Bell Helicopter TEXTRON
Division of Textron Inc

INTER-OFFICE MEMO

8 March 1977
81:JSD:jo-543

Memo To: Mr. L. Hochreiter

Copies To: Messrs. W. Byrd, T. Eidson, G. Grimes, L. Lortz, O.K. McCaskill, W. Thatcher

Subject: Tailboom to Fuselage Attachment Holes - Maximum Allowable Diameter

Enclosure: Table I

The various helicopter maintenance manuals do not have a consistent baseline for allowable hole sizes for the four tailboom to fuselage attachment holes. Also, the manner in which these allowables are presented is not consistent.

For field maintenance purposes, the Airframe Structures Group recommends the maximum allowable hole size be set at .015 above the blue print minimum hole diameter for these attachment holes. It is suggested that these allowable hole sizes be presented in the tailboom section of the various maintenance manuals and hole allowables for both the tailboom and fuselage be addressed in this section.

The Airframe Structures Group also recommends that bushing of these holes by Material Review Board (MRB) disposition be limited to holes that exceed one half (½) the difference between the maximum hole diameter allowed for wear and the maximum blue print hole diameter.

Table I presents the recommended maximum hole diameters for field maintenance purposes and MRB action.

Will you please present the allowable wear limits from Table I to the Service Engineering Group and request that they revise the maintenance manuals accordingly.

J. S. Dulaney
J. S. Dulaney
Airframe Structures Group

Orville Baker
O. Baker
Senior Group Engineer
Airframe Structures

0.568
0.006
0.574

TABLE I

TAILBOOM TO FUSELAGE ATTACHMENT HOLES - MAXIMUM ALLOWABLE HOLE SIZE

Model	Drawing No. (Ref.)	Hole Location	B/P Hole Size	Maximum Allowable Hole Size (Field Maintenance)	Maximum Allowable Hole Size (MRB Action)
205	205-030-713 205-031-801	Upr L.H. & R.H. Lwr L.H. & R.H.	.501/.506 .376/.381	.516 .391	.511 .386
206	206-031-003 206-031-004	Upr & Lwr (4 Holes)	.376/.378	.391	.384
206L	206-033-003 206-033-004	Upr & Lwr (4 Holes)	.376/.378	.391	.384
212	212-030-156 212-030-128	Upr L.H. Upr R.H. Lwr L.H. & R.H.	.563/.568 .501/.506 .376/.381	.578 .516 .391	.573 .511 .386
214	214-030-248 214-030-314 214-030-315	Upr L.H. Upr R.H. Lwr L.H. & R.H.	.750/.756 .625/.631 .625/.631	.765 .640 .640	.760 .635 .635
AH-1G	209-030-113	Upr & Lwr (4 Holes)	.501/.506	.516	.511
AH-1J	209-031-113 209-031-876 209-031-800	Upr L.H. & R.H. Lwr L.H. & R.H.	.625/.631 .563/.568	.640 .578	.635 .573
AH-1S	209-033-002 209-033-114 209-033-810 209-033-811	Upr & Lwr (4 Holes)	.501/.506	.516	.511
AH-1T	209-031-227 209-032-800	Upr L.H. & R.H. Lwr L.H. & R.H.	.625/.631 .563/.568	.640 .578	.635 .573

Bell Helicopter TEXTRON
Division of Textron Inc

INTER-OFFICE MEMO

9 March 1978
81:GRA:jo-705

Memo To: Production Airframe Stress

Copies To: Messrs. O. Baker, P. Bauer, L. Lortz, O. K. McCaskill, J. McGuigan, D. Poster, E. Scroggs, R. Scoma

Subject: MINIMUM Thickness of Aluminum Airframe Parts

The purpose of this memo is to establish the procedure to use when specifying thicknesses and conducting analyses on Aluminum Airframe parts. Airframe Design prefers that we sepcify the MINIMUM thickness required for structural consideration. Therefore, the procedure to use for analysis of all Aluminum Airframe parts with the exception of basic extrusion (shapes), drawn tubing, and sheet stock, shall be the same as defined for chem milled Aluminum parts in BHT IOM 81:GRA:jo-693 dated 3 February 1978 and repeated below. Analyses of basic extrusion (shapes), drawn tubing and sheet stock should be conducted using NOMINAL dimensions.

Calculate the required thickness for structural consideration based on standard analysis procedures. Specify the MINIMUM thickness to be the required thickness minus the tolerance shown below:

Thickness Range	Tolerance
.012 to .036	.002
.037 to .045	.003
.046 to .096	.004
.097 to .140	.005
.141 to .172	.008
.173 to .203	.010
.204 to .249	.011
.250 to .320	.013

Note: If it is possible to hold tighter tolerances than shown above, use NOMINAL thickness in the analysis.

The above table is based on the standard tolerances specified for 36 to 48 inch wide aluminum sheet stock shown in ANSI H35.2 - 1975. It is standard practice to use nominal sheet stock thicknesses for analyses with the above tolerances. Therefore, the same tolerance is acceptable for analysis of other aluminum parts.

9 March 1978
81:GRA:jo-705

As an example,

> If you calculate t_{reqd} = .035 in., you should specify t_{min} = .035 - .002 = .033 in. The t_{max} will be based on Design and Manufacturing considerations. The final analysis will be based on t_{min} + tolerance from the above table. For our example, the analysis would be based on $t_{min} + t_{tol}$ = .033 + .002 = .035 in.

It is in our best interest to try to influence Design and Manufacturing to maintain as tight a tolerance as possible in order to minimize weight.

G. R. Alsmiller, Jr.
Group Engineer
Production Airframe Stress

1/804

Bell Helicopter TEXTRON

A Subsidiary of Textron Inc

INTER-OFFICE MEMO

19 June 1985
81:MEG:glv-215

MEMO TO: T. Attridge, C. Baskin, R. Battles, J. Braswell, M. Ernest, W. Koch, L. Lortz, M. Lufkin, R. Murray. P. Patel, G. Perry, J. Reynolds, E. Schellhase, W. Sundland, K. Tessnow, R. Wardlaw.

COPIES TO: R. Alsmiller, R. Barrett, F. House, W. Fountain, L. Lynch, E. Roseler, D. Sims, W. Thomas

SUBJECT: Diagram for Metallurgical Test Locations on Forging/ Casting Drawings

Effective immediately, all new forging and designated casting drawings shall include a diagram which defines locations for metallurgical tests. Locations for grain flow, tensile bars, and microstructure evaluations, as applicable, shall be shown in the diagram.

For all forgings, a "Prototype Forging Test Diagram", shall be required. This diagram will be similar to the x-ray diagram required for most castings. Two or three reduced size views of the part will be required to show grain flow, tensile bar, and microstructure test locations.

Castings which make a critical "MAC" or primary "MAP" part shall have an "X-RAY AND FOUNDRY CONTROL TEST DIAGRAM". Since these castings already require an x-ray diagram, tensile bar and microstructure test locations can be shown on the views of the diagram.

The Metal Materials Group will coordinate with Stress, Design Group, and the Metallurgical Laboratory to determine and designate the appropriate areas to be tested. The Metals Group will mark the test locations on a check print for the Design Group to incorporate on the engineering drawing prior to submitting to the Check Group.

This information on the engineering drawing will eliminate problems encountered over the past years.

- Metallurgical test locations designated with design and stress input will insure that critical or highly stressed areas are adequately evaluated. This will eliminate requesting additional tests after reviewing prototype forging or foundry control casting reports.

- o Test locations on the drawing will assure that parts are evaluated the same when produced by two vendors or when vendors are changed.

- o Charges for destructive tests by vendors can be compared and controlled at time of quote. Charges for metallurgical evaluations vary considerably from vendor to vendor depending on the number of tests they perform. When the same tests are specified for each vendor, direct comparison of costs can be made.

The requirements for the forging and casting test diagram will be incorporated into the DRM at next revision.

M. E. Greene
M. E. Greene
Group Engineer
Metal Materials

F. Wagner
F. Wagner
Director
Vehicle Design

BELL HELICOPTER COMPANY

Inter-Office Memo

5 December 1974
81:LL:sw-3013

Memo to: Airframe Design Group

Copies to: Messrs. R. Alsmiller, O. Baker, D. Braswell, B. DeLorme, T. Eidson, E. Fischer, D. Higby, L. Hochreiter, R. Lynn, O. McCaskill, J. McGuigan, D. Poster

Subject: STRUCTURAL NUTPLATE POLICY IN AIRFRAME DESIGN

It has recently been established that the airframe drawings have an inconsistency in types of nutplates/hole sizes in both basic structure and removable panels. This memo is written to clarify the airframe groups position pertaining to structural applications and standardization of nutplates. For all future designs the following parameters will be used for hole sizes and nutplate selection except where structural requirements dictate a closer tolerance hole to guarantee integrity of the airframe design. The use of reduced spacing nutplates and regular fixed nutplates will be used only with the approval of airframe supervision.

For 3/16 inch threaded fasteners using nutplates:

a. In the base structure (frame caps, doublers, etc.), use a .193/.198 diameter hole and a floating type nutplate. Do not attach nutplates to materials less than .032 thickness for structural application.

b. In the removable panel/door/structure, use a .203/.208 diameter hole. A larger hole might be authorized if stress levels permit.

For 1/4 inch threaded fasteners using nutplates:

a. In the base structure, use a .256/.262 diameter hole and a floating type nutplate. Do not attach nutplates to materials less than .040 inch thickness for structural applications.

b. In the removable panel/door/structure, usa a .264/.270 diameter hole. A larger hole might be authorized if stress levels permit.

For 5/16 inch threaded fasteners using nutplates:

a. In the base structure, use a .316/.322 diameter hole and a floating type nutplate. Do not attach nutplates to materials less than .050 inch thickness for structural applications.

b. In the removable panel/door/structure, use a .327/.333 diameter hole. A larger hole might be authorized if stress levels permit.

5 December 1974
81:LL:sw-3013

For 3/8 inch threaded fasteners using nutplates:

a. In the base structure, use a .378/.384 diameter hole and a floating type nutplate. Do not attach nutplates to materials less than .050 inch thickness for structural applications.

b. In the removable panel/door/structure, use a .391/397 diameter hole. A larger hole might be authorized if stress levels permit.

L. Lortz, Assistant Group Engineer
Airframe Design Group

Bell Helicopter TEXTRON
Division of Textron Inc

INTER-OFFICE MEMO

3 February 1978
81:GRA:jo-693

Memo To: Production Airframe Stress Group

Copies To: Messrs. O. Baker, P. Bauer, L. Lortz, O. K. McCaskill, J. McGuigan, D. Poster, E. Scroggs, R. Scoma

Subject: Minimum Thickness on Chem Milled Aluminum Sheet or Wrought Alloys

Due to the increased use of chem milling for weight and cost reduction, it is necessary to clarify the Structures Group position on the chem milled thickness to use in analyses and reports. The following procedure should be used when specifying thicknesses and conducting analyses on sheet or wrought aluminum alloys.

Calculate the required thickness for structural consideration based on standard analysis procedures. Specify the minimum thickness to be the required thickness minus the tolerance shown below:

Thickness Range	Tolerance
.012 to .036	.002
.037 to .045	.003
.046 to .096	.004
.097 to .140	.005
.141 to .172	.008
.173 to .203	.010
.204 to .249	.011
.250 to .320	.013

Note: If it is possible to hold tighter tolerances than shown above, it is permissible to use nominal thickness.

The above table is based on the standard tolerances specified for 36 to 48 inch wide aluminum sheet stock shown in ANS1 H35.2 - 1975. It is standard practice to use nominal sheet stock thicknesses for analyses with the above tolerances. Therefore, the same tolerance is acceptable for analysis of chem milled aluminum parts.

3 February 1978
81:GRA:jo-693

As an example,

> If you calculate t_{reqd} = .035 in., you should specify t_{min} = .035 - .002 = .033 in. The t_{max} will be based on Design and Manufacturing considerations. The final analysis will be based on t_{min} + tolerance from the above table. For our example, the analysis would be based on $t_{min} + t_{tol}$ = .033 + .002 = .035 in.

It is in our best interest to try to influence Design and Manufacturing to maintain as tight a tolerance as possible in order to minimize weight.

G. R. Alsmiller, Jr.
Group Engineer
Production Airframe Stress

Bell Helicopter TEXTRON

10th May 1988

MEMO TO: ALL STRUCTURES AND DESIGN (INCLUDING LIAISON) PERSONNEL, G.M. GRITZKA, R. FEWS

FROM: A. WATERHOUSE

SUBJECT: **USE OF 7075-T73 FOR PRIMARY AND CRITICAL PARTS**

A copy of the attached memo must be inserted in the relevant place(s) of all copies of Structures and Design manuals, i.e. the Airframe Design Manual, Rotor Systems Design Manual, Landing Gear Design Manual, Structural Design Manual, Fatigue Design Handbook, Rotor Stress Group Manual, and any other design or analysis guidelines used.

A. Waterhouse

A. Waterhouse
Chief of Structures

Attachment: IOM F. Wagner to Vehicle Design Managers, "Shot-Peening Requirements for Primary and Critical Parts made from 7075-T73", 27 April 1984.

AW/dd/19/031

Bell Helicopter TEXTRON
Division of Textron Inc.

INTER-OFFICE MEMO

April 27,.1984

Memo to: Messrs. B. Alapic, O. Baker, W. Cresap, R. Duppstadt, S. Roberts, E. Roseler

Copies to: Messrs. C. Davis, M. Glass, R. Lynn, G. Rodriguez, S. Viswanathan

Subject: Shot-Peening Requirements for Primary and Critical Parts made from 7075-T73

The following normal design procedure is to be implemented on all future designed 7075-T73 parts as a result of a recently completed fatigue committee study:

> All primary and critical parts made from 7075-T73 will require BPS 4409 shot-peening.

Since this is a structural shot-peening as compared to Mil Spec peening, those parts which require fatigue testing will be tested in the peened condition. Field repairs of damage will only be authorized in locations on the subject parts which would not require a peening of the repaired area.

You are requested to inform your designers and insert a copy of this memo into the Design Manual of each appropriate Design Group.

F. J. Wagner
Director
Vehicle Design

THREADED CONNECTIONS

The following rules are stated for the case of threaded connections between two tubes subjected to axial tensile force, P. For bolt-and-nut assemblies, the same rules apply, by substituting $D_i = 0$.

Investigate the following stresses:

(1) Tensile stress in outer tube at root section of thread

$$f_t = \frac{P}{\frac{\pi}{4}\left[D_0^2 - (D_N + f)^2\right]}$$

(2) Tensile stress in outer tube at relief groove

$$f_t = \frac{P}{\frac{\pi}{4}\left[D_0^2 - D_{R_0}^2\right]}$$

(3) Tensile stress in inner tube at root section of thread

$$f_t = \frac{P}{\frac{\pi}{4}\left[(D_N - e)^2 - D_I^2\right]}$$

THREADED CONNECTIONS Cont'd

(4) Tensile stress in inner tube at relief groove

$$f_t = \frac{P}{\frac{\pi}{4}\left[D_{R_1}^2 - D_1^2\right]}$$

(5) Shear stress across threads, along a cylindrical surface having a diameter equal to the minimum pitch diameter.

$$f_S = \frac{PF}{\frac{\pi}{2}(D_N - c)\,L} \approx \frac{4P}{\pi D L}$$

$$F = \frac{\text{UNLOADED TOOTH HEIGHT}}{\text{LOADED TOOTH HEIGHT}}$$

Unloaded Tooth Height = Minimum major diameter of inner tube minus maximum minor diameter of outer tube.

Loaded Tooth Height = Unloaded tooth height minus Δ_0 minus Δ_1

(6) Bearing stress on the surface of contact between inner and outer threads.

$$f_{br} = \frac{P}{\frac{nL\pi}{4}\left[(D_N - a - \Delta_1)^2 - (D_N - b - \Delta_0)^2\right]}$$

where:

L = length of engagement

Δ_1 = elastic decrease in diameter of inner tube

Δ_0 = elastic increase in diameter of outer tube

THREADED CONNECTIONS Cont'd

(7) Hoop compression stress in inner tube

$$f_c = \frac{P}{\sqrt{3}\,\pi L\left(D_N - c - D_I\right)}$$

The corresponding elastic decrease in diameter of inner tube is

$$\Delta_I = \frac{f_c}{E}\left(D_N - c\right)$$

(8) Hoop tension stress in outer tube

$$f_t = \frac{P}{\sqrt{3}\,\pi L\left(D_O - D_N + d\right)}$$

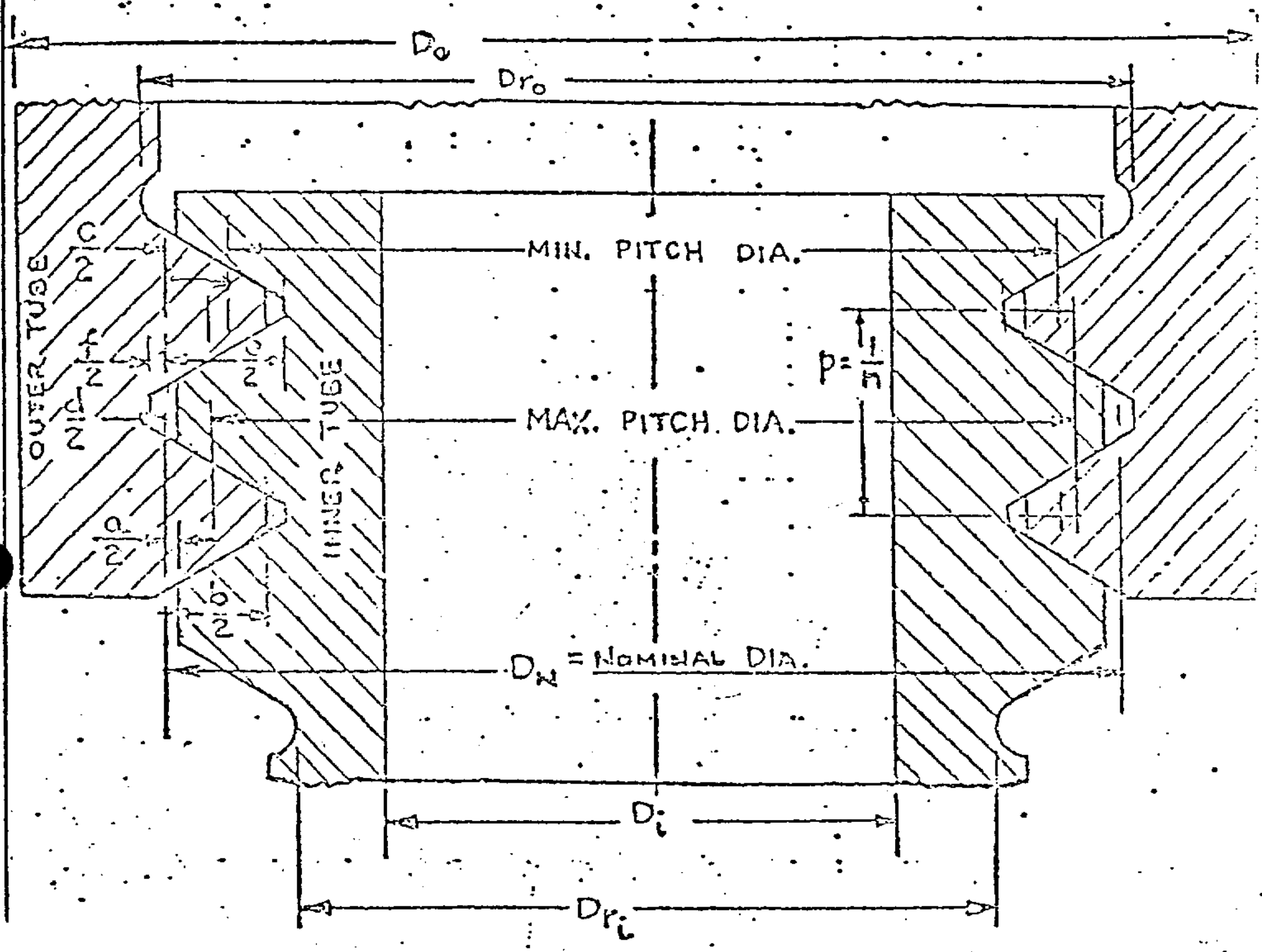
The corresponding elastic increase in diameter of outer tube is

$$\Delta_O = \frac{f_t}{E}\left(D_N - d\right)$$

REVISED ______ REPORT ______

REVISED ______ MODEL ______

THREADED CONNECTIONS



a = Truncation of external thread root

p = Pitch

c = Difference between maximum major and pitch diameters of internal thread.

d = Height of internal thread and depth of thread engagement

e = Double height of external thread

f = Addendum of external thread

$(D_n - a)$ = Minimum major diameter of inner tube

$(D_n - b)$ = Maximum minor diameter of outer tube

$(D_n - c)$ = Minimum pitch diameter

$(D_n - d)$ = Maximum pitch diameter

$(D_n - e)$ = Minimum root diameter of thread on inner tube

$(D_n + f)$ = Maximum root diameter of thread on outer tube

UNF THREADS [1]
BASIC DATA

Threads per Inch	a	b	c	d	e	f
80	0.00180	0.012500	0.00992	0.00677	0.01534	0.00406
72	.00200	.013889	.01103	.00752	.01704	.00451
64	.00226	.015625	.01240	.00846	.01917	.00507
56	.00258	.017857	.01418	.00967	.02191	.00580
48	.00301	.020833	.01654	.01128	.02556	.00677
44	.00328	.022727	.01804	.01230	.02788	.00738
40	.00361	.025000	.01985	.01353	.03067	.00812
36	.00401	.027778	.02205	.01504	.03408	.00902
32	.00451	.031250	.02481	.01691	.03834	.01015
28	.00515	.035714	.02835	.01933	.04382	.01160
27	.00535	.037037	.02940	.02005	.04544	.01203
24	.00601	.041667	.03308	.02255	.05112	.01353
20	.00722	.050000	.03969	.02706	.06134	.01624
18	.00802	.055556	.04410	.03007	.06816	.01804
16	.00902	.062500	.04962	.03383	.07668	.02030
14	.01031	.071429	.05670	.03866	.08763	.02320
13	.01110	.076923	.06107	.04164	.09437	.02498
12	.01203	.083333	.06615	.04511	.10224	.02706
11 1/2	.01255	.086957	.06903	.04707	.10668	.02824
11	.01312	.090909	.07217	.04921	.11153	.02952
10	.01443	.100000	.07939	.05413	.12269	.03248
9	.01604	.111111	.08821	.06014	.13632	.03608
8	.01804	.125000	.09923	.06766	.15336	.04059
7	.02062	.142857	.11341	.07732	.17527	.04639
6	.02406	.166667	.13231	.09021	.20448	.05413
5	.02887	.200000	.15877	.10825	.24537	.06495
4 1/2	.03208	.222222	.17641	.12028	.27264	.07217
4	.03608	.250000	.19846	.13532	.30672	.08119

[1] 1963 supplement to screw-thread standards for Federal Services Handb[illegible] H28, Page 4, Table III.1

DATE ____________

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REPORT ____________

MODEL ____________

UNJ THREADS [1]
BASIC DATA

Threads Per Inch	a	b	c	d	e	f
80	0.00135	0.012500	0.00992	0.00609	0.01443	0.0040
72	.00150	.013889	.01103	.00677	.01604	.0045
64	.00169	.015625	.01240	.00761	.01804	.0050
56	.00193	.017857	.01418	.00870	.02062	.0058
48	.00226	.020833	.01654	.01015	.02406	.0067
44	.00246	.022727	.01804	.01107	.02624	.0073
40	.00271	.025000	.01985	.01218	.02887	.0081
36	.00301	.027778	.02205	.01353	.03208	.0090
32	.00338	.031250	.02481	.01522	.03608	.0101
28	.00387	.035714	.02835	.01740	.04124	.0116
24	.00451	.041667	.03308	.02030	.04811	.0135
20	.00541	.050000	.03969	.02436	.05774	.0162
18	.00601	.055556	.04410	.02706	.06415	.0180
16	.00677	.062500	.04962	.03045	.07217	.0203
14	.00773	.071429	.05670	.03480	.08248	.0232
13	.00833	.076923	.06107	.03747	.08882	.0249
12	.00902	.083333	.06615	.04059	.09622	.0270
11	.00984	.090909	.07217	.04429	.10497	.0295
10	.01083	.100000	.07939	.04871	.11547	.0324
9	.01203	.111111	.08821	.05413	.12830	.0360
8	.01353	.125000	.09923	.06089	.14434	.0405
7	.01546	.142857	.11341	.06959	.16496	.0463
6	.01804	.166667	.13231	.08119	.19245	.0541
5	.02165	.200000	.15877	.09743	.23094	.0649
4.5	.02406	.222222	.17641	.10825	.25660	.0721
4	.02706	.250000	.19846	.12178	.28868	.0811

[1] MIL-S-8879A 8 December 1965

DATE

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REVISED

REPORT

MODEL

UNJ THREADS [1]

BASIC DATA

Threads Per Inch	a	b	c	d	e	f
80	0.00135	0.012500	0.00992	0.00609	0.01443	0.00[illegible]
72	.00150	.013889	.01103	.00677	.01604	.004[illegible]
64	.00169	.015625	.01240	.00761	.01804	.00[illegible]
56	.00193	.017857	.01418	.00870	.02062	.00[illegible]
48	.00226	.020833	.01654	.01015	.02406	.006[illegible]
44	.00246	.022727	.01804	.01107	.02624	.007[illegible]
40	.00271	.025000	.01985	.01218	.02887	.00[illegible]
36	.00301	.027778	.02205	.01353	.03208	.00[illegible]
32	.00338	.031250	.02481	.01522	.03608	.01[illegible]
28	.00387	.035714	.02835	.01740	.04124	.011[illegible]
24	.00451	.041667	.03308	.02030	.04811	.01[illegible]
20	.00541	.050000	.03969	.02436	.05774	.01[illegible]
18	.00601	.055556	.04410	.02706	.06415	.0[illegible]
16	.00677	.062500	.04962	.03045	.07217	.02[illegible]
14	.00773	.071429	.05670	.03480	.08248	.02[illegible]
13	.00833	.076923	.06107	.03747	.08882	.024[illegible]
12	.00902	.083333	.06615	.04059	.09622	.027[illegible]
11	.00984	.090909	.07217	.04429	.10497	.02[illegible]
10	.01083	.100000	.07939	.04871	.11547	.03[illegible]
9	.01203	.111111	.08821	.05413	.12830	.03[illegible]
8	.01353	.125000	.09923	.06089	.14434	.04[illegible]
7	.01546	.142857	.11341	.06959	.16496	.0[illegible]
6	.01804	.166667	.13231	.08119	.19245	.0[illegible]
5	.02165	.200000	.15877	.09743	.23094	.0[illegible]
4.5	.02406	.222222	.17641	.10825	.25660	.0[illegible]
4	.02706	.250000	.19846	.12178	.28868	.0[illegible]

[1] MIL-S-8879A, 9 December 1965

STANDARD DESIGN BEND RADII ⚠

MATERIAL AND CONDITION FORMED AT ROOM TEMP		MATERIAL THICKNESS															
		.012	.016	.020	.025	.032	.040	.050	.063	.071	.080	.090	.100	.125	.160	.190	.250
ALUMINUM	2024-0	.03	.03	.06	.06	.06	.09	.09	.12	.16	.16	.18	.25	.25	.31	.38	[illegible]
	2024-T( )	.06	.06	.06	.09	.12	.16	.18	.25	.31	.31	.37	.37	.50	.75	.90	1.25
	5052-0	.03	.03	.03	.03	.06	.06	.06	.06	.09	.09	.09	.12	.16	.19	.25	.38
	5052-H34	.03	.03	.06	.06	.06	.09	.09	.12	.16	.16	.18	.25	.25	.31	.38	.56
	5052-H34*	.02	.02	.02	.03	.03	.03	.06	.06	.09	.09	.09	.12				
	6061-0	.03	.03	.03	.03	.06	.06	.06	.06	.09	.09	.09	.12	.16	.18	.25	.38
	6061-T( )	.06	.06	.06	.09	.12	.16	.18	.25	.31	.31	.37	.37	.50	.75	.88	1.25
	7075-0	.06	.06	.06	.06	.06	.09	.09	.12	.16	.16	.18	.25	.31	.37	.50	.62
	7075-T( )	.09	.09	.09	.12	.16	.25	.25	.31	.37	.37	.50	.50	.75	.87	1.00	1.50
MAG	AZ31B-0	.09	.09	.09	.16	.18	.25	.25	.31	.37	.50	.75	.75	.87			
	AZ31B-H24	.18	.18	.18	.25	.31	.50	.50	.62	.75	.87	1.00	1.50	1.75			
STEEL	300 SERIES ANL	.03	.03	.03	.03	.06	.06	.06	.09	.09	.09	.12	.12				
	300 SERIES 1/4 HARD	.03	.03	.06	.06	.06	.09	.09	.12	.16	.16	.19	.25				
	300 SERIES 1/2 HARD	.06	.06	.06	.09	.09	.12	.16	.25	.25	.38	.38	.50				
	300 SERIES 3/4 HARD	.09	.09	.09	.12	.16	.19	.22	.25	.28	.31	.38	.47	.50	.50	.75	1.00
	300 SERIES HARD	.09	.09	.12	.16	.19	.22	.22	.25	.28	.31	.38	.47	.50	.50	.75	1.00
	4130 ANL & LOW CARBON STL			.06	.06	.06	.09	.09	.12	.16	.16	.19	.22	.25	.28	.34	.40
	4130 NORM & 8630 NORM		.06	.09	.09	.12	.12	.16	.19	.22	.25	.31	.31	.31	.38	.50	.62
TITANIUM	TYPE I, COMP A	.024	.032	.040	.050	.054	.080	.100	.126	.177	.200	.225	.250	.312			
	TYPE I, COMP B	.030	.040	.050	.062	.080	.100	.125	.157	.213	.240	.270	.300	.375			
	TYPE I, COMP C	.024	.032	.040	.050	.064	.080	.100	.126	.177	.200	.225	.250	.312			
	TYPE II, COMP A	.048	.064	.080	.100	.128	.160	.200	.252	.319	.360	.405	.450	.562			
	TYPE III, COMP C	.054	.072	.090	.112	.144	.180	.225	.283	.355	.400	.450	.500	.625			
	Ⓐ TYPE III, COMP D	.054	.072	.090	.112	.144	.180	.225	.283	.355	.400	.450	.500	.625			

*FOR NON-STRUCTURAL USE ONLY

TITANIUM CONDITION	COMMON DESIGNATION
TYPE I, COMP A TYPE I, COMP B TYPE I, COMP C	COMMERCIALLY PURE
TYPE II, COMP A	Ti-5Al-2.5Sn
TYPE III, COMP C	Ti-6Al-4V
TYPE III, COMP D	Ti-6Al-4V ELI

APPROVED Ⓐ 11-30-73 11-7-11

REVISIONS

Ⓐ SHEET 1 REVISED.

BELL DESIGN STANDARD

CODE IDENT NO. 97499

DRAFT	DATE	TITLE	Drawing Number
Willard E. Tombs	11-6-70	BEND RADII-STANDARD DESIGN	160-001
APPROVAL David Miller	11-7-70		
CUSTOMER Don L. Parma	11-18-70		SHEET 2 OF 2

Bell Helicopter TEXTRON

STRUCTURAL INFORMATION MEMO NO TBD

July 26th, 1993

MEMO TO: BHTI/BHTC AIRFRAME STRUCTURES GROUPS

COPIES TO: G.R. ALSMILLER, K.M. STEVENSON

SUBJECT: **MINIMUM THICKNESS FOR STRUCTURAL STATIC ANALYSIS OF AIRFRAME PARTS.**

The purpose of this memo is to establish the policy with respect to the thickness to be used in structural static analysis and reports.

Analyses of all airframe forgings, castings, machinings and chem-mill parts shall be conducted using minimum drawing thicknesses. Analyses of stock material (such as extrusion, drawn tubing, sheet, plate, bar, etc) should be conducted using nominal thicknesses.

Airframe Structures
BHTI

Technology
BHTC

GL/kd

Bell Helicopter TEXTRON
A Subsidiary of Textron Inc

INTER-OFFICE MEMO

81:JCS:jw-2110
22 October 1987

MEMO TO: Holders of Structural Design Manual

SUBJECT: **REVISION F**

Please insert the following pages in your copy of the Structural Design Manual and insert this memo preceding the Table of Contents in Volume I as a summary of the changes of Revision F.

Volume I changes:

Title Page, Table of Contents, Section 1, Section 2, 3-54, 3-77 to 3-83, 4-9, 4-11, 6-6, 6-58, 7-9, 7-11, 7-71.

Volume II changes:

Title Page, Table of Contents, 10-26, 10-41, 10-47, 11-28, 13-7.

J. C. Shu
Structural Methods

Approved:

D. R. Lindsay
Group Engineer, Structural Methods

Bell Helicopter TEXTRON
A Subsidiary of Textron Inc

INTER-OFFICE MEMO

81:JCS:jw-2110
22 October 1987

MEMO TO: Holders of Structural Design Manual

SUBJECT: <u>REVISION F</u>

Please insert the following pages in your copy of the Structural Design Manual and insert this memo preceding the Table of Contents in Volume I as a summary of the changes of Revision F.

Volume I changes:

Title Page, Table of Contents, Section 1, Section 2, 3-54, 3-77 to 3-83, 4-9, 4-11, 6-6, 6-58, 7-9, 7-11, 7-71.

Volume II changes:

Title Page, Table of Contents, 10-26, 10-41, 10-47, 11-28, 13-7.

J. C. Shu
Structural Methods

Approved:

D. R. Lindsay
Group Engineer, Structural Methods

G.M. Pollock
FM-10-19-84 From JM

TABLE 7.7 Thickness Tolerances①

ALLOYS 2014, 2024, 2036, 2124, 2219, 3004, 5052, 5083, 5086, 5154, 5252, 5254, 5454, 5456, 5652, 6061, 7075, 7079, 7178, AND BRAZING SHEET NOS. 11, 12, 21, 22, 23 AND 24.

NOTE: ALSO APPLICABLE TO THE ALLOYS LISTED WHEN SUPPLIED AS ALCLAD.

SPECIFIED THICKNESS in.	SPECIFIED WIDTH—in.														
	Up thru 18	Over 18 thru 36	Over 36 thru 48	Over 48 thru 54	Over 54 thru 60	Over 60 thru 66	Over 66 thru 72	Over 72 thru 78	Over 78 thru 84	Over 84 thru 90	Over 90 thru 96	Over 96 thru 132	Over 132 thru 144	Over 144 thru 156	Over 156 thru 168
	TOLERANCE—in. plus and minus														
0.006-0.010	.001	.0015	.0025	.0025	....	....	....	....	....	....	....	....	....	....	....
0.011-0.017	.0015	.0015	.0025	.0035	....			....	....	....	....	....	....	....	....
0.018-0.028	.0015	.002	.0025	.0035	.004	.004	.004		....	....	....	....	....	....	....
0.029-0.036	.002	.002	.0025	.004	.005	.005	.005	.006	.006	.007	.009	....	....	....	....
0.037-0.045	.002	.0025	.003	.004	.005	.005	.005	.006	.006	.007	.011	....	....		....
0.046-0.068	.0025	.003	.004	.005	.006	.006	.006	.007	.007	.008	.012	.013	....	....	....
0.069-0.076	.003	.003	.004	.005	.006	.006	.006	.007	.007	.012	.012	.016	....	....	....
0.077-0.096	.0035	.0035	.004	.005	.006	.006	.006	.007	.007	.012	.012	.016	....	....	....
0.097-0.108	.004	.004	.005	.005	.007	.007	.007	.008	.008	.016	.018	.020	....	....	....
0.109-0.125	.0045	.0045	.005	.005	.007	.007	.007	.008	.008	.016	.018	.020	....	....	....
0.126-0.140	.0045	.0045	.005	.005	.007	.010	.012	.013	.014	.016	.018	.020	....	....	....
0.141-0.172	.006	.006	.008	.008	.009	.012	.014	.015	.016	.017	.019	.023	....	....	....
0.173-0.203	.007	.007	.010	.010	.011	.014	.016	.017	.017	.017	.022	.026	....	....	....
0.204-0.249	.009	.009	.011	.011	.013	.016	.018	.018	.018	.018	.024	.028	....	....	....
0.250-0.320	.013	.013	.013	.013	.015	.018	.020	.020	.020	.020	.025	.030	.035	.042	.053
0.321-0.438	.019	.019	.019	.019	.020	.020	.023	.023	.025	.025	.026	.033	.038	.045	.057
0.439-0.625	.025	.025	.025	.025	.025	.025	.025	.030	.030	.030	.035	.035	.043	.049	.067
0.626-0.875	.030	.030	.030	.030	.030	.030	.030	.037	.037	.037	.045	.045	.054	.059	.077
0.876-1.125	.035	.035	.035	.035	.035	.035	.035	.045	.045	.045	.055	.055	.065	.070	.088
1.126-1.375	.040	.040	.040	.040	.040	.040	.040	.052	.052	.052	.065	.065	.075	.080	.098
1.376-1.625	.045	.045	.045	.045	.045	.045	.045	.060	.060	.060	.075	.075	.085	.090	.108
1.626-1.875	.052	.052	.052	.052	.052	.052	.052	.070	.070	.070	.088	.088	....	....	....
1.876-2.250	.060	.060	.060	.060	.060	.060	.060	.080	.080	.080	.100	.100	....	....	....
2.251-2.750	.075	.075	.075	.075	.075	.075	.075	.100	.100	.100	.125	.125	....	....	....
2.751-3.000	.090	.090	.090	.090	.090	.090	.090	.120	.120	.120	.150	.150	....	....	....
3.001-4.000	.110	.110	.110	.110	.110	.110	.110	.140	.140	.140	.160	.:60	....	....	....
4.001-5.000	.125	.125	.125	.125	.125	.125	.125	.150	.150	.150	.160	....	....	....	....
5.001-6.000	.135	.135	.135	.135	.135	.135	.135	.160	.160	.160	.170	....	....	....	....

TABLE 7.9 Width Tolerances SHEARED FLAT SHEET AND PLATE

SPECIFIED THICKNESS in.	SPECIFIED WIDTH—in.					
	Up thru 6	Over 6 thru 24	Over 24 thru 60	Over 60 thru 96	Over 96 thru 132	Over 132 thru 168
	TOLERANCE③—in.					
0.006-0.124	±1/16	±3/32	±1/8	±1/8	±5/32	—
0.125-0.249	±3/32	±3/32	±1/8	±5/32	±3/16	—
0.250-0.499	+1/4	+5/16	+3/8	+3/8	+7/16	+1/2

TABLE 7.10 Length Tolerances SHEARED FLAT SHEET AND PLATE

SPECIFIED THICKNESS in.	SPECIFIED LENGTH—in.							
	Up thru 30	Over 30 thru 60	Over 60 thru 120	Over 120 thru 240	Over 240 thru 360	Over 360 thru 480	Over 480 thru 600	Over 600 thru 720
	TOLERANCES③—in.							
0.006-0.124	±1/16	±3/32	±1/8	±5/32	±3/16	±7/32	±9/32	—
0.125-0.249	±3/32	±3/32	±1/8	±5/32	±7/32	±1/4	±5/16	—
0.250-0.499	+1/4	+3/8	+7/16	+1/2	+9/16	+5/8	+11/16	+3/4

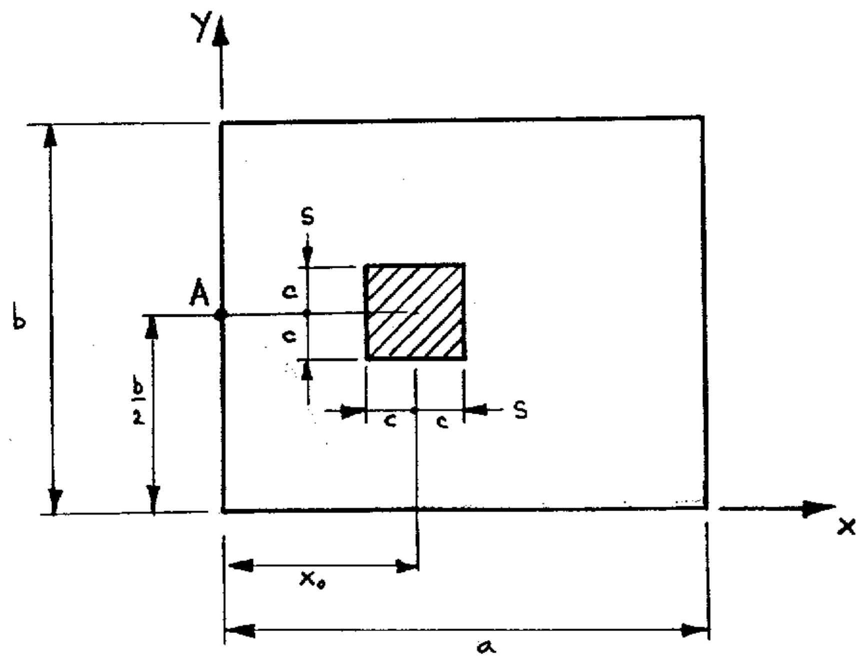
For all numbered footnotes, see page 120

Reaction at the edge of a plate

Problem

The problem is to compute the reaction (in lb/in or N/m) R at the edge of a simply supported plate submitted to a Load P located at $(x_o , b/2)$.

The figure below shows all the relevant parameters. The load P is *uniformly* distributed on the shaded area $S \times S$ ($S \equiv 2c$, $q = P/S^2$). "R" is the reaction at A.



Solution

Using the method explained in ref 1, it is found that:

$$\frac{R}{qa} = \frac{16}{\pi^3} \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x_0}{a}\right) \sin\left(\frac{m\pi c}{a}\right) \sum_{n=1,3,5\ldots}^{\infty} \frac{\left(m^2 + (2-\nu)\left(\frac{a}{b}\right)^2 n^2\right)}{n\left(m^2 + \left(\frac{a}{b}\right)^2 n^2\right)^2} \sin\left(\frac{n\pi c}{b}\right)$$

or knowing that $q = P/s^2$

$$\frac{Rb}{P} = \frac{(a/b)}{(s/b)^2} \times \frac{16}{\pi^3} \sum \ldots \sum \ldots$$

Tables I, III and V give:

$$\frac{Rb}{P} = f\left(\frac{a}{b}, \frac{s}{b}, \frac{x_0}{a}, \nu = 0.3\right)$$

Tables II, IV, VI give the maximum error on $\frac{Rb}{P}$ values obtained in Tables I, III, V respectively.

Table I

$$Rb/P = f\left(\frac{s}{b}=0.2, \frac{a}{b}, \frac{x_o}{a}, \nu = 0.3\right)$$

S/b= 0.20

$\frac{a}{b}$	$\frac{x_o}{a}$=0.05	0.10	0.15	0.20	0.30	0.40	0.50
0.25	******	******	******	******	******	2.61	2.12
0.50	******	******	******	3.04	2.31	1.73	1.29
0.75	******	******	2.94	2.43	1.68	1.21	0.89
1.00	******	3.12	2.46	1.93	1.27	0.89	0.64
1.25	******	2.60	2.06	1.57	1.00	0.68	0.47
1.50	******	2.47	1.74	1.29	0.79	0.52	0.35
1.75	******	2.16	1.49	1.09	0.64	0.40	0.25
2.00	3.11	1.95	1.30	0.92	0.52	0.31	0.18
2.25	2.99	1.74	1.14	0.80	0.43	0.23	0.13
2.50	2.79	1.57	1.00	0.69	0.35	0.18	
2.75	2.64	1.43	0.89	0.60	0.28		
3.00	2.48	1.30	0.80	0.52	0.23		
3.25	2.32	1.19	0.72	0.45	0.19		
3.50	2.18	1.08	0.64	0.39			
3.75	2.06	1.00	0.57	0.34			
4.00	1.95	0.92	0.52	0.30			

Table II

Error on Rb/P values given in Table I

S/b= 0.20

a/b	$\frac{x_o}{a}$=0.05	0.10	0.15	0.20	0.30	0.40	0.50
0.25	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.50	0.00	0.00	0.00	0.01	0.01	0.01	0.01
0.75	0.00	0.00	0.01	0.01	0.01	0.01	0.01
1.00	0.00	0.02	0.02	0.02	0.02	0.01	0.01
1.25	0.00	0.02	0.02	0.02	0.02	0.02	0.02
1.50	0.00	0.03	0.03	0.02	0.02	0.02	0.02
1.75	0.00	0.03	0.03	0.03	0.03	0.03	0.03
2.00	0.04	0.04	0.03	0.03	0.03	0.03	0.03
2.25	0.05	0.04	0.04	0.04	0.04	0.03	0.03
2.50	0.05	0.05	0.04	0.04	0.04	0.04	
2.75	0.06	0.05	0.05	0.04	0.04		
3.00	0.07	0.06	0.05	0.05	0.05		
3.25	0.07	0.06	0.06	0.05	0.05		
3.50	0.08	0.06	0.06	0.06			
3.75	0.08	0.07	0.06	0.06			
4.00	0.09	0.07	0.07	0.06			

Table III

$Rb/P = f(S/b = 0.1,\ \frac{a}{b},\ \frac{x_0}{a},\ \nu = 0.3)$

S/b= 0.10

$\frac{a}{b}$ \ $\frac{x_0}{a}$ =	0.05	0.10	0.15	0.20	0.30	0.40	0.50
0.25	******	******	******	6.09	4.61	3.45	2.58
0.50	******	6.29	4.94	3.89	2.59	1.84	1.34
0.75	******	5.00	3.57	2.71	1.75	1.23	0.90
1.00	6.31	3.99	2.74	2.04	1.30	0.90	0.65
1.25	5.67	3.27	2.19	1.62	1.01	0.68	0.47
1.50	5.02	2.74	1.82	1.32	0.80	0.52	0.35
1.75	4.46	2.36	1.53	1.10	0.64	0.40	0.26
2.00	4.00	2.05	1.32	0.94	0.52	0.30	
2.25	3.60	1.81	1.15	0.81	0.42	0.23	
2.50	3.27	1.62	1.01	0.69	0.35		
2.75	2.99	1.47	0.90	0.60	0.29		
3.00	2.74	1.33	0.81	0.52	0.24		
3.25	2.53	1.21	0.73	0.46			
3.50	2.35	1.10	0.65	0.40			
3.75	2.20	1.01	0.59	0.35			
4.00	2.07	0.93	0.53				

Table IV

Error on Rb/P values given in Table III

S/b= 0.10

$\frac{a}{b}$ \ $\frac{x_0}{a}$ =	0.05	0.10	0.15	0.20	0.30	0.40	0.50
0.25	0.00	0.00	0.00	0.01	0.01	0.01	0.01
0.50	0.00	0.01	0.01	0.01	0.01	0.01	0.01
0.75	0.00	0.02	0.02	0.02	0.02	0.02	0.02
1.00	0.03	0.03	0.03	0.03	0.02	0.02	0.02
1.25	0.04	0.04	0.03	0.03	0.03	0.03	0.03
1.50	0.05	0.04	0.04	0.04	0.04	0.04	0.03
1.75	0.06	0.05	0.05	0.04	0.04	0.04	0.04
2.00	0.07	0.06	0.05	0.05	0.05	0.05	
2.25	0.08	0.06	0.06	0.06	0.05	0.05	
2.50	0.08	0.07	0.07	0.06	0.06		
2.75	0.09	0.08	0.07	0.07	0.07		
3.00	0.10	0.09	0.08	0.08	0.07		
3.25	0.11	0.09	0.09	0.08			
3.50	0.12	0.10	0.09	0.09			
3.75	0.13	0.11	0.10	0.10			
4.00	0.14	0.11	0.11				

Table V

$Rb/p = f\left(\frac{s}{b} = 0.05,\ \frac{a}{b},\ \frac{X_0}{a},\ \nu = 0.3\right)$

S/b= 0.05

a/b =	X_0/a = 0.05	0.10	0.15	0.20	0.30	0.40	0.50
0.25	******	12.61	9.87	7.78	5.17		
0.50	12.68	8.00	5.49	4.10	2.64		
0.75	10.07	5.54	3.72	2.76	1.77		
1.00	8.03	4.20	2.79	2.06	1.31		
1.25	6.60	3.36	2.21	1.62	1.00		
1.50	5.57	2.80	1.83	1.32	0.80		
1.75	4.80	2.39	1.53	1.10	0.64		
2.00	4.21	2.08	1.33	0.93	0.53		
2.25	3.73	1.83	1.17	0.80	0.43		
2.50	3.37	1.62	1.03	0.70	0.34		
2.75	3.08	1.45	0.91	0.61			
3.00	2.82	1.33	0.81	0.54			
3.25							
3.50							
3.75							
4.00							

Table VI

Error on Rb/p values given in Table V

S/b= 0.05

a/b =	X_0/a =0.05	0.10	0.15	0.20	0.30	0.40	0.50
0.25	0.00	0.01	0.01	0.01	0.01		
0.50	0.03	0.03	0.03	0.02	0.02		
0.75	0.05	0.04	0.04	0.04	0.04		
1.00	0.07	0.06	0.05	0.05	0.05		
1.25	0.08	0.07	0.06	0.06	0.06		
1.50	0.10	0.08	0.08	0.07	0.07		
1.75	0.11	0.10	0.09	0.09	0.08		
2.00	0.13	0.11	0.10	0.10	0.09		
2.25	0.15	0.12	0.12	0.11	0.11		
2.50	0.16	0.14	0.13	0.12	0.12		
2.75	0.18	0.15	0.14	0.14			
3.00	0.20	0.16	0.15	0.15			
3.25							
3.50							
3.75							
4.00							

Gérard Carrier

ref 1 : L.G. Laeger "Elementary Theory of elastic Plates"
Pergamon Press, London, 1964

Revision F
#177 I

STRUCTURAL DESIGN MANUAL

VOLUME I

Bell Helicopter TEXTRON

Bell Helicopter TEXTRON
Division of Textron Inc.

INTER-OFFICE MEMO

27 October 1977
81:GLJ:jo-666

Memo To: Holders of Structural Design Manual

Subject: Revision A

Changes to the Structural Design Manual made by Revision A are listed below. Please remove superseded pages and add revised pages and new pages to your copy.

Revised pages: 2-1, 2-5, 2-6, 3-18, 3-19, 3-27, 3-57, 4-13, 4-17, 6-1, 6-2, 6-44, 6-52, 6-58, 6-63 thru 6-67, 6-80, 6-81, 6-82, 7-50, 8-11, 9-9, 9-24, 10-4, 10-20, 10-22 thru 10-26, 10-37, 10-39 thru 10-42, 10-44 thru 10-47, 10-49, 10-50, 10-52, 10-57, 10-58, 10-71, 10-73, 11-9, 11-10, 11-11, 11-73, 11-74, 12-93, 12-94, 13-5, 13-7, 13-57, 14-10.

New pages: 10-20a, 10-20b, 13-6a.

G. L. Jordan
G. L. Jordan
Experimental Airframe Stress

Approved:

M. J. McGuigan Jr.
M. J. McGuigan, Jr.
Chief of Structures Technology

Bell Helicopter TEXTRON
Division of Textron Inc.

INTER-OFFICE MEMO

12 May 1978
81:GLJ:jo-726

Memo To: Holders of Structural Design Manual

Subject: Revision B

Changes to the Structural Design Manual made by Revision B are listed below. Please remove superseded pages and add revised pages and new pages to your copy. Insert this memo preceding the Table of Contents in Volume I.

Volume I revised pages: Title Page, v, vi, viii, 2-2, 3-4, 3-11, 3-57, 3-79, 4-16, 4-22, 4-23, 4-34, 4-46, 6-11, 6-15, 6-18, 6-24, 6-32 thru 6-36, 6-38, 6-41, 6-61, 6-62, 6-64 thru 6-67, 6-80, 6-82, 6-87, 6-88, 8-5, 9-18, 9-19, 9-34, 9-38.

Volume I new pages: 6-65a, 6-65b, 6-67a, 6-67b, 6-89 thru 6-94.

Volume II revised pages: Title Page, v, vi, viii, 10-7, 10-22, 10-58, 10-70, 10-71, 10-75, 11-74, 12-8.

G. L. Jordan
G. L. Jordan
Experimental Airframe Stress

Approved:

M. J. McGuigan, Jr.
M. J. McGuigan, ~~Jr.~~
Chief of Structures Technology

Bell Helicopter TEXTRON
Division of Textron Inc.

INTER-OFFICE MEMO

Brach

9 October 1979
81:ELB:jo-908

Memo To: Holders of Structural Design Manual

Subject: Revision C

Please insert the following revised pages in your copy of the Structural Design Manual and insert this memo preceding the Table of Contents in Volume I as a summary of the changes of Revision C.

Additional corrections and suggested inclusions may be submitted to the undersigned for incorporation in the next revision.

Volume I changes:

Revised Title Page, vi, 2-1, 2-2, 2-3, 2-4, 2-5, 2-6, 3-2, 3-64, 4-24, 5-2, 6-7, 6-8, 6-54, 7-4, 7-5, 8-5, 8-6, 9-26, 9-58
Added Pages 6-93, 6-94, 6-95/6-96
Changed Page number 6-93, 6-94 to 6-97, 6-98

Volume II changes:

Revised Title Page, vi, 10-15, 11-18, 11-25, 11-30, 11-31, 11-32, 11-33, 11-75, 11-101, 12-3, 13-21, 13-23, 13-47

E. L. Brach
E. L. Brach
Experimental Airframe Stress

APPROVED:

M. J. McGuigan, Jr.
M. J. McGuigan, Jr.
Chief of Structures Technology

Bell Helicopter TEXTRON
A Subsidiary of Textron Inc

INTER-OFFICE MEMO

2 September 1983
81:DL:wc-1475

MEMO TO: Holders of Structural Design Manual

SUBJECT: REVISION D

Please insert the following pages in your copy of the Structural Design Manual and insert this memo preceding the Table of Contents in Volume I as a summary of the changes of Revision D.

Volume I changes:

Revised Title Page, iii, 2-1, 2-4 thru 2-12.

Added Pages:

2-13, 2-14, 2-15.

Volume II changes:

Revised Title Page, page iii.

D. Lindsay
Airframe Structures

Approved:

O. Baker
Manager of Structural Analysis

Zuy

Bell Helicopter TEXTRON
A Subsidiary of Textron Inc

INTER-OFFICE MEMO

17 April 1985
81:DL:db-1475

MEMO TO: Holders of Structural Design Manual

SUBJECT: REVISION E

Please insert the following pages in your copy of the Structural Design Manual and insert this memo preceding the Table of Contents in Volume I as a summary of the changes of Revision E.

Volume I changes:

Revised Title Page, 3-17, 4-11, 4-24, 5-18, 6-13, 6-39, 6-52, 6-56, 6-59, 6-64, 6-73, 6-82, 6-68, 6-69, 7-19, 7-27, 7-31, 8-3, 8-15, 9-52, 9-113.

Volume II changes:

Revised Title Page, 10-19, 10-21, 10-24, 10-26, 10-28, 10-31, 10-35, 10-39, 10-40, 10-41, 10-45, 10-47, 10-70, 10-71, 11-82, 11-101, 11-112, 11-113, 12-20, 12-21.

D. Lindsay
Airframe Structures

Approved:

G. R. Alsmiller, Jr.
Manager Structures and Materials

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INTRODUCTION

The purpose of the Bell Helicopter Structural Design Manual is to provide a source of structural analysis methods, material data, procedures and policies applicable to structural design. The data contained herein are largely condensations of information obtained from the Federal Government, universities, textbooks, technical publications and Bell reports and memoranda. As much as possible the sources are noted and shown in a list of references at the beginning of the manual.

This manual is intended to provide the structural designer with necessary design information in the form of equations, curves, tables and step-by-step procedures. Derivations are purposely omitted to make the manual easier to use. Comments and suggestions regarding this manual are encouraged. They should be directed to the Chief of Structural Technology, Bell Helicopter Textron.

SECTION 1

PROCEDURES

1.1 General

The Structures Technology Section of Engineering has the primary responsibility of insuring structural integrity of all Bell Helicopter products at minimum weight and cost. To this end, the following procedures are outlined.

1.2 Design Coverage

The Structures Engineer must follow a design from its inception. His requirements and suggestions must be submitted to the Design Engineer as the design progresses. The Structures Engineer must frequently review the work of the designer and if possible be ready to sign the drawing when it is complete. In most cases, no changes should have to be made to a drawing after it has been submitted to the Structures Group for signature.

Structures Engineers are responsible for obtaining concurrence, and signatures where required, of Materials Technology and Fatigue Groups.

Components should be substantiated by using established and recognized stress analysis methods or by comparison to existing test data. If, for specific problems design support test data are needed, Structures Engineers should initiate test requests.

1.2.1 Critical Parts

All parts of the helicopter structure which are subjected to significant oscillatory loads, or which are of prime importance to safety of flight, will come under the heading of critical parts and will be treated as follows:

- All critical parts will be evaluated for static design loading, fatigue, fretting, corrosion, residual stresses, stress corrosion, corrosion fatigue and other environmental conditions.

- Analysis should include primary and secondary modes of failures as well as the effects of bearing friction or stiffness under load. All such analysis should be kept in a drawing check notebook.

- any significant design changes, alternate parts or materials will be reviewed with special care. The requirement for requalification tests will be a joint decision of the Structures Group Engineer and Project Engineer (and D.E.R. where applicable).

- In all tests conducted, the objectives of the test will be stated prior to the test. where possible, the method of interpretation of test results will be defined prior to the test. Test results that are unexpected or unexplained will be pursued to a solution.

- Flight and laboratory test requests should be accompanied by a brief writeup of expected results where possible and Lab and Flight Test personnel will be requested to alert affected groups of any unexpected results.

- All Structures personnel will work with the Fatigue Group, Materials Technology Group, Laboratory personnel and the appropriate design group in the design of new parts to reduce stress concentrations and stress corrosion susceptibility and to improve fretting characteristics.

- In all new designs, maximum use will be made of service experience from similar parts of structural configurations used on previous models.

1.2.2 Flight Safety Parts

A flight safety part is defined as any part, assembly or installation whose failure, malfunction or absence could cause loss or serious damage to the aircraft and/or serious injury or death to the occupants or group support personnel.

A part is considered a FSP if item 1.2.2.1 is affirmative and any one of item 1.2.2.2 is affirmative.

1.2.2.1 Primary failure or malfunction affects the safe operation of the aircraft.

1.2.2.2

a. The part has a predicted or demonstrated finite life.

b. A 10% reduction in laboratory working strength would result in an unlimited life becoming a finite life.

c. Loss of function could occur due to improper assembly or operation.

d. Fabrication of the part involves a manufacturing process which, if improperly performed, has high probability of changing material properties significantly impacting life.

Flight Safety Parts are further governed by BHTI Requirements Specification 299-947-478. Each FSP must show at least one critical characteristic.

1.2.3 Check List for Drawing Review

The following check list is a guide for drawing review:

General Considerations

- Are critical net sections okay?

- Take into account area reductions for stretched formed parts.

- Be aware of clamp-up stresses and associated stress corrosion problems.
- Account for all local discontinuities and centroid shifts.
- Consider all secondary effects such as tension field loads.
- Avoid abrupt changes in area.
- Use no one-rivet shear clips.
- Check for column stability on all compression members.
- Stress relieve and heat treat after welding.
- Specify machine finishes.
- Design for thermal stresses and strains.
- Reduce allowables for temperature.
- Check for no yield at limit load as well as no failure at ultimate since some materials have low yield/ultimate relationship.
- For material allowables, use "A" values with all statically determinate structures. Use "B" values with redundant structures and structures designed for crash conditions, if specifically approved by the procuring agency.
- Investigate stiffness requirements.
- Check strain compatibility of joined structures.
- Check structural deflection.
- When necessary to use dissimilar metals, design for their use, especially in joints.
- Design for wear, abrasion and fretting.
- Design for corrosion and stress corrosion due to residual stresses.
- Account for friction.
- Check for surface treatments such as paint, chrome, hard anodizing, cadmium, etc.
- Proof load parts per specification requirements (such as hoists, control systems, hydraulics, etc.).
- Make the structure as light as possible.

- Verify that all processes necessary to the strength of the structure are properly called out.

Fasteners

- Verify that bolt grip lengths are adequate to prevent threads in bearing.
- Avoid using rivets in primary tension applications.
- Make joint critical in sheet bearing rather than shear of fasteners.
- Specify torque requirements on all bolts. Do not use bolts less than ¼ inch diameter in load carrying capacity without specific approval of Structures Group Engineer.
- Avoid using blind fasteners in engine inlets or by the tail rotor side of a fin.
- Do not use nutplates with reduced rivet spacing.
- Avoid using rivets less than 1/8 inch in diameter in structural applications.
- Avoid mixing bolts and rivets in shear joints.
- Avoid using tension and shear fasteners as load sharing attachments in joint design.
- Examine hole tolerances and fits.
- Avoid using screws in primary tension applications with repeated loads.
- NAS quality bolts shall be installed in the movable portion of all control system joints.
- Commercial applications require dual locking devices on threaded fasteners or analysis to show fail safe with one fastener missing.

Composite Structures

- Verify the use of appropriate adhesive and supporting BPS.
- Complete and check the destructive test diagram for sandwich construction.
- Assure that appropriate extensions or cutoffs are included for destructive test of laminates.
- Verify that the structure carries the appropriate classification.

- Identify critical areas on the destructive test diagram.
- Verify that processes called out will produce the desired structure.
- Verify that the part or assembly carries the appropriate classification. See the DRM, Section 2H-13 for definitions.

Castings

- Verify that the casting carries the appropriate classification.
- Review for approval of weld repair and the associated reduction in strength. (Reference SIM No. 9)
- Verify that x-ray standards are appropriate for the casting classification.
- Identify critical areas on the x-ray diagram.

1.2.4 Envelope, Source Control, and Specification Control Drawings

All Structures personnel who have occasion to sign Envelope, Source Control, or Specification Control Drawings shall, as a minimum, establish that the drawing adequately defines the following:

- Configuration
- Mounting and mating dimensions
- Dimensional limitations (interferences)
- Performance (loads, environment, life, etc.)
- Weight limitations
- Reliability requirements
- Interchangeability requirements
- Test requirements
- Verification requirements (analysis or test)
- Material limitations (example, no castings allowed, etc.)
- Casting classification if allowed (also casting factor)
- Primary Part designation
- Reference to applicable specification.

Also, if special inspections and tests such as x-rays and static tests are required, the Project Engineer should be alerted so that plans can be made to procure parts for the required tests.

"Approved Sources of Supply" or "Suggested Sources of Supply" shall be approved by Structures only if the proposed vendor item meets all structural requirements. This may mean vendors must submit stress analyses of their design or test data as a part of their proposal.

On all Primary Parts or other items with significant structural requirements, the Structures Engineer shall retain a copy of the approved design, vendor stress analysis and test data and file this information in the proper drawing check notebook.

1.3 Stress Analysis

A stress analysis of each component of the helicopter structure is required. The analysis should be in the following format:

- $8\frac{1}{2}$ x 11 paper (BHTI stress pad)
- Analyst's name (no initials) and date at the top of each page
- Part number of the part being analyzed at the top right hand corner of each page
- Number all pages (1 of 10, etc.).

Each analysis must contain the following information:

- Sketch of the part being analyzed. Should be to scale but must contain enough dimensions to derive loads or calculate critical sections.
- Free body diagram. Must contain all loads for a particular condition. Must be in static equilibrium in all views.
- All the loads cases necessary to determine the critical cases for the part. Various sections of the part may be designed by different loads conditions. Be careful to identify loads as limit or ultimate.
- Material of which the part is made.
- Material allowable stresses and source. The source must be approved by the procuring agency.
- Step by step procedure for the stress analysis of all failure modes. If equations being used are not widely recognizable standard stress equations (i.e., P/A, Mc/I, Vq/It, etc.) the source of the equation must be stated.

- Margin of safety calculation.

- Any information which was used in the design of the part must be included.

- Specify applicable factors (casting, fitting, etc.).

- Assumptions that you made in order to "idealize" a structure, unless they would be clearly understood by another person.

- Calculations that are superseded by a redesign must be clearly marked "void" or "obsolete" and referenced to the new calculations.

The stress notes must be kept up to date by the Structures Engineer until the time the engineering drawing is approved by the Structures Group. At this time the stress notes are filed in a master file from which they will not be removed. This file is maintained in each project area by the Lead Structures Engineer.

Drawing Tolerances for Section Properties and Stress Analysis

For analyses of basic extrusion (shapes), drawn tubing and sheet stock, nominal dimensions will be used. Nominal dimensions are defined as those from which the tolerance is added or subtracted. If a dimension is given as an upper and lower limit, the nominal dimension is the mean value.

For aluminum airframe parts other than those specified above where it is desirable to specify a min-max thickness, the following procedure will be used:

- Calculate the required thickness for structural consideration based on standard analysis methods. Specify the minimum thickness to be the required thickness minus the tolerance shown below:

Thickness Range	Tolerance	Example
.012 to .036	.002	If t (required) = .120 in.
.037 to .045	.003	Specify: t (min) = .120-.005
.046 to .096	.004	= .115 in.
.097 to .140	.005	t (max) may be .125, .130, 132...
.141 to .172	.008	depending on the material
.173 to .203	.010	thickness tolerance.
.204 to .249	.011	
.250 to .320	.013	

NOTE: If it is possible to hold tighter tolerances than shown above, use nominal thickness in the analysis. The Structures Engineer should seek the closest tolerance possible commensurate with cost considerations in order to minimize weight.

1.4 Structures Report

The following information is submitted as a format for structures reports. It is necessarily a general outline but should be adhered to in order to produce a document that has clearly defined subject matter, is accurate in technical aspects and is clearly readable as well as reproducible.

1.4.1 Introductory Data

Preceding the technical content or body of the report is:

a. Title Page - Must contain all authors, group and project approval, DER approval if commercial, contract number if military and revision status.

b. Revision Status Listing (for a multi-volume report)

c. Revision Summary Page

d. Proprietary Rights Notice

e. A page of individual author signatures (when too numerous for the Title Page)

f. Table of Contents - must contain all the major section headings and basic divisions within each section. Each topic will have the appropriate page number. If the report is multi-volume, each volume will contain a full Table of Contents for all volumes.

g. References - (See General Information)

h. List of Figures (not included in reports having less than 200 pages)

i. List of Tables (not included in reports having less than 200 pages)

j. Definitions and Symbols

k. Three-View Drawing (basic design criteria, loads and fuselage reports)

l. Basic Lines Data (loads report and airframe analysis)

m. Sign Convention

n. Table of Minimum Margins of Safety - On a part with multiple margins, both high and low, no margin greater than .25 will be shown. Minimum margins of safety include only the minimums on each part analyzed and generally speaking, margins greater than 1.0 are not shown. Discretion must be used and margins greater than 1.0 may be shown on critical parts or system such as controls if all margins are greater than 1.0.

o. Introduction - The report introduction should always refer to the Basic Design Criteria Report as a source of helicopter data, horsepower, C.G. vs G. Wt., load conditions, load factors, etc.

1.4.2 Body of the Report

The body of the report is the structural analysis and should be broken down into major components, parts or system and conform to the outline as shown in the Table of Contents. Pages will be numbered sequentially within major sections; i.e., 1.001 . . ., 2.001 . . ., etc. Page numbers for all pages preceding the body of the report shall be small Roman numerals, as i, iv, xii . . .

Stress analysis of components, details or systems:

a. Should be written on Bell stress pad (originals should always be used in a report and copies filed in drawing checks or elsewhere).

b. A discussion should precede the analysis of each major component, part or system. This should include a description of the system or structures being analyzed, a reference to the loading conditions or design criteria, a summary of the methods of analysis used, the assumptions made and should often include statements regarding parts that are not analyzed as "being not critical." Load configurations or conditions not analyzed should also be noted as "not critical."

c. Page headings will include nomenclature and part number.

d. A sketch and/or geometry of the part or component being analyzed with the critical load condition and load direction shown. Reactions shall be shown and the part or component will be statically balanced. When possible, the axis shown should conform to the helicopter sign convention in the preface of the report, location of the part should be identified in a positive manner such as W.L.'s, B.L.'s or STA's and when not clear "up", "fwd", etc. should be noted. Should the analysis to follow include several details, sections, or panels, etc., these should be lettered for identification.

e. Identification of material(s) and properties. Note the allowable stresses and/or loads and a reference to the source by page number or show the computed allowable.

f. Special factors accounting for stress concentrations, cast materials, fatigue considerations, etc. should be identified together with their source. These factors will appear not in loads but in margin of safety calculations.

g. The analysis is generally written for ultimate loads unless it is necessary to show compliance with limit load or fatigue criteria.

h. A margin of safety or fatigue life calculation concludes the analysis. The correct margin should be shown. If a part is loaded in tension and bending, stress ratios for tension and bending modulus should be calculated and a margin of safety based on the sum of the stress ratios shown. Show the margin calculation at the extreme right hand side of the page.

i. When the analysis is extensive and it is deemed advisable to summarize results in a table, shown the tabulated results with reference to pages from which the results were obtained.

This table of "Summary of Results" should precede the analysis. The entire picture is, therefore, shown in the "Discussion", the "Geometry and Loads", and the "Summary of Results".

1.4.3 <u>General Information</u>

a. Since our reports are reproduced, write with a soft lead for maximum reproducibility.

b. Extensive usage of flag notes and footnotes is not advised.

c. Adequately reference what you put down; i.e., do not assume the reader knows where this information came from. Many of our readers are in foreign countries. When making the reference, give page number as well as the title.

d. Use of the Tenses - Computations appearing in the report should be referred to in the present tense. The past tense is used when referring to work not appearing in the report, but which was done as a prerequisite to data in the report. Do not slip into the present imperative or past or future tenses. Use the third person throughout.

e. Be neat; do not overcrowd the page.

f. Avoid the usage of 11 x 17 pages, if practical. Proper planning will minimize this.

g. A "List of References", "Proprietary Rights Notice", a "Revision" page and a "Distribution List" will be in all volumes.

h. Coordinate with the project office and/or contractual data to determine who is on the distribution list.

i. References should list Bell reports first, generally beginning with the "Basic Design Criteria", other reports, textbooks such as "Peery", "Bruhn", "Timoshenko", etc., MIL-HBDKs, NACA Technical notes and vendor data, i.e., honeycomb data, bearing catalogs, etc., in the order shown.

j. Discussions, references, general data, table of contents, etc., on all reports shall be typed.

1.5 Structural Information Memos

1.5.1 Purpose

Structural Information Memos (SIM) are distributed by the Director of Structures Technology to make new and unique structural design information available to members of the Structures groups and appropriate design groups.

1.5.2 Preparation

Each new SIM submitted for approval shall include a cover memo addressed to the Director of Structures Technology, giving a brief synopsis of the Material. The memo shall be signed by the originator and approved by the SIM coordinator. The originator of each SIM shall be responsible for establishing the credibility and accuracy of his information and for preparing the SIM for distribution. Each SIM shall "stand on its own" and be thoroughly checked and referenced. Format of the material is left to the discretion of the originator.

1.5.3 Published SIM's

Previously published SIM's are included in the following pages. The signatures have been removed, but the originals of the SIM's are available from structures technology.

STRUCTURES INFORMATION MEMO NO. 1

June 15, 1972

SUBJECT: PROCEDURE FOR STRUCTURES INFORMATION MEMO (SIM)

REFERENCES: (a) As required
(b) As required

ENCLOSURES: (a) SIM Index
(b) SIM Distribution

This memo is written to establish a procedure for making new and unique structural design information available to members of the Structures groups and appropriate design groups. Much useful information is either generated or collected by members of the Structures groups during the normal performance of their duties. This information is usually available to a limited number of persons and is often filed away and forgotten. In order to prevent valuable information from becoming useless and forgotten, the Structures Information Memo is hereby established as the vehicle for conveying this information.

The Methods and Materials Structures Group Engineer will be the coordinator for all SIM's and will assist in determining what information is valuable enough to publish. He will retain all originals, will assign SIM index number, and update the index and distribution list as required.

Each SIM shall include a cover memo, addressed to the Chief of Structural Design, giving a brief synopsis of the material. The memo shall be signed by the originator and approved by the SIM coordinator. The originator of each SIM shall be responsible for establishing the credibility and accuracy of his information and for preparing the SIM for distribution. Each SIM shall "stand on its own" and be thoroughly checked and referenced. Format of the material is left to the discretion of the originator, however, it should be remembered that all SIM's will be considered for incorporation in a Structures Manual to be issued at a later date. Similar significant structural information originating in any design group will be welcomed and handled in the same manner. Any additions or deletions to the distribution list should be directed to the SIM coordinator.

STRUCTURES INFORMATION MEMO NO. 2

August 27, 1973

SUBJECT: REPORT FORMAT FOR STRUCTURAL ANALYSIS

The following information is submitted in an effort to clarify questions regarding the above subject by persons involving in writing stress Analysis reports. It is necessarily a general outline but should be adhered to in order to produce a document that has clearly defined subject matter, is accurate in technical aspects and is clearly readable as well as reproducible. This is our product.

1. Preceding the technical content or body of the report is:

   a. Title Page
   b. Proprietary Rights Notice
   c. Revision Status Listing (when a report has multiple volumes)
   d. A revision page (blank in a new report except for report number and volume number in place of page number)
   e. A page of individual author signatures (when too numerous to be listed on the Title Page).
   f. Table of Contents (the first numbered page)
   g. References (see General Information)
   h. List of Figures (not included in reports having less than 200 pages)
   i. List of Tables (" " " " " " " " ")
   j. Definitions and Symbols
   k. Three-View Drawing (basic design criteria, loads and fuselage reports)
   l. Basic Lines Data (loads report and airframe analysis)
   m. Sign Convention
   n. Table of Minimum Margins of Safety
      On a part with multiple margins, both high and low, no margin greater than .25 will be shown. Minimum margins of safety include only the minimums on each part analyzed and generally speaking, margins greater than 1.0 are not shown. Discretion must be used and margins greater than 1.0 may be shown on critical parts or system such as controls if all margins are greater than 1.0.
   o. Introduction
      The report introduction should always refer to the Basic Design Criteria Report as a source of helicopter data, horsepower, C.G. vs G. Wt. load conditions, load factors, etc.

2. Body of the report:

   The body of the report is the structural analysis and should be broken down into major components, parts or systems and conform to the outline as shown in the Table of Contents. Pages will be numbered sequentially within major sections; i.e., 1.001. . ., 2.001 . . ., etc. Page numbers for all pages preceding the body of the report shall be small roman numerals, as i, iv, xii . . .

3. Stress analysis of components, details or systems:

a. Should be written on Bell stress pad (see example on page 1-12; originals should always be used in a report and copies filed in drawing checks or elsewhere).

b. A discussion should precede the analysis of each major component, part or system. This should include a description of the system or structure being analyzed, a reference to the loading conditions or design criteria. a summary of the methods of analysis used, the assumptions made and should often include statements regarding parts that are not analyzed as "being not critical". Load configurations or conditions not analyzed should also be noted as "not critical".

c. Page headings will include nomenclature and part number. (See example on page 1-16).

d. A sketch and/or geometry of the part or component being analyzed with the critical load condition and load direction shown. Reactions shall be shown and the part or component will be statically balanced. When possible the axis shown should conform to the helicopter sign convention in the preface of the report, location of the part should be identified in a positive manner such as W.L.'s, B.L.'s or STA's and when not clear "up", "fwd", etc. should be noted. Should the analysis to follow include several details, sections, or panels, etc., these should be lettered for identification.

e. Identification of material(s) and properties. Note the allowable stresses and/or loads and a reference to the source by page number or show the computed allowable.

f. Special factors accounting for stress concentrations, cast materials, fatigue considerations, etc. should be identified together with their source. These factors will appear not in loads but in margin of safety calculations.

g. The analysis is generally written for ultimate loads unless it is necessary to show compliance with limit load or fatigue criteria.

h. A margin of safety or fatigue life calculation concludes the analysis. Show the margin calculation at the extreme right hand side of the page. No negative margins of safety may be shown; however, zero margins are acceptable.

i. When the analysis is extensive and it is deemed advisable to summarize results in a table, show the tabulated results with reference to pages from which the results were obtained. This table of "Summary of Results" should <u>precede</u> the analysis. The entire picture is therefore shown in the "Discussion", the "Geometry and Loads", and the "Summary of Results".

4. General Information

a. Since our reports are reproduced, write with a soft lead, such as "F" grade for maximum reproducibility.

b. Extensive usage of flag notes and footnotes is not advised.

c. Adequately reference what you put down; i.e., don't assume the reader knows where this information came from. Many of our readers are in foreign countries.

d. Be neat, dont' overcrowd the page.

e. Avoid the usage of 11 x 17 pages, if practical. Proper planning will minimize this.

f. For multi-volume reports, a complete table of contents will be shown for all volumes in Volume I.

g. A table of contents for that volume will be shown in volumes other than Volume I.

h. A "List of References," "Proprietary Rights Notice," a "Revision" page and a "Distribution List" will be in all volumes.

i. Co-ordinate with the project office and/or contractual data to determine who is on the distribution list.

j. References should list Bell reports first, generally beginning with the "Basic Design Criteria," other reports, textbooks such as "Peery," "Bruhn," "Timoshenko," etc., MIL-HDBK's, NACA Technical notes and vendor data i.e. Hexcel data, bearing catalogs, etc. in the order shown. Note: Structures manuals from other companies are <u>not</u> legitimate references.

i. Discussions, references, general data, table of contents, etc. on all reports shall be typed.

BY Your Name Here	Bell Helicopter TEXTRON	MODEL ____ PAGE ____
CHECKED ____	POST OFFICE BOX 482 • FORT WORTH, TEXAS 76101	Leave Blank on Com. RPT ____

Add dwg. number part being analyzed in this box.

TITLE HERE

Leave space for revision letter later

DETAIL PART NOMENCLATURE HERE

State geometry, loads, detail and location or reference Sect. A., Pg. ____ where this is shown. See #3.

Compute the actual ultimate stress level, generally from limit loads, referencing a report or page number for the loads.

It may be necessary to compute section properties, loads on the section being analyzed or to determine and show a static balance prior to computation of the stress level.

Compute an allowable or state a reference for the allowables being used.

State limit loads and yield allowables when these are used to prove structure is non-yielding at limit load.

State the margin of safety.

This is the purpose and conclusion of the analysis. Be sure to include fitting factors, casting factors in the M.S. and so state, i.e., "using 1.15 fitting factor". State which formula is used such as,

$$M.S. = \frac{1}{(R_1 + R_2)(\text{Factor})} - 1 = +.xx$$

Confine the analysis within these limits and thereby preserve the neatness of the report

STRUCTURES INFORMATION MEMO NO. 3

August 28, 1973

SUBJECT: LOAD SHEETS FOR STATIC TEST OF CASTINGS

This memo is written to standardize the data furnished on the Load Sheets prepared for the Mechanical Laboratory and used by them in the static tests of castings.

The following information, as a minimum, should be included on all Load Sheets.

1. Title, part no., and name, thus:

   CASTING LOAD SHEET
   Part No. 204-XXX-XXX-X, Bellcrank, Cyclic Control

2. Indicate loads as "limit" or "ultimate". Ultimate preferred.

3. Draw sketches of part showing the external load application, direction and magnitude, and the reactions (usually designed by , , and). Sufficient views shall be used to completely define the critical loading condition. Each view shall show the reactions necessary to place the part, with its applied external load(s), in a state of static equilibrium. The loads and reactions shall be the same as those used in the structural analysis to insure that the part will be tested in the same manner as it was analyzed. Where moment vectors ( ) are used a note shall be included to indicate whether the right or left hand rule is applicable.

4. When available the report number from which loads and reactions were obtained shall be referenced, thus:

   Ref. Report 205-XXX-XXX

5. A note shall give a brief description of the loading condition, thus:

   Loading condition: 8G Forward Crash

6. A note shall indicate the casting factor, thus:

   Loads include a 1.33 casting factor

   Where the casting factor is unity, so indicate, thus:

   Casting factor of 1.00 is applicable

7. Any other special information necessary to assure that the casting will be tested as it was analyzed.

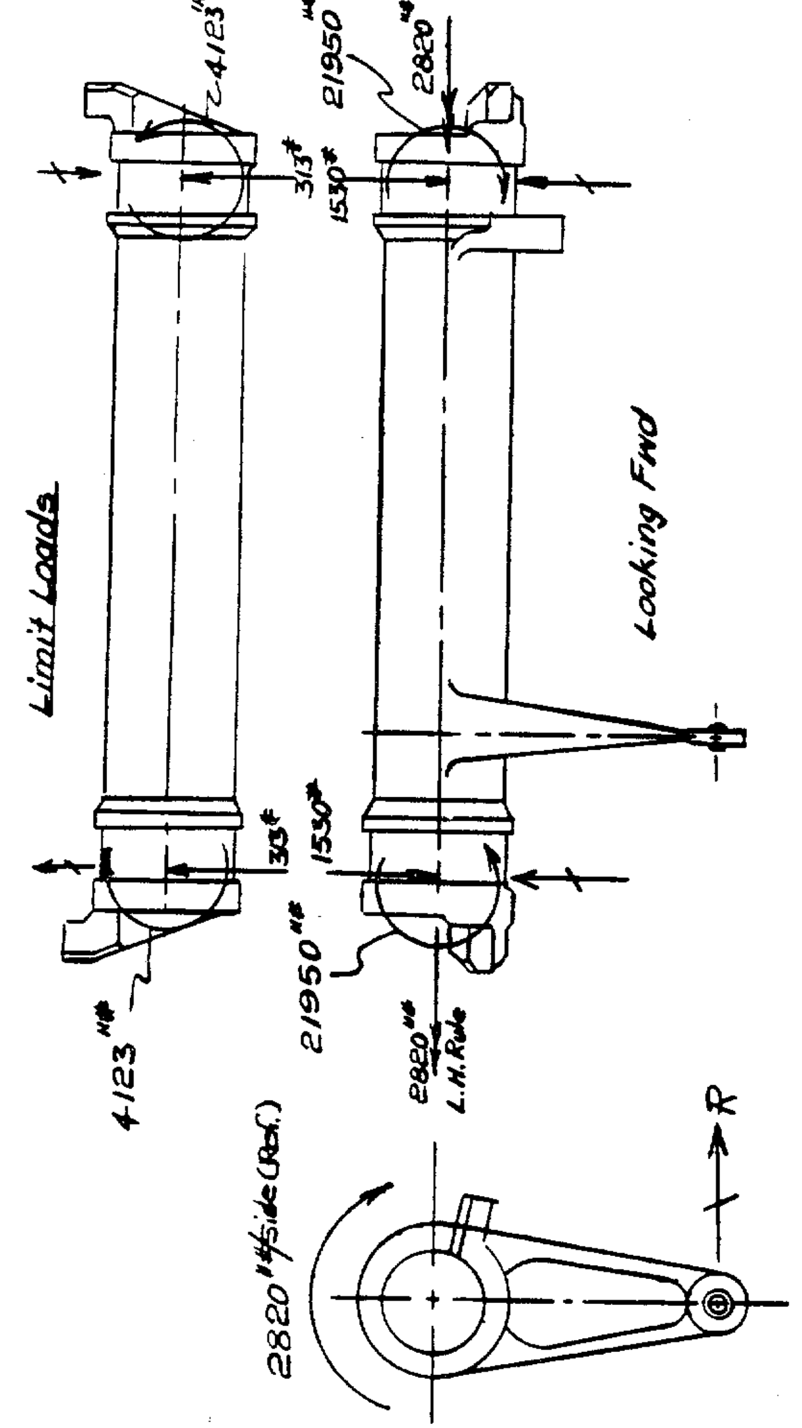
8. All Load Sheets shall be prepared on stress pad paper.

An example Load Sheet is attached. Note that the rule for the moment vectors was omitted.

BY	Bell Helicopter TEXTRON A Subsidiary of Textron Inc.	MODEL	PAGE
CHECKED	POST OFFICE BOX 482 • FORT WORTH, TEXAS 76101	RPT	

CASTING LOAD SHEET

209-001-908 HORN-ELEVATOR CONTROLS



Loads Are Based on $C_{L_{MAX}}$ @ 220 knots.

A 1.33 Casting Factor is Included in the Loads Shown.

Ref. 209-099-056

STRUCTURES INFORMATION MEMO NO. 4

27 February 1974

SUBJECT: BOLTS IN MOVEABLE CONTROL SYSTEM JOINTS

In order to avoid the possibility of installing an understrength bolt and to provide increase resistance to repeated loads, the following policy shall be implemented on the Model 409, Model D306, Model 301; the production series of the Model 214 and Model 206L; and all future designs.

- NAS quality bolts shall be installed in the movable portion of all control system joints.

STRUCTURES INFORMATION MEMO NO. 5

12 March 1974

SUBJECT: JUMP TAKEOFF LOADS

Recently, it has come to my attention that we are not addressing the rotor tilt for the jump takeoff conditions in a consistent manner. In order to provide a uniform approach, the following procedure shall be followed:

- Assume the helicopter has landed on a slope of specified magnitude in any direction (normally 6°) and executes a vertical takeoff at maximum load factor for this condition. The rotor tilt will be that which is necessary to execute this maneuver.

STRUCTURES INFORMATION MEMO NO. 6

2 August 1974

SUBJECT: DETERMINATION OF FAILURE MODES

ENCLOSURE: Suggested Form for Recording Failure Modes

Beginning with the Model 222, and effective for all future design activity, the Airframe and Dynamic Structures Groups will establish and maintain a notebook which shows the first and second predicted failure modes for all structural elements. The maintenance of these notebooks will be the responsibility of the lead structures engineer for each project.

The determination of these failure modes will consider static and dynamic loads along with other contributing factors, such as temperature, corrosion, and fabrication effects. The primary control will be maintained at the subassembly level (i.e., engine mount, bulkhead, main beam, etc.). Primary and secondary failure modes for static and fatigue loading will be determined for each subassembly. For those elements which are subjected to static or fatigue testing, the results of those tests will be entered in the notebook. In addition, any service problems encountered in the production cycle of the element will be entered. A suggested form for these records is enclosed.

To aid the designer in his determination of these failure modes, the structural design groups will supply the designer with the critical loads for the structural element under consideration. These will be supplied in the form of a sketch or free body of the element with the applied loads and reactions. These loads will be updated as the mathematical model is refined during the design process.

The establishment and maintenance of these records can mean much in establishing the rationale for a particular design, tracking its performance and guiding similar designs in the future. Your cooperation in implementing the procedure is essential to its success.

DWG. NO.	DESCRIPTION	PREDICTED FAILURE MODE		TEST RESULTS	SERVICE HISTORY
		PRIMARY	SECONDARY		

STRUCTURES INFORMATION MEMO NO. 7

8 August 1974

SUBJECT: STRUCTURES APPROVAL OF ENVELOPE, SOURCE CONTROL, AND SPECIFICATION CONTROL DRAWINGS

It has recently come to my attention that some of the subject type drawings do not always contain adequate information to allow us to properly validate the item to the government or the FAA. For example, castings may be purchased from a Source Control Drawing without proper inspection or test requirements being fully met within the company. The Source Control Drawing may make no reference to x-ray requirements, static test requirements or any other special inspections required on castings.

Therefore, all Structural Design personnel who have occasion to sign Envelope, Source Control, or Specification Control Drawings shall, as a minimum, establish that the drawing adequately defines the following:

- Configuration
- Mounting and mating dimensions
- Dimensional limitations (interferences)
- Performance (loads, environment, life, etc.)
- Weight limitations
- Reliability requirements
- Interchangeability requirements
- Test requirements
- Verification requirements (analysis or test)
- Material limitations (example, no castings allowed, etc.)
- Casting classification if allowed (also casting factor)
- Primary Part designation
- Reference to applicable specification

Also, if special inspections and tests such as x-rays and static tests are required, Project should be alerted so that plans can be made to procure parts for the required tests.

"Approved Sources of Supply" or "Suggested Sources of Supply" shall not be approved by Structures until we are completely satisfied that the proposed vendor item does meet all structural requirements. This may mean vendors must submit stress analyses of their design or test data as a part of their proposal.

On all Primary Parts or other items with significant structural requirements, the Structures Engineer shall retain a copy of the approved design, vendor stress analysis and test data and file this information in the proper Drawing Check Notebook.

It is hoped that other design groups will use this or some other check list for processing these type drawings.

STRUCTURES INFORMATION MEMO NO. 8

26 February 1975

SUBJECT: EDGE DISTANCE REQUIREMENTS FOR NAS 1738 AND NAS 1739 BLIND RIVET INSTALLATION

As stated in MIL-HDBK-5B, paragraph 8.1.4, Blind Fasteners, "The strength values were established from test data and are applicable to joints having values of e/D equal to or greater than 2.0. Where e/D values less than 2.0 are used, tests to substantiate yield and ultimate strengths must be made." On page 1-11 of MIL-HDBK-5B, e is defined as the distance from a hole centerline to the edge of the sheet and D is the hole diameter.

The ultimate and yield strength values for NAS 1738 locked spindle blind rivets are based on a hole diameter of 0.144 for a 1/8 rivet, 0.177 for a 5/32 rivet, and 0.2055 for a 3/16 rivet, reference MIL-HDBK-5B, Table 8.1.4.1.2(d). The shank diameter for the NAS 1738 and NAS 1739 rivets are 0.140 for a 1/8 rivet, 0.173 for a 5/32 rivet, and 0.201 for a 3/16 rivet.

Loft and sometimes Engineering Design will dimension edge distances and parts for the NAS 1738 blind rivet based on two times the 5/32 value (.31) rather than two times the 0.177 MIL-HDBK-5B value (.36), for example. This practice results in a rivet edge distance of less than 2.0; therefore, the MIL-HDBK-5B strength values in Table 8.1.4.1.2(1) for NAS 1738B rivets are not applicable.

In conclusion, to ensure the correct edge distance is used when planned patterns of NAS 1738 and NAS 1739 rivets are installed, Structures Group recommends that the correct edge distance dimension be specified on the face of the drawing for rivet patterns rather than using the drawing note that states rivet e/D is equal to two times the rivet shank diameter. Also, special attention must be given to skin overlaps, and bulkhead and stiffener flange dimensioning. The edge distance for the countersunk NAS 1739 rivet of 2.5 times the rivet shank diameter is valid because MIL-HDBK-5B values for the NAS 1739 rivet are based on two times the hole diameter. The table below summarizes the recommended minimum nominal edge distance values for NAS 1738 and NAS 1739 blind spindle locked rivets.

Rivet	EDGE DISTANCE	
Size	NAS 1738	NAS 1739
1/8	.29	.32
5/32	.36	.39
3/16	.41	.47

STRUCTURES INFORMATION MEMO NO. 9

4 May 1976

SUBJECT: MECHANICAL PROPERTIES REDUCTION FACTORS FOR CASTINGS WITH FOUNDRY WELD REPAIR

REFERENCES: a) BHC Report 599-233-909, "The Effect of Weld Repair on the Static and Fatigue Strengths of Various Cast Alloys"

b) BPS FW 4470 - In Process Welding of Castings

c) ASM Technical Report W 6-6.3, "Static and Fatigue Properties of Repair Welded Aluminum and Magnesium Premium Quality Castings"

Future casting drawings should have a note that permits the in process welding of castings per BPS 4470. To allow for this weld repair, parts should be analyzed using the following reductions in allowables.

Material	Reduction Factors for Foundry Weld Repair			
	Ultimate Tensile	Yield Tensile	Elongation	Endurance Limit
356-T6	10%	5%	0	10%
A356-T6	10%	13%	0	10%
AZ91-T6	25%	22%	50%	10%
ZE41-T5	10%	0	50%	10%
17-4 PH	0	2%	30%	10%

In those circumstances where the part cannot be sized to allow for weld repair throughout the part, a weld map should be provided on the drawing to indicate those areas which may receive weld repair.

If the entire part is so critical that no weld repair can be permitted and the part cannot be redesigned, the drawing and all analysis should be clearly marked "No Weld Repair Allowed". All 201 Aluminum Alloy castings shall be marked "No Weld Repair Allowed".

STRUCTURES INFORMATION MEMO NO. 10

9 March 1978

SUBJECT: DESIGN CRITERIA FOR DOORS AND HATCHES

Unless otherwise specified in a Detail Specification or Structural Design Criteria Report, the Structural Design Criteria presented herein should be used on new designs for the following:

1. Access doors
2. Hinged or sliding canopies
3. Sliding doors
4. Passenger doors
5. Crew doors
6. Cargo compartment doors
7. Emergency doors
8. Escape hatches

All loads associated with the use and operation of doors and hatches terminated in the latches and hinges and their attachment to the airframe. The sources of these loads are:

1. Open canopy during approach or taxi operation
2. Gusts
3. Outward push from personnel
4. Air loads
5. Rough handling

1. Open canopy during approach or taxi operation

If a sliding or hinged canopy is used, it should be designed to withstand an air load from taxi operations of up to 60 kt.

2. Gusts

All doors that are subject to damage by ground gusts and wind loads from other helicopters being run up or taxied nearby or flown close overhead, should be provided with a means to absorb the energy resulting from a 40 kt ground gust occurring during opening or closing. Doors and access doors or panels that have a positive hold-open feature should be capable of withstanding gust loads to 65 kt when the door or panel is in the open position and unattended.

3. Outward push from personnel

Due to possible inadvertent loading by personnel, passenger doors should be capable of withstanding an outward load of 200 lb. without opening. Also, doors between occupied compartments shall be capable of withstanding a load of 200 lb. in either direction without opening. These loads are assumed to be applied upon a 10 sq. in. area at any point on the surface of the door. Yielding and excessive deflections are permitted but the door must not open.

4. Air loads

The air loads on doors and hatches for helicopters probably are minimal when compared to the many personnel-oriented loads. The air loads, however, should be investigated, including the application of the appropriate gust criteria. All doors should be capable of withstanding air loads up to V_D in the closed position. All sliding cargo and passenger doors should be capable of withstanding air loads up to 120 knots in the full open position and up to 80 knots in any partially open position.

5. Rough handling

All doors and hatches that are likely to receive rough handling during their lifetime should be capable of withstanding loads they are expected to receive in operation. Passenger and crew doors should withstand a 150 pound load applied downward at the most critical location without permanent deformation. All other doors that are unlikely to be stepped on or used as a handhold or which are marked with a "NO STEP" or "NO HANDHOLD" decal should withstand a 50 pound load parallel to the hinge pin axes and a 50 pound load perpendicular to the surface without permanent deformation.

STRUCTURES INFORMATION MEMO NO. 11

16 January 1980

SUBJECT: EMERGENCY FLOAT KIT LOADS

In addition to the existing design conditions, emergency float kit loads must be developed for the following conditions:

1. Floats in the water at 0.8 bag buoyancy and combined with salt water drag for 20 knots forward speed. These loads will be treated as limit loads. These loads will be applied at angles corresponding to the righting moments, but not to exceed 20°.

2. For skid mounted floats;

   a) A computer drop will be done in a tail down attitude for limit sink speed. Skids will be checked for a positive M.S. at yield.

   b) Crosstubes will not yield with the helicopter in the water, floats inflated and no rotor lift.

STRUCTURES INFORMATION MEMO NO. 12

7 August 1981

SUBJECT: 7050-T73 RIVETS IN LIEU OF 2024-T31 (ICE BOX) RIVETS

7050-T73 rivets will be utilized in lieu of 2024-T31 "DD" (ice box) rivets as of 20 July 1981. The 7050 rivets can be stored at room temperature, thereby eliminating numerous problems that exist with the 2024 rivets.

The following policy will be implemented.

1. Manufacturing will utilize the 7050-T73 rivets to supersede the MS 20246DD and MS 20470DD rivets (Reference, the SUPER SESSION LIST, BHT Standard 170-001, Revision "G"), effective the target date of 20 July 1981.

2. The 100° flush and protruding head 7050 aluminum alloy rivets are delineated in BHT Standards 110-174 and 110-175, respectively.

3. All new drawings initiated after this date will call out the 7050 rivets for 3/16 and 1/4 inch diameters. Approval for other diameters MUST be obtained from applicable Structures and Design Group Engineers prior to utilization.

4. The 7050 rivets "will not" be utilized to replace "AD" rivets (generally used in 5/32 inch diameters and smaller) at this time (not cost effective).

5. The driven shear strengths for both the 7050 and 2024 rivets are established for an F_{SU} = 41 ksi. Until MIL-HDBK-5 allowables are available, 2024-T31 MIL-HDBK-5 data in the 3/16 and 1/4 inch diameters, for both protruding and 100° flush heads, are acceptable for 7050-T73 installations and should be so identified for report referencing. It is anticipated that 7050-T73 MIL-HDBK-5 allowables will be available during the 1981-1982 time frame.

STRUCTURES INFORMATION MEMO NO. 13

31 August 1981

SUBJECT: FITTING FACTORS, THEIR DEFINITION AND APPLICABILITY

Reference: FAR 29.623, 29.619

A fitting factor is a 1.15 load factor, applied to limit loads, and is in addition to the 1.50 factor of safety. It accounts for uncertanties such as deterioration in service manufacturing process variables and unaccountability in the inspection processes.

For design considerations, a fitting shall be defined as part(s) used in a primary structural load path whose principal function is to provide a load path through the joint of one member to another. The connecting means is generally a single fastener.

A fitting factor is applicable to the fitting, the fastener bearing on the joined members, as well as the attachments joining the fitting(s) to the structure. It is particularly considered when failure of such fitting should not allow load redistribution in a manner that would provide continued safe flight and that load redistribution cannot be verified by analysis or test. Obviously then, a fitting factor is applied to non-redundant connecting members in primary load path applications the failure of which may affect safety of the aircraft and its occupants. It is applied until the load is distributed into the surrounding back-up structure to which the fitting is attached.

A fitting factor is not applicable to:

a) Crash load factors that are the only design condition and/or crash load factors that exceed limit load factors x 1.5 x 1.15.

b) A continuous riveted joint(s) in basic structure when section properties remain consistent throughout the joint and the joint consists of approved practices and methods such as splices of main beam caps - riveted door post caps to bulkheads, riveted skin splice doublers, continuous riveted skins to longerons, continuous riveted structure such as bulkheads to beams or intercostals, or frames, etc.

c) An integral fitting beyond the point where section properties become typical of the part. Example, integrally fabricated lug on a forging, or machining.

d) Welded joints.

e) To a member when a larger load factor is used such as a larger special bearing factor, a 1.25 casting factor, a 1.33 fatigue factor, a 1.33 retention factor of seats and safety belts.

f) Systems or structure when they are verified by limit or ultimate load tests. The fixed control system is an example of this exception.

g) Bonded inserts and/or fittings in sandwich panels.

h) A fitting in redundant connecting members.

STRUCTURES INFORMATION MEMO NO. 14

25 January 1982

SUBJECT: STRUCTURAL APPROVAL POLICY

Reference: Structures Information Memo No. 7 - "Structures Approval of Envelope, Source Control and Specification Control Drawings"

Structures Group approval of any drawing is defined as structural approval of all parts called out on that drawing regardless of whether or not they are Bell designed parts.

It is therefore the responsibility of the Structures Engineer who signs a drawing to satisfy himself that all components of that drawing, including vendor part numbers, standard parts and specification controlled items, meet Bell's structural requirements for that particular installation. For components that are defined by Bell Procurement Specification, Specification Control Drawing or Source Control Drawing, the guidelines of SIM No. 7, as amplified here, are to be followed. The Structures Engineer must be assured that the controlling Bell specification or drawing contains adequate requirements for vendor stress analysis and/or structural test proposal and results report to assure that strength requirements are met. Provision should be made for FAA conformity and for Bell witness of testing, if required.

In the case of a product defined entirely by vendor's drawings and procured by their part number, the Structures Engineer must notify the Project Engineer in writing of the extent of structural substantiation by analysis or testing required from the vendor. Provision should be made for FAA conformity and for Bell witness of testing, if required. It must be made clear that drawing approval is contingent upon successful completion of analysis or testing and submittal of these data for structures approval. If Bell testing is indicated, EWAs and schedules must be written to establish these tests.

STRUCTURES INFORMATION MEMO NO. 15

(This memo is not published)

STRUCTURES INFORMATION MEMO NO. 16

22 February 1983

SUBJECT: PROPERTIES FOR CRES 17-4 PH CASTINGS

Reference: (a) MIL-HDBK-5, 61st Meeting Agenda, Item 79-21, "Design Allowables (Derived Properties) for 17-4PH (H1000) Castings," April 1981, Pg. 2-166 Attached

MIL-HDBK-5 does not currently contain shear and bearing properties for 17-4PH castings. Properties to be used for analysis of 17-4PH castings shall be as shown in Table I. This table includes properties for the two recommended and most often used tensile strength ranges at BHTI. Properties for 17-4PH castings in the tensile range of 150-170 KSI per Reference (a) have been derived for and approved by the MIL-HDBK-5 committee. These properties are scheduled for inclusion in MIL-HDBK-5, Revision D.

The properties in Table I were established or derived as follows:

(1) Tensile Ultimate - Minimum tensile strengths and tensile strength ranges are those most commonly used at BHTI.

(2) Tensile Yield - Tensile yield strengths were established by comparing yield to ultimate ratios obtained from a large quantity of tensile test results. Test data from separately cast test bars and bars cut from castings were evaluated. These data came primarily from lot certifications submitted to BHTI by casting suppliers. Tensile yield strengths shall be 20 KSI lower than the ultimate tensile for tensile strengths equal to or less than 155 KSI minimum and 25 KSI for tensile strengths greater than 155 KSI minimum.

(3) Tensile Range of 155-175 KSI - Properties for compressive yield, shear and bearing for the tensile range of 155-175 KSI in Table I were derived by direct ratio of Reference (a) properties with ultimate tensile strengths as reference.

(4) Tensile Range of 170-200 KSI - Properties for compressive yield, shear and bearing for the tensile range of 170-200 KSI in Table I were derived by direct ratio of the 155-175 KSI tensile range properties, as shown in Table I, plus a conservative 5% reduction with ultimate tensile strength as reference.

Properties for tensile ranges not shown in Table I shall be derived in accordance with Items (2) and (4) except properties for the tensile range of 150-170 KSI shall be as identified in the attached Table 2.6.9.0(j) of MIL-HDBK-5D. Properties shown in Table I and the procedures herein shall also be applicable to 15-5PH castings.

Applicable casting factors or minimum margins of safety shall be maintained per appropriate Design Criteria. Note the design approach to minimize or eliminate static test of castings due to cost and schedule impacts.

TABLE I

*PROPERTIES FOR CRES 17-4PH CASTINGS

Tensile Range 155 - 175 KSI		Tensile Range 170 - 200 KSI	
Ftu	- 155	Ftu	- 170
Fty	- 135	Fty	- 145
Fcy	- 136	Fcy	- 141
Fsu	- 101	Fsu	- 105
Fbru(1.5)	- 262	Fbru(1.5)	- 272
Fbru(2.0)	- 340	Fbru(2.0)	- 354
Fbry(1.5)	- 195	Fbry(1.5)	- 203
Fbry(2.0)	- 229	Fbry(2.0)	- 239
e	- 8%	e	- 6%

*Some property values may be higher than those previously derived and shown in MIL-HDBK-5C for wrought products.

TABLE 2.6.9.0(j). *Design and Physical Properties of 17-4PH Stainless Steel Casting*

Specification	AMS 5355				AMS 5398
Form	Investment casting				Sand casting
Condition	H900	H925	H1000	H1100	H925
Thickness, in.	...	...	...	...	...
Basis	S[a]	S[a]	S[a]	S[a]	S[a]
Mechanical properties:					
F_{tu}, ksi	180	180	150	130	180
F_{ty}, ksi	160	150	130	120	150
F_{cy}, ksi	...	...	132	...	...
F_{su}, ksi	...	...	98	...	...
F_{bru},[b] ksi:					
(e/D=1.5)	...	...	254	...	...
(e/D=2.0)	...	...	329	...	...
F_{bry},[b] ksi:					
(e/D=1.5)	...	...	189	...	...
(e/D=2.0)	...	...	222	...	...
e, percent	6	6	8	8	6
RA, percent	15	15	20	15	12
E, 10^3 ksi	28.5				
E_c, 10^3 ksi	30.0				
G, 10^3 ksi	12.7				
μ	0.27				
Physical properties:					
ω, lb/in.^{3}	0.282 (H900)				
C, K, and α	See Figure 2.6.9.0				

[a] For separately cast bars. Properties of test specimens machined from castings shall be as agreed upon by purchase and supplier.
[b] Bearing values are "dry pin" values per Section 1.4.7.1.

(1) This Table is schedule for publication in the future MIL-HDBK-5, Revision D.

STRUCTURES INFORMATION MEMO NO. 17

24 January 1984

SUBJECT: LATERAL LOAD CRITERIA FOR COLLECTIVE CONTROL

To preclude inadvertent damage from handling, the following additional criteria will be met on all future collective control systems:

> 170 pound limit load applied separately in a horizontal plane inboard or outboard at the center of the collective handgrip.

SECTION 2

COMPUTER PROGRAMS

2.1 GENERAL

This section presents the computer aided analyses available to the structure engineers at Bell Helicopter. It provides a brief description of the computer program systems and associated computer programs. More detailed information can be obtained from relevant documentation available in the Structural Methods Group.

2.2 Computer Facilities

The major computer facilities used by structure engineers are the IBM mainframe computing system, which utilizes an IBM 3081 and an IBM 3090 computers, and the TSO (Time Sharing Option) operating system. TSO video terminals, supporting text and graphics, are located in all engineering departments. CADAM scopes, the special purpose graphics terminals, are also available for access to the Computer Aided Design package CADAM.

Depending on the setup of the computer programs, a computer job can be run in an interactive mode or a batch mode. The resulting printout and plotting will, by default, be sent to the computing center for process, but can also be routed to one of the local printers if preferred. Permanent datasets are stored and managed by the PANVALET data base management system. Magnetic tapes can be used to store large or inactive data, or for exchange of information with outside sources (approval required).

A number of VAX super mini-computers and IBM PC's are available for engineering use. They are usually installed for the purpose of running certain special purpose programs.

2.3 Finite Element and Supplementary Programs

The finite element method has become one of the most important techniques for determining structural loads and performing stress analysis for sophisticated aircraft structures. The two major and very time-consuming steps in the finite element analysis are the model generation and output interpretation. Many finite element preprocessors and postprocessors have been developed for these purposes. The preprocessors are used for automatic finite element mesh generation. The postprocessors are used for checking and plotting the model, as well as sorting, plotting, and processing the results for a variety of purposes. This section will introduce the finite element programs and their preprocessors and postprocessors available at Bell.

2.3.1 Finite Element Programs

NASTRAN - General Purpose Finite Element Analysis

NASTRAN was originally developed by NASA. However, many enhanced versions of NASTRAN have been developed thereafter. Bell has two versions: MSC/NASTRAN and COSMIC/NASTRAN.

The MSC/NASTRAN is developed and maintained by MacNeal-Schwendler Corporation (MSC). It is a large-scale general purpose digital computer program which solves a wide variety of engineering problems, including static and dynamic structural analyses, acoustics, etc., by the finite element method. MSC/NASTRAN has made many additions and changes to the original NASTRAN; such as additional elements, composite laminate analysis, improved dynamic analysis methods, etc.

The COSMIC/NASTRAN is the NASTRAN version currently maintained by the COSMIC. It is more like the original NASTRAN and does not have many features of MSC/NASTRAN.

ANSYS - Engineering Analysis System

The ANSYS software is developed and maintained by Swanson Analysis Systems, Inc. It is a large-scale general purpose finite element program. Analysis capabilities include static and dynamic; elastic, plastic, creep and swelling; buckling; small and large deflections; and other engineering analyses.

SECR02 - Static Finite Element Program (RPT 599-272-001)

This program utilizes the stiffness approach to perform a static finite element analysis of a structure. Contained in the program is a fracture mechanics element that calculates the stress intensity factor of a crack.

2.3.2 Finite Element Preprocessors and Postprocessors

PATRAN - Finite Element Preprocessor and Postprocessor

The PATRAN software is developed and maintained by PDA Engineering. PATRAN has extensive geometric modeling and graphic capabilities. Its capabilities are numerous but include the finite element preprocessing and postprocessing; such as creating a finite element model and presenting the output graphically. PATRAN consists of many processing modules; including Conceptual Solid Modeling, Advanced Geometry Modeling, Finite Element Modeling, Linear Statics and Dynamics Analysis, Composite Materials Design and Analysis, Results Evaluation, X-Y Plotting, Engineering Animation, etc. PATRAN itself does not contain a finite element solver. PATRAN data have to be translated to and from finite element programs (including NASTRAN and ANSYS) through interface programs.

ANASYS - Preprocessor (PREP7) and Postprocessor

The ANSYS program has a preprocessor (PREP7) and a postprocessor of its own. The preprocessor contains mesh generation capability and geometric plotting. The postprocessor routines will plot distorted geometries, stress contours, safety factor contours, temperature contours, mode shapes, time history graphs, and stress-strain curves. The postprocessor also has the routines for algebraic modification, differentiation, integration of calculated results, and other functions.

CADAM - Computer Aided Design Package

The CADAM software is primarily a computer aided design tool. However, its CADAM/FEM module is capable of creating and editing a 3-D finite element model. CADAM and NASTRAN data can be translated through the interface programs. The most important advantage of this approach is that the finite element model of a structure can be created directly from design drawings so that accurate modeling and time savings can be obtained.

GIGN01 (old program NASPLOT) - NASTRAN Model Plotter

This program is a NASTRAN postprocessor and will process NASTRAN data deck directly. It can plot the finite element model grid points and element numbers for a specified subsections and arbitrary views. The program must be executed on a graphics terminal.

SECR64 - Model Generation and Analysis of Cutouts in Composite Materials (RPT 599-162-919)

This program generates a NASTRAN finite element model of a rectangular orthotropic plate containing popular cutout shapes in order to calculate the strains and margins of safety for each lamina.

SECR64 - Model Generator for Rectangular Plates with Circular Patches (RPT 599-162-922)

This program generates a NASTRAN finite element model of a flat composite rectangular panel containing a circular hole with a flat circular composite patch.

SESN04 - Finite Element Model Data Generator for Shells (RPT 599-162-912)

This program generates a NASTRAN finite element model for any shell type structure which can be mathematically defined.

SESN05 - Finite Element Model Data Generator for Rectangular Plates, Solids, and Laminated Plates (RPT 599-162-913)

This program generates a NASTRAN finite element model for a rectangular plate, doubler stack, or a laminated plate.

SESN06 - Finite Element Data Generator for Pin-Loaded Lugs
(RPT 599-162-917)

This program generates a NASTRAN finite element model for a pin-loaded lug. The hole in the lug can be either concentric or eccentric. The lug can be modeled with a bushing and a pin.

SE1703 - Finite Element Data Generator for Truss Tailboom

This program generates a NASTRAN finite element model of a truss tailboom. Arbitrary station input and number of longerons are allowed.

SDSB01 - Unit Inertia Forces and Weight Generator (RPT 299-099-252)

This program uses a WAVES weights file and a NASTRAN grid deck to obtain an inertia representation of the helicopter. The option is given to have either weights or equivalent unit inertia forces.

SESB12 - NASTRAN Critical Loading Selector (RPT 299-099-252)

This program is used with SESB10 and compares the forces and stresses calculated during the execution of SESB10. It selects the critical load and loading condition for each member in the finite element model. The critical loads selected are printed in report format and saved on tape.

SESN09 - Shear and Bending Moment Diagrams Generated from NASTRAN Force Input
(RPT 299-099-252)

This program uses the inertia loads and corresponding applied loads for a NASTRAN airframe finite element model to create the shear forces and bending moment diagram associated with each given design load condition and gross weight configuration. The program generates the diagram as Calcomp plots and tabular tables.

SESN13 - Element Grouping Program

This program groups the NASTRAN output for a particular area of the structure regardless of the element numbering sequence. It is particularly useful for bulkheads, skins, stringers, tailboom, etc.

SESB01 - Automatic NASTRAN Element Sizing

This program will generate a NASTRAN model with more representative element size early in a project design phase. A NASTRAN model with unit element areas is run against a set of design conditions. Selected elements are sized for these internal loads. New NASTRAN property cards are punched for rod, bar and shear panel elements.

SESB10 - Basic NASTRAN Airframe Output Option (RPT 299-099-252)

This program transforms NASTRAN output into a form more suitable for structural analysis. Shear flows, end loads and moments are generated.

2.4 Approved Structural Analysis Methodology (ASAM)

The computer programs approved for structural analysis at Bell are collected in the ASAM (Approved Structural Analysis Methodology) system. The documentation for the system overview and individual programs are on file in the Structural Methods Group. These documents are also on line of the computer and can be obtained by request thru a self instructive procedure on a TSO terminal.

The ASAM software currently contains three computer systems; which are LODAM, CASA, and CPS2TSO in addition to separate individual programs categorized under the title "Program". The ASAM system is not static; it is constantly being revised and expanded. The user should consult the active system for the latest system capabilities.

ASAM is a TSO based system. The user can access the system by logging onto TSO and entering the command "ASAM"; this will bring the primary analysis menu to the screen. ASAM is a menu directed system, i.e. by selecting a menu option the system will display successive menus, for option selection, until the desired program or function is reached.

It is suggested that before using the system the user accesses ASAM to request for the documentations for system overview and to get familiar with its menus and structure. In addition, he should request the documentation for the program of his interest prior to attempting the analysis. The documentation of the program includes an explanation of the theory/methodology, technical references, and major analytical equations used, if practical. It also contains user instructions for proper execution of the program, along with example problems with their associated input and output data. The ASAM documentation can be referenced in the Bell's official reports to meet government agency's or customer's requirements.

2.4.1 The LODAM System

The LODAM is a system for structural loads development. This system is currently under development and will be available for use at a future date.

2.4.2 The CASA System

The CASA (Computer Aided Stress Analysis) system contains a group of matured analysis programs. It provides the structures engineer with a consolidated stress information system. The primary goal of CASA is to increase the efficiency of engineers by reducing the manhours required to perform structural analysis and to produce reports. The description of the system and its features, as given next, are general in nature.

2.4.2.1 CASA System Features. The Primary CASA system features are as follows:

1. A computer stored stress information system accessed through TSO

   The CASA System collects the stress discipline into a computer stored stress information system. It uses computing facilities to provide the computations, the utility linkages, and the high speeds required to process the information efficiently. It eliminates time consuming data manipulations which were previously done by hand. It accomplishes these tasks within the CASA System itself and also allows access to external programs and systems such as NASTRAN, WAVES, and CADAM.

2. Automated Modular Stress Analysis Programs

   The CASA application programs are a collection of automated stress analysis programs with the following features:

   - They are structured to be used by both entry level and experienced stress analysts.
   - They all are interfaced by the user through interactive menus and prompting on TSO terminals.
   - The inputs to the programs require the minimum amount of manual calculations.
   - The programs create an input dataset when they are run. This dataset can be edited and used to run the program without the user having to answer each prompt on an individual basis.
   - The programs provide the user with options as to the format of the output data. He may choose either temporary output format or the report format (Form 8441).
   - The output from the calculations performed by the programs are presented on standard formats that are concise and easily understood.
   - All application programs provide the user with an automated system for the permanent saving of input datasets on the CASA system database.
   - The flexibility and applicability of the programs are maintained by specifying general solutions of basic theory such as tension, compression, and shear. When combined with appropriate utilities and interface, the analysis of airframe fuselage sections, tailbooms, bulkheads, and mechanical systems can be attained.

3. CASA has self contained tutorial and documentation functions.

   The CASA system provides an up to date tutorial and set of documentation for each application program. The tutorial function presents the user with an introduction, a set of program capabilities and limitations, and a detailed step by step set of user instructions on how to execute the program. The documentation function provides the user with a printed copy of the methodology/description of each option and includes the necessary graphics.

4. Report ready formatted output.

   Where possible, the CASA programs perform analysis to the compilation of margin of safety summaries on an output report format. This format is the standard stress pad format (including heading, border, proprietary note, etc.). Where graphics are required space is left on the pages. This report ready format (8.5 inches by 11.0 inches with holes punched) is ready for immediate cataloging and inclusion into stress reports. Each application program presents its output in a standardized, concise, and readable form.

5. Secured and protected data and programs.

   The CASA system provides the user with an automated system to permanently identify and save all input datasets that have been used to produce final report analysis. Once an analysis is designated as a final report analysis the security function automatically labels it with all the information required to save it on a permanent database. This dataset can be recalled and used to rerun the analysis in the future to obtain the same results. The CASA programs in all revisions are also saved. In addition, all current programs are protected from unauthorized changes to the source code. The final report datasets that are saved on the database are protected from any alteration to their contents. The database automatically names its datasets to prevent duplication or overwriting.

6. Uses structural design manual methods.

   The application programs utilize BHT approved methodologies and/or structural design manual (SDM) methods. SDM variable identification is maintained, where possible, for ease of reference and identification.

7. Produces reports and documentation suitable for submittal to regulating and procuring agencies.

   The documentation and analysis produced by the CASA system for both reports and methodology has been formatted and produced in a manner so that it is suitable for submittal to the FAA and Military authorities. The methodology for all CASA programs can be found in BHT Report No. 299-099-252, Volume IV.

2.4.2.2 <u>CASA System Architecture</u>. The CASA System architecture (see Figure 1) gives the CASA user an overview of the capabilities of the present (Phase IV) and future systems. Each item on the figure can be associated with software that is used by the CASA System in performing its functions. The

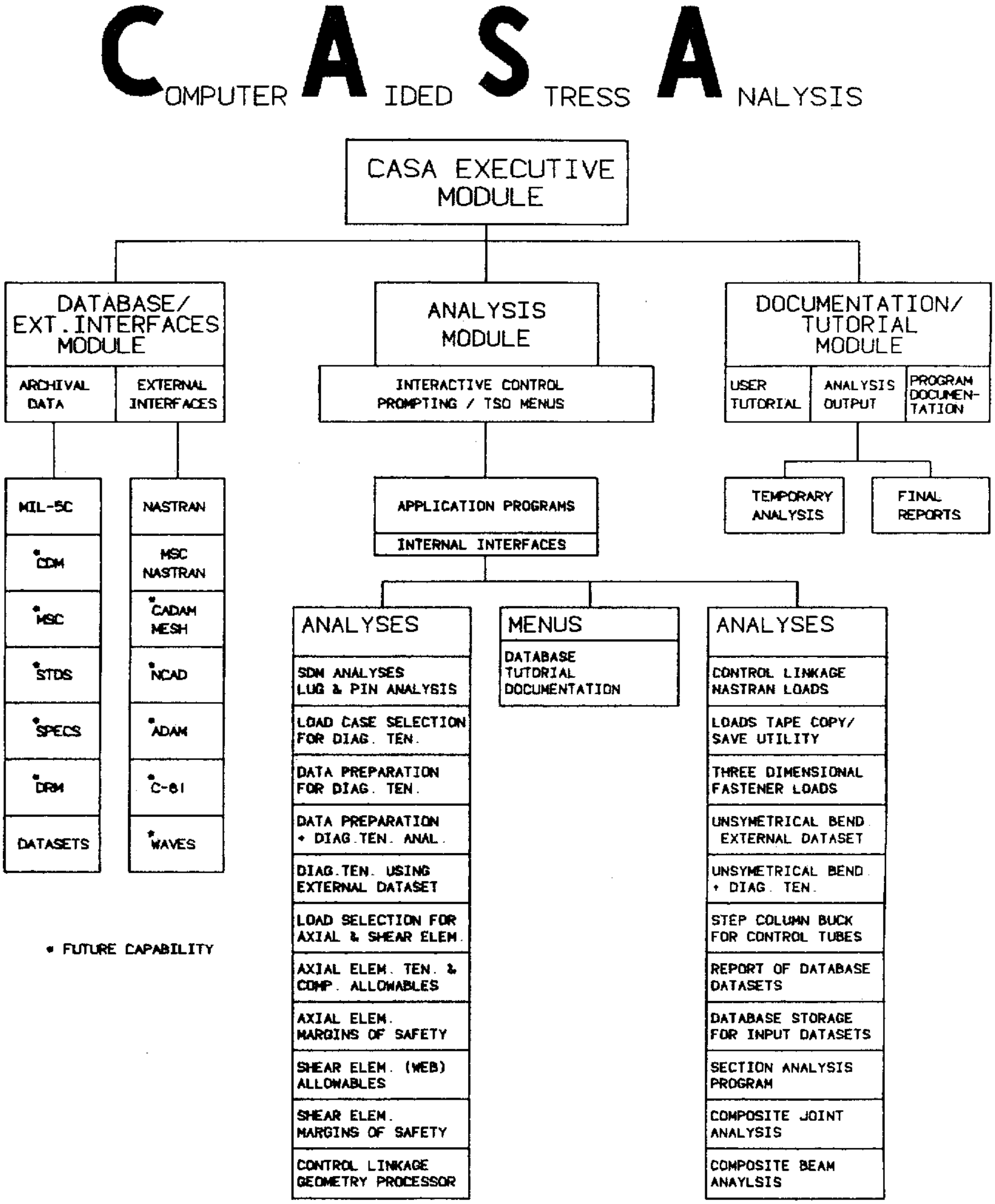


Figure 1. Phase IV CASA System Architecture

executive module provides the user with interactive control and interfaces to the three primary modules: the analysis module, the documentation/tutorial module, and the database/external interface module. The executive software transforms CASA into a modern stress information system that will allow future expansion to be done in a timely and cost effective manner. New application programs and functions can be plugged into the system and use the existing software in many of their options.

