MATHEMATICS AND MAGIC SQUARES

8	3	4
1	5	9
6	7	2

4	9	2
3	5	7
8	1	6

6	1	8
7	5	3
2	9	4

2	7	6
9	5	1
4	3	8

1	2	3
4	5	6
7	8	9

2	9	4
7	5	3
6	1	8

8	1	6
3	5	7
4	9	2

4	3	8
9	5	1
2	7	6

6	7	2
1	5	9
8	3	4

THE ASSOCIATION OF
MATHEMATICS TEACHERS OF INDIA

A Book for Junior Mathematicians

MATHEMATICS AND MAGIC SQUARES

TALKS ON MATHEMATICAL THINKING ABOUT AND THROUGH THIRD ORDER MAGIC SQUARES

(Age Group 15-18)

P.K. SRINIVASAN

The Association of
Mathematics Teachers of India

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BETWEEN US

Dear Reader,

We are happy to place in your hands the latest publication of the AMTI for the year 2005-06, which of course is a reprint of an earlier publication authored by the late lamented Legendary, Sri P.K.Srinivasan.

Details about the author may be found elsewhere in the following pages, a perusal of which will inspire any creative thinker to emulate his example.

The subject of Magic Squares, which is fascinating in Itself, is made more fascinating by his lucid presentation and motivated learning experiences provided.

We offer this publication as a homage to the great Mathematics Education Consultant whose memory will be ever reverberating in the minds of prospective readers turned admirers.

It is my duty to place on record our thanks to Sri G. Narayanan for type setting the same, M/S M.K.Graphics and J.V.Printers for bringing out this book in record time.

We do look forward to your valuable encouragement and suggestions for our ongoing projects of this type.

With best wishes/kind regards,
Yours sincerely,

(M.MAHADEVAN) 13/1/'05
General Secretary

About The Author

Shri P.K.Srinivasan was born on November 4, 1924. After graduating from Loyola College, Chennai and obtaining a degree in Education from Teacher's College, Saidapet, Chennai, he joined the Muthialpet Higher Secondary School, Chennai from where he retired in 1981 as Head Master. Even while working as a teacher he also obtained a Masters Degree in Education. He had served as Fulbright Exchange teacher of Mathematics in the United States of America and also as a Senior Federal Education Officer and a Senior Lecturer in Mathematics in Nigeria. He was also a lecturer in NCERT Study Group during 1968-70. After retirement he had offered his services as Mathematics Education Consultant to a number of government agencies, both State and Central and also to many private and public educational institutions all over India. He had been regularly contributing to leading dailies like the Hindu and the Dinamani, popular articles of mathematical interest, which were widely appreciated by children, parents and general public. He had been regularly participating in the meetings of International Congress on Mathematics Education held once in four years.

Shri P.K.Srinivasan had great admiration for the mathematical prodigy Shri Srinivasa Ramanujan. He collected avidly and painstakingly a number of letters and manuscripts relating to Sri Ramanujan and brought out a book, ``A compilation of letters and reminiscences on Srinivasa Ramanujan", in two volumes. He was a member of the Ramanujan Museum Committee of Madras University. He was till recently the Curator of Ramanujan Museum and Mathematics Education Centre, Royapuram, Chennai.

Shri P.K.Srinivasan was the originator of conduct of Mathematics Expos and Fairs and interschool Mathematics Project expositions all over India and also abroad, in USA and Nigeria. He had been crusading for acceptance of Mathematics in cultural programmes. He fostered math-kit culture among young students and devised several practical experiments for mathematical laboratories in schools.

He was the recipient of several awards. During the Silver Jubilee Year, AMTI honoured him with the Best Innovative Teacher Award. In 1991 he was given the National Science Award for Best Effort in Popularizasation of Mathematics among children.

He had authored numerous enrichment books for children. Some of the popular titles are: Number Fun with a Calendar, Romping in Number Land, Math Quiz 300, Introduction to Creativity of Ramanujan, Instruction Guides for Teachers (Primary School, Middle School and High School levels), Mathetic Muse, TAK Numbers, Magic Squares and Development of Mathematical Thinking. Many of these books are AMTI publications.

Shri P.K.Srinivasan was a simple and lovable person imbibed with patriotism and national fervour. He always wore khadi and sported `Gandhi Cap' (even when he visited foreign countries). He was popular for his lucid way of presenting his ideas in a convincing, but at the same time, authoritative tone.

For almost six decades Shri P.K.S. had been ceaselessly striving for making mathematics interesting to children. That he left us on the 20th of June 2005 is yet an unbelievable event for his numerous admirers.

CONTENTS

WELCOME

I am sure you know what a magic square means. It is a square array of numbers so arranged that the *row sums*, the *column sums* and the *two diagonal sums are all equal.* The *common sum is* called the *sum* and the magic square is usually of the **additive** or **arithmetic** type.

The most familiar one as you know is made up of digits 1 to 9 and arranged in three rows of 3 digits each such that entries in each row, each column or each diagonal total to 15. You would have checked this to be true and learnt to build a magic square with any 9 consecutive numbers. Since you have not been provoked or encouraged to examine its structural properties, your interest would have flagged. But you don't know how much you have missed.

Many great mathematicians like Euler and Ramanujan have shown fascination for magic squares and the wonderful questions they raise. Magic squares have a hoary past with the ancient Chinese and Indians taking the credit for their development and popularisation. They were first used as talismans or media for instant astrological predictions. Magic squares form a big chunk in the lore of recreational mathematics made respectable by the contributions of pioneers like Dudeney, Sam Loyd and recently Martin Gardner. This area of recreational mathematics continues to have its unsolved problems challenging enough for mathematicians, even in this computer age.

The third order magic square, the magic square of any order for that matter, has exciting structural properties and poses interesting issues to be settled by mathematical reasoning or logic. Since the third order basic square,that is the one built with numbers 1 to 9, is **unique**, taking up mathematical questions arising from it provides a delightful avenue to learn and enjoy mathematical ways of thinking.

In building a magic square, there are, no doubt, **techniques** but mere knowledge of technique without understanding the mathematical questions that get raised is not at all illuminating and insightful and hence less rewarding.

We shall confine ourselves to the third order, that is 9 cell magic square and discuss the mathematics of it in such a manner you could acquire the know-how to study magic squares of higher orders, particularly the 4th, that comes next. From the magic square of the 4th order onwards the the structural ramifications and properties get wider and deeper. It is no wonder that even today one can come across some persons devoting their life time to unfolding the beauties of magic squares.

So you are welcome to flex your mathematical muscles in getting a better understanding of the magic in magic squares. To provide a sense of completeness and have a greater insight, not only will the additive or arithmetic magic squares with magic sums be considered, but also the multiplicative or geometric magic squares with magic products taken up as an exercise in appreciation of **duality** in mathematical thinking.

Towards the end a passage to group theory is made to fire your imagination and rouse your creativity.

Note: The contents of this book are based on the talks given by the author to students and teachers in the **NBHM** sponsored workshops conducted on 'Creativity in Mathematics Teaching' at Coimbatore, Bhubaneswar, Guwahati and Rajkot in 1987-88 to mark *Ramanujan Centenary Celebrations.*

Thanks go to Prof. M.S. Rangachari and Prof. G. Rangan of the Ramanujan Institute for Advanced Study in Mathematics, Madras, Prof. Phoolan Prasad of the Indian Institute of Science, Bangalore and Prof. K.R. Parthasarathy of the Indian Statistical Institute, Delhi for providing me incentive and encouragement in making this contribution of mine.

AUTHOR

PART-A

1.0 WARMING UP

1.0 A magic square, as everybody knows, is a square array usually of natural numbers so arranged or positioned as to get the totals along each row, each column and each diagonal the same and their common total is usually called the **magic sum**. No number is repeated. So it can also be explicitly called non-repeating magic square.

Since any magic square can be generated by multiplication and / or addition from a basic magic square made up of consecutive numbers starting with 1 and ending with a square number such as 9, 16, 25 etc., we shall first of all consider basic magic squares.

The first order basic magic square is very trivial as the magic sum is simply 1.

Consider the second order, 2 by 2 or 4 cell basic magic square formed with numbers 1 to 4

1	2
3	4

The diagonal total are 1+4=2+3=5. But the row totals and the column totals are all different from 5. Try as you may in positioning the digits in different ways, which are finite in number, you have to conclude that there is **no second order magic square,** unless all the entries are the same in which case it is the most trivial type of magic square violating the condition that no two entries could be the same.

1.1 The next move has to be to consider the square array formed with numbers 1 to 9.

1	2	3
4	5	6
7	8	9

As you strike the row totals, column totals and the diagonal totals, you notice the interesting situation of the diagonal totals the same as the middle row total and the middle column total.

$$\frac{3+5+7=1+5+9}{\text{diagonal totals}} = \frac{4+5+6}{\text{middle row total}} = \frac{2+5+8}{\text{middle column total}}$$

You connect fail to notice also that 1 & 9, 2 & 8, 3 & 7, 4 & 6 are complements of 10 and 5 is its own ten-complement.

Our curiosity eggs us on to alter the positions of the digits to see if a magic square gets formed. What happens if (2,5,8) and (4,5,6) are positioned diagonal wise and the empty cells suitably filled with the complement pairs (1 & 9) & (3 & 7).

Try and see:

4		2
	5	
8		6

4	9	2
3	5	7
8	1	6

Lo! The magic square does appear.
What about other arrangements?

