*On the Most Perfect Forms of Magic Squares, with Methods for their Production.**

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PART I.—*The Method of Uniform Steps.*

1. The square A is *magic* because each row, column, and diagonal has the same sum, 175; it is *pandiagonal* because not only the two main diagonals, but also

A.

10	5	49	37	32	27	15
41	29	24	19	14	2	46
16	11	6	43	38	33	28
47	42	30	25	20	8	3
22	17	12	7	44	39	34
4	48	36	31	26	21	9
35	23	18	13	1	45	40

the twelve broken diagonals, such for example as $49 + 29 + 16 + 3 + 39 + 26 + 13$, have each the same sum; it is *symmetrical* because any number added to the one centrally opposite makes 50, as for example $11 + 39$, so that any three numbers and their opposites, plus the central number, will have the same sum, 175; and it is a *knight's path* square because the numbers 1, 2, 3, ... 48, 49, 1 will be found to follow such a path, it being presupposed that the knight can leave either of the four sides freely to re-enter upon the side opposite, exactly as though such sides were adjacent. Thus, from 1, the step to 2 may be divided into two steps down, through 32 to 14, and one step to the right; or from 9, two down, through 40 to 15, then one on the right, arriving at 10.

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2. The forms of magic squares most familiarly known have been symmetrical, though not pandiagonal. It is not to be presumed that symmetrical squares should have been known so long, possessing the property, for the square of 7, that two centrally opposite numbers shall have 50 for their sum, and in general, for the square of $2n+1$, that two centrally opposite numbers shall have $(2n+1)^2+1$ for their sum, without some one noticing that any n numbers, plus their opposites, plus the central number, making $2n+1$ in all, must form a symmetrical group having the same sum as that of any row or column. Granting that this observation must have been made, unknown to the present writer, it would yet appear that this property of summing symmetrical groups has been slighted beyond measure and without cause. In A, for example, there are 28 ways of summing 175 by rows, columns, and diagonals, while by reference to the symmetric groups hundreds upon hundreds of other ways may be exhibited. If dealing with trifles like magic squares is worthy of an Euler or a Cayley; if in short it is legitimate, it is because amusement has value. Any one who writes a symmetrical square indelibly upon some substance permitting instant removal of temporary marks can excite much interest by marking the numbers composing a symmetrical group, and subsequently exhibiting in like manner many other such groups, all having the same sum. It will be assumed in this paper that symmetry is essential in every case.

3. Pandiagonal squares, not symmetrical, have also been known for centuries. It is a property of pandiagonal squares, long known and easily recognized, that a row or column on one of the four sides can be transposed to the opposite side without destroying the magic and pandiagonal attributes of the square. As the operation may be repeated indefinitely, any number of rows or columns taken together can be transposed with the same freedom. By reason of these remarkable properties pandiagonal squares have received from some writers, beginning with La Hire, the name "perfect," and from others, beginning with Lucas, the name "diabolic." It is therefore possible to bring any given number in such a square, by not more than two transpositions, into the middle or any other given position. The square A, for example, is only one of 49 commutative forms, and of each of these forms there are 8 varieties, because any square whatever may be turned upside down or sideways, or reflected in a mirror, without losing its identity. The other 48 forms are not symmetrical, but any one of them may be regarded as "capable of symmetry," requiring only such transpo-

sitions as shall bring the middle number to the middle place. In most cases pandiagonal squares are not capable of symmetry.

4. A special property of pandiagonal squares has been variously stated by analysts. The following description will probably be found as simple as the matter permits of. Choose two numbers in the square, certain combinations excepted, and note their relative location, observing how many places up or down, how many to the right or left, the second number stands from the first, and call this measurement the "step" between the two. It is forbidden to choose two numbers which, when divided severally by the root of the square, that is, the number of places on its side, say r, shall have either the same integral quotient or the same remainder; provided, as regards the division, that when there is no remainder the quotient must be reduced by 1, leaving r as the remainder. Repeat the "step" by passing from the second number to a third, and so on until stopped by meeting the first number in the way. The cycle of numbers thus found will have the "magic" sum. Thus, in A, $5+38+22+13+46+30+21=175$. This property may be called "step summation."

5. To produce a knight's path square which shall also be both symmetrical and pandiagonal, the root r being any odd number not divisible by 3 or 5, say $2n+1$, we may begin in the middle, with the middle number, and write n numbers succeeding, following the knight's path downward and to the right. When we have set down the n^{th} number from the middle, that is, having made n steps, this number being the first one reached which is divisible by r [in A, 28], we must make a cross step before setting down the next number. This cross step may either be two places to our left and one down, or two to our right and one up, the general direction being at right angles to that of the previous steps, the knight's step always preserving its original character or "bend," viz. in any direction which it may take, two places forward and one to the left. [In A, the second of the two cross steps indicated is chosen; otherwise 29 would appear where 20 is.] Proceeding now again in the original direction, until further progress is obstructed [in A, 35 is such a stopping place, since the next step leads to the place occupied by 29], we find that a second cross step is necessary, similar to the one chosen for the first, and so on till the last number is written, when the cross step leads to the place where 1 must be written, after which the process is to be continued till the square is completed. [There is therefore a second

square having the properties of A, which the reader will, on laying out the 49 spaces, have no trouble in filling up off-hand.] In this manner we may produce two squares, possessed of this combination of properties, for each odd root not divisible by 3 or 5. Or, we may begin by placing 1 in the uppermost row, just to the left of the middle column, if the form of cross step "to our left" is chosen, and write the numbers directly in their order, making a cross step when obstructed, till the square is completed; or again, by placing 1 in the lowest row, just to the right of the middle column, and using the other form of cross step throughout. [See A.] A square of any size, however large, may be filled out in this way for the mere trouble of writing the numbers in their natural order. The rule has been given in both forms, starting from the middle and from 1 respectively, the former method giving more insight into the principles involved, the latter being derived from the former with a little assistance from algebra. Again, we may start from 1 in any position and produce either square in a non-symmetrical form, requiring one or two transpositions, as already explained, to bring the middle number to the middle.