2.4.2.3 CASA Programs Description

CV004P(A) - Structural Design Manual Lug and Pin Analysis

This program performs a stress analysis for a lug or lug and bolt/pin combination according to the procedures and methodology presented in section 6 of this manual.

CV005P - Load Case Selection for Diagonal Tension Analysis

This utility program is used to copy selected load cases from an Airframe Options Loads Tape to a temporary file. The file will be used as the NASTRAN input loads for a diagonal tension analysis.

CV006P - Data Preparation for Diagonal Tension Analysis

This program prepares an input dataset for CASA program CV008P (Diagonal Tension). It prepares the geometric data, material properties, and the loads from a NASTRAN Airframe Options Tape for input into the Diagonal Tension program.

CV007P - Data Preparation with Diagonal Tension Analysis

This program prepares an input dataset for CASA program CV008P (Diagonal Tension). It prepares the geometric data, material properties, and the loads from a NASTRAN Airframe Options Tape and performs a diagonal tension analysis of the structure.

CV008P - Diagonal Tension

This program uses the methods outlined in NACA TN2661 to analyze a flat or curved panel that has developed incomplete diagonal tension. The program is set up to analyze a multi-bay structure with single or multiple loading conditions. It does not require NASTRAN generated loads.

CV009P - Load Case Selection for Axial and Shear Element Summaries

This program is used to create a critical loads dataset from an Airframe Options Tape. These loads are used by programs CV011P and CV013P (Axial and Shear Elements Summary) to calculate margins of safety for the structure.

CV010P - Axial Elements Tension and Compression Allowables

This program produces the axial element allowable loads for tension, compression, crippling, compressive yield, and interrivet buckling for airframe structures. It processes the axial element IDs, geometry, and material allowables for use by program CV011P (Axial Elements Allowables and Margins of Safety).

CV011P - Axial Elements Allowables and Margins of Safety Summary

This program produces an axial elements' margins of safety summary for airframe structures. It processes the axial element allowables from program CV010P and combines them with the NASTRAN loads from program CV009P to produce the margins of safety.

CV012P - Standard, Lightened Hole, and Beaded Shear Web Allowables

This program produces the shear element allowable loads for airframe structure. It processes the shear element IDs, geometry, and material allowables for use by program CV013P (Shear Elements Allowables and Margins of Safety).

CV013P - Shear Elements Allowables and Margins of Safety Summary

This program produces a shear elements' margins of safety summary for airframe structures. It processes the shear element allowables from program CV012P and combines them with the NASTRAN loads from program CV009P to produce the margins of safety.

CV014P - Control Linkage Geometry Processor

This program is used as a geometry processor to analyze the kinematics of a control linkage. It produces the necessary NASTRAN bulk data to represent the movement of the control linkage. The linkages are modeled as cranks, idlers, slides, actuators, rods, jackshafts, and mixers by using modeling operators.

CV015P - Control Linkage Geometry - NASTRAN Loads - Airframe Options

This program is used as a geometry processor to analyze the kinematics of a control linkage. It produces the necessary NASTRAN bulk data to represent the movement of the control linkage. The linkage is modeled as cranks, idlers, slides, actuators, rods, jackshafts, and mixers by using modeling operators. In addition, it calculates the internal loads and reactions in the control system due to specified input loading conditions using COSMIC/NASTRAN as a solver.

CV017P - 3-D Fastener Pattern Loads

This program calculates the three dimensional loads on a fastener pattern using rigid body mechanics. It allows weighing of the fastener areas in proportion to the stiffness that the individual fastener and its backup structure provides in a given direction.

CV018P - Section Properties and Unsymmetrical Bending Analysis

This program calculates the section properties, unsymmetrical bending stresses, element loads, and shear flows for a single or multiple cell torque box. It is applicable to standard stiffener/skin and sandwich constructions.

CV019P(A) - Load Case Selection for Unsymmetrical Bending and Diagonal Tension Analysis

This program is used to copy selected load cases from a Shears and Moments History Tape to a TSO dataset. This dataset contains the NASTRAN loads which will be used by program CV019(B) to prepare a load input dataset for the program CV019P(C) (Unsymmetric Bending Analysis).

CV019P(B) - Data Preparation for Unsymmetrical Bending Analysis

This program prepares an input dataset for program CV019P(C) (Unsymmetrical Bending Analysis) using input dataset produced by program CV019P(A) and the geometry data of the structure.

CV019P(C) - Section Properties and Unsymmetrical Bending Analysis

This program calculates the section properties, unsymmetrical bending stresses, element loads, and shear flows for a single or multiple cell torque box. It is applicable to standard stiffener/skin and sandwich constructions. It should be used when a NASTRAN model is available for the structure and when a diagonal tension analysis follows the unsymmetrical bending analysis. This program requires a different input dataset from program CV018P and produces different output.

CV019P(D) - Data Preparation for Diagonal Tension Analysis

This program produces an input dataset for program CV019P(E) (Diagonal Tension Analysis). It combines the geometry inputs with the loads output from program CV019P(C) to produce the desired input dataset.

CV019P(E) - Diagonal Tension Analysis

This program uses the methods outlined in NACA TN2661 to analyze a flat or curved panel that has developed incomplete diagonal tension. The program is set up to analyze a multi-bay structure with single or multiple loading conditions. It accounts for the increase or decrease in the web buckling allowable due to the compression or tension stresses in the web.

CV020P - Step Column Buckling for Control Tubes

This program calculates the critical buckling loads for a stepped and pinned end control tube or column. The program includes empirical factors to adjust the analysis for control tubes. These factors can be omitted for columns that are not control tubes. The program only checks for buckling and not for local instability. It uses the Tangent Modulus Theory and allows the user to define his own materials.

CV023P - Section Analysis

This program can be used to perform a section analysis on cross section made from metallic and certain composite materials. It calculates the section properties, the resultant loads on the section, the resulting stresses/strains of general polygonal sections. It is based on beam theory as opposed to plate theory.

CS024P - Bolted Joint Stress Analysis

This program computes stress distributions on a lamina or laminate basis for unloaded or loaded (bolt bearing) holes in isotropic or ansitropic materials and predicts failure based on lamina properties and user selected failure criterion. The program should be used for preliminary design only. The margins of safety it calculates are approximate because of the many assumptions made.

CS025P - Composite Beam Analysis

This program computes section properties and stresses for a beam of composite materials with irregular cross-sections using a finite element method. Section properties include cross-sectional area, centroids, moments of inertia, location of the shear center, shear coefficients, torsional constants, and warping constant. The stresses include the normal stress due to axial force and bending moments and shear stress due to transverse shear forces and twisting moment.

2.4.3 The CPS2TSO System

Most programs, developed at Bell or acquired from outside sources, are not normally installed into CASA directly. After being checked out and approved for use, they are normally included in the CPS2TSO system. The CPS2TSO system does not have all the features of CASA. Each CPS2TSO program has documentation, but does not produce the results on the final report form. A CPS2TSO program will be transformed to CASA when it is fully checked for its reliability and accuracy. The programs currently contained in the CPS2TSO system are described in the following:

CPS01P (old program AIRFACTORS) - Airload Distribution on the Fuselage

The external loadings needed for the structural analysis of airframe structure include the distributed loads representing the airload pressure distribution. These airload distributions for various maneuvers are normally developed form wind tunnel data. The C-81 program determines the contributing inertia force of each mass/weight item of the helicopter and the applied airloads necessary to balance a given maneuver. These airloads are applied at the C-81 model reference point. Program SD5001, or similar programs, calculates a unit airload distribution for a unit force and/or movement at a reference point. Then CPS01P is used to determine the scale factors to apply the unit load datasets produced by programs like SD5001, to produce loads that are equivalent to the applied airloads from C-81.

CPS02P (old program DL3301) - Crack Growth Program

This program analytically calculates the crack growth rates and crack sizes for flat plates and cylinders under axial loading.

CPS03P (old program NFSN05) - Inertia Scale Factors For Point Loads

This program calculates an inertia scale factor for each of the six degrees of freedom for each airframe loading condition. In addition, the forces and moments about the helicopter center of gravity are computed. These scale factors and the appropriate inertia forces are combined on a NASTRAN load card.

CPS04P (old program SCAV17) - Composite Laminate Analysis

This program performs a linear point stress analysis for a composite laminate. It predicts the initial and progressive failures using one of the selected failure theories. Applied loads include inplane forces and moments, transverse shear, and thermal effects. A library for commonly used materials has been built into the program in addition to an option that allows the user to input materials. The output contains laminate and ply-by-ply strains and stresses, and margins of safety. It can also produce NASTRAN composite plate data cards, failure envelope plots, and carpet plots for laminate properties.

CPS05P (old program SFCR02) - Section Properties and Unsymmetrical Bending Analysis

This program determines the section properties, unsymmetrical bending stresses, element loads and shear flows for a single cell torque box. It accounts for sandwich materials, monocoque construction, local stiffening of element, and allows for a safety factor to be used in determining the loads.

CPS07P (old program SD5001) - Representation of Forward and Aft Pressure Distributions by Panel Point Loads

Once the airload distribution acting on a helicopter has been determined, the task of representing this distribution by means of concentrated loads at selected panel points remains. This program distributes the airloads to the panel points for various loading conditions at a reference point, which is usually the center of gravity location.

CPS08P (old program SE5002/SE1713) - Engine Loads Program

This program calculates the forces and moments at the engine center of gravity due to flight maneuvers and gyroscopic coupling of the rotating components. It uses the engine mass properties and aircraft accelerations to calculate the engine loads. The program can be used for single or twin engine aircraft.

CPS10P (old program SLHW02) - Dynamic Landing Gear Analysis

This program develops the loads in a crosstube landing gear for vertical and inclined plane landings. Both conventional crosstube attachments and the three point attachment with stops are allowed. The user has the option to have the program develop the load deflection curve or input it himself. The rotor lift, gross weight, and C.G. may be varied to analyze more than one landing load case at a time. After the loads are developed, a stress analysis is given that can be used to help size the crosstube.

CPS11P (old program PB2801) - Plastic Bending Analysis

This program develops the load deflection characteristics and the internal plastic bending stresses of a cantilevered or a symmetrically loaded simply supported beam with a tubular cross section. Arbitrary end loading is allowed with either large deflection or small deflection theory.

CPS12P (old program SP5009) - Pylon Loads Computer Programs

This program calculates the forces on a conventional transmission pylon system. The forces and moments about the lift links are calculated, these moments are used to determine which, if any, pylon mounts are bottomed. Based on the stop geometry, the loads on the mounts are calculated. The program can be executed in a mode that accounts for the deflections of the pylon system in calculating the moments about the lift link; or the moments can be calculated assuming the deflections are negligible.

CPS13P (old program ST4101) - Tension Beam Analysis (for Blade Fatigue Specimens and Masts)

This program serves as an aid in statically determining an initial set of loads for a blade fatigue specimen. For a given axial load the tip moment and shear are determined so that the bending moment distribution in the specimen approximates a flight loads moment distribution as closely as a least squares curve fit will permit.

CPS14P (old program TW3301) - Torsional Analysis of Flexures with Rectangular & I Sections

This program performs an analysis for flexures having rectangular or I cross sections that vary arbitrarily with spanwise distance. The stiffening effects due to centrifugal force, elongation of the outer fibers, and warping restraint due to end fixity are considered and the maximum shear stress at each section is calculated.

CPS15P (old program NLAM01) - Simultaneous Linear Equations (High Accuracy Solution)

This program solves a set of simultaneous linear equations using IMSL routine LEQT2F, which is the linear equation solution full-storage-mode-high accuracy solution routine. The program solves the equations to the desired accuracy and informs the user of the accuracy achieved.

CPS16P (old program LD2112) - One Dimensional Joint Analysis

This program calculates the distribution of an axial load thru a one dimensional fastener pattern. It is applicable to a mechanical joint composed of two similar or dissimilar elastic materials.

CPS17P (old program SRAN01) - Frame Energy Solution

This program calculates the internal shears and moments in a frame or bulkhead when a set of balanced shear flows and/or concentrated loads are applied. The solution is based upon the assumption that the elastic energy causing deformation of the frame produces consistent deformation at any point.

CPS18P (old program OB1) - Composite Tube Analysis

This program performs an analysis for composite or metal tubes and columns. It calculates the ultimate buckling loads and the natural frequencies for the tube or column assuming pin ends. It allows each segment of the tube to have its own material properties and for this reason, it is preferred for composites. It does not check for local failures, this should be done independently.

CPS19P (old program SDAN08) - Design Loads (with a Panel Point Weight Distribution)

This program combines a set of external loads and balancing load factors with the panel point unit shears and moments to produce airframe design loads.

CPS23P (old program SESE01) - NASTRAN Model Description Report Generator

This program selects and sorts structural member data from a NASTRAN Bulk Data Deck. The selected structure is associated with its defining grids, section properties, and material data. A complete definition of this data is produced on a report format. The structural members may be selected by the following means:

Members totally in a region defined by a rectangular parallelpiped

Members which are completely defines by an input grid list

Members whose element IDs are contained in a list

CPS24P (old program SP2803) - Pylon Support Loads (A Laminate Analysis)

This program calculates the loads in the pylon supports and the resulting applied airframe loads. The rotor loads and airframe accelerations are used to determine the axial bi-pod loads, the loads applied the airframe attachments, the mount, and the stop.

CPS25P (old program SESB13) - Shear and Moments Envelope Program

This program plots shear and moment envelopes using specified cases selected from a SESN09 Shears and Moments History Tape. The envelope is a plot of the maximum and minimum values at each station location. This program is similar to program SESB13 but it allows two additional features; a plot heading and a figure designation for the plot.

CPS26P - Composite Laminate Buckling Delamination Analysis

This program is designed to assess the capability of a laminate to resist near surface delamination growth. The delamination is assumed to exist in a laminate. The strain energy release rate and related buckling and threshold strains are then calculated and plotted against the delamination sizes so that a more delamination resistant laminate can be designed.

CPS27P - The Crippling Strength of Compression Members

This program determines the crippling strength of a compression member with a simple or complex cross section based on a set of empirical design curves. The user may select from a number of different design curves obtained from Bell Design Manuals and U.S. Government Reports. In addition, the program can develop crippling design curves from the users own experimental data.

CPS28P - Determination of Aircraft Tiedown Loads

This program is developed to analyze a tiedown configuration and to determine the load distribution in the tiedown cables. The significant features of the analytical model include using a rigid body fuselage with a flexible tailboom and landing gear structures, and using one-directional load carrying springs for cables. This methodology is generalized to be applicable for structures of any configuration possessing these characteristics.

2.4.4 Other Programs

Individual Programs not appropriate or unable to be incorporated in the above systems will be included in this category. Possible examples are those programs of other disciplines or programs without source code and those that are unable to be transformed to a form compatible with any of the ASAM systems. This part of ASAM is currently under development.

2.5 Miscellaneous Programs

Other miscellaneous programs used by structure engineers are described in the following:

2.5.1 Load Programs Unrelated to Finite Element Analysis

SDAN02 - Unit Panel Point Shears and Moments (RPT 299-099-252)

This program uses panel point weight distribution from computer program SDCS01 to create unit shear and moment data.

SDAN39 - Shear and Moment Plotting Program (RPT 299-099-252)

This program plots the panel point shear and moment data created by SDAN08.

SDCS01 - Panel Point Unit Weight Distribution (RPT 299-099-252)

Unit panel point weight distributions are created using SDCS01. The program inputs a weights tape and outputs weight distributions to be used as input to SDAN02.

2.5.2 Dynamic Structures Analysis

CBCR02 - Rotor Blade Section Mass and Stiffness Properties (RPT 299-099-749)

Airfoil coordinates are extracted from tape storage and used in calculating mass and stiffness properties for various rotor blade cross sections.

CSCR02 - Stress Maps for Rotor Blade Cross Sections (RPT 299-099-749)

Bending plus axial stresses are plotted for steady and oscillatory loads along the outer surface of rotor blade cross sections using airfoil coordinates from a tape.

2.5.3 Fatigue Evaluation

FFAM06 - Flight Test Data Generator

This program converts measured flight loads in scalor units from a Xerox 530 GDC tape into engineering units. This data can be saved on a file tape for use with other programs. Output includes span plots, load vs. airspeed plots, and tab listings of mean and oscillatory loads depending on the program option chosen.

FFST01 - Loads-Airspeed Comparison Plot Program

This program allows "Loads vs. Airspeed" plots from different flights, or different models, to be plotted together for comparison. Loads are taken from FECS01 or FECS09 file tapes.

FECR22 - Flight Data Organization Program

This program produces a listing of sorted loads for a given Item Code, using FECS01 file tape as input. The user can request oscillatory loads and/or maximum and minimum peak loads. Each gross weight and C.G. is sorted separately and the condition number and altitude "Line" is given with each column.

DLCR04 - Fatigue Life Calculation

This program reads loads from FECS01 file tape, computes a stress set, and determines the fatigue life of a component. Histogram plots, with various selection options which apply to the plots only, can also be generated from this program. Part II of the program is used to create, update, delete and list frequency of occurrence spectra used for calculation of the fatigue life.

DLCR21 - Cycle Counted Fatigue Life Calculation

This program calculates the fatigue life of a component by considering the damage caused by each rotor revolution. This is used with loads from FFCR03 file tape. This program may also be used as a cycle count program, producing a cycle count listing and override cards for program DLCR04.

DACR62 - Harmonic Fatigue Damage

The purpose of this program is to obtain a summation of damage caused by various harmonics (up to 9/Rev) within a rotor cycle for a given record. The program reads a time history tape digitized continuously at high rate.

DLCS10 - Span Plot-Regression

This program reads FECS01 file tape and creates span plots for main rotor and tail rotor blades. A curvilinear regression up to a fourth order is performed after interrogating the data. Output includes plots and punched cards for loads at a given span station.

FLASH - Fatigue Life Analysis System for Helicopter

This is a computer software system that was developed to speed up fatigue life evaluation for helicopter certification. The system uses the computer to manipulate the data and perform analysis in the fatigue life evaluation process which consists of measuring the fatigue loads in flight, comparing these loads to fatigue test data, and then making computations of the expected fatigue life.

SECTION 3

GENERAL

3.0 GENERAL

This section, for the most part, deals with sections. Properties of various types of sections are shown. In addition, conversion factors, graphical integration, Rockwell hardness, strain gages, etc., are described.

3.1 PROPERTIES OF AREAS

In structural analysis, certain properties of areas are needed, such as location of the centroid; first moment of the area and the second moment or moment of inertia of the area with respect to an axis either perpendicular to the area or lying in the plane of the area and the product of inertia of the area with respect to a set of perpendicular axes lying in the area. These properties are defined in this section.

3.1.1 Areas and Centroids

The area of a generalized shape, as shown in Figure 3.1, is the sum of all of the incremental areas, dA.

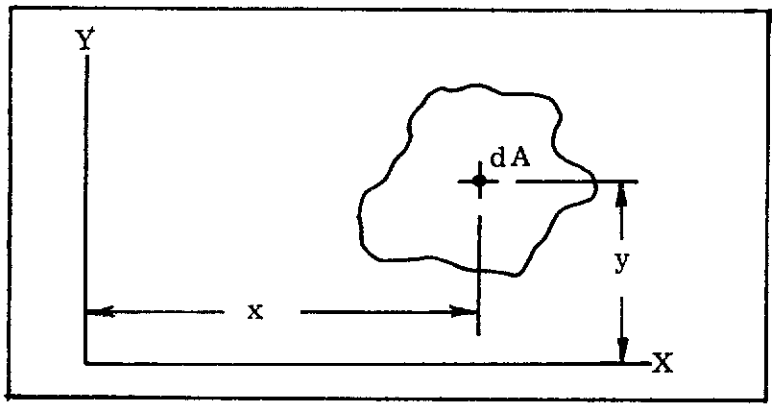


FIGURE 3.1 GENERALIZED AREA

$$A = \int dA = \sum_{i=1}^{n} A_i \qquad 3.1$$

The centroid of an area is that point in the plane of the area about any axis through which the moment of the area is zero; it coincides with the center of gravity of a body with the same shape having an infinitely thin homogeneous thickness. Equation 3.2 and 3.3 define the centroids of an area:

$$\overline{x} = \frac{\int x dA}{A} = \frac{\sum_{i=1}^{n} x_i A_i}{\sum_{i=1}^{n} A_i} \qquad 3.2$$

$$\overline{y} = \frac{\int y dA}{A} = \frac{\sum_{i=1}^{n} y_i A_i}{\sum_{i=1}^{n} A_i} \qquad 3.3$$

3.1.2 Moments of Inertia

The moment of inertia is the second moment of area. The moment of inertia of an element of area such as dA in Figure 3.1 with respect to a given axis is defined as the product of the area and the square of the distance from the axis to the element. It is shown mathematically as:

$$dI_y = x^2 dA \qquad 3.4$$

The sum of the moments of inertia of all the elements in a generalized area is defined as the moment of inertia of the area, that is,

$$I_y = \int x^2 dA = \sum_{i=1}^{n} (x_i)^2 A_i \qquad 3.5$$

$$I_x = \int y^2 dA = \sum_{i=1}^{n} (y_i)^2 A_i \qquad 3.6$$

The subscripts x and y indicate the axis about which the moment of inertia is taken.

The moment of inertia of any area about any axis is equal to the moment of inertia of the area about an axis through the centroid of the area and parallel to the given axis plus the product of the area and the square of the distance between the two parallel axes; that is

$$I_x = I_o + (\overline{y})^2 A \qquad 3.7$$

$$I_y = I_o + (\overline{x})^2 A \qquad 3.8$$

These equations can accomplish transformation of moment of inertia of plane areas only between parallel axes and one of the two parallel axes must pass through the centroid as shown in Figure 3.2.

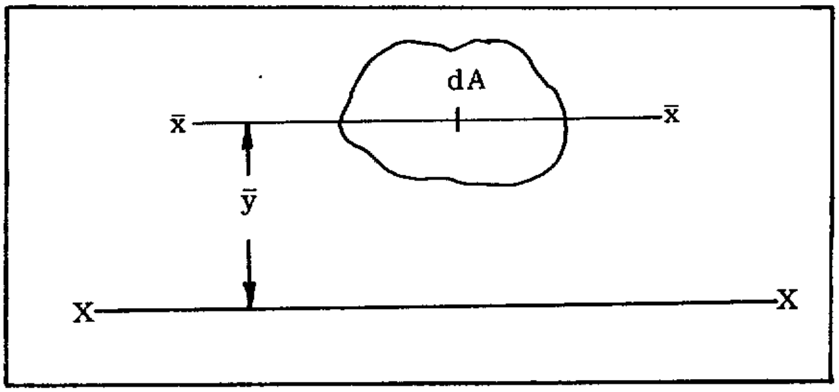


FIGURE 3.2 PARALLEL AXIS TRANSFORMATION

The term I_o in equations 3.7 and 3.8 is the moment of inertia of the area about its own centroidal axis. Figure 3.4 shows an example of a typical moment of inertia calculation by the tabular method.

3.1.3 Polar Moment of Inertia

The polar moment of inertia is the moment of inertia of an area about an axis perpendicular to the plane of the area. The elementary area, dA, in Figure 3.3 lying in the plane xy has

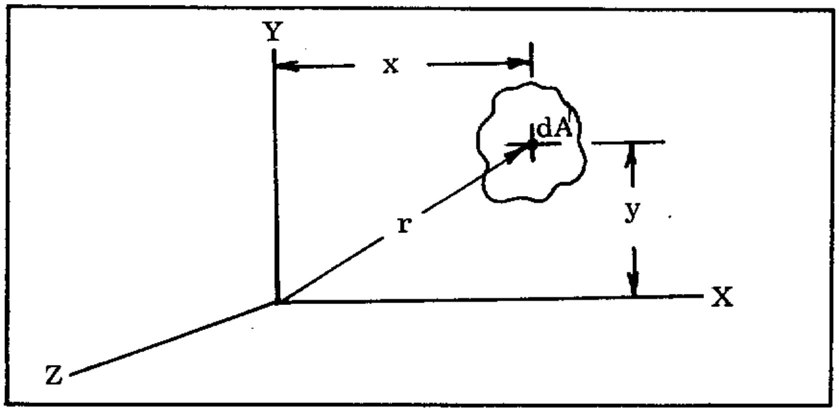


FIGURE 3.3 POLAR MOMENT OF INERTIA

$\bar{Y} = \Sigma AY / A = .0833/.1204 = .6919$ in.

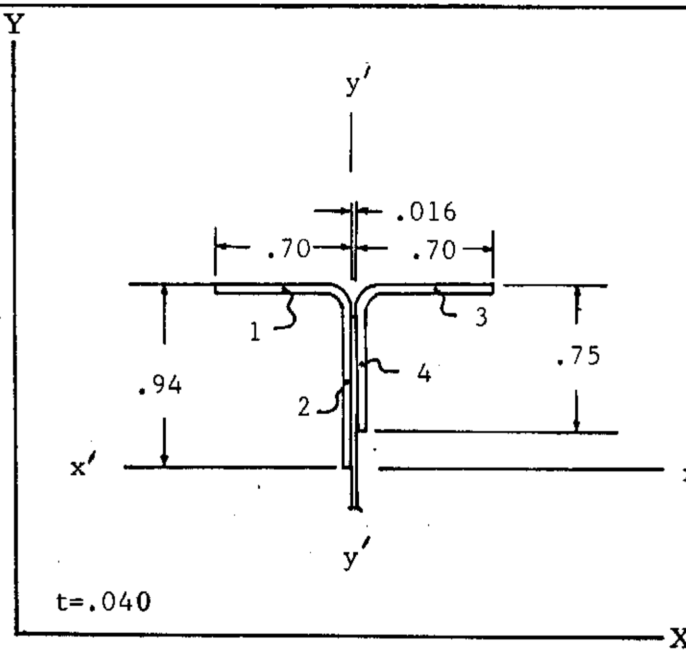
$\bar{X} = \Sigma AX / A = .0007/.1204 = .0058$ in.

$I_{\bar{X}} = \Sigma AY^2 + \Sigma I_{OX} - \bar{Y}^2 \Sigma A = .0631 + .0036 - .1204\,(.6919)^2$
$= .0091$ in.^{4}

$I_{\bar{Y}} = \Sigma AX^2 + \Sigma I_{OY} - \bar{X}^2 \Sigma A = .0072 + .0022 - .1204\,(.0058)^2$
$= .0094$ in.^{4}

$I_{\bar{X}\bar{Y}} = \Sigma AXY + \Sigma I_{OXY} - \bar{X}\bar{Y}\Sigma A = .0007 + 0 - (.0058)(.6919)(.1204)$
$= .0002$ in.^{4}

$\tan 2\beta = -2 I_{\bar{X}\bar{Y}} / (I_{\bar{X}} - I_{\bar{Y}}) = -2(.0002)/(.0091-.0094)$
$= 1.3333$



ELE	A	Y	Y^2	AY	AY^2	I_{OX}	X	X^2	AX	AX^2	I_{OY}	AXY
1	.028	.92	.846	.0258	.0237	--	-.35	.1225	-.0098	.0034	.0011	-.0090
2	.036	.45	.203	.0162	.0073	.0024	-.02	.0004	-.0007	.0000	--	-.0003
3	.028	.92	.846	.0258	.0237	--	.366	.1340	.0102	.0038	.0011	.0094
4	.0284	.545	.297	.0155	.0084	.0012	.036	.0013	.0010	.0000	--	.0006
Σ	.1204			.0833	.0631	.0036			.0007	.0072	.0022	.0007

FIGURE 3.4 EXAMPLE OF A MOMENT OF INERTIA CALCULATION

$$I_x = \int y^2 dA, \text{ (Equation 3.5)}$$

$$I_y = \int x^2 dA, \text{ (Equation 3.6)}$$

The moment of inertia about the z axis is

$$I_z = \int r^2 dA \qquad 3.9$$

but

$$r^2 = x^2 + y^2 \qquad 3.10$$

then

$$\int r^2 dA = \int x^2 dA + \int y^2 dA \qquad 3.11$$

The polar moment of inertia of an area is therefore equal to the sum of the moment of inertia of the area about two mutually perpendicular axes. Thus

$$I_z = I_x + I_y = I_p \qquad 3.12$$

3.1.4 Product of Inertia

The elementary area, dA, in Figure 3.1 is located at a distance x from the y axis and y from the x axis. The product of the area multiplied by the coordinate distances is then, xydA and is called the product of inertia. This term is a mathematical property that is dependent upon the area itself and its location relative to two mutually perpendicular axes. The value of the product of inertia for the entire area is

$$I_{xy} = \int xy dA = \sum_{i=1}^{n} X_i Y_i A_i \qquad 3.13$$

The subscripts of I serve to define the pair of axes of reference. Unlike the moments of inertia, products of inertia involve the first powers of the coordinate distance and such products may be positive, negative or zero. An example is shown in Figure 3.4.

Products of inertia can be transferred between axes. Thus the product of inertia of an area about any pair of mutually perpendicular axes is equal to the sum of the products of inertia of that area about a parallel mutually perpendicular pair through the centroid plus the product of the distances between the axis times the area as shown in Figure 3.5.

$$I_{xy} = I_{\bar{x}\bar{y}} + \bar{x}\bar{y}A \qquad 3.14$$

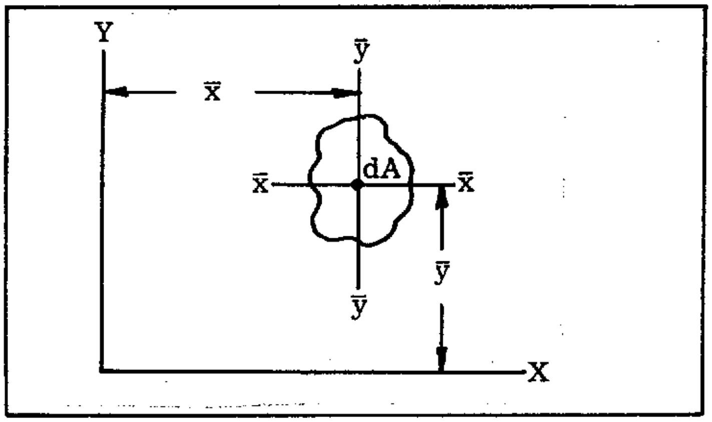


FIGURE 3.5 PRODUCT OF INERTIA

3.1.5 Moments of Inertia About Inclined Axes

Unsymmetrical sections are quite common and it is often necessary to find the moments of inertia about an inclined axis. The general procedure is to first find the moment of inertia about some set of rectangular axes through the centroid and transfer to a second set of axes also through the centroid making an angle Θ with the first set of axes using the following relationships and Figure 3.6.

$$I_u = I_x \cos^2\theta + I_y \sin^2\theta - I_{xy} \sin 2\theta \qquad 3.15$$

$$I_v = I_x \sin^2\theta + I_y \cos^2\theta + I_{xy} \sin 2\theta \qquad 3.16$$

$$I_{uv} = 1/2(I_x - I_y) \sin 2\theta + I_{xy} \cos 2\theta \qquad 3.17$$

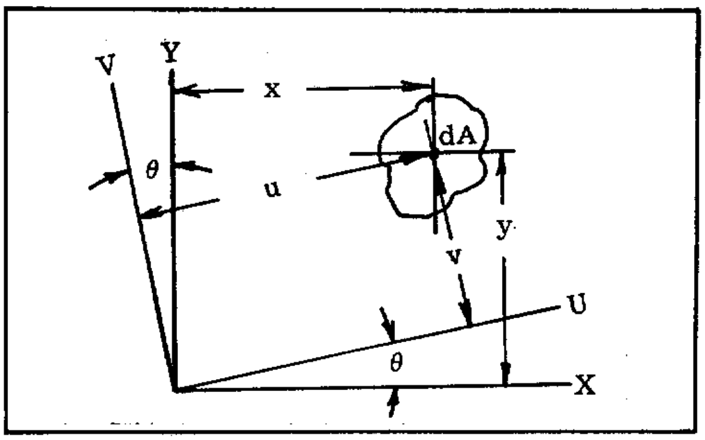


FIGURE 3.6 MOMENTS OF INERTIA ABOUT INCLINED AXES

The angle θ is positive when produced by a counterclockwise motion from the original axes.

3.1.6 Principal Axes

It is often necessary to determine the maximum and minimum values of moments of inertia. These occur about an axis passing through the centroid of the section. The axes are called the principal axes and the orientation is such that the moment of inertia is either greater than or less than for any other axis passing through the centroid.

The principal axes location makes an angle with the original axes of

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} \qquad 3.18$$

There are two values of 2θ differing by 180°, having the same tangent. Then there are two values of θ differing by 90°. About one axis at the angle θ from the original x axis, the value of the moment of inertia will be maximum and about the other axis, inclined at 90° to the first, the value of the moment of inertia will be minimum.

Substitution of the values of θ into equations 3.15 and 3.16 produces the principal moments of inertia.

$$I_{up} = 1/2\ (I_x + I_y) + \sqrt{{I_{xy}}^2 + 1/4\ (I_x - I_y)^2} \qquad 3.19$$

$$I_{vp} = 1/2\ (I_x + I_y) - \sqrt{{I_{xy}}^2 + 1/4\ (I_x - I_y)^2} \qquad 3.20$$

The sum of the principal moments of inertia, like the sum of any two moments of inertia about mutually perpendicular axes, is the polar moment of inertia.

It should be noted that the product of inertia of an area about a pair of axes which are principal axes of inertia is zero. The product of inertia about a pair of axes, one of which is an axis of summetry, is also equal to zero. Thus the axes of symmetry are principal axes of inertia.

3.1.7 Radius of Gyration

The radius of gyration of an area with respect to a given inertia axis may be defined as a distance to the point where the area would be concentrated in order to produce the same amount of inertia. Thus

$$\rho = \sqrt{I/A} \qquad 3.21$$

The subscript for I determines the inertia axis for the respective radius of gyration.

3.2 MOHR'S CIRCLE FOR MOMENTS OF INERTIA

The equations for the moments and products of inertia about inclined axes may be found using a semigraphic solution which aids in visualizing the relationship between the moments of inertia about various axes. The method is called Mohr's circle. Using Figure 3.7, the following procedure describes Mohr's circle for inertia.

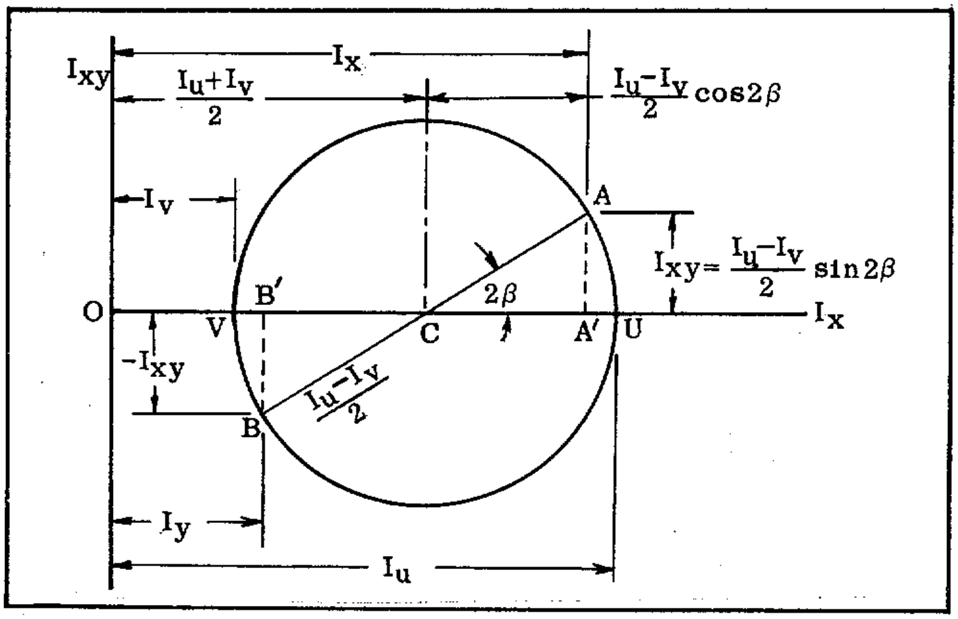


FIGURE 3.7 MOHR'S CIRCLE FOR MOMENTS OF INERTIA

(1) Calculate I_x, I_y and I_{xy} for the section. The x and y axes can be at any orientation but it should be located so that I_x is greater than I_y.

(2) Draw a set of rectangular axes. Label the horizontal axis I_x and the vertical axis I_{xy}. This is shown in Figure 3.7.

(3) Moments of inertia (second moments of areas) are always positive but products of inertia can be positive or negative. Positive moments of inertia are plotted to the right of the origin. Positive products of inertia (I_{xy}) are plotted above the I_x axis and negative values below.

(4) Lay off the distance OA′ along the I_x axis equal to I_x and A′A parallel to the I_{xy} axis equal to I_{xy}. Label the point (I_x, I_{xy}) as point A.

(5) In a similar manner locate point B by making OB′ equal to I_y and B′B equal to $-I_{xy}$, the value of I_{xy} with the algebraic sign reversed.

(6) Draw the line AB intersecting the I_x axis at C and draw a circle of diameter AB. This circle is Mohr's circle for moments of inertia and each point on the circle represents I_u and I_{uv} for any orientation of the u and v axes. The abscissa represents I_u and the ordinate I_{uv}.

The following relationships can be developed using Figure 3.7.

Maximum moment of inertia:

$$I_{up} = 1/2\ (I_x+I_y) + \sqrt{I_{xy}^2 + 1/4\ (I_x-I_y)^2} \qquad 3.22$$

Minimum moment of inertia:

$$I_{vp} = 1/2\ (I_x+I_y) - \sqrt{I_{xy}^2 + 1/4\ (I_x-I_y)^2} \qquad 3.23$$

These are the moments of inertia about the principal axes.

The angle of the principal axis with respect to the reference axis is

$$\tan 2\beta = \frac{2I_{xy}}{I_x-I_y} \qquad 3.24$$

The sign of β is taken as positive for counterclockwise movement from the reference x axis (I_x in Figure 3.7).

The moment and product of inertia (I_u, I_v, I_{uv}) may be determined at any angle 2θ from the principal axes. In the cross section the angle is θ while in Mohr's circle it is plotted as 2θ. From Figure 3.7 the values can be derived to be

$$I_u = I_x \cos^2\theta + I_y \sin^2\theta - I_{xy} \sin 2\theta \qquad 3.25$$

$$I_v = I_x \sin^2\theta + I_y \cos^2\theta + I_{xy} \sin 2\theta \qquad 3.26$$

$$I_{uv} = I_{xy} \cos 2\theta + \left(\frac{I_x-I_y}{2}\right) \sin 2\theta \qquad 3.27$$

3.3 MASS MOMENTS OF INERTIA

The inertia resistance to rotational acceleration is that property of a body which is commonly known as mass moment of inertia. If a body of mass m is allowed to rotate about an axis at an angular acceleration a, an element of this mass dm, will have a component of acceleration tangent to the circular path of ra, with the tangential force on the element being radm. Since the distance to the element is r, the resulting moment of the force equals r^2adm. Integrating the elements of the body gives

$$I = \int r^2 dm \qquad 3.28$$

This expression is known as the mass moment of inertia of the body, where a is dropped because it is constant for a given rigid body.

If the body is of constant mass density, the differential, dm, may be replaced with ρdV, since $dm = \rho dV$, and the following expression results

$$I = \rho \int r^2 dV \qquad 3.29$$

The units of mass moment of inertia are commonly expressed as lb-ft-sec^2 or slug-ft^2.

3.4 SECTION PROPERTIES OF SHAPES

Tables 3.1 through 3.5 show section properties of various sections. Table 3.6 presents the properties of standard tubings.

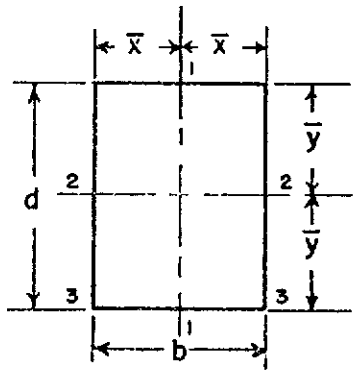
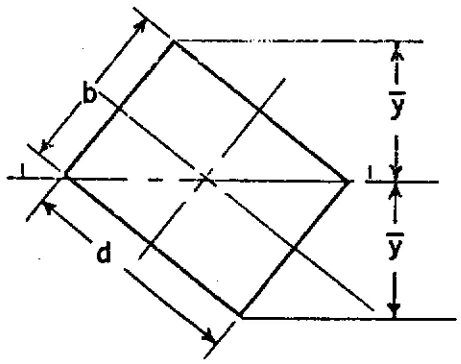
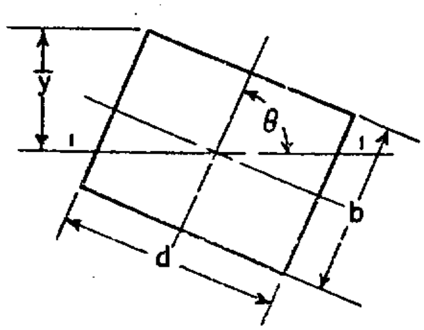
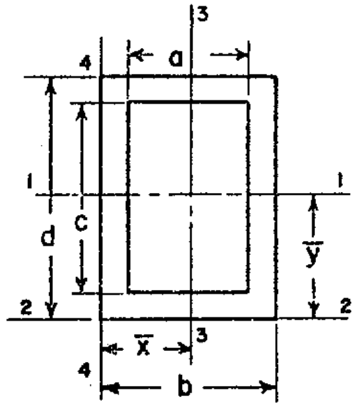
RECTANGLE 	$A = bd$ $\bar{x} = b/2$ $\bar{y} = d/2$ $I_{1-1} = \frac{db^3}{12}$ $I_{2-2} = \frac{bd^3}{12}$ $I_{3-3} = \frac{bd^3}{3}$ $\frac{I_{1-1}}{\bar{x}} = \frac{db^2}{6}$ $\frac{I_{2-2}}{\bar{y}} = \frac{bd^2}{6}$ $\rho_{1-1} = 0.2887b$ $\rho_{2-2} = 0.2887d$ $\rho_{3-3} = 0.5773d$
RECTANGLE 	$A = bd$ $\bar{y} = \frac{bd}{\sqrt{b^2 + d^2}}$ $I_{1-1} = \frac{b^3\ d^3}{6(b^2 + d^2)}$ $\frac{I_{1-1}}{\bar{y}} = \frac{b^2\ d^2}{6\sqrt{b^2 + d^2}}$ $\rho_{1-1} = \frac{bd}{\sqrt{6(b^2 + d^2)}}$
RECTANGLE 	$A = bd$ $\bar{y} = \frac{b\ SIN\theta + d\ COS\theta}{2}$ $I_{1-1} = \frac{bd\ (b^2\ SIN^2\ \theta + d^2\ COS^2\ \theta)}{12}$ $\frac{I_{1-1}}{\bar{y}} = \frac{bd(b^2\ SIN^2\ \theta + d^2\ COS^2\theta)}{6(b\ SIN\theta + d\ COS\theta)}$ $\rho_{1-1} = \sqrt{\frac{b^2\ SIN^2\ \theta + d^2\ COS^2\theta}{12}}$

TABLE 3.1 PROPERTIES OF COMMON SECTIONS

HOLLOW RECTANGLE

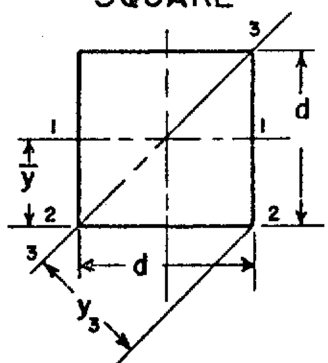


$$A = bd - ac$$
$$\bar{x} = 0.5b$$
$$\bar{y} = 0.5d$$
$$I_{1-1} = \frac{bd^3 - ac^3}{12}$$
$$I_{2-2} = \frac{bd^3}{3} - \frac{ac\,(3d^2 + c^2)}{12}$$
$$I_{3-3} = \frac{b^3 d - a^3 c}{12}$$
$$I_{4-4} = I_{3-3} + \left[\frac{bd - ac}{4}\right] b^2$$

$$\rho_{1-1} = \sqrt{\frac{bd^3 - ac^3}{12\,(bd - ac)}} \qquad \rho_{3-3} = \sqrt{\frac{b^3 d - a^3 c}{12\,(bd - ac)}}$$

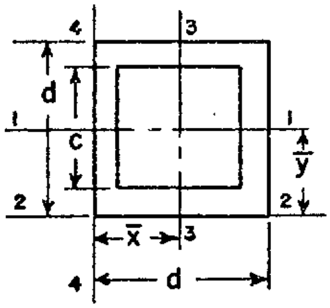
$$\rho_{2-2} = \sqrt{\frac{I_{2-2}}{(bd - ac)}} \qquad \rho_{4-4} = \sqrt{\frac{I_{4-4}}{(bd - ac)}}$$

SQUARE



$$A = d^2$$
$$\bar{y} = 0.5d$$
$$y_3 = 1.414d$$
$$I_{2-2} = \frac{d^4}{3}$$
$$I_{3-3} = \frac{d^4}{12} = I_{1-1}$$
$$\frac{I_{1-1}}{\bar{y}} = \frac{d^3}{6}$$
$$\rho_{1-1} = 0.2887d$$

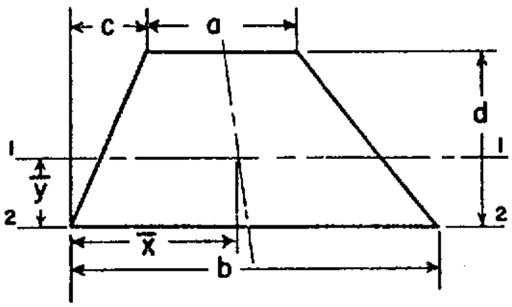
HOLLOW SQUARE



$$A = d^2 - c^2$$
$$\bar{x} = \bar{y} = 0.5d$$
$$I_{1-1} = I_{3-3} = \frac{d^4 - c^4}{12}$$
$$I_{2-2} = I_{4-4} = \frac{4d^4 - 3d^2c^2 - c^4}{12}$$
$$\rho_{1-1} = \rho_{3-3} = 0.289\sqrt{d^2 + c^2}$$
$$\rho_{2-2} = \rho_{4-4} = \sqrt{\frac{I_{1-1}}{d^2 - c^2}}$$

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

TRAPEZOID



$$A = \frac{(a+b)d}{2}$$

$$\bar{x} = \frac{b}{2} + \frac{\bar{y}}{d}\left[c - \frac{b-a}{2}\right]$$

$$\bar{y} = \frac{d\,(b+2a)}{3\,(a+b)}$$

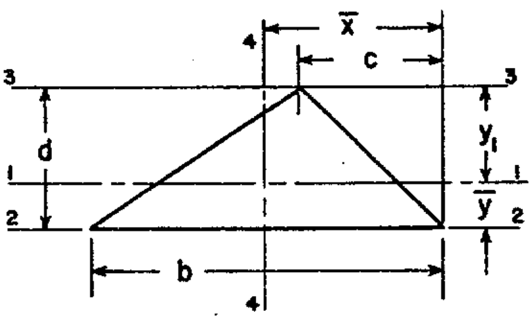
$$I_{1-1} = \frac{d^3\,(b^2+4ab+a^2)}{36\,(a+b)}$$

$$I_{2-2} = \frac{d^3\,(b+3a)}{12}$$

$$\rho_{1-1} = \frac{d}{6(a+b)}\sqrt{2\,(b^2+4ab+a^2)}$$

$$\rho_{2-2} = d\sqrt{\frac{3a+b}{6(a+b)}}$$

OBLIQUE TRIANGLE



$$A = \frac{bd}{2}$$

$$\bar{y}_2 = \frac{d}{3}$$

$$\bar{x} = \frac{b+c}{3}$$

$$I_{1-1} = \frac{bd^3}{36}$$

$$I_{2-2} = \frac{bd^3}{12}$$

$$I_{3-3} = \frac{bd^3}{4}$$

$$I_{4-4} = \frac{bd}{36}(b^2+c^2-bc)$$

$$\frac{I_{1-1}}{\bar{y}} = \frac{bd^2}{12}$$

$$\frac{I_{1-1}}{y_1} = \frac{bd^2}{24}$$

$$\rho_{1-1} = .236d$$

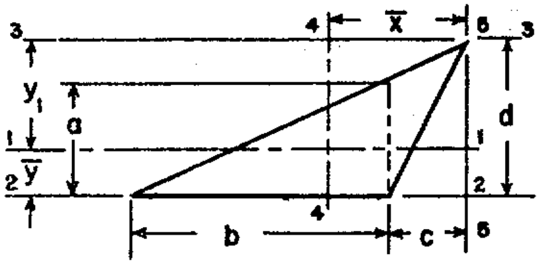
$$\rho_{2-2} = .408d$$

$$\rho_{3-3} = .707d$$

$$\rho_{4-4} = .236\sqrt{b^2+c^2-bc}$$

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

OBTUSE TRIANGLE



$$A = \frac{bd}{2} \qquad \frac{I_{1-1}}{\bar{y}} = \frac{bd^2}{12}$$

$$\bar{y} = \frac{d}{3} \qquad \frac{I_{1-1}}{y_1} = \frac{bd^2}{24}$$

$$\bar{x} = \frac{b+2c}{3} \qquad \rho_{1-1} = .236\,d$$

$$a = \frac{bd}{b+c} \qquad \rho_{2-2} = .408\,d$$

$$I_{1-1} = \frac{bd^3}{36} \qquad \rho_{3-3} = .707\,d$$

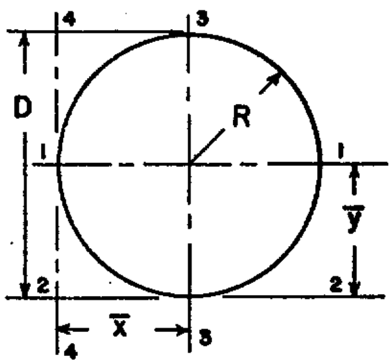
$$I_{2-2} = \frac{bd^3}{12} \qquad \rho_{4-4} = .236\sqrt{b^2+c^2+bc}$$

$$I_{3-3} = \frac{bd^3}{4} \qquad \rho_{5-5} = .408\sqrt{b^2+c^2+3bc}$$

$$I_{4-4} = \frac{bd}{36}\left(b^2+bc+c^2\right)$$

$$I_{5-5} = \frac{bd}{12}\left(b^2+3bc+3c^2\right)$$

CIRCLE



$$A = \frac{\pi}{4} D^2 = .7854\,D^2 = 3.1416\,R^2$$

$$\bar{y} = \bar{x} = \frac{D}{2} = R$$

$$I_{1-1} = I_{3-3} = \frac{\pi}{64} D^4 = .04909\,D^4 = .7854\,R^4$$

$$I_{2-2} = I_{4-4} = \frac{5}{4}\pi R^4$$

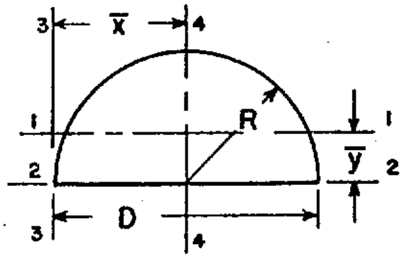
$$\frac{I_{1-1}}{\bar{y}} = \frac{I_{3-3}}{\bar{x}} = .09818\,D^3 = 0.7854\,R^3$$

$$\rho_{1-1} = \rho_{3-3} = \frac{D}{4}$$

$$\rho_{2-2} = \rho_{4-4} = 1.118\,D$$

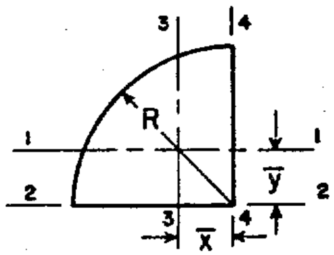
TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

SEMI - CIRCLE



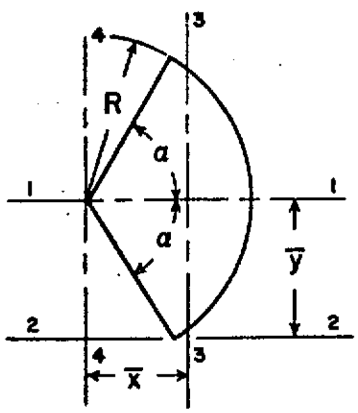
$$A = \frac{\pi}{2} R^2 = 1.5708\, R^2$$
$$\bar{y} = \frac{4R}{3\pi} = 0.4244\, R$$
$$\bar{x} = R$$
$$I_{1-1} = R^4 \left[\frac{\pi}{8} - \frac{8}{9\pi} \right] = 0.1097\, R^4$$
$$I_{2-2} = .3927\, R^4 \qquad \rho_{1-1} = .2643\, R$$
$$I_{4-4} = .3927\, R^4 \qquad \rho_{2-2} = .50\, R$$
$$I_{3-3} = 1.9637\, R^4 \qquad \rho_{3-3} = 1.118\, R$$
$$\rho_{4-4} = .50\, R$$

QUARTER CIRCLE



$$A = .7854\, R^2$$
$$\bar{x} = \bar{y} = .4244\, R$$
$$I_{1-1} = .0549\, R^4 = I_{3-3}$$
$$I_{2-2} = I_{4-4} = .1964\, R^4$$

CIRCULAR SECTOR

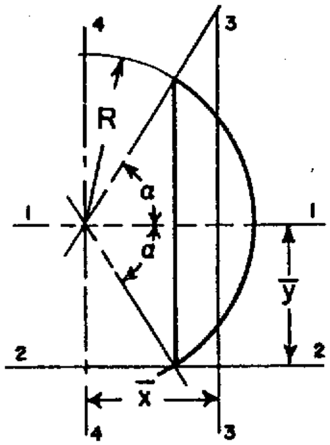


α IN RADIANS

$$A = R^2 \alpha$$
$$\bar{x} = \frac{2}{3} \left[\frac{R \sin\alpha}{\alpha} \right]$$
$$\bar{y} = R \sin\alpha$$
$$I_{1-1} = 0.25R^4 (\alpha - \sin\alpha \cos\alpha)$$
$$I_{2-2} = 0.25R^4 (\alpha - \sin\alpha \cos\alpha) + R^4 \alpha \sin^2\alpha$$
$$I_{3-3} = 0.25R^4 \left(\alpha - \frac{16 \sin^2\alpha}{9\alpha} + \frac{\sin 2\alpha}{2} \right)$$
$$I_{4-4} = 0.25R^4 (\alpha + \sin\alpha \cos\alpha)$$
$$\rho_{1-1} = 0.50R \sqrt{1 - \frac{\sin\alpha \cos\alpha}{\alpha}}$$
$$\rho_{2-2} = \sqrt{\frac{I_{2-2}}{R^2 \alpha}}$$
$$\rho_{3-3} = 0.50R \sqrt{1 + \frac{\sin\alpha \cos\alpha}{\alpha} - \frac{16 \sin^2\alpha}{9\alpha^2}}$$
$$\rho_{4-4} = 0.50R \sqrt{1 + \frac{\sin\alpha \cos\alpha}{\alpha}}$$

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

CIRCULAR SEGMENT



α IN RADIANS

$$A = 0.50R^2(2\alpha - \mathrm{SIN}\,2\alpha)$$

$$\bar{x} = \frac{4R\ \mathrm{SIN}^3\alpha}{3(2\alpha - \mathrm{SIN}\,2\alpha)}$$

$$\bar{y} = R\,\mathrm{SIN}\,\alpha$$

$$I_{1-1} = 0.25AR^2\left[1 - \frac{2}{3}\left(\frac{\mathrm{SIN}^3\alpha\,\mathrm{COS}\,\alpha}{\alpha - \mathrm{SIN}\,\alpha\,\mathrm{COS}\,\alpha}\right)\right]$$

$$I_{2-2} = I_{1-1} + 0.50R^4(2\alpha - \mathrm{SIN}\,2\alpha)(\mathrm{SIN}^2\alpha)$$

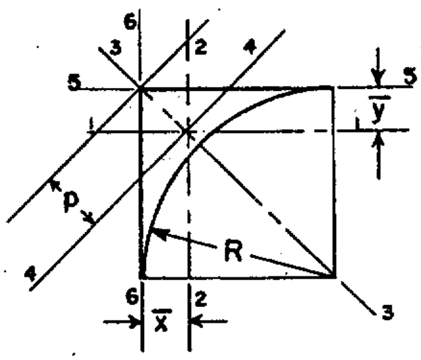
$$I_{3-3} = 0.25AR^2\left[1 + \frac{2\,\mathrm{SIN}^3\alpha\,\mathrm{COS}\,\alpha}{\alpha - \mathrm{SIN}\,\alpha\,\mathrm{COS}\,\alpha}\right] - \frac{4R^6\,\mathrm{SIN}^6\alpha}{9A}$$

$$I_{4-4} = 0.25AR^2\left[1 + \frac{2\,\mathrm{SIN}^3\alpha\,\mathrm{COS}\,\alpha}{\alpha - \mathrm{SIN}\,\alpha\,\mathrm{COS}\,\alpha}\right]$$

$$\rho_{1-1} = 0.50R\sqrt{1 - \frac{2\,\mathrm{SIN}^3\alpha\,\mathrm{COS}\,\alpha}{3(\alpha - \mathrm{SIN}\,\alpha\,\mathrm{COS}\,\alpha)}}$$

$$\rho_{3-3} = \sqrt{\frac{2I_{3-3}}{R^2(2\alpha - \mathrm{SIN}\,\alpha)}}$$

CIRCULAR COMPLEMENT



$$A = 0.2146R^2$$

$$\bar{x} = \bar{y} = 0.2234R$$

$$p = 0.3159R$$

$$I_{1-1} = I_{2-2} = 0.007545R^4$$

$$I_{3-3} = 0.01198R^4$$

$$I_{4-4} = 0.003106R^4$$

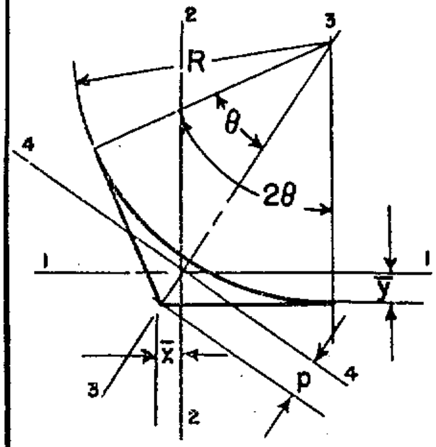
$$I_{5-5} = I_{6-6} = 0.0182R^4$$

$$\rho_{1-1} = \rho_{2-2} = 0.187R$$

$$\rho_{5-5} = \rho_{6-6} = 0.292R$$

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

OBLIQUE FILLET



$$A = (\tan\theta - \theta)R^2$$

$$\bar{y} = \left[1 - \frac{\sin^2\theta \tan\theta}{3(\tan\theta - \theta)}\right]R$$

$$\bar{x} = \left[\tan\theta - \frac{\sin^2\theta \tan^2\theta}{3(\tan\theta - \theta)}\right]R$$

$$p = \left[\sec\theta - \frac{\sin^2\theta \tan^2\theta}{3(\tan\theta - \theta)}\right]R$$

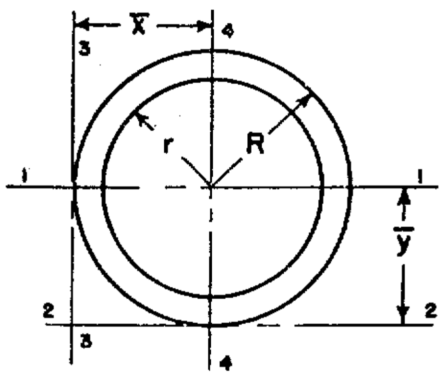
$$I_{1-1} = \left[\frac{\sin^4\theta \tan\theta}{3} + \frac{\sin\theta\cos\theta(1+2\sin^2\theta) - \theta}{4} - \frac{\sin^4\theta \tan^2\theta}{9(\tan\theta - \theta)}\right]R^4$$

$$I_{2-2} = \left[\frac{\tan\theta(\tan^2\theta + \sin^2\theta)}{6} - \frac{3\theta - \sin\theta\cos\theta(3 - 2\sin^2\theta)}{12} - \frac{\sin^4\theta \tan^4\theta}{9(\tan\theta - \theta)}\right]R^4$$

$$I_{3-3} = \left[\frac{\sin^3\theta}{6\cos\theta} - \frac{\theta - \sin\theta\cos\theta}{4}\right]R^4$$

$$I_{4-4} = \left[\frac{\sec^3\theta - \cos^3\theta}{6\sin\theta} - \frac{\theta + \sin\theta\cos\theta}{4} - \frac{\sin^2\theta \tan^4\theta}{9(\tan\theta - \theta)}\right]R^4$$

HOLLOW CIRCLE



$$A = 3.1416(R^2 - r^2)$$

$$\bar{x} = \bar{y} = R$$

$$I_{1-1} = I_{4-4} = 0.7854(R^4 - r^4)$$

$$I_{2-2} = I_{3-3} = 0.7854(5R^4 - 4R^2r^2 - r^4)$$

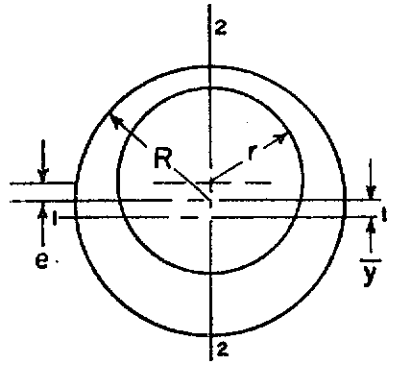
$$\rho_{1-1} = \rho_{4-4} = 0.5(R^2 + r^2)^{1/2}$$

$$\rho_{2-2} = \rho_{3-3} = \sqrt{\frac{5R^2 + r^2}{2}}$$

$$\frac{I_{1-1}}{\bar{y}} = \frac{I_{4-4}}{\bar{x}} = \frac{.7854(R^4 - r^4)}{R}$$

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

ECCENTRIC HOLLOW CIRCLE



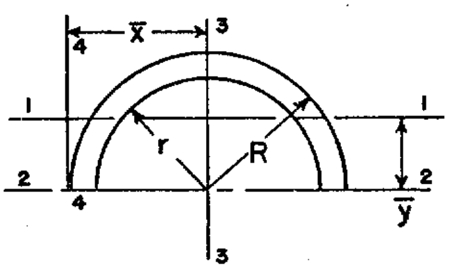
$$A = \pi (R^2 - r^2)$$

$$\bar{y} = \frac{e r^2}{R^2 - r^2}$$

$$I_{1-1} = .7854 (R^4 - r^4) - \frac{\pi e^2 R^2 r^2}{(R^2 - r^2)}$$

$$I_{2-2} = .7854 (R^4 - r^4)$$

HOLLOW SEMI-CIRCLE



$$A = 1.5708 (R^2 - r^2)$$

$$\bar{x} = R$$

$$\bar{y} = 0.4244 \left(R + \frac{r^2}{R + r}\right)$$

$$I_{1-1} = 0.3927 (R^4 - r^4) - 1.5708 (R^2 - r^2) \bar{y}^2$$

$$I_{2-2} = 0.3927 (R^4 - r^4)$$

$$I_{3-3} = 0.3927 (R^4 - r^4)$$

$$I_{4-4} = 0.3927 (R^2 - r^2)(5R^2 + r^2)$$

$$\rho_{1-1} = \sqrt{\frac{2 I_{1-1}}{\pi (R^2 - r^2)}}$$

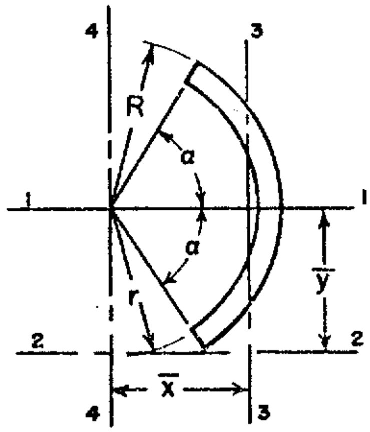
$$\rho_{2-2} = 0.5 \sqrt{R^2 + r^2}$$

$$\rho_{3-3} = 0.5 \sqrt{R^2 + r^2}$$

$$\rho_{4-4} = \sqrt{\frac{2 I_{4-4}}{\pi (R^2 - r^2)}}$$

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

HOLLOW CIRCULAR SECTOR



$$A = \alpha (R^2 - r^2)$$

$$\bar{x} = \frac{2 \sin\alpha (R^3 - r^3)}{3\alpha (R^2 - r^2)}$$

$$\bar{y} = R \sin\alpha$$

$$I_{1-1} = 0.25\,\alpha (R^4 - r^4)\left[1 - \frac{\sin\alpha \cos\alpha}{\alpha}\right]$$

$$I_{2-2} = 0.25\,\alpha (R^4 - r^4)\left[1 - \frac{\sin\alpha \cos\alpha}{\alpha}\right] + \alpha (R^4 - r^2 R^2) \sin^2\alpha$$

$$I_{3-3} = 0.25\,\alpha (R^4 - r^4)\left[1 + \frac{\sin\alpha \cos\alpha}{\alpha}\right] - \frac{1}{\alpha (R^2 - r^2)}\left[\frac{2 \sin\alpha (R^3 - r^3)}{3}\right]^2$$

$$I_{4-4} = 0.25\,\alpha (R^4 - r^4)\left[1 + \frac{\sin\alpha \cos\alpha}{\alpha}\right]$$

$$\rho_{1-1} = \sqrt{\left(\frac{R^2 + r^2}{4}\right)\left[1 - \frac{\sin\alpha \cos\alpha}{\alpha}\right]}$$

$$\rho_{2-2} = \sqrt{\frac{I_{2-2}}{(R^2 - r^2)\alpha}}$$

$$\rho_{3-3} = \sqrt{\frac{I_{3-3}}{(R^2 - r^2)\alpha}}$$

$$\rho_{4-4} = \sqrt{\left(\frac{R^2 + r^2}{4}\right)\left[1 + \frac{\sin\alpha \cos\alpha}{\alpha}\right]}$$

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

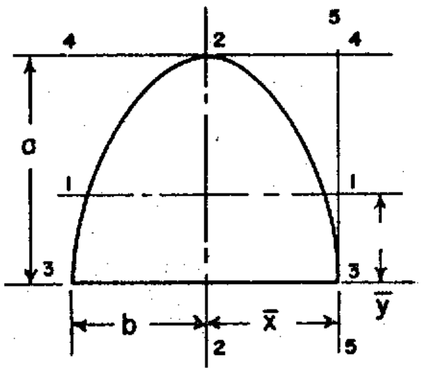
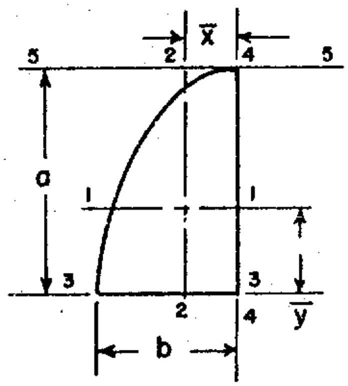
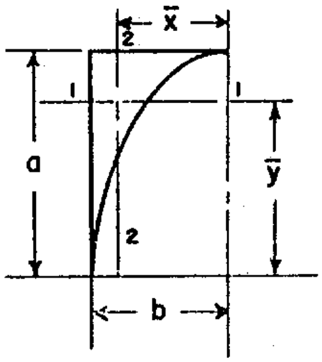
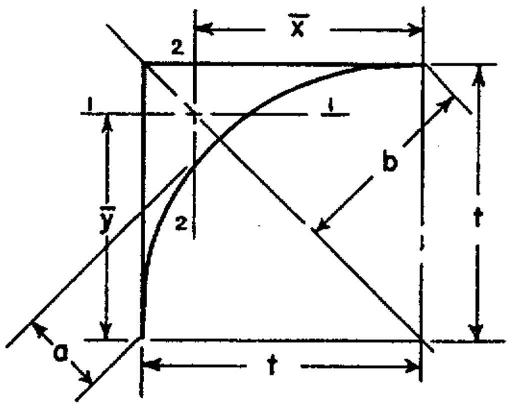
Section	Properties
PARABOLA 	$A = 4/3\ ab$ $\bar{x} = b$ $\bar{y} = 0.40a$ $I_1 = 0.09143\, a^3 b$ $I_2 = 0.2667\, ab^3$ $I_3 = 0.3048\, a^3 b$ $I_{4-4} = .5714\, a^3 b$ $\quad I_{5-5} = 1.6\, ab^3$ $\rho_{1-1} = .2619a$ $\quad \rho_{4-4} = .6546a$ $\rho_{2-2} = .4472b$ $\quad \rho_{5-5} = 1.095b$
HALF PARABOLA 	$A = 2/3\ ab$ $\bar{y} = 0.40a$ $\bar{x} = 0.375b$ $I_1 = 0.04571\, a^3 b$ $I_2 = 0.03958\, ab^3$ $I_3 = 0.15238\, a^3 b$ $I_4 = 0.1333\, ab^3$ $I_{5-5} = .2857\, a^3 b$ $\rho_{1-1} = .2619a$ $\quad \rho_{4-4} = .4472b$ $\rho_{2-2} = .2437b$ $\quad \rho_{5-5} = .6546a$
COMPLEMENT OF HALF PARABOLA 	$A = 1/3\ ab$ $\bar{y} = 0.70a$ $\bar{x} = 0.75b$ $I_1 = 0.01762\, a^3 b$ $I_2 = 0.0125\, ab^3$ $\rho_{1-1} = .2299a$ $\rho_{2-2} = .1937b$

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

PARABOLIC FILLET IN RIGHT ANGLE



$a = 0.3536t$

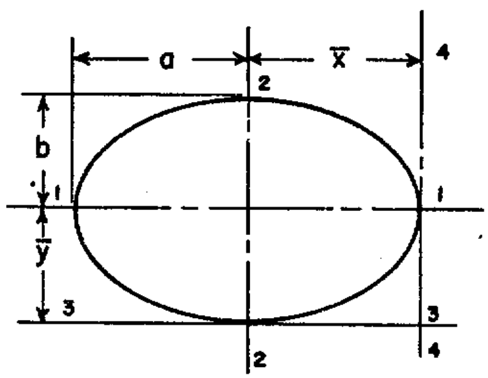
$b = 0.7071t$

$A = 0.1667t^2$

$\bar{x} = \bar{y} = 0.800t$

$I_1 = I_2 = 0.05238t^4$

ELLIPSE



$A = 3.1416ab$

$\bar{x} = a$

$\bar{y} = b$

$I_1 = 0.7854ab^3$

$I_2 = 0.7854a^3b$

$\rho_{1-1} = b/2$

$\rho_{3-3} = 1.118b$

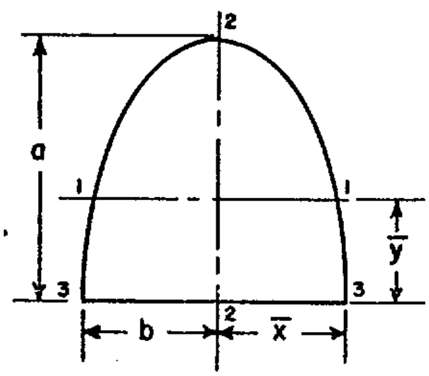
$\rho_{2-2} = a/2$

$\rho_{4-4} = 1.118a$

$I_{3-3} = 5/4\,\pi ab^3 = 3.927ab^3$

$I_{4-4} = 5/4\,\pi b^3a = 3.927a^3b$

HALF ELLIPSE



$A = 1.5708b$

$\bar{x} = b$

$\bar{y} = 0.4244a$

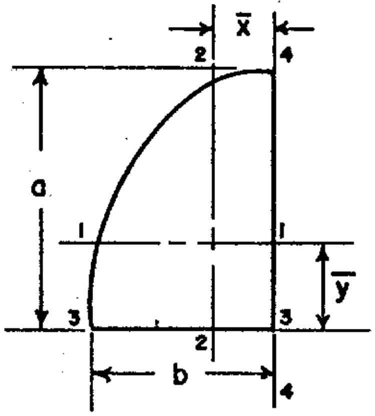
$I_1 = 0.1098a^3b$

$I_2 = 0.3927ab^3$

$I_3 = 0.3927a^3b$

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

QUARTER ELLIPSE



$A = 0.7854\,ab$

$\bar{y} = 0.4244\,a$

$\bar{x} = 0.4244\,b$

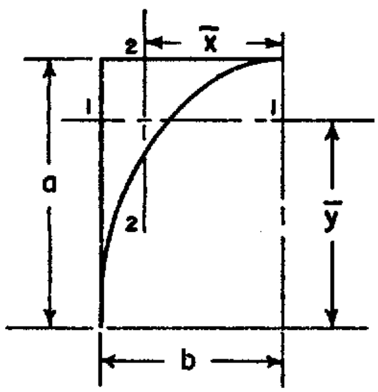
$I_1 = 0.05488\,a^3 b$

$I_2 = 0.05488\,ab^3$

$I_3 = 0.1963\,a^3 b$

$I_4 = 0.1963\,ab^3$

ELLIPTIC COMPLEMENT



$A = 0.2146\,ab$

$\bar{y} = 0.7766\,a$

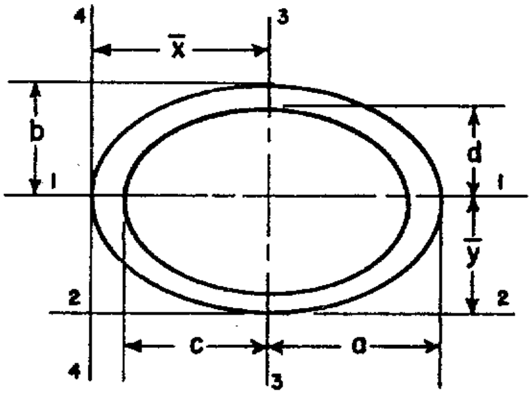
$\bar{x} = 0.7766\,b$

$I_1 = 0.00754\,a^3 b$

$I_2 = 0.00754\,ab^3$

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

HOLLOW ELLIPSE



$A = 3.1416\,(ab - cd)$

$\bar{x} = a$

$\bar{y} = b$

$I_{1-1} = 0.7854\,(ab^3 - cd^3)$

$I_{2-2} = 0.7854\,(ab^3 - cd^3) + 3.1416\,(ab - cd)b^2$

$I_{3-3} = 0.7854(a^3b - c^3d)$

$I_{4-4} = 0.7854(a^3b - c^3d) + \pi\,(ab - cd)a^2$

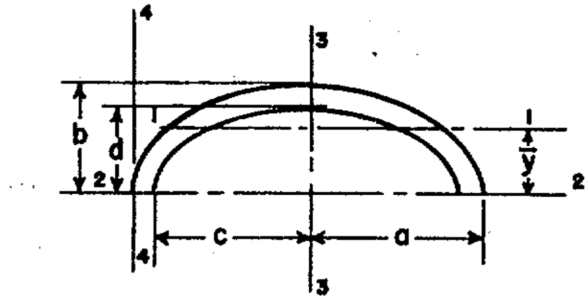
$\rho_{1-1} = \sqrt{\dfrac{ab^3 - cd^3}{4(ab - cd)}}$

$\rho_{2-2} = \sqrt{\dfrac{I_{2-2}}{\pi(ab - cd)}}$

$\rho_{3-3} = \sqrt{\dfrac{a^3b - c^3d}{4(ab - cd)}}$

$\rho_{4-4} = \sqrt{\dfrac{I_{4-4}}{\pi(ab - cd)}}$

HOLLOW SEMI-ELLIPSE



$A = 1.5708\,(ab - cd)$

$\bar{x} = a$

$\bar{y} = \dfrac{4}{3\pi}\left[\dfrac{ab^2 - cd^2}{ab - cd}\right]$

$I_{1-1} = \dfrac{\pi}{8}(ab^3 - cd^3) - \dfrac{\pi(ab - cd)}{2}\left[\dfrac{4(ab^2 - cd^2)}{3\pi(ab - cd)}\right]^2$

$I_{2-2} = \dfrac{\pi}{8}(ab^3 - cd^3)$

$I_{3-3} = \dfrac{\pi}{8}(a^3b - c^3d)$

$I_{4-4} = \dfrac{\pi}{8}(a^3b - c^3d) + \dfrac{\pi a^2(ab - cd)}{2}$

$\rho_{1-1} = \sqrt{\dfrac{2\,I_{1-1}}{\pi(ab - cd)}}$

$\rho_{2-2} = \sqrt{\dfrac{2\,I_{2-2}}{4(ab - cd)}}$

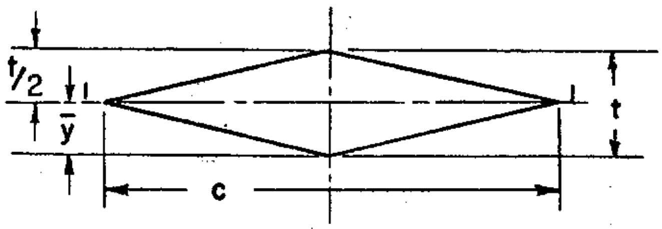
$\rho_{3-3} = \sqrt{\dfrac{2I_{3-3}}{\pi(ab - cd)}}$

$\rho_{4-4} = \sqrt{\dfrac{2\,I_{4-4}}{\pi(ab - cd)}}$

NOTE: - ABOVE PROCEDURES IMPLY A NON-UNIFORM WALL THICKNESS

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

DOUBLE WEDGE SECTION



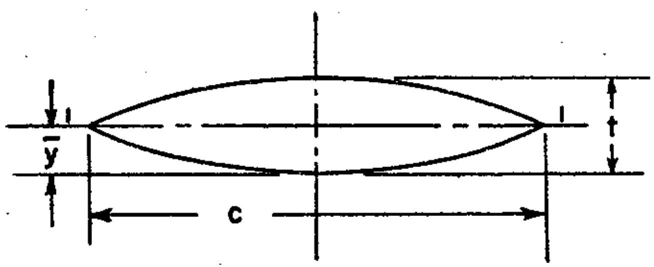
$$A = \frac{ct}{2}$$

$$\bar{y} = \frac{t}{2}$$

$$I_{1-1} = \frac{ct^3}{48} = .0208\,ct^3$$

$$\frac{I_{1-1}}{y} = \frac{ct^2}{24} = .0416\,ct^2$$

BI-CONVEX SECTION



$$A = \frac{2}{3}\,ct$$

$$\bar{y} = \frac{t}{2}$$

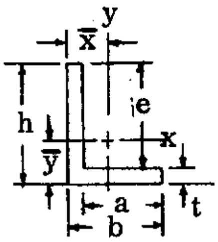
$$I_{1-1} = \frac{4}{105}\,ct^3 = .0381\,ct^3$$

$$\frac{I_{1-1}}{y} = \frac{8ct^2}{105} = .0761\,ct^2$$

VALUES OF A AND I ARE BASED ON BI-PARABOLIC SECTION.
ERROR IS NEGLIGIBLE WHEN $t/c < 0.10$.

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

ANGLE



$$A = t(b + e)$$

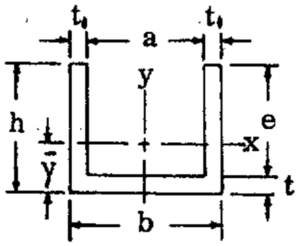
$$\bar{x} = \frac{t(2a + h) + a^2}{2(a + h)} \qquad \bar{y} = \frac{t(b + 2e) + e^2}{2(b + e)}$$

$$I_x = \frac{t(h - \bar{y})^3 + b\bar{y}^3 - a(\bar{y} - t)^3}{3}$$

$$I_y = \frac{t(b - \bar{x})^3 + h\bar{x}^3 - e(\bar{x} - t)^3}{3}$$

$$I_{xy} = \pm \frac{abeht}{4(b + e)}$$

CHANNEL



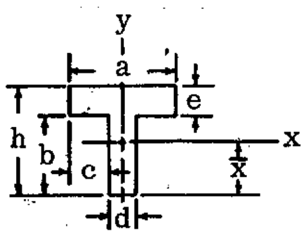
$$A = 2ht_1 + at$$

$$\bar{y} = \frac{h^2t_1 + .5t^2a}{2ht_1 + at}$$

$$I_x = \frac{2t_1h^3 + at^3}{3} - \frac{(h^2t_1 + .5t^2a)^2}{2ht_1 + at}$$

$$I_y = \frac{hb^3 - ea^3}{12}$$

T SECTION



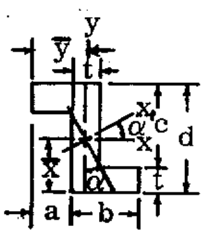
$$A = ae + bd$$

$$\bar{x} = h - \tfrac{1}{2}\left(\frac{dh^2 + 2ce^2}{dh + 2ce}\right)$$

$$I_x = \frac{a(h - \bar{x})^3 - 2c(b - \bar{x})^3 + d\bar{x}^3}{3}$$

$$I_y = \frac{ea^3 + bd^3}{12}$$

Z SECTION



$$A = t(d + 2a)$$

$$\bar{x} = d/2 \qquad \bar{y} = \frac{2b - t}{2}$$

$$I_x = \frac{1}{12}\left[bd^3 - a(d - 2t)^3\right]$$

$$I_y = \frac{1}{12}\left[d(b + a)^3 - 2a^3c - 6ab^2c\right]$$

$$I_{x_1} = \frac{I_x \cos^2\alpha - I_y \sin^2\alpha}{\cos 2\alpha}$$

$$\tan 2\alpha = \frac{(dt - t^2)(b^2 - bt)}{I_x - I_y}$$

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

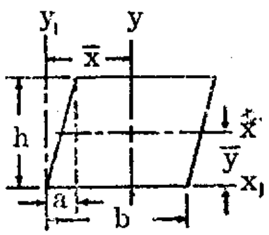
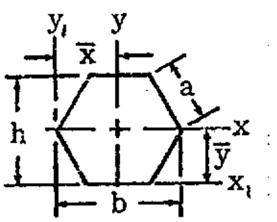
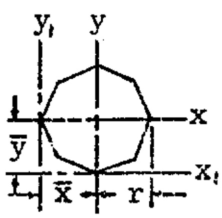
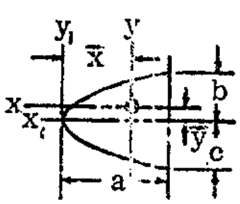
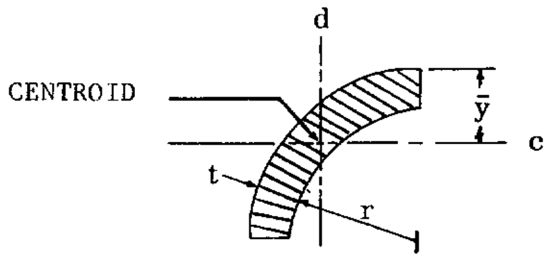
PARALLELOGRAM 	$A = bh \quad \bar{x} = \frac{a+b}{2} \quad \bar{y} = \frac{h}{2}$ $I_x = \frac{bh^3}{12} \quad I_y = \frac{bh(a^2+b^2)}{12}$ $I_{x_1} = \frac{bh^3}{3} \quad I_{y_1} = \frac{bh}{6}(2a^2 + 2b^2 + 3ab)$ $I_p = \frac{bh}{12}(a^2 + b^2 + h^2)$
REGULAR HEXAGON 	$A = .866h^2$ $\bar{x} = b/2 = a \quad \bar{y} = h/2$ $I_x = I_y = .0601h^4$ $I_{x_1} = .2766h^4 \quad I_{y_1} = .3969h^4$ $I_p = .1203h^4$
REGULAR OCTAGON 	$A = 2.8286r^2$ $\bar{x} = \bar{y} = r$ $I_x = I_y = .6381\ r^4$ $I_{x_1} = I_{y_1} = 3.4667\ r^4$ $I_p = 1.2763\ r^4$
NOSE RIB  x_1 = chord line. Based on parabolic segment.	$A = 2/3a(b + c)$ $\bar{x} = .6a \quad \bar{y} = .375(b - c)$ $I_x = \frac{a(b+c)}{480}(19b^2 + 26bc + 19c^2)$ $I_{x_1} = .1333(ab + ac)(b^2 - bc + c^2)$ $I_y = .0418\ a^3(b + c)$ $I_{y_1} = .2857\ a^4(b + c)^2 \quad I_p = I_x + I_y$

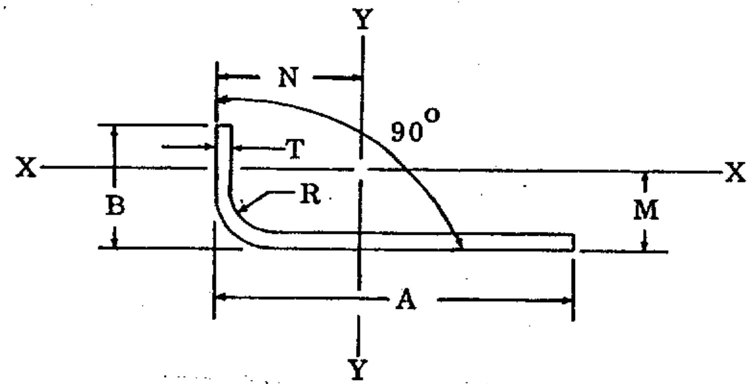
TABLE 3.1 (CONT'D) - PROPERTIES OF COMMON SECTIONS



t	r	Area	$\bar{y}$	I_c
.012	.03	.00679	.01887	$.0889\times10^{-6}$
	.06	.001244	.0299	.524 "
.016	.03	.000955	.0215	.1446 "
	.06	.001709	.0325	.773 "
.020	.03	.001257	.0240	.219 "
	.06	.00220	.0351	1.071 "
	.09	.00314	.0461	3.05 "
.025	.06	.00285	.0384	1.519 "
	.09	.00403	.0494	4.15 "
	.13	.00560	.0640	10.96 "
.032	.06	.00382	.0429	2.31 "
	.13	.00734	.0686	15.25 "
	.16	.00885	.0727	26.5 "
.040	.06	.00503	.0480	3.50 "
	.09	.00691	.0592	8.55 "
	.13	.00942	.0739	20.9 "
	.22	.01508	.1059	83.7 "
.050	.09	.00903	.0656	12.60 "
	.13	.01217	.0805	29.5 "
	.16	.01453	.0915	49.2 "
	.25	.0216	.1244	157.8 "
.063	.13	.01598	.0889	43.1 "
	.22	.0249	.1221	154.7 "
	.31	.0338	.1550	381 "
.071	.13	.01846	.0940	$.532\times10^{-4}$
	.16	.0218	.1052	.853 "
	.25	.0318	.1383	2.55 "
	.38	.0463	.1859	7.71 "
.080	.13	.0214	.0998	.662 "
	.19	.0289	.1221	1.554 "
	.28	.0402	.1552	4.05 "
	.41	.0565	.203	11.05 "
.090	.13	.0247	.1061	.832 "
	.19	.0332	.1286	1.891 "
	.31	.0502	.1728	6.22 "
	.47	.0728	.231	18.62 "

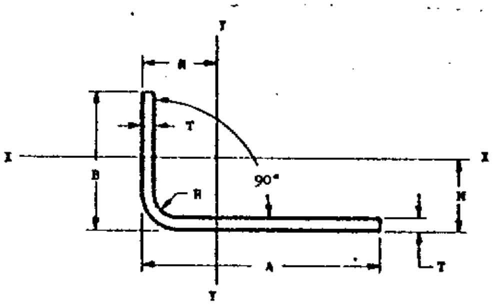
t	r	Area	$\bar{y}$	I_c
.100	.16	.0330	.1238	1.566×10^{-4}
	.22	.0424	.1462	3.17 "
	.38	.0675	.205	12.22 "
	.56	.0958	.271	34.3 "
.125	.19	.0496	.1510	3.43 "
	.28	.0672	.1845	8.07 "
	.44	.0987	.243	24.5 "
	.69	.1478	.335	80.6 "
.160	.25	.0829	.1958	9.76 "
	.38	.1156	.244	24.9 "
	.56	.1609	.310	64.8 "
	.88	.241	.427	214×10^{-4}
.190	.31	.1209	.237	.00213
	.44	.1597	.286	.00466
	.75	.252	.400	.01757
	1.13	.366	.539	.0527
.250	.50	.245	.347	.00996
	.63	.296	.395	.01707
	1.00	.442	.531	.0545
	1.50	.638	.713	.1619
.313	.63	.387	.436	.0248
	.81	.475	.502	.0447
	1.25	.692	.664	.1335
	2.00	1.060	.938	.473
.375	.75	.552	.520	.0504
	1.00	.670	.613	.0991
	1.50	.994	.796	.276
	2.50	1.583	1.161	1.096
.500	1.00	.982	.694	.1593
	1.50	1.374	.878	.418
	2.00	1.767	1.062	.873
	3.25	2.75	1.518	3.23

TABLE 3.2 - AREA, CENTROID & MOMENT OF INERTIA OF 90° BENDS



A	B	T	R	AREA	M	N	I_{XX}	I_{YY}	ρ_{XX}	ρ_{YY}
9/16	1/4	.020	1/16	.015	.048	.211	.0001	.0005	.068	.185
		.032	3/32	.023	.054	.221	.0001	.0008	.066	.180
5/8	1/4	.020	1/16	.016	.045	.240	.0001	.0007	.066	.205
		.032	3/32	.025	.051	.250	.0001	.0010	.064	.200
11/16	1/4	.020	1/16	.018	.043	.269	.0001	.0009	.064	.223
		.032	3/32	.028	.049	.279	.0001	.0013	.062	.220
		.040	1/8	.033	.052	.266	.0001	.0016	.062	.216
3/4	5/16	.020	1/16	.020	.056	.281	.0001	.0012	.064	.245
		.032	3/32	.031	.062	.209	.0002	.0018	.083	.242
		.040	1/8	.038	.066	.299	.0003	.0022	.082	.240
		.051	5/32	.048	.072	.309	.0003	.0027	.081	.237
7/8	5/16	.020	1/16	.023	.051	.340	.0001	.0018	.081	.285
		.032	3/32	.035	.056	.350	.0002	.0028	.080	.282
		.040	1/8	.043	.061	.358	.0003	.0034	.078	.279
		.051	5/32	.054	.066	.368	.0003	.0041	.077	.276
1	5/16	.020	1/16	.025	.047	.399	.0002	.0026	.077	.324
		.032	3/32	.039	.052	.409	.0002	.0041	.076	.321
		.040	1/8	.048	.057	.418	.0003	.0049	.075	.316
		.051	5/32	.060	.062	.426	.0003	.0060	.075	.314
1 1/8	5/16	.040	1/8	.053	.053	.478	.0003	.0068	.073	.356
		.051	5/32	.067	.050	.489	.0003	.0083	.072	.352
		.064	3/16	.082	.065	.501	.0004	.0094	.071	.347
1 1/4	5/16	.040	1/8	.058	.050	.539	.0003	.0091	.071	.394
		.051	5/32	.073	.056	.550	.0004	.0111	.069	.390
		.064	3/16	.090	.062	.562	.0004	.0133	.068	.384
		.072	7/32	.099	.066	.572	.0005	.0144	.068	.380
1 3/8	5/16	.040	1/8	.063	.048	.600	.0003	.0118	.069	.432
		.051	5/32	.079	.053	.611	.0004	.0146	.067	.428
		.064	3/16	.098	.060	.624	.0004	.0174	.066	.422
		.072	7/32	.108	.054	.634	.0005	.0189	.065	.417
1 1/2	5/16	.040	1/8	.068	.046	.661	.0005	.0151	.066	.470
		.051	5/32	.086	.051	.672	.0004	.0186	.065	.465
		.064	3/16	.106	.058	.685	.0004	.0223	.064	.459
		072	7/32	.117	.062	.696	.0005	.0242	.064	.454
1 3/4	3/8	.051	5/32	.102	.057	.771	.0006	.0306	.079	.548
		.064	3/16	.126	.063	.784	.0008	.0370	.079	.542
		.072	7/32	.140	.068	.794	.0009	.0405	.078	.538
		.081	1/4	.155	.073	.804	.0009	.0441	.078	.533
1 7/8	3/8	.051	5/32	.108	.055	.833	.0007	.0381	.078	.586
		.064	3/16	.134	.062	.845	.0008	.0450	.077	.579
		.072	7/32	.149	.066	.856	.0009	.0493	.076	.575
		.081	1/4	.166	.071	.866	.0009	.0528	.076	.570
2	1/2	.051	5/32	.121	.074	.848	.0016	.0490	.114	.636
		.064	3/16	.150	.080	.860	.0019	.0598	.112	.631
		.072	7/32	.167	.084	.870	.0021	.0658	.112	.629
		.081	1/4	.186	.089	.880	.0023	.0723	.111	.624
2 1/4	1/2	.064	3/16	.166	.076	.982	.0019	.0830	.106	.706
		.072	7/32	.186	.080	.992	.0021	.0915	.107	.705
		.081	1/4	.206	.084	1.002	.0023	.1000	.106	.699
		.091	9/32	.229	.089	1.013	.0026	.1105	.106	.694

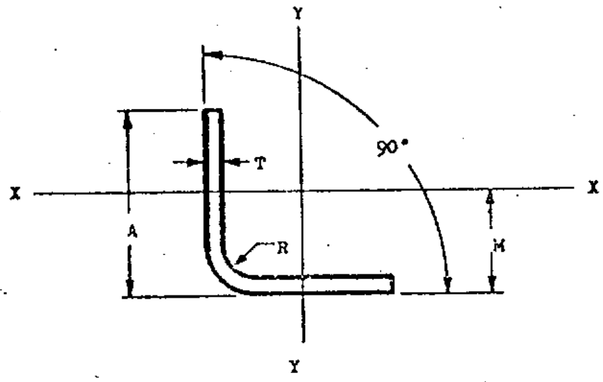
TABLE 3.3 - PROPERTIES OF ANGLES



A	B	T	R	AREA	M	N	I_{XX}	I_{YY}	ρ_{XX}	ρ_{YY}
9/16	7/16	.025	3/32	.025	.111	.176	.0004	.0008	.134	.182
		.032	3/32	.029	.113	.179	.0005	.0010	.133	.181
		.040	1/8	.036	.118	.186	.0006	.0012	.132	.180
5/8	1/2	.032	3/32	.034	.125	.194	.0008	.0013	.153	.201
		.040	1/8	.041	.134	.200	.0009	.0016	.152	.199
		.051	5/32	.051	.141	.208	.0012	.0020	.152	.199
11/16	7/16	.032	3/32	.034	.101	.232	.0006	.0017	.125	.224
		.040	1/8	.041	.106	.239	.0007	.0020	.128	.221
		.051	5/32	.051	.113	.248	.0008	.0024	.127	.220
	9/16	.032	3/32	.038	.144	.209	.0011	.0018	.173	.222
		.040	1/8	.046	.149	.215	.0014	.0022	.173	.222
		.051	5/32	.057	.156	.222	.0017	.0027	.171	.219
3/4	1/2	.032	3/32	.037	.117	.246	.0008	.0022	.149	.244
		.040	1/8	.046	.122	.255	.0010	.0027	.148	.242
		.051	5/32	.057	.128	.262	.0012	.0033	.147	.240
		.064	3/16	.070	.135	.271	.0015	.0040	.145	.239
		.072	7/32	.077	.141	.279	.0016	.0042	.143	.234
	5/8	.032	3/32	.041	.160	.224	.0016	.0024	.194	.242
		.040	1/8	.051	.165	.230	.0019	.0030	.193	.241
		.051	5/32	.063	.172	.238	.0023	.0036	.191	.239
		.064	3/16	.078	.179	.246	.0028	.0043	.188	.236
		.072	7/32	.086	.186	.253	.0030	.0047	.187	.234
7/8	1/2	.040	1/8	.051	.112	.308	.0011	.0041	.144	.284
		.051	5/32	.064	.118	.316	.0013	.0050	.143	.282
		.064	3/16	.076	.125	.326	.0015	.0061	.141	.279
		.072	7/32	.086	.130	.334	.0017	.0065	.140	.276
		.081	1/4	.095	.135	.342	.0018	.0072	.139	.274
		.091	9/32	.104	.142	.351	.0020	.0077	.133	.271
	5/8	.040	1/8	.056	.152	.281	.0020	.0045	.159	.285
		.051	5/32	.070	.158	.289	.0024	.0056	.157	.282
		.064	3/16	.086	.165	.300	.0029	.0066	.185	.278
		.072	7/32	.095	.171	.306	.0032	.0072	.184	.276
		.081	1/4	.105	.177	.313	.0035	.0079	.182	.274
		.091	9/32	.115	.183	.322	.0038	.0085	.180	.272
	3/4	.040	1/8	.061	.196	.261	.0033	.0046	.233	.281
		.051	5/32	.076	.202	.268	.0041	.0059	.232	.279
		.064	3/16	.094	.210	.276	.0050	.0072	.230	.275
		.072	7/32	.104	.215	.283	.0054	.0079	.228	.275
		.081	1/4	.115	.222	.290	.0059	.0086	.227	.275
		.091	9/32	.127	.228	.297	.0065	.0093	.225	.271
1	1/2	.040	1/8	.056	.103	.364	.0011	.0059	.140	.327
		.051	5/32	.070	.109	.375	.0013	.0075	.130	.324
		.064	3/16	.086	.116	.384	.0016	.0086	.137	.320
		.072	7/32	.095	.121	.391	.0018	.0095	.136	.317
		.081	1/4	.105	.126	.400	.0018	.0105	.135	.314
		.091	9/32	.115	.132	.408	.0020	.0111	.133	.310
	5/8	.040	1/8	.061	.141	.336	.0021	.0065	.185	.326
		.051	5/32	.076	.147	.344	.0025	.0080	.183	.325
		.064	3/16	.094	.154	.353	.0031	.0096	.181	.320
		.072	7/32	.104	.159	.360	.0034	.0105	.180	.318
		.081	1/4	.115	.165	.369	.0037	.0115	.178	.316
		.091	9/32	.127	.171	.377	.0040	.0124	.176	.313
	3/4	.040	1/8	.066	.183	.312	.0035	.0069	.230	.326
		.051	5/32	.082	.188	.319	.0043	.0086	.226	.322
		.064	3/16	.102	.195	.328	.0052	.0104	.226	.320
		.072	7/32	.113	.201	.335	.0057	.0114	.225	.318
		.081	1/4	.125	.207	.342	.0063	.0125	.223	.318
		.091	9/32	.139	.214	.348	.0067	.0136	.221	.314
	7/8	.040	1/8	.071	.227	.282	.0053	.0073	.273	.324
		.051	5/32	.089	.233	.299	.0066	.0091	.272	.320

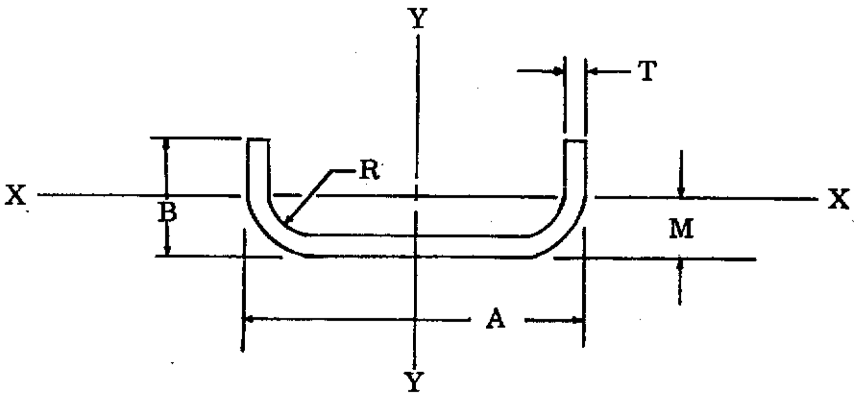
A	B	T	R	AREA	M	N	I_{XX}	I_{YY}	ρ_{XX}	ρ_{YY}
1	7/8	.064	3/16	.110	.241	.307	.0080	.0111	.271	[illegible]
		.072	7/32	.122	.247	.313	.0088	.0122	.269	.317
		.081	1/4	.135	.252	.320	.0097	.0134	.267	.315
		.091	9/32	.150	.259	.327	.0105	.0146	.265	.313
1 1/8	5/8	.040	1/8	.066	.132	.391	.0021	.0089	.181	[illegible]
		.064	3/16	.102	.145	.408	.0032	.0135	.176	[illegible]
		.081	1/4	.125	.155	.426	.0038	.0160	.172	[illegible]
		.102	11/32	.152	.168	.444	.0045	.0187	.170	[illegible]
	3/4	.040	1/8	.071	.171	.364	.0036	.0096	.224	[illegible]
		.064	3/16	.110	.184	.382	.0054	.0144	.222	[illegible]
		.081	1/4	.135	.194	.395	.0065	.0179	.219	[illegible]
		.102	11/32	.165	.204	.414	.0077	.0205	.219	[illegible]
	7/8	.040	1/8	.076	.213	.342	.0059	.0101	.269	.363
		.064	3/16	.118	.226	.357	.0085	.0154	.266	.361
		.081	1/4	.145	.238	.371	.0101	.0186	.265	.359
		.102	11/32	.178	.251	.386	.0120	.0221	.260	.355
	1	.040	1/8	.081	.258	.322	.0080	.0106	.314	.362
		.064	3/16	.125	.272	.337	.0121	.0162	.310	.359
		.081	1/4	.156	.283	.350	.0148	.0194	.307	.357
		.102	11/32	.190	.297	.365	.0176	.0235	.304	.351
1 1/4	5/8	.040	1/8	.071	.125	.448	.0024	.0119	.185	.409
		.064	3/16	.110	.136	.465	.0033	.0175	.173	.404
		.081	1/4	.135	.146	.482	.0039	.0214	.171	.398
		.102	11/32	.165	.159	.501	.0048	.0252	.167	.391
	3/4	.040	1/8	.076	.161	.419	.0039	.0127	.225	.409
		.064	3/16	.118	.174	.438	.0056	.0193	.217	.405
		.081	1/4	.145	.184	.450	.0067	.0235	.219	.402
		.102	11/32	.178	.197	.468	.0079	.0276	.211	.396
	7/8	.040	1/8	.081	.201	.395	.0059	.0135	.269	.408
		.064	3/16	.125	.214	.411	.0086	.0205	.262	.406
		.072	7/32	.140	.202	.320	.0093	.0232	.258	.407
		.081	1/4	.156	.225	.424	.0105	.0249	.259	.400
		.102	11/32	.190	.238	.441	.0125	.0297	.256	.395
	1	.040	1/8	.086	.244	.373	.0084	.0141	.313	.405
		.064	3/16	.134	.257	.386	.0126	.0216	.307	.402
		.081	1/4	.168	.258	.401	.0155	.0263	.304	.398
		.102	11/32	.205	.282	.417	.0183	.0315	.300	.394
	1 1/8	.040	1/8	.091	.289	.353	.0116	.0147	.356	.402
		.064	3/16	.142	.303	.366	.0175	.0226	.351	.397
		.081	1/4	.176	.314	.380	.0211	.0276	.348	.396
		.102	11/32	.215	.328	.395	.0257	.0331	.345	.392
1 3/8	3/4	.040	1/8	.081	.152	.475	.0040	.0165	.221	.451
		.064	3/16	.125	.165	.492	.0057	.0251	.213	.448
		.081	1/4	.156	.174	.508	.0069	.0303	.211	.445
		.102	11/32	.190	.187	.526	.0082	.0361	.208	.439
	7/8	.040	1/8	.086	.191	.448	.0060	.0175	.265	.451
		.064	3/16	.134	.203	.464	.0088	.0267	.257	.446
		.081	1/4	.166	.214	.479	.0108	.0324	.255	.442
		.102	11/32	.201	.227	.500	.0129	.0384	.252	.436
	1	.040	1/8	.091	.232	.425	.0087	.0185	.309	.449
		.064	3/16	.142	.244	.440	.0130	.0281	.302	.445
		.081	1/4	.176	.255	.454	.0158	.0343	.299	.441
		.102	11/32	.215	.270	.473	.0189	.0409	.296	.436
	1 1/8	.040	1/8	.096	.275	.403	.0113	.0190	.351	.445
		.064	3/16	.150	.290	.419	.0151	.0293	.347	.442
		.081	1/4	.186	.299	.431	.0220	.0359	.344	.439
		.102	11/32	.227	.314	.449	.0266	.0431	.341	.435
	1 1/4	.040	1/8	.101	.320	.384	.0157	.0198	.395	.443
		.064	3/16	.158	.334	.399	.0242	.0305	.391	.439
		.081	1/4	.196	.345	.411	.0297	.0373	.389	.437

TABLE 3.3 (CONT'D) - PROPERTIES OF ANGLES



A	T	R	AREA	M	Ixx	ρ_{XX}
1/2	.020	3/32	.019	.139	.0005	.159
	.025	3/32	.023	.141	.0006	.158
	.032	3/32	.030	.144	.0007	.157
9/16	.025	3/32	.026	.162	.0008	.173
	.032	3/32	.033	.159	.0011	.177
	.040	1/8	.041	.165	.0013	.176
5/8	.032	3/32	.037	.175	.0015	.198
	.040	1/8	.046	.181	.0018	.197
	.051	5/32	.057	.188	.0022	.197
11/16	.032	3/32	.042	.191	.0020	.218
	.040	1/8	.051	.196	.0024	.216
	.051	5/32	.063	.203	.0029	.216
3/4	.032	3/32	.045	.207	.0026	.238
	.040	1/8	.056	.212	.0031	.237
	.051	5/32	.070	.218	.0039	.236
	.064	3/16	.086	.227	.0047	.234
	.072	7/32	.095	.233	.0051	.233
13/16	.040	1/8	.061	.228	.0040	.257
	.051	5/32	.076	.234	.0080	.256
	.064	3/16	.094	.242	.0060	.254
	.072	7/32	.104	.248	.0066	.252
	.081	1/4	.115	.254	.0072	.250
7/8	.040	1/8	.066	.242	.0051	.278
	.051	5/32	.083	.249	.0063	.277
	.064	3/16	.102	.257	.0076	.274
	.072	7/32	.113	.264	.0084	.273
	.081	1/4	.125	.270	.0091	.270
1	.040	1/8	.076	.274	.0077	.318
	.051	5/32	.095	.280	.0096	.317
	.064	3/16	.118	.288	.0116	.314
	.072	7/32	.131	.294	.0128	.313
	.081	1/4	.145	.300	.0141	.312
	.091	9/32	.161	.307	.0154	.310
1 1/8	.040	1/8	.086	.305	.0109	.356
	.051	5/32	.108	.311	.0138	.357
	.064	3/16	.134	.319	.0169	.356
	.072	7/32	.149	.325	.0185	.355
	.081	1/4	.166	.331	.0221	.353
	.091	9/32	.184	.337	.0226	.351
	.102	11/32	.203	.346	.0247	.349
1 1/4	.040	1/8	.096	.337	.0154	.400
	.051	5/32	.221	.343	.0190	.397
1 1/4	.064	3/16	.150	.350	.0234	.395
	.072	7/32	.167	.356	.0258	.393
	.081	1/4	.186	.362	.0285	.392
	.091	9/32	.206	.368	.0317	.392
	.102	11/32	.229	.376	.0346	.391
1 3/8	.040	1/8	.106	.367	.0206	.441
	.051	5/32	.134	.374	.0256	.438
	.064	3/16	.166	.381	.0315	.435
	.072	7/32	.185	.387	.0349	.435
	.081	1/4	.206	.392	.0387	.433
	.091	9/32	.229	.399	.0426	.431
	.102	11/32	.254	.407	.0468	.429
1 1/2	.051	5/32	.146	.405	.0334	.478
	.064	3/16	.182	.413	.0411	.475
	.072	7/32	.203	.418	.0459	.475
	.081	1/4	.226	.423	.0509	.474
	.091	9/32	.252	.430	.0563	.473
	.102	11/32	.280	.438	.0615	.468
1 5/8	.051	5/32	.150	.435	.0430	.519
	.064	3/16	.198	.443	.0528	.517
	.072	7/32	.221	.448	.0586	.515
	.081	1/4	.248	.453	.0652	.514
	.091	9/32	.275	.461	.0727	.513
	.102	11/32	.305	.468	.0796	.511
1 3/4	.051	5/32	.172	.467	.0537	.559
	.064	3/16	.214	.475	.0666	.558
	.072	7/32	.239	.479	.0739	.556
	.081	1/4	.267	.483	.0823	.555
	.091	9/32	.297	.493	.0911	.554
	.102	11/32	.341	.500	.1008	.553
1 7/8	.051	5/32	.184	.498	.0652	.594
	.064	3/16	.230	.506	.0829	.597
	.072	7/32	.257	.512	.0917	.597
	.081	1/4	.287	.518	.1018	.596
	.091	9/32	.320	.523	.1125	.596
	.102	11/32	.356	.529	.1252	.593
2	.051	5/32	.197	.529	.0809	.640
	.064	3/16	.246	.537	.0998	.637
	.072	7/32	.275	.543	.1114	.636
	.081	1/4	.307	.548	.1241	.636
	.091	9/32	.342	.555	.1382	.635
	.102	11/32	.382	.562	.1527	.633

TABLE 3.3 (CONT'D) - PROPERTIES OF ANGLES



A	B	T	R	AREA	M	I_{xx}	I_{yy}	ρ_{xx}	ρ_{yy}
5/8	1/4	.032	3/32	.031	.074	.0002	.0016	.075	.226
		.040	1/8	.037	.080	.0002	.0016	.073	.218
3/4	1/4	.032	3/32	.035	.068	.0002	.0025	.072	.267
		.040	1/8	.042	.073	.0002	.0028	.071	.259
7/8	1/4	.032	3/32	.039	.062	.0002	.0037	.070	.308
		.040	1/8	.047	.068	.0002	.0042	.069	.299
	3/8	.032	3/32	.047	.105	.0006	.0051	.115	.330
		.040	1/8	.057	.111	.0007	.0059	.113	.323
1	1/4	.032	3/32	.043	.058	.0002	.0052	.068	.347
		.040	1/8	.052	.063	.0002	.0059	.067	.330
	3/8	.040	1/8	.062	.103	.0008	.0082	.112	.385
		.051	5/32	.076	.110	.0009	.0096	.110	.356
		.064	3/16	.092	.118	.0011	.0110	.108	.346
1 1/8	1/4	.032	3/32	.047	.054	.0002	.0071	.066	.386
		.040	1/8	.057	.059	.0002	.0081	.066	.377
	1/2	.040	1/8	.077	.141	.0018	.0140	.154	.427
		.051	5/32	.095	.148	.0022	.0167	.152	.419
		.064	3/16	.116	.157	.0026	.0194	.150	.410
1 1/4	3/8	.040	1/8	.072	.092	.0008	.0144	.108	.447
		.051	5/32	.089	.098	.0010	.0170	.106	.438
		.064	3/16	.108	.106	.0012	.0196	.104	.427
1 3/8	3/8	.040	1/8	.077	.087	.0009	.0182	.106	.487
		.051	5/32	.095	.093	.0010	.0217	.104	.478
		.064	3/16	.116	.100	.0012	.0252	.102	.467
1 1/2	1/2	.051	5/32	.114	.128	.0025	.0339	.146	.544
		.064	3/16	.140	.135	.0029	.0399	.145	.534
		.072	7/32	.154	.141	.0032	.0427	.143	.527
1 3/4	1/2	.051	5/32	.125	.118	.0026	.0496	.142	.625
		.064	3/16	.156	.125	.0031	.0589	.141	.615
		.072	7/32	.172	.130	.0033	.0633	.140	.607
2	5/8	.051	5/32	.153	.147	.0051	.0816	.183	.731
		.064	3/16	.188	.154	.0061	.0978	.181	.721
		.081	1/4	.230	.165	.0073	.1147	.179	.706
2 1/2	3/4	.064	3/16	.236	.174	.0111	.1946	.217	.909
		.072	7/32	.262	.178	.0122	.2128	.216	.901
		.091	9/32	.322	.190	.0147	.2521	.214	.885
3	3/4	.064	3/16	.268	.157	.0117	.3059	.209	1.069
		.091	9/32	.367	.172	.0156	.4004	.206	1.044
		.102	11/32	.404	.180	.0168	.4279	.204	1.030

TABLE 3.4 - PROPERTIES OF CHANNELS

DIAMETER (IN.)		AREA	MOMENT OF	CIRCUMFERENCE	RADIUS OF
FRACTION	DECIMAL	(SQ. INCHES)	INERTIA (I)	(INCHES)	GYRATION (ρ)
3/16	.1875	.02761	6.066×10^{-5}	.5891	.0469
13/64	.2031	.03241	8.358×10^{-5}	.6381	.0508
7/32	.2188	.03758	.0001125	.6872	.0547
15/64	.2344	.04314	.0001481	.7363	.0586
1/4	.2500	.04909	.0001918	.7854	.0625
9/32	.2813	.06213	.0003072	.8836	.0703
5/16	.3125	.07670	.0004682	.9818	.0781
11/32	.3438	.09281	.0006854	1.0799	.0860
3/8	.3750	.1105	.0009710	1.1781	.0938
13/32	.4063	.1296	.001337	1.2763	.1016
7/16	.4375	.1503	.001798	1.3744	.1094
15/32	.4688	.1726	.002370	1.4726	.1172
1/2	.5000	.1964	.003068	1.5708	.1250
17/32	.5313	.2217	.003910	1.6690	.1328
9/16	.5625	.2485	.004914	1.7671	.1406
19/32	.5938	.2769	.006101	1.8653	.1485
5/8	.6250	.3068	.007490	1.9635	.1563
21/32	.6563	.3382	.009104	2.0617	.1641
11/16	.6875	.3712	.01097	2.1598	.1719
23/32	.7188	.4057	.01310	2.2580	.1797
3/4	.7500	.4418	.01553	2.3562	.1875
25/32	.7813	.4794	.01829	2.4544	.1953
13/16	.8125	.5185	.02139	2.5525	.2031
27/32	.8438	.5591	.02488	2.6507	.2110
7/8	.8750	.6013	.02878	2.7489	.2188
29/32	.9063	.6450	.03311	2.8471	.2246
15/16	.9375	.6903	.03792	2.9452	.2344
31/32	.9688	.7371	.04323	3.0434	.2422
1	1.0000	.7854	.04908	3.1416	.2500
1 1/16	1.0625	.8866	.06256	3.3380	.2656
1 1/8	1.1250	.9940	.07863	3.5343	.2813
1 3/16	1.1875	1.1075	.09761	3.7307	.2969
1 1/4	1.2500	1.2272	.1198	3.9270	.3125
1 5/16	1.3125	1.3530	.1457	4.1234	.3281
1 3/8	1.3750	1.4849	.1755	4.3197	.3438
1 7/16	1.4375	1.6230	.2096	4.5161	.3594
1 1/2	1.5000	1.7671	.2485	4.7124	.3750
1 9/16	1.5625	1.9175	.2926	4.9088	.3906
1 5/8	1.6250	2.0739	.3423	5.1051	.4063
1 11/16	1.6875	2.2365	.3980	5.3015	.4219
1 3/4	1.7500	2.4053	.4603	5.4978	.4375
1 3/16	1.8125	2.5802	.5298	5.6942	.4531
1 7/8	1.8750	2.7612	.6067	5.8905	.4688
1 15/16	1.9375	2.9483	.6917	6.0869	.4844
2	2.0000	3.1416	.7854	6.2832	.5000

TABLE 3.5 - PROPERTIES OF CIRCLES

DIAMETER (IN.) FRACTION	DECIMAL	AREA	MOMENT OF INERTIA (I)	CIRCUMFERENCE (INCHES)	RADIUS OF GYRATION (ρ)
2 1/16	2.0625	3.3410	.8883	6.4796	.5156
2 1/8	2.1250	3.5466	1.0009	6.6759	.5313
2 3/16	2.1875	3.7583	1.1240	6.8723	.5469
2 1/4	2.2500	3.9761	1.2580	7.0686	.5625
2 5/16	2.3125	4.2000	1.4038	7.2650	.5781
2 3/8	2.3750	4.4301	1.5618	7.4613	.5938
2 7/16	2.4375	4.6664	1.7328	7.6577	.6094
2 1/2	2.5000	4.9087	1.9175	7.8540	.6250
2 9/16	2.5625	5.1572	2.1165	8.0504	.6406
2 5/8	2.6250	5.4119	2.3307	8.2467	.6563
2 11/16	2.6875	5.6727	2.5607	8.4431	.6719
2 3/4	2.7500	5.9396	2.8074	8.6394	.6850
2 13/16	2.8125	6.2126	3.0714	8.8358	.7031
2 7/8	2.8750	6.4918	3.3537	9.0321	.7188
2 15/16	2.9375	6.7771	3.6549	9.2285	.7344
3	3.0000	7.0686	3.9761	9.4248	.7500
3 1/16	3.0625	7.3662	4.3179	9.6212	.7656
3 1/8	3.1250	7.6699	4.6813	9.8175	.7813
3 3/16	3.1875	7.9798	5.0673	10.0139	.7969
3 1/4	3.2500	8.2958	5.4765	10.2102	.8125
3 5/16	3.3125	8.6179	5.9101	10.4066	.8281
3 3/8	3.3750	8.9462	6.3689	10.6029	.8438
3 7/16	3.4375	9.2806	6.8540	10.7993	.8594
3 1/2	3.5000	9.6211	7.3662	10.9956	.8750
3 9/16	3.5625	9.9678	7.9066	11.1920	.8906
3 5/8	3.6250	10.321	8.4765	11.3883	.9063
3 11/16	3.6875	10.680	9.0764	11.5847	.9219
3 3/4	3.7500	11.045	9.7072	11.7810	.9375
3 13/16	3.8125	11.416	10.371	11.9774	.9531
3 7/8	3.8750	11.793	11.067	12.1737	.9688
3 15/16	3.9375	12.177	11.799	12.3701	.9844
4	4.0000	12.566	12.556	12.5664	1.0000
4 1/8	4.1250	13.364	14.214	12.9591	1.0313
4 1/4	4.2500	14.186	16.025	13.3518	1.0625
4 3/8	4.3750	15.033	17.992	13.7445	1.0938
4 1/2	4.5000	15.904	20.128	14.1372	1.1250
4 5/8	4.6250	16.800	22.550	14.5299	1.1563
4 3/4	4.7500	17.721	25.124	14.9226	1.1875
4 7/8	4.8750	18.666	27.738	15.3153	1.2188
5	5.0000	19.635	30.675	15.7080	1.2500
5 1/8	5.1250	20.629	33.852	16.1006	1.2813
5 1/4	5.2500	21.648	37.321	16.4933	1.3125
5 3/8	5.3750	22.691	40.980	16.8860	1.3438
5 1/2	5.5000	23.758	44.915	17.2787	1.3750

TABLE 3.5 (CONT'D) - PROPERTIES OF CIRCLES

O.D. Inches	Wall Decl. Inches	Wall Gage No.	Area	I	Z	ρ
3/16	.022	24	.0114	.00004	.0004	.0590
	.028	22	.0140	.00005	.0005	.0573
	.035	20	.0168	.00005	.0005	.0553
1/4	.022	24	.0158	.0001	.0008	.0809
	.028	22	.0195	.0001	.0009	.0791
	.035	20	.0236	.0001	.0011	.0770
5/16	.022	24	.0201	.0002	.0013	.1030
	.028	22	.0250	.0002	.0016	.1010
	.035	20	.0305	.0002	.0019	.0988
3/8	.022	24	.0244	.0003	.0020	.1250
	.028	22	.0305	.0004	.0024	.1230
	.035	20	.0374	.0005	.0029	.1208
	.049	18	.0502	.0006	.0036	.1165
7/16	.022	24	.0287	.0006	.0028	.1471
	.028	22	.0360	.0007	.0034	.1451
	.035	20	.0443	.0009	.0041	.1428
	.049	18	.0598	.0011	.0052	.1384
1/2	.022	24	.0330	.0010	.0040	.1691
	.028	22	.0415	.0012	.0048	.1672
	.035	20	.0511	.0014	.0056	.1649
	.049	18	.0694	.0018	.0072	.1604
	.058	17	.0805	.0020	.0080	.1576
9/16	.022	24	.0374	.0014	.0050	.1913
	.028	22	.0470	.0017	.0060	.1892
	.035	20	.0580	.0020	.0071	.1869
	.049	18	.0790	.0026	.0092	.1824
	.058	17	.0919	.0030	.0107	.1796
5/8	.022	24	.0417	.0019	.0061	.2133
	.028	22	.0525	.0023	.0074	.2113
	.035	20	.0649	.0028	.0090	.2090
	.049	18	.0887	.0037	.0118	.2044
	.058	17	.1033	.0042	.0134	.2015
11/16	.022	24	.0460	.0025	.0073	.2354
	.028	22	.0580	.0032	.0093	.2334
	.035	20	.0717	.0038	.0111	.2310
	.049	18	.0983	.0050	.0145	.2264
	.058	17	.1147	.0057	.0166	.2235
3/4	.022	24	.0503	.0033	.0088	.2575
	.028	22	.0635	.0041	.0109	.2555
	.035	20	.0786	.0050	.0133	.2531
	.049	18	.1079	.0067	.0179	.2484
	.058	17	.1261	.0076	.0203	.2455
	.065	16	.1399	.0083	.0221	.2433
7/8	.022	24	.0590	.0054	.0123	.3017
	.028	22	.0745	.0067	.0153	.2996
	.035	20	.0924	.0082	.0187	.2972
	.049	18	.1272	.0109	.0249	.2925
	.058	17	.1489	.0125	.0286	.2896
	.065	16	.1654	.0137	.0313	.2873
	.083	14	.2065	.0164	.0375	.2814
1	.028	22	.0855	.0101	.0202	.3438
	.035	20	.1061	.0124	.0248	.3414
	.049	18	.1464	.0166	.0332	.3367
	.058	17	.1716	.0191	.0382	.3337
	.065	16	.1909	.0210	.0420	.3314
	.083	14	.2391	.0253	.0506	.3255
	.095	13	.2701	.0280	.0560	.3217
1 1/8	.028	22	.0965	.0145	.0258	.3880
	.035	20	.1199	.0178	.0316	.3856
	.049	18	.1656	.0240	.0427	.3808
	.058	17	.1944	.0277	.0492	.3778
	.065	16	.2165	.0305	.0542	.3755
	.083	14	.2717	.0371	.0660	.3696
	.095	13	.3074	.0411	.0731	.3657
1 1/4	.028	22	.1075	.0201	.0322	.4322
	.035	20	.1336	.0247	.0395	.4297
	.049	18	.1849	.0334	.0534	.4250
	.058	17	.2172	.0387	.0619	.4219
	.065	16	.2420	.0426	.0682	.4196
	.083	14	.3042	.0521	.0834	.4136
	.095	13	.3447	.0578	.0925	.4097
	.120	11	.4260	.0688	.1101	.4018
1 3/8	.028	22	.1185	.0269	.0391	.4763
	.035	20	.1473	.0331	.0481	.4739
	.049	18	.2041	.0449	.0653	.4691
	.058	17	.2400	.0521	.0758	.4661
	.065	16	.2675	.0575	.0835	.4637
	.083	14	.3369	.0706	.1027	.4577
	.095	13	.3820	.0787	.1145	.4538
	.120	11	.4731	.0940	.1367	.4457
1 1/2	.028	22	.1295	.0351	.0468	.5205
	.035	20	.1611	.0432	.0576	.5181
	.049	18	.2234	.0589	.0785	.5133
	.058	17	.2627	.0684	.0912	.5103
	.065	16	.2930	.0756	.1006	.5079
	.083	14	.3695	.0931	.1241	.5019
	.095	13	.4193	.1039	.1385	.4979
	.120	11	.5202	.1248	.1664	.4898
1 5/8	.028	22	.1405	.0448	.0551	.5647
	.035	20	.1748	.0553	.0681	.5623
	.049	18	.2426	.0754	.0928	.5575
	.058	17	.2855	.0878	.1081	.5544
	.065	16	.3186	.0970	.1194	.5520
	.083	14	.4021	.1198	.1474	.5459
	.095	13	.4566	.1341	.1650	.5420
	.120	11	.5674	.1617	.1990	.5338
1 3/4	.035	20	.1886	.0694	.0793	.6065
	.049	18	.2618	.0948	.1083	.6017
	.058	17	.3083	.1105	.1263	.5986
	.065	16	.3441	.1514	.1398	.5962
	.083	14	.4347	.1514	.1730	.5901
	.095	13	.4939	.1697	.1939	.5861
	.120	11	.6145	.2052	.2345	.5779
	.156	5/32	.7823	.2508	.2866	.5662
1 7/8	.035	20	.2023	.0857	.0914	.6507
	.049	18	.2811	.1172	.1250	.6458
	.058	17	.3311	.1368	.1459	.6427
	.065	16	.3696	.1516	.1617	.6404
	.083	14	.4673	.1880	.2005	.6343
	.095	13	.5312	.2110	.2251	.6302
	.120	11	.6616	.2559	.2730	.6220
	.156	5/32	.8437	.3141	.3350	.6102
2	.035	20	.2161	.1043	.1043	.6949
	.049	18	.3003	.1430	.1430	.6900
	.058	17	.3539	.1670	.1670	.6869
	.065	16	.3951	.1851	.1851	.6845

TABLE 3.6 - PROPERTIES OF ROUND TUBING

O.D. Inches	Wall Decl. Inches	Wall Gage No.	Area	I	Z	ρ
2	.083	14	.4999	.2300	.2300	.6784
	.095	13	.5685	.2586	.2585	.6744
	.120	11	.7087	.3144	.3143	.6660
	.156	5/32	.9050	.3873	.3873	.6542
2 1/8	.049	18	.3196	.1723	.1622	.7342
	.058	17	.3766	.2013	.1895	.7311
	.065	16	.4207	.2234	.2103	.7287
	.083	14	.5325	.2780	.2616	.7226
	.095	13	.6059	.3128	.2944	.7185
	.120	11	.7559	.3812	.3588	.7102
	.156	5/32	.9664	.4712	.4435	.6983
2 1/4	.049	18	.3388	.2053	.1825	.7784
	.058	17	.3994	.2401	.2134	.7753
	.065	16	.4462	.2665	.2369	.7729
	.083	14	.5650	.3322	.2953	.7667
	.095	13	.6432	.3741	.3325	.7627
	.120	11	.8030	.4568	.4060	.7543
	.156	5/32	1.0278	.5663	.5034	.7423
	.188	3/16	1.2149	.6514	.5790	.7322
2 3/8	.049	18	.3581	.2423	.2040	.8225
	.058	17	.4222	.2835	.2387	.8195
	.065	16	.4717	.3149	.2652	.8171
	.083	14	.5976	.3930	.3309	.8109
	.095	13	.6805	.4429	.3730	.8068
	.120	11	.8501	.5419	.4563	.7984
	.156	5/32	1.0891	.6735	.5672	.7864
	.188	3/16	1.2885	.7764	.6538	.7762
2 1/2	.049	18	.3773	.2834	.2267	.8667
	.058	17	.4450	.3319	.2655	.8636
	.065	16	.4972	.3688	.2950	.8612
	.083	14	.6302	.4608	.3686	.8550
	.095	13	.7178	.5198	.4158	.8510
	.120	11	.8972	.6369	.5095	.8425
	.156	5/32	1.1505	.7935	.6348	.8305
	.188	3/16	1.3622	.9165	.7332	.8202
2 3/4	.049	18	.4158	.3793	.2759	.9551
	.058	17	.4905	.4445	.3233	.9520
	.065	16	.5483	.4944	.3596	.9496
	.083	14	.6954	.6189	.4501	.9434
	.095	13	.7924	.6991	.5084	.9393
	.120	11	.9915	.8590	.6247	.9308
	.156	5/32	1.2732	1.0746	.7815	.9187
	.188	3/16	1.5094	1.2456	.9059	.9084
3	.049	18	.4543	.4946	.3297	1.0435
	.058	17	.5361	.5802	.3868	1.0404
	.065	16	.5993	.6457	.4305	1.0379
	.083	14	.7606	.8097	.5398	1.0317
	.095	13	.8670	.9156	.6104	1.0276
	.120	11	1.0857	1.1276	.7517	1.0191
	.156	5/32	1.3959	1.4153	.9435	1.0069
	.188	3/16	1.6567	1.6454	1.0969	.9966
	.210	7/32	1.9113	1.8595	1.2397	.9864
3 1/4	.049	18	.4928	.6313	.3885	1.1319
	.058	17	.5816	.7410	.4560	1.1287
	.065	16	.6504	.8251	.5078	1.1263
	.083	14	.8258	1.0360	.6375	1.1203
	.095	13	.9416	1.1727	.7217	1.1160
	.120	11	1.1800	1.4471	.8905	1.1074
	.156	5/32	1.5186	1.8216	1.1210	1.0952
	.188	3/16	1.8040	2.1228	1.3063	1.0848
	.219	7/32	2.0831	2.4051	1.4601	1.0745

O.D. Inches	Wall Decl. Inches	Wall Gage No.	Area	I	Z	ρ
3 1/2	.065	16	.7014	1.0349	.5914	1.2147
	.083	14	.8910	1.3012	.7435	1.2085
	.095	13	1.0162	1.4739	.8422	1.2043
	.120	11	1.2742	1.8220	1.0411	1.1958
	.156	5/32	1.6414	2.2990	1.3137	1.1835
	.188	3/16	1.9512	2.6848	1.5342	1.1730
	.219	7/32	2.2550	3.0483	1.7418	1.1627
	.250	1/4	2.5525	3.3901	1.9372	1.1524
3 3/4	.065	16	.7525	1.2777	.6814	1.3031
	.083	14	.9562	1.6080	.8576	1.2968
	.095	13	1.0908	1.8228	.9722	1.2927
	.120	11	1.3685	2.2565	1.2035	1.2841
	.156	5/32	1.7641	2.8533	1.5218	1.2718
	.188	3/16	2.0985	3.3383	1.7804	1.2613
	.219	7/32	2.4268	3.7971	2.0251	1.2509
	.250	1/4	2.7489	4.2307	2.2564	1.2406
4	.065	16	.8035	1.5557	.7779	1.3914
	.083	14	1.0214	1.9597	.9799	1.3852
	.095	13	1.1655	2.2228	1.1114	1.3810
	.120	11	1.4627	2.7552	1.3776	1.3724
	.156	5/32	1.8867	3.4903	1.7452	1.3601
	.188	3/16	2.2457	4.0902	2.0451	1.3495
	.219	7/32	2.5985	4.6598	2.3299	1.3391
	.250	1/4	2.9452	5.2002	2.6001	1.3288
	.313	5/16	3.6202	6.1975	3.0988	1.3084
4 1/4	.065	16	.8546	1.8714	.8897	1.4798
	.083	14	1.0866	2.3593	1.1103	1.4736
	.095	13	1.2400	2.6775	1.2600	1.4694
	.120	11	1.5570	3.3224	1.5635	1.4608
	.156	5/32	2.0095	4.2158	1.9839	1.4484
	.188	3/16	2.3930	4.9473	2.3281	1.4378
	.219	7/32	2.7703	5.6442	2.6561	1.4274
	.250	1/4	3.1416	6.3077	2.9683	1.4170
	.313	5/16	3.8656	7.5387	3.5476	1.3965
4 1/2	.065	16	.9056	2.2271	.9898	1.5682
	.083	14	1.1517	2.8098	1.2488	1.5619
	.095	13	1.3147	3.1902	1.4179	1.5578
	.120	11	1.6512	3.9627	1.7612	1.5491
	.156	5/32	2.1322	5.0354	2.2380	1.5367
	.188	3/16	2.5403	5.9166	2.6296	1.5261
	.219	7/32	2.9421	6.7585	3.0038	1.5156
	.250	1/4	3.3379	7.5625	3.3611	1.5052
	.313	5/16	4.1111	9.0612	4.0272	1.4846
	.375	3/8	4.8597	10.4217	4.6319	1.4644
4 3/4	.065	16	.9567	2.6253	1.1054	1.6566
	.083	14	1.2169	3.3143	1.3955	1.6503
	.095	13	1.3893	3.7645	1.5851	1.6462
	.120	11	1.7455	4.6803	1.9707	1.6375
	.156	5/32	2.2550	5.9550	2.5074	1.6251
	.188	3/16	2.6875	7.0049	2.9494	1.6145
	.219	7/32	3.1139	8.0107	3.3729	1.6039
	.250	1/4	3.5343	8.9738	3.7764	1.5934
	.313	5/16	4.3565	10.7764	4.5374	1.5728
	.375	3/8	5.1542	12.4224	5.2305	1.5525
5	.083	14	1.2821	3.8758	1.5503	1.7387
	.095	13	1.4639	4.4012	1.7617	1.7345
	.120	11	1.8397	5.4798	2.1919	1.7259
	.156	5/32	2.3777	6.9803	2.7921	1.7134
	.188	3/16	2.8348	8.2192	3.2877	1.7028
	.219	7/32	3.2858	9.4089	3.7636	1.6922
	.250	1/4	3.7306	10.5507	4.2203	1.6817
	.313	5/16	4.6019	12.6958	5.0783	1.6610
	.375	3/8	5.4467	14.6647	5.8659	1.6406

TABLE 3.6 (CONT'D) - PROPERTIES OF ROUND TUBING

3.5 BEND RADII

The minimum bend radii for sheet materials are given in Tables 3.7 and 3.8. Table 3.7 shows the minimum radii obtainable by cold forming the sheet while Table 3.8 gives the minimum radii by hot forming the sheet. Figure 3.8 shows the minimum flange width. Shorter flanges may be obtained by trimming after forming, but this is expensive and shall not be specified unless additional tooling and processing costs can be justified.

