-1}(\Omega)$. Then $E \in \mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$ and as

$$\forall\, \tilde{r} > 0: \quad \left\|E_{\pi(n)} - E\right\|_{\mathsf{L}^2(\Omega(\tilde{r}))} \le (1+\tilde{r})^{1/2} \cdot \left\|E_{\pi(n)} - E\right\|_{\mathsf{L}^2_{-1}(\Omega)}\,,$$

we obtain $(E_{\pi(n)})_{n\in\mathbb{N}} \longrightarrow E$ in $\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$.

Similar arguments to those corresponding to (b) show the assertion for (c). □

As shown by Bauer, Pauly, and Schomburg [1, Theorem 4.7], bounded weak Lipschitz domains satisfy Weck's selection theorem and by Lemma 3.3 (a) this directly implies the following.

Theorem 3.4. *Exterior weak Lipschitz domains satisfy Weck's local selection theorem.*

Returning to our initial question, a first step to a solution theory for (3.2) is the following observation.

Theorem 3.5. *The* Maxwell operator

$$\mathcal{M} : \mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega) \subset \mathsf{L}^2_\Lambda(\Omega) \longrightarrow \mathsf{L}^2_\Lambda(\Omega),\ u \longmapsto \mathrm{M}\,u\,,$$

is self-adjoint and reduced by the closure of its range

$$\overline{\mathcal{R}(\mathcal{M})} = \varepsilon^{-1}\overline{\operatorname{rot}\mathbf{R}_{\Gamma_2}(\Omega)} \times \mu^{-1}\overline{\operatorname{rot}\mathbf{R}_{\Gamma_1}(\Omega)}\,.$$

We note that here, in the case of an exterior domain Ω, the respective ranges are not closed.

Proof. The proof is straightforward using Lemma 2.2, i. e., the equivalence of the definition of weak and strong boundary conditions. □

Thus $\sigma(\mathcal{M}) \subset \mathbb{R}$, meaning that every $\omega \in \mathbb{C} \setminus \mathbb{R}$ is contained in the resolvent set of $\mathcal{M}$ and for given $f \in \mathsf{L}^2_\Lambda(\Omega)$ we obtain a unique solution of (3.2) by

$$u := (\mathcal{M} - \omega)^{-1} f \in \mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega).$$

Moreover, using the resolvent estimate $\|(\mathcal{M} - \omega)^{-1}\| \leq |\operatorname{Im}\omega|^{-1}$ and the differential equation, we get

$$\|u\|_{\mathbf{R}(\Omega)} \leq c \cdot \Big(\|u\|_{\mathsf{L}^2_\Lambda(\Omega)} + \|f\|_{\mathsf{L}^2_\Lambda(\Omega)} + |\omega|\,\|u\|_{\mathsf{L}^2_\Lambda(\Omega)} \Big) \leq c \cdot \frac{1+|\omega|}{|\operatorname{Im}\omega|} \cdot \|f\|_{\mathsf{L}^2_\Lambda(\Omega)}\,.$$

Theorem 3.6. *For $\omega \in \mathbb{C} \setminus \mathbb{R}$, the solution operator*

$$\mathcal{L}_\omega := (\mathcal{M} - \omega)^{-1} : \mathsf{L}^2_\Lambda(\Omega) \longrightarrow \mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega)$$

is continuous with $\|\mathcal{L}_\omega\|_{\mathsf{L}^2_\Lambda(\Omega),\mathbf{R}(\Omega)} \leq c \cdot \frac{1+|\omega|}{|\operatorname{Im}\omega|}$, where c is independent of ω and f.

Remark 3.7. Let $\omega \in \mathbb{C} \setminus \mathbb{R}$. By Lemma 2.3, the following statements are equivalent to the boundedness of $\mathcal{L}_\omega$:

– (Friedrichs/Poincaré-type estimate) There exists $c > 0$ such that

$$\|u\|_{\mathbf{R}(\Omega)} \leq c \cdot \|(\mathrm{M} - \omega)u\|_{\mathsf{L}^2_\Lambda(\Omega)} \qquad \forall\, u \in \mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega)\,.$$

– (Closed range) The range

$$\mathcal{R}(\mathcal{M} - \omega) = (\mathcal{M} - \omega)\big(\mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega)\big)$$

is closed in $\mathsf{L}^2_\Lambda(\Omega)$.

The case $\omega \in \mathbb{R} \setminus (0)$ is much more challenging, since we want to solve in the continuous spectrum of the Maxwell operator. Clearly, this cannot be done for every $f \in \mathsf{L}^2_\Lambda(\Omega)$, since otherwise we would have $\mathcal{R}(\mathcal{M} - \omega) = \mathsf{L}^2_\Lambda(\Omega)$ and, therefore, $(\mathcal{M} - \omega)^{-1}$ would be continuous (cf. Lemma 2.3) or in other words $\omega \notin \sigma(\mathcal{M})$. Thus we have to restrict ourselves to certain subspaces of L^2_Λ or generalize our solution concept. Actually, we will do both and show existence as well as uniqueness of weaker, so-called *"radiating solutions,"* by switching to data $f \in \mathsf{L}^2_s(\Omega)$ for some $s > 1/2$.

Definition 3.8. Let $\omega \in \mathbb{R} \setminus (0)$ and $f \in \mathsf{L}^2_{\mathrm{loc}}(\Omega)$. We call u (radiating) solution of (3.2), if

$$u \in \mathbf{R}_{<-\frac{1}{2},\Gamma_1}(\Omega) \times \mathbf{R}_{<-\frac{1}{2},\Gamma_2}(\Omega)$$

and

$$(\mathrm{M} - \omega)u = f\,, \tag{3.5}$$

$$(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u \in \mathsf{L}^2_{>-\frac{1}{2}}(\Omega)\,. \tag{3.6}$$

Remark 3.9. Since

$$(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u = \Lambda_0\Big(E - \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\xi \times H\,,\, H + \sqrt{\frac{\varepsilon_0}{\mu_0}}\xi \times E\Big)\,,$$

the last condition is just the classical Silver–Müller radiation condition which describes the behavior of the electro-magnetic field at infinity and is needed to distinguish outgoing from incoming waves (interchanging signs would yield incoming waves).

In order to construct such a radiating solution u, we use the *"limiting absorption principle"* introduced by Eidus and approximate u by solutions $(u_n)_{n\in\mathbb{N}}$ associated with frequencies $(\omega_n)_{n\in\mathbb{N}} \subset \mathbb{C} \setminus \mathbb{R}$ converging to $\omega \in \mathbb{R} \setminus (0)$. This leads to statement (4) of our main result Theorem 3.10, where the following abbreviations are used:

$$\begin{aligned} \mathcal{N}_{\mathrm{gen}}(\mathcal{M} - \omega) &:= \{u \,|\, u \text{ is a radiating solution of } (\mathrm{M} - \omega)u = 0\} \\ &\qquad (\textit{generalized kernel of } \mathcal{M} - \omega)\,, \\ \sigma_{\mathrm{gen}}(\mathcal{M}) &:= \{\,\omega \in \mathbb{C} \setminus (0) \,|\, \mathcal{N}_{\mathrm{gen}}(\mathcal{M} - \omega) \neq (0)\} \\ &\qquad (\textit{generalized point spectrum of } \mathcal{M})\,. \end{aligned}$$

Theorem 3.10 (Fredholm alternative). *Let $\Omega \subset \mathbb{R}^3$ be an exterior weak Lipschitz domain with boundary Γ and weak Lipschitz boundary parts Γ_1 and $\Gamma_2 = \Gamma \setminus \overline{\Gamma}_1$. Furthermore, let $\omega \in \mathbb{R} \setminus (0)$ and ε, μ be κ-admissible with $\kappa > 1$. Then:*

(1) $\mathcal{N}_{\mathrm{gen}}(\mathcal{M} - \omega) \subset \bigcap_{t\in\mathbb{R}} \big(\mathbf{R}_{t,\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{t,\Gamma_2}(\Omega)\big) \times \big(\mathbf{R}_{t,\Gamma_2}(\Omega) \cap \mu^{-1}{}_0\mathsf{D}_{t,\Gamma_1}(\Omega)\big)$.[3]
(2) $\dim \mathcal{N}_{\mathrm{gen}}(\mathcal{M} - \omega) < \infty$.
(3) *$\sigma_{\mathrm{gen}}(\mathcal{M}) \subset \mathbb{R} \setminus (0)$ and $\sigma_{\mathrm{gen}}(\mathcal{M})$ has no accumulation point in $\mathbb{R} \setminus (0)$.*
(4) *For all $f \in \mathsf{L}^2_{>\frac{1}{2}}(\Omega)$ there exists a radiating solution u of (3.2), if and only if*

$$\forall\, v \in \mathcal{N}_{\mathrm{gen}}(\mathcal{M} - \omega): \quad \langle f\,, v\rangle_{\mathsf{L}^2_\Lambda(\Omega)} = 0\,. \tag{3.7}$$

3 We even have

$$\mathcal{N}_{\mathrm{gen}}(\mathcal{M} - \omega) \subset \bigcap_{t\in\mathbb{R}} \big(\mathbf{R}_{t,\Gamma_1}(\Omega) \cap \varepsilon^{-1} \operatorname{rot} \mathbf{R}_{t,\Gamma_2}(\Omega)\big) \times \big(\mathbf{R}_{t,\Gamma_2}(\Omega) \cap \mu^{-1} \operatorname{rot} \mathbf{R}_{t,\Gamma_1}(\Omega)\big)\,.$$

Moreover, we can choose u such that

$$\forall v \in \mathcal{N}_{\mathrm{gen}}(\mathcal{M}-\omega): \quad \langle u, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} = 0 . \tag{3.8}$$

Then u is uniquely determined.

(5) *For all s, −t > 1/2, the solution operator*

$$\mathcal{L}_\omega : \mathsf{L}^2_s(\Omega) \cap \mathcal{N}_{\mathrm{gen}}(\mathcal{M}-\omega)^{\perp_\Lambda} \longrightarrow \big(\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)\big) \cap \mathcal{N}_{\mathrm{gen}}(\mathcal{M}-\omega)^{\perp_\Lambda}$$

defined by (4) *is continuous.*

Remark 3.11. Under the conditions of Theorem 3.10, the following statements are equivalent to the boundedness of $\mathcal{L}_\omega$ (cf. Lemma 2.3 and Remark 3.7):

– (Friedrichs/Poincaré-type estimate) For all $s, -t > 1/2$, there exists $c > 0$ such that

$$\|u\|_{\mathbf{R}_t(\Omega)} \le c \cdot \big\|(\mathrm{M}-\omega)u\big\|_{\mathsf{L}^2_s(\Omega)}$$

holds for all $u \in \big(\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)\big) \cap \mathcal{N}_{\mathrm{gen}}(\mathcal{M}-\omega)^{\perp_\Lambda}$ satisfying the radiation condition.

– (Closed range) For all $s, -t > 1/2$, the range

$$\mathcal{R}(\mathcal{M}-\omega) = (\mathcal{M}-\omega)\big(\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)\big)$$

is closed in $\mathsf{L}^2_s(\Omega)$.

By the same indirect arguments as in [15, Corollary 3.9] (see also [14, Section 4.9]), we get even stronger estimates for the solution operator $\mathcal{L}_\omega$.

Corollary 3.12. *Let $\Omega \subset \mathbb{R}^3$ be an exterior weak Lipschitz domain with boundary Γ and weak Lipschitz boundary parts Γ_1 and $\Gamma_2 := \Gamma \setminus \overline{\Gamma}_1$. Furthermore, let $s, -t > 1/2$, ε, μ be κ-admissible with $\kappa > 1$ and $K \Subset \mathbb{C}_+ \setminus (0)$ with $\overline{K} \cap \sigma_{\mathrm{gen}}(\mathcal{M}) = \emptyset$. Then:*

(1) *There exist constants $c > 0$ and $\hat{t} > -1/2$ such that for all $\omega \in \overline{K}$ and $f \in \mathsf{L}^2_s(\Omega)$*

$$\|\mathcal{L}_\omega f\|_{\mathbf{R}_t(\Omega)} + \big\|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)\mathcal{L}_\omega f\big\|_{\mathsf{L}^2_{\hat{t}}(\Omega)} \le c \cdot \|f\|_{\mathsf{L}^2_s(\Omega)}$$

holds, implying that $\mathcal{L}_\omega : \mathsf{L}^2_s(\Omega) \longrightarrow \mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)$ is equicontinuous w. r. t. $\omega \in \overline{K}$.

(2) *The mapping*

$$\begin{aligned} \mathcal{L} : \overline{K} &\longrightarrow \mathrm{B}\big(\mathsf{L}^2_s(\Omega), \mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)\big) \\ \omega &\longmapsto \mathcal{L}_\omega \end{aligned}$$

is uniformly continuous.

4 Polynomial decay and a priori estimate

As stated before, we will construct a solution u in the case of $\omega \in \mathbb{R} \setminus (0)$ by solving (3.2) for $\omega_n = \omega + i\sigma_n \in \mathbb{C}_+ \setminus \mathbb{R}$ and sending $\sigma_n \longrightarrow 0$ (using $(\omega_n)_{n\in\mathbb{N}} \in \mathbb{C}_- \setminus \mathbb{R}$ instead will lead to "incoming" solutions). The essential ingredients to generate convergence are the polynomial decay of eigensolutions, an a priori estimate for solutions corresponding to nonreal frequencies and Weck's local selection theorem. While the latter one is already satisfied (cf. Theorem 3.4), we obtain the first two in the spirit of [27] using the following decomposition Lemma introduced in [14] (see also [15, 16]).

Lemma 4.1. *Let $\omega \in K \Subset \mathbb{C} \setminus (0)$, ε, μ be κ-admissible with $\kappa \geq 0$ and $s, t \in \mathbb{R}$ such that $0 \leq s \in \mathbb{R} \setminus \mathbb{I}$ and $t \leq s \leq t + \kappa$. Moreover, assume that $u \in \mathbf{R}_t(\Omega)$ satisfies the equation $(\mathrm{M} - \omega)\, u = f \in \mathsf{L}^2_s(\Omega)$. Then*

$$f_1 := (\mathsf{C}_{\mathrm{Rot},\check{\eta}} - i\omega\check{\eta}\hat{\Lambda})u - i\check{\eta}\Lambda f \in \mathsf{L}^2_s$$

and, by decomposing

$$f_1 = f_{\mathsf{R}} + f_{\mathsf{D}} + f_{\mathcal{S}} \in {}_0\mathsf{R}_s \dotplus {}_0\mathsf{D}_s \dotplus \mathcal{S}_s$$

according to [26, Theorem 4]*, it holds*

$$f_2 := f_{\mathsf{D}} + \frac{i}{\omega}\,\widetilde{\Lambda}_0^{-1}\ \mathrm{Rot} f_{\mathcal{S}} \in {}_0\mathsf{D}_s\,.$$

Additionally, u may be decomposed into

$$u = \eta u + u_1 + u_2 + u_3\,,$$

where

(1) $\eta u \in \mathsf{R}_{\mathrm{vox}}(\Omega)$ *and for all* $\hat{t} \in \mathbb{R}$

$$\|\eta u\|_{\mathsf{R}_{\hat{t}}(\Omega)} \leq c \cdot \Big(\ \|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)}\ \Big)\,;$$

(2) $u_1 := -\frac{i}{\omega}\Lambda_0^{-1}(f_{\mathsf{R}} + f_{\mathcal{S}}) \in \mathsf{R}_s$ *and*

$$\|u_1\|_{\mathsf{R}_s} \leq c \cdot \|f_1\|_{\mathsf{L}^2_s}\ ;$$

(3) $u_2 := \mathcal{F}^{-1}\big(\rho^{-2}(1 - ir\Xi)\mathcal{F}(f_2)\big) \in \mathbf{H}^1_s \cap {}_0\mathsf{D}_s$ *and*

$$\|u_2\|_{\mathbf{H}^1_s} \leq c \cdot \|f_2\|_{\mathsf{L}^2_s}\ ;$$

(4) $u_3 := \tilde{u} - u_2 \in \mathbf{H}^2_t \cap {}_0\mathsf{D}_t$ *and for all* $\hat{t} \leq t$

$$\|u_3\|_{\mathbf{H}^2_{\hat{t}}} \leq c \cdot \Big(\ \|u_3\|_{\mathsf{L}^2_{\hat{t}}} + \|u_2\|_{\mathbf{H}^1_{\hat{t}}}\ \Big)\,,$$

where $\tilde{u} := i\omega^{-1}\Lambda_0^{-1}\big(\,\mathrm{Rot}\,\check{\eta}u - f_{\mathsf{D}}\big) \in \mathbf{H}^1_t \cap {}_0\mathsf{D}_t$

with constants $c \in (0,\infty)$ *independent of* u, f *or* ω. *These fields solve the following equations:*

$$(\mathrm{Rot} + i\omega\Lambda_0)\check{\eta}u = f_1\,, \quad (\mathrm{Rot} + i\omega\Lambda_0)\tilde{u} = f_2\,, \quad (\mathrm{Rot} + i\omega\Lambda_0)u_3 = (1 - \omega\Lambda_0)u_2\,,$$
$$(\Delta + \omega^2\varepsilon_0\mu_0)u_3 = (1 - i\omega\widetilde{\Lambda}_0)f_2 - (1 + \omega^2\varepsilon_0\mu_0)u_2\,.$$

Moreover, the following estimates hold for all $\hat{t} \le t$ *and uniformly w. r. t.* $\lambda \in K$, u *and* f*:*
- $\|f_2\|_{\mathsf{L}^2_s} \le c \cdot \|f_1\|_{\mathsf{L}^2_s} \le c \cdot \big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)} \big)$
- $\|u\|_{\mathbf{R}_{\hat{t}}(\Omega)} \le c \cdot \big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)} + \|u_3\|_{\mathsf{L}^2_{\hat{t}}} \big)$
- $\big\|(\Delta + \omega^2\varepsilon_0\mu_0)u_3\big\|_{\mathsf{L}^2_s} \le c \cdot \big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)} \big)$
- $\big\|(\mathrm{Rot} - i\lambda\sqrt{\varepsilon_0\mu_0}\,\Xi)u\big\|_{\mathsf{L}^2_{\hat{t}}} \le c \cdot \big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)} + \big\|(\mathrm{Rot} - i\lambda\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\big\|_{\mathsf{L}^2_{\hat{t}}} \big)$

Here, $\mathcal{S}_s$ *is a finite dimensional subspace of* $\mathring{\mathsf{C}}^\infty(\mathbb{R}^3)$, $\mathcal{F}$ *the Fourier transformation and*

$$\mathsf{C}_{A,B} := AB - BA$$

the commutator of A *and* B.

Basically, this lemma allows us to split u into two parts. One part (consisting of ηu, u_1 and u_2) has better integrability properties and the other part (consisting of u_3) is more regular and satisfies a Helmholtz equation in the whole of $\mathbb{R}^3$. Thus we can use well-known results from the theory for Helmholtz equation (cf. Appendix, Section B) to establish corresponding results for Maxwell's equations. We start with the polynomial decay of solutions, especially of eigensolutions, which will lead to assertions (1)–(3) of our main theorem. Moreover, this will also show, that the solution u we are going to construct, can be chosen to be perpendicular to the generalized kernel of the time-harmonic Maxwell operator. As in the proof of [16, Theorem 4.2], we obtain (see also Appendix, Section C) the following.

Lemma 4.2 (Polynomial decay of solutions). *Let* $J \subset \mathbb{R} \setminus (0)$ *be some interval,* $\omega \in J$, ε, μ *be* κ*-admissible with* $\kappa > 1$, *and* $s \in \mathbb{R} \setminus \mathbb{I}$ *with* $s > 1/2$. *If*

$$u \in \mathbf{R}_{>-\frac{1}{2}}(\Omega) \quad \textit{satisfies} \quad (\mathrm{M} - \omega)u =: f \in \mathsf{L}^2_s(\Omega)\,,$$

then

$$u \in \mathbf{R}_{s-1}(\Omega) \quad \textit{and} \quad \|u\|_{\mathbf{R}_{s-1}(\Omega)} \le c \cdot \Big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2(\Omega(\delta))} \Big)\,,$$

with $c, \delta \in (0,\infty)$ *independent of* ω, u *and* f.

In short: If a solution u satisfies $u \in \mathbf{R}_t(\Omega)$ for some $t > -1/2$ and the right-hand side $f = (\mathrm{M} - \omega)u$ has better integrability properties, meaning $f \in \mathsf{L}^2_s(\Omega)$ for some $s > 1/2$, then also u is better integrable, i. e., $u \in \mathbf{R}_{s-1}(\Omega)$. Especially, if

$$u \in \mathbf{R}_{>-\frac{1}{2}}(\Omega) \quad \text{and} \quad f \in \mathsf{L}^2_s(\Omega) \quad \forall\, s \in \mathbb{R}\,,$$

then $u \in \mathbf{R}_s(\Omega)$ for all $s \in \mathbb{R}$, which is called *"polynomial decay."*

Corollary 4.3. *Let $\omega \in \mathbb{R} \setminus (0)$ and assume ε, μ to be κ-admissible with $\kappa > 1$ and*

$$u \in \mathbf{R}_{<-\frac{1}{2},\Gamma_1}(\Omega) \times \mathbf{R}_{<-\frac{1}{2},\Gamma_2}(\Omega)$$

to be a radiating solution (*cf. Definition* 3.8) *of* $(\mathcal{M} - \omega)u = 0$. *Then:*

$$u \in \bigcap_{t \in \mathbb{R}} \Big(\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega) \Big) .$$

Proof. According to Lemma 4.2, it suffices to show $u \in \mathbf{R}_t(\Omega)$ for some $t > -1/2$. Therefore, remember that u is a radiating solution, the radiation condition (3.6) holds and there exists $\hat{t} > -1/2$ such that

$$(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u \in \mathsf{L}^2_{\hat{t}}(\Omega) . \tag{4.1}$$

On the other hand, we have

$$\begin{aligned} &\|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u\|^2_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))} \\ &\quad = \|\Lambda_0 u\|^2_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))} + 2\sqrt{\varepsilon_0\mu_0}\,\operatorname{Re}\langle \Xi u, \Lambda_0 u\rangle_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))} + \varepsilon_0\mu_0\,\|\Xi u\|^2_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))} \end{aligned}$$

and using Lemma A.3 (cf. Appendix, Section A) with

$$\phi(s) := (1+s^2)^{\hat{t}}, \quad \Phi := \phi \circ r, \quad \psi(\sigma) = \int_{\max\{r_0,\sigma\}}^{\tilde{r}} \phi(\tau)\,d\tau, \quad \Psi = \psi \circ r ,$$

as well as the differential equation, we conclude

$$\begin{aligned} \operatorname{Re}\langle \Xi u, \Lambda_0 u\rangle_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))} &= \operatorname{Re}\langle \Phi\,\Xi u, \Lambda_0 u\rangle_{\mathsf{L}^2(\mathrm{G}(r_0,\tilde{r}))} \\ &= \operatorname{Re}\Big(\langle \Psi \operatorname{Rot} u, \Lambda_0 u\rangle_{\mathsf{L}^2(\Omega(\tilde{r}))} + \langle \Psi u, \widetilde{\Lambda}_0 \operatorname{Rot} u\rangle_{\mathsf{L}^2(\Omega(\tilde{r}))} \Big) \\ &= \operatorname{Re}\Big(\langle -i\omega\Psi\Lambda u, \Lambda_0 u\rangle_{\mathsf{L}^2(\Omega(\tilde{r}))} + \langle \Psi u, -i\omega\widetilde{\Lambda}_0\Lambda u\rangle_{\mathsf{L}^2(\Omega(\tilde{r}))} \Big) \\ &= \operatorname{Re} \underbrace{i\omega\langle \Psi\Lambda u, (\widetilde{\Lambda}_0 - \Lambda_0)u\rangle_{\mathsf{L}^2(\Omega(\tilde{r}))}}_{\in i\mathbb{R}} = 0 , \end{aligned}$$

hence

$$\|u\|_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))} \le c \cdot \|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u\|_{\mathsf{L}^2_{\hat{t}}(\mathrm{G}(r_0,\tilde{r}))}$$

with $c \in (0,\infty)$ independent of $\tilde{r}$. Now the monotone convergence theorem and (4.1) show

$$\|u\|_{\mathsf{L}^2_{\hat{t}}(\check{\mathrm{U}}(r_0))} \le c \cdot \|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u\|_{\mathsf{L}^2_{\hat{t}}(\check{\mathrm{U}}(r_0))} < \infty ,$$

which already implies $u \in \mathsf{L}^2_{\hat{t}}(\Omega)$ and completes the proof. □

The next step is an a priori estimate for solutions corresponding to nonreal frequencies, which will later guarantee that our solution satisfies the radiation condition (3.6) and has the proper integrability. The proof of it is practically identical with the proof of [16, Lemma 6.3] (cf. Appendix, Section C).

Lemma 4.4 (A priori estimate for Maxwell's equations). *Let $J \Subset \mathbb{R} \setminus (0)$ be some interval, $-t, s > 1/2$ and ε, μ be κ-admissible with $\kappa > 1$. Then there exist constants $c, \delta \in (0, \infty)$ and some $\hat{t} > -1/2$, such that for all $\omega \in \mathbb{C}_+$ with $\omega^2 = \lambda^2 + i\lambda\sigma$, $\lambda \in J$, $\sigma \in (0, \sqrt{\varepsilon_0\mu_0}^{-1}]$ and $f \in \mathsf{L}^2_s(\Omega)$*

$$\|\mathcal{L}_\omega f\|_{\mathbf{R}_t(\Omega)} + \|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)\mathcal{L}_\omega f\|_{\mathsf{L}^2_{\hat{t}}(\Omega)} \leq c \cdot \Big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|\mathcal{L}_\omega f\|_{\mathsf{L}^2(\Omega(\delta))} \Big).$$

5 Proof of the main result

Before we start with the proof of Theorem 3.10, we provide some Helmholtz-type decompositions, which will be useful in the following. These are immediate consequences of the projection theorem and Lemma 2.2.

Lemma 5.1. *It holds*

$$\begin{aligned}
\mathsf{L}^2_\varepsilon(\Omega) &= \overline{\nabla \mathbf{H}^1_{\Gamma_1}(\Omega)} \oplus_\varepsilon \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Omega)\,,\\
\mathsf{L}^2_\mu(\Omega) &= \overline{\nabla \mathbf{H}^1_{\Gamma_2}(\Omega)} \oplus_\mu \mu^{-1}{}_0\mathsf{D}_{\Gamma_1}(\Omega)\,,\\
\mathbf{R}_{\Gamma_1}(\Omega) &= \overline{\nabla \mathbf{H}^1_{\Gamma_1}(\Omega)} \oplus_\varepsilon \Big(\mathbf{R}_{\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Omega)\Big)\,,\\
\mathbf{R}_{\Gamma_2}(\Omega) &= \overline{\nabla \mathbf{H}^1_{\Gamma_2}(\Omega)} \oplus_\mu \Big(\mathbf{R}_{\Gamma_2}(\Omega) \cap \mu^{-1}{}_0\mathsf{D}_{\Gamma_1}(\Omega)\Big)\,,
\end{aligned}$$

where the closures are taken in $\mathsf{L}^2(\Omega)$.

Proof. Let $\gamma \in \{\varepsilon, \mu\}$ and $i, j \in \{1, 2\}$ with $i \neq j$. The linear operator

$$\nabla_i : \mathbf{H}^1_{\Gamma_i}(\Omega) \subset \mathsf{L}^2(\Omega) \longrightarrow \mathsf{L}^2_\gamma(\Omega)$$

is densely defined and closed with adjoint (cf. Lemma 2.2)

$$-\operatorname{div}_j \gamma : \gamma^{-1}\mathbf{D}_{\Gamma_j}(\Omega) \subset \mathsf{L}^2_\gamma(\Omega) \longrightarrow \mathsf{L}^2(\Omega)\,.$$

The projection theorem yields

$$\mathsf{L}^2_\gamma(\Omega) = \overline{\mathcal{R}(\nabla_i)} \oplus_\gamma \mathcal{N}(\operatorname{div}_j \gamma)\,.$$

The remaining assertion follows by $\nabla \mathbf{H}^1_{\Gamma_i}(\Omega) \subset \mathsf{R}_{\Gamma_i}(\Omega)$. □

Proof of Theorem 3.10. Let $\omega \in \mathbb{R} \setminus (0)$ and ε, μ be κ-admissible for some $\kappa > 1$.

(1): The assertion follows by Corollary 4.3 and the differential equation

$$(\mathrm{M} - \omega)\, u = 0 \iff u = i\omega^{-1}\Lambda^{-1}\,\mathrm{Rot}\, u\,,$$

using the fact that (cf. Lemma 2.2)

$$\mathrm{rot}\,\mathbf{R}_{\mathrm{t},\Gamma_1}(\Omega) \subset {}_0\mathsf{D}_{\mathrm{t},\Gamma_1}(\Omega) \quad \text{resp.} \quad \mathrm{rot}\,\mathbf{R}_{\mathrm{t},\Gamma_2}(\Omega) \subset {}_0\mathsf{D}_{\mathrm{t},\Gamma_2}(\Omega)\,.$$

(2): Assume $\dim \mathcal{N}_{\mathrm{gen}}(\,\mathcal{M} - \omega\,) = \infty$. Using (1) there exists a L^2_Λ-orthonormal sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}_{\mathrm{gen}}(\,\mathcal{M} - \omega\,)$ converging weakly in $\mathsf{L}^2(\Omega)$ to 0. By the differential equation this sequence is bounded in

$$\big(\mathbf{R}_{\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Omega)\big) \times \big(\mathbf{R}_{\Gamma_2}(\Omega) \cap \mu^{-1}{}_0\mathsf{D}_{\Gamma_1}(\Omega)\big).$$

Hence, due to Weck's local selection theorem, we can choose a subsequence, $(u_{\pi(n)})_{n\in\mathbb{N}}$ converging to 0 in $\mathsf{L}^2_{\mathrm{loc}}(\overline{\Omega})$ ($(u_{\pi(n)})_{n\in\mathbb{N}}$ also converges weakly on every bounded subset). Now let $1 < s \in \mathbb{R} \setminus \mathbb{I}$. Then Lemma 4.2 guarantees the existence of $c, \delta \in (0, \infty)$ independent of $(u_{\pi(n)})_{n\in\mathbb{N}}$ such that

$$1 = \|u_{\pi(n)}\|_{\mathsf{L}^2_\Lambda(\Omega)} \leq c \cdot \|u_{\pi(n)}\|_{\mathbf{R}_{s-1}(\Omega)} \leq c \cdot \|u_{\pi(n)}\|_{\mathsf{L}^2(\Omega(\delta))} \xrightarrow{n\to\infty} 0$$

holds; a contradiction.

(3): $\mathcal{M}$ is a self-adjoint operator, hence we clearly have $\sigma_{\mathrm{gen}}(\mathcal{M}) \subset \mathbb{R}\setminus(0)$. Now assume $\widetilde{\omega} \in \mathbb{R}\setminus(0)$ is an accumulation point of $\sigma_{\mathrm{gen}}(\mathcal{M})$. Then we can choose a sequence $(\omega_n)_{n\in\mathbb{N}} \subset \mathbb{R} \setminus (0)$ with $\omega_n \neq \omega_m$ for $n \neq m$, $\omega_n \longrightarrow \widetilde{\omega}$ and a corresponding sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \in \mathcal{N}_{\mathrm{gen}}(\,\mathcal{M} - \omega_n\,) \setminus (0)$. As $\mathcal{M}$ is self-adjoint, eigenvectors associated to different eigenvalues are orthogonal provided they are well enough integrable (which is given by (1)), and thus by normalizing $(u_n)_{n\in\mathbb{N}}$ we end up with an L^2_Λ-orthonormal sequence. Continuing as in (2), we again obtain a contradiction.

(4): First of all, if a solution u satisfies (3.8), it is uniquely determined as for the homogeneous problem $u \in \mathcal{N}_{\mathrm{gen}}(\,\mathcal{M} - \omega\,)$ together with (1) and (3.8) implies $u = 0$. Moreover, using Lemma 2.2 and (1), we obtain

$$\langle f, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} = \langle (\mathrm{M} - \omega)\, u, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} = \langle u, (\mathrm{M} - \omega)\, v \rangle_{\mathsf{L}^2_\Lambda(\Omega)} = 0 \quad \forall\, v \in \mathcal{N}_{\mathrm{gen}}(\,\mathcal{M} - \omega\,),$$

meaning (3.7) is necessary. In order to show, that (3.7) is also sufficient, we use Eidus' principle of limiting absorption. Therefore, let $s > 1/2$ and $f \in \mathsf{L}^2_s(\Omega)$ satisfy (3.7). We take a sequence $(\sigma_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $\sigma_n \longrightarrow 0$ and construct a sequence of frequencies

$$(\omega_n)_{n\in\mathbb{N}}, \quad \omega_n := \sqrt{\omega^2 + i\sigma_n\omega} \in \mathbb{C}_+ \setminus \mathbb{R}\,,$$

converging to ω. Since $\mathcal{M}$ is a self-adjoint operator we obtain (cf. Section 3) a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$, $u_n := \mathcal{L}_{\omega_n} f \in \mathbf{R}_{\Gamma_1}(\Omega) \times \mathbf{R}_{\Gamma_2}(\Omega)$ satisfying

$(\mathrm{M}-\omega_n)\,u_n = f$. Now our aim is to show that this sequence or at least a subsequence is converging to a solution u. By Lemma 5.1, we decompose

$$u_n = \hat{u}_n + \tilde{u}_n \quad \text{and} \quad f = \hat{f} + \tilde{f}\,,$$

with

$$\begin{aligned} \hat{u}_n, \hat{f} &\in \overline{\nabla \mathbf{H}^1_{\Gamma_1}(\Omega)} \times \overline{\nabla \mathbf{H}^1_{\Gamma_2}(\Omega)} \subset {}_0\mathsf{R}_{\Gamma_1}(\Omega) \times {}_0\mathsf{R}_{\Gamma_2}(\Omega)\,, \\ \tilde{u}_n, \tilde{f} &\in \left(\mathbf{R}_{\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{\Gamma_2}(\Omega)\right) \times \left(\mathbf{R}_{\Gamma_2}(\Omega) \cap \mu^{-1}{}_0\mathsf{D}_{\Gamma_1}(\Omega)\right). \end{aligned} \tag{5.1}$$

Inserting these (orthogonal) decompositions in the differential equation, we end up with two equations

$$-\omega_n \hat{u}_n = \hat{f} \quad \text{and} \quad (\mathrm{M}-\omega_n)\,\tilde{u}_n = \tilde{f}\,,$$

noting that the first one is trivial and implies L^2-convergence of $(\hat{u}_n)_{n\in\mathbb{N}}$. For dealing with the second equation, we need the following additional assumption on $(u_n)_{n\in\mathbb{N}}$, which we will prove in the end:

$$\forall\, t < -1/2 \quad \exists\, c \in (0,\infty) \quad \forall\, n \in \mathbb{N}: \qquad \|u_n\|_{\mathsf{L}^2_t(\Omega)} \le c \tag{5.2}$$

Let $\hat{t} < -1/2$ and $c \in (0,\infty)$ such that (5.2) holds. Then, by construction and $(5.1)_2$, the sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ is bounded in $\big(\mathbf{R}_{\hat{t},\Gamma_1}(\Omega) \cap \varepsilon^{-1}{}_0\mathsf{D}_{\hat{t},\Gamma_2}(\Omega)\big) \times \big(\mathbf{R}_{\hat{t},\Gamma_2}(\Omega) \cap \mu^{-1}{}_0\mathsf{D}_{\hat{t},\Gamma_1}(\Omega)\big)$. Hence (Theorem 3.4 and Lemma 3.3), $(\tilde{u}_n)_{n\in\mathbb{N}}$ has a subsequence $(\tilde{u}_{\pi(n)})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2_{\tilde{t}}(\Omega)$ for some $\tilde{t} < \hat{t}$ and by the equation even in $\mathbf{R}_{\tilde{t},\Gamma_1}(\Omega) \times \mathbf{R}_{\tilde{t},\Gamma_2}(\Omega)$. Consequently, the entire sequence $(u_{\pi(n)})_{n\in\mathbb{N}}$ converges in $\mathbf{R}_{\tilde{t}}(\Omega)$ to some u satisfying

$$u \in \mathbf{R}_{\tilde{t},\Gamma_1}(\Omega) \times \mathbf{R}_{\tilde{t},\Gamma_2}(\Omega) \quad \text{and} \quad (\mathrm{M}-\omega)\,u = f\,.$$

Additionally, with Corollary 4.3 and Lemma 2.2 we obtain for $n \in \mathbb{N}$ and arbitrary $v \in \mathcal{N}_{\mathrm{gen}}(\mathcal{M}-\omega)$,

$$\begin{aligned} 0 = \langle f, v\rangle_{\mathsf{L}^2_\Lambda(\Omega)} &= \langle (\mathrm{M}-\omega_{\pi(n)})\,u_{\pi(n)}, v\rangle_{\mathsf{L}^2_\Lambda(\Omega)} \\ &= \langle u_{\pi(n)}, (\mathrm{M}-\overline{\omega}_{\pi(n)})\,v\rangle_{\mathsf{L}^2_\Lambda(\Omega)} = (\omega-\omega_{\pi(n)})\cdot\langle u_{\pi(n)}, v\rangle_{\mathsf{L}^2_\Lambda(\Omega)}\,. \end{aligned}$$

Hence $\langle u_{\pi(n)}, v\rangle_{\mathsf{L}^2_\Lambda(\Omega)} = 0$ and as $\langle\,\cdot\,, v\rangle_{\mathsf{L}^2_\Lambda(\Omega)}$ is continuous on $\mathsf{L}^2_{\tilde{t}}(\Omega) \times \mathsf{L}^2_{\tilde{t}}(\Omega)$ by (1), we obtain

$$\langle u, v\rangle_{\mathsf{L}^2_\Lambda(\Omega)} = \lim_{n\to\infty}\langle u_{\pi(n)}, v\rangle_{\mathsf{L}^2_\Lambda(\Omega)} = 0\,.$$

Thus, up to now, we have constructed a vector field $u \in \mathcal{N}_{\mathrm{gen}}(\mathcal{M}-\omega)^{\perp_\Lambda}$, which has the right boundary conditions and satisfies the differential equation. But for being a radiating solution, it still remains to show, that $u \in \mathbf{R}_{<-\frac{1}{2}}(\Omega)$ and enjoys the radiation condition (3.6). For that, let $t < -1/2$. Then, by Lemma 4.4, there exist $c, \delta \in (0, \infty)$ and some $\check{t} > -1/2$, such that for $n \in \mathbb{N}$ large enough we obtain uniformly in $\sigma_{\pi(n)}$, $u_{\pi(n)}$, f and $\tilde{r} > 0$:

$$\|u_{\pi(n)}\|_{\mathbf{R}_t(\Omega(\tilde{r}))} + \|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u_{\pi(n)}\|_{\mathsf{L}^2_{\check{t}}(\Omega(\tilde{r}))} \leq c \cdot \Big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u_{\pi(n)}\|_{\mathsf{L}^2(\Omega(\delta))} \Big).$$

Sending $n \longrightarrow \infty$ and afterwards $\tilde{r} \longrightarrow \infty$ (monotone convergence), we obtain

$$\|u\|_{\mathbf{R}_t(\Omega)} + \|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u\|_{\mathsf{L}^2_{\check{t}}(\Omega)} \leq c \cdot \Big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2(\Omega(\delta))} \Big) < \infty, \tag{5.3}$$

yielding

$$u \in \mathbf{R}_{<-\frac{1}{2}}(\Omega) \quad \text{and} \quad (\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u \in \mathsf{L}^2_{>-\frac{1}{2}}(\Omega).$$

This completes the proof of existence, if we can show (5.2). To this end, we assume it to be wrong, i. e., there exists $t < -1/2$ and a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)$, $u_n := \mathcal{L}_{\omega_n} f$ with $\|u_n\|_{\mathsf{L}^2_t(\Omega)} \longrightarrow \infty$ for $n \longrightarrow \infty$. Defining

$$\check{u}_n := \|u_n\|^{-1}_{\mathsf{L}^2_t(\Omega)} \cdot u_n \quad \text{and} \quad \check{f}_n := \|u_n\|^{-1}_{\mathsf{L}^2_t(\Omega)} \cdot f,$$

we have

$$\|\check{u}_n\|_{\mathsf{L}^2_t(\Omega)} = 1, \quad \check{f} \longrightarrow 0 \text{ in } \mathsf{L}^2_s(\Omega) \quad \text{and} \quad (\mathrm{M} - \omega_n)\,\check{u}_n = \check{f}_n.$$

Then, repeating the arguments from above, we obtain some $\check{t} < t$ and a subsequence $(\check{u}_{\pi(n)})_{n\in\mathbb{N}}$ converging in $\mathsf{L}^2_{\check{t}}(\Omega)$ to some $\check{u} \in \mathcal{N}_{\text{gen}}(\mathcal{M} - \omega) \cap \mathcal{N}_{\text{gen}}(\mathcal{M} - \omega)^{\perp_\Lambda}$, hence $\check{u} = 0$. But Lemma 4.4 ensures the existence of $c, \delta \in (0, \infty)$ (independent of $\sigma_{\pi(n)}$, $\check{u}_{\pi(n)}$ and $\check{f}_{\pi(n)}$) such that

$$1 = \|\check{u}_{\pi(n)}\|_{\mathsf{L}^2_t(\Omega)} \leq c \cdot \Big(\|\check{f}_{\pi(n)}\|_{\mathsf{L}^2_s(\Omega)} + \|\check{u}_{\pi(n)}\|_{\mathsf{L}^2(\Omega(\delta))} \Big) \xrightarrow{n\to\infty} 0$$

holds; a contradiction.