It appears as though there can be eight arrangements as presented below:

4	9	2
3	5	7
8	1	6

8	3	4
1	5	9
6	7	2

6	1	8
7	5	3
2	9	4

2	7	6
9	5	1
4	3	8

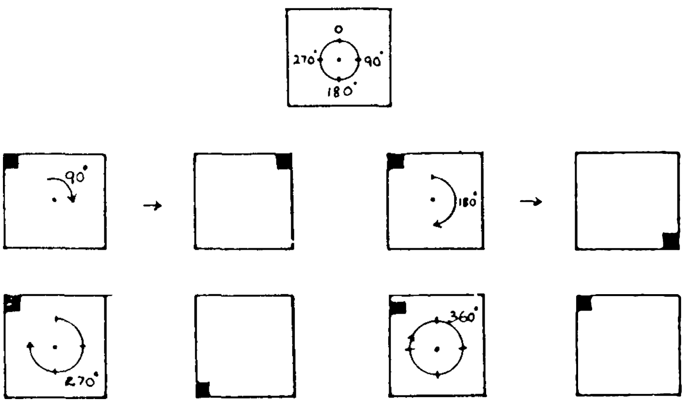
2	9	4
7	5	3
6	1	8

8	1	6
3	5	7
4	9	2

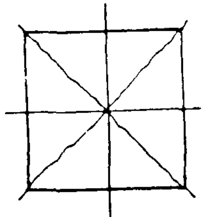
4	3	8
9	5	1
2	7	6

6	7	2
1	5	9
8	3	4

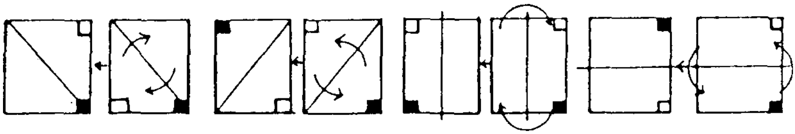
But given one of these, the rest can be obtained by quarter turn, half turn or three quarter turn, flipping about the middle row, middle column or either of the diagonals. Recall that the square has central or rotational symmetry and four axial or reflexive symmetries.



Central or rotational symmetries



Axial or reflexive symmetries



Flipping about the horizontal median

Flipping about the vertical medina

Flipping about the main diagonal

Flipping about the other diagonal

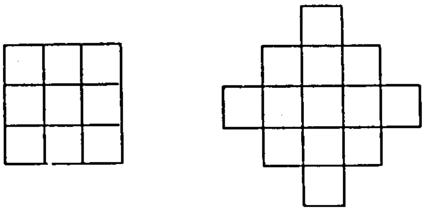
So these are not to be considered as eight different arrangements but eight variations of the same arrangement and hence the 3rd order basic magic square, any 3rd order magic square for that matter, is **unique**.

2.0 DISCOVERIES OF TECHNIQUES

2.1 It is natural to stumble upon clever techniques in building magic squares while observing the positional behaviour of entries in the cells. Some techniques are demanding and some are almost routine and simple. We shall begin with the most popular technique which any one can follow and build up a 9 cell magic square.

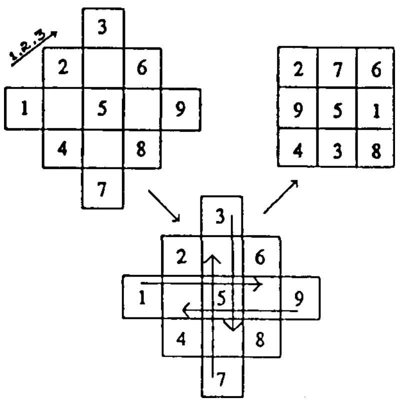
2.2 Technique 1

Take a serrated third order square as shown below:

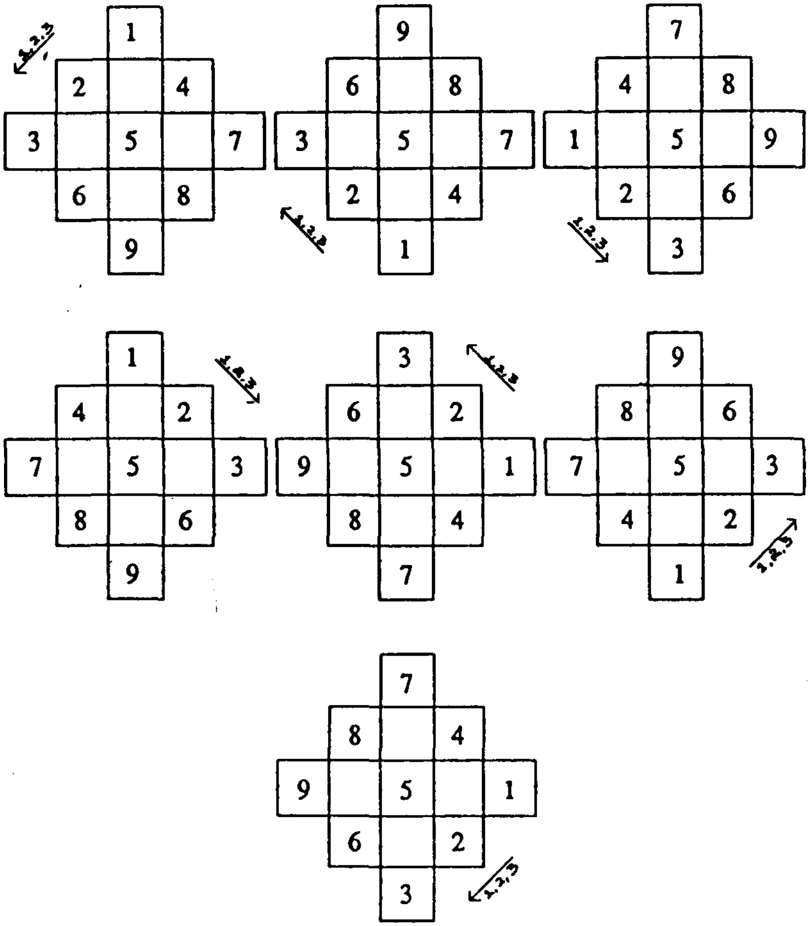


Square Serrated Square

Write the numbers 1 to 9 in order diagonally as shown below. Find out to which empty cell each of the numbers outside the square should be taken. It is easily seen that no number moves to the neighboring empty cell but to the opposite empty cell to get the magic square formed.



This diagonalwise marking of entries and their subsequent movements can be done in 7 other variations as shown below. It is left to you to complete the magic square in each variation.



Since there cannot be any more variations, you can say by virtue if the method of proof by exhaustion, that there could only be eight variations of the third order basic magic square and of any third order magic square for that matter.

You cannot fail to make some observations as mentioned below regarding the positional behaviour of numbers.

(1) 1 gets placed only in the mid cell of the top or bottom row, left or right column and not in any corner cell. Can you prove it by exhaustion?

(2) 5 which is the middle of the numbers
1,2,3,4,5,6,7,8,9
always occupies the central cell. Why?
The reason as you can see is that 5 its own complement and except the central cell, every other cell has a counter part along the column, row or diagonal. We shall also try to prove it generally later.

(3) The entries placed symmetrically with respect to the central cell are in A.P.

2.3 Technique 2

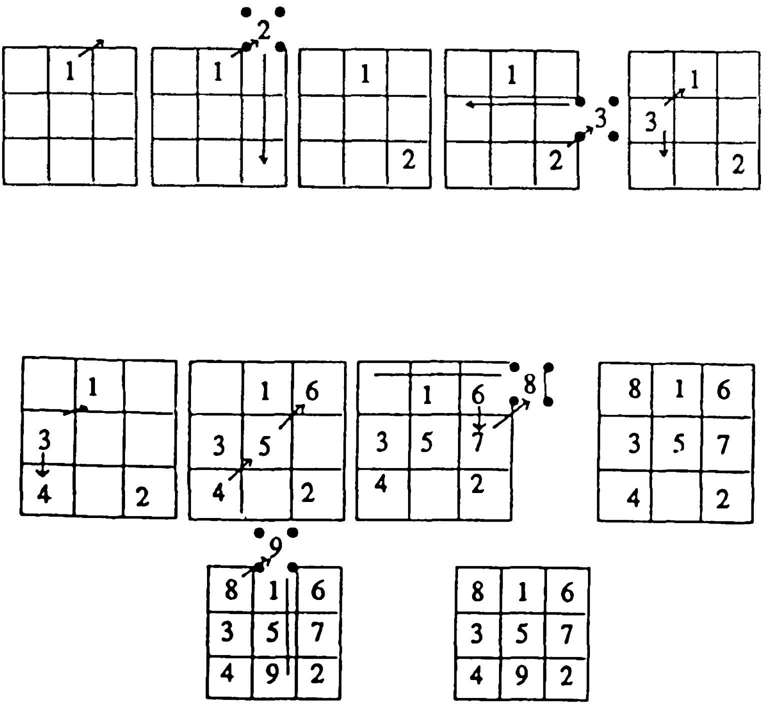
Since 4,5,6 get placed crosswise, what would happen if we place 1 in the middle cell of the top or bottom row, the left or right column and move crosswise? Notice that with 1 in the middle of the top row, a crosswise move or a cross-jump means one step to the right followed by one step up.

As you go about following this technique, you discover two situations:

(1) Whenever you have to go outside, the entry gets made in the end cell of the corresponding row or column.

(2) Whenever you get blocked by meeting with a cell already filled or facing the corner of a cell, the entry gets made in the neighboring cell located opposite to the side for crosswise movement.

See the illustrations below showing move by move.



For practice fix 1 in any other mid cell and proceed by this cross-jump method to fill up the cells to have the magic square formed.

1		

At this juncture if you can visualise the bottom row as lying next to the top row and the left column as lying next to the right column, outside squares need not be considered. (Imagine if need be the square wrapped round a cylinder to facilitate the above consideration.)