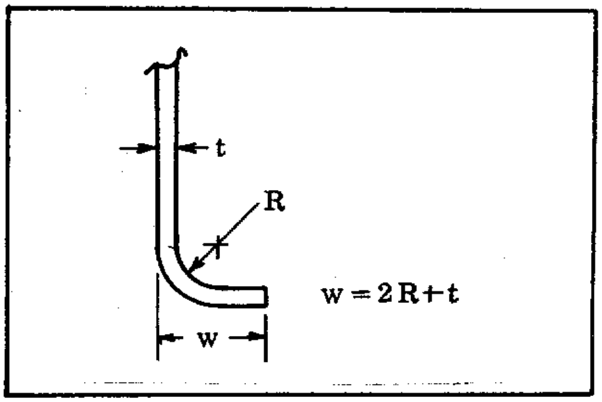


FIGURE 3.8 STANDARD DESIGN BEND RADII

3.6 HARDNESS CONVERSIONS

Table 3.9 presents the conversions for hardness numbers to ultimate tensile strength. In this table the ultimate strength values are in the range 50 to 304 ksi. The corresponding hardness number is given for each of three hardness machines; Vickers, Brinell and three scales of the Rockwell machine.

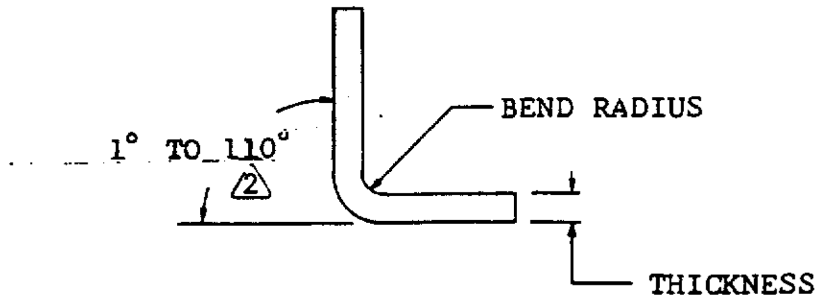
3.7 GRAPHICAL INTEGRATION BY SCOMEANO METHOD

It is often necessary to integrate curves of unusual or irregular shapes. There is a convenient method of integrating even the most complex shapes. It is called graphical integration and is attributed to Scomeano.

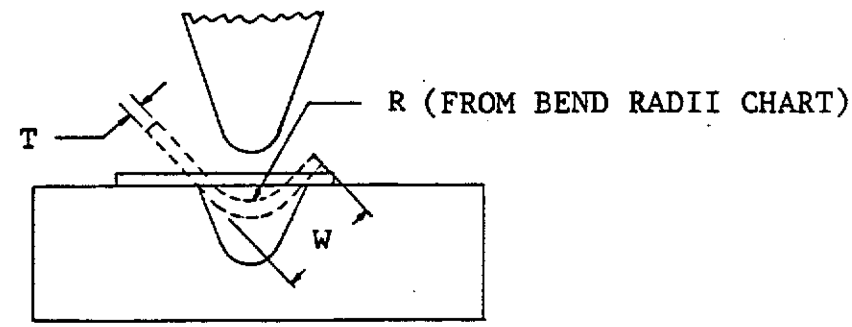
For the purposes of this discussion, assume the curve to be integrated has the form of $y=f(x)$. The curve could be composed of several different shapes; such as, $f_1(x)$, for $x=0$ to n_1; $f_2(x)$, for $x=n_1$ to n_2; $f_3(x)$, for $x=n_2$ to n_3, and so forth. Figure 3.9 shows a typical problem for which graphical integration will be used.

Since the integral of the curve $y=f(x)$ is equal to the area under the curve, the maximum ordinate of the first integral of $f(x)$ will be the area under the $y=f(x)$ curve. It is necessary to chose a scale for the ordinate of the integral curve so that the maximum value of the ordinate will lie in the working area of the graph. The area under the $y=f(x)$ curve is estimated. In the example in Figure 3.9, the area can easily be calculated since the $f(x)$ curves are straight lines. It is 134 and the scale for the first integral is chosen to give easily read divisions and to show the maximum value within the limits of the paper.

STANDARD DESIGN BEND RADII



MINIMUM FLANGE WIDTH



W (MINIMUM FLANGE WIDTH) = 2R + T ⚠1

NOTES: ⚠1 SHORTER FLANGES MAY BE OBTAINED BY TRIMMING AFTER FORMING, BUT THIS OPERATION IS EXPENSIVE AND SHALL NOT BE SPECIFIED UNLESS ADDITIONAL TOOLING AND PROCESSING COSTS CAN BE JUSTIFIED.

⚠2 BEND RADII LISTED ARE FOR ANGLES BETWEEN 1° AND 110° INCLUSIVE. RADII FOR LARGER ANGLES MUST BE COORDINATED WITH PRODUCTION ENG.

⚠3 TITANIUM PER MIL-T-9046

Ⓐ 4. THE STANDARD BEND RADII TOLERANCE IS ± .02.

REVISIONS: APPROVED Ⓐ 11-30-73; CUSTOMER Ⓐ 11-30-73

BELL HELICOPTER COMPANY

POST OFFICE BOX 482 · FORT WORTH, TEXAS 76101 · A TEXTRON COMPANY

DESIGN STANDARD

STANDARD DESIGN BEND RADII ⚠A

MATERIAL AND CONDITION FORMED AT ROOM TEMP		MATERIAL THICKNESS .012	.016	.020	.025	.032	.040	.050	.063	.071	.080	.090	.100	.125	.160	.190	.250
ALUMINUM	2024-0	.03	.03	.06	.06	.06	.09	.09	.12	.16	.16	.18	.25	.25	.31	.38	.56
	2024-T( )	.06	.06	.06	.09	.12	.16	.18	.25	.31	.31	.37	.37	.50	.75	.30	1.25
	5052-0	.03	.03	.03	.03	.06	.06	.06	.06	.09	.09	.09	.12	.16	.18	.25	.38
	5052-H34	.03	.03	.06	.06	.06	.09	.09	.12	.16	.16	.18	.25	.25	.31	.38	.56
	5052-H34*	.02	.02	.02	.03	.03	.03	.06	.06	.09	.09	.09	.12				
	6061-0	.03	.03	.03	.03	.06	.06	.06	.06	.09	.09	.09	.12	.16	.18	.25	.38
	6061-T( )	.06	.06	.06	.09	.12	.16	.18	.25	.31	.31	.37	.37	.50	.75	.88	1.25
	7075-0	.06	.06	.06	.06	.06	.09	.09	.12	.16	.16	.18	.25	.31	.37	.50	.62
	7075-T( )	.09	.09	.09	.12	.16	.25	.25	.31	.37	.37	.50	.50	.75	.87	1.00	1.50
MAG	AZ31B-0	.09	.09	.09	.16	.18	.25	.25	.31	.37	.50	.75	.75	.57			
	AZ31B-H24	.18	.18	.18	.25	.31	.50	.50	.62	.75	.87	1.00	1.50	1.75			
STEEL	300 SERIES ANL	.03	.03	.03	.03	.06	.06	.06	.09	.09	.09	.12	.12				
	300 SERIES 1/4 HARD	.03	.03	.06	.06	.06	.09	.09	.12	.16	.16	.19	.25				
	300 SERIES 1/2 HARD	.06	.06	.06	.09	.09	.12	.16	.25	.25	.38	.38	.50				
	300 SERIES 3/4 HARD	.09	.09	.09	.12	.16	.19	.22	.25	.28	.31	.38	.47	.50	.50	.75	1.00
	300 SERIES HARD	.09	.09	.12	.16	.19	.22	.22	.25	.28	.31	.38	.47	.50	.50	.75	1.00
	4130 ANL & LOW CARBON STL			.06	.06	.06	.09	.09	.12	.16	.16	.19	.22	.25	.28	.34	.40
	4130 NORM & 8630 NORM		.06	.09	.09	.12	.12	.16	.19	.22	.25	.31	.31	.31	.38	.50	.62
TITANIUM	TYPE I, COMP A	.024	.032	.040	.050	.064	.080	.100	.126	.177	.200	.225	.250	.312			
	TYPE I, COMP B	.030	.040	.050	.062	.080	.100	.125	.157	.213	.240	.270	.300	.375			
	TYPE I, COMP C	.024	.032	.040	.050	.064	.080	.100	.126	.177	.200	.225	.250	.312			
	TYPE II, COMP A	.048	.064	.080	.100	.128	.160	.200	.252	.319	.360	.405	.450	.562			
	TYPE III, COMP C	.054	.072	.090	.112	.144	.180	.225	.283	.355	.400	.450	.500	.625			
	⚠A TYPE III, COMP D	.054	.072	.090	.112	.144	.180	.225	.283	.355	.400	.450	.500	.625			

*FOR NON-STRUCTURAL USE ONLY

TITANIUM CONDITION	COMMON DESIGNATION
TYPE I, COMP A TYPE I, COMP B TYPE I, COMP C	COMMERCIALLY PURE
TYPE II, COMP A	Ti-5Al-2.5Sn
TYPE III, COMP C	Ti-6Al-4V
TYPE III, COMP D	Ti-6Al-4V ELI

REVISIONS

APPROVED Ⓐ 11-30-73 ... 11-7-71

Ⓐ SHEET 1 REVISED.

BELL DESIGN STANDARD

CODE IDENT NO. 97499

DRAFT	DATE	TITLE	Drawing Number
James E. Jones	11-6-70	BEND RADII-STANDARD DESIGN	160-001
APPROVAL: David Miller	DATE: 11-7-70		
CUSTOMER: Don R. Payne	DATE: 11-18-70		SHEET 2 OF 2

7842 55577 REV-369

	MATERIAL AND CONDITION FORMED AT ROOM TEMP	MATERIAL THICKNESS .012	.016	.020	.025	.032	.040	.050	.063	.071	.080	.090	.100	.125	.160
ALUMINUM	2024-0	.03	.03	.06	.06	.06	.09	.09	.12	.16	.16	.18	.25	.25	.31
	2024-T( )	.06	.06	.06	.09	.12	.16	.18	.25	.31	.31	.37	.37	.50	.75
	5052-0	.03	.03	.03	.03	.06	.06	.06	.06	.09	.09	.09	.12	.16	.18
	5052-H34	.03	.03	.06	.06	.06	.09	.09	.12	.16	.16	.18	.25	.25	.31
	5052-H34*	.02	.02	.02	.03	.03	.03	.06	.06	.09	.09	.09	.12		
	6061-0	.03	.03	.03	.03	.06	.06	.06	.06	.09	.09	.09	.12	.16	.18
	6061-T( )	.06	.06	.06	.09	.12	.16	.18	.25	.31	.31	.37	.37	.50	.75
	7075-0	.06	.06	.06	.06	.06	.09	.09	.12	.16	.16	.18	.25	.31	.37
	7075-T( )	.09	.09	.09	.12	.16	.25	.25	.31	.37	.37	.50	.50	.75	.87
MAG	AZ31B-0	.09	.09	.09	.16	.18	.25	.25	.31	.37	.50	.75	.75	.87	
	AZ31B-H24	.18	.18	.18	.25	.31	.50	.50	.62	.75	.87	1.00	1.50	1.75	
STEEL	300 SERIES ANL	.03	.03	.03	.03	.06	.06	.06	.09	.09	.09	.12	.12		
	300 SERIES ¼ HARD	.03	.03	.06	.06	.06	.09	.09	.12	.16	.16	.19	.25		
	300 SERIES ½ HARD	.06	.06	.06	.09	.09	.12	.16	.25	.25	.38	.38	.50		
	300 SERIES 3/4 HARD	.09	.09	.09	.12	.16	.19	.22	.25	.28	.31	.38	.47	.50	.50
	300 SERIES HARD	.09	.09	.12	.16	.19	.22	.22	.25	.28	.31	.38	.47	.50	.50
	4130 ANL & LOW CARBON STL			.06	.06	.06	.09	.09	.12	.16	.16	.19	.22	.25	.28
	4130 NORM & 8630 NORM		.06	.09	.09	.12	.12	.16	.19	.22	.25	.31	.31	.31	.38
TITANIUM	TYPE I, COMP A	.024	.032	.040	.050	.064	.080	.100	.126	.177	.200	.225	.250	.312	
	TYPE I, COMP B	.030	.040	.050	.062	.080	.100	.125	.157	.213	.240	.270	.300	.375	
	TYPE I, COMP C	.024	.032	.040	.050	.064	.080	.100	.126	.177	.200	.225	.250	.312	
	TYPE II, COMP A	.048	.064	.080	.100	.128	.160	.200	.252	.319	.360	.405	.450	.562	
	TYPE III, COMP C	.054	.072	.090	.112	.144	.180	.225	.283	.355	.400	.450	.500	.625	
	TYPE III, COMP D	.054	.072	.090	.112	.144	.180	.225	.283	.355	.400	.450	.500	.625	

TITANIUM CONDITION	COMMON DESIGNATION
TYPE I, COMP A TYPE I, COMP B TYPE I, COMP C	COMMERCIALLY PURE
TYPE II, COMP A	Ti-5Al-2.5Sn
TYPE III, COMP C	Ti-6Al-4V
TYPE III, COMP D	Ti-6Al-4V ELI

Note:
1. Bend radii listed are for angles between 1° and 110° inclusive. Radii for larger angles must be coordinated with production engineering.
2. The standard bend radii tolerance is $\pm$ 0.015.
3. Reference Bell Design Standard 160 - 002.

TABLE 3.7 - STANDARD DESIGN BEND RADII (COLD FORMING)

MATERIAL AND CONDITION		MATERIAL THICKNESS													
		.012	.016	.020	.025	.032	.040	.050	.063	.071	.080	.090	.100	.125	.160
ALUMINUM	7075-T( )@ 300° F	.06	.06	.06	.06	.09	.12	.16	.19	.25	.31	.38	.41	.50	.69
MAG	AZ31B-0 @ 350° F	.06	.06	.06	.06	.06	.09	.09	.16	.16	.18	.25	.25	25	
MAG	AZ31B-H24 @ 350° F	.06	.09	.09	.09	.16	.18	.25	.31	.37	.50	.50	.50	.62	
TITANIUM	TYPE I, COMP A @ 400°-600° F	.018	.024	.030	.037	.048	.080	.100	.126	.142	.160	.180	.200	.250	
TITANIUM	TYPE I, COMP B @ 400°-600° F	.018	.024	.030	.037	.048	.080	.100	.126	.142	.160	.180	.200	.250	
TITANIUM	TYPE I, COMP C @ 400°-600° F	.018	.024	.030	.037	.048	.080	.100	.126	.142	.160	.180	.200	.250	
TITANIUM	TYPE II, COMP A @ 400°-600° F	.036	.048	.060	.075	.096	.120	.150	.189	.248	.280	.315	.350	.437	
TITANIUM	TYPE III, COMP C @ 400°-600° F	.024	.032	.040	.050	.064	.080	.100	.126	.142	.160	.180	.200	.250	
TITANIUM	TYPE III, COMP D @ 400°-600° F	.024	.032	.040	.050	.064	.080	.100	.126	.142	.160	.180	.200	.250	

TITANIUM CONDITION	COMMON DESIGNATION
TYPE 1, COMP A TYPE 1, COMP B TYPE 1, COMP C	COMMERCIALLY PURE
TYPE II, COMP A	Ti-5Al-2.5Sn
TYPE III, COMP C	Ti-6Al-4V
TYPE III, COMP D	Ti-6Al-4V ELI

Note: 1. Bend radii listed are for angles between 1° and 110° inclusive. Radii for larger angles must be coordinated with Production Engineering.
2. The standard bend radii tolerance is ± 0.015.
3. Reference Bell Design Standard 160 - 002.

TABLE 3.8 - STANDARD DESIGN BEND RADII (HOT FORMING)

Tensile Strength	Vickers-Firth Diamond	Brinell 3000 kg 10mm Stl Ball	Rockwell		
			A Scale	B Scale	C Scale
ksi	Hardness Number	Hardness Number	60 kg 120 deg Diamond Cone	100 kg 1/16 in. Dia Stl Ball	150 kg 120 deg Diamond Cone
50	104	92	--	58	--
52	108	96	--	61	--
54	112	100	--	64	--
56	116	104	--	66	--
58	120	108	--	68	--
60	125	113	--	70	--
62	129	117	--	72	--
64	135	122	--	74	--
66	139	127	--	76	--
68	143	131	--	77.5	--
70	149	136	--	79	--
72	153	140	--	80.5	--
74	157	145	--	82	--
76	162	150	--	83	--
78	167	154	51	84.5	--
80	171	158	52	85.5	--
82	177	162	53	87	--

TABLE 3.9 - HARDNESS CONVERSION TABLE

Tensile Strength	Vickers-Firth Diamond	Brinell 3000 kg 10mm Stl Ball	Rockwell		
			A Scale	B Scale	C Scale
ksi	Hardness Number	Hardness Number	60 kg 120 deg Diamond Cone	100 kg 1/16 in. Dia Stl Ball	150 kg 120 deg Diamond Cone
83	179	165	53.5	87.5	--
85	186	171	54	89	--
87	189	174	55	90	--
89	196	180	56	91	--
91	203	186	56.5	92.5	--
93	207	190	57	93.5	--
95	211	193	57	94	--
97	215	197	57.5	95	--
99	219	201	57.5	95.5	--
102	227	210	59	97	--
104	235	220	60	98	19
107	240	225	60.5	99	20
110	245	230	61	99.5	21
112	250	235	61.5	100	22
115	255	241	62	101	23
118	261	247	62.5	101.5	24
120	267	253	63	102	25

TABLE 3.9 (CONT'D) - HARDNESS CONVERSION TABLE

Tensile Strength	Vickers-Firth Diamond	Brinell 3000 kg 10mm Stl Ball	Rockwell		
			A Scale	B Scale	C Scale
ksi	Hardness Number	Hardness Number	60 kg 120 deg Diamond Cone	100 kg 1/16 in. Dia Stl Ball	150 kg 120 deg Diamond Cone
123	274	259	63.5	103	26
126	281	265	64	--	27
129	288	272	64.5	--	28
132	296	279	65	--	29
136	304	286	65.5	--	30
139	312	294	66	--	31
142	321	301	66.5	--	32
147	330	309	67	--	33
150	339	318	67.5	--	34
155	348	327	68	--	35
160	357	337	68.5	--	36
165	367	347	69	--	37
170	376	357	69.5	--	38
176	386	367	70	--	39
181	396	377	70.5	--	40
188	406	387	71	--	41
194	417	398	71.5	--	42

TABLE 3.9 (CONT'D) - HARDNESS CONVERSION TABLE

Tensile Strength	Vickers-Firth Diamond	Brinell 3000 kg 10mm Stl Ball	Rockwell		
			A Scale	B Scale	C Scale
ksi	Hardness Number	Hardness Number	60 kg 120 deg Diamond Cone	100 kg 1/16 in. Dia Stl Ball	150 kg 120 deg Diamond Cone
201	428	408	72	--	43
208	440	419	72.5	--	44
215	452	430	73	--	45
221	465	442	73.5	--	46
231	479	453	74	--	47
237	493	464	75	--	48
246	508	476	75.5	--	49
256	523	488	76	--	50
264	539	500	76.5	--	51
273	556	512	77	--	52
283	573	524	77.5	--	53
294	592	536	78	--	54
304	611	548	78.5	--	55

TABLE 3.9 (CONT'D) - HARDNESS CONVERSION TABLE

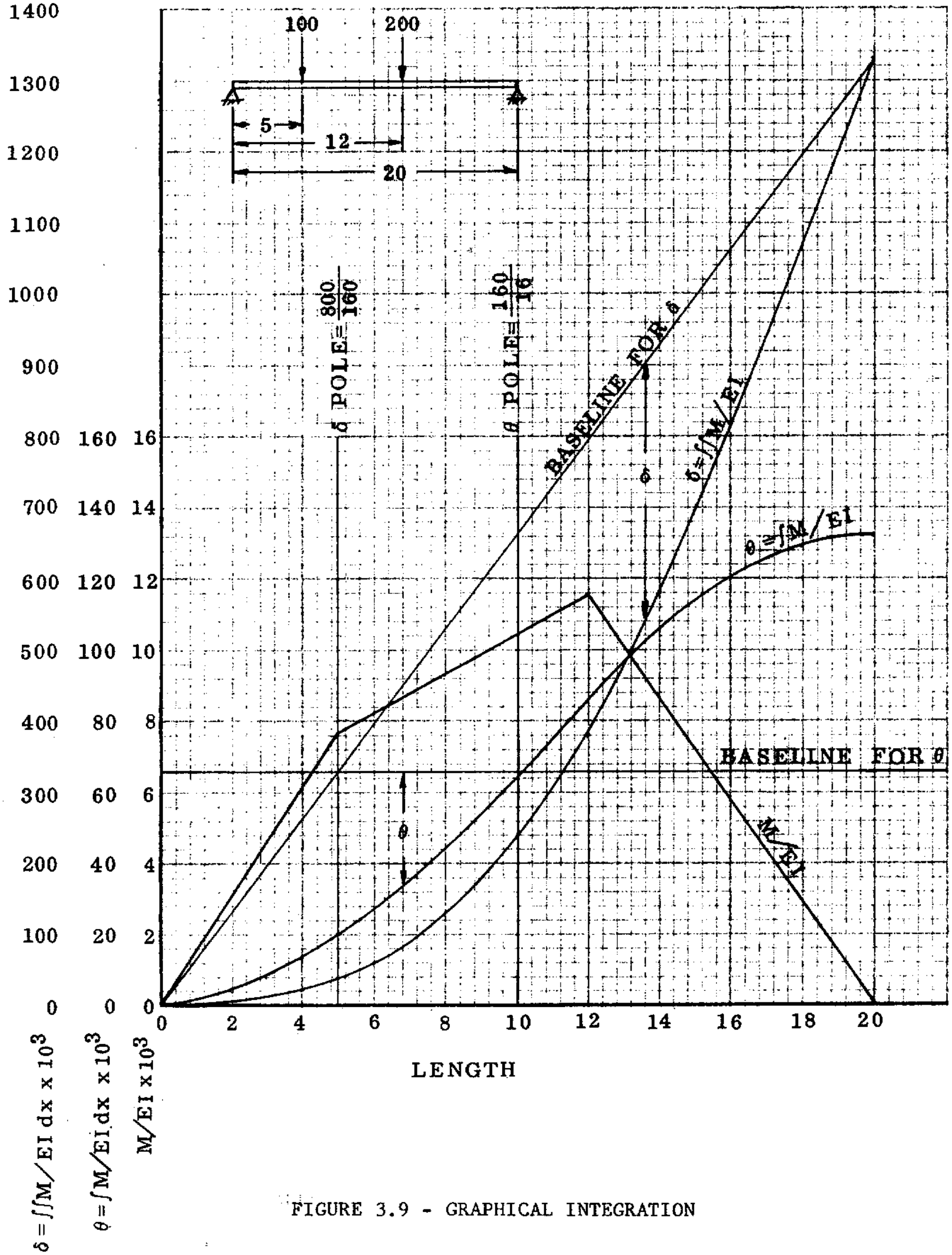


FIGURE 3.9 - GRAPHICAL INTEGRATION

The next step is to locate a "pole." This is accomplished by dividing any ordinate of the first integral curve by the corresponding ordinate of the y=f(x) curve. For the case in Figure 3.9, select the first integral ordinate of 120. The corresponding y=f(x) ordinate is 12. The pole distance from the origin is then 120/12=10. Now it is necessary to determine from which side the integration is to be made. It is customary to integrate from the left but integration from either side is possible. If from the left, draw a vertical line at x=10 from the origin. If from the right, draw a vertical line at x=10 to the left of the maximum x value of the y=f(x) curve. In the example the integration is from the left and the pole is located at x=10 from the origin.

The next step is to determine the incremental x, (Δx), to be used. In the example, y=f(x) is a straight line throughout, so Δx=1 is arbitrarily selected. The smaller the Δx, the smoother the integral curve. Δx can be changed as the integration progresses if more accuracy is needed.

The first step in the integration is to lay out Δx=1 from the origin. Locate the median point on the y=f(x) within this Δx increment. For a straight line, this will be the midpoint but if the y=f(x) is a curve within the Δx, then the median must be estimated. This is done by assuming a location. Draw a horizontal line through the point. Check the areas above the horizontal line and below the horizontal bounded by y=f(x) within the Δx increment. When these two areas are equal, the median is the point where the horizontal line crosses the y=f(x) curve.

Project the horizontal line to the vertical pole. Draw a line between this intersection and the origin of the y=f(x) curve. Mark a point where this line crosses the x=1 vertical line. Lay out another increment of Δx. This time it will extend between x=1 and x=2. Locate the median as before and project a horizontal line to the vertical pole. Determine the slope from this intersection to the origin. Transfer this slope to the point previously located at x=1. Draw a line with this slope from x=1 to x=2.

Continue the above procedure until the limit of the y=f(x) curve has been reached. The curve formed by the many slopes of the projected lines is the integral curve of y=f(x). The same procedure can be used to get the second integral by integrating the $\int f(x)dx$ curve.

A baseline for δ and θ must be located so that the true deflections and rotations can be determined. The δ baseline connects two points of known deflection. In the example in Figure 3.9, the ends of the beam have zero deflection, so the δ baseline is located by connecting the two end points of the δ curve with a straight line.

To locate the baseline for θ, take the ordinate value of the δ curve at x=L and divide this value by L. In Figure 3.9, let δ=1329 and L=20, then θ_{base} = 1329/20 = 66.45. Draw a horizontal line at θ=66.45. This is the θ baseline.

The deflection and rotation at any point on the beam is the vertical distance from the respective curve to the baseline.

3.8 CONVERSION FACTORS

Table 3.10 shows conversion factors for most technical units. Equations for converting from one unit of temperature to the other are shown below:

$^{o}R=1.8(^{o}K-273.16)+491.69$ 3.30
$^{o}R=1.8^{o}C+491.69$ 3.31
$^{o}R=^{o}F+459.69$ 3.32

$^{o}K=5/9(^{o}R-491.69)+273.16$ 3.33
$^{o}K=5/9(^{o}F-32)+273.16$ 3.34
$^{o}K=^{o}C+273.16$ 3.35

$^{o}C=5/9(^{o}R-491.69)$ 3.36
$^{o}C=5/9(^{o}F-32)$ 3.37
$^{o}C=^{o}K-273.16$ 3.38

$^{o}F=1.8(^{o}K-273.16)+32$ 3.39
$^{o}F=1.8^{o}C+32$ 3.40
$^{o}F=^{o}R-459.69$ 3.41

3.9 THE INTERNATIONAL SYSTEM OF UNITS (SI)

The purpose of this section is to acquaint the engineer with the inevitable, the Metric System. The International System of Units, or Systeme Internationale (SI), is sometimes referred to, in less precise terms, as the Meter-Kilogram-Second-Ampere (MKSA) system. The SI should be considered as the definitive metric system, since it is much broader in scope and purpose than any previous system.

3.9.1 Basic SI Units

The following are the basic units for the SI:

meter, m	ampere, A
kilogram, kg	degree Kelvin, ^{o}K
second, s	candela, cd

In addition, the amount of a substance is treated as a basic quantity. The basic unit is the mole, symbol: mol. The mole (mol), a unit of quantity in chemistry, is defined as the amount of a substance in grams (gram mole, gram molecular weight; pound mole, pound molecular weight) which corresponds to the sum of the atomic weights of all the atoms constituting the molecule.

3.9.2 Symbols and Notation

When using the SI, the following rules apply:

TO CONVERT	INTO	MULTIPLY BY
acres	sq. feet	43,560.0
acres	sq. meters	4,047.0
atmospheres	kgs/sq. cm	1.0333
atmospheres	pounds/sq. in.	14.70
atmospheres	newton/sq. meter	1.013×10^{5}
Btu	foot-lbs	778.3
Btu	joules	1,054.8
Btu	kilowatt-hrs	2.928×10^{-4}
centimeters	feet	3.281×10^{-2}
centimeters	inches	0.3937
centimeters	meters	0.01
centimeter-grams	cm-dynes	980.7
centimeter-grams	meter-kgs	10^{-5}
centimeter-grams	pound-feet	7.233×10^{-5}
centimeters/sec	feet/min	1.1969
centimeters/sec	feet/sec	0.03281
centimeters/sec	kilometers/hr	0.036
centimeters/sec	knots	0.1943
centimeters/sec	meters/min	0.6
centimeters/sec	miles/hr	0.02237
centimeters/sec/sec	feet/sec/sec	0.03281
centimeters/sec/sec	kms/hr/sec	0.036
centimeters/sec/sec	meters/sec/sec	0.01
centimeters/sec/sec	miles/hr/sec	0.02237
coulombs	faradays	1.036×10^{-5}
coulombs/sq/in	coulombs/sq. meter	1,550.
cubic centimeters	cu. inches	0.06102
cubic centimeters	liters	0.001
cubic feet	cu. meters	0.02832
cubic feet	gallons (U.S. liq.)	7.48052
cubic feet	liters	28.32
cubic inches	cu. meters	16,387.06
cubic inches	liters	0.01639
cubic inches	quarts (U.S. liq.)	0.01732
cubic meters	cu. feet	35.31
cubic meters	cu. inches	61,023.0
cubic meters	cu. yards	1.308
cubic meters	gallons (U.S. liq.)	264.2
cubic meters	liters	1,000.0
degrees (angle)	radians	0.01745
degrees/sec	revolutions/min	0.1667
drams (U.S., fluid or apoth.)	cubic cm.	3.6967
drams	grams	1.7718
drams	grains	27.3437
drams	ounces	0.0625

TABLE 3.10 - CONVERSION FACTORS

TO CONVERT	INTO	MULTIPLY BY
feet	meters	0.3048
feet of water	atmospheres	0.02950
feet of water	in. of mercury	0.8826
feet of water	kgs/sq/meter	304.8
feet of water	pounds/sq. ft	62.43
feet of water	pounds/sq. in.	0.4335
feet/min	cms/sec	0.5080
feet/min	kms/hr	0.01829
feet/min	miles/hr	0.01136
feet/sec	cms/sec	30.48
feet/sec	kms/hr	1.097
feet/sec	knots	0.5921
feet/sec	miles/hr	0.6818
foot-pounds	Btu	1.286×10^{-3}
foot-pounds	gram-calories	0.3238
foot-pounds	joules	1.356
foot-pounds	kg-calories	3.24×10^{-4}
foot-pounds	kg-meters	0.1383
foot-pounds	kilowatt-hrs	3.766×10^{-7}
foot-pounds	newton-meters	1.356
foot-pounds/min	horsepower	3.030×10^{-5}
foot-pounds/sec	Btu/hr	4.6263
foot-pounds/sec	horsepower	1.818×10^{-3}
foot-pounds/sec	kilowatts	1.356×10^{-3}
gallons	cu. cms.	3,785.0
gallons	cu. feet	0.1337
gallons	cu. inches	231.0
gallons	liters	3.785
gallons (liq. Br. Imp.)	gallons (U.S. liq.)	1.20095
gallons (U.S.)	gallons (Imp.)	0.83267
gallons of water	pounds of water	8.3453
gallons/min	cu. ft/sec	2.228×10^{-3}
gallons/min	liters/sec	0.06308
grains (troy)	grains (avdp)	1.0
grains (troy)	grams	0.06480
grams	dynes	980.7
grams	grains	15.43
grams	joules/meter (newtons)	9.807×10^{-3}
grams	ounces (avdp)	0.03527
grams	ounces (troy)	0.03215
grams	poundals	0.07093
grams/cu. cm.	pounds/cu. ft.	62.43
grams/cu. cm.	pounds/cu. in.	0.03613
grams/liter	grains/gal	58.417
grams/liter	pounds/1,000 gal	8.345
grams/liter	pounds/cu. ft.	0.062427
grams/sq. cm.	pounds/sq. ft.	2.0481

TABLE 3.10 (CONT'D) - CONVERSION FACTORS

TO CONVERT	INTO	MULTIPLY BY
horsepower	Btu/min	42.44
horsepower	foot-lbs/min	33,000.
horsepower	foot-lbs/sec	550.0
horsepower (550 ft lb/sec)	horsepower(metric)(542.5 ft lb/sec)	1.014
horsepower	kilowatts	0.7457
horsepower	watts	745.7
horsepower-hrs	Btu	2,547.
horsepower-hrs	foot-lbs	1.98×10^4
inches	centimeters	2.540
inches of mercury	atmospheres	0.03342
inches of mercury	feet of water	1.133
inches of mercury	kgs/sq. meter	345.3
inches of mercury	pounds/sq. ft.	70.73
inches of mercury	pounds/sq. in.	0.4912
inches of water (at $4^{\circ}C$)	inches of mercury	0.07355
inches of water (at $4^{\circ}C$)	pounds/sq. ft.	5.204
inch-pounds	newton-meters	0.11298
joules	kg-meters	0.1020
joules/cm	poundals	723.3
joules/cm	pounds	22.48
kilograms	poundals	70.93
kilograms	pounds	2.205
kilograms/cu. meter	pounds/cu. ft.	0.06243
kilograms/meter	pounds/ft.	0.6720
kilograms/sq. cm.	atmospheres	0.9678
kilograms/sq. cm.	pounds/sq. in.	14.22
kilogram-calories	Btu	3.968
kilogram-calories	foot-pounds	3,088.
kilogram-calories	kg-meters	426.9
kilogram meters	foot-pounds	7.233
kilometers	feet	3,281.
kilometers	miles	0.6214
kilometers/hr	cms/sec	27.78
kilometers/hr	feet/min	54.68
kilometers/hr	feet/sec	0.9113
kilometers/hr	knots	0.5396
kilometers/hr	meters/min	16.67
kilowatts	Btu/min	56.92
kilowatts	foot-lbs/sec	737.6
kilowatts	horsepower	1.341
kilowatt-hrs	Btu	3,413.
kip	kilonewton	4.4482
kips/sq. in.	megapascals	6.8948
knots	kilometers/hr	1.8532
knots	nautical miles/hr	1.0
knots	statute miles/hr	1.151
knots	feet/sec	1.689
knots	meters/sec	0.5148

TABLE 3.10 (CONT'D) - CONVERSION FACTORS

TO CONVERT	INTO	MULTIPLY BY
liters	cu. cm.	1,000.0
liters	cu. feet	0.03531
liters	cu. inches	61.02
liters	quarts (U.S. liq.)	1.057
Megapascal	pounds/sq. in.	145.039
Megapascal	newton/sq. mm	1.0
meters	feet	3.281
meters	inches	39.37
meters/min	cms/sec	1.667
meters/min	feet/sec	0.05468
meters/min	knots	0.03238
meters/min	miles/hr	0.03728
meters/sec	feet/min	196.8
meters/sec	kilometers/hr	3.6
meters/sec	miles/hr	2.237
meters/sec	miles/min	0.03728
meter-kilograms	pound-feet	7.233
miles (naut.)	feet	6,080.27
miles (naut.)	kilometers	1.853
miles (naut.)	miles (statute)	1.1516
miles (statute)	feet	5,280.
miles (statute)	kilometers	1.609
miles (statute)	miles (naut.)	0.8684
miles/hr	feet/sec	1.467
miles/hr	knots	0.8684
miles/hr	meters/sec	0.4470
millimeters	inches	0.03937
mils	inches	0.001
Newton	pounds	0.22481
Newton	Dynes	1×10^5
Newton-meter	inch-pound	8.8507
Newton/sq. mm	Megapascal	1.0
ounces	grains	437.5
ounces	grams	28.3495
ounces	ounces (troy)	0.9115
poundals	grams	14.10
poundals	pounds	0.03108
pounds	grams	453.5924
pounds	kilograms	0.4536
pounds	newtons	4.4482
pounds	ounces	16.0
pounds	ounces (troy)	14.5833
pounds	poundals	32.17
pounds	pounds (troy)	1.21528
pounds of water	cu. feet	0.01602
pounds of water	gallons	0.1198
pound-feet	cm-grams	13,825.
pound-feet	meter-kgs	0.1383
pounds/cu. ft.	kgs/cu. meter	16.02

TABLE 3.10 (CONT'D) - CONVERSION FACTORS

TO CONVERT	INTO	MULTIPLY BY
pounds/cu. in.	kgs/cu. meter	2.768×10^4
pounds/ft	kgs/meter	1.488
pounds/in.	gms/cm	178.6
pounds/sq.in.	atmospheres	0.06804
pounds/sq.in.	feet of water	2.307
pounds/sq.in.	inches of mercury	2.036
pounds/sq.in.	kgs/sq. meter	703.1
pounds/sq.in.	megapascals	6.8948×10^{-3}
quadrants (angle)	degrees	90.0
quadrants (angle)	radians	1.571
radians	degrees	57.30
radians	quadrants	0.6366
radians/sec.	revolutions/min	9.549
radians/sec.	revolutions/sec	0.1592
revolutions	radians	6.283
revolutions/min	degrees/sec	6.0
revolutions/min	radians/sec	0.1047
slug	kilogram	14.5939
slug	pounds	32.17
square centimeters	sq. inches	0.1550
square feet	sq. meters	0.09290
square inches	sq. cms.	6.452
square kilometers	sq. miles	0.3861
square meters	sq. feet	10.76
square millimeters	sq. inches	1.550×10^{-3}
temperature (°C) + 273	absolute temperature (°C)	1.0
temperature (°C) + 17.78	temperature (°F)	1.8
temperature (°F) + 460	absolute temperature (°F)	1.0
temperature (°F) - 32	temperature (°C)	5/9
tons (long)	kilograms	1,016.
tons (long)	pounds	2,240.
tons (long)	tons (short)	1.120
tons (metric)	kilograms	1,000.
tons (metric)	pounds	2,205.
tons (short)	pounds	2,000.
tons (short)	pounds (troy)	2,430.56
watts	Btu/hr	3.4192

TABLE 3.10 (CONT'D) - CONVERSION FACTORS

(1) Symbols for units of physical quantities shall be printed in Roman upright type.

(2) Symbols for units shall not contain a period and shall remain singular; e.g., 7cm, not 7cms.

(3) Symbols for units shall be printed in lower case Roman upright type. However, the symbol for a unit derived from a proper name shall start with a capital Roman letter; e.g.: m (meter); A (ampere); Wb (weber); Hz (hertz).

(4) The following prefixes shall be used to indicate decimal fractions or multiples of a unit.

Prefix	Equiv.	Symbol
deci	(10^{-1})	d
centi	(10^{-2})	c
milli	(10^{-3})	m
micro	(10^{-6})	μ
nano	(10^{-9})	n
pico	(10^{-12})	p
femto	(10^{-15})	f
atto	(10^{-18})	a
deka	(10^{1})	da
hecto	(10^{2})	h
kilo	(10^{3})	k
mega	(10^{6})	M
giga	(10^{9})	G
tera	(10^{12})	T

(5) The use of double prefixes shall be avoided when single prefixes are available.

(6) When a prefix symbol is placed before a unit symbol, the combination shall be considered as a new symbol. A numerical prefix shall never be used before a unit symbol which is squared, thus cm^2 is never written, nor is it written $0.01(m^2)$ but as $(0.01m)^2$.

(7) The following are SI units for various commonly used factors.

	SI Unit	Symbol
acceleration	meter/second squared	m/s^2
area	square meter	m^2
density	kilogram/cu meter	kg/m^3
energy	joule	J=N·M
energy/area time	watt/sq meter	W/m^2
force	newton	$N= kgm/s^2$
length	meter	m
mass	kilogram	kg
power	watt	W=J/s
pressure	newton/sq meter	N/m^2
speed	meter/second	m/s
time	second (mean solar)	s
viscosity	newton second/sq meter	Ns/m^2
volume	cu meter	m^3

Table 3.10 shows the alphabetical listing of conversions from the English system to SI.

3.10 WEIGHTS

Weights of aircraft structural materials are shown in MIL-HDBK-5.

3.11 SHEAR CENTERS

The shear center is defined as the point on the cross-section of a beam through which the transverse load on the beam must pass in order that no rotation of the beam will occur. A pure twist with no moment produces no deflection at the shear center, only rotation.

In the case of a beam of variable cross-section, the shear center may be determined for each section, but these points will not connect to form a straight axis. For instance, a cantilever beam of non-uniformly varying cross-section may have a load so placed on the end that the end section will not rotate, but all other sections of the beam may rotate. The axis formed by connecting all of the shear centers is called the axis of rotation or the elastic axis.

For any doubly symmetrical section or a section with point symmetry, as a zee, the shear center is at the center of gravity.

For any singly symmetrical section, the shear center is somewhere on the axis of symmetry.

For any section made up of two intersecting plates, for example an angle or a tee, the shear center is at the point of intersection of the plates.

3.11.1 Shear Centers of Open Sections

The coordinates of the shear center position for the general case of an open section are defined by equation 3.42 and 3.43 as shown in Figure 3.10.

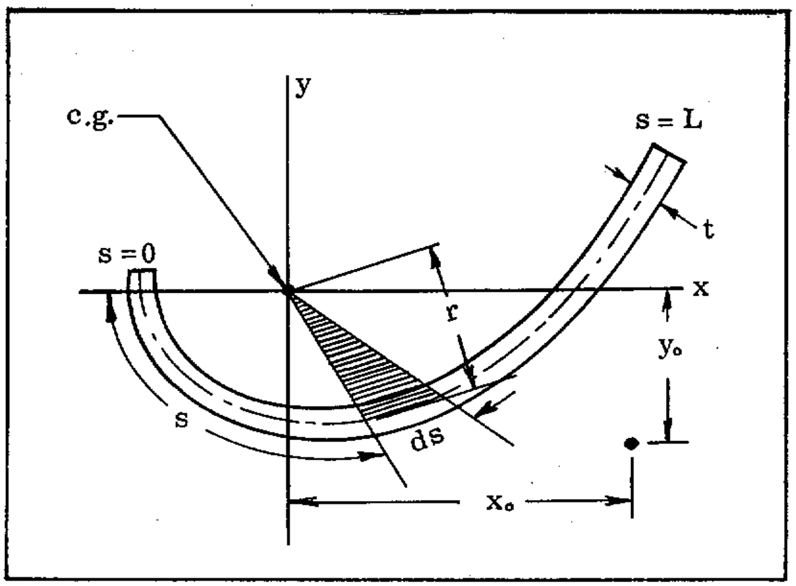


FIGURE 3.10 SHEAR CENTER OF OPEN SECTIONS

$$x_o = \frac{1}{I_x I_y - (I_{xy})^2} \left[I_y \int_0^L w_s\, y t ds - I_{xy} \int_0^L w_s\, x t ds \right] \qquad 3.42$$

$$y_o = \frac{1}{I_x I_y - (I_{xy})^2} \left[-I_x \int_0^L w_s\, x t ds + I_{xy} \int_0^L w_s\, y t ds \right] \qquad 3.43$$

where

$w_s = \int_0^s r ds$; The double area swept by the radius vector when the distance along the midline increases from s=0 to s. w_s is taken as positive when the area is swept in a counter-clockwise direction.

x_o, y_o; Coordinates of the shear center with respect to the axis through the section centroid.

I_x, I_y, I_{xy}; Inertias of the section.

L; The limit of s or the developed length of the cross section.

r; Radius vector measured from the section centroid to median line of the cross section.

s; Circumferential distance measured along the median line of the section.

t; Thickness.

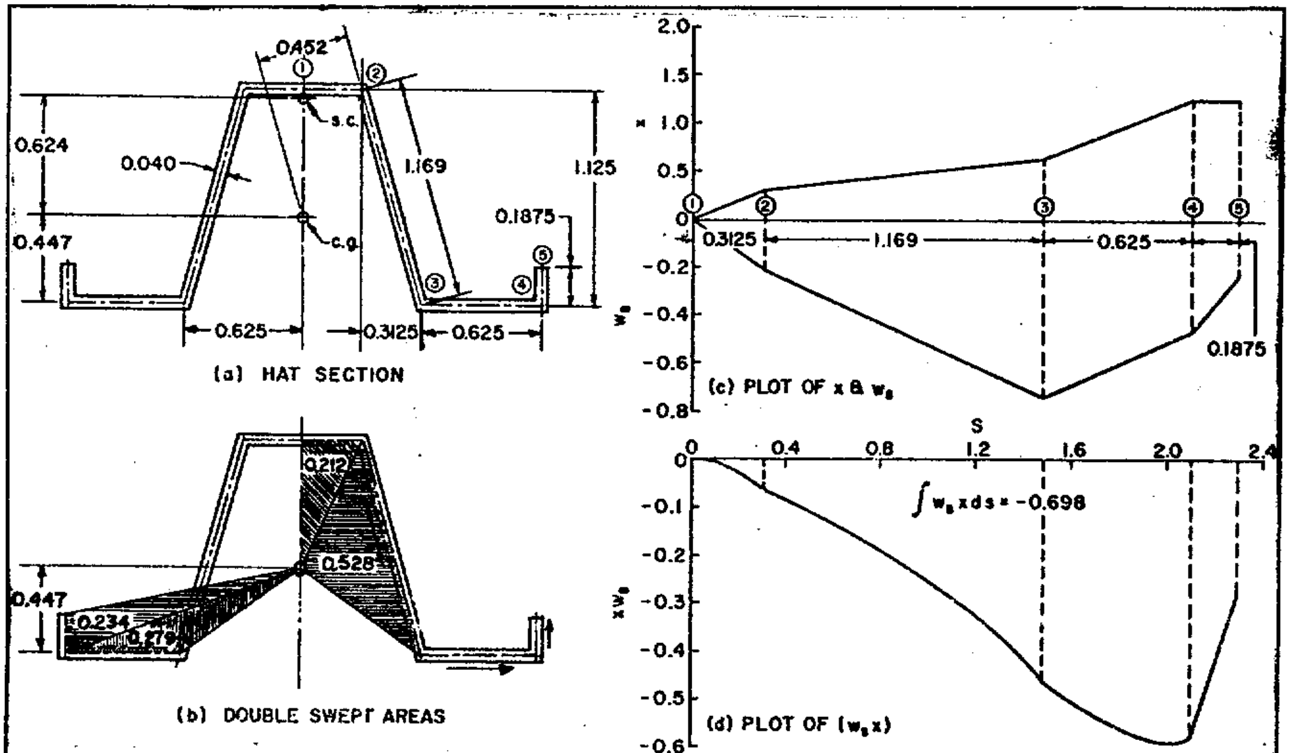
If x and y are the principal axes of the section, $I_{xy}=0$ and equations 3.42 and 3.43 become:

$$x_o = 1/I_x \int_0^L w_s y t ds \qquad 3.44$$

$$y_o = -1/I_y \int_0^L w_s x t ds \qquad 3.45$$

Examples of the previous procedure are shown in Figures 3.11 and 3.12. Table 3.11 and Figures 3.13 through 3.17 give shear center locations for some common sections.

3.11.1.1 Example of Shear Center Location for Hat Section



$A = 0.1835$ in.^{2}

$I_x = 0.0345$ in.^{4}

$I_y = 0.0894$ in.^{4}

$\bar{y} = 0.447$ in.

	Double Swept Area	w_s
1	0	0
2	-0.212	-0.212
3	-0.528	-0.740
4	+0.279	-0.461
5	+0.234	-0.227

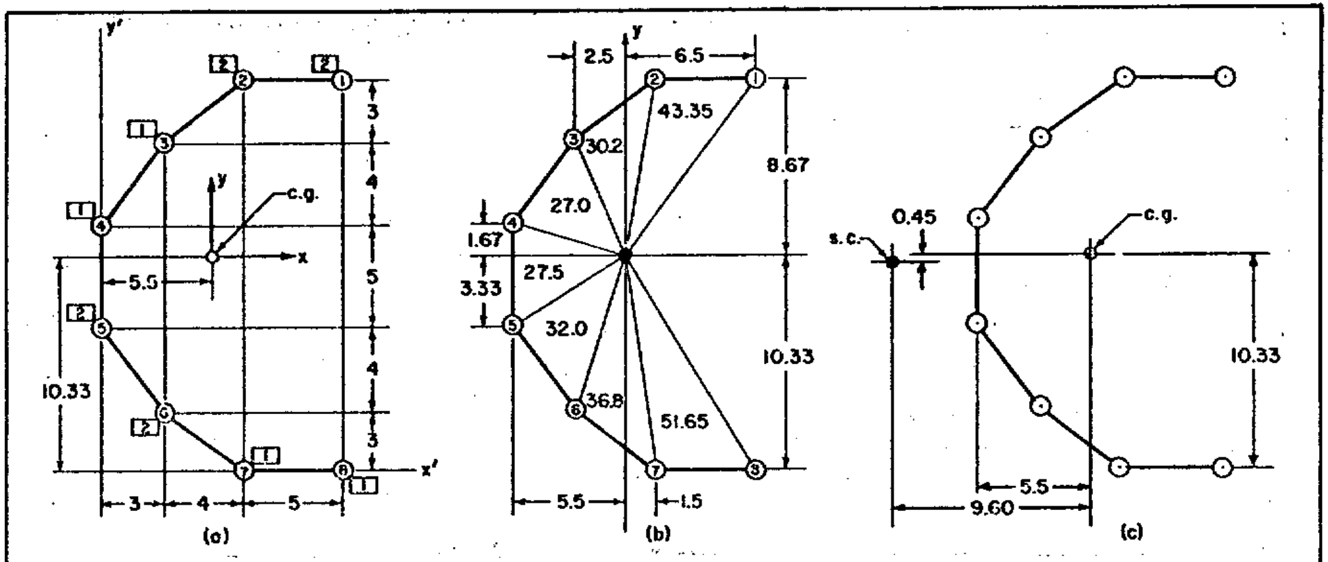
FIGURE 3.11 - EXAMPLE OF SHEAR CENTER LOCATION FOR HAT SECTION

The section has one axis of symmetry and consequently the shear center will be on this axis. The plot in (c) above shows the variation of w_s and x along the developed length of the hat section. Due to symmetry only one half of the cross section is considered. The result will be multiplied by two. The expression $w_s x$ is shown in (d) and the area under this curve is $\int w_s x ds$. Then

$$y_o I_y = -\int_0^L w_s x t ds = 2(0.698)(.040) = 0.0558$$

$$y_o = .0558/.0894 = .624$$

3.11.1.2 Example of Shear Center Location for Open Shell



SECTION PROPERTIES

1	2	3	4	5	6	7	8	9
Item	A_n	x'	$A_n x'$	y'	$A_n y'$	$A_n (x')^2$	$A_n (y')^2$	$A_n x'y'$
			(3)x(4)		(2)x(5)	(3)x(4)	(5)x(6)	(4)x(5)
1	2	12	24	19	38	288	722	456
2	2	7	14	19	38	98	722	266
3	1	3	3	16	16	9	256	48
4	1	0	0	12	12	0	144	0
5	2	0	0	7	14	0	98	0
6	2	3	3	3	6	18	18	18
7	1	7	14	0	0	49	0	0
8	1	12	24	0	0	144	0	0
Σ	12	5.5	66	10.33	124	606	1960	788

SHEAR CENTER LOCATION

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Item	Double Swept Area	w_s	y	A_n	$w_s y A_n$	x	$w_s x A_n$
		Σ (2)			(3)(4)(5)		(3)(5)(7)
1	0	0	+8.57	2	0	+6.5	0
2	43.35	43.35	+8.67	2	+751	+1.5	+130
3	30.20	73.55	+5.67	1	+417	-2.5	-184
4	27.00	100.55	+1.67	1	+176	-5.5	-554
5	27.50	128.05	-3.33	2	-854	-5.5	-1410
6	32.00	160.05	-7.33	2	-2350	-2.5	-800
7	36.80	196.85	-10.33	1	-2030	+1.5	+295
8	51.65	248.50	-10.33	1	-2570	+6.5	+1615
Σ				12	-6460		-908

FIGURE 3.12 - SHEAR CENTER LOCATION FOR UNSYMMETRICAL OPEN SKIN-STIFFENED SHELL

In this example, the method is applied to an unsymmetrical open section composed of stiffeners. The dimensions of the cross-section are indicated in (a) above. The numerals within the squares indicate the cross-sectional areas of the stiffeners. The skin has been assumed to be ineffective in bending for the subject problem.

The sketch in (b) above shows the double areas swept by a radius vector as it moves counter-clockwise from element to element.

Due to the introduction of the stiffener areas, Equations 3.44 and 3.45 have to be modified. For this case, tds is replaced by A_n, the stiffener area.

$$\bar{y} = \frac{124}{12} = 10.33$$

$$\bar{x} = \frac{66}{12} = 5.5$$

$$I_y = 606 - 12(5.5)^2 = 243$$

$$I_x = 1960 - 12(10.33)^2 = 680$$

TABLE 3.11 - LOCATION OF SHEAR CENTER

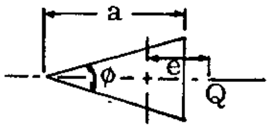
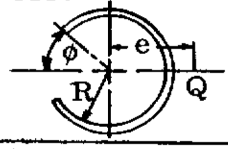
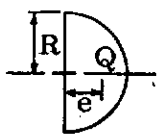
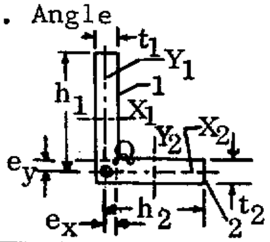
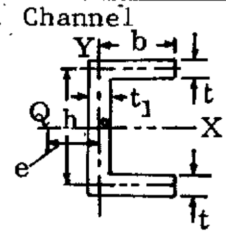
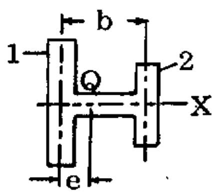
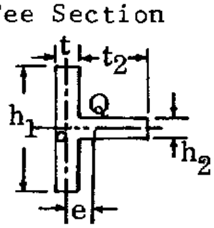
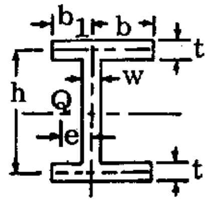
FORM OF SECTION	LOCATION OF SHEAR CENTER, Q, FOR SECTIONS HAVING ONE AXIS OF SYMMETRY
1. Triangle 	$e = 0.47\ a$ for narrow triangle ($\phi < 12^{o}$) approx.
2. Sector of thin circular tube 	$e = \frac{2R}{(\pi - \phi) + \text{Sin}\phi\text{Cos}\phi} \left[(\pi - \phi)\text{Cos}\phi + \text{Sin}\phi \right]$
3. Semicircular area 	$e = \left(\frac{8}{15\pi}\ \frac{3 + 4\mu}{1 + \mu}\right)R$ Q is to right of centroid
4. Angle 	I_1 = moment of inertia of leg 1 about Y_1 (central axis) I_2 = moment of inertia of leg 2 about Y_2 (central axis) Leg 1 = t_1h_1 Leg 2 = t_2h_2 $e_y = \frac{1}{2}\ h_1\left(\frac{I_1}{I_1 + I_2}\right)$ (for e_x use X_1 and X_2 central axes) If t_1 & t_2 are small, $e_x = e_y = 0$ (practically) and Q is at 0
5. Channel 	$e = h\left(\frac{H_{xy}}{I_x}\right)$ where H_{xy} = product of inertia of the half section (above x) with respect to axes X and Y; and I_x = moment of inertia of whole section with respect to axis X. If $t = t_1$, $e = \frac{b^2h^2t}{4\ I_x}$
6. I Section 	$e = b\left(\frac{I_2}{I_1 + I_2}\right)$ where I_1 and I_2, respectively, denote moments of inertia about X-axis of flange 1 and flange 2
7. Tee Section 	$e = \frac{1}{2}(t_1 + t_2)\left[\frac{1}{1 + \frac{h_1{}^3t_1}{h_2{}^3t_2}}\right]$ For a T - beam of ordinary proportions, Q may be assumed to be at 0

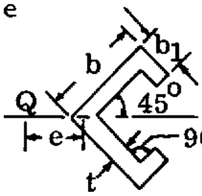
TABLE 3.11 (CONT'D) - LOCATION OF SHEAR CENTER

8. I with unequal legs



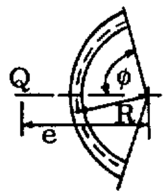
$$e = \frac{3\left[b^2 - b_1{}^2\right]}{(w/t)h + 6(b + b_1)} \quad ; \; b_1 < b$$

9. Right angle with lips



$$e = \frac{\left[b(b_1)^2\right] \quad (3b - 2b_1)}{\sqrt{2} \quad \left[2b^3 - (b - b_1)^3\right]}$$

10. Sector of arc

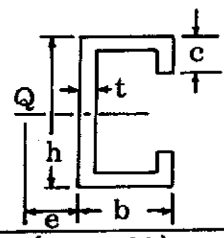


$$e = 2\,R\frac{\sin\phi - \phi\cos\phi}{\phi - \sin\theta\cos\theta}$$

when $\theta = \pi/2$ (semicircle)

$$e = \frac{4R}{\pi}$$

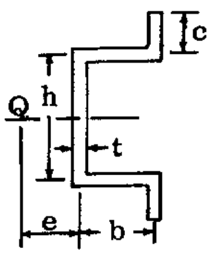
11. Lipped channel (t small)



Values of e/h

c/h \ b/h	1.0	0.8	0.6	0.4	0.2
0	0.430	0.330	0.236	0.141	0.055
0.1	0.477	0.380	0.280	0.183	0.087
0.2	0.530	0.425	0.325	0.222	0.115
0.3	0.575	0.470	0.365	0.258	0.138
0.4	0.610	0.503	0.394	0.280	0.155
0.5	0.621	0.517	0.405	0.290	0.161

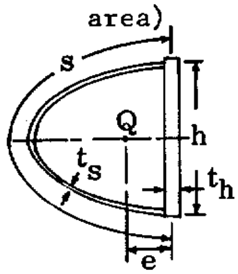
12. Hat section (t small)



Values of e/h

c/h \ b/h	1.0	0.8	0.6	0.4	0.2
0	0.430	0.330	0.236	0.141	0.055
0.1	0.464	0.367	0.270	0.173	0.080
0.2	0.474	0.377	0.280	0.182	0.090
0.3	0.453	0.358	0.265	0.172	0.085
0.4	0.410	0.320	0.235	0.150	0.072
0.5	0.355	0.275	0.196	0.123	0.056
0.6	0.300	0.225	0.155	0.095	0.040

13. D-Section (A = enclosed area)



Values of e(h/A)

t_h/t_s \ s/h	1	1.5	2	3	4	5	6	7
0.5	-----	-----	1.000	0.800	0.665	0.570	0.500	0.445
0.6	-----	-----	0.910	0.712	0.588	0.498	0.434	0.386
0.7	-----	0.980	0.831	0.641	0.525	0.443	0.384	0.338
0.8	-----	0.910	0.770	0.590	0.475	0.400	0.345	0.305
0.9	-----	0.850	0.710	0.540	0.430	0.360	0.310	0.275
1.0	1.0	0.800	0.662	0.500	0.400	0.330	0.285	0 250
1.2	0.905	0.715	0.525	0.380	0.304	0.285	0.244	0.215
1.6	0.765	0.588	0.475	0.345	0.270	0.221	0.190	0.165
2.0	0.660	0.497	0.400	0.285	0.220	0.181	0.155	0.135
3.0	0.500	0.364	0.285	0.200	0.155	0.125	0.106	0.091

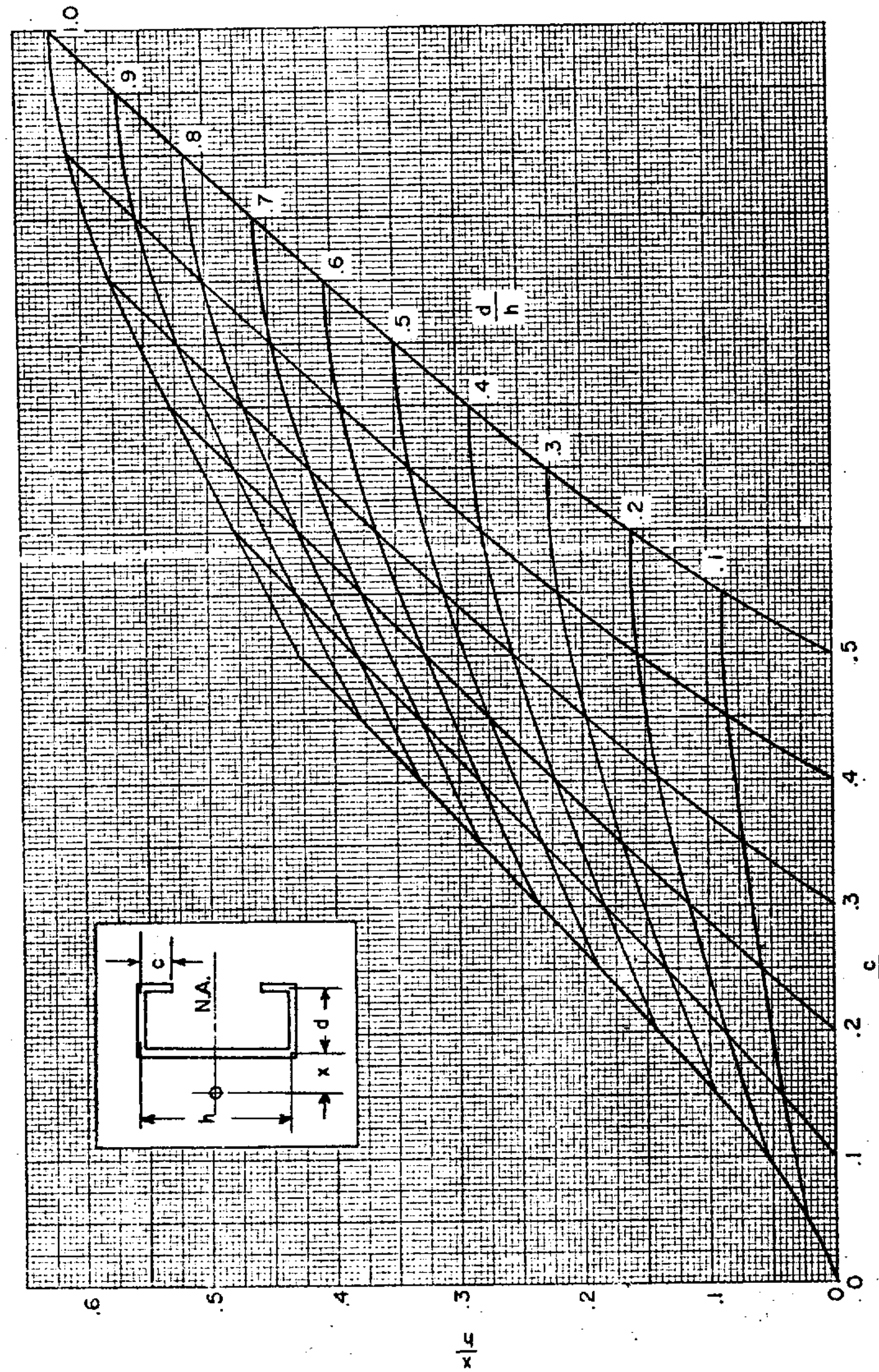


FIGURE 3.13 - SHEAR CENTER OF A LIPPED CHANNEL SECTION

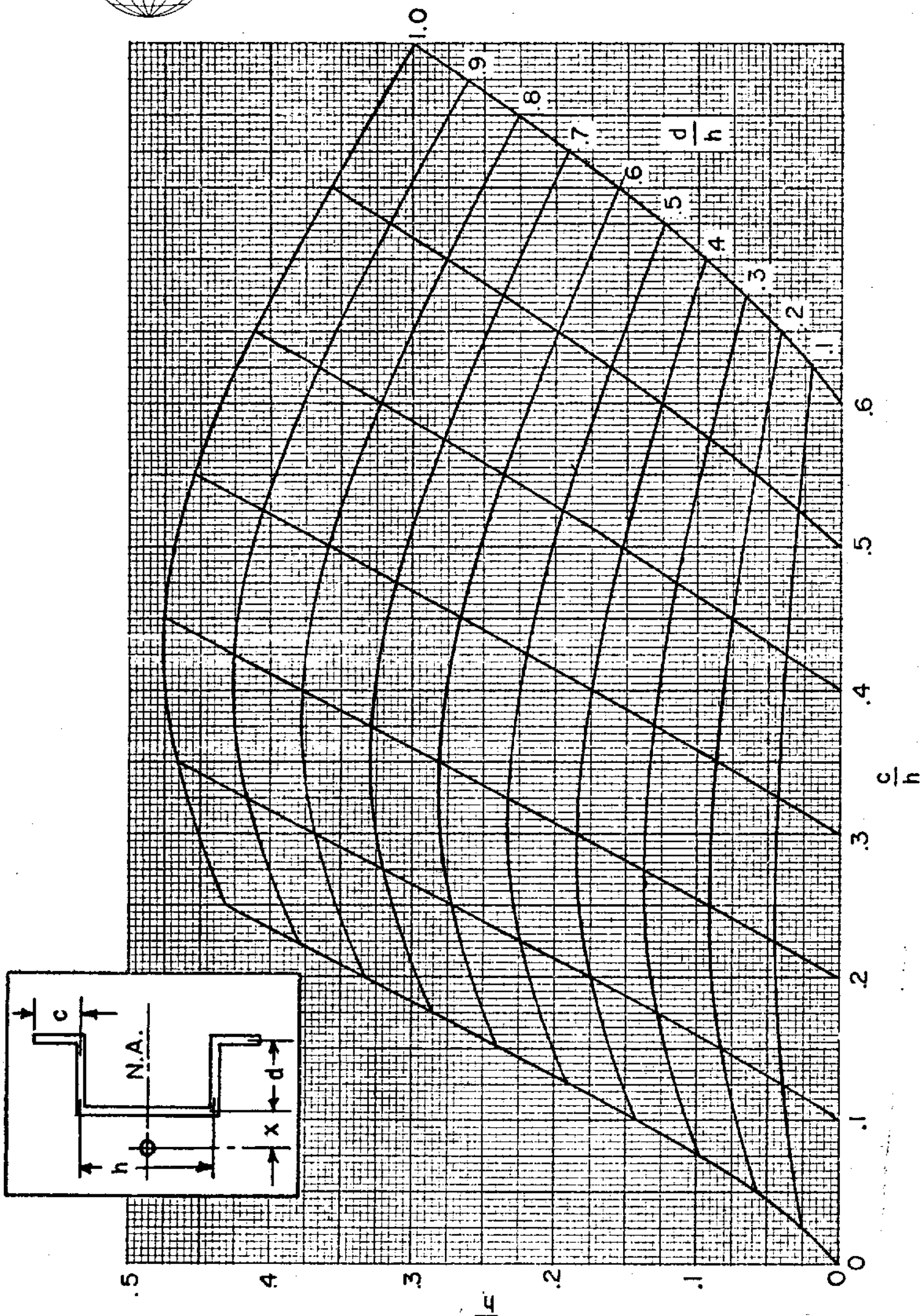


FIGURE 3.14 - SHEAR CENTER OF A HAT SECTION

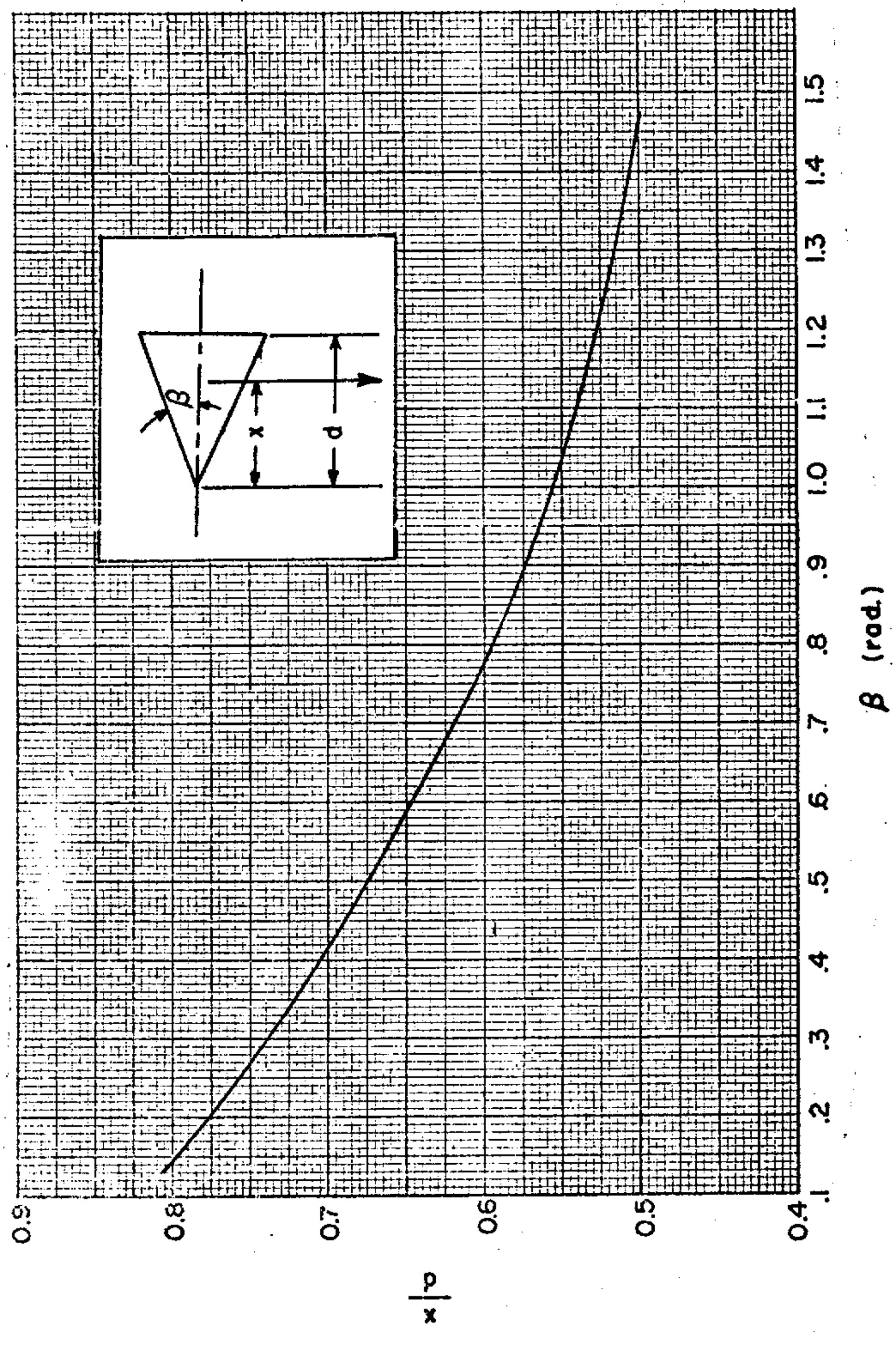


FIGURE 3.15 - SHEAR CENTER FOR A TRIANGULAR SECTION

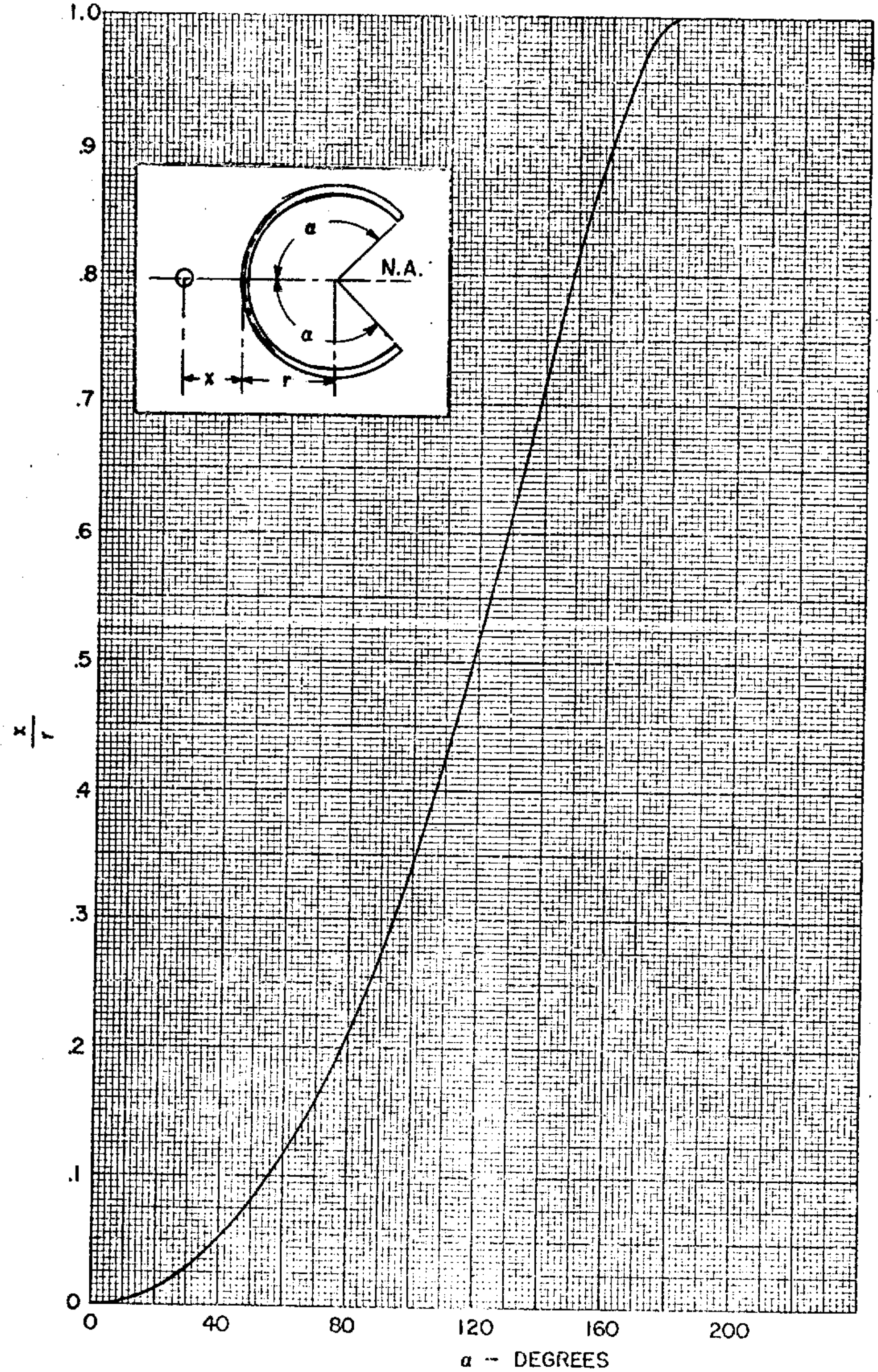


FIGURE 3.16 - SHEAR CENTER OF CIRCULAR ARC SECTION

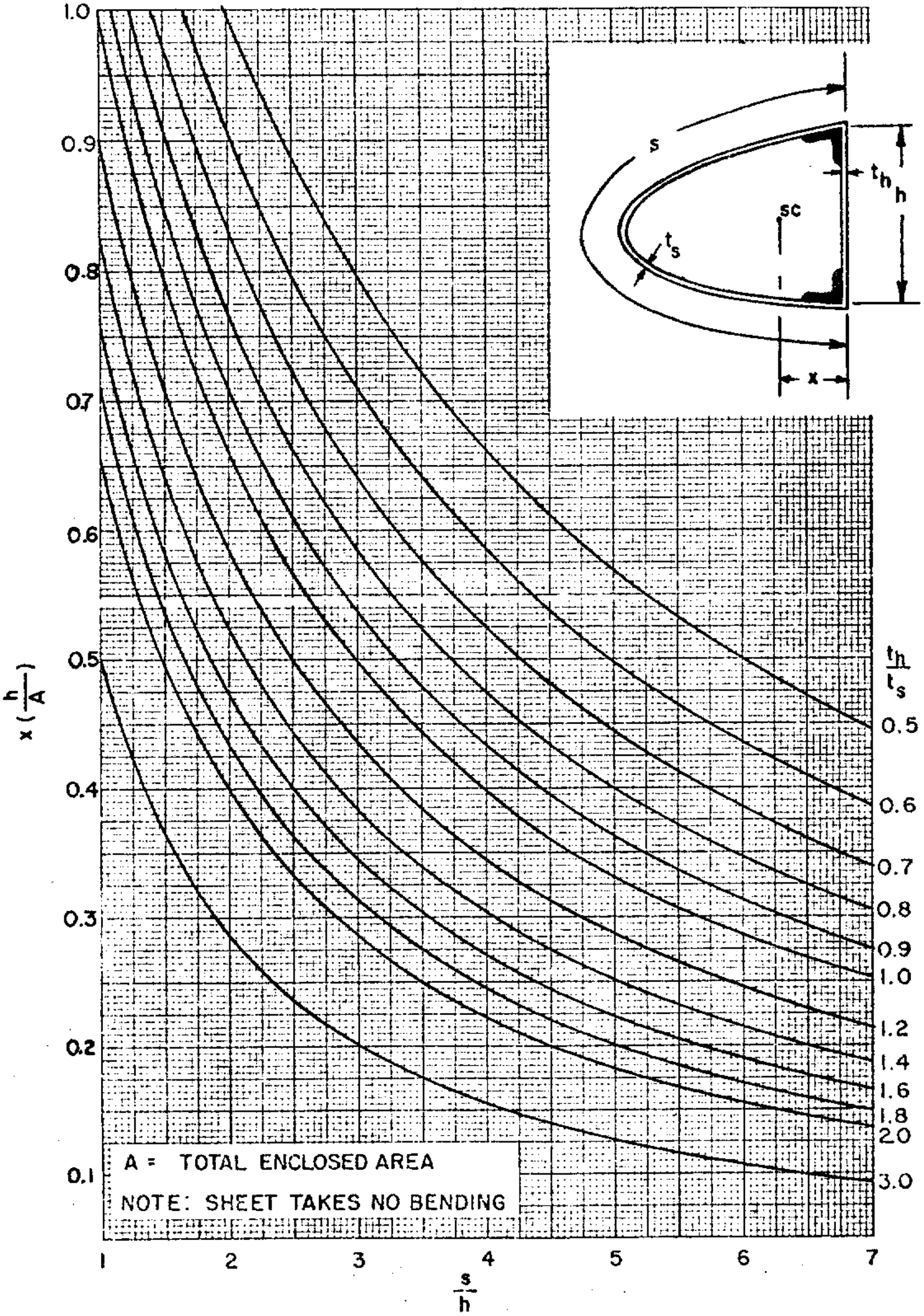


FIGURE 3.17 - SHEAR CENTER OF D-SECTIONS

$I_{xy} = 788 - 12(5.5)(10.33) = 106$

Using Equations 3.44 and 3.45 the following shear center location is determined:

$$x_o = \frac{1}{I_x I_y - (I_{xy})^2} \left| I_y \Sigma(w_s y A_n) - I_{xy} \Sigma(w_s x A_n) \right|$$

$$x_o = \frac{1}{(680)(243)-(106)^2} \left| (243)(-6460)-(106)(-908) \right| = -9.60$$

$$y_o = \frac{1}{I_x I_y - (I_{xy})^2} \left| -I_x \Sigma(w_s x A_n) + I_{xy}(w_s y A_n) \right|$$

$$y_o = \frac{1}{(680)(243)-(106)^2} \left| (-680)(-908) + (106)(-6460) \right| = -0.45$$

3.11.1.3 Shear Center of an Open Cell Box Beam

The shear center of an open cell box beam such as the one shown in Figure 3.18 is found by determining the internal loads

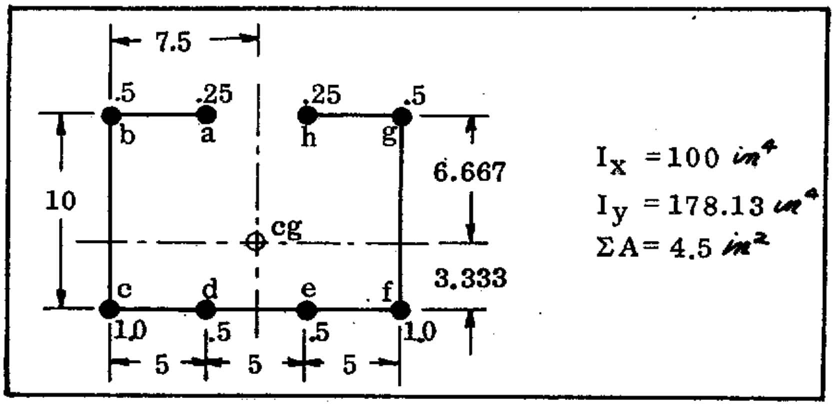


FIGURE 3.18 OPEN SINGLE CELL BOX BEAM

distribution for an arbitrary applied load first vertically and then horizontally and then equating the internal moments to the external. The shear center is the point at which the load is placed to create zero in-plane moment or zero rotation.

It is assumed that the beam is composed of axial members capable of carrying tension and compression loads and thin skins capable of shear only. The beam can be any length with the members changing area and thickness. Figure 3.19 shows a typical cross section with a load of 100 lbs. applied. The axial stress per inch of span is calculated using:

$$f = \frac{-Vy}{I} \qquad 3.46$$

where

V = arbitrary applied load
y = distance from centroid to axial member
I = moment of inertia resisting the bending created by V

The axial load is then obtained by multiplying equation 3.46 by the axial area. Figure 3.19 shows the distribution of axial loads for the 100 lb. shear. The loads are calculated as follows:

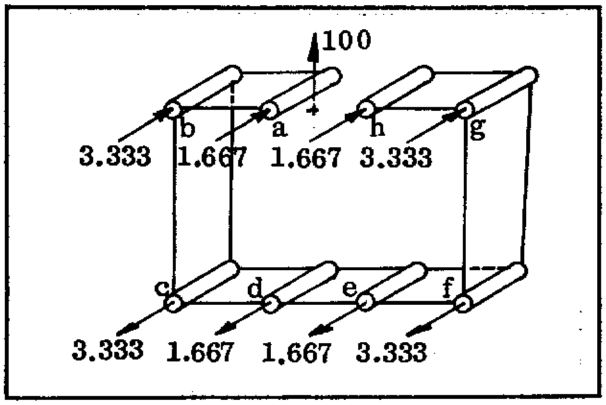


FIGURE 3.19 AXIAL LOADS DISTRIBUTION

$$P_a = P_h = \frac{-Vy_a A_a}{I_x} = \frac{-100(6.667)(.25)}{100} = -1.667 \text{ lb/in}$$

$$P_b = P_g = \frac{-Vy_b A_b}{I_x} = \frac{-100(6.667)(.5)}{100} = -3.333 \text{ lb/in}$$

$$P_c = P_f = \frac{-Vy_c A_c}{I_x} = \frac{-100(-3.333)(1.0)}{100} = 3.333 \text{ lb/in}$$

$$P_d = P_e = \frac{-Vy_d A_d}{I_x} = \frac{-100(-3.33)(.5)}{100} = 1.667 \text{ lb/in}$$

The shear flow distribution is determined from the axial loads by

$$q = \Delta P/L \qquad 3.47$$

where

ΔP = the change in axial load
L = the length over which ΔP occurs

Beginning at point "a" and summing forces to put each element in static equilibrium:

$q_{ab} = -P_a/L = -1.667/1.0 = -1.667$ lb/in.

$q_{bc} = -P_b/L + q_{ab} = -3.333/1.0 - 1.667 = -5.0$ lb/in.

$q_{cd} = P_c/L + q_{bc} = 3.333/1.0 - 5.0 = -1.667$ lb/in.

$q_{de} = P_d/L + q_{cd} = 1.667/1.0 - 1.667 = 0$

This procedure is completed around the cell until the internal loads as shown in Figure 3.20 are all calculated.

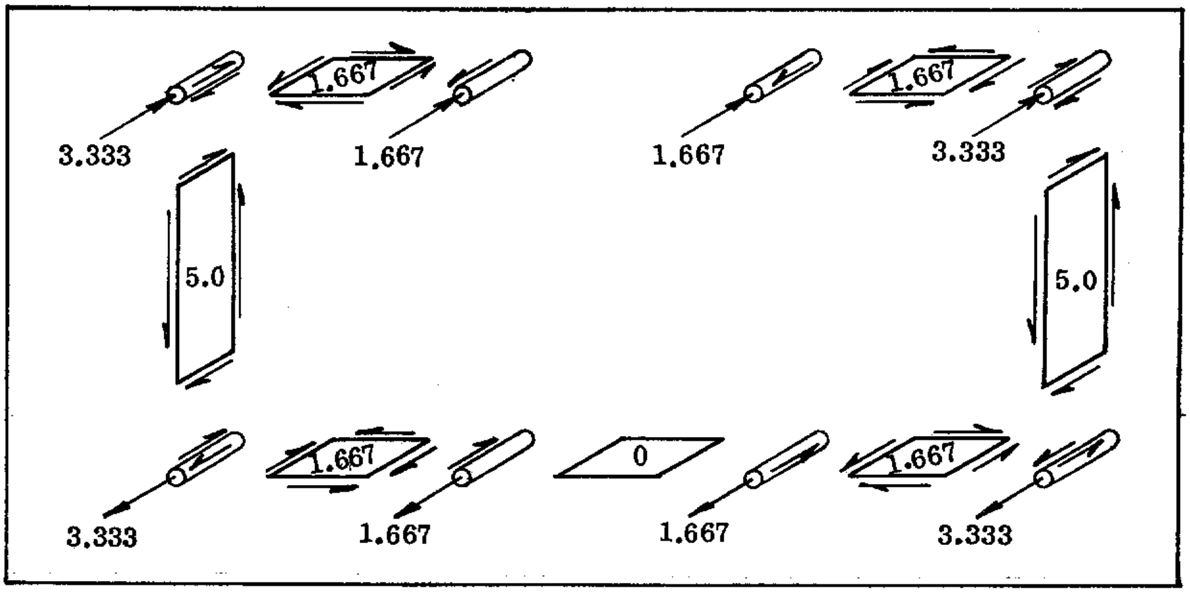


FIGURE 3.20 INTERNAL LOADS DISTRIBUTION

The sign of the shear flows is determined by their direction on the forward face of the cup.

If q is clockwise, it is positive.

The next step is to check the internal balance by ΣFx, ΣFy and ΣFz. If these sum to zero, proceed. If they do not sum to zero, there is an error and it must be found before proceeding.

Sum moments about any point. The lower LH corner is convenient.

$\Sigma M_c = 0$

$$\Sigma M_c = -100e-10(5)(1.667)+10(5)(1.667)+15(5)(10) = 0$$
$$= -100e+750 = 0$$
$$e = 7.5$$

The horizontal location of the shear center is then 7.5 inches to the right of point c. This is an axis of symmetry and the previous calculations could have been avoided by recognizing the axis of symmetry. They were shown to demonstrate the method.

The vertical location of the shear center is calculated in the same manner. A horizontal load is applied and the shear flows are determined. Figure 3.21 shows the shear flow distribution for the horizontally applied 100 lbs. The load must be

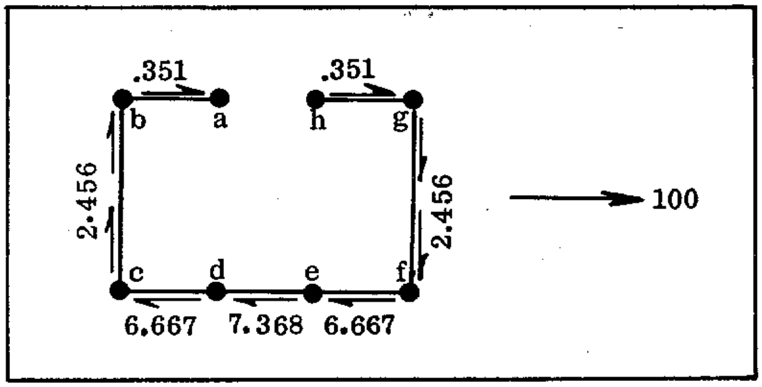


FIGURE 3.21 SHEAR FLOW DISTRIBUTION FOR HORIZONTAL LOAD

applied so that no rotation occurs.

$$\Sigma M_c = 0 = -100e+10(5)(.351)+10(5)(.351)+15(10)(2.456)$$
$$100e = 403.49$$
$$e = 4.035$$

The shear center is located as shown in Figure 3.22

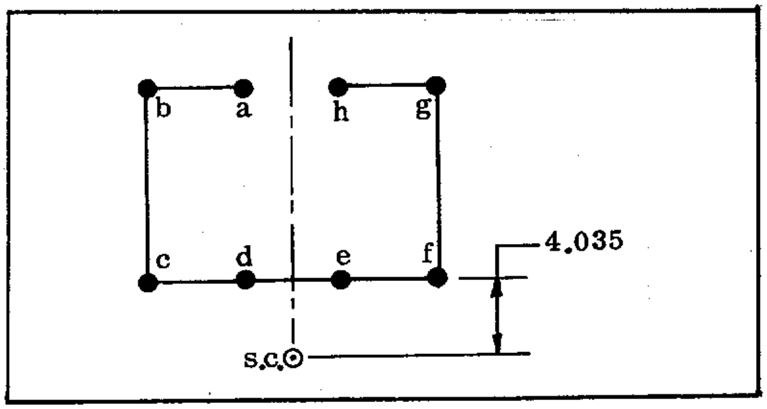


FIGURE 3.22 SHEAR CENTER LOCATION

3.11.2 Shear Center of Closed Cells

Figure 3.23 shows a typical closed cell. It is the same as the Fig. 3.18 example except the cell is closed. The first step is to assume that one web is cut. Web

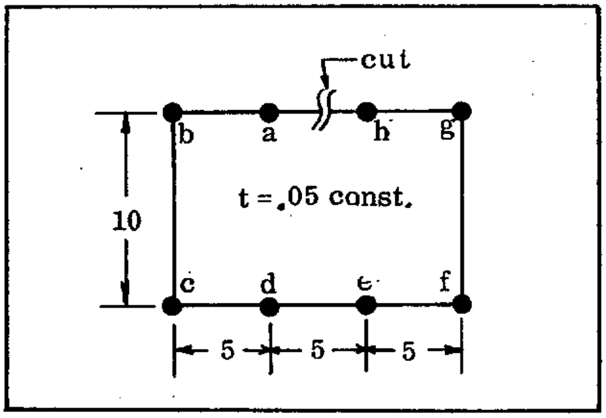


FIGURE 3.23 CLOSED SINGLE CELL BOX BEAM

ah will be cut since much of the previous solution can be used. The horizontal position of the shear center lies on the axis of symmetry. If no axis of symmetry existed, the procedure would be the same as for the forthcoming analysis.

With web ah cut, the resisting shear flow distribution for a horizontal load is shown in Figure 3.21. The web is then assumed to be closed and a constant shear flow of q_o is applied arbitrarily in the counter clockwise direction. The shear flow at any point in the cell is then defined as:

$$q = q_o + q' \tag{3.48}$$

where

q_o = the balancing shear flow

q' = the shear flow with one web cut

The shear flow q_o is that which will make the angle of twist, θ, equal zero. The twist is:

$$\theta = \sum \frac{q \Delta s L}{2AtG} = 0 \tag{3.49}$$

where L = 1 inch and 2AG is constant for all webs and may be taken outside the summation and then cancelled. Equation 3.49 becomes:

$$\theta = \sum q \Delta s/t = 0 \tag{3.50}$$

Substituting equation 3.48 into equation 3.50 and taking q_o outside the summation sign because it is constant for all webs, the following equation is obtained:

$$q_o \sum \Delta s/t + \sum q' \Delta s/t = 0 \tag{3.51}$$

The numerical solution can be put into table form as shown in Table 3.12.

Web	Δs	$\Delta s/t$	q' (1)	$q'\Delta s/t$	2A (2)	$2Aq'$
a-b	5	100	.351	35.1	50	17.55
b-c	10	200	2.456	491.2	0	0
c-d	5	100	6.667	666.7	0	0
d-e	5	100	7.368	736.8	0	0
e-f	5	100	6.667	666.7	0	0
f-g	10	200	2.456	491.2	150	368.4
g-h	5	100	.351	35.1	50	17.55
h-a	5	100	0	0	50	0
Total	--	1000	--	3122.8	300	403.5

(1) Figure 3.21

(2) Figure 3.24

TABLE 3.12 - NUMERICAL SOLUTION FOR SINGLE CELL CLOSED BOX

Substituting values from Table 3.15 into equation 3.51

$$1000q_o + 3122.8 = 0$$

$$q_o = -3.123 \text{ lbs/in}$$

The moment about any point can be obtained from the relation

$$T = \Sigma 2Aq \qquad 3.52$$

Substituting equation 3.48 into 3.52 yields

$$T = q_o \Sigma 2A + \Sigma 2Aq' \qquad 3.53$$

where A is the area enclosed by a web and the lines joining the end points of the web and the center of moments as shown in Figure 3.24.

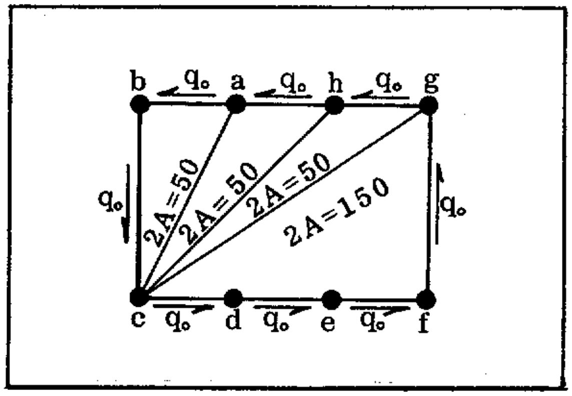


FIGURE 3.24 ENCLOSED AREA DEFINITIONS

The values from Table 3.12 can be substituted into equation 3.53 with moments being taken about point c.

$$100e+300q_0+403.5 = 0$$

$$100e+300(-3.123)+403.5 = 0$$

$$100e = 533.4$$

$$e = 5.334$$

The shear center is 5.334 inches above point c on the vertical axis of symmetry.

3.12 STRAIN GAGES

A strain gage is a small device which is attached to a structure to measure the strain. They are usually attached to test articles but in some cases they are permanent fixtures on operational aircraft.

A strain gage measures the change in length of the structure over the length of the gage. The gage itself changes length along with the structure and the electrical resistivity of the gage changes. This resistance change is measured and by proper calibration the resistance can be related to strain in the structure.

The strain gages most commonly used are (1) the wire gage, (2) the foil gage and (3) the weldable gage.

3.12.1 The Wire Strain Gage

The wire gage as shown in Figure 3.25 is made of a grid of very fine wires. The wires are usually made of copper-nickel alloy and bonded to a lacquered paper base which is subjected to a slight initial tension. The paper base is bonded to the specimen. A felt cover is placed over the grid in the longer size gages for protection. The felt cover also helps to minimize temperature changes. This type of gage is called a "Duco" gage and is the least expensive and most convenient gage to use.

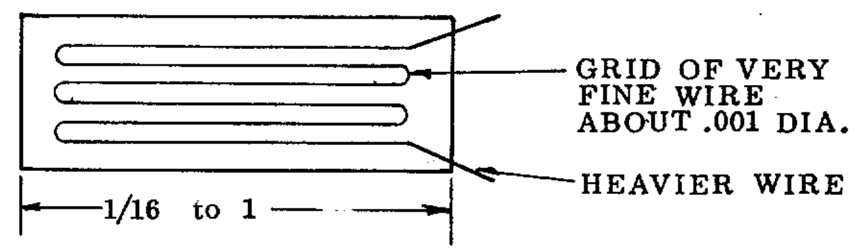


FIGURE 3.25 WIRE STRAIN GAGE

Another type of wire strain gage has the wire grid molded in a thermosetting phenol resin. It is called a "bakelite" gage and the molding process makes this type of gage much more time consuming. Bakelite gages are useful though, when temperatures are between 150°F and 450°F and when humidity presents stability problems to Duco gages. Otherwise the Duco gage is sufficient.

3.12.2 The Foil Strain Gage

The foil strain gage is shown in Figure 3.26. It is etched from a foil sheet. The width of the element is increased at the end of the loops to reduce the effect of the transversely oriented parts of the conductors. The gage is mounted in a thin cement which is applied to the specimen. Adhesives are available which permit the use of the gage up to 700°F. Gage lengths are available from 1/64 inch. These gages are easy to mount, inexpensive and very accurate.

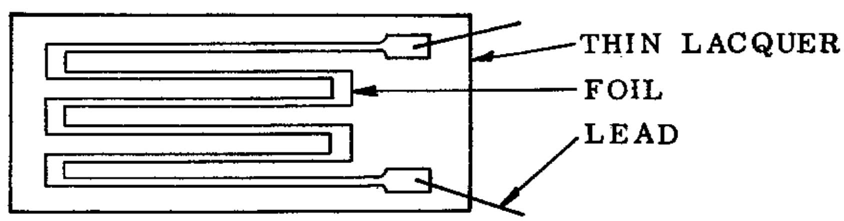


FIGURE 3.26 FOIL STRAIN GAGE

3.12.3 The Weldable Strain Gage

The weldable strain gage is a length of fine wire surrounded by high temperature insulation encased in a flanged metal tube. It can be spot welded to a test specimen in less time than any of the other gages can be installed. It can be located on a curved surface. Very small gage lengths are available and they can be used in static tests up to 850°F and dynamic tests up to 1600°F.

3.12.4 The Strain Gage Rosette

When the direction of the principal strain is known, it is usually sufficient to mount a single gage along the axis of the strain. If the direction of the principal strain is unknown or if it is not axial, it is necessary to make several measurements along different axes or to use a "rosette" consisting of three strain gages on the same paper or bakelite backing. A 45° rosette is shown in Figure 3.27. In this rosette, three gages are mounted with their axes intersecting at a common

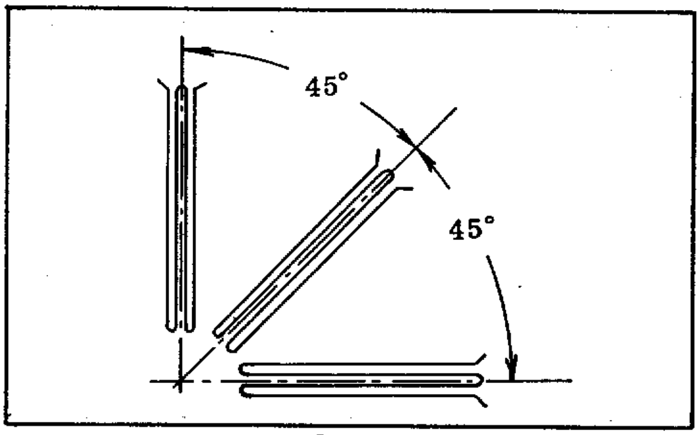


FIGURE 3.27 45° STRAIN GAGE ROSETTE

centerpoint and are 45° apart. Each gage is electrically insulated so that the effect is that of three separate gages.

When the direction of the principal stresses are known, a "Visette" which is shown in Figure 3.28, is used. It consists of two gages mounted 90° apart on a single mount.

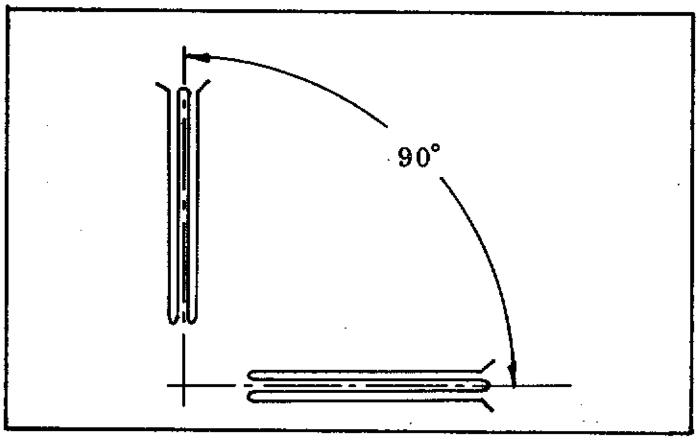


FIGURE 3.28 90° STRAIN GAGE ROSETTE (VISETTE)

When the direction of the principal strain is unknown, the delta rosette can be used. It is shown in Table 3.13. It has the maximum possible angle between gage axes. The angles are 0, 60 and 120 degrees.

A T-delta rosette, shown in Table 3.13, is identical to a delta rosette except that a fourth gage is mounted at right angles on top of the other gages. This fourth gage is used as a check.

3.12.5 Strain Gage Temperature Compensation

Two methods are available for eliminating the strain due to thermal expansion; the dummy gage and the self compensating gage.

The dummy gage consists of a gage mounted on a separate bar and a gage mounted on the specimen. The separate bar is made of the same material as the specimen. The two gages are connected to the readout equipment in such a way that equal resistance changes in the two gages will cancel. If the gages are identical and the bar is subject to the same temperature conditions as the test specimen, the temperature effects will cancel and the readout will be in terms of actual mechanical strain in the test specimen.

The self compensating gage is made in such a way that they are uneffected by thermal strain. This gage is expensive and each gage is limited to one temperature range and one material. Its advantage is simplicity.

3.12.6 Stress Determination From Strain Measurements

The purpose of strain measurements is to obtain stress levels in a structure. If the directions of the principal stresses are known, only two strain measurements are required. It can be shown by Hooke's law that if the directions are known, the stresses can be found using the following equations:

$$\sigma_1 = \frac{E}{1-\mu^2} (\epsilon_1 + \mu\epsilon_2) \qquad 3.54$$

$$\sigma_2 = \frac{E}{1-\mu^2} (\epsilon_2 + \mu\epsilon_1) \qquad 3.55$$

where

μ = Poisson's ratio

σ_1 = Principal stress in one direction

σ_2 = Principal stress at right angles to σ_1

ϵ_1 = Strain in the direction of σ_1

ϵ_2 = Strain in the direction of σ_2

E = Modulus of elasticity

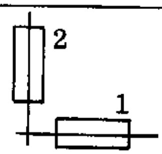
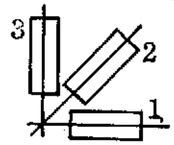
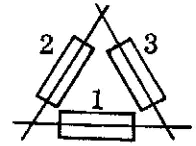
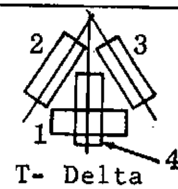
Rosette Types / Required Solution	 Two-Gage	 Rectangular	 Delta	 T- Delta
Max. Normal Stress σ_{max}	$\frac{E}{1-\mu^2}(\epsilon_1+\mu\epsilon_2)$	$\frac{E}{2}\left[\frac{\epsilon_1+\epsilon_3}{1-\mu}+\frac{1}{1+\mu}\sqrt{(\epsilon_1-\epsilon_3)^2+\left[2\epsilon_2-(\epsilon_1+\epsilon_3)\right]^2}\right]$	$E\left[\frac{\epsilon_1+\epsilon_2+\epsilon_3}{3(1-\mu)}+\frac{1}{1+\mu}\sqrt{\left(\epsilon_1-\frac{\epsilon_1+\epsilon_2+\epsilon_3}{3}\right)^2+\left(\frac{\epsilon_2-\epsilon_3}{\sqrt{3}}\right)^2}\right]$	$\frac{E}{2}\left[\frac{\epsilon_1+\epsilon_4}{1-\mu}+\frac{1}{1+\mu}\sqrt{(\epsilon_1-\epsilon_4)^2+\frac{4}{3}(\epsilon_2-\epsilon_3)^2}\right]$
Min. Normal Stress σ_{min}	$\frac{E}{1-\mu^2}(\epsilon_2+\mu\epsilon_1)$	$\frac{E}{2}\left[\frac{\epsilon_1+\epsilon_3}{1-\mu}-\frac{1}{1+\mu}\sqrt{(\epsilon_1-\epsilon_3)^2+\left[2\epsilon_2-(\epsilon_1+\epsilon_3)\right]^2}\right]$	$E\left[\frac{\epsilon_1+\epsilon_2+\epsilon_3}{3(1-\mu)}-\frac{1}{1+\mu}\sqrt{\left(\epsilon_1-\frac{\epsilon_1+\epsilon_2+\epsilon_3}{3}\right)^2+\left(\frac{\epsilon_2-\epsilon_3}{\sqrt{3}}\right)^2}\right]$	$\frac{E}{2}\left[\frac{\epsilon_1+\epsilon_4}{1-\mu}-\frac{1}{1+\mu}\sqrt{(\epsilon_1-\epsilon_4)^2+\frac{4}{3}(\epsilon_2-\epsilon_3)^2}\right]$
Max. Shearing Stress τ_{max}	$\frac{E}{2(1+\mu)}(\epsilon_1-\epsilon_2)$	$\frac{E}{2(1+\mu)}\sqrt{(\epsilon_1-\epsilon_3)^2+\left[2\epsilon_2-(\epsilon_1+\epsilon_3)\right]^2}$	$\frac{E}{1+\mu}\sqrt{\left(\epsilon_1-\frac{\epsilon_1+\epsilon_2+\epsilon_3}{3}\right)^2+\left(\frac{\epsilon_2-\epsilon_3}{\sqrt{3}}\right)^2}$	$\frac{E}{2(1+\mu)}\sqrt{(\epsilon_1-\epsilon_4)^2+\frac{4}{3}(\epsilon_2-\epsilon_3)^2}$
Angle From Gage 1 Axis To Max. Normal Stress Axis, θp	0	$\frac{1}{2}\tan^{-1}\left[\frac{2\epsilon_2-(\epsilon_1+\epsilon_3)}{\epsilon_1-\epsilon_3}\right]$	$\frac{1}{2}\tan^{-1}\left[\frac{\frac{1}{\sqrt{3}}(\epsilon_2-\epsilon_3)}{\epsilon_1-\frac{\epsilon_1+\epsilon_2+\epsilon_3}{3}}\right]$	$\frac{1}{2}\tan^{-1}\left[\frac{2(\epsilon_2-\epsilon_3)}{3(\epsilon_1-\epsilon_4)}\right]$

TABLE 3.13 - RELATIONS BETWEEN STRAIN ROSETTE READINGS AND PRINCIPAL STRESSES

The maximum shearing stress will occur at 45° to the principal stresses and is found as follows:

$$\tau_{max} = \frac{E}{2(1+\mu)} (\epsilon_1 - \epsilon_2) \qquad 3.56$$

If the directions of the principal stresses are unknown, the problem is more complex. The axes of an element are shown in Figure 3.29.