(5): Let $-t, s > 1/2$. By (4) the solution operator

$$\mathcal{L}_\omega : \underbrace{\mathsf{L}^2_s(\Omega) \cap \mathcal{N}_{\text{gen}}(\mathcal{M} - \omega)^{\perp_\Lambda}}_{=:\mathcal{D}(\mathcal{L}_\omega)} \longrightarrow \underbrace{\Big(\mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)\Big) \cap \mathcal{N}_{\text{gen}}(\mathcal{M} - \omega)^{\perp_\Lambda}}_{=:\mathcal{R}(\mathcal{L}_\omega)}$$

is well defined. Furthermore, due to the polynomial decay of eigensolutions, $\mathcal{D}(\mathcal{L}_\omega)$ is closed in $\mathsf{L}^2_s(\Omega)$. Thus, the assertion follows from the closed graph theorem, if we can show that $\mathcal{L}_\omega$ is closed. Therefore, take $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{L}_\omega)$ with

$$f_n \longrightarrow f \text{ in } \mathsf{L}^2_s(\Omega) \quad \text{and} \quad u_n := \mathcal{L}_\omega f_n \longrightarrow u \text{ in } \mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega).$$

Then clearly $f \in \mathcal{D}(\mathcal{L}_\omega)$, $u \in \mathcal{R}(\mathcal{L}_\omega)$ and as $(\mathrm{M} - \omega)\, u_n = f_n$, we obtain $(\mathrm{M} - \omega)\, u = f$. Now estimate (5.3) (along with monotone convergence) shows as before

$$u \in \mathbf{R}_{<-\frac{1}{2}}(\Omega) \quad \text{and} \quad (\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u \in \mathsf{L}^2_{>-\frac{1}{2}}(\Omega)\,,$$

meaning u is a radiating solution, i. e., $u = \mathcal{L}_\omega f$, which completes the proof. □

Remark 5.2. During the discussion at AANMPDE10 (10th Workshop on Analysis and Advanced Numerical Methods for Partial Differential Equations), M. Waurick and S. Trostorff pointed out that it is sufficient to use weakly convergent subsequences for the construction of the (radiating) solution. This is in fact true (the radiation condition and regularity properties follow from Lemma 4.4 by the boundedness of the sequence and the weak lower semicontinuity of the norms), but it should be noted, that Weck's local selection theorem is still needed to prove (5.2), since here norm convergence is indispensable in order to generate a contradiction. Anyway, we thank both for the vivid discussion and constructive criticism.

Appendix A. Technical tools

Lemma A.1. *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary exterior domain and $s, t, \theta \in \mathbb{R}$ with $t < s$ and $\theta > 0$. Then there exist constants $c, \delta \in (0, \infty)$ such that*

$$\|w\|_{\mathsf{L}^2_t(\Omega)} \le c \cdot \|w\|_{\mathsf{L}^2(\Omega(\delta))} + \theta \cdot \|w\|_{\mathsf{L}^2_s(\Omega)}$$

holds for all $w \in \mathsf{L}^2_s(\Omega)$.

Proof. Let $\mathbb{R}^3 \setminus \Omega \subset \mathrm{U}(r_0)$. For $\tilde{r} \ge r_0$, we obtain

$$\begin{aligned} \|w\|^2_{\mathsf{L}^2_t(\Omega)} &= \|w\|^2_{\mathsf{L}^2_t(\Omega(\tilde{r}))} + \|w\|^2_{\mathsf{L}^2_t(\breve{\mathrm{U}}(\tilde{r}))} \\ &\le (1 + \tilde{r}^2)^{\max\{0,t\}} \cdot \|w\|^2_{\mathsf{L}^2(\Omega(\tilde{r}))} + (1 + \tilde{r}^2)^{t-s} \cdot \|w\|^2_{\mathsf{L}^2_s(\breve{\mathrm{U}}(\tilde{r}))} \\ &\le (1 + \tilde{r}^2)^{\max\{0,t\}} \cdot \|w\|^2_{\mathsf{L}^2(\Omega(\tilde{r}))} + (1 + \tilde{r}^2)^{t-s} \cdot \|w\|_{\mathsf{L}^2_s(\Omega)}\,. \end{aligned}$$

Since $t < s$, we can choose $\tilde{r}$ such that $(1 + \tilde{r}^2)^{t-s} \le \theta^2$, which completes the proof. □

Lemma A.2. *For $\tilde{r} > 0$ and $f \in \mathsf{L}^1(\mathbb{R}^n)$, it holds*

$$\liminf_{r\to\infty} r \int_{\mathrm{S}(r)} |f|\, d\lambda_s^{n-1} = 0\,.$$

Proof. Otherwise, there exists $\hat{r} > 0$ and $c > 0$ such that

$$\int_{\mathrm{S}(r)} |f|\, d\lambda_s^{n-1} \ge \frac{c}{r} \quad \forall\, r \ge \hat{r}$$

and using Fubini's theorem, we obtain

$$\|f\|^2_{\mathsf{L}^1(\mathbb{R}^n)} \geq \int_{\breve{\mathrm{U}}(\hat{r})} |f|\, d\lambda^n = \int_{\hat{r}}^{\infty} \int_{\mathrm{S}(r)} |f|\, d\lambda_s^{n-1}\, dr \geq c \cdot \int_{\hat{r}}^{\infty} \frac{1}{r}\, dr = \infty\,,$$

a contradiction. □

Lemma A.3. *Let $\Omega \subset \mathbb{R}^3$ be an exterior weak Lipschitz domain with boundary Γ and weak Lipschitz boundary parts Γ_1 and $\Gamma_2 = \Gamma \setminus \overline{\Gamma}_1$. Furthermore, let $\hat{r}, \tilde{r} \in \mathbb{R}_+$ with $\tilde{r} > \hat{r}$ and $\mathbb{R}^3 \setminus \Omega \subset \mathrm{U}(\hat{r})$ as well as $\phi \in \mathsf{C}^0([\hat{r}, \tilde{r}], \mathbb{C})$. If $u \in \mathbf{R}_{t,\Gamma_1}(\Omega) \times \mathbf{R}_{t,\Gamma_2}(\Omega)$ for some $t \in \mathbb{R}$, it holds*

$$\langle\, \Phi \Xi u\,, \Lambda_0 u\,\rangle_{\mathsf{L}^2(\mathrm{G}(\hat{r},\tilde{r}))} = \langle\, \Psi \operatorname{Rot} u\,, \Lambda_0 u\,\rangle_{\mathsf{L}^2(\Omega(\tilde{r}))} + \langle\, \Psi u\,, \operatorname{Rot} \Lambda_0 u\,\rangle_{\mathsf{L}^2(\Omega(\tilde{r}))}\,, \tag{A.1}$$

where $\Phi := \phi \circ r$, $\Psi := \psi \circ r$, and

$$\psi : [0, \tilde{r}] \longrightarrow \mathbb{C},\ \sigma \longmapsto \int_{\max\{\hat{r},\sigma\}}^{\tilde{r}} \phi(\tau)\, d\tau.$$

Proof. As $\mathsf{C}^\infty_{\Gamma_1}(\Omega)$ respectively $\mathsf{C}^\infty_{\Gamma_2}(\Omega)$ is dense in $\mathbf{R}_{t,\Gamma_1}(\Omega)$ respectively $\mathbf{R}_{t,\Gamma_2}(\Omega)$ by definition it is enough to show equation (A.1) for $u = (u_1, u_2) \in \mathsf{C}^\infty_{\Gamma_1}(\Omega) \times \mathsf{C}^\infty_{\Gamma_2}(\Omega) \subset \mathring{\mathsf{C}}^\infty(\mathbb{R}^3)$. Observing that the support of products of u_1 and u_2 is compactly supported in some $\Theta \subset \overline{\Theta} \subset \Omega$, we may choose a cut-off function $\varphi \in \mathring{\mathsf{C}}^\infty(\Omega) \subset \mathring{\mathsf{C}}^\infty(\mathbb{R}^3)$ with $\varphi|_\Theta = 1$ and replace u by $\varphi u =: v =: (E, H)$. Without loss of generality we assume $\mathbb{R}^3 \setminus \Theta \subset \mathrm{U}(\hat{r})$. Using Gauss's divergence theorem we compute

$$\begin{aligned}
\langle\, \Phi \Xi u\,, \Lambda_0 u\,\rangle_{\mathsf{L}^2(\mathrm{G}(\hat{r},\tilde{r}))} &= \int_{\hat{r}}^{\tilde{r}} \phi(r) \langle\, \Xi u\,, \Lambda_0 u\,\rangle_{\mathsf{L}^2(\mathrm{S}(r))}\, dr = \int_{\hat{r}}^{\tilde{r}} \phi(r) \langle\, \Xi v\,, \Lambda_0 v\,\rangle_{\mathsf{L}^2(\mathrm{S}(r))}\, dr \\
&= \int_{\hat{r}}^{\tilde{r}} \phi(r) (\mu_0 \langle\, \xi \times E\,, H\,\rangle_{\mathsf{L}^2(\mathrm{S}(r))} - \varepsilon_0 \langle\, \xi \times H\,, E\,\rangle_{\mathsf{L}^2(\mathrm{S}(r))}\,)\, dr \\
&= \int_{\hat{r}}^{\tilde{r}} \phi(r) \int_{\mathrm{S}(r)} (\mu_0 \xi \cdot (E \times \overline{H}) - \varepsilon_0 \xi \cdot (H \times \overline{E}))\, d\lambda_s^2\, dr \\
&= \int_{\hat{r}}^{\tilde{r}} \phi(r) \int_{\mathrm{U}(r)} (\mu_0 \operatorname{div}(E \times \overline{H}) - \varepsilon_0 \operatorname{div}(H \times \overline{E}))\, d\lambda^3\, dr\,.
\end{aligned}$$

Note, that

$$\begin{aligned}
&\mu_0 \operatorname{div}(E \times \overline{H}) - \varepsilon_0 \operatorname{div}(H \times \overline{E}) \\
&\quad = \mu_0 (\overline{H} \operatorname{rot} E - E \operatorname{rot} \overline{H}) - \varepsilon_0 (\overline{E} \operatorname{rot} H - H \operatorname{rot} \overline{E})
\end{aligned}$$

$$= \big((\mu_0\overline{H})\,\mathrm{rot}\,E - (\varepsilon_0\overline{E})\,\mathrm{rot}\,H\big) + \big(H\,\mathrm{rot}\,(\varepsilon_0\overline{E}) - E\,\mathrm{rot}\,(\mu_0\overline{H})\big)$$
$$= \overline{\Lambda_0 v}\cdot \mathrm{Rot}\,v + v\cdot \mathrm{Rot}\,\overline{\Lambda_0 v}.$$

Hence, by Fubini's theorem, we see

$$\begin{aligned}\langle \Phi\Xi u\,,\Lambda_0 u\rangle_{\mathrm{L}^2(\mathrm{G}(\hat{r},\tilde{r}))} &= \int_{\hat{r}}^{\tilde{r}} \phi(r)\,\big(\langle \mathrm{Rot}\,v\,,\Lambda_0 v\rangle_{\mathrm{L}^2(\mathrm{U}(r))} + \langle v\,,\mathrm{Rot}\,\Lambda_0 v\rangle_{\mathrm{L}^2(\mathrm{U}(r))}\big)\,dr\\
&= \int_{\hat{r}}^{\tilde{r}} \phi(r) \int_0^r \big(\langle \mathrm{Rot}\,v\,,\Lambda_0 v\rangle_{\mathrm{L}^2(\mathrm{S}(\sigma))} + \langle v\,,\mathrm{Rot}\,\Lambda_0 v\rangle_{\mathrm{L}^2(\mathrm{S}(\sigma))}\big)\,d\sigma\,dr\\
&= \int_0^{\tilde{r}} \int_{\max\{\hat{r},\sigma\}}^{\tilde{r}} \phi(r)\,\big(\langle \mathrm{Rot}\,v\,,\Lambda_0 v\rangle_{\mathrm{L}^2(\mathrm{S}(\sigma))} + \langle v\,,\mathrm{Rot}\,\Lambda_0 v\rangle_{\mathrm{L}^2(\mathrm{S}(\sigma))}\big)\,dr\,d\sigma\\
&= \int_0^{\tilde{r}} \psi(\sigma)\,\big(\langle \mathrm{Rot}\,v\,,\Lambda_0 v\rangle_{\mathrm{L}^2(\mathrm{S}(\sigma))} + \langle v\,,\mathrm{Rot}\,\Lambda_0 v\rangle_{\mathrm{L}^2(\mathrm{S}(\sigma))}\big)\,d\sigma\\
&= \langle \Psi\,\mathrm{Rot}\,v\,,\Lambda_0 v\rangle_{\mathrm{L}^2(\mathrm{U}(\tilde{r}))} + \langle \Psi v\,,\mathrm{Rot}\,\Lambda_0 v\rangle_{\mathrm{L}^2(\mathrm{U}(\tilde{r}))}\\
&= \langle \Psi\,\mathrm{Rot}\,v\,,\Lambda_0 v\rangle_{\mathrm{L}^2(\Omega(\tilde{r}))} + \langle \Psi v\,,\mathrm{Rot}\,\Lambda_0 v\rangle_{\mathrm{L}^2(\Omega(\tilde{r}))}\\
&= \langle \Psi\,\mathrm{Rot}\,u\,,\Lambda_0 u\rangle_{\mathrm{L}^2(\Omega(\tilde{r}))} + \langle \Psi u\,,\mathrm{Rot}\,\Lambda_0 u\rangle_{\mathrm{L}^2(\Omega(\tilde{r}))}\,,\end{aligned}$$

where the last line follows by construction of v. □

We end this section with a lemma, which will be needed to prove the polynomial decay and a priori estimate for the Helmholtz equation and can be shown by elementary partial integration.

Lemma A.4. *Let $w \in \mathbf{H}^2_{\mathrm{loc}}(\mathbb{R}^n)$, $0 \notin \operatorname{supp} w$, $m \in \mathbb{R}$ and $\tilde{r} > 0$. Then with $\partial_r := \xi\cdot\nabla$:*

(1) $$\mathrm{Re}\int_{\mathrm{U}(\tilde{r})} r^{m+1}\Delta w\,\partial_r\overline{w} = \frac{1}{2}\int_{\mathrm{U}(\tilde{r})} r^m\Big((n+m-2)|\nabla w|^2 - 2m|\partial_r w|^2\Big) + \int_{\mathrm{S}(\tilde{r})} r^{m+1}\Big(|\partial_r w|^2 - \frac{1}{2}|\nabla w|^2\Big)$$

(2) $$\mathrm{Re}\int_{\mathrm{U}(\tilde{r})} r^m\Delta w\overline{w} = -\int_{\mathrm{U}(\tilde{r})} r^m\Big(|\nabla w|^2 - \frac{m}{2}\,(n+m-2)\,r^{-2}|w|^2\Big) + \int_{\mathrm{S}(\tilde{r})} r^m\Big(\mathrm{Re}\,(\partial_r w\overline{w}) - \frac{m}{2}\,r^{-1}|w|^2\Big)$$

(3) $$\mathrm{Im}\int_{\mathrm{U}(\tilde{r})} r^m\Delta w\overline{w} = -m\int_{\mathrm{U}(\tilde{r})} r^{m-1}\mathrm{Im}\,(\partial_r w\overline{w}) + \frac{1}{2}\int_{\mathrm{S}(\tilde{r})} r^{m+1}|w|^2$$

(4) $$\mathrm{Re}\int_{\mathrm{U}(\tilde{r})} r^{m+1}w\,\partial_r\,\overline{w} = -\frac{1}{2}\int_{\mathrm{U}(\tilde{r})} r^m\,(n+m)|w|^2 + \frac{1}{2}\int_{\mathrm{S}(\tilde{r})} r^{m+1}|w|^2$$

Appendix B. Polynomial decay and a-priori estimate for the Helmholtz equation

In this section we present well-known results for the Helmholtz equation, which we will use to achieve similar results for Maxwell's equations. We start with a regularity result (cf. [27, Lemma 4]) and the polynomial decay (cf. [27, Lemma 5]).

Lemma B.1. *Let $t \in \mathbb{R}$. If $w \in \mathsf{L}^2_t(\mathbb{R}^n)$ and $\Delta w \in \mathsf{L}^2_t(\mathbb{R}^n)$, it holds $w \in \mathbf{H}^2_t(\mathbb{R}^n)$ and*

$$\|w\|_{\mathbf{H}^2_t(\mathbb{R}^n)} \leq c \cdot \Big(\|\Delta w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} \Big)$$

with $c \in (0, \infty)$ independent of w and Δw.

Proof. For $t = 0$, we have $w, \Delta w \in \mathsf{L}^2(\mathbb{R}^n)$ and using Fourier transformation, we obtain

$$\begin{aligned}\|\Delta w\|^2_{\mathsf{L}^2(\mathbb{R}^n)} + \|w\|^2_{\mathsf{L}^2(\mathbb{R}^n)} &= \big\|r^2 \mathcal{F}(w)\big\|^2_{\mathsf{L}^2(\mathbb{R}^n)} + \|\mathcal{F}(w)\|^2_{\mathsf{L}^2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} (r^4 + 1)|\mathcal{F}(w)|^2 \geq \frac{1}{2} \cdot \big\|(1 + r^2)\mathcal{F}(w)\big\|^2_{\mathsf{L}^2(\mathbb{R}^n)}, \end{aligned} \tag{B.1}$$

yielding $w \in \mathbf{H}^2(\mathbb{R}^n)$ and the desired estimate. So let us switch to $t \neq 0$. Then, using a well-known result concerning inner regularity (e. g., [3, Chapter VII, *Section* 3.2, Theorem 1]), we already have $w \in \mathbf{H}^2_{\mathrm{loc}}(\mathbb{R}^n)$. Now let $\tilde{r} > 1$ and define $\eta_{\tilde{r}} \in \mathring{\mathsf{C}}^\infty(\mathbb{R}^n)$ by $\eta_{\tilde{r}}(x) := \rho^t \eta(r(x)/\tilde{r})$. Then $\eta_{\tilde{r}} w \in \mathbf{H}^2(\mathbb{R}^n)$,

$$|\nabla \eta_{\tilde{r}}| \leq c \cdot \rho^{t-1} \quad \text{with} \;\; c = c(t) > 0\,,$$

and

$$\begin{aligned}\langle\, \nabla(\eta_{\tilde{r}} w)\,, \nabla(\eta_{\tilde{r}} w)\,\rangle_{\mathsf{L}^2(\mathbb{R}^n)} &= \mathrm{Re}\,\langle\, \nabla w\,, \nabla(\eta^2_{\tilde{r}} w)\,\rangle_{\mathsf{L}^2(\mathbb{R}^n)} + \big\|(\nabla \eta_{\tilde{r}}) w\big\|^2_{\mathsf{L}^2(\mathbb{R}^n)} \\ &\leq c \cdot \Big(\|\eta_{\tilde{r}} \Delta w\|_{\mathsf{L}^2(\mathbb{R}^n)}\, \|\eta_{\tilde{r}} w\|_{\mathsf{L}^2(\mathbb{R}^n)} + \|w\|^2_{\mathsf{L}^2_{t-1}(\mathbb{R}^n)} \Big) \\ &\leq c \cdot \Big(\|\Delta w\|^2_{\mathsf{L}^2_t(\mathbb{R}^n)} + \|w\|^2_{\mathsf{L}^2_t(\mathbb{R}^n)} \Big), \end{aligned}$$

with $c = c(n, t) \in (0, \infty)$, hence

$$\|\nabla w\|_{\mathsf{L}^2_t(\mathrm{B}(\tilde{r}))} \leq \|\nabla(\eta_{\tilde{r}} w) - (\nabla \eta_{\tilde{r}}) w\|_{\mathsf{L}^2(\mathbb{R}^n)} \leq c(n, t) \cdot \Big(\|\Delta w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} \Big).$$

Sending $\tilde{r} \longrightarrow \infty$ (monotone convergence) shows $w \in \mathbf{H}^1_t(\mathbb{R}^n)$ and

$$\|w\|_{\mathbf{H}^1_t(\mathbb{R}^n)} \leq c(n, t) \cdot \Big(\|\Delta w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} \Big). \tag{B.2}$$

Moreover,

$$\Delta(\rho^t w) = t\Big(n + (t - 2)\frac{r^2}{1 + r^2}\Big)\rho^{t-2} w + 2r\rho^{t-2}\partial_r w + \rho^t \Delta w\,,$$

such that with (B.2) we obtain

$$\left\|\Delta(\rho^t w)\right\|_{\mathsf{L}^2(\mathbb{R}^n)} \le c \cdot \Big(\|\Delta w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} \Big), \tag{B.3}$$

with $c \in (0,\infty)$ independent of w and Δw. Hence $\Delta(\rho^t w) \in \mathsf{L}^2(\mathbb{R}^n)$ and we may apply the first case. This shows $\rho^t w \in \mathbf{H}^2(\mathbb{R}^n)$ and using (B.1), (B.2) and (B.3), we obtain (uniformly w. r. t. w and Δw)

$$\begin{aligned}
\|w\|_{\mathbf{H}^2_t(\mathbb{R}^n)} &\le c \cdot \Big(\left\|\rho^t w\right\|_{\mathbf{H}^2(\mathbb{R}^n)} + \left\|(\nabla\rho^t)\nabla w\right\|_{\mathsf{L}^2(\mathbb{R}^n)} + \left\|(\nabla\rho^t) w\right\|_{\mathsf{L}^2(\mathbb{R}^n)} + \sum_{|\alpha|=2} \left\|(\partial^\alpha \rho^t) w\right\|_{\mathsf{L}^2(\mathbb{R}^n)} \Big) \\
&\le c \cdot \Big(\left\|\Delta(\rho^t w)\right\|_{\mathsf{L}^2(\mathbb{R}^n)} + \left\|\rho^t w\right\|_{\mathsf{L}^2(\mathbb{R}^n)} + \|\nabla w\|_{\mathsf{L}^2_{t-1}(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_{t-1}(\mathbb{R}^n)} \Big) \\
&\le c \cdot \Big(\|\Delta w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_t(\mathbb{R}^n)} \Big)
\end{aligned}$$

yielding $w \in \mathbf{H}^2_t(\mathbb{R}^n)$ and the required estimate. □

Lemma B.2 (Polynomial decay). *Let $J \Subset \mathbb{R} \setminus (0)$ be some interval, $\gamma \in J$ and $s,t \in \mathbb{R}$ with $t > -1/2$ and $t \le s$. If $w \in \mathsf{L}^2_t(\mathbb{R}^n)$ and $g := (\Delta + \gamma^2)\, w \in \mathsf{L}^2_{s+1}(\mathbb{R}^n)$, it holds*

$$w \in \mathbf{H}^2_s(\mathbb{R}^n) \quad \textit{and} \quad \|w\|_{\mathbf{H}^2_s(\mathbb{R}^n)} \le c \cdot \Big(\|g\|_{\mathsf{L}^2_{s+1}(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_{s-1}(\mathbb{R}^n)} \Big)$$

with $c = c(n,s,J) \in (0,\infty)$ not depending on γ, g or w.

Proof. The assertion follows directly from Lemma B.1, if we can show

$$w \in \mathsf{L}^2_s(\mathbb{R}^n) \quad \text{with} \quad \|w\|_{\mathsf{L}^2_s(\mathbb{R}^n)} \le c \cdot \Big(\|g\|_{\mathsf{L}^2_{s+1}(\mathbb{R}^n)} + \|w\|_{\mathsf{L}^2_{s-1}(\mathbb{R}^n)} \Big).$$

Therefore, let $v := \check{\chi} w$, where $\check{\chi} \in \mathsf{C}^\infty(\mathbb{R}^n)$ with $\check{\chi} = 1$ on $\check{\mathrm{U}}(1)$ and vanishing in a neighborhood of the origin. By assumption, we already have $w \in \mathbf{H}^2_t(\mathbb{R}^n)$ (cf. Lemma B.1), hence $v \in \mathbf{H}^2_{\mathrm{loc}}(\mathbb{R}^n)$ and we may apply the partial integration rules from Lemma A.4 to

$$\operatorname{Re} \int_{\mathrm{G}(\hat{r},\tilde{r})} (\Delta w + \gamma^2 w)(r^{2t+1}\partial_r \overline{w} + \beta r^{2t}\overline{w}) = \operatorname{Re} \int_{\mathrm{G}(\hat{r},\tilde{r})} (\Delta v + \gamma^2 v)(r^{2t+1}\partial_r \bar{v} + \beta r^{2t}\bar{v}) = \cdots,$$

with $\tilde{r} > \hat{r} \ge 1$ and

$$\beta := \max\{(n-1)/2\,, t + (n-1)/2\}.$$

After some rearrangements, this leads to

$$\begin{aligned}
&\int_{\mathrm{G}(\hat{r},\tilde{r})} r^{2t}\Big((\beta - (n+2t-2)/2)|\nabla w|^2 + ((n+2t)/2 - \beta)\gamma^2 |w|^2 \Big) \\
&\quad + 2t \int_{\mathrm{G}(\hat{r},\tilde{r})} r^{2t} |\partial_r w|^2 + \int_{\mathrm{S}(\tilde{r})} \tilde{r}^{2t+1} |\nabla w|^2
\end{aligned}$$

$$= -\mathrm{Re} \int\limits_{G(\hat{r},\tilde{r})} (\Delta w + \gamma^2 w)(r^{2t+1}\partial_r \overline{w} + \beta r^{2t}\overline{w}) + t(n+2t-2)\beta \int\limits_{G(\hat{r},\tilde{r})} r^{2t-2}|w|^2$$
$$+ \int\limits_{S(\hat{r})} \hat{r}^{2t+1}\Big(\beta t\hat{r}^{-2}|w|^2 - \beta\hat{r}^{-1}\mathrm{Re}\,(\partial_r w\overline{w}) - |\partial_r w|^2\Big) \tag{B.4}$$
$$+ \int\limits_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(|\partial_r w|^2 + \beta\tilde{r}^{-1}\mathrm{Re}\,(\partial_r w\overline{w}) - \beta t\tilde{r}^{-2}|w|^2\Big)$$
$$+ \frac{1}{2}\int\limits_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(|\nabla w|^2 + \gamma^2|w|^2\Big) + \frac{1}{2}\int\limits_{S(\hat{r})} \hat{r}^{2t+1}\Big(|\nabla w|^2 - \gamma^2|w|^2\Big).$$

Let us first have a look on the left-hand side. For $t \geq 0$ (i. e., $\beta = t + (n-1)/2$), we skip the second and third integral to obtain

$$\int\limits_{G(\hat{r},\tilde{r})} r^{2t}\Big((\beta - (n+2t-2)/2)|\nabla w|^2 + ((n+2t)/2 - \beta)\gamma^2|w|^2\Big)$$
$$+ 2t \int\limits_{G(\hat{r},\tilde{r})} r^{2t}|\partial_r w|^2 + \int\limits_{S(\tilde{r})} \tilde{r}^{2t+1}|\nabla w|^2$$
$$\geq \frac{1}{2}\int\limits_{G(\hat{r},\tilde{r})} r^{2t}\Big((2\beta - (n+2t-2))|\nabla w|^2 + ((n+2t) - 2\beta)\gamma^2|w|^2\Big)$$
$$= \frac{1}{2}\int\limits_{G(\hat{r},\tilde{r})} r^{2t}\Big(|\nabla w|^2 + \gamma^2|w|^2\Big),$$

while in the case of $t < 0$ (i. e., $\beta = (n-1)/2$) we just skip the third integral and end up with

$$\int\limits_{G(\hat{r},\tilde{r})} r^{2t}\Big((\beta - (n+2t-2)/2)|\nabla w|^2 + ((n+2t)/2 - \beta)\gamma^2|w|^2\Big)$$
$$+ 2t \int\limits_{G(\hat{r},\tilde{r})} r^{2t}|\partial_r w|^2 + \int\limits_{S(\tilde{r})} \tilde{r}^{2t+1}|\nabla w|^2$$
$$\geq \int\limits_{G(\hat{r},\tilde{r})} r^{2t}\Big((\beta - (n+2t-2)/2 + 2t)|\nabla w|^2 + ((n+2t)/2 - \beta)\gamma^2|w|^2\Big)$$
$$= (\frac{1}{2} + t) \int\limits_{G(\hat{r},\tilde{r})} r^{2t}(|\nabla w|^2 + \gamma^2|w|^2),$$

since $|\partial_r w| \leq |\nabla w|$. Thus for arbitrary $t \in \mathbb{R}$ the left-hand side of (B.4) can be estimated from below by

$$\min\{\frac{1}{2}, \frac{1}{2} + t\} \int\limits_{G(\hat{r},\tilde{r})} r^{2t}\Big(|\nabla w|^2 + \gamma^2|w|^2\Big).$$

For the right-hand side, we have ($\tilde{r} > 1$)

$$\int_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(|\partial_r w|^2 + \beta\tilde{r}^{-1}\operatorname{Re}(\partial_r w\overline{w}) - \beta t\tilde{r}^{-2}|w|^2\Big)$$
$$\leq \int_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(|\partial_r w|^2 + \beta|\partial_r w\overline{w}| + \beta|t||w|^2\Big) \leq c \cdot \int_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(|\nabla w|^2 + |w|^2\Big),$$

as well as

$$\int_{S(\hat{r})} \hat{r}^{2t+1}\Big(\beta t\hat{r}^{-2}|w|^2 - \beta\hat{r}^{-1}\operatorname{Re}(\partial_r w\overline{w}) - |\partial_r w|^2\Big)$$
$$\leq \int_{S(\hat{r})} \hat{r}^{2t+1}\Big(\beta|t|\hat{r}^{-2}|w|^2 + \beta\hat{r}^{-1}|\partial_r w\overline{w}|\Big) \leq c \cdot \int_{S(\hat{r})} \hat{r}^{2t+1}\Big(\hat{r}^{-2}|w|^2 + |\nabla w|^2\Big),$$

such that equation (B.4) becomes

$$\min\{\frac{1}{2}, \frac{1}{2} + t\} \int_{G(\hat{r},\tilde{r})} r^{2t}\Big(|\nabla w|^2 + \gamma^2|w|^2\Big)$$
$$\leq \int_{G(\hat{r},\tilde{r})} r^{t+1}|g|\Big(r^t|\nabla w| + \beta r^{t-1}|w|\Big) + \beta\big|t(n+2t-2)\big| \int_{G(\hat{r},\tilde{r})} r^{2t-2}|w|^2$$
$$+ c(n,t) \cdot \left(\int_{S(\hat{r})} \hat{r}^{2t+1}\Big(\hat{r}^{-2}|w|^2 + |\nabla w|^2 - \gamma^2|w|^2\Big) + \int_{S(\tilde{r})} \tilde{r}^{2t+1}\Big(|\nabla w|^2 + |w|^2\Big)\right).$$

By assumption, we have $w \in \mathbf{H}^2_t(\mathbb{R}^n)$, such that according to Lemma A.2 the lower limit for $\tilde{r} \longrightarrow \infty$ of the last boundary integral vanishes. Hence we may replace $G(\hat{r},\tilde{r})$ by $\breve{U}(\hat{r})$ and in addition use Young's inequality to obtain

$$\begin{aligned}
&\left\|r^t\nabla w\right\|^2_{L^2(\breve{U}(\hat{r}))} + \gamma^2\left\|r^t w\right\|^2_{L^2(\breve{U}(\hat{r}))} \\
&\leq c(n,t) \cdot \left(\left\|r^{t+1}g\right\|^2_{L^2(\breve{U}(\hat{r}))} + \left\|r^{t-1}w\right\|^2_{L^2(\breve{U}(\hat{r}))} + \int_{S(\hat{r})} \hat{r}^{2t+1}\Big(|\nabla w|^2 - \gamma^2|w|^2 + \hat{r}^{-2}|w|^2\Big)\right) \qquad \text{(B.5)} \\
&\leq c(n,t) \cdot \left(\|g\|^2_{L^2_{t+1}(\mathbb{R}^n)} + \|w\|^2_{L^2_{t-1}(\mathbb{R}^n)} + \int_{S(\hat{r})} \hat{r}^{2t+1}\Big(|\nabla w|^2 - \gamma^2|w|^2 + \hat{r}^{-2}|w|^2\Big)\right).
\end{aligned}$$