2.4 Technique 3

Examine

1	2	3
4	5	6
7	8	9

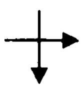
and write the second and third row entries as sums of 2 numbers involving 1,2 & 3, as shown below:

0+1 0+2 0+3
3+1 3+2 3+3
6+1 6+2 6+3

Now this can be given in the form of an addition table:

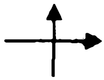
+	1	2	3
0	1	2	3
3	4	5	6
6	7	8	9

This suggests building up two **auxiliary** repeating type of magic squares of the same order with 2 sets of triads in A.P. or Arithmetic Progression and then forming a composite square by adding the entry in each cell of one latin square to corresponding cell of the other latin square. Of course the composite square should have to be a magic square. This requires care in constructing the two auxiliary squares. This repeating type of magic square is called a **latin square.**



	1	
1	2	3
	3	

3	1	2
1	2	3
2	3	1



	6	
0	3	6
	0	

3	6	0
0	3	6
6	0	3

Now adding the entries cell wise, we get the magic square

6	7	2
1	5	9
8	3	4

This approach reveals that with 2 sets of three numbers in Arithmetic Progression, 2 latin squares can be formed suitably so as to build the magic suare by cell-to-cell addition.

2.5 General Treatment

Instead of examining a number of particular cases, don't you feel mathematically inclined to dispose of the same questions about magic squares generally? If so, let us proceed as follows:

Let A, B, C and P, Q, R be the two sets of numbers such that $B - A = C - B = d$ or $A + C = 2B$ and $Q - P = R - Q = d'$ or $P + R = 2Q$. Better start with the addition table formed by A, B, C and P, Q, R.

$+$	A	B	C
P	$A+P$	$B+P$	$C+P$
Q	$A+Q$	$B+Q$	$C+Q$
R	$A+R$	$B+R$	$C+R$

Take the sequence of the nine entries:
$A+P,\ B+P,\ C+P,\ A+Q,\ B+Q,\ C+Q,\ A+R,\ B+R$

and $C + R$, and form a third order magic square, using any of the techniques seen so far. Now you get

$B+R$	$A+P$	$C+Q$
$C+P$	$B+Q$	$A+R$
$A+Q$	$C+R$	$B+P$

Can you prove that this is a magic square with the hypothesis that A, B, C are in A.P. and P, Q, R also in A.P.?

The magic sum is obviously

$$A+B+C+P+Q+R$$

Each of the row totals and column totals is seen to be equal to the magic sum.

What remains to be proved is that each diagonal total is also equal to the magic sum. Now the two diagonal totals are:

$$A+B+C+3Q \text{ and } 3B+P+Q+R$$

We have to prove that

$$3B+P+Q+R = A+B+C+P+Q+R$$

Since $2Q = P + R$ (P, Q, R are in A.P) $3Q = P + Q + R$

$$A+B+C+3Q = A+B+C+P+Q+R$$

Similarly the other diagonal sum can also be shown to be equal to the magic sum.

What is the converse of this theorem?

If the entries $A + P$, $B + P$, $C + P$, $A + Q$, $B + Q$,

$C+Q$, $A+R$, $B+R$, $C+R$, form a magic square, then the two triads, one of A, B, C and the other of P, Q, R are each in Arithmetic Progression.

As before we find that each of the row totals and the column totals is $A+B+C+P+Q+R$. Since this is a magic square

$$A+B+C+3Q = A+B+C+P+Q+R$$

$$2Q = P+R$$

That is $P, Q \& R$ are in A.P. Similarly $2B = A+C$ showing A, B and C are in A.P.

Incidentally, you can see that the entries need not be consecutive as now

(i) $B-A$ and $C-B$ need not be equal to 1 and $P=0$, $Q=3$ and $R=6$.

(ii) Only when $B=A+1$, $C=B+1$, $P=0$, $Q=3$, $R=6$, the nine entries will be consecutive; see the table given below:

+	A	$A+1$	$A+2$
0	$A+0$	$A+1$	$A+2$
3	$A+3$	$A+4$	$A+5$
6	$A+6$	$A+7$	$A+8$

When $A=1$, you get the basic magic square.

(iii) Also, if $P \neq 0, Q \neq 3, R \neq 6$ then rowwise the entries will be contiguous.

+	A	$A+1$	$A+2$
P	$A+P$	$A+P+1$	$A+P+2$
Q	$A+Q$	$A+Q+1$	$A+Q+2$
R	$A+R$	$A+R+1$	$A+R+2$

As an interesting example, consider the 3rd order square entries on a calendar sheet and you can always build a magic square with the entries.

2	3	4
9	10	11
16	17	18

9 cell entries on a calendar sheet

17	2	11
4	10	16
9	18	3

built into a magic square.

The structure can be seen, if it is presented in the form of an addition table:

+	2	2+1	2+2
0	2+0	2+0+1	2+0+2
7	2+7	2+7+1	2+7+2
14	2+14	2+14+1	2+14+2

2.6 There is a general question to be answered.

If

a	b	c
d	e	f
g	h	i

forms a magic square with the magic sum $3m$, then

$al+k$	$bl+k$	$cl+k$
$dl+k$	$el+k$	$fl+k$
$gl+k$	$hl+k$	$il+k$

can be written as

$$\begin{array}{|ccc|}\hline a & b & c \\ d & e & f \\ g & h & i \\ \hline\end{array} + \begin{array}{|ccc|}\hline a & b & c \\ d & e & f \\ g & h & i \\ \hline\end{array} + \begin{array}{|ccc|}\hline a & b & c \\ d & e & f \\ g & h & i \\ \hline\end{array} + \ldots l \text{ times}$$

added to

$$\begin{array}{|ccc|}\hline k & k & k \\ k & k & k \\ k & k & k \\ \hline\end{array}$$

the trivial form of a magic square, the proposition gets proved.

What is the magic sum? It is $3(lm + k)$.

This shows that with one magic square, any number of magic squares can be generated.

2.7 Problem: Prove that

$(a+k)l$	$(b+k)l$	$(c+k)l$
$(d+k)l$	$(e+k)l$	$(f+k)l$
$(g+k)l$	$(h+k)l$	$(i+k)l$

is a magic square, given

a	b	c
d	e	f
g	h	i

is a magic square.

3.0 SOME THEOREMS ABOUT ENTRIES

3.1 Central cell entry

If $a, b, c, d, e, f, g, h, i$, with $a < b < c < d < e < f < g < h < i$, or the other way, are the nine entries in a magic square with the magic sum m, it can be shown that e is the central cell entry. Given that

a	b	c
d	e	f
g	h	i

is a magic square with the magic sum m,

$$\begin{array}{ll} a+b+c=m, & a+e+i=m \\ a+d+g=m, & b+e+h=m \\ c+f+i=m, & c+e+g=m \\ g+h+i=m, & d+e+f=m \\ \text{(Set 1)} & \text{(Set 2)} \end{array}$$

By addition in Set 2,

$$a+b+c+d+e+f+g+h+i+3e=4m$$
$$\Rightarrow \quad m+m+m+3e=4m \text{ (from Set 1)}$$
$$\Rightarrow \quad e=\frac{m}{3}$$

Since,

$$\frac{a+e+i}{3}=\frac{m}{3}=e \text{ etc.}$$

e is the mean of each triad and thus is the central cell entry.

3.2 Some more interesting properties can also be seen. Since $a+e+i=3e$, $a+i=2e$ and so a,e,i and in A.P. Similarly b,e,h; c,e,g& d,e,f are in A.P. Also

$$a+d=h+i, d+g=b+c, g+h=c+f, f+i=a+b$$

3.3 The cell for the least entry:

First of all, show that the least number of the sequence 1,2,3,4,5,6,7,8 & 9 cannot occupy any position except the midcell of the border row or column. Without loss of generality, it is enough if the mid cell of top row alone is considered, since the third order magic square is unique by virtue of invariance of triads through rotation and reflection. If the least number 1 does not occupy the mid cell, it should occupy either of the corner cells of the top row. It is enough to show that it cannot occupy the left corner cell or the right corner cell.

Suppose it occupies the left corner cell. Since it has been shown that 5 has to be in the central cell, two cells are seen occupied. Now, the bottom right corner cell and the top left corner cell are symmetrically placed about the central cell. Every pair of such cells are occupied by additive complements of 10. Since 9 is the additive complement of 1,9 goes into the bottom right corner cell. The magic sum has to be 15.

1		
	5	
		?

1		
	5	
		9

Consider the possible entries in the top mid cell. It cannot be 2,3 or 4 as the entry in the top row right corner will then exceed 9 in order to get the magic sum 15. Suppose it is 6. Then the top row corner cell will get filled up by 8 and the bottom row mid cell by 4. But the sum of the two entries in the right end column exceeds 15. So 6 gets disallowed. On similar grounds 7 and 8 get disallowed. When it is 7, it gets repeated also. All the possibilities have been exhausted. So 1 cannot go into a corner cell.

1	2 3 4	?
	5	
		9

1	~~2~~ ~~3~~ ~~4~~	~~12~~ ~~11~~ ~~10~~
	5	
		9

1	6	
	5	
		9

1	6	8
	5	
	4	9

1	7	7
	5	
	4	9

3.4 A problem set

1. If A, B, C form an A.P., show that $As + t, Bs + t, Cs + t$ form an A.P.

2. Show that

$m - p$	$m + p - q$	$m + q$
$m + p + q$	m	$m - p - q$
$m - q$	$m - p + q$	$m + p$

forms a magic square with the magic sum $3m$. Write the entries as a sequence of numbers from the least to the greatest. Give this sequence in the form of an addition table. For what values of m, p, q can you get the magic square?