6. What is here presented as a "knight's path square," meaning a square containing the numbers in their natural order arranged along a path consisting throughout of knight's steps having the same bend, is but a special case, one of a class of squares having similar properties, a class which may be referred to as "uniform step squares." The square B, wherein $r = 11$, is a second illustration. In A, the uniform step was "two places forward, one to the left," or left-bend knight's step. In B it is three places forward, whatever the direction may be,

B.

90	54	7	81	34	119	72	25	110	63	16
85	38	112	76	29	103	56	20	94	47	11
69	33	107	60	13	98	51	4	78	42	116
64	17	91	55	8	82	35	120	73	26	100
48	1	86	39	113	77	30	104	57	21	95
43	117	70	23	108	61	14	99	52	5	79
27	101	65	18	92	45	9	83	36	121	74
22	96	49	2	87	40	114	67	31	105	58
6	80	44	118	71	24	109	62	15	89	53
111	75	28	102	66	19	93	46	10	84	37
106	59	12	97	50	3	88	41	115	68	32

then two to the left. The regular step, say from 61 in the middle to 62, or from 1 to 2, is three places down and two to our right, as we look at the square, our "right" being "left" to any one looking downward from the starting point. The cross step, say from 66 to 67, or from 11 to 12, is three places to the right and two up. A second square having the same properties may be produced by taking the opposite cross step, three places to the left and two down. If we count x places to the right and y places up, the regular step in B is defined by $x = 2$, $y = -3$, and the cross step (let us say the cross step to our right) by $x = 3$, $y = 2$; and the cross step (to our left) for the other square would be defined by $x = -3$, $y = -2$. The general rule, when r is prime and greater than 5, for producing a "uniform step square" is, first to take for the regular step $x = a$, $y = -b$, and for the cross step either $x = b$, $y = a$, or $x = -b$, $y = -a$, where b is any number from 2 to n inclusive, and a is any number above 0 and less than b, provided that $a^2 + b^2$ is prime to r. Here, as before, n is such that $r = 2n + 1$. (Otherwise we may, subject to the same rare exceptions, define the choice of steps as from the middle place to any place in the south-southeast eighth of the square, using the points of the compass as in a map; any place, that is to say, included between the middle column and the diagonal, below and to the right of the middle place. This space may at pleasure be transferred to any other desired eighth of the square by turning the square around or using a real or imaginary looking-glass, so that the restriction of movement from the middle to a small fraction of the whole space does not impair the generality of the method. The same remark holds concerning the use of steps having only the left bend, which in the mirror would of course appear as having the opposite bend, such variations not affecting the identity of the square.) Having chosen the steps, we must now proceed exactly as explained for the knight's path square in the preceding paragraph, except that another place must be found for 1, if it be desired to begin with 1 instead of with the middle number. For the knight's step, $a = 1$, $b = 2$, and the place of beginning (with cross step to our right) is [see A] in the first row, counting from the bottom, and in the first column to the right of the middle column. When the cross step is to our left, the place of beginning is in all cases symmetrically, that is, centrally, opposite to the place of beginning when the cross step is to our right, so that it is unnecessary to discuss both cases. The latter case, "to our right," is presupposed in what follows. For all steps in which $a = 1$, the place of beginning is in the first row, counting upward from the bottom; if $a = 3$, the second row; if $a = 5$, the third row, and so on. If $a = 2$, it is in the first row above the middle; if $a = 4$, the second, and so on. For all steps in which $b = 2$, the place of beginning is in the first column from the middle, counting to the right; if $b = 4$, the second, and so on. If $b = 3$, the place of beginning is in the second column, counting from the left; if $b = 5$, the third, and so on. Knowing these things, it is usually possible to begin the writing of the square by putting 1 in any designated place not in line with the middle place. For example, let it be demanded, in a square of 13 on a side, that 1 be placed in the fourth column, and in the second row from the top. Here $b = 6$, $a = 3$, and the cross step is to our left. Or, let the assigned place be in the first column, and in the second row from the top. This requires that the direction of the steps be changed. We may treat the first column as if it were the top row, and, thus giving the square an imaginary turn of 90°, we shall regard the row next to the top as if it were the column next to that on the right. The problem now is to place 1 in the top row and in the second column counting from the right. For the directions thus altered, we have $b = 3$, $a = 1$, cross step to the left. In reality, the regular step will be three places to the right, one upward, and the cross step will be three places down, one to the right.

7. For "uniform step squares," regarded more generally, in which the step is defined by coördinates x and y, without restricting the direction of the step, we have for the cross steps the coördinates $x' = \mp y$, $y' = \pm x$, the upper sign referring to what has been spoken of as "the cross step to our right." Taking the middle place as the origin of coördinates, the location of the first multiple of r which is reached in order, after taking n steps from the middle number, is nx, ny, or rather the remainder of these after division by r; that of the number next higher, after making the cross step, is $nx \mp y$, $ny \pm x$, and that of 1, readily deduced from these, $x'' = \pm ny$, $y'' = \mp nx$, both expressions being likewise subject to reduction by any multiple of r, because as a coördinate $r = 0$. Thus, as before, if $x = a = 1$ and $y = -b = -2$, the location of 1 is $x'' = \mp 2n = \pm 1$, $y'' = \mp n$; that is, column ± 1, row $\mp n$, as in the preceding paragraph. From these two general formulæ for the location of 1, namely, $x'' = \pm ny$, $y'' = \mp nx$, we learn that to produce a square after assigning 1 to column x'', row y'', measured from the middle of the square, we have merely to add as many times

r respectively to x'' and y'' as shall make them each divisible by n, thus deriving x and y at once. Thus, if $r = 2n + 1 = 13$, $x'' = n - 1 = 5$, $y'' = -3$, the step for this location of 1 will be shown by $x = (y'' + 3r)/(\mp n) = \mp 6$, $y = (x'' + r)/(\pm n) = \pm 3$, as in paragraph 6. Since the signs of x and y when determined in this way depend on that of the cross step, two squares can always—subject to certain exceptions which will be noted—be developed from any location not in line with the middle which may be assigned to 1, one by the direct step, the other by the same step backwards taken with the opposite cross step. No step is available for producing a pair of "uniform step squares," in which the numbers follow the steps in their natural order, including of course under this name the class of knight's path squares first described, unless r is prime to both x and y, and also prime to both the sum and the difference of the natural numbers indicated by those letters, as well as to the sum of their squares.