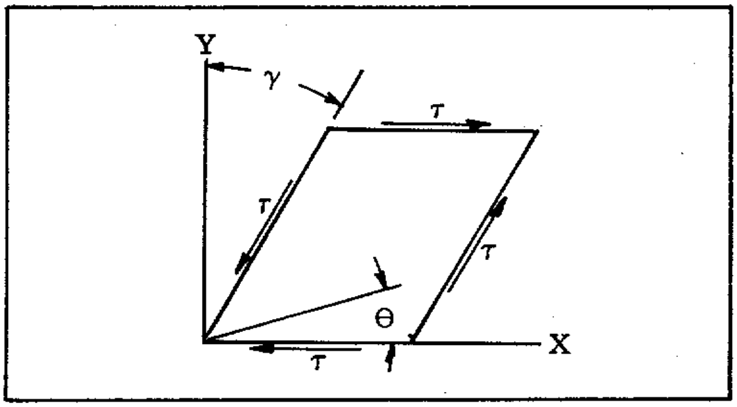


FIGURE 3.29 STRESSED ELEMENT

The general equation for strain at an angle θ using the results of a 45° rosette is

$$\epsilon_\theta = \frac{\epsilon_x + \epsilon_y}{2} + \left(\frac{\epsilon_x - \epsilon_y}{2}\right) \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \qquad 3.57$$

where

γ_{xy} = the shearing strain

In Figure 3.30 the strain measured along the (A), (B) and (C) axes can be used to

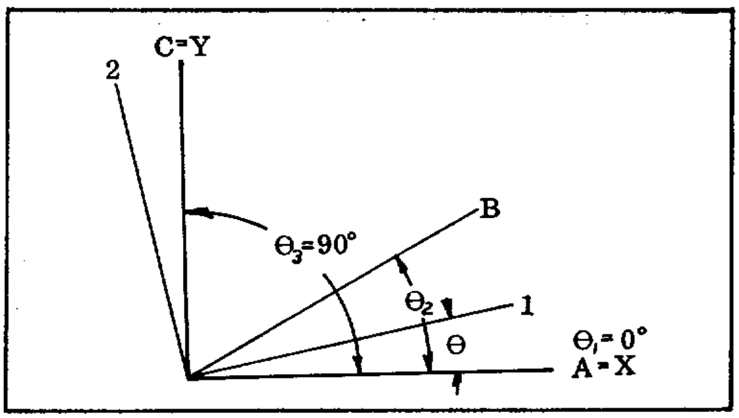


FIGURE 3.30 STRAIN AXES

calculate the principal stresses which act along axes (1) and (2). When a 45^{o} rosette is used, the strain along (B) is measured on an axis at an angle of 45^{o} to the (A) and (C) axes. The angles (AOB) and (BOC) are 45^{o}. Axis (2) is at an angle of 90^{o} to axis (1). The strains along axes (A), (B) and (C) will have the following relationships with the (x) and (y) axes of Figure 3.28.

$$\epsilon_A = \frac{\epsilon_x+\epsilon_y}{2} + \frac{\epsilon_x-\epsilon_y}{2} = \epsilon_x \qquad 3.58$$

$$\epsilon_B = \frac{\epsilon_x+\epsilon_y}{2} + \frac{\gamma_{xy}}{2} \qquad 3.59$$

$$\epsilon_C = \frac{\epsilon_x+\epsilon_y}{2} - \frac{\epsilon_x-\epsilon_y}{2} = \epsilon_y \qquad 3.60$$

or

$$\epsilon_x = \epsilon_A \qquad 3.61$$

$$\epsilon_y = \epsilon_C \qquad 3.62$$

$$\gamma_{xy} = 2\epsilon_B-(\epsilon_A+\epsilon_C) \qquad 3.63$$

By using the equation for maximum shear and Hooke's law, the principal stresses can be obtained from the following:

$$\sigma_1 = \frac{E}{2}\left[\frac{\epsilon_A+\epsilon_C}{1-\mu} + \frac{1}{1+\mu}\sqrt{2(\epsilon_A-\epsilon_B)^2 + 2(\epsilon_B-\epsilon_C)^2}\right] \qquad 3.64$$

$$\sigma_2 = \frac{E}{2}\left[\frac{\epsilon_A+\epsilon_C}{1-\mu} - \frac{1}{1+\mu}\sqrt{2(\epsilon_A-\epsilon_B)^2 + (\epsilon_B-\epsilon_C)^2}\right] \qquad 3.65$$

$$\tau_{MAX} = G\sqrt{2(\epsilon_A-\epsilon_B)^2 + 2(\epsilon_B-\epsilon_C)^2} \qquad 3.66$$

$$\theta = 1/2\ \mathrm{Tan}^{-1}\left[\frac{2\epsilon_B - (\epsilon_A-\epsilon_C)}{\pm(\epsilon_A-\epsilon_C)}\right] \qquad 3.67$$

3.13 ACOUSTICS AND VIBRATIONS

Most periodic waves, regardless of the form, can be represented by two or more sine waves. Most waves can be reduced to simple harmonic or sine wave components which generally form harmonic series. They have frequencies which are integral multiples of the lowest frequency. The lowest frequency is called the fundamental and the higher ones are called harmonic.

The frequency of a vibrating body is the number of cycles of motion in a unit time.

The period of a wave is the time elapsed while the motion repeats itself. It is the reciprocal of the frequency.

The amplitude of a wave is the maximum distance the vibrating particles of the medium in the path of the wave are displaced from their position of equilibrium.

The wavelength of a wave is the shortest distance between two particles along the wave which differ in phase by one cycle.

The number of independent coordinates necessary to describe the motion of a system is called degrees of freedom. Examples of systems with various degrees of freedom are shown in Figure 3.31. Example (d) in Figure 3.31 is a single degree of freedom with a mass "m" supported on frictionless and massless rollers attached to a spring and a dashpot. This is a representation of a fairly common situation occurring in aircraft and helicopters.

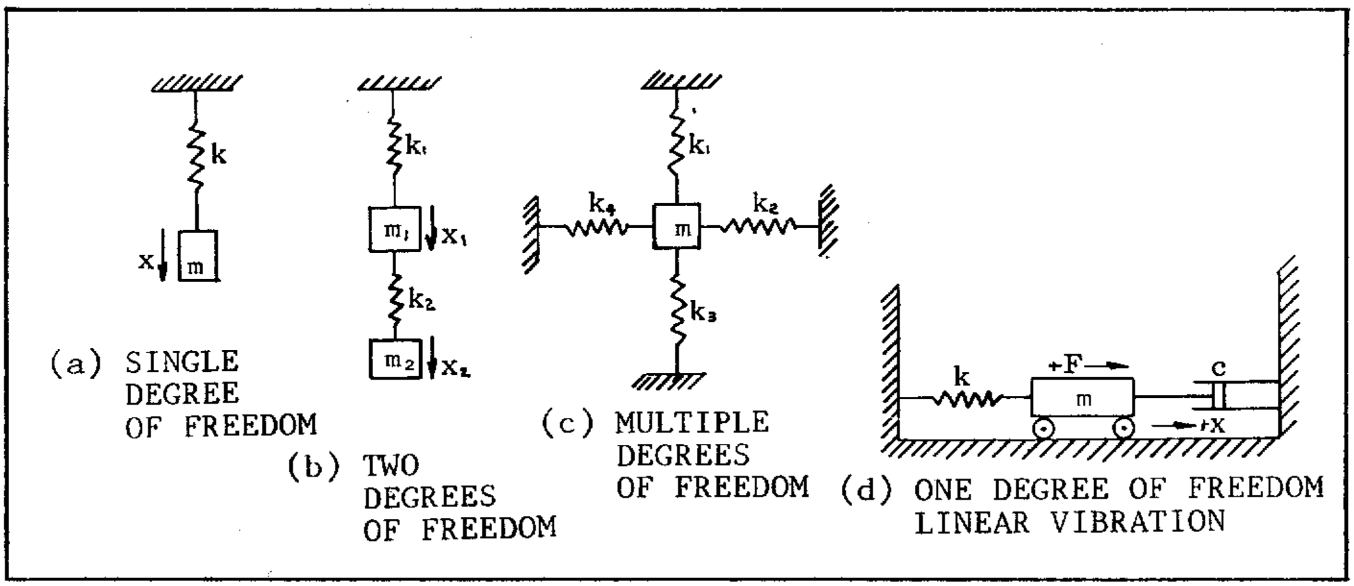


FIGURE 3.31 EXAMPLES OF DEGREES OF FREEDOM

If a force "F" which is a function of time "t" acts on the mass, the differential equation

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F(t) \qquad 3.68$$

must be satisfied at all times,

where

m = the mass of the system
c = the coefficient of viscous damping
k = the spring constant
x = the displacement from rest
$F(t)$ = the external force as a function of time

If after an initial displacement of the system the external force ceases to act, the equation becomes:

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = 0 \qquad 3.69$$

If the quantity $c^2/4m^2 > k/m$, the mass "m" will not oscillate but will gradually return to its rest position. If $c^2/4m^2 < k/m$, there will result a decaying oscillation of circular frequency,

$$\omega_n = \sqrt{k/m - c^2/4m^2}, \quad \text{radians/sec} \qquad 3.70$$

for which the corresponding linear frequency will be

$$f_n = 1/2\pi \sqrt{k/m - c^2/4m^2}, \quad \text{cycles/sec} \qquad 3.71$$

where "n" denotes natural frequency with damping, and

$$\omega_n = 2\pi f_n \qquad 3.72$$

If $c^2/4m^2 = k/m$, this is the limiting case for which no oscillation occurs and the system is said to be critically damped. This particular value of c is designated c_{cr} where

$$c_{cr} = 2m\sqrt{k/m} = 2\sqrt{mk} = 2m\omega_n = 2k/\omega_n \qquad 3.73$$

If the driving force is sinusoidal

$$F(t) = F_o \sin\omega t \qquad 3.74$$

equation 3.68 becomes

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F_o \sin\omega t \qquad 3.75$$

where F_o is the maximum value of the sinusoidal force and ω is the forced frequency. The general solution to equation 3.75 is

$$x = e^{\left(\frac{-ct}{2m}\right)} (A\sin\omega_n t + B\cos\omega_n t) + \frac{F_o \sin(\omega t - \phi)}{(c\omega)^2 + (k - m\omega^2)^2} \quad 3.76$$

where ϕ is the phase angle and A and B are arbitrary constants depending on the initial conditions. As before:

$$\omega_n = \sqrt{k/m - c^2/4m^2}\ \text{, radians/sec.} \quad 3.70$$

and

$$\phi = \tan^{-1} \left| c\omega/(k - m\omega^2) \right| \quad 3.77$$

The first term on the right hand side of equation 3.76 vanishes in time due to the fact that the term, $e^{(-ct/2m)}$, constantly diminishes and is called the transient term. The second term gives the amplitude of the forced vibration in terms of the system constants and driving force and is called the steady state term. The amplitude of the steady state vibration is

$$x = F_o / \sqrt{(c\omega)^2 + (k - m\omega^2)^2} \quad 3.78$$

This is also expressed in the convenient form

$$x = (F_o/k) / \sqrt{[1 - (\omega/\omega_n)^2]^2 + [2(c/c_{cr})(\omega/\omega_n)]^2} \quad 3.79$$

where F_o/k is the displacement that would be produced by a static force F_o.

Following are equations which predict deflections and mode patterns for beams. They are based on simple beam theory and are accurate for beams having a length to depth ratio of the order of 10 or more.

3.13.1 Uniform Beams

Uniform Bar With Free Ends

The equation for finding the deflection for different mode patterns is as follows:

$$y = 1/2.04\left(-\sin\alpha_n x + 1.02\cos\alpha_n x - \sinh\alpha_r x + 1.02\ \cosh\alpha_n x\right) \quad 3.80$$

where α_n is the characteristic number for the n^{th} mode and is the root of the equation $\cos\alpha_n \cosh\alpha_n = 1$. The characteristic numbers for the first three modes of this beam are: 4.73, 7.853 and 10.996. Frequencies of higher modes for this beam are given in Figure 3.32, in which w is the weight per unit length of the beam and $g = 386\ \text{in/sec}^2$.

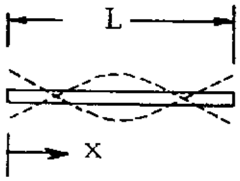
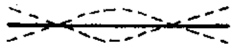
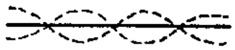
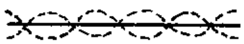
FIGURE	AMPLITUDE PROFILE	MODE	NATURAL CIRCULAR FREQUENCY
		2 NODES	$\omega_1 = \frac{22.4}{L^2}\sqrt{\frac{EIg}{w}}$
		3 NODES	$2.77\,\omega_1$
		4 NODES	$5.44\,\omega_1$
		5 NODES	$9.00\,\omega_1$

FIGURE 3.32 UNIFORM BAR WITH FREE ENDS

Uniform Bar With Simple Supports at Ends

The equation of deflection for the fundamental mode is

$$y = \sin(\pi x/L) \tag{3.81}$$

if the amplitude is taken as unity at the center. Frequencies of higher modes for this beam are shown in Figure 3.33.

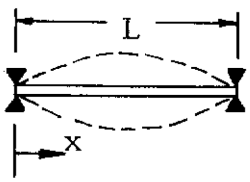
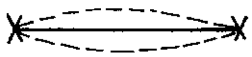
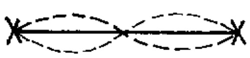
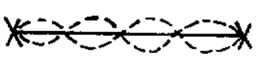
FIGURE	AMPLITUDE PROFILE	MODE	NATURAL CIRCULAR FREQUENCY
		2 NODES	$\omega_1 = \frac{\pi^2}{L^2}\sqrt{\frac{EIg}{w}}$
		3 NODES	$4\,\omega_1$
		4 NODES	$9\,\omega_1$
		5 NODES	$16\,\omega_1$

FIGURE 3.33 UNIFORM BAR WITH SIMPLE SUPPORTS

Uniform Cantilever Beam

The frequencies of modes for this beam are shown in Figure 3.34.

Uniform Beam With Clamped Ends

The frequencies of various modes for this beam are shown in Figure 3.35.

Hinged Fixed Uniform Beam

The frequencies of various modes for this beam are shown in Figure 3.36.

Uniform Cantilever Beam With Mass at the End

See Figure 3.37.

Uniform Beam Simply Supported With Central Mass

See Figure 3.38.

3.13.2 Rectangular Plates

The general equation for the frequency of a plate with simply supported edges is

$$\omega = \pi^2 \left(\frac{M^2}{a^2} + \frac{N^2}{b^2}\right) \sqrt{gEt^3/(12(1-\mu^2)w)} \qquad 3.82$$

where $g = 386$ in/sec^2, μ is Poisson's ratio. W is the weight per unit area of the plate, and M and N are integers depending on the number of nodal lines. Figure 3.39 shows normal modes of rectangular panels with values for M and N.

3.13.3 Columns

The equation for the natural frequency for an axially load member is given by

$$\omega_1 = \omega_0 \sqrt{1-P/P_{cr}}\text{, radians/sec.} \qquad 3.83$$

where

P = axial load
P_{cr} = Euler buckling load = $\pi^2 EI/L^2$
ω_0 = natural frequency for zero load

For the pin-ended column, ω_0 is given by ω_1 for an uniform bar with simple support at ends. For the fixed-end column, ω_0 is given by ω_1 for an uniform beam with champed ends.

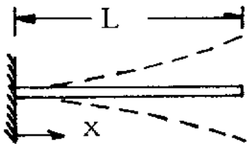
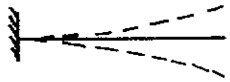
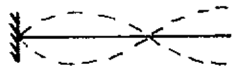
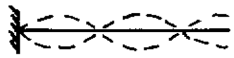
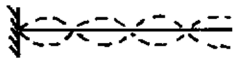
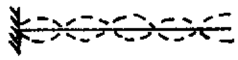
FIGURE	AMPLITUDE PROFILE	MODE	NATURAL CIRCULAR FREQUENCY
		1 NODES	$\omega_1 = \frac{.36\pi^2}{L^2}\sqrt{\frac{EIg}{W}}$
		2 NODES	$6.27\omega_1$
		3 NODES	$17.6\omega_1$
		4 NODES	$34.4\omega_1$
		5 NODES	$56.8\omega_1$

FIGURE 3.34 - UNIFORM CANTILEVER BEAM

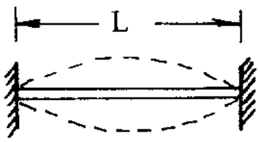
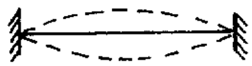
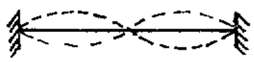
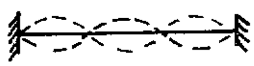
FIGURE	AMPLITUDE PROFILE	MODE	NATURAL CIRCULAR FREQUENCY
		2 NODES	$\omega_1 = \frac{22.4}{L^2}\sqrt{\frac{EIg}{W}}$
		3 NODES	$2.77\omega_1$
		4 NODES	$5.44\omega_1$
		5 NODES	$9.00\omega_1$

FIGURE 3.35 - UNIFORM BEAM WITH CLAMPED ENDS

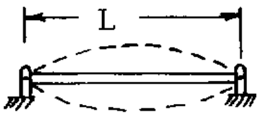
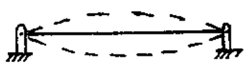
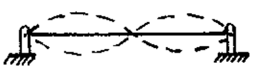
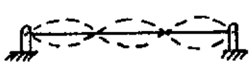
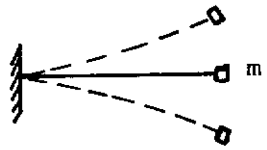
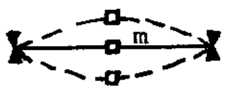
FIGURE	AMPLITUDE PROFILE	MODE	NATURAL CIRCULAR FREQUENCY
		2 NODES	$\omega_1 = \frac{15.4}{L^2}\sqrt{\frac{EIg}{W}}$
		3 NODES	$3.24\omega_1$
		4 NODES	$6.76\omega_1$

FIGURE 3.36 - HINGED FIXED UNIFORM BEAM



$$\omega_1 = .56\pi\sqrt{\frac{EI}{(m + \frac{33}{140}\frac{g}{W})L^3}}$$

FIGURE 3.37 - CANTILEVER BEAM WITH MASS AT END



$$\omega_1 = 2.2\pi\sqrt{\frac{EI}{(m + \frac{17}{35}\frac{g}{W}L^3}}$$

FIGURE 3.38 - SIMPLY SUPPORTED BEAM WITH CENTRAL MASS

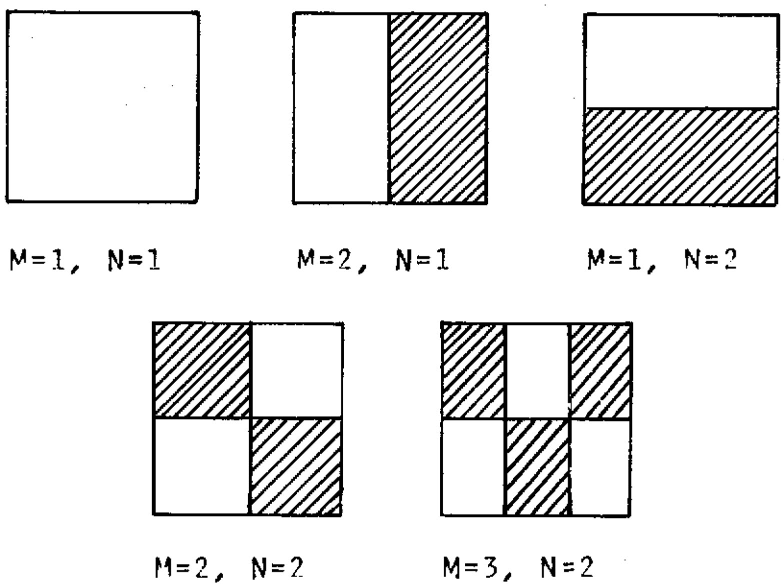


FIGURE 3.39 - NORMAL MODES FOR RECTANGULAR PANELS

3.13.4 Stress and Strain in Vibrating Plates

The determination of stress and strain in this section is based on a rectangular plate, perfectly elastic, homogeneous, isotropic with uniform thickness small with respect to its other dimensions. It is assumed that the deflections are small compared to the thickness with no stretching of the mid plane. The strain in a thin layer indicated by the shaded area in Figure 3.40 located a distance Z from the mid plane is given by equations 3.84 through 3.86.

$$e_{xx} = z/R_1 = -z\frac{\partial^2\delta}{\partial x^2} \qquad 3.84$$

$$e_{yy} = z/R_2 = -z\frac{\partial^2\delta}{\partial y^2} \qquad 3.85$$

$$\gamma_{xy} = -2z\frac{\partial^2 y}{\partial x\ \partial y} \qquad 3.86$$

e_{xx} and e_{yy} are unit deflections in the x and y directions, γ_{xy} is shear deformation in the xy plane and δ is the deflection of the plate. R_1 and R_2 are curvatures of the plate in the xz and yz plane.

Stress is obtained by

$$fx = \frac{E}{1-\mu^2}\ (e_{xx} + \mu e_{yy}) \qquad 3.87$$

$$fy = \frac{E}{1-\mu^2}\ (e_{yy} + \mu e_{xx}) \qquad 3.88$$

$$\tau = Ge_{xx} \qquad 3.89$$

where μ = Poisson's ratio

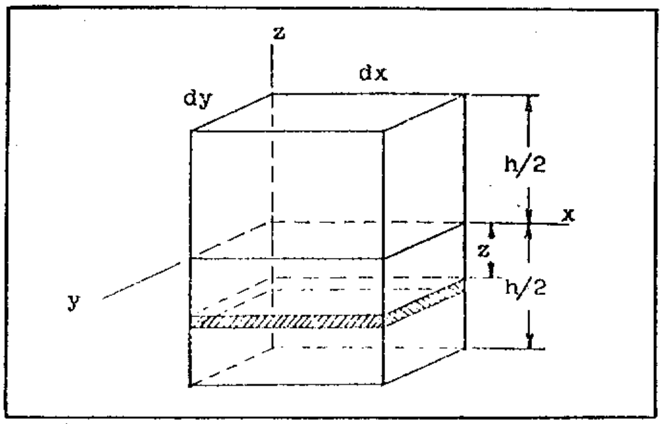


FIGURE 3.40 - ELEMENT FOR STRAIN DETERMINATION

3.14 BELL PROCESS STANDARDS

The list of subjects covered by Bell Process Standards (BPS) is given in Table 3.16. The BPS should be consulted because most likely procedures have already been established for a particular aspect of structures design.

A	
Acrylic Lacquer, Application of	FW 4386
Acrylic Plastic: Working and Maintenance of	FW 4302
Adhesion Promoter, Application of	FW 4398
Adhesive Bonding (Nonstructural) of Silicones	FW 4392
Adhesive Bonding (Nonstructural) with Film Type Cloth Supported Epoxy Adhesive	FW 4429
Adhesive Bonding (Nonstructural) with Rubber Phenolic Adhesives	FW 4401
Adhesive Bonding of Nameplates	FW 4335
Adhesive Bonding of Polycarbonates	FW 4434
Adhesive Bonding (Structural) with Elastomer Modified Epoxy	FW 4415
Adhesive Bonding (Structural) with Epoxy Resin-Based Adhesives	FW 4402
Adhesive Bonding (Structural) with Film Type Modified Epoxy Adhesive	FW 4408
Adhesive Bonding (Structural) Using Intermediate Temperature Curing Modified Epoxy Film	FW 4423
Adhesive Bonding (Structural) Using Intermediate - High Performance Supported and Unsupported Adhesive Film	FW 4458
Adhesive Bonding (Structural) with Rubber Phenolic Adhesive	FW 4400
Adhesive Bonding (Structural) with Vinyl Phenolic Base Adhesives	FW 4328
Adhesive Bonding Using Epoxy Based Adhesives	FW 4403
Adhesive System (Structural) for Honeycomb Sandwich Construction	FW 4449
Alcoholic Phosphoric Treatment	FW 4300
Aluminum Foil Identification Plates, Application of	FW 4168
Anaerobic Sealants	FW 4421
Anaerobic Sealants, for Bearing Retention	FW 4426
Anodizing, Chromic Acid	FW 4001
Anodizing, Hard	FW 4387
Anti-Fretting Treatments for Titanium	FW 4456
Anti-Seize Compounds, Use of	FW 4396
Anti-Static Coating, Epoxy, Application of	FW 4413
Application of Adhesion Promoter	FW 4398
Application of Flame-Resistant Silicone	FW 4447
Application of Powdered Coatings	FW 4465
Application of Suede Coatings	FW 4479
Application of Urethane Coating	FW 4464
B	
Barrel Finishing of Metals	FW 4326
Black Oxide Treatment of Steels	FW 4084
Blind High Strength Steel Fasteners, Pull Type, Installation of	FW 4472
Bonding Clutch Linings	FW 4121
Bonding, Composite	FW 4446
Bonding, Nonstructural	FW 4171
Bonding Structural, with High Temperature Resistant Epoxy Phenolics	FW 4448
Brazing, Silver	FW 4098
Brush Cadmium Plating	FW 4312
C	
Cable, Control, Fabrication and Testing	FW 4108
Cables and Terminals, Electrical, Preparation and Installation of	FW 4332
Cadmium Coating (Vacuum Deposited)	FW 4436
Cadmium, Fluoborate, Plating, High Strength Steels	FW 4466
Cadmium Plating, Brush	FW 4312
Cadmium Plating (Electrodeposited)	FW 4006
Carburizing and Heat Treatment of Carburized Parts	FW 4420
Casting Impregnation, Process for	FW 4432
Castings, Aircraft	FW 4163
Castings, In-Process Welding of	FW 4470
Chemical Cleaning of Aircraft Materials	FW 4138
Chemical Film Treatment	FW 4182
Chemical Machining of Metals	FW 4389
Cleaning and Preparation of Materials for Resistance Welding	FW 4113
Cleaning, Mechanical, of Metals	FW 4343
Clutch Linings, Bonding of	FW 4121
Coating, Urethane, Application of	FW 4464
Coating, Walkway, Nonslip	FW 4306
Coatings, Powdered, Application of	FW 4465
Coatings, Suede, Application of	FW 4479
Coatings, Tungsten Carbide, Deposition of	FW 4463
Color Identification of Rivets, Bolts, Nuts and Washers	FW 4064
Composite Bonding	FW 4446
Compounds, Corrosion Preventive, for Aircraft Assemblies	FW 4362
Compression Molding of Plastic Parts	FW 4469
Copper Plating	FW 4110
Corrosion and Abrasion Resistant Coatings (2216 Mix)	FW 4435
Corrosion Preventive Compounds, for Aircraft Assemblies	FW 4362
Countersunk or Dimpled Screws, Installation and Inspection of	FW 4039
Covering for Model 47 Fuel Tanks	FW 4433
Cyclewelding Metal to Wood	4028
D	
Decal, Application of	FW 4158
Dimpling, Hot Coin, of Aluminum, Magnesium, Stainless Steel, and Titanium	FW 4135

TABLE 3.14 - BELL PROCESS STANDARDS

Process	Standard
Dopes to Fabric Surfaces, Application of	4024
Dual Purpose Laminating Glass Cloth	FW 4437
Dynatube Fluid System Fittings, Installation and Assembly of	FW 4473
Dzus Fasteners, Installation of	FW 4365
E	
Edgings, Thermoplastic, Bonding to Acrylic Assemblies	FW 4351
Elastomer Modified Epoxy, Bonding with	FW 4415
Electrical Wiring; Installation of	FW 4332
Electrical Connectors, Potting of	FW 4311
Electrical Connections, Soldering of	FW 4439
Embrittlement and Stress Relief for Plated and Pickled Parts	FW 4044
Epoxy Based Adhesives, Adhesive Bonding with	FW 4403
Epoxy Primer Surfacer, Application of	FW 4471
Epoxy Resin Based Adhesives, Adhesive Bonding with	FW 4402
External Roll Threading of Metals	FW 4445
F	
Fabrication of Heat Resistant Structural Components of Glass Fabric Reinforced Plastic Materials	FW 4366
Fabrication of Structural Components of Glass Fabric Reinforced Polyester Materials	FW 4364
Fairing Compound, Application of	FW 4380
Fasteners, Dzus, Installation of	FW 4365
Fasteners, Installation of Blind, High Strength Steel, Pull Type	FW 4472
Film Type Modified Epoxy Adhesive, Structural Adhesive Bonding with	FW 4408
Fittings, Dynatube Fluid System, Installation and Assembly of	FW 4473
Fittings, Permaswage, Installation of	FW 4474
Fittings, Rosan Fluid System, Installation of	FW 4468
Fluoborate Cadmium Plating, High Strength Steels	FW 4466
Fluorescent Penetrant Inspection	FW 4089
Forgings and Wrought Stock, Ultrasonic Inspection of	FW 4399
Forgings for Aircraft Application	FW 4017
G	
Glass Fabric, Preimpregnated, Fabrication of	FW 4366
Glass Fabric Reinforced Polyester Materials, Fabrication of	FW 4364
Glycol, Polyethylene, Use of	FW 4444
H	
Hard Anodizing of Aluminum Alloys	FW 4387
Hardness Testing and Requirements of Metals	FW 4467
Heat Treating (Annealing) of Titanium	4212
Heat Treatment of Carburized Parts	FW 442[illegible]
Heat Treatment of Steel	FW 4140
High Visibility Paint System, Application of	FW 4397
Honeycomb Core Material; Handling, Cutting, Forming, and Adhesive Priming of	FW 4339
Honeycomb Cores, Splicing of	FW 4425
Hydraulic Fluids, Handling of	FW 4305
I	
Impregnation of Castings	FW 4432
Inert Gas Tungsten Arc Welding of Aluminum Alloy	FW 4404
Inserts, Installation of	FW 442[illegible]
Internal Threading of Swaged Aluminum Tubing	FW 441[illegible]
L	
Lacquers, Acrylic; Application of	FW 4386
Lacquers, Cellulose Nitrate, Application of	FW 4441
Linseed Oil Treatment of Closed Steel Tubular and Hollow Parts	FW 4317
Low Density Insert Material in Conjunction with Honeycomb Sandwich Construction, Application of	FW 4383
Lubricants, Solid Film	FW 4310
M	
Magnesium Alloy; Chemical Treatments for Corrosion Resistance of	FW 400[illegible]
Magnetic Particle Inspection	FW 407[illegible]
Marking Aircraft Parts	FW 405[illegible]
Materials for Adhesive Bonding, Surface Preparation of	FW 4352
Mechanical Cleaning of Metals	FW 4343
Metal-Cals, Application of	FW 4168
Metal to Wood, Cycleweldіng of	4028
N	
Nameplates - Adhesive Bonding of	FW 4335
Nital Etch Process	FW 4092
Nitriding of Steel	FW 4304
Nondestructive Testing of Bonded Components	FW 4424
Nonstructural Bonding, Using Rubber Base Cements	FW 4171
O	
Oil, Hot Linseed, Application of	FW 431[illegible]
P	
Packaging and/or Preservation of Aircraft Parts for Stock	FW 4102
Paint, Corrosion Protective Coating	FW 4428
Paint, Epoxy Enamel, Application of	FW 4427
Paint Finishing of Polycarbonates	FW 4438
Paint Stripping of Metal Parts and Assemblies	FW 4357
Paint System, High Visibility, Application of	FW 4397
Passivation	FW 4007

TABLE 3.14 (CONT'D) - BELL PROCESS STANDARDS

Penetrant Inspection	FW 4089
Permaswage Fittings, Installation of	FW 4474
Phosphatizing of Steels	FW 4384
Plastic, Acrylic; Working and Maintenance of	FW 4302
Plastic Parts, Compression Molding of	FW 4469
Plating, Cadmium, Electrodeposited	FW 4006
Plating, Cadmium, Vacuum Deposited	FW 4436
Plating, Copper	FW 4110
Plating, Fluoborate Cadmium, High Strength Steels	FW 4466
Plating, Selective Brush Cadmium	FW 4312
Plexiglas Panels, Fabrication of	FW 4351
Polycarbonates, Adhesive Bonding of	FW 4434
Polycarbonates, Paint Finishing of	FW 4438
Polyethylene Glycol, Use of	FW 4444
Polyurethane, Abrasion and Corrosion Resistant Coating	FW 4457
Polyurethane Enamel - Application of	FW 4442
Polyurethane Striping Material - Application of	FW 4452
Potting of Electrical Connectors	FW 4311
Powdered Coatings, Application of	FW 4465
Preparation and Application of Fuel Resistant Sealant Compounds	FW 4407
Primer, Epoxy, Application of	FW 4325
Primer, Epoxy Polyamide, Application of	FW 4451
Primer Surfacer, Epoxy, Application of	FW 4471
Primer, Zinc Chromate, Application of	FW 4367
Proof Testing of Lines and Tanks	4020
Protective Coating (Strippable)	FW 4381
Protective Compound (Strippable) Hot Dip	FW 4055
R	
Radiographic Inspection	FW 4309
Rivets, Aluminum and Aluminum Alloy, Processing, Use, and Storage of	FW 4316
Rivets, Blind (Hollow and Self-Plugging Type) Riveting Procedure for	FW 4036
Rivets, Chobert Blind, Riveting Procedure for	FW 4315
Rivets, Hi-Shear, Riveting Procedure for	4038
Rivets, Universal, Round Head, and 100 Degrees Flush, Riveting Procedure for	FW 4019
Roll Threading, External, of Metals	FW 4445
Rosan Fluid System Fittings, Installation of	FW 4468
Rotor Blades, Wood Control and Bonding Procedure for	4083
Rubber Phenolic Adhesives, Non-Structural Adhesive Bonding with	FW 4401
Rubber Phenolic Adhesives, Structural Adhesive Bonding with	FW 4400
S	
Safety Wiring Methods	FW 4043
Scotchcal Film, Application of	FW 4358
Screws, Dimpled or Countersunk, Installation and Inspection of	FW 4039
Sealing Compound, Fuel Resistant, Application of	FW 4407
Shot Peening of Steel, Aluminum, and Titanium	FW 4409
Shrink Fit Units, Method of Assy.	FW 4012
Silver Brazing	FW 4098
Soldering of Electrical Connections	FW 4439
Solid Film Lubricants	FW 4310
Spotwelding Aircraft Parts	FW 4115
Spray-Up Fabrication of Structural Components	FW 4440
Staking of Bearings	FW 4162
Strippable Protective Coating, Application of	FW 4381
Strippable Protective Compound, Hot Dip, Application of	FW 4055
Suede Coatings, Application of	FW 4479
Surface Preparation of Materials for Adhesive Bonding	FW 4352
Surfacer, Epoxy Primer, Application of	FW 4471
Swaged Aluminum Tubing, Internal Threading of	FW 4419
T	
Tanks, Fuel, Covering for	FW 4433
Testing, Conductivity, to Determine Heat Treat Condition of Wrought Aluminum Alloys	FW 4453
Threading, Roll, External, of Metals	FW 4445
Tightening Procedures for Threaded Fasteners and Fittings	FW 4018
Titanium, Fusion Welding of	4207
Titanium, Heat Treatment (Annealing) of	4212
Titanium Sheet, Forming of	FW 4462
Tubing Assemblies; Metal, Fabrication and Installation of	FW 4149
Tungsten Carbide Coatings, Deposition of	FW 4463
U	
Ultrasonic Inspection of Forgings and Wrought Stock	FW 4399
Ultrasonic Inspection of Rotor Blades	FW 4424
Urethane Coating, Application of	FW 4464
V	
Vacuum Cadmium Coating	FW 4436
Vibratory and Barrel Finishing of Metals	FW 4326
Vinyl Phenolic Base Adhesive, Structural Adhesive Bonding with	FW 4328
W	
Walkway Coating, Nonslip	FW 4306
Waterproofing Electrical Connectors Using Flexible Compound	FW 4311
Welding, Electron Beam	FW 4455

TABLE 3.14 (CONT'D) - BELL PROCESS STANDARDS

TABLE 3.14 (CONT'D) - BELL PROCESS STANDARDS

SECTION 4

INTERACTION

4.0 GENERAL

When a structural element is subjected to combined loadings, such as tension, compression and shear, it is often necessary to detemine the resultant maximum stresses and their respective principal axes. When a body is subjected to a combination of tensile and compressive stresses, it is usually designated as a biaxial or triaxial stress condition. When a combination of tensile, compressive and shear stresses are present, the body is usually referred to be in a combined stress state. The material at the inner surface of a thick pressure vessel is subjected to triaxial stress (radial compression, longitudinal tension and circumferential tension); a shaft bent and twisted is subjected to combined stresses (longitudinal tension or compression and torsional shear).

4.1 Material Failures

Fracture of a material is very complex. Generally, failures can be grouped into two categories, ductile or brittle, depending upon the state of stress and the environment. Metals with high strength exhibit low ductility prior to failure. Failure can occur after elongation of the metal over a relatively large uniform length or after a concentrated elongation in a short length. Shear deformation will also vary depending on the metal and the stress state. Because of these variations in magnitude and mode of deformation, the ductility of a metal can have a profound effect on the ability of a part to withstand applied loads.

Brittle failures are characteristic of the high-strength materials and are accompanied by little or no plasticity. The lack of plastic strain generally results in the brittle failure occurring without warning, and in service can lead to catastrophic results. The fracture surface is usually distinguished by a cleavage or rough crystalline texture which appears bright and granular. The fracture surface of steels will have a herringbone appearance with the chevrons pointing to the origin of the fracture. A micro analysis of the fracture surface shows a direct separation of the crystalline planes with the plane of separation normal to the applied load.

Ductile failures will show substantial amounts of plastic strain. The load-deflection curve will show the failure occurring well out of the linear region. The fracture surface is generally distinguished by a "hilly" appearance with some reduction in cross section. A micro analysis will show that the fracture is a result of slippage between crystalline planes with the plane of failure oblique to the applied load. The ductile failure is, thus, an action of shear stresses. There are ductile materials whose fractures appear brittle and brittle materials demonstrate ductile failures. Fracture surfaces often have the appearance of both brittle and ductile.

There are no precise equations to define the mechanism of fracture. High or low temperatures affect the mechanism of failure. The strain history and rate of load application will affect the fracture.

4.2 Theories of Failure

When one principal stress exists at a point, the stress is frequently referred to as a uniaxial or one-dimensional stress; when two principal stresses occur, the state of stress is frequently called biaxial stress, or two-dimensional stress, or plane stress, and when all three principal stresses exist, the state of stress is triaxial or three-dimensional stress.

It is not always possible to produce in a test the exact biaxial or triaxial stress which exists during operation. Sometimes it is not even possible to test at all. In such eventsrational analysis procedures must be used. Such procedures are called theories of failure and require that the general mode of failure of the member under the assumed service conditions be determined or assumed (failure is usually yielding or fracture) and that a quantity (stress, strain, energy) be chosen which is associated with the failure. This means there is a maximum or critical value of the quantity selected which limits the loads that can be applied to the member. Generally, an ultimate load test of the material with the resulting stress-strain curve is a suitable test for determining the quantities associated with theories of failure.

The six main theories of failure for a material that is considered to fail by yielding under static loading are:

(1) Maximum principal stress theory - Rankines theory - Inelastic action at any point in a material at which any state of stress exists begins only when the maximum principal stress at the point reaches a value equal to the tensile (or compressive) elastic limit or yield strength of the material, regardless of the normal or shearing stresses that occur on other planes through the point.

(2) Maximum shearing stress theory - Coulomb's or Guest's law - Inelastic action at any point in a body at which any state of stress exists begins only when the maximum shearing stress on some plane through the point reaches a value equal to the maximum shearing stress in a tension specimen when yielding starts.

(3) Maximum strain theory - St. Venants theory - Inelastic action at a point in a body at which any state of stress exists begins only when the maximum strain at the point reaches a value equal to that which occurs when inelastic action begins in the material under a uniaxial state of stress as occurs in a specimen in a tension test.

(4) Total energy theory - Beltrami and Haigh theory - Inelastic action at any point in a body due to any state of stress begins only when the energy per unit volume absorbed at the point is equal to the energy absorbed by the material when subjected to the elastic limit under a uniaxial state of stress as occurs in a simple tensile test.

(5) Energy of distortion theory - Inelastic action at any point in a body under any combination of stresses begins only when the strain energy of distortion per unit volume absorbed at the point is equal to the strain energy of distortion absorbed per unit volume at any point in a bar stressed to the elastic limit under a state of uniaxial stress as occurs in a simple tension or compression test.

(6) Octahedral shearing stress theory - Inelastic action at any point in a body under any combination of stresses begins only when the octahedral shearing stress becomes equal to 0.47 times the tensile elastic strength of the material as determined from the standard tension test.

4.3 Determination of Principal Stresses

When an element is subjected to combined stresses such as tension, compression and shear, it is often necessary to determine resultant maximum stress values and their respective principal axes. These stresses and their angles may be obtained by the use of Mohr's circle. This is a convenient graphical representation of the relation between principal stresses at a point and the shearing and normal stresses at the same point on planes inclined to the planes of principal stresses.

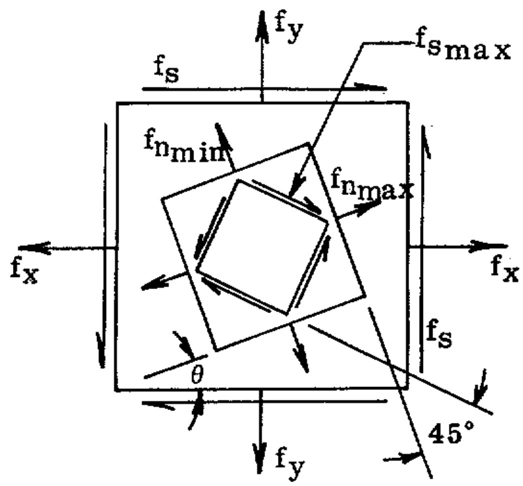


FIGURE 4.1 STATE OF STRESS AT A POINT

The following are definitions and sign conventions for terms in Figure 4.1:

- f_x, f_y: applied normal stresses
- f_s: applied shear stress
- $f_{n_{max}}$, $f_{n_{min}}$: resulting principal normal stresses
- $f_{s_{max}}$: resulting principal shear stress
- θ: angle of principal axes
- Sign Convention:
  - Tensile stress is positive (+)
  - Compressive stress is negative (-)
  - Shear stress is positive (+) if its action is a tendency to rotate the element clockwise
  - Positive θ is counterclockwise

The procedure for constructing Mohr's circle is as follows:

(1) Make a sketch of an element for which the normal and shearing stresses are known and indicate on it the proper senses of the stresses. Such a sketch is shown in Figure 4.2.

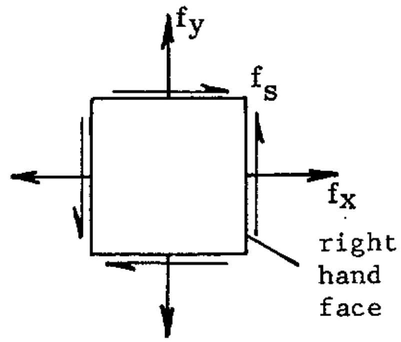


FIGURE 4.2 TYPICAL ELEMENT IN BIAXIAL STRESS

(2) Set up a rectangular coordinate system of axes where the horizontal axis is the normal stress axis and the vertical axis is the shearing stress axis. Directions of positive axes are taken as upward and to the right.

(3) Locate the center of the circle, which is on the horizontal axis at a distance $(f_x + f_y)/2$ from the origin. Tensile stresses are positive and compression stresses are negative. See Figure 4.3.

(4) From the right-hand face of the element prepared in step (1) read the values for f_x and f_s and plot as point "A". The coordinate distances to this point are measured from the origin. The sign of f_x is positive if tensile, negative if compression; f_s is positive clockwise.

(5) Draw a circle with center as found in step (3) through point "A" found in step (4). The two points of intersection of the circle with the normal stress axis (fn) give the magnitudes and sign of the two principal stresses. If the intercept is positive, the principal stress is tensile and vice versa.

(6) The circle will also pass through point "B" which has coordinates of f_y and f_s taken from the upper surface of the element in step (1).

(7) Construct a line through the center of the circle connecting points A and B. The angle formed by this line and the normal stress (f_n) axis (abscissa) is 2θ, twice the angle of the principal axes.

(8) Construct the biaxial state of stress at the point in question as shown in Figure 4.4.

The following equations can now be written using Figures 4.3 and 4.4.

$$f_{n_{max}} = \frac{f_x + f_y}{2} + \sqrt{\left(\frac{f_x - f_y}{2}\right)^2 + f_s^2} \qquad 4.1$$

$$f_{n_{min}} = \frac{f_x + f_y}{2} - \sqrt{\left(\frac{f_x - f_y}{2}\right)^2 + f_s^2} \qquad 4.2$$

$$\tan 2\theta = \frac{2f_s}{f_x - f_y}: \quad \left[\begin{array}{l}\text{The solution results in} \\ \text{two angles representing the} \\ \text{principal axes of } f_{max} \,\&\, f_{min}\end{array}\right] \qquad 4.3$$

$$f_{s_{max}} = \sqrt{\left(\frac{f_x - f_y}{2}\right)^2 + f_s^{\,2}} \qquad 4.4$$

The angles located on Mohr's circle are twice the angles on the biaxial stress element. The maximum principal stress ($f_{n_{max}}$) occurs at point I in Figure 4.3. Point I is located at a 2θ counterclockwise rotation (about point G) from applied stress point A. In the element, Figure 4.4, face I is located on angle θ counterclockwise (about point O) rotation from face A. The minimum principal stress ($f_{n_{min}}$) occurs at point H in Figure 4.3. It is located 2θ counterclockwise from point B and 180° counterclockwise from point I. In the element, face H is 90° from face I or θ counterclockwise from face B.

The maximum positive and negative shear stresses ($f_{s_{max}}$) occur at points J and K in Figure 4.3 and their magnitudes are equal to the radius of the circle. Point J is the maximum positive shear stress and is located (90° - 2θ) clockwise from point B or 90° clockwise from point H. Face J, in the element, Figure 4.4, is located (45° - θ) clockwise from face B or 45° clockwise from face H. The shear stress on face J is positive which is shown producing a clockwise rotation of the element. Point K is the maximum negative shear stress and is located in the same manner as point J. The planes of maximum shearing stress are always at 45° to the principal planes, regardless of the applied stress conditions.

Figure 4.5 shows some typical loading conditions with the resulting Mohr's circle and the state of stress on an element in the body.

4.4 Interaction of Stresses

The means of predicting structural failure under combined loading without determining principal stresses is known as the interaction method. The critical strength of structural members with a single type of load is generally defined. That is, the yield, ultimate or buckling of a member load in tension, compression, shear or bending can be determined. The critical strength of a member subjected to simultaneously applied combinations of loads is often difficult to determine. This is especially true if local or overall stability, plastic bending or torsion are involved. The interaction method was developed for predicting ultimate strength of members subjected to combined loads. It is the most satisfactory method of predicting structural failure without determining principal stresses.

The basis for the interaction method is:

(1) The allowable strength for each simple loading condition (tension, shear, bending, buckling, etc.) is determined by test or theory.

(2) Each load of the combined load conditions is represented by a ratio (R) of applied load or stress to allowable load or stress.

(3) The interaction relationship is the effect of one condition on another (or others) and is determined by theory, test or both.

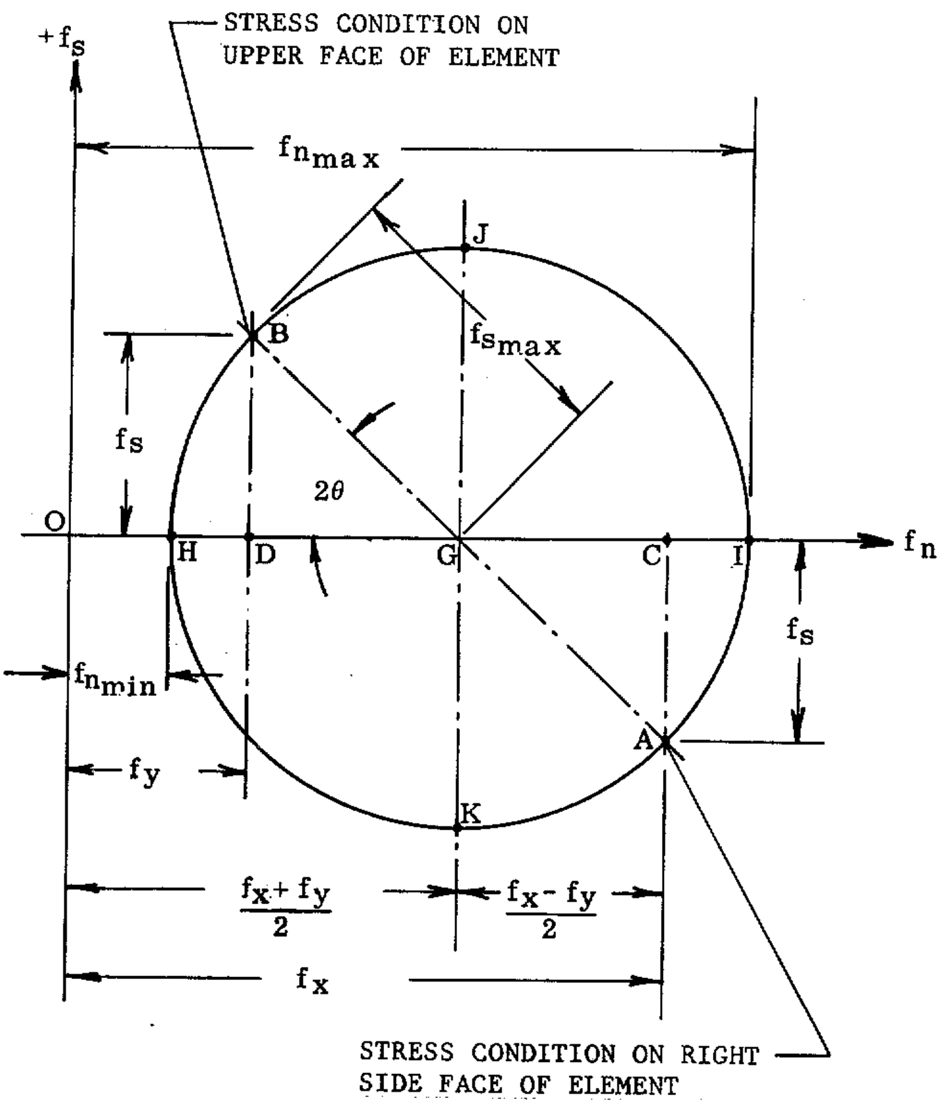


FIGURE 4.3 MOHR'S CIRCLE REPRESENTATION OF BIAXIAL STRESS AT POINT "Q" IN FIGURE 4.4

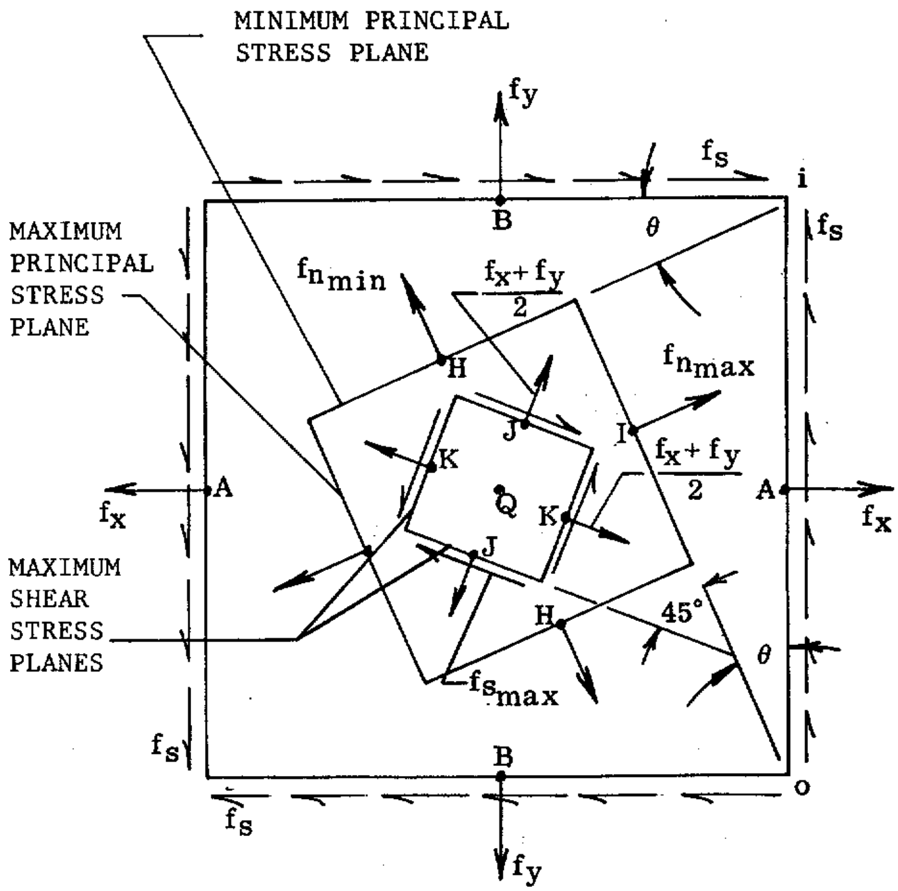


FIGURE 4.4 BIAXIAL STATE OF STRESS AT POINT "Q"

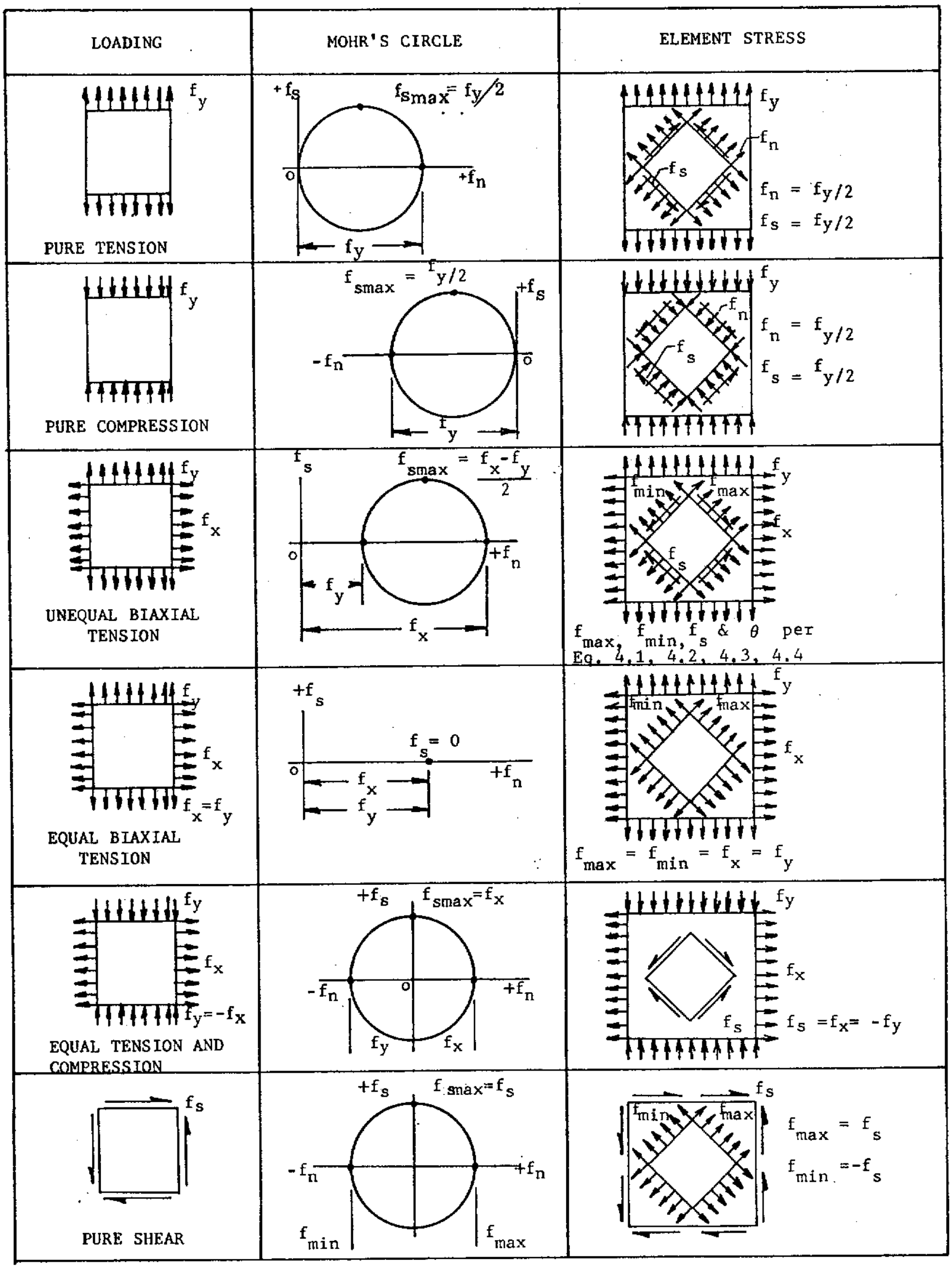


FIGURE 4.5 - ELEMENT STRESSES

The stress ratio, R, is expressed as:

$$R = \frac{\text{applied load or stress}}{\text{allowable load or stress}} \qquad 4.5$$

The margin of safety is then

$$MS = 1/R - 1 \qquad 4.6$$

Generally, for a combined system of loadings the interaction relationship is expressed as

$$R_1^{\,x} + R_2^{\,y} + R_3^{\,z} + \cdots = 1 \qquad 4.7$$

where R_1, R_2 and R_3 are stress ratios and x, y and z are exponents defining interaction relationships.

When only two loading conditions exist, such as bending and torsion, equation 4.7 can be plotted as a single interaction curve of R_b and R_s. When three or more loading conditions exist, the equation of interaction becomes a surface and can be plotted as a family of curves. When the exponents are equal, the interaction curve is a straight line and indicates the maximum interaction. This might occur when bending is present with tension or compression. Making one exponent equal 2 gives a parabola. With both exponents equal 2 the interaction curve is a circle. Complete independence or zero interaction is obtained when the exponents are infinite.

The amount of interaction between two loads is determined by theory or test. The analyst must use good engineering judgment and common sense to determine the relationship of one load to the other. For instance, if torsion and bending are present and torsion is the predominant stress, then the interaction equation using the maximum shear stress theory should be used. If bending had been the dominant stress, then the interaction equation based on maximum principal stress should be used. The end points of the interaction curves are always correct; at least they represent failure under simple loading. This reduces the probable error when one type of loading dominates. The prime advantage of this method is that it yields good results when any one loading condition dominates and exact results when only one loading condition is present.

The effect of one loading R_1 on another simultaneous loading R_2 is represented by an equation or interaction curve like that shown in Figure 4.6. This curve represents all the possible combinations of R_1 and R_2 that will cause failure. The curve is used as follows:

(1) Let the value of R_1 and R_2 locate point "a". A positive margin of safety is indicated because it is inside the curve.

(2) Failure can occur at three points

a. Point "d" by a proportionate increase in R_1 and R_2
b. Point "h" by an increase in R_1 with R_2 constant
c. Point "g" by an increase in R_2 with R_1 constant

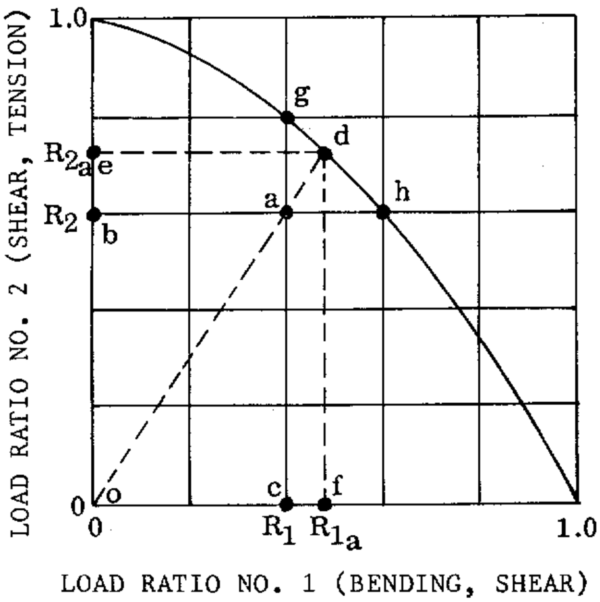


FIGURE 4.6 TYPICAL TWO-LOADS-ACTING INTERACTION

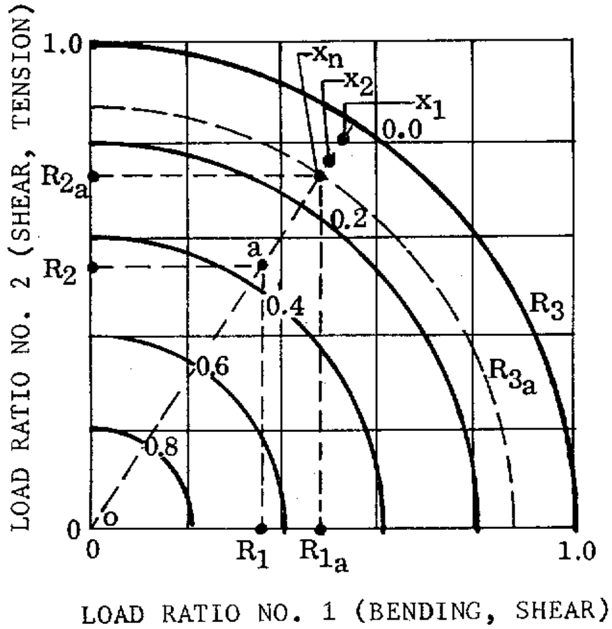


FIGURE 4.7 TYPICAL THREE-LOADS-ACTING INTERACTION

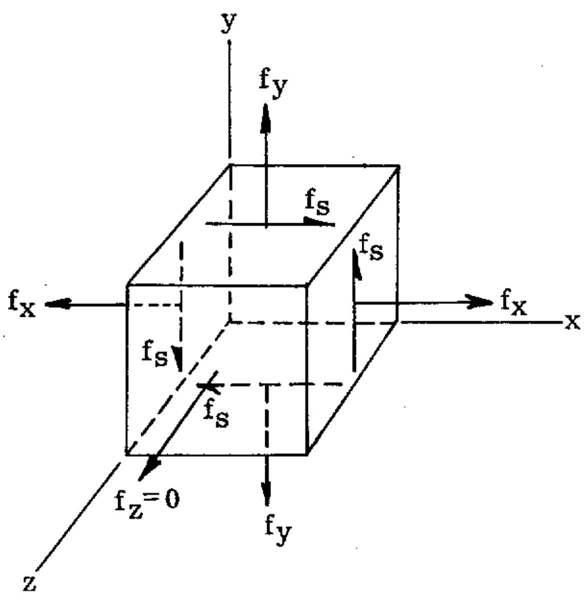


FIGURE 4.8 BIAXIAL STRESS CONDITION

(3) The margin of safety for the loading represented by point "a" can be found in three ways

a. MS = od/oa - 1
b. MS = bh/ba - 1
c. MS = cg/ca - 1

Values of od, bh and cg are referred to as allowables (load or stress) and oa, ba and ca are applied load or stress. Using this procedure and equation 4.7 procedures for two loads acting and three loads acting can be determined.

4.4.1 Procedure for Margin of Safety for Two Loads Acting

(1) Using buckling, yield or ultimate criteria and equation 4.5, calculate the stress ratio for each load acting alone.

(2) Using the calculated stress ratios locate point "a" on the proper interaction curve (using Figure 4.6 as an example).

(3) Draw a straight line from the origin "o" through point "a" and intersect the interaction curve at point "d". Read the stress ratios R_{1a}(ed) and R_{2a}(fd).

(4) Compute the applied stress ratios R_1(ba) and R_2(ca).

(5) Compute the margin of safety

$$MS = R_{1a}/R_1 - 1 = R_{2a}/R_2 - 1 \qquad 4.8$$

4.4.2 Procedure for Margin of Safety for Three Loads Acting

(1) Using buckling, yield or ultimate criteria and equation 4.5, calculate the stress ratios for each load acting above.

(2) Using the appropriate interaction family of curves locate point "a" corresponding to the calculated stress ratios R_1 and R_2 as shown in Figure 4.7.

(3) Draw a straight line from the origin "o" through point "a".

(4) Extend this line to locate the allowable point "x" which must satisfy the following relationships:

$$R_1/R_{1a} = R_2/R_{2a} = R_3/R_{3a} \qquad 4.9$$

or

$$R_{3a} = \left(R_3/R_1\right) R_{1a} \qquad 4.10$$

Point "x" is obtained by trial and error in the following manner:

(a) Select an arbitrary value of R_{1a}.

(b) Calculate R_{3a} from equation 4.10 using the known value of R_1 and R_3 and the arbitrary value of R_{1a}.

(c) Locate point "x" on the line "oa" using the calculated R_{3a} from step (b) and compare the corresponding R_{1a} with the assumed R_{1a}.

(d) Repeat steps (a) through (c) until the assumed R_{1a} and the "x" value of R_{1a} converge. At convergence, R_{1a}, R_{2a} and R_{3a} will be at a common point on line "oa".

(5) Compute the margin of safety

$$MS = R_{1a}/R_1 - 1 = R_{2a}/R_2 - 1 = R_{3a}/R_3 - 1 \qquad 4.11$$

4.5 Compact Structures

A compact structure is one in which failure does not occur by crippling or buckling. This section presents interaction criteria for compact structures with biaxial stress in a rectangular volume such as in plates, membranes and shells and with uniaxial stress in a plane such as in beams, round bars and bolts.

4.5.1 Biaxial Stress Interaction Relationships

Tests have been conducted to determine the failure theories of biaxially loaded isotropic ductile materials. The maximum shear stress theory and the octahedral shear stress theory adequately predict the yield and ultimate strengths. There are a few cases where convenient margin of safety calculations are possible. These are shown in Table 4.3. A general interaction method is required. It is shown in Figure 4.8. The method is applicable to stress conditions which combine in a two-dimensional manner like that shown in Figure 4.8. This condition exists in a rectangular volume and not on a single plane. Tension is positive, compression is negative. The interaction equations and curves are applicable for ultimate and yield by use of the parameters given in Table 4.1.

The interaction equations contain certain factors which relate one stress to the other. They are defined as follows:

The constant relating interaction in terms of tension or shear strength allowables:

$$K = F_{su}/F_{tu} \qquad 4.12$$

Tests show this value to vary from 0.5 to 0.75.

The transverse shear and torsional stress ratios combine as

$$R_s = R_{ss} + R_{st} \qquad 4.13$$

The directional tension and bending stress ratios combine as

$$R_x = R_{tx} + R_{bx} \qquad 4.14$$

$$R_y = R_{ty} + R_{by} \qquad 4.15$$

The directional compression and bending stress ratios combine as

$$R_x = R_{cx} + R_{bx} \qquad 4.16$$

$$R_y = R_{cy} + R_{by} \qquad 4.17$$

Using equations 4.1 and 4.2 and substituting the previous relationships the following is derived.

$$R_{n\,max} = \frac{R_x + R_y}{2} + \sqrt{\left(\frac{R_x - R_y}{2}\right)^2 + K^2 R_s^{\,2}} \qquad 4.18$$

$$R_{n\,min} = \frac{R_x + R_y}{2} - \sqrt{\left(\frac{R_x - R_y}{2}\right)^2 + K^2 R_s^{\,2}} \qquad 4.19$$

4.5.1.1 Maximum Shear Stress Theory Interaction Equations

The maximum shear stress theory states that yielding or fracture occurs when the maximum shear stress in a combined stress element equals the maximum shear stress in a pure tension test specimen subjected to the yielding or ultimate stress of the material. The substitution of $f_x = F_{tu}$, $f_y = o$ and $f_s = o$ into equation 4.4 results in the maximum shear stress in a pure tension specimen at fracture equal to 1/2 F_{tu}. Dividing equation 4.4 by F_{tu}, substituting equations 4.12 and 4.13 and parameters contained in Table 4.1 and setting $f_{s\,max} = 1/2\ F_{tu}$ results in

$$R_x^{\,2} + R_y^{\,2} - 2R_xR_y + 4K^2R_s^{\,2} = 1 \qquad 4.20$$

Using K = 1/2, which is the theoretical value of K determined for a specimen loaded in tension, the equation 4.20 yields

$$R_x^{\,2} + R_y^{\,2} - 2R_xR_y + R_s^{\,2} = 1 \qquad 4.21$$

This equation is plotted as the dashed lines in Figure 4.9. The maximum shear stress is obtained in terms of the three principal triaxial stresses (f_1, f_2 and f_3) as:

$$f_{s\,max} = \pm\left(\frac{f_1 - f_2}{2}\right) \qquad 4.22$$

$$f_{s\,max} = \pm\left(\frac{f_2 - f_3}{2}\right) \qquad 4.23$$

$$f_{s\,max} = \pm\left(\frac{f_1 - f_3}{2}\right) \qquad 4.24$$

Using Figure 4.9 and testing equations 4.22, 4.23 and 4.24 in each quadrant for the biaxial stress state ($f_3 = o$) results in the maximum shear stress theory interactions. These are shown in Table 4.2.

4.5.1.2 Octahedral Shear Stress Theory Interaction Equations

The octahedral shear stress theory states that yielding or fracture occurs when the octahedral shear stress in a combined stress element equals the octahedral shear stress in a pure tension test specimen subjected to the yielding or fracture stress

TABLE 4.1 STRESS RATIO PARAMETERS

STRESS TYPE	STRESS RATIO PARAMETER	
	YIELD CONDITION	ULTIMATE CONDITION
Tension "x" Direction	$R_{tx}=f_{tx}/F_{ty}$	$R_{tx}=f_{tx}/F_{tu}$
Tension "y" Direction	$R_{ty}=f_{ty}/F_{ty}$	$R_{ty}=f_{ty}/F_{tu}$
Compression "x" Direction	$R_{cx}=f_{cx}/F_{cy}$	$R_{cx}=f_{cx}/F_{cu}$ (1)
Compression "y" Direction	$R_{cy}=f_{cy}/F_{cy}$	$R_{cy}=f_{cy}/F_{cu}$ (1)
Bending "x" Direction	$R_{bx}=f_{bx}/F_{by}$	$R_{bx}=f_{bx}/F_{bu}$
Bending "y" Direction	$R_{by}=f_{by}/F_{by}$	$R_{by}=f_{by}/F_{bu}$
Transverse Shear	$R_{ss}=f_{ss}/F_{sy}$ (2)	$R_{ss}=f_{ss}/F_{su}$
Torsional Shear	$R_{st}=f_{st}/F_{sty}$ (2)	$R_{st}=f_{st}/F_{stu}$

Subscripts:

t = tension
c = compression
b = bending
ss = transverse shear
st = torsional shear
s = shear

Subscripts x and y on R and f refer to x and y directions, respectively.
Subscripts y and u on F refer to yield and ultimate strength conditions, respectively.

Notes: (1) Assume $F_{cu} = F_{tu}$

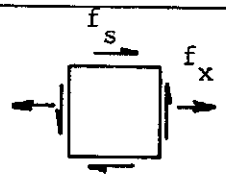
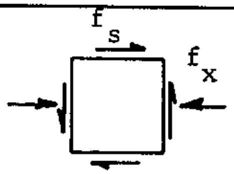
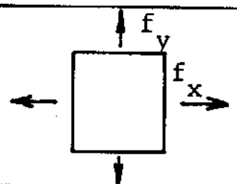
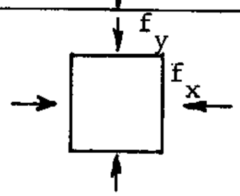
(2) Assume $F_{sy} = F_{sty} = KF_{ty}$

K = 0.5-0.75 for most isotropic ductile materials

TABLE 4.2. MAXIMUM SHEAR STRESS THEORY INTERACTION EQUATIONS

QUADRANT (Figure 4.9)	CONDITIONS	$f_{s_{max}}$ EQUATION	INTERACTION EQUATION
I	$f_{n_{max}} = +$, $f_{n_{min}} = +$ $f_{n_{max}} > f_{n_{min}}$	$+f_{n_{max}}/2$	$R_{n_{max}} = 1$
I	$f_{n_{max}} = +$, $f_{n_{min}} = +$ $f_{n_{min}} > f_{n_{max}}$	$+f_{n_{min}}/2$	$R_{n_{max}} = 1$
II	$f_{n_{max}} = -$, $f_{n_{min}} = +$	$\dfrac{-f_{n_{max}} - f_{n_{min}}}{2}$	$R_{n_{max}} - R_{n_{min}} = -1$
III	$f_{n_{max}} = -$, $f_{n_{min}} = -$ $\lvert f_{n_{max}}\rvert > \lvert f_{n_{min}}\rvert$	$-f_{n_{max}}/2$	$R_{n_{max}} = -1$
III	$f_{n_{max}} = -$, $f_{n_{min}} = -$ $\lvert f_{n_{min}}\rvert > \lvert f_{n_{max}}\rvert$	$-f_{n_{min}}/2$	$R_{n_{min}} = -1$
IV	$f_{n_{max}} = +$, $f_{n_{min}} = -$	$\dfrac{+f_{n_{max}} - f_{n_{min}}}{2}$	$R_{n_{max}} - R_{n_{min}} = -1$

TABLE 4.3 COMPACT STRUCTURES-BIAXIAL INTERACTION CRITERIA (NO CRIPPLING OR BUCKLING) - YIELD AND ULTIMATE CONDITIONS OF STRENGTH

CASE	LOADING PICTURE	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY EQUATION	REMARKS
1		Uniaxial Tension + Shear	$R_x^2 + R_s^2 = 1$	4.9 or 4.10	$\frac{1}{\sqrt{R_x^2 + R_s^2}} - 1$	
2		Uniaxial Compression + Shear	$R_x^2 + R_s^2 = 1$	4.9 or 4.10	$\frac{1}{\sqrt{R_x^2 + R_s^2}} - 1$	
3		Biaxial Tension		4.9		
4		Biaxial Compression		4.9		
5	All other states of biaxial stress			4.9		Refer to section 4.5.1 and use: (1) Table 4.1 (2) Equations 4.18 and 4.19 (3) Figure 4.9

NOTES:

(1) Tension is positive, compression is negative.

(2) $R_s = R_{ss} + R_{st}$

(3) $R_x = R_{tx} + R_{bx}$ (tension); $R_x = R_{cx} + R_{bx}$ (compression)

(4) $R_y = R_{ty} + R_{by}$ (tension); $R_y = R_{cy} + R_{by}$ (compression)

(5) $R_t = f_t/F_{ty}$, $R_c = f_c/F_{ty}$, $R_b = f_b/F_{by}$, $R_{ss} = f_s/F_{sy}$, $R_{st} = f_{st}/F_{sty}$ (YIELD)

(6) $R_t = f_t/F_{tu}$, $R_c = f_c/F_{tu}$, $R_b = f_b/F_{bu}$, $R_{ss} = f_s/F_{su}$, $R_{st} = f_{st}/F_{stu}$ (ULTIMATE)

of the material. By substituting $f_x = F_{tu}$, $f_y = o$, $f_s = o$ into the octahedral shear stress equation

$$f_{s_{oct}} = \sqrt{2}/3 \sqrt{f_x^2 + f_y^2 - f_x f_y + 3f_s^2}$$

the octahedral shear stress in a pure tension specimen at fracture equals $\sqrt{2}/3\ F_{tu}$. By proceeding as described previously the octahedral shear stress theory interaction equation becomes

$$R_x^2 + R_y^2 - R_x R_y + 3K^2 R_s^2 = 1 \qquad 4.25$$

Using $K = 1/\sqrt{3}$ equation 4.25 becomes

$$R_x^2 + R_y^2 - R_x R_y + R_s^2 = 1 \qquad 4.26$$

Equation 4.26 can be rearranged to form

$$(R_{n_{max}})^2 + (R_{n_{min}})^2 + (R_{n_{max}} - R_{n_{min}})^2 = 2 \qquad 4.27$$

This equation is plotted as the solid line in Figure 4.9.

4.5.1.3 Margin of Safety Determination

The procedure for determining the margin of safety where orientation of applied stress with respect to the grain direction is unknown is as follows:

(1) Assume the allowable stresses in each direction of applied stress are the same as in the weaker direction.

(2) Evaluate $K' = F_{su}/F_{tu}$.

(3) If $0.5 \leq K' \leq .577$, use $K = 0.5$ in calculating the principal stress ratios (equations 4.18 and 4.19) and use the maximum shear stress interaction curve, Figure 4.9.

(4) If $K' \geq 0.577$, use $K = 0.577$ in calculating the principal stress ratios (equations 4.18 and 4.19) and use the octahedral shear stress interaction curve, Figure 4.9.

(5) In evaluating R_x and R_y, the allowable stress is taken to be F_{ty} (yield) or F_{tu} (ultimate) regardless of whether the applied stress is tension or compression. Applied tension is positive, applied compression is negative.

(6) Calculate the margin of safety using the two-parameter procedure outlined in section 4.4.1.

The procedure for determining the margin of safety where orientation of applied stress with respect to the grain direction is known is as follows:

(1) Determine applied stresses on an element with sides parallel and perpendicular to the grain direction.

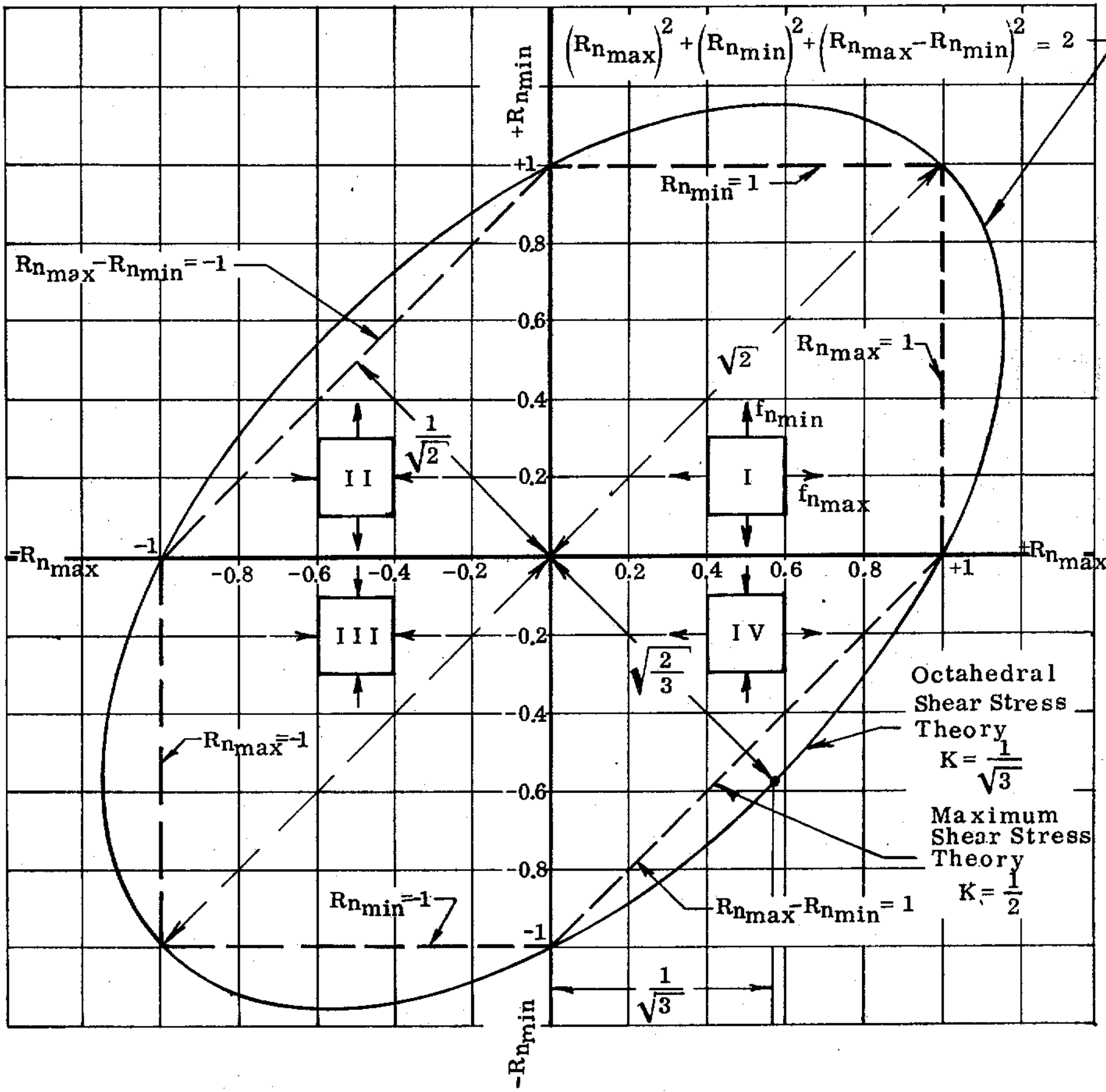


FIGURE 4.9 INTERACTION CURVES FOR BIAXIALLY STRESSED STRUCTURES

(2) Evaluate :

$$K' = \frac{F_{su}}{(F_{tu})_T} \sqrt{\frac{1 + \left[\frac{(F_{tu})_T}{(F_{tu})_L}\right]^2}{2}}$$

where: T and L refer to the transverse and longitudinal grain direction.

(3) If $0.5 \leq K' \leq .577$, use K = 0.5 and the appropriate transverse and longitudinal allowable stresses in calculating the principal stress ratios (equations 4.18 and 4.19) and use the maximum shear stress interaction curve, Figure 4.9.

(4) If $K' \geq 0.577$, use K = 0.577 and the appropriate transverse and longitudinal allowable stresses in calculating the principal stress ratios (equations 4.18 and 4.19) and use the octahedral shear stress interaction curve, Figure 4.9.

(5) In evaluating R_x and R_y the allowable axial stress is taken to be F_{ty} (yield) or F_{tu} (ultimate) regardless of whether the applied stress is tension or compression. Applied tension is positive, applied compression is negative.

(6) Calculate the margin of safety using the two-parameter interaction procedure outlined in section 4.4.1.

4.5.2 Uniaxial Stress Interaction Relationships

When compact structures such as beams are loaded by axial load, bending moment and shear, methods other than those presented in section 4.5.1 must be used. Such conditions where shear does not combine in a simple two-dimensional manner like that shown in Figure 4.8 and conditions where shear, tension and bending must be combined in the plastic region. Table 4.4 shows conservative interaction equations to be used when combining these stresses. It is sometimes convenient to combine the maximum bending stress and maximum shear stress even though these stresses do not occur at the same point. It is recommended that several points in the section be checked for their actual conditions of stress to reduce the conservatism.

4.5.3 Thick Walled Tubular Structures

The interaction stress ratios for the design and analysis of thick walled tubular structures must be determined from critical tube strength and stability criteria. Bending stress ratios must include the effects of secondary bending, if any, and compressive stress ratios must be based on column stability criteria. For cases involving combined tension and bending the exponents of 1.5 is conservative. Also for combined shear and bending the exponent of 2 is generally conservative. Table 4.5 shows the applicable interaction equations and margin of safety equations along with the applicable interaction curve.

4.5.4 Unstiffened Panels

The interaction stress ratios for flat rectangular and curved unstiffened panels are based on elastic initial buckling. If a panel is subjected to direct axial stress, tension is considered as negative compression using the critical compression allowable. Table 4.6 shows the interaction relations.

4.5.5 Unstiffened Cylindrical Shells

The shell structures for which interaction is shown must have a radius to thickness ratio greater than 10. Otherwise the interaction relationships given for tubes should be used. The interaction stress ratios must be based on initial buckling criteria. If direct axial stresses are present, tension is treated as negative compression using the compression buckling allowable. Table 4.7 shows the interaction relations.

4.5.6 Stiffened Structures

Table 4.8 shows combined loads interaction data for the design and analysis of stiffened panel and cylindrical shell structures. The interaction stress ratios must be based on stability criteria.

TABLE 4.4 COMPACT STRUCTURES-UNIAXIAL INTERACTION CRITERIA (NO CRIPPLING OR BUCKLING)-YIELD AND ULTIMATE CONDITIONS OF STRENGTH

Axial, bending (simple and complex), and shear (simple and complex) on beam cross section (symmetrical and unsymmetrical)

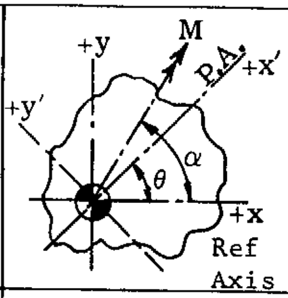
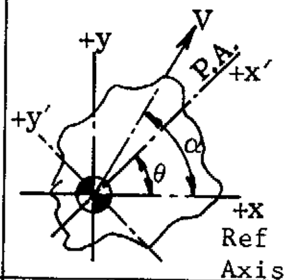
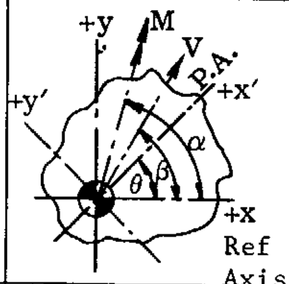
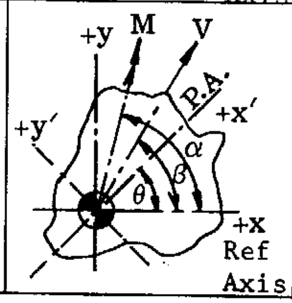
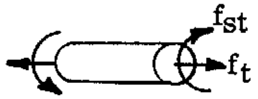
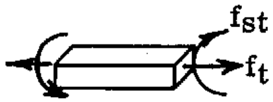
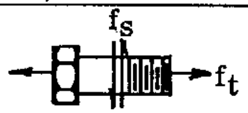
CASE	LOADING	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY EQUATION	REMARKS
1		Axial Tension or Compression + Bending	$R_a + R_b = 1$	4.10	$\frac{1}{R_a + R_b} - 1$	
2		Axial Tension or Compression + Shear	$R_a^2 + R_s^2 = 1$	4.10	$\frac{1}{\sqrt{R_a^2 + R_s^2}} - 1$	
3		Bending + Shear	$R_b^2 + R_s^2 = 1$	4.10	$\frac{1}{\sqrt{R_b^2 + R_s^2}} - 1$	
4		Axial Tension or Compression + Bending + Shear	$(R_a + R_b)^2 + R_s^2 = 1$	4.10	$\frac{1}{\sqrt{(R_a + R_b)^2 + R_s^2}} - 1$	

TABLE 4.4 COMPACT STRUCTURES-UNIAXIAL INTERACTION CRITERIA (NO CRIPPLING OR BUCKLING)-YIELD AND ULTIMATE CONDITIONS OR STRENGTH (CONCLUDED)

Axial, bending (simple and complex), and shear (simple and complex) on beam cross section (symmetrical and unsymmetrical)

CASE	LOADING	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY EQUATION	REMARKS
ROUND BARS						
5		Tension + Torsion	$R_t^2 + R_{st}^2 = 1$	4.10	$\frac{1}{\sqrt{R_t^2 + R_{st}^2}} - 1$	
RECTANGULAR BARS						
6		Tension + Torsion	$R_t^2 + R_{st}^2 = 1$	4.10	$\frac{1}{\sqrt{R_t^2 + R_{st}^2}} - 1$	
BOLTS (FINGER-TIGHT NUTS)						
7		Tension + Shear	$R_t^2 + R_s^3 = 1$	4.24		

NOTES:

(1) $R_b = R_{bx'} + R_{by'}$

(2) $R_s = \sqrt{R_{sx'}^2 + R_{sy'}^2}$

(3) $\theta = 1/2 \tan^{-1}\left(\frac{2I_{xy}}{I_x - I_y}\right)$

(4) $M_x' = M(\cos\alpha \cos\theta + \sin\alpha \sin\theta)$
$M_y' = M(-\cos\alpha \sin\theta + \sin\alpha \cos\theta)$

(5) $V_x' = V(\cos\beta \cos\theta + \sin\beta \sin\theta)$
$V_y' = V(-\cos\beta \sin\theta + \sin\beta \cos\theta)$

(6) $R_t = \frac{f_t}{F_{ty}}$, $R_c = \frac{f_c}{F_{cy}}$, $R_{bx'} = \frac{M_{x'}}{M_{x'y}}$, $R_{by'} = \frac{M_{y'}}{M_{y'y}}$, $R_{sx'} = \frac{f_{sx'y}}{F_{sy}}$, $R_{sy'} = \frac{f_{sy'y}}{F_{sy}}$ (YIELD)

(7) $R_t = \frac{f_t}{F_{tu}}$, $R_c = \frac{f_c}{F_{tu}}$, $R_{bx'} = \frac{M_{x'}}{M_{x'u}}$, $R_{by'} = \frac{M_{y'}}{M_{y'u}}$, $R_{sx'} = \frac{f_{sx'}}{F_{su}}$, $R_{sy'} = \frac{f_{sy'}}{F_{su}}$ (ULTIMATE)

(8) $M_{x'y}$, $M_{x'u}$, $M_{y'y}$, and $M_{y'u}$ are yield and ultimate allowable bending moments.

(9) $f_{sx'y}$, $f_{sx'u}$, $f_{sy'}$, and $f_{sy'u}$ are yield and ultimate plastic bending shear stresses.

TABLE 4.5 THICK-WALLED TUBULAR STRUCTURES - INTERACTION CRITERIA-YIELD AND ULTIMATE CONDITIONS OF STRENGTH, INCLUDING THE EFFECTS OF COLUMN STABILITY

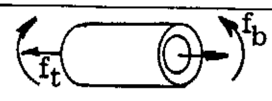
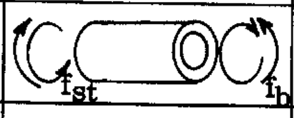
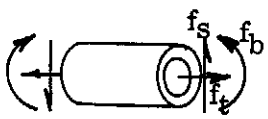
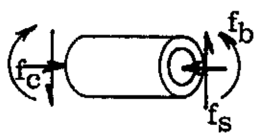
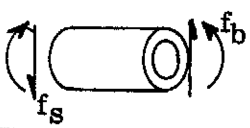
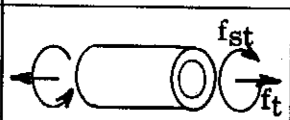
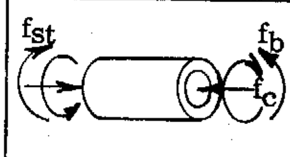
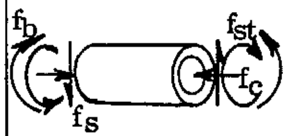
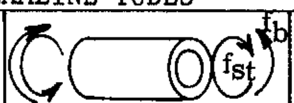
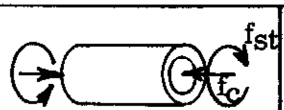
CASE	LOADING PICTURE	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY EQUATION	REMARKS
ROUND TUBES						
1		Compression + Bending	$R_c + R_b = 1$	4.10	$\frac{1}{R_c + R_b} - 1$	
2		Tension + Bending	$R_t^{1.5} + R_b = 1$	4.23		Let $R_t = R_1$, $R_b = R_2$
3		Bending + Torsion	$R_b^2 + R_{st}^2 = 1$	4.10	$\frac{1}{\sqrt{R_b^2 + R_{st}^2}} - 1$	
4		Tension + Bending + Shear	$R_{bt}^2 + R_s^2 = 1$	4.10	$\frac{1}{\sqrt{R_{bt}^2 + R_s^2}} - 1$	$R_{bt} = R_t/R_{t_a} = R_b/R_{b_a}$, where R_{t_a} and R_{b_a} are obtained from figure 4.23 ($R_t = R_1$, $R_b = R_2$, n = 1.5) by the two-loads-acting procedure as outlined in section 4.4.1
5		Compression + Bending + Shear	$(R_c + R_b)^2 + R_s^2$	4.10	$\frac{1}{\sqrt{(R_c + R_b)^2 + R_s^2}} - 1$	Let $R_c + R_b = R_1$, $R_s = R_2$
6		Bending + Shear	$R_b^2 + R_s^2 = 1$	4.10	$\frac{1}{\sqrt{R_b^2 + R_s^2}} - 1$	
7		Tension + Torsion	$R_t^2 + R_{st}^2 = 1$	4.10	$\frac{1}{\sqrt{R_t^2 + R_{st}^2}} - 1$	
8		Compression + Bending + Torsion	$R_b^2 + R_{st}^2 = (1-R_c)^2$	4.11	$\frac{1}{R_c + \sqrt{R_b^2 + R_{st}^2}} - 1$	In using figure 4.11 follow two-loads-acting procedures as outlined in section 4.9.1

TABLE 4.5 THICK-WALLED TUBULAR STRUCTURES-INTERACTION CRITERIA-YIELD AND ULTIMATE CONDITIONS OF STRENGTH, INCLUDING THE EFFECTS OF COLUMN STABILITY (CONCLUDED)

CASE	LOADING PICTURE	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY EQUATION	REMARKS
9		Compression + Bending + Torsion + Shear	$R_c + R_{st} + \left[R_b^2 + R_s^2\right]^{.5} = 1$	4.12	$\frac{1}{Rc+R_{st}+\sqrt{R_b^2+R_s^2}}-1$	In using figure 4.12, follow two-loads-acting procedure as outlined in section 4.4.1
10		Tension + Torsion + Internal Pressure	$R_t^2 + R_{st}^2 + R_p^2 = 1$	4.13	$\frac{1}{\sqrt{R_t^2 + R_{st}^2 + R_p^2}}-1$	
STREAMLINE TUBES						
11		Bending + Torsion	$R_b + R_{st} = 1$	4.10	$\frac{1}{\sqrt{R_b + R_{st}}} -1$	
SQUARE TUBES						
12		Compression + Torsion		4.14		Let $R_c = R_1$ $R_s = R_2$

NOTES:

(1) R_c must be based on the tube column allowable.

(2) R_t must be based on the material strength allowable.

(3) R_b and R_{st} must be based on tube strength allowables.

(4) R_b must include the effects of secondary bending.

(5) For shear-bending analysis use $f_s = f_{smax}$ and $f_b = f_{bmax}$ even though the locations of the two maxima do not coincide. The allowable transverse shear stress is equal to the lower of 1.20 times the allowable torsional shear stress and the material allowable shear stress.

(6) $R_p = pd/2t\ F_{tu}$, d = tube mean diameter, t = wall thickness.

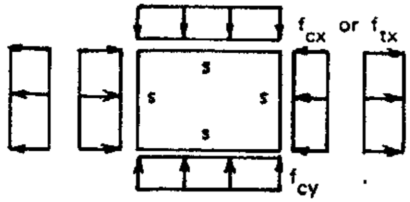
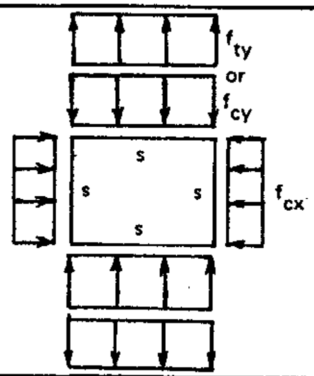
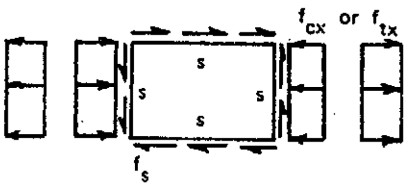
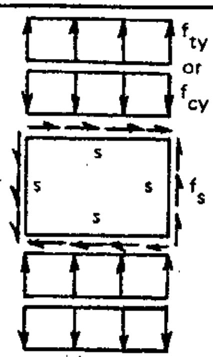
CASE	LOADING	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY	REMARKS
1		Longitudinal Compression or Tension + Transverse Shear		4.15		
2		Longitudinal Compression + Transverse Compression or Tension		4.15		
3		Longitudinal Compression or Tension + Shear	$R_{cx} + R_s^2 = 1$ or $R_{tx} + R_s^2 = 1$	4.14 (1)	$\frac{2}{R_{cx} + \sqrt{R_{cx}^2 + 4R_s^2}} - 1$ or $\frac{2}{R_{tx} + \sqrt{R_{tx}^2 + 4R_s^2}} - 1$	R_{cx} or $R_{tx} = R_1$ $R_s = R_2$
4		Transverse Compression or Tension + Shear		4.16		

TABLE 4.6. UNSTIFFENED PANEL INTERACTION CRITERIA (INITIAL BUCKLING)

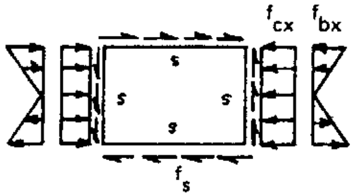
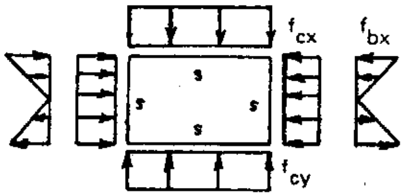
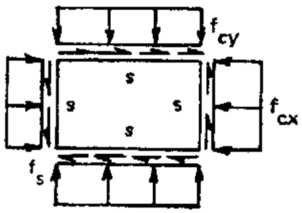
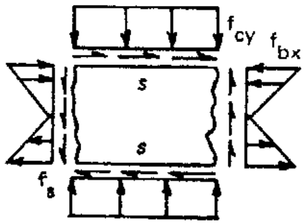
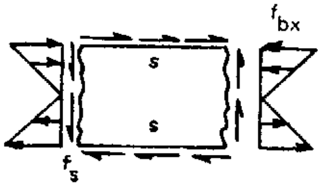
CASE	LOADING	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY	REMARKS
5		Longitudinal Compression + Longitudinal Bending + Shear		4.17		In using Figure 4.17, follow two-loads-acting as outlined in Section 4.4.1.
6		Longitudinal Compression + Transverse Compression + Longitudinal Bending		4.18		In using Figure 4.18, follow three-loads-acting as outlined in Section 4.4.2.
7		Longitudinal Compression + Transverse Compression + Shear		4.19		
8		Transverse Compression + Longitudinal Bending + Shear		4.20		In using Figure 4.20, follow three-loads-acting as outlined in Section 4.4.2.
9		Longitudinal Bending + Shear		4.20		In using Figure 4.20, use $R_{cy}/R_s = 0$ curve and follow two-loads-acting as outlined in Section 4.4.1.

TABLE 4.6 (CONT'D). UNSTIFFENED PANEL INTERACTION CRITERIA (INITIAL BUCKLING).

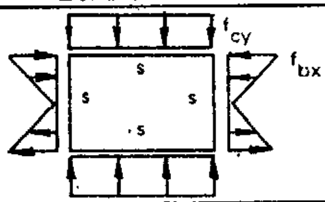
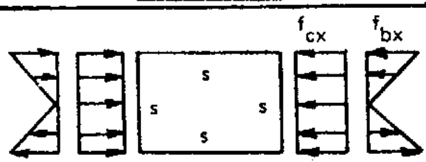
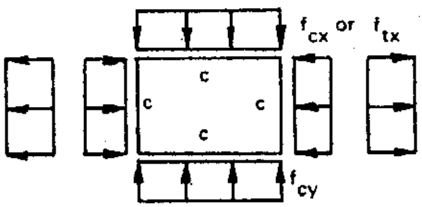
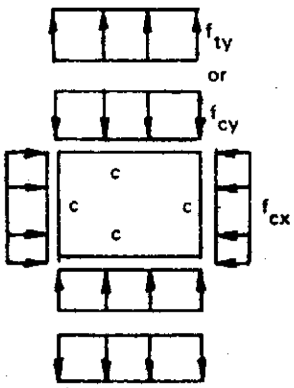
CASE	LOADING	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY	REMARKS
10		Longitudinal Bending + Transverse Compression		4.18		In using Figure 4.18, use $R_{cx} = 0$ and follow two-loads-acting of Section 4.4.1.
11		Longitudinal Compression + Longitudinal Bending	$R_{cx} + R_{bx}^{1.75} = 1$	4.23		$R_{bx} = R_1$ $R_{cx} = R_2$
12		Longitudinal Compression or Tension + Transverse Compression		4.21		
13		Longitudinal Compression + Transverse Compression or Tension		4.21		

TABLE 4.6 (CONT'D). UNSTIFFENED PANEL INTERACTION CRITERIA (INITIAL BUCKLING)

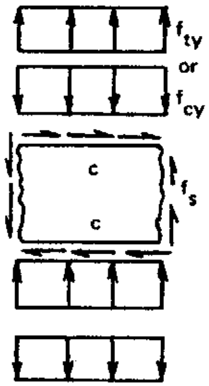
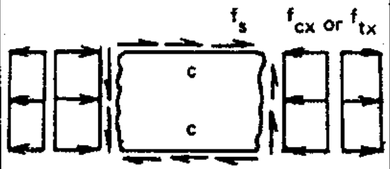
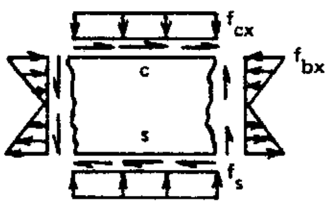
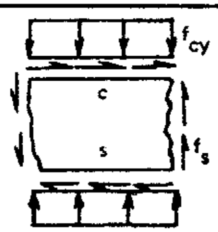
CASE	LOADING	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY	REMARKS
14		Transverse Compression or Tension + Shear		4.14 (2)		$R_{cy} = R_1$ $R_s = R_2$
15		Longitudinal Compression or Tension + Shear	$R_{cx} + R_s^2 = 1$ or $R_{tx} + R_s^2 = 1$	4.14 (1)	$\dfrac{2}{R_{cx} + \sqrt{R_{cx}^2 + 4R_s^2}} - 1$ or $\dfrac{2}{R_{tx} + \sqrt{R_{tx}^2 + 4R_s^2}} - 1$	$R_{cx} = R_1$ $R_s = R_2$ or $R_{tx} = R_1$ $R_s = R_2$
16		Longitudinal Bending + Transverse Compression + Shear		4.22		
17		Transverse Compression + Shear	$R_{cy} + R_s^2 = 1$	4.23	$\dfrac{2}{R_{cy} + \sqrt{R_{cy}^2 + 4R_s^2}} - 1$	$R_s = R_1$ $R_{cy} = R_2$

TABLE 4.6 (CONT'D). UNSTIFFENED PANEL INTERACTION CRITERIA (INITIAL BUCKLING)

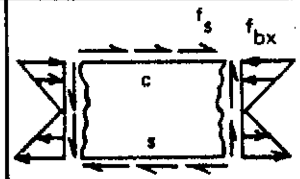
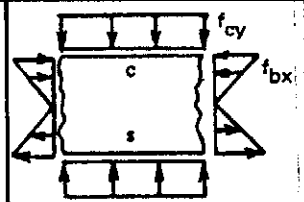
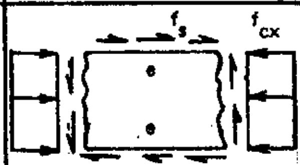
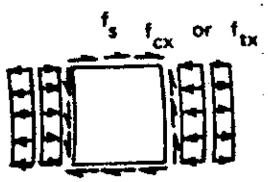
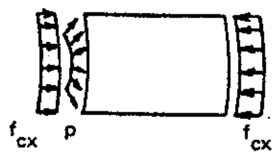
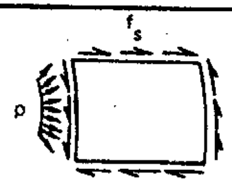
CASE	LOADING	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY	REMARKS
18		Longitudinal Bending + Shear		4.22		In using Figure 4.22, $R_{cy} = 0$ and follow two-loads-acting as outlined in Section 4.4.1.
19		Longitudinal Bending + Transverse Compression		4.14 (3)		$R_{cy} = R_1$ $R_{bx} = R_2$
20		Longitudinal Compression + Shear	$R_{cx} + R_s^2 = 1$	4.23	$\frac{2}{R_{cx} + \sqrt{R_{cx}^2 + 4R_s^2}} - 1$	$R_s = R_1$ $R_{cx} = R_2$
CURVED RECTANGULAR PANELS						
21		Longitudinal Compression or Tension + Shear	$R_{cx} + R_s^2 = 1$ or $R_{tx} + R_s^2 = 1$	4.14 (1)	$\frac{2}{R_{cx} + \sqrt{R_{cx}^2 + 4R_s^2}} - 1$ or $\frac{2}{R_{tx} + \sqrt{R_{tx}^2 + 4R_s^2}} - 1$	$R_{cx} = R_1$ $R_s = R_2$ or $R_{tx} = R_1$ $R_s = R_2$

TABLE 4.6 (CONT'D). UNSTIFFENED PANEL INTERACTION CRITERIA (INITIAL BUCKLING)

CASE	LOADING	LOADING DESCRIPTION	INTERACTION CURVE		MARGIN-OF-SAFETY	REMARKS
			EQUATION	FIGURE		
22		Longitudinal Compression + Internal Pressure	$R_{cx}^2 - \lvert R_p \rvert = 1$	4.24		$R_{cx} = R_1$ $R_p = R_2$
23		Shear + Internal Pressure	$R_s^2 - \lvert R_p \rvert = 1$	4.24		$R_s = R_1$ $R_p = R_2$

NOTES:

(1) R_c, R_b, and R_s must be based on panel initial buckling allowables.

(2) R_t is negative and is based on compression allowable.

(3) R_p is negative and is based on external collapsing pressure.

(4) S = simple supports, C = clamped supports, e = elastic supports.

(5) Dimensions; a = long side, b = short side.

TABLE 4.6 (CONT'D). UNSTIFFENED PANEL INTERACTION CRITERIA (INITIAL BUCKLING)

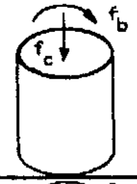
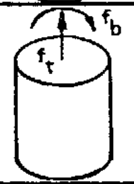
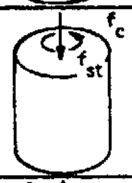
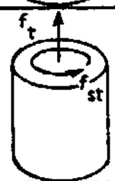
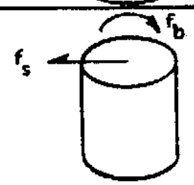
CASE	LOADING	LOADING DESCRIPTION	INTERACTION CURVE		MARGIN-OF-SAFETY	REMARKS
			EQUATION	FIGURE		
1		Longitudinal Compression + Longitudinal Bending	$R_c + R_b = 1$	4.10	$\frac{1}{R_c + R_b} - 1$	
2		Longitudinal Tension + Longitudinal Bending	$R_b + 0.9R_t = 1$	4.25		$R_t = R_1$ $R_b = R_2$
3		Longitudinal Compression + Torsion	$R_c + R_{st}^2 = 1$	4.23	$\frac{2}{R_c + \sqrt{R_c^2 + 4R_{st}^2}} - 1$	$R_{st} = R_1$ $R_c = R_2$
4		Longitudinal Tension + Torsion	$R_{st}^3 - \lvert R_t \rvert = 1$	4.24		$\lvert R_t \rvert < 0.8$ $R_{st} = R_1$ $\lvert R_t \rvert = R_2$
5		Longitudinal Bending + Torsion	$R_b^{1.5} + R_{st}^2 = 1$	4.24		$R_b = R_1$ $R_{st} = R_2$
6		Longitudinal Bending + Transverse Shear	$R_b^3 + R_s^3 = 1$	4.10		

TABLE 4.7. UNSTIFFENED CYLINDRICAL SHELL STRUCTURES (INITIAL BUCKLING)

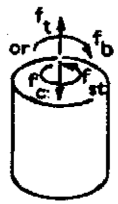
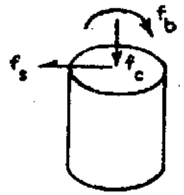
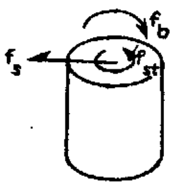
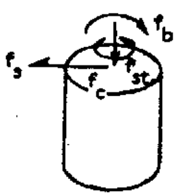
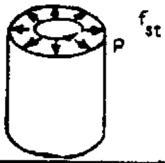
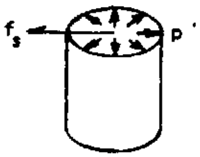
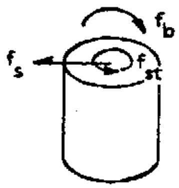
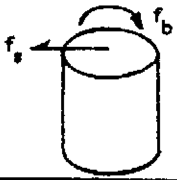
CASE	LOADING	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY	REMARKS
7		Longitudinal Compression or Tension + Longitudinal Bending + Torsion	$R_c + R_b + R_{st}^2 = 1$ or $R_t + R_b + R_{st}^2 = 1$	4.14 (1)	$\frac{2}{R_c + R_b + \sqrt{(R_c + R_b)^2 + 4R_{st}^2}} - 1$ or $\frac{2}{R_t + R_b + \sqrt{(R_c + R_b)^2 + 4R_{st}^2}} - 1$	$R_c + R_b = R_1$ $R_{st} = R_2$ or $R_t + R_b = R_1$ $R_{st} = R_2$
8		Longitudinal Compression + Longitudinal Bending + Transverse Shear	$R_c + \sqrt[3]{R_b^3 + R_s^3} = 1$			If $R_c + \sqrt[3]{R_b^3 + R_s^3} < 1$ MS = +
9		Longitudinal Bending + Torsion + Transverse Shear	$R_b^P + (R_s + R_{st})^2$ $= 1$	4.23		For $R_{st} \geq R_s$ $P = 1$, $R_{st} < R_s$ $P = 1.5$
10		Longitudinal Compression + Longitudinal Bending + Torsion + Transverse Shear	$R_c + R_{st}^2$ $+ \sqrt[3]{R_b^3 + R_s^3}$ $= 1$			If $R_c + R_{st}^2$ $+ \sqrt[3]{R_b^3 + R_s^3} < 1$ MS = +

TABLE 4.7 (CONT'D). UNSTIFFENED CYLINDRICAL SHELL STRUCTURES (INITIAL BUCKLING)

CASE	LOADING	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY	REMARKS
11		Torsion + Internal Pressure	$R_{st}^2 + \lvert R_p \rvert = 1$	4.24		$R_{st} + R_1$ $R_p = R_2$
12		Transverse Shear + Internal Pressure	$R_s - \lvert R_p \rvert = 1$	4.24		$R_s = R_1$ $R_p = R_2$
ELLIPTIC CYLINDER						
13		Longitudinal Bending + Torsion + Transverse Shear	$R_b^2 + (R_s + R_{st})^2 = 1$	4.10	$\frac{1}{\sqrt{R_b^2 + (R_s + R_{st})^2}} - 1$	$R_b = R_1$ $R_s + R_{st} = R_2$
14		Longitudinal Bending + Transverse Shear	$R_b^2 + R_s^2 = 1$	4.10	$\frac{1}{R_b^2 + R_x^2} - 1$	
15		Longitudinal Bending + Torsion	$R_b^2 + R_{st}^2 = 1$	4.10	$\frac{1}{\sqrt{R_b^2 + R_{st}^2}} - 1$	

NOTES:

(1) R_c, R_b, R_{st}, and R_s must be based on cylindrical shell initial buckling allowables.

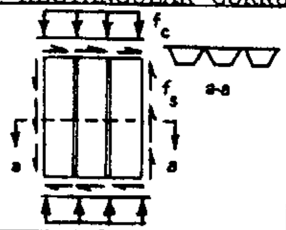
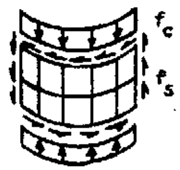
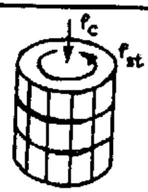
(2) R_t is negative and is based on compression buckling allowable.

(3) For shear-bending analysis, use $f_s = f_{smax}$ and $f_b = f_{bmax}$ even though the locations of the maxima do not coincide. The transverse shear buckling allowable stress is equal to 1.2 times the torsional buckling allowable.

(4) The interaction equations are applicable to both internally pressurized and unpressurized cylinders.

TABLE 4.7 (CONT'D). UNSTIFFENED CYLINDRICAL SHELL STRUCTURES (INITIAL BUCKLING)

TABLE 4.8 STIFFENED STRUCTURES (STABILITY)

CASE	LOADING	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY	REMARKS
FLAT RECTANGULAR CORRUGATED PANELS						
1		Longitudinal Compression + Shear	$R_c + R_s^{1.7} = 1$	4.23		
CURVED PANELS						
2		Longitudinal Compression + Shear	$R_c + R_s^2 = 1$	4.23	$\frac{2}{R_c + \sqrt{R_c^2 + 4R_s^2}} - 1$	
CIRCULAR CYLINDERS						
3		Longitudinal Bending + Torsion	$R_b + R_{st}^2 = 1$	4.23	$\frac{2}{R_b + \sqrt{R_b^2 + 4R_{st}^2}} - 1$	
4		Longitudinal Bending + Shear	$R_b^{\infty} + R_s^{\infty} = 1$ (Zero Interaction)	4.10	$\frac{1}{R_b} - 1$ or $\frac{1}{R_s} - 1$	
5		Longitudinal Compression + Torsion	$R_c + R_{st}^{1.5} = 1$	4.23		

NOTES:

(1) R_c, R_p, and R_{st} must be based on stiffened structures stability criteria.

(2) For shear-bending analysis of circular cylinders use $f_s = f_{smax}$ and $f_b = f_{bmax}$ even though the locations of the maxima do not coincide. For circular cylinder general instability failure criteria the allowable trans[illegible]se shear stress assumed [illegible]be 0.8 times the allowa[illegible] torsional shear stre[illegible]

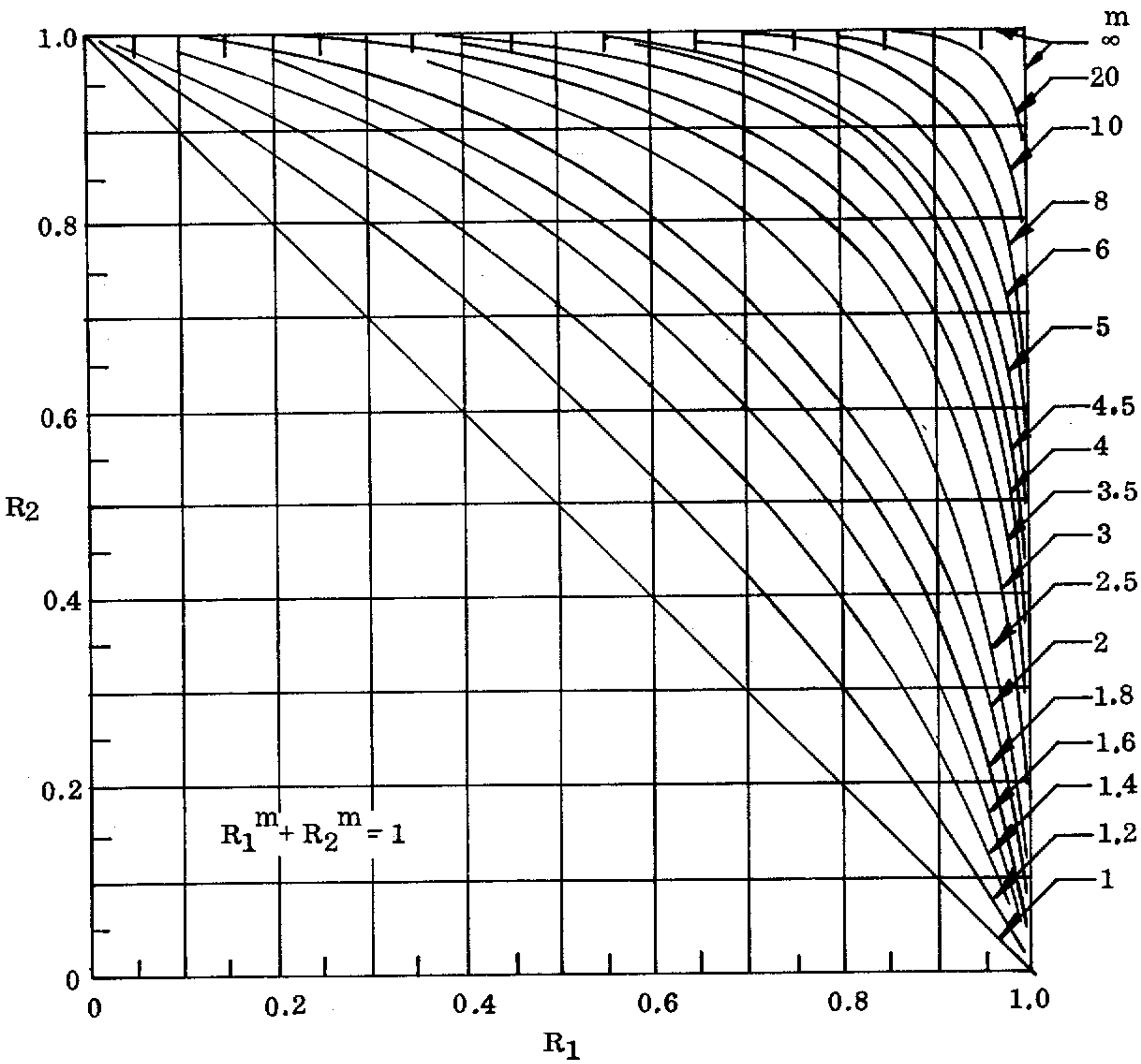


FIGURE 4.10 GENERAL INTERACTION CURVES

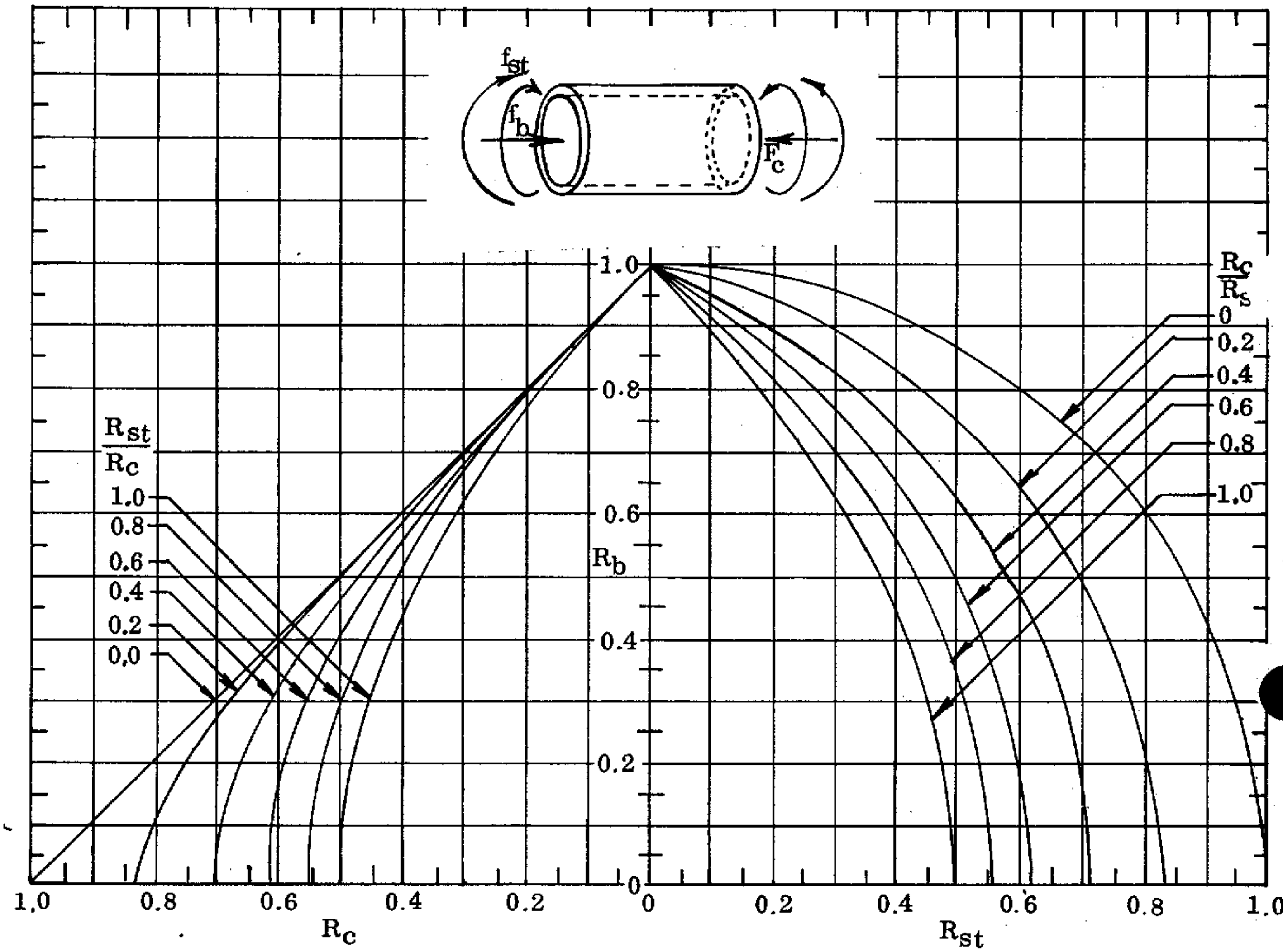


FIGURE 4.11. INTERACTION CURVES FOR THICK-WALLED ROUND TUBES - COMPRESSION, BENDING, AND TORSION (REF. TABLE 4.5, CASE 8).