Now suppose that $s = t$. Then the assertion simply follows by choosing $\hat{r} := 1$ as the trace theorem bounds the surface integral by $\|w\|^2_{\mathbf{H}^2(U(1))}$ and with Lemma B.1

$$\begin{aligned}
\|w\|_{\mathbf{H}^1_t(\mathbb{R}^n)} &\leq c(n,s,J) \cdot \Big(\|g\|_{L^2_{t+1}(\mathbb{R}^n)} + \|w\|_{L^2_{t-1}(\mathbb{R}^n)} + \|w\|_{\mathbf{H}^2(U(2))} \Big) \\
&\leq c(n,s,J) \cdot \Big(\|g\|_{L^2_{t+1}(\mathbb{R}^n)} + \|w\|_{L^2_{t-1}(\mathbb{R}^n)} + \|w\|_{\mathbf{H}^2_{t-1}(\mathbb{R}^n)} \Big)
\end{aligned}$$

$$\leq c(n,s,J)\cdot\Big(\,\|g\|_{\mathsf{L}^2_{t+1}(\mathbb{R}^n)}+\|w\|_{\mathsf{L}^2_{t-1}(\mathbb{R}^n)}+\|\Delta w\|_{\mathsf{L}^2_{-1}(\mathbb{R}^n)}\Big)$$
$$\leq c(n,s,J)\cdot\Big(\,\|g\|_{\mathsf{L}^2_{t+1}(\mathbb{R}^n)}+\|w\|_{\mathsf{L}^2_{t-1}(\mathbb{R}^n)}\Big)$$

holds. For the case $w \notin \mathsf{L}^2_{\mathsf{s}}(\mathbb{R}^n)$, let $\hat{s} := \sup\{m \in \mathbb{R}\,|\,u \in \mathsf{L}^2_{\mathsf{m}}(\mathbb{R}^n)\}$. Then, w. l. o. g.,[4] we may assume

$$\hat{s}-1/2 < t < \hat{s} < s \leq t+1/2\,,$$

hence $\delta := 1-2(s-t) \in (0,1)$. Multiplying (B.5) with $\hat{r}^{-\delta}$ and integrating from 1 to some $\check{r} > 1$ leads to:

$$\int_1^{\check{r}} \hat{r}^{-\delta} \int_{\breve{\mathrm{U}}(\hat{r})} r^{2t}\Big(|\nabla w|^2+\gamma^2|w|^2\Big)\,d\hat{r} \leq c(n,t)\cdot\Bigg(\int_1^{\check{r}} \hat{r}^{-\delta}\int_{\breve{\mathrm{U}}(\hat{r})} r^{2t+2}|g|^2+r^{2t-2}|w|^2\,d\hat{r} + \int_{\mathrm{G}(1,\check{r})} r^{2t+1-\delta}\Big(|\nabla w|^2-\gamma^2|w|^2+r^{-2}|w|^2\Big)\Bigg) \tag{B.6}$$

By Fubini's theorem, we have for arbitrary $h \in \mathsf{L}^1(\mathbb{R}^n)$

$$\int_1^{\check{r}} \hat{r}^{-\delta}\int_{\breve{\mathrm{U}}(\hat{r})} h\,d\hat{r} = \int_1^{\check{r}}\int_{\hat{r}}^{\infty}\int_{\mathrm{S}(\sigma)} \hat{r}^{-\delta}\,h\,d\sigma\,d\hat{r} = \int_1^{\infty}\int_1^{\min\{\sigma,\check{r}\}} \hat{r}^{-\delta}\int_{\mathrm{S}(\sigma)} h\,d\hat{r}\,d\sigma$$
$$= \int_1^{\infty}(1-\delta)^{-1}\min\{\sigma^{1-\delta}-1,\check{r}^{1-\delta}-1\}\int_{\mathrm{S}(\sigma)} h\,d\sigma$$
$$= \int_1^{\infty}\int_{\mathrm{S}(\sigma)} \underbrace{(1-\delta)^{-1}\min\{r^{1-\delta}-1,\check{r}^{1-\delta}-1\}}_{=:\theta_{\check{r}}}\,h\,d\sigma = \int_{\breve{\mathrm{U}}(1)} \theta_{\check{r}}\,h,$$

such that (B.6) becomes (note that $\theta_{\check{r}} \leq (1-\delta)^{-1}\cdot r^{1-\delta}$ and $1-\delta = 2(s-t)$)

$$\int_{\breve{\mathrm{U}}(1)} \theta_{\check{r}}\,r^{2t}\Big(|\nabla w|^2+\gamma^2|w|^2\Big) \tag{B.7}$$
$$\leq c(n,t)\cdot\Bigg(\int_{\breve{\mathrm{U}}(1)}\theta_{\check{r}}\Big(r^{2t+2}|g|^2+r^{2t-2}|w|^2\Big)+\int_{\mathrm{G}(1,\check{r})} r^{2t+1-\delta}\Big(|\nabla w|^2-\gamma^2|w|^2+r^{-2}|w|^2\Big)\Bigg)$$
$$\leq c(n,s)\cdot\Bigg(\|g\|^2_{\mathsf{L}^2_{s+1}(\mathbb{R}^n)}+\|w\|^2_{\mathsf{L}^2_{s-1}(\mathbb{R}^n)}+\int_{\mathrm{G}(1,\check{r})} r^{2s}\Big(|\nabla w|^2-\gamma^2|w|^2\Big)\Bigg).$$

4 Otherwise, we replace s and t by $t_k := t + k/4$ respectively $s_k := t_{k+1}$, $k = 0,1,2,\dots$ and obtain the assertion after finitely many steps of the type $t_k < s_k \leq t_k + 1/2$ (cf. Appendix, Section C, Proof of Lemma 4.1).

Finally, look at

$$\operatorname{Re} \int_{G(1,\check r)} r^{2s} g\overline{w} = \operatorname{Re} \int_{G(1,\check r)} r^{2s} g\bar v\,.$$

Applying Lemma A.4, we obtain (after some rearrangements)

$$\begin{aligned}
&\int_{G(1,\check r)} r^{2s}\Big(|\nabla w|^2 - \gamma^2|w|^2\Big)\\
&\quad = -\operatorname{Re} \int_{G(1,\check r)} r^{2s} g\overline{w} + s(n+2s-2) \int_{G(1,\check r)} r^{2s-2}|w|^2\\
&\qquad + \int_{S(\check r)} \check r^{2s}\Big(\operatorname{Re}(\partial_r w\overline{w}) - s\check r^{-1}|w|^2\Big) - \int_{S(1)} \Big(\operatorname{Re}(\partial_r w\overline{w}) - s|w|^2\Big)\\
&\quad \le c(n,s)\cdot\bigg(\int_{G(1,\check r)} \Big(r^{2s+2}|g|^2 + r^{2s-2}|w|^2\Big) + \int_{S(1)} \Big(|\nabla w|^2 + |w|^2\Big) + \int_{S(\check r)} \check r^{2s}\Big(|\nabla w|^2 + |w|^2\Big)\bigg),
\end{aligned}$$

hence (using the trace theorem and Lemma B.1)

$$\begin{aligned}
&\int_{G(1,\check r)} r^{2s}\Big(|\nabla w|^2 - \gamma^2|w|^2\Big)\\
&\quad \le c(n,s)\cdot\bigg(\|g\|^2_{L^2_{s+1}(\mathbb{R}^n)} + \|w\|^2_{L^2_{s-1}(\mathbb{R}^n)} + \|w\|^2_{H^2(U(1))} + \int_{S(\check r)} \check r^{2s}\Big(|\nabla w|^2 + |w|^2\Big)\bigg) \qquad \text{(B.8)}\\
&\quad \le c(n,s,J)\cdot\bigg(\|g\|^2_{L^2_{s+1}(\mathbb{R}^n)} + \|w\|^2_{L^2_{s-1}(\mathbb{R}^n)} + \int_{S(\check r)} \check r^{2s}\Big(|\nabla w|^2 + |w|^2\Big)\bigg)
\end{aligned}$$

and inserting (B.8) into (B.7) we end up with

$$\int_{\check U(1)} \theta_{\check r}\, r^{2t}(|\nabla w|^2 + \gamma^2|w|^2) \le c(n,s,J)\cdot\bigg(\|g\|^2_{L^2_{s+1}(\mathbb{R}^n)} + \|w\|^2_{L^2_{s-1}(\mathbb{R}^n)} + \int_{S(\check r)} \check r^{2s}\Big(|\nabla w|^2 + |w|^2\Big)\bigg).$$

Again the lower limit for $\check r \longrightarrow \infty$ of the boundary integral vanishes (cf. Lemma A.2 and observe that $w \in H^2_{s-\frac{1}{2}}(\mathbb{R}^n)$, since $0 < s-t \le 1/2$ by assumption), such that passing to the limit on a suitable subsequence, we obtain

$$\begin{aligned}
\|w\|^2_{L^2_s(\mathbb{R}^n)} &\le c(n,s,J)\cdot\bigg(\int_{\check U(1)} (1-\delta)^{-1} r^{2t+1-\delta}\Big(|\nabla w|^2 + \gamma^2|w|^2\Big) + \|w\|^2_{L^2_s(U(1))}\bigg)\\
&\le c(n,s,J)\cdot\bigg(\lim_{\check r\to\infty} \int_{\check U(1)} \theta_{\check r} r^{2t}\Big(|\nabla w|^2 + \gamma^2|w|^2\Big) + \|w\|^2_{L^2_{s-1}(U(1))}\bigg)\\
&\le c(n,s,J)\cdot\Big(\|g\|^2_{L^2_{s+1}(\mathbb{R}^n)} + \|w\|^2_{L^2_{s-1}(\mathbb{R}^n)}\Big),
\end{aligned}$$

showing $w \in L^2_s(\mathbb{R}^n)$ and the required estimate. □

Lemma B.3 (A priori estimate). *Let $n \in \mathbb{N}$, $t < -1/2$, $1/2 < s < 1$, and let $J \Subset \mathbb{R}\setminus(0)$ be an interval. Then there exist $c, \delta \in (0, \infty)$, such that for all $\beta \in \mathbb{C}_+$ with $\beta^2 = \nu^2 + i\nu\tau$, $\nu \in J$, $\tau \in (0, 1]$ and $g \in \mathsf{L}^2_s(\mathbb{R}^n)$*

$$\begin{aligned}&\left\|(\Delta+\beta)^{-1}g\right\|_{\mathsf{L}^2_t(\mathbb{R}^n)} + \left\|\exp(-i\nu r)(\Delta+\beta)^{-1}g\right\|_{\mathsf{H}^1_{s-2}(\mathbb{R}^n)}\\ &\leq c\cdot\Big(\|g\|_{\mathsf{L}^2_s(\mathbb{R}^n)} + \left\|(\Delta+\beta)^{-1}g\right\|_{\mathsf{L}^2(\Omega(\delta))}\Big)\end{aligned} \tag{B.9}$$

holds.

Ikebe and Saito [6] proved this estimate for the space dimension $n = 3$ and with $t = -s$, which already shows the result also for any $t < -1/2$ as the norms depend monotonic on the parameters s and t. For arbitrary space dimensions, we follow the proof of Vogelsang [21, Satz 4].

Proof. First of all, observe that

$$\Delta : \mathbf{H}^2(\mathbb{R}^n) \subset \mathsf{L}^2(\mathbb{R}^n) \longrightarrow \mathsf{L}^2(\mathbb{R}^n),\ w \longmapsto \Delta w$$

is self-adjoint and, therefore, $w := (\Delta+\beta)^{-1}g \in \mathbf{H}^2(\mathbb{R}^n)$ is well-defined. Moreover, due to the monotone dependence of the norms on the parameters s and t, it is enough to concentrate on the case $t = -s$. With $w_e := \exp(-i\nu r)w$ and $g_e := \exp(-i\nu r)g$, we have $w_e \in \mathbf{H}^2(\Omega)$ and

$$\Delta w_e + i\nu\Big(\tau w_e + \frac{n-1}{r}w_e + 2\partial_r w_e\Big) = g_e\,.$$

Applying Lemma A.4 to

$$\operatorname{Re}\int_{G(1,\tilde r)} g_e\Big(r^{2s-1}\partial_r\overline{w}_e + \frac{n-1}{2}r^{2s-2}\overline{w}_e + \frac{\tau}{2}r^{2s-1}\overline{w}_e\Big) = \cdots,$$

with $\tilde r > 1$ and using the same techniques as in the proof of Lemma B.2 we obtain

$$\begin{aligned}&\frac12\int_{G(1,\tilde r)} r^{2s-2}\Big((4s-4)|\partial_r w_e|^2 - (2s-3)|\nabla w_e|^2\Big) + \frac12\int_{G(1,\tilde r)} r^{2s-1}\tau|\nabla w_e|^2\\ &= -\operatorname{Re}\int_{G(1,\tilde r)} g_e\Big(r^{2s-1}\partial_r\overline{w}_e + \frac{n-1}{2}r^{2s-2}\overline{w}_e + \frac{\tau}{2}r^{2s-1}\overline{w}_e\Big)\\ &\quad + \frac{n-1}{2}(s-1)(n+2s-4)\int_{G(1,\tilde r)} r^{2s-4}|w_e|^2 + \frac{\tau}{4}(2s-1)(n+2s-3)\int_{G(1,\tilde r)} r^{2s-3}|w_e|^2\\ &\quad + \frac12\int_{S(\tilde r)} \tilde r^{2s-1}\Big(2|\partial_r w_e|^2 + \tau\operatorname{Re}(\partial_r w_e\overline{w}_e) - |\nabla w_e|^2 - \frac{(2s-1)}{\tilde r}\tau|w_e|^2\Big)\\ &\quad + \frac{(n-1)}{2}\int_{S(\tilde r)} \tilde r^{2s-2}\Big(\operatorname{Re}(\partial_r w_e\overline{w}_e) - \frac{s-1}{\tilde r}|w_e|^2\Big)\end{aligned}$$

$$-\frac{1}{2}\int_{S(1)}\Big(2|\partial_r w_e|^2+\tau\operatorname{Re}(\partial_r w_e\overline{w}_e)-|\nabla w_e|^2-(2s-1)\,\tau|w_e|^2\Big)$$
$$-\frac{(n-1)}{2}\int_{S(1)}\Big(\operatorname{Re}(\partial_r w_e\overline{w}_e)-(s-1)|w_e|^2\Big).$$

Since $4s-4<0$ and $|\partial_r w_e|\le|\nabla w_e|$, the left-hand side can be estimated from below

$$\frac{1}{2}\int_{G(1,\tilde r)} r^{2s-2}\Big((4s-4)|\partial_r w_e|^2-(2s-3)|\nabla w_e|^2\Big)+\frac{1}{2}\int_{G(1,\tilde r)} r^{2s-1}\tau|\nabla w_e|^2$$
$$\ge\frac{1}{2}\int_{G(1,\tilde r)} r^{2s-2}\Big((4s-4)-(2s-3)\Big)|\nabla w_e|^2=\Big(s-\frac{1}{2}\Big)\int_{G(1,\tilde r)} r^{2s-2}|\nabla w_e|^2,$$

while for the right-hand side we obtain

$$-\operatorname{Re}\int_{G(1,\tilde r)} g_e\Big(r^{2s-1}\partial_r\overline{w}_e+\frac{n-1}{2}r^{2s-2}\overline{w}_e+\frac{\tau}{2}r^{2s-1}\overline{w}_e\Big)+\cdots$$
$$\le\int_{G(1,\tilde r)} r^s|g_e|\Big(r^{s-1}|\nabla w_e|+\frac{n-1}{2}r^{s-2}|w_e|+\frac{\tau}{2}r^{s-1}|w_e|\Big)$$
$$+c\cdot\Bigg(\int_{G(1,\tilde r)} r^{2s-4}|w_e|^2+\tau\int_{G(1,\tilde r)} r^{s-2}|w_e|r^{s-1}|w_e|$$
$$+\int_{S(1)}\Big(|\nabla w_e|^2+|\partial_r w_e\overline{w}_e|+|w_e|^2\Big)+\int_{S(\tilde r)}\tilde r^{2s-1}\Big(|\nabla w_e|^2+|\partial_r w_e\overline{w}_e|+|w_e|^2\Big)\Bigg),$$

yielding

$$\Big(s-\frac{1}{2}\Big)\int_{G(1,\tilde r)} r^{2s-2}|\nabla w_e|^2$$
$$\le\int_{G(1,\tilde r)} r^s|g_e|\Big(r^{s-1}|\nabla w_e|+\frac{n-1}{2}r^{s-2}|w_e|+\frac{\tau}{2}r^{s-1}|w_e|\Big)+c\cdot\Bigg(\int_{G(1,\tilde r)} r^{2s-4}|w_e|^2$$
$$+\tau\int_{G(1,\tilde r)} r^{2s-2}|w_e|+\int_{S(1)}\Big(|\nabla w_e|^2+|w_e|^2\Big)+\int_{S(\tilde r)}\tilde r^{2s-1}\Big(|\nabla w_e|^2+|w_e|^2\Big)\Bigg).$$

Here, as well as in the sequel, $c\in(0,\infty)$ denotes a generic constant independent of ν, τ, w and g. According to Lemma A.2, the lower limit for $\tilde r\longrightarrow\infty$ of the last boundary integral vanishes. Thus we may omit it and replace $G(1,\tilde r)$ by $\breve{U}(1)$, such that using Young's inequality we end up with

$$\big\|r^{s-1}\,\nabla w_e\big\|^2_{L^2(\breve{U}(1))}$$

$$\leq c \cdot \Big(\|r^s g_e\|^2_{L^2(\check{U}(1))} + \tau \left\| r^{s-1} w_e \right\|^2_{L^2(\check{U}(1))} + \left\| r^{s-2} w_e \right\|^2_{L^2(\check{U}(1))} + \int\limits_{S(1)} \left(|\nabla w_e|^2 + |w_e|^2 \right) \Big)$$

$$\leq c \cdot \Big(\|g_e\|^2_{L^2(\mathbb{R}^n)} + \tau \|w_e\|^2_{L^2_{s-1}(\mathbb{R}^n)} + \|w_e\|^2_{L^2_{s-2}(\mathbb{R}^n)} + \int\limits_{S(1)} \left(|\nabla w_e|^2 + |w_e|^2 \right) \Big) .$$

In addition, the surface integral is bounded by $\|w_e\|^2_{\mathbf{H}^2(U(1))}$ (trace theorem) and Lemma B.1 yields

$$\|w_e\|_{\mathbf{H}^2(U(2))} \leq c \cdot \|w_e\|_{\mathbf{H}^2_{-s}(\mathbb{R}^n)} \leq c \cdot \left(\|g_e\|_{L^2_s(\mathbb{R}^n)} + \|w_e\|_{L^2_{-s}(\mathbb{R}^n)} \right) ,$$

showing

$$\begin{aligned} \|\nabla w_e\|^2_{L^2_{s-1}(\mathbb{R}^n)} &\leq c \cdot \Big(\|g\|^2_{L^2(\mathbb{R}^n)} + \tau \|w_e\|^2_{L^2_{s-1}(\mathbb{R}^n)} + \|w_e\|^2_{L^2_{s-2}(\mathbb{R}^n)} + \|w_e\|^2_{\mathbf{H}^2(U(1))} \\ &\leq c \cdot \Big(\|g\|^2_{L^2(\mathbb{R}^n)} + \tau \|w\|^2_{L^2_{s-1}(\mathbb{R}^n)} + \|w\|^2_{L^2_{s-2}(\mathbb{R}^n)} + \|w\|^2_{L^2_{-s}(\mathbb{R}^n)} \Big) . \end{aligned}$$

By the differential equation we see

$$\|g\|_{L^2(\mathbb{R}^n)} \|w\|_{L^2(\mathbb{R}^n)} \geq \left| \operatorname{Im} \langle g , w \rangle_{L^2(\mathbb{R}^n)} \right| = \left| i\nu\tau \langle w , w \rangle_{L^2(\mathbb{R}^n)} \right| = \tau |\nu| \, \|w\|^2_{L^2(\mathbb{R}^n)} ,$$

hence ($-s > s - 2$)

$$\begin{aligned} \|\exp(-i\nu r) w\|_{H^1_{s-2}(\mathbb{R}^n)} &\leq c \cdot \Big(\|w_e\|_{L^2_{s-2}(\mathbb{R}^n)} + \|\nabla w_e\|_{L^2_{s-1}(\mathbb{R}^n)} \Big) \\ &\leq c \cdot \Big(\|g\|^2_{L^2_s(\mathbb{R}^n)} + \tau \|w\|^2_{L^2_{s-1}(\mathbb{R}^n)} + \|w\|^2_{L^2_{-s}(\mathbb{R}^n)} \Big) \\ &\leq c \cdot \Big(\|g\|_{L^2_s(\mathbb{R}^n)} + \|w\|_{L^2_{-s}(\mathbb{R}^n)} \Big) , \end{aligned} \tag{B.10}$$

and it remains to estimate $\|w\|_{L^2_{-s}(\mathbb{R}^n)}$. For that, we calculate

$$\operatorname{Im} \int\limits_{G(1,\tilde{r})} g_e \overline{w}_e = \operatorname{Im} \int\limits_{G(1,\tilde{r})} \Delta w_e \overline{w}_e + \int\limits_{G(1,\tilde{r})} \nu \left(\tau w_e + \frac{n-1}{r} w_e \right) \overline{w}_e + 2\nu \operatorname{Re} \int\limits_{G(1,\tilde{r})} \partial_r w_e \overline{w}_e = \cdots ,$$

using Lemma A.4 and obtain

$$\begin{aligned} &\nu \int\limits_{G(1,\tilde{r})} r^{-2s} \Big((2s-1)|w_e|^2 + \tau r |w_e|^2 \Big) \\ &= \operatorname{Im} \int\limits_{G(1,\tilde{r})} r^{1-2s} g_e \overline{w}_e - (2s-1) \int\limits_{G(1,\tilde{r})} r^{-2s} \operatorname{Im} (\partial_r w_e \overline{w}_e) \\ &\quad + \int\limits_{S(\tilde{r})} r^{1-2s} \Big(\tau |w_e|^2 + \operatorname{Im} (\partial_r w_e \overline{w}_e) \Big) - \int\limits_{S(1)} \Big(\tau |w_e|^2 + \operatorname{Im} (\partial_r w_e \overline{w}_e) \Big) \\ &\leq \int\limits_{G(1,\tilde{r})} r^s |g_e| r^{1-3s} |w_e| + (2s-1) \int\limits_{G(1,\tilde{r})} r^{s-1} |\partial_r w_e| r^{1-3s} |w_e| \end{aligned}$$

$$+ c \cdot \left(\int_{S(\tilde r)} r^{1-2s}\left(|w_e|^2 + |\partial_r w_e \overline{w}_e|\right) + \int_{S(1)} \left(|w_e|^2 + |\partial_r w_e \overline{w}_e|\right) \right)$$
$$\leq \left(\left\|r^s g_e\right\|_{L^2(G(1,\tilde r))} + (2s-1)\left\|r^{s-1}\nabla w_e\right\|_{L^2(G(1,\tilde r))} \right) \cdot \left\|r^{1-3s} w_e\right\|_{L^2(G(1,\tilde r))}$$
$$+ c \cdot \left(\int_{S(1)} \left(|w_e|^2 + |\nabla w_e|^2\right) + \int_{S(\tilde r)} \tilde r^{1-2s}\left(|w_e|^2 + |\nabla w_e|^2\right) \right).$$

As before, the lower limit for $\tilde r \longrightarrow \infty$ of the last boundary integral vanishes (cf. Lemma A.2 and observe that $w_e \in \mathbf{H}^2(\Omega)$, $s > 0$), such that we may omit it and replace $G(1, \tilde r)$ by $\breve{U}(1)$, yielding (with (B.10))

$$\left\|r^{-s} w_e\right\|^2_{L^2(\breve{U}(1))}$$
$$\leq c \cdot \left(\left(\left\|r^s g_e\right\|_{L^2(\breve{U}(1))} + \left\|r^{s-1}\nabla w_e\right\|_{L^2(\breve{U}(1))} \right) \cdot \left\|r^{1-3s} w_e\right\|_{L^2(\breve{U}(1))} + \int_{S(1)} \left(|w_e|^2 + |\nabla w_e|^2\right) \right)$$
$$\leq c \cdot \left(\left(\|g_e\|_{L^2_s(\mathbb{R}^n)} + \|\nabla w_e\|_{L^2_{s-1}(\mathbb{R}^n)} \right) \cdot \|w_e\|_{L^2_{1-3s}(\mathbb{R}^n)} + \int_{S(1)} \left(|w_e|^2 + |\nabla w_e|^2\right) \right).$$

As the surface integral is bounded by $\|w_e\|^2_{\mathbf{H}^2(U(1))}$ (trace theorem) and with (B.10) we obtain

$$\|w_e\|^2_{L^2_{-s}(\mathbb{R}^n)} \leq c \cdot \left(\left(\|g_e\|_{L^2_s(\mathbb{R}^n)} + \|\nabla w_e\|_{L^2_{s-1}(\mathbb{R}^n)} \right) \cdot \|w_e\|_{L^2_{1-3s}(\mathbb{R}^n)} + \|w_e\|^2_{\mathbf{H}^2(U(2))} \right)$$
$$\leq c \cdot \left(\left(\|g_e\|_{L^2_s(\mathbb{R}^n)} + \|w_e\|_{L^2_{-s}(\mathbb{R}^n)} \right) \cdot \|w_e\|_{L^2_{1-3s}(\mathbb{R}^n)} + \|w_e\|^2_{\mathbf{H}^2(U(2))} \right),$$

hence (Young's inequality)

$$\|w_e\|^2_{L^2_{-s}(\mathbb{R}^n)} \leq c \cdot \left(\|g_e\|_{L^2_s(\mathbb{R}^n)} + \|w_e\|_{L^2_{1-3s}(\mathbb{R}^n)} + \|w_e\|^2_{\mathbf{H}^2(U(2))} \right),$$

Finally, using once again Lemma B.1 we arrive at

$$\|w_e\|^2_{L^2_{-s}(\mathbb{R}^n)} \leq c \cdot \left(\|g_e\|_{L^2_s(\mathbb{R}^n)} + \|w_e\|_{L^2_{1-3s}(\mathbb{R}^n)} \right),$$

which together with (B.10) and Lemma A.1 implies

$$\|w\|_{L^2_{-s}(\mathbb{R}^n)} + \|\exp(-i\nu r)\, w\|_{H^1_{s-2}(\mathbb{R}^n)} \leq c \cdot \left(\|g\|_{L^2_s(\mathbb{R}^n)} + \|w\|_{L^2(\Omega(\delta))} \right) \qquad \text{(B.11)}$$

with $c, \delta > 0$ independent of ν, τ, w and g. □

Appendix C. Proofs in the case of the time-harmonic Maxwell equations

This section deals with the proofs of the decomposition lemma, the polynomial decay, and the a priori estimate, which we skipped in the main part.

Proof of Lemma 4.1. We start with $u = \eta u + \check{\eta} u$, noting that $\check{\eta} u \in \mathbf{R}_t$. Moreover,

$$\operatorname{Rot} \check{\eta} u = \mathrm{C}_{\operatorname{Rot},\check{\eta}} u + \check{\eta} \operatorname{Rot} u = \mathrm{C}_{\operatorname{Rot},\check{\eta}} u - i\check{\eta}\Lambda f - i\omega\check{\eta}\Lambda u$$

and we have

$$(\operatorname{Rot} + i\omega\Lambda_0)\check{\eta} u = (\mathrm{C}_{\operatorname{Rot},\check{\eta}} - i\omega\check{\eta}\hat{\Lambda})u - i\check{\eta}\Lambda f = f_1 \in \mathsf{L}^2_{\mathrm{s}},$$

since $\operatorname{supp} \nabla\check{\eta}$ is compact and $t + \kappa \geq s$. According to [26, Theorem 4],

$$f_1 = f_{\mathsf{R}} + f_{\mathsf{D}} + f_{\mathcal{S}} \in {}_0\mathsf{R}_{\mathrm{s}} \dotplus {}_0\mathsf{D}_{\mathrm{s}} \dotplus \mathcal{S}_s$$

holds and we obtain

$$i\omega\check{\eta}\Lambda_0 u = f_1 - \operatorname{Rot} \check{\eta} u = f_{\mathsf{D}} - \operatorname{Rot} \check{\eta} u + f_{\mathsf{R}} + f_{\mathcal{S}}.$$

Defining
- $u_1 := -\dfrac{i}{\omega}\Lambda_0^{-1}(f_{\mathsf{R}} + f_{\mathcal{S}}) \in \mathsf{R}_{\mathrm{s}}$;
- $\tilde{u} := \check{\eta} u - u_1 = \dfrac{i}{\omega}\Lambda_0^{-1}(\operatorname{Rot} \check{\eta} u - f_{\mathsf{D}}) \in \mathbf{R}_{\mathrm{t}} \cap {}_0\mathsf{D}_{\mathrm{t}}$,

[8, Lemma 4.2] shows $\tilde{u} \in \mathbf{H}^1_{\mathrm{t}}$ and we have

$$(\operatorname{Rot} + i\omega\Lambda_0)\tilde{u} = \operatorname{Rot}(\check{\eta} u - u_1) + i\omega\Lambda_0\tilde{u} = f_{\mathsf{D}} + \frac{i}{\omega}\,\tilde{\Lambda}_0^{-1} \operatorname{Rot} f_{\mathcal{S}} = f_2 \in {}_0\mathsf{D}_{\mathrm{s}}.$$

Next, we solve $(\operatorname{Rot} + 1)u_2 = f_2$. Using Fourier transformation, we look at

$$\hat{u}_2 := (1 + r^2)^{-1}(1 - ir\,\Xi)\,\mathcal{F}(f_2)$$

Since $s > 1/2$ and $f_2 \in \mathsf{L}^2_{\mathrm{s}}$, we obtain $\hat{u} \in \mathsf{L}^2_1$, hence $u_2 := \mathcal{F}^{-1}(\hat{u}_2) \in \mathbf{H}^1$. Moreover, $\mathcal{F}(\mathcal{F}(f_2)) = \mathcal{P}(f_2) \in \mathsf{L}^2_{\mathrm{s}}$ ($\mathcal{P}$: parity operator) yielding $\mathcal{F}(f_2) \in \mathbf{H}^{\mathrm{s}}$ and as product of an $\mathbf{H}^{\mathrm{s}}$-field with bounded C^∞-functions, $\hat{u} \in \mathbf{H}^{\mathrm{s}}$ (cf. [30, Lemma 3.2]), hence $u_2 \in \mathsf{L}^2_{\mathrm{s}}$. In addition a straight forward calculation shows $\mathcal{F}((\operatorname{Rot} + 1)u_2) = \mathcal{F}(f_2)$, which by [8, Lemma 4.2] implies

$$(\operatorname{Rot} + 1)u_2 = f_2 \quad \text{and} \quad u_2 \in \mathbf{H}^1_{\mathrm{s}} \cap {}_0\mathsf{D}_{\mathrm{s}}.$$

Then $(t \leq s)$

$$u_3 := \tilde{u} - u_2 \in \mathbf{H}^1_{\mathrm{t}} \cap {}_0\mathsf{D}_{\mathrm{t}}$$

satisfies

$$(\operatorname{Rot} + i\omega\Lambda_0)u_3 = (\operatorname{Rot} + i\omega\Lambda_0)\tilde{u} - (\operatorname{Rot} + i\omega\Lambda_0)u_2$$

$$= f_2 - (\operatorname{Rot} + 1)u_2 + (1 - i\omega\Lambda_0)u_2 = (1 - i\omega\Lambda_0)u_2 \in \mathsf{H}^1_{\mathrm{s}} \cap {}_0\mathsf{D}_{\mathrm{s}}$$

and using once more [8, Lemma 4.2] we get

$$u_3 \in \mathsf{H}^2_{\mathrm{t}} \cap {}_0\mathsf{D}_{\mathrm{t}} .$$

Finally

$$\begin{aligned} \Delta u_3 &= \operatorname{Rot}(\operatorname{Rot} u_3) = (1 - i\omega\widetilde{\Lambda}_0)\operatorname{Rot} u_2 - i\omega\widetilde{\Lambda}_0 \operatorname{Rot} u_3 \\ &= (1 - i\omega\widetilde{\Lambda}_0)(f_2 - u_2) - i\omega\widetilde{\Lambda}_0((1 - i\omega\Lambda_0)u_2 - i\omega\Lambda_0 u_3) \\ &= (1 - i\omega\widetilde{\Lambda}_0) f_2 - (1 + \omega^2\varepsilon_0\mu_0)u_2 - \omega^2\varepsilon_0\mu_0 u_3 \end{aligned}$$

holds, and hence

$$(\Delta + \omega^2\varepsilon_0\mu_0)u_3 = (1 - i\omega\widetilde{\Lambda}_0) f_2 - (1 + \omega^2\varepsilon_0\mu_0)u_2.$$

The asserted estimates follow by straightforward calculations using [8, Lemma 4.2] and the continuity of the projections from $\mathsf{L}^2_{\mathrm{s}}$ into ${}_0\mathsf{R}_{\mathrm{s}}$, ${}_0\mathsf{D}_{\mathrm{s}}$ and $\mathcal{S}_s$. □

Proof of Lemma 4.2. As for $t \geq s - 1$ there is nothing to prove, we concentrate on

$$u \in \mathbf{R}_{\mathrm{t}}(\Omega) \quad \text{with} \quad -1/2 < t < s - 1 .$$

Therefore, assume first that in addition

$$s - \kappa < t \quad \Longrightarrow \quad t < s < t + \kappa .$$

Then we may apply Lemma 4.1 and decompose the field u in

$$u = \eta u + u_1 + u_2 + u_3 ,$$

with $\eta u + u_1 + u_2 \in \mathbf{R}_{\mathrm{s}}(\Omega)$ and $u_3 \in \mathsf{H}^2_{\mathrm{t}}$ satisfying $(\Delta + \omega^2\varepsilon_0\mu_0)u_3 \in \mathsf{L}^2_{\mathrm{s}}$. Thus the polynomial decay for the Helmholtz equation (cf. Lemma B.2) shows

$$u_3 \in \mathsf{H}^2_{\mathrm{s}-1} \quad \text{and} \quad \|u_3\|_{\mathsf{H}^2_{\mathrm{s}-1}} \leq c \cdot \Big(\big\|(\Delta + \omega^2\varepsilon_0\mu_0)u_3\big\|_{\mathsf{L}^2_{\mathrm{s}}} + \|u_3\|_{\mathsf{L}^2_{\mathrm{s}-2}} \Big),$$

$c = c(s, J) > 0$, yielding $u = \eta u + u_1 + u_2 + u_3 \in \mathbf{R}_{\mathrm{s}-1}(\Omega)$. Moreover, using the estimates of Lemma 4.1 we obtain uniformly with respect to ω, u, and f

$$\begin{aligned} \|u\|_{\mathbf{R}_{\mathrm{s}-1}(\Omega)} &\leq c \cdot \Big(\|f\|_{\mathsf{L}^2_{\mathrm{s}}(\Omega)} + \|u\|_{\mathsf{L}^2_{\mathrm{s}-\kappa}(\Omega)} + \|u_3\|_{\mathsf{L}^2_{\mathrm{s}-1}} \Big) \\ &\leq c \cdot \Big(\|f\|_{\mathsf{L}^2_{\mathrm{s}}(\Omega)} + \|u\|_{\mathsf{L}^2_{\mathrm{s}-\kappa}(\Omega)} + \big\|(\Delta + \omega^2\varepsilon_0\mu_0)u_3\big\|_{\mathsf{L}^2_{\mathrm{s}}} + \|u_3\|_{\mathsf{L}^2_{\mathrm{s}-2}} \Big) \\ &\leq c \cdot \Big(\|f\|_{\mathsf{L}^2_{\mathrm{s}}(\Omega)} + \|u\|_{\mathsf{L}^2_{\mathrm{s}-\mathrm{m}}(\Omega)} \Big), \end{aligned}$$

where $m := \min\{\kappa, 2\}$ and applying Lemma A.1 we end up with

$$\|u\|_{\mathsf{R}_{s-1}(\Omega)} \leq c \cdot \Big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2(\Omega(\delta))} \Big),$$

for $c, \delta \in (0, \infty)$ independent of ω, u and f. So let us switch to the case

$$t \leq s - \kappa \quad \Longrightarrow \quad t + \kappa \leq s\,.$$

Here, the idea is to approach s by overlapping intervals to which the first case is applicable. For that, we choose some $\hat{k} \in \mathbb{N}$, such that with $\gamma := (\kappa - 1)/2 > 0$ we have

$$t + \kappa + (\hat{k} - 1) \cdot \gamma \leq s \leq t + \kappa + \hat{k} \cdot \gamma\,,$$

and for $k = 0, 1, \ldots, \hat{k}$ we define

$$t_k := t + k \cdot \gamma \quad \text{as well as} \quad s_k := t_{k+1} + 1 = t_k + (\kappa + 1)/2\,.$$