8	1	6
3	5	7
4	9	2

3. Find the condition for the entries in the two addition tables

+	$10x$	$10y$	$10z$
a			
b			
c			

+	$10a$	$10b$	$10c$
x			
y			
z			

to give magic squares with the same magic sum, assuming that the two triads a, b, c and x, y, z are each in A.P. Construct two such examples of magic squares with numbers. List all possible values of (x, y, z) and (a,b,c).

4. Show that if $a, b, c, d, e, f, g, h, i$ with $a < b < c < d < e < f < g < h < i$ form a magic square, the least entry a cannot occupy a corner cell.

4.0 A QUEST FOR MORE MAGICAL PROPERTIES

Suppose the magic sum is to be zero, how can the magic square be formed?

4.1 Start with the magic square

8	1	6
3	5	7
4	9	2

Subtract 5 from each entry and since it is to be subtracted from three entries in each row, column or diagonal, the magic sum becomes zero. But negative numbers appear. You can see the exciting role played by negative numbers in revealing the structure of the magic square.

3	-4	1
-2	0	2
-1	4	-3

This may be called the Skeleton Magic Square. You cannot fail to notice that the end entries about the central cell along any row, column or diagonal are **opposites** of each other. The absolute values of entries in the top row and the bottom row, the left column and the right column are the same, showing up the hidden magical property that not only the top row total and the bottom row total are the same but also **totals got by adding the squares of the entries in them are the same.**

4.2 This skeleton magic square provokes us to view the general form of the magic square in another way as shown below presented in two stages.

p	$-q$	r
$-s$	0	s
$-r$	q	$-p$

Stage 1

$x+p$	$x-q$	$x+r$
$x-s$	$x+0$	$x+s$
$x-r$	$x+q$	$x-p$

Stage 2

Can you find the condition that p, q, r, s should satisfy for the square to be zero magic?

4.3 Problem set

1. Rebuild the magic square with q and s written in terms of p and r.

2. Show that
$(x+p)^2+(x-q)^2+(x+r)^2 = (x-r)^2+(x+q)^2+(x-p)^2$
given that $p + r = q$.

3. Show that

$(x+p)^2+(x-s)^2+(x-r)^2 = (x+r)^2+(x+s)^2+(x-p)^2$
given that $r - p = -s$.

4. Show that $(x-r)^2 + x^2 + (x+r)^2 \neq (x+p)^2 + x^2 + (x-p)^2$.

5. Show that the sum of the squares of the entries either in the top row or the bottom row is $3m^2 + 26$ where m is the entry in the central cell.

6. Show that the sum of the squares of the entries either in the left column or the right column is $3m^2 + 14$ with the usual meaning for m.

7. If the common difference of nine entries to form a magic square is k, show that the sum of the squares of the entries either in the top row or the bottom row is $3m^2 + 26k^2$ and the sum of the squares of the entries either in the left column or the right column is $3m^2 + 14k^2$.

8. Consider the magic square

$b+r$	$a+p$	$c+q$
$c+p$	$b+q$	$a+r$
$a+q$	$c+r$	$b+p$

Where $a + c = 2b$ and $p + r = 2q$.

Show that

$$\begin{aligned} & (b+r)^2 + (a+p)^2(c+q)^2 \\ = \; & (a+q)^2 + (c+r)^2 + (b+p)^2 \end{aligned}$$

and

$$\begin{aligned} & (b+r)^2 + (c+p)^2 + (a+q)^2 \\ = \; & (c+q)^2 + (a+r)^2 + (b+p)^2 \end{aligned}$$

4.4 Projects:

1. Find if an expression can be formed for the sum of the squares of entries in the top row or bottom row, in terms of m where $m = b+q, t$ the common difference of a, b, c and s the common difference of p, q, r.

2. Examine if there could be a similar expression for the sum of squares of entries in the left column or right column.

5.0 MULTIPLICATIVE MAGIC SQUARE

So far you have acquired the experience to handle third order additive or arithmetic magic squares and get their properties. It is natural therefore to look for the experience of handling third order multiplicative or geometric magic squares and getting their properties.

5.1 Multiplicative law of indices suggests a way out immediately. Treat the entries of the additive magic square as the indices for a non-zero base and easily the multiplicative magic square is formed.

8	1	6
3	5	7
4	9	2

Additive magic square with the magic sum 15

a^8	a^1	a^6
a^3	a^5	a^7
a^4	a^9	a^2

multiplicative magic square with the magic product a^{15}

5.2 Consider the geometric magic square formed with $a=2$.

2^8	2^1	2^6
2^3	2^5	2^7
2^4	2^9	2^2

Fig. *A*

256	2	64
8	32	128
16	512	4

Fig. *B*

Do you know what number plays a role in multiplication akin to the role played by 0 in addition? It is 1. You have seen how to build an arithmetic magic square with 0 as th magic sum, using **integers.** You would naturally be interested in building a geometric magic square with 1 as the magic product, using **fractions.**

Divide the central cell entry in fig. *A* by 2^5 and in fig. *B* by 32.

What do you get? Watch the duality.

2^3	2^{-4}	2^1
2^{-2}	2^0	2^2
2^{-4}	2^4	2^{-3}

8	$\frac{1}{16}$	2
$\frac{1}{4}$	1	4
$\frac{1}{2}$	16	$\frac{1}{8}$

Where you get opposites in an arithmetic magic square, you get reciprocals in a geometric magic square. Where you get the additive identity 0 in an arithmetic magic square, you get the multiplicative identity 1 in a geometric magic square. When 0 is the magic sum, 1 is the magic product.

So the duality consists in replacing + by $\times$, $-$ by $\div$ and $\times$ by raising to power. This would help you in stating corresponding theorems in respect of geometric magic squares and proving them without effort, almost mechanically.

Now you can straight away state the corresponding theorems in respect of the geometric magic square. For convenience, the theorems pertaining to the arithmetic magic square are repeated for you to give their duals relating to the geometric magic square. Proving is left to you.

Do you like to have the geometric magic square also starting with 1 in the top middle cell? Divide each entry of fig. *B* by 2.

128	1	32
4	16	64
8	256	2

Here is that geometric magic square. The magic product is 4096. The pair of entries in the cells symmetrically placed with respect to 16 are multiplicative complements of 16^2.

Notice that where as 3 times the entry in the central cell in the arithmetic magic square is taken to get the magic sum, the magic product becomes the cube of the entry in the central cell. The symmetrically positioned cell entry pairs are additive complements of twice the entry in the central cell in arithmetic magic square whereas they are multiplicative complements of the square of the entry in the central cell.

$+$	A	B	C
P	$A+P$	$B+P$	$C+P$
Q	$A+Q$	$B+Q$	$C+Q$
R	$A+R$	$B+R$	$C+R$

$A+C=2B;\ P+R=2Q$

Arthmetic array

$\times$	A	B	C
P	AP	BP	CP
Q	AQ	BQ	CQ
R	AR	BR	CR

$AC=B^2,\ PR=Q^2$

Geonetric array

1) Prove that the above 9 entries can be used to form am arithmetic magic square with the magic sum $A+B+C+P+Q+R$.

Its converse has also been proved.

Prove that the above 9 entries can be used to form a geometric magic square with the magic product $ABCDPQR$.

Prove also its converse.

2)

+	$A+0$	$A+1$	$A+2$
0	$A+0$	$A+1$	$A+2$
3	$A+3$	$A+4$	$A+5$
6	$A+6$	$A+7$	$A+8$

Prove that the above 9 entices can from a magic square with the magic sum $3A+12$ or $3(A+4)$.

$\times$	A^0	A^1	A^2
1	A^0	A^1	A^2
3	$3A^0$	$3A^1$	$3A^2$
9	$9A^0$	$9A^1$	$9A^2$

Prove that the above 9 entries can be used to form a geometric magic square with the magic product $27A^3$ or $A^3 \times 27$ or $(a \times 3)^3$

3)

$+q$	$-p$	$+r$
$-s$	0	$+s$
$-r$	$+p$	$-q$

Prove that this is a **zero** magic sum magic square where $p = q + r$ and $q = s + r$

q	$\frac{1}{p}$	r
$\frac{1}{s}$	1	s
$\frac{1}{r}$	p	$\frac{1}{q}$

Prove that the above 9 entries form a **one** magic product magic square where $p = qr$ and q=sr

4) If

a	b	c
d	e	f
g	h	i

is an arithmetic magic square with the sum $3m$, then

$al+k$	$bl+k$	$cl+k$
$dl+k$	$el+k$	$fl+k$
$gl+k$	$hl+k$	$il+k$

forms also an arithmetic magic square with the magic sum $3(lm+k)$.

If

a	b	c
d	e	f
g	h	i

is a geometric magic square with the magic product m^3, then

ka^1	kb^1	kc^1
kd^1	ke^1	kf^1
kg^1	kh^1	ki^1

forms a geometric square with the magic product $(km^1)^3$.

5) If A, B, C form an A.P. it is prove that $As+t$, $Bs+t$, $Cs+t$ also form an A.P.

If A, B, C form a G.P., prove that tA^s, tB^s, tC^s form a G.P.

6) Prove that

$m-p$	$m+p-q$	$m+q$
$m+p+q$	m	$m-p-q$
$m-q$	$m-p+q$	$m+p$

forms an arithmetic magic square with the magic sum $3m$.

Prove that

$\frac{m}{p}$	$\frac{mp}{q}$	mq
mpq	m	$\frac{m}{pq}$
$\frac{m}{p}$	$\frac{mq}{p}$	mp

forms a geometic magic square with the magic product m^3.