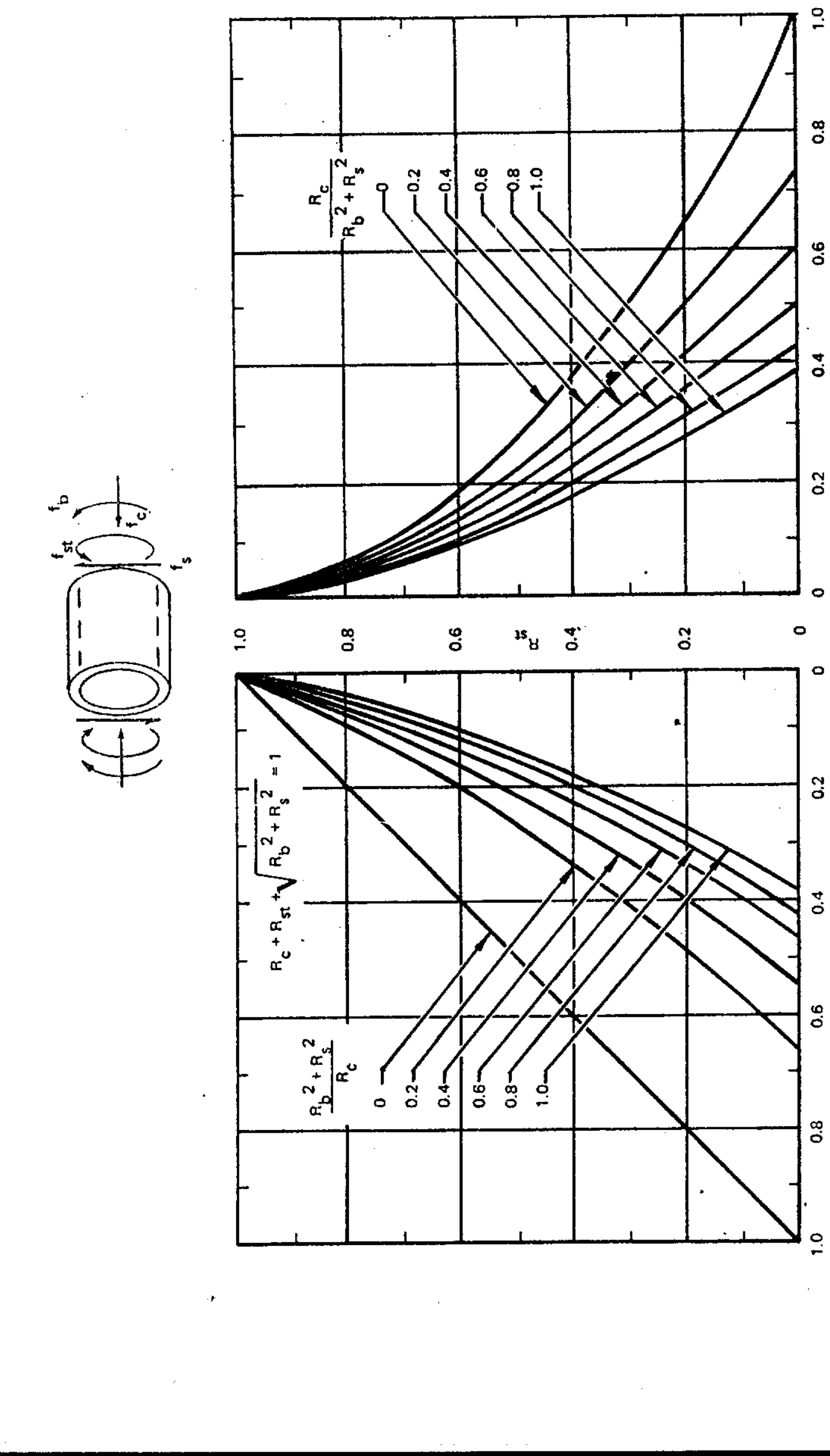


FIGURE 4.12. INTERACTION CURVES FOR THICK-WALLED ROUND TUBES: COMPRESSION, BENDING, TORSION AND SHEAR (REF. TABLE 4.5, CASE 9).

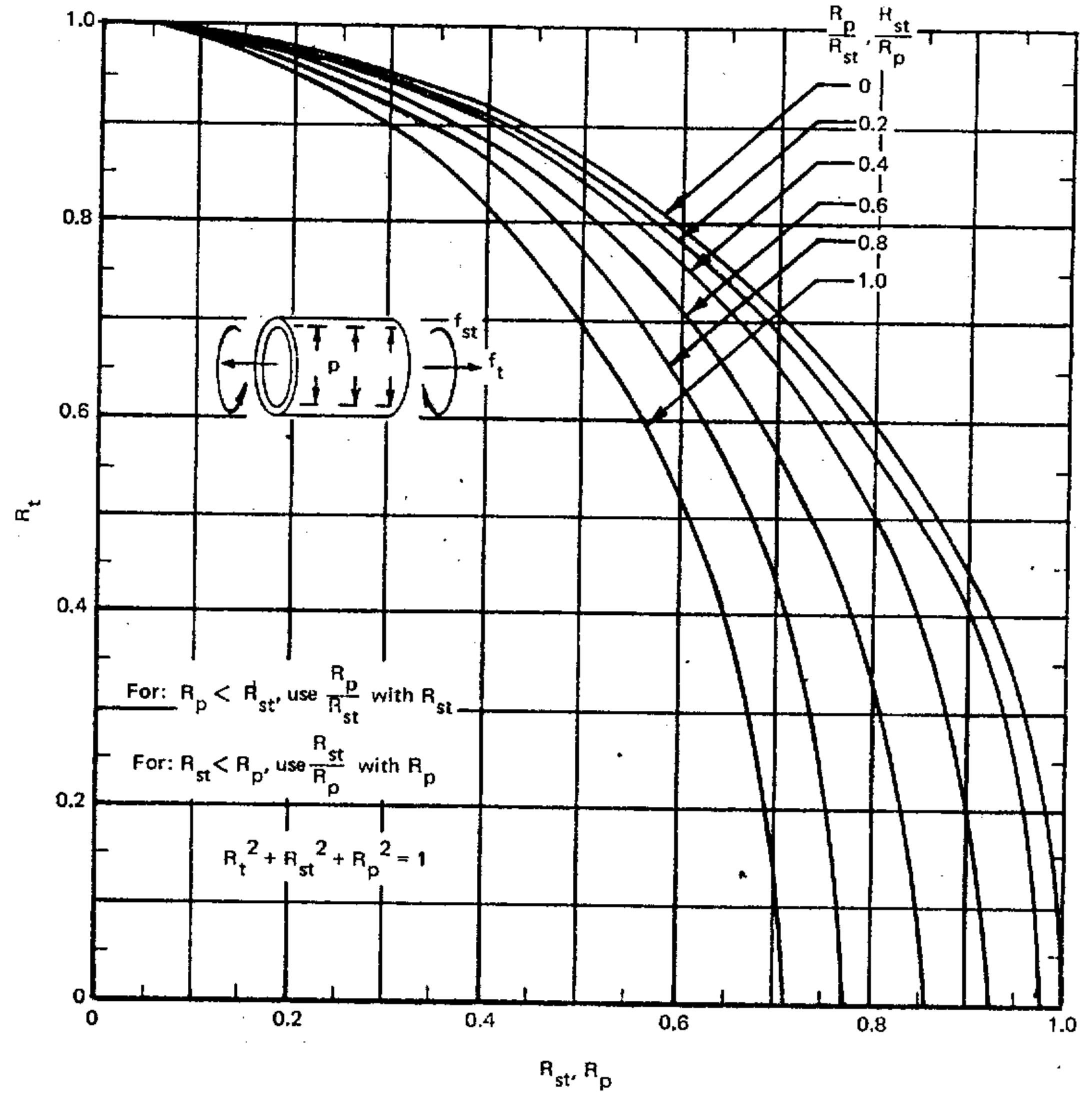
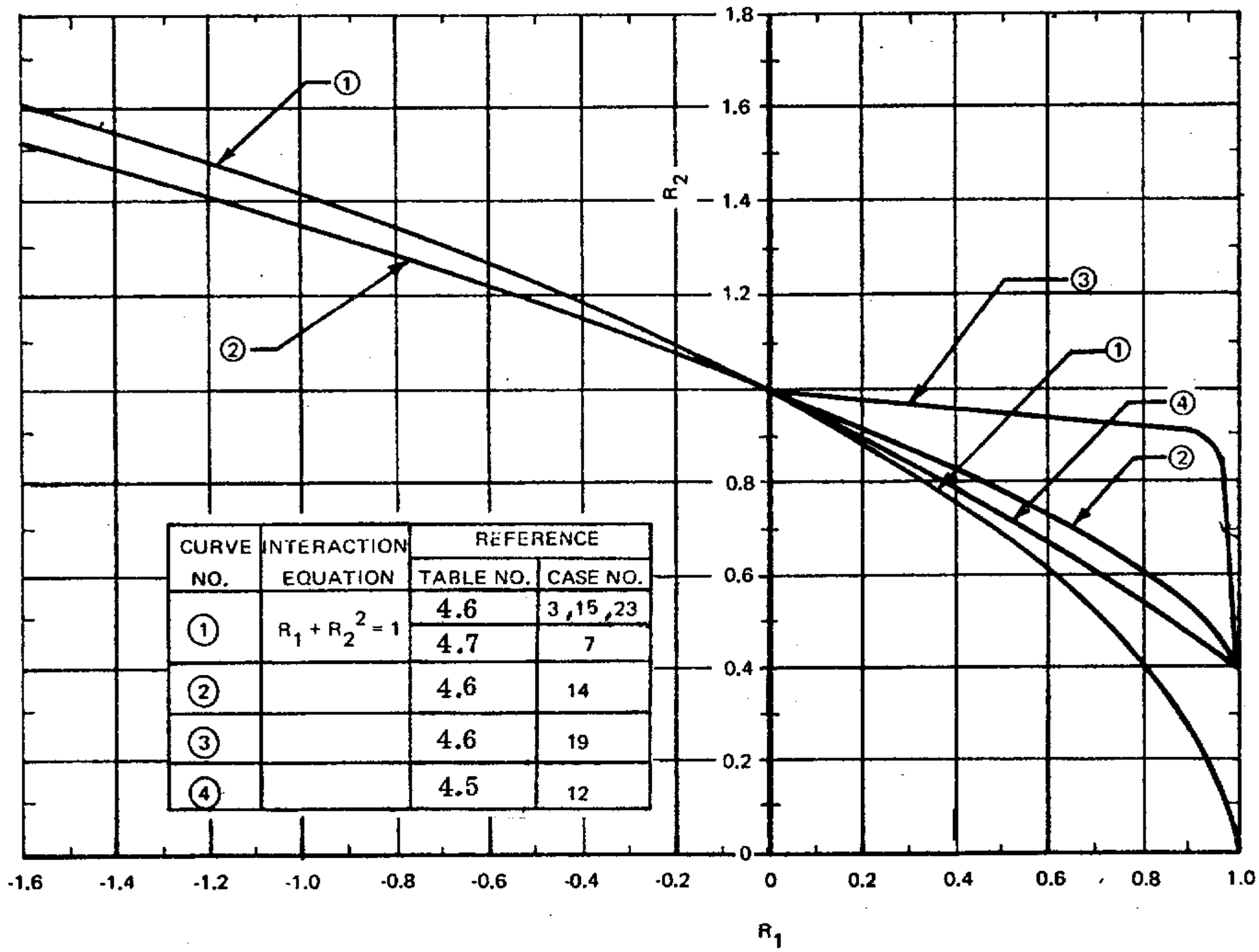


FIGURE 4.13 INTERACTION CURVES FOR THICK-WALLED ROUND TUBES-TENSION, TORSION AND INTERNAL PRESSURE (REF. TABLE 4.5, CASE 10)



CURVE NO.	INTERACTION EQUATION	REFERENCE TABLE NO.	REFERENCE CASE NO.
①	$R_1 + R_2^2 = 1$	4.6	3, 15, 23
		4.7	7
②		4.6	14
③		4.6	19
④		4.5	12

FIGURE 4.14. GENERAL INTERACTION CURVES

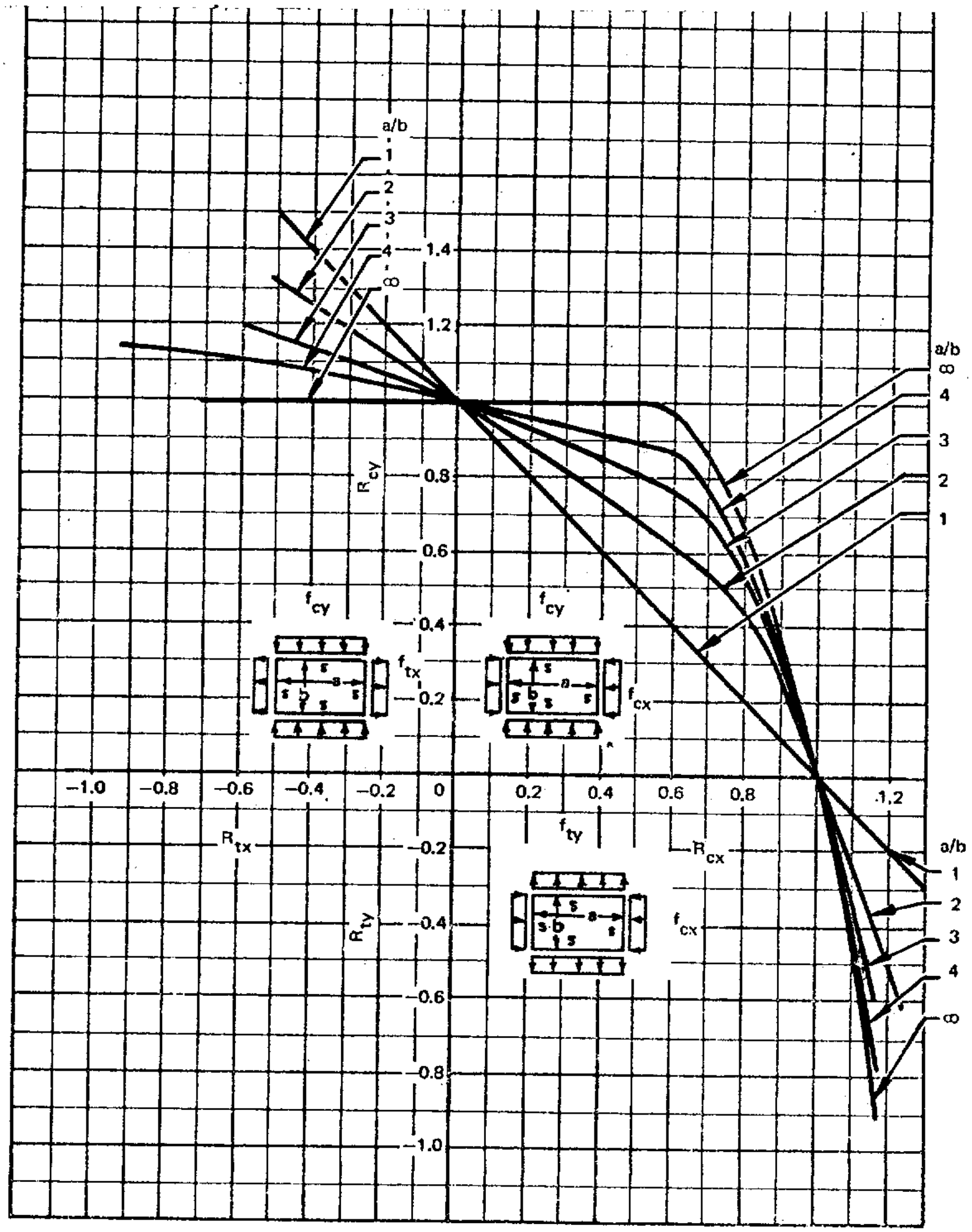


FIGURE 4.15. INTERACTION CURVES FOR FLAT RECTANGULAR PANELS (REF. TABLE 4.6, CASES 1 AND 2)

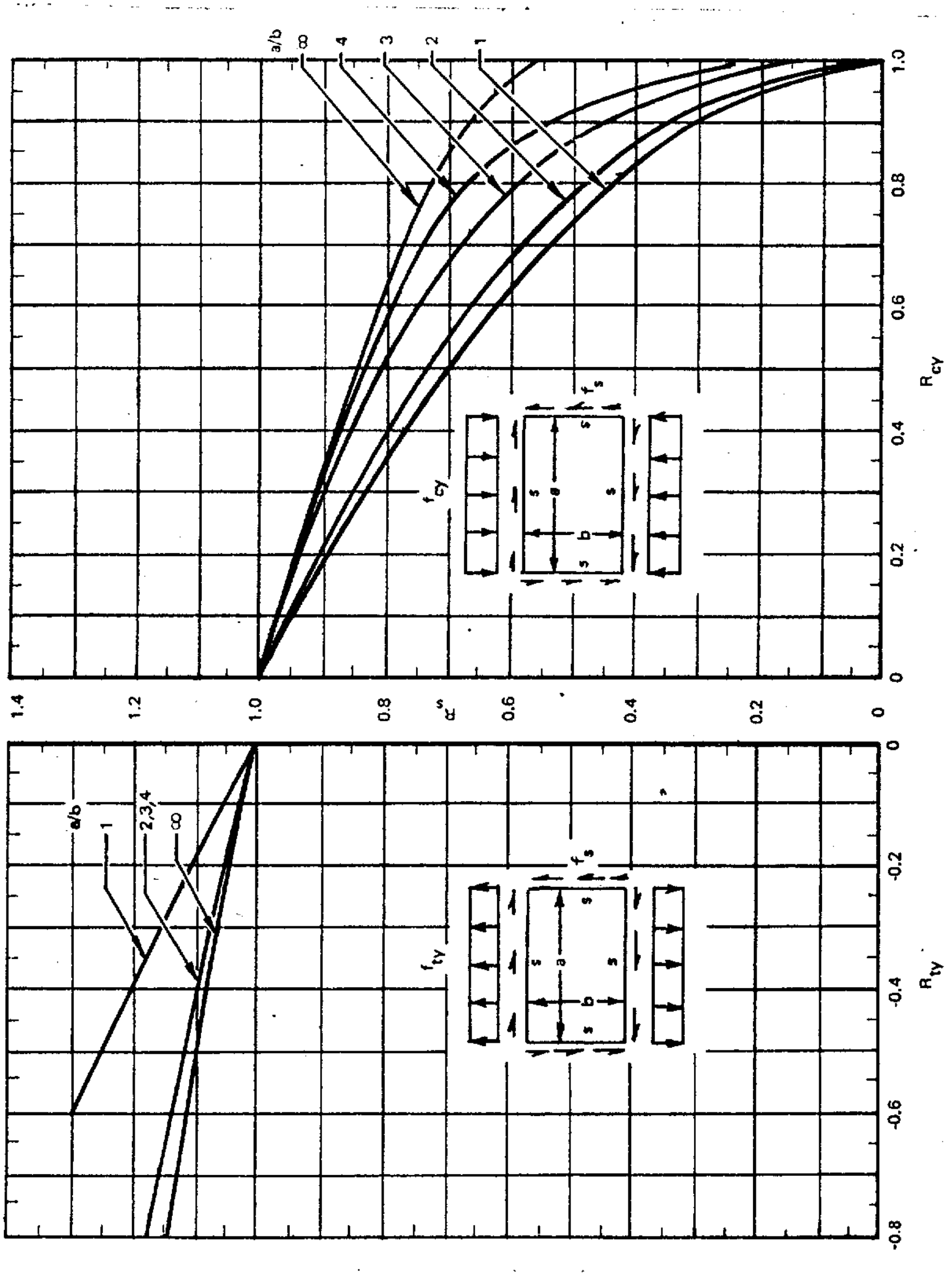


FIGURE 4.16. INTERACTION CURVES FOR FLAT RECTANGULAR PANELS (REF. TABLE 4.6, CASE 4)

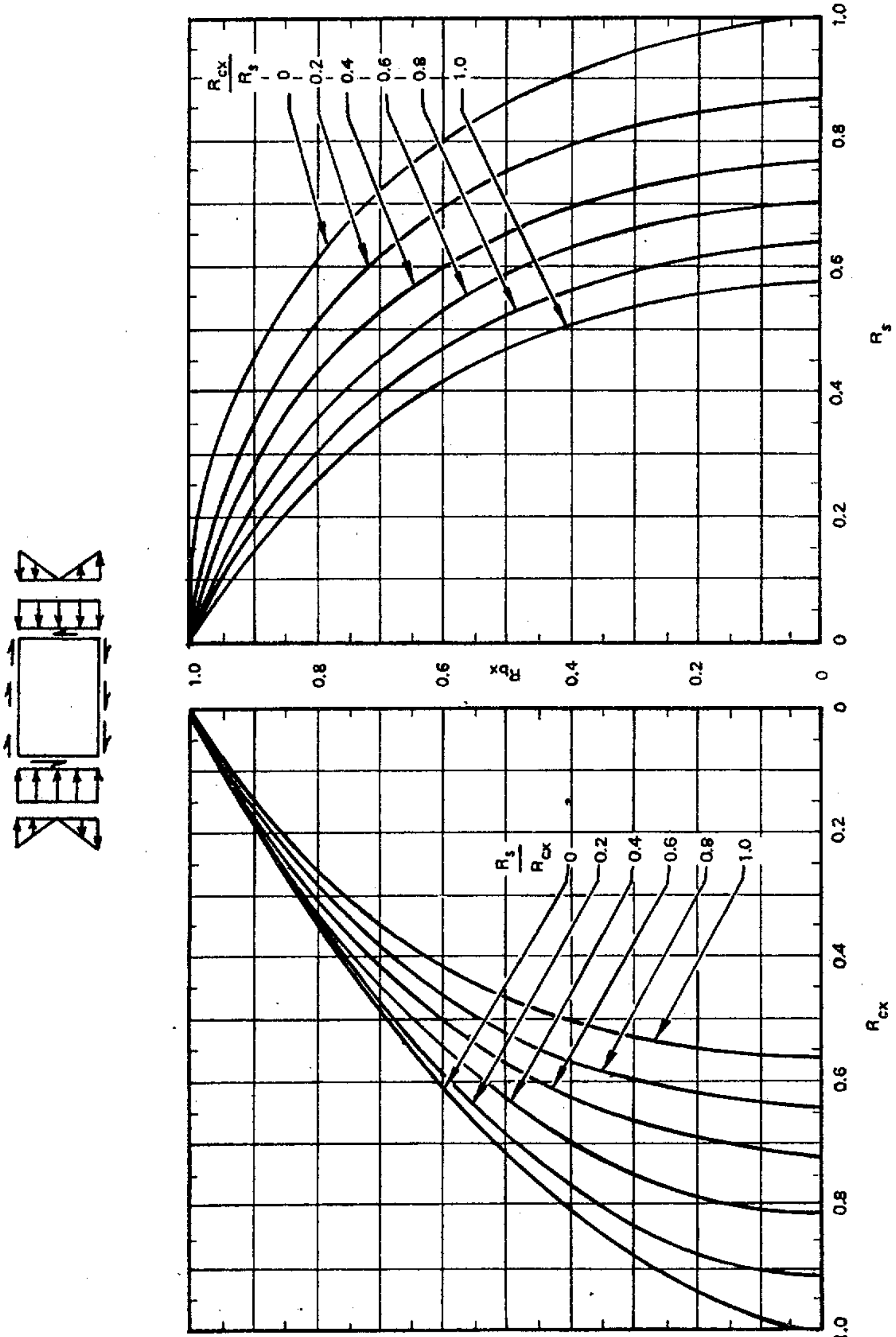


FIGURE 4.17. INTERACTION CURVES FOR FLAT RECTANGULAR PANELS (REF. TABLE 4.6, CASE 5)

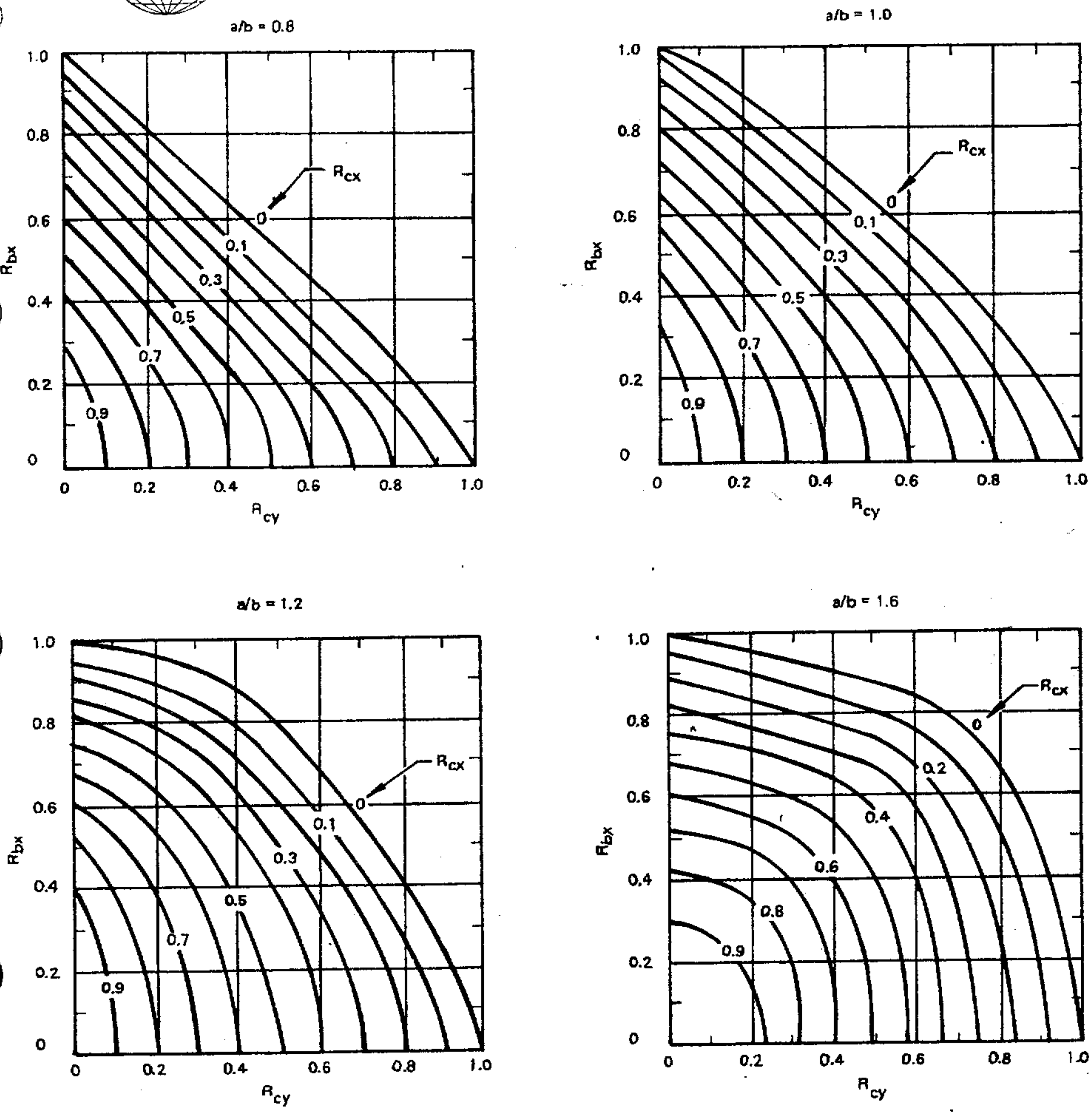


FIGURE 4.18. INTERACTION CURVES FOR FLAT RECTANGULAR PANELS
(REF. TABLE 4.6, CASES 6 AND 10)

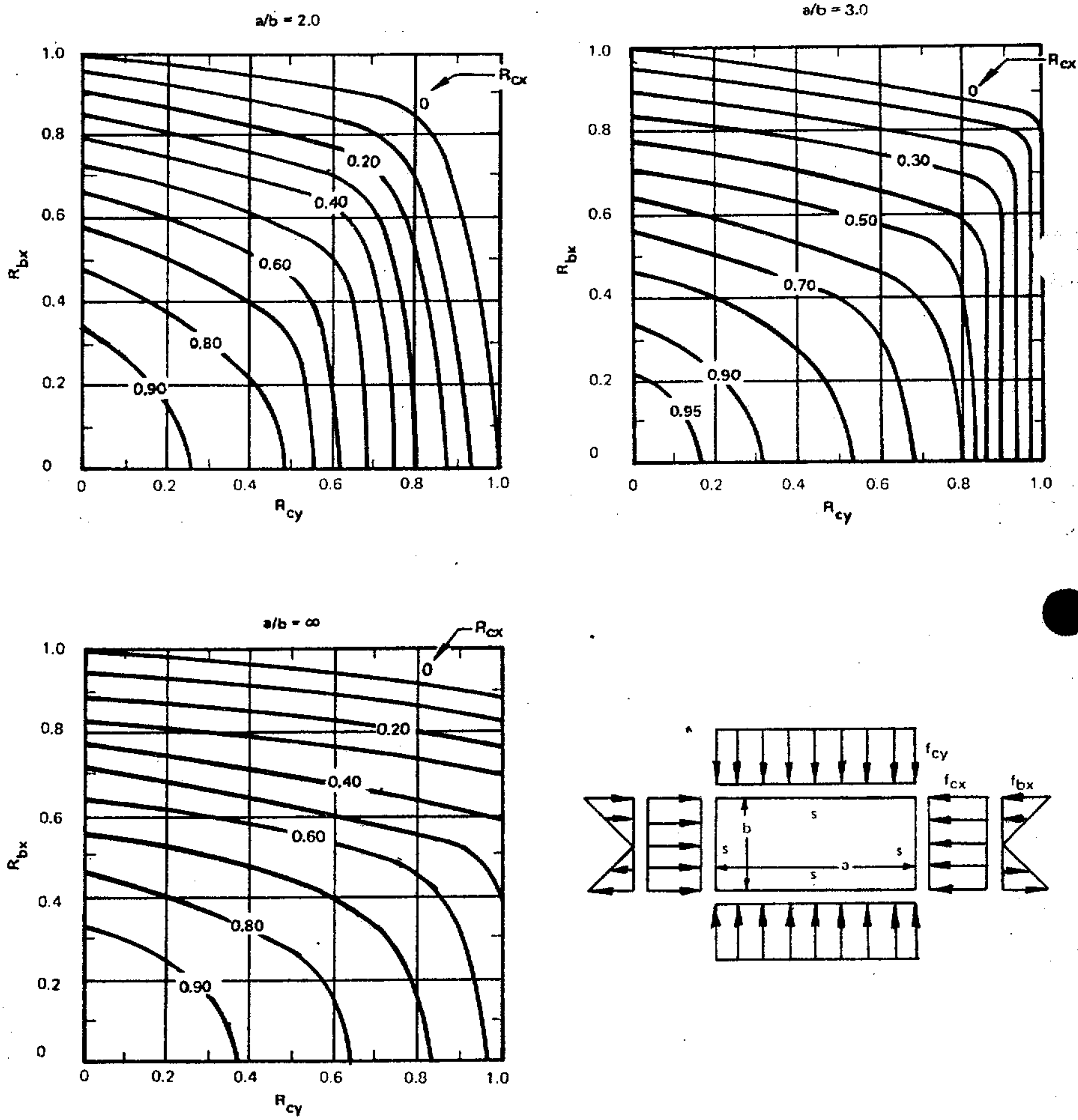


FIGURE 4.18.

(CONT'D) INTERACTION CURVES FOR FLAT RECTANGULAR PANELS - (REF. TABLE 4.(CASES 6 AND 10)

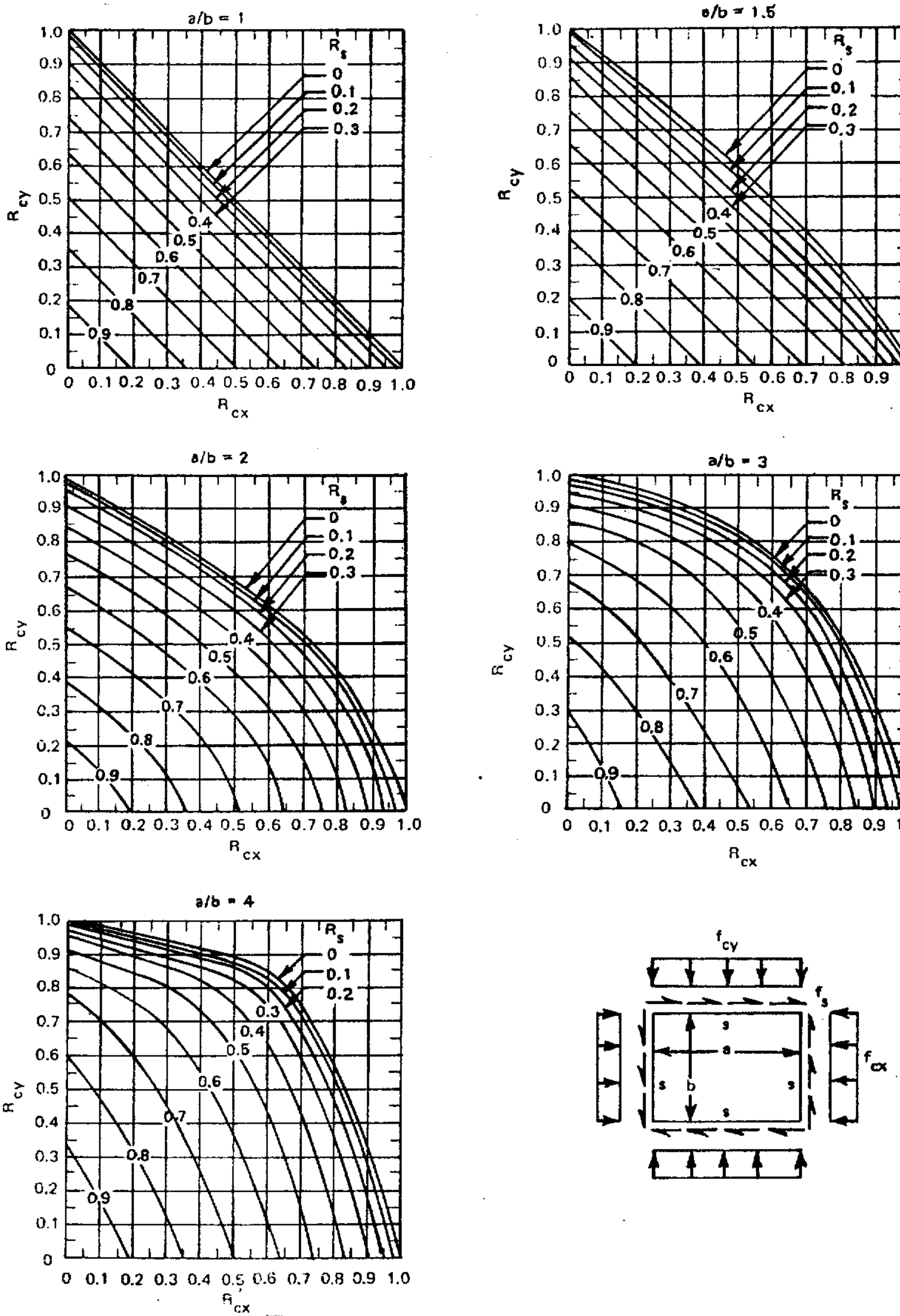


FIGURE 4.19. INTERACTION CURVES FOR FLAT RECTANGULAR PANELS (REF. TABLE 4.6, CASE 7)

Revision B

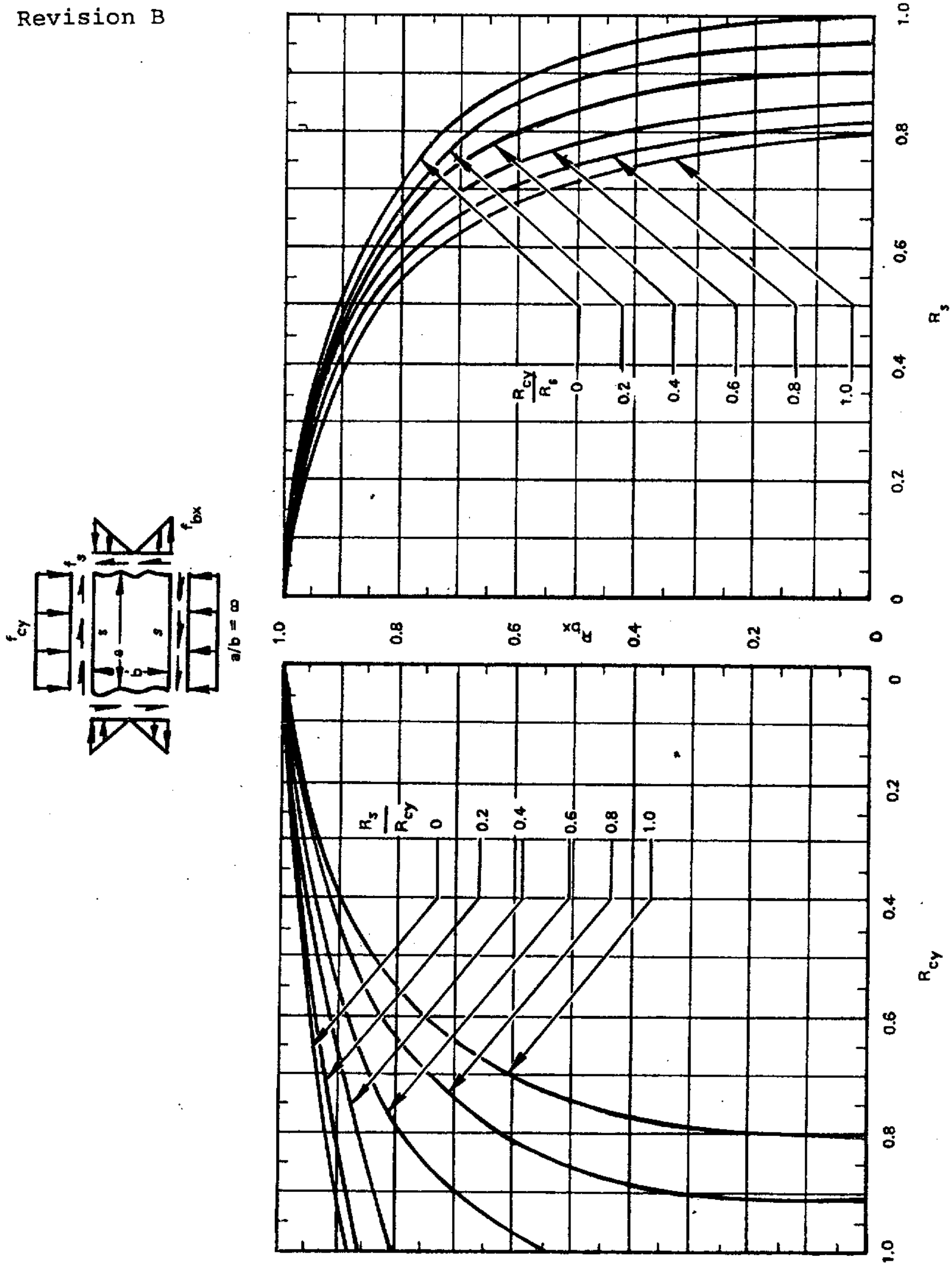


FIGURE 4.20. INTERACTION CURVES FOR FLAT RECTANGULAR PANELS (REF. TABLE 4.6, CASES 8 AND 9)

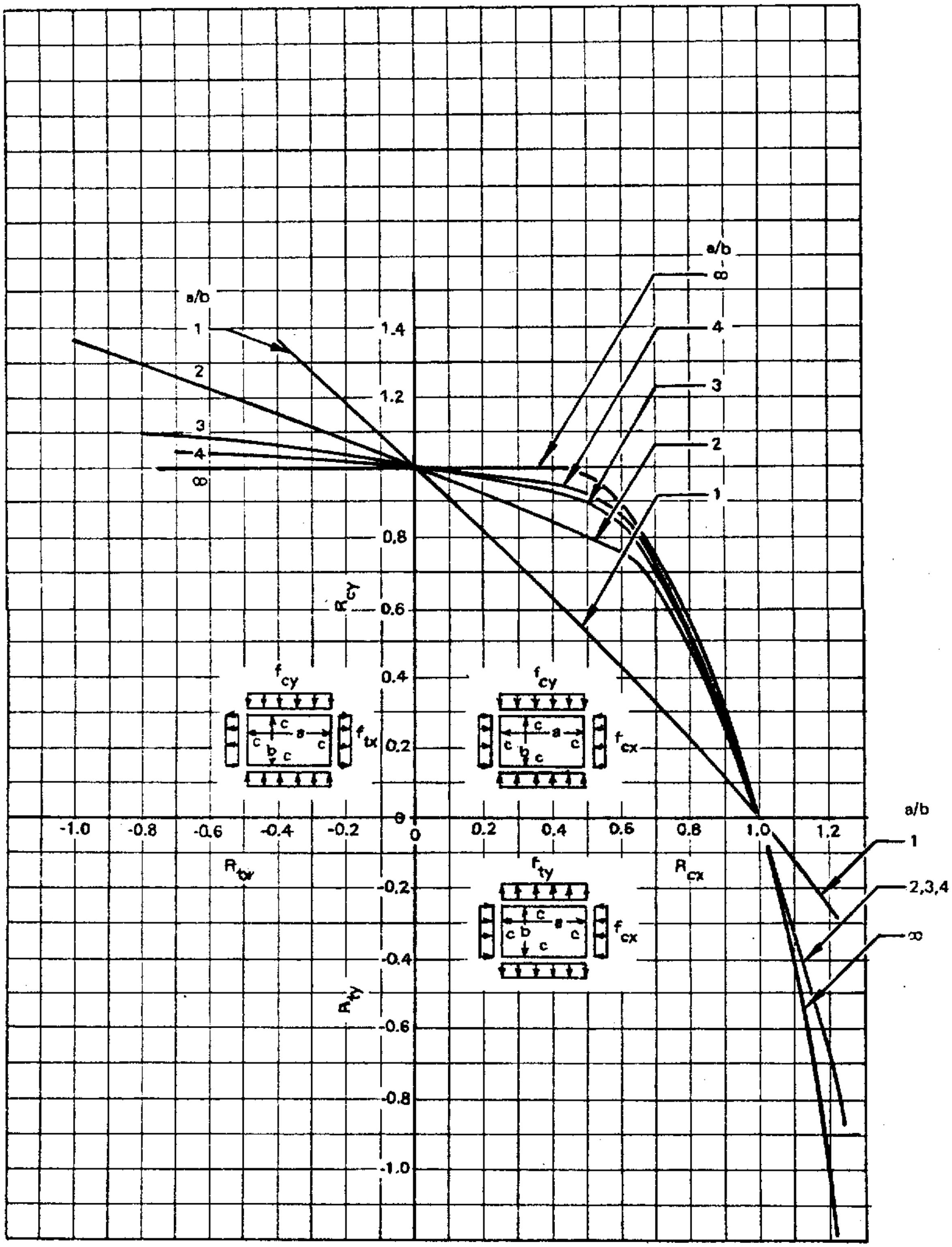
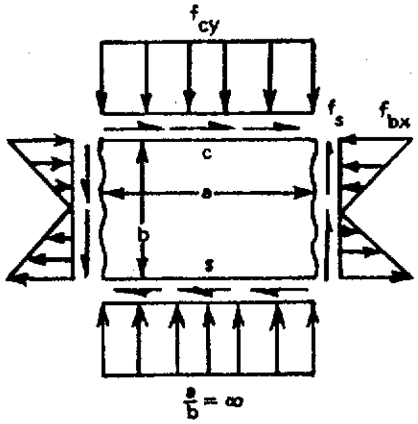


FIGURE 4.21. INTERACTION CURVES FOR FLAT RECTANGULAR PANELS
(REF. TABLE 4.6, CASES 12 AND 13)



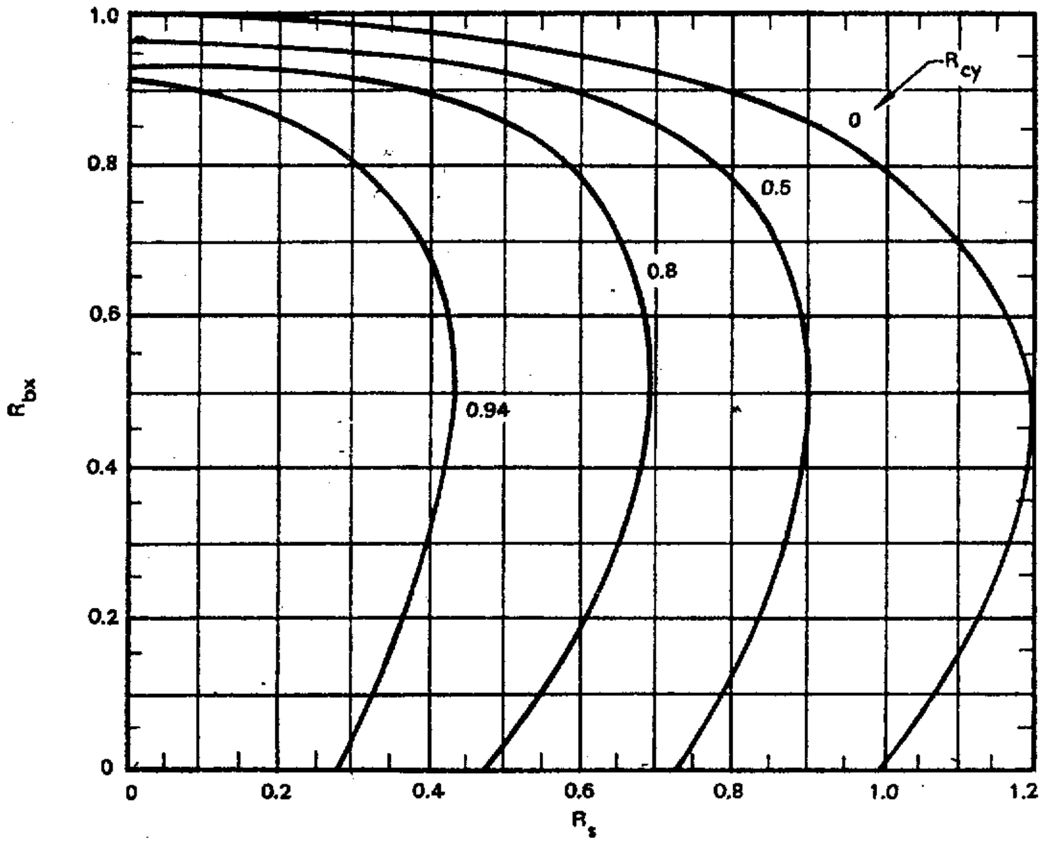


FIGURE 4.22 INTERACTION CURVES FOR FLAT RECTANGULAR PANELS.
(REF. TABLE 4.6, CASES 16 AND 18)

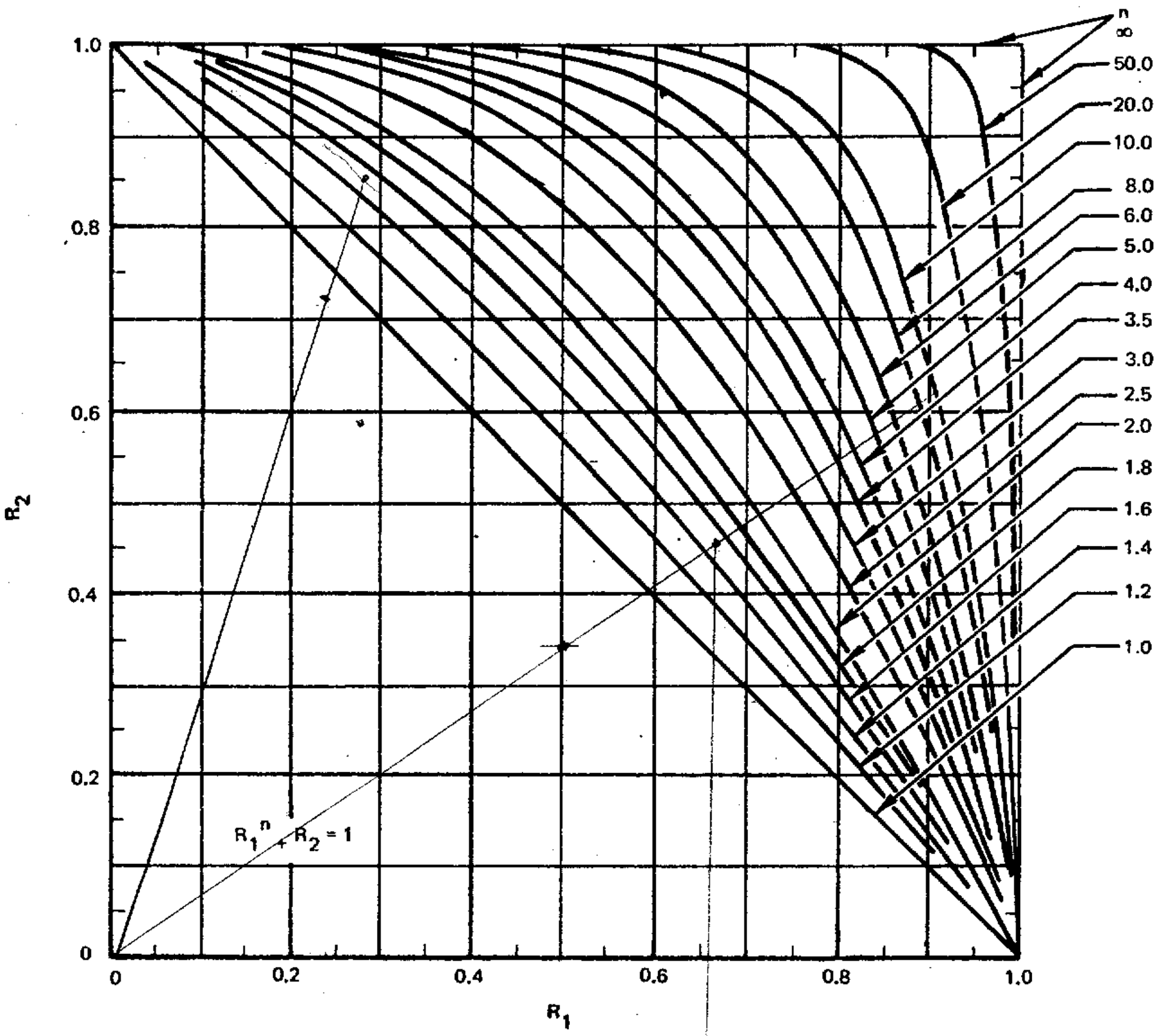


FIGURE 4.23. UNSYMMETRICAL GENERAL INTERACTION CURVES

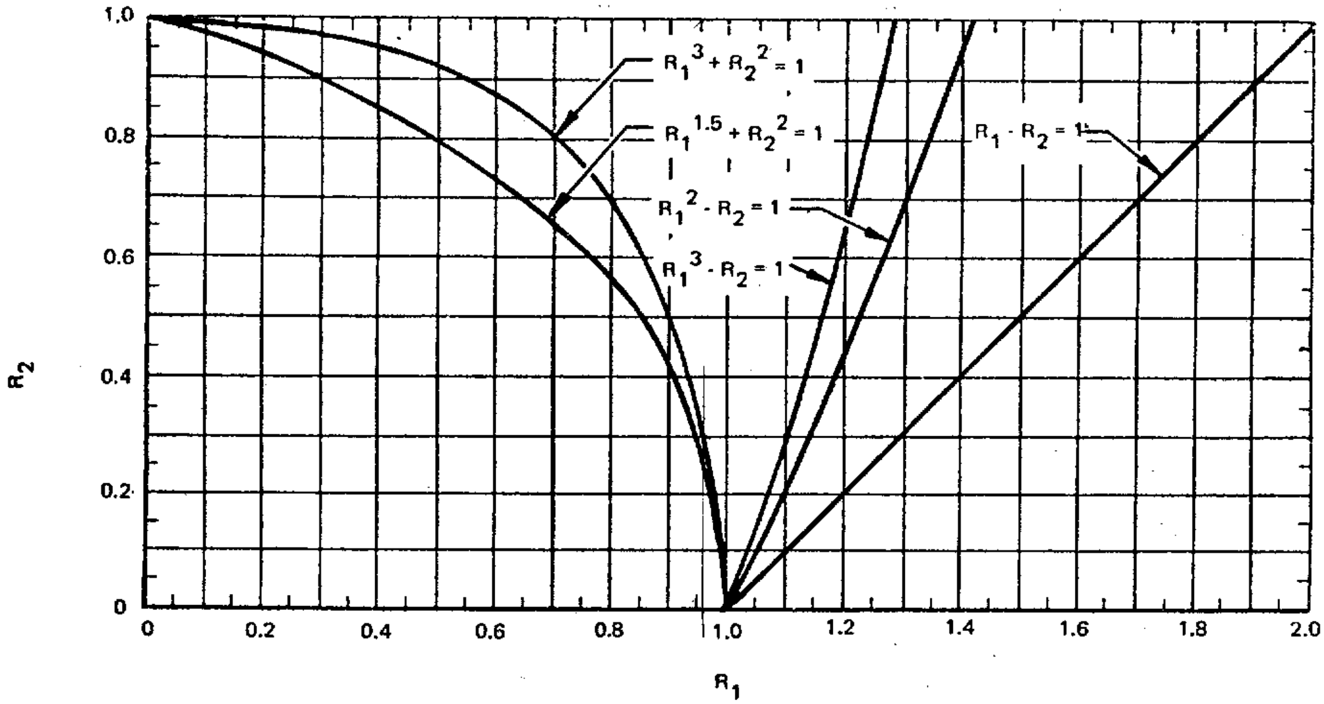


FIGURE 4.24. SPECIAL INTERACTION CURVES

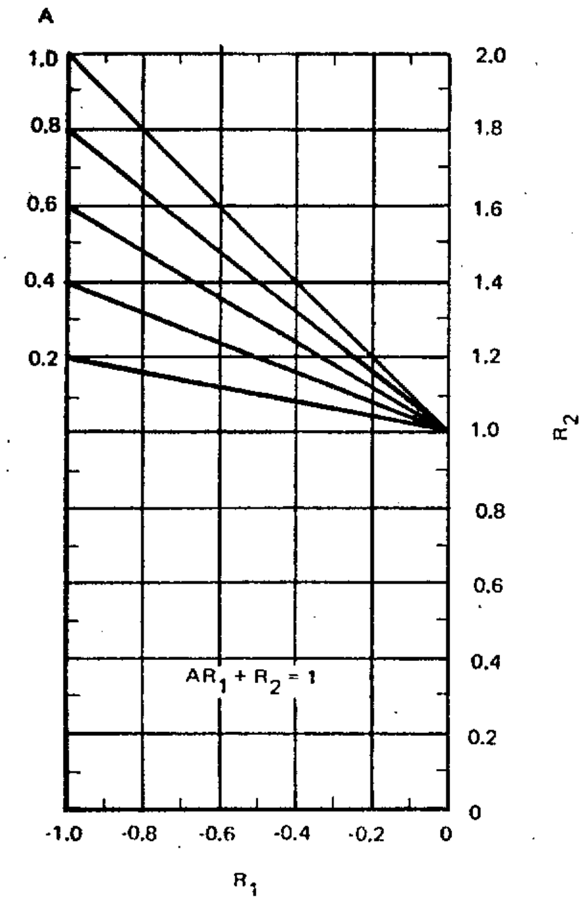


FIGURE 4.25. LINEAR INTERACTION CURVES.

SECTION 5
MATEPIALS

5.1 GENERAL

This section describes the materials commonly used at Bell Helicopter. The forms, temper designations, properties, design limiations and testing methods are discussed. The information contained herein is in agreement with MIL-HDBK-5 and MIL-HDBK-17. For information on materials not shown in this section the appropriate material specification or the previously referenced handbooks should be consulted.

5.1.1 Material Properties

The physical properites of some materials are shown in Table 5.1. This is a summary of commercially pure elements and is presented for comparison only.

Primary strength properties of metallic materials are minimum values at room temperature, established on an A, B, C or S basis.

Elongation and reduction in area properties presented in the referenced handbook property tables are minimum values at room temperature, established on an A or S basis. Elongation and reduction in area at other temperatures, as well as elastic properties (E, Ec, G and μ), physical properties (w, C, K and α), creep properites and fracture toughness properties are average unless otherwise specified.

A Basis - The mechanical-property value indicated is the value above which at least 99 percent of the population of values is expected to fall, with a confidence of 95 percent.

B Basis - The mechanical-property value indicated is the value above which at least 90 percent of the population of values is expected to fall, with a confidence of 95 percent.

S Basis - The mechanical-property value indicated is usually the specified minimum value of the appropriate Government specification, or SAE Aerospace Material Specification for this material. The statistical assurance associated with this value is not known.

C Basis - The mechanical property value indicated is the value developed by tests at Bell Helicopter.

5.1.2 Selection of Design Allowables

Specification MIL-S-8698 contains the general requirements which in combination with specific model detail specifications set the requirements for structural design, analysis and test of helicopters. It is required by these specifications that minimum guaranteed values (A values) be used with nominal dimensions. Nominal is the average between tolerances.

The use of B values must be approved by the procuring agency. If conditions such as crash, rollover or impact produce loads in excess of maneuver design loads, consideration should be given to requesting permission for use of B values.

MATERIAL	ATOMIC WGT.	DENSITY (#/in^3 @ 68°F)	MELTING TEMP. ($^\circ$F)	BOILING TEMP. ($^\circ$F)	HEAT OF FUSION (g-CAL/g)	SPECIFIC HEAT (BTU/LB @ 68°F)	THERMAL CONDUCTIVITY ($\frac{10^{-6} BTU-IN}{in^2 . SEC ^\circ F}$ @ 68°F)	THERMAL COEFFICIENT OF EXPANSION (x10-6 in/in$^\circ$F)	ELECTRICAL RESISTIVITY (μ OHM/CM)	MODULUS OF ELASTICITY (x10^{6}PSI)
ALUMINUM	27	.098	1215	3733	168	.226	2912	13.1	2.66	10
BERYLLIUM	9	.067	2345	5036	581	.425	-	6.3	18.5	42
CADMIUM	112	.313	608	1411	24	.055	1215	27.8	7.6	10
CALCIUM	40	.056	1564	2719	143	.157	-	12.9	4.6	-
CHROMIUM	52	.258	2822	4500	58	.12	924	4.2	13.1	36
COBALT	59	.322	2696	5252	107	.099	924	6.2	9.7	30
COPPER	64	.323	1981	4703	92	.092	5169	8.5	1.7	16
GOLD	197	.697	1945	5371	29	.031	3959	7.4	2.4	12
GRAPHITE	12	.081	6740	8730	-	.17	1800	1.2	78.0	.7
INCONEL	56	.300	2600	-	-	.109	200	6.2	98.0	31
IRON	56	.284	2795	5438	119	.108	1064	6.1	9.8	29.7
LEAD	207	.410	621	3171	11	.03	1652	15.2	20.7	2.6
MAGNESIUM	24	.063	1204	2025	118	.249	2106	13.2	4.5	6.3
MANGANESE	55	.269	2268	3904	118	.107	-	11.7	185	23
MERCURY	201	.489	-38	675	5	.033	112	-	95.8	-
MOLYBDENUM	96	.369	4748	8677	-	.065	1938	2.8	4.8	-
NICKEL	59	.320	2646	5252	135	.112	784	7.1	6.9	30.
PLATINUM	195	.775	3224	7932	50	.032	932	4.5	9.8	24.2
SILVER	108	.379	1761	3634	44	.056	5454	9.7	1.6	11.2
TIN	119	.264	449	4118	26	.548	879	11.5	11.5	4.0
TITANIUM	48	.163	3272	9212	-	.142	223	3.7	3.0	15.5
TUNGSTEN	184	.697	6098	10701	80	.034	2666	2.3	5.5	51.4
URANIUM	238	.676	3075	6332	-	.028	-	-	60.0	-
VANADIUM	51	.205	3110	5432	-	.115	373	5	26.0	19
ZINC	65	.258	787	1665	44	.093	1501	23	6.2	17.0

TABLE 5.1 - COMPARISON OF PHYSICAL PROPERTIES OF ELEMENTS

C values must also be approved by the procuring agency. These values should be developed in accordance with the procedure outlined in MIL-HDBK-5 for allowable development.

Regardless of the values used the structure must sustain static ultimate load without failure. The use of any value in excess of A values does not justify static test failures.

5.1.3 Structural Design Criteria

It is the responsibility of the Airframe Structures group to prepare a Structural Design Criteria report for each model helicopter. This document is based on the requirements of the detail specification for the helicopter. It contains the structural criteria to which the helicopter will be designed and tested. This report should specify the materials, limitations and allowable basis.

5.2 MATERIAL FORMS

The metallic materials most commonly used at Bell Helicopter come in a variety of forms. They are available as plate, sheet, bar, extrusion, forging and casting. The selection of a form for a particular part should be based on material properties, machining costs, compatibility with part shape and manufacturing methods, total manufacturing cost and availability.

All of the rolled, drawn, extruded and forged forms exhibit anisotropic properties. These properties often differ considerably along the principal axes particularly in forgings. The directional characteristics are produced by moving the material during forming. Castings, since they are formed in the molten state, do not exhibit these directional properties. The anisotropic properties are defined as longitudinal (L), long transverse (LT) and short transverse (ST). These terms define the direction the grain is formed within the material. Figure 5-1 shows typical examples of grain directions in various forms of materials.

5.2.1 Extruded, Rolled and Drawn Forms

These forms are produced by rolling and by forcing or drawing metal through dies to the proper shape. In these processes, the grains are elongated in the direction of extrusion or rolling and are parallel to the longitudinal direction of the finished product as it comes from the mill. These processes can be used to make many different products. Some common products and forming methods are defined as follows:

Sheet - rolled flat products less than .250 inch thick (most materials)

Clad - a thin coating of metal bonded to the base alloy by cold rolling. The purpose is to improve the corrosion resistance of the basic metal. Thickness of the clad is generally 2½ - 5% of the total thickness per side.

Plate - rolled flat products generally .250 inch or greater in thickness.

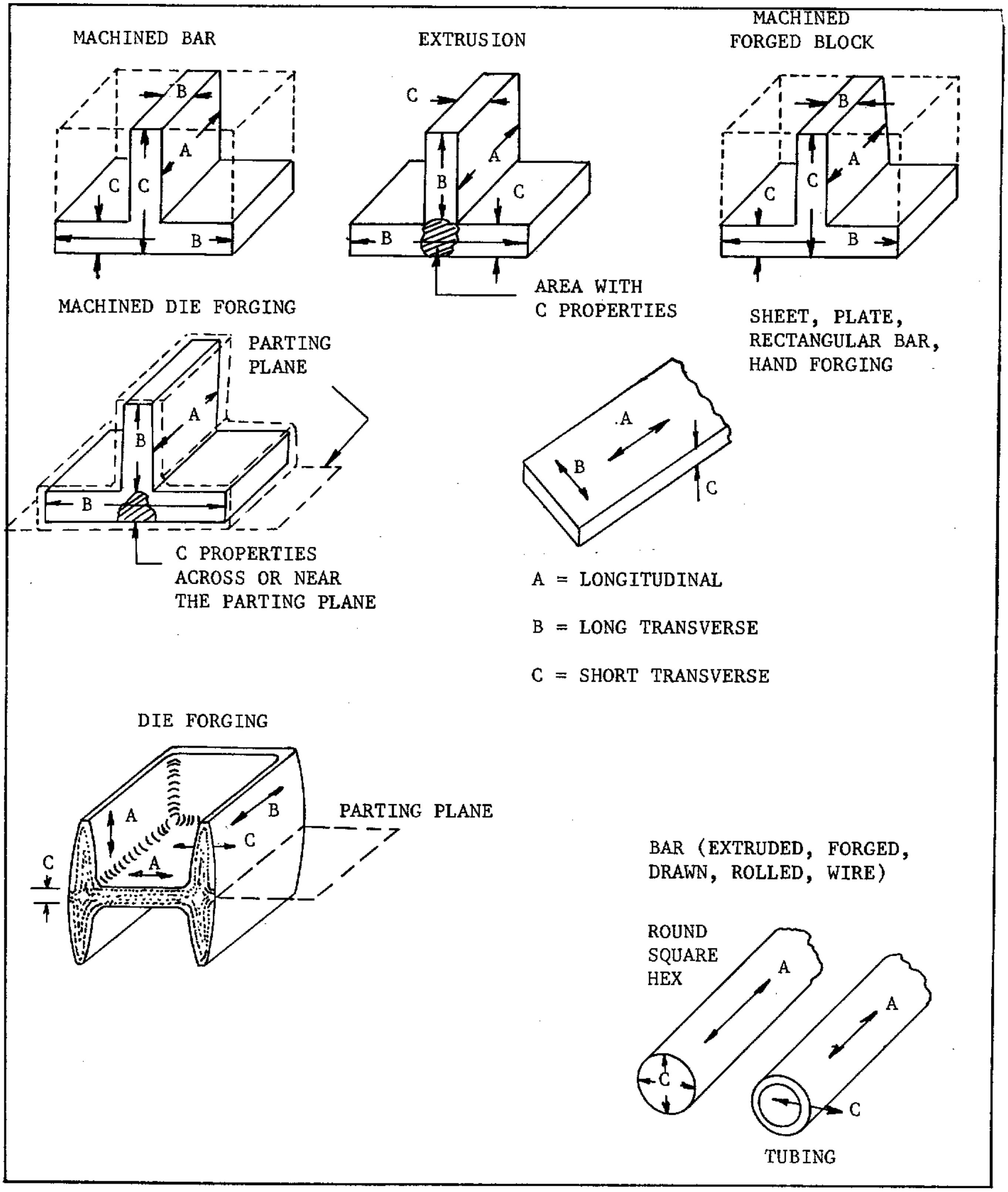


FIGURE 5.1 - EXAMPLES OF GRAIN DIRECTIONS IN VARIOUS MATERIAL FORMS

Extrusion - formed by forcing metal at elevated temperature through a die.

Stepped extrusion - a single product with one or more abrupt changesin cross section.

Impact extrusion - a product formed by a punch striking unheated metal in a confining die and consequently extruded through an opening or around the punch.

Hot impact extrusion - the same as the previous impact extrusion except the metal is preheated.

Drawing - forming the cross section by pulling it through a die.

Wire - a drawn form with a diameter or width across the flats of less than .375 inch.

Rod - a drawn form with a diameter greater than .375 inch.

Bar - a drawn form with a width across the flats greater than .375 inch.

5.2.2 Forged Forms

Forged forms are produced by impacting or pressing the material into a predetermined configuration. Pre-forge stock has generally been pressed, rolled, hammered or extruded to produce a well wrought material. Pressure is then applied to force the pre-forge stock into the desired shape. Pre-forms and multiple dies are often necessary to produce the desired configuration and tolerances in die forgings. The mechanical properties of forgingsare maximum parallel to the grain flow. This is the primary advantage of a forging. Parts should be designed to fully utilize these properties. However, the disadvantage of a forging is that the mechanical properties transverse to the grain flow and parallel to the compression forces exerted on the material are minimum. These are called the short transverse properties and the part should be designed to minimize the occurrence of areas where short transverse can occur.

Terms normally used to describe forgings are as follows:

Forging - plastically deforming metal into desired shapes. Dies may or may not be used.

Hand forgings - a product formed by hot working the material, usually between simple flat dies, into the desired shape. This process requires the least expensive dies but most expensive machining. These forgings come in two types:

Hand forged billet - a forged block with basically unidirectional properties. Must be completely machined to shape.

Shaped hand forgings - a product forged into a shape that generally outlines the basic contour of the desired final configuration.

Blocker die forgings - the final forging resembles the machined configuration and is generally machined on all surfaces. Approximately .25 to .50 inch excess material is allowed on all surfaces of the blocker forging depending on part size, configuration and material. Low die cost. High material cost.

Conventional die forgings - made in closed dies. The final forging more closely resembles the machined part than the blocker die forging. Usually some "as forged" surfaces are left on the finished part. Medium die cost, medium machine cost.

Close-to-form die forging - the forging approximates the finished part as closely as possible. Machining is minimized and most surfaces are used "as forged". Highest die cost. Lowest machine cost.

Flash - metal which is extruded through the space between die halves along the parting line.

Mismatch - the die halves will not match perfectly. The offset of the dies at the parting line.

As Forged - the term "as forged" is used to describe the surfaces that are not machined. Forged surfaces of steel and titanium are usually chem-milled to remove decarburization and alpha case but are still referred to as "as forged" surfaces.

All forgings produced at Bell Helicopter must satisfy the requirements of Bell Process Specification BPS FW4017. This specification must be shown on the drawing.

5.2.3 Cast Forms

A casting is a product made by pouring molten material into a mold of predetermined shape. The material is allowed to solidify and then removed from the mold.

There are generally four types of castings available:

Sand casting - a mold of compacted sand is made for each individual part to be poured. The mold is broken away from the part after solidification.

Permanent mold casting - a mold of high strength steel alloy is most commonly used. The mold is reusable. Castings produced in these molds usually yield high quality parts due to the chilling action of the mold and core.

Investment casting - a mold of plaster is formed around an expendable or plastic pattern. The pattern is burned out of the mold leaving the desired cavity into which the molten metal is poured. More intricate parts can be cast by this method.

Die casting - molten metal is injected into metal dies under pressure. Die castings are not used in structural applications.

Casting strengths are not significantly directional but they do vary from part to part, lot to lot, and even within the individual part. This is usually caused by cooling rate differences due to thickness changes. It can also be caused by poor manufacturing techniques; such as improper location of chill blocks, risers and gates. Cooling rate can be controlled by use of chill blocks which tend to improve the structural quality of the casting.

Bell Process Specification BPS FW4163 should be specified on the drawing of all castings except die castings. The drawing should also contain the casting classification. The Structures Engineer should make sure the stress analysis contains the casting factors specified in the Structural Design Criteria and check to see that the casting is properly classified according to one of the following:

Class IA - A class IA casting is a casting, the single failure of which would result in the loss of the aircraft, one of its major components, or loss of control.

Class IB - Class IB includes all critical castings which are not included in Class IA and which would cause unintentional release of or inability to release any armament/store, failure of gun installation components, or failure of which may cause significant injury to occupants of the aircraft.

Class IIA - Class IIA castings are those not included in either Class IA or Class IB, which have a margin of safety of 200 percent or less ($MS \leq 2.0$).

Class IIB - Class IIB castings are those castings having a margin of safety greater than 200 percent ($MS > 2.0$), or those for which no stress analysis is required.

The Structures Engineer is responsible for insuring that an X-ray diagram is shown on the drawing. All engineering drawings for Classes IA, IB, and IIA shall contain an X-ray diagram. This diagram shall contain the following information:

(a) Areas of casting that may or may not be weld repaired per BPS 4470 when applicable.

(b) Location of critically stressed areas. (The critically stressed areas shall be shown by encircling those areas with phantom lines.)

(c) Designate X-ray views required. The X-ray laboratory shall specify the required views and shall initial the diagram if acceptable for X-ray views.

The Structures Engineer shall make certain that the Structural Materials Group and the X-Ray Laboratory have approved the casting drawing. This should be accomplished before the Structures Engineer approves the casting drawing.

Casting materials commonly used at Bell Helicopter are steel and aluminum. Although magnesium is readily castable, its properties vary significantly. Magnesium should not be used without the approval of the procuring agency.

5.3 ALUMINUM ALLOYS

Aluminum is a lightweight structural material that can be strengthened through alloying and, dependent upon composition, further strengthened by heat treatment and/or cold working. Among its advantages for specific applications are: low density, high strength-to-weight ratio, good corrosion resistance, ease of fabrication and diversity of form.

Wrought and cast alloys are identified by a four-digit number, the first digit of which generally identifies the major alloying element as shown in Table 5.2. For casting alloys, the fourth digit is separated from the first three digits by a decimal point and indicates the form, i.e., casting or ingot.

Wrought Alloys		Cast Alloys	
Alloy Number	Major Alloying Element	Alloy Number	Major Alloying Element
1XXX	99% Min. Aluminum	1XX.X	99% Min. Aluminum
2XXX	Copper	2XX.X	Copper
3XXX	Manganese	3XX.X	Silicon with added copper and/or magnesium
4XXX	Silicon	4XX.X	Silicon
5XXX	Magnesium	5XX.X	Magnesium
6XXX	Magnesium and Silicon	6XX.X	Unused series
7XXX	Zinc	7XX.X	Zinc
8XXX	Other Elements	8XX.X	Tin
9XXX	Unused series	9XX.X	Other Elements

TABLE 5.2 BASIC DESIGNATION FOR WROUGHT AND CAST ALUMINUM ALLOYS

5.3.1 Basic Aluminum Temper Designations

The temper designation appears as a hyphenated suffix to the basic alloy number. An example would be 7075-T73 where -T73 is the temper designation. Four basic temper designations are used for aluminum alloys. They are -F: as fabricated; -O: annealed; -H: strain hardened and -T: thermally treated. A fifth designation, -W, is used to describe an as-quenched condition between solution heat treatment and artificial or room temperature aging. Following is a list of tempers which define aluminum alloys.

-O: annealed. Applies to wrought products which are fully annealed.

-F: as fabricated: No special control over thermal conditions or strain-hardening is employed. For wrought products, there are no mechanical property limits.

-H: strain-hardened (wrought products only). Applies to products which have their strength increased by strain-hardening, with or without supplementary thermal treatments to produce some reduction in strength. The -H is always followed by two or more digits. The first digit following the H indicates the specific combination of basic operations as follows:

-H1: strain-hardened only. Applies to products which are strain-hardened to obtain the desired strength without supplementary thermal treatment. The number following this designation indicates the degree of strain-hardening. The numeral 8 has been assigned to indicate tempers having an ultimate tensile strength equivalent to that achieved by a cold reduction of approximately 75 percent following full anneal. Tempers between 0 and 8 are designated by numbers 1 through 7. Materials having an ultimate tensile strength about half way between that of the 0 temper (annealed) and the 8 temper are designated by the number 4; about midway between 4 and 8 by 6; and midway between 0 and 4 by 2. Any of the odd number designations can be obtained in the same manner, i.e., midway between the adjacent designations. The following generally defines the two-digit tempers:

-H10: annealed
-H12: strain-hardened to 1/4 hard
-H14: strain-hardened to 1/2 hard
-H16: strain-hardened to 3/4 hard
-H18: strain-hardened to full hard

-H2: strain-hardened and partially annealed. Applies to products which are strain-hardened more than the desired final amount and then reduced in strength to the desired level by partial annealing. The second digit indicates the same as in the -H1 tempers. Temper -H24 would be strain-hardened and partially annealed to 1/2 hard.

-H3: strain-hardened and stabilized. Applies to products which are strain-hardened and then stabilized by a low-temperature thermal treatment to increase ductility and prevent stress corrosion (applies only to alloys containing magnesium). The same second digit rules as described for -H1 tempers apply here also.

The third digit, when used, indicates a variation of the two-digit temper to which it was added. The minimum ultimate tensile strength of a three-digit H temper is at least as close to that of the corresponding two-digit H temper as it is to the adjacent two-digit H tempers.

-H111: Applies to products which are strain-hardened less than the amount required for a controlled H11 temper.

-H112: Applies to products which acquire some temper from shaping processes not having special control over the amount of strain-hardening or thermal treatment, but for which there are mechanical property limits.

The following H temper designations have been assigned for wrought products in alloys containing over a nominal 4 percent magnesium.

-H311: Applies to products which are strain-hardened less than the amount for a controlled H31 temper.

-H321: Applies to products which are strain-hardened less than the amount for a controlled H32 temper.

-H323: Applies to products which are specially fabricated to have acceptable resistance to stress corrosion cracking.

Products which are thermally treated with or without supplementary strain-hardening are designated with a -T temper. The T is followed by a digit or digits which designate the specific thermal treatment. Temper designations for aluminum alloys are as follows:

-T1: Cooled from an elevated temperature shaping process and naturally aged to a substantially stable condition.

-T2: Annealed (cast products only).

-T3: Solution heat treated and then cold worked. Applies to products which are cold worked to improve strength or in which the effect of cold work in flattening or straightening is recognized in mechanical property limits.

-T31: Solution heat treated and then cold worked by flattening or stretching. Applies to 2219 and 2024 sheet and plate per MIL-A-8920. Also applies to rivets driven cold immediately after solution heat treatment or cold storage. 2024 rivets are an example.

-T351: Solution heat treatment and stress relieved by stretching. This is equivalent to -T4 condition. It applies to 2024 plate and rolled bar and 2219 plate per MIL-A-8920.

-T3511: Solution heat treated and stress relieved by stretching with minor stretching allowed. This is equivalent to -T4 condition and applies to 2024 extrusions.

-T36: Solution heat treated and then cold worked by a reduction of 6 percent. Applies to 2024 sheet and plate.

-T37: Solution heat treated and then cold worked by a reduction of 8 percent. Applies to 2219 sheet and plate.

-T4: Solution heat treated and naturally aged to a substantially stable condition. Applies to products which are not cold worked after solution heat treatment, or in which the effect of cold work in flattening or straightening may not be recognized in mechanical property limits.

-T42: Solution heat treated and naturally aged by the user to a substantially stable condition. Applies to 2014-0 and 2024-0 plate and extrusions which are heat treated by the user from the annealed condition.

-T451: Solution heat treated and stress relieved by stretching. Equivalent to -T4 and applies to plate and rolled bar stock except 2024 and 2219.

-T4511: Solution heat treated and stress relieved by stretching with minor straightening allowed. Equivalent to -T4 and applies to all extrusions except 2024 and 2219.

-T5: Cooled from an elevated temperature shaping process and then artificially aged.

-T51: Cooled from an elevated temperature shaping process, stress-relieved by stretching and then artificially aged.

-T52: Cooled from an elevated temperature shaping process, stress-relieved by compressing and then artificially aged.

-T54: Cooled from an elevated temperature shaping process, stress-relived by stretching and compressing and then artificially aged. Applies to die forgings which are stress-relieved by restriking cold in the finish die.

-T6: Solution heat treated and then artificially aged. Mechanical property limits not affected by cold working. Most alloys in the -W and -T4 conditions artificially aged to -T6.

-T61: Solution heat treated and then artificially aged. Applies to forgings which receive a boiling water quench to avoid internal quenching stress. Applies to solution heat treated and artificially aged castings when more than one aging cycle is available for that alloy.

-T611: Solution heat treated and artificially aged. Applies only to 7079 forgings which are quenched in 175° to 185°F water.

-T62: Solution heat treated and then artificially aged by the user. Applies to any temper which has been heat treated and aged by user which attains mechanical properties different from those of the -T6 condition.

-T651: Solution heat treated, stress-relieved by stretching and artificially aged. Equivalent to -T6 and applies to plate and rolled bar except 2219.

-T6510: Solution heat treated, stress-relieved by stretching and artificially aged with no hard straightening after aging. Applies to extruded rod, bar and shapes except 2024.

-T6511: Solution heat treated, stress-relieved by stretching and artificially aged with minor straightening. Equivalent to -T6 and applies to extruded rod, bar and shapes except 2024.

-T652: Solution heat treated, stress-relieved by compressive deformation and artificially aged. Equivalent to -T6 and applies to hard forged squares, rectangles and simply shaped die forgings except 2219.

-T7: Solution heat treated and then stabilized. Applies to products which are stabilized to carry them beyond the point of maximum strength to provide control of growth and residual stress.

-T73: Solution heat treated and then specially artificially aged. Applies to 7075 alloys which have been specially aged to make the material resistent to stress-corrosion.

-T7351: Solution heat treated and specially artificially aged. Applies to 7075 alloy sheet and plate which have been specially aged to make the material resistant to stress-corrosion.

-T73511: Solution heat treatment and specially artificially aged. Applies to 7075 alloy extrusions which have been specially aged to make the material resistant to stress-corrosion.

-T7352: Solution heat treated and specially artificially aged. Applies to 7075 alloy forgings which have both compression-stress relief and special aging to make the material resistant to stress-corrosion.

-T8: Solution heat treated, cold worked and then artificially aged. Applies to products which are cold worked to improve strength, or in which the effect of cold work in flattening or straightening is recognized in the mechanical property limits.

-T81: Solution heat treated, cold worked and then artificially aged. Applies to 2024-T3 artificially aged to T-81.

-T851: Solution heat treated, stress-relieved by stretching and artificially aged. Applicable to plate, rolled bar and rod.

-T8511: Solution heat treated, stress-relieved by stretching and artificially aged. Applies to 2024 extrusions and 2219.

-T86: Solution heat treated, cold worked by a thickness reduction of 6 percent and then artificially aged. Applies to 2024 sheet and plate.

-T87: Solution heat treated, cold worked by a thickness reduction of 10 percent and then artifically aged. Applies to 2219 sheet and plate.

-T9: Solution heat treated, artificially aged and then cold worked. Applies to products which are cold worked to improve strength.

-T10: Cooled from an elevated temperature shaping process, artificially aged and then cold worked. Applies to products which are artificially aged after cooling from an elevated temperature shaping process, such as casting or extrusion and then cold worked to further improve strength.

5.3.2 Aluminum Alloy Processing

The processes through which various aluminum alloys must be subjected to achieve a particular temper are shown in Figures 5.2 through 5.4.

5.3.3 Fracture Toughness of Aluminum Alloys

Typical values of plane-strain fracture toughness, K_{I_C}, for several aluminum alloys are shown in Table 5.3. These are average values for the alloys and tempers for which valid data are available and are thus representative of the various products. They do not have the statistical reliability of the room temperature mechanical properties shown in subsequent sections.

5.3.4 Resistance to Stress-Corrosion of Aluminum Alloys

The high strength heat treatable wrought aluminum alloys in certain tempers are susceptible to stress corrosion to some degree, dependent upon product, section size, direction and magnitude of stress. These alloys include 2014, 7075, 7079 and 7178 in the T6 tempers and 2014, 2024 and 2219 in the T3 and T4 tempers. Other alloy temper combinations, notably 2024 and 2219 in the T6 or T8 tempers and 7049, 7075 and 7175 in the T73 tempers, are decidedly more resistant and sustained tensile stresses of 50 to 75 percent of the minimum yield strength may be permitted without concern about stress-corrosion cracking. The T76 temper of 7075 and 7178 provides an intermediate degree of resistance to stress-corrosion cracking, i.e., superior to that of the T6 temper, but not as good as that of the T73 temper of 7075. A measure of the degree of susceptibility of various products of these alloys and tempers is given in Table 5.4.

Where short times at elevated temperatures of 150° to 500°F may be encountered, the precipitation heat-treated tempers of 2024 and 2119 alloys are recommended over the naturally aged tempers.

Alloys 5083, 5086 and 5456 should not be used under high constant applied stress for continuous service at temperatures exceeding 150°F, because of the hazard of developing susceptibility to stress corrosion cracking.

In general, the H34 through H38 tempers of 5086 and the H32 through H38 tempers of 5083 and 5456 are not recommended, because these tempers can become susceptible to stress corrosion cracking.

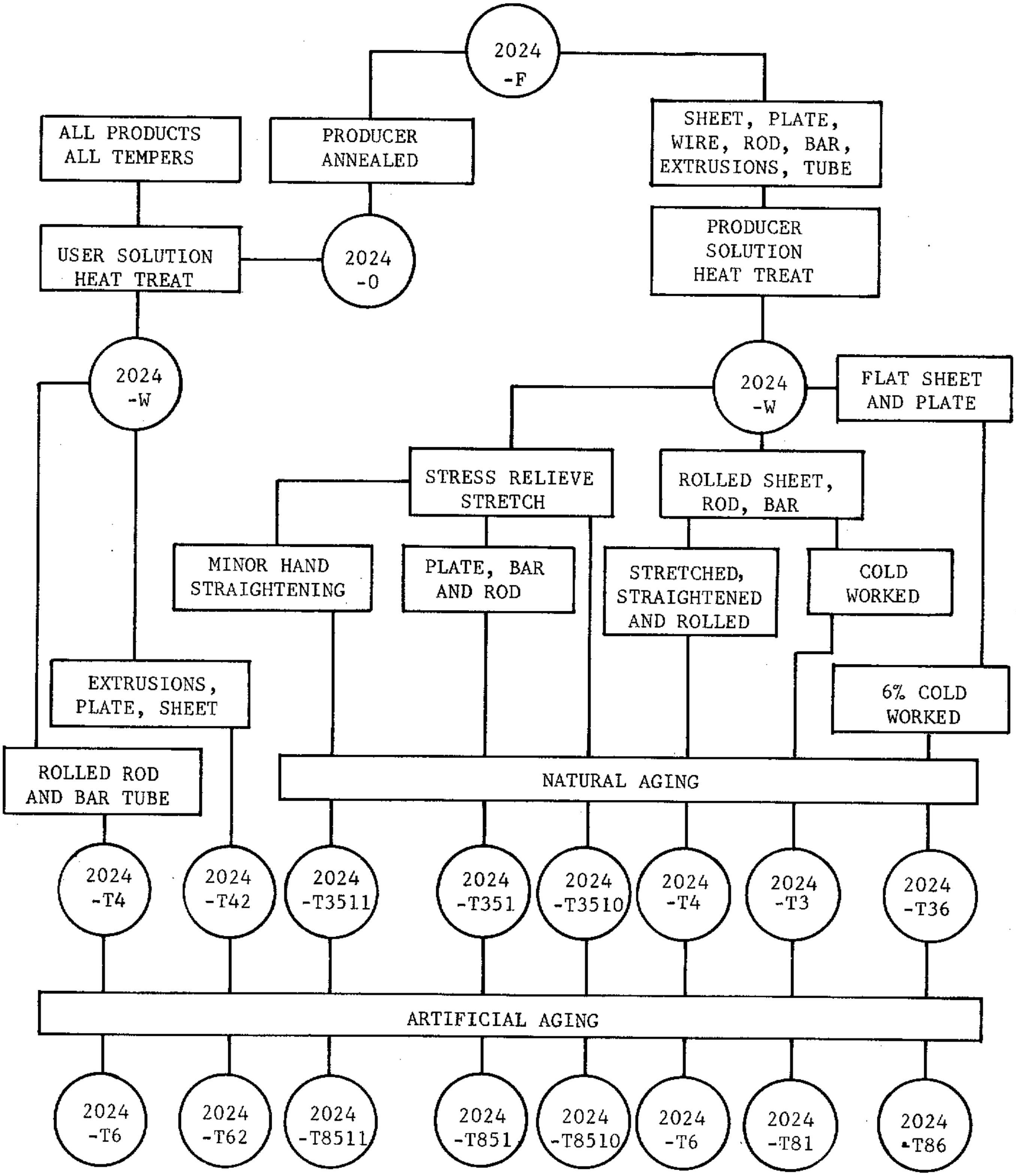


Figure 5.2--Temper Processing Chart for 2024 Alloys

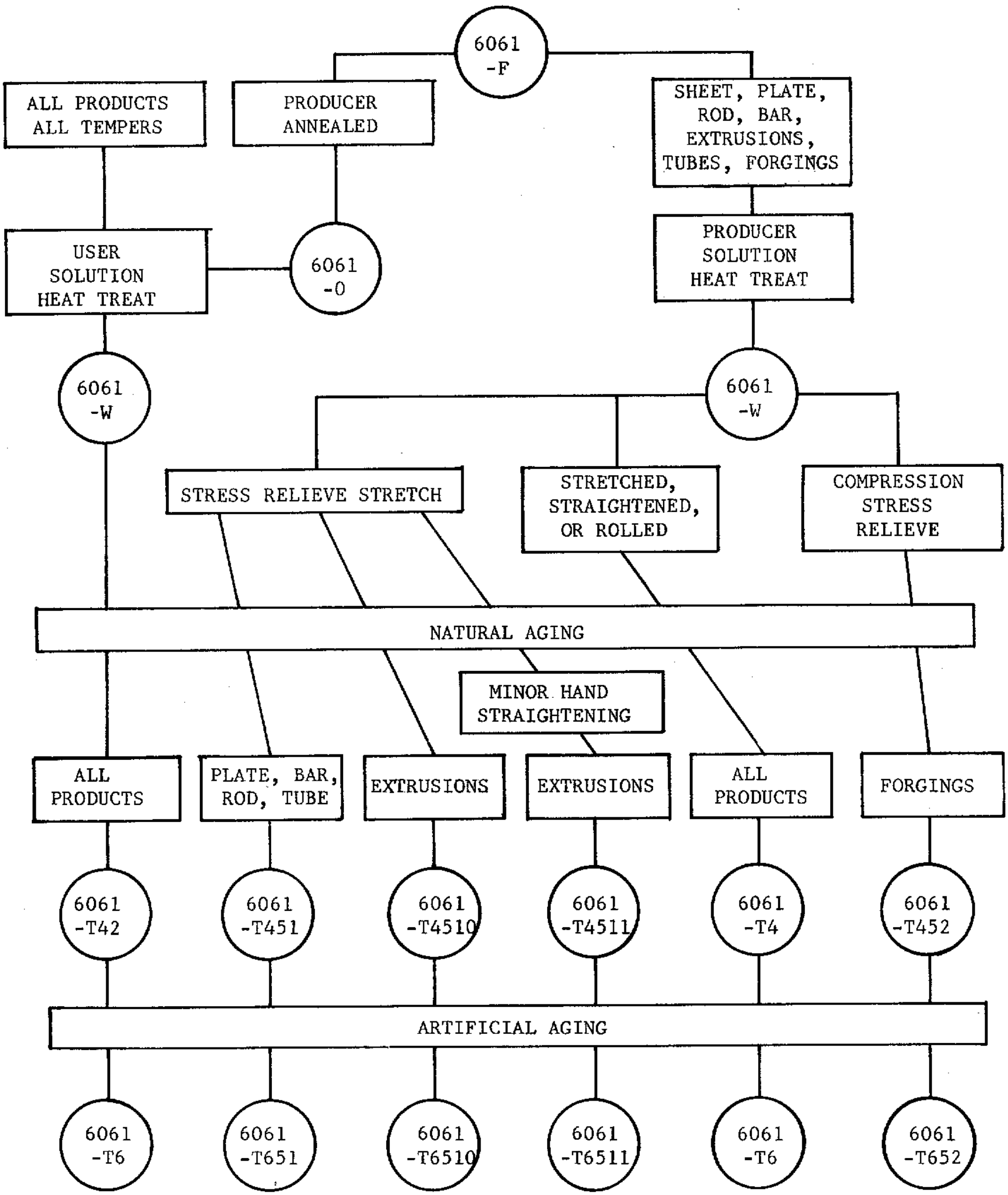


Figure 5.3--Temper Processing Chart for 6061 Alloys

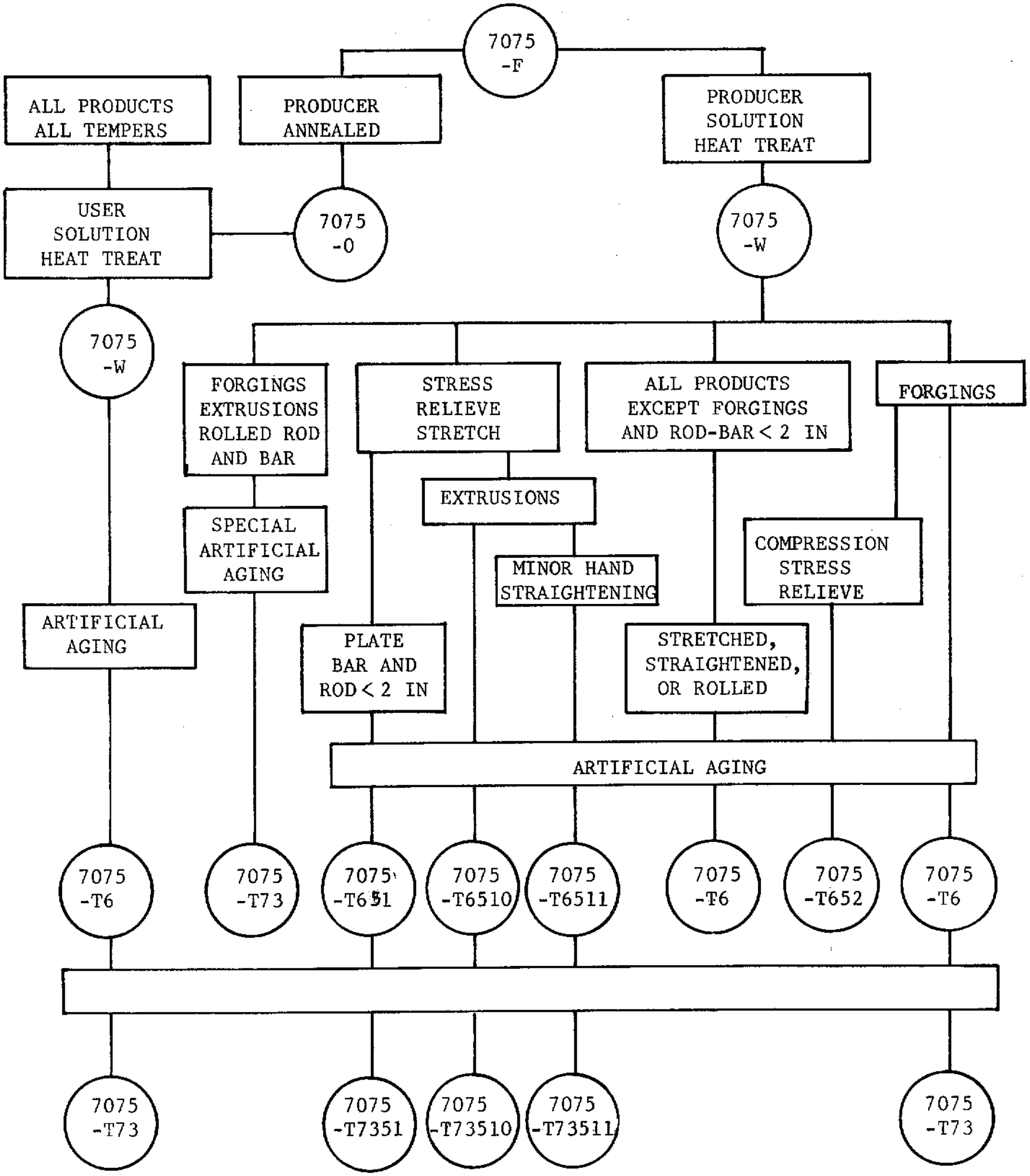


Figure 5.4--Temper Processing Chart for 7075 Alloys

Alloy	Product	Temper	Product Thickness Range, Inch	Plane-Strain Fracture Toughness, K_{Ic}, ksi $\sqrt{in.}$														
				(L)(T)					(T)(L)					(ST)(L)				
				No. of Lots	Thick. Inch	Min	Avg	Max	No. of Lots	Thick. Inch	Min	Avg	Max	No. of Lots	Thick. Inch	Min	Avg	Max
2014	Plate	T651	1 - 2	3	1	22	23	24	4	1	19	21	23	1	1/2	---	18	---
	Forgings	T652	2 - 6	4	3/4	25	29	34	4	3/4	19	23	30	2	1/2	18	19	19
2024	Plate	T351	1-1/2	---	1-1/2	---	---	---	---		---	---	---	2	1	20	23	26
	Extruded Shapes	T3510,1	1-1/2- 2	2	1-1/2	---	46	---	---		---	---	---	---		---	---	---
	Plate	T851	1 - 2	4	1-3/8	22	23	25	2	1-3/8	20	20	20	1	1	---	17	---
	Extruded Shapes	T8510,1	3/4- 4	5	3/4	22	28	32	4	3/4	16	17	18	---		---	---	---
	Forgings	T852	2 - 6	5	3/4	23	26	30	4	3/4	17	18	20	3	1/4	16	16	17
2219	Plate	T851	1 - 2	2	1-3/8	31	33	36	2	1	29	30	30	1	3/4	---	20	---
		T87	3/4- 1	3	3/4	26	27	28	---		---	---	---	1	1/2	---	20	---
7075	Plate	T651	1/2- 2	6	1/2	25	26	27	4	1/2	20	22	23	2	1/2	15	16	18
	Extruded Shapes	T6510,1	1/2- 4	10	1/2	26	28	32	10	1/2	19	22	26	4	1/4	18	19	22
	Forgings	T652	2 - 6	2	1/2	24	26	28	1	1/2	---	23	---	1	1/2	---	17	---
	Plate	T7351	1-3/8	1	1-3/8	---	30	---	2	1	25	29	33	2	1/2	19	20	21
	Extruded Shapes	T73510,1	1/2- 4	2	5/8	31	33	34	5	1/2	22	24	28	1	1	---	20	---
	Forgings	T7352	1 - 5	5	3/4	27	31	35	3	3/4	23	25	26	3	1/2	19	21	25
7079	Plate	T651	1 - 3	3	1	27	29	30	2	1	24	24	24	1	1/2	---	16	---
	Forgings	T652	2 - 6	2	3/4	28	30	31	3	3/4	21	23	25	3	1/4	17	18	18
7178	Plate	T651	1/2- 2	3	1	22	23	26	4	1/2	19	21	23	1	1/2	---	15	---
	Extruded Shapes	T6510,1	1/2- 1-1/2	1	1	---	25	11	4	1/2	16	19	20	1	1	---	14	---
	Plate	T7671	1/2- 2	3	1/2	26	29	30	2	1/2	22	22	22	1	1/2	---	17	---
	Extruded Shapes	T7650,1	1/2- 2	5	1/2	26	29	31	3	5/8	18	22	28	1	1/2	---	16	---

TABLE 5.3 - TYPICAL VALUES OF ROOM TEMPERATURE PLANE-STRAIN FRACTURE TOUGHNESS OF ALUMINUM ALLOYS (REF 1)

Revision E

Alloy and Type of Temper	Estimate of Highest Sustained Tension Stress (ksi) at Which Test Specimens of Different Orientations to the Grain Structure Would Not Fail in the 3½% NaCl Alternate Immersion Test in 84 Days					
	Test Direction	Plate	Rolled Bar and Bar	Extruded Shapes Section Thickness, Inch 0.25-1	1-2	Hand Forgings
2014-T6	L	45	45	50	45	30
	LT	30	..	27	22	25
	ST	8	15	..	8	8
2219-T8	L	40	..	35	35	38
	LT	38	..	35	35	38
	ST	38	..	..	35	38
2024-T3, T4	L	35	30	50	50	..
	LT	20	..	37	18	..
	ST	8	10	..	8	..
2024-T8	L	50	47	60	60	43
	LT	50	..	50	50	43
	ST	30	43	..	45	15
7075-T6	L	50	50	60	60	35
	LT	45	..	50	32	25
	ST	8	15	..	8	8
7075-T76	L	49	..	52	..	..
	LT	49	..	49	..	..
	ST	25	..	25	..	..
7075-T73	L	50	50	54	53	50
	LT	48	48	48	48	48
	ST	43	43	46	46	43
7079-T6	L	55	..	60	60	50
	LT	40	..	50	35	30
	ST	8	..	..	8	8
7178-T6	L	55	..	65	65	..
	LT	38	..	45	25	..
	ST	8	..	..	8	..
7178-T76	L	52	..	55	..	..
	LT	52	..	52	..	..
	ST	25	..	25	..	..

TABLE 5.4--COMPARISON OF THE RESISTANCE TO STRESS CORROSION OF VARIOUS ALUMINUM ALLOYS (REF. 1)

In the recommended tempers, H113 and H321 for sheet and plate, cold forming of 5083 and 5456 should be held to a minimum radius of 5T. Hot forming of the 0 temper alloys 5083 and 5456 is recommended and is preferred for the H113 and H321 tempers in order to avoid excessive cold work and high residual stress. If the H113, H321, H323 and H343 tempers are heated for hot forming a slight decrease in mechanical properties, particularly yield strength may result.

In order to avoid stress-corrosion cracking, practices, such as the use of press or shrink fits; taper pins; clevis joints in which tightening of the bolt imposes a bending load on lugs; and straightening or assembly operations; which result in sustained surface tensile stresses, should be avoided in these alloys: 2014-T451, T4, T6, T651; 2024-T3, T351, T4; 7075-T6, T651, T652 and 7178-T6 and T651.

Where straightening or forming of heat-treated materials is necessary, it should be performed when the material is in the freshly quenched condition, or at an elevated temperature to minimize the residual stresses induced. Where elevated temperature forming is performed on 2014-T4, T451 or 2024-T3, T351, a subsequent precipitation heat treatment to produce the T6 or T651, T81 or T851 temper is recommended.

Specific guidance on safe stress levels for avoiding stress-corrosion cracking is shown in Table 5.4. These stresses represent the algebraic sum of all the continuous tension and compression surface stresses resulting from any source such as quenching, forming, assembly and design. These stresses should be kept below those given in Table 5.4. It is particularly important to consider clamp-up stresses and pressfit stresses. If stress levels cannot be kept within the Table 5.4 limits, the Airframe Structures Group Engineer should be consulted.

5.3.5 Mechanical Properties of Aluminum Alloys

The mechanical properites of aluminum alloys are specified in MIL-HDBK-5. A description of some common aluminum alloys follows:

2014 is an Al-Cu alloy available in a wide variety of product forms. It is useful for application over the range from cryogenic to elevated temperatures. Resistance to stress corrosion is discussed in Section 5.4.4.

2024 is a heat-treatable Al-Cu alloy which is available in a wide variety of product forms and tempers. The T3 and T4 tempers have high toughness while the T6 and T8 tempers have high strengths. The T6 and T8 tempers also offer good resistance to stress corrosion cracking, while the T3 and T4 tempers should be considered in light of the guidelines in Section 5.3.4. 2024 alloy is not weldable by commercial practices.

5052 is a low-strength Al-Mg alloy. It is very ductile and very readily welded and is usually used in applications where these characteristics are more important than strength. It is highly resistant to corrosion. It is extremely tough at low as well as room temperatures.

6061 is a very readily weldable Al-Mg-Si alloy available in a wide range of product forms. It has high resistance to corrosion.

7075 is a high-strength Al-Zn-Mg-Cu alloy and is available in a wide variety of product forms. It is available in several types of tempers, the T6, T73 and T76 types. T6 has the highest strength and lowest toughness, and is susceptible to stress-corrosion cracking. Since toughness decreases with a decrease in temperature, the T76 temper is not generally recommended for cryogenic applications. The T73 temper has the lowest strength, but is relatively tough and very resistant to stress-corrosion cracking and exfoliation attack. The T76 temper is a compromise providing higher strength than the T73 temper and higher resistance to stress corrosion than the T6 temper. 7075 is not commercially weldable.

201.0 is a high-strength, heat-treatable Al-Cu-Ag casting alloy. It is readily weldable. In the T6 (aged) temper, it possesses high strength and good ductility, but is not recommended for use in environments conducive to stress-corrosion cracking. In the T7 (over-aged) temper, it possesses a high strength and moderate ductility and optimum resistance to stress-corrosion cracking.

224.0 is a heat-treatable Al-Cu-Zr casting alloy. When solution heat treated and over-aged, it possesses excellent mechanical properties at elevated temperatures, good fatigue properties and toughness.

295.0 is a heat-treatable Al-Cu casting alloy with high strength at elevated temperatures. Casting characteristics are only fair and it is very readily welded.

354.0 is a heat treatable Al-Si-Mg alloy having among the highest strength of commercial casting alloys. It has good casting characteristics and is readily weldable. Its use is generally restricted to permanent mold castings.

355.0 is a heat-treatable Al-Si-Mg alloy that is readily cast, very readily weldable and has good pressure tightness.

C355.0 is an Al-Si-Mg alloy similar to 355.0 but has impurities controlled to lower limits resulting in higher strengths. It is very readily weldable and has good casting characteristics.

356.0 is among the easiest of alloys to cast by a variety of techniques. It is heat treatable, has intermediate strengths, is very readily weldable and has high resistance to corrosion.

A356.0 is an Al-Si-Mg alloy similar to 356.0 but with impurities controlled to lower limits resulting in higher strengths and ductility. It is very readily weldable, has good casting characteristics and high resistance to corrosion.

A357.0 is an Al-Si-Mg alloy generally used for permanent mold and premium quality castings in which special properties are developed by careful control of casting and chilling techniques. It has excellent casting characteristics, is heat treatable and provides the highest strengths available in commercial castings, together with good toughness. The alloy also has excellent corrosion resistance and is very readily welded.

359.0 is a relatively high-strength, permanent mold casting alloy. It is heat treatable, very readily weldable, and has good corrosion resistance.

5.4 STEEL ALLOYS

One of the major factors contributing to the general utility of steels is the wide range of mechanical properties they are capable of attaining. Softness and good ductility may be required during fabrication of a part and very high strength during its service life. Both sets of properties are obtainable in the same material.

All steels can be softened to a greater or lesser degree by annealing, depending on the chemical composition of the specific steel. Annealing is achieved by heating the steel to an appropriate temperature, holding, then cooling it at the proper rate. Likewise, steels may be hardened or strengthened by means of cold working, heat treating or a combination of these.

The basic classifications of steels are as follows:

A. Plain carbon steels - contain carbon as the only major alloying element.

B. Alloy steels - contain small percentages of alloying elements, thus modifying properties of the steel. Alloy steels include: The AISI alloy steels, the alloy tool steels, the high-strength steels and silicon steels.

C. Corrosion resistant steels - contain significant additions of chromium (greater than 12 percent by weight) plus other additions. The corrosion resistant steels are further broken down into: Martensitic, Ferritic, Austenitic and Precipitation Hardening. The martensitic and precipitation hardening grades are hardenable by heat treatment, the austenitic grades are hardenable by cold work and the ferritic grades are essentially unhardenable. By definition, steels containing 14 percent chromium or more are classified as corrosion resistant. Alloys with lesser chromium content require protective finishing.

D. Maraging and 9 Ni-4Co - New alloys containing substantial additions of nickel, cobalt and molybdenum. The maraging alloys are machined and formed in the solution treated condition, then hardened by aging at approximately 900°F. Fabrication and heat treatment procedures for 9 Ni-4Co are similar to those used for alloy steels. All usage of these steels should be used only with the guidance of the Structural Materials Technology Group.

Cold working is the method used to strengthen both the low-carbon unalloyed steels and the highly alloyed austenitic stainless steels. Only moderately high strength levels can be attained in the former but the latter can be cold rolled to quite high strength levels, or "tempers". These are commonly supplied to minimum strength levels.

Heat treating is the principal method for strengthening the remainder of the steels (the low-carbon and the austenitic steels cannot be strengthened by heat treatment). The heat treatment of steel may be of three types: martensitic hardening, age hardening and austempering. Carbon and alloy steels are martensitic-hardened by heating them to a high temperature, or "austenitizing", then cooling at a recommended rate, often by quenching in oil or water. This is followed by "tempering" which consists of reheating to an intermediate temperature to relieve internal stresses and to improve toughness.

A relatively new class of steel is strengthened by age hardening. This heat treatment is designed to dissolve certain constituents in the steel, then precipitate them in some preferred particle size and distribution. Special combinations of working and heat treating are being employed to further enhance the mechanical properties of certain steels.

Another process used in the heat treatment of steels is austempering. In this process ferrous steels are austenitized, quenched rapidly to avoid transformation of the austenite to a temperature below the pearlite and above the martensite formation ranges, allowed to transform isothermally at that temperature to a completely bainitic structure and finally cooled at room temperature. The purpose of austempering is to obtain increased ductility or notch toughness at high hardness levels, or to decrease the likelihood of cracking and distortion that might occur in conventional quenching and tempering.

Steel bars, billets, forgings and thick plates, especially when heat treated to high strength levels, exhibit variations in mechanical properties with location and direction. In particular, elongation, reduction of area, toughness and notched strength are likely to be lower in either of the transverse directions than in the longitudinal direction. In applications where transverse properties are critical, requirements should be discussed with the steel supplier and properties in critical locations should be substantiated by appropriate testing.

5.4.1 Basic Heat Treatments of Steel

The mechanical properties of alloy steels are largely dependent on the use of proper thermal treatment. Some of these treatments and a description follow.

A. Annealing - A heating process which places the material in its softest condition to enhance formability and produce a desirable microstructure. Annealing consists of heating to temperatures of 1500-1600°F and slow cooling.

B. Normalizing - A homogenizing treatment used to improve machinability and response to hardening treatments. Normalizing consists of heating to 1600-1700°F and cooling in air.

C. Hardening (Quench)- A controlled cooling of parts heated above the transformation temperature to produce martensite. Parts are normally quenched in oil from a temperature of 1550-1600°F. The quenched material is extremely hard and brittle, so it must be tempered prior to use.

D. Tempering - Reheating a quench hardened or normalized part to a temperature below the transformation range to restore ductility and toughness. Hardness is reduced during tempering, so the temperature selected is a compromise to yield the optimum combination of strength and ductility.

E. Stress Relieving - A heating process which reduces the residual stresses occurring during machining, grinding, forming, etc. and reduces distortion during hardening.

F. Heat Treat Range - Steels used at BHT are usually heat treated between 125 ksi and 200 ksi tensile strength. The strength range usually has a 20 ksi spread, e.g., 145-165 ksi, 180-200 ksi, etc. When alloys are to be used above 200 ksi tensile, the Structural Materials Technology Group should be consulted.

5.4.2 Fracture Toughness of Steel Alloys

Steels when processed to obtain high strength or when tempered or aged within certain critical temperature ranges may become more sensitive to the presence of small flaws. The usefulness of high strength steels for certain applications is largely dependent on their toughness. It is generally noted that the fracture toughness of a given alloy product decreases relative to increases in the yield strength. Typical values of plane-strain fracture toughness, K_{IC}, for several high-strength alloy steels are presented in Table 5.5.

Alloy	Product	F_{TY} (KSI)	t (in.)	K_{IC}, KSI-in.$^{1/2}$ (L)(LT)[(a)]	K_{IC}, KSI-in.$^{1/2}$ (LT)(L)[(b)]
4340	Plate	260	3/8	53	53
5-Cr-Mo-V	Plate	260	1/2	34	32
5-Cr-Mo-V	Bar	275	1	23	--
17-4 PH	Plate	190 (H900)	1/2	42	38
17-4 PH	Bar	190 (H900)	5/8	52	--

(a) Longitudinal grain direction normal to the crack plane and long transverse grain direction parallel to the fracture direction.
(b) Long transverse grain direction normal to the crack plane and longitudinal grain direction parallel to the fracture direction.

TABLE 5.5 --TYPICAL VALUES OF ROOM TEMPERATURE PLANE STRAIN FRACTURE TOUGHNESS FOR AIR MELTED ALLOY STEELS. (REF. 1)

5.4.3 Mechanical Properties of Steel Alloys

Table 5.6 shows the maximum diameters to which various alloy steels may be through hardened consistently by quenching. The values shown are based on through hardening to at least 90 percent martensite at center.

Table 5.7 shows temperature exposure limits for various alloy steels.

Material specifications for alloy steels are shown in Table 5.8. Mechanical and physical properties are specified in MIL-HDBK-5.

5.5 MAGNESIUM ALLOYS

Magnesium is a lightweight structural metal which can be strengthened greatly by alloying, and in some cases, by heat treatment or cold work or both. Magnesium alloys are highly susceptible to corrosion and proper protection must be included in all designs. Proper drainage must be provided to prevent entrapment of fluids. Dissimilar metal joints must be properly and completely insulated. Magnesium alloys must not be used in elevated temperature applications since annealing can result after exposure to elevated temperatures. The use of magnesium must be approved by Structural Materials Technology Group.

Mechanical and physical properties of magnesium alloys are specified in MIL-HDBK-5. Standard temper designations for magnesium alloys are shown in Table 5.9.

F_{tu}	DIAMETER OF ROUND OR EQUIVALENT ROUND, inch						
	0.5	0.8	1.0	1.7	2.5	3.5	5.0
280 ksi	→	→	→	→	→	→	300M 98BV40
260 ksi	→	→	→	AISI 4340	AISI 4340	AISI 4340	......
220 ksi	→	→	→	AMS Grades	AMS Grades	D6AC	D6AC
200 ksi	→	AISI 8740	AISI 4140	AISI 4340 AMS Grades	AISI 4340 AMS Grades	AISI 4340 D6AC	D6AC
180 ksi and lower	AISI 4130 and 8630	AISI 8735 and 8740	AISI 4140	AISI 4340 AMS Grades	AISI 4340 AMS Grades	AISI 4340 D6AC	D6AC

Table 5.6 - Maximum Round Diameters for Alloy Steel Bars (Ref 1)

Alloy	F_{tu} ksi							
	125	150	180	200	220	240	260	280
AISI 4130 and 8630	925	775	575	...	...	...	...	...
AISI 4140 and 8740	1025	875	725	625	...	...	...	...
AISI 4340	1100	950	800	700	...	...	350	...
AISI 8735	975	825	675	...	...	...	...	...
D6AC	1150	1075	1000	950	900	...	500	...
AMS 6418	875	750	650	550	450	...	...	...
4330Si and 4330V	925	850	775	700	500	...	...	...
4335V	975	875	775	700	500	...	...	...
98BV40	...	...	...	...	...	...	...	400
300M	...	...	...	...	...	...	...	475

Table 5.7 - Temperature Exposure Limits for Alloy Steels (Ref 1)

Alloy	TYPE OF PRODUCT		
	Sheet, strip, and plate	Bars and forgings	Tubing
4130......	MIL-S-18729............	MIL-S-6758.............	MIL-T-6736.......
4140......	....	MIL-S-5626.............	AMS 6381, 6390...
4340......	AMS 6359..............	MIL-S-5000............. MIL-S-8844	AMS 6415......... MIL-S-8844.......
8630......	MIL-S-18728............	MIL-S-6050.............	MIL-T-6732....... MIL-T-6734.......
8735......	MIL-S-18733............	MIL-S-6098.............	MIL-T-6733.......
8740......	AMS 6358..............	MIL-S-6049.............	AMS 6323.........
D6AC......	MIL-S-8949............	MIL-S-8949.............	....
4330 Si...	....	AMS 6407...............	....
AMS 6418..	....	AMS 6418...............	....
4330V.....	....	AMS 6427...............	....
4335V.....	AMS 6434..............	AMS 6428...............	....
300M......	....	MIL-S-8844.............	MIL-S-8844.......
98BV40....	....	AMS 6423...............	AMS 6423.........

Table 5.8 - Material Specifications for Alloy Steels (Ref 1)

Temper	Definition
F	As fabricated
O	Annealed, recrystallized (wrought products only)
H	Strain hardened (wrought products only)
H2, plus one or more digits	Strain hardened and then partially annealed
T	Treated to produce stable tempers other than F, O, or H
T4	Solution heat-treated
T5	Cooled from an elevated temperature shaping process and then artificially aged
T6, T61	Solution heat-treated (T4) and then artificially aged
T7	Solution heat-treated (T4) and then stabilized
T8, T81	Solution heat-treated (T4) cold worked, and then artificially aged

TABLE 5.9 --TEMPER DESIGNATIONS FOR MAGNESIUM ALLOYS

5.6 TITANIUM ALLOYS

Titanium is a relatively lightweight, corrosion-resistant structural material that can be strengthened greatly through alloying and in some cases by heat treatment. It has good strength-to-weight ratios, low density, low coefficient of thermal expansion, good corrosion resistance, good oxidation resistance at intermediate temperatures and good notch toughness as well as other metallurgical advantages.

The material properties of titanium and its alloys are determined mainly by their alloy content and heat treatment, both of which are influential in determining the allotropic form in which this material will be found. Under equilibrium conditions, pure titanium has an "alpha" structure up to 1620°F, above which it transforms to a "beta" structure. The properties of these two are quite different. Through alloying and heat treatment one or the other, or a combination of these two structures, can be made to exist at service temperatures.

Titanium is susceptible to creep deformation in its unalloyed form below 300°F and above 700°F at stresses above 50 percent F_{ty}. This stress level should be avoided. Alloyed titanium at stresses above 60 percent F_{ty} will also be susceptible to creep deformation.

Mechanical and physical properties of titanium are specified in MIL-HDBK-5. Description of some commonly used titanium alloys follow:

Commercially Pure Titanium is unalloyed and is available in a familiar product form and is noted for its excellent formability. Unalloyed titanium is readily welded or brazed. It is used mainly where strength is not a requirement since it cannot be heat treated to high strength levels. Property degradation can be experienced after severe forming if as-received material properties are not restored by re-annealing.

Ti-8Al-1Mo-1V is a near-alpha composition alloy with improved creep resistance and thermal stability up to about 500°F. It is available as billet, bar, plate, sheet, strip and extrusions and is usually used in the single annealed or duplex annealed condition. Room temperature forming is difficult, and for severe operations hot forming is required. It can be fusion welded with inert gas protection and spotwelded without protection.

Ti-6Al-4V is an alpha-beta alloy and is available in all mill product forms as well as in castings and powder metallurgy forms. It can be used in either the annealed or solution treated plus aged (STA) conditions and is weldable. For maximum toughness, Ti-6Al-4V should be used in the annealed or duplex annealed condition whereas for maximum strength, the STA condition should be used. The full strength of this alloy is not available in thicknesses greater than 1 inch. This alloy can be fusion welded and spotwelded, but stress relief annealing after welding is recommended.

5.7 STRESS-STRAIN CURVES

Many useful properties are obtained from a stress-strain diagram of a material. Figure 5.5 shows a typical stress-strain diagram for a metal with no definite yield stress. Such metals as aluminum, magnesium and some steels fall into this category.

5.7.1 Typical Stress-Strain Diagram

The curve in Figure 5.5 is composed of two regions; the straight-line portion up to the proportional limit where the stress varies linearly with strain; and the remaining portion where the stress is not proportional to strain. Most analysis methods assume stresses to be elastic below the ultimate tensile stress (F_{tu}); however, some analyses will employ a plasticity reduction factor to correct for the nonlinearity of the plastic range.

The properties shown in Figure 5.5 are described below:

E — Modulus of elasticity; average ratio of stress to strain for stresses below the proportional limit. In Figure 5.5, $E = \tan\phi$.

E_s — Secant modulus; slope of the stress-strain curve at any point; reduces to E in the proportional range. In Figure 5.5, $E_s = \tan\phi_1$.

E_t — Tangent modulus; slope of the stress-strain curve at any point; reduces to E in the proportional range. In Figure 5.5, $E_t = df/d\epsilon = \tan\phi_2$.

F_{ty}, F_{cy} — Tensile or compressive yield stress; since many materials do no exhibit a definite yield point, the yield stress is determined by the 0.2% offset method. A straight line is constructed with a slope E passing through a point of zero stress and a strain of 0.002 in./in. The intersection of the stress-strain curve and the constructed straight line defines the magnitude of the yield stress.

F_{tp}, F_{cp} — Proportional limit stress in tension or compression; the stress at which the stress ceases to vary linearly with strain.

F_{tu} — Ultimate tensile stress; the maximum stress reached in tensile tests of standard specimens.

F_{cu} — Ultimate compressive stress; taken as F_{tu} unless governed by instability.

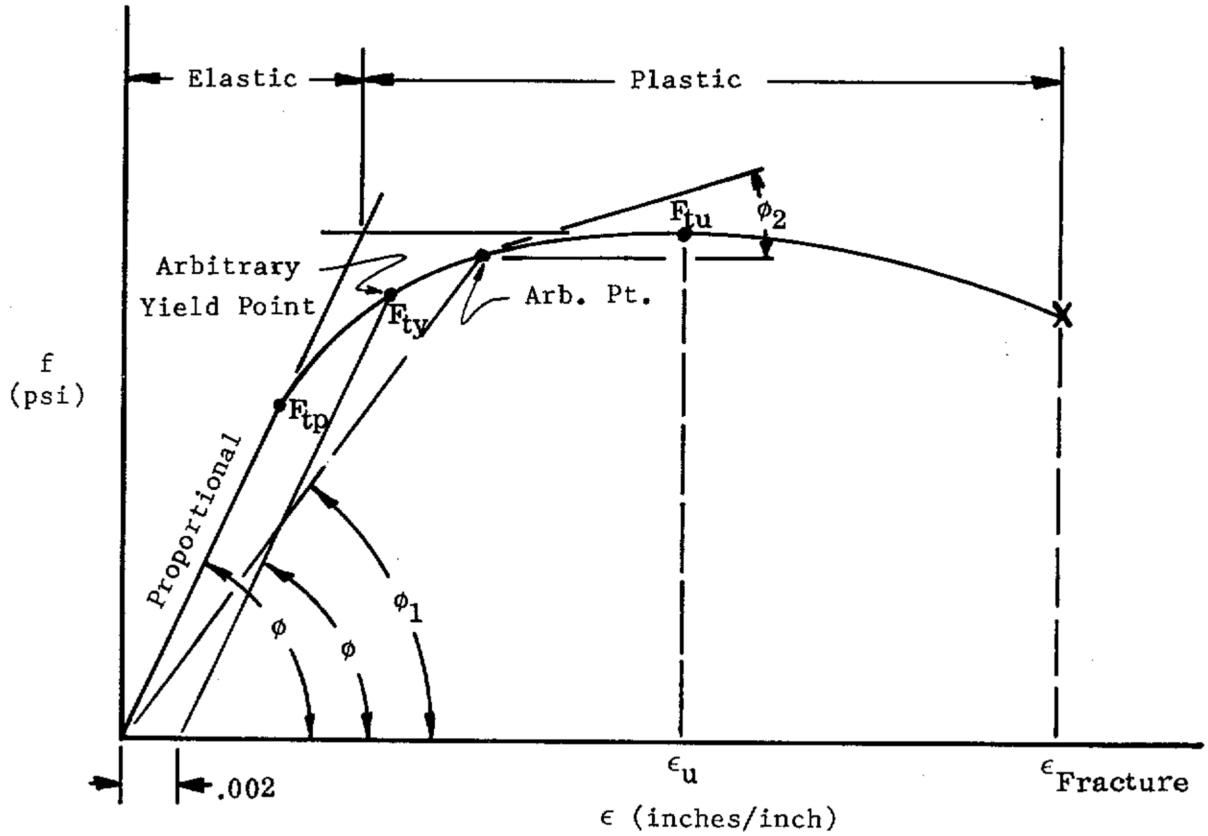


FIGURE 5.5 - A TYPICAL STRESS-STRAIN DIAGRAM

ϵ_u The strain corresponding to F_{tu}.