Then (as $\kappa > 1$)

$$\begin{aligned} t_{k+1} < s_k &= t_{k+1} + 1 = t + \kappa + (k-1) \cdot \gamma \leq s, \\ t_k < t_{k+1} + 1 = s_k &= t_k + (\kappa + 1)/2 < t_k + \kappa, \end{aligned}$$

such that we can successively apply the first case, ending up with $u \in \mathsf{R}_{s_{\hat{k}}-1}(\Omega)$. If $s = s_{\hat{k}}$, we are done. Otherwise, we choose $t_{\hat{k}+1} := s_{\hat{k}} - 1$ and apply the first case once more, since

$$t_{\hat{k}+1} < s_{\hat{k}} < s \leq t + \kappa + \hat{k} \cdot \gamma = t_{\hat{k}+1} + \kappa\,.$$

Either way, we obtain $u \in \mathsf{R}_{s-1}(\Omega)$ and now the estimate follows as in the first case. □

Proof of Lemma 4.4. Without loss of generality, we may assume $s \in (1/2, 1)$. Then we have $s \in \mathbb{R} \setminus \mathbb{I}$ with $0 < s < \kappa$ and we can apply Lemma 4.1 (with $t = 0$) to decompose $u := \mathcal{L}_\omega f \in \mathsf{R}_{\Gamma_1}(\Omega)$ into

$$u = \eta u + u_1 + u_2 + u_3$$

with $u_3 \in \mathsf{H}^2$ solving

$$(\Delta + \omega^2 \varepsilon_0 \mu_0) u_3 = (1 - i\omega \widetilde{\Lambda}_0) f_2 - (1 + \omega^2 \varepsilon_0 \mu_0) u_2 =: f_3 \in \mathsf{L}^2_s\,,$$

where f_2 is defined as in Lemma 4.1. Moreover, the estimates from Lemma 4.1 along with

$$(\operatorname{Rot} - i\omega \sqrt{\varepsilon_0 \mu_0}\, \Xi) u = -i\Lambda f - i\omega(\Lambda_0 + \sqrt{\varepsilon_0 \mu_0}\, \Xi) u - i\omega \hat{\Lambda} u$$

yield

$$\begin{aligned} &\|u\|_{\mathsf{R}_t(\Omega)} + \big\|(\Lambda_0 + \sqrt{\varepsilon_0\mu_0}\,\Xi)u\big\|_{\mathsf{L}^2_{s-1}(\Omega)} \\ &\leq c\cdot\Big(\,\|u\|_{\mathsf{R}_t(\Omega)} + \big\|(\mathrm{Rot} - i\omega\sqrt{\varepsilon_0\mu_0}\,\Xi)u\big\|_{\mathsf{L}^2_{s-1}(\Omega)} + \|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)}\Big) \\ &\leq c\cdot\Big(\,\|u_3\|_{\mathsf{L}^2_t} + \big\|(\mathrm{Rot} - i\omega\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\big\|_{\mathsf{L}^2_{s-1}} + \|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)}\Big), \end{aligned} \tag{C.1}$$

with $c = c(s,t,J) > 0$. Due to the monotonicity of the norms with respect to t and s, we may assume t and s to be close enough to $-1/2$ respectively $1/2$ such that $1 < s - t < \kappa$ holds. Hence, the assertion follows by (C.1) and Lemma A.1, if we can show

$$\|u_3\|_{\mathsf{L}^2_t} + \big\|(\mathrm{Rot} - i\omega\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\big\|_{\mathsf{L}^2_{s-1}} \leq c\cdot\Big(\,\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)}\Big),$$

with $c \in (0,\infty)$ independent of ω, u and f. Therefore, note that the self-adjointness of the Laplacian $\Delta : \mathsf{H}^2 \subset \mathsf{L}^2 \longrightarrow \mathsf{L}^2$ yields $(\Delta + \omega^2\varepsilon_0\mu_0)^{-1} f_3 = u_3$ and applying Lemma B.3 componentwise, we obtain

$$\|u_3\|_{\mathsf{L}^2_t} + \big\|\exp(-i\lambda\sqrt{\varepsilon_0\mu_0}r)u_3\big\|_{\mathsf{H}^1_{s-2}} \leq c\cdot\Big(\,\|f_3\|_{\mathsf{L}^2_s} + \|u_3\|_{\mathsf{L}^2(\Omega(\delta))}\Big).$$

With $\mathrm{Rot}(\exp(-i\lambda\sqrt{\varepsilon_0\mu_0}r)u_3) = \exp(-i\lambda\sqrt{\varepsilon_0\mu_0}r)(\mathrm{Rot} - i\lambda\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3$ this leads to

$$\begin{aligned} &\|u_3\|_{\mathsf{L}^2_t} + \big\|(\mathrm{Rot} - i\lambda\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\big\|_{\mathsf{L}^2_{s-1}} \\ &\leq \|u_3\|_{\mathsf{L}^2_t} + \big\|\mathrm{Rot}\,(\exp(-i\lambda\sqrt{\varepsilon_0\mu_0}r)u_3)\big\|_{\mathsf{L}^2_{s-1}} \\ &\leq \|u_3\|_{\mathsf{L}^2_t} + \big\|\exp(-i\lambda\sqrt{\varepsilon_0\mu_0}r)u_3\big\|_{\mathsf{H}^1_{s-2}} \leq c\cdot\Big(\,\|f_3\|_{\mathsf{L}^2_s} + \|u_3\|_{\mathsf{L}^2(\Omega(\delta))}\Big), \end{aligned} \tag{C.2}$$

where $c > 0$ is not depending on ω, u_3 and f_3. But, actually, we would like to estimate $(\mathrm{Rot} - i\omega\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3$. For that, we need some additional arguments, starting with the observation that

$$\omega = |\lambda|(1 + (\sigma/\lambda)^2)^{1/4}\cdot\begin{cases} \exp(i\varphi/2) & \text{for } \lambda > 0 \\ \exp(i(\varphi/2 + \pi)) & \text{for } \lambda < 0 \end{cases}$$
$$\text{with}\quad \varphi := \arctan(\sigma/\lambda) \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big),$$

hence $|\mathrm{Re}\,(\omega)| \geq \sqrt{2}/2\cdot|\lambda|$. Then $|\omega + \lambda| \geq \sqrt{3/2}\cdot|\lambda|$ and we have

$$|\omega - \lambda|^2 = \left|\frac{\omega^2 - \lambda^2}{\omega + \lambda}\right|^2 = \left|\frac{i\sigma\lambda}{\omega + \lambda}\right|^2 \leq \frac{2}{3}\cdot\sigma^2.$$

From this and the resolvent estimate,

$$\|f_3\|_{\mathsf{L}^2} = \big\|(\Delta + \omega^2\varepsilon_0\mu_0)\,u_3\big\|_{\mathsf{L}^2} \geq |\mathrm{Im}\,(\omega^2\varepsilon_0\mu_0)|\cdot\|u_3\|_{\mathsf{L}^2} = \varepsilon_0\mu_0\sigma|\lambda|\cdot\|u_3\|_{\mathsf{L}^2},$$

we obtain ($s > 1/2$)

$$\left\|(\operatorname{Rot} - i\omega\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\right\|_{\mathsf{L}^2_{s-1}} \le \left\|(\operatorname{Rot} - i\lambda\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\right\|_{\mathsf{L}^2_{s-1}} + \left\|(\omega-\lambda)\sqrt{\varepsilon_0\mu_0}\,\Xi u_3\right\|_{\mathsf{L}^2_{s-1}}$$
$$\le \left\|(\operatorname{Rot} - i\lambda\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\right\|_{\mathsf{L}^2_{s-1}} + c\cdot|\lambda|^{-1}\,\|f_3\|_{\mathsf{L}^2_s}\,,$$

such that with (C.2) and the estimates from Lemma 4.1 uniformly with respect to ω, u and f

$$\|u_3\|_{\mathsf{L}^2_t} + \left\|(\operatorname{Rot} - i\omega\sqrt{\varepsilon_0\mu_0}\,\Xi)u_3\right\|_{\mathsf{L}^2_{s-1}} \le c\cdot\Big(\|f\|_{\mathsf{L}^2_s(\Omega)} + \|u\|_{\mathsf{L}^2_{s-\kappa}(\Omega)}\Big). \qquad \square$$

Bibliography

[1] S. Bauer, D. Pauly, and M. Schomburg, The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions, *SIAM J. Math. Anal.*, **48**(4) (2016), 2912–2943.

[2] M. Costabel, A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains, *Math. Methods Appl. Sci.*, **12**(4) (1990), 365–368.

[3] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology: Volume 2 – Functional and Variational Methods*, Springer-Verlag, 2000.

[4] D. M. Eidus, The principle of limiting absorption, *Transl. Am. Math. Soc. (2)*, **47** (1965), 157–191.

[5] P. Fernandes and G. Gilardi, Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions, *Math. Models Methods Appl. Sci.*, **07**(7) (1997), 957–991.

[6] T. Ikebe and Y. Saito, Limiting absorption method and absolute continuity for the Schrödinger operator, *J. Math. Kyoto Univ.*, **12**(3) (1972), 513–542.

[7] F. Jochmann, A compactness result for vector fields with divergence and curl in $L^2(\Omega)$ involving mixed boundary conditions, *Appl. Anal.*, **66**(1) (1997), 189–203.

[8] P. Kuhn and D. Pauly, Regularity results for generalized electro-magnetic problems, *Analysis*, **30**(3) (2010), 225–252.

[9] R. Leis, Zur Theorie elektromagnetischer Schwingungen in anisotropen inhom. Medien, *Math. Z.*, **106** (1968), 213–224.

[10] R. Leis, Aussenraumaufgaben in der Theorie der Maxwellschen Gleichungen, in *Topics in Analysis*, pp. 237–247, Springer, Berlin, Heidelberg, 1974.

[11] R. Leis, *Initial Boundary Value Problems in Mathematical Physics*, Courier Corporation, 2013.

[12] C. Müller, Randwertprobleme der Theorie elektromagnetischer Schwingungen, *Math. Z.*, **56**(3) (1952), 261–270.

[13] C. Müller, On the behavior of the solutions of the differential equation $\Delta u = f(x,u)$ in the neighborhood of a point, *Commun. Pure Appl. Math.*, **7**(3) (1954), 505–515.

[14] D. Pauly, *Niederfrequenzasymptotik der Maxwell-Gleichung im inhomogenen und anisotropen Außengebiet*, Dissertation, Universität Duisburg-Essen, Fakultät für Mathematik, 2003.

[15] D. Pauly, Low frequency asymptotics for time-harmonic generalized Maxwell equations in nonsmooth exterior domains, *Adv. Math. Sci. Appl.*, **16**(2) (2006), 591–622.

[16] D. Pauly, On polynomial and exponential decay of eigen-solutions to exterior boundary value problems for the generalized time-harmonic Maxwell system, *Asymptot. Anal.*, **79**(1) (2012), 133–160.

[17] D. Pauly, Solution theory, variational formulations, and functional a posteriori error estimates for general first order systems with applications to electro-magneto statics and more, *Numer. Funct. Anal. Optim.* (2019), https://arxiv.org/abs/1611.02993.
[18] R. Picard, *Ein vereinheitlichter Zugang für eine Klasse linearer Wellenausbreitungsphänomene*, Technical Report 489, SFB 72, Universität Bonn, 1982.
[19] R. Picard, An elementary proof for a compact imbedding result in generalized electromagnetic theory, *Math. Z.*, **187** (1984), 151–164.
[20] R. Picard, N. Weck, and K. J. Witsch, Time-harmonic Maxwell equations in the exterior of perfectly conducting, irregular obstacles, *Analysis (Munich)*, **21**(3) (2001), 231–264.
[21] V. Vogelsang, Die absolute Stetigkeit des positiven Spektrums der Schwingungsgleichungen mit oszillierendem Hauptteil, *Math. Z.*, **181** (1982), 201–214.
[22] C. Weber, A local compactness theorem for Maxwell's equations, *Math. Methods Appl. Sci.*, **2**(1) (1980), 12–25.
[23] C. Weber, Regularity theorems for Maxwell's equations, *Math. Methods Appl. Sci.*, **3**(1) (1981), 523–536.
[24] N. Weck, Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries, *J. Math. Anal. Appl.*, **46**(2) (1974), 410–437.
[25] N. Weck and K. J. Witsch, Complete low frequency analysis for the reduced wave equation with variable coefficients in three dimensions, *Commun. Partial Differ. Equ.*, **17**(9) (1992), 1619–1663.
[26] N. Weck and K. J. Witsch, Generalized spherical harmonics and exterior differentiation in weighted Sobolev spaces, *Math. Methods Appl. Sci.*, **17**(13) (1994), 1017–1043.
[27] N. Weck and K. J. Witsch, Generalized linear elasticity in exterior domains. I: Radiation problems, *Math. Methods Appl. Sci.*, **20**(17) (1997), 1469–1500.
[28] P. Werner, Randwertprobleme für die zeitunabhängigen Maxwellschen Gleichungen mit variablen Koeffizienten, *Arch. Ration. Mech. Anal.*, **18**(3) (1965), 167–195.
[29] H. Weyl, Die natürlichen Randwertaufgaben im Außenraum für Strahlungsfelder beliebiger Dimension und beliebigen Ranges, *Math. Z.*, **56**(2) (1952), 105–119.
[30] J. Wloka, *Partial Differential Equations*, Cambridge University Press, 1987.

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11 On an electro-magneto-elasto-dynamic transmission problem

Abstract: We consider a coupled system of Maxwell's equations and the equations of elasticity, where the coupling occurs not via material properties but through an interaction on an interface separating the two regimes. Evolutionary well-posedness in the sense of Hadamard well-posedness supplemented by causal dependence is shown for a natural choice of generalized interface conditions. The results are obtained in a Hilbert space setting incurring no regularity constraints on the boundary and the interface of the underlying regions.

Keywords: Elasticity equations, Maxwell equations, transmission problem, boundary interaction, mother/descendant mechanism

MSC 2010: 35Q60, 74F15, 74B05, 46N20

1 Introduction

Similarities between various initial boundary value problems of mathematical physics have been noted as general observations throughout the literature. Indeed, the work by K. O. Friedrichs [2, 3] already showed that the classical linear phenomena of mathematical physics belong – in the static case – to his class of *symmetric positive hyperbolic partial differential equations*, later referred to as *Friedrichs systems*, which are of the abstract form

$$(M_1 + A)u = f, \tag{1}$$

with A at least formally, i. e., on C_∞-vector fields with compact support in the underlying region Ω, a skew-symmetric differential operator and the L^∞-matrix-valued multiplication-operator M_1 satisfying the condition

$$\operatorname{sym}(M_1) \coloneqq \frac{1}{2}(M_1 + M_1^*) \geq c > 0$$

for some real number c. Indeed, a typical choice for the domain of A is to incorporate a boundary condition into $D(A)$, so that A is skew self-adjoint (A quasi-m-accretive would be sufficient). Problem (1) can be considered as the static problem associated

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with the dynamic problem (∂_0 denotes the time-derivative)

$$\partial_0 M_0 + M_1 + A \tag{2}$$

with M_0 a self-adjoint L^∞-multiplication-operator and $M_0 \geq 0$, which were also addressed in [3]. It is noteworthy that even the temporal exponential weight factor, which plays a central role in our approach, is introduced as an ad hoc formal trick to produce a suitable M_1 for a well-posed static problem. For the so-called time-harmonic case, where ∂_0 is replaced by $\mathrm{i}\omega$, $\omega \in \mathbb{R}$, we replace A simply by $\mathrm{i}\omega M_0 + A$ to arrive at a system of the form (1).

Operators of the Friedrichs type (2), can be generalized to obtain a fully time-dependent theory allowing for operator-valued coefficients, indeed, in the time-shift invariant case, for systems of the general form

$$(\partial_0 M(\partial_0^{-1}) + A)U = F \tag{Evo-Sys}$$

where A is – for simplicity – skew self-adjoint and M an operator-valued – say – rational function as an abstract coefficient. The meaning of $M(\partial_0^{-1})$ is in terms of a suitable function calculus associated with the (normal) operator ∂_0, [13, Chapter 6]. We shall refer to such systems as evolutionary equations, *evo-systems* for short, to distinguish them from the special subclass of classical (explicit) evolution equations.

In this paper, we intend to study a particular transmission problem between two physical regimes, electro-magneto-dynamics and elasto-dynamics, within this general framework and establish its well-posedness, which for evo-systems entails not only Hadamard well-posedness, i. e., *uniqueness*, *existence* and *continuous dependence*, but also the crucial property of *causality*.

The peculiarity of the problem we shall investigate is that the interaction between the two regimes is solely via the interface, not via material interactions as in piezo-electrics; compare, e. g., [7] for the latter type of effects.

After properly introducing evo-systems in the next section, we shall establish the equations of electro-magneto-dynamics and elasto-dynamics, respectively, as such systems in Section 3. Finally, in Section 4 we establish a particular interface coupling problem between the two regimes in adjacent regions via a mother-descendant mechanism; see the survey [15]. We emphasize that our setup allows for arbitrary open sets as underlying domains with no additional constraints on boundary regularity.

2 A short introduction to a class of evo-systems

2.1 Basic ideas

We shall approach solving (Evo-Sys) by looking at the equation as a space-time operator equation in a suitable Hilbert space setting. Without loss of generality, we may[1] and will assume that all Hilbert spaces are real.

Solutions will be discussed in a weighted L^2-space $H_\nu(\mathbb{R}, H)$, constructed by completion of the space $\mathring{C}_1(\mathbb{R}, H)$ of differentiable H-valued functions with compact support w. r. t. $\langle\,\cdot\,|\,\cdot\,\rangle_{\nu,H}$ (norm: $|\cdot|_{\nu,H}$)

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \langle \varphi(t) | \psi(t) \rangle_H \exp(-2\nu t)\, dt.$$

Here, H denotes a generic real Hilbert space. We introduce time differentiation ∂_0 as a closed operator in $H_\nu(\mathbb{R}, H)$ defined as the closure of

$$\begin{aligned} \mathring{C}_1(\mathbb{R}, H) \subseteq H_\nu(\mathbb{R}, H) &\to H_\nu(\mathbb{R}, H), \\ \varphi &\mapsto \varphi'. \end{aligned}$$

The operator ∂_0 is normal in $H_\nu(\mathbb{R}, H)$. For $\nu_0 \in\,]0, \infty[$, $\nu \in\,]\nu_0, \infty[$, we have

$$\operatorname{sym}(\partial_0) \coloneqq \overline{\frac{1}{2}(\partial_0 + \partial_0^*)} = \nu \geq \nu_0 > 0, \tag{3}$$

i. e.

$$\begin{aligned} \partial_0 \text{ is a strictly (and uniformly w. r. t. } \nu \in\,]\nu_0, \infty[) \text{ positive definite} \\ \text{(i. e., m-accretive) operator.} \end{aligned}$$

---

1 Every complex Hilbert space X is a real Hilbert space choosing only real numbers as multipliers and

$$(\phi, \psi) \mapsto \mathfrak{Re}\langle \phi | \psi \rangle_X$$

as new inner product. Note that with this choice ϕ and $\mathrm{i}\phi$ are always orthogonal. Moreover, for any skew-symmetric operator A we have

$$x \perp Ax$$

for all $x \in D(A)$.
Indeed, since $\langle x|y\rangle - \langle y|x\rangle = 0$ (symmetry) we have

$$\langle x|Ax\rangle - \langle Ax|x\rangle = 0$$

or by skew-symmetry

$$\begin{aligned} 0 &= \langle x|Ax\rangle - \langle Ax|x\rangle \\ &= 2\langle x|Ax\rangle \end{aligned}$$

for all $x \in D(A)$.

This core observation can be lifted to a larger class of more complex problems involving operator-valued coefficients and systems of the general form

$$(\partial_0 M(\partial_0^{-1}) + A)U = F \tag{Evo-Sys}$$

where A is – for simplicity – skew self-adjoint and M an operator-valued – say – rational function as abstract coefficient.

In many practical cases, skew self-adjointness of A is evident from its structure as a block operator matrix of the form

$$A = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix},$$

with $H = H_0 \oplus H_1$ and $C : D(C) \subseteq H_0 \to H_1$ a densely defined, closed linear operator.

2.2 Well-posedness for evo-systems

Since reasonable well-posedness requires closed operators, we describe our problem class more rigorously as of the form

$$\overline{(\partial_0 M(\partial_0^{-1}) + A)}U = F. \tag{Evo-Sys}$$

For a convenient special class, more than sufficient for our purposes here, we record the following general well-posedness result; see [10, 11, 15].

Theorem 2.1. *Let $z \mapsto M(z)$ be a rational $\mathcal{L}(H,H)$-valued function in a neighborhood of 0 such that $M(0)$ is self-adjoint and*[2]

$$\nu M(0) + \operatorname{sym}(M'(0)) \geq \eta_0 > 0 \tag{4}$$

for some $\eta_0 \in \mathbb{R}$ and all $\nu \in \,]\nu_0, \infty[$, $\nu_0 \in \,]0, \infty[$ sufficiently large, and let A be skew self-adjoint. Then well-posedness of (Evo-Sys) *follows for all $\nu \in \,]\nu_0, \infty[$. Moreover, the solution operator $\overline{(\partial_0 M(\partial_0^{-1}) + A)}^{-1}$ is causal in the sense that*

$$\chi_{]-\infty,0]}\overline{(\partial_0 M(\partial_0^{-1}) + A)}^{-1} = \chi_{]-\infty,0]}\overline{(\partial_0 M(\partial_0^{-1}) + A)}^{-1}\chi_{]-\infty,0]}.$$

2 Here, we use sym in an analogous meaning to (3), i. e.,

$$\operatorname{sym}(B) := \frac{1}{2}\overline{(B + B^*)},$$

which is equal to $\frac{1}{2}(B + B^*)$ since B is continuous..

Indeed, apart from occasional side remarks we will simply have

$$M(\partial_0^{-1}) = M_0 + \partial_0^{-1} M_1$$

and since ∂_0, A can be *continuously* extended to suitable extrapolation spaces, it is justified[3] to drop the closure bar, which we shall do henceforth.

3 Maxwell's equations and the equations of linear elasticity as evo-systems

3.1 Maxwell's equations as an evo-system

James Clerk Maxwell developed his new ideas on electro-magnetic waves in 1861–1864 resulting in his famous two volume publication: A Treatise on Electricity and Magnetism, [6]. His ingenious contribution to what we nowadays call Maxwell's equations is to amend Ampere's law with a so-called *displacement current* term. Heaviside and Gibbs have given the system in its now familiar form as

$$\begin{aligned} \partial_0 D + \sigma E - \operatorname{curl} H &= -j_{\mathrm{ext}}, \quad \text{(Ampere's law)} \\ \partial_0 B + \operatorname{curl} E &= 0, \quad \text{(Faraday's law of induction)} \\ D &= \varepsilon E, \\ B &= \mu H. \end{aligned}$$

The usually included divergence conditions are redundant, since the two equations together with the material relations can be seen to be leading already to a well-posed initial boundary value problem. The so-called *six-vector* block matrix *form*:

$$\left(\partial_0 \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \mathring{\operatorname{curl}} & 0 \end{pmatrix}\right) \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -j_{\mathrm{ext}} \\ 0 \end{pmatrix}$$

brings us already close to our initial goal to formulate the equations as an evo-system. Here, $\mathring{\operatorname{curl}}$ denotes the L^2-closure of the classical curl defined on $C_1(\mathbb{R}^3)$-vector fields vanishing outside closed, bounded subsets of $\mathbb{R}^3$. Moreover, $\operatorname{curl} := \mathring{\operatorname{curl}}^*$ and so the spatial Maxwell operator is skew self-adjoint in $L^2(\mathbb{R}^3, \mathbb{R}^6)$. In case of a domain Ω with

3 Albeit this being sometimes confusing and misleading, it is a common practice in the field of partial differential equations. For example, one frequently writes

$$\Delta = \partial_1^2 + \partial_2^2$$

although $\phi \in D(\Delta)$ does in general not – as the notation appears to suggest – allow for $\phi \in D(\partial_1^2) \cap D(\partial_2^2)$.

boundary, we take $\mathring{\text{curl}}$ constructed analogously with $C_1(\Omega)$-vector fields vanishing outside closed, bounded sets contained in Ω, where Ω is a non-empty open set in $\mathbb{R}^3$ (**strong** definition of $\mathring{\text{curl}}$) and define

$$\text{curl} \coloneqq \mathring{\text{curl}}^* \tag{5}$$

(**weak**[4] definition of curl). Thus we arrive indeed at the evo-system[5]

$$(\partial_0 M(\partial_0^{-1}) + A)\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} -j_{\text{ext}} \\ k_{\text{ext}} \end{pmatrix}$$

with $M(\partial_0^{-1}) = M(0) + \partial_0^{-1} M'(0)$ and here specifically

$$M(0) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}, \quad M'(0) = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -\text{curl} \\ \mathring{\text{curl}} & 0 \end{pmatrix}, \tag{6}$$

which satisfies the well-posedness constraint if we assume ε, μ selfadjoint and (compare (3) and (2))

$$\nu\varepsilon + \text{sym}(\sigma), \quad \mu \geq \eta_0 > 0, \tag{7}$$

for all sufficiently large $\nu \in]0, \infty[$. Note that with this assumption also ε having a nontrivial null space, the so-called eddy current problem, can be handled without further adjustments. Of course, in the spirit of Theorem 2.1 we could consider more general media. More recently, so-called electro-magnetic *metamaterials* have come into focus, which are media, where $M'' \neq 0$ or $M(z)$ is *not* block-diagonal. To classify some prominent cases, there are for example:

– Bi-anisotropic media, characterized by

$$M(0) = \begin{pmatrix} \varepsilon & \kappa^* \\ \kappa & \mu \end{pmatrix}, \quad \kappa \neq 0.$$

Since, due to (4), we must have $M(0) \geq 0$, we get $\varepsilon \geq 0$ and

$$\left|\mu^{-1/2}\kappa\varepsilon^{-1/2}\right| \leq 1.$$

Note that this is a strong smallness constraint on the off-diagonal entry κ. For example in homogeneous, isotropic media $c_0 = \varepsilon^{-1/2}\mu^{-1/2}$ is the speed of light and the above condition yields

$$|\kappa| \leq \frac{1}{c_0}.$$

4 Of course, "weak equals strong." It is $C_1(\Omega) \cap D(\text{curl})$ dense in $D(\text{curl})$ by T. Kasuga's argument (see [4], [5, Section 2.1]), the strong definition of curl as the closure $\overline{\text{curl}\,|_{C_1(\Omega)\cap D(\text{curl})}}$ equals its weak definition. Consequently, also $\mathring{\text{curl}} = \text{curl}^* = (\text{curl}\,|_{C_1(\Omega)\cap D(\text{curl})})^*$, which confirms "weak equals strong" for $\mathring{\text{curl}}$ as well.

5 Here, we have thrown in an extra magnetic external source term, since mathematically it is no obstacle to treat $k_{\text{ext}} \neq 0$.

– Chiral media:

$$M'(0) = \begin{pmatrix} 0 & -\chi \\ \chi & 0 \end{pmatrix}, \quad \chi \neq 0 \text{ selfadjoint.}$$

– Omega media:

$$M'(0) = \begin{pmatrix} 0 & \chi \\ \chi & 0 \end{pmatrix}, \quad \chi \neq 0 \text{ skew-selfadjoint.}$$

3.2 The equations of linear elasto-dynamics as an evo-system

Linear elasto-dynamics is usually discussed in a symmetric tensor-valued L^2-setting for the stress T, i. e., $T \in L^2(\Omega, \operatorname{sym}[\mathbb{R}^{3\times 3}])$, and a vector L^2-setting for the displacement $u \in L^2(\Omega, \mathbb{R}^3)$. Here, sym is the (orthogonal) projector onto real-symmetric-matrix-valued L^2-functions. More precisely, we extend sym to the matrix-valued case by letting

$$\begin{aligned} \operatorname{sym} : L^2(\Omega, \mathbb{R}^{3\times 3}) &\to L^2(\Omega, \mathbb{R}^{3\times 3}), \\ W &\mapsto \frac{1}{2}(W + W^*), \end{aligned}$$

where the adjoint W^* is taken pointwise by the standard Frobenius inner product

$$(T, S) \mapsto \operatorname{trace}(T^\top S)$$

for 3×3-matrices, such that

$$\begin{aligned} \mathbb{R}^{3\times 3} &\to \mathbb{R}^6 \\ \begin{pmatrix} T_{00} & T_{01} & T_{02} \\ T_{10} & T_{11} & T_{12} \\ T_{20} & T_{21} & T_{22} \end{pmatrix} &\mapsto \begin{pmatrix} T_{00} \\ T_{11} \\ T_{22} \\ T_{12} \\ T_{20} \\ T_{01} \\ T_{21} \\ T_{02} \\ T_{10} \end{pmatrix} \end{aligned}$$

is unitary. Then with

$$\begin{aligned} \iota_{\mathrm{sym}} : L^2(\Omega, \operatorname{sym}[\mathbb{R}^{3\times 3}]) &\to L^2(\Omega, \mathbb{R}^{3\times 3}) \\ T &\mapsto T, \end{aligned}$$

denoting the canonical embedding of the subspace $L^2(\Omega, \mathrm{sym}[\mathbb{R}^{3\times3}])$ in $L^2(\Omega, \mathbb{R}^{3\times3})$ we have

$$\begin{aligned} \iota_{\mathrm{sym}}^* : L^2(\Omega, \mathbb{R}^{3\times3}) &\to L^2(\Omega, \mathrm{sym}[\mathbb{R}^{3\times3}]) \\ W &\mapsto \mathrm{sym}\, W \end{aligned}$$

and so we have the useful factorization

$$\mathrm{sym} = \iota_{\mathrm{sym}} \iota_{\mathrm{sym}}^*.$$

With this observation, we can now approach the standard equations of elasticity theory. The dynamics of elastic processes is commonly captured in a second-order formulation for the displacement u by

$$\varrho_* \partial_0^2 u - \mathrm{Div}\, C\, \mathrm{Grad}\, u = f,$$

where

$$\begin{aligned} \mathrm{Grad}\, u &:= \iota_{\mathrm{sym}}^* (\nabla u) \\ \mathrm{Div}\, T &:= (\nabla^\top T)^\top \end{aligned}$$

for symmetric T, i. e., $T \in L^2(\Omega, \mathrm{sym}[\mathbb{R}^{3\times3}])$. The elasticity "tensor," i. e., rather the mapping

$$C : L^2(\Omega, \mathrm{sym}[\mathbb{R}^{3\times3}]) \to L^2(\Omega, \mathrm{sym}[\mathbb{R}^{3\times3}])$$

and the mass density operator

$$\varrho_* : L^2(\Omega, \mathbb{R}^3) \to L^2(\Omega, \mathbb{R}^3)$$

are assumed to be self-adjoint and strictly positive definite.

The origin, from which the above second-order system is derived, is naturally a system of algebraic and first-order differential equations. The original system can be easily reconstructed by reintroducing the relevant physical quantities velocity $v := \partial_0 u$ and stress $T := C\, \mathrm{Grad}\, u$. Thus, we arrive at the system

$$\begin{aligned} \varrho_* \partial_0 v - \mathrm{Div}\, T &= f, \\ T &= C\, \mathrm{Grad}\, \partial_0^{-1} v, \end{aligned}$$

in the unknowns v and T. Differentiating the second equation with respect to time, we end up with a system of the block operator matrix form

$$\left(\partial_0 \begin{pmatrix} \varrho_* & 0 \\ 0 & C^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{Div} \\ -\mathrm{Grad} & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ T \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Choosing now, for example, a homogeneous Dirichlet boundary condition, i. e., we replace Grad by[6]

$$\mathring{\mathrm{Grad}} := \overline{\iota_{\mathrm{sym}}^{*}\mathring{\mathrm{grad}}},$$

where $\mathring{\mathrm{grad}}$ is the closure of differentiation for vector fields (the Jacobian matrix) with compact support in Ω as a mapping from $L^2(\Omega,\mathbb{R}^3)$ to $L^2(\Omega,\mathbb{R}^{3\times 3})$, and

$$\mathrm{Div} := \mathrm{div}\,\iota_{\mathrm{sym}}$$

so that

$$\mathring{\mathrm{Grad}} = -\,\mathrm{Div}^{*},$$

we are led to consider an evo-system of the form

$$\left(\partial_0\begin{pmatrix}\varrho_* & 0\\ 0 & C^{-1}\end{pmatrix}+\begin{pmatrix}0 & -\,\mathrm{Div}\\ -\mathring{\mathrm{Grad}} & 0\end{pmatrix}\right)\begin{pmatrix}v\\ T\end{pmatrix}=\begin{pmatrix}f\\ g\end{pmatrix}. \tag{8}$$

Remark 3.1. We note that also here we have "weak equals strong" following the same rationale as in the electro-magneto-dynamics case; compare Footnote 4.

In the light of (4), the well-posedness results from assuming that

$$\varrho_*, C \geq \eta_0 > 0 \tag{9}$$

for some real constant η_0.

4 An interface coupling mechanism

After the above preliminary considerations, we are now ready to consider the situation, where the electro-magnetic field in one region interacts with elastic media in another region via some common interface. Rather than basing our choice of transmission constraints on the interface by physical arguments, we shall explore a deep connection between electro-magneto-dynamics and elasto-dynamics to arrive at natural transmission conditions built into the construction of the evo-system. This construction will utilize the idea of a *mother-descendant* construction introduced in [12]; see [14] for a more viable version, which we will briefly recall.

6 Korn's inequality shows that the closure bar is superfluous

$$\mathring{\mathrm{Grad}} = \iota_{\mathrm{sym}}^{*}\mathring{\mathrm{grad}}.$$

4.1 Mother operators and their descendants

We recall from [12] the following simple but crucial lemma.

Lemma 4.1. *Let $C : D(C) \subseteq H \to Y$ be a closed densely-defined linear operator between Hilbert spaces H, Y. Moreover, let $B : Y \to X$ be a continuous linear operator into another Hilbert space X. If $C^* B^*$ is densely defined, then*

$$\overline{BC} = \left(C^* B^*\right)^*.$$

Proof. It is

$$C^* B^* \subseteq (BC)^*.$$

If $\phi \in D((BC)^*)$, then

$$\langle BCu|\phi\rangle_X = \left\langle u|(BC)^*\phi\right\rangle_H$$

for all $u \in D(C)$. Thus, we have

$$\left\langle Cu|B^*\phi\right\rangle_Y = \langle BCu|\phi\rangle_X = \left\langle u|(BC)^*\phi\right\rangle_H$$

for all $u \in D(C)$ and we read off that $B^*\phi \in D(C^*)$ and

$$C^* B^* \phi = (BC)^* \phi.$$

Thus we have

$$(BC)^* = C^* B^*.$$

If now $C^* B^*$ is densely defined, we have for its adjoint operator

$$\left(C^* B^*\right)^* = \overline{BC}. \qquad \square$$

As a consequence, we have that the *descendant*

$$\overline{\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}}\begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix} = \begin{pmatrix} 0 & -C^* B^* \\ \overline{BC} & 0 \end{pmatrix}$$

indeed inherits its skew self-adjointness from its *mother* $\left(\begin{smallmatrix} 0 & -C^* \\ C & 0 \end{smallmatrix}\right)$ (with C replaced by $\overline{BC}$). Moreover, we record the following result on the stability of well-posedness in the mother-descendant process.