8. Some attention may for a moment be drawn to those failing cases wherein r is not prime to $a^2 + b^2$. It is not practically necessary to speak of composite values of r, the lowest of which, under the restrictions stated, is 49. Let us glance at the successive prime numbers 7, 11, 13, 17, 19, 23. For every value of r, if there were no failing cases, there would be $\frac{1}{8}(r - 1)(r - 3)$ steps, each producing two squares with opposite cross steps. There are 3 such steps for $r = 7$, namely, $a = 1$, $-b = 2$ or 3; $a = 2$, $-b = 3$. There is likewise no failing case for $r = 11$, the steps being 10 in number, namely, $a = 1$, $-b = 2, 3, 4,$ or 5; $a = 2$, $-b = 3, 4,$ or 5; $a = 3$, $-b = 4$ or 5, $a = 4$, $-b = 5$. For $r = 13$, the failing cases are $a = 1$, $-b = 5$; $a = 2$, $-b = 3$; $a = 4$, $-b = 6$; leaving 12 available steps. For $r = 17$, the failing cases are $a = 1$, $-b = 4$; $a = 2$, $-b = 8$; $a = 3$, $-b = 5$; $a = 6$, $-b = 7$; leaving 24 available steps. For $r = 19$ and $r = 23$ there are respectively 36 and 55 available steps without failing cases.

9. The most important novel element in the knight's path method, and in the more general uniform step method of which the knight's path method is a special case, consists in the exhibition of uniform steps by which the numbers are written down throughout in order, perhaps starting offhand with 1 in a place arbitrarily assigned. If for any purpose it be desired to follow the same series of steps while employing a series of numbers not in their natural order, it is possible to do so, and still to produce a square both symmetrical and pandiago-

nal, the path pursued, however, being no longer evident after completion of the process. The way of doing this will be shown most readily by an example, in which $r = 7$, the same process being obviously applicable when r has other prime values. Let the numbers from 1 to 7 be arranged in any order, subject to certain conditions. The middle number, 4, must not be changed. The two numbers next to it on either side must have 8, that is, $3 + 5$, the same as before, for their sum. The two numbers next adjacent must have the same sum, and so on. Thus we may perhaps reach some such order as this: 3, 7, 2, 4, 6, 1, 5. Call this series S_0. Form another series, S_1, by adding 7 to each term of S_0; then another, S_2, by adding 14 to each term of S_0, and so on till 7 series, ending with S_6, have been written down. Now rearrange the letters S_0, S_1, ... S_6, retaining S_3 in the middle, in any order, provided the sum of the subscripts of any two equidistant from S_3 shall remain 6 as before, and suppose the result is S_5, S_0, S_4, S_3, S_2, S_6, S_1. If for these letters we substitute the numbers which they represent, we shall have as the result the numbers from 1 to 49 arranged thus: 38, 42, 37, 39, 41, 36, 40, 3, 7, 2, 4, 6, 1, 5, 31, 35, 30, 32, 34, 29, 33, 24, 28, 23, 25, 27, 22, 26, 17, 21, 16, 18, 20, 15, 19, 45, 49, 44, 46, 48, 43, 47, 10, 14, 9, 11, 13, 8, 12. If of the three possible steps for $r = 7$ we

C.

2	41	12	49	18	22	31
43	17	23	34	5	42	11
35	4	36	10	44	20	26
13	47	21	25	29	3	37
24	30	6	40	14	46	15
39	8	45	16	27	33	7
19	28	32	1	38	9	48

choose the knight's path, and of the two cross steps choose the one to our right, we shall follow the order of the steps shown in A, but by using the numbers in their new order we shall produce the square C, which is both symmetrical and pandiagonal, but in which the knight's path is not obtrusive to the eye. Some other uniform step, however, less known than the knight's path, should be chosen if complete disguise is desired.

10. It is not difficult to prove that the method of uniform steps, for which one of the steps which may be chosen is the knight's step, must produce squares

both symmetrical and pandiagonal, provided the numbers are written in what may be called "symmetrical order," that is to say, either in their natural order, or rearranged symmetrically in r series of r numbers each as prescribed in the foregoing paragraph. Any square so formed must be symmetrical, because the middle number of the middle series is the middle number of all and is set in the middle place of the square; the two numbers next it either way have the uniform sum $r^2 + 1$, one of them being located by a step forward, the other by a step backward, so as to occupy places symmetrically opposite each other; the next pair of numbers are similarly opposite each other, and so on by pairs throughout.

11. Any square so formed must be pandiagonal, because it satisfies La Hire's requirement for what he called "perfect" squares. He divided the numbers into r series of r numbers each, without reference to symmetry, regard for which appears always to have been slighted, and nearly always unthought of, by writers on pandiagonal squares. His system of dividing the numbers into series, which has been fruitful in the hands of subsequent writers, was to regard every number as the sum of two constituents, say an elementary number p such as 1, 2, ... $(r-1)$, r, and a base number q such as 0, r, $2r$, ... $(r-1)r$. The elementary numbers, after being arranged in any order, may be designated in their new order as p_1, p_2, ... p_r, and the base numbers, similarly arranged in any order, as q_0, q_1, q_2 .... q_{r-1}. Any one of the original numbers, from 1 to r^2, is known now as the sum of its two constituents, say $q_k + p_m$. La Hire observed that if a square were first formed of q's, each q being repeated r times, in such a manner that the same q did not appear twice in the same line, that is, the same row, column, or diagonal, whole or broken, the square would be pandiagonal; and that if another square were similarly formed of p's, each p being repeated r times, but arranged in some different order from that followed in the q square, this also would be pandiagonal; and that the two squares might be superimposed, the constituents falling together being added so as to produce a "perfect" square. It is enough for us if the same series (see paragraph 9) is not represented twice in the same line, and that two numbers of the same rank in different series do not appear together in the same line.