6.0 BUILDING THE MAGIC SQUARE WITH INCOMPLETE ENTRIES

6.1 You have seen that of the six data that determine a triangle, three if properly chosen are enough to build the triangle. The sides should satisfy the condition that the total length of any two of them should be greater than the third and the three data cannot be the three angles as the third angle can be determined, once any two angles are given, the sum of the three angles of a triangle being 2 right angles.

Likewise you can study the minimum number of cells to be filled to determine or complete a magic square.

6.2 Given the entries in any row or column, the magic square can be completed. If the entries are to be natural numbers, the sum of the entries should be a multiple of 3. Now complete the magic square with entries given in the first row, say a, b and c.

Step 1

a	b	c

Since this is a magic square, the row sum is the magic sum. Let it be 3m

Step 2

a	b	c
	m	

Since thrice the central cell entry is $3m$, the magic sum, m goes into the central cell

Step 3

a	b	c
	m	
$2m-c$		$2m-a$

Now the bottom corner cell entries can be made as they are along 3 cell diagonals, each with 2 cells already filled up.

Step 4

a	b	c
$m-a+c$	m	$m-c+a$
$2m-c$	$2m-b$	$2m-a$

Now the remaining cells in the border columns and bottom row can easily be filled up.

It is easily seen the bottom row sums up to $3m$.
$2m-c+2m-b+2m-a=6m-(a+b+c)=6m-3m=3m$
Note that the solution is unique, as the entries involve only $a+b+c$ and a,b,c which are given.

6.3 Given the entries along a diagonal, the magic square can be completed.

If this were to be a magic square with natural number entries, then $a+c=2b$. In other words a,b,c are in A.P.

a		
	b	
		c

Note that a comer cell entry appears in three sums. So the corner cells can be called VIP cells. So filling them gets precedence.

Step 2

a		$b-d$
	b	
$b+d$		c

Since the entries in the diagonal have to be in A.P., put $b-d$ in the right corner cell of top row and $b+d$ in the left corner cell of the bottom row, such that $d < b$.

Now the rest of the cells can be easily filled with the magic sum in view.

Step 3

a	$2b-a+b$	$b+d$
$2b-a-d$	b	$2b+d-c$
$b+d$	$2b-c-d$	c

Check the row and column sums, as the enties in either of two diagonals sum up to the magic sum 3b.

The sum of the entries in the middle row or column is $5b-(a+c) = 5b-2b = 3b$. Hence this is a magic square.

6.4 Given the central cell entry, and a corner cell entry, the magic square can be formed.

This is a refinement of § 6.3. Since the entries in any diagonal cell have to be in A.P. the filling can be done as in § 6.3, once the bottom row right corner cell is filled up as shown overleaf:

Step 1

a		
	b	

(i)

Step 2

a		
	b	
		$2b - a$

(ii)

common difference:

$a < b$

Step 3

a		$b - d$
	b	
$b + d$		$2b - a$

(iii)

$b - a$

Step 4

a	$2b - a + d$	$b - d$
	b	
$b + d$	$a - d$	$2b - a$

(iv)

Step 5

a	$2b - a + d$	$b - d$
	b	$a + d$
$b + d$	$a - d$	$2b - a$

(v)

$d < b$

Step 6

a	$2b - a + d$	$b - d$
$2b - a - d$	b	$a + d$
$b + d$	$a - d$	$2b - a$

(vi)

6.5 Given the entries in middle row or middle column, the magic square can be completed.

As before the diagonal cells are to be taken care of. Besides, the entries along any diagonal have to be in A.P. Since the entries can be made in different ways, **the solution cannot be unique.** Completing a magic square in steps is shown below. Checking up the sums is also shown below:

Step 1

	a	
	b	
	c	

Step 2

$b-e$	a	$b-d$
	b	
$b+d$	c	$b-e$

Step 3

$b-e$	a	$b-d$
$b-d+e$	b	$b+d-e$
$b+d$	c	$b+e$

$$e < d < b; b + d + e = a; a + c = 2b$$

6.6 Given the central cell entry alone, the magic square can be formed. This is a refinement of § 6.3 taken to a higher stage. Since the diagonal entries have to be in A.P. and the magic sum thrice the entry in the central cell, the completion of a magic square is easily done. But the magic square will **not be unique.**

Completing the magic square in stages is shown below. Checking up the sum poses no difficulty.

Step 1

	a	

Step 2

$a-e$		$a-d$
	a	
$a+d$		$a+e$

Step 3

$a-e$	$a+e+d$	$a-d$
$a-d+e$	a	$a+d-e$
$a+d$	$a-e-d$	$a+e$

$$e, d < a$$

So the minimum number of entries to be given to complete a magic square is 1, provided the entry is in the central cell.

6.7 Problem set:

1) Complete the magic squares:

(i)

5		
8		
17		

(ii)

12		
	16	

(iii)

	11	

2) Build a magic square with the nine numbers:
1,7,13; 31,37,43; 61, 67 and 73.

6.8 Projects

1. Examine if a magic square can be formed given the corner cell entries in a border row (or column) and the entry in the mid cell in the opposite border row (or column).

   Complete the magic square below if you can.

7		12
	10	

2. A number which is a multiple of 3 is given to be the magic sum. Assuming it to be $3m$, show that the least entry is $\frac{3m-12}{3}$, if the entries are to be consecutive. Show also how to build the magic square of the third order to have the given number as the magic sum.

3. Make a study similar to projects 1&2 in respect of the geometric magic square of the third order.

4. Complete the magic square arithmetically, given the four corner cell entries.

5. Find the values that *the magic sum can take in an* arithmetic magic square, when the entries are from different number systems.

6. Find the values that the magic product can take in a geometric magic square, when the entries are from different number systems.

7.0 IRREGULAR MAGIC SQUARES

An **anti magic square,** as you can expect, should have all its row totals, column totals and the diagonal totals **different**.

It is interesting to discover that the solution for building an anti magic square is **not unique.**

7.1 Here are a few models of anti-magic square built with numbers 1 to 9 for your study.

7	8	9	— 24
6	1	2	— 9
5	4	3	— 12

15 18 13 14 11

1	5	2	— 8
7	8	9	— 24
4	3	6	— 13

14 12 16 17 15

The first one is of the spiraling type.

7.2 If more than one row, column or diagonal but not all have their entry totals the same, **semi magic squares** are formed. A few models of semi magic squares built with numbers 1 to 9 are given below:

1	6	4	— 11
8	2	9	— 19
5	7	3	— 15

11 14 15 16 6

1	2	3	— 6
4	5	6	— 15
7	8	9	— 24

15 12 15 18 15

5	1	6	— 12
8	4	9	— 21
2	7	3	— 12

12 15 12 18 12

7.3 Projects

Anti magic squares and semi magic squares are exciting areas for investigation for getting an excellent exposure to mathematical ways of thinking. You can restrict your investigation to squares of the arithmetic type if you so desire.

1. Study the conditions for the row sums, the column sums and the diagonal sums to be all different.

2. Study the conditions for the required number of row sums, the column sums and the diagonal sums to be the same, while the rest are different.

8.0 MATHEMATICAL BY-PRODUCTS FROM MAGIC SQUARES

Surprisingly enough, mathematical ideas developed in a certain context get applied in solving some problems which appear to be intractable in other areas in mathematics.

The third order magic square has a fascinating application in some problems of number theory.

8.1 Consider the problem of finding six **square numbers such that the sum of three of them is equal to the sum of the rest.** Any third degree magic square gives the solution **instantly.** But the solution can never be unique, as it relates to an indeterminate equation of the second degree in six unknowns. An indeterminate equation has no unique solution or solutions. The number of solutions is not finite either, unless some conditions are imposed. In symbols, the problem can be presented as follows:

8.2 Find a, b, c, p, q, r such that $a^1 + b^2 + c^2 = p^2 + q^2 + r^2$.

Solution: Take the basic third order magic square itself. This gives $a = 8$, $b = 1$, $c = 6$ and $p = 4$, $q = 9$, $r = 2$.

For more solutions through magic squares related to this

8	1	6
3	5	7
4	9	2

basic one see § 4.2 and § 4.3, Problem 2. For more solutions you can take $a = 8n$, $b = n$, $c = 6n$ and $p = 4n$, $q = 9n$, $r = 2n$, $n \in N$.

For more general solutions, you can take

$a = 8n + k$ or $n(8 + k)$ $\qquad p = 4n + k$ or $n(4 + k)$
$b = n + k$ or $n(1 + k)$ $\qquad q = 9n + k$ or $n(9 + k)$
$c = 6n + k$ or $n(6 + k)$ $\qquad r = 2n + k$ or $n(2 + k)$

where $n, k \in N$.

8.3 PROBLEM SET

1. Prove
$$(8n + k)^2 + (n + k)^2 + (6n + k)^2 = (4n + k)^2 + (9n + k)^2 + (2n + k)^2$$
and give six solutions for the indeterminate equation:
$$a^2 + b^2 + c^2 = p^2 + q^2 + r^2$$

2. Show that
$$(x+5)^2+(x-2)^2+(x+3)^2 = (x+1)^2+(x+6)^2+(x-1)^2$$
and give the identity in more general form.

3. Show that
$$(x+5)^2+x^2+(x+1)^2 = (x+3)^2+(x+4)^2+(x-1)^2$$
is also an identity. Use it to solve the indeterminate equation.
$$a^2 + b^2 + c^2 = p^2 + q^2 + r^2$$

4. For what values of $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$, for $0 < a_1 < 10 (i = 1, 2, 3)$ is
$$(100a_1 + 10b_1 + c_1) + (100a_2 + 10b_2 + c_2) + (100a_3 + 10b_3 + c_3) = (100c_1 + 10b_1 + a_1) + (100c_2 + 10b_2 + a_2) + (100c_3 + 10b_3 + a_3) \ ?$$

PART - B

9.0 PREDICTING THE TOP ROW ENTRIES

In section 2, you have seen 8 variations of the basic third order magic square and how they are related to the rotational and reflective symmetries of a square. This provides you an opening to enjoy a prediction game with your friends through compositions of rotation and reflection. Give your friend a square card board showing the magic square on the right.