ϵ_e Elastic strain.

ϵ_p Plastic strain.

$\epsilon_{fracture}$ Fracture strain; a relative indication of ductility of the material.

There are other properties and terminology used by the Structures Engineer which are not shown in Figure 5.5. These are defined below.

F_{bry}
F_{bru} Yield and ultimate bearing stress; determined in a manner similar to those for tension and compression. A load deformation curve is plotted where the deformation is the change in hole diameter. Bearing yield (F_{bry}) is defined by an offset of 2% of the hole diameter while bearing ultimate (F_{bru}) is the actual failing stress divided by 1.15.

F_{su} Ultimate shear stress.

F_{sp} Proportional limit in shear; usually taken as 0.577 times the proportional limit in tension for ductile materials.

μ Poisson's ratio; the ratio of transverse strain to axial strain in tension or compression. For materials stressed in the elastic range, μ may be taken as a constant but for inelastic strains, μ becomes a function of axial strain.

μ_p Plastic Poisson's ratio; unless otherwise stated, μ_p may be taken as 0.5.

G Modulus of rigidity or shearing modulus of elasticity for pure shear in isotropic materials. $G = E/2\ (1 + \mu)$.

Isotropic Elastic properties are the same in all directions.

Anisotropic Elastic properties are different in different directions.

Orthotropic Distinct material properties in mutually perpendicular planes.

Stress-strain curves for various materials can be found in MIL-HDBK-5 (Ref. 1).

5.7.2 Ramberg-Osgood Method of Stress-Strain Diagrams

Many structural problems involve inelastic instability. The solutions require information from a compressive stress-strain curve. Often it is desirable to represent this curve analytically. A method has been developed by Walter Ramberg and William R. Osgood and reported in NACA TN902 (Ref. 9).

The Ramberg-Osgood method uses three parameters to represent the stress-strain relations in the inelastic range. The resulting equations are:

$$\frac{E_e}{F_{0.7}} = \frac{f}{F_{0.7}} + \frac{3}{7}\left(\frac{f}{F_{0.7}}\right)^n \quad \ldots\ldots(1)$$

$$n = 1 + \ln(17/7)/\ln(F_{0.7}/F_{0.85}) \quad \ldots\ldots(2)$$

where:

e = strain

E = modulus of elasticity

f = stress

$F_{0.7}$ = the stress at which a line of slope 0.7E drawn from the origin intersects the stress-strain curve (see Figure 5.6)

$F_{0.85}$ = the stress at which a line of slope 0.85E drawn from the origin intersects the stress-strain curve (see Figure 5.6)

The curves expressed by Equations 1 and 2 are plotted in Figure 5.7 and 5.8. Consult the stress strain curves of Ref. 1 for $F_{0.7}$ and $F_{0.85}$ for various materials.

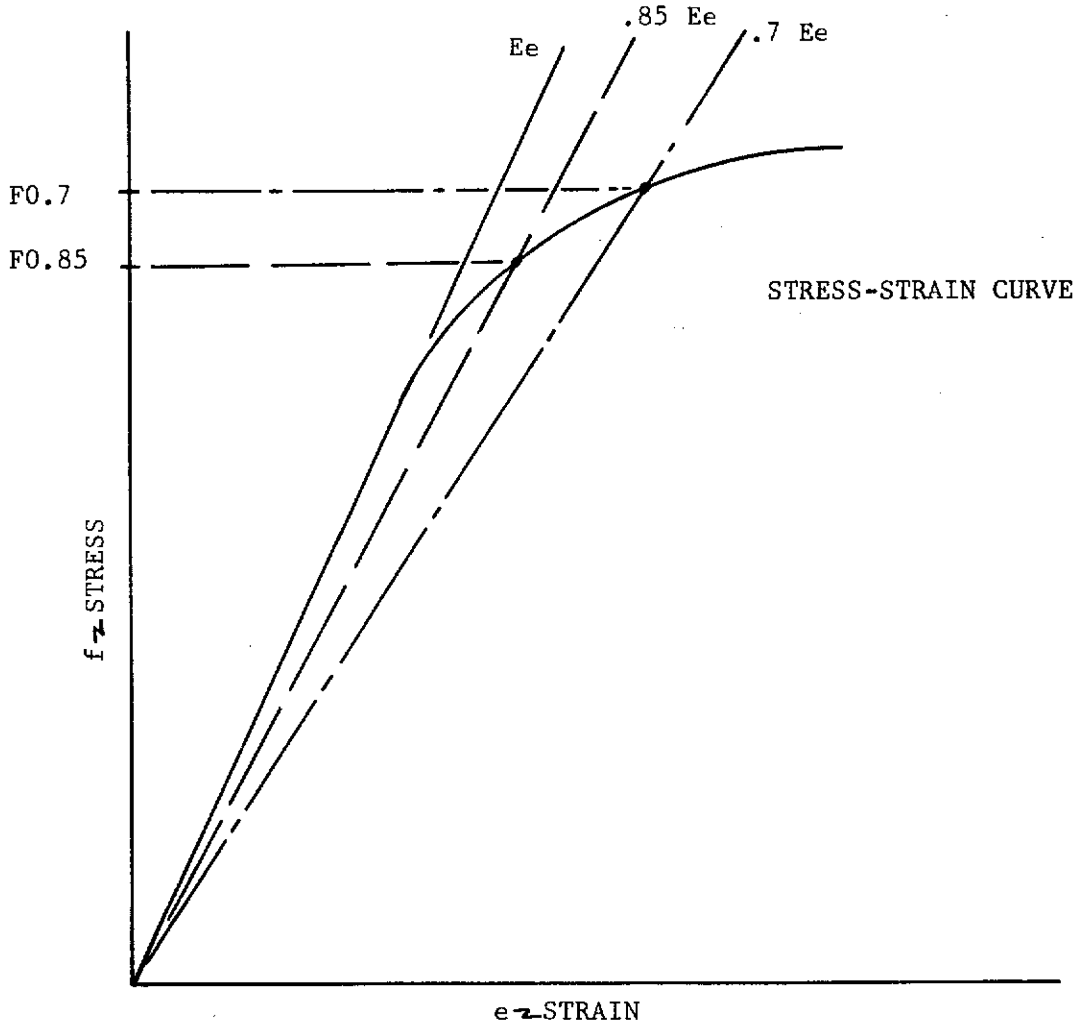


Figure 5.6 - Ramberg-Osgood Parameters

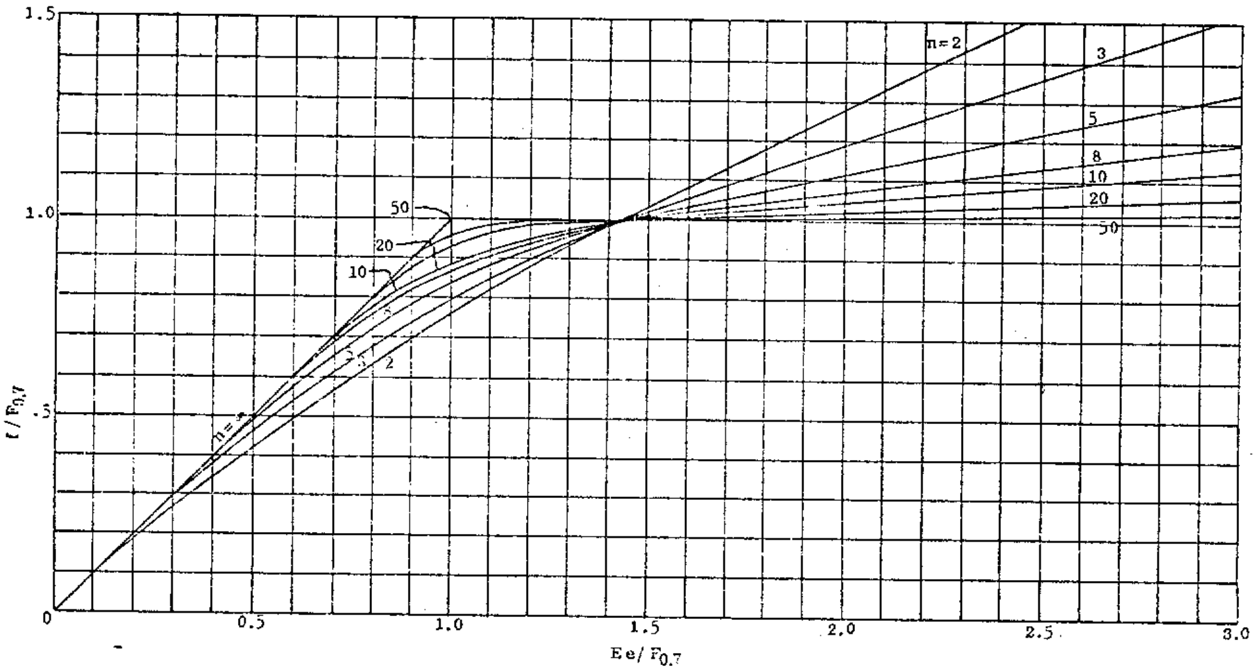


Figure 5.7 - Ramberg-Osgood Constant, n, as a function of $f/F_{0.7}$

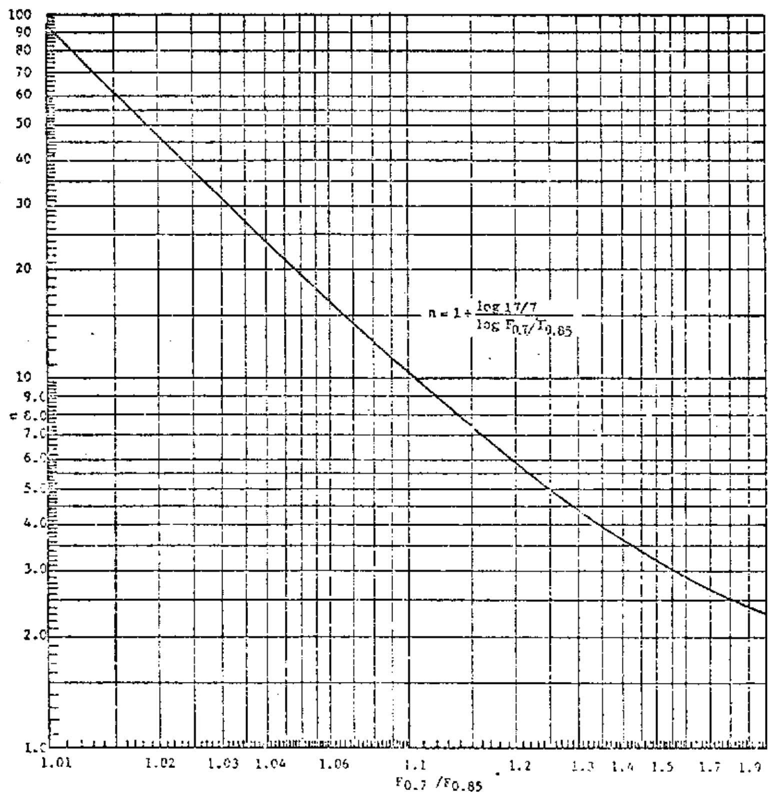


Figure 5.8 - Relation between n and $F_{0.7}/F_{0.85}$

SECTION 6

FASTENERS AND JOINTS

6.1 GENERAL

This section presents BHT policy on design allowables for mechanical fasteners, metallurgical joints and mechanical joints. Mechanical fasteners include solid and blind rivets and nuts, bolts and pins. Spotwelds and fusion welds are shown in the metallurgical joint section, while lugs, sockets, bearings and bonding are discussed in mechanical joints.

6.2 Mechanical Fasteners

The actual state of stress in a joint is complex. Such items as stress concentrations at the edges of the holes, non-uniform distribution of shear stress across the section of the fastener, and bearing stress between fastener and plate (installation stresses) are generally ignored in the sizing of fasteners. Simplifying assumptions are made for riveted and short bolted (no bending present) joints and are summarized below:

(1) The applied load is assumed to be transmitted entirely by the fasteners; friction between connected plates is ignored.

(2) When the center of the cross-sectional area of each of the fasteners is on the line of action of the load, or when the centroid of the total fastener area is on this line, the fasteners of the joint are assumed to carry equal parts of the load if of the same size; otherwise loaded proportionally to their section areas.

(3) The shear stress is assumed to be uniformly distributed across the fastener section.

(4) The bearing stress between the plate and fastener is assumed to be uniformly distributed over an area equal to the fastener diameter (hole diameter for rivets) times the plate thickness.

(5) The stress in a tension fastener is assumed to be uniformly distributed over the net area.

(6) The stress in a compression fastener is assumed to be uniformly distributed over the gross area.

No matter how well structural components are designed to carry their intended loads, a poor use of fasteners joining the components can cause the entire structure to fail with catastrophic results.

Joint failures can be grouped in three general categories or combinations of the three:

(1) Fastener shear

(2) Sheet bearing

(3) Sheet tearout

The first category is primarily a failure of the fastener, whereas the last two are primarily failures of the material being fastened. The material being fastened can have an effect on the shear strength of a fastener and the fastener geometry can have an effect on the bearing strength of the sheet being fastened.

6.2.1 Joint Geometry

In addition to the more obvious considerations of fastener and sheet material, an equally important consideration, joint geometry, is necessary to provide a joint capable of developing the fastener and sheet strengths. The three geometry parameters to consider are:

D, fastener hole diameter

t, sheet thickness

e, edge distance (center of hole to edge of sheet)

Joint allowables are influenced by e/D and D/t. Small values of e/D will produce shear tearout failures and lower bearing allowables. In addition, where fatigue is a consideration small values of e/D will result in fatigue cracking. For design, e/D shall not be less than 2.0. In fatigue critical areas, e = 2D + .06 should be maintained. For repair and manufacturing discrepancies the e/D ratio shall not go lower than 1.5.

The D/t ratio influences the sheet bearing stress distribution and fastener shear allowable. A high D/t indicates a fastener is too large for the sheets being joined. Above D/t = 3 the bearing stress distribution changes significantly enough to require a reduction of the basic allowable because the sheets tend to cut into the rivet.

A low D/t indicates a fastener that is too small for the sheets being joined. This situation would produce fastener shear failures rather than sheet bearing failures. This is a very undesirable situation. The joint can literally "zip" open if a failure of a single fastener should occur. If the joint is properly designed the sheet will fail in bearing before the fasteners shear. As a joint is loaded the end fasteners in a pattern will load first. As the first line loads the sheet deflects at each fastener in the line and a portion of the load transfers to the next line and so on until the whole pattern is carrying the load. If the joint is shear critical, the first line of fasteners will shear, then the second will overload and shear and so on. If the joint is bearing critical, a load path will remain after the sheet yields in bearing.

The D/t ratio shall not exceed 5.5 nor be less than 1.0.

Fasteners shall not be designed closer than 4D nor farther apart than 8D. For repair and manufacturing discrepancies spacing can be reduced to 3.5D.

Design for sheet bearing to be reached prior to fastener shear.

Do not mix non hole filling fasteners with hole filling fasteners. If a mixture cannot be avoided the non hole filling fastener should be installed in an interference fit hole.

6.2.2 Mechanical Fastener Allowables

MIL-HDBK-5 allowables should be used for all mechanical fasteners. A description of fastener types follows.

6.2.2.1 Protruding - Head Solid Rivets

The load per rivet at which the shear or bearing type of failure occurs is separately calculated and the lower of the two governs the design. Table 6.1 shows the standard rivet hole drill size and nominal hole diameter.

Determination of the design bearing stress for hole filling fasteners is based on the nominal hole diameter as specified in Table 6.1. The yield and ultimate bearing stresses for various materials are given in MIL-HDBK-5, and are applicable to riveted joints where cylindrical holes are used and where $D/t < 5.5$. Where $D/t > 5.5$, tests to substantiate yield and ultimate bearing strengths must be performed in accordance with MIL-HDBK-5. These bearing stresses are applicable only for the design of rigid joints where there is no possibility of relative motion of the parts joined without deformation of the parts.

In computing the design shear strength of a protruding head rivet, the shear strength allowable should be multiplied by a correction factor for sheet thickness. This compensates for the reduction in rivet shear strength resulting from high bearing stresses on the rivet and D/t ratios in excess of 3.0 for single shear joints and 1.5 for double shear joints.

A further shear reduction factor is required if the fasteners are exposed to elevated temperatures. Figure 6.1 shows the reduction factors applicable to protruding head rivets at elevated temperatures.

6.2.2.2 Flush-Head Solid Rivets

Ultimate and yield allowables are specified in MIL-HDBK-5 for both machine countersunk and dimpled sheet using solid flush rivets with a head angle of 100°. These strength-values are applicable when the edge distance is equal to or greater than two times the nominal rivet diameter ($e \geq 2D$). Other strength values for different edge distances must be substantiated by test per MIL-HDBK-5. The yield allowable is the average load at which the following permanent set across the joint is developed:

DRILL SIZE	DEC EQUIV	DRILL SIZE	DEC EQUIV	DRILL SIZE	DEC EQUIV	DRILL SIZE	DEC EQUIV
80	.0135	43	.089	8	.199	25/64	.3906
79	.0145	42	.0935	7	.201	X	.397
1/64	.0156	3/32	.0938	13/64	.2031	Y	.404
78	.016	41	.096	6	.204	13/32	.4062
77	.018	40	.098	5	.2055	Z	.413
76	.020	39	.0995	4	.209	27/64	.4219
75	.021	38	.1015	3	.213	7/16	.4375
74	.0225	37	.104	7/32	.2188	29/64	.4531
73	.024	36	.1065	2	.221	15/32	.4688
72	.025	7/64	.1094	1	.228	31/64	.4844
71	.026	35	.110	A	.234	1/2	.500
70	.028	34	.111	15/64	.2344	33/64	.5156
69	.0292	33	.113	B	.238	17/32	.5312
68	.031	32	.116	C	.242	35/64	.5469
1/32	.0312	31	.120	D	.246	9/16	.5625
67	.032	1/8	.125	1/4(E)	.250	37/64	.5781
66	.033	30	.1285	F	.257	19/32	.5938
65	.035	29	.136	G	.261	39/64	.6094
64	.036	28	.1405	17/64	.2656	5/8	.625
63	.037	9/64	.1406	H	.266	41/64	.6406
62	.038	27	.144	I	.272	21/32	.6562
61	.039	26	.147	J	.277	43/64	.6719
60	.040	25	.1495	K	.281	11/16	.6875
59	.041	24	.152	9/32	.2812	45/64	.7031
58	.042	23	.154	L	.290	23/32	.7188
57	.043	5/32	.1562	M	.295	47/64	.7344
56	.0465	22	.157	19/64	.2969	3/4	.750
3/64	.0469	21	.159	N	.302	49/64	.7656
55	.052	20	.161	5/16	.3125	25/32	.7812
54	.055	19	.166	O	.316	51/64	.7969
53	.0595	18	.1695	P	.323	13/16	.8125
1/16	.0625	11/64	.1719	21/64	.3281	53/64	.8281
52	.0635	17	.173	Q	.332	27/32	.8438
51	.067	16	.177	R	.339	55/64	.8594
50	.070	15	.180	11/32	.3438	7/8	.875
49	.073	14	.182	S	.348	57/64	.8906
48	.076	13	.185	T	.358	29/32	.9062
5/64	.0781	3/16	.1875	23/64	.3594	59/64	.9219
47	.0785	12	.189	U	.368	15/16	.9375
46	.081	11	.191	3/8	.375	61/64	.9531
45	.082	10	.1935	V	.377	31/32	.9688
44	.086	9	.196	W	.386	63/64	.9844

FOR HOLES DRILLED WITH A DRILLING MACHINE USING SUITABLE JIGS AND FIXTURES, THE HOLE TOLERANCES DEPEND UPON THE DIAMETER OF THE HOLE AND INCREASE AS THE HOLE DIAMETER INCREASES. THE FOLLOWING ARE STANDARD TOLERANCES FOR GENERAL MACHINE WORK AND APPLY IN ALL CASES EXCEPT WHERE GREATER OR LESSER ACCURACY IS REQUIRED BY THE DESIGN:

HOLE DIA	TOLERANCES
.0135 THRU .125	+.004 -.001
.126 THRU .250	+.005 -.001
.251 THRU .500	+.006 -.001
.501 THRU .750	+.008 -.001
.751 THRU 1.000	+.010 -.001
1.001 THRU 2.000	+.012 -.001

(a)

REFERENCE AND 10387

TABLE 6.1 - STANDARD DRILL SIZES AND DRILLED HOLE TOLERANCES

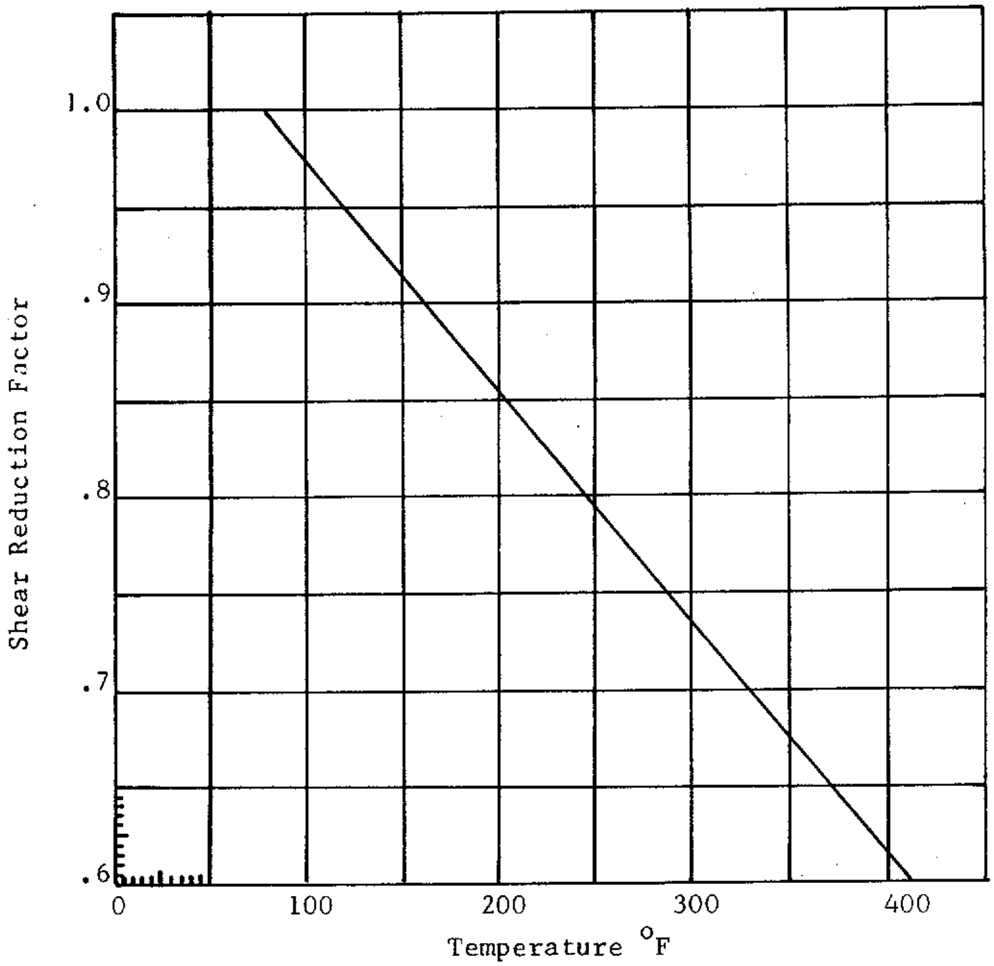


Figure 6.1 REDUCTION FACTOR FOR ALLOWABLES OF PROTRUDING HEAD, MS20470AD RIVETS AT ELEVATED TEMPERATURES FOR FIVE MINUTES

(1) 0.005 inch, up to and including 3/16 diameter rivets.

(2) 2.5 percent of the rivet diameter for rivet sizes larger than 3/16 diameter.

6.2.2.3 Solid Rivets in Tension

Solid rivets are not to be used as a primary tension load path. They are to be used in shear. A tension load will loosen the rivet. Tests have shown this loosening will occur at a low load level. When loosening occurs the shear value for the rivet no longer applies since this value was obtained for a tight joint. The joint with loose fasteners is highly susceptible to fatigue and can fail at a low number of cycles of the design shear.

Often it is impossible to keep tension loads out of rivets. If this condition is unavoidable it should be held to a minimum. When secondary axial loads are imposed on a protruding head aluminum rivet such that it is loaded in tension and shear, the margin of safety is based on the the following interaction:

$$R_s^3 + R_t^2 = 1$$

where R_s = applied shear/allowable shear

R_t = applied tension/allowable tension

6.2.2.4 Threaded Fasteners

Bolts, screws, nuts and nutplates commonly used at Bell Helicopter are shown in Tables 6.2 and 6.3. In shear joints the load per fastener at which the shear or bearing type of failure occurs is separately calculated, and the lower of the two values designs the joint.

Two types of thread forms are available on threaded fasteners. Cut threads, which conform to MIL-S-7742, are generally acceptable for use in shear applications or areas where high tension or repeated loads are not present. Rolled threads which conform to MIL-S-8879, controlled root radius threads, are acceptable for use in any application. They should be used where tension loads design the joint or where fatigue is present. Tensile strengths are based on the basic minor diameter of the thread.

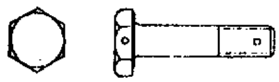
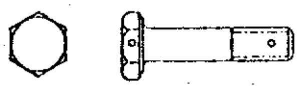
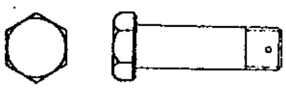
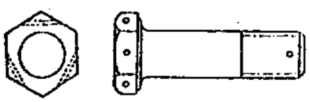
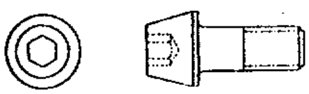
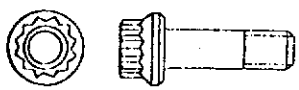
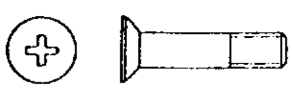
FASTENER	DESCRIPTION	MATERIAL	RECOMMENDED USAGE
 AN3-AN17	HEX HEAD BOLT DRILLED HEAD/SHANK OPT. STD. TOLERANCE	ALUMINUM: Ftu=62ksi, Fsu=35ksi STEEL, CRES: Ftu=125ksi,Fsu=75ksi	SHEAR ALUM 300°F STEEL 450°F CRES 800°F
 AN23-AN27	CLEVIS BOLT DRILLED SHANK OPT. STD. TOLERANCE	STEEL: Ftu=125ksi,Fsu=75ksi	SHEAR 450°F
 AN173-AN185	HEX HEAD BOLT DRILLED HEAD/SHANK OPT. CLOSE TOLERANCE	ALUMINUM: Ftu=62ksi,Fsu=35ksi STEEL, CRES Ftu=125ksi,Fsu=75ksi	SHEAR ALUM 300°F STEEL 450°F CRES 800°F
 NAS 464	HEX HEAD BOLT DRILLED SHANK OPT. CLOSE TOLERANCE SHORT THREAD HIGH STRENGTH	STEEL: Ftu=160ksi,Fsu=95 ks Fsu=95ksi	SHEAR 450°F
 NAS1303-NAS1316	HEX HEAD BOLT DRILLED HEAD/SHANK OPT. CLOSE TOLERANCE	STEEL: Ftu=160ksi, Fsu=95ksi	SHEAR AND TENSION 450°F
 MS20004-MS20017	INTERNAL WRENCHING HIGH STRENGTH DRILLED HEAD OPT.	STEEL: Ftu=160ksi, Fsu=95ksi	TENSION 450°F
 MS21250	EXTERNAL WRENCHING HIGH STRENGTH	STEEL: Ftu=180ksi, Fsu=108ksi	TENSION 450°F
 NAS333-NAS340	100°CSK HEAD SCREW PHILLIPS OR HEX RECESS CLOSE TOLERANCE HIGH STRENGTH	STEEL: Ftu=160ksi, Fsu=95ksi	SHEAR 450°F
 NAS517	100°CSK HEAD SCREW PHILLIPS RECESS CLOSE TOLERANCE	STEEL: Ftu=160ksi, Fsu=95ksi	SHEAR 450°F

TABLE 6.2 GENERAL DESCRIPTION OF BOLTS, SCREWS AND NUTS

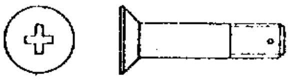
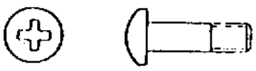
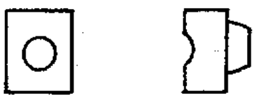
FASTENER	DESCRIPTION	MATERIAL	RECOMMENDED USAGE
 NAS1203-NAS1210	100° CSK HEAD SCREW PHILLIPS RECESS CLOSE TOLERANCE DRILLED SHANK OPT. SHORT HEAD	STEEL: Ftu=160ksi, Fus=95ksi	SHEAR 450°F
 NAS623	PAN HEAD SCREW PHILLIPS RECESS STANDARD TOLERANCE SHORT THREAD	STEEL: Ftu=160ksi, Fus=95ksi	SHEAR 450°F
 AN310	PLAIN NUT CASTELLATED	ALUMINUM STEEL CRES	TENSION ALUM 300°F STEEL 450°F CRES 450°F
 AN320	PLAIN NUT CASTELLATED	ALUMINUM STEEL CRES	SHEAR ALUM 300°F STEEL 450°F CRES 450°F
 NAS577	BARREL NUT SELF LOCKING FLOATING	STEEL	HIGH STRENGTH 450°F
 NAS1291	SELF LOCKING NUT LOW HEIGHT LIGHT WEIGHT	STEEL CRES	SUPPLEMENTS MS21042 & MS21043 HIGH STRENGTH 450°F
 MS21042	SELF LOCKING NUT REDUCED HEX REDUCED HEIGHT	STEEL	HIGH STRENGTH 450°F
 MS21043	SELF LOCKING NUT REDUCED HEX REDUCED HEIGHT	CRES	HIGH STRENGTH HIGH TEMPERATURE 800°F
 MS21044	SELF LOCKING NON-METALLIC INSERT REGULAR HEIGHT	ALUMINUM STEEL	TENSION 250°F

TABLE 6.2 (CONT'D) GENERAL DESCRIPTION OF BOLTS, SCREWS AND NUTS

	MS21047	MS21048	MS21049	MS21050	MS21051	MS21052	MS21053	MS21054	MS21055	MS21056	MS21057
TYPE	TWD LUG	TWO LUG	100° CSK TWO LUG	100° CSK TWO LUG	ONE LUG	ONE LUG	100° CSK ONE LUG	100° CSK ONE LUG	CORNER	CORNER	100° CSK CORNER
RIVET SPACING	STANDARD	STANDARD	STANDARD	STANDARD	STANDARD	STANDARD	STANDARD	STANDARD	STANDARD	STANDARD	STANDARD
MATERIAL	STEEL	CRES	STEEL	CRES	STEEL	CRES	STEEL	CRES	STEEL	CRES	STEEL
MAX. TEMP.	450°	450°&800°	450°	450°&800°	450°	450°&800°	450°	450°&800°	450°	450°&800°	450°
SIZE RANGE	#4 -7/16	#4 -7/16	#8 -5/16	#8 -5/16	#6 -7/16	#6 -7/16	#8 -5/16	#8 -5/16	#6 -7/16	#6- 7/16	#8 -5/16
FIXED	X	X	X	X	X	X	X	X	X	X	X
FLOATING											

	MS21058	MS21059	MS21060	MS21061	MS31062	MS21069	MS21070	MS21071	MS21072	MS21073	MS21074
TYPE	100° CSK CORNER	TWO LUG	TWO LUG	ONE LUG	ONE LUG	TWO LUG	TWO LUG	ONE LUG	ONE LUG	CORNER	CORNER
RIVET SPACING	STANDARD	STANDARD	STANDARD	STANDARD	STANDARD	REDUCED	REDUCED	REDUCED	REDUCED	REDUCED	REDUCED
MATERIAL	CRES	STEEL	CRES	STEEL	CRES	STEEL	CRES	STEEL	CRES	STEEL	CRES
MAX. TEMP.	450°&800°	450°	450°&800°	450°	450°&800°	450°	450°&800°	450°	450°&800°	450°	450°&800°
SIZE RANGE	#8 -5/16	#4 -3/8	#4 -3/8	#4 -5/16	#4 -5/16	#2 -3/8	#2 -3/8	#2 -3/8	#2 -3/8	#2 -3/8	#2 -3/8
FIXED	X					X	X	X	X	X	X
FLOATING		X	X	X	X						

	MS21075	MS21076	MS21086	MS21087	NAS1473	NAS1474	90-008	90-033	90-036	90-037	90-038
TYPE	TWO LUG	TWO LUG	SIDE BY SIDE	SIDE BY SIDE	DOME	DOME	DOME	DOME	DOME	DOME	DOME
RIVET SPACING	REDUCED	REDUCED	REDUCED	REDUCED	STANDARD	REDUCED	REDUCED	STANDARD	REDUCED	STANDARD	REDUCED
MATERIAL	STEEL	CRES	STEEL	CRES	STEEL,CRES	STEEL,CRES	STEEL	STEEL	STEEL	STEEL	STEEL
MAX. TEMP.	450°	450°&800°	450°	450°&800°	450°&800°	450°&800°	450°	450°	450°	450°	450°
SIZE RANGE	#4 -5/16	#4 - 5/16	#8 -3/8	#8 -3/8	#8 -5/16	#4 - 1/4	#6 -5/16	#8 -5/16	#6 -5/16	#6 -3/8	#6 -5/16
FIXED			X	X			X		X	X	X
FLOATING	X	X			X	X		X			

TABLE 6.3 - GENERAL DESCRIPTION OF NUTPLATES

In addition the nut or nut plate allowable must be used to determine the tension capability of the joint. The nut or nut plate will generally be the limiting item in a nut/bolt joint.

Determination of the design bearing stress for non-hole filling fasteners is based on nominal shank diameter of the fastener. The design bearing stresses for various materials are given in MIL-HDBK-5 and are applicable to joints with fasteners in cylindrical holes and where $D/t < 5.5$. Where $D/t > 5.5$, tests to substantiate yield and ultimate bearing strengths must be performed in accordance with MIL-HDBK-5. These bearing stresses are applicable only for the design of rigid joints where there is no possibliity of relative motion of the parts joined without deformation of the parts.

6.2.2.5 Blind Rivets

MIL-HDBK-5 gives the ultimate and yield allowable single shear strengths for protruding and flush head blind rivets. These strengths are applicable only when the grip lengths and hole tolerances are as recommended by the respective manufacturers and may be substantially reduced if oversize holes or improper grip lengths are used.

The strengths have been established by test using e/D equal to or greater than 2. Where e/D values less than 2 are used, tests to substantiate yield and ultimate strengths must be made.

In view of the wide variance in dimpling methods and tolerances for aluminum and magnesium alloys, no standard or uniform load allowables are recommended. Allowables for ultimate and shear strengths of blind rivets in double-dimpled or dimpled, machine countersunk application should be established on the basis of specific tests. In the absence of such data, allowables for blind rivets in machine-countersunk sheet may be used.

Blind rivets are primarily shear type fasteners. They should not be used where appreciable tension loads on the rivets will exist. They should not be used to attach heavily loaded fittings, in engine air inlets or on the tail rotor side of vertical fins.

6.2.2.6 Swaged Collar Fasteners

The ultimate allowable shear and tensile strengths of "Hi-Shear" rivets, lockbolts and lockbolt stumps may be obtained from MIL-HDBK-5. For all lockbolts under combined loading of shear and tension installed in material having a thickness large enough to make the shear cutoff strength critical for shear loading, the following interaction equation is applicable:

$$\text{Steel lockbolts, } R_t + R_s^{10} = 1.0$$

where R_t and R_s are ratios of applied load to allowable load in tension and shear, respectively.

6.2.2.7 Lockbolts

Lockbolts and lockbolt stumps shall be installed in properly drilled holes. The ultimate allowable shear and tensile strengths for protruding and flush head lockbolts and lockbolt stumps are shown in MIL-HDBK-5. When both tension and shear are present the lockbolts should be designed to the same interaction criteria defined in Section 6.2.2.6.

6.2.2.8 Torque Values for Threaded Fasteners

Proper installation torque is of utmost importance if a threaded fastener joint is to function properly. The torque values to be applied to threaded fasteners and fittings are specified in Bell Standard 160-007. Torque values for commonly used nuts and bolts are shown in Tables 6.4 through 6.6. All drawings showing installations of threaded fasteners should specify "torque per Bell Standard 160-007."

Two types of torque values are given; one for shear nuts and one for tension nuts. Type III (shear nut) values are based on developing a nominal 24,000 psi in the shank of the bolt while Type IV values produce 40,000 psi in the bolt. Table 6.6 shows maximum tightening values. These produce 54,000 psi and 90,000 psi in shanks of shear and tension bolts respectively. When repeated tensile or bending stresses are present, preload should be applied to the fastener by torqueing. The amount of preload should equal the maximum tensile stress expected. This eliminates cycling of the stress in the bolt since the prestress will remain constant. Often the preload will need to be greater than that developed by the torques in Table 6.6. An equation has been developed which gives a reasonable estimate of the torque necessary to produce a preload. That equation is:

$$T = .2\ DL$$

where T is torque, D is the mean diameter of the thread and L is the preload produced by the torque. This is an empirical equation for dry threads and is a function of many variables. For lubricated threads the coefficient may reduce as much as 50 per cent.

For large threaded connections, such as mast nuts, the following equation has been developed.

$$T = .5L\left[D_r\gamma + d_p\left(\frac{\tan(\beta + \phi)}{\cos\alpha}\right)\right]$$

where D_r is the diameter of the contact ring, γ is the friction coefficient, d_p is pitch diameter, β is the lead angle (lead/πd_p), ϕ = arc tan γ, and α = ½ thread profile angle.

TYPE III	
BOLT	NUT
AN3-AN20 AN42-AN49 AN173-AN186 AN525 MS20004-MS20024 MS20033-MS20046 MS70073-MS20081 MS24694 MS27039 NAS144-NAS158 NAS333-NAS340 NAS464 NAS517 NAS583-NAS590 NAS623 NAS1003-NAS1020 NAS1202-NAS1210 NAS1218 NAS1297 NAS1303-NAS1320 NAS1351 (NON-LOCKING) NAS1352 (NON-LOCKING) ALL THREADED STUDS	AN316 AN320 AN345 AN150401-AN150425 MS25082 MS35650 MS35691 MS51968 NAS1022

TYPE III CONSISTS OF ANY COMBINATION
OF NUT AND BOLT SHOWN

REFERENCE BELL STD 160-007

TABLE 6.4 - TYPE III FASTENERS

TYPE IV			
BOLT	NUT		
AN3-AN20 AN42-AN49 AN173-AN186 AN525 MS20033-MS20046 MS20073-MS20081 MS24694 MS27039 NAS333-NAS340 NAS517 NAS623 NAS1003-NAS1020 NAS1202-NAS1210 NAS1297 NAS1352 (NON-LOCKING) ALL THREADED STUDS	AN256 AN310 AN315 MS9358 MS20365 MS20500 MS21042 MS21043 MS21044 MS21045 MS21047 MS21048 MS21049 MS21051 MS21052 MS21053 MS21054 MS21055 MS21056 MS21058 MS21059 MS21060	MS21061 MS21062 MS21069 MS21070 MS21071 MS21072 MS21073 MS21074 MS21075 MS21076 MS21083 MS21086 MS21208 MS21209 MS21991 MS122076 thru MS122275 MS124651 thru MS124850 NAS509	NAS577 NAS1291 NAS1329 NAS1330 NAS1473 NAS1474 80-006 80-007 80-013 90-002 90-003 110-061 110-062

TYPE IV CONSISTS OF ANY COMBINATION OF NUT AND BOLT SHOWN

REFERENCE BELL STD 160-007

TABLE 6.5 - TYPE IV FASTENERS

NUT AND BOLT THREAD SIZE	Torque, In-Lbs			
	TYPE III		TYPE IV	
	SHEAR		TENSION	
	Recommended Installation Torque Range (a)	Max Allowable Tightening Torque (b)	Recommended Installation Torque Range (c)	Max Allowable Tightening Torque (d)
10-32	12-15	25	20-25	40
1/4-28	30-40	60	50-70	100
5/16-24	60-85	140	100-140	225
3/8-24	95-110	240	160-190	390
7/16-20	270-300	500	440-500	840
1/2-20	288-408	660	480-700	1100
9/16-18	480-600	960	800-1000	1600
5/8-18	660-780	1400	1100-1300	2400
3/4-16	1300-1500	3000	2300-2500	5000
7/8-14	1500-1800	4200	2500-3000	7000
1 - 12	2200-3300	6000	3700-5500	10000
1 1/8-12	3000-4200	9000	5000-7000	15000
1 1/4-12	5400-6600	15000	9000-11000	25000

(a) TYPE III RECOMMENDED TORQUE RANGE IS BASED ON A NOMINAL STRESS OF 24 KSI IN THE BOLT.

(b) TYPE III MAX ALLOWABLE TORQUE IS BASED ON A STRESS OF 54 KSI IN THE BOLT.

(c) TYPE IV RECOMMENDED TORQUE RANGE IS BASED ON A NOMINAL STRESS OF 40 KSI IN THE BOLT.

(d) TYPE IV MAX ALLOWABLE TORQUE IS BASED ON A STRESS OF 90 KSI IN THE BOLT.

REFERENCE BELL STD 160-007

TABLE 6.6 - TORQUE VALUES FOR THREADED FASTENERS AND FITTINGS

6.3 METALLURGICAL JOINTS

In the design of metallurgical joints, the strength of the joining material (weld metal) and the adjacent parent material must be considered. Wherever possible, joints should be designed so that the welds will be loaded in shear.

The allowable strength for both the adjacent parent metal and the weld metal is given in MIL-HDBK-5. Two types of metallurgical joints are discussed; welding and brazing. Welding consists of joining two or more pieces of metal by applying heat,pressure or both, with or without filler material, to produce a localized union through fusion or recrystallization across the joint. Common welding processes include: fusion (inert gas, shielded arc with non-consumable tungsten electrode - TIG and inert gas shielded metal arc welding using consumable electrodes - MIG), resistance (spot and seam) and flash.

Brazing consists of joining metals by the application of heat causing the flow of a thin layer, capillary thickness of nonferrous filler metal into the space between pieces. Bonding results from the intimate contact produced by the dissolution of a small amount of base metal in the molten filler metal without fusion of the base metal.

6.3.1 Fusion Welding - Arc and Gas

In the design of welded joints, the strength of both the weld metal and the adjacent parent metal must be considered. For materials heat treated after welding, the allowable strength in the parent metal near a welded joint may equal the allowable strength for the material in the heat treated condition; however, it should be noted that the weld metal allowables are based on 85 percent of these values.

6.3.2 Flash and Pressure Welding

The ultimate tensile allowable strength and bending allowable modulus of rupture for flash and pressure welds are specified in MIL-HDBK-5.

6.3.3 Spot and Seam Welding

Design shear strength allowables for spot welds in various alloys are given in MIL-HDBK-5; the thickness ratio of the thickest sheet to the thinnest outer sheet in the combination should not exceed 4:1. Table 6.7 gives the minimum allowable edge distance for spot welds and seam welds. Combinations of aluminum alloys suitable for spot welding are given in Table 6.8.

6.3.4 Effect of Spot Welds on Parent Metal

In applications of spot welding where ribs, intercostals, or doublers are attached to sheet, either at splices or at other points on the sheet panels, the allowable ultimate strength of the spot welded sheet shall be determined by multiplying the ultimate tensile sheet strength by the approximate efficiency factor shown in MIL-HDBK-5.

Nominal thickness of thinner sheet, in.	Edge distance, E, in.
0.016	3/16
0.020	3/16
0.025	7/32
0.032	1/4
0.036	1/4
0.040	9/32
0.045	5/16
0.050	5/16
0.063	3/8
0.071	3/8
0.080	13/32
0.090	7/16
0.100	7/16
0.125	9/16
0.160	5/8

(a) Intermediate gages will conform to the requirement for the next thinner gage shown.

(b) For edge distances less than those specified above, appropriate reductions in the spot-weld allowable loads shall be made.

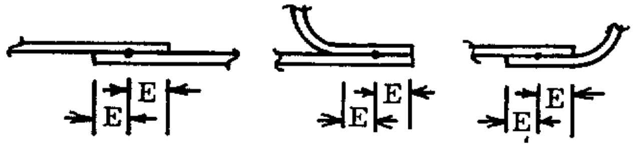


TABLE 6.7 - MINIMUM EDGE DISTANCES FOR SPOT-WELDED JOINTS

Specification		QQ-A-250/1	AMS-4029[b]	QQ-A-4014[b]	QQ-A-250/3	QQ-A-250/4[b]	QQ-A-250/5	QQ-A-250/2	QQ-A-250/8	QQ-A-250/11	QQ-A-250/12	QQ-A-250/13
Material	Material	1100	Bare 2014	Bare 2014	Clad 2014	Bare 2024	Clad 2024	3003	5052	6061	Bare 7075	Clad 7075
QQ-A-250/1	1100	...	...	...	...	...	...	...	...	...	...	...
AMS-4029	Bare 2014(b)	...	*	*	*	*	...	...	...	...	*	...
AMS-4014	Bare 2014(b)	...	*	*	*	*	...	...	...	...	*	...
OO-A-250/3	Clad 2014	...	*	*	...	*	...	...	...	...	*	...
QQ-A-250/4	Bare 2024(b)	...	*	*	*	*	...	...	...	...	*	...
QQ-A-250/5	Clad 2024	...	...	...	...	...	...	...	...	...	...	...
QQ-A-250/2	3003	...	...	...	...	...	...	...	...	...	...	...
QQ-A-250/8	5052	...	...	...	...	...	...	...	...	...	...	...
QQ-A-250/11	6061	...	...	...	...	...	...	...	...	...	...	...
QQ-A-250/12	Bare 7075(b)	...	*	*	*	*	...	...	...	...	*	...
QQ-A-250/13	Clas 7075(c)	...	...	...	...	...	...	...	...	...	...	...

(a) The various aluminum and aluminum-alloy materials referred to in this table may be spot welded in any combinations except the combinations indicated by the asterisk (*) in the table. The combinations indicated by the asterisk (*) may be spot welded only with the specific approval of the procuring or certifying agency.

(b) This table applies to construction of land- and carrier-based aircraft only. The welding of bare, high-strength alloys in construction of seaplanes and amphibians is prohibited unless specifically authorized by the procuring or certifying agency.

(c) Clad heat-treated and aged 7075 material in thicknesses less than 0.020 inch shall not be welded without specific approval of the procuring or certifying agency.

TABLE 6.8 - ACCEPTABLE ALUMINUM AND ALLOY COMBINATIONS (a) FOR SPOT AND SEAM WELDING

Revision B

6.3.5 Welding of Castings

In-process repair welding of rough castings is permissible when accomplished in accordance with drawing requirements or other authorized documentation. BHT designed castings are repair welded to BPS 4470 procedures. These basic procedures, as listed below, shall also be considered when supplier designed castings are to be repair welded.

A. Defects shall be carefully and completely removed by grinding, routing or filing and reworked area shall be thoroughly cleaned.

B. Fluorescent penetrant or radiographic inspection shall be used to insure complete defect removal prior to welding.

C. Acceptable heat treat conditions of castings at time of welding shall be in accordance with BPS 4470 or as otherwise authorized by the BHT Metallurgical Laboratory.

D. Welding method shall be tungsten inert gas (TIG) and shall be specified per specifications MIL-W-8604 for aluminum alloys, MIL-W-8611 for ferrous alloys and MIL-W-18326 for magnesium alloys when applied to supplier design parts.

E. Applicable filler material shall be specified if parts are not welded per BPS 4470.

F. Zoned sketches shall be required when specific areas or defect sizes to be welded are limited. Procedures shall conform to BPS 4470 requirements.

G. Welding shall be performed by welders qualified in accordance with BPS 4470 test requirements or as otherwise authorized by BHT Metallurgical Laboratory.

H. Welded castings shall be properly identified to indicate that the casting was repair welded.

I. Casting shall be completely heat treated after all welding is completed.

J. All weld repaired castings, except Class IIB, shall be 100% radiographically inspected after all welding and heat treatment is completed.

K. When supplier designed castings are to be weld repaired, the Structural Materials Technology Group should be consulted for specific requirements.

6.4 MECHANICAL JOINTS

This section contains design information for lugs, sockets, pins, cables, pulleys, bearings, bonding, etc.

6.4.1 Joint Load Analysis

The analysis in this section is a thermo-elastic analysis of mechanical joints based on the procedures set forth in Reference 10, WADD-TR-60-517. The analysis presented is directly applicable to problems where the stress levels lie in the elastic range. Certain aspects of the analysis are shown to be of value in the solution of problems with stress levels in the inelastic and plastic ranges.

6.4.1.1 One-Dimensional Compatibility

The problem is simplified if the overall geometry of the joint does not allow the joint to bend out of plane. Such a case is shown in Figure 6.2. This results in joint displacements and attachment loads essentially dependent on axial flexibility of the joint components in the direction of the applied external loading.

The load distribution is obtained by satisfying compatibility conditions for the joint displacements and the equilibrium equation.

This analyis is applicable to a mechanical joint composed of two dissimilar elastic materials. It is assumed that the attachments initially fill the holes and that each attachment hole combination deforms elastically under load. The presence of "slop" and its influence upon the load distribution is then considered.

Figure 6.3 shows a longitudinal section through a typical joint. Denoting the shear load at the j^{th} attachment by P_j, equilibrium of forces requires that

$$X = \sum_{j=1}^{N} P_j \qquad (1)$$

where a tensile applied load X and attachment loads acting to the right on the upper sheet are positive.

As shown in Figure 6.4, compatibility of displacements requires that the axial contraction or expansion of the plate material at the common surface, measured from a datum (defined by the unloaded, unheated spacing between the centerlines of adjacent pins) must be identical for the upper and lower plates. For the j^{th} general bay the compatibility equation is

$$\Delta j_T = \Delta j_B \qquad (2)$$

where T and B denote top and bottom sheet. For the one-dimensional case with no "slop" there are basicially two types of deformation which contribute to the Δ_{js} of the joint.

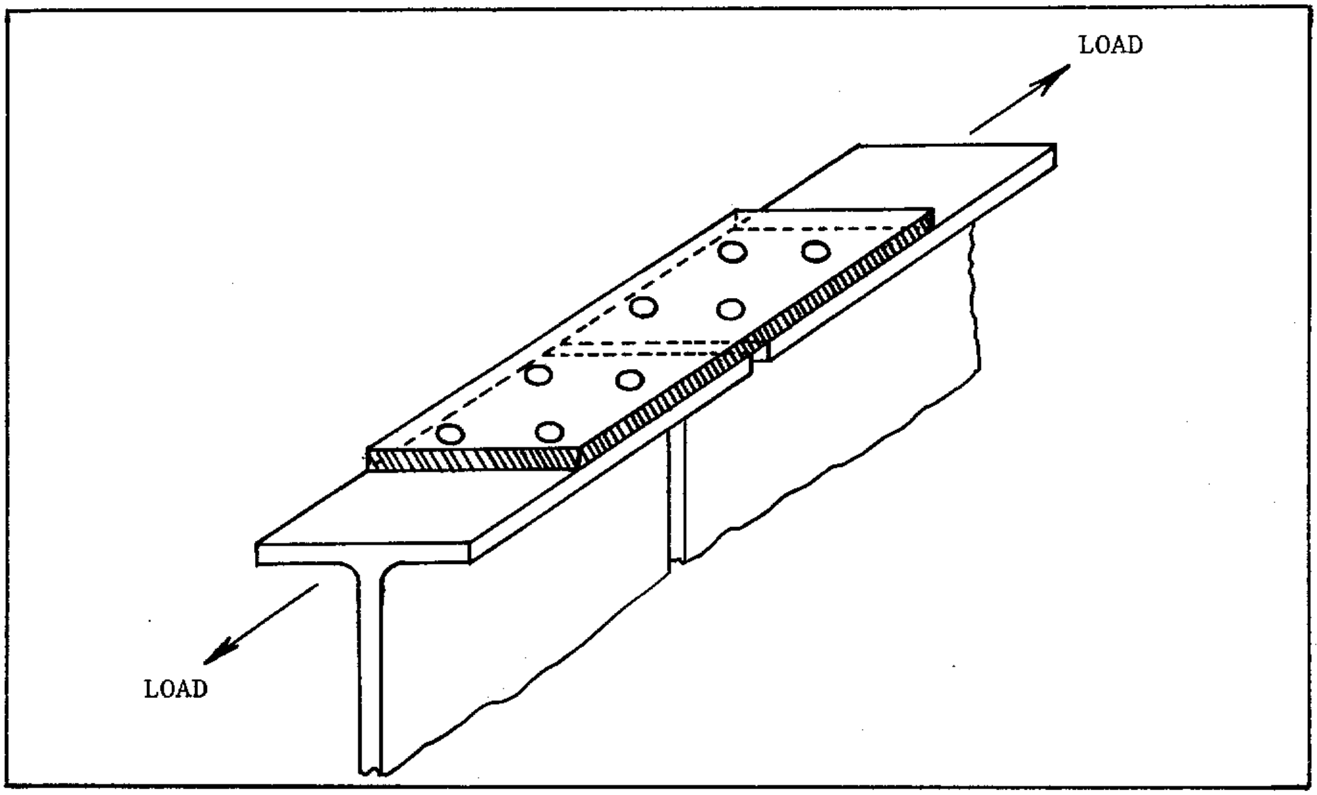


FIGURE 6.2 TYPICAL SPLICE FOR RIGID BEAM

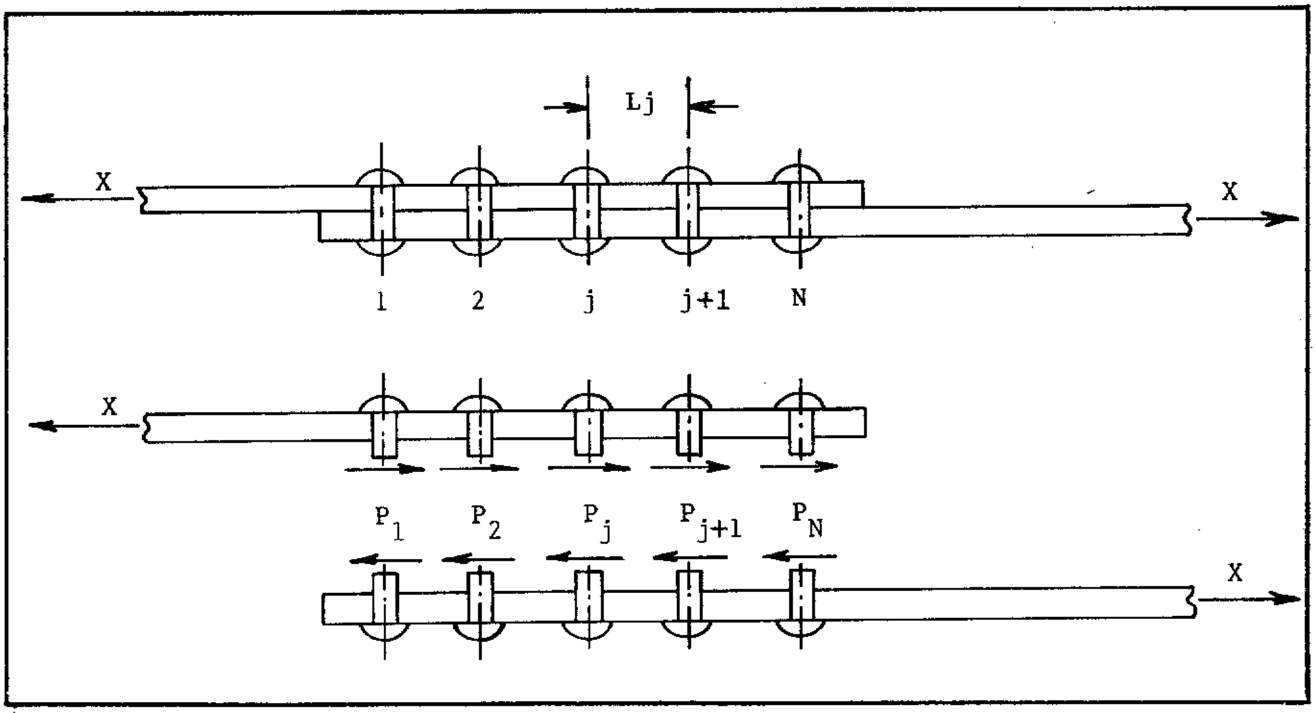


FIGURE 6.3 ONE DIMENSIONAL JOINT EQUILIBRIUM

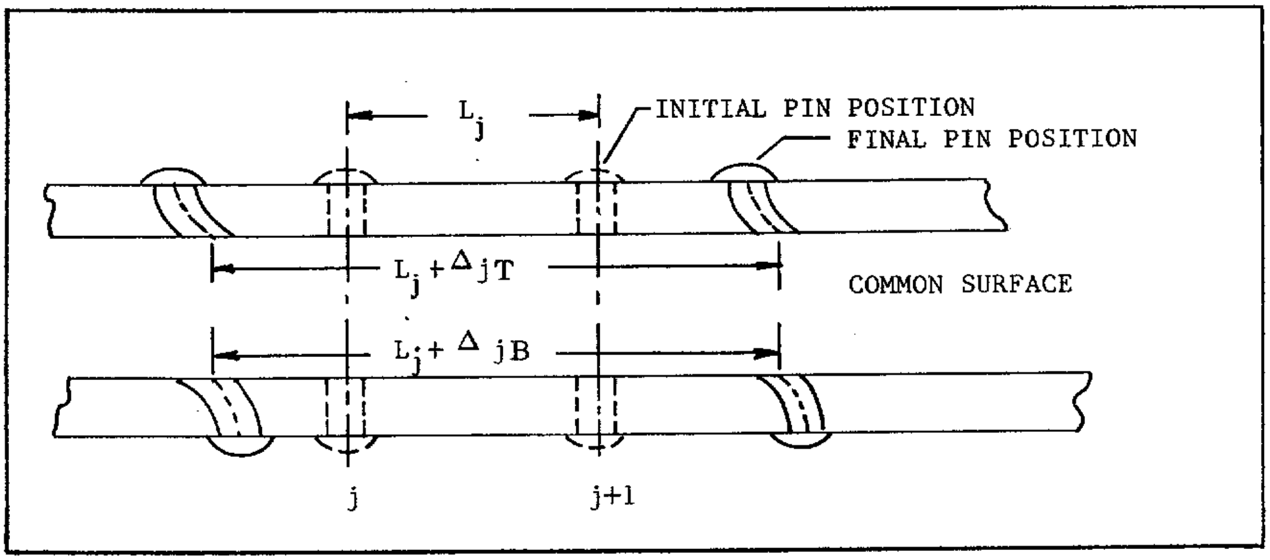


FIGURE 6.4 ONE DIMENSIONAL COMPATIBILITY

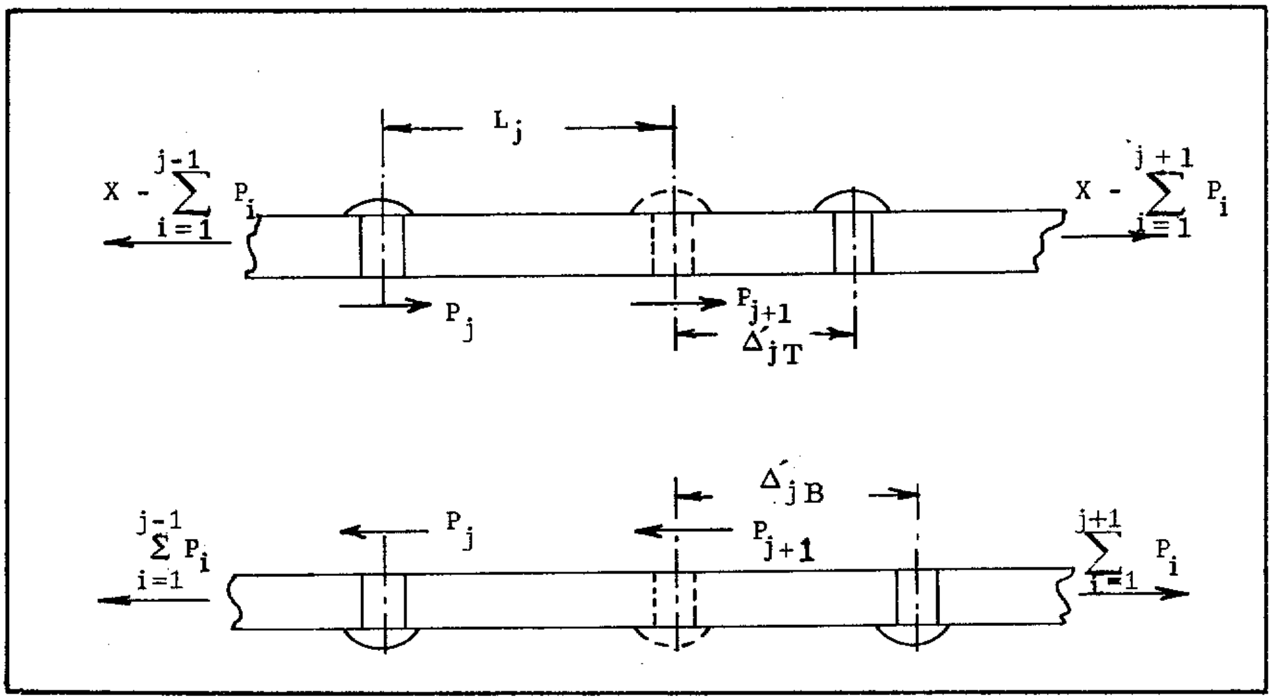


FIGURE 6.5 DEFORMATION OF THE JOINT DUE TO LOCAL DISTORTIONS OF THE HOLES AND ATTACHMENTS

The first type is uniaxial stretching or contraction of the sheets due to the combined effects of temperature and mechanical loading. From Figure 6.5 the uniaxial stretching for the j^{th} bay is

$$\Delta'_{jT} = \left(X - \sum_{i=1}^{j} P_i\right)\left(\frac{L}{AE}\right)_{jT} + \int_0^{Lj} \bar{\epsilon}_{jT} dx \tag{3a}$$

$$\Delta'_{jB} = \left(\sum_{i=1}^{j} P_i\right)\left(\frac{L}{AE}\right)_{jB} + \int_0^{Lj} \bar{\epsilon}_{jB} dx \tag{3b}$$

where a positive Δ increases the spacing between adjacent attachments and

$$\bar{\epsilon} = \frac{\int_0^h E\alpha T\, dy}{\int_0^h E\, dy}$$

If the thermal gradient is linear through the thickness, then $\bar{\epsilon}$ is approximately equal to the value αT at the plate midplane.

The second basic type of joint deformation occurs because the internal joint loads create local distortions of the holes and attachments as shown in Figure 6.6 . The deformation is expressed in terms of an experimentally determined attachment hole flexibility factor f for the given attachment-sheet flexibility (see Section 6.4.3). Thus, for the top sheet

$$\Delta''_{jT} = P_{j+1} f_{(j+1)T} - P_j f_{jT} \tag{4a}$$

and for the bottom sheet

$$\Delta''_{jB} = -P_{j+1} f_{(j+1)B} + P_j f_{jB} \tag{4b}$$

Substituting $\Delta_{jT} = \Delta'_{jT} + \Delta''_{jT}$ and $\Delta_{jB} = \Delta_{jB} + \Delta''_{jB}$ from equations (3) and (4) into equation (2) yields

$$\left[\left(\frac{L}{AE}\right)_{jT} + \left(\frac{L}{AE}\right)_{jB}\right]\left[\sum_{i=1}^{j} P_i\right] = \Delta\phi_j - P_j f_j + P_{j+1} f_{j+1} + X\left(\frac{L}{AE}\right)_{jT} \tag{5}$$

where $$\Delta\phi_j = \int_0^{L_j} (\bar{\epsilon}_{jT} - \bar{\epsilon}_{jB})\, dx$$

and $$f_j = f_{jT} + f_{jB}$$

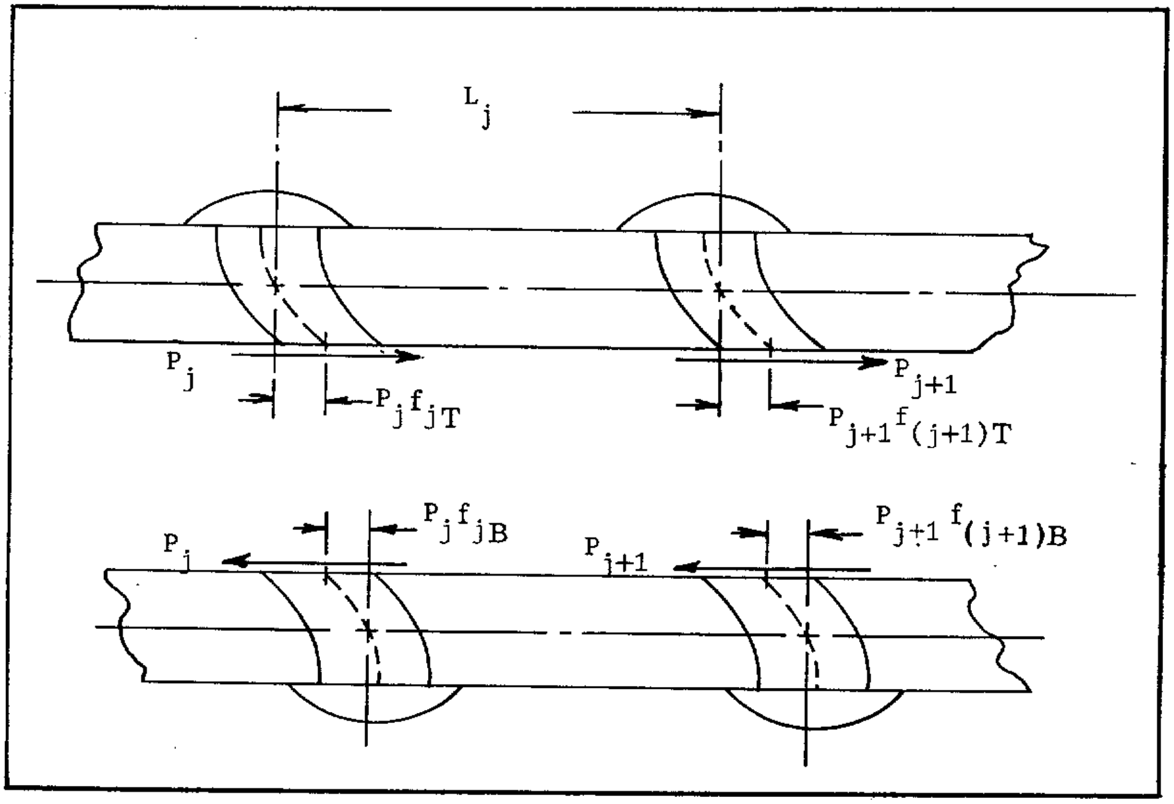


FIGURE 6.6 DEFORMATION OF THE JOINT DUE TO LOCAL DISTORTIONS OF THE HOLES AND ATTACHMENTS

The compatibility equation (5) which constitutes N-1 equations together with the equilibrium equation (1) provides N linear algebraic simultaneous equations for N unknown attachment loads, P_j. Since equation (5) is in the form of a recurrence equation, it can be used to express all the attachment loads in terms of P_1, the load in the first attachment. Solutions can also be obtained by solving the simultaneous equations directly or by iteration and relaxation techniques. Often in design, the sheet thicknesses are constant; the attachments are all of the same type, size and spacing and the temperature through the splice thickness does not vary appreciably in the direction of mechanical loading. This case of constant bay properties alters equation (5) somewhat. The coefficients of the P_j's become constant and a general solution takes the form of

$$\left[\left(\frac{L}{AE}\right)_T + \left(\frac{L}{AE}\right)_B\right]\left[\sum_{i=1}^{j} P_i\right] = \Delta\phi - P_j f + P_{j+1} f + X\left(\frac{L}{AE}\right)_T \qquad (6)$$

$$(j=1,2,\ldots,\ N-1)$$

where

$$\Delta\phi = \int_0^L (\bar{\epsilon}_T - \bar{\epsilon}_B)\, dx$$

and

$$f = f_T + f_B$$

Combining equations (1) and (6) yields the following:

$$P_{jN} = \left[A_{jN} + B_{jN}\left(\frac{L}{AE}\right)_T\left(\frac{1}{f}\right)\right] X + B_{jN}\left(\frac{\Delta\phi}{f}\right) \qquad (7)$$

where the subscript jN refers to the j^{th} attachment in a joint of N attachments. Values of the coefficients A_{jN}, B_{jN} and Z are plotted in Figures 6.7 through 6.20, where

$$Z = \left[\left(\frac{L}{AE}\right)_T + \left(\frac{L}{AE}\right)_B\right]\left(\frac{1}{f}\right)$$

By interchanging the designation of the top and bottom sheets, the curves of Figures 6.7 through 6.20 can be used to obtain all of the loads in joints having as many as ten attachments. When the total number of attachments exceeds ten, the curves give the loads in the first five attachments from either end of the splice.

The first term on the right-hand side of equation (7) represents the contribution of mechanical loading and the second term the contribution of thermal loading. For constant bay properties the thermal load at the center of the joint must be zero because of symmetry. Thus, $B_{23} = B_{35} = B_{47} = \ldots = 0$. In addition the thermal loads, in joints with constant bay properties, are symmetrical about the center of the splice, $B_{1N} = -B_{NN}$; $B_{2N} = -B_{(N-1)N}$; etc.

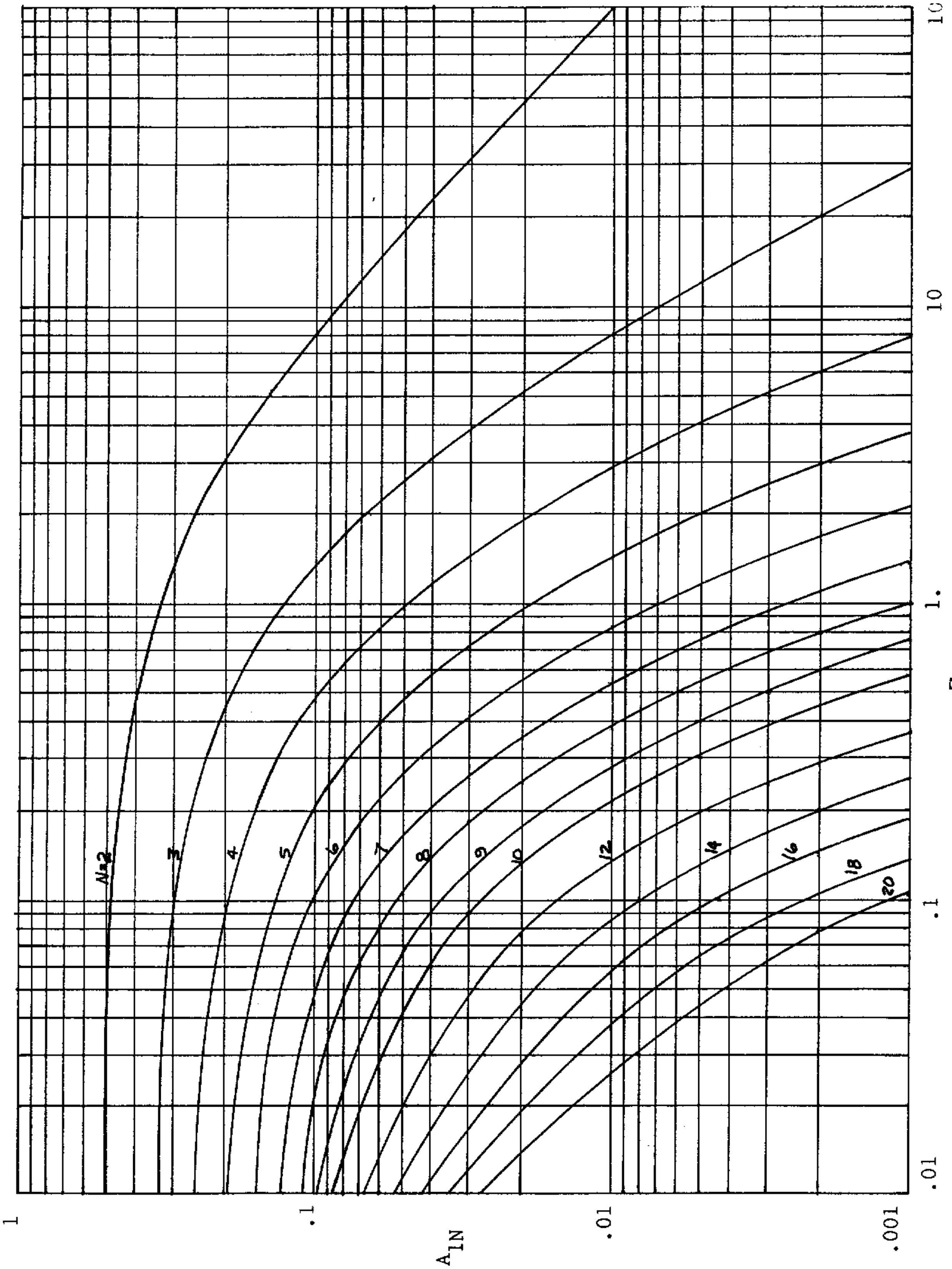


FIGURE 6.7 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (A_{1N}, j=1)

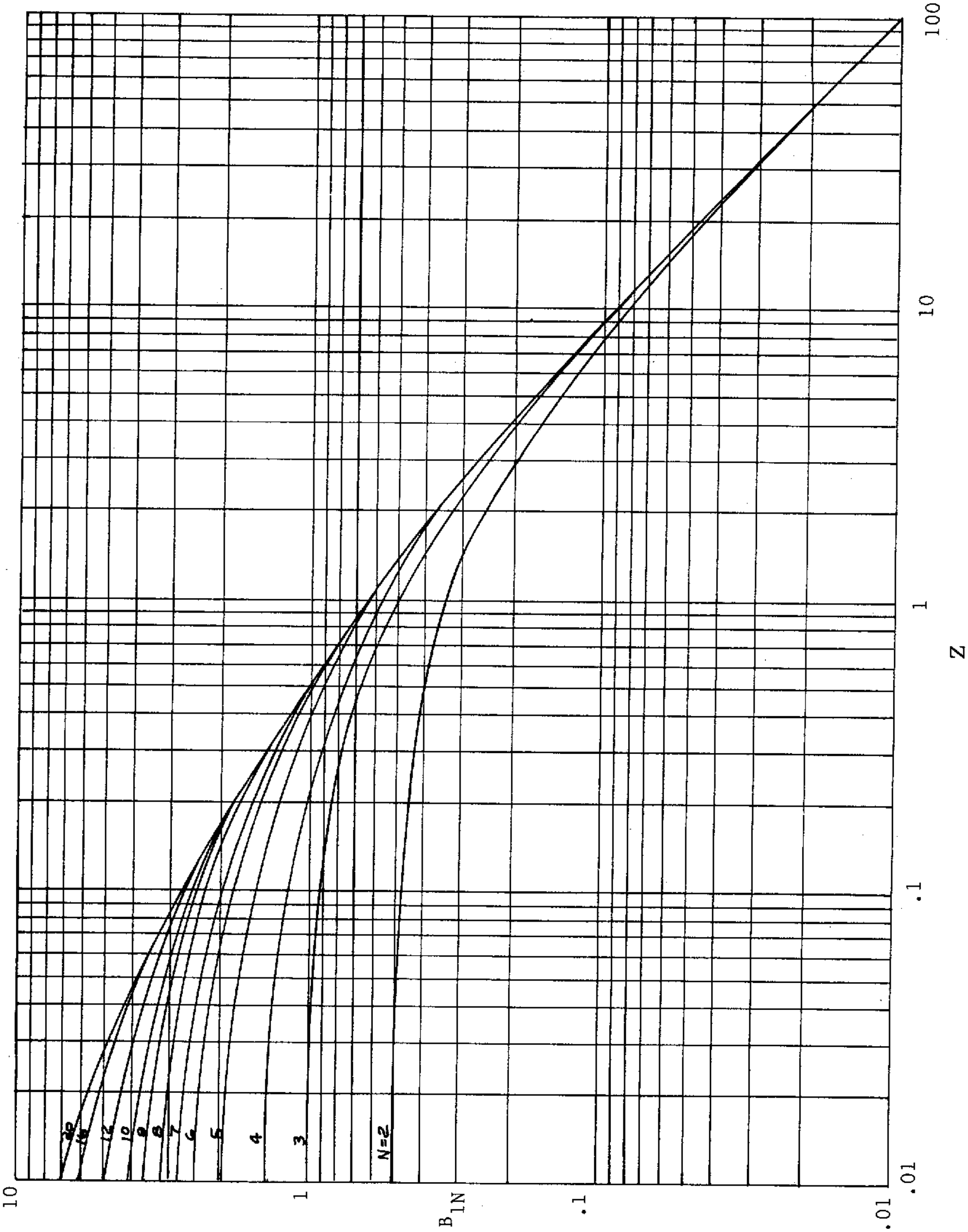


FIGURE 6.8 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (B_{1N}, j=1)

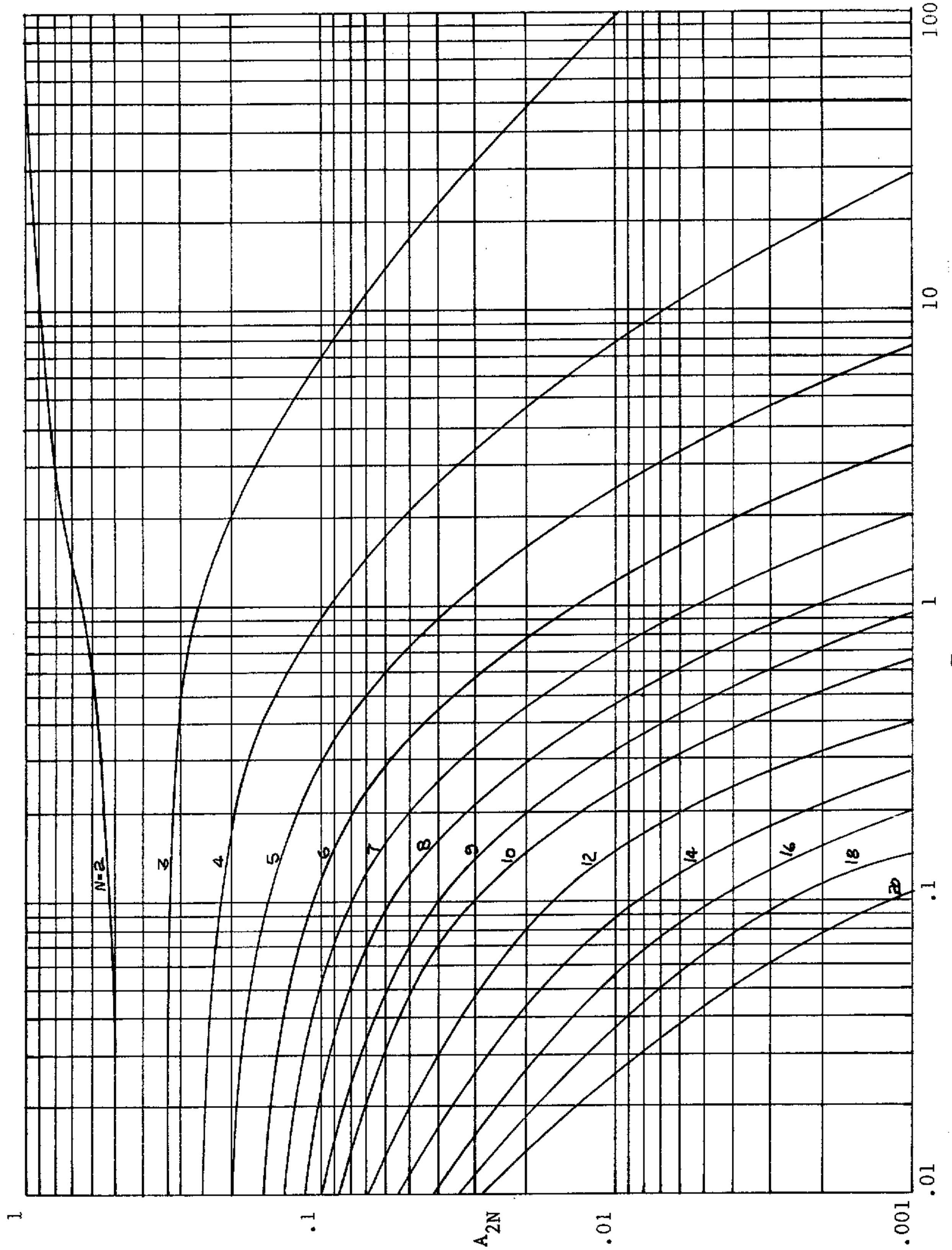


FIGURE 6.9 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (A_{2N}, j=2)

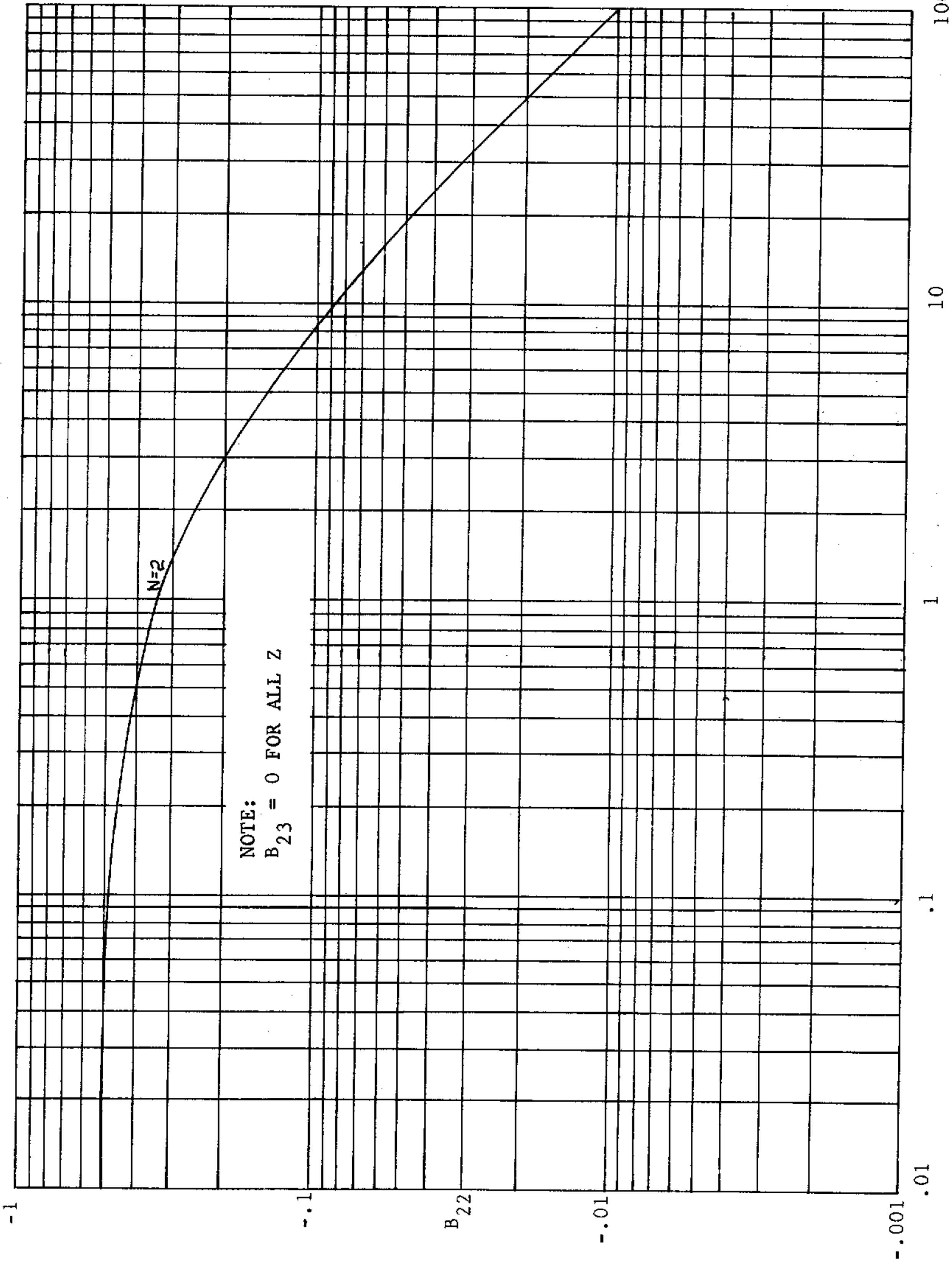


FIGURE 6.10 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (B_{22}, j =2, N=2)

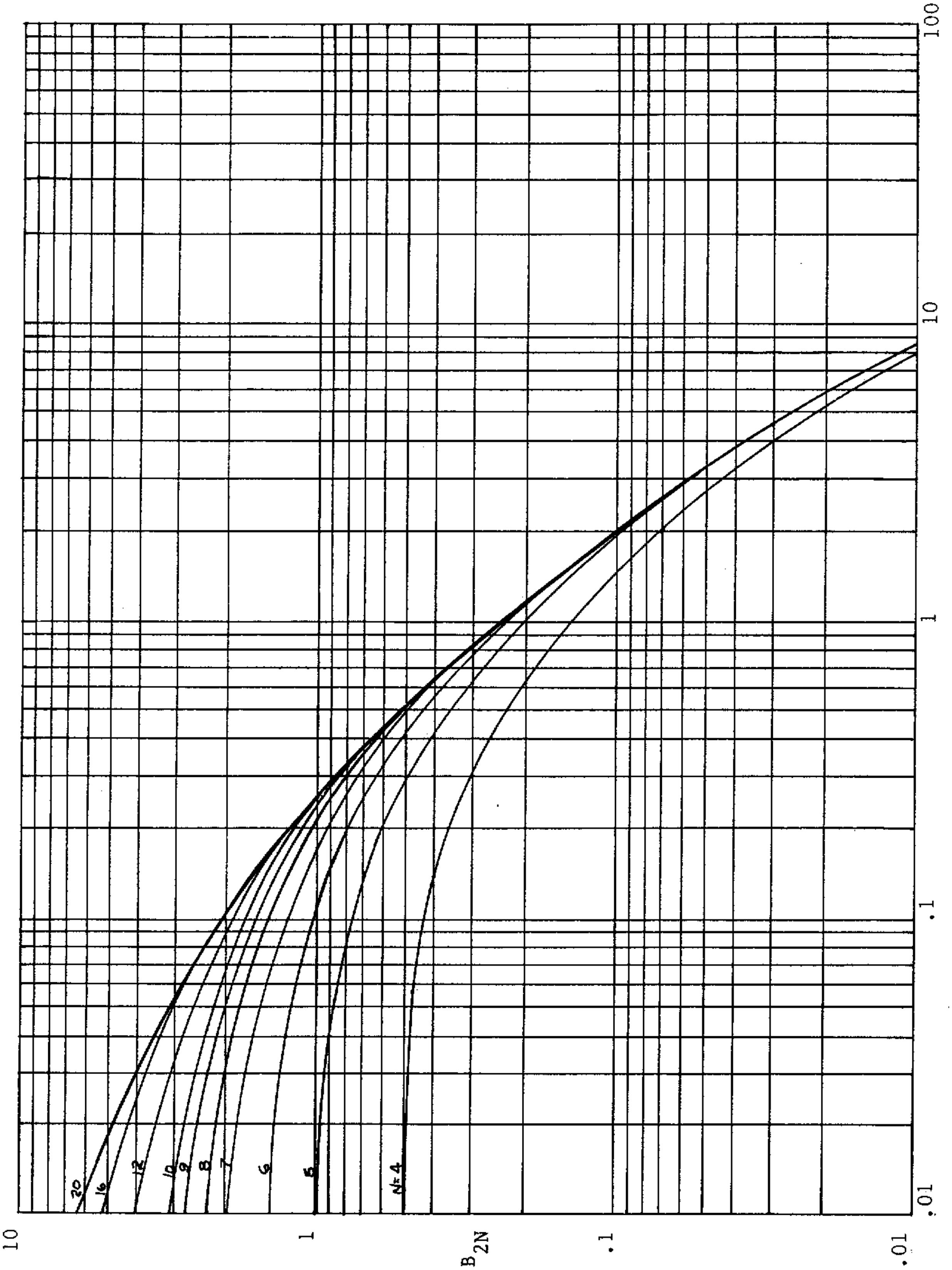


FIGURE 6.11 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (B_{2N}, j=2)

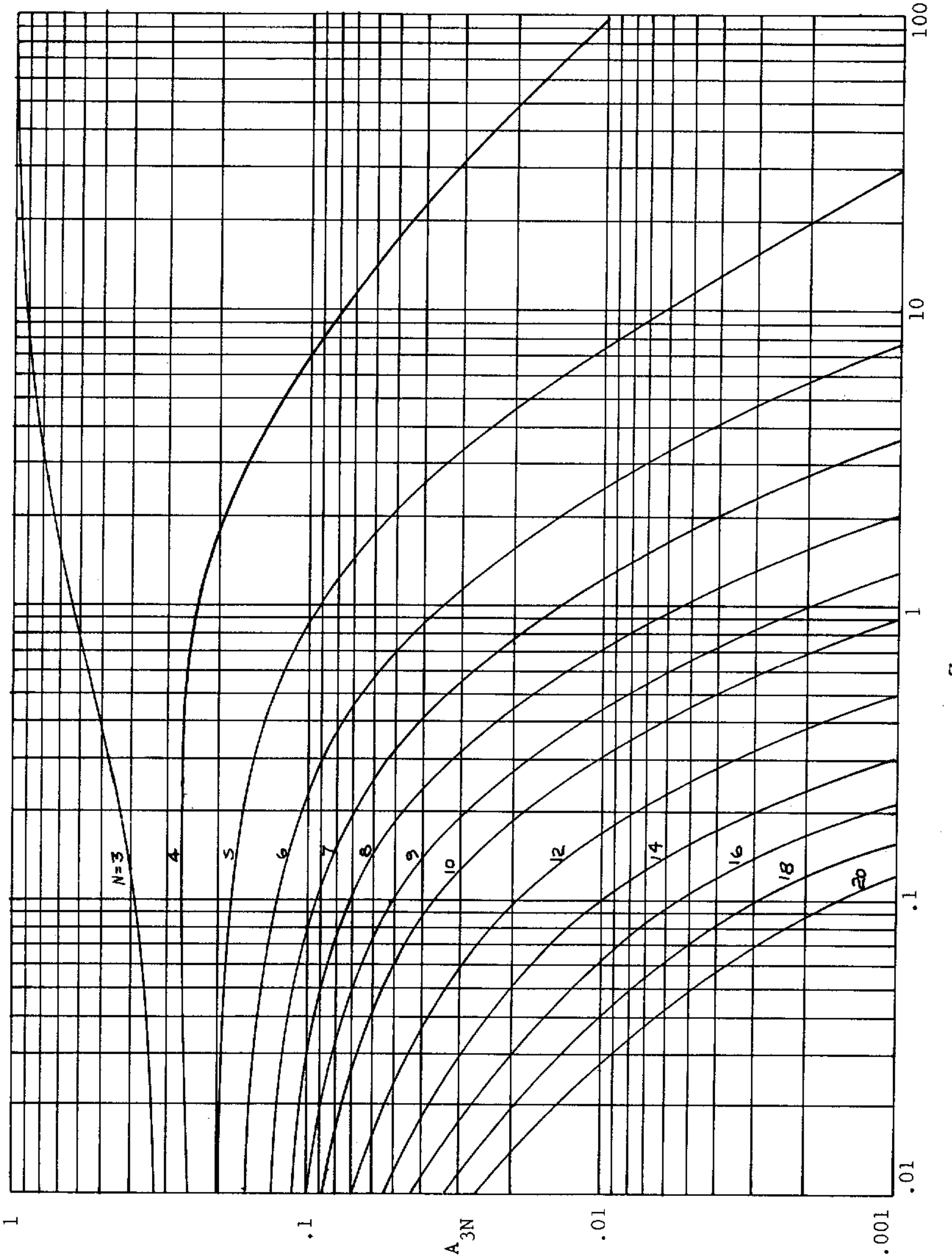


FIGURE 6.12 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (A_{3N}, j=3)

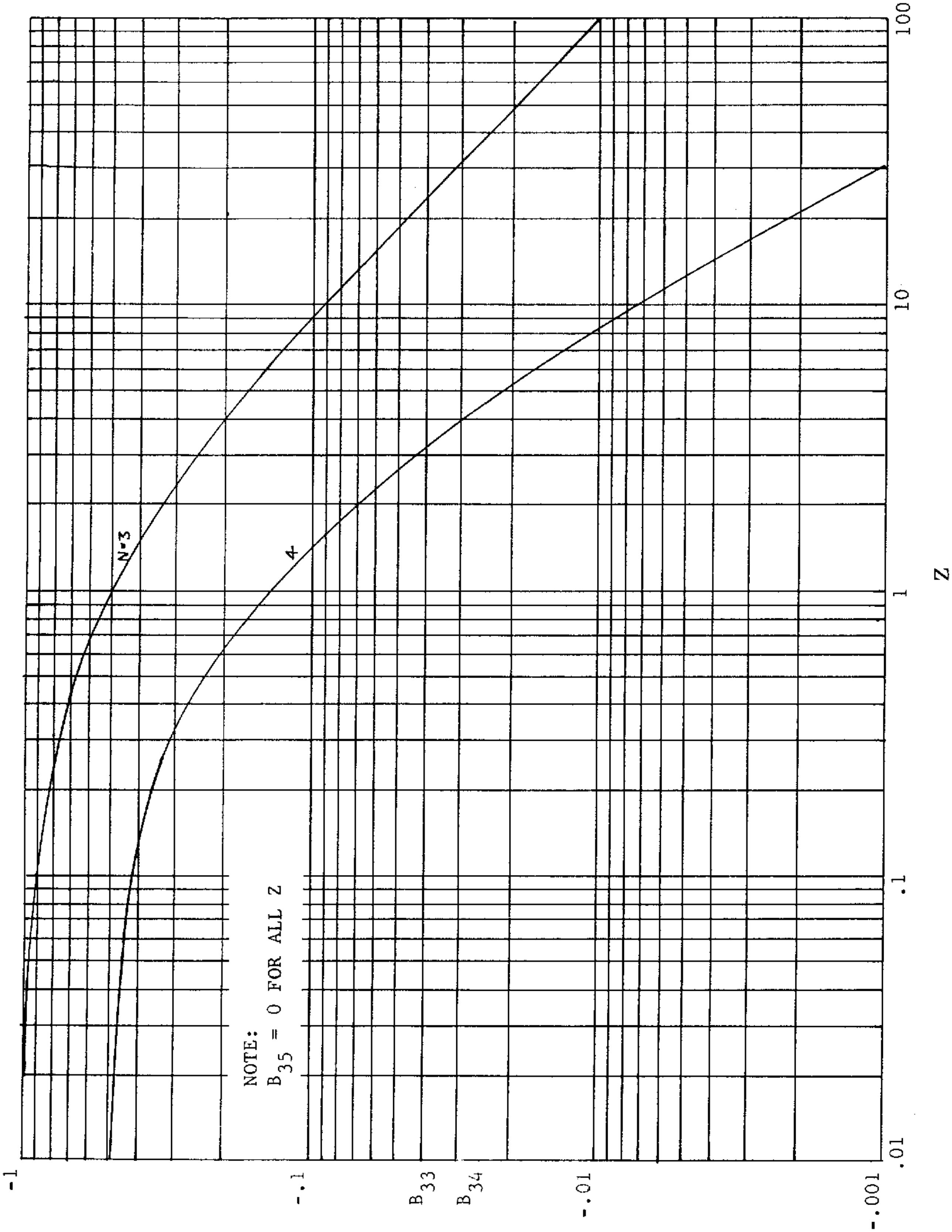


FIGURE 6.13 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (B_{33}, B_{34}, j=3)

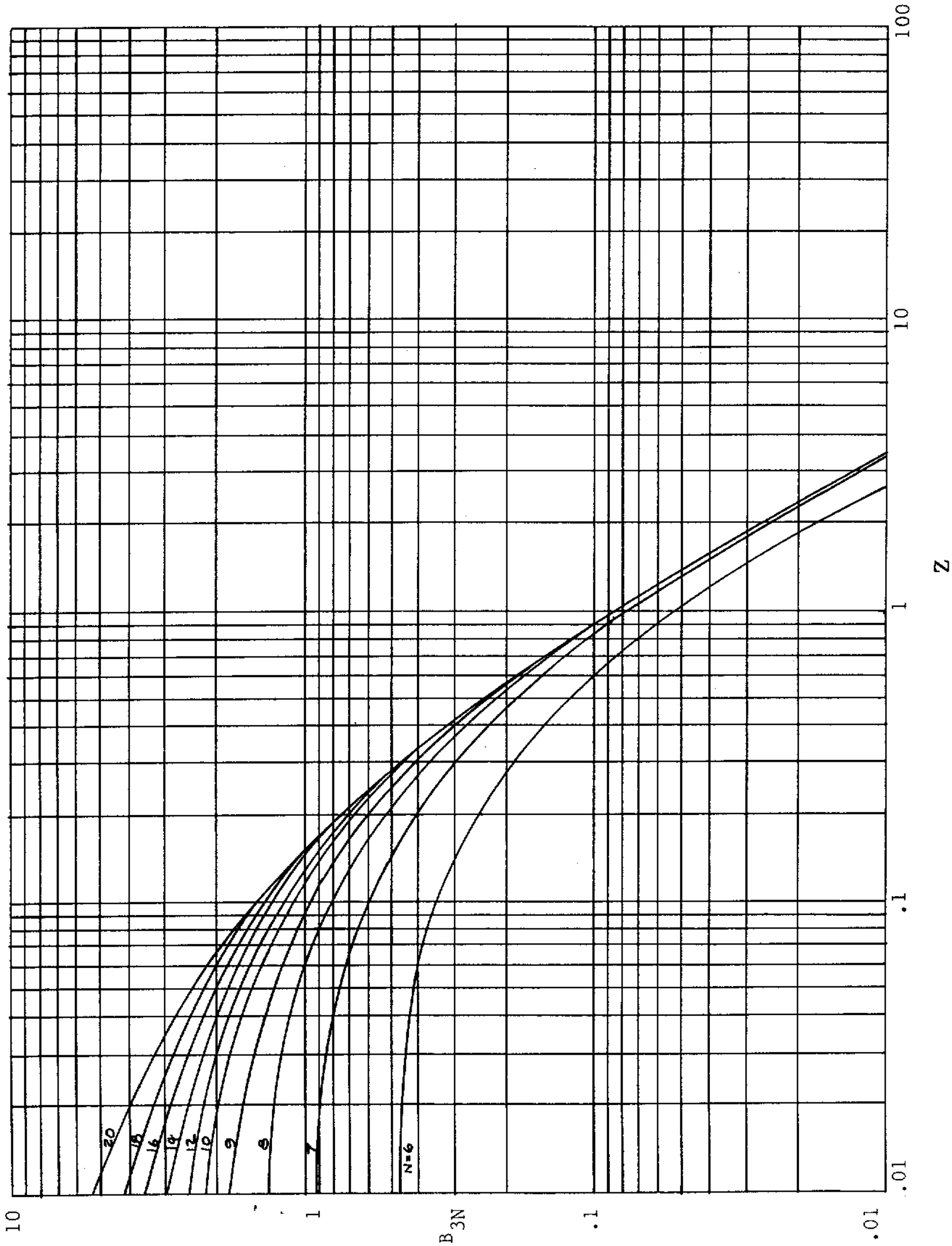


FIGURE 6.14 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (B_{3N}, j=3)

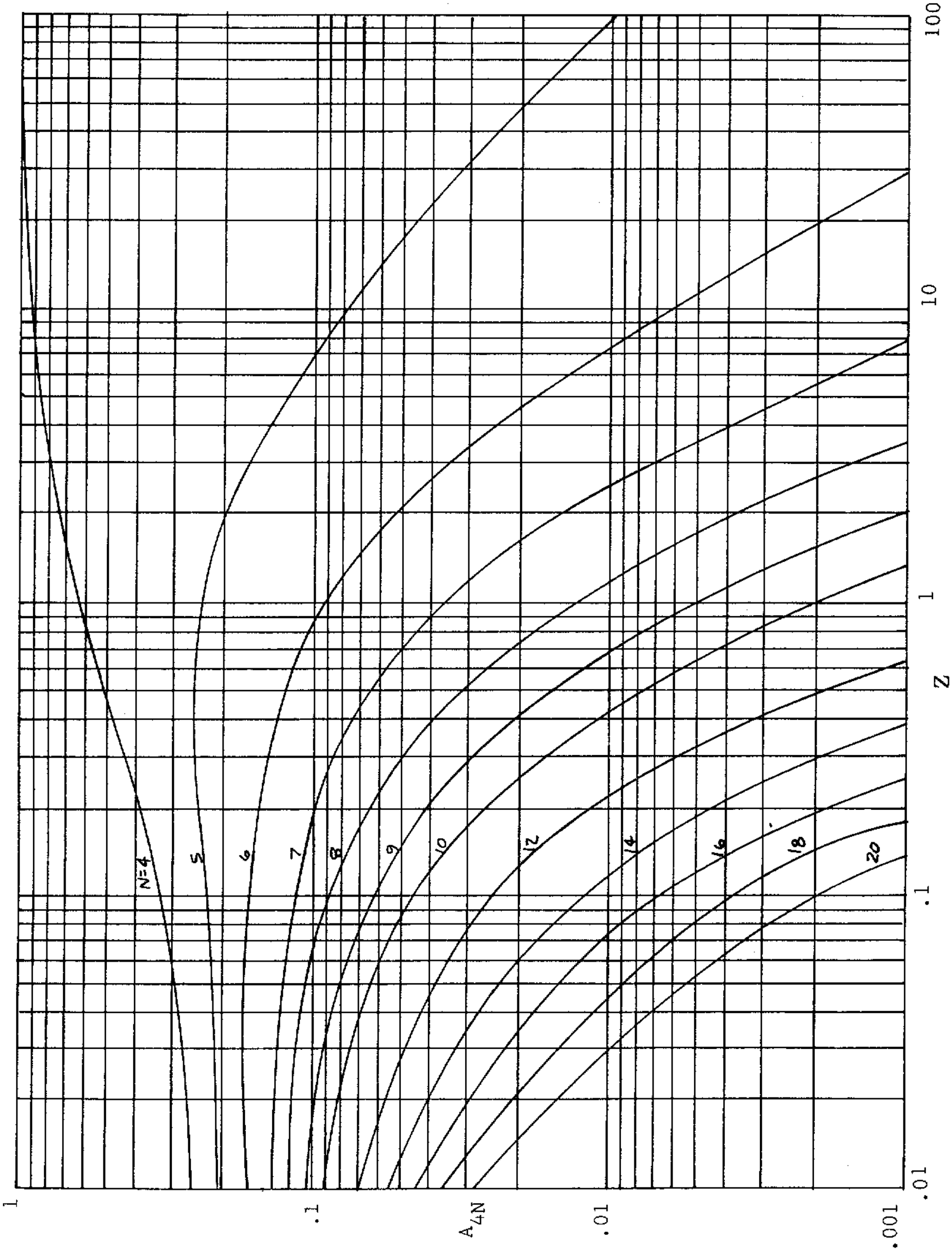


FIGURE 6.15 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (A_{4N}, j=4)

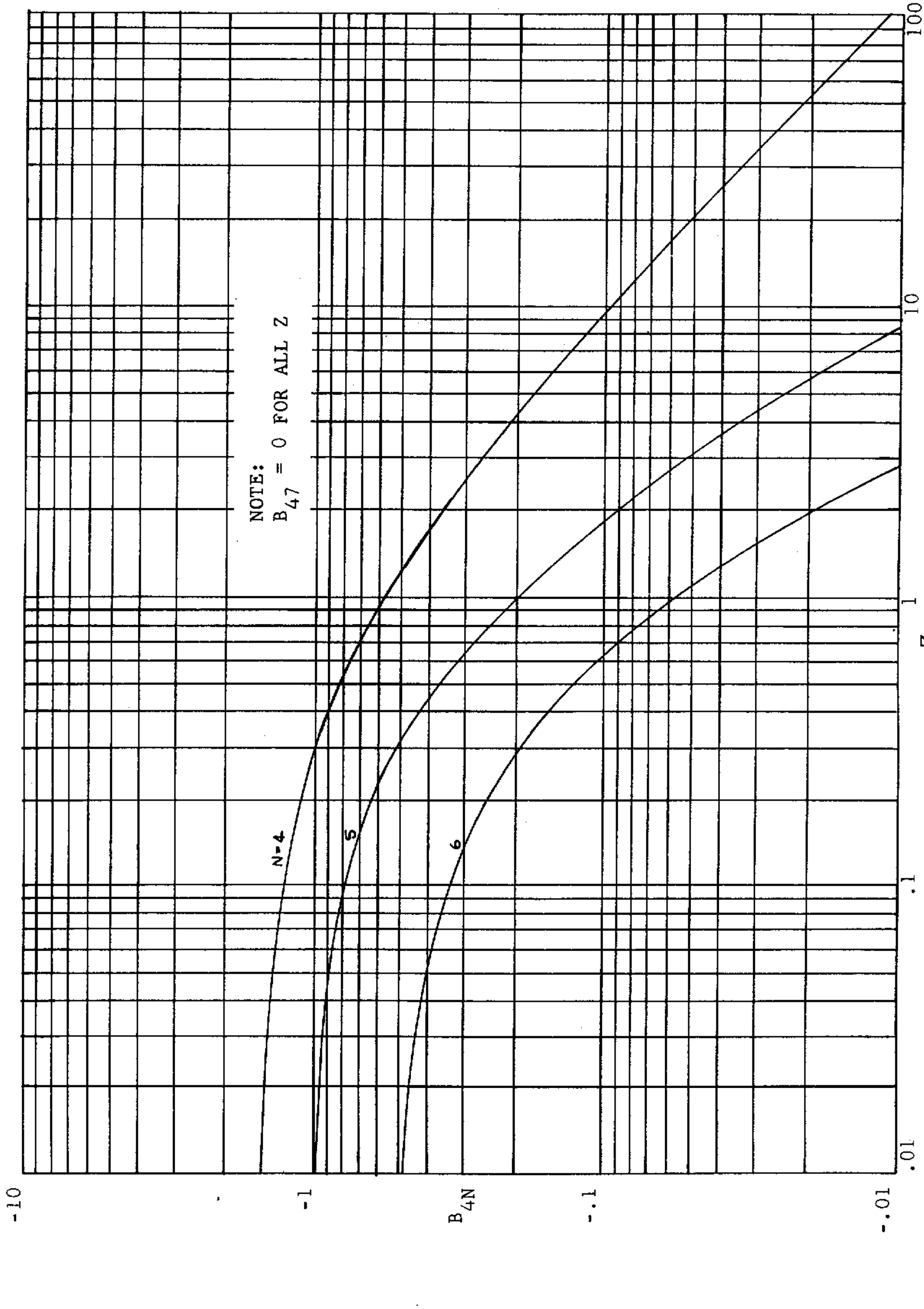


FIGURE 6.16 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (B_{44}, B_{45}, B_{46}, j=4)

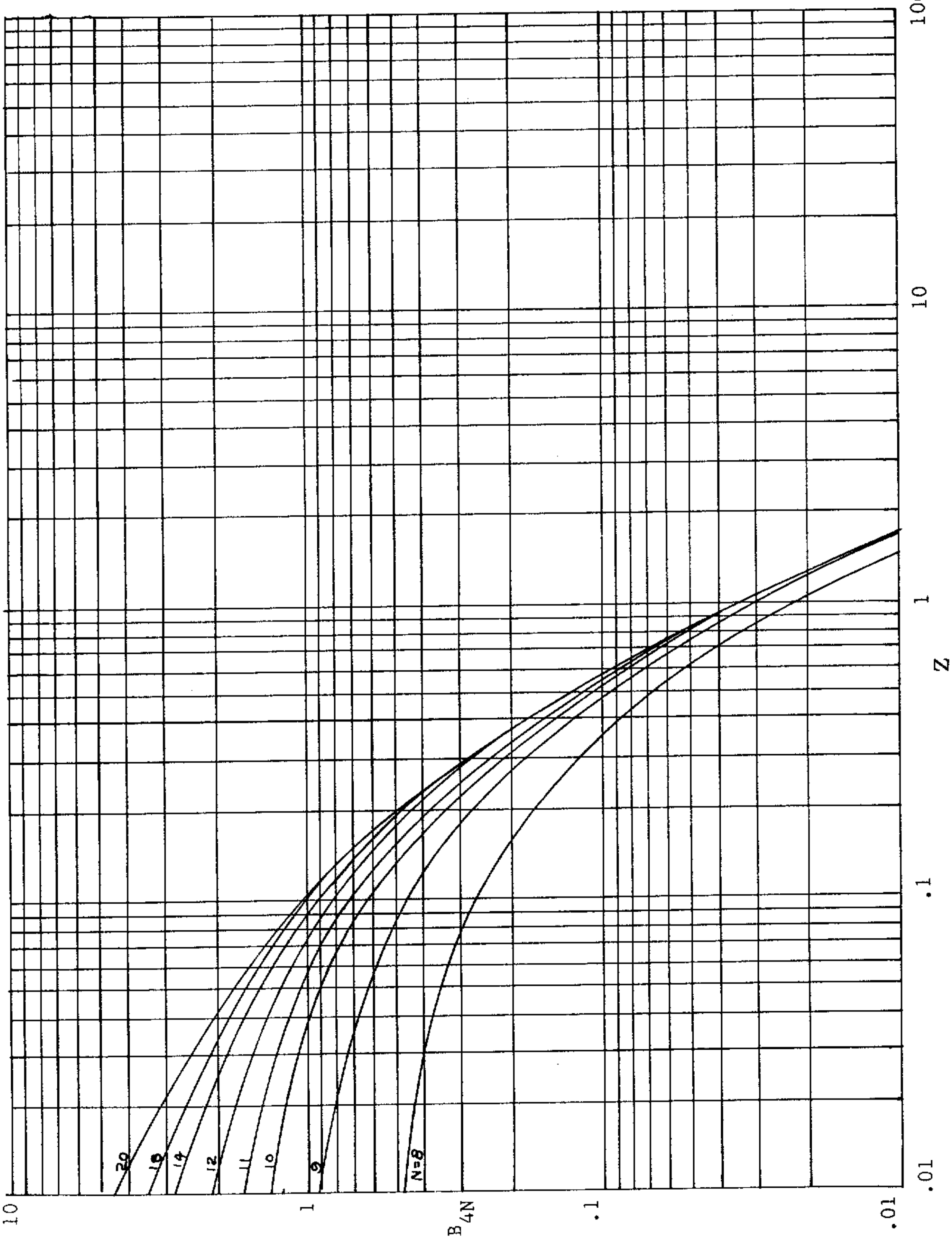


FIGURE 6.17 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (B_{4N}, j=4)

Revision B

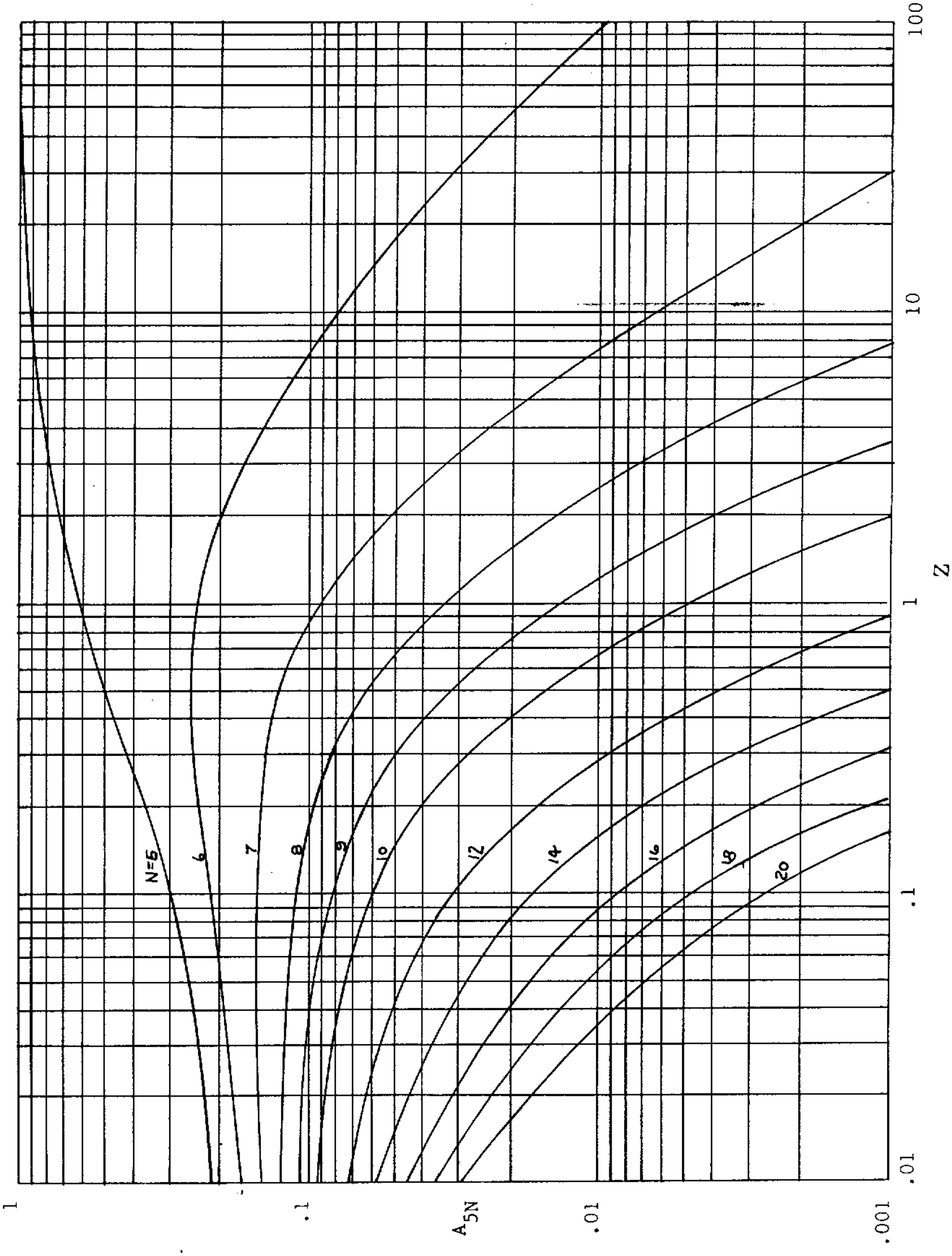


FIGURE 6.18 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (A_{5N}, j=5)

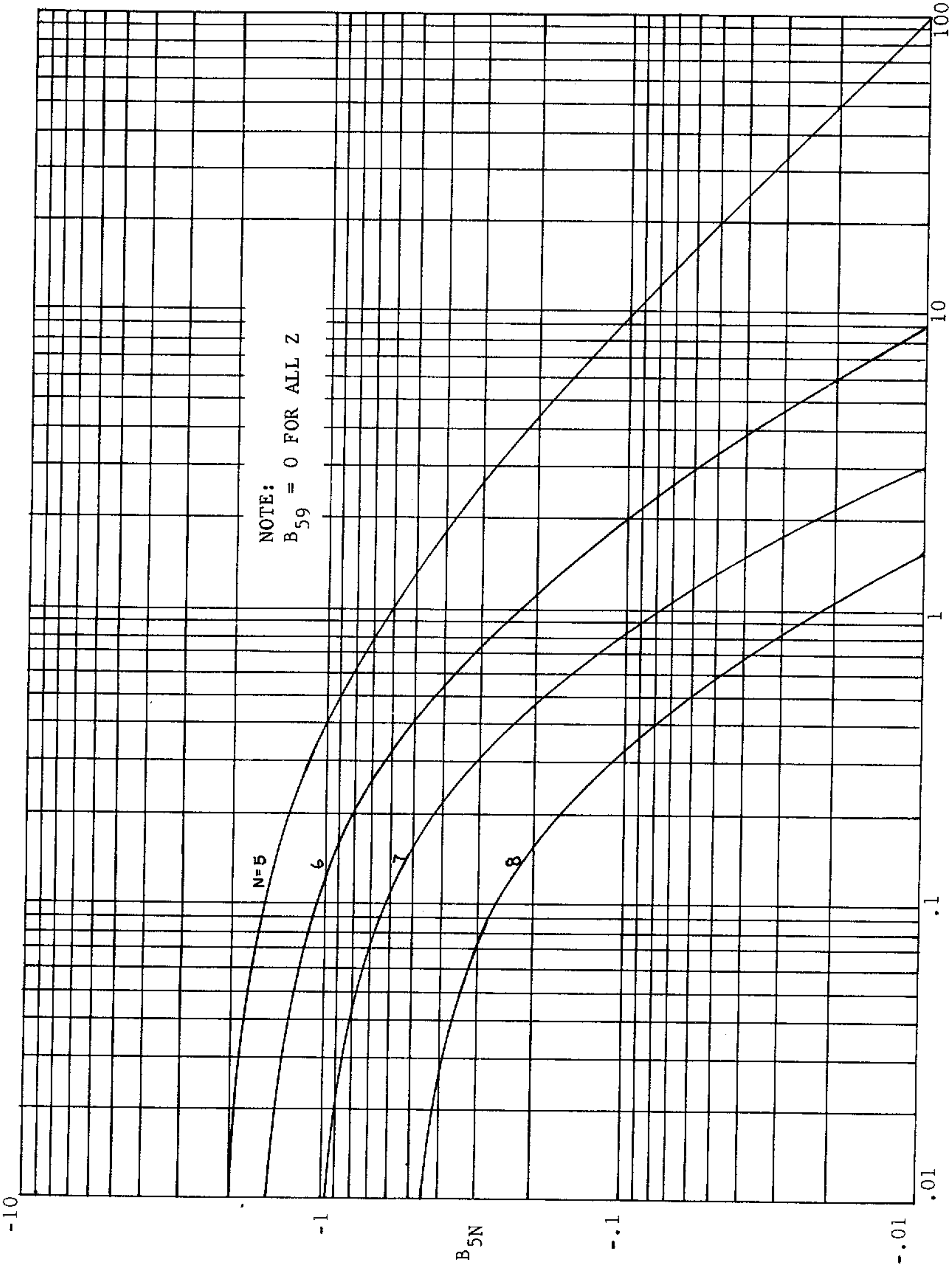


FIGURE 6.19 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (B_{5N}, j=5)

Revision B

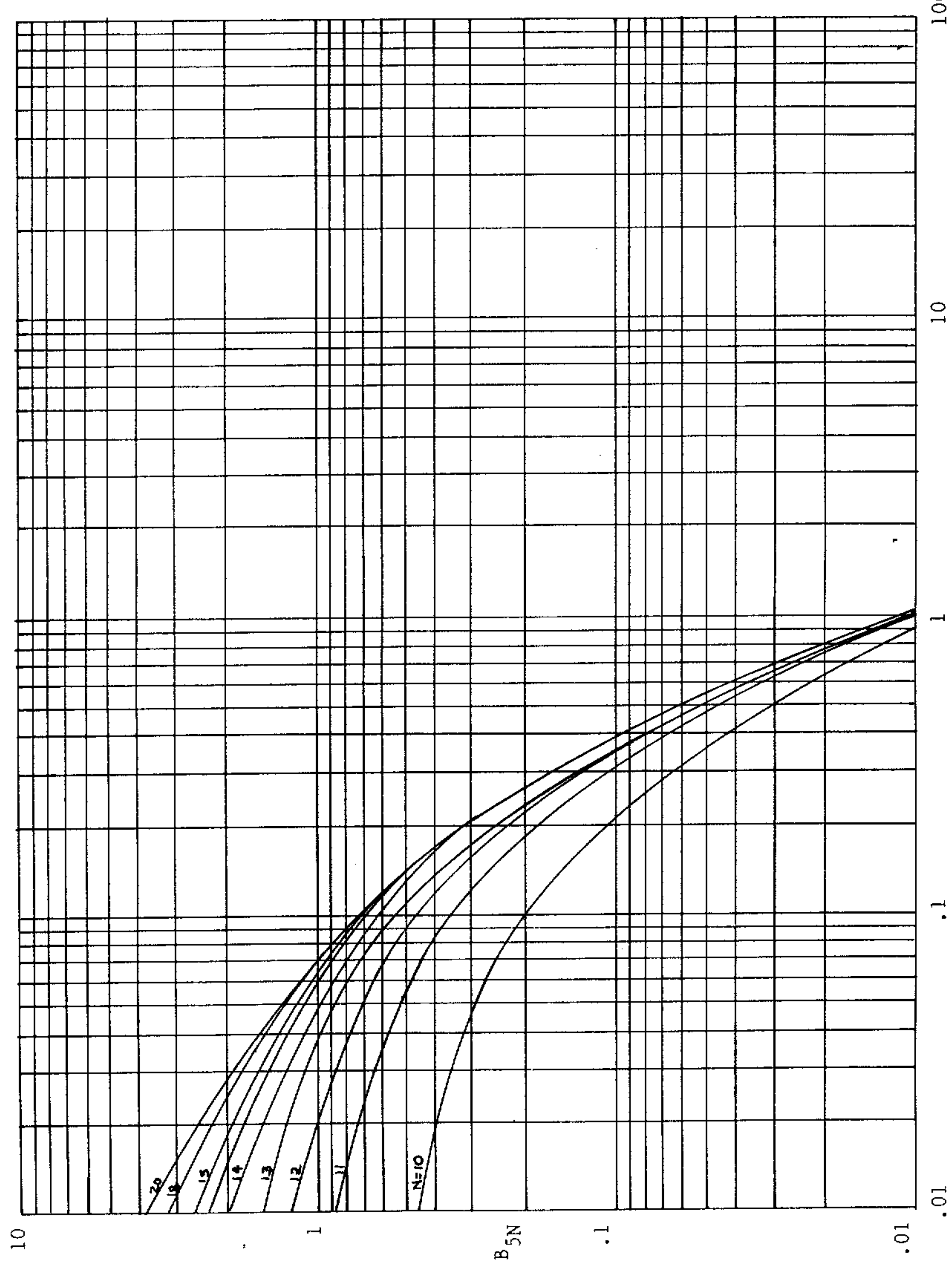


FIGURE 6.20 JOINT COEFFICIENTS FOR CONSTANT BAY PROPERTIES (B_{5N}, j=5)

An example problem best illustrates the procedure. Figure 6.21 shows a bimetallic splice, titanium and aluminum sheets joined by six steel bolts. The titanium and aluminum have uniform temperature rises of 300°F and 70°F respectively.