12. That the same series is not represented twice in the same row is plain, because each series of r numbers is located by r steps of uniform character, each

measuring b places down, a places to the right, and b is taken prime to r. It is not represented twice in the same column, similarly, because a is prime to r. And it is not represented twice in the same diagonal, because each step leads to the $(b+a)^{\text{th}}$ diagonal whose direction is downward to the left and to the $(b-a)^{\text{th}}$ diagonal whose direction is downward to the right, and both $b+a$ and $b-a$ are prime to r. Again, two numbers of the same rank in different series cannot appear in line together unless the two leading numbers of those series are in line together, for the several series march in, so to speak, parallel order with equal steps. We must therefore examine the steps by which the leading numbers of the several series follow one upon the other, steps which from the nature of the whole network must be uniform. It is sufficient to consider only the cross step to our right, the same reasoning sufficing for the case in which the other cross step is chosen. The regular step from 1 to 2, assuming for brevity that the numbers are in their natural order, is $y=-b$, $x=a$. The step backward, therefore, from 1 to r, is $y=b$, $x=-a$. The cross step from r to $r+1$ is $y=a$, $x=b$. The step from 1 to $r+1$ is therefore $y=b+a$, $x=b-a$. Since these numbers and their sums and differences are prime to r, no two of the leading terms $1, r+1, 2r+1, \ldots\ldots,$ can be in line together; and it follows, as stated, that no two numbers of the same rank in different series can be in line together, so that any square produced by the method of uniform steps is pandiagonal.

13. The method must obviously produce a square whenever the step chosen is such as not to interfere with itself, so to speak, an interference which must happen whenever a succession, less than r in number, of cross steps, each defined by $y=a$, $x=b$, leads to a place which is likewise to be reached from the same starting point by a succession of less than r regular steps, each defined by $y=-b$, $x=a$. Let us suppose such a place to be reached by p regular steps, and by q cross steps. Its location, taking the starting point as the origin, is $y=-pb=qa$, and $x=pa=qb$, each expression when greater than r being reducible by the subtraction of the arithmetical value of r or some multiple of it. Thus, assigning due values to j and k, we have $jr-pb=qa$, and $qb=pa+kr$. From these by multiplying we derive $jbr-pb^2=pa^2+kar$, whence $a^2+b^2=mr$, where $m=(jb-ka)/p$, and p is less than r. Interference is therefore avoided (see paragraph 8) when a^2+b^2 is prime to r. Sup-

pose, for an example of interference, $r = 13$, $a = 1$, $b = 5$. Here $m = 2$, and the simplest values suitable are $p = 5$, $q = 1$, $j = 2$, $k = 0$; the first cross step clashes with the fifth direct. Again, suppose an attempt made to write a knight's path square with $r = 25$, $a = 1$, $b = 2$. Here $m = 1/5$, according with $p = 10$, $q = 5$, $j = 1$, $k = 0$; the fifth cross step clashes with the tenth direct. When r is prime it is sufficient to assume $q = 1$, $k = 0$.

14. In addition to uniform step squares, other forms may be produced which shall be both symmetrical and pandiagonal, proceeding as before by a regular step for the first series of r numbers, arranged symmetrically as in paragraph 9 if not in their natural order, but using a different cross step. If for brevity we call the first of the first series 1 and the last of the same series r, the rule, when r is prime and > 3, may be laid down that the cross step from r to $r + 1$, the first of the next series, may be taken to any place not in line with 1 and not already occupied by a number of the first series. To prove this it is only necessary to show that the steps 1, 2, 3, ... and 1, $r + 1$, $2r + 1$, ... cannot lead to a common place of meeting before their return to the place of beginning where 1 is located. If one series of steps be denoted by x, y, the first place will be located by x, y; the second by $2x$, $2y$; the third by $3x$, $3y$; and so on to rx, ry, which is the same as 0, 0, the place of beginning. Let us suppose a second series of steps, each denoted by $2x$, $2y$; these will reach the same places, r in number, though in different order, namely, $2x$, $2y$; $4x$, $4y$; .... $(r - 1)x$, $(r - 1)y$; $(r + 1)x$, $(r + 1)y$, that is, x, y; then $3x$, $3y$; .... rx, ry, as before; and this must hold good whether $2x$ or $2y$, if greater than r, is counted in full or reduced by r, since multiples of r will not affect locations on the square. In the same way we shall see that a step from 1 to any one of the places in question, repeated r times, must reach each other of the same places and no other place. The like is true of the other series of steps, leading successively to the series of places of 1, $r + 1$, $2r + 1$, ... Since $r + 1$ was assigned to a place not occupied by any one of the numbers 1, 2, ... r, the two paths are therefore wholly distinct. The work may either be begun with the middle number in the middle place, or a square "capable of symmetry" may be produced by beginning with 1 in any position. No discussion is here contemplated concerning the formation of squares of this irregular sort when r is not prime. It will be remarked that this variation of the method must be used when $r = 5$, whenever

a square both symmetrical and pandiagonal is desired, since in this case r cannot be prime to $a^2 + b^2$, so that uniformity of step throughout is not possible.