Since the top row entries determine the magic square, we shall mention only the top row entries, and represent each position, got by turning and flipping, by means of a letter as detailed below:

8	1	6
3	5	7
4	9	2

Rotating clockwise positions:

8	1	6
3	5	7
4	9	2

i

8	**1**	**6**

4	3	8
9	5	1
2	7	6

a

4	**3**	**8**

obtained by turning through a right angle (or a quarter turn clockwise)

2	9	4
7	5	3
6	1	8

b

2	**9**	**4**

the **b** position obtained through turning further through a right angle clockwise, that is, turning on the whole through 2 right angles (or a half turn) clockwise.

6	7	2
1	5	9
8	3	4

c

6	**7**	**2**

the **c** position obtained through turning still further through a right angle clockwise, that is, turning on the whole through 3 right angles (or three quarter turn) clockwise.

Notice that when the square is still further more turned through a right angle clockwise, that is, turning, on the whole, through 4 right angles (or full turn) clockwise, the original position showing

i

8	**1**	**6**

or **i** is got.

Flipping positions:

8	1	6
3	5	7
4	9	2

Flipping the basic magic square **i** position about the middle column giving

6	1	8
7	5	3
2	9	4

p

6	**1**	**8**

So the **p** position is obtained through flipping the entries of the basic magic square taken as **i** about the **vertical axis of symmetry**; that is, interchange of columns 1 & 3 from the left.

Flipping the basic magic square **i** position about the middle row gives

8	1	6
3	5	7
4	9	2

4	9	2
3	5	7
8	1	6

q

4	**9**	**2**

So that **q** position is obtained through flipping the entries of the basic magic square taken as **i** about the **horizontal axis of symmetry,** that is, interchange of rows 1 & 3 from the top.

Now flipping the basic magic square **i** about the main diagonal gives

8	1	6
3	5	7
4	9	2

2	7	6
9	5	1
4	3	8

r

2	**7**	**6**

So the **r** position is obtained through flipping the entries of the basic magic square taken as **i** about the **main diagonal** along the left bottom corner to the right top corner.

Again flipping the basic magic square **i** about the other diagonal gives

8	1	6
3	5	7
4	9	2

8	3	4
1	5	9
6	7	2

s

8	**3**	**4**

So the **s** position is obtained through flipping the entries of the basic magic square taken as **i** about the **other**

diagonal along the left top corner to the right bottom corner.

Prepare a set of magic square cards showing the eight variations. Since one rotation or flipping can be followed by another rotation or flipping, it will help to use the cards and build a table of compositions of two movements for playing the prediction game. While reading the tables, take an entry for movement in the left marginal column first and then an entry in the top marginal row next. Make your own table and tally the entries given here with yours.

Some examples:

1. Taking the position obtained by turning the original position **i** through three quarters of a turn and then flipping it about the vertical axis of symmetry i.e. **cp** gives the position that would be obtained by flipping the original position about the main diagonal **i.e. cp=r**.

2. **c** (in the left marginal column) followed by **p** (in top marginal row) is not **r.** Note **pc** $\neq$ **r** but s. Hence the order gets stressed. **The composition is not commutative always.**

Second

POSITION TABLE

First move followed by second move		no turn or a clockwise full turn i	clockwise quarter turn a	clockwise half turn b	clockwise three quarter turn c	flipping about the vertical column of symmetry p	flipping about the horizontal row of symmetry q	flipping about the main diagonal (left bottom to right top) r	flipping about the other diagonal (left top to right bottom) s
no turn or clock wise full turn	i	i	a	b	c	p	q	r	s
clockwise quarter turn	a	a	b	c	i	s	r	p	q
clockwise half turn	b	b	c	i	a	q	p	s	r
clockwise three quarter turn	c	c	i	a	b	r	s	q	p
flipping about the vertical axis/ column of symmetry	p	p	r	q	s	i	b	a	c
flipping about the horizontal axis/row of symmetry	q	q	s	p	r	b	i	c	a
flipping about the main diagonal (left bottom to right top)	r	r	q	s	p	c	a	i	b
flipping about the other diagonal (left top to right bottom)	s	s	p	r	q	a	c	b	i

For the prediction about the finial position use the table given below:

i	a	b	c	p	q	r	s
8 | 1 | 6	4 | 3 | 8	2 | 9 | 4	6 | 7 | 2	6 | 1 | 8	4 | 9 | 2	2 | 7 | 6	8 | 3 | 4

A model game is presented to get started.

Ask your friend to make the following movement in order.

Start with | 4 | 3 | 8 | in a position.

Your friend's moves on Your instructions	**Your simultaneous calculations using the table.**
Rotate it clockwise half turn	**ab = c**
Flip it next along the main diagonal	**cr = q**
Rotate it next clockwise quarter turn	**qa = s**
Flip it next about the horizontal and stop	**sq = c**

Now you can tell him the top row entries.

It is **c** or | 6 | 7 | 2 |

10.0 AN ELEMENTARY INTRODUCTION TO GROUP THEORY

We now attempt a mathematical connection between the magic square of the third order and the abstract concept of a finite group.

The game provides the backdrop for exposure to an elementary introduction to the theory of groups, a model of mathematical excellence in axiomatic thinking. It is dealt with here because of its extensive applications whenever symmetry is an underlying factor such as crystallography, particle physics, fields of a force, quantum mechanics, besides its natural application in algebra and geometry and art in particular. While resorting to group theoretic approach, physicists found themselves compelled to predict the existence of an unobserved particle and even give its characteristics. This happened in the sixties. What wonder it was when the existence of the particle and its property were confirmed by later experiments! Incidentally, it will be realised how the prediction game can be played without the table, once certain transformations are remembered.

The group as a mathematical entity was identified and recognised in the eighteenth century while attempting to find the general solution for the polynomial equation of the 5th degree, quintic equation as it is called. This led to triggering a lot of research in developing this theory. Galois (1811-1832), Able (1802-1829) and Lagrange (1736-1813) are some of the immoral names associated with the theory of groups.

A set of elements with a binary operation on it is called a closed system, when the outcome of the operation on two elements of the set is an element of the system. Recall the various number systems:

$\mathbf{N}$ = the system of natural or counting numbers 1,2,3, $\cdots$

$\mathbf{W}$ = $\{0,1,2,3,\cdots\}$

$\mathbf{Q}_0^+$ = the system of fractional or measuring numbers of the form $\frac{a}{b}, a \in \mathbf{W}, b \in \mathbf{N}$.

$\mathbf{Z}$ = the system of integers or motion numbers $0, \pm 1, \pm 2, \cdots$

$\mathbf{Q}$ = the system of rational numbers of the form $\frac{a}{b}, a, b \in Z, b \neq 0$ and so on.

10.1 The basic characteristics of the group surface wile solving the first degree polynomial equation $ax + b = 0$.

Let us consider some particular equations and find out why they do not have solution in some of the systems.

(1) Consider the equation $x + 3 = 3$. Equations of this type have no solution in $\mathbf{N}$. Why? There is no natural number which when added to 3 gives the sum 3. But it has solution in other systems as 0 is an element in them. Since addition of 0 to a number does not change the number, it is called the **identity element, additive identity** for that matter.

(2) Consider the equation $x + 5 = 0$. Equations of this type have no solution in $\mathbf{N}, \mathbf{W}$ and $\mathbf{Q}^+$. Why? There is no number which when added to 5 given

the sum 0. But it has solution in **Z** and **Q**. For $(-5)+5=0$. -5 is the **opposite** or the **additive inverse** of 5. As you know every number in **Z** and **Q** has its additive inverse. Additive inverse of (-1) is 1 and so on.

We shall solve this equation step by step observe the properties used

	$x+5=0$	
$\Longrightarrow$	$(x+5)+(-5)=0+(-5)$	Additive property of the equation.
$\Longrightarrow$	$x+[5+(-5)]=0+(-5)$	Associative property of addition.
$\Longrightarrow$	$x+0=0+(-5)$	Property of additive inverses.
$\Longrightarrow$	$x=-5$	Additive identity property

So that solution set is $\{-5\}$.

Thus we see that even to solve an equation of the type $x+b=0$ in a number system, the number system should have

1) Closure for addition (i.e. addition should be a binary operation)

2) Associative property for addition

3) Identity for addition

4) Each number should have its additive inverse.

Now consider

(3) $4x = 4$. Equations of this type have solutions in the number systems mentioned earlier. Why? 1 is in each system and 4 times 1 is 4. The role played by 1 in multiplication is akin to the role played by 0 in addition and hence 1 is called the **multiplicative identity** in each system.

(4) $4x = 1$. Equations of this type have no solutions in $\mathbf{N}, \mathbf{W}$ and $\mathbf{Z}$. Why? 4 multiplied by $\frac{1}{4} = 1$ but $\frac{1}{4}$ is neither a natural number, a whole number nor an integer. In other words, in these systems no number (of course zero is out of consideration) in $\mathbf{W}\&\mathbf{Z}$ has a **reciprocal** or **multiplicative inverse.**

As before, we shall solve this equation step by step and observe the properties used.

$4x = 1$

$\Longrightarrow$	$\frac{1}{4}(4x) = \frac{1}{4} \times 1$	Multiplicative property of the equation
$\Longrightarrow$	$(\frac{1}{4} \times 4)x = \frac{1}{4} \times 1$	Associative property of multiplication
$\Longrightarrow$	$1 \cdot x = \frac{1}{4} \times 1$	Property of multiplicative inverses.
$\Longrightarrow$	$x = \frac{1}{4}$	Multiplicative identity property.