Theorem 4.2. *Let $C : D(C) \subseteq H \to Y$ be a closed densely-defined linear operator between Hilbert spaces H, Y. Moreover, let $B : Y \to X$ be a continuous linear operator* *into another Hilbert space X with a closed range $B[Y]$ such that C^*B^* is densely defined. Then, if*

$$\left(\partial_0 M(\partial_0^{-1}) + \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}\right)\begin{pmatrix} U_0 \\ U_1 \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}$$

with data $\left(\begin{smallmatrix} F_0 \\ F_1 \end{smallmatrix}\right) \in H_\nu(\mathbb{R}, H \oplus X)$ and a solution $\left(\begin{smallmatrix} U_0 \\ U_1 \end{smallmatrix}\right) \in H_\nu(\mathbb{R}, H \oplus X)$ is a well-posed evo-system (satisfying in particular (4)), so is the descendant problem

$$(\partial_0 \widetilde{M}(\partial_0^{-1}) + \widetilde{A})U = \begin{pmatrix} F_0 \\ G_1 \end{pmatrix} \in H_\nu(\mathbb{R}, H \oplus B[Y]),$$

where

$$\widetilde{M}(\partial_0^{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} M(\partial_0^{-1}) \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix},$$

$$\widetilde{A} = \overline{\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}}\begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix}.$$

Proof. The positive-definiteness condition (4) carries over to the new material law operator in the following way. If

$$\nu M(0) + \operatorname{sym}(M'(0)) \geq c_* > 0$$

for all $\nu \in [\nu_0, \infty[$ and some $\nu_0 \in \,]0, \infty[$, then

$$\nu \widetilde{M}(0) + \operatorname{sym}(\widetilde{M}'(0)) = \nu \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} M(0) \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix} + \operatorname{sym}\left(\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} M'(0) \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix}\right)$$

and we estimate for $(V_0, V_1) \in H \oplus B[Y]$

$$\begin{aligned}
&\nu \left\langle \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} M(0) \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right\rangle_{H\oplus B[Y]} \\
&\quad + \left\langle \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \operatorname{sym}(M'(0)) \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right\rangle_{H\oplus B[Y]} \\
&= \nu \left\langle \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \middle| M(0) \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right\rangle_{H\oplus Y} \\
&\quad + \left\langle \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \middle| \operatorname{sym}(M'(0)) \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right\rangle_{H\oplus Y}, \\
&\geq c_* \left\langle \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix}\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right\rangle_{H\oplus Y}
\end{aligned}$$

$$\geq \widetilde{c}_* \left\langle \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \middle| \begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \right\rangle_{H\oplus B[Y]}$$

Indeed, since by the closed range assumption $B[Y]$ and $B^*[X]$ are Hilbert spaces and by the closed graph theorem the operator

$$\begin{pmatrix} 1 & 0 \\ 0 & B^*\iota_{B[Y]} \end{pmatrix} : H \oplus B[Y] \to H \oplus B^*[X]$$
$$\begin{pmatrix} V_0 \\ V_1 \end{pmatrix} \mapsto \begin{pmatrix} V_0 \\ B^* V_1 \end{pmatrix}$$

has a continuous inverse, we have

$$\begin{aligned} \left|\begin{pmatrix} V_0 \\ V_1 \end{pmatrix}\right|_{H\oplus B[Y]} &= \left|\begin{pmatrix} 1 & 0 \\ 0 & B^*\iota_{B[Y]} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix}\right|_{H\oplus B[Y]} \\ &\leq \left\|\begin{pmatrix} 1 & 0 \\ 0 & B^*\iota_{B[Y]} \end{pmatrix}^{-1}\right\| \left|\begin{pmatrix} 1 & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \end{pmatrix}\right|_{H\oplus Y} \end{aligned}$$

and so we may choose

$$\widetilde{c}_* = c_* \left\|\begin{pmatrix} 1 & 0 \\ 0 & B^*\iota_{B[Y]} \end{pmatrix}^{-1}\right\|^{-2}$$

to confirm that

$$\nu\widetilde{M}(0) + \operatorname{sym}(\widetilde{M}'(0)) \geq \widetilde{c}_* > 0$$

for all $\nu \in [\nu_0, \infty[$ and some $\nu_0 \in]0, \infty[$. □

As a particular instance of this construction, we can take B specifically as ι_S^*, where $\iota_S : S \to H$, $x \mapsto x$, is the canonical embedding of the closed subspace S in H. Then

$$\overline{\begin{pmatrix} 1 & 0 \\ 0 & \iota_S^* \end{pmatrix} \begin{pmatrix} 0 & -C \\ C^* & 0 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & \iota_S \end{pmatrix} = \begin{pmatrix} 0 & -C\iota_S \\ \overline{\iota_S^* C^*} & 0 \end{pmatrix}$$

is skew self-adjoint if $C\iota_S : D(C) \cap S \subseteq S \to Y$, the restriction of $C : D(C) \subseteq H \to Y$ to the closed subspace $S \subseteq H$ is densely defined in S. This is the construction we shall employ to approach our specific problem. First, we observe that both physical regimes do indeed have the same *mother*.

4.2 Two descendants of non-symmetric elasticity

As a convenient mother to start from, we take the theory of non-symmetric elasticity, W. Nowacki, [8, 9], leading to an evo-system of the form

$$\left(\partial_0 M_0 + M_1 + \begin{pmatrix} 0 & -\operatorname{div} \\ -\mathring{\operatorname{grad}} & 0 \end{pmatrix}\right) \begin{pmatrix} v \\ T \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

We shall now discuss two particular descendants:

1. Classical symmetric elasticity theory can be considered as a descendant of the form

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & \iota_{\mathrm{sym}}^* \end{pmatrix} M_0 \begin{pmatrix} 1 & 0 \\ 0 & \iota_{\mathrm{sym}} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \iota_{\mathrm{sym}}^* \end{pmatrix} M_1 \begin{pmatrix} 1 & 0 \\ 0 & \iota_{\mathrm{sym}} \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{Div} \\ -\mathring{\operatorname{Grad}} & 0 \end{pmatrix}\right) \begin{pmatrix} v \\ T_{\mathrm{sym}} \end{pmatrix} = \begin{pmatrix} f \\ g_{\mathrm{sym}} \end{pmatrix},$$

where

$$\mathring{\operatorname{Grad}} \coloneqq \overline{\iota_{\mathrm{sym}}^* \mathring{\operatorname{grad}}}$$

and

$$\operatorname{Div} \coloneqq \operatorname{div} \iota_{\mathrm{sym}}.$$

Note that the assumptions of Theorem 4.2 are clearly satisfied since smooth elements with compact support are already a dense sub-domain of $\operatorname{div} \iota_{\mathrm{sym}}$. In the classical situation, which we shall assume for simplicity, we have $M_1 = 0$ and

$$M_0 = \begin{pmatrix} \varrho_* & 0 \\ 0 & C^{-1} \end{pmatrix}.$$

2. Maxwell's equation are obtained in a sense by the opposite construction. If we denote analogously

$$\begin{aligned} \operatorname{skew} : L^2(\Omega, \mathbb{R}^{3\times 3}) &\to L^2(\Omega, \mathbb{R}^{3\times 3}), \\ W &\mapsto \frac{1}{2}(W - W^*), \end{aligned}$$

then with

$$\begin{aligned} \iota_{\mathrm{skew}} : L^2(\Omega, \operatorname{skew}[\mathbb{R}^{3\times 3}]) &\to L^2(\Omega, \mathbb{R}^{3\times 3}) \\ T &\mapsto T, \end{aligned}$$

denoting the canonical embedding of $L^2(\Omega, \operatorname{skew}[\mathbb{R}^{3\times 3}])$ in $L^2(\Omega, \mathbb{R}^{3\times 3})$ we find

$$\begin{aligned} \iota_{\mathrm{skew}}^* : L^2(\Omega, \mathbb{R}^{3\times 3}) &\to L^2(\Omega, \operatorname{skew}[\mathbb{R}^{3\times 3}]) \\ W &\mapsto \operatorname{skew} W. \end{aligned}$$

With this, we may now construct the Maxwell evo-system as

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{2} I^* \iota_{\mathrm{skew}}^* \end{pmatrix} M_0 \begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{2} \iota_{\mathrm{skew}} I \end{pmatrix}\right.$$

$$+\begin{pmatrix}1 & 0\\ 0 & -\sqrt{2}I^{*}\iota_{\mathrm{skew}}^{*}\end{pmatrix}M_1\begin{pmatrix}1 & 0\\ 0 & -\sqrt{2}\iota_{\mathrm{skew}}I\end{pmatrix}$$

$$+\begin{pmatrix}0 & -\operatorname{curl}\\ \mathring{\operatorname{curl}} & 0\end{pmatrix}\Bigg)\begin{pmatrix}E\\ H\end{pmatrix}=\begin{pmatrix}f\\ -I^{*}g_{\mathrm{skew}}\end{pmatrix},$$

where

$$I:\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}\mapsto\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -\alpha_3 & \alpha_2\\ \alpha_3 & 0 & -\alpha_1\\ -\alpha_2 & \alpha_1 & 0\end{pmatrix}$$

is a unitary transformation and so is its inverse

$$I^{*}:\begin{pmatrix}0 & -\alpha_3 & \alpha_2\\ \alpha_3 & 0 & -\alpha_1\\ -\alpha_2 & \alpha_1 & 0\end{pmatrix}\mapsto\sqrt{2}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}.$$

Again, for simplicity we focus on the classical choice of (6). We calculate

$$\begin{aligned}I^{*}\iota_{\mathrm{skew}}^{*}\operatorname{grad}v &= \frac{1}{2}I^{*}\begin{pmatrix}0 & \partial_2 v_1-\partial_1 v_2 & \partial_3 v_1-\partial_1 v_3\\ \partial_1 v_2-\partial_2 v_1 & 0 & \partial_3 v_2-\partial_2 v_3\\ \partial_1 v_3-\partial_3 v_1 & \partial_2 v_3-\partial_3 v_2 & 0\end{pmatrix}\\ &= -\frac{1}{\sqrt{2}}\begin{pmatrix}\partial_3 v_2-\partial_2 v_3\\ \partial_1 v_3-\partial_3 v_1\\ \partial_2 v_1-\partial_1 v_2\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\partial_2 v_3-\partial_3 v_2\\ \partial_3 v_1-\partial_1 v_3\\ \partial_1 v_2-\partial_2 v_1\end{pmatrix}\\ &=: \frac{1}{\sqrt{2}}\operatorname{curl}v\end{aligned}$$

and also confirm that

$$\operatorname{div}\ \iota_{\mathrm{skew}}I=-\frac{1}{\sqrt{2}}\operatorname{curl}.$$

In other terms, we have the congruence to a descendant

$$\begin{aligned}&\begin{pmatrix}0 & -\operatorname{curl}\\ \mathring{\operatorname{curl}} & 0\end{pmatrix}\\ &=\begin{pmatrix}1 & 0\\ 0 & -\sqrt{2}I^{*}\end{pmatrix}\begin{pmatrix}0 & -\operatorname{div}\iota_{\mathrm{skew}}\\ -\overline{\iota_{\mathrm{skew}}^{*}\mathring{\operatorname{grad}}} & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -\sqrt{2}I\end{pmatrix},\end{aligned}$$

where we have used that

$$\mathring{\operatorname{curl}}=\sqrt{2}I^{*}\,\overline{\iota_{\mathrm{skew}}^{*}\mathring{\operatorname{grad}}}.$$

Note that again smooth elements with compact support are a dense sub-domain of $\operatorname{div}\iota_{\mathrm{skew}}$ and so the assumptions of Theorem 4.2 are clearly satisfied. Motivated by the observation that Maxwell's equations and the (symmetric) elasto-dynamic equations are both descendants from the asymmetric elasto-dynamics equations of Nowacki, [8, 9], we will now discuss boundary interactions between both systems.

4.3 An application to interface coupling

Motivated by a paper of F. Cakoni and G. C. Hsiao, [1], where the time-harmonic isotropic homogeneous case of electro-dynamics and elasticity, respectively, is studied via transmission conditions across a separating interface, we consider the corresponding time-dependent case. We assume $\Omega_0 \cup \Omega_1 \subseteq \Omega$, such that the orthogonal decompositions

$$\begin{aligned} L^2(\Omega, \mathbb{R}^{3\times 3}) &= L^2(\Omega_0, \mathbb{R}^{3\times 3}) \oplus L^2(\Omega_1, \mathbb{R}^{3\times 3}) \\ L^2(\Omega, \mathbb{R}^{3}) &= L^2(\Omega_0, \mathbb{R}^{3}) \oplus L^2(\Omega_1, \mathbb{R}^{3}) \end{aligned} \tag{10}$$

hold, and let $I_0 \coloneqq \begin{pmatrix} \iota_{L^2(\Omega_0, \mathrm{sym}[\mathbb{R}^{3\times 3}])} & -\iota_{L^2(\Omega_1, \mathrm{skew}[\mathbb{R}^{3\times 3}])} \sqrt{2} I \end{pmatrix}$, i. e.,

$$I_0 \begin{pmatrix} S \\ v \end{pmatrix} = \iota_{L^2(\Omega_0, \mathrm{sym}[\mathbb{R}^{3\times 3}])} S - \iota_{L^2(\Omega_1, \mathrm{skew}[\mathbb{R}^{3\times 3}])} \sqrt{2} I v$$

with the respective canonical embeddings into $L^2(\Omega, \mathbb{R}^{3\times 3})$. Then

$$\begin{aligned} I_0^* : L^2(\Omega, \mathbb{R}^{3\times 3}) &\to L^2(\Omega_0, \mathrm{sym}[\mathbb{R}^{3\times 3}]) \oplus L^2(\Omega_1, \mathbb{R}^{3}), \\ T &\mapsto \begin{pmatrix} \iota^*_{L^2(\Omega_0, \mathrm{sym}[\mathbb{R}^{3\times 3}])} T \\ -\sqrt{2} I^* \iota^*_{L^2(\Omega_1, \mathrm{skew}[\mathbb{R}^{3\times 3}])} T \end{pmatrix}, \end{aligned}$$

and so

$$I_0^* = \begin{pmatrix} \iota^*_{L^2(\Omega_0, \mathrm{sym}[\mathbb{R}^{3\times 3}])} \\ -\sqrt{2} I^* \iota^*_{L^2(\Omega_1, \mathrm{skew}[\mathbb{R}^{3\times 3}])} \end{pmatrix}.$$

With this, we get a congruence to a descendant construction as

$$A = \overline{\begin{pmatrix} 1 & 0 \\ 0 & I_0^* \end{pmatrix} \begin{pmatrix} 0 & -\operatorname{div} \\ -\mathring{\operatorname{grad}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & I_0 \end{pmatrix}} \tag{11}$$

$$\subseteq \begin{pmatrix} 0 & \begin{pmatrix} -\operatorname{Div}_{\Omega_0} & -\operatorname{curl}_{\Omega_1} \end{pmatrix} \\ \begin{pmatrix} -\operatorname{Grad}_{\Omega_0} \\ \operatorname{curl}_{\Omega_1} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \tag{12}$$

and

$$M(0) = \begin{pmatrix} \varrho_{*,\Omega_0} + \varepsilon_{\Omega_1} & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} C_{\Omega_0}^{-1} & 0 \\ 0 & \mu_{\Omega_1} \end{pmatrix} \end{pmatrix} \tag{13}$$

$$M'(0) = \begin{pmatrix} \sigma_{\Omega_1} & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}. \tag{14}$$

The indexes Ω_k, $k = 0, 1$, are used to denote the respective supports of the quantities. The coefficients are – as a matter of simplification labeled in the same meaning as in (6) and (8), just with the support information added.[7] The unknowns are now

$$\begin{pmatrix} v_{\Omega_0} + E_{\Omega_1} \\ \begin{pmatrix} T_{\Omega_0} \\ H_{\Omega_1} \end{pmatrix} \end{pmatrix} \in H = L^2(\Omega, \mathbb{R}^3) \oplus (L^2(\Omega_0, \operatorname{sym}[\mathbb{R}^{3\times3}]) \oplus L^2(\Omega_1, \mathbb{R}^3)),$$

where the first component is to be understood in the sense of (10). Note that the assumptions of Theorem 4.2 are clearly satisfied since smooth elements with compact support in Ω_0 and Ω_1, respectively, are already a dense sub-domain as in the separate cases of Subsection 4.2. From the inclusion (11), (12), we read off that the resulting evo-system

$$(\partial_0 M(0) + M'(0) + A) \begin{pmatrix} v_{\Omega_0} + E_{\Omega_1} \\ \begin{pmatrix} T_{\Omega_0} \\ H_{\Omega_1} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} f_{\Omega_0} - j_{\mathrm{ext},\Omega_1} \\ \begin{pmatrix} g_{\mathrm{sym},\Omega_0} \\ k_{\mathrm{ext},\Omega_1} \end{pmatrix} \end{pmatrix} \tag{15}$$

indeed yields

$$\partial_0(\varrho_{*,\Omega_0} + \varepsilon_{\Omega_1})(v_{\Omega_0} + E_{\Omega_1}) - \operatorname{Div}_{\Omega_0} T_{\Omega_0} - \operatorname{curl}_{\Omega_1} H_{\Omega_1} = f_{\Omega_0} - j_{\mathrm{ext},\Omega_1},$$

which in turn splits into

$$\begin{aligned} \partial_0 \varrho_{*,\Omega_0} v_{\Omega_0} - \operatorname{Div}_{\Omega_0} T_{\Omega_0} &= f_{\Omega_0}, \\ \partial_0 \varepsilon_{\Omega_1} E_{\Omega_1} - \operatorname{curl}_{\Omega_1} H_{\Omega_1} &= -j_{\mathrm{ext},\Omega_1}. \end{aligned}$$

The second block row yields another pair of equations

$$\begin{aligned} \partial_0 C^{-1} T_{\Omega_0} - \operatorname{Grad} v_{\Omega_0} &= g_{\mathrm{sym},\Omega_0}, \\ \partial_0 \mu_{\Omega_1} H_{\Omega_1} + \operatorname{curl} E_{\Omega_1} &= k_{\mathrm{ext},\Omega_1}. \end{aligned}$$

The actual system models now natural transmission conditions on the common boundary part $\dot{\Omega}_0 \cap \dot{\Omega}_1$ and the homogeneous Dirichlet boundary condition on $\dot{\Omega}_0 \setminus \dot{\Omega}_1$

7 Although we consider for convenience and physical relevance this evo-system in its own right, a formal mother material law – without physical meaning – could be easily given:

$$\begin{pmatrix} \varrho_{*,\Omega_0} + \varepsilon_{\Omega_1} + \partial_0^{-1}\sigma_{\Omega_1} & 0 \\ 0 & m_{11} \end{pmatrix}$$

with, for example,

$$m_{11} = \iota^*_{\mathrm{sym},\Omega_0} C^{-1}_{\Omega_0} \iota_{\mathrm{sym},\Omega_0} + \iota^*_{\mathrm{skew},\Omega_0} \iota_{\mathrm{skew},\Omega_0} + \iota^*_{\mathrm{skew},\Omega_1} \mu_{\Omega_1} \iota_{\mathrm{skew},\Omega_1} + \iota^*_{\mathrm{sym},\Omega_1} \iota_{\mathrm{sym},\Omega_1}.$$

Then the described mother-descendant mechanism would lead to a descendant, which in turn would be congruent to the described interface system.

and the standard homogeneous electric boundary condition on $\dot{\Omega}_1 \setminus \dot{\Omega}_0$ without assuming any smoothness of the boundary.

On the contrary, assuming sufficient regularity of the boundary, one can see that the model yields a generalization of the classical transmission conditions on $\dot{\Omega}_0 \cap \dot{\Omega}_1$:

$$\begin{aligned} T_{\Omega_0} n &= n \times H_{\Omega_1}, \\ n \times v_{\Omega_0} &= n \times E_{\Omega_1}, \end{aligned} \tag{16}$$

where n is a smooth unit normal field on $\dot{\Omega}_0 \cap \dot{\Omega}_1$. Indeed, with

$$\begin{pmatrix} v_{\Omega_0} + E_{\Omega_1} \\ \begin{pmatrix} T_{\Omega_0} \\ H_{\Omega_1} \end{pmatrix} \end{pmatrix} \in D(A)$$

we have (noting for the smooth exterior unit normal vector fields $n_{\dot{\Omega}_0}$, $n_{\dot{\Omega}_1}$ on the boundaries of Ω_0 and Ω_1, respectively, that $n_{\dot{\Omega}_0} = -n_{\dot{\Omega}_1}$ on $\dot{\Omega}_0 \cap \dot{\Omega}_1$) with

$$\widetilde{A} = \begin{pmatrix} 0 & \begin{pmatrix} -\operatorname{Div}_{\Omega_0} & -\operatorname{curl}_{\Omega_1} \end{pmatrix} \\ \begin{pmatrix} -\operatorname{Grad}_{\Omega_0} \\ \operatorname{curl}_{\Omega_1} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

that

$$\begin{aligned} 0 &= \left\langle \begin{pmatrix} v_{\Omega_0} + E_{\Omega_1} \\ \begin{pmatrix} T_{\Omega_0} \\ H_{\Omega_1} \end{pmatrix} \end{pmatrix} \middle| A \begin{pmatrix} v_{\Omega_0} + E_{\Omega_1} \\ \begin{pmatrix} T_{\Omega_0} \\ H_{\Omega_1} \end{pmatrix} \end{pmatrix} \right\rangle_H \\ &= \left\langle \begin{pmatrix} v_{\Omega_0} + E_{\Omega_1} \\ \begin{pmatrix} T_{\Omega_0} \\ H_{\Omega_1} \end{pmatrix} \end{pmatrix} \middle| \widetilde{A} \begin{pmatrix} v_{\Omega_0} + E_{\Omega_1} \\ \begin{pmatrix} T_{\Omega_0} \\ H_{\Omega_1} \end{pmatrix} \end{pmatrix} \right\rangle_H \\ &= -\langle v_{\Omega_0} \,|\, \operatorname{Div} T_{\Omega_0} \rangle_{L^2(\Omega_0,\mathbb{R}^3)} - \langle T_{\Omega_0} \,|\, \operatorname{Grad}_{\Omega_0} v_{\Omega_0} \rangle_{L^2(\Omega_0,\mathbb{R}^{3\times 3})} \\ &\quad + \langle H_{\Omega_1} \,|\, \operatorname{curl}_{\Omega_1} E_{\Omega_1} \rangle_{L^2(\Omega_1,\mathbb{R}^3)} - \langle E_{\Omega_1} \,|\, \operatorname{curl}_{\Omega_1} H_{\Omega_1} \rangle_{L^2(\Omega_1,\mathbb{R}^3)} \\ &= -\int_{\dot{\Omega}_0 \cap \dot{\Omega}_1} v_{\Omega_0}^{\top} T_{\Omega_0} n_{\dot{\Omega}_0} \, do + \int_{\dot{\Omega}_0 \cap \dot{\Omega}_1} n_{\dot{\Omega}_1}^{\top} (E_{\Omega_1} \times H_{\Omega_1}) \, do \\ &= -\int_{\dot{\Omega}_0 \cap \dot{\Omega}_1} v_{\Omega_0}^{\top} T_{\Omega_0} n_{\dot{\Omega}_0} \, do + \int_{\dot{\Omega}_0 \cap \dot{\Omega}_1} E_{\Omega_1}^{\top} (n_{\dot{\Omega}_0} \times H_{\Omega_1}) \, do. \end{aligned}$$

Since $(v_{\Omega_0} + E_{\Omega_1}) \in D(\mathring{\operatorname{grad}})$ is by construction admissible, we may choose $v_{\Omega_0} = E_{\Omega_1}$ on the interface and conclude that

$$T_{\Omega_0} n_{\dot{\Omega}_0} = n_{\dot{\Omega}_0} \times H_{\Omega_1} \tag{17}$$

is a needed transmission condition. In particular, we see

$$n_{\dot{\Omega}_0}^{\top} T_{\Omega_0} n_{\dot{\Omega}_0} = 0.$$

Inserting the explicit transmission condition (17) now yields

$$\begin{aligned} -\int\limits_{\dot{\Omega}_0 \cap \dot{\Omega}_1} & (n_{\dot{\Omega}_0} \times (n_{\dot{\Omega}_0} \times (v_{\Omega_0} - E_{\Omega_1})))^{\top} (n_{\dot{\Omega}_0} \times H_{\Omega_1}) do \\ = \int\limits_{\dot{\Omega}_0 \cap \dot{\Omega}_1} & (v_{\Omega_0} - E_{\Omega_1})^{\top} (n_{\dot{\Omega}_0} \times H_{\Omega_1}) do = 0, \end{aligned}$$

which, with $n_{\dot{\Omega}_0} \times H_{\Omega_1}$ for $H_{\Omega_1} \in D(\mathrm{curl}_{\Omega_1})$ being sufficiently arbitrary, now implies

$$n_{\dot{\Omega}_0} \times v_{\Omega_0} = n_{\dot{\Omega}_0} \times E_{\Omega_1}$$

i. e., the continuity of the tangential components

$$v_{\Omega_0,\mathrm{t}} = E_{\Omega_1,\mathrm{t}},$$

as a complementing transmission condition. These more or less heuristic considerations motivate to take the above evo-system as an appropriate generalization to cases, where the boundary does *not* have a reasonable normal vector field.

All in all, we summarize our findings in the following well-posedness result.

Theorem 4.3. *The evo-system* (15) *is well-posed if* ϱ_{*,Ω_0}, C_{Ω_0} *and* ε_{Ω_1}, μ_{Ω_1} *are self-adjoint, non-negative, continuous operators on* $L^2(\Omega_0, \mathbb{R}^3)$, $L^2(\Omega_0, \mathrm{sym}[\mathbb{R}^{3\times 3}])$ *and on* $L^2(\Omega_1, \mathbb{R}^3)$, *respectively,* σ_{Ω_1} *is continuous and linear on* $L^2(\Omega_1, \mathbb{R}^3)$ *and such that*

$$\varrho_{*,\Omega_0}, C_{\Omega_0}, \mu_{\Omega_1} \geq \eta_0 > 0,$$

as well as

$$\nu \varepsilon_{\Omega_1} + \mathrm{sym}(\sigma_{\Omega_1}) \geq \eta_0 > 0$$

for some real number η_0 *and all sufficiently large* ν.

Remark 4.4.

1. If we formally transcribe the time-harmonic case into its time dependent form, the transmission conditions of [1] are actually

$$\begin{aligned} T_{\Omega_0} n &= n \times \partial_0^{-1} H_{\Omega_1}, \\ n \times \partial_0^{-1} v_{\Omega_0} &= n \times E_{\Omega_1}. \end{aligned} \tag{18}$$

   Although these obviously differ from (16), we give preference to our choice above for several reasons. For one, the energy balance requirement of [1, formula (5)], which reads as

$$v_{\Omega_0}^{\top} T_{\Omega_0} n = n^{\top} (H_{\Omega_1} \times E_{\Omega_1}), \tag{19}$$

is satisfied by (16) but not by (18). With the latter transmission conditions, we obtain instead

$$v_{\Omega_0}^\top T_{\Omega_0} n = (\partial_0 E_{\Omega_1})^\top (n \times (\partial_0^{-1} H_{\Omega_1})) = n^\top ((\partial_0^{-1} H_{\Omega_1}) \times (\partial_0 E_{\Omega_1})).$$

The problem seems to be that the difference to (19) becomes unnoticeable in the formal time-harmonic transcription of [1], since there ∂_0 is formally replaced by $\mathrm{i}\omega\sqrt{\varepsilon_0\mu_0}$ and so algebraic cancellation essentially makes the product rule for differentiation disappear, erroneously suggesting that the energy balance[8] is satisfied.

2. In the notation above, (11), (13), (14), if $M(0)$ is already strictly positive definite, we can construct a fundamental solution as a small perturbation of the fundamental solution of $\partial_0 + \sqrt{M(0)}^{-1} A \sqrt{M(0)}^{-1}$, which in turn is obtained from the unitary group

$$(\exp(-t\,\sqrt{M(0)}^{-1} A \sqrt{M(0)}^{-1}))_{t\in\mathbb{R}}$$

by cut-off as

$$(\chi_{[0,\infty[}(t)\ \exp(-t\,\sqrt{M(0)}^{-1} A \sqrt{M(0)}^{-1}))_{t\in\mathbb{R}}.$$

The restriction of the fundamental solution to $[0,\infty[$ yields the family

$$(\exp(-t\,\sqrt{M(0)}^{-1} A \sqrt{M(0)}^{-1}))_{t\in[0,\infty[}$$

commonly referred to as the associated one-parameter semi-group. In general, however, a fundamental solution may be complicated or impossible to construct.

3. We note that beyond eddy current type behavior, which is actually a change of type situation from hyperbolic to parabolic, and beyond the possibility of including, for example, piezo-electric effects via a more complex material law, we may actually allow for completely general rational material laws as long as condition (4) is warranted.

Bibliography

[1] F. Cakoni and G. C. Hsiao, Mathematical model of the interaction problem between electromagnetic field and elastic body, in *Acoustics, Mechanics, and the Related Topics of Mathematical Analysis. Proceedings of the International Conference to Celebrate Robert P. Gilbert's 70th Birthday, Frejus, France, June 18–22, 2002*, pp. 48–54, World Scientific, River Edge, NJ, 2002.

8 The correct energy balance in the time-harmonic case would actually involve temporal convolution products.

[2] K. O. Friedrichs, Symmetric hyperbolic linear differential equations, *Commun. Pure Appl. Math.*, **7** (1954), 345–392.

[3] K. O. Friedrichs, Symmetric positive linear differential equations, *Commun. Pure Appl. Math.*, **11**(3) (1958), 333–418.

[4] T. Kasuga, On Sobolev–Friedrichs generalisation of derivatives, *Proc. Jpn. Acad.*, **33** (1957), 596–599.

[5] R. Leis, *Initial Boundary Value Problems in Mathematical Physics*, John Wiley & Sons Ltd. and B. G. Teubner, Stuttgart, 1986.

[6] J. C. Maxwell, *A Treatise on Electricity and Magnetism*, A Treatise on Electricity and Magnetism, Bd. 1, Clarendon Press, 1873.

[7] A. J. Mulholland, R. Picard, S. Trostorff, and M. Waurick, On well-posedness for some thermo-piezoelectric coupling models, *Math. Methods Appl. Sci.*, **39**(15) (2016), 4375–4384.

[8] W. Nowacki, Some theorems of asymmetric thermoelasticity, *J. Math. Phys. Sci.*, **2** (1968), 111–122.

[9] W. Nowacki, *Theory of Asymmetric Elasticity*, transl. from the Polish by H. Zorski, Pergamon Press, Oxford, etc., PWN – Polish Scientific Publishers, Warszawa, VIII, 383 p., 1986.

[10] R. Picard, A structural observation for linear material laws in classical mathematical physics, *Math. Methods Appl. Sci.*, **32**(14) (2009), 1768–1803.

[11] R. Picard, An elementary hilbert space approach to evolutionary partial differential equations, *Rend. Ist. Mat. Univ. Trieste*, **42**(suppl.) (2010), 185–204.

[12] R. Picard, Mother operators and their descendants, *J. Math. Anal. Appl.*, **403**(1) (2013), 54–62.

[13] R. Picard and D. F. McGhee, *Partial Differential Equations: A Unified Hilbert Space Approach*, De Gruyter Expositions in Mathematics, vol. 55, De Gruyter, Berlin, New York, 518 p., 2011.

[14] R. Picard, S. Trostorff, and M. Waurick, On some models for elastic solids with micro-structure, *Z. Angew. Math. Mech.*, **95**(7) (2015), 664–689.

[15] R. Picard, S. Trostorff, and M. Waurick, Well-posedness via monotonicity – an overview, in *Operator Semigroups Meet Complex Analysis, Harmonic Analysis and Mathematical Physics*, Operator Theory – Advances and Applications, vol. 250, pp. 397–452, Springer International Publishing, Cham, 2015.

Marcus Waurick

12 Continuous dependence on the coefficients for a class of non-autonomous evolutionary equations

Abstract: The continuous dependence of solutions to certain equations on the coefficients is addressed. The class of equations under consideration has only recently be shown to be well posed. We give criteria that guarantee that convergence of the coefficients in the weak operator topology implies weak convergence of the respective solutions. We discuss three examples: A homogenization problem for a Kelvin–Voigt model for elasticity, the discussion of continuous dependence of the coefficients for acoustic waves with impedance type boundary conditions and a singular perturbation problem for a mixed type equation. By means of counterexamples, we show optimality of the results obtained.

Keywords: Homogenization, *G*-convergence, non-autonomous, evolutionary problems, integro-differential algebraic equations, singular perturbations, continuous dependence on the coefficients

MSC 2010: 35F45, 35M10, 35Q99, 46N20, 47N20, 74Q15

1 Introduction

In this article, we discuss the continuous dependence of solutions to evolutionary equations on the coefficients. In particular, we provide a hands-on approach to some results that can be deduced from the more elaborate exposition in [42]. Moreover, in comparison to [42] we shall present more involved applications.

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The usual method of choice to discuss issues of evolution equations is the semi-group approach. In fact, many evolutionary problems in mathematical physics can be described by the abstract Cauchy problem

$$u' = Au \quad u(0) = u_0$$

with A being a generator of a strongly continuous semi-group in a certain Banach space X, $u_0 \in X$. Thus, in the semi-group language, we are led to consider

$$u_n' = A_n u_n \quad u_n(0) = u_0$$

for a suitable sequence of generators $(A_n)_n$ in a common Banach space X. Now, we address whether the sequence $(u_n)_n$ of solutions converges in a particular sense. If the sequence of solutions converge to some u, we further ask whether there exists an operator A such that the following holds:

$$u' = Au \quad u(0) = u_0.$$

Within the semi-group perspective, there are several issues to be taken care of: Variable domains of the generators A_n, non-reflexivity of the space in which the solutions $(u_n)_n$ are obtained and the generator property for A. To illustrate the latter, we discuss a simple example with bounded generators: Take a bounded measurable function $a\colon \mathbb{R} \to \mathbb{R}$ and consider the sequence $(u_n)_n$ of solutions to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u_n(t,x) + a(nx)u_n(t,x) = 0 \quad u(0,x) = u_0(x) \quad \big((t,x) \in (0,\infty) \times \mathbb{R}\big),$$

for some given $u_0 \in L^2(\mathbb{R})$. The Cauchy problem can be formulated in the state space $X = L^2(\mathbb{R})$. It can be shown—assuming for instance the periodicity of a—that the limit equation is *not* of the type discussed above. Indeed, the resulting equation is of integro-differential type; see, for instance, [30, Chapter 23]. In particular, we cannot expect the limit equation to be of the form of the abstract Cauchy problem described above. Hence, the semi-group perspective to this kind of equation cannot be utilized. The very reason for this shortcoming is that the convergence of $(a(n\cdot))_n$ is too weak, [35]. In fact, it can be shown that $(a(n\cdot))_n$ converges in the weak star topology of $L^\infty(\mathbb{R})$ to the integral mean over the period, or, equivalently, the sequence of associated multiplication operators in $L^2(\mathbb{R})$ converges in the weak operator topology to the identity times the integral mean over the period of a. We refer to [8], where subtleties with regards to the Trotter product formula and the weak operator topology are highlighted. Due to the non-closedness of abstract Cauchy problems with regards to the convergences under consideration, we shall use semi-group theory here.

To the best of the author's knowledge, besides the author's work [35, 34, 40, 36, 37, 41, 42], there are very few studies (if any) of continuous dependence on the coefficients of a general problem class under the weak operator topology. However, there are some results for particular equations and/or with stronger topologies with respect to which the convergence of the coefficients is considered in: In [45], a particular nonlinear equation is considered and the continuous dependence of the solution on some scalar factors is addressed. Similarly, in [6, 10, 32, 17, 16], the so-called Brinkman–Forchheimer equation is discussed with regards to continuous dependence on some bounded functions under the sup-norm. The local sup-norm has been considered in [5], where the continuous dependence on the (non-linear) constitutive relations for particular equations of fluid flow in porous media is discussed. A weak topology for the coefficients is considered in [12]. However, the partial differential equations considered are of a specific form and the underlying spatial domain is the real line. Dealing with time-dependent coefficients in a boundary value problem of parabolic type, the author of [18] shows continuous dependence of the associated evolution families on the coefficients. In [18], the coefficients are certain functions considered with the C^1-norm. The author of [33] studies the continuous dependence of diffusion processes under the C^0-norm of the coefficients. Also with regards to strong topologies, the authors of [13, 14] studied continuous dependence results for a class of stochastic partial differential equations.

We also refer to [30, 7, 4], where the continuous dependence of the coefficients has been addressed in the particular situation of homogenization problems. See also the references in [37]. Due to the specific structure of the problem semi-group theory could be applied for a homogenization problem for thermo-elasticity [9].

As indicated above, the main observation for discussing homogenization problems is that the coefficients might only converge in a rather weak topology. A possible choice modeling this is the weak operator topology [36, 35]. Thus, motivated by the problems in homogenization theory, we investigate the continuous dependence of solutions of evolutionary problems on the coefficients, where the latter are endowed with the weak operator topology. Aiming at an abstract result and having sketched the drawbacks of semi-group theory in this line of problems, we need to consider a different class of evolutionary equations. We focus on a certain class of integro-differential algebraic partial differential equations. Recently, a well-posedness result could be obtained for this class [38, 42]. Moreover, generalizations of the results in [36, 40, 37] need the development of other techniques.