15. The reasoning of the preceding paragraph may also manifestly be applied to cases where the cross step is uniform with the direct step. It is to be remarked that it is not possible to take a cross step which shall have a different "bend" while uniform with the direct step in other respects, because such a step would lead to a place for $r+1$ in line with the place of 1, which is forbidden. The direct step being $x = a$, $y = -b$, the position of r, measured backward from 1, is $x = -a$, $y = b$. The cross step with the other bend from r to $r+1$ being $x = \mp b$, $y = \pm a$, the position of $r+1$, measured from 1, is found to be $x = -(a \pm b)$, $y = \pm(a \pm b)$, showing that $r+1$ and 1 are in line diagonally with each other. This explains, for example, why no attempt has here been made to produce a knight's path square with a cross step having a different bend from that of the direct step. It is almost unnecessary to observe that if $r+1$ were in line with 1 the line could not have the magic sum $\frac{1}{2}r(r^2+1)$, since its sum would be $1 + r + 1 + 2r + 1 + \ldots + (r-1)r + 1 = \frac{1}{2}r(r^2 - r + 2)$. When a different cross step is used, as in paragraph 14, it is not impossible to produce squares both symmetrical and pandiagonal for odd values of r, such as 15 or 25, which do not permit the off-hand formation of uniform step squares.

PART II.—*The Figure-of-Eight Method.*

16. Symmetry, when the root r is even, is less useful a quality than when the root is odd, as there is no middle place from which to measure distances. The pandiagonal quality is still essential, when r is divisible by 4. It is not feasible for other even values of r. Let us assume without further repetition that r is divisible by 4. Pandiagonal squares of the best form for such values of r—let us call them "complete" squares—possess the following combined properties: first, they possess all their properties without diminution however much the rows and columns may be transposed (see paragraph 3), differing in this respect from symmetrical pandiagonal squares for odd values of r; second, they possess additional magic summations by blocks of four, any small square of four being chosen as a block, and enough blocks being chosen, overlapping or otherwise, to make up r numbers in all; third, each number is complementary to the one distant from it $\frac{1}{2}r$ places in the same diagonal. The second property pro-

D.

1	63	3	61	12	54	10	56
16	50	14	52	5	59	7	57
17	47	19	45	28	38	26	40
32	34	30	36	21	43	23	41
53	11	55	9	64	2	62	4
60	6	58	8	49	15	51	13
37	27	39	25	48	18	46	20
44	22	42	24	33	31	35	29

duces a fourth, that of alternate equivalent couplets. For example, the square D is one in which $r = 8$; in which every block of four has the sum 130, so that any two blocks have the magic sum 260; and in which every number and its diagonal fourth have the sum 65. The sum of any two overlapping blocks being equal, it follows that all alternate couplets have equal sums. Thus $1+16 = 3+14$, $50+47 = 52+45$, $63+3 = 47+19$, and so on throughout without exception, both vertically and horizontally. A fifth property is an easy consequence of the fourth. The alternate couplets being equivalent, the four corners of any rectangle whatever, having an even number of places on each side, constitute a block again possessed of half the magic sum, so that any $\frac{1}{4}r$ such blocks, however different in size or shape, whether apart or overlapping, will have the magic sum. The magic and pandiagonal properties themselves follow necessarily in these squares from the third and fourth: as regards the whole and broken diagonals, directly from the third, or perhaps rather from a sixth property which is a corollary of the third, namely, that any selected $\frac{1}{2}r$ numbers in the square added to the $\frac{1}{2}r$ numbers complementary to them in the same diagonals respectively, distant each from its complement $\frac{1}{2}r$ places, will have the magic sum. Of each row or column, one-half is composed of the complements of couplets which are alternate with and equivalent to the couplets composing the other half, so that the row or column again has the magic sum. What is obviously to be desired is a simple method of producing squares possessed of the second and third properties, from which all the others are thus seen to follow. The problem is in fact to distribute $\frac{1}{2}r^2$ non-complementary numbers in $\frac{1}{2}r$ adjacent rows or columns, forming one-half of the square, so as to exhibit the second or "blocks of four" property throughout the whole square when it is completed by adding the complementary numbers.

17. The square D was devised by what may be called the figure-of-eight method, because the order in which the rows are first written bears some resemblance to the figure 8 laid on one side, the usual sign for infinity. The upper half of the square was first filled as indicated below. By following the

1	2	3	4	12	11	10	9
16	15	14	13	5	6	7	8
17	18	19	20	28	27	26	25
32	31	30	29	21	22	23	24

order of the numbers from 1 to 32, the reason for using the phrase in question will readily be seen. The numbers in every alternate column, second, fourth, etc., were then replaced by their complements, and this supplied the upper half of D, the lower half being added by writing in the complements as indicated in the last paragraph. The rule therefore for producing "complete" squares is to write the first $\frac{1}{2}r$ numbers in the first row, then drop to the second row, returning backwards along the first row and dropping to the second so as to complete both rows in what we may call the figure-of-eight manner. The next two rows must come next in the same way, and so on till half the square is filled, when every alternate column is to be replaced by the complementary numbers, after which the rest of the square is to be completed by writing down the complement of each number in the same diagonal, $\frac{1}{2}r$ places lower down. The numbers may be arranged in their natural order, or in an appropriate artificial order, as will be seen later, but no other variation is proposed.

18. Let us see what happens upon taking the odd numbers first, as seen below. If we replace the first, third, fifth, and seventh columns by writing in

1	3	5	7	23	21	19	17
31	29	27	25	9	11	13	15
33	35	37	39	55	53	51	49
63	61	59	57	41	43	45	47

lieu of them the complements of the numbers composing them, and supply the lower half, we obtain the "complete" square E. We might begin also by

E.

64	3	60	7	42	21	46	17
34	29	38	25	56	11	52	15
32	35	28	39	10	53	14	49
2	61	6	57	24	43	20	47
23	44	19	48	1	62	5	58
9	54	13	50	31	36	27	40
55	12	51	16	33	30	37	26
41	22	45	18	63	4	59	8

writing the even numbers, but the result would be the same square, upside down, written backwards, and transposed. It is also to be remarked generally that it makes no real difference which set of columns is selected for replacement, whether the first, third, etc., or the second, fourth, etc. If, for example, the other set of columns had been replaced in this case by the complementary numbers, the resulting square would have been what E becomes after such transpositions as are required to bring 1 to the upper corner on the left hand. The reader will find on trial that the numbers may also be taken at intervals of 4, viz. 1, 5, 9, ... 61, followed by 2, 6, 10, ... 62. Illustrations of like results for larger squares, as where $r = 12$, $r = 16$, etc., may be multiplied to any extent.