So the solution set is $\{\frac{1}{4}\}$.

We can stop this exploration at this stage and state that a system which has the following structural properties:

(1) *a binary operation or closure for an operation*

(2) *associative property for the operation*

(3) *identity for the operation and*

(4) *inverse for each element with respect to the operation* is defined to constitute a **group**. With these four fundamental assumptions, mathematicians have built up a magnificent edifice of hundreds of theorems proved with remarkable elegance thereby giving aesthetic joy to the cultivated mind in mathematics and logic.

It is important to understand that groups do not just deal with numbers but also entities that have number-like characteristics.

We have seen that $\mathbf{Z}$ is a group for addition; $\mathbf{Q}^+$ for multiplication and these are infinite groups. We shall not consider in this elementary introduction infinite groups but only finite groups. When the number of elements of a group is a natural number, it is a finite group and the order of the group is the number of its elements.

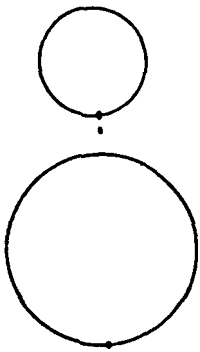
A word about binary operation.

Consider

$$\begin{aligned} 3 \times 2 &= 6 \\ 5 + 4 &= 9 \\ 8 - 1 &= 7 \\ 10 + 2 &= 5 \end{aligned}$$

What do we find? With two numbers a unique third number is associated in each case. This can be put in general form thus a $\square\, b = c$ where $\square$ means some binary operation. Let us start with an example of a group of one element.

10.2 A group of 1st order (a group of 1 element)



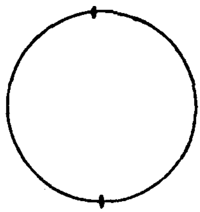
Consider a point on the rim of a ring. Through what turn should we rotate the ring in its plane, say, clockwise to get back to this position of the ring. Obviously only 1 full turn. This therefore turns out to be the identity position. Not only that; it is also the inverse. This can be shown in the form of a table.

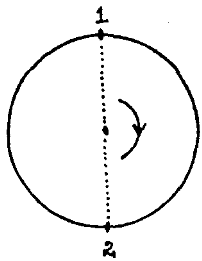
□	i
i	i

This show that $i \,\square i = i$.

$i \,\square\, i = i$ where i is the identity element. Note that i is its own inverse.

10.3 A group of 2nd order





Consider two points opposite to each other on the rim of a ring. Now, through what turn should we rotate the ring, say, clockwise to get back this appearance of the ring. Not only will full turn secure it but also half turn. Numbering the points 1 & 2, what happens when clockwise half turn is made? 1 goes to 2 and 2 goes to 1. Next half turn sees 2 going to 1 and 1 to 2. So we can denote half turn as

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

and full turn and no turn as $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

Denoting $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ as a and $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ as **i** we can give here the table of transformations of positions as follows:

□	i	a
i	i	a
a	a	i

10.4 It is interesting to view the outcome of two transformations as a product of transformations. Consider

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

which stands for half turn followed by half turn or **a** □ **a** or simply **aa**

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

We can spell it out thus: 1 goes to 2 the first $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and 2 goes 1 in the second $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ giving 1 going to 1.2 goes to 1 in the first $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and 1 goes to 2 in the second $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ giving 2 going to 2. So we get the outcome as 1 going to 1 and 2 going to 2, that is $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ which is i. Thus **a**□**a** is **i**.

This can also be shown as below:

$$\begin{array}{ccccc} & \mathbf{a} & & \mathbf{a} & \\ 1 & \longrightarrow & 2 & \longrightarrow & 1 \\ 2 & \longrightarrow & 1 & \longrightarrow & 2 \end{array}$$

and so **aa** is $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

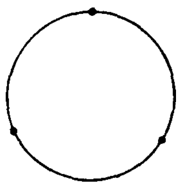
that is $\mathbf{aa} = \mathbf{i}$.

10.5 A group of the 3rd order. Consider three points forming the vertices of an inscribed equilateral triangle in the ring. Not only does the rotation full turn restore this initial positional appearance but also rotations through $\frac{1}{3}$ rd turn (or $120°$) and $\frac{2}{3}$ rd turn (or $240°$). After naming the points clockwise, these turns of $\frac{1}{3}, \frac{2}{3}$ and $\frac{3}{3}$ can be represented respectively by the following transformations

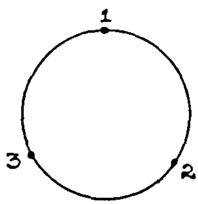
$$\mathbf{a} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \ \mathbf{i} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Working out the product of two transformations as before we can give the table

□	*i*	*a*	b
i	*i*	*a*	*b*
a	*a*	*b*	*i*
b	*b*	*i*	*a*



(i) Find **b a a** using the table and interpret it in terms of turns.

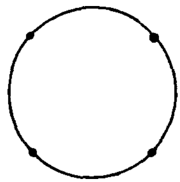


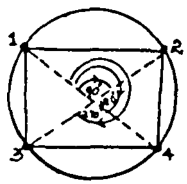
(ii) Give the transformation as ordered pairs.

(iii) Show

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

10.6 A group of 4th order. Consider four points forming the vertices of an inscribed square in the ring. Not only does the full turn restore the initial positional appearance but alos $\frac{1}{4}$ turn (90°), $\frac{1}{2}$ turn (or 180°), $\frac{3}{4}$ turn (or 270°). After naming the points clockwise, these turns of $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}$, and $\frac{4}{4}$ can be represented respectively by the following transformations





$$\mathbf{a} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\mathbf{c} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \quad \mathbf{i} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

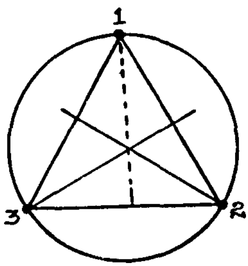
Working out the product of two transformations as before, we can get the table:

o	i	a	b	c
i	i	a	b	c
a	a	b	c	i
b	b	c	i	a
c	c	i	a	b

Find **aabcca** and interpret it in terms of turns.

10.7 Since an equilateral triangle has 3 axial symmetries as well, besides 3 rotational symmetries, we can extend the table in §10.5 to include the following:

Flipping or reflecting about the axis to side 2-3 can be represented by $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \mathbf{p}$



Flipping or reflecting about the axis to side 3-1 can be represented by $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \mathbf{q}$

Reflecting about the axis to side 1-2 can be represented by

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \mathbf{r}$$

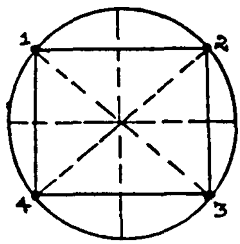
Now the enlarged table of transformations forms a group of the 6th order.

□	i	a	b	p	q	r
i	i	a	b	p	q	r
a	a	b	i	q	r	p
b	b	i	a	r	p	q
p	p	r	q	i	b	a
q	q	p	r	a	i	b
r	r	q	p	b	a	i

10.8 Since a square has 4 axial symmetries, besides 4 rotational or centre symmetries, we can extend the table in §10.6 include the following.

Flipping about the **vertical** axis of symmetry can be represented by the transformation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \mathbf{p}$$



Flipping about the **horizontal** axis of symmetry can be represented by the transformation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \mathbf{q}$$

Flipping about the other diagonal (line joining the corners

4 & 2) can be represented by the transformation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = \mathbf{r}$$

Flipping about the other diagonal (line joining the corners 1 & 3) can be represented by

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = \mathbf{s}$$

Now the enlarged table of transformations forms a group of the 8th order.

□	i	a	b	c	p	q	r	s
i	i	a	b	c	p	q	r	s
a	a	b	c	i	s	r	p	q
b	b	c	i	a	q	p	s	r
c	c	i	a	b	r	s	q	p
p	p	r	q	s	i	b	a	c
q	q	s	p	r	b	i	c	a
r	r	q	s	p	c	a	i	b
s	s	p	r	q	a	c	b	i

Since these are all finite groups, show that each of the tables hither-to built up shows that the elements indeed form a group by establishing properties: (1) closure, (2) associative, (3) identity, (4) inverse.

Dissociating the situational meaning here of **i, a, b, c,** etc. and determining their relations by means of composition table helps consideration of abstract groups.

10.9 It is interesting to notice how permutations can be viewed to form a transformation group.

This introduction will whet your appetite to study more about the group and its fascinating characteristics and dramatic uses. We can now associate the appearance of triads in each position of the magic square with a transformation.

8	1	6	$\longleftrightarrow \mathbf{i} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

no turn or clockwise full turn

4	3	8	$\longleftrightarrow \mathbf{a} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

clockwise $\frac{1}{4}$ turn

2	9	4	$\longleftrightarrow \mathbf{b} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

clockwise $\frac{1}{2}$ turn

6	7	2	$\longleftrightarrow \mathbf{c} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

clockwise $\frac{3}{4}$ turn

6	1	8	$\longleftrightarrow \mathbf{p} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

flipping about the vetical
line of symmetry

4	9	2

$\longleftrightarrow \mathbf{q} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

flipping about the horizontal line of symmetry

2	7	6

$\longleftrightarrow \mathbf{r} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

flipping about the main diagonal (left bottom corner to right top corner)

8	3	4

$\longleftrightarrow \mathbf{s} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$

flipping about the other diagonal (left top corner to right bottom corner)

By remembering this table of correspondences for relating the top row triad of each magic square to a transformation, prediction can be done through computation without referring to the table and that will make you a wizard.