The class of equations under consideration is roughly described as follows. Consider

$$(\mathcal{M}u)' + \mathcal{A}u = f, \tag{1.1}$$

where $\mathcal{M}$ is a bounded linear operator acting in space-time, $(\mathcal{M}u)'$ denotes the time-derivative of $\mathcal{M}u$ and $\mathcal{A}$ is a (unbounded, linear) maximal monotone operator (see, e. g., [27]) in space-time, which is invariant under time-translations, f is a given forcing term and u is to be determined. The underlying Hilbert space setting will be described in Section 2. Though (1.1) seems to be an evolution equation in any case, it is possible to choose $\mathcal{M}$ in the way that (1.1) does not contain any time-derivative at all. Indeed, in the Hilbert space framework developed below the time-derivative becomes a continuously invertible operator. Thus, as $\mathcal{M}$ acts in space-time, we can choose $\mathcal{M}$ as the inverse of the time-derivative times some bounded linear operator M, such that (1.1) amounts to be $Mu + \mathcal{A}u = f$. In view of the latter observation and in order not to exclude the algebraic type equation $Mu + \mathcal{A}u = f$, we are led to consider (1.1) with no initial data. However, imposing sufficient regularity for the initial conditions, one can formulate initial value problems equivalently into problems of the type (1.1); see, e. g., [22, Section 6.2.5].

There are many standard equations from mathematical physics fitting in the abstract form described by (1.1). These are, for instance, the heat equation ([22, Section 6.3.1], [36, Theorem 4.5]), the wave equation ([21, Section 3], [31, Section 4.2]), Poisson's equation (see Section 4), the equations for elasticity [31, Section 4.2] or Maxwell's equations ([20, Section 4.1], [40, Section 5]). Coupled phenomena such as the equation for thermo-elasticity ([22, Section 6.3.2], [36, Theorem 4.10]) or the equations for thermo-piezo-electro-magnetism [22, Section 6.3.3], or equations with fractional derivatives like subdiffusion or superdiffusion problems ([36, Section 4], [25, Section 4]) can be dealt with in the general framework of (1.1). We note here that the operator $\mathcal{M}$ in (1.1) needs not to be time-translation invariant. Thus, the coefficients may not only contain memory terms, but they may also explicitly depend on time; see [26, Section 3].

In order to have an idea of the form of the operators $\mathcal{M}$ and $\mathcal{A}$, we give three more concrete examples. All these three examples are considered in a three-dimensional spatial domain Ω. Written in block operator matrix form with certain source term f, a first-order formulation of the heat equation reads as

$$\left(\frac{\partial}{\partial t}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & \kappa(t,x)^{-1}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}\\ \operatorname{grad} & 0\end{pmatrix}\right)\begin{pmatrix}\theta(t,x)\\ q(t,x)\end{pmatrix} = \begin{pmatrix}f(t,x)\\ 0\end{pmatrix},$$

where θ is the temperature, q is the heat flux and κ is the conductivity matrix, which is assumed to be continuously invertible. Note that the second line of the system is Fourier's law. Hence, in this case

$$\mathcal{M} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \left(\frac{\partial}{\partial t}\right)^{-1}\begin{pmatrix}0 & 0\\ 0 & \kappa^{-1}\end{pmatrix}$$

and

$$\mathcal{A} = \begin{pmatrix}0 & \operatorname{div}\\ \operatorname{grad} & 0\end{pmatrix},$$

which is skew self-adjoint if suitable boundary conditions are imposed.

Similarly, we find for the wave equation

$$\left(\frac{\partial}{\partial t}\begin{pmatrix}1 & 0\\ 0 & \kappa(t,x)^{-1}\end{pmatrix} + \begin{pmatrix}0 & \operatorname{div}\\ \operatorname{grad} & 0\end{pmatrix}\right)\begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix} = \begin{pmatrix}f(t,x)\\ 0\end{pmatrix},$$

for some suitable coefficient matrix κ, the relations

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ 0 & \kappa^{-1} \end{pmatrix}$$

and

$$\mathcal{A} = \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}.$$

Maxwell's equations read as

$$\left(\frac{\partial}{\partial t}\begin{pmatrix} \varepsilon(t,x) & 0 \\ 0 & \mu(t,x) \end{pmatrix} + \begin{pmatrix} \sigma(t,x) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix}\right)\begin{pmatrix} E(t,x) \\ H(t,x) \end{pmatrix} = \begin{pmatrix} J(t,x) \\ 0 \end{pmatrix},$$

where ε, μ and σ are the material coefficients electric permittivity, magnetic permeability and the electric conductivity, respectively. J is a given source term and (E, H) is the electro-magnetic field. We have

$$\mathcal{M} = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \left(\frac{\partial}{\partial t}\right)^{-1}\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\mathcal{A} = \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix},$$

which is skew self-adjoint for instance under the electric boundary condition. Well-posedness conditions for the above equations are suitable strict positive definiteness conditions for ε, μ and κ. Moreover, the derivative with respect to time needs to be uniformly bounded. The precise conditions can be found in [26, Condition (2.3)].

Now, we turn to discuss the main contribution, Theorem 3.1, of the present article. Take a sequence of bounded linear operators $(\mathcal{M}_n)_{n\in\mathbb{N}}$ in space-time converging in the weak operator topology τ_w to some bounded linear operator $\mathcal{M}$. The $\mathcal{M}_n$'s are assumed to satisfy suitable conditions (see Theorem 2.3 or [38, 42]) such that the respective equations as in (1.1) are well posed in the sense that the (closure of the) operator $u \mapsto (\mathcal{M}u)' + \mathcal{A}u$ is continuously invertible in space-time. Let f be a given right-hand side. For $n \in \mathbb{N}$, let u_n solve

$$(\mathcal{M}_n u_n)' + \mathcal{A}u_n = f. \tag{1.2}$$

The main result now states that if the sequence of the commutator of the $\mathcal{M}_n$'s with time-differentiation is a bounded sequence of bounded linear operators[1] and if the

1 If $\mathcal{M}_n$ is given by multiplication by some function κ_n depending on both the temporal and spatial variables, the commutator with time-differentiation is given by the operator of multiplying with the resolvent of $\mathcal{A}$ satisfies a certain compactness condition, we have $u_n \rightharpoonup u$, i. e., $(u_n)_n$ weakly converges to u, where u satisfies

$$(\mathcal{M}u)' + \mathcal{A}u = f.$$

It should be noted that the operator $\mathcal{A}$ may only be skew self-adjoint. In particular, the equations under consideration may not have maximal regularity. Moreover, the freedom in the choice of the sequence $(\mathcal{M}_n)_n$ also allows for the treatment of differential-algebraic equations, which have applications in control theory, [24, 23]. Since we only assume convergence in the weak operator topology for the operator sequence, the result particularly applies to norm-convergent sequences or sequences converging in the strong operator topology (see also Section 4). However, for the latter two cases the results are certainly not optimal. For the case of convergence in the weak operator topology, we give two examples (Examples 3.5 and 3.4) that the assumptions in our main theorem cannot be dropped.

In order to proceed in equation (1.2) to the limit as $n \to \infty$, the main difficulty to overcome is to find conditions such that $(\mathcal{M}_n u_n)_n$ converges to the product of the limits. This is where a compactness condition for the resolvent of $\mathcal{A}$ comes into play. With this, it is then possible to apply the compact embedding theorem of Aubin–Lions (see Theorem 5.1 below) in order to gain a slightly better convergence of (a subsequence of) the u_n's.

As it will be demonstrated in Section 4, the results have applications to homogenization theory. In a different situation, where certain time-translation invariant operators were treated, the latter has also been observed and exemplified in [36, 40]. We shall also mention applications to problems of mixed type; see [41].

We build up the Hilbert space setting mentioned above in Section 2. Section 3 is devoted to state and briefly discuss the main result of the paper, which will be applied in Section 4 to a homogenization problem in visco-elasticity, a wave equation with impedance type boundary conditions and a singular perturbation problem. The concluding section is devoted to the proof of Theorem 3.1. Any Hilbert space treated here is a complex Hilbert space.

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derivative of κ_n with respect to time. Thus, the boundedness of the sequence of commutators under consideration is warranted if, for instance, $(\kappa_n)_n$ is a C^1-bounded sequence considered as the sequence of mappings

$$\left(\mathbb{R} \ni t \mapsto \kappa_n(t, \cdot) \in L^\infty\right)_n.$$

2 Preliminaries

We summarize some findings of [22, 39, 38, 42]. In the whole section, let H be a Hilbert space. We introduce the time-derivative operator ∂_0 as an operator in

$$L^2_\nu(\mathbb{R}; H) := L^2(\mathbb{R}, \exp(-2\nu(\cdot))\lambda; H),$$

λ denoting the 1-dimensional Lebesgue measure, for some $\nu > 0$ as follows:

$$\partial_0 \colon H_{\nu,1}(\mathbb{R}; H) \subseteq L^2_\nu(\mathbb{R}; H) \to L^2_\nu(\mathbb{R}; H), \quad \phi \mapsto \phi',$$

where $H_{\nu,1}(\mathbb{R}; H)$ is the space of weakly differentiable $L^2_\nu(\mathbb{R}; H)$-functions with weak derivative also lying in the exponentially weighted L^2-space. For the scalar product in the latter space, we occasionally write $\langle\cdot,\cdot\rangle_\nu$. One can show that ∂_0 is one-to-one and that for $f \in L^2_\nu(\mathbb{R}; H)$ we have for all $t \in \mathbb{R}$ the Bochner-integral representation

$$\partial_0^{-1} f(t) = \int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau.$$

The latter formula particularly implies $\|\partial_0^{-1}\| \leqq \frac{1}{\nu}$, and thus, $0 \in \rho(\partial_0)$; see, e. g., [11, Theorem 2.2 and Corollary 2.5] for the elementary proofs. From the integral representation for ∂_0^{-1}, we also read off that $\partial_0^{-1} f$ vanishes up to some time $a \in \mathbb{R}$, if so does f. This fact may roughly be described as causality. A possible definition is the following.

Definition ([39]). Let $M \colon D(M) \subseteq L^2_\nu(\mathbb{R}; H) \to L^2_\nu(\mathbb{R}; H)$. We say that M is *causal* if for all $R > 0$, $a \in \mathbb{R}$, $\phi \in L^2_\nu(\mathbb{R}; H)$ the mapping

$$\begin{aligned}(B_M(0,R), |\mathbf{1}_{\mathbb{R}_{\leqq a}}(m_0)(\cdot - \cdot)|) &\to (L^2_\nu(\mathbb{R}; H), |\langle \mathbf{1}_{\mathbb{R}_{\leqq a}}(m_0)(\cdot - \cdot), \phi\rangle|)\\ f &\mapsto Mf,\end{aligned}$$

is uniformly continuous, where $B_M(0,R) := \{f \in D(M); |f| + |Mf| < R\}$.

Remarks 2.1.

(a) For closed linear operators M, we have shown in [39, Theorem 1.6] that M is causal if and only if for all $a \in \mathbb{R}$ and $\phi \in D(M)$ the implication

$$\mathbf{1}_{\mathbb{R}_{<a}}(m_0)\phi = 0 \Rightarrow \mathbf{1}_{\mathbb{R}_{<a}}(m_0)M\phi = 0$$

holds. The latter, in turn, is equivalent to

$$\mathbf{1}_{\mathbb{R}_{<a}}(m_0)M\,\mathbf{1}_{\mathbb{R}_{<a}}(m_0) = \mathbf{1}_{\mathbb{R}_{<a}}(m_0)M \quad (a \in \mathbb{R})$$

provided that $\mathbf{1}_{\mathbb{R}_{<a}}(m_0)[D(M)] \subseteq D(M)$ for all $a \in \mathbb{R}$.

(b) Assume that $M\colon D(M) \subseteq L_\nu^2(\mathbb{R};H) \to L_\nu^2(\mathbb{R};H)$ is continuous and for all $a \in \mathbb{R}$ the set

$$D(M\,\mathbf{1}_{\mathbb{R}_{\leqq a}}(m_0)) \cap D(M) \subseteq L_\nu^2(\mathbb{R};H)$$

is dense.[2] If for all $a \in \mathbb{R}$, we have

$$\mathbf{1}_{\mathbb{R}_{<a}}(m_0) M\, \mathbf{1}_{\mathbb{R}_{<a}}(m_0) = \mathbf{1}_{\mathbb{R}_{<a}}(m_0) M$$

on $D(M\,\mathbf{1}_{\mathbb{R}_{<a}}(m_0)) \cap D(M)$, then both M and $\overline{M}$ are causal. Indeed, by continuity, the latter equality implies that

$$\begin{aligned}\mathbf{1}_{\mathbb{R}_{<a}}(m_0)\overline{M}\,\mathbf{1}_{\mathbb{R}_{<a}}(m_0) &= \overline{\mathbf{1}_{\mathbb{R}_{<a}}(m_0) M\, \mathbf{1}_{\mathbb{R}_{<a}}(m_0)} \\ &= \overline{\mathbf{1}_{\mathbb{R}_{<a}}(m_0) M} = \mathbf{1}_{\mathbb{R}_{<a}}(m_0)\overline{M} \quad (a \in \mathbb{R}).\end{aligned}$$

Hence, by (a) $\overline{M}$ is causal, implying causality for M.

Remarks 2.2. A prototype of causal operators are particular functions of ∂_0^{-1}.[3] Though being of independent interest, we need this class of operators to properly formulate the examples in Section 4. We use the explicit spectral theorem for ∂_0^{-1} given by the *Fourier–Laplace transformation* $\mathcal{L}_\nu$. Here, $\mathcal{L}_\nu$ is the unitary transformation from $L_\nu^2(\mathbb{R})$ onto $L^2(\mathbb{R})$ such that

$$\mathcal{L}_\nu f = \left(x \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy-\nu y} f(y)\,\mathrm{d}y\right)$$

for continuous functions f with compact support. Then one can show that

$$\partial_0^{-1} = \mathcal{L}_\nu^* \frac{1}{im_0 + \nu} \mathcal{L}_\nu,$$

where $(\frac{1}{im_0+\nu}\phi)(x) := \frac{1}{ix+\nu}\phi(x)$ for $\phi \in L^2(\mathbb{R})$, $x \in \mathbb{R}$. Now, any M belonging to the Hardy space $\mathcal{H}^\infty(B(r,r))$ of bounded and analytic functions $B(r,r) \to \mathbb{C}$ for some $r > \frac{1}{2\nu}$ leads to a causal, time-translation invariant operator $M(\partial_0^{-1})$ in the way that

$$M(\partial_0^{-1}) := \mathcal{L}_\nu^* M\left(\frac{1}{im_0 + \nu}\right)\mathcal{L}_\nu.$$

2 The latter happens to be the case if, for instance, $D(M) \supseteq C_{\infty,c}(\mathbb{R};H)$, the space of indefinitely differentiable functions with compact support.

3 In [37, Section 4], [35, 25] examples for this kind of operators are given. Some of these are convolutions with suitable L^1-functions or the time-shift.

We endow $\mathcal{H}^\infty(B(r,r))$ with the supremum norm. Moreover, note that the definitions made can readily be extended to the vector-valued case, i. e., if $\mathcal{H}^\infty(B(r,r);L(H))$ denotes the Hardy space of bounded analytic functions with values in the space of bounded linear operators, we can define for $M \in \mathcal{H}^\infty(B(r,r);L(H))$ the operator

$$M(\partial_0^{-1}) \coloneqq \mathcal{L}_\nu^* M\left(\frac{1}{im_0+\nu}\right)\mathcal{L}_\nu, \tag{2.1}$$

acting in the Hilbert space $L_\nu^2(\mathbb{R};H)$, where we re-used $\mathcal{L}_\nu$ to denote the extension of the scalar-valued Fourier–Laplace transformation to the H-valued one. Thus, (2.1) should be read in the strong sense. With the help of the Paley–Wiener theorem, it is possible to show causality for $M(\partial_0^{-1})$; see, e. g., [20].

In [38, 42], we have shown the following well-posedness result, which comprises a large class of linear partial integro-differential algebraic equations of mathematical physics as it has been demonstrated in [22, 26] (see also Section 1); it does, however, not quite supersede the stochastic variant of evolutionary equations; see [29]. Before we state the well-posedness theorem, we introduce the notion of a bounded commutator.

Definition (Bounded commutator). Let $B \in L(H)$ and let $A\colon D(A) \subseteq H \to H$ be a densely defined linear operator. Then B is said to have a *bounded commutator with* A, if there exist $C \in L(H)$ such that $BA \subseteq AB + C$. In the latter case, we shall write $[B,A] \coloneqq -[A,B] \coloneqq C$. A sequence $(B_n)_n$ of bounded linear operators is said to have *bounded commutators with* A, if for all $n \in \mathbb{N}$ the operator B_n has a bounded commutator with A and the sequence $([B_n,A])_n$ is bounded.

Theorem 2.3 ([38, Theorem 2.7 and 2.4]). *Let H Hilbert space, $\mathcal{M} \in L(L_\nu^2(\mathbb{R};H))$. Assume that $\mathcal{M}$ has a bounded commutator with ∂_0. Let $\mathcal{A}\colon D(\mathcal{A}) \subseteq L_\nu^2(\mathbb{R};H) \to L_\nu^2(\mathbb{R};H)$ be linear, maximal monotone and such that $\partial_0(\mathcal{A}+1) = (\mathcal{A}+1)\partial_0$, i. e., $\mathcal{A}$* commutes with ∂_0. *Moreover, assume the positive definiteness conditions*

$$\Re\langle \partial_0\mathcal{M}u, \mathbf{1}_{\mathbb{R}_{\leqq a}}(m_0)u\rangle \geqq c\langle u, \mathbf{1}_{\mathbb{R}_{\leqq a}}(m_0)u\rangle, \quad \Re\langle \mathcal{A}u, \mathbf{1}_{\mathbb{R}_{\leqq 0}}(m_0)u\rangle \geqq 0 \tag{2.2}$$

for all $u \in D(\partial_0)\cap D(\mathcal{A})$, $a \in \mathbb{R}$, and some $c > 0$.

Then $0 \in \rho(\overline{\partial_0\mathcal{M}+\mathcal{A}})$ and the operator $(\overline{\partial_0\mathcal{M}+\mathcal{A}})^{-1}$ is causal.

Remarks 2.4.
(a) The operator $\mathcal{A}$ in the latter theorem is assumed to be maximal monotone. By this, we mean that $\mathcal{A}$ is maximal monotone as a relation and still being an operator. This implies closedness of the operator $\mathcal{A}$, as well as that $\mathcal{A}$ is densely defined; see [19, Lemma 1.1.3]. Moreover, $\mathcal{A}^*$ is also maximal monotone; see [19, Theorem 1.1.2].
(b) The fact that $\mathcal{A}$ commutes with ∂_0 implies in particular that $\partial_0^{-1}\mathcal{A} \subseteq \mathcal{A}\partial_0^{-1}$. So, with [42, Proposition 3.2.8], we deduce that $\mathcal{A}$ commutes with time-translation, so that, in particular, $\Re\langle \mathcal{A}u, \mathbf{1}_{\mathbb{R}_{\leqq 0}}(m_0)u\rangle \geqq 0$ implies $\Re\langle \mathcal{A}u, \mathbf{1}_{\mathbb{R}_{\leqq a}}(m_0)u\rangle \geqq 0$ for *all* $a \in \mathbb{R}$ and $u \in D(\mathcal{A})$.

For the following, we need to record some continuity estimates. In order to do so, we briefly recall the concept of Sobolev lattices discussed in [22, Chapter 2].

Definition ((Short) Sobolev lattice). Let C_1, C_2 be two densely defined closed linear operators in H with $0 \in \rho(C_1) \cap \rho(C_2)$ and $C_1C_2 = C_2C_1$. Then, for $k, \ell \in \{-1, 0, 1\}$ we define $H_{k,\ell}(C_1, C_2)$ as the completion of $D(C_1C_2)$ with respect to the norm $\phi \mapsto |C_1^k C_2^\ell \phi|$. The family $(H_{k,\ell}(C_1, C_2))_{k,\ell\in\{-1,0,1\}}$ is called *(short) Sobolev lattice*.

Remarks 2.5.
(a) We have continuous embeddings $H_{k,\ell}(C_1, C_2) \hookrightarrow H_{k',\ell'}(C_1, C_2)$ provided that $k \geqq k'$ and $\ell \geqq \ell'$. Moreover, $H_{k,\ell}(C_1, C_2) = H_{\ell,k}(C_2, C_1)$ for all $k, k', \ell, \ell' \in \{-1, 0, 1\}$.
(b) The operators $C_1^{\pm 1}$ can be established as unitary operators from $H_{k,\ell}(C_1, C_2)$ into $H_{k\mp 1,\ell}(C_1, C_2)$ for all $k, \ell \in \{-1, 0, 1\}$ such that $k \mp 1 \in \{-1, 0, 1\}$ and similarly for C_2.
(c) In the special case of $C_2 = 1$, we write $H_k(C_1) \coloneqq H_{k,1}(C_1, 1)$ for all $k \in \{-1, 0, 1\}$.
(d) In the special case of $C_1 = \partial_0$ and $C_2 = 1$, we write $H_{\nu,k}(\mathbb{R}; H) \coloneqq H_k(\partial_0)$ for all $k \in \{-1, 0, 1\}$.

Remark 2.6. With the help of the Sobolev lattice construction stated, we can drop the closure bar in $\overline{\partial_0\mathcal{M} + \mathcal{A}}$ and compute in the Sobolev lattice associated with $(\partial_0, \mathcal{A} + 1)$. In order to make this more precise, we denote here the extensions of ∂_0 and $\mathcal{A}$ to the Sobolev lattice (with the common domain $H_{0,0}(\partial_0, \mathcal{A} + 1) = L_\nu^2(\mathbb{R}; H)$) by ∂_0^{e} and $\mathcal{A}^{\mathrm{e}}$, respectively. Now, let $u \in D(\overline{\partial_0\mathcal{M} + \mathcal{A}}) \subseteqq L_\nu^2(\mathbb{R}; H)$. Then, by definition, there exists a sequence $(u_n)_n$ in $D(\partial_0\mathcal{M}) \cap D(\mathcal{A})$ such that $u_n \to u$ and $v_n \coloneqq (\partial_0\mathcal{M} + \mathcal{A})u_n \to \overline{(\partial_0\mathcal{M} + \mathcal{A})}u \eqqcolon v$ in $L_\nu^2(\mathbb{R}; H)$ as $n \to \infty$. On the other hand, the continuity of ∂_0^{e} and $\mathcal{A}^{\mathrm{e}}$ implies $\partial_0^{\mathrm{e}}\mathcal{M}u_n \to \partial_0^{\mathrm{e}}\mathcal{M}u$ and $\mathcal{A}^{\mathrm{e}}u_n \to \mathcal{A}^{\mathrm{e}}u$ in $H_{-1,0}(\partial_0, \mathcal{A} + 1)$ and $H_{0,-1}(\partial_0, \mathcal{A} + 1)$, respectively, as $n \to \infty$. From $L_\nu^2(\mathbb{R}; H) \hookrightarrow H_{-1,-1}(\partial_0, \mathcal{A} + 1)$ and

$$v_n = \partial_0^{\mathrm{e}}\mathcal{M}u_n + \mathcal{A}^{\mathrm{e}}u_n \overset{n\to\infty}{\to} \partial_0^{\mathrm{e}}\mathcal{M}u + \mathcal{A}^{\mathrm{e}}u \in H_{-1,-1}(\partial_0, \mathcal{A} + 1)$$

it follows that $v = \partial_0^{\mathrm{e}}\mathcal{M}u + \mathcal{A}^{\mathrm{e}}u$. Thus,

$$D(\overline{\partial_0\mathcal{M} + \mathcal{A}}) \subseteqq \{u \in L_\nu^2(\mathbb{R}; H); \partial_0^{\mathrm{e}}\mathcal{M}u + \mathcal{A}^{\mathrm{e}}u \in L_\nu^2(\mathbb{R}; H)\} \eqqcolon D$$

and $\overline{(\partial_0\mathcal{M} + \mathcal{A})}u = \partial_0^{\mathrm{e}}\mathcal{M}u + \mathcal{A}^{\mathrm{e}}u$ for $u \in D(\overline{\partial_0\mathcal{M} + \mathcal{A}})$.

On the other hand, if $u \in D$ then one can show that $(1 + \varepsilon\partial_0^{-1})u \in D(\partial_0) \cap D(\mathcal{A}) \subseteqq D(\partial_0\mathcal{M} + \mathcal{A})$ for every $\varepsilon > 0$; see [38, Lemma 4.2] or [26, Lemma 2.9]. Moreover, from the lemmas stated, it also follows that $((\partial_0\mathcal{M} + \mathcal{A})(1 + \varepsilon\partial_0^{-1})u)_{\varepsilon>0}$ is weakly convergent in $L_\nu^2(\mathbb{R}; H)$ as $\varepsilon \to 0+$. As the strong closure of linear operators coincides with the weak closure, we deduce that $u \in D(\overline{\partial_0\mathcal{M} + \mathcal{A}})$.

With the observations made in the latter remark, we henceforth omit the closure bar in $\overline{\partial_0\mathcal{M} + \mathcal{A}}$, use the continuous extensions of ∂_0 and $\mathcal{A}$ to the Sobolev lattice, reuse the respective notation and agree that $D(\partial_0\mathcal{M} + \mathcal{A}) = \{u \in L_\nu^2(\mathbb{R}; H); \partial_0\mathcal{M}u + \mathcal{A}u \in L_\nu^2(\mathbb{R}; H)\}$. Now, we are in the position to state the continuity estimates.

Corollary 2.7. *In the situation of Theorem* 2.3*, let* $f \in L^2_\nu(\mathbb{R};H)$ *and* $u \in L^2_\nu(\mathbb{R};H)$ *with* $(\partial_0\mathcal{M} + \mathcal{A})u = f$*. Then* $u \in H_{-1,1}(\partial_0, \mathcal{A}+1)$ *and*

$$|u|_{H_{-1,1}(\partial_0,\mathcal{A}+1)} \leqq \left(\frac{1}{\nu} + \|\mathcal{M}\|_{L(L^2_\nu(\mathbb{R};H))}\frac{1}{c} + \frac{1}{c\nu}\right)|f|_{L^2_\nu(\mathbb{R};H)}$$

Proof. In the Sobolev lattice associated to $(\partial_0, \mathcal{A}+1)$, we compute that

$$(\mathcal{A}+1)u = f - \partial_0\mathcal{M}u + u \in H_{-1,0}(\partial_0, \mathcal{A}+1).$$

Thus,

$$u = (\mathcal{A}+1)^{-1}(f - \partial_0\mathcal{M}u + u) \in H_{-1,1}(\partial_0, \mathcal{A}+1)$$

and

$$\begin{aligned}
|u|_{-1,1} &= \left|\partial_0^{-1}(\mathcal{A}+1)u\right|_{0,0} \\
&= \left|\partial_0^{-1}(\mathcal{A}+1)(\mathcal{A}+1)^{-1}(f - \partial_0\mathcal{M}u + u)\right|_{0,0} \\
&\leqq \left|\partial_0^{-1}f\right|_{0,0} + |\mathcal{M}u|_{0,0} + \left|\partial_0^{-1}u\right|_{0,0} \\
&\leqq \frac{1}{\nu}|f|_{0,0} + \|\mathcal{M}\|\frac{1}{c}|f|_{0,0} + \frac{1}{c\nu}|f|_{0,0} \\
&\leqq \left(\frac{1}{\nu} + \|\mathcal{M}\|\frac{1}{c} + \frac{1}{c\nu}\right)|f|_{0,0}
\end{aligned}$$

□

3 The basic convergence theorem

We recall the concept of G-convergence.

Definition (G-convergence, [46, p. 74], [40]). Let H be a Hilbert space. Let $(A_n : D(A_n) \subseteqq H \to H)_n$ be a sequence of continuously invertible linear operators onto H and let $B : D(B) \subseteqq H \to H$ be linear and one-to-one. We say that $(A_n)_n$ *G-converges to B* if $(A_n^{-1})_n$ converges in the weak operator topology to B^{-1}, i. e., for all $f \in H$ the sequence $(A_n^{-1}(f))_n$ converges weakly to some u, which satisfies $u \in D(B)$ and $B(u) = f$. B is called the[4] *G-limit* of $(A_n)_n$ and we write $A_n \xrightarrow{G} B$.

Our main theorem reads as follows.

Theorem 3.1. *Let* H *be a Hilbert space,* $\nu > 0$*. Let* $(\mathcal{M}_n)_n$ *be a bounded sequence in* $L(L^2_\nu(\mathbb{R};H))$ *with bounded commutators with* ∂_0*. Moreover, let* $\mathcal{A}: D(\mathcal{A}) \subseteqq L^2_\nu(\mathbb{R};H) \to$

4 Note that the G-limit is uniquely determined; cf. [40, Proposition 4.1].

$L^2_\nu(\mathbb{R};H)$ linear and maximal monotone commuting with ∂_0 and assume that $\mathcal{M}_n$ is causal, $n \in \mathbb{N}$. Moreover, assume the positive definiteness conditions

$$\Re\langle \partial_0 \mathcal{M}_n u, \mathbf{1}_{\mathbb{R}_{<a}}(m_0)u\rangle \geqq c\langle u, \mathbf{1}_{\mathbb{R}_{<a}}(m_0)u\rangle, \quad \Re\langle \mathcal{A}u, \mathbf{1}_{\mathbb{R}_{<0}}(m_0)u\rangle \geqq 0$$

for all $u \in D(\partial_0) \cap D(\mathcal{A})$, $a \in \mathbb{R}$, $n \in \mathbb{N}$ and some $c > 0$.

Assume that there exists a Hilbert space K such that $K \hookrightarrow\hookrightarrow H$, i. e., K is compactly embedded into H, $H_1(\mathcal{A}+1) \hookrightarrow L^2_\nu(\mathbb{R};K)$ and that $(\mathcal{M}_n)_n$ converges in the weak operator topology to some $\mathcal{M}$.

Then $\partial_0\mathcal{M} + \mathcal{A}$ is continuously invertible in $L^2_\nu(\mathbb{R};H)$ and

$$(\partial_0\mathcal{M}_n + \mathcal{A}) \xrightarrow{G} (\partial_0\mathcal{M} + \mathcal{A}) \quad \text{as } n \to \infty.$$

Remarks 3.2. It should be noted that it is possible to show another continuity property. Namely, if $(f_n)_n$ in $L^2_\nu(\mathbb{R};H)$ is a weakly convergent sequence with[5] $\inf_n \inf \operatorname{spt} f_n > -\infty$ and $(u_n)_n$ is the sequence of solutions to

$$(\partial_0\mathcal{M}_n + \mathcal{A})u_n = f_n,$$

then $(u_n)_n$ weakly converges to the solution u of

$$(\partial_0\mathcal{M} + \mathcal{A})u = \text{w-}\lim_{n\to\infty} f.$$

In view of the well-posedness theorems [26, Theorem 2.13] and [38, Theorem 2.4], there is a more adapted version of Theorem 3.1:

Corollary 3.3. *Let H be a Hilbert space, $\nu > 0$. Let $(\mathcal{M}_n)_n$, $(\mathcal{N}_n)_n$ be bounded sequences of causal operators in $L(L^2_\nu(\mathbb{R};H))$ having bounded commutators with ∂_0 and $\mathcal{A}\colon D(\mathcal{A}) \subseteq L^2_\nu(\mathbb{R};H) \to L^2_\nu(\mathbb{R};H)$ linear, maximal monotone commuting with ∂_0. Assume the positive definiteness conditions*

$$\Re\langle (\partial_0\mathcal{M}_n + \mathcal{N}_n)u, \mathbf{1}_{\mathbb{R}_{<a}}(m_0)u\rangle \geqq c\langle u, \mathbf{1}_{\mathbb{R}_{<a}}(m_0)u\rangle \quad (a \in \mathbb{R})$$
$$\Re\langle \mathcal{A}u, \mathbf{1}_{\mathbb{R}_{<0}}(m_0)u\rangle \geqq 0$$

for all $u \in D(\partial_0) \cap D(\mathcal{A})$, $n \in \mathbb{N}$ and some $c > 0$.

Assume that there exists a Hilbert space K such that $K \hookrightarrow\hookrightarrow H$ and $H_1(\mathcal{A}+1) \hookrightarrow L^2_\nu(\mathbb{R};K)$ and that $(\mathcal{M}_n)_n$, $(\mathcal{N}_n)_n$ converges in the weak operator topology to some $\mathcal{M}$ and $\mathcal{N}$, respectively.

Then $\partial_0\mathcal{M} + \mathcal{N} + \mathcal{A}$ is continuously invertible in $L^2_\nu(\mathbb{R};H)$ and

$$(\partial_0\mathcal{M}_n + \mathcal{N}_n + \mathcal{A}) \xrightarrow{G} (\partial_0\mathcal{M} + \mathcal{N} + \mathcal{A}) \quad \text{as } n \to \infty.$$

5 We denote the support of a function $v\colon \mathbb{R} \to X$ with values in some topological vector space X by $\operatorname{spt} v$.

Proof. It suffices to verify the assumptions in Theorem 3.1 on $(\mathcal{M}_n)_n$ for the operator sequence $(\mathcal{M}_n + \partial_0^{-1}\mathcal{N}_n)_n$. This, however, is easy to see. □

We remark here that in order to prove well-posedness for equations of the form

$$(\partial_0\mathcal{M} + \mathcal{N} + \mathcal{A})u = f$$

for suitable $\mathcal{M}$, $\mathcal{N}$, $\mathcal{A}$ in [26, Theorem 2.13] or [38, Theorem 2.4], we did not need any assumptions on the commutator of $\mathcal{N}$ and ∂_0. Thus, one might wonder, whether the boundedness for the commutators of $(\mathcal{N}_n)_n$ with ∂_0 is needed in Corollary 3.3. The next example shows that this boundedness assumption is needed to compute the limit equation in the way it is done in Corollary 3.3.

Example 3.4 (On the boundedness of $([\mathcal{N}_n, \partial_0])_n$). Let $\nu > 0$. Consider for $n \in \mathbb{N}$ the operator

$$\sin(nm_0): L_\nu^2(\mathbb{R}) \to L_\nu^2(\mathbb{R}), f \mapsto (\sin(n\cdot)f(\cdot)).$$

Define for $n \in \mathbb{N}$ the operators $\mathcal{M}_n = 0$, $\mathcal{N}_n := \sin(nm_0)+2$ and $\mathcal{A}: \mathbb{C} \to \mathbb{C}$, $x \mapsto x$. Then, clearly, the (uniform) positive definiteness condition is satisfied and $\mathcal{A}$ has compact resolvent. For $f \in C_{\infty,c}(\mathbb{R})$, consider the problem of finding $u_n \in L_\nu^2(\mathbb{R})$ such that

$$(\partial_0\mathcal{M}_n + \mathcal{N}_n + \mathcal{A})u_n = f,$$

which is the same as to say that

$$((\sin(nm_0) + 2) + 1)u_n = f.$$

We get that $u_n = \frac{1}{\sin(nm_0)+3}f$. By periodicity of sin we get with the help of [7, Theorem 2.6], that

$$u_n \rightharpoonup \int_{-\pi}^{\pi} \frac{1}{\sin(t) + 3}\,\mathrm{d}t f = \frac{\pi}{\sqrt{2}}f =: u,$$

as $n \to \infty$. Moreover, it is easy to see that

$$\mathcal{N}_n \overset{\tau_w}{\to} \int_{-\pi}^{\pi} \sin(t) + 2\,\mathrm{d}t = 4\pi =: \mathcal{N}.$$

Thus, if the representation formulas for the limit equation remain true also in this case, we would obtain that u satisfies the equation

$$4\pi\frac{\pi}{\sqrt{2}}f + 1\frac{\pi}{\sqrt{2}}f = \mathcal{N}u + \mathcal{A}u = f,$$

which is not true, since $4\pi+1 \neq \frac{\sqrt{2}}{\pi}$. A reason for this is that we cannot deduce that the weak limit of the sequence $(\mathcal{N}_n u_n)_n$ equals the product of the respective limits. Indeed, we have

$$\mathcal{N}u = 4\pi\frac{\pi}{\sqrt{2}}f \neq \text{w-}\lim_{n\to\infty}\mathcal{N}_n u_n = \left(\int_{-\pi}^{\pi}\frac{\sin(t)+2}{\sin(t)+3}\,\mathrm{d}t\right)f = \pi\left(2-\frac{1}{\sqrt{2}}\right)f.$$

Though the latter example does not fit into the scheme developed above, it fits well into the theory established in [37], where we did not need the assumptions on the sequence having bounded commutator with ∂_0.

We recall [40, Example 4.9] to show that the compactness condition on $\mathcal{A}$ is also needed to compute the limit in the way it is done in Corollary 3.3.