19. Since other ways of arranging the numbers in order are doubtless available, while certainly the numbers cannot be arranged at random, it becomes necessary to examine the principle underlying this method, so as to ascertain the limits within which the order of the numbers can be changed. Let the first r numbers in the required artificial order be $a_1, a_2, \ldots\ldots a_r$; the second r numbers $b_1, b_2, \ldots\ldots b_r$, and so on. Let the sum of any vertical couplet of the first and second row, as first arranged, be s_1; of the second and third row, s_2, and so on. This is then the first arrangement:

$$
\begin{array}{llllllll}
a_1 & a_2 & \ldots\ldots\ldots\ldots & a_{\frac{1}{2}r} & b_{\frac{1}{2}r} & \ldots\ldots\ldots\ldots & b_2 & b_1 \\
b_r & b_{r-1} & \ldots\ldots\ldots\ldots & b_{\frac{1}{2}r+1} & a_{\frac{1}{2}r+1} & \ldots\ldots\ldots\ldots & a_{r-1} & a_r \\
c_1 & c_2 & \ldots\ldots\ldots\ldots & c_{\frac{1}{2}r} & d_{\frac{1}{2}r} & \ldots\ldots\ldots\ldots & d_2 & d_1 \\
d_r & d_{r-1} & \ldots\ldots\ldots\ldots & d_{\frac{1}{2}r+1} & c_{\frac{1}{2}r+1} & \ldots\ldots\ldots\ldots & c_{r-1} & c_r \\
.. & ... & & ..... & .... & & .... & .. \\
m_r & m_{r-1} & \ldots\ldots\ldots\ldots & m_{\frac{1}{2}r+1} & l_{\frac{1}{2}r+1} & \ldots\ldots\ldots\ldots & l_{r-1} & l_r.
\end{array}
$$

The relations which are required for our purpose are: $a_x + b_{r+1-x} = s_1$, $b_x + c_{r+1-x} = s_2$, ... $l_x + m_{r+1-x} = s_l$, and also, as will be shown immediately, $m_{\frac{1}{2}r+x} - b_x = l_{\frac{1}{2}r+x} - a_x = t$, another constant sum, positive or negative. Also, no two of the numbers in this scheme, representing the first arrangement of the upper half of the proposed square, can be complementary, that is, their sum must not be $r^2 + 1$. Let us represent $r^2 + 1$ by ρ. It is immaterial whether the alternate columns be replaced by the complementary numbers before or after the lower half of the square is filled out complementarily; for the moment, we may assume the lower half filled first. The two expressions here given for t correspond to those preceding them, to this extent, that if we denote by s_m the constant sum of numbers in the row beginning with m_r and those respectively below them in the first complementary row, viz. $\rho - b_{\frac{1}{2}r}$, ..., we shall have $m_{\frac{1}{2}r+x} + \rho - b_x = s_m$, so that t represents $s_m - \rho$. The foregoing relations are sufficient, because when they exist any two adjacent vertical couplets must have the same sum, say k, and when one of these two couplets is replaced by its complementary couplet, the sum of which is $2\rho - k$, the block of four thus formed has the required sum 2ρ. Since any two adjacent rows have constant sums, each row and the second, or fourth, etc., row from it must have constant differences, so that $l_x - a_x = m_x - b_x$ is constant for values of x from $\frac{1}{2}r + 1$ to r inclusive; or let us say that $l_{\frac{1}{2}r+x} - a_{\frac{1}{2}r+x} = m_{\frac{1}{2}r+x} - b_{\frac{1}{2}r+x} = g$, a constant. Then, $g = t + a_x - a_{\frac{1}{2}r+x} = t + b_x - b_{\frac{1}{2}r+x}$. From this, if we take $t - g = u_1$, we have this special relation, $a_{\frac{1}{2}r+x} = u_1 + a_x$, $b_{\frac{1}{2}r+x} = u_1 + b_x$. For the next two rows, similarly, $c_{\frac{1}{2}r+x} = u_2 + c_x$, $d_{\frac{1}{2}r+x} = u_2 + d_x$, and so on for every pair of rows. Conversely, if the rows are thus arranged, the final relation first stated, containing t, and involving the first complementary row, will follow. We may therefore choose $\frac{1}{2}r$ numbers, $a_1, a_2, \ldots. a_{\frac{1}{2}r}$, and by adding u_1 to each derive successively $a_{\frac{1}{2}r+1}$ to a_r; then $b_1 = s - a_r$, $b_2 = s - a_{r-1}$, ...; then $c_1 = a_1 + p$, $c_2 = a_2 + p$, ... $c_{\frac{1}{2}r} = a_{\frac{1}{2}r} + p$; then $c_{\frac{1}{2}r+1} = c_1 + u_2$, ...; then $d_1 = b_1 + p$ and so on. For e_1 we must introduce another constant, say q, such that $e_1 = a_1 + q$; and for $e_{\frac{1}{2}r+1}$ another, say u_3; and so on. Thus, if a represent any one of the original $\frac{1}{2}r$ numbers chosen, the others will follow as in the schedule marked F, b being derived by subtracting the proper a from s. The choice of the original numbers, a_1 to $a_{\frac{1}{2}r}$, is restricted by the requirement that they, with all the numbers derived from them by the assignment of s and of the differences p, q, ..., u_1, u_2, ..., F.