Projects

1. Find all the permutations of 1 2 3 4 and treating each as a transformation of 1 2 3 4 build the table of binary products of transformations. Examine the table and find if the elements constitute a group? What is its order?

2. If a set with an operation satisfies the four characteristics of a group, it becomes a group. If a subset of the group also satisfies the four characteristics under the same operation, the subset becomes a subgroup

   Consider the finite groups of order 1,2,3,4,6 and 8 developed in this Section 10 and explore the existence of subgroups in each order. Discover the relation between the order of a subgroup of a finite group and the order of the group. If you succeed you would have rediscovered a theorem of Lagrange:

   The order of any subgroup of a finite group divides the order of the group.

PART - C

11.0 EQUI MAGIC SQUARES

11.1 Consider the skeleton form of the third order Magic Square

fig. 1

p	$-p-q$	q
$-p+q$	0	$p-q$
$-q$	$p+q$	$-p$

Row totals, column totals and diagonal totals give the magic sum 0. If the magic sum is a multiple of 3, then the third order magic square can be built thus:

fig. 2

$\frac{m}{3}+p$	$\frac{m}{3}-p-q$	$\frac{m}{3}+q$
$\frac{m}{3}-p+q$	$\frac{m}{3}$	$\frac{m}{3}+p-q$
$\frac{m}{3}-q$	$\frac{m}{3}+p+q$	$\frac{m}{3}-p$

This shows that for the same magic sum, more than one magic square can be formed depending on the values of p and q.

You have seen earlier seen earlier that the magic square can be built with entries in a Cayley table got by summing two arithmetic progressions.

fig. 3

+	1	2	3
0	1	2	3
3	4	5	6
6	7	8	9

On altering the positions you know how to fix the third order magic square with the magic sum 15.

fig. 4

8	1	6
3	5	7
4	9	2

You know also that this can be presented in 8 different ways (axial symmetry 4, rotational symmetry 4). Incidentally, it is worth noticing that the skeleton magic square for this third order magic square can be formed by taking $p = 1$ and $q = 3$ in fig. 1. p and q are nothing but the common differences of their row wise arithmetic progression 1,2,3 and the column wise arithmetic progression 0,3,6. Adding 5 to each entry fig. 4 is obtained.

3	−4	1
−2	0	2
−1	4	−3

So by studying the values for p and q, you can build magic squares having the same sum, otherwise called **equi magic squares.**

11.2 First of all examine the skeleton magic square in fig 1 and find out for all values of p and q, the magic square will cease to be a perfect one, that is one without repeated entries. Obviously $p \neq q$.

Next p and q cannot be consecutive numbers 1 & 2. If

$p = 1$ and $q = 2$, then $-p + q = -1 + 2 = 1$. Then two cells will have the same entries leading to a trivial magic square, or to state more generally $p \neq 2q$ or $q \neq 2p$. Otherwise $-p + q = -2q + q$ or q. But q is already an entry resulting in repeated entries which is not permissible for a perfect magic square.

Finally, $(p, q) \neq \left(\frac{m}{3} - a, a\right)$. Otherwise

$$-p - q = -\tfrac{m}{3} + a - a = -\tfrac{m}{3}$$

(skeleton magic square) and

$$\tfrac{m}{3} - p - q = \tfrac{m}{3} - \tfrac{m}{3} = 0.$$

In a perfect magic square no entry can be 0. Of course $(p, q) = (m, n)$ and $(p, q) = (n, m)$ given the same magic square. With the above mentioned conditions, study how to build equimagic squares for the same sum. Starting with 5 the magic sum for the third order magic square with numbers 1 to 9, the central entry is 5, build Cayley table with 3 term arithmetic progressions having respectively p and q for their common difference.

$p = 1$

$q = 3$

+	1	2	3
0	1	2	3
3	4	5	6
6	7	8	9

Table

6	1	8
7	5	3
2	9	4

Magic square

11.3 A better approach will be to construct the skeleton square first with p and q and add $\frac{m}{3}$ to each entry in the skeleton square which would fix the desired magic square.

$p = 1, \ q = 3$

1	−4	3
2	0	−2
−3	4	−1

Adding 5 to each entry you get the magic square with the magic sum 15.

6	1	8
7	5	3
2	9	4

It can be easily seen that (p, q) can only be (1,3) for the magic sum 15 or the central entry 5.
For, $(p, q) \neq (1, 1), (1, 2)$ and (1,4).

So 15 sum magic squares can be had in a unique way with its presentation in 8 different ways involving axial and point symmetries of the group of 8 elements whose multiplication table was discussed in detail earlier.

11.4 Enumeration

So the number of equimagic squares for a given multiple of 3 depends on the values that p and q can take. Consider the magic sum 18. The central value is 6. $(p, q) = (1, 3), (1, 4)$ and (2,3). So there can be only 3 different equimagic squares having the magic sum 18. Build them. Are you getting the following?

7	2	9
8	6	4
3	10	5

7	1	10
9	6	3
2	11	5

8	1	9
7	6	5
3	11	4

You would have seen by now a short cut in building the magic square without building the skeleton square in full. Fix $\frac{1}{3}$ of the magic sum 18, that is 6 in the central cell. Add 6 to 1 & 3 and fix them in the corner squares in the top row. Then fix the entries in the corresponding opposite corners to make up the sum 12. The remaining entries rowwise and columnwise are fixed to get the magic sum 18.

Consider the number of different equimagic squares for a few more magic sums 21, 24, 27 and 30, say.

Magic sum 21, *Central entry* 7

$$(p, q) = (1,3), (1,4), (1,5), (2,3)$$

Magic sum 24, *Central entry* 8

$$(p, q) = (1,3), (1,4), (1,5), (1,6); \quad (2,3), (2,5), (3,4)$$

Magic sum 27, *Central entry* 9

$$(p, q) = (1,3), (1,4), (1,5), (1,6), (1,7); \quad (2,3), (2,5), (3,4), (3,5)$$

Magic sum 30, *Central entry* 10

$$(p, q) = (1,3), (1,4), (1,5), (1,6), (1,7), (1,8); \quad (2,3), (2,5), (2,6), (2,7), (3,4), (3,5), (4,5)$$

It is interesting to tabulate the squares with entry 1, entry 2 etc., in the top left corner of each of the equimagic squares for a given magic sum.

Magic sum and Central entry		Number of squares with top left corner entry in the skeleton magic squares							Total number of different equimagic squares
		1	2	3	4	5	6	7	
(15)	5	1							1
(18)	6	2	1						3
(21)	7	3	1						4
(24)	8	4	2	1					7
(27)	9	5	3	2					10
(30)	10	6	4	2	1				13
(33)	11	7	5	3	2				17
(36)	12	8	6	4	3	1			22
(39)	13	9	7	5	4	2			27
(42)	14	10	8	6	4	3	1		32
(45)	15	11	9	7	5	4	2		38
(48)	16	12	10	8	6	4	3	1	44

Note: Compare the above table with the table in §9.0. the eight different moves used in the prediction game form a group. If we know one magic square of third order with central entry 5, all the other magic squares are obtained form this by making the 8th order group of §10.0 to act on this square. Don't you observe the surprising connection now. On the other hand, if the central entry of the third order magic square is different from 5, it is not generally the case that any magic square is obtainable from any other by a transformation pertaining to the group just now mentioned. However, this group plays a crucial role in the case of these magic squares too. In this case all the magic squares which are not obtainable by transformations in the group are to be treated as constituting a block of magic squares. The action of the group on this block exhausts all possible magic squares. The action of the group on this block exhausts all possible magic squares. When the central entry $n = 5$ this block only consists of one element. When $n = 6$ it consists of 4 elements and so on. A distribution of the blocks is shown by the above table.

11.5 Problem: Show that when $(p, q) = (1, 3)$, the entries are consecutive for any magic sum.

11.6 Project: Find a formula as well as recurrence formula for finding the number of equimagic squares for a given magic sum.

ACKNOWLEDGEMENT

Development of Part C was provoked by the questions posed by Master Amber Rao, a class VI student of the Valley School, KFI, Bangalore - 560 062 while participating in the MATH EXPO DIALOGUES held this year.

GOOD BYE

I hope you have enjoyed the mathematical experiences in handling third order magic squares and getting to know the basic elements of group theory. You would certainly have enjoyed much if you had done a lot of work on your own.

This gives you adequate background to handle 4th order magic squares and study more of groups, finite as well as infinite. Unlike the basic third order magic square with entries from 1 to 9, the basic fourth order magic square with entries from 1 to 16 is not unique. An analogy will help you in your eagerness to pursue further this line of study.

Having swimming experience in a pond, you can confidently take to swimming in a lake. But swimming in an ocean is different.

Remember working in the field of magic squares of higher orders is akin to swimming in an ocean. With all the formation shocks and surprises and the challenging projects that suggest themselves, you can find an excellent use for your mathematics tools introduced in the classrooms by your teachers. Magic squares can have a place in sophisticated mathematics as well (See The Mathematical Intelligencer, Vol. 14, No.3, 1992). You can acquire more mathematics tools by studying groups and their behavior. For one who would like to have a healthy pastime all through one's life at least expense, magic squares claim preference. For one who would like to keep company with world class mathematicians group theory is a must.

Wish you an exciting time!

Books for further enlightenment and enjoyment:

1) **W.H. Benson and O. Jacoby**: New Recreations with Magic Squares, Dover Publications, New York, 1976.

2) **Irving Adler**: Groups In The New Mathematics, The John Day Company, New York, 1967.

3) **Anthony Zee**: Fearful Symmetry, Collier Macmillan, New York, 1980.

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