Example 3.5 (Compactness assumption does not hold). Let $\nu, \varepsilon > 0$. Consider the mapping $a : \mathbb{R} \to \mathbb{R}$ given by

$$a(x) := \mathbf{1}_{[0,\frac{1}{2})}(x-k) + 2\mathbf{1}_{[\frac{1}{2},1]}(x-k)$$

for all $x \in [k, k+1)$, where $k \in \mathbb{Z}$. Define the corresponding multiplication operator in $L_2(\mathbb{R})$, i. e. for $\phi \in C_{\infty,c}(\mathbb{R})$, $a(n\cdot m)\phi := (x \mapsto a(nx)\phi(x))$ for $n \in \mathbb{N}$. Note that $a(x+k) = a(x)$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Let $f \in L^2_\nu(\mathbb{R}; L^2(\mathbb{R}))$. We consider the evolutionary equation with $(\mathcal{M}_n)_n := (0)_n$, $(\mathcal{N}_n)_n := (a(n\cdot m))_n$ and $\mathcal{A} = i : L^2(\mathbb{R}) \to L^2(\mathbb{R}) : \phi \mapsto i\phi$. By [7, Theorem 2.6], we deduce that

$$\mathcal{N}_n \to \frac{3}{2}$$

as $n \to \infty$. If the assertion of Theorem 3.1 remains true in this case, then $(\mathcal{N}_n + \mathcal{A})_n$ G-converges to $\frac{3}{2} + i$. For $n \in \mathbb{N}$, let $u_n \in L^2_\nu(\mathbb{R}; L^2(\mathbb{R}))$ be the unique solution of the equation

$$(\mathcal{N}_n + \mathcal{A})u_n = (a(nm) + i)u_n = f. \tag{3.1}$$

Observe that by [7, Theorem 2.6]

$$u_n = (a(nm) + i)^{-1}f \rightharpoonup \left(\int_0^1 (a(x) + i)^{-1}\,\mathrm{d}x\right)f =: u,$$

as $n \to \infty$. We integrate

$$\int_0^1 (a(x) + i)^{-1}\,\mathrm{d}x = \frac{1}{2}(1+i)^{-1} + \frac{1}{2}(2+i)^{-1}.$$

Inverting the latter equation yields

$$\left(\int_0^1 (a(x) + i)^{-1}\,\mathrm{d}x\right)^{-1} = \left(\frac{1}{2}(1+i)^{-1} + \frac{1}{2}(2+i)^{-1}\right)^{-1} = \frac{18}{13} + \frac{14}{13}i.$$

Hence, u satisfies

$$\left(\frac{3}{2}+i\right)u=f \quad \text{and} \quad \left(\frac{18}{13}+\frac{14}{13}i\right)u=f,$$

which of course is a contradiction.

4 Examples

4.1 A time-dependent Kelvin–Voigt model

We discuss an example from [1] (we also refer to [3]), where some convergence estimates have been established. For showing that this example fits into our abstract scheme, we introduce some operators first. Let $\Omega \subseteq \mathbb{R}^3$ be open and bounded. Denote the weak symmetrized gradient acting on square-integrable vector fields in $L^2(\Omega)^3$ with (generalized) Dirichlet boundary condition by $\mathring{\mathrm{Grad}}$. Korn's inequality implies $D(\mathring{\mathrm{Grad}}) = H_{1,0}(\Omega)^3$. By definition, for $v \in D(\mathring{\mathrm{Grad}})$ the mapping $\mathring{\mathrm{Grad}}v$ is an element of $H_{\mathrm{sym}}(\Omega)$, the space of square-integrable symmetric 3×3-matrices. Endowing the latter space with the inner product

$$(\Phi,\Psi) \mapsto \int_\Omega \operatorname{trace}\big(\Phi(x)^*\Psi(x)\big)\,\mathrm{d}x,$$

we realize that $\mathring{\mathrm{Grad}} = -\operatorname{Div}$, where the latter operator is the weak row-wise divergence with maximal domain. Note that by Korn's inequality and Rellich's selection theorem, we have that $D(\mathring{\mathrm{Grad}}) \hookrightarrow\hookrightarrow L^2(\Omega)$, where the first space is endowed with the graph-norm of $\mathring{\mathrm{Grad}}$. From Poincare's inequality, we see that $\mathring{\mathrm{Grad}}$ has closed range. Denote by $\iota_R\colon R(\mathring{\mathrm{Grad}}) \to H_{\mathrm{sym}}(\Omega)$ the canonical injection. As a consequence, the operator ι_R^* is the orthogonal projection onto the range of $\mathring{\mathrm{Grad}}$; see, e. g., [25, Lemma 3.2].

In order to treat the problem class properly, we need to recall some notions from [38] and [37].

Definition (Evolutionary mappings, [37, Definition 2.1]). Let $\nu_1 > 0$. For Hilbert spaces H_0, H_1, we call a linear mapping

$$M\colon D(M) \subseteq \bigcap_{\nu\geqq\nu_1} L^2_\nu(\mathbb{R};H_0) \to \bigcap_{\nu\geqq\nu_1} L^2_\nu(\mathbb{R};H_1) \tag{4.1}$$

evolutionary (at ν_1) if $D(M) \subseteq L^2_\nu(\mathbb{R};H_0)$ is dense and $M\colon D(M) \subseteq L^2_\nu(\mathbb{R};H_0) \to L^2_\nu(\mathbb{R};H_1)$ is closable for all $\nu \geqq \nu_1$. We say M is *bounded*, if, in addition, $M_\nu := \overline{M} \in L(L^2_\nu(\mathbb{R};H_0),$

$L^2_\nu(\mathbb{R};H_1))$ for all $\nu \geqq \nu_1$ such that[6]

$$\limsup_{\nu\to\infty} \|M\|_{L(L^2_\nu)} < \infty.$$

We define

$$L_{\mathrm{ev},\nu_1}(H_0,H_1) \coloneqq \{M; M \text{ is as in (4.1), is evolutionary at } \nu_1 \text{ and bounded}\}$$

and abbreviate $L_{\mathrm{ev},\nu_1}(H_0) \coloneqq L_{\mathrm{ev},\nu_1}(H_0,H_0)$. We call $\mathfrak{M} \subseteqq L_{\mathrm{ev},\nu_1}(H_0,H_1)$ *bounded* if $\limsup_{\nu\to\infty}\sup_{M\in\mathfrak{M}}\|M\|_{L(L^2_\nu)} < \infty$. A family $(M_\iota)_{\iota\in I}$ in $L_{\mathrm{ev},\nu_1}(H_0,H_1)$ is called *bounded* if $\{M_\iota; \iota\in I\}$ is bounded.

In [37, 38], we gave several examples for evolutionary mappings. Multiplication operators are a particular subclass of these. Moreover, the operator

$$\mathcal{A} \coloneqq \begin{pmatrix} 0 & \operatorname{Div}\iota_R \\ \iota_R^*\mathring{\operatorname{Grad}} & 0 \end{pmatrix}$$

(defined in space-time) is also evolutionary for every $\nu > 0$ and even bounded evolutionary in $L_{\mathrm{ev},\nu}(D(\iota_R^*\mathring{\operatorname{Grad}}) \oplus D(\operatorname{Div}\iota_R); L^2(\Omega)^3 \oplus R(\mathring{\operatorname{Grad}}))$. Trivially, $\mathcal{A}$ is causal. For bounded evolutionary mappings, we recall the following result.

Lemma 4.1 ([38, Lemma 3.3]). *Let $\nu \geqq \nu_1 \geqq \nu_0$, H_0, H_1 Hilbert spaces. Let $M \in L_{\mathrm{ev},\nu_0}(H_0,H_1)$ be causal. Then M_ν and M_{ν_1} coincide on $L^2_{\nu_1}(\mathbb{R};H_0)\cap L^2_\nu(\mathbb{R};H_0)$.*

In view of the latter lemma, we omit the subscript in the notation of the closures for causal, evolutionary mappings for different values of ν, if there is no risk of confusion.

Now, take $\nu > 0$ and let $\rho \in L_{\mathrm{ev},\nu}(L^2(\Omega)^3)$, $A, B \in L_{\mathrm{ev},\nu}(L^2(\Omega)^{3\times 3})$. The model treated in [1], can be written as

$$\partial_0\rho\partial_0 u - \operatorname{Div} B\mathring{\operatorname{Grad}}\partial_0 u - \operatorname{Div} A\mathring{\operatorname{Grad}} u = f.$$

Abbreviating $v \coloneqq \partial_0 u$ and using $\operatorname{Div} B\mathring{\operatorname{Grad}} = \operatorname{Div}\iota_R\iota_R^* B\iota_R\iota_R^*\mathring{\operatorname{Grad}}$ (see, e. g., [36]), we arrive at

$$\partial_0\rho v - \operatorname{Div}\iota_R(\iota_R^*(B + A\partial_0^{-1})\iota_R)\iota_R^*\mathring{\operatorname{Grad}} v = f.$$

Now, if B_ν is *strictly positive definite (uniformly for all large ν)* in the sense that

$$\Re\langle B_\nu u, \mathbf{1}_{\mathbb{R}_{<a}}(m_0)u\rangle \geqq c\langle u, \mathbf{1}_{\mathbb{R}_{<a}}(m_0)u\rangle$$

6 For a bounded linear operator A from $L^2_\nu(\mathbb{R};H_0)$ to $L^2_\nu(\mathbb{R};H_1)$, we denote its operator norm by $\|A\|_{L(L^2_\nu(\mathbb{R};H_0),L^2_\nu(\mathbb{R};H_1))}$. If the spaces H_0 and H_1 are clear from the context, we shortly write $\|A\|_{L(L^2_\nu)}$.

for some $c > 0$ and all sufficiently large ν, $u \in D(B)$, $a \in \mathbb{R}$ and ν is chosen large enough, we end up with ($q := (\iota_R^*(B + A\partial_0^{-1})\iota_R)\iota_R^*\mathring{\mathrm{Grad}}v$):

$$\left(\partial_0\begin{pmatrix}\rho & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & (\iota_R^*(B + A\partial_0^{-1})\iota_R)^{-1}\end{pmatrix} - \begin{pmatrix}0 & \mathrm{Div}\,\iota_R\\ \iota_R^*\mathring{\mathrm{Grad}} & 0\end{pmatrix}\right)\begin{pmatrix}v\\ q\end{pmatrix} = \begin{pmatrix}f\\ 0\end{pmatrix}.$$

Assuming that ρ, A, B have bounded commutators with ∂_0 in the sense of Theorem 2.3.[7] If, in addition, $\partial_0\rho_\nu$ is strictly positive definite (uniformly for all large ν), then it is easy to see that the aforementioned Kelvin–Voigt model for visco-elasticity is well posed in the sense of Theorem 2.3. Moreover, it is easy to see that if A, B and ρ are thought of as being multiplication operators, the assumption on the boundedness of the commutator follows if one assumes that the respective functions are Lipschitz continuous and almost every where strongly differentiable (with respect to the temporal variable). For the latter, see [26, 38]. Thus,

$$\left(\partial_0\begin{pmatrix}\rho & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & (\iota_R^*(B + A\partial_0^{-1})\iota_R)^{-1}\end{pmatrix} - \begin{pmatrix}0 & \mathrm{Div}\,\iota_R\\ \iota_R^*\mathring{\mathrm{Grad}} & 0\end{pmatrix}\right)$$

is continuously invertible in the underlying Hilbert space $L_\nu^2(\mathbb{R}; L^2(\Omega)^3 \oplus R(\mathring{\mathrm{Grad}}))$. As well-posedness issues are not the focus of the present article, we now apply our abstract homogenization theorem:

Theorem 4.2. *Let $\nu > 0$, $(\rho_n)_n$, $(B_n)_n$, $(A_n)_n$ be bounded sequences of causal operators in $L_{\mathrm{ev},\nu}(L^2(\Omega)^3)$, $L_{\mathrm{ev},\nu}(H_{\mathrm{sym}}(\Omega))$, and $L_{\mathrm{ev},\nu}(H_{\mathrm{sym}}(\Omega))$, respectively. Assume that the respective sequences have bounded commutators with ∂_0. Moreover, assume there exists $c > 0$ such that*

$$\begin{aligned}\mathbb{R}\langle B_n u, \mathbf{1}_{\mathbb{R}_{\leqq a}}(m_0)u\rangle_{\nu'} &\geqq c\langle \phi, \mathbf{1}_{\mathbb{R}_{\leqq a}}(m_0)\phi\rangle_{\nu'},\\ \mathbb{R}\langle \partial_0\rho_n u, \mathbf{1}_{\mathbb{R}_{\leqq a}}(m_0)u\rangle_{\nu'} &\geqq c\langle u, \mathbf{1}_{\mathbb{R}_{\leqq a}}(m_0)u\rangle_{\nu'}\end{aligned}$$

for all $\nu' \geqq \nu$ and $\phi \in L_\nu^2(\mathbb{R}; H_{\mathrm{sym}}(\Omega))$ and $u \in H_{\nu,1}(\mathbb{R}; L^2(\Omega)^3)$, $a \in \mathbb{R}$.

Then there exists a subsequence $(n_k)_k$ such that

$$\begin{aligned}&\left(\partial_0\begin{pmatrix}\rho_{n_k} & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & (\iota_R^*(B_{n_k} + A_{n_k}\partial_0^{-1})\iota_R)^{-1}\end{pmatrix} - \begin{pmatrix}0 & \mathrm{Div}\,\iota_R\\ \iota_R^*\mathring{\mathrm{Grad}} & 0\end{pmatrix}\right)\\ &\qquad\xrightarrow{G}\left(\partial_0\begin{pmatrix}\rho & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & \sum_{\ell=0}^{\infty}\mathcal{M}_\ell\end{pmatrix} - \begin{pmatrix}0 & \mathrm{Div}\,\iota_R\\ \iota_R^*\mathring{\mathrm{Grad}} & 0\end{pmatrix}\right),\end{aligned}$$

7 Note that the boundedness of the commutator of A and ∂_0 is not needed to ensure the well-posedness of the respective equation. For general well-posedness conditions for this particular equation, we refer to the concluding section in [26].

where the latter operator is continuously invertible and

$$\mathcal{M}_\ell = \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty}\left(-(\iota_R^* B_{n_k}\iota_R)^{-1}\iota_R^* A_{n_k}\iota_R\partial_0^{-1}\right)^\ell (\iota_R^* B_{n_k}\iota_R)^{-1}$$

and $\rho = \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty}\rho_{n_k}$.

Proof. The proof follows with a Neumann series expansion of

$$\left(\iota_R^*(B_{n_k} + A_{n_k}\partial_0^{-1})\iota_R\right)^{-1},$$

the fact that for a sequence $(T_n)_n$ converging in the weak operator topology in some Hilbert space H, we have $\|\tau_{\mathrm{w}}\text{-}\lim_{n\to\infty} T_n\| \leqq \liminf_{n\to\infty}\|T_n\|$, that for a separable Hilbert space H norm-bounded subsets of $L(H)$ are relatively compact and metrizable with respect to the weak operator topology and Theorem 3.1. □

Remarks 4.3.

(a) We give some more explicit formulae for $\mathcal{M}_\ell$ for particular situations:

(i) In the particular case, where $A_n = 0$ and B_n is time-independent, i. e., for every $n \in \mathbb{N}$ there exists $b_n \in L(H_{\mathrm{sym}}(\Omega))$ such that B_n is the (canonical) extension of b_n to $L_\nu^2(\mathbb{R}; H_{\mathrm{sym}}(\Omega))$, then $\mathcal{M}_\ell = 0$ ($\ell \in \mathbb{N}_{>0}$) and $\mathcal{M}_0 = \lim_{k\to\infty}(\iota_R^* b_{n_k}\iota_R)^{-1}$. One can show that in the special case of $b_n = d(n\cdot)$, where d is a matrix of suitable size with entries in the space of $[0,1]^3$-periodic $L^\infty(\mathbb{R}^3)$-functions, the result coincides with the classical limit; see also [42, Theorem 5.5.3] or [43, Theorem 1.2].

(ii) Assume that $(B_n)_n = (b_n)_n$, where $(b_n)_n$ is a bounded sequence of causal operators in $L_{\mathrm{ev},\nu}(\mathbb{C})$ such that $(b_n)_\nu \geqq c$ for all $n \in \mathbb{N}$ and some $c > 0$ and such that the sequence of respective commutators with ∂_0 is bounded as well. Furthermore, assume that $(A_n)_n = (a_n)_n$, where $a_n = d(n\cdot)$ for a function d as in the previous part. Now, if $b_n \to b$ strongly[8] for some $b \in L_{\mathrm{ev},\nu}(\mathbb{C})$, then

$$\mathcal{M}_\ell = \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty}\left(-\iota_R^* A_{n_k}\iota_R\right)^\ell \left(b^{-1}\partial_0^{-1}\right)^\ell b^{-1} \quad (\ell \in \mathbb{N}).$$

(iii) Assume that both $(A_n)_n$ and $(B_n)_n$ satisfy the structural assumption on B_n as in (ii) being representable as operators only acting in time. Assume, in addition, that B_n is uniformly strictly positive (as in (ii)) and that $((A_n)_n, (B_n^{-1})_n, (\partial_0^{-1})_n)$ has the *product convergence property* (see [37, Definition 5.1]), then

$$\mathcal{M}_\ell = \iota_R^*\Big(\tau_{\mathrm{w}}\text{-}\lim_{n\to\infty}\left(-B_n^{-1}A_n\partial_0^{-1}\right)^\ell (B_n)^{-1}\Big)\iota_R \quad (\ell \in \mathbb{N}).$$

8 Here, strong convergence means that there exists $b \in L_{\mathrm{ev},\nu}(\mathbb{C})$ such that for any $\nu' \geqq \nu$ we have that $(b_n)_{\nu'} \to (b)_{\nu'}$ as $n \to \infty$ in the strong operator topology.

(b) We shall note here that the considerations above can be done similarly for the case of $B_n = 0$ and A_n self-adjoint and (uniformly) strictly positive definite. The homogenization result then coincides with the classical one in the sense of part (a)(i).
(c) There is also a possibility to treat the cases (a)(i) and (b) in a unified way. The resulting formulas, however, become more involved. We refer to the concluding section in [26] for a unified treatment of the cases (a)(i) and (b) with regards to well-posedness issues.

4.2 The wave equation with impedance type boundary conditions

We recall the setting in [21, Section 3] or [31, Section 4]. We let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set such that $H_1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$,[9] i. e., the maximal domain of the distributional gradient grad defined on $L^2(\Omega)$ endowed with the graph norm of grad is compactly embedded into $L^2(\Omega)$. Analogously let div be the distributional divergence on $L^2(\Omega)^n$ with maximal domain. The respective skew-adjoints will be denoted by $\mathring{\operatorname{div}}$ and $\mathring{\operatorname{grad}}$, as these operators encode homogeneous Neumann and Dirichlet boundary conditions, respectively.

Formally, the equations treated in [21, Section 3] (or in [31, Section 4]) read as

$$\left(\partial_0 \mathcal{M} + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}\right) U = F,$$

for some given F and $\mathcal{M}$. We address the continuous dependence on the coefficient $\mathcal{M}$. Imposing additional structure on $\mathcal{M}$ and the right-hand side F, we may rewrite the latter system into a more common form. Indeed, if $F = (f, 0)$ and $\mathcal{M} = \operatorname{diag}(\mathcal{M}_1, \mathcal{M}_2)$ with respect to the block structure of $\left(\begin{smallmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{smallmatrix}\right)$ we obtain with $U = (u_1, u_2)$:

$$\partial_0 \mathcal{M}_1 u_1 + \operatorname{div} u_2 = f \quad \text{and} \quad \partial_0 \mathcal{M}_2 u_2 + \operatorname{grad} u_1 = 0,$$

which leads to

$$\partial_0 \mathcal{M}_1 u_1 - \operatorname{div} \mathcal{M}_2^{-1} \partial_0^{-1} \operatorname{grad} u_1 = f. \tag{4.2}$$

Choosing an appropriate domain for $\left(\begin{smallmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{smallmatrix}\right)$ in *space-time*, which will be done below, it is possible to show that $\partial_0^{-1} \operatorname{grad} u = \operatorname{grad} \partial_0^{-1} u$ for suitable u. Thus, equation (4.2) reads

$$\partial_0 \mathcal{M}_1 u_1 - \operatorname{div} \mathcal{M}_2^{-1} \operatorname{grad} \partial_0^{-1} u_1 = f.$$

9 There is a vast literature on compact embedding theorems for the space of weakly differentiable $L^2(\Omega)$-functions into $L^2(\Omega)$. In order to maintain such compact embedding, one has to assume some 'regularity' property of the boundary of Ω; see, e. g., [2, 44].

Substituting $u := \partial_0^{-1} u_1$, we arrive at

$$\partial_0 \mathcal{M}_1 \partial_0 u - \operatorname{div} \mathcal{M}_2^{-1} \operatorname{grad} u = f, \tag{4.3}$$

which may be regarded as the wave equation in a more familiar form.

Before we address continuous dependence of the solution on the coefficients, we comment on the choice of the domain for $\left(\begin{smallmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{smallmatrix}\right)$.

Let $\nu > 0$. As in [21, Section 3], we take a time-translation-invariant subspace of the maximal domain of

$$\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} \subseteqq \left(L_\nu^2(\mathbb{R}; L^2(\Omega) \oplus L^2(\Omega)^n)\right)^2,$$

such that the respective operator satisfies the conditions imposed on $\mathcal{A}$ in Theorem 2.3. For this, we let $r > \frac{1}{2\nu}$ and $a\colon B(r,r) \to L^\infty(\Omega)^n$ bounded, analytic. Similar to Remark 2.2, a gives rise to an operator in $L(L_\nu^2(\mathbb{R}; L^2(\Omega)), L_\nu^2(\mathbb{R}; L^2(\Omega)^n))$ in the way that if

$$a(z) = \sum_{k=0}^{\infty} a_{k,r}(m)(z-r)^k \quad (z \in B(r,r)) \tag{4.4}$$

is the power series expression for a in r for suitable $L^\infty(\Omega)^n$-elements $a_{k,r}$, we define

$$\begin{aligned} a(\partial_0^{-1})\phi &:= \sum_{k=0}^{\infty} a_{k,r}(m)(\partial_0^{-1} - r)^k \phi \\ &:= \sum_{k=0}^{\infty} ((x,t) \mapsto a_{k,r}(x))((\partial_0^{-1} - r)^k \phi)(t,x) \end{aligned}$$

for $\phi \in C_{\infty,c}(\mathbb{R} \times \Omega)$.

Throughout, we assume the following smoothness conditions on the coefficients in (4.4): The mappings

$$\begin{aligned} z &\mapsto \operatorname{div} a(z) := \sum_{k=0}^{\infty} (\operatorname{div} a_{k,r})(m)(z-r)^k \\ z &\mapsto \operatorname{curl} a(z) := \sum_{k=0}^{\infty} (\operatorname{curl} a_{k,r})(m)(z-r)^k \end{aligned}$$

are bounded, analytic with $\operatorname{div} a_{k,r}$ and $\operatorname{curl} a_{k,r}$ being measurable and bounded functions (the latter condition of course only in the case $n = 3$).

Now, we are in the position to define the domain mentioned above:[10]

$$D(\mathcal{A}) := \left\{ (\phi, \psi) \in D\left(\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}\right); a(\partial_0^{-1})\phi - \psi \in L_\nu^2(\mathbb{R}; D(\mathring{\operatorname{div}})) \right\},$$

10 We shall note here that in [21, p. 541] the condition $a(\partial_0^{-1})\phi - \psi \in L_\nu^2(\mathbb{R}; D(\mathring{\operatorname{div}}))$ is replaced by $a(\partial_0^{-1})\phi - \partial_0^{-1}\psi \in L_\nu^2(\mathbb{R}; D(\mathring{\operatorname{div}}))$.

with $\mathcal{A} = \left(\begin{smallmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{smallmatrix}\right)$ on $D(\mathcal{A})$. We shall note here that the boundary conditions introduced include the Robin boundary conditions or boundary conditions with temporal convolutions at the boundary; cf. [21, p. 542]. By the definition, we see that $\mathcal{A}$ is time-translation invariant. Henceforth, we will also impose the sign-constraint [21, formula (3.3)] on $a(\partial_0^{-1})$:

$$\Re \int_{-\infty}^{0} (\langle \operatorname{grad} p, a(\partial_0^{-1})p\rangle(t) + \langle p, \operatorname{div} a(\partial_0^{-1})p\rangle(t))e^{-2\nu t}\, \mathrm{d}t \geqq 0 \tag{4.5}$$

for all $p \in L^2_\nu(\mathbb{R}; D(\operatorname{grad}))$. We have the following.

Theorem 4.4. *The operator $\mathcal{A}$ is maximal monotone. Moreover, we have*

$$\Re \langle \mathcal{A}u, \mathbf{1}_{\mathbb{R}_{<0}}(m_0)u\rangle \geqq 0 \quad (u \in D(\mathcal{A})).$$

If, in addition, we have that

$$\Re \int_{\mathbb{R}} (\langle \operatorname{grad} p, a(\partial_0^{-1})p\rangle(t) + \langle p, \operatorname{div} a(\partial_0^{-1})p\rangle(t))e^{-2\nu t}\, \mathrm{d}t = 0 \tag{4.6}$$

for all $p \in L^2_\nu(\mathbb{R}; D(\operatorname{grad}))$, then $\mathcal{A}$ is skew self-adjoint.

Before the proof, we record the following fact communicated by Sascha Trostorff.

Proposition 4.5. *Let H be a Hilbert space, $A\colon D(A) \subseteq H \to H$ linear. Assume that both A and $-A$ are maximal monotone. Then A is skew self-adjoint.*

Proof. From $\Re\langle Au, u\rangle \geqq 0$ and $\Re\langle -Au, u\rangle \geqq 0$, it follows that $\Re\langle Au, u\rangle = 0$ for all $u \in D(A)$. Thus, by polarization, $-A \subseteq A^*$. The maximal monotonicity of A implies the (maximal) monotonicity for A^*. The maximality of $-A$ yields $-A = A^*$. □

Proof of Theorem 4.4. [21, Proposition 3.2] shows the inequality stated and the closedness of $\mathcal{A}$. Time-translation invariance together with [21, Proposition 3.3], which for $u \in D(A^*)$ asserts that

$$\Re \langle \mathcal{A}^* u, \mathbf{1}_{\mathbb{R}_{<0}}(m_0)u\rangle \geqq 0,$$

yields

$$\Re\langle \mathcal{A}u, u\rangle, \Re\langle \mathcal{A}^* v, v\rangle \geqq 0 \quad (u \in D(\mathcal{A}), v \in D(\mathcal{A}^*)).$$

The latter together with the closedness of $\mathcal{A}$ implies the maximal monotonicity for $\mathcal{A}$.

Now, assume the validity of (4.6). Then the above reasoning shows that both $\mathcal{A}$ and $-\mathcal{A}$ are maximal monotone. The assertion follows from Proposition 4.5. □

In view of Theorem 2.3, we also need the following result.

Proposition 4.6. *Let H be a Hilbert space, $\nu > 0$ and $\mathcal{B}: D(\mathcal{B}) \subseteq L^2_\nu(\mathbb{R}; H) \to L^2_\nu(\mathbb{R}; H)$ densely defined, closed, linear with $0 \in \rho(\mathcal{B})$. Assume that $\tau_h \mathcal{B} = \mathcal{B}\tau_h$ for all $h \in \mathbb{R}$ on $D(\mathcal{B})$, where $\tau_h \in L(L^2_\nu(\mathbb{R}; H))$ with $\tau_h f := f(\cdot + h)$. Then $\partial_0^{-1}(\mathcal{B})^{-1} = (\mathcal{B})^{-1}\partial_0^{-1}$.*

Proof. For $h \in \mathbb{R} \setminus \{0\}$ and $u \in L^2_\nu(\mathbb{R}; H)$, we have

$$\frac{1}{h}(\tau_h - 1)(\mathcal{B})^{-1}\partial_0^{-1}u = (\mathcal{B})^{-1}\frac{1}{h}(\tau_h - 1)\partial_0^{-1}u.$$

Now, since $\partial_0^{-1}u \in D(\partial_0)$ and $(\mathcal{B})^{-1}$ is continuous the right-hand side converges to $(\mathcal{B})^{-1}u$ as $h \to 0$. Thus, the left-hand side is bounded, weak compactness of L^2_ν now implies that the left-hand side converges weakly, the limit equals $\partial_0(\mathcal{B})^{-1}\partial_0^{-1}u$. The assertion follows. □

Now, from $\partial_0^{-1}(\mathcal{A} + 1)^{-1} = (\mathcal{A} + 1)^{-1}\partial_0^{-1}$ and $0 \in \rho(\partial_0) \cap \rho(\mathcal{A} + 1)$ it follows that $\partial_0(\mathcal{A} + 1) = (\mathcal{A} + 1)\partial_0$; see, e. g., [22, p. 56], [34, Lemma 1.1.1].

In order to show a continuous dependence result on the coefficients, we need to warrant the compactness condition for the operator $\mathcal{A}$ in Theorem 3.1. For higher dimensions, the null space of the operator $\mathcal{A}$ discussed in this section is infinite-dimensional. Thus, if we want to apply Theorem 3.1, we have to consider the reduced operator $\iota_N^*\mathcal{A}\iota_N$, where $\iota_N: N(\mathcal{A})^\perp \to L^2_\nu(\mathbb{R}; L^2(\Omega) \oplus L^2(\Omega)^n)$ is the canonical embedding from the orthogonal complement of the null space of $\mathcal{A}$ into $L^2_\nu(\mathbb{R}; L^2(\Omega) \oplus L^2(\Omega)^n)$. The latter procedure of course is not needed if we restrict ourselves to the one-dimensional case.

Theorem 4.7. *Let $\nu > 0$. Assume that Ω is a bounded, open interval, and let $(\mathcal{M}_k)_k$ be a sequence of causal operators in $L^2_\nu(\mathbb{R}; L^2(\Omega) \oplus L^2(\Omega))$ converging in the weak operator topology such that the sequence has bounded commutators with ∂_0. If, in addition, there exists $c > 0$ such that*

$$\Re\langle \partial_0\mathcal{M}_n u, \mathbf{1}_{\mathbb{R}_{<a}}(m_0)u\rangle \geqq c\langle u, \mathbf{1}_{\mathbb{R}_{<a}}(m_0)u\rangle \quad (n \in \mathbb{N}, a \in \mathbb{R}, u \in D(\partial_0))$$

then

$$\partial_0\mathcal{M}_n + \mathcal{A} \xrightarrow{G} \partial_0\mathcal{M} + \mathcal{A}$$

in $L^2_\nu(\mathbb{R}; L^2(\Omega)^2)$.

Proof. For the proof, note that the Hilbert space $D(\partial_1) \oplus D(\partial_1) = D(\text{grad}) \oplus D(\text{div}) = H_1(\Omega)^2$ is compactly embedded into $L_2(\Omega)^2$. Moreover, the validity of the conditions in Theorem 3.1 are easily checked with the help of Theorem 4.4 and Proposition 4.6. Thus, Theorem 3.1 applies. □

Remarks 4.8. With the second-order formulation of equation (4.3), we consider

$$\partial_0\mathcal{M}_{1,n}\partial_0 u_n - \operatorname{div}\mathcal{M}_{2,n}^{-1}\operatorname{grad} u_n = f,$$

for $(\mathcal{M}_{1,n})_n$, $(\mathcal{M}_{2,n})_n$ being such that $\mathcal{M}_n \coloneqq \operatorname{diag}(\mathcal{M}_{1,n}, \mathcal{M}_{2,n})$ satisfies the assumptions of Theorem 4.7. It follows that $\mathcal{M}_{j,n} \xrightarrow{\tau_w} \mathcal{M}_j$ for some $\mathcal{M}_j$, $j \in \{1, 2\}$. The limit equation would then be the following:

$$\partial_0 \mathcal{M}_1 \partial_0 u - \operatorname{div} \mathcal{M}_2^{-1} \operatorname{grad} u = f.$$

We note here that at first one computes the limit of $(\mathcal{M}_{2,n})_n$ and after that one inverts the limit to get the latter equation. In classical terms, i. e., under certain structural and periodicity assumptions, $\mathcal{M}_2^{-1}$ is the harmonic mean of the $\mathcal{M}_{2,n}^{-1}$'s.

Next, we discuss whether the compactness property assumed in Theorem 3.1 for $\mathcal{A}$ holds in the case of dimension $n = 3$, which will be assumed in the remainder of this section. Recall that our strategy relies on considering the reduced operator $\iota_N^* \mathcal{A} \iota_N$. We state a first important consequence.

Proposition 4.9. *The operator $\iota_N^* \mathcal{A} \iota_N$ is maximal monotone. If $\mathcal{A}$ is skew self-adjoint, then so is $\iota_N^* \mathcal{A} \iota_N$.*

Proof. It is plain that the operator is monotone. Thus, by Minty's theorem, it suffices to show that $1 + \iota_N^* \mathcal{A} \iota_N$ is onto. For this let $y \in N(\mathcal{A})^\perp$. By the maximal monotonicity of $\mathcal{A}$, there exists $x \in D(\mathcal{A})$ such that $x + \mathcal{A}x = y$. We multiply the latter equality by ι_N^*, which gives $\iota_N^* x + \iota_N^* \mathcal{A} x = \iota_N^* y = y$. Decomposing $x = x_1 + x_2$ for some $x_1 \in N(\mathcal{A})^\perp$ and $x_2 \in N(\mathcal{A})$, we get that $\iota_N^* x = x_1$ and $\mathcal{A}x = \mathcal{A}(x_1 + x_2) = \mathcal{A}x_1 = \mathcal{A}\iota_N \iota_N^* x$. Hence, $\iota_N^* x$ is the desired element in the domain of $1 + \iota_N^* \mathcal{A} \iota_N$ mapped to y. The last assertion of the proposition, follows from Proposition 4.5. □

As a next step, we need to verify that $\iota_N^* \mathcal{A} \iota_N$ satisfies the assumptions in our main homogenization theorem. For this, however, we need to impose additional regularity of the boundary of Ω. With additional effort, these regularity requirements can certainly be relaxed. Since we are only interested in providing a class of examples rich enough, we do not follow the way of presenting a streamlined version of a particular compactness result.

Theorem 4.10. *Assume, in addition, that Ω is of class C_5. Then we have that*

$$H_1(1 + \iota_N^* \mathcal{A} \iota_N) \hookrightarrow L_\nu^2(\mathbb{R}; H_1(\Omega)^4),$$

Before we go into the proof of the theorem, we state the main ingredient: Gaffney's inequality. For the latter, recall the operator curl being the distributional curl defined on $L^2(\Omega)^3$ with values in $L^2(\Omega)^3$ with maximal domain. We also use the canonical extension of curl to space-time and re-use the notation. It will become clear from the context which operator is used.