The results of the example show that the maximum load occurs in the first attachment and that the two end attachments carry more than half of the total applied mechanical load. When plastic deformations occur in the vicinity of the bolt holes, the bolts tend to carry equal loads.

Example Problem:

$f = .900 \times 10^{-6}$ in/lb (Section 6.4.3)

$(L/AE)_T = 1/(1.5)(.125)(15)(10^6) = .356\ (10^{-6})$ in/lb

$(L/AE)_B = 1/(1.5)(.250)(10)(10^6) = .267\ (10^{-6})$ in/lb

$\Delta\phi = (\alpha\ \Delta T)_T - (\alpha\ \Delta T)_B$

$\Delta\Phi = \left((6.5)(300) - 12(70)\right)(1)(10^{-6}) = 1110(10^{-6})$ in.

Substituting into equation (7)

$P_{jn} = \left(A_{jn} + B_{jn}\ (.356/.900)\right)(20{,}000) + B_{JN}\ (1110/.900)$

$P_{jn} = 20{,}000\ A_{jn} + 9135\ B_{jn}$

The coefficients A_{jn} and B_{jn} are now determined from Figure 6.7 through Figure 6.20.

$$Z = \left[\left(\frac{L}{Ae}\right)_T + \left(\frac{L}{Ae}\right)_B\right]\left(\frac{1}{f}\right) = (.356 + .267)\ (1/.900) = .692$$

N = 6

A_{16} = .0140; Figure 6.7 , B16 = .8000; Figure 6.8

A_{26} = .0239; Figure 6.9 , B26 = .3200; Figure 6.11

A_{36} = .0500; Figure 6.12, B36 = .0880; Figure 6.14

A_{46} = .1090; Figure 6.15, B46 = -.0860; Figure 6.16

A_{56} = .2450; Figure 6.18, B56 = -.3200; Figure 6.19

The curves give values of A_{jn} and B_{jn} up to j = 5, but the splice under consideration has 6 fasteners. In order to obtain the coefficients for the last attachment, the designation of the top and bottom plates must be interchanged as shown.

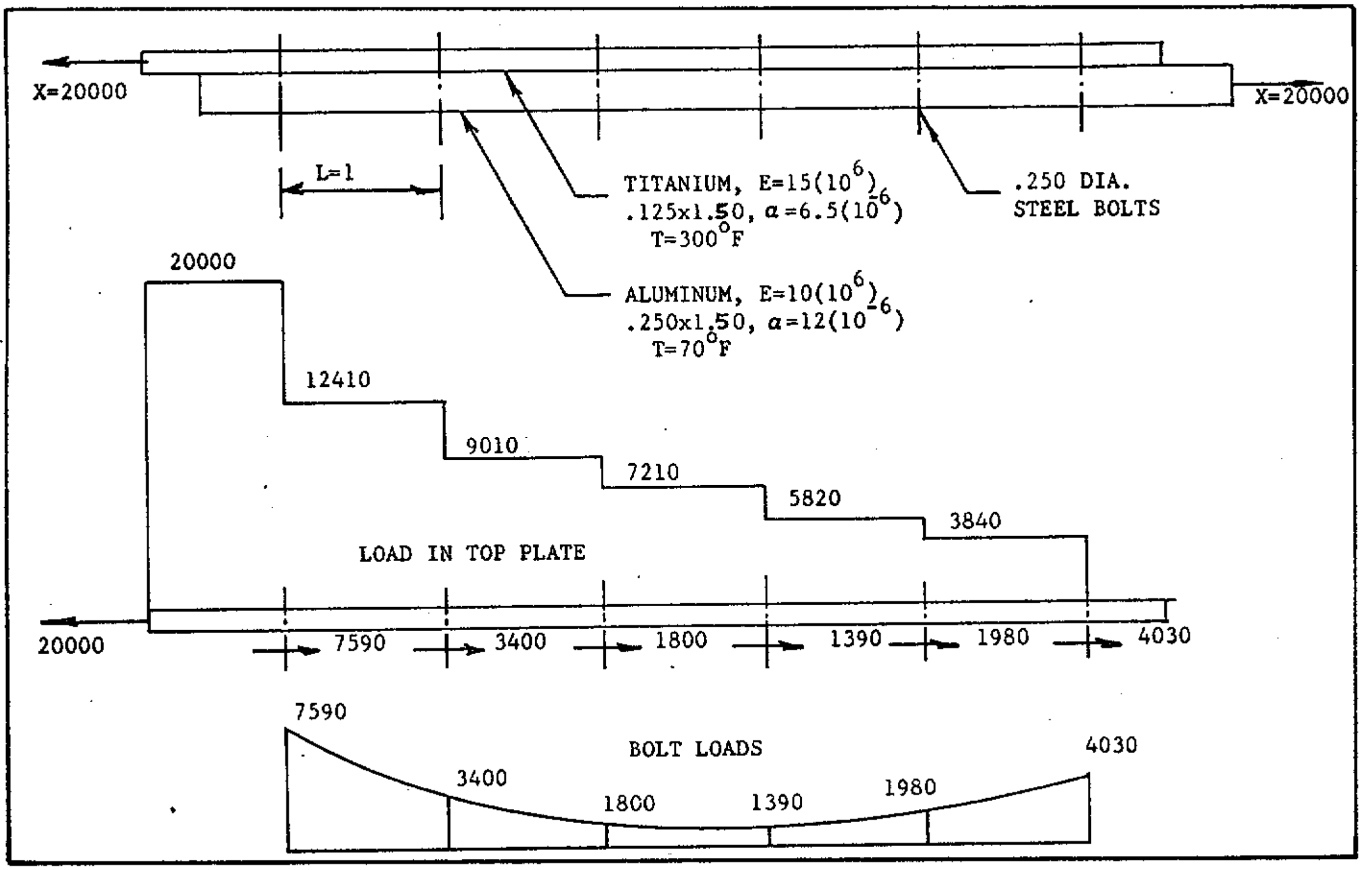


FIGURE 6.21 - EXAMPLE PROBLEM, COMPATIBILITY

As shown above, the last attachment (j = 6) in the original designation becomes the first attachment (j = 1) in the interchanged position.

$$f' = f = .900\ (10^{-6})\ \text{in/lb}$$

$$(L/AE)'_T = (L/AE)_B = .267\ (10^{-6})\ \text{in/lb}$$

$$(L/AE)'_B = (L/AE)_T = .356\ (10^{-6})\ \text{in/lb}$$

$$Z' = Z = .692 \qquad \Delta\phi' = -\ \Delta\phi = -1110(10^{-6})$$

from equation (7)

$$P'_{jn} = (A'_{jn} + B'_{jn}\ (.267/.900))(20{,}000) = B'_{jn}\ (1110/.900)$$

$$P'_{jn} = 20{,}000\ A'_{jn} + 4693\ B'_{jn}$$

from Figures 6.7 and 6.8

$$A'_{16} = A_{16} = .0140 \quad B'_{16} = B_{16} = .8000$$

Substituting coefficients A_{jn} and B_{jn} into the equations for P_{jn}

P_{16} = 20,000 (.0140) + 9135 (.8000) = 7590 lbs.

P_{26} = 20,000 (.0239) + 9135 (.3200) = 3400 lbs.

P_{36} = 20,000 (.0500) + 9135 (.0880) = 1800 lbs.

P_{46} = 20,000 (.1090) + 9135 (-.0860) = 1390 lbs.

P_{56} = 20,000 (.2450) + 9135 (-.3200) = 1980 lbs.

P_{66} = $P'16$ = 20,000 (.0140) + 4693 (.8000) = 4030 lbs.

Equilibrium Check

$$\sum_{j=1}^{n} P_j = 7590 + 3400 + 1800 + 1390 + 1980 + 4030 = 20190 \text{ lbs.}$$

6.4.1.2 Constant Bay Properties - Rigid Sheets

The special case in which the sheets have negligible axial deformation as compared to the deformations caused by local distortions of the holes and attachment will be considered now. This condition is possible when the sheets are thick and the attachments are small or when local yielding causes the effective attachment/hole flexibility to become large as compared to the axial flexibility of the sheets. In this case

$$(L/AE)_T \rightarrow 0 \text{ and } (L/AE)_B \rightarrow 0$$

and the compatibility equation (5) in Section 6.4.1.1 reduces to

$$P_{j+1} = P_j - \Delta\phi/f$$

$$P_j = P_1 - (j-1)\,\Delta\phi/f \qquad (8)$$

P_1 is obtained by summing equation (1) over the total number of fasteners:

$$\sum_{j=1}^{N} P_j = X = NP_1 - \phi/f \left[\sum_{j=1}^{N} (j-1)\right] = NP_1 - N/2(N-1)\,\Delta\phi/f$$

$$P_1 = X/N + (N-1)/2\,(\Delta\phi/f) \qquad (9)$$

Substituting (9) into (8)

$$P_j = X/N + (N+1/2 - j)\,\Delta\phi/f \qquad (10)$$

The solution shows that for the case of constant bay properties and infinitely rigid sheets, the mechanical load distributes equally to the fasteners while attachment loads due to thermal effects vary symmetrically about the transverse centerline of the joint with magnitudes inversely proportional to the attachment/hole flexibility.

For high loads which cause extensive plastic deformation in the vicinity of the attachment holes, the effective attachment/hole flexibility may become large as compared to the sheet flexibility, in which case the solution of equation (10) is approached. If these plastic effects become large enough, the increase in f tends to wipe out the effects of thermal loading with the result that

$$P_j \sim X/N$$

This indicates that near the failure of ductile materials the mechanical load tends to distribute equally to the attachments regardless of temperature distribution.

6.4.1.3 <u>Constant Bay Properties - Rigid Attachments</u>

The attachments may be considered rigid if the attachment/hole flexibility is negligible when compared to the axial flexibility of the sheet. This situation seldom occurs in practice since it represents the limiting case of $f \rightarrow 0$.

6.4.1.4 <u>The Influence of "Slop"</u>

The presence of "slop" due to manufacturing tolerance and differential thermal expansions between the plate holes and attachments affects load distribution through the basic joint compatibility equation. The slop at each attachment is indicated by the difference in diameters of the plate hole and attachment and is expressed by

$$e = e_{mfg} + e_{temp}; \ (e > 0)$$

where

e_{mfg} = initial room temperature manufacturing tolerance (clearance or interference. Interference is negative.)

and

e_{temp} = thermal slop (clearance or interference due to differential thermal expansion between plate holes and attachments)

$$e_{temp} = \left[(\alpha T)_{sheet} - (\alpha T)_{attach}\right] D_{hole} \qquad (11)$$

The joint displacements (for compatibility purposes) have been measured from a datum defined by the initial spacing between the centerlines of adjacent attachments. When slop is present and thermal and mechanical loads are applied to the joint, the attachments are displaced from the centers of the holes until they bear up against sheet material as shown in Figure 6.22. The algebraic sign of the slop displacements depends on the direction of the joint loads. For the top sheet, the displacement between adjacent attachments is given by

$$\delta_{jT} = \left[\frac{-e_{jT}}{2}\right]\left[\frac{P_j}{|P_j|}\right] + \left[\frac{e_{(j+1)T}}{2}\right]\left[\frac{P_{j+1}}{|P_{j+1}|}\right]$$

and for the bottom sheet

$$\delta_{jB} = \left[\frac{e_{jB}}{2}\right]\left[\frac{P_j}{|P_j|}\right] - \left[\frac{e_{(j+1)B}}{2}\right]\left[\frac{P_{j+1}}{|P_{j+1}|}\right]$$

where a positive δ increases the spacing between adjacent attachments and

$\frac{P}{|P|}$ = +1 for a positive attachment load and -1 for a negative attachment load

The incompatibility due to slop is therefore

$$\Delta\delta_j = \delta_{jT} - \delta_{jB} = \tfrac{1}{2}(e_T + e_B)_{j+1}\left[\frac{P_{j+1}}{|P_{j+1}|}\right] - \tfrac{1}{2}(e_T + e_B)\left[\frac{P_j}{|P_j|}\right] \qquad (12)$$

Equation (12) is significant in that it brings out the very nature of the slop problem. Consider, for example, that the slop is the same at all attachments. In this case equation (12) reduces to

$$\Delta\delta_j = \tfrac{1}{2}(e_T + e_B)\left[\frac{P_{j+1}}{|P_j|} - \frac{P_j}{|P_j|}\right] \qquad (13)$$

As the mechanical loads applied to the joint increase, all the attachment loads tend to act in the same direction (opposite to the externally applied load) or

$$\frac{P_{j+1}}{|P_{j+1}|} = \frac{P_j}{|P_j|} = \pm 1$$

The bracket quantity in equation (13) becomes zero and

$$\Delta\delta_j = 0 \qquad (14)$$

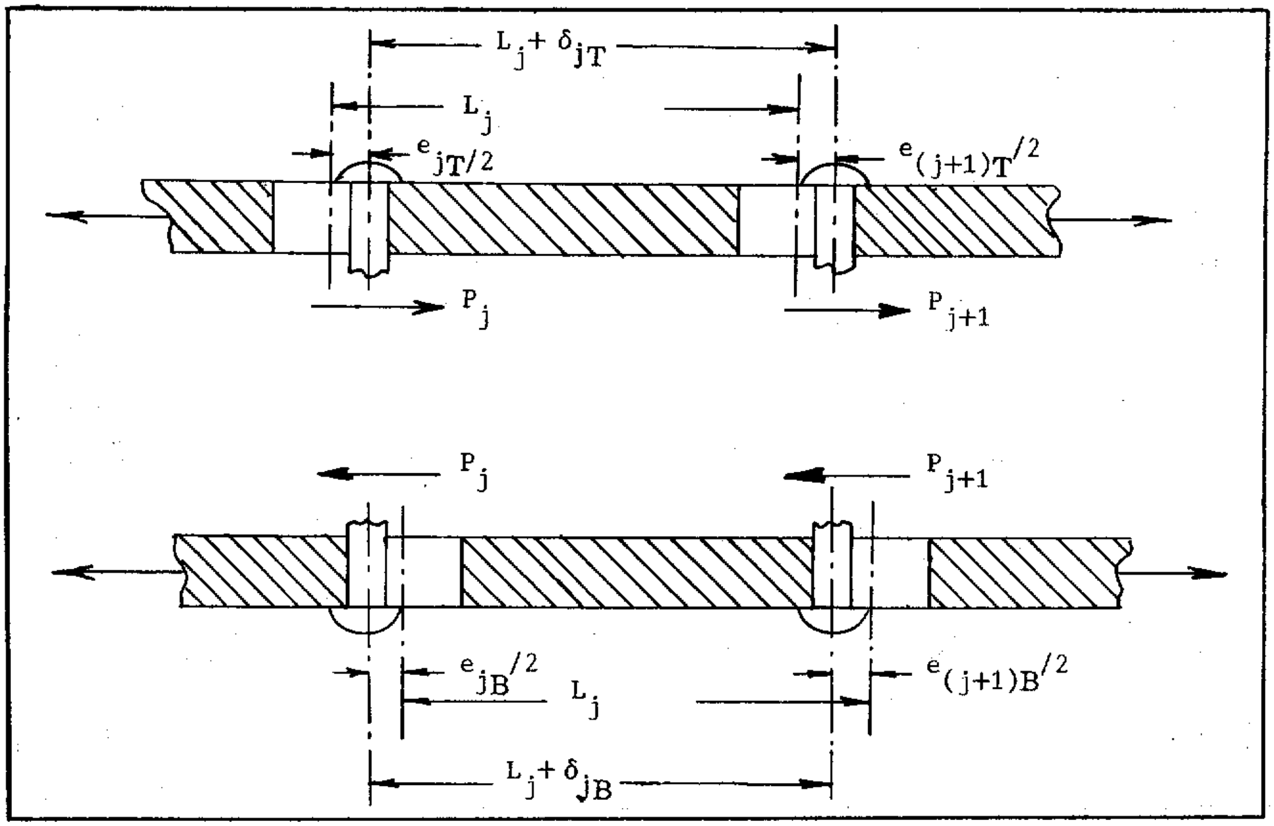


FIGURE 6.22 - ATTACHMENT DISPLACEMENTS DUE TO SLOP

Since the $\Delta\delta$'s determine the influences of slop on the load distribution, equation (14) indicates that for high joint loadings the effects of uniform slop are eliminated. To solve for the load distribution with slop, the basic one-dimensional compatibility expression equation (5) must be modified by the addition of $\Delta\delta_j$. The compatibility equations become:

$$\left[\left(\frac{L}{AE}\right)_{jT} + \left(\frac{L}{AE}\right)_{jB}\right]\left[\sum_{i=1}^{j} P_i\right] = (\Delta\phi_j + \Delta\delta_j) - P_j f_j + P_{(j+1)} f_{(j+1)} + X\left(\frac{L}{AE}\right)_{jT} \quad \ldots\ldots(15)$$

Equation (15) and equation (1) provide N equations for the N required attachment loads P_j. The solution of these equations, however, involves more than simply solving a set of simultaneous algebraic equations. The values of $\Delta\delta_j$ on the right side of equation (15) are given by (12) from which, in order to determine the $\Delta\delta_j$'s, the sign of the attachment loads (positive or negative) must be determined. But this is not known in advance. This presents one of the major difficulties of the slop problem. The method is as follows:

(1) Assume a set of directions for the attachment loads.

(2) Determine the $\Delta\delta_j$'s from equation (12) and solve the simultaneous capability and equilibrium equations (1) and (15).

(3) If the directions of the attachment loads as obtained from the solution agree with the initially assumed directions, the solution is correct.

(4) If the directions of the attachment loads as obtained from the solution do not agree with the initially assumed directions, the solution is incorrect. The procedure must be repeated with a new set of attachment load directions, preferably the ones obtained from the solution.

The solution is the correct one when the assumed set of attachment load directions yields a solution with the same set of directions. An example best illustrates the procedure. Figure 6.23 shows scarfed steel and aluminum plates bolted together with three attachments. The steel and aluminum plates are subjected to uniform temperature rises of 640°F and 80°F respectively and a mechanical load of 5000 pounds is applied. The manufacturing tolerance is e_{mfg} = 0.0003 inch for all bolts.

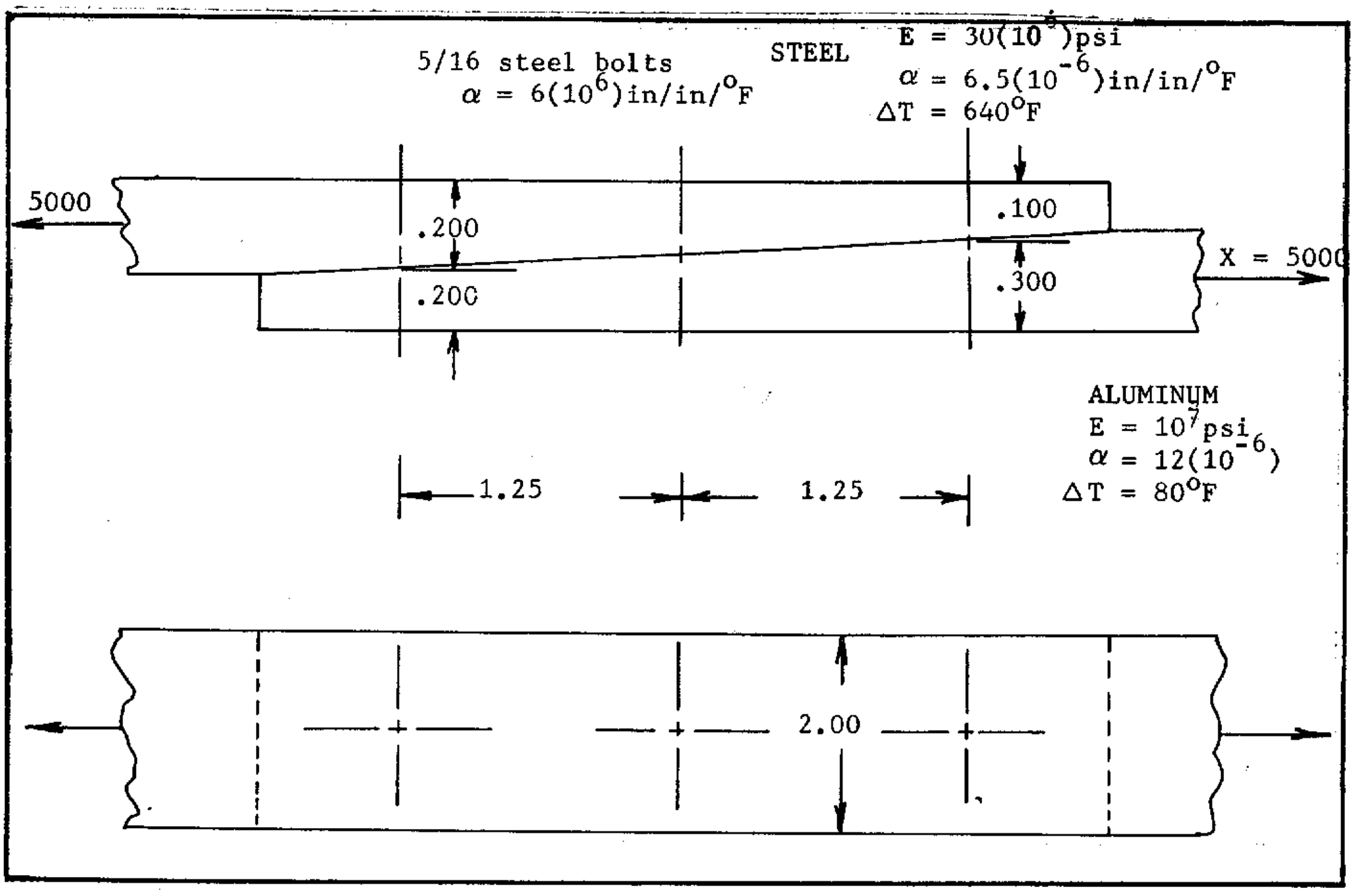


FIGURE 6.23 - SCARFED SPLICE

Example Problem:

$f_1 = 1.300\ (10^{-6})$ in/lb

$f_2 = 1.200\ (10^{-6})$ in/lb

$f_3 = 1.300\ (10^{-6})$ in/lb

Using average thicknesses for each bay;

$$(L/AE)_{1T} = \frac{1.25}{(.175)(2)(30)(10^6)} = .119(10^{-6})\ \text{in/lb}$$

$$(L/AE)_{2T} = \frac{1.25}{(.125)(2)(30)(10^6)} = .167(10^{-6})\ \text{in/lb}$$

$$(L/AE)_{1B} = \frac{1.25}{(.225)(2)(10)(10^6)} = .278(10^{-6})\ \text{in/lb}$$

$$(L/AE)_{2B} = \frac{1.25}{(.275)(2)(10)(10^6)} = .227(10^{-6})\ \text{in/lb}$$

Since the temperature rise in each bay is uniform, the incompatibilities due to unrestrained thermal expansion is

$$\Delta\phi_1 = \Delta\phi_2 = [(\alpha\Delta T)_T - (\alpha\Delta T)_B]\ L$$

$$= ((6.5)(640) - (12)(80))(1.25)(10^{-6}) = 4000(10^{-6})\ \text{in.}$$

Assuming the temperature of each bolt to be the same as the surrounding sheet material, the slops due to temperature are given by

$$(e_{temp})_{top} = ((\alpha\Delta T)_{top\ sheet} - (\alpha\Delta T)_{top\ of\ attach})\ D_{hole}$$

$$= ((6.5)(640) - (6)(640))\ (.3125)(10^{-6})$$

$$= 100\ (10^{-6})\ \text{in.}$$

$$(e_{temp})_{bottom} = ((\alpha\Delta T)_{bottom\ sheet} - (\alpha\Delta T)_{bottom\ of\ attach})\ D_{hole}$$

$$= ((12)(80) - (6)(80))(.3125)(10^{-6})$$

$$= 150\ (10^{-6})\ \text{in.}$$

Since $e_{mfg} = 300\ (10^{-6})$ inch, the total slop is given by

$$e_{1T} = e_{2T} = e_{3T} = (e_{mfg} + e_{temp})_{top}$$

$$= (300 + 100)\ (10)^{-6}$$

$$= 400\ (10^{-6})\ \text{in.}$$

$$e_{1B} = e_{2B} = e_{3B} = (e_{mfg} + e_{temp})_{bottom}$$

$$= (300 + 150)\ (10^{-6})$$

$$= 450\ (10^{-6})\ \text{in.}$$

The incompatibilities due to slop, equation (12) are

$$\Delta\delta_1 = .5\ (400 + 450)\ P_2/|P_2| - .5\ (400 + 450)\ P_1/|P_1|$$

$$= 425\ (P_2/|P_2| - P_1/|P_1|)$$

$$\Delta\delta_2 = 425\ (P_3/|P_3| - P_2/|P_2|)$$

Substituting into equation (15)

$$1.697\ P_1 - 1.200\ P_2 = 4595 + 425(P_2/|P_2| - P_1/|P_1|)$$

$$.394\ P_1 + 1.594\ P_2 - 1.300\ P_3 = 4835 + 425(P_3/|P_3| - P_2/|P_2|$$

and for equilibrium

$$P_1 + P_2 + P_3 = 5000$$

1st. Assume all attachment loads are positive

$$1.697\ P_1 - 1.200\ P_2 = 4595$$

$$.394\ P_1 + 1.594\ P_2 - 1.300\ P_3 = 4835$$

$$P_1 + P_2 + P_3 = 5000$$

$$P_1 = 3880,\ P_2 = 1850,\ P_3 = -530$$

The above solution would be correct if no slop were present, however; since joint slop is present, the solution contradicts the initial assumption that the bolt loads are all positive and it is therefore incorrect.

2nd. Assume that P_1 & P_2 are positive and P_3 is negative

$$1.697\ P_1 - 1.200\ P_2 = 4595$$

$$.394\ P_1 + 1.594\ P_2 - 1.300\ P_3 = 3985$$

$$P_1 + P_2 + P_3 = 5000$$

$$P_1 = 3730,\ P_2 = 1440,\ P_3 = -170$$

This is a correct solution since the directions agree with those assumed.

6.4.1.5 Two-Dimensional Compatibility

When the boundary conditions are such that the joint is allowed to bow out of its own plane, the solution is much more complicated. Additional factors such as rotational and out-of-plane displacements, beam column effects, moments at the attachments, etc., enter into the solution. An exact analytical solution will not be attempted here. Instead, an analysis method is presented which obtains the first approximation to the solution of the two-dimensional problem by modifying the equations of the one-dimensional solution.

Bowing of the joint, Figure 6.24, occurs due to the combined effects of non-uniform temperatures and externally applied mechanical loadings. The solution presented gives the shear loads in the attachments for a known set of applied mechanical and thermal loads where the following simplifying assumptions are made:

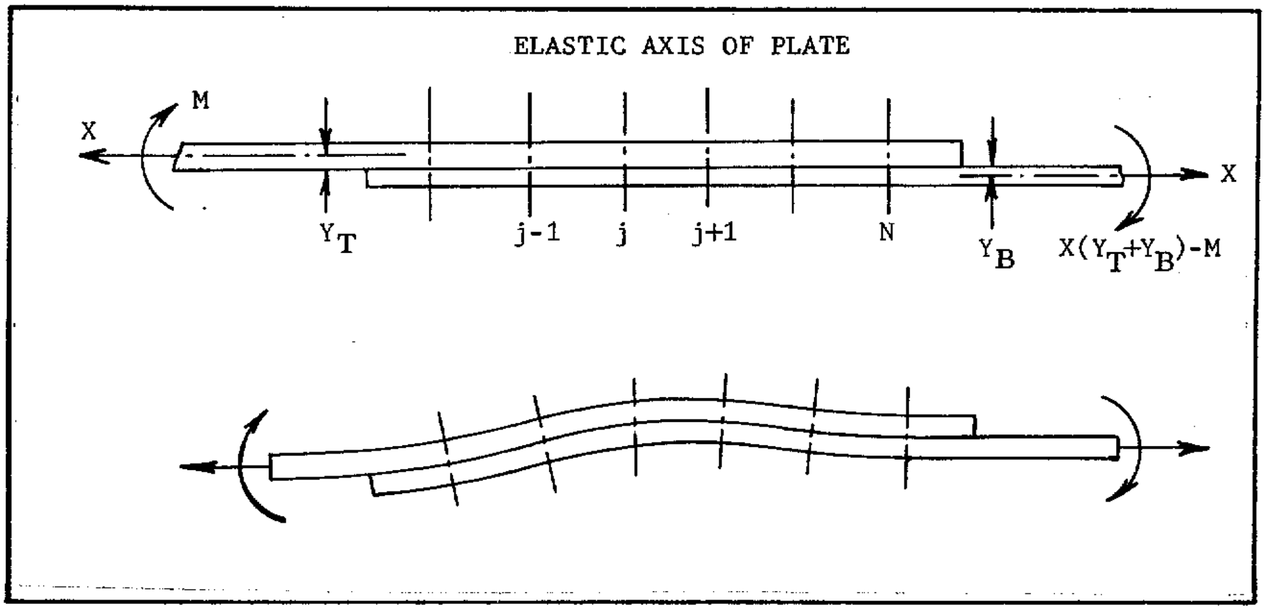


FIGURE 6.24 - JOINT WITH BOWED CONFIGURATION

(1) The bay properties are constant (sheet thicknesses, attachment size and spacing, stiffnesses, etc., are the same for each bay). The thermal loading is assumed not to vary in the longitudinal direction but may vary through its thickness.

(2) Vertical out-of-plane deflections and clamping loads are assumed to have a negligible effect on the load distribution (negligible beam column effects).

(3) Moments at the attachments have a negligible effect on, or are included in the attachment/hole flexibility.

(4) The contact faces of the top and bottom plates of the joint are initially plane; the external axial loading is applied parallel to this plane in the direction of the line of attachments.

(5) As in the one-dimensional case, the joint materials are assumed to deform elastically under load.

Under the above assumptions, the requirements of compatibility at the attachments yield the following

$$f_A \sum_{i=1}^{j} P_i = \overline{\Delta\phi} - P_j f + P_{j+1} f + X f_{AT} \quad (j = 1, 2, \ldots N-1) \tag{16}$$

where

$$f_A = \left(\frac{L}{AE}\right)_T + \left(\frac{L}{AE}\right)_B + \frac{L\,(y_T + y_B)^2}{\overline{EI}_T + \overline{EI}_B} \tag{17}$$

$$\overline{\Delta\phi} = \Delta\phi - L\left[\frac{\overline{EI}_T W_T + \overline{EI}_B W_B}{\overline{EI}_T + \overline{EI}_B}\right](y_T + y_B) \tag{18}$$

$$f_{AT} = \left(\frac{L}{AT}\right)_T + \frac{ML}{X}\left[\frac{y_T + y_B}{\overline{EI}_T + \overline{EI}_B}\right] \tag{19}$$

and W is the curvature due to temperature. If the thermal gradient is linear through the thickness, then W approximately equals $\alpha\Delta T/h$ where $\Delta T/h$ is the linear thermal gradient through the plate thickness (positive for higher temperatures) on the upper face of the plate. Equation (16) and equation (1) combine to form N equations and N unknowns.

A comparison of equation (16) with the one-dimensional compatibility equation, equation (6), shows that the two forms are identical. Thus, when the bay properties are constant, the procedure for the two-dimensional solution is exactly the same as for the one-dimensional case if the one-dimensional coefficients

$$\left[\left(\frac{L}{AE}\right)_T + \left(\frac{L}{AE}\right)_B\right],\ \Delta\phi,\ \left(\frac{L}{AE}\right)_T$$

are replaced by the expressions on the right side of equations (17), (18), and (19), respectively. Coefficients A_{jN} and B_{jN} can then be obtained, from Figures 6.7 through 6.20.

6.4.2 Joint Load Distribution - Semi-Graphical Method

This is a semi-graphical method for determining the load distribution in a joint. This method is based strictly on geometry. If a more precise load distribution is required, the method of strains in Section 6.4.1 should be used.

6.4.2.1 Fastener Pattern Center of Resistance

Locate the center of resistance of the fastener pattern, G, on the basis of bearing or shear area. If the fasteners are critical in sheet bearing, the bearing area should be used. If the fasteners are shear critical, use the shear area.

Equal Areas

Figure 6.25 shows a typical fastener pattern. Assume each fastener has equal areas. Connect any two of the fasteners and bisect this line. The point of bisection is the centroid of the first two fasteners. Join this centroid with the third fastener and locate a point one-third of the line distance from the previous centroid to obtain the centroid of the three fasteners. Join this centroid to a fourth fastener and locate a point one-fourth of the distance from the previous centroid. Continue adding a fastener at a time until all areas have been included.

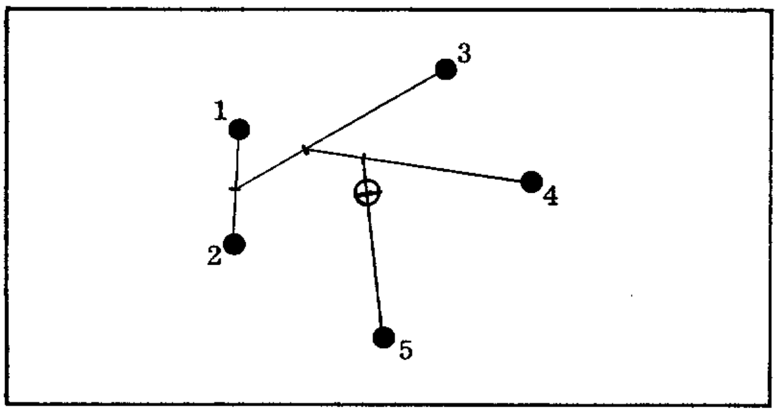


FIGURE 6.25 - FASTENER PATTERN CENTER OF RESISTANCE

Unequal Areas

Add a fastener at a time as described previously. At any stage where the centroid of n bolts has been found and is joined to the (n+1) fastener, the fractional part of the connecting line measured from the previous centroid is

$$\frac{A_{n+1}}{A_1 + A_2 + \cdots\cdots + A_n + A_{n+1}}$$

6.4.2.2 Load Determination

Figure 6.26 shows a typical joint with an applied load P and three fasteners A_1, A_2 and A_3. Draw the joint to scale and locate the center of resistance G. Extend the line of action of the applied load P, and from this line erect a perpendicular that passes through the centroid G and extends a distance GQ away from P, so that

$$GQ = \frac{\Sigma Ar^2}{e \Sigma A}$$

where

A = area of fastener in shear or bearing
r = radial distance from G to fastener
e = distance from G to line of action of P

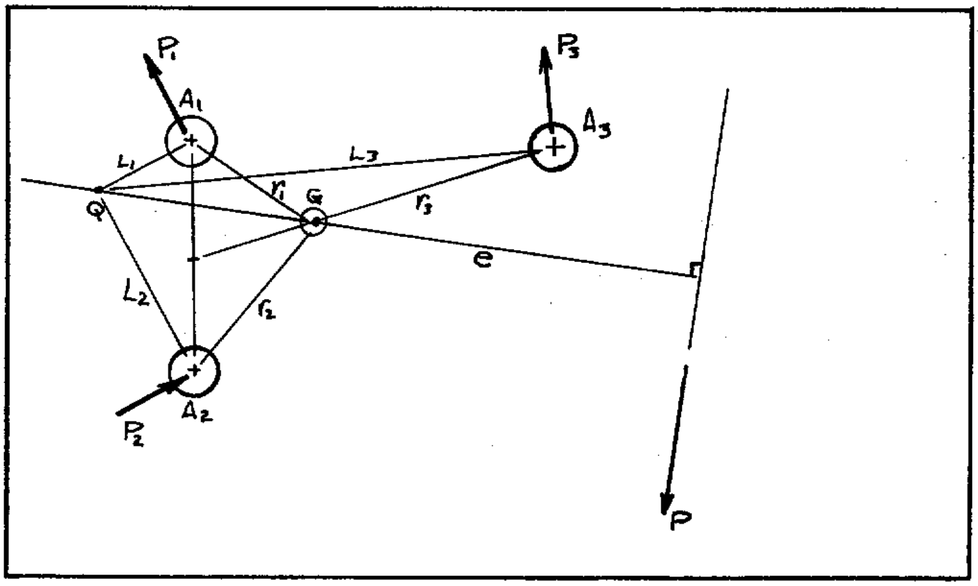


FIGURE 6.26 - TYPICAL JOINT

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Next determine the radial distance L_1 of the number one fastener from Q. The load P on that bolt is

$$P_1 = \frac{PeA_1L_1}{\Sigma_{Ar^2}}$$

and is directed perpendicular to radial line L_1.

Repeat this procedure until the loads for all fasteners are determined.

6.4.3 Attachment Flexibility

The flexibility of an attachment/sheet combination should be determined experimentally. If load-deflection curves for a particular fastener/sheet combination are available, the flexibility is the slope of the curve at the estimated load level.

If load-deflection test data is not available for the exact fastener/sheet combination, two methods can be used to determine a spring rate.

6.4.3.1 Method I - Generalized Test Data

Some test data is available to develop generalized stiffness curves. Figure 6.27 shows a curve of t/D versus K for a single shear joint with a steel fastener. The procedure for determining joint stiffness is as follows:

DIA	1/8	5/32	3/16	1/4	5/16	3/8	7/16	1/2	9/16	5/8
	$SR\times10^{-6}$									
ALUM	1.21	1.51	1.81	2.42	3.02	3.63	4.20	4.83	5.17	6.03
STEEL	3.62	4.53	5.44	7.25	9.06	10.9	12.6	14.5	15.5	18.1
TITAN	1.93	2.42	2.90	3.87	4.83	5.81	6.72	7.73	8.27	9.65
OTHER	(Eother/Esteel)xSRsteel									
SHEET SPRING RATE = K x SR JOINT SPRING RATE = 1/(1/SRu + 1/SR1)										

TABLE 6.9 - BASIC SPRING RATES

1. Calculate t/D for upper sheet

2. Calculate t/D for lower sheet

3. From Figure 6.27 determine K for upper sheet

4. From Figure 6.27 determine K for lower sheet

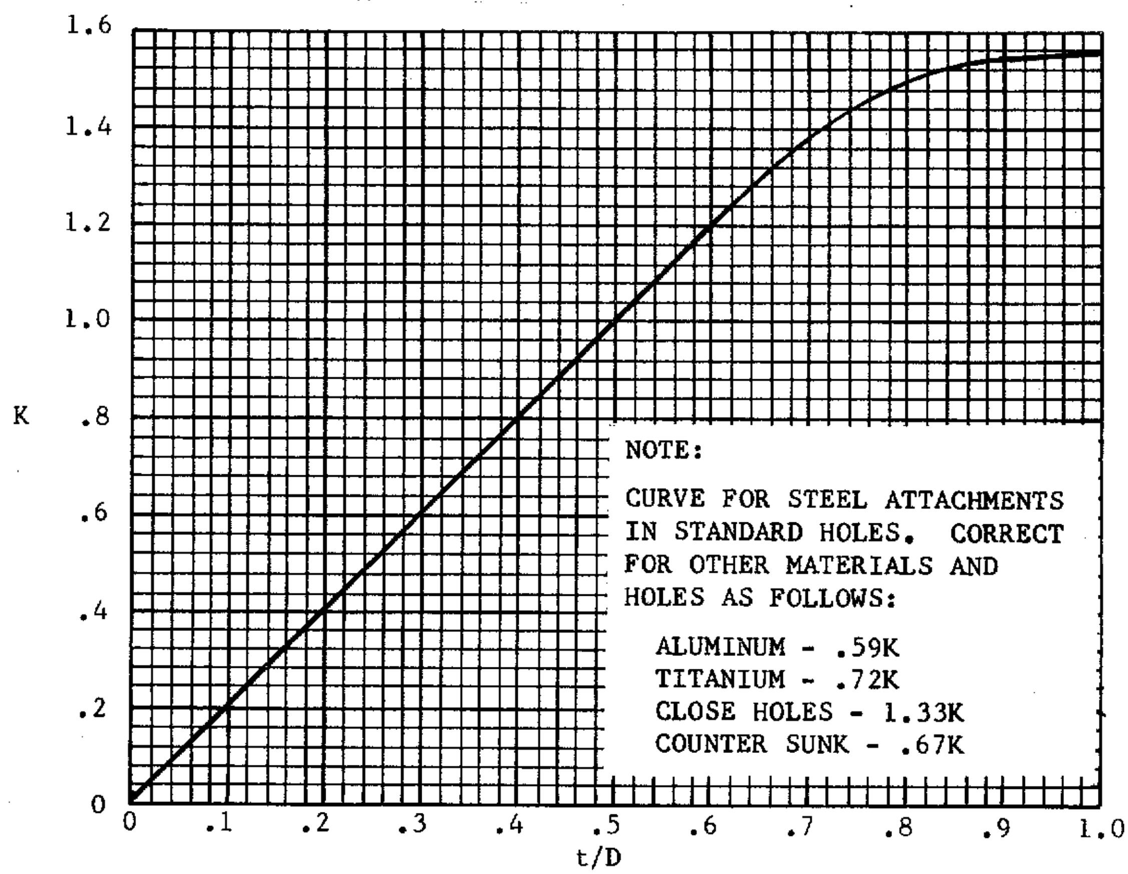

FIGURE 6.27 - EFFECTIVE SPRING RATES FOR STEEL PINS IN SINGLE SHEAR

5. If the fastener is steel and in a hole drilled to normal tolerance (Table 6.1), proceed to step 6. Modify the K factors of step A by the following factors

   Aluminum Fastener, K x .59
   Titanium Fastener, K x .72
   Close Tolerance Hole, K x 1.33
   Countersunk Hole, K x .67

   Example:

   Titanium fastener in close tolerance hole.
   From step 4: K
   Correct K: .72 x 1.33 x K

6. Determine SR from Table 6.9 . Calculate spring rate for each sheet by

   $$k_n = K_n \times SR$$

   where:

   k_n = spring rate of sheet
   K_n = constant from step 4 or 5
   SR = value from Table 6.9

7. Calculate joint spring rate

   $$k_{joint} = \frac{1}{1/k_u + 1/k_L}$$

6.4.3.2 Method II - Bearing Criteria

If load deflection data is not available, the limit bearing load criteria of Reference 1 may be used to obtain an estimate of the attachment-hole flexibility. These criteria result in an overestimate of the attachment-hole flexibility and an underestimate of the maximum attachment load at load levels below yield.

As an example of the way the criteria of MIL-HDBK-5B (Reference 1) may be used to determine the attachment-hole flexibility factor, consider a joint in which the bolt diameter is 0.25 inch, the upper sheet is 0.125 inch titanium and the lower sheet is 0.25 inch aluminum. Assuming that the aluminum is 2024-T6 and the titanium is 6 Al-4V, the respective bearing yield stress allowables from Reference 1 are 78,000 psi and 198,000 psi. The yield loads are then calculated to be

$$P_{al} = 78{,}000(.250)(.250) = 4875 \text{ lbs}$$

$$P_{titanium} = 198{,}000(.125)(.250) = 6200 \text{ lbs}$$

The average load is then

$$P_{avg} = (6200 + 4875)/2 = 5590 \text{ lbs.}$$

The flexibility is calculated for a deformation of 2 percent of the hole diameter per Reference 1.

$$f_{avg} = \Delta/P_{avg} = (.02)(.250)/5590 \approx 900(10^{-9}) \text{in./lb}$$

6.4.4 Lug Design

This section presents a basic method of analysis and procedure for the design of lug-pin combinations loaded axially, obliquely or transversely.

An accurate analysis of a lug-pin combination under load is difficult because the actual distributions of stresses in the lug and pin involve a combination of shear, bending and tension of varying amounts, which are a function of the ratio of lug edge distance and thickness to pin diameter, shape of lug, number of lugs in a joint, material properties, stress concentrations, rigidity of adjacent structure, etc.

The various modes of failure for a lug are:

1. Bearing of pin, lug or bushing
2. Tension across minimum net section. The full P/A_{net} stress cannot be carried because of the stress concentration around the hole.
3. Hoop tension failure of the lug across the section in line with the load.
4. Shear tearout failure of the lug.
5. Shear and bending of the pin.

Shear tearout and bearing are closely related and are covered by shear-bearing calculations based on empirical data. Also, the shear-bearing criteria precludes hoop tension failures.

Yielding of the lug is also a consideration. It is considered excessive at a permanent set of 0.02 times the pin diameter. This condition must always be checked as it is frequently reached at a lower load than would be anticipated from the ratio of the yield stress, F_{ty}, to the ultimate stress, F_{tu}, for the material.

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Since lugs are elements having severe stress concentrations, the ductility and/or impact strength of the material is of importance. For this reason, attention should be paid to the longitudinal, long transverse and short transverse material properties.

Lugs are a small weight portion of a structure and are prone to fabrication errors and service damage. Since their weight is usually insignificant relative to their importance, the following criteria should be used.

1. Design lugs for a minimum margin of safety of 0.15 in both yield and ultimate.

2. If no bushing is included in the original design, design the lug so that one can be inserted in the future; however, express margins of safety with no bushings.

6.4.4.1 Nomenclature

F_{tu} = Ultimate tensile strength; F_{tuw} with grain, F_{tux} cross grain. When the plane of the lug contains both long and short transverse grain directions, F_{tux} is the smaller of the two.

F_{ty} = Tensile yield strength; F_{tyw} with grain, F_{tyx} cross grain. When the plane of the lug contains both long and short transverse grain directions, F_{tyx} is the smaller of the two.

F_{cy} = Compressive yield strength

P_u = Ultimate load

P_y = Yield load

M_{max} = Maximum bending moment on pin

P'_u = Allowable ultimate load

P'_{bru} = Allowable ultimate shear-bearing load

P'_{bry} = Allowable yield bearing load on bushing

P'_{tu} = Allowable ultimate tensile load

P'_{tru} = Allowable ultimate transverse load

P'_y = Allowable yield load of lug

A = Area; A_{br} projected bearing area, A_t minimum net section for tension, A_{av} weighted average area for transverse load.

K = Efficiency factor; K_{br} for shear-bearing, K_t for tension, K_{tru} for transverse load (ultimate), K_{try} for transverse load (yield).

c = Yield factor

R = Load ratio; R_a for axial, R_{tr} for transverse

t = Thickness of lug

L, T, ST = Grain direction; (L) longitudinal, (T) transverse and (ST) short transverse.

α = Angle of oblique load; α = 0 for axial, α = 90 for transverse and $0 < \alpha < 90$ oblique.

γ = Pin bending moment reduction factor for peaking

r = $[e - D/2]/t$

6.4.4.2 Analysis of Lugs with Axial Loads ($\alpha = 0^\circ$)

The determination of the allowable ultimate and yield axial loads for lugs of the type shown in Figure 6.28 is as follows:

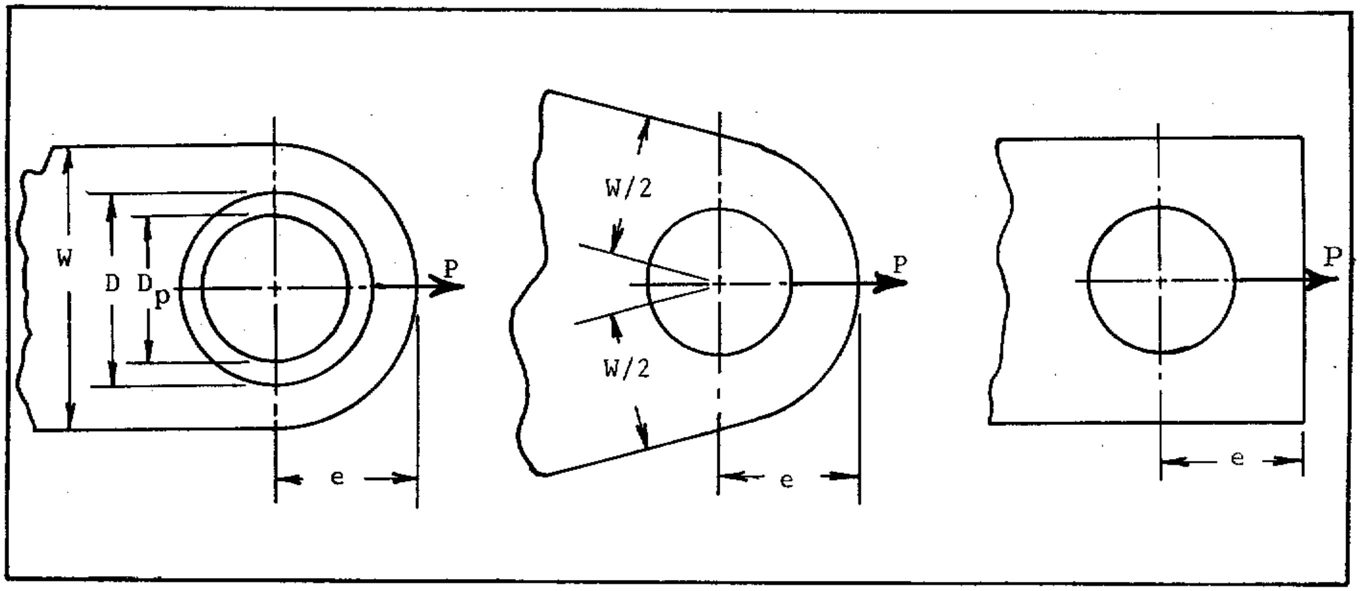

FIGURE 6.28 - AXIALLY LOADED LUGS

A. Compute: e/D, D/t, W/D, $A_{br} = Dt$, $A_t = (W - D)\ t$

B. P'_{bru} = allowable ultimate shear-bearing load

1. Enter Figure 6.30 with e/D and D/t and obtain K_{br}.
2. $P'_{bru} = K_{br}A_{br}F_{tux}$

C. P'_{tu} = allowable ultimate tension load

1. Enter Figure 6.32 with W/D and obtain K_t for proper material.
2. $P'_{tu} = K_t A_t F_{tu}$

D. P'_y = allowable yield load of lug

1. Enter Figure 6.31 with e/D and obtain K_{bry}.
2. $P'_y = K_{bry}A_{br}F_{ty}$

E. P'_{bry} = allowable yield bearing load on bushing

1. $P'_{bry} = 1.85\ F_{cy}A_{br_b}$

   Where A_{br_b} is the smaller of bearing area of bushing on pin or bearing area of bushing on lug. Latter value may be smaller due to effect of external chamfer of bushing.

F. Margins of safety

1. Minimum M.S. = .15 for ultimate shear-bearing and ultimate tension
2. Minimum M.S. = 0 for yield of lug and bushing.

6.4.4.3 Analysis of Lugs with Transverse Loads ($\alpha = 90^o$)

The determination of the allowable ultimate and yield transverse loads for lugs of the type shown in Figure 6.29 is as follows:

A. Compute: A_1, A_2, A_3, A_4

$$A_{br} = Dt$$

$$A_{av} = \frac{6}{3/A_1 + 1/A_2 + 1/A_3 + 1/A_4}$$

$$A_{av}/A_{br}$$

(1) A_1, A_2 and A_4 are measured on the planes indicated in Figure 6.29(a), A_1 and A_4 should be measured perpendicular to the local centerline.

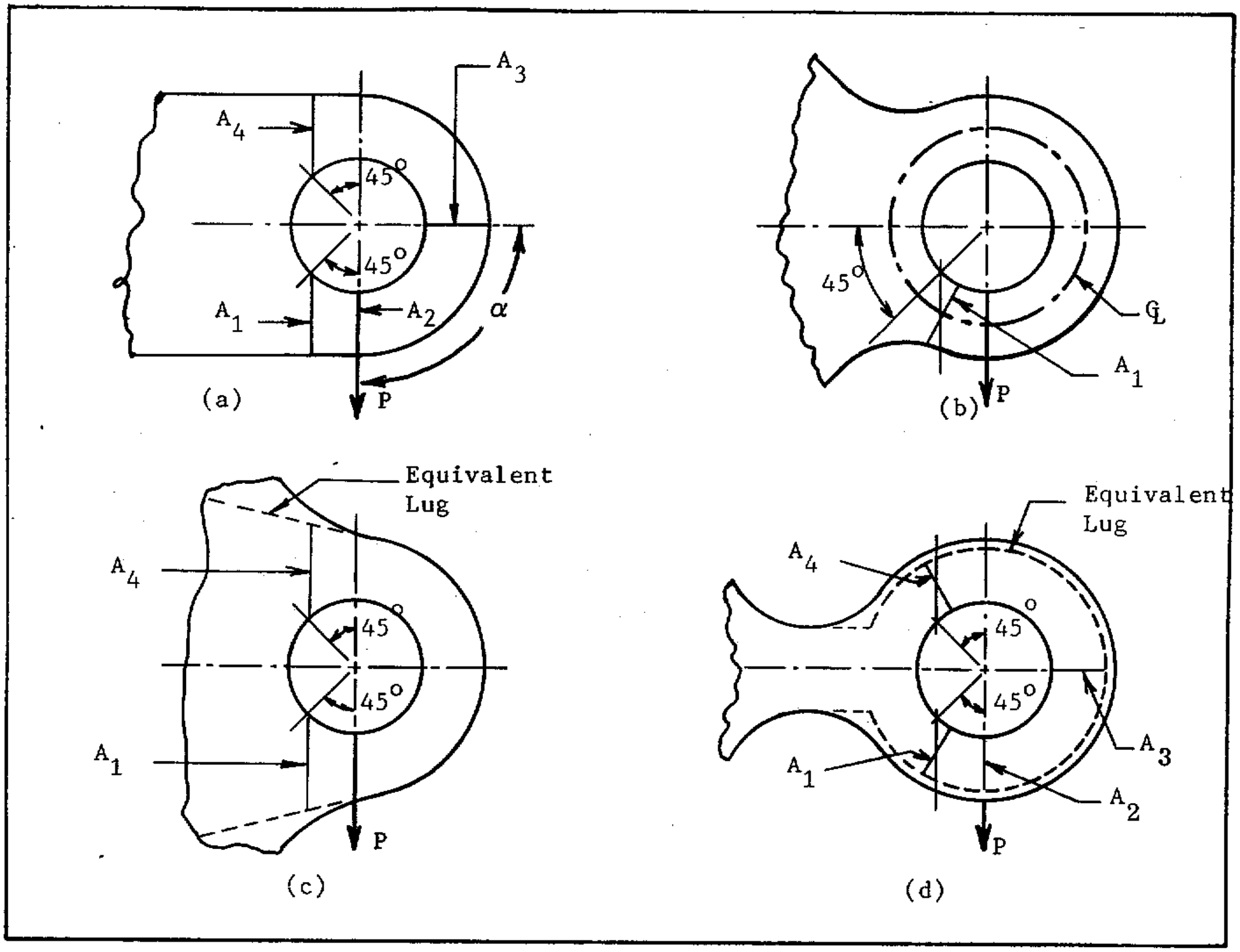

FIGURE 6.29 - TRANSVERSELY LOADED LUGS

(2) A_3 is the least area on any radial section around the hole.

(3) A_1, A_2, A_3 and A_4 should adequately reflect the strength of the lug. For lugs of unusual shape, such as severe necking or other sudden changes in cross section, an equivalent lug should be used such as shown in Figure 6.29(c) and (d).

B. P'_{tru} = Allowable ultimate load for lug failure

1. Enter Figure 6.33 with A_{av}/A_{br} and obtain K_{tru}.
2. $P'_{tru} = K_{tru} A_{br} F_{tux}$

C. P'_y = Allowable yield load of lug

1. Enter Figure 6.33 with A_{av}/A_{br} and obtain K_{try}.
2. $P'_y = K_{try} A_{br} F_{tyx}$

D. Check bushing yield per 6.4.4.2(E).

E. Margins of Safety

1. Minimum M.S. = .15 for ultimate transverse load
2. Minimum M.S. = 0 for yield of the lug and bushing

6.4.4.4 Analysis of Lugs with Oblique Loads ($0<\alpha<90^{\circ}$)

In analyzing lugs with oblique loading it is necessary to resolve the loading into axial and transverse components (denoted by the subscripts "a" and "tr" respectively), analyze the two cases separately and then combine the results using the interaction equation. The interaction equation:

$$R_a^{1.6} + R_{tr}^{1.6} = 1$$

where, for ultimate load,

$$R_a = \frac{\text{Axial component of applied ultimate load}}{\text{Smaller of } P'_{bru} \text{ or } P'_{tu} \text{ (6.4.4.2 B or C)}}$$

$$R_{tr} = \frac{\text{Transverse component of applied ultimate load}}{P'_{tru} \text{ (6.4.4.3.B)}}$$

and for yield load

$$R_a = \frac{\text{Axial component of applied yield load}}{P'_y \text{ (6.4.4.2D)}}$$

$$R_{tr} = \frac{\text{Transverse component of applied yield load}}{P'_y \text{ (6.4.4.3C)}}$$

The margin of safety should be 0.15 minimum and is calculated using the following equation:

$$MS = \frac{1}{\left(R_a^{1.6} + R_{tr}^{1.6}\right)^{0.625}} - 1$$

6.4.4.5 Analysis of Pins

The ultimate strength for a pin in a single lug/clevis joint as shown in Figure 6.34 will be analyzed first.

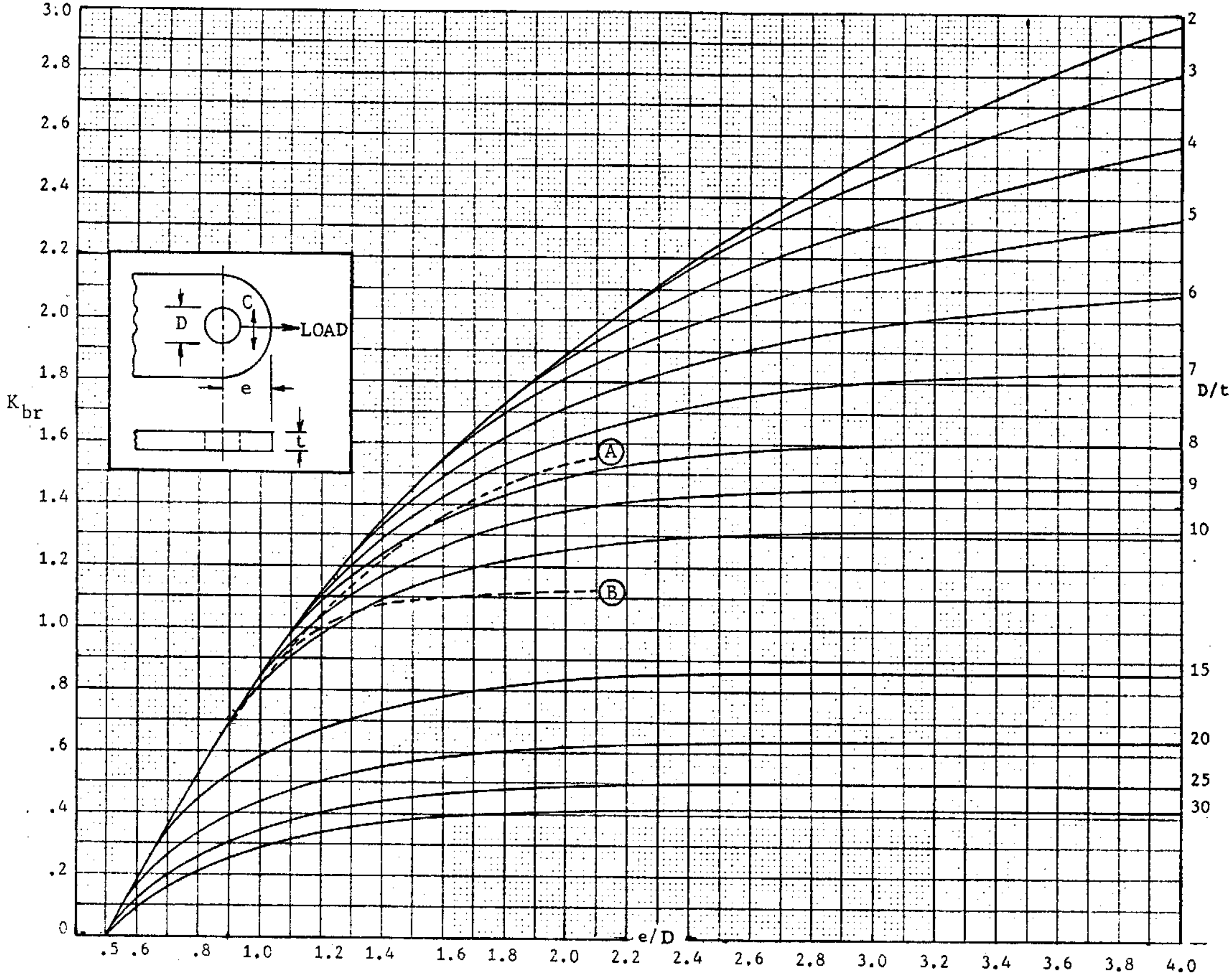

FIGURE 6.30 - SHEAR - BEARING EFFICIENCY FACTORS OF AXIALLY LOADED LUGS

Curve (A) is a cutoff to be used for all aluminum alloy handforged billet when the long transverse grain direction has the general direction "C" in the sketch.

Curve (B) is a cutoff to be used for all aluminum alloy plate, bar, and handforged billet when the short transverse grain direction has the general direction "C" in the sketch, and for die forgings when the lug contains the parting plane in a direction approximately normal to the direction "C".

NOTE: In addition to the limitations provided by curves (A) and (B), in no event shall a K_{br} greater than 2.00 be used for lugs made from .5" thick or thicker aluminum alloy plate, bar, or handforged billet.

FIGURE 6.30 (CONT'D) - SHEAR - BEARING EFFICIENCY FACTORS OF AXIALLY LOADED LUGS

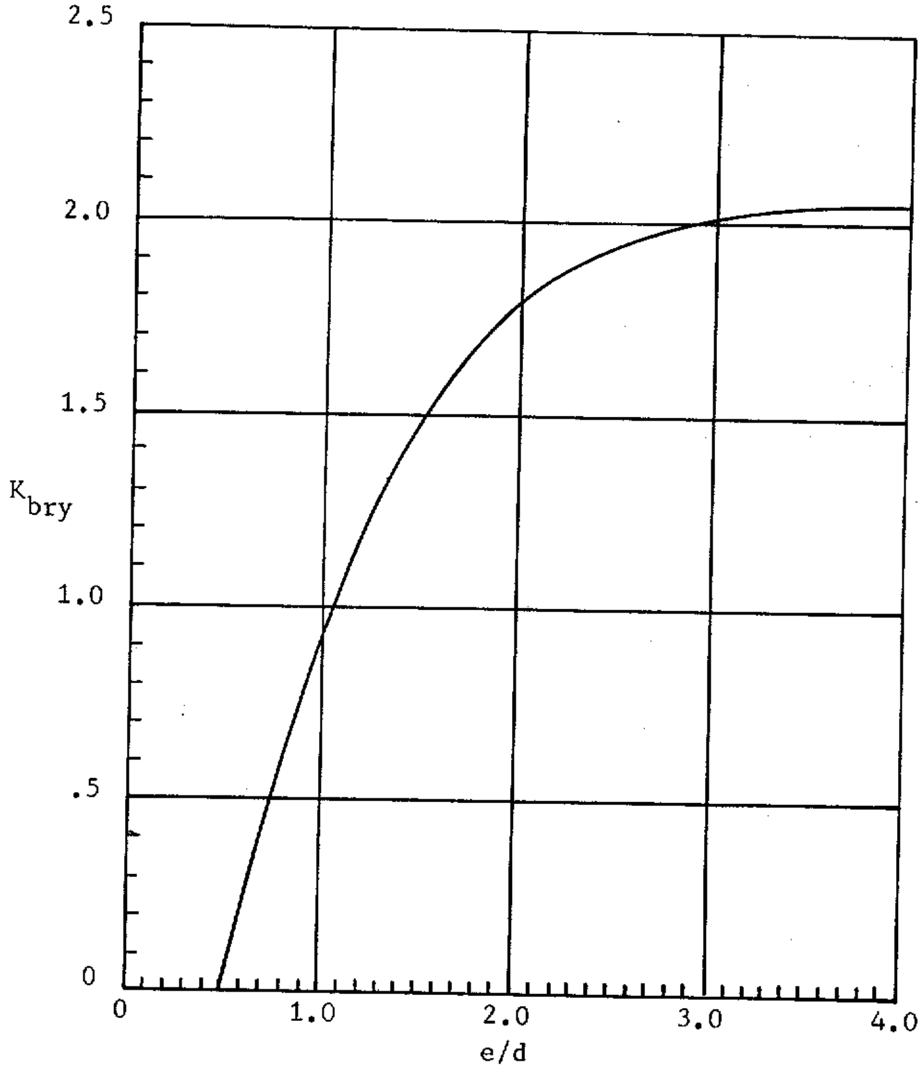

FIGURE 6.31 - BEARING YIELD EFFICIENCY FACTORS FOR AXIALLY LOADED LUGS

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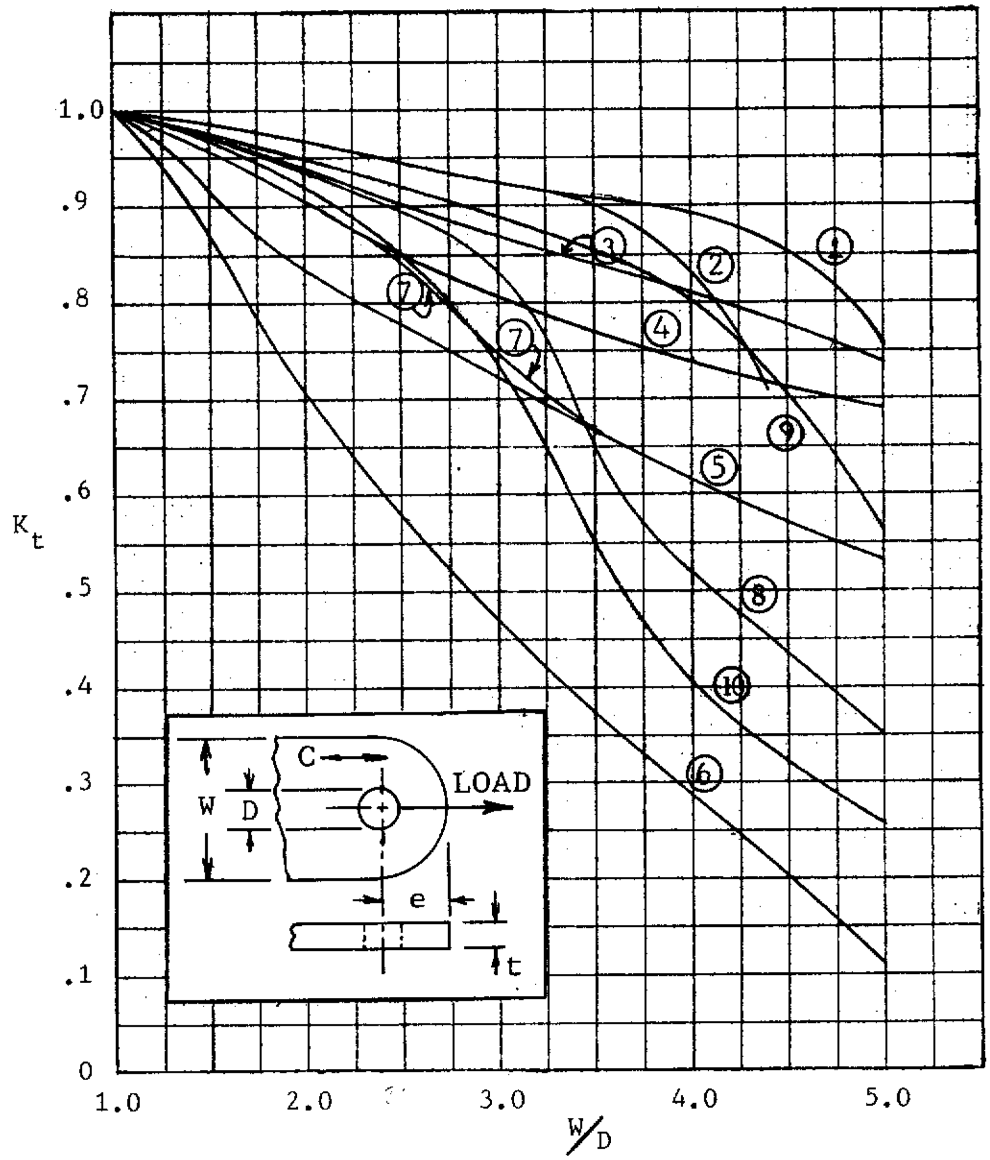

FIGURE 6.32a - TENSION EFFICIENCY FACTORS FOR AXIALLY LOADED ALUMINUM AND STEEL LUGS

L, T and ST indicate grain in the "C" direction.

Material	Curve
4130, 4140, 4340 and 8630 steel	1
2014-T6 & 7075-T6 plate ≤ .5" (L, T)	1
7075-T6 bar and extrusion (L)	1
2014-T6 handforged billet ≤ 144 sq in (L)	1
2014-T6 & 7075-T6 die forgings (L)	1
2014-T6 & 7075-T6 plate > .5" ≤ 1" (L, T)	2
7075-T6 extrusion (T, ST)	2
7075-T6 handforged billet ≤ 36 sq in (L)	2
2014-T6 handforged billet > 144 sq in (L)	2
2014-T6 handforged billet ≤ 36 sq in (T)	2
2014-T6 & 7075-T6 die forgings (ST)	2
17-4PH & 17-7 PH-THD	2
2024-T6 plate (L, T)	3
2024-T4 & 2024-T42 extrusion (L, T, ST)	3
2024-T4 plate (L, T)	4
2024-T3 plate (L, T)	4
2014-T6 & 7075-T6 plate > 1" (L, T)	4
2024-T4 bar (L, T)	4
7075-T6 handforged billet > 36 sq in (L)	4
7075-T6 handforged billet ≤ 16 sq in (T)	4
195-T6, 220-T4 & 356-T6 aluminum castings	5
7075-T6 handforged billet > 16 sq in (T)	5
2014-T6 handforged billet > 36 sq in (T)	5
Aluminum alloy plate, bar, handforged billet & die forging (ST). NOTE: ST direction exists only at parting plane	6
7075-T6 bar (T)	6
18-8 stainless steel, annealed	7
18-8 stainless steel, full hard. NOTE: for ¼, ½ & 3/4 hard interpolate between Curves 7 and 8	8
7075-T73 Die Forging (L) ≤3"	9
7075-T73 Die Forging (ST) ≤3"	10

FIGURE 6.32a (CONT'D) - TENSION EFFICIENCY FACTORS FOR AXIALLY LOADED ALUMINUM AND STEEL LUGS

Revision B

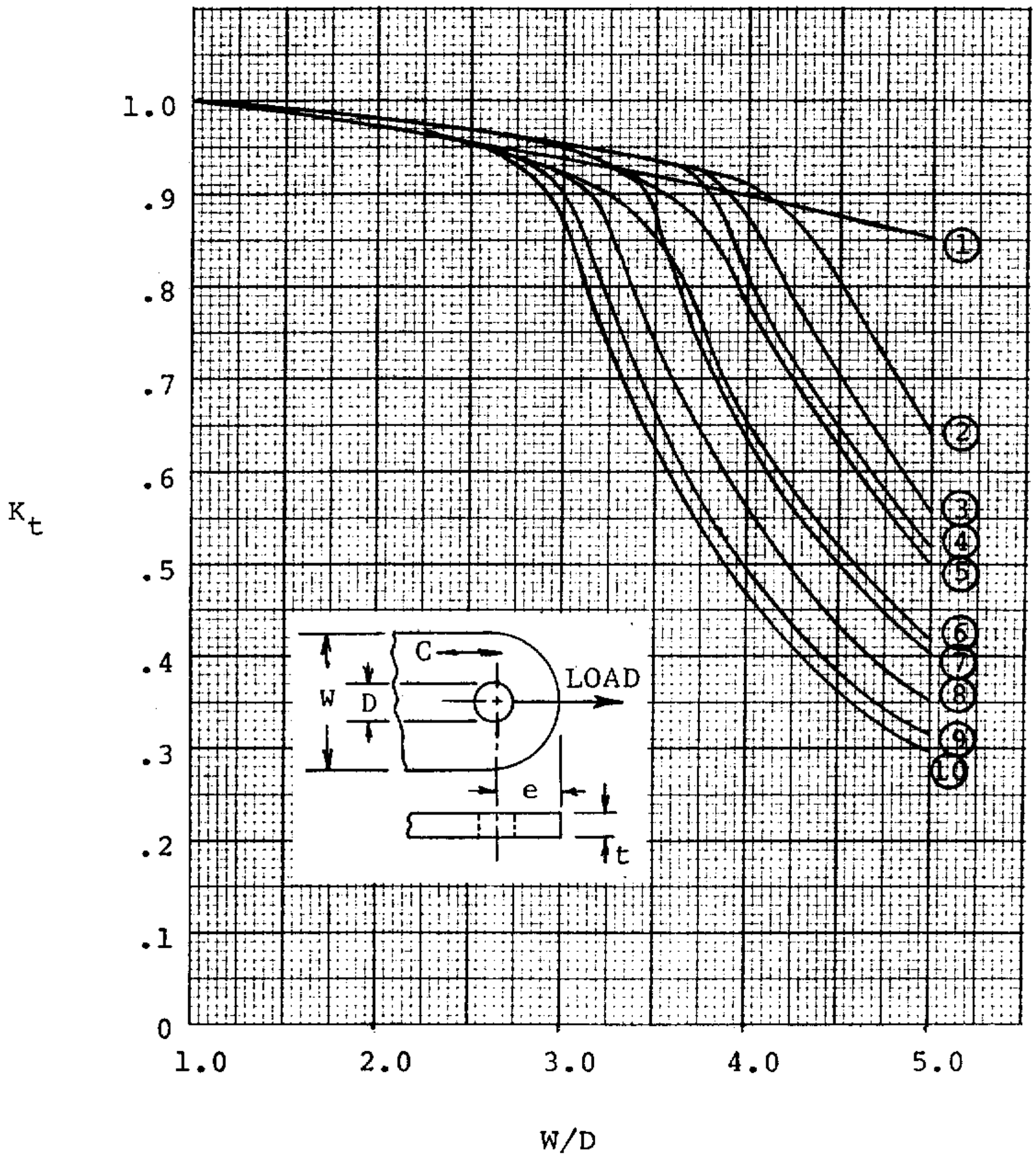

FIGURE 6.32b - TENSION EFFICIENCY FACTORS FOR AXIALLY LOADED TITANIUM LUGS

Material	Curve
C.P. Ti Type I Comp. B t ≤ 1.0	1
Ti-6Al-4V Mill Ann. Plate t ≤ 4.0	2
Ti-6Al-4V Mill Ann. B&F t ≤ 3.0	2
Ti-6Al-4V Ann. Ext.	2
Ti-6Al-4V STA B&F .5 < t ≤ 2.0	3
Ti-6Al-4V STA B&F t ≤ .5	4
Ti-6Al-4V STA B&F 2.0 < t < 3.0	5
Ti-6Al-4V STA Plate t ≤ .75	6
Ti-6Al-4V STA Plate .75 < t ≤ 2.0	8
Ti-6Al-6V-2Sn Cond. A-1 Die Forging t ≤ 2.0	4
Ti-6Al-6V-2Sn Ann. Ext. t ≤ 2.0	4
Ti-6Al-6V-2Sn Mill Ann. Plate t ≤ 2.0	7
Ti-6Al-6V-2Sn STA Plate t ≤ 1.5	7
Ti-6Al-6V-2Sn Cond. A-1 Die Frg. 2.0 < t ≤ 4.0	7
Ti-6Al-6V-2Sn STA Die Frg. t ≤ 1.0	8
Ti-6Al-6V-2Sn STA Plate 1.5 < t ≤ 2.5	9
Ti-6Al-6V-2Sn STA Die Frg. 1.0 < t ≤ 3.0	9
Ti-6Al-6V-2Sn Cond. A-1 Die Frg. 4.0 < t ≤ 8.0	9
Ti-6Al-6V-2Sn STA Die Frg. 3.0 < t ≤ 4.0	10

FIGURE 6.32b (CONT'D) - TENSION EFFICIENCY FACTORS FOR AXIALLY LOADED TITANIUM LUGS

Revision B

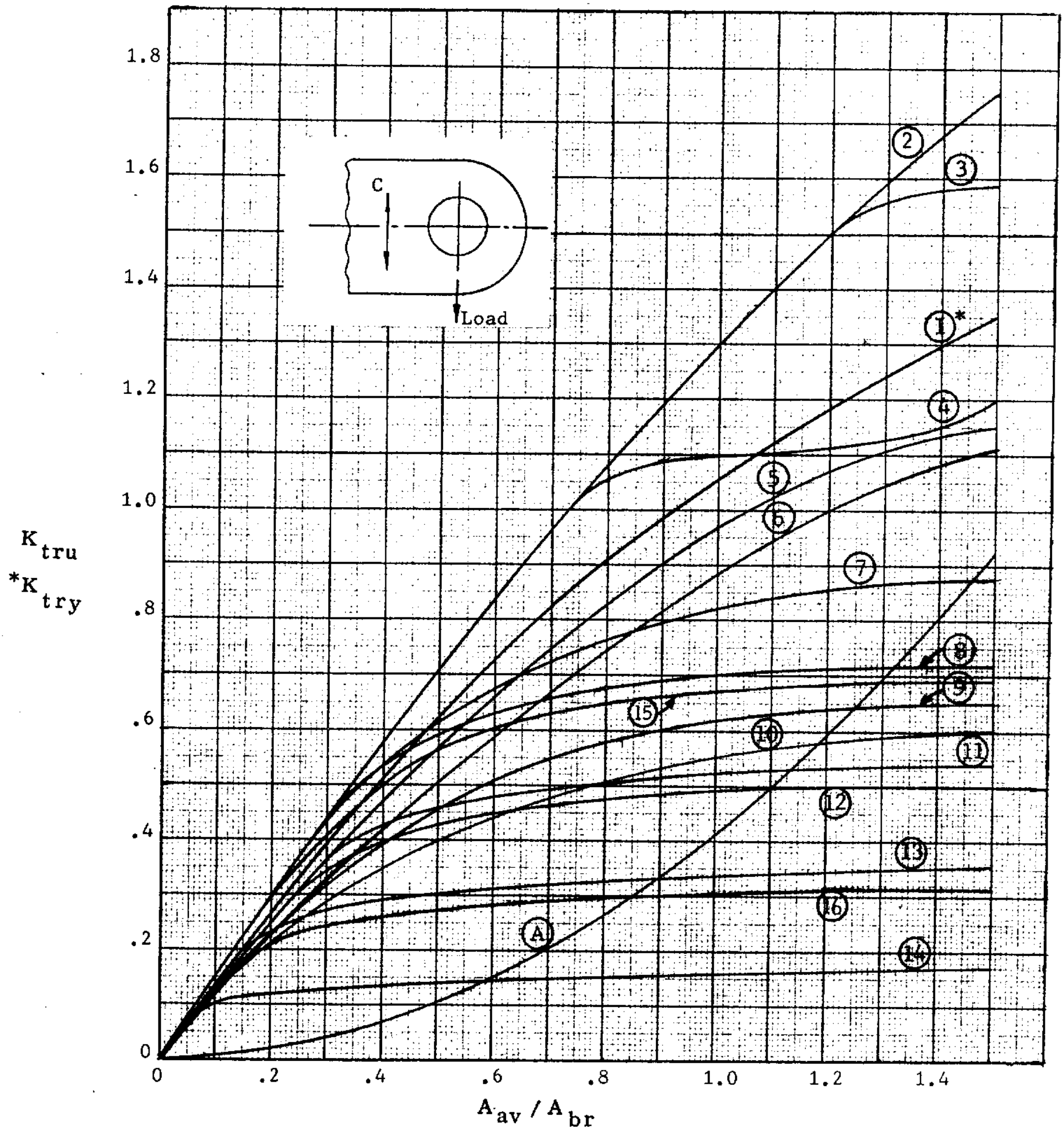

FIGURE 6.33a - EFFICIENCY FACTORS FOR TRANSVERSELY LOADED ALUMINUM AND STEEL LUGS

Revision B

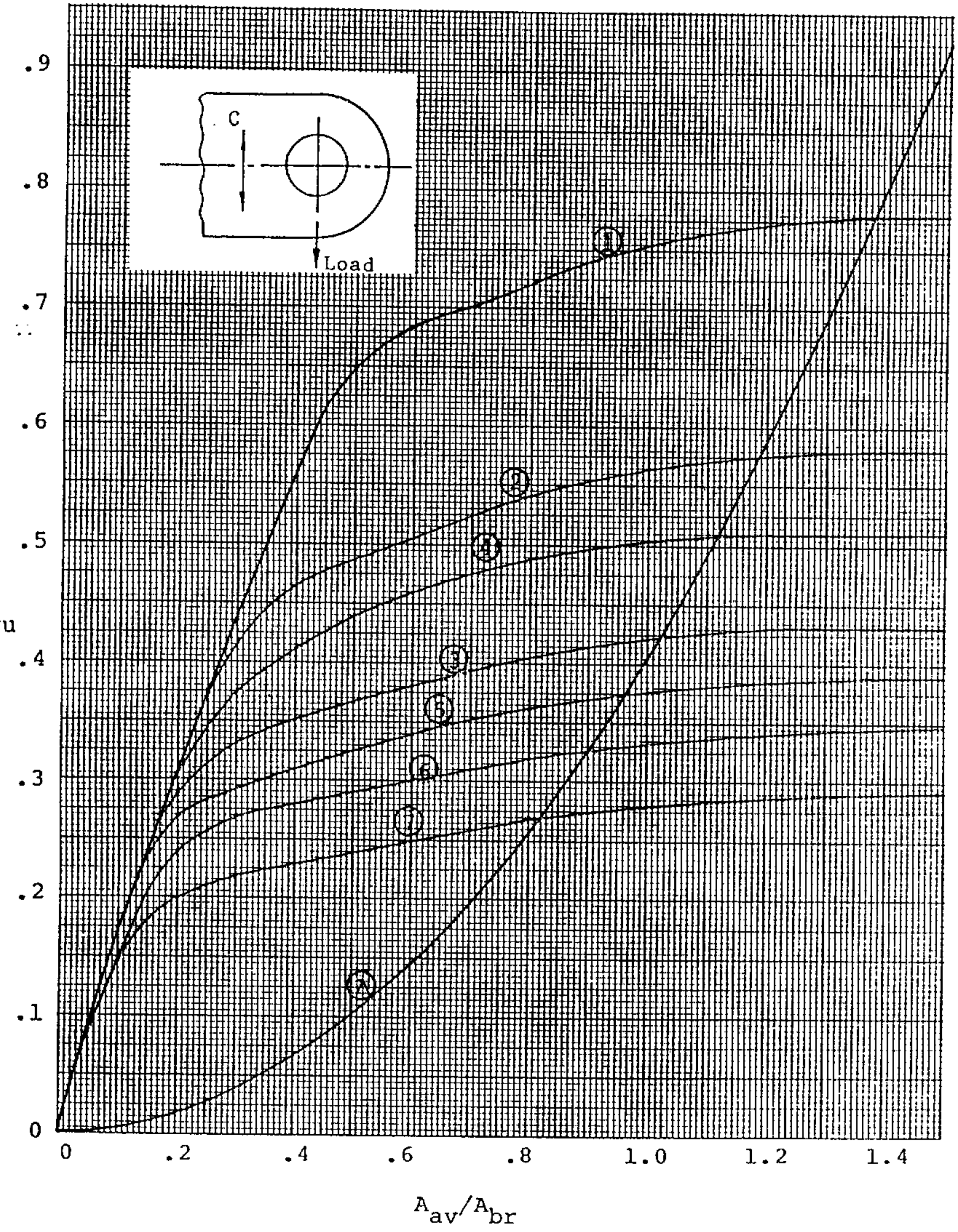

FIGURE 6.33b - EFFICIENCY FACTORS FOR TRANSVERSELY LOADED TITANIUM LUGS

L, T, and ST indicate grain in the "C" direction

Material	Curve
17-7PH-THD	7
2014-T6 plate ≤ .5"	8
2014-T6 plate > .5" ≤ 1"	11
2014-T6 plate > 1"	13
2014-T6 die forgings	11
2014-T6 handforged billet ≤ 36 sq in	11
2014-T6 handforged billet > 36 sq in	14
2024-T3 plate ≤ .5"	5
2024-T3 plate > .5"	9
2024-T4 plate ≤ .5"	5
2024-T4 plate > .5"	9
2024-T4 bar	9
2024-T4 & 2024-T42 extrusion	12
2024-T6 plate	12
7075-T6 plate ≤ .5"	8
7075-T6 plate > .5 ≤ 1"	11
7075-T6 plate > 1"	13
7075-T6 extrusion	11
7075-T6 die forgings	11
7075-T6 handforged billet ≤ 16 sq in	13
7075-T6 handforged billet > 16 sq in	14
195-T6 & 356-T6 aluminum casting	10
220-T4 aluminum casting	6
4130, 4140, 4340 & 8630, Ftu = 125 ksi	2
4130, 4140, 4340 & 8630, Ftu = 150 ksi	3
4130, 4140, 4340 & 8630, Ftu = 180 ksi	4
7075-T73 Die Forging (L) ≤3"	15
7075-T73 Die Forging (ST) ≤3"	16
Ktry, all materials	1

All curves are for Ktru except the one noted as Ktry.

In no case should the ultimate transverse load be taken as less than that which could be carried by cantilever beam action of the portion of the lug under the load. The load that can be carried by cantilever beam action is indicated approximately by Curve A. Should Ktru be below Curve A, separate calculation as a cantilever beam is necessary.

FIGURE 6.33a (CONT'D) - EFFICIENCY FACTORS FOR TRANSVERSELY LOADED ALUMINUM AND STEEL LUGS

L, T and ST indicate grain in the "C" direction

Material	Curve
Ti-6Al-4V Ann. Cond. A Die Forging (T) t ≤ 5.0	1
Ti-6Al-4V Ann. Cond. A Hand Forging (T) A ≤ 16	1
Ti-6Al-4V Ann. Cond. A Hand Forging (T) A > 16	2
Ti-6Al-4V STA Die Forging (L) t ≤ 5.0	2
Ti-6Al-4V STA Die Forging (T) t ≤ 1.0	2
Ti-6Al-4V STA Die Forging (T) 1.0 < t ≤ 3.0	3
Ti-6Al-4V STA Hand Forging (L,T) t ≤ 2.0	1
Ti-6Al-4V STA Hand Forging (T) 2.0 < t ≤ 3.0	2
Ti-6Al-6V-2Sn Ann. Plate (T) t ≤ 2.0	4
Ti-6Al-6V-2Sn Ann. Die Frg. (ST) t ≤ 2.0	4
Ti-6Al-6V-2Sn Ann. Hand Frg. (T) t ≤ 2.0	4
Ti-6Al-6V-2Sn Ann. Plate (T) 2.0 < t ≤ 4.0	5
Ti-6Al-6V-2Sn Ann. Die Frg. (ST) 2.0 < t ≤ 4.0	5
Ti-6Al-6V-2Sn Ann. Hand Frg. (T) 2.0 < t ≤ 4.0	5
Ti-6Al-6V-2Sn STA Die Forg. (L) All	6
Ti-6Al-6V-2Sn STA Die Forging (T) All	7
Ti-6Al-6V-2Sn STA Hand Forging (L,T) t ≤ 4.0	6
Ti-6Al-6V-2Sn STA Hand Forging (T) t > 4.0	7

In no case should the ultimate transverse load be taken as less than that which could be carried by cantilever beam action of the portion of the lug under the load. The load that can be carried by cantilever beam action is indicated approximately by Curve A. Should Ktru be below Curve A, separate calculation as a cantilever beam is necessary.

FIGURE 6.33b (CONT'D) - EFFICIENCY FACTORS FOR TRANSVERSELY LOADED TITANIUM LUGS

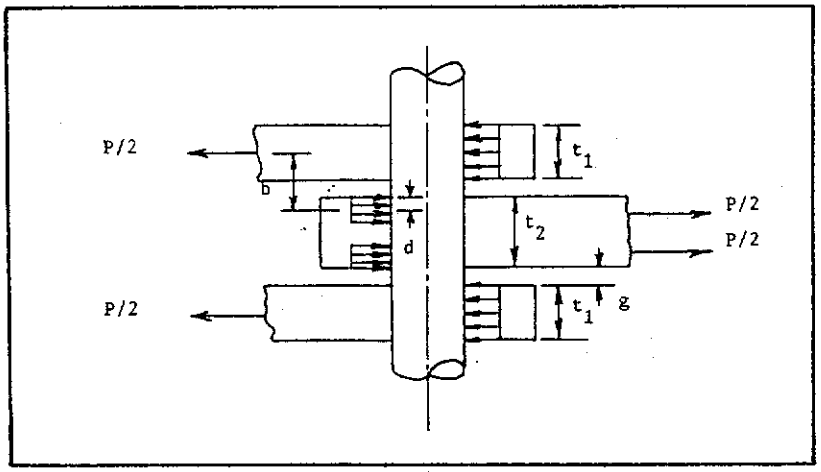

FIGURE 6.34 - SINGLE LUG/CLEVIS JOINT

A. Obtain moment arm "b". For the inner lug of Figure 6.34 calculate $r = \{(e/D)-\frac{1}{2}\}D/t_2$. Determine e and (P') min as noted below, and compute $(P'_u)\ min/A_{br}F_{tux}$. Enter Figure 6.36 and obtain the reduction factor "γ" which compensates for the "peaking" of the distributed pin bearing load near the shear plane. Calculate

$$b = (t_1/2) + g + \gamma\ (t_2/4)$$

where "g" is the gap between lugs as shown in Figure 6.34 and may be zero. Note that the peaking reduction factor applies only to the inner lugs.

Determination of e and (P'_u) min

1. For lugs loaded axially.

   Take the smaller of P'_{bru} and P'_{tu} for the inner lug as (P'_u)min and e as the edge distance at α = 90 degrees.

2. For lugs loaded transversely

   Take (P'_u) min = P'_{tru} and e as the edge distance at α = 90 degrees.

3. For lugs loaded obliquely

   Take $(P'_u)\ min = \dfrac{P}{(R_\alpha^{1.6} + R_{tr}^{1.6})^{0.625}}$

   and e as the edge distance at the value of α corresponding to the direction of load on the lug.

B. Calculate maximum pin bending moment, "M", from the equation

$$M = P(b/2)$$

C. Calculate bending stress assuming a MC/I distribution.

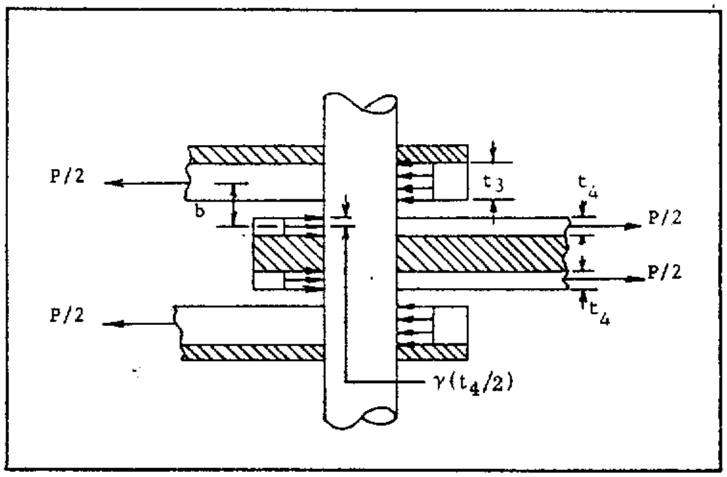

FIGURE 6.35 – ACTIVE LUG THICKNESS

D. Obtain the ultimate strength of the pin in bending by use of Section 9.4. If the analysis should show inadequate pin bending strength it may be possible to take advantage of any excess lug strength as follows.

E. Consider a portion of the lugs to be inactive as indicated by the shaded area of Figure 6.35. The portion of the thickness to be considered active may have any desired value sufficient to carry the load and should be chosen by trial and error to give approximately equal margins of safety for the lugs and pin.

F. Recalculate all lug margins of safety with allowable loads reduced in the ratio of active thickness to actual thickness.

G. Recalculate pin bending moment, $M = P(b/2)$ and margin of safety using value of "b" which is obtained as follows:

$$r = \{(e/D) - \tfrac{1}{2}\}D/2t_4.$$

Take the smaller of P'_{bru} and P'_{tu} for the inner lug, based based upon the active thickness, as (P'_u) min and compute $(P'_u)\text{min}/A_{br}\ F_{tux}$ where $A_{br}=2t_4D$. Enter Figure 6.36 and obtain "γ". Then

$$b = t_3/2 + g + \gamma(t_4/2).$$

This reduced value of "b" should not be used if the resulting eccentricity of load on the outer lugs introduces excessive bending stresses in the adjacent structure. In such cases pins must be strong enough to distribute the load uniformly across the entire lug.

Lug-pin combinations having multiple shear connections such as those shown in Figure 6.37 are analyzed as follows.

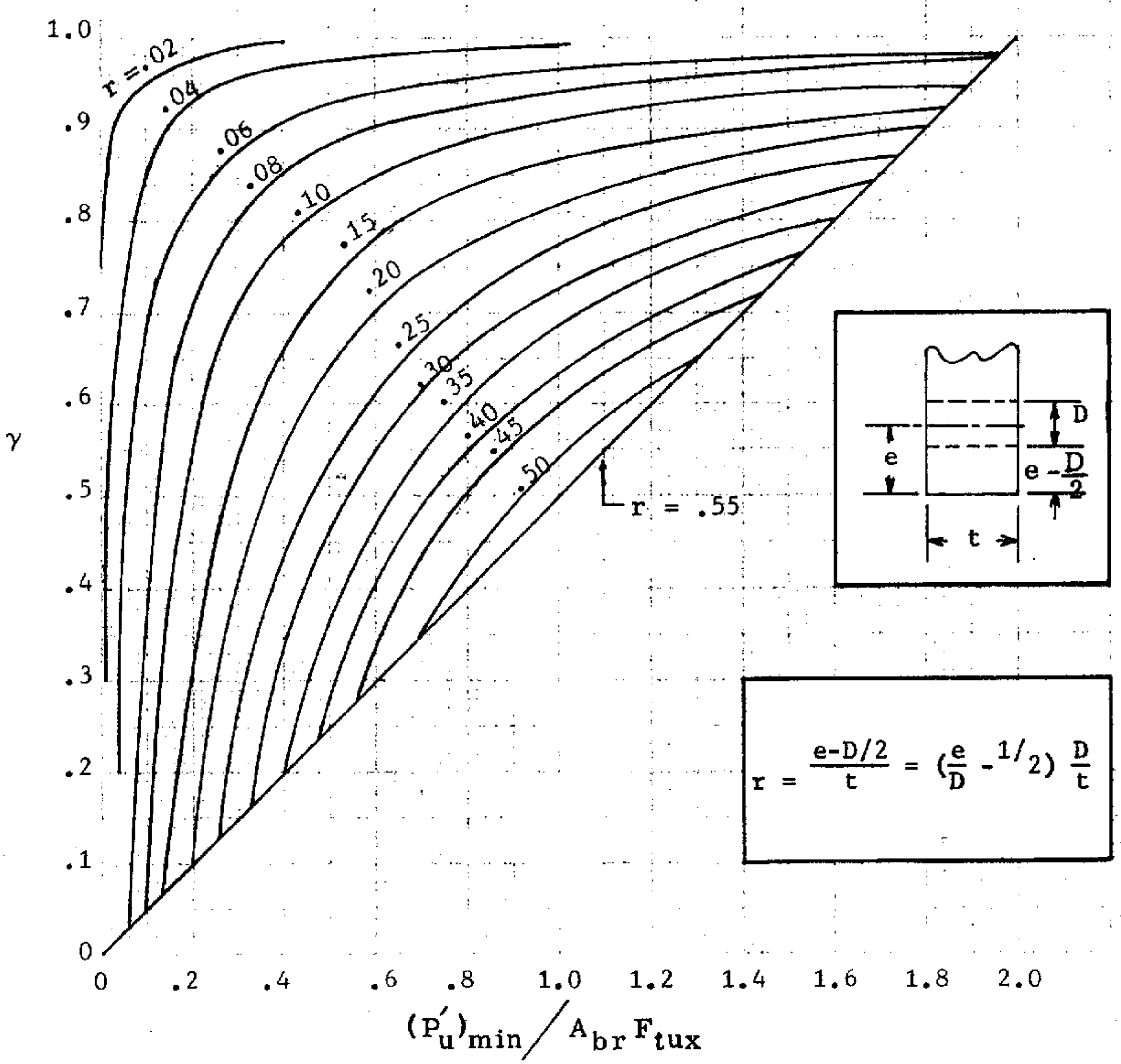

FIGURE 6.36 - REDUCTION FACTOR FOR PEAKING OF BEARING LOADS ON PINS

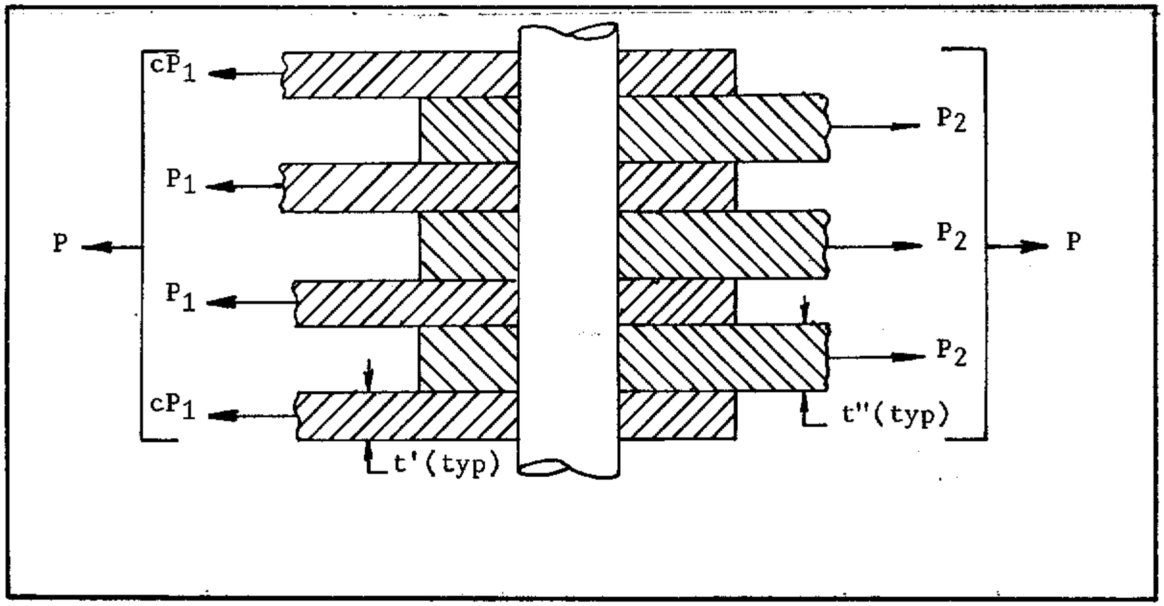

FIGURE 6.37 - MULTIPLE SHEAR JOINT

A. The load carried by each lug is determined by distributing the total applied load "P" among the lugs as shown in Figure 6.37 and the value of "C" is obtained from Table 6.10.

B. The maximum shear load on the pin is given in Table 6.10.

C. The maximum bending moment in the pin is given by: $M = P_1\ b/2$ where "b" is given in Table 6.10.

6.4.4.6 Lugs with Eccentrically Located Hole

If the hole is located as in Figure 6.38 (e_1 less than e_2), the ultimate load and yield lug loads are determined by obtaining P'_{bru}, P'_{tu} and P'_y for the equivalent lug shown and multiplying by the factor $(e_1 + e_2 + 2D)/(2e_2 + 2D)$.

6.4.4.7 Lubrication Holes In Lugs

When lubrication holes are present, the lug may be analyzed as follows.

A. Axially loaded lugs. Modify the calculation of P'_{tu} or P'_{bru} or both, depending upon the location of the hole. (Fig. 6.39)

If P'_{tu} requires modification, obtain the net tension area using a thickness given by t-minus lube hole diameter.

If P'_{bru} requires modification, obtain A_{br} using a thickness given by t minus the lube hole diameter. Obtain K_t from Figure 6.31 for W/D = 1.75 using the weakest grain direction occurringin the plane of the lug. Then

$$P'_{bru} = K_{bru}\ K_t\ A_{br}\ F_{tux}$$

Total number of lugs including both sides	C	Pin Shear	b
5	.35	$.50P_1$	$.28\left(\frac{t' + t''}{2}\right)$
7	.40	$.53P_1$	$.33\left(\frac{t' + t''}{2}\right)$
9	.43	$.54P_1$	$.37\left(\frac{t' + t''}{2}\right)$
11	.44	$.54P_1$	$.39\left(\frac{t' + t''}{2}\right)$
∞	.50	$.50P_1$	$.50\left(\frac{t' + t''}{2}\right)$

TABLE 6.10 - PIN BENDING FACTORS

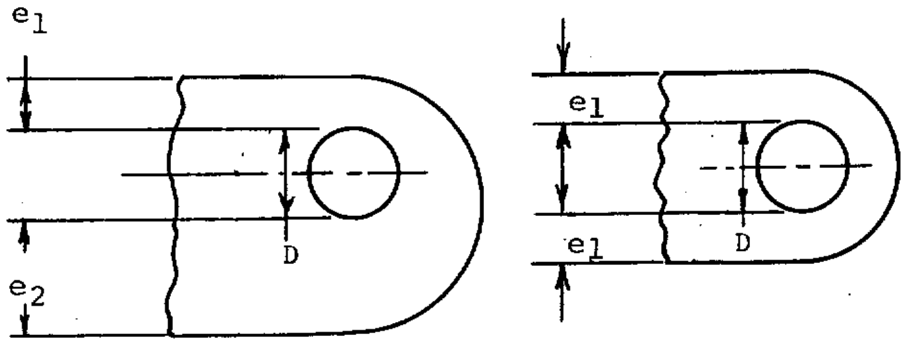

FIGURE 6.38 - LUGS WITH ECCENTRICALLY LOCATED HOLES

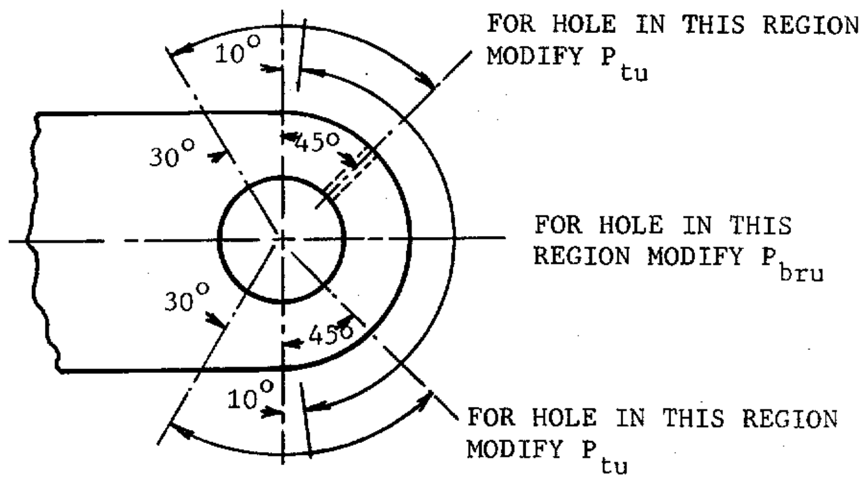

FIGURE 6.39 - LUBRICATION HOLES IN LUGS

B. Transversely loaded lugs. Obtain P'_{tru} neglecting lube hole and multiply by 0.9 $(1 - \frac{\text{lube hole diameter}}{t})$.

C. Obliquely loaded lugs. Obtain P'_{tu}, P'_{bru}, and P'_{tru} according to A and B above. Then proceed according to Section 6.4.4.4.

6.4.5 Stresses Due to Press Fit Bushings

Pressure between a lug and bushing assembly having negative clearance can be determined from consideration of the radial displacements. After assembly, the increase in inner radius of the ring (lug) plus the decrease in outer radius of the bushing equals the difference between the radii of the bushing and ring before assembly.

$$\delta = u_{ring} - u_{bushing}$$

where

δ = difference between outer radius of bushing and inner radius of the ring

u = radial displacement, positive away from the axis of ring or bushing.

Radial displacement at the inner surface of a ring subjected to internal pressure p is

$$u = \frac{D_p}{E_{ring}} \left[\frac{C^2 + D^2}{C^2 - D^2} + \mu_{ring} \right]$$

Radial displacement at the outer surface of a bushing subjected to external pressure p is

$$u = - \frac{B_p}{E_{bush}} \left[\frac{B^2 + A^2}{B^2 - A^2} - \mu_{bush} \right]$$

where:

A = inner radius of bushing
B = outer radius of bushing
C = outer radius of ring (lug)
D = inner radius of ring (lug)
E = modulus of elasticity
μ = Poisson's ratio

Substitution of the previous two equations into the first yields:

$$p = \frac{\delta}{\frac{D}{E_{ring}}\left(\frac{C^2 + D^2}{C^2 - D^2} + \mu_{ring}\right) + \frac{B}{E_{bush}}\left(\frac{B^2 + A^2}{B^2 - A^2} - \mu_{bush}\right)}$$

Maximum radial and tangential stresses for a ring (lug) subjected to internal pressure occur at the inner surface of the ring (lug) and are

$$f_r = -p, \; f_t = p\left(\frac{C^2 + D^2}{C^2 - D^2}\right).$$

Positive sign indicates tension. The maximum shear stress at this point is

$$f_s = \frac{f_t - f_r}{2}$$

The maximum radial stress for a bushing subjected to external pressure occurs at the outer surface of the bushing and is

$$f_r = -p$$

The maximum tangential stress for a bushing subjected to external pressure occurs at the inner surface of the bushing and is

$$f_t = -\frac{2\,p\,B^2}{B^2 - A^2}$$

The allowable press fit stress should be based on stress corrosion, static fatigue, fatigue life, and the ultimate strength. Any questions concerning the limits of the press fit stress should be directed to the Airframe Structures Group Engineer.

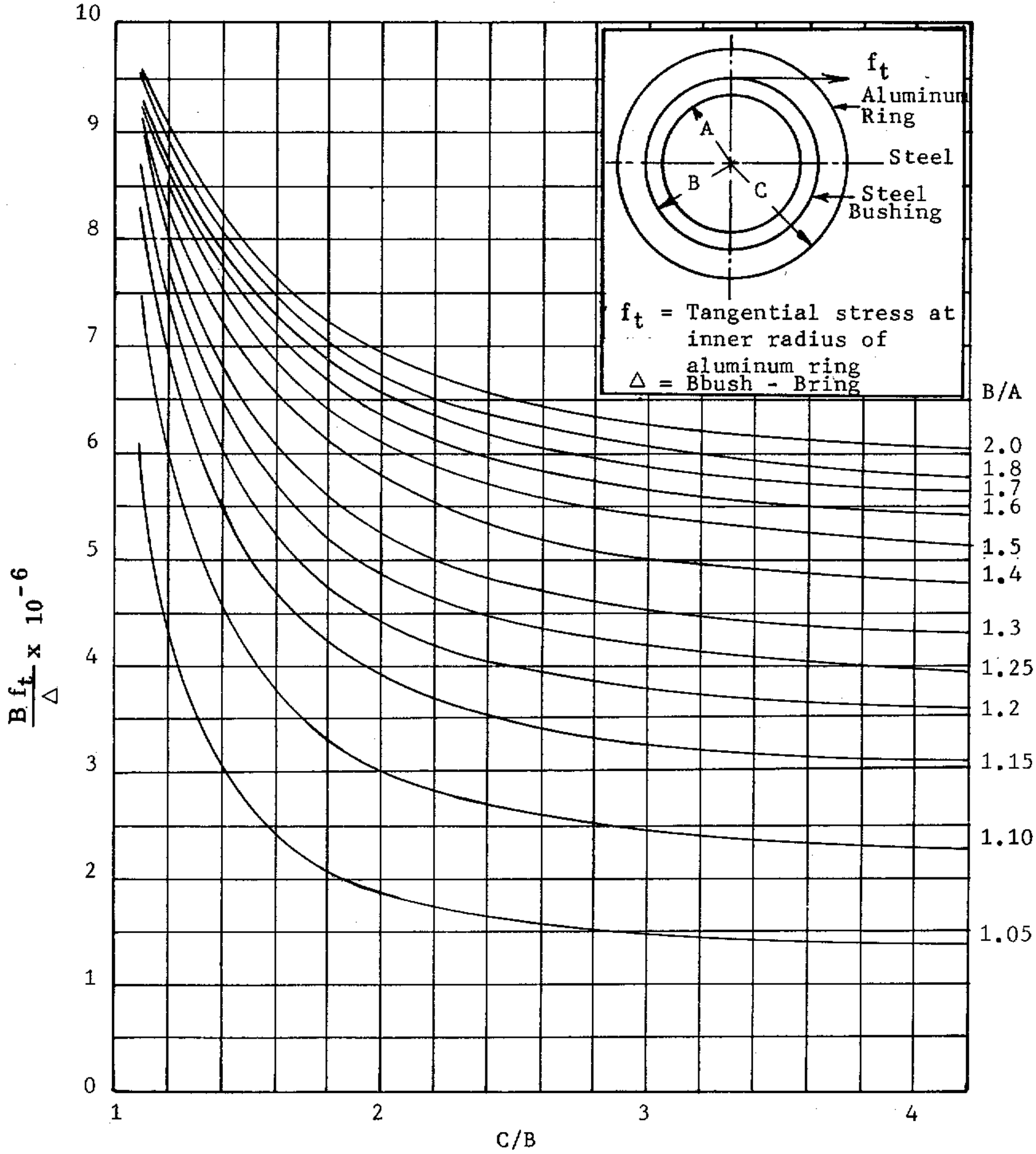

FIGURE 6.40 - TANGENTIAL STRESSES FOR PRESSED STEEL BUSHINGS IN ALUMINUM RINGS

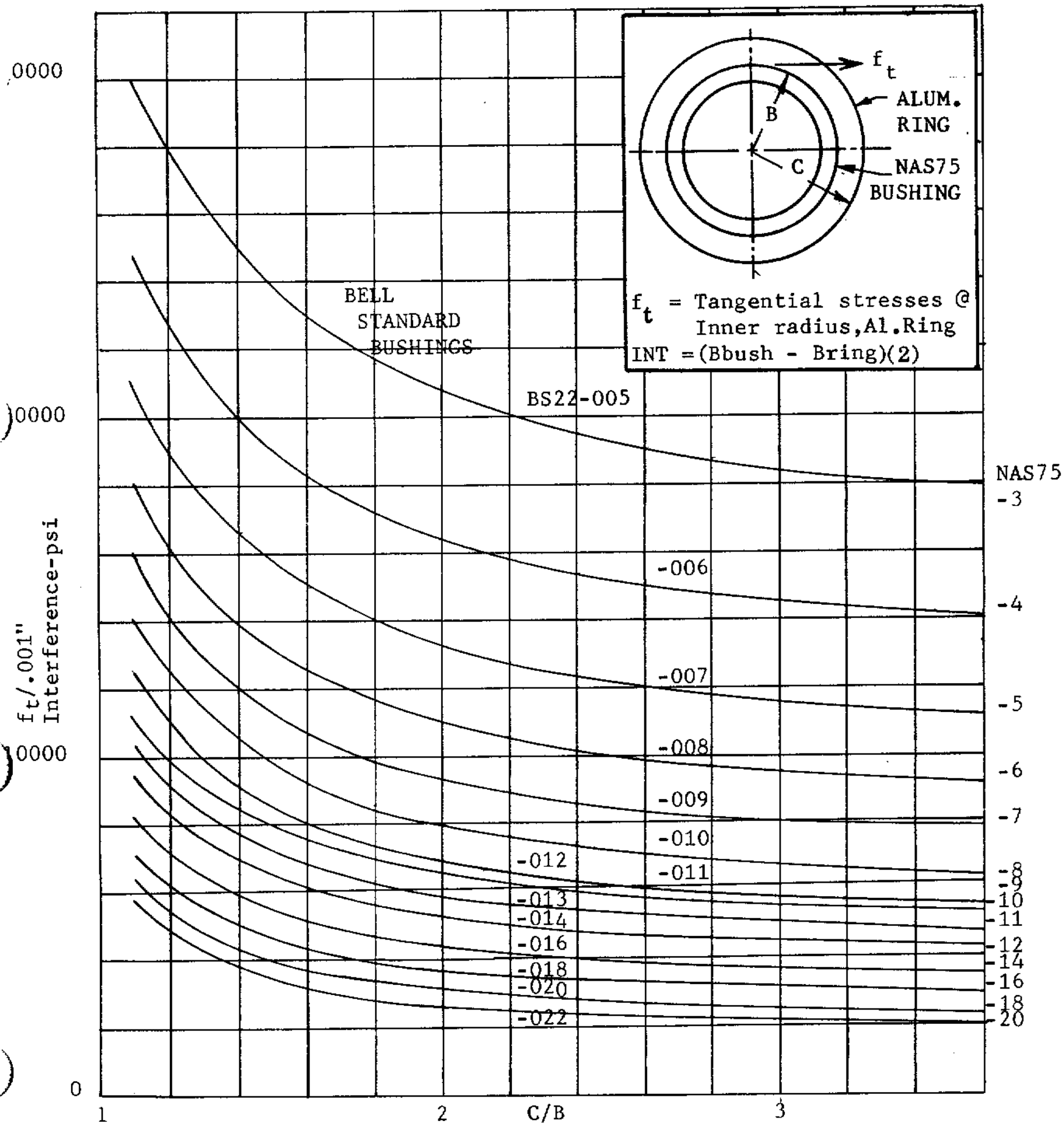

FIGURE 6.41 TANGENTIAL STRESSES FOR PRESSED NAS75 BUSHINGS

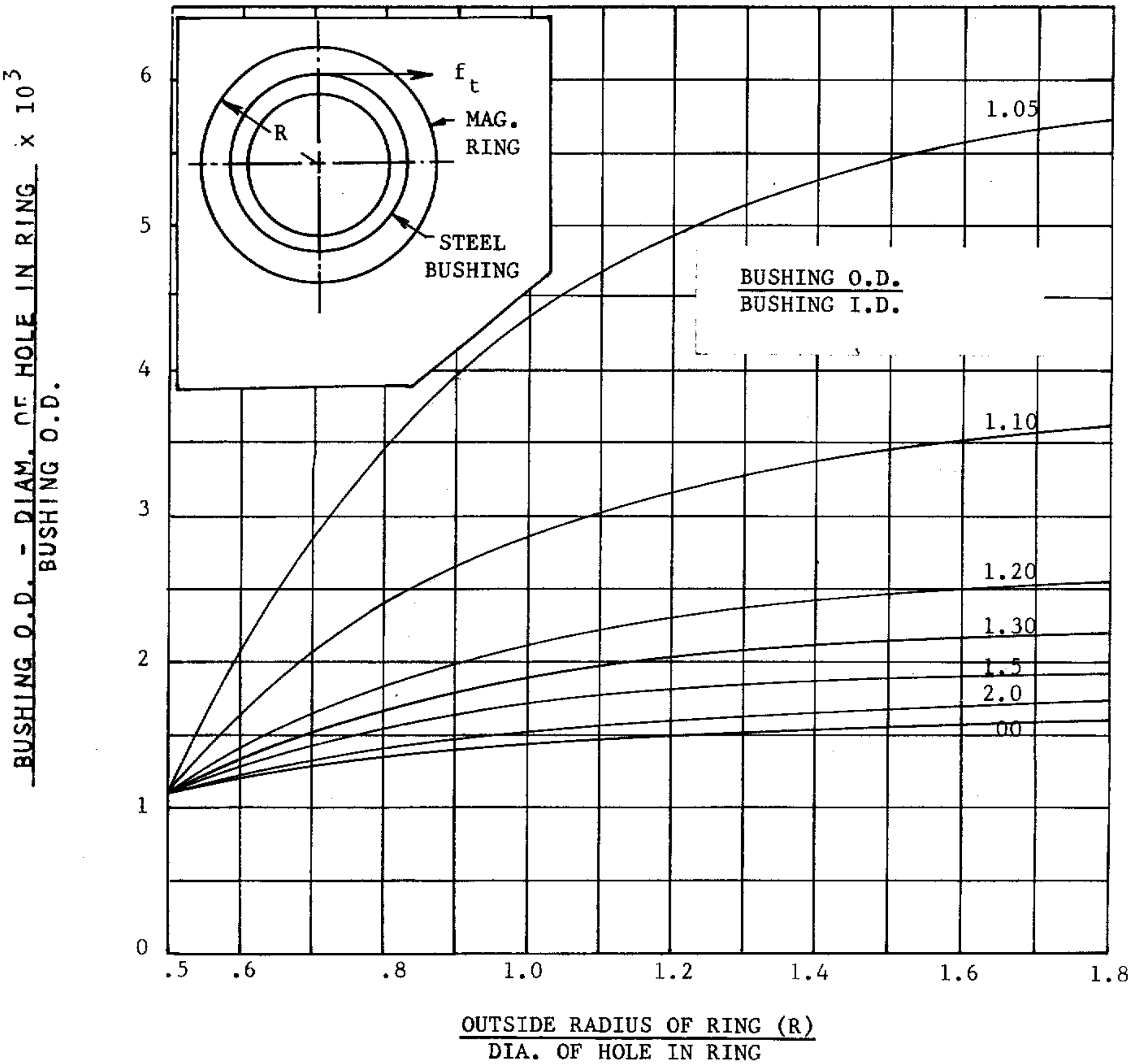

FIGURE 6.42 MAXIMUM INTERFERENCE FITS OF STEEL BUSHINGS IN MAGNESIUM ALLOY RINGS

The presence of hard brittle coatings in holes that contain a press fit bushing or bearing can cause premature failure by cracking of the coating or by high press fit stresses caused by build-up of coating. Therefore, hardcoat or HAE coatings should not be used in holes that will subsequently contain a press fit bushing or bearing.

Figures 6.40 and 6.41 permit determining the tangential stress, f_t, for bushings pressed into aluminum rings. Figure 6.40 presents data for general steel bushings and Figure 6.41 presents data for NAS75 class bushings. Figure 6.42 gives limits for maximum interference fits for steel bushings in magnesium alloy rings.

6.4.6 Stresses Due to Clamping of Lugs

Joints which are clamped should be checked for the residual stresses developed in the lugs. This "clamp-up" stress can be determined in the following manner. Figure 6.43 shows a typical lug/clevis joint subject to clamping.