	$x=1$ to $x=\frac{1}{2}r$.	$x=\frac{1}{2}r$ to $x=r$.
a	a	$a+u_1$,
c	$a+p$	$a+p+u_2$
d	$b+p$	$b+p+u_2$
e	$a+q$	$a+q+u_3$
f	$b+q$	$b+q+u_3$

whether such differences be positive or negative, making $\frac{1}{2}r^2$ numbers in all, shall be non-complementary throughout.

a_1	1	1	1
a_2	2	2	2
s	9	13	9
u	1	1	4

20. The simplest illustration is derived, of course, from the case $r=4$. The three available sets of values corresponding to the three pandiagonal squares of 4, all of which are necessarily "complete," are shown in the margin. Other values of a_1, etc., reproduce the same squares. It would involve much study to determine the number of possible complete squares of 8 and assign the values corresponding. In the simple case D, where the numbers are taken in natural order, we have $s=2r+1=17$, $u_1=u_2=\frac{1}{2}r=4$, $p=2r=16$; and in general, for the natural order in all cases where $r=4n$, we have $u_1=u_2=\ldots\ldots=2n$, $p=q=\ldots\ldots=2r$, $s=2r+1$. An obvious variation is obtained by changing the order of the numbers a_1, a_2, ... $a_{\frac{1}{2}r}$, while retaining the same values of s, u_1, etc., and this sort of variation is available for every complete square, however obtained. The result is to interchange the columns in like order, prior to the complementary substitution. In any complete square the odd-numbered columns, first, third, etc., of the left half may therefore be interchanged in any order, provided those of the right half are interchanged in like order, and the like is true of the even-numbered columns among themselves. By turning the square, columns become rows, so that the like is true of rows. It is easy to show algebraically that for squares turned partly around, or written backwards, or both, the numbers in their new relative positions are subject to the same rules of formation. For example, if the square D be so turned that the top row reads 44, 37, 60, ..., we can have $a_1=44$, $a_2=65-37=28$, $a_3=60$, $a_4=12$, $s=66$, $u_1=-10$, $p=-2$, $u_2=-6$.

21. No square can be "complete" which cannot be analyzed according to this method of formation, as shown in F. Each block of four, which must have the sum 2ρ, is composed of two pairs of numbers, or couplets, whose sums respectively are, let us say, α and β. If the second couplet is assumed to have been derived from an earlier one by complementary substitution, the sum of the earlier couplet is $2\rho - \beta = \alpha$, so that any two adjacent couplets must, prior to the substitution, have had the same sum α. This, together with the stated properties of the complete square, is all that was presupposed in paragraph 19.

22. There are many ways more or less formal, of arranging the order of the numbers in applying the general method, besides those simplest ways mentioned in paragraphs 17 and 18. It will often be convenient to begin by selecting $2r$ numbers in arithmetical progression such that if to each be added p, positive or negative, for $r = 8$; p and q separately, for $r = 12$; three such constants, for $r = 16$, and so on; the numbers so found, including the original $2r$ numbers, making $\frac{1}{2}r^2$ in all, shall all be different and non-complementary. The $2r$ numbers so taken may then be arranged in r pairs having a uniform sum s, the largest being paired with the smallest, and so on. Then, taking only one number from a pair, it is necessary to choose $\frac{1}{2}r$ numbers for a_1, a_2, etc., such that the other $\frac{1}{2}r$ shall severally differ from them respectively by a constant difference u_1. As an illustration, let $r = 12$, and let us choose the first 24 for our $2r$ numbers, in pairs, having $s = 25$, viz. 1–24, 2–23, 3–22, 4–21, 5–20, 6–19, 7–18, 8–17, 9–16, 10–15, 11–14, 12–13. We may elect to take for the first six a's 1, 2, 3, 7, 8, 9, since by adding 12 to each, say $u_1 = 12$, we reach the other six pairs as required, giving the other six a's. If now for the utmost simplicity we choose $p = q = 24$, $u_2 = u_3 = 12$, the numbers throughout assume this order, when arranged in figures-of-eight:

1	2	3	7	8	9	12	11	10	6	5	4
24	23	22	18	17	16	13	14	15	19	20	21
25	26	27	31	32	33	36	35	34	30	29	28
48	47	46	42	41	40	37	38	39	43	44	45
49	50	51	55	56	57	60	59	58	54	53	52
72	71	70	66	65	64	61	62	63	67	68	69

Part III.—*Previous Approaches to these Methods.*

23. The annexed squares G and H were produced some four centuries ago

G.

22	47	16	41	10	35	4
5	23	48	17	42	11	29
30	6	24	49	18	36	12
13	31	7	25	43	19	37
38	14	32	1	26	44	20
21	39	8	33	2	27	45
46	15	40	9	34	3	28

H.

38	14	32	1	26	44	20
5	23	48	17	42	11	29
21	39	8	33	2	27	45
30	6	24	49	18	36	12
46	15	40	9	34	3	28
13	31	7	25	43	19	37
22	47	16	41	10	35	4

H.

λη	ιδ	λβ	α	κ6	μδ	κ
ε	κγ	μη	ιζ	μβ	ια	κθ
κα	λθ	η	λγ	β	κζ	με
λ	6	κδ	μθ	ιη	λ6	ιβ
μ6	ιε	μ	θ	λδ	γ	κη
ιγ	λα	ζ	κε	μγ	ιθ	λζ
κβ	μζ	ι6	μα	ι	λε	δ

by Moschopulus of Constantinople.* His original Greek form of H is given here as a matter of interest. For the printer's convenience the cursive digamma representing 6 is replaced by 6. It will be seen that G is symmetrical and that H is pandiagonal and "capable of symmetry." Only a single author, so far as the writer's knowledge extends,† has noticed that certain pandiagonal squares are capable of symmetry; and it is most remarkable, for example, that the possibility of producing from H a square both pandiagonal and symmetrical by removing the two upper rows to the bottom should have escaped, if indeed it has escaped, the attention of the many acute computers, including a number of excellent mathematicians, who have dealt with this subject. The author referred to is the Rev. A. H. Frost,‡ who rediscovered the second rule of Moschopulus, unaware of its history, and indeed reproduced H in a varied form, and announced that squares derived by that rule could be made symmetrical. His object, however, was to produce pandiagonal squares, and in speaking of symmetry he referred only to the location of complementary numbers in opposite places.‖

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* See Günther, "Vermischte Untersuchungen," Leipzig, 1876, for many historical details concerning magic squares, including a reprint of the essay of Moschopulus. The squares G and H had already been reprinted, in our notation, with an account of the methods of Moschopulus, by Mollweide.