Theorem 4.11 (Gaffney's inequality; see, e. g., [15, below Theorem 8.6, p. 157]). *Let Ω belong to the class C_5. Then there exists $c > 0$ such that for all $u \in D(\operatorname{curl}) \cap D(\mathring{\operatorname{div}})$ we have*

$$|u|_{H_1(\Omega)} \leqq c\big(|u|_{L^2(\Omega)} + |\operatorname{div} u|_{L^2(\Omega)} + |\operatorname{curl} u|_{L^2(\Omega)}\big).$$

Proof of Theorem 4.10. At first, we observe that

$$\begin{pmatrix} 0 & \operatorname{div}|_{C_{\infty,c}(\Omega)^n} \\ \operatorname{grad}|_{C_{\infty,c}(\Omega)} & 0 \end{pmatrix} \subseteqq \mathcal{A} \subseteqq \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}.$$

Now, $\mathcal{A}$ is maximal monotone. Thus, $\mathcal{A}$ is closed and we get that

$$\begin{pmatrix} 0 & \mathring{\operatorname{div}} \\ \mathring{\operatorname{grad}} & 0 \end{pmatrix} \subseteqq \mathcal{A} \subseteqq \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix}.$$

The latter implies

$$\iota_N^* \begin{pmatrix} 0 & \mathring{\operatorname{div}} \\ \mathring{\operatorname{grad}} & 0 \end{pmatrix} \iota_N \subseteqq \iota_N^* \mathcal{A} \iota_N \subseteqq \iota_N^* \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{pmatrix} \iota_N.$$

From $\begin{pmatrix} 0 & \mathring{\operatorname{div}} \\ \mathring{\operatorname{grad}} & 0 \end{pmatrix} \subseteqq \mathcal{A}$, it follows that

$$N(\mathcal{A})^\perp \subseteqq N(\mathring{\operatorname{grad}})^\perp \oplus N(\mathring{\operatorname{div}})^\perp.$$

Thus,

$$\iota_N^* \mathcal{A} \iota_N \subseteqq \iota_N^* \begin{pmatrix} 0 & \operatorname{div}|_{N(\mathring{\operatorname{div}})^\perp} \\ \operatorname{grad}|_{N(\mathring{\operatorname{grad}})^\perp} & 0 \end{pmatrix} \subseteqq \begin{pmatrix} 0 & \operatorname{div}|_{N(\mathring{\operatorname{div}})^\perp} \\ \operatorname{grad} & 0 \end{pmatrix}.$$

Now, let $(\phi, \psi) \in D(\iota_N \mathcal{A} \iota_N^*)$. The latter inclusion shows that it suffices to estimate the norm of ψ in the space $L_\nu^2(\mathbb{R}; H_1(\Omega))$. Moreover, we also read off that $\psi \perp N(\mathring{\operatorname{div}})$. Thus, $\psi \in R(\operatorname{grad})$, which implies that ψ takes almost everywhere values in the domain of curl and that $\operatorname{curl} \psi = 0$. Recall

$$a(\partial_0^{-1})\phi - \psi \in L_\nu^2(\mathbb{R}; D(\mathring{\operatorname{div}})).$$

Using the smoothness assumptions on a, we compute

$$\begin{aligned} \operatorname{curl}(a(\partial_0^{-1})\phi - \psi) &= \operatorname{curl}(a(\partial_0^{-1})\phi) \\ &= \operatorname{curl}(a(\partial_0^{-1}))\phi + a(\partial_0^{-1}) \times \operatorname{grad} \phi \end{aligned}$$

and

$$\operatorname{div}(a(\partial_0^{-1})\phi) = (\operatorname{div} a(\partial_0^{-1}))\phi + a(\partial_0^{-1}) \operatorname{grad} \phi.$$

Hence, with Theorem 4.11, we estimate pointwise almost everywhere

$$\begin{aligned} &|\psi|_{H_1} - |a(\partial_0^{-1})\phi|_{H_1} \\ &\quad \leqq |\psi - a(\partial_0^{-1})\phi|_{H_1} \end{aligned}$$

$$
\begin{aligned}
&\leqq c(|\psi - a(\partial_0^{-1})\phi|_{L^2} + |\operatorname{div}(\psi - a(\partial_0^{-1})\phi)|_{L^2} + |\operatorname{curl}(\psi - a(\partial_0^{-1})\phi)|_{L^2})\\
&\leqq c\Big(|\psi|_{L^2} + |a(\partial_0^{-1})\phi|_{L^2}\\
&\quad + |\operatorname{div}\psi|_{L^2} + |(\operatorname{div} a(\partial_0^{-1}))\phi|_{L^2} + |a(\partial_0^{-1})\operatorname{grad}\phi|_{L^2}\\
&\quad + |\operatorname{curl}(a(\partial_0^{-1}))\phi|_{L^2} + |a(\partial_0^{-1})\times\operatorname{grad}\phi|_{L^2}\Big).
\end{aligned}
$$

Thus, we get for some constant $c' > 0$ that

$$
\begin{aligned}
|\psi|_{L^2_\nu(\mathbb{R};H^1(\Omega))} \leqq c'\Big(&|a(\partial_0^{-1})\phi|_{L^2_\nu(\mathbb{R};H_1(\Omega))} + |\psi|_{L^2_\nu(\mathbb{R};L^2(\Omega))}\\
&+ |a(\partial_0^{-1})\phi|_{L^2_\nu(\mathbb{R};L^2(\Omega))} + |\operatorname{div}\psi|_{L^2_\nu(\mathbb{R};L^2(\Omega))} + |(\operatorname{div} a(\partial_0^{-1}))\phi|_{L^2_\nu(\mathbb{R};L^2(\Omega))}\\
&+ |a(\partial_0^{-1})\operatorname{grad}\phi|_{L^2_\nu(\mathbb{R};L^2(\Omega))} + |\operatorname{curl}(a(\partial_0^{-1}))\phi|_{L^2_\nu(\mathbb{R};L^2(\Omega))}\\
&+ |a(\partial_0^{-1})\times\operatorname{grad}\phi|_{L^2_\nu(\mathbb{R};L^2(\Omega))}\Big).
\end{aligned}
$$

The smoothness assumptions on a yield the assertion. □

Now, we are in the position to formulate the continuous dependence result. For simplicity, we only treat the case, where the operators in the material law do not depend on the spatial variables. The full homogenization problem will be discussed in future work. We adopt the strategy described in [40, Section 1]. More specifically, we will treat the case of the particular class of operators being functions of ∂_0^{-1} as discussed in Remark 2.2.

Theorem 4.12. *Let $\nu > 0$, $r > \frac{1}{2\nu}$. Assume that $\Omega \subseteq \mathbb{R}^3$ is of class C_5 and such that $H_1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$. Let $(M_k)_k$ be a bounded sequence in $\mathcal{H}^\infty(B(r,r))$ and denote $\mathcal{M}_k \coloneqq M_k(\partial_0^{-1})$, $k \in \mathbb{N}$. If the conditions* (4.6) *and* (4.5) *hold and, in addition, there exists $c > 0$ such that*

$$\mathbb{R}\langle z^{-1}M_k(z)u, u\rangle \geqq c\langle u, u\rangle \quad (k \in \mathbb{N},\ z \in B(r,r),\ u \in L^2(\Omega)^4),$$

then there is a subsequence $(n_k)_k$ of $(n)_n$ such that

$$\partial_0\mathcal{M}_{n_k} + \mathcal{A} \xrightarrow{G} \partial_0\mathcal{M} + \mathcal{A}$$

in $L^2_\nu(\mathbb{R}; L^2(\Omega)^4)$, where

$$\mathcal{M} = \begin{pmatrix} \tau_{\mathrm{w}}\text{-}\lim_{k\to\infty} \iota_N^*\mathcal{M}_{n_k}\iota_N & 0\\ 0 & \partial_0^{-1}(\tau_{\mathrm{w}}\text{-}\lim_{k\to\infty}(\partial_0\kappa_N^*\mathcal{M}_{n_k}\kappa_N)^{-1})^{-1}\end{pmatrix},$$

with $\kappa_N: N(\mathcal{A}) \to L^2_\nu(\mathbb{R}; L^2(\Omega) \oplus L^2(\Omega)^3)$ being the canonical injection.

Proof. At first, we use Theorem [20, Lemma 3.5] to deduce that $(\partial_0\mathcal{M}_k(\partial_0^{-1}))^{-1}$ is causal. Thus, from [31, Lemma 3.8], we get that $\mathcal{M}_k$ satisfies the positive definiteness condition imposed in Theorem 3.1, $k \in \mathbb{N}$.

Let $f \in L^2_\nu(\mathbb{R}; L^2(\Omega)\oplus L^2(\Omega)^3)$ and consider the sequence $(u_n)_n$ in $L^2_\nu(\mathbb{R}; L^2(\Omega)\oplus L^2(\Omega)^3)$ satisfying

$$(\partial_0 \mathcal{M}_n + \mathcal{A})u_n = f.$$

The latter equation then reads as (note that being functions of $\mathcal{A}$ the operators κ_N, ι_N commute with ∂_0):

$$\left(\partial_0 \begin{pmatrix} \iota_N^* \mathcal{M}_n \iota_N & \iota_N^* \mathcal{M}_n \kappa_N \\ \kappa_N^* \mathcal{M}_n \iota_N & \kappa_N^* \mathcal{M}_n \kappa_N \end{pmatrix} + \begin{pmatrix} \iota_N^* \mathcal{A} \iota_N & \iota_N^* \mathcal{A} \kappa_N \\ \kappa_N^* \mathcal{A} \iota_N & \kappa_N^* \mathcal{A} \kappa_N \end{pmatrix}\right) \begin{pmatrix} \iota_N^* u_n \\ \kappa_N^* u_n \end{pmatrix} = \begin{pmatrix} \iota_N^* f \\ \kappa_N^* f \end{pmatrix}.$$

Now, since ∂_0^{-1} commutes with $\mathcal{A}$, the $\mathcal{M}_n$'s commute with κ_N and ι_N. Moreover, the skew self-adjointness of $\mathcal{A}$ implies that $\mathcal{A}$ reduces $N(\mathcal{A})^\perp$. Thus, the latter system may be written as

$$\left(\partial_0 \begin{pmatrix} \iota_N^* \mathcal{M}_n \iota_N & 0 \\ 0 & \kappa_N^* \mathcal{M}_n \kappa_N \end{pmatrix} + \begin{pmatrix} \iota_N^* \mathcal{A} \iota_N & 0 \\ 0 & 0 \end{pmatrix}\right) \begin{pmatrix} \iota_N^* u_n \\ \kappa_N^* u_n \end{pmatrix} = \begin{pmatrix} \iota_N^* f \\ \kappa_N^* f \end{pmatrix}.$$

The latter gives the two (decoupled) equations:

$$(\partial_0 \iota_N^* \mathcal{M}_n \iota_N + \iota_N^* \mathcal{A} \iota_N)\iota_N^* u_n = \iota_N^* f$$

and

$$\partial_0 \kappa_N^* \mathcal{M}_n \kappa_N \kappa_N^* u_n = \kappa_N^* f$$

For the first equation, we use Theorem 3.1, the convergence of the equation in the stated manner follows from sequential compactness of bounded subsets of bounded linear operators in the weak operator topology. □

Remarks 4.13. If, in the latter theorem, we restrict ourselves to the Hilbert space $N(\mathcal{A})^\perp$, i. e., using right-hand sides, which are in $N(\mathcal{A})^\perp$, then the term involving κ vanishes.

4.3 Applications to a singular perturbation problem

To illustrate the applicability of Theorem 2.3, we give the following example of an elliptic/parabolic type equation, which is adopted from an example given in [26]. For this, let $\Omega \subseteq \mathbb{R}^n$ be open, bounded and connected and let $-\Delta$ be the Dirichlet–Laplacian in $L^2(\Omega)$. Then $-\Delta$ is continuously invertible with compact resolvent. Let $\lambda \in (0, \lambda_1)$ for λ_1 being the smallest eigenvalue of $-\Delta$. Then, in particular, the operator $-\Delta - \lambda$ is maximal monotone. Now, let $\Omega_p, \Omega_e \subseteq \Omega$ be disjoint, measurable and such that $\Omega_p \cup \Omega_e = \Omega$. We let $\phi\colon \mathbb{R} \to \mathbb{R}$ be such that $\phi|_{(-\infty,0]} = 0$, $\phi|_{(0,1)} = \mathrm{id}_{(0,1)}$ and $\phi|_{[1,\infty)} = 1$. In $L^2_\nu(\mathbb{R}; L^2(\Omega))$, we consider for $\varepsilon > 0$ and given $f \in L^2_\nu(\mathbb{R}; L^2(\Omega))$ the problem of finding u_ε such that

$$(\varepsilon \partial_0 \phi(m_0) \mathbf{1}_{\Omega_p}(m) + \mathbf{1}_{\Omega_e}(m)(1 - \phi(m_0))\tau_{-\varepsilon} - \Delta)u_\varepsilon = f, \tag{4.7}$$

where $\tau_{-\varepsilon}$ denotes the time-shift operator $\tau_{-\varepsilon}g := g(\cdot - \varepsilon)$ for suitable g. At first, note that the latter problem is clearly well posed. Indeed, this follows from

$$\Re\langle(\varepsilon\partial_0\phi(m_0)\,\mathbf{1}_{\Omega_{\mathrm{p}}}(m) + \mathbf{1}_{\Omega_{\mathrm{e}}}(m)(1-\phi(m_0))\tau_{-\varepsilon} + \lambda)u, \mathbf{1}_{\mathbb{R}_{\leqq a}}(m_0)u\rangle$$
$$\geqq \lambda'\langle u, \mathbf{1}_{\mathbb{R}_{\leqq a}}(m_0)u\rangle \quad (a \in \mathbb{R})$$

for $u \in D(\partial_0)$, ν large enough and some $\lambda' \in (0,\lambda)$. On Ω_p the equation (4.7) is of parabolic type and on Ω_e it is of elliptic type with an additional temporal variable. With

$$\mathcal{M}_\varepsilon := \varepsilon\phi(m_0)\,\mathbf{1}_{\Omega_{\mathrm{p}}}(m) + \partial_0^{-1}\,\mathbf{1}_{\Omega_{\mathrm{e}}}(m)(1-\phi(m_0))\tau_{-\varepsilon} + \partial_0^{-1}\lambda,$$

we get that $\mathcal{M}_\varepsilon \xrightarrow{\tau_{\mathrm{s}}} \mathcal{M}_0 = \partial_0^{-1}\,\mathbf{1}_{\Omega_{\mathrm{e}}}(m)(1-\phi(m_0)) + \partial_0^{-1}\lambda$, where we denoted by τ_{s} the strong operator topology. As strong convergence implies convergence in the weak operator topology, we infer with the help of Theorem 3.1 that $(u_\varepsilon)_{\varepsilon>0}$ weakly converges as $\varepsilon \to 0$ to the solution u_0 of the problem

$$(\mathbf{1}_{\Omega_{\mathrm{e}}}(m)(1-\phi(m_0)) - \Delta)u_0 = f,$$

which itself is of pure elliptic type.

5 Proof of Theorem 3.1

For the proof, we need several preparations.

Theorem 5.1 (Theorem of Aubin–Lions, [28, p. 67, 2°]). *Let H, K be Hilbert spaces, $I \subseteqq \mathbb{R}$ bounded, open interval. Assume that $K \hookrightarrow\hookrightarrow H$. Then*

$$H_1(I;H) \cap L^2(I;K) \hookrightarrow\hookrightarrow L^2(I;H).$$

Lemma 5.2. *Let H be a Hilbert space, $\nu > 0$. Let $(\mathcal{M}_n)_n$ be τ_{w}-convergent sequence in $L(L^2_\nu(\mathbb{R};H))$ with limit $\mathcal{M}$. If $\mathcal{M}_n$ is causal for all $n \in \mathbb{N}$ then so is $\mathcal{M}$.*

Proof. It suffices to observe that $\mathbf{1}_{\mathbb{R}_{<a}}(m_0) \in L(L^2_\nu(\mathbb{R};H))$ for all $a \in \mathbb{R}$. Thus, the equation

$$\mathbf{1}_{\mathbb{R}_{<a}}(m_0)\mathcal{M}_n = \mathbf{1}_{\mathbb{R}_{<a}}(m_0)\mathcal{M}_n\,\mathbf{1}_{\mathbb{R}_{<a}}(m_0)$$

carries over to the limit as $n \to \infty$. □

Theorem 5.3 (Weak-strong principle). *Let H, K be Hilbert spaces and with $K \hookrightarrow\hookrightarrow H$. Let $\nu > 0$ and $(v_n)_n$ be a weakly convergent sequence in $L^2_\nu(\mathbb{R};K) \cap H_{\nu,1}(\mathbb{R};H)$. Assume further that $\inf_{n\in\mathbb{N}} \inf \operatorname{spt} v_n > -\infty$. If $(\mathcal{M}_n)_n$ is a τ_{w}-convergent sequence of causal operators in $L(L^2_\nu(\mathbb{R};H))$, then*

$$\text{w-}\lim_{n\to\infty} \mathcal{M}_n v_n = \Big(\tau_{\mathrm{w}}\text{-}\lim_{n\to\infty} \mathcal{M}_n\Big)\Big(\text{w-}\lim_{n\to\infty} v_n\Big) \in L^2_\nu(\mathbb{R};H).$$

Proof. The uniform boundedness principle implies that both $(v_n)_n$ and $(\mathcal{M}_n)_n$ are bounded sequences in $L^2_\nu(\mathbb{R};K) \cap H_{\nu,1}(\mathbb{R};H)$ and $L(L^2_\nu(\mathbb{R};H))$, respectively. Thus, there exists a subsequence $(\mathcal{M}_{n_k}v_{n_k})_k$ of $(\mathcal{M}_n v_n)_n$ which weakly converges to some $w \in L^2_\nu(\mathbb{R};H)$. It suffices to identify w. For this, let $\phi \in C_{\infty,c}(\mathbb{R};H)$ and define $a \coloneqq \sup\operatorname{spt}\phi$. Choose $\psi \in C_\infty(\mathbb{R})$ such that $0 \leqq \psi \leqq 1$, $\psi = 1$ on $\mathbb{R}_{<a+1}$ and $\psi = 0$ on $\mathbb{R}_{>a+2}$. We denote $v \coloneqq \text{w-}\lim_{n\to\infty} v_n$ and $\mathcal{M} \coloneqq \tau_{\mathrm{w}}\text{-}\lim_{n\to\infty}\mathcal{M}_n$. Now, by Theorem 5.1, we deduce that $(\psi(m_0)v_n)_n$ converges to $\psi(m_0)v$ in $L^2_\nu(\mathbb{R};H)$. For $n \in \mathbb{N}$, we compute

$$\begin{aligned}
\langle \mathcal{M}_n v_n, \phi\rangle_{\nu,0} &= \langle \mathcal{M}_n v_n, \mathbf{1}_{\mathbb{R}_{<a+1}}(m_0)\phi\rangle_{\nu,0} \\
&= \langle \mathbf{1}_{\mathbb{R}_{<a+1}}(m_0)\mathcal{M}_n v_n, \phi\rangle_{\nu,0} \\
&= \langle \mathbf{1}_{\mathbb{R}_{<a+1}}(m_0)\mathcal{M}_n \mathbf{1}_{\mathbb{R}_{<a+1}}(m_0) v_n, \phi\rangle_{\nu,0} \\
&= \langle \mathbf{1}_{\mathbb{R}_{<a+1}}(m_0)\mathcal{M}_n \mathbf{1}_{\mathbb{R}_{<a+1}}(m_0)\psi(m_0) v_n, \phi\rangle_{\nu,0} \\
&= \langle \mathbf{1}_{\mathbb{R}_{<a+1}}(m_0)\mathcal{M}_n \psi(m_0) v_n, \phi\rangle_{\nu,0} \\
&= \langle \mathcal{M}_n \psi(m_0) v_n, \mathbf{1}_{\mathbb{R}_{<a+1}}(m_0)\phi\rangle_{\nu,0} \\
&\to \langle \mathcal{M} \psi(m_0) v, \mathbf{1}_{\mathbb{R}_{<a+1}}(m_0)\phi\rangle_{\nu,0} \\
&= \langle \mathcal{M} v, \phi\rangle_{\nu,0},
\end{aligned}$$

where we have used that the $\mathcal{M}_n$'s and $\mathcal{M}$ are causal; see also Lemma 5.2. Hence,

$$\langle w, \phi\rangle_{\nu,0} = \langle \mathcal{M} v, \phi\rangle_{\nu,0}$$

for all $\phi \in C_{\infty,c}(\mathbb{R};H)$. Thus, $w = \mathcal{M}v$. □

Remark 5.4. The support condition for the v_n's is needed to make Theorem 5.1 applicable.

Lemma 5.5. *Let H be a Hilbert space, $\mathcal{D}$ densely defined, closed, linear operator in H with $0 \in \rho(\mathcal{D})$. Let $(M_n)_n$ be a sequence in $L(H)$ converging in the weak operator topology to some M and having bounded commutators with $\mathcal{D}$. Then*

$$[M_n, \mathcal{D}] \to \overline{M\mathcal{D} - \mathcal{D}M} \quad (n \to \infty)$$

in the weak operator topology. In particular, $M\mathcal{D} - \mathcal{D}M$ extends to a bounded linear operator, M has a bounded commutator with $\mathcal{D}$ and

$$\mathcal{D}M_n u \rightharpoonup \mathcal{D}Mu \quad (n \to \infty,\ u \in D(\mathcal{D})).$$

Proof. For $x, y \in H$, $n \in \mathbb{N}$, we compute

$$\begin{aligned}
\langle [M_n, \mathcal{D}]\mathcal{D}^{-1}x, (\mathcal{D}^{-1})^* y\rangle &= \langle (M_n\mathcal{D} - \mathcal{D}M_n)\mathcal{D}^{-1}x, (\mathcal{D}^{-1})^* y\rangle \\
&= \langle \mathcal{D}^{-1}(M_n\mathcal{D} - \mathcal{D}M_n)\mathcal{D}^{-1}x, y\rangle \\
&= \langle (\mathcal{D}^{-1}M_n - M_n\mathcal{D}^{-1})x, y\rangle
\end{aligned}$$

$$\begin{aligned}&= \langle \mathcal{D}^{-1}M_n x, y\rangle - \langle M_n \mathcal{D}^{-1}x, y\rangle\\ &\to \langle \mathcal{D}^{-1}Mx, y\rangle - \langle M\mathcal{D}^{-1}x, y\rangle\\ &= \langle [M,\mathcal{D}]\mathcal{D}^{-1}x, (\mathcal{D}^{-1})^* y\rangle.\end{aligned}$$

By the boundedness of $([M_n, \mathcal{D}])_n$ and the density of both $D(\mathcal{D})$ and $D(\mathcal{D}^*)$, we get the first convergence result. In order to see the last convergence result, we compute for $n \in \mathbb{N}$ and $u \in \mathcal{D}$:

$$\mathcal{D}M_n u = [\mathcal{D}, M_n]u + M_n\mathcal{D}u \rightharpoonup [\mathcal{D}, M]u + M\mathcal{D}u = \mathcal{D}Mu. \qquad \square$$

Lemma 5.6. *Let H be Hilbert space, $\nu > 0$. Let $M \in L(L^2_\nu(\mathbb{R}; H))$ be causal and such that M has a bounded commutator with ∂_0. Then $[M, \partial_0]$ is causal.*

Proof. Let $a \in \mathbb{R}$ and $\phi \in C_{\infty,c}(\mathbb{R}; H)$ be such that $\operatorname{spt}\phi \geqq a$. Thus, $\operatorname{spt} M\phi \geqq a$ and $\operatorname{spt}\phi' \geqq a$. Now, since $M\phi \in D(\partial_0)$ since $M[D(\partial_0)] \subseteqq D(\partial_0)$ we further get $\operatorname{spt}\partial_0 M\phi \geqq a$. Hence, we arrive at $\operatorname{spt}[M, \partial_0]\phi = \operatorname{spt}(M\partial_0 - \partial_0 M)\phi \geqq a$. The continuity of $[M, \partial_0]$ together with Remark 2.1 imply the assertion. □

Corollary 5.7. *Let K, H be Hilbert spaces, $\nu > 0$. Let $(\mathcal{M}_n)_n$ be a bounded sequence of causal mappings in $L(L^2_\nu(\mathbb{R}; H))$ converging in the weak operator topology to some $\mathcal{M}$ and having bounded commutators with ∂_0. Assume that $K \hookrightarrow\hookrightarrow H$. Let $(u_n)_n$ be a weakly convergent sequence in $H_{\nu,-1}(\mathbb{R}; K) \cap L^2_\nu(\mathbb{R}; H)$ with limit u and such that $\inf_{n\in\mathbb{N}} \inf \operatorname{spt} u_n > -\infty$. Then $\mathcal{M}_n u_n \rightharpoonup \mathcal{M}u$ in $L^2_\nu(\mathbb{R}; H)$ as $n \to \infty$.*

Proof. At first, note that $(\partial_0^{-1}u_n)_n$ is weakly convergent in $L^2_\nu(\mathbb{R}; K) \cap H_{\nu,1}(\mathbb{R}; H)$. Moreover, $[\mathcal{M}_n, \partial_0]$ is causal by Lemma 5.6 for all $n \in \mathbb{N}$. Furthermore, $[\mathcal{M}_n, \partial_0] \overset{\tau_w}{\to} [\mathcal{M}, \partial_0]$, by Lemma 5.5. Thus, for $n \in \mathbb{N}$ we deduce with the help of Theorem 5.3 that

$$\begin{aligned}\mathcal{M}_n u_n = \mathcal{M}_n\partial_0\partial_0^{-1}u_n &= [\mathcal{M}_n, \partial_0]\partial_0^{-1}u_n + \partial_0\mathcal{M}_n\partial_0^{-1}u_n\\ &\rightharpoonup [\mathcal{M}, \partial_0]\partial_0^{-1}u + \partial_0\mathcal{M}\partial_0^{-1}u = \mathcal{M}u \in L^2_\nu(\mathbb{R}; H),\end{aligned}$$

where we have used that ∂_0^{-1} is weakly continuous and causal. □

Lemma 5.8. *Let H be a Hilbert space, $\nu > 0$. Let $\mathcal{A}$ be a densely defined, closed, linear operator in $L^2_\nu(\mathbb{R}; H)$ with $0 \in \rho(\mathcal{A})$. Assume that $\partial_0^{-1}\mathcal{A}^{-1} = \mathcal{A}^{-1}\partial_0^{-1}$. Let $(u_n)_n$ be a bounded sequence in $H_{-1,1}(\partial_0, \mathcal{A}) \cap L^2_\nu(\mathbb{R}; H)$, which weakly converges in $L^2_\nu(\mathbb{R}; H)$ to $u \in L^2_\nu(\mathbb{R}; H)$. Then $u \in H_{-1,1}(\partial_0, \mathcal{A})$ and*

$$\mathcal{A}u_n \rightharpoonup \mathcal{A}u \in H_{-1,0}(\partial_0, \mathcal{A}).$$

Proof. Let $(u_{n_k})_k$ be a weakly convergent subsequence of $(u_n)_n$ in $H_{-1,1}(\partial_0, \mathcal{A})$. Denote its limit by w. Note that $\partial_0^{-1}u_n \rightharpoonup \partial_0^{-1}u \in H_{\nu,1}(\mathbb{R}; H) \hookrightarrow L^2_\nu(\mathbb{R}; H)$, by unitarity of ∂_0^{-1}. Moreover, by Remark 2.5, $\partial_0^{-1}\colon H_{-1,1}(\partial_0, \mathcal{A}) \to H_{0,1}(\partial_0, \mathcal{A})$ is unitary. Hence, we get that

$\partial_0^{-1}u_{n_k} \rightharpoonup \partial_0^{-1}w \in H_{0,1}(\partial_0, \mathcal{A}) \hookrightarrow L_\nu^2(\mathbb{R}; H)$. Hence, $\partial_0^{-1}w = \partial_0^{-1}u$. Thus, $u = w$ and $(u_n)_n$ weakly converges in $H_{-1,1}(\partial, \mathcal{A})$. Now, by Remark 2.5, the operator $\mathcal{A}: H_{-1,1}(\partial_0, \mathcal{A}) \to H_{-1,0}(\partial_0, \mathcal{A})$ is continuous. Thus, we deduce the asserted convergence. □

Proof of Theorem 3.1. The well-posedness of the limiting equation, i. e., continuous invertibility and causality of $(\partial_0\mathcal{M} + \mathcal{A})$ in $L_\nu^2(\mathbb{R}; H)$ follows from Lemma 5.5 together with Theorem 2.3.

Now, we prove the version, which is asserted in Remark 3.2. Let $(f_n)_n$ in $L_\nu^2(\mathbb{R}; H)$ be a weakly convergent sequence with $\inf_n \inf \operatorname{spt} f_n > -\infty$; we denote its limit by f. For $n \in \mathbb{N}$, we define

$$u_n := (\partial_0\mathcal{M}_n + \mathcal{A})^{-1} f_n.$$

By causality (see Theorem 2.3 and Remark 2.1), we get that

$$\inf_{n\in\mathbb{N}} \inf \operatorname{spt} u_n \geqq \inf_n \inf \operatorname{spt} f_n > -\infty.$$

Moreover, $(u_n)_n$ is bounded in $L_\nu^2(\mathbb{R}; H) \cap H_{-1,1}(\partial_0, \mathcal{A} + 1)$ by Corollary 2.7 and the uniform boundedness principle applied to $(f_n)_n$. Now, let $(u_{n_k})_k$ be a $L_\nu^2(\mathbb{R}; H)$-weakly convergent subsequence of $(u_n)_n$. We denote the respective limit by u. Now, for $k \in \mathbb{N}$ we have

$$\partial_0\mathcal{M}_{n_k} u_{n_k} + \mathcal{A}u_{n_k} = f_{n_k} \tag{5.1}$$

in $H_{-1,0}(\partial_0, \mathcal{A} + 1)$. Now, by Corollary 5.7 for the first term and Lemma 5.8 for the second term on the left side of equation (5.1), we may let $k \to \infty$ in (5.1). We arrive at

$$\partial_0\mathcal{M}u + \mathcal{A}u = f$$

in $H_{-1,0}(\mathbb{R}; H)$. Moreover, by construction, $u \in L_\nu^2(\mathbb{R}; H)$ and $(\partial_0\mathcal{M} + \mathcal{A})u = f \in H_{0,0}(\partial_0, \mathcal{A} + 1)$. Thus, $u \in D(\partial_0\mathcal{M} + \mathcal{A})$, by Remark 2.6. Now, since $(\partial_0\mathcal{M} + \mathcal{A})$ is continuously invertible in $L_\nu^2(\mathbb{R}; H)$ the sequence $(u_n)_n$ itself weakly converges.

In order to see that Theorem 3.1 holds, apply the previous part to constant sequences $(f_n)_n = (f)_n$ for some $f \in C_{\infty,c}(\mathbb{R}; H)$. It remains to observe that $C_{\infty,c}(\mathbb{R}; H)$ is dense in $L_\nu^2(\mathbb{R}; H)$ and that $((\partial_0\mathcal{M}_n + \mathcal{A})^{-1})_n$ is bounded. □

Bibliography

[1] Z. Abdessamad, I. Kostin, G. Panasenko, and V. P. Smyshlyaev, Memory effect in homogenization of a viscoelastic Kelvin–Voigt model with time-dependent coefficients, *Math. Models Methods Appl. Sci.*, **13**(9) (2009), 1603–1630.

[2] R. Adams and J. Fournier, *Sobolev Spaces*, Elsevier, 2003.

[3] A. Amosov, I. Kostin, G. Panasenko, and V. P. Smyshlyaev, Homogenization of a thermo-chemo-viscoelastic Kelvin–Voigt model, *J. Math. Phys.*, **54** (2013), 081501.

[4] A. Bensoussan, J. L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North-Holland, Amsterdam, 1978.

[5] I. Borsi, A. Farina, R. Gianni, and M. Primicerio, Continuous dependence on the constitutive functions for a class of problems describing fluid flow in porous media, *Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl.*, **20**(1) (2009), 1–24.

[6] A. O. Çelebi, V. K. Kalantarov, and D. Uğurlu, On continuous dependence on coefficients of the Brinkman–Forchheimer equations, *Appl. Math. Lett.*, **19**(8) (2006), 801–807.

[7] D. Cioranescu and P. Donato, *An Introduction to Homogenization*, Oxford University Press, New York, 2010.

[8] T. Eisner and A. Serény, On the weak analogue of the Trotter–Kato theorem, *Taiwan. J. Math.*, **14**(4) (2010), 1411–1416.

[9] G. A. Francfort, Homogeneization and linear thermoelasticity, *SIAM J. Math. Anal.*, **14** (1983), 696–708.

[10] F. Franchi and B. Straughan, Continuous dependence and decay for the Forchheimer equations, *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.*, **459**(2040) (2003), 3195–3202.

[11] A. Kalauch, R. Picard, S. Siegmund, S. Trostorff, and M. Waurick, A Hilbert space perspective on ordinary differential equations with memory term, J. Dyn. Differ. Equ., to appear, http://arxiv.org/abs/1204.2924.

[12] H. Kim, On continuous dependence of solution of parabolic equations on coefficients, *Asymptot. Anal.*, **62**(3–4) (2009), 147–162.

[13] M. Kunze and J. van Neerven, Approximating the coefficients in semilinear stochastic partial differential equations, *J. Evol. Equ.*, **11**(3) (2011), 577–604.

[14] M. Kunze and J. van Neerven, Continuous dependence on the coefficients and global existence for stochastic reaction diffusion equations, *J. Differ. Equ.*, **253**(3) (2012), 1036–1068.

[15] R. Leis, *Initial Boundary Value Problems in Mathematical Physics*. B. G. Teubner/John Wiley VIII, Stuttgart/Chichester, New York, etc., 1986.

[16] Y. Liu, Convergence and continuous dependence for the Brinkman–Forchheimer equations, *Math. Comput. Model.*, **49**(7–8) (2009), 1401–1415.

[17] L. E. Payne and B. Straughan, Convergence and continuous dependence for the Brinkman–Forchheimer equations, *Stud. Appl. Math.*, **102**(4) (1999), 419–439.

[18] F. D. Penning, Continuous dependence results for parabolic problems with time-dependent coefficients, *Quaest. Math.*, **14**(1) (1991), 33–49.

[19] R. S. Phillips, Dissipative operators and hyperbolic systems of partial differential equations, *Commun. Pure Appl. Math.*, **12**(2) (1959), 249–276.

[20] R. Picard, A structural observation for linear material laws in classical mathematical physics, *Math. Methods Appl. Sci.*, **32** (2009), 1768–1803.

[21] R. Picard, A class of evolutionary problems with an application to acoustic waves with impedance type boundary conditions, in W. Arendt et al.(eds.), *Spectral Theory, Mathematical System Theory, Evolution Equations, Differential and Difference Equations. Selected Papers of 21st International Workshop on Operator Theory and Applications, IWOTA10, Berlin, Germany, July 12–16, 2010*, pp. 533–548, Operator Theory: Advances and Applications, vol. 221, Birkhäuser, Basel, 2012.

[22] R. Picard and D. McGhee, *Partial Differential Equations: A unified Hilbert Space Approach*, volume 55 of *Expositions in Mathematics*, De Gruyter, Berlin, 2011.

[23] R. Picard, S. Trostorff, and M. Waurick, A note on a class of conservative, well-posed linear control systems, in M. Reissig and M. Ruzhansky (eds.), *Progress in Partial Differential Equations: Asymptotic Profiles, Regularity and Well-Posedness*, Springer Proceedings in Mathematics and Statistics, vol. 44, pp. 261–286, Springer, Heidelberg, 2013.

[24] R. Picard, S. Trostorff, and M. Waurick, On a class of boundary control problems, *Oper. Matrices*, **8**(1) (2014), 185–204.
[25] R. Picard, S. Trostorff, and M. Waurick, On evolutionary equations with material laws containing fractional integrals, *Math. Methods Appl. Sci.*, **38**(15) (2015), 3141–3154.
[26] R. Picard, S. Trostorff, M. Waurick, and M. Wehowski, On non-autonomous evolutionary problems, *J. Evol. Equ.*, **13** (2013), 751–776.
[27] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, American Mathematical Society, 1997.
[28] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl., IV. Ser.*, **146** (1987), 65–96.
[29] A. Süß and M. Waurick, Solution theory for a general class of SPDEs, *Stoch. Partial Differ. Equ., Anal. Computat.*, doi:10.1007/s40072-016-0088-8 (2016).
[30] L. Tartar, *The General Theory of Homogenization: A Personalized Introduction*, Springer, Heidelberg, 2009.
[31] S. Trostorff, Autonomous evolutionary inclusions with applications to problems with nonlinear boundary conditions, *Int. J. Pure Appl. Math.*, **85**(2) (2013), 303–338.
[32] H. Tu and C. Lin, Continuous dependence for the Brinkman equations of flow in double-diffusive convection, *Electron. J. Differ. Equ.*, **2007** (2007), 9.
[33] C. Tudor, Continuous dependence on the coefficients for the diffusions on a bounded domain, *Lith. Math. J.*, **16**(4) (1976), 630–633.
[34] M. Waurick, *Limiting Processes in Evolutionary Equations – A Hilbert Space Approach to Homogenization*, Dissertation, TU Dresden, 2011, http://nbn-resolving.de/urn:nbn:de:bsz:14-qucosa-67442.
[35] M. Waurick, A Hilbert space approach to homogenization of linear ordinary differential equations including delay and memory terms, *Math. Methods Appl. Sci.*, **35** (2012), 1067–1077.
[36] M. Waurick, Homogenization of a class of linear partial differential equations, *Asymptot. Anal.*, **82** (2013), 271–294.
[37] M. Waurick, *G*-convergence of linear differential operators, *J. Anal. Appl.*, **33**(4) (2014), 385–415.
[38] M. Waurick, On non-autonomous integro-differential-algebraic evolutionary problems, *Math. Methods Appl. Sci.*, **38**(4) (2014), 665–676.
[39] M. Waurick, A note on causality in reflexive Banach spaces, *Indag. Math.*, **26**(2) (2015), 404–412.
[40] M. Waurick, On the homogenization of partial integro-differential-algebraic equations, *Oper. Matrices*, **10**(2) (2016), 247–283.
[41] M. Waurick, Stabilization via homogenization, *Appl. Math. Lett.*, **60** (2016), 101–107.
[42] M. Waurick, *On the Continuous Dependence on the Coefficients of Evolutionary Equations*, Habilitation Thesis, TU Dresden, 2016, arXiv:1606.07731.
[43] M. Waurick, G-convergence and the weak operator topology, *PAMM*, **16** (2016), 883–884.
[44] J. Wloka, *Partielle Differentialgleichungen*, B. G. Teubner, Stuttgart, 1982.
[45] M. Yaman and S. Gür Continuous dependence for the damped nonlinear hyperbolic equation, *Math. Comput. Appl.*, **16**(2) (2011), 437–442.
[46] V. V. Zhikov, S. M. Kozlov, O. A. Oleinik, and K. T'en Ngoan, Averaging and G-convergence of differential operators, *Russ. Math. Surv.*, **34** (1979), 69–147.

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