† This saving clause, which for convenience will be suppressed in what follows, will kindly be understood and supplied by the reader concerning every other historical statement herein contained. It is needed, for very many have written on this subject in all sorts of odd ways and places.

‡ Quarterly Journal, XV, 43, dated by author February, 1877.

‖ See paragraph 2, *ante*. A previous paper by Mr. Frost will be mentioned later.

24. The second rule of Moschopulus, illustrated by H, is merely that special case of the method of paragraph 14 wherein the direct step is the knight's, two places down and one to the right, and the cross step four places down. Another rule involving a cross step in our sense, that is, for example, when $r = 7$, the step from 7 to 8 and not the step from 1 to 8, was given by Mr. Frost in an earlier paper,* in which, with the knight's step down as the direct step, he prescribed one place up for what is here called the cross step, and remarked that squares so produced are capable of symmetry. Cross steps were treated freely by the late President Barnard of Columbia College,† who prescribed analytic tests by which to learn whether any given cross step is permissible in connection with any given direct step. The simple criterion of paragraph 14 could not have occurred to him; and in fact in his discussion of cross steps he had always in view the transition from 1 to 8—using the same example—rather than that from 7 to 8. It is perhaps owing to so many writers having followed La Hire in attending to the relation between 1 and 8, and to so few having followed Moschopulus in attending to that between 7 and 8, that the idea of a cross step uniform with the direct step has not heretofore been brought forward. Like other writers, Barnard appears not to have thought of the possibility of producing odd squares both symmetrical and pandiagonal.

25. Apart from Frost, the only writer known to have produced, even casually, in isolated cases, odd squares at once symmetrical and pandiagonal is Frolow,‡ who showed that the symmetrical square G becomes pandiagonal if its rows be written in the order 6, 3, 7, 4, 1, 5, 2, making a new square which we see to be the same as H with the two upper rows written below the others; and that a single square formed just like G for each other prime value of r above 3 becomes likewise pandiagonal by a similar commutation of rows. Like Frost, Frolow failed to extend the notion of symmetry beyond the bare remark that two complementary numbers lie opposite each other throughout.

26. The property of "step summation" explained in paragraph 4 was described fully by Barnard as pertaining to all "perfect," here called pandiag-

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* Quarterly Journal, VII, 97, dated by author August, 1864.

† Johnson's Cyclopædia, first edition, article "Magic Squares"; an able and comprehensive treatise. The preface of this volume bears date August, 1876; the title, 1877.

‡ "Le Problème d'Euler," St. Petersburg, 1884.

onal, squares. From his subsequently referring to the later part of his treatise as original, it may be presumed that this point was drawn from some earlier author. It was independently discovered and published in 1877 by Frost, and again published by him later in the article "Magic Squares" of the Encyclopædia Britannica. Unaware of its earlier discovery, he gave it the name of "nasical summation," from the village of Nasik in India, where he resided when first engaged upon this subject.

27. No uniform step square, whether by the knight's step or any other, appears to have been produced heretofore having the pandiagonal property. A symmetrical knight's path square was devised by Euler, for the case $x = 5$. It may be found in the Encyclopædia Britannica.

28. The proof in paragraph 14 of the independence of two paths is identical in principle with that suggested by Frost for the independence of the various "normal paths" existing within a square of given dimension. The criterion of paragraph 14, thus proved, may also be drawn from the analytic data of Barnard's treatise, which would have aided materially any one to whom it had occurred to experiment with uniform steps, though it gives no hint towards originating that idea.*

29. The "blocks of four" property, for $r > 4$, seems due to Benjamin Franklin, whose square of 16 is reprinted by Günther. It is not pandiagonal, nor does it follow any uniform law, but it is so ingeniously put together that every "block of four" without exception has the uniform sum 514. Three errors, obviously typographical, require correction.

30. The "blocks of four" property is to be found in an incomplete form in many known pandiagonal squares, that is to say, it holds good for many blocks of the square, but not universally. In one at least, namely, the "magic square of squares" set forth by Barnard, both properties are universal, but the square

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* This idea, in point of fact, occurred to the writer when he was examining, on page 209 of Günther's treatise, a square of 13 produced by the second rule of Moschopulus. He had, while unacquainted with the papers of Barnard and Frost, been working up all possible symmetrical pandiagonal squares of 5 by diophantine methods. A chance observation that the published square of 13 was what is here called "capable of symmetry" led to an investigation of this method of Moschopulus, and to its extension as now shown.

fails in what is here stated as the first property of a complete square, that any row or column or any number of rows or columns may be transposed without changing in any manner the properties of the square. The special square referred to is not arranged complementarily in the manner defined here to be essential.

31. But one complete square appears to have been published heretofore, and that unconsciously, the existence of the "blocks of four" property not being mentioned. It was given as a pandiagonal square by Frost in his paper of 1877. Determining its elements according to the present method, they appear, in their simplest form, to be: $r=8$, $a_1=1$, $a_2=7$, $a_3=3$, $a_4=5$, $s=17$, $u_1=8$, $p=16$, $u_2=8$.

32. Writers on magic squares have always recognized peculiar difficulties in producing pandiagonal squares when r is odd and divisible by 3, and have confessed its impossibility when r is even and not divisible by 4. The two methods now brought forward deal with all classes of cases in which general methods for producing pandiagonal squares are supposed to be possible, and introduce for the first time the summation of symmetrical groups as a main object together with the pandiagonal property. In addition to this element which characterizes both methods, that method which relates to odd squares introduces the further novel element of "uniform steps," with an easy rule for steps not uniform; and that which relates to even squares combines the further element, due separately to Franklin, which is needed for what is here called the "complete" square, a square produced at once by the simple process of the "figure of eight."