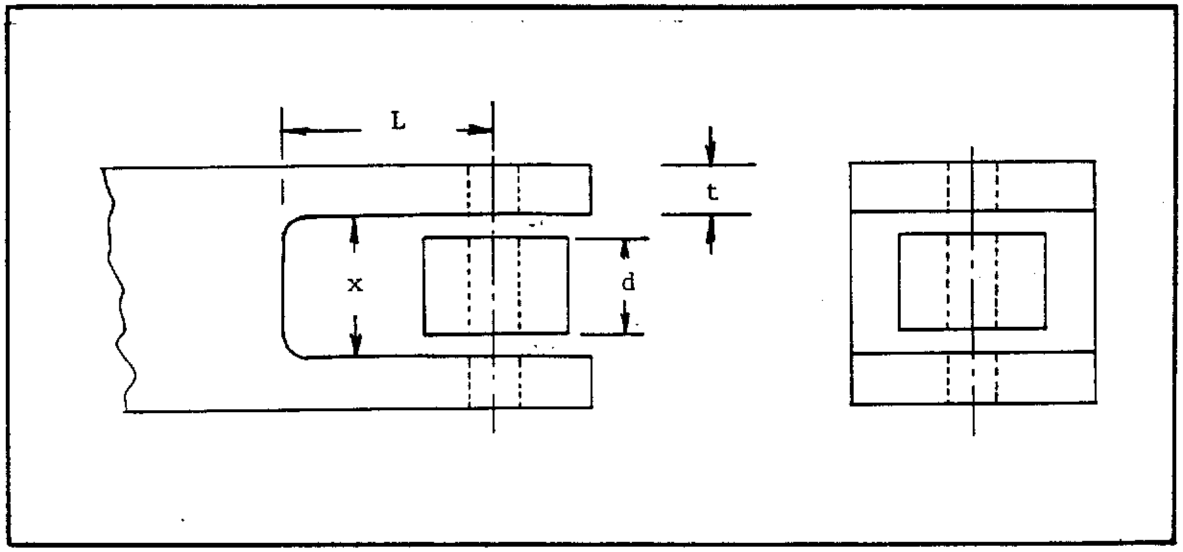

FIGURE 6.43 - TYPICAL LUG/CLEVIS JOINT   s/be $f = \frac{3E\delta t}{2L^2}$. *

The stress produced in the lugs by clamping is $f = 3E\delta t/L^2$, where $\delta = (x - d)/2$. This assumes that the total clearance x - d is equally divided between each lug. If some other distribution of clearance is required, the stress in each lug must be calculated.

The allowable stress, $F_{all} = F_{ty}/2$, should not be exceeded in order to minimize the possibility of stress corrosion failures.

6.4.7 Single Shear Joints

In single shear joints lug and pin bending are more critical than in double shear joints. The amount of bending can be significantly affected by bolt

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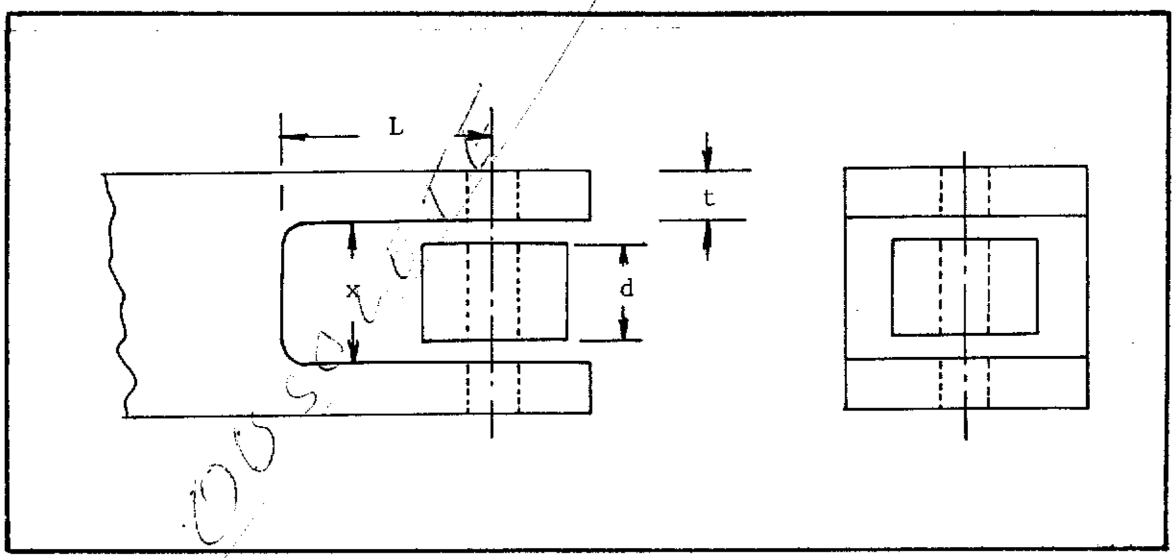

FIGURE 6.43 - TYPICAL LUG/CLEVIS JOINT

The stress produced in the lugs by clamping is $f = 3E\delta t/L^2$, where $\delta = (x - d)/2$. This assumes that the total clearance x - d is equally divided between each lug. If some other distribution of clearance is required, the stress in each lug must be calculated.

The allowable stress, $F_{all} = F_{ty}/2$, should not be exceeded in order to minimize the possibility of stress corrosion failures.

6.4.7 Single Shear Joints

In single shear joints lug and pin bending are more critical than in double shear joints. The amount of bending can be significantly affected by bolt clamping. In the case considered in this section, no bolt clamping is assumed and the bending moment in the pin is reacted by socket action in the lugs. Therefore, even though Figure 6.44 shows a bolt/bushing arrangement, the analysis is applicable to solid pin/sockets.

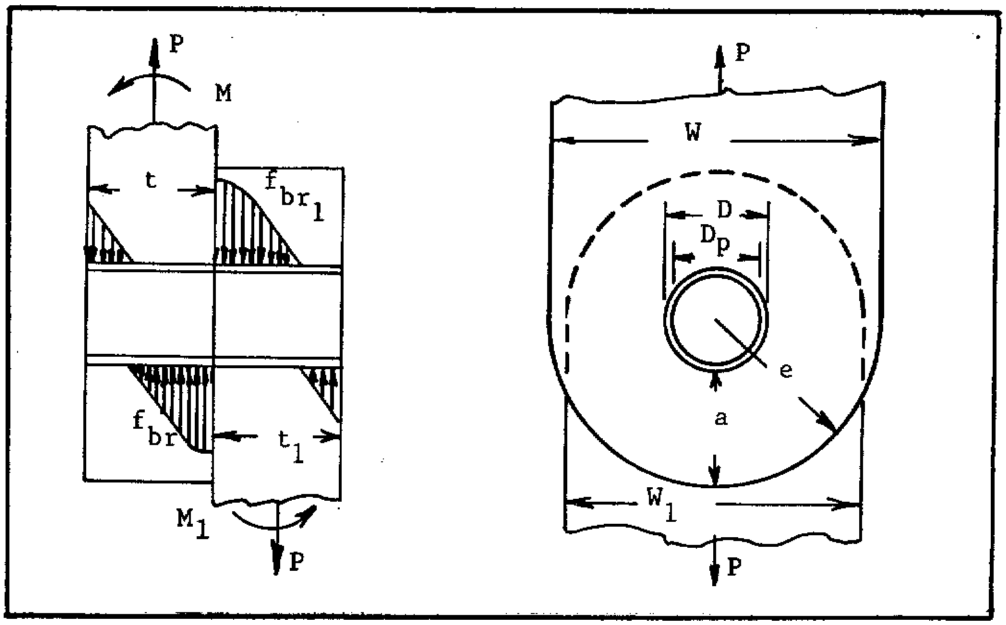

FIGURE 6.44 - SINGLE SHEAR LUG JOINT

In Figure 6.44 a representative single shear joint is shown with centrally applied load (P) in each lug and bending moments M and M_1 that keep the system balanced. Assuming no gap between the lugs, $M + M_1 = P(t_1 + t)/2$. The individual values of M and M_1 are determined from the loading of the lugs as modified by the deflection, if any, of the lugs.

The following analysis procedure is applicable to either lug. The joint strength is determined by the lowest margin of safety of the various failure modes.

The bearing stress distribution between lug and bushing is assumed to be similar to the stress distribution that would be obtained in a rectangular cross section of width, D, and depth, t, subject to load, P, and moment, M. At ultimate load the maximum lug bearing stress, f_{br}, is approximated by

$$f_{br} = P/Dt + 6M/k_{br}\ Dt^2$$

where k_{br} is the plastic bearing coefficient for both the lug material and is assumed to be the same as the plastic bending coefficient for a rectangular section which may be found in Section 9.6. The ultimate allowable is found by the methods defined in Section 6.4.4 for shear bearing.

The ultimate tensile stress in the outer fibers in the lug net section is approximately

$$f_t = P/(W - D)t + 6M/k_b(W - D)t^2$$

where k_b is the plastic bending coefficient for the lug net section. The allowable ultimate is found by the methods defined in Section 6.4.4 for axial tension.

The bearing stress distribution between bushing and pin is assumed to be similar to that between the lug and bushing. At ultimate bushing load the maximum bushing bearing stress is approximated by

$$f_{br} = P/Dpt + 6M/k_{br}\, Dpt^2$$

where k_{br}, the plastic bearing coefficient, is assumed the same as the plastic bending coefficient for a rectangular section. The allowable ultimate value is F_{cy} for the bushing material.

The maximum value of pin shear can occur either within the lug or at the common shear face of the two lugs, depending upon the value of M/Pt. At the lug ultimate load, the maximum pin shear stress (fs) is approximated by

$$fs = 1.273\ P/Dp^2; \quad (M/Pt \leq 2/3)$$

$$fs = \frac{1.273\ P}{Dp^2} \quad \frac{\left(\sqrt{(2M/Pt)^2 + 1} - 1\right)}{\left((2M/Pt)+1 - \sqrt{(2M/Pt)^2 + 1}\right)}\ ; \quad (M/Pt > 2/3)$$

The first equation above defines the case where the maximum pin shear is obtained at the common shear face of the lugs. The second equation defines the case where the maximum pin shear occurs away from the shear face. The allowable ultimate is F_{su} of the pin material.

The maximum pin bending moment can occur within the lug or at the common shear faces of the two lugs, depending on the value of M/Pt. At the lug ultimate load, the maximum pin bending stress (f_{bu}) is approximated by

$$f_{bu} = \frac{10.19\ M}{k_b\ Dp^3} \quad \left(\frac{Pt}{2M} - 1\right)\ ; \quad (M/Pt \leq 3/8)$$

$$f_{bu} = \frac{10.19\ M}{k_b\ Dp^3} \quad \frac{\left(\sqrt{(2M/Pt)^2 + 1} - 1\right)}{2M/Pt}; \quad (M/Pt > 3/8)$$

where k_b is the plastic bending coefficient for the pin.

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The equation for ($M/Pt \leq 3/8$) defines the case where the maximum pin bending moment is obtained at the common shear face of the lugs and the equation for ($M/Pt > 3/8$) defines the case where the maximum pin bending moment occurs away from the shear face, where the pin shear is zero. The allowable ultimate value is F_{bu} for the pin or if deflection or fatigue is critical F_{tu} should be used.

6.4.8 Socket Analysis

The method presented here applies to sockets or sleeves made of aluminum or steel alloys. It is based on the assumption that the socket or sleeve walls (section cut by a plane parallel to the beam or pin centerline) are rectangular or nearly rectangular.

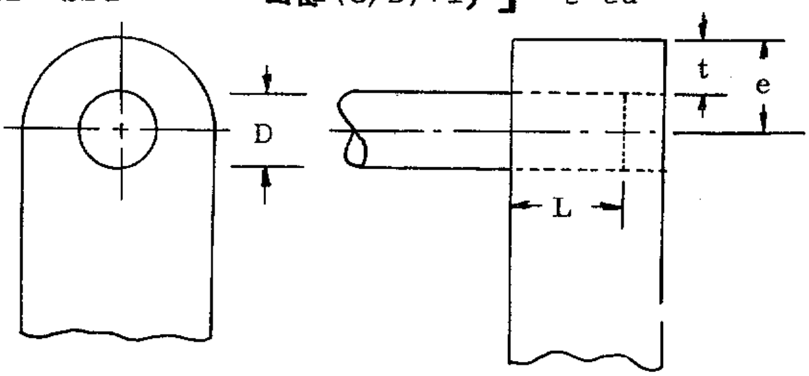
The method for obtaining bearing pressures within the socket or sleeve is also applicable to sockets or sleeves whose wall cross-sections vary appreciably from rectangular. An analysis suitable to the wall configuration must be used for the determination of the wall strengths.

This method may also be used for the analysis of single shear lug joints by considering the lug as a socket and the bolt as the beam.

The maximum allowable wall strengths of sockets or sleeves having rectangular or nearly rectangular wall cross sections (section cut by a plane parallel to the beam or pin center-line) may be determined from the following equations. Note that when e/D approaches 1.0 Pall may be larger than the allowable lug load as determined by Section 6.4.4. In all cases the lesser of the two allowables should be used for the margin of safety.

$$P_{all} = K_{bru}\left[4(e/D)^2 - 1\right] D K_t F_{tu}; \qquad (D/t \leq 10)$$

$$P_{all} = K_{bru}\left[16(e/D)\right]\left[\frac{2(e/D)-1}{(2(e/D)+1)^2}\right] D K_t F_{tu}; \qquad (D/t > 10)$$



the above result in pounds per inch

e = edge distance of socket, inches
D = diameter of beam or bolt, inches
K_t = tension efficiency factor, Figure 6.32
K_{bru} = bearing rupture factor, Figure 6.45
F_{tu} = ultimate tensile strength, psi
t = wall thickness of socket, inch

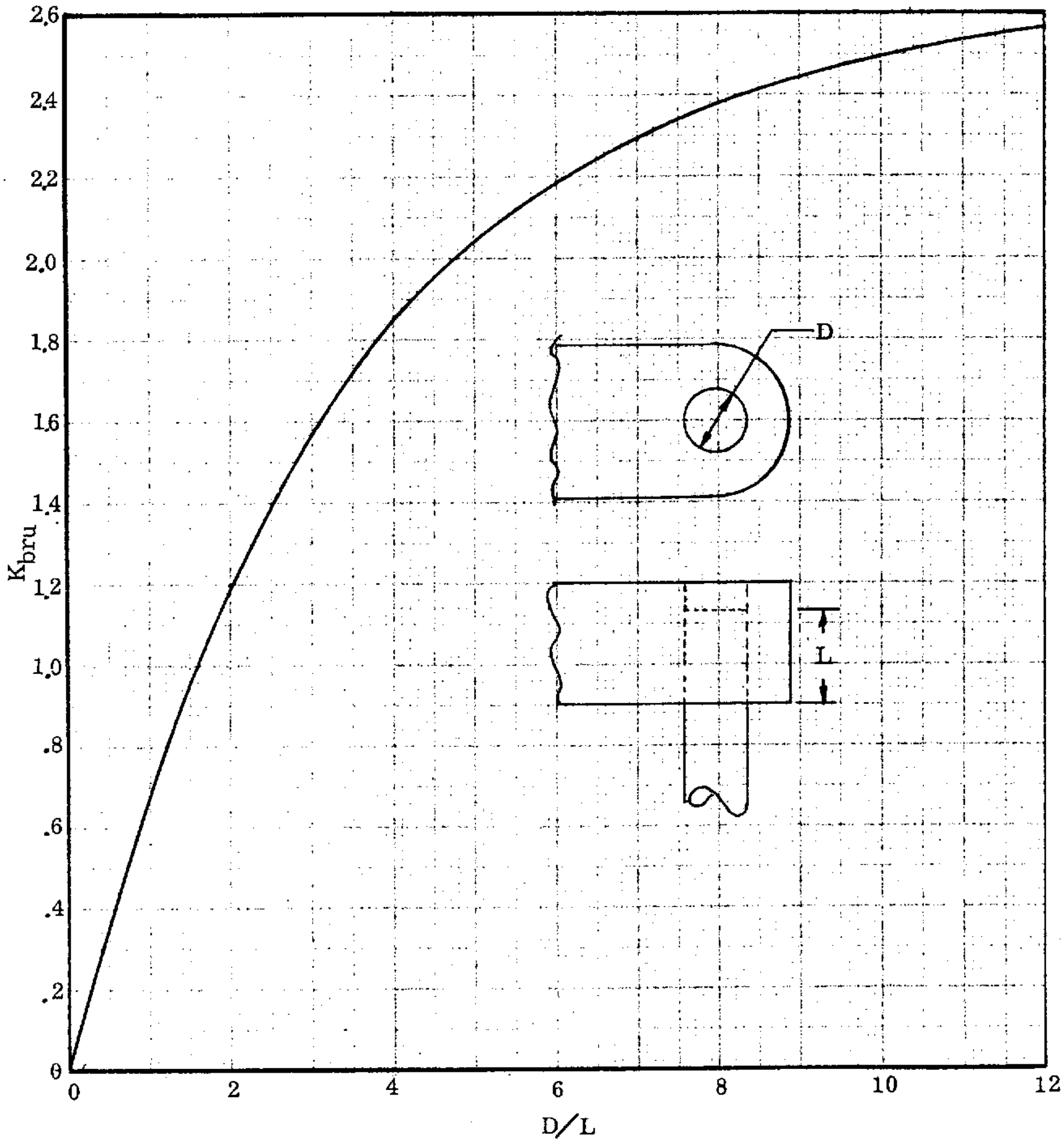


FIGURE 6.45-BEARING RUPTURE FACTOR ~ K_{bru}

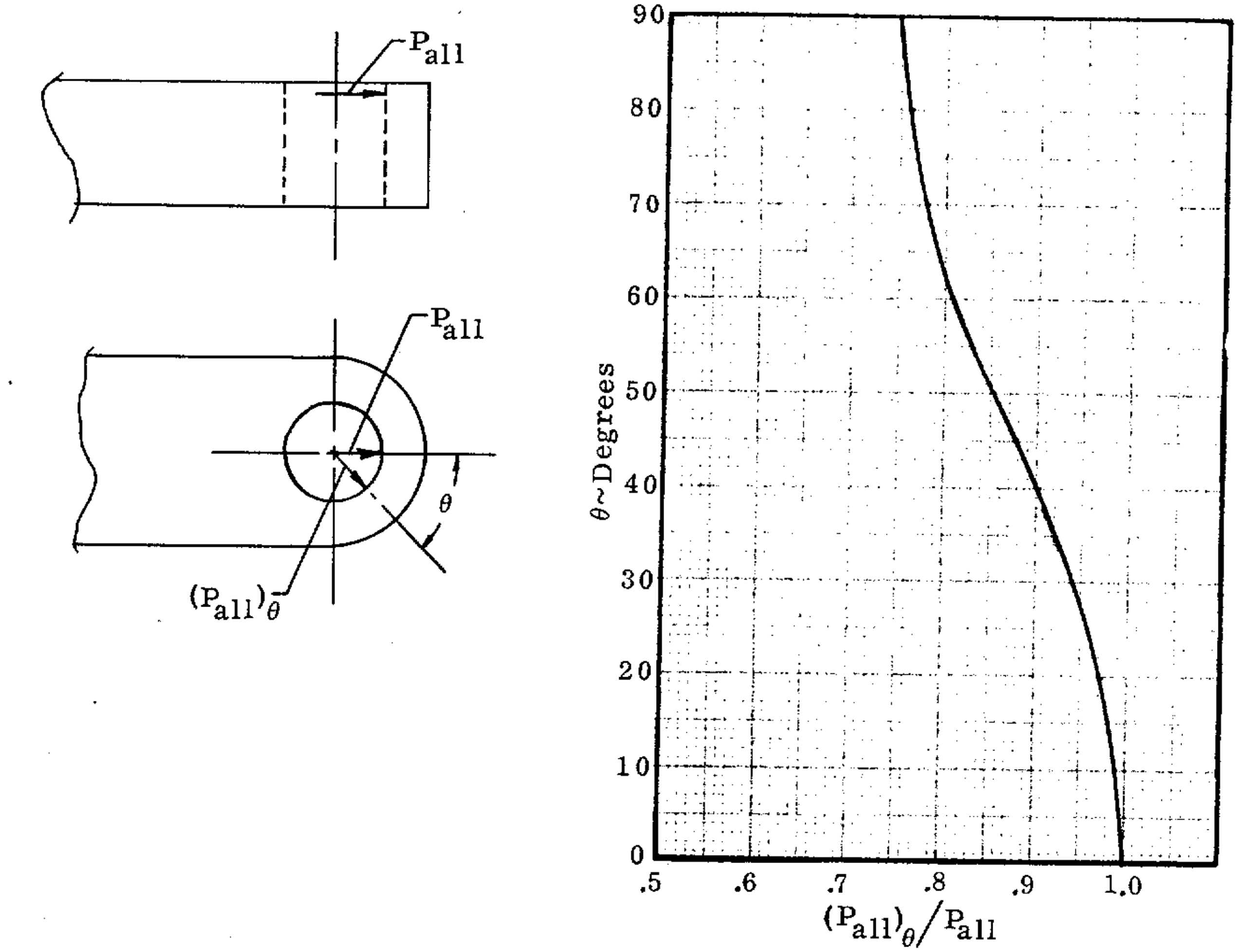
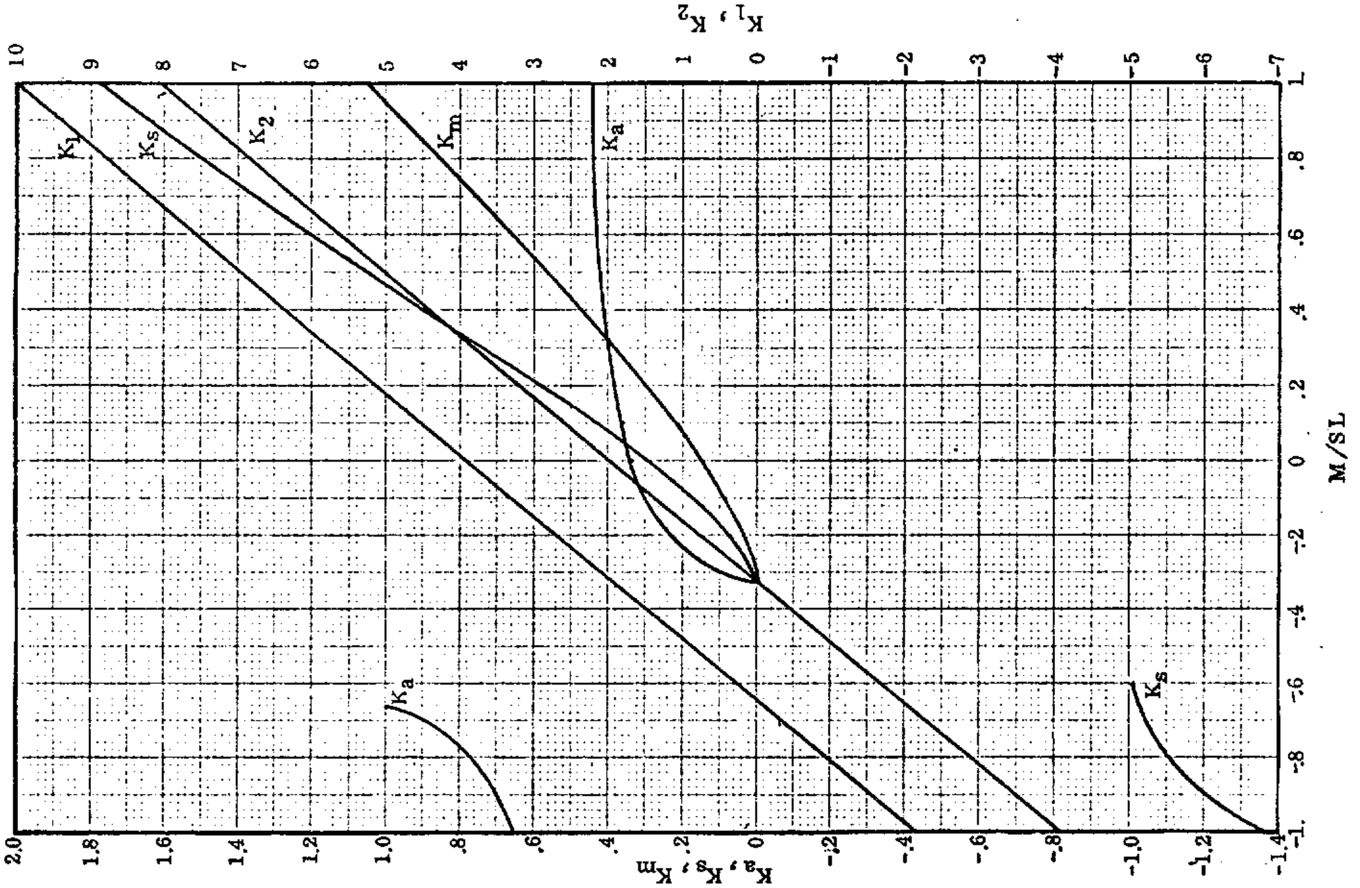
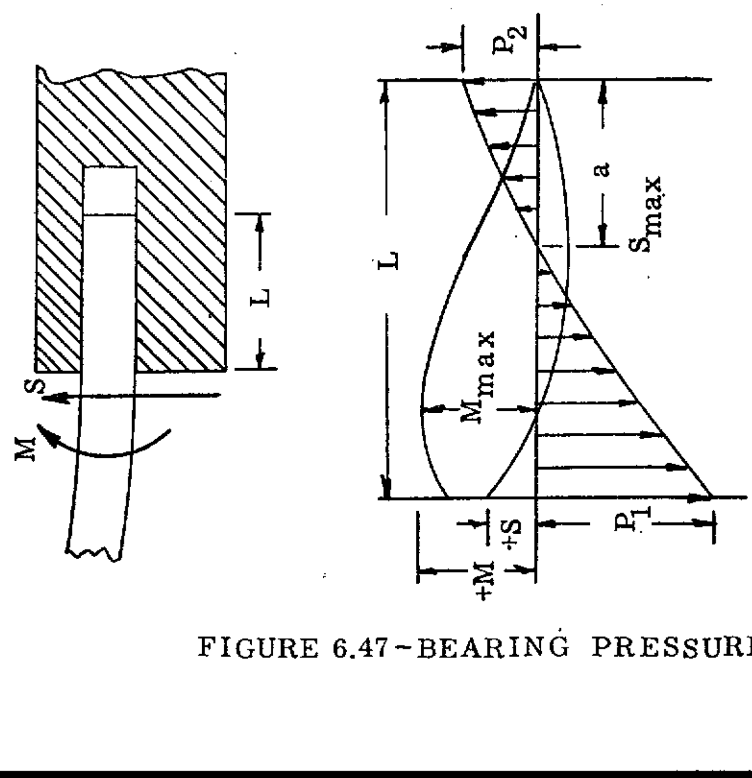


FIGURE 6.46 ALLOWABLE LATERAL LOAD





P ~ unit load ~ kips/in
S ~ shcar ~ kips
M ~ moment ~ in-kips at end of socket

$P_1 = K_1 S / L$

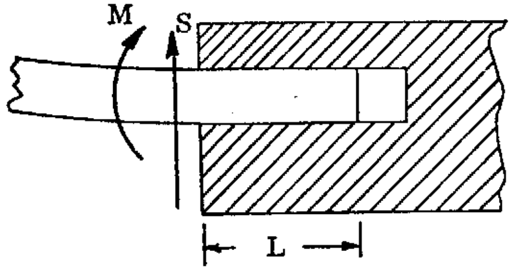
$P_2 = K_2 S / L$

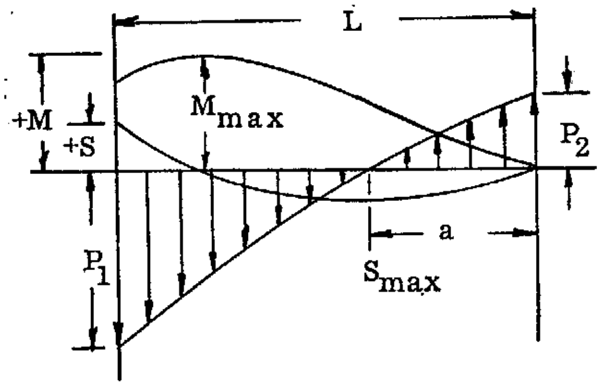
$a = K_a L$

$S_{max} = -K_s S$

$M_{max} = K_m S L$

FIGURE 6.47 ~ BEARING PRESSURE LOADS ~ $M/SL \leq 1$

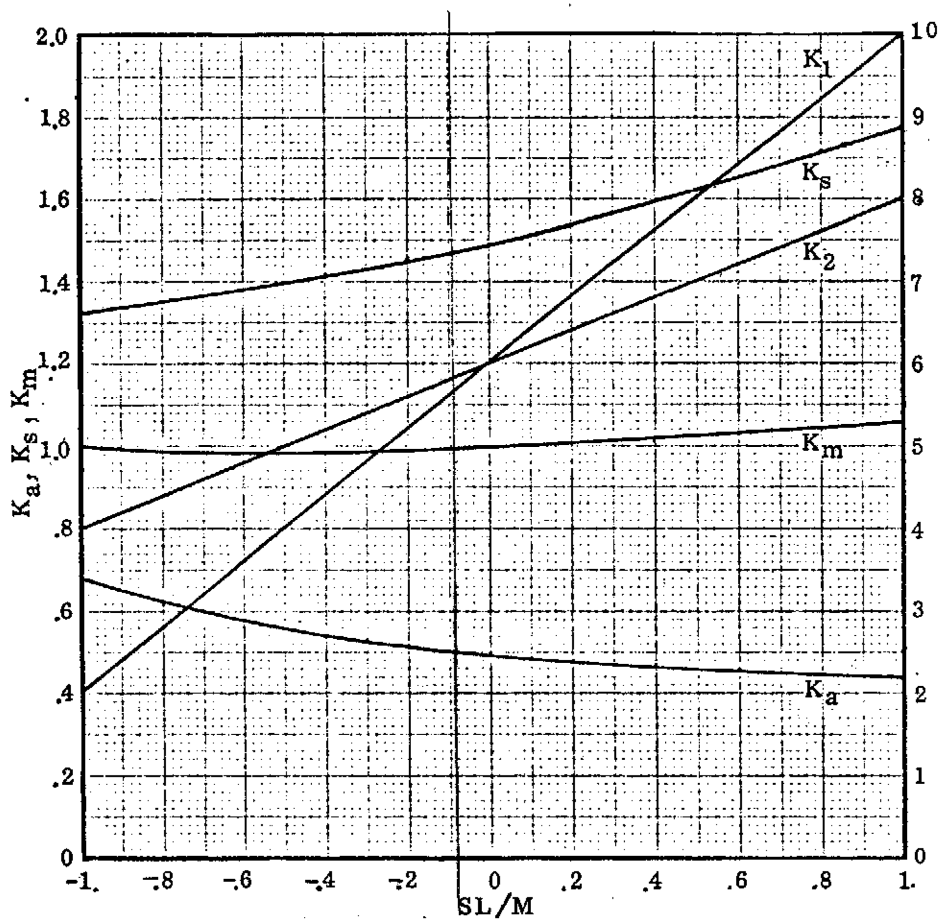




P ~ unit load ~ kips/in

S ~ shear ~ kips

M ~ moment ~ in-kips at end of socket



$P_1 = K_1 M/L^2$

$P_2 = K_2 M/L^2$

$a = K_a L$

$S_{max} = -K_s M/L$

$M_{max} = K_m M$

FIGURE 6.48 - BEARING PRESSURE LOAD ~ SL/M ≤ 1

Figure 6.46 shows the correction factor for obliquely loaded sockets. Multiplication of the axial allowable loading of the socket or sleeve by this correction factor gives the allowable loading when the load is applied at an angle to the axis of symmetry.

Figures 6.47 and 6.48 show the methods for determining the bearing pressure loadings in the beam - socket.

6.4.9 Tension Fittings

The analysis methods presented here apply to the most common types of tension fittings. The fitting design should also comply with the following design practices:

A. Bolts highly loaded in tension should be assembled with a washer under both the head and the nut.
B. Eccentricities in fitting attachments should be kept to a minimum.
C. In order to keep deflections to a minimum and to increase fatigue life, it is desirable that the fitting end pad thickness be not less than the bolt diameter for aluminum alloy fittings, nor less than .70 of the bolt diameter for steel fittings.
D. Fitting and casting factors as specified in Structural Design Criteria shall always be used in the analysis. If in any application both a fitting and a casting factor are applicable, they shall not be multiplied, but only the larger shall be used.

6.4.9.1 Wall Analysis

Tension and bending stresses in the fitting wall may be determined by conventional methods.

6.4.9.2 End Pad Analysis

Tension fitting end pads should be analyzed for both shear and bending. The shear surface area is the bolt head periphery through the end pad. Then the shear stress is

$$f_s = \frac{P}{2\pi r_o t_e}$$

where r_o is the bolt head outer radius and t_e is the end pad thickness. End pad bending in common types of tension fittings may be analyzed as shown in Figures 6.49 through 6.55.

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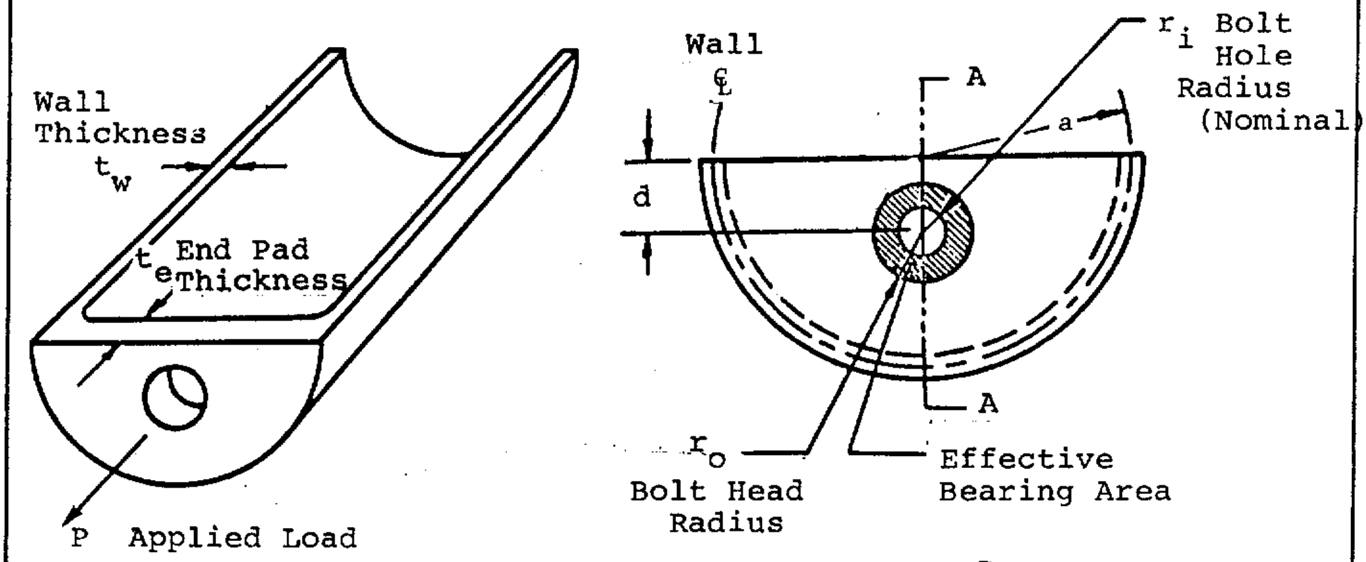
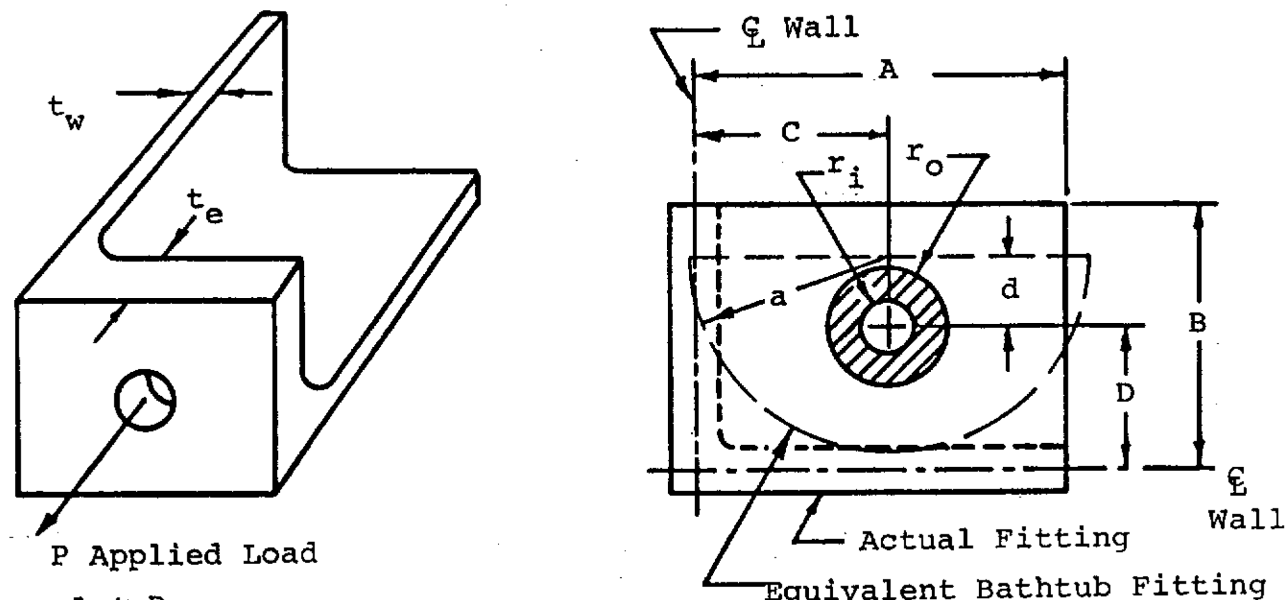


$$f_{b_{A-A}} = \frac{P}{t_e^2} K_1 K_2$$

Ref. Figures 6.51 and 6.52

FIGURE 6.49 BATHTUB FITTING - END PAD BENDING ANALYSIS



$$a = \frac{A + B}{\pi}$$

$$d = a - \left(\frac{C + D}{2}\right)$$

$$f_b = \frac{P}{t_e^2} K_1 K_2$$

Ref. Figures 6.51 and 6.52

FIGURE 6.50 ANGLE FITTING - END PAD BENDING ANALYSIS

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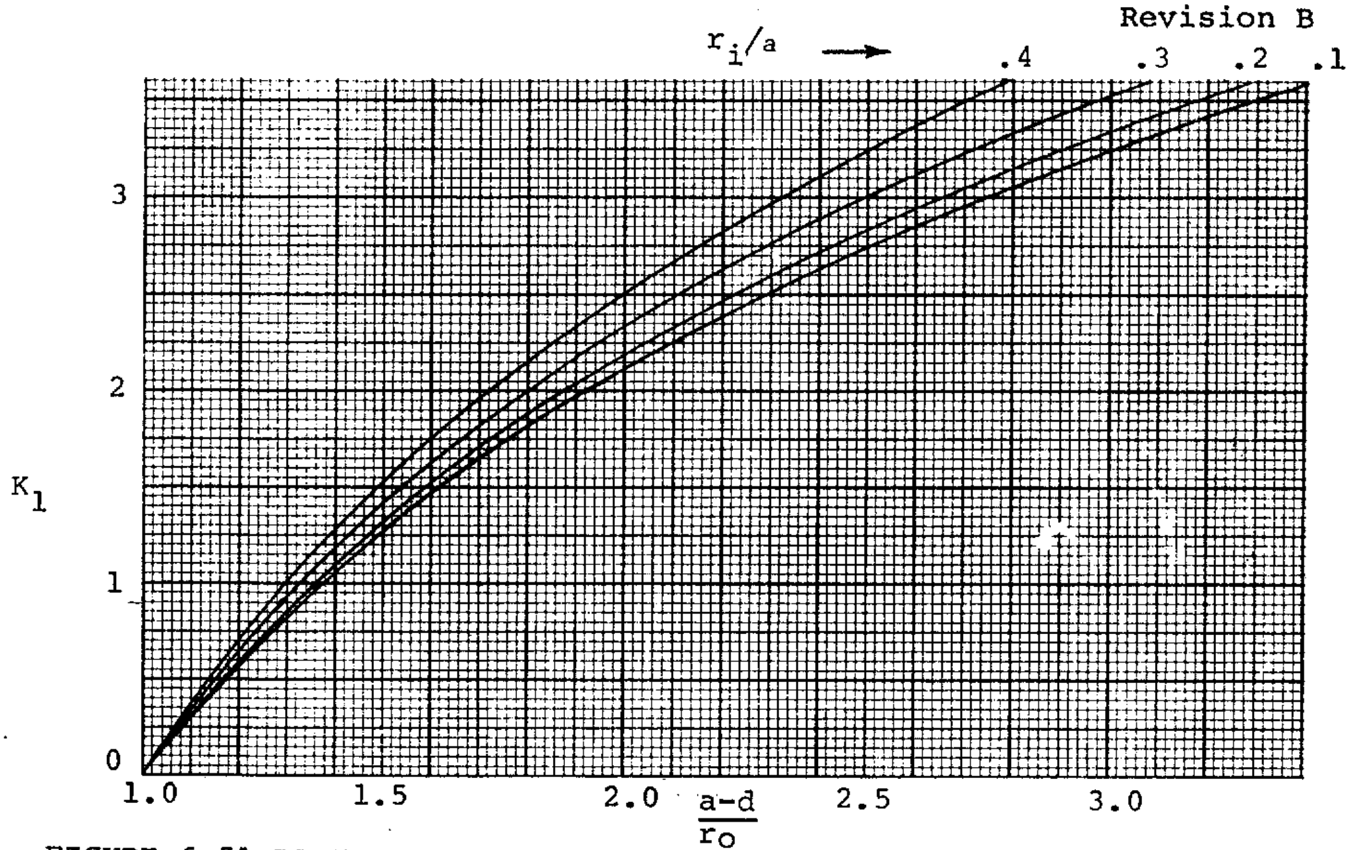

FIGURE 6.51 BATHTUB FITTING - END PAD K_1 VALUES

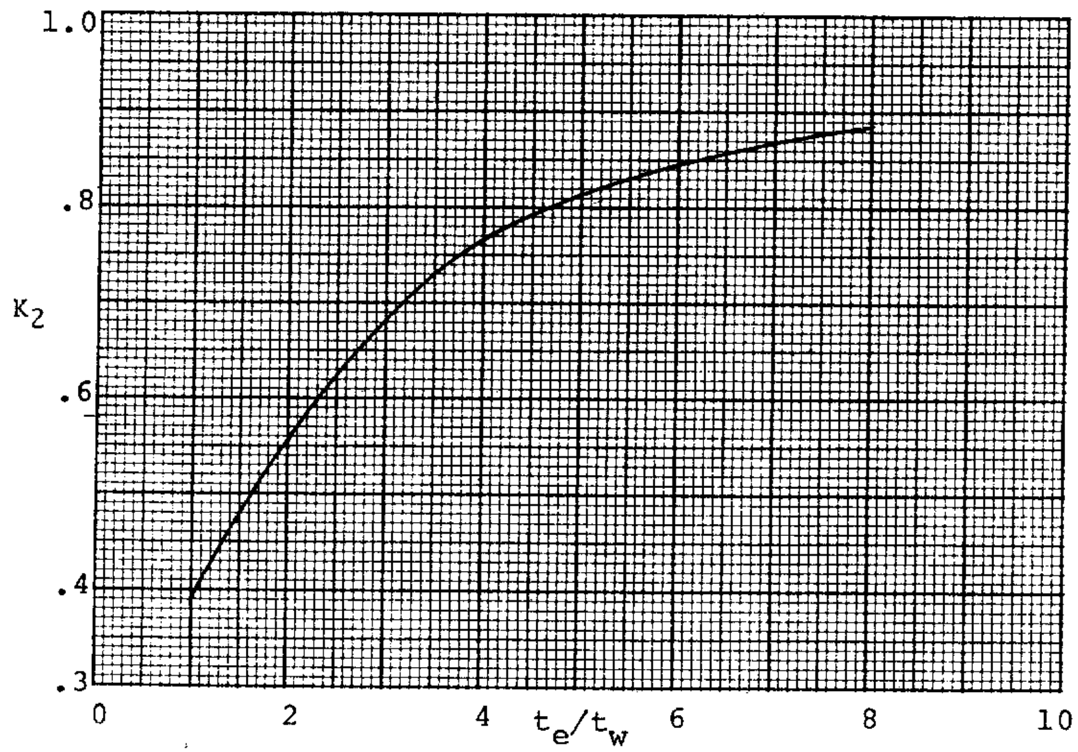

FIGURE 6.52 BATHTUB FITTING - END PAD K_2 VALUES

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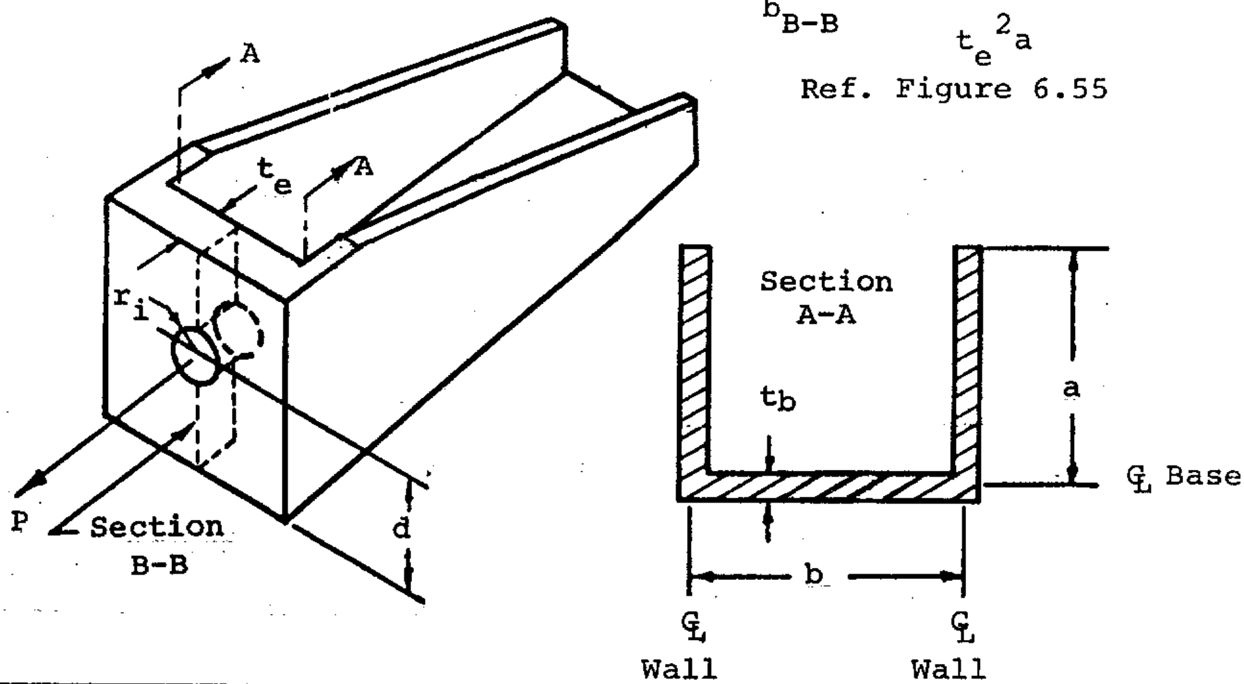

FIGURE 6.53 CHANNEL FITTING - END PAD BENDING ANALYSIS

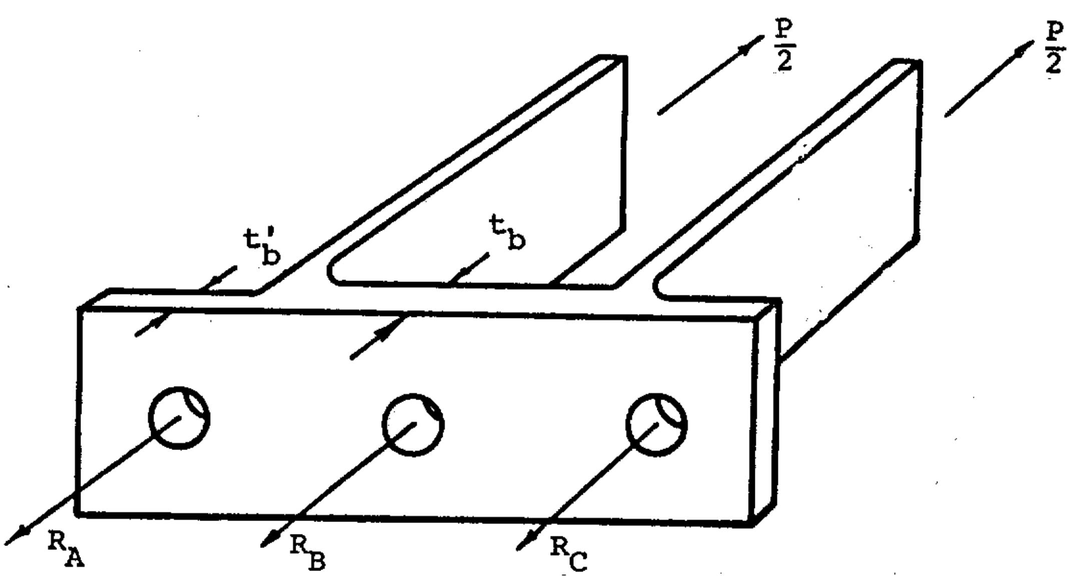

FIGURE 6.54 PI FITTING

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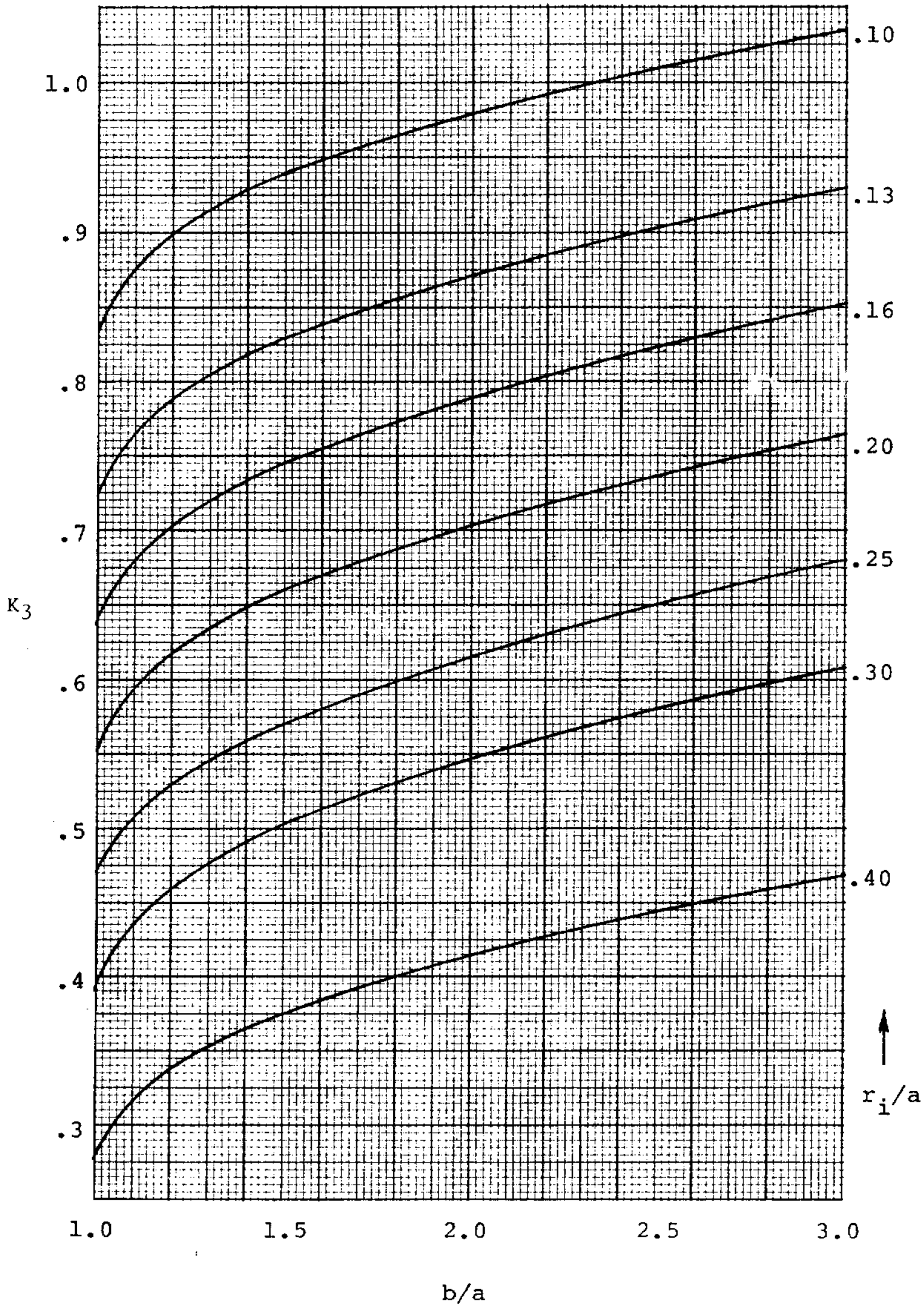
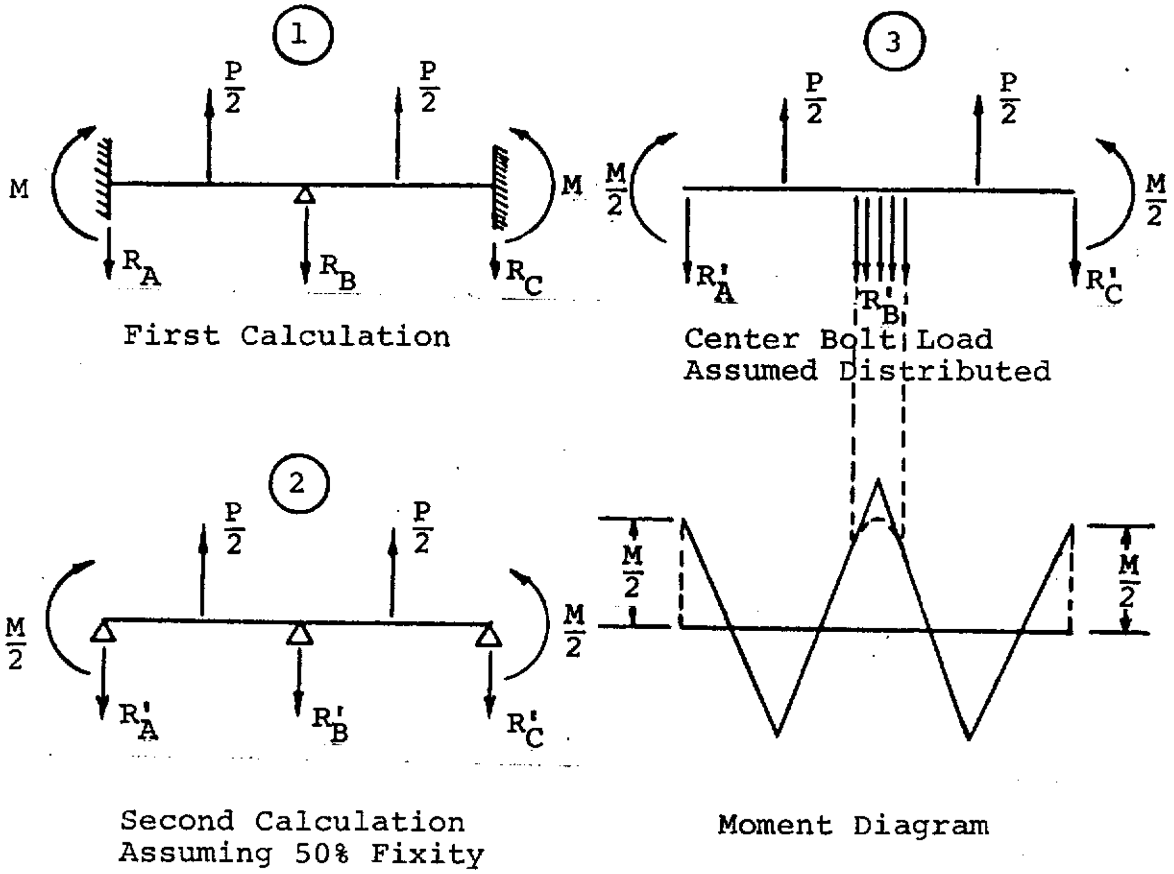


FIGURE 6.55 CHANNEL FITTING - END PAD K_3 VALUES

Ref. Figure 6.54



1. Determine the fixed end moment for a beam continuous over three supports assuming all loads as concentrated.

2. Assume 50% of the above values as end moments and determine bolt loads (concentrated).

   Analyze the end bolts for the combined loading (moment plus tension) and the center bolt for direct tension.

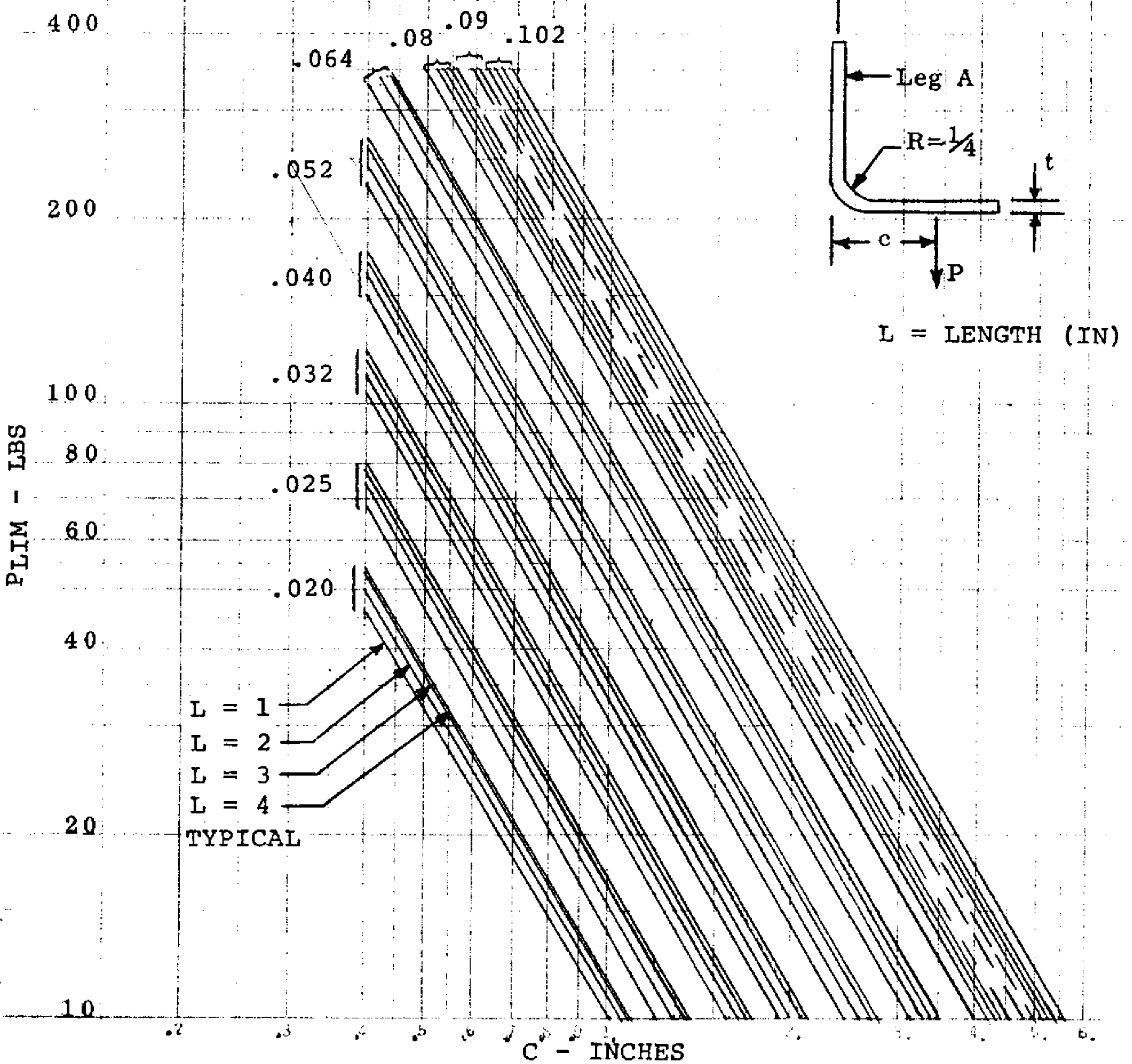
3. To determine the bending moment curve, assume the center bolt load computed in (2) is uniformly distributed over the bolt head flat.

FIGURE 6.56 PI FITTING - END PAD BENDING ANALYSIS

ALLOWABLE *LOADS FOR SINGLE ANGLES

24S-T

$$P = 2300\left(\frac{t}{c}\right)^{1.7}(3.3 + \log_{10} L)$$



Notes: *These are limit loads. This design is not recommended where the angles are subjected to repeated loads which produce high stresses. These curves are for leg A restrained, for little restraint use half these values. Beaded corners increase the failing BM approx. 6.5 times. Deflection is decreased about 50%. Ref. LAC SM 63 p 2. For thick angles, the bolt may be critical.

Ref. Rpt. R-900, pg. 11
Vega A.C.

Figure 6-57

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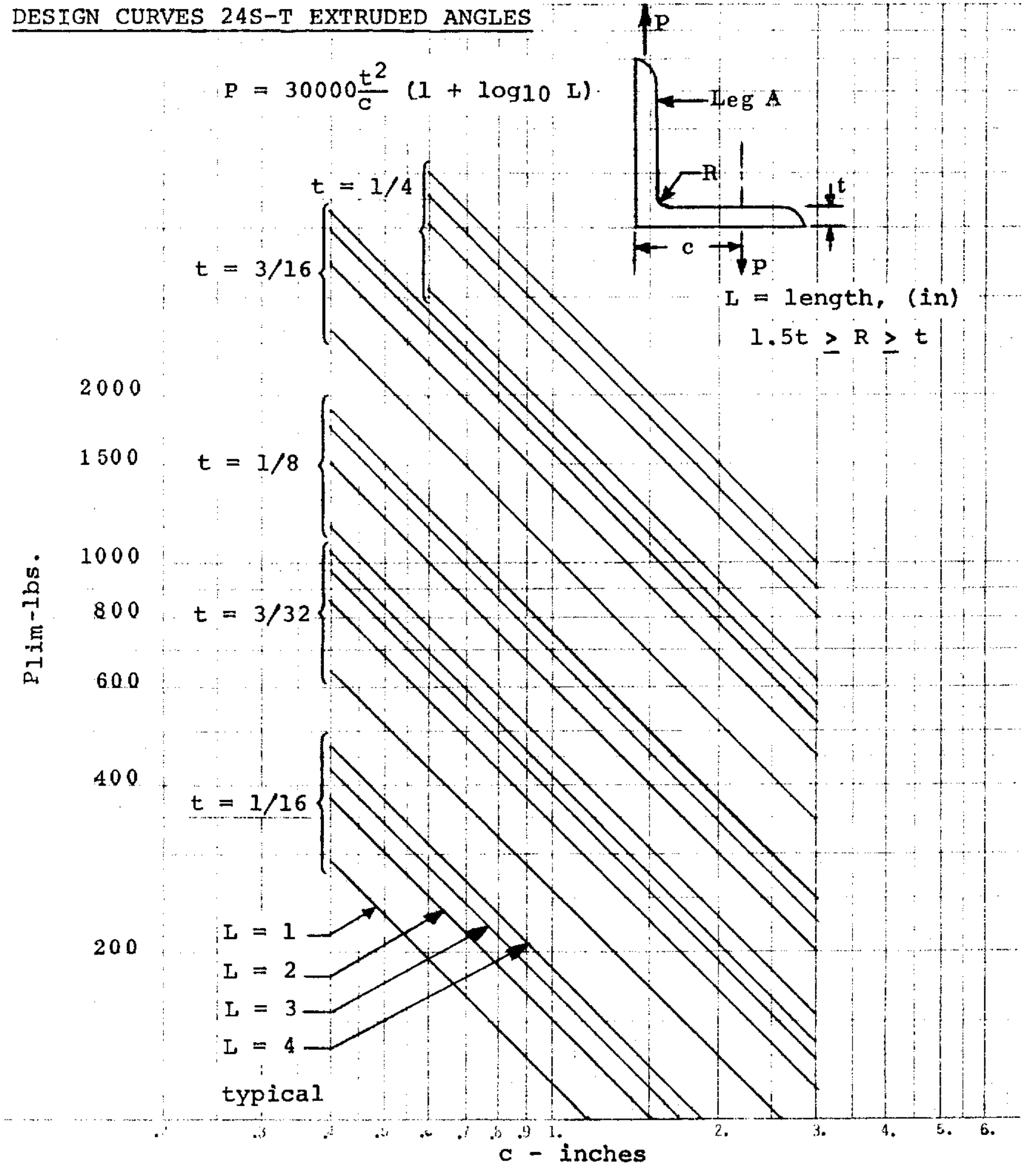
DESIGN CURVES 24S-T EXTRUDED ANGLES



Note: These curves are for leg A restrained. For little restraint use half these values.

Ref. Rpt. R-900, pg. 10
Vega A.C.

Figure 6-58

ALLOWABLE COMPRESSION IN LEGS OF FORMED ALUMINUM CHANNELS

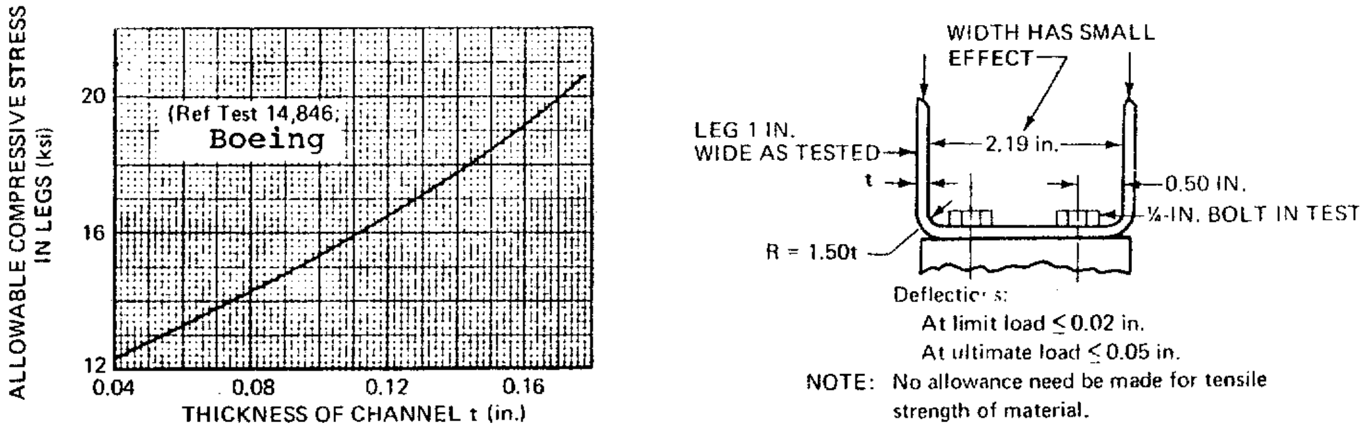
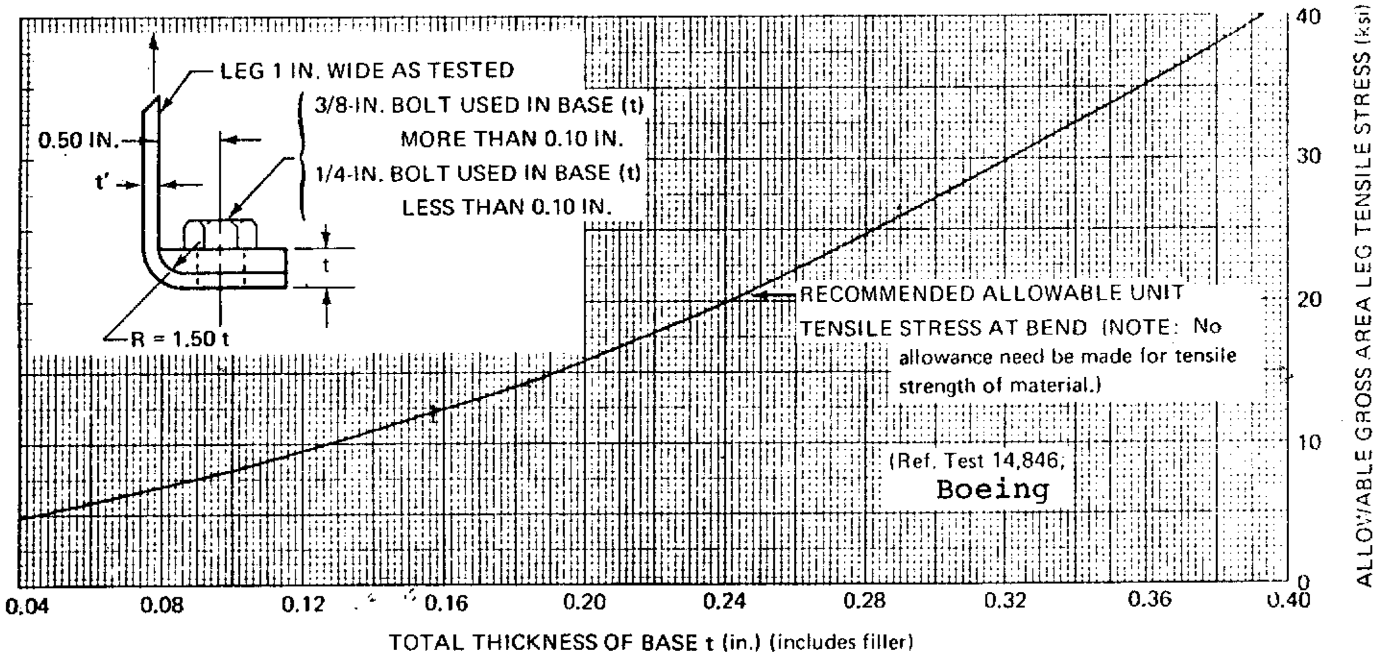


FIGURE 6-59

ALLOWABLE TENSION IN LEGS OF FORMED ALUMINUM CHANNELS



NOTE: Deflections at limit load up to 0.025 in. per channel; up to 0.07 in. at ultimate load (applicable through range of chart).

For radius larger than 1.5t, increased bolt spacing or, bases or fillers less than required for minimum deflections, conservative allowances must be made.

6.5 CABLES AND PULLEYS

The strength of MIL-W-83420 flexible wire rope is shown in Table 6.11 for various size cables. The load/deflection relationship of these cables is shown in Figure 6.49.

NOM. DIA.	TYPE OF CONST.	MIL-W-83420, TYPE I (NON-JACKETED)		
		COMPOSITION A CARBON STEEL	COMPOSITION B CRES	APPROXIMATE WEIGHT LBS/100 FT
		MIN. BREAKING STRENGTH (LBS)	MIN. BREAKING STRENGTH (LBS)	
1/32	3x7	110	110	.16
3/64	3x7	230	230	.33
3/64	7x7	270	270	.42
1/16	7x7	480	480	.75
1/16	7x19	480	480	.75
3/32	7x7	920	920	1.60
3/32	7x19	1000	920	1.74
1/8	7x19	2000	1760	2.90
5/32	7x19	2800	2400	4.50
3/16	7x19	4200	3700	6.50
7/32	7x19	5600	5000	8.60
1/4	7x19	7000	6400	11.00
9/32	7x19	8000	7800	13.90
5/16	7x19	9800	9000	17.30
11/32	7x19	12500	--	20.70
3/8	7x19	14000	12000	24.30

Cable Terminal Efficiency (% of Cable Strength)

Five-Tuck and Nicopress Type Splice-Flexible Cable....75%
Shackle and Ferrule Loop Terminal - Hard Wire.........85%
Wrapped and Soldered Splice - 19 Strand Wire..........90%
Swaged Ends...100%

TABLE 6.11 STRENGTH OF STEEL CABLE

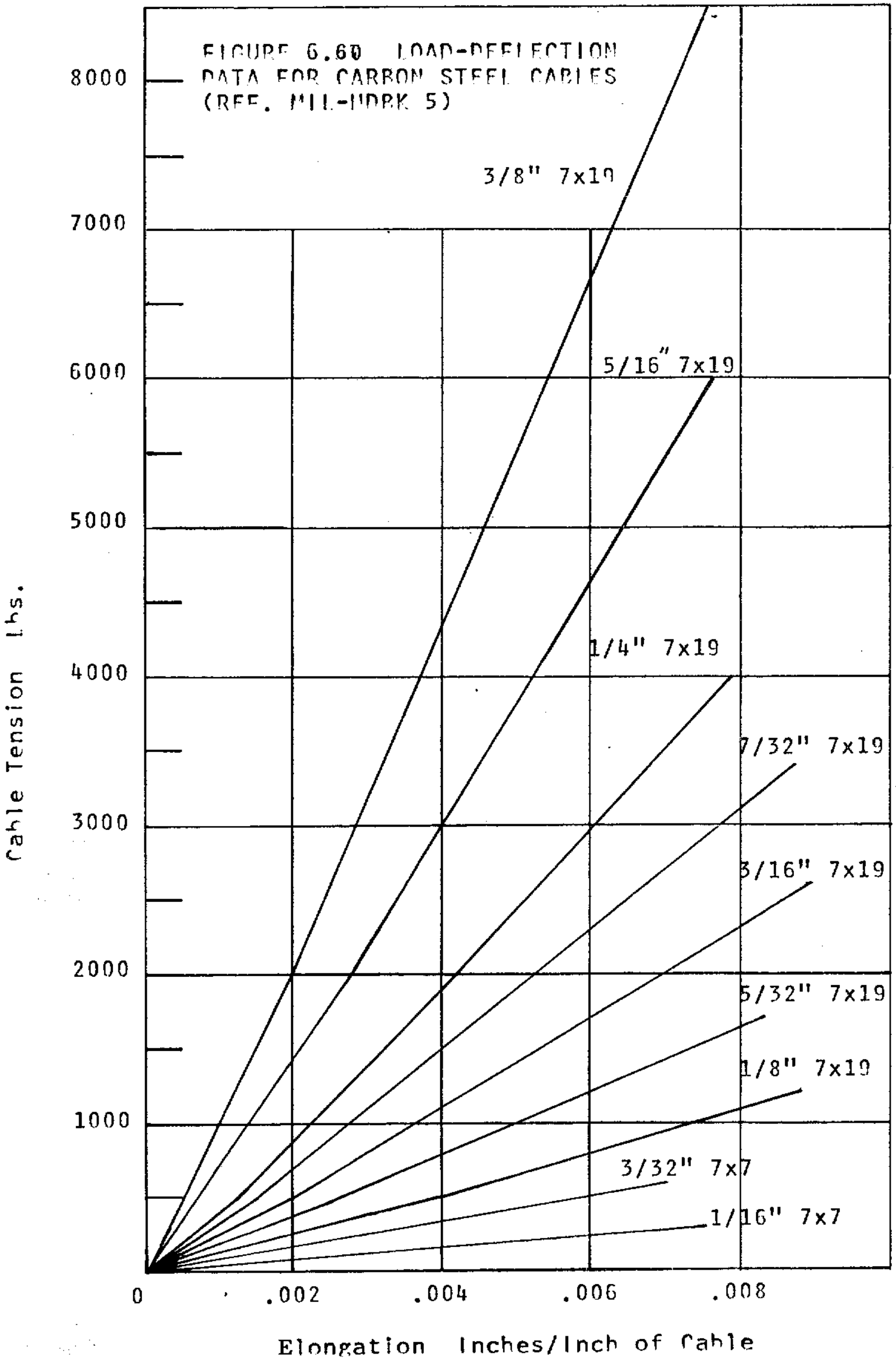


FIGURE 6.60 LOAD-DEFLECTION DATA FOR CARBON STEEL CABLES (REF. MIL-HDBK 5)
8000
7000
6000
5000
4000
3000
2000
1000
0
.002
.004
.006
.008
3/8" 7x19
5/16" 7x19
1/4" 7x19
7/32" 7x19
3/16" 7x19
5/32" 7x19
1/8" 7x19
3/32" 7x7
1/16" 7x7
Cable Tension lbs.
Elongation Inches/Inch of Cable

SECTION 7
PLATES AND MEMBRANES

7.1 Introduction to Plates

This section covers the analysis of plates as commonly used in aircraft structures. In general, such plates are classified as thin; that is deflections are small in comparison with the plate thickness ($y \leq t/2$). These plates are capable of carrying compression, bending, and shear loads; however, critical values of these loads produce a wrinkling or buckling of the plate. Such buckling produces unwanted aerodynamic effects on the surface of the aircraft, and also may result in the redistribution of loads to other structural members, causing critical stresses to develop. Thus, it is essential that the initial buckling stress of the plate be known. In addition, if the buckling stress is above the proportional limit, the panel will experience ultimate failure very soon after buckling.

The critical buckling of a plate depends upon the type of loading, plate dimensions, material, temperature, and conditions of edge support.

This section considers the various loadings of both flat and curved plates, with and without stiffeners. Single loadings are considered first followed by a discussion of combined loadings.

7.2 Nomenclature for Analysis of Plates

a	plate length
A_{st}	stiffener area
b	plate width
b_{ei}	effective panel width
c	core thickness, signifies clamped edge
C	compressive buckling coefficient for curved plates
e	strain
E	modulus of elasticity
E_s	secant modulus
E_t	tangent modulus
$\bar{E}_s$, $\bar{E}_t$	secant and tangent moduli for clad plates
f	ratio of cladding thickness to total plate thickness
F	stress
$F_{0.7}$, $F_{0.85}$	secant yield stress at 0.7E and 0.85E
F_{cr}	critical normal stress
$\bar{F}_{cr}$	critical normal stress, clad plates
F_{crs}	critical shear stress
F_{pl}	stress at proportional limit
F_{cl}	proportional limit of cladding
F_{cy}	compressive yield stress
F_f	crippling stress
FR	free (refers to edge fixity)

g	number of cuts plus number of flanges
k	buckling coefficient
k_c	compressive buckling coefficient
k_s	shear buckling coefficient
$\overline{k}_c$	equivalent compressive buckling coefficient
L'	effective column length
n	shape parameter, number of half waves in buckled plate
p	rivet pitch
P	total concentrated load
r	radius of curvature
R	stress ratio
ss	simply supported
t	thickness
t_s	skin thickness
t_w	web thickness
t_1	total cladding thickness
w	unit load
W	total load, potential energy
y	deflection
Z_b	length range parameter $b^2(1 - \nu_e^2)^{\frac{1}{2}}/rt$
β	ratio of cladding yield stress to core stress
β_g	crippling coefficient
ϵ	ratio of rotational rigidity of plate edge stiffness
η	plasticity reduction factor
$\overline{\eta}$	cladding reduction factor
λ	buckle half wavelength
ν	inelastic Poisson's ratio
ν_e	elastic Poisson's ratio
ν_p	plastic Poisson's ratio
ρ	radius of gyration

7.3 Axial Compression of Flat Plates

The compressive buckling stress of a rectangular flat plate is given by Equation (7-1).

$$F_{cr} = \eta \overline{\eta} \left(\frac{k \pi^2 E}{12\left(1 - \nu_e^2\right)} \right) \left(\frac{t}{b} \right)^2 \qquad (7\text{-}1)$$

The relation is applicable to various types of loadings in both the elastic and the inelastic ranges and for various conditions of edge fixity.

The case of unstiffened plates is treated first and then stiffened plates are discussed.

The edge constraints which are considered vary from simply supported to fixed. A simply supported edge is constrained to remain straight at all loads up to and including the buckling load, but is free to rotate about the center line of the edge. A fixed edge is constrained to remain straight and to resist all rotation.

These two conditions define the limits of torsional restraint and are represented by $\epsilon = 0$ simply supported edges and $\epsilon = \infty$ for fixed edges.

Plates are frequently loaded so that the stresses are beyond the proportional limit of the material. If such is the case, the critical buckling stress is reduced by the plasticity reduction factor η, which accounts for changes in k, E, and ν. This allows the values of k, E, and ν to always be the elastic values.

The second reduction factor in Equation (7-1) is the cladding factor $\overline{\eta}$. In order to obtain desirable corrosion resistance, the surface of some aluminum alloys are coated or clad with a material of lower strength, but of better corrosion resistance. The resultant panel may have lower mechanical properties than the basic core material and allowance must be made. Values for the factor $\overline{\eta}$ are given in Table 7.1.

Loading	$F_{cl} < F_{cr} < F_{pl}$	F_{crs} or $F_{cr} > F_{pl}$
Short plate columns	$\frac{1 + (3\beta f/4)}{1 + 3f}$	$\frac{1}{1 + 3f}$
Long plate columns	$\frac{1}{1 + 3f}$	$\frac{1}{1 + 3f}$
Compression and shear panels	$\frac{1 + 3\beta f}{1 + 3f}$	$\frac{1}{1 + 3f}$

Table 7.1 - Simplified Cladding Reduction Factors

7.3.1 Buckling of Unstiffened Flat Plates in Axial Compression

The buckling coefficients and reduction factors of Equation (7-1) applicable to flat rectangular plates in compression are presented in this section.

Figures 7-1, 7-2, and 7-3 show the buckling coefficient k_c as a function of the ratio a/b and the type of edge restraint; and, in the case of Figure 7-2, the buckle wave length and number of half waves. Figure 7-4 shows k_c for infinitely long flanges and plates as a function of the edge restraint only. The edge restraint ratio ϵ is the ratio of the rotational rigidity of plate edge support to the rotational rigidity of the plate.

The condition of unequal rotational support can be treated by Equation (7-2).

$$k_c = (k_{c1}\, k_{c2})^{\frac{1}{2}} \qquad (7\text{-}2)$$

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The coefficients k_{c1} and k_{c2} are obtained by using each value of ϵ independently.

Figures 7-5, 7-6, and 7-7 present k_c for flanges. A flange is considered to be a long rectangular plate with one edge free.

The plasticity reduction factor η for a long plate with simply supported edges is given by Equation (7-3).

$$\eta = \left[\left(\frac{E_s}{E}\right)\left(\frac{1-\nu_e^2}{1-\nu^2}\right)\right]\left\{0.500 + 0.250\left[1 + \left(\frac{3E_t}{E_s}\right)\right]\right\}^{\frac{1}{2}} \qquad (7\text{-}3)$$

For a long plate with clamped edges, the factor is given by Equation (7-4).

$$\eta = \left[\left(\frac{E_s}{E}\right)\left(\frac{1-\nu_e^2}{1-\nu^2}\right)\right]\left\{0.352 + 0.324\left[1 + \left(\frac{3E_t}{E_s}\right)\right]\right\}^{\frac{1}{2}} \qquad (7\text{-}4)$$

The value of the inelastic Poisson ratio ν is given by Equation (7-5).

$$\nu = \nu_p - (\nu_p - \nu_e)\left(\frac{E_s}{E}\right) \qquad (7\text{-}5)$$

The tangent and secant moduli can be determined from the Ramberg-Osgood relation as shown in Equations (7-6) and (7-7).

$$\frac{E}{E_s} = 1 + \left(\frac{3}{7}\right)\left(\frac{F}{F_{0.7}}\right)^{n-1} \qquad (7\text{-}6)$$

$$\frac{E}{E_t} = 1 + \left(\frac{3}{7}\right)n\left(\frac{F}{F_{0.7}}\right)^{n-1} \qquad (7\text{-}7)$$

Figure 7-8 shows the characteristics of stress-strain curves used to determine the shape factor n. Table 11.1 lists values of E, F0.7, and n for various materials.

Figures 7-9 and 7-10 present values of k_c for plates restrained by stiffeners. This data is included here instead of in the section on stiffened plates because the stiffeners are not a part of the plate. To be noted is the effect of torsional rigidity of the stiffener on the buckling coefficient of the plate.

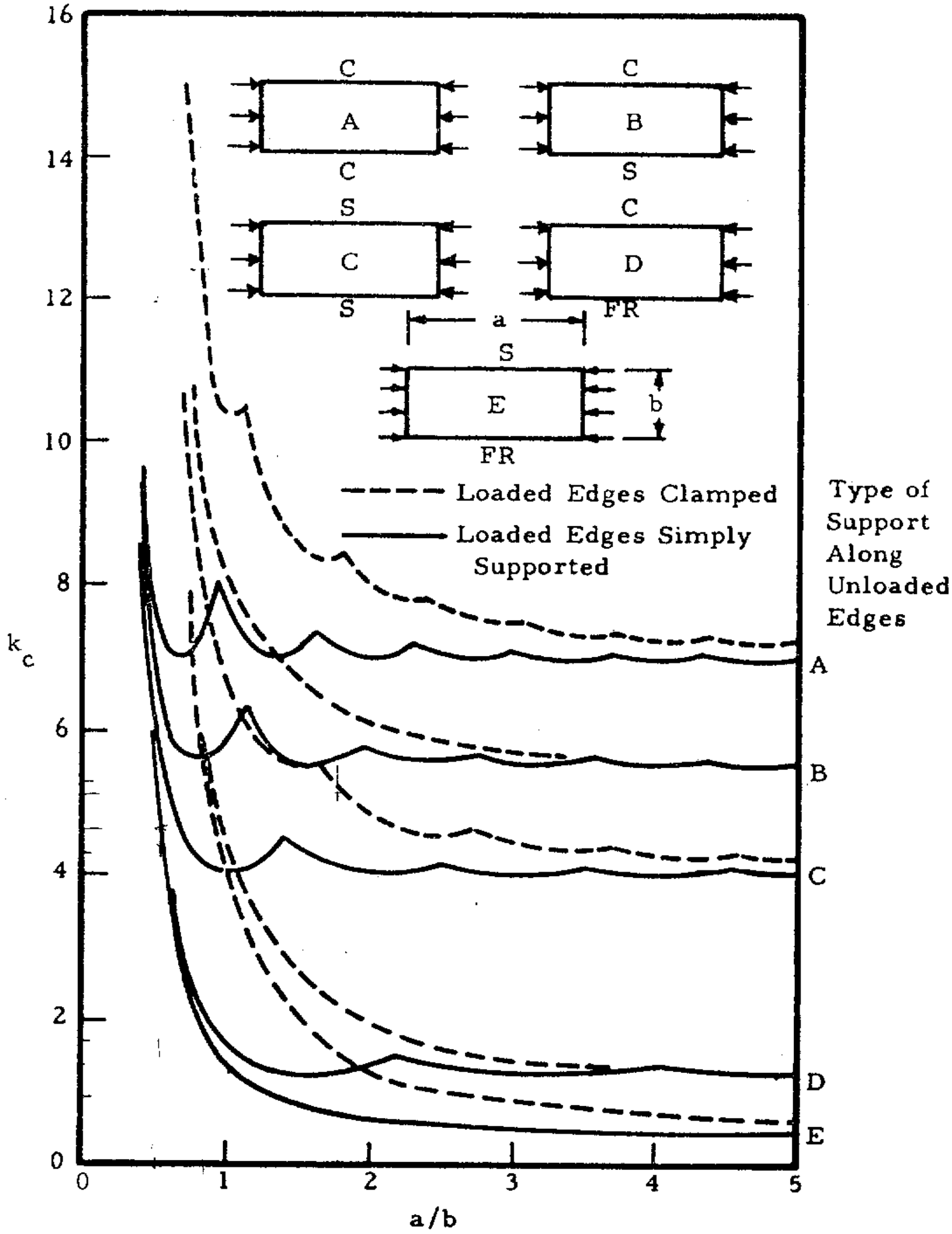


FIGURE 7.1 COMPRESSIVE BUCKLING COEFFICIENTS FOR FLAT RECTANGULAR PLATES

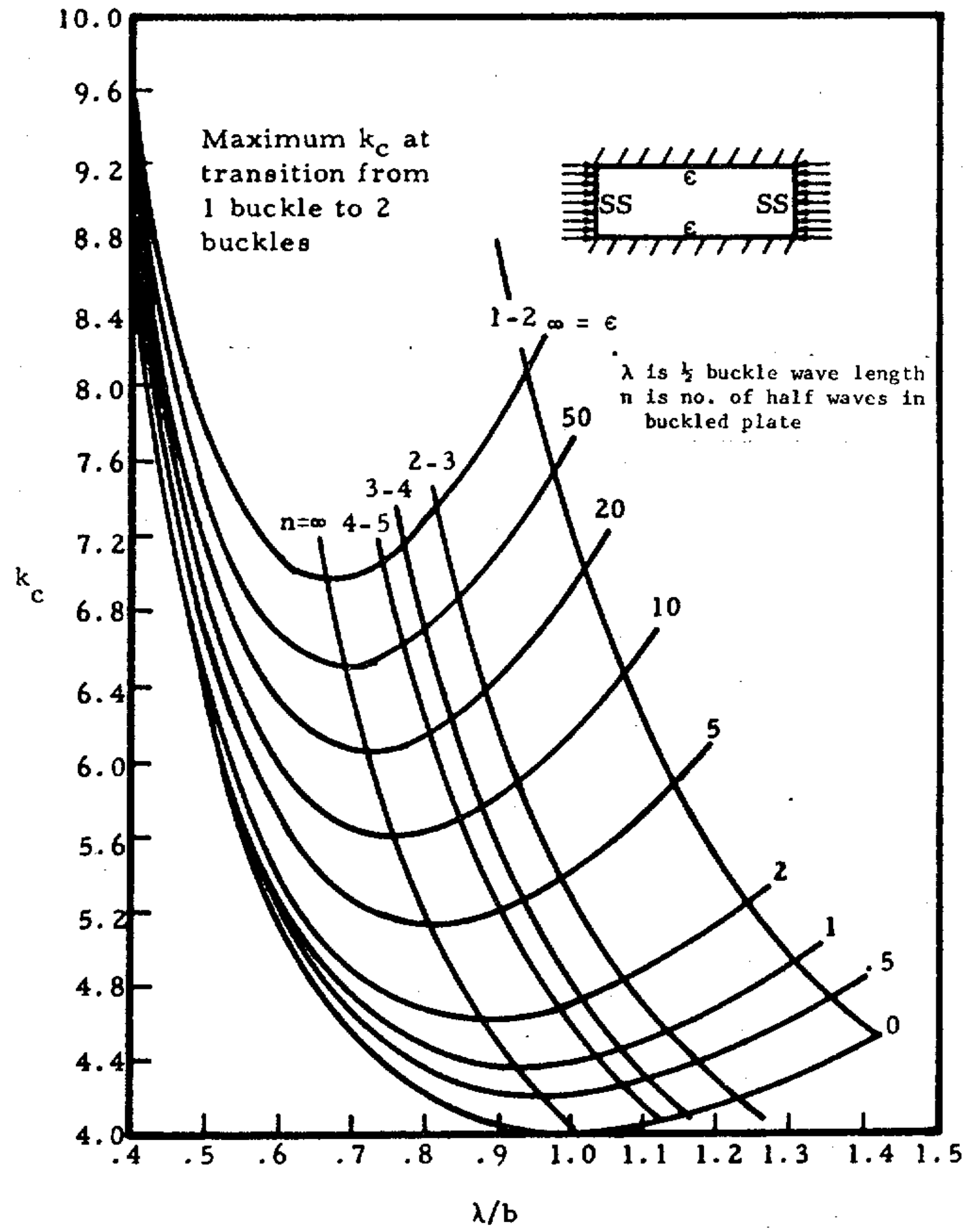


FIGURE 7.2 COMPRESSIVE BUCKLING COEFFICIENTS OF PLATES AS A FUNCTION OF λ/b FOR VARIOUS AMOUNTS OF EDGE ROTATIONAL RESTRAINT

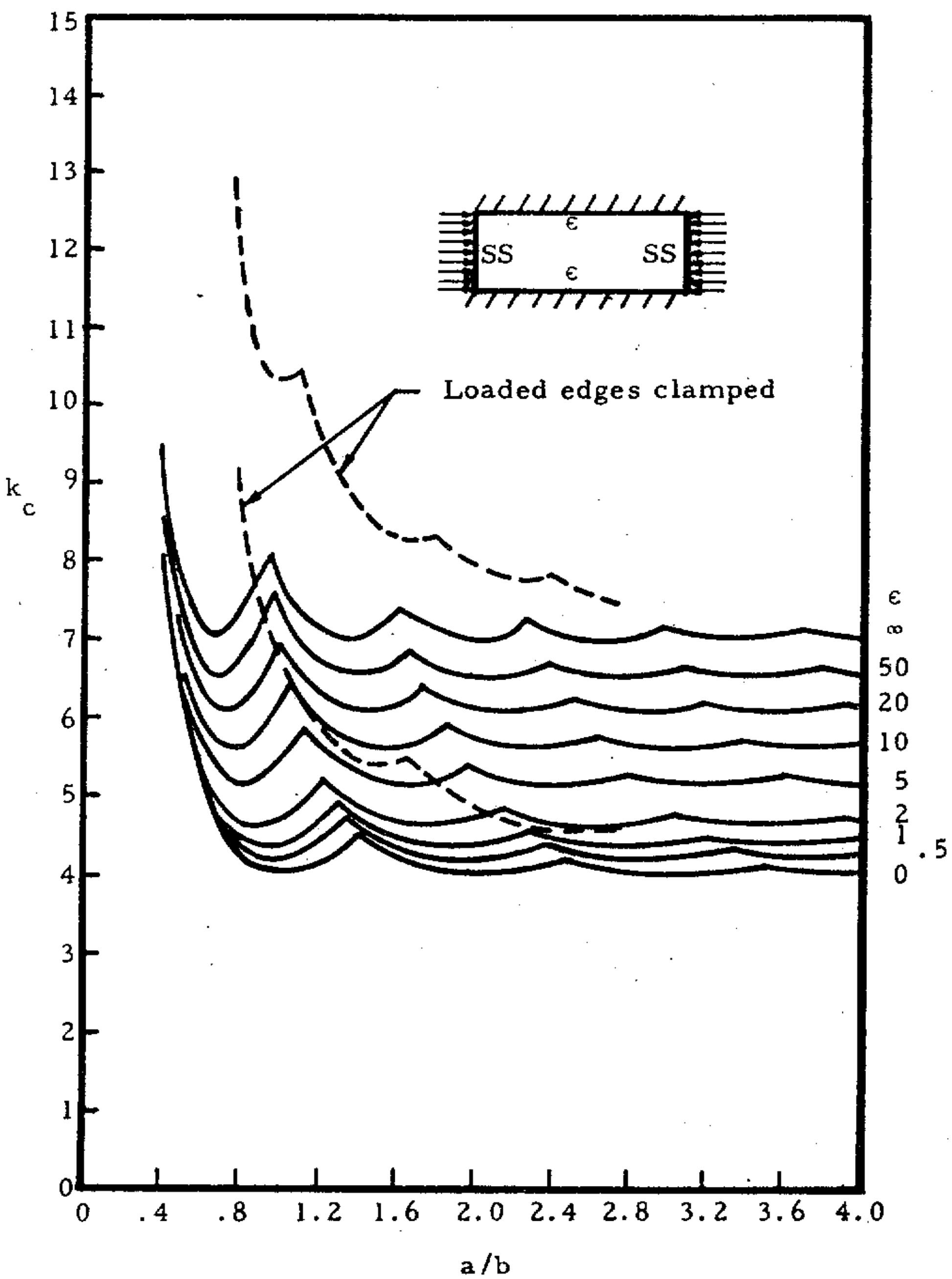


FIGURE 7.3 COMPRESSIVE BUCKLING COEFFICIENTS OF PLATES AS A FUNCTION OF a/b FOR VARIOUS AMOUNTS OF EDGE ROTATIONAL RESTRAINTS

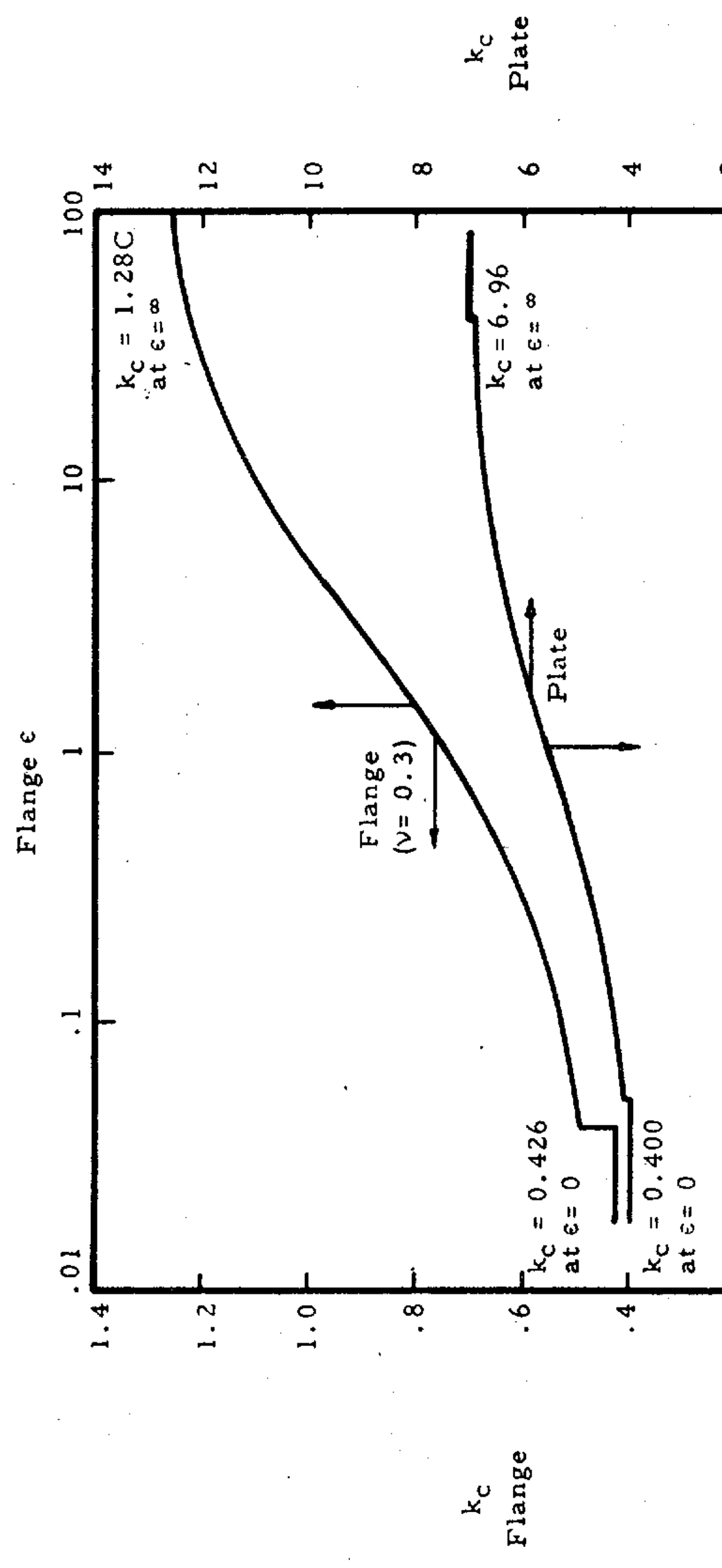


FIGURE 7.4 COMPRESSIVE BUCKLING COEFFICIENTS FOR INFINITELY LONG FLANGES AND PLATES AS FUNCTIONS OF EDGE ROTATIONAL RESTRAINTS

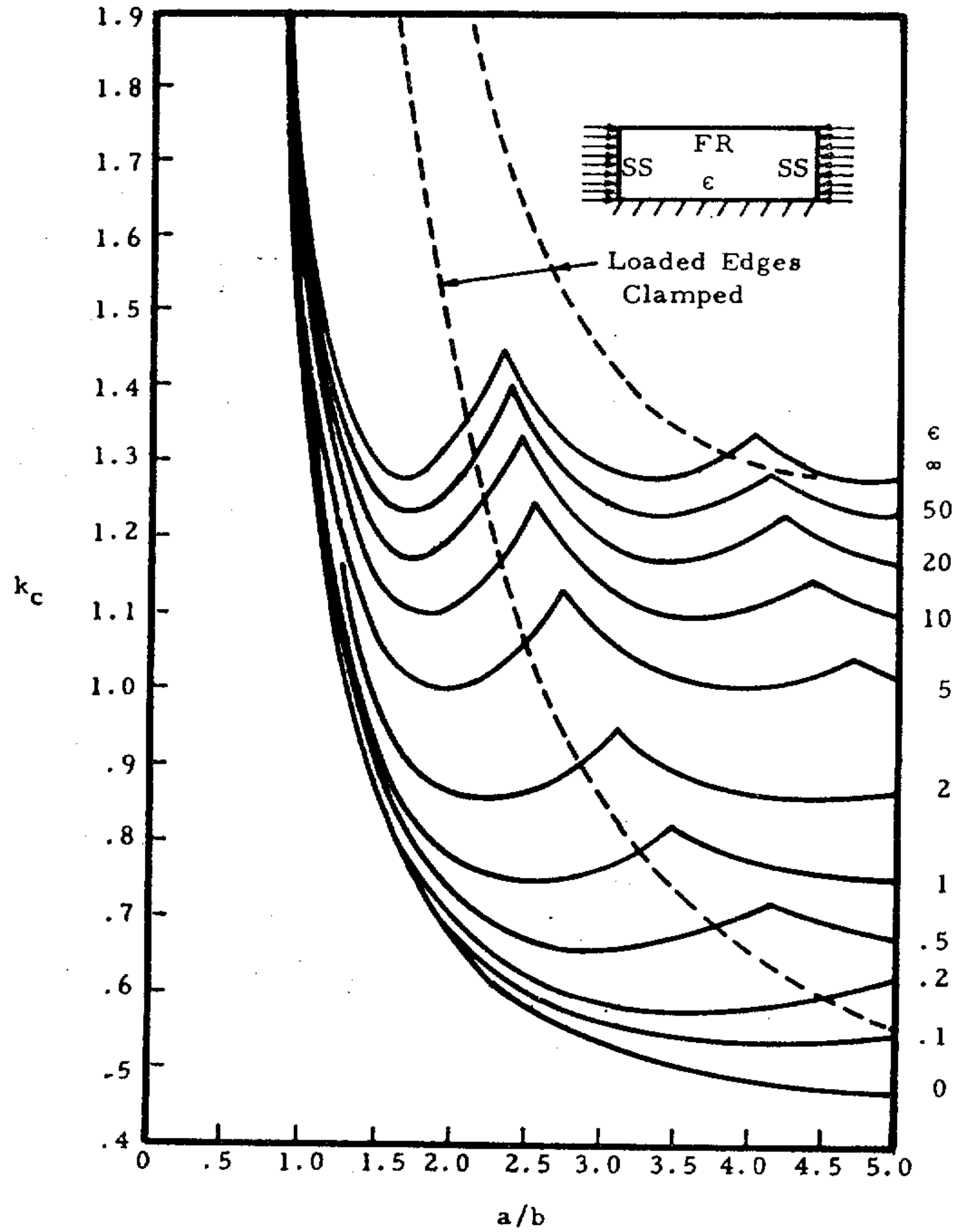


FIGURE 7.5 COMPRESSIVE BUCKLING COEFFICIENTS OF FLANGES AS A FUNCTION OF a/b FOR VARIOUS AMOUNTS OF EDGE ROTATIONAL RESTRAINTS

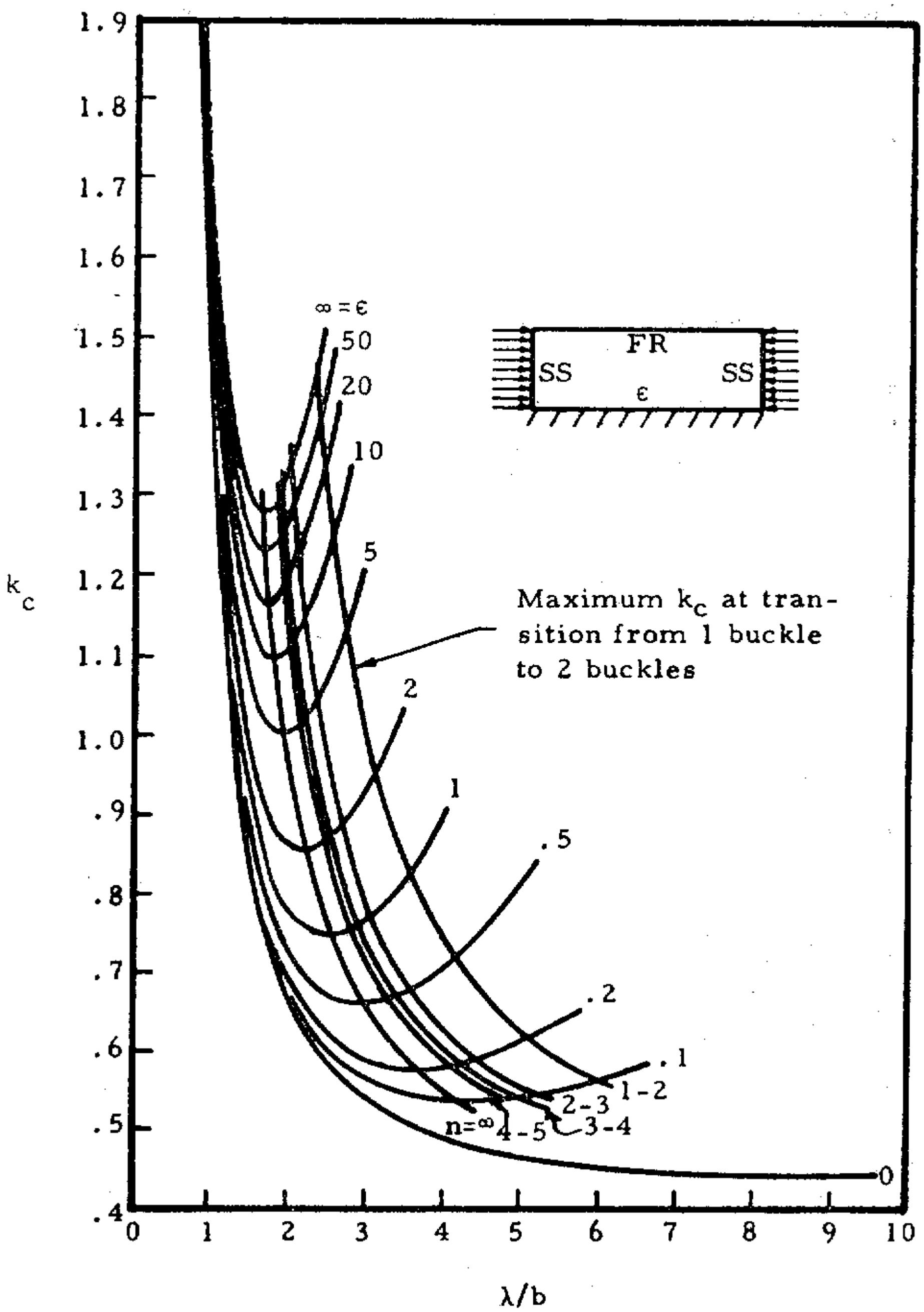
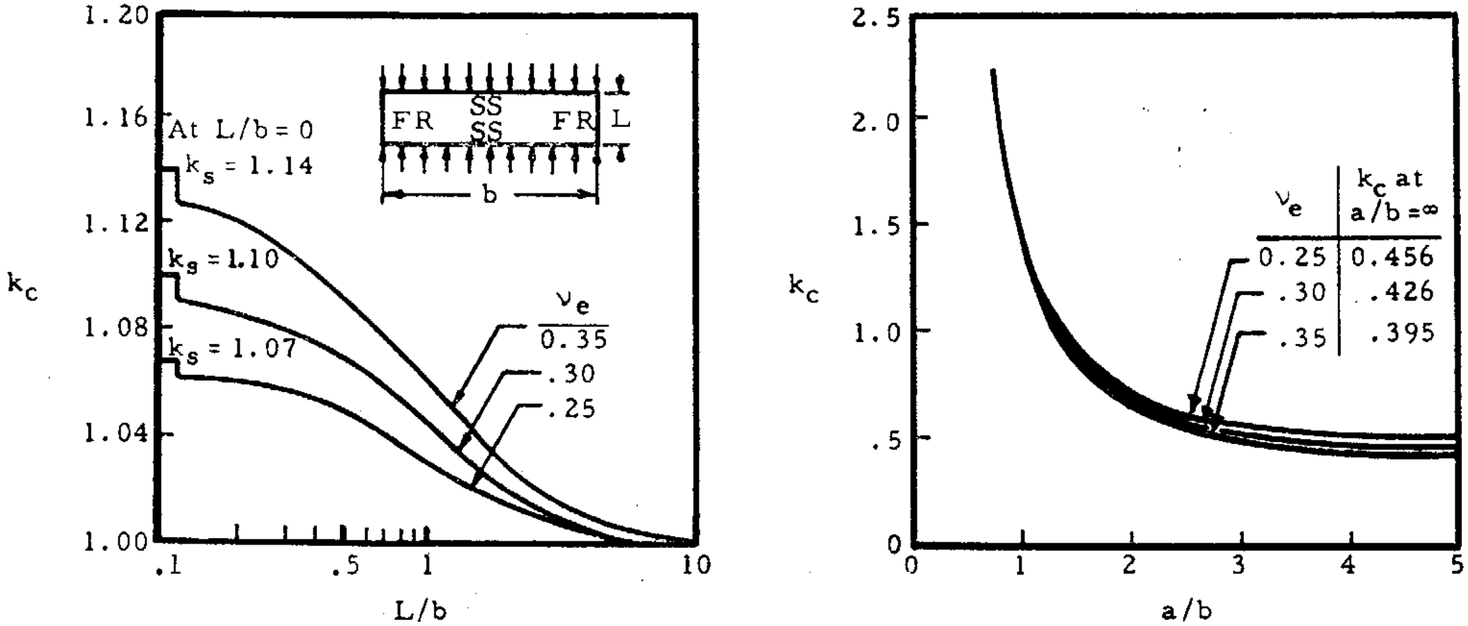


FIGURE 7.6 COMPRESSIVE BUCKLING COEFFICIENTS OF FLANGES AS A FUNCTION OF λ/b FOR VARIOUS AMOUNTS OF EDGE ROTATIONAL RESTRAINTS

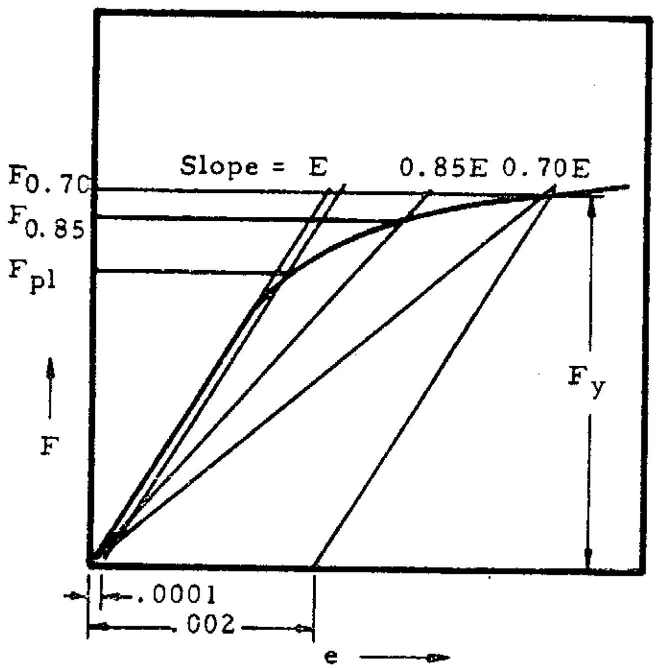


(a) Plate columns with hinged loaded edges

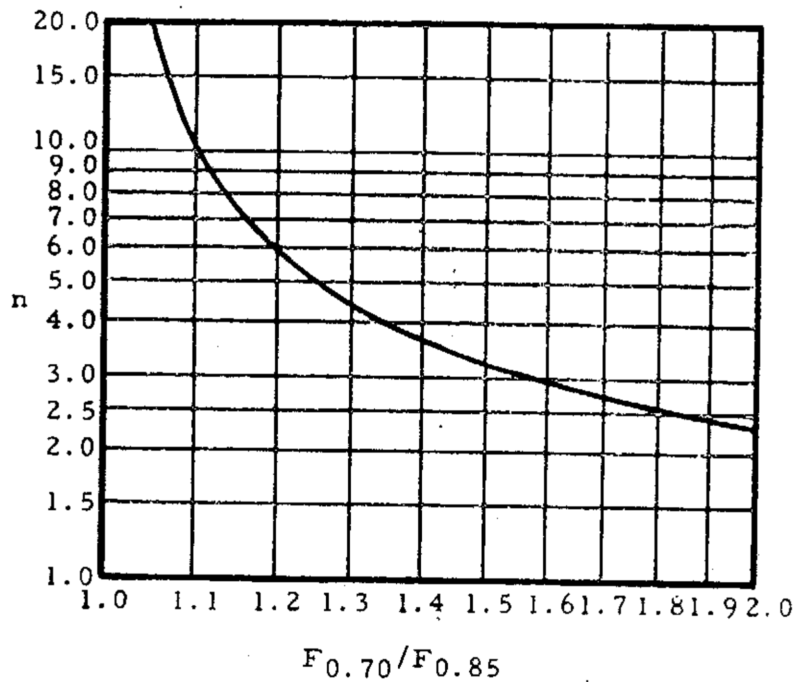
$$F_{cr} = \frac{k_c}{12} \pi^2 E/(L/t)^2$$

(b) Hinged flanges

FIGURE 7.7 COMPRESSIVE BUCKLING COEFFICIENTS OF PLATE COLUMNS AND FLANGES AS A FUNCTION OF POISSON'S RATIO



(a) Significant stress quantities on a typical stress-strain curve



(b) Dependence of shape factor on ratio $F_{0.70}/F_{0.85}$

$$n = 1 + \log_e (17/7)/\log_e(F_{0.70}/F_{0.85})$$

FIGURE 7.3 CHARACTERISTICS OF STRESS-STRAIN CURVES FOR STRUCTURAL ALLOYS DEPICTING QUANTITIES USED IN THE THREE PARAMETER METHOD

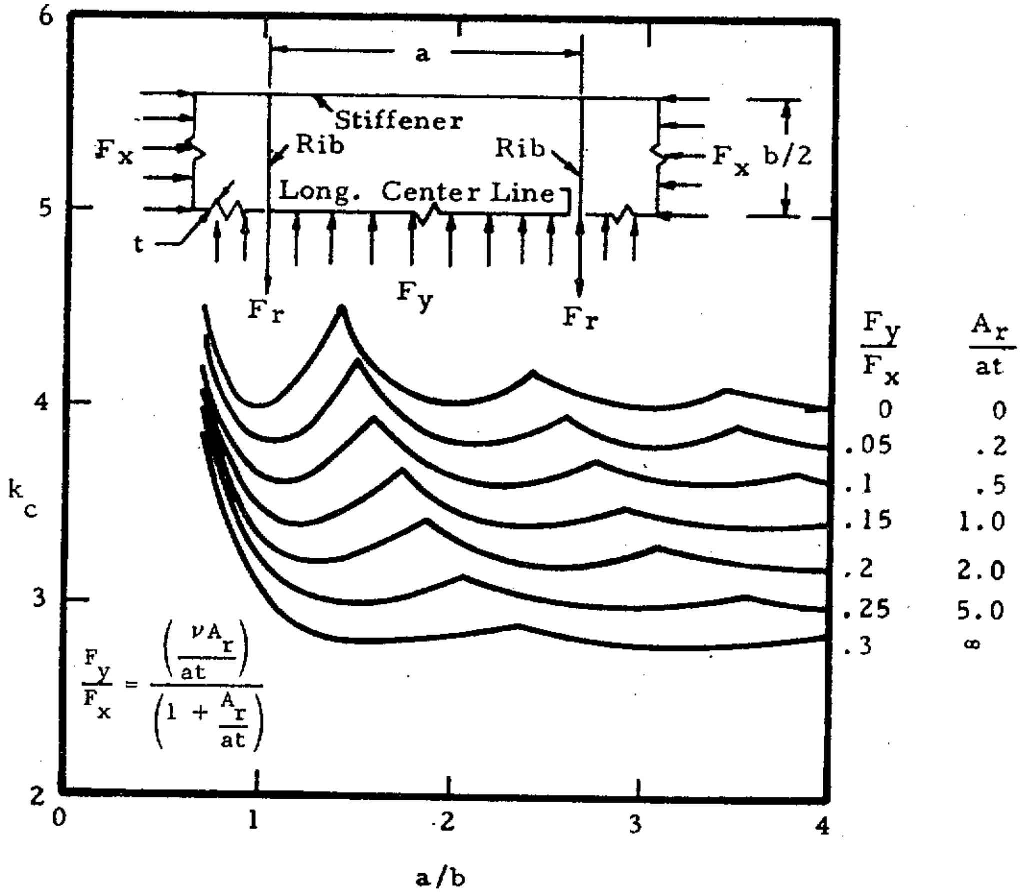


FIGURE 7.9 COMPRESSIVE BUCKLING COEFFICIENT OF FLAT PLATES RESTRAINED AGAINST LATERAL EXPANSION (Poisson's ratio = 0.3)

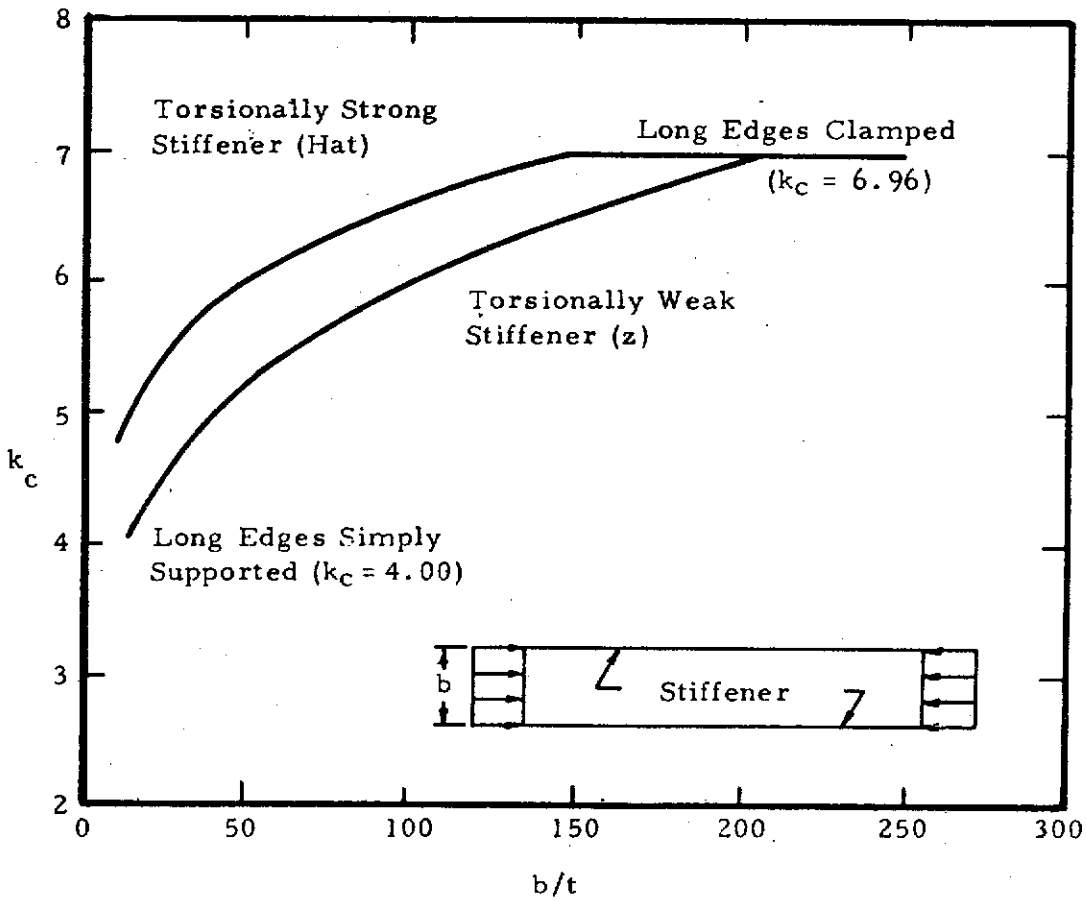


FIGURE 7.10 COMPRESSIVE BUCKLING COEFFICIENT FOR LONG RECTANGULAR STIFFENED PANELS AS A FUNCTION OF b/t AND STIFFENER TORSIONAL RIGIDITY

7.3.2 Buckling of Stiffened Flat Plates in Axial Compression

The treatment of stiffened flat plates is the same as that of unstiffened plates except that the buckling coefficient, k, is now also a function of the stiffener geometry. Equation (7-1) is the basic analysis tool for the critical buckling stress.

As the stiffener design is a part of the total design, Figures 7-11 and 7-12 present buckling coefficients for various types of stiffeners. The applicable critical buckling equation is indicated on each figure.

A plasticity reduction factor η applicable to channel and Z- section stiffeners is given by Equation (7-8), which is taken from Reference 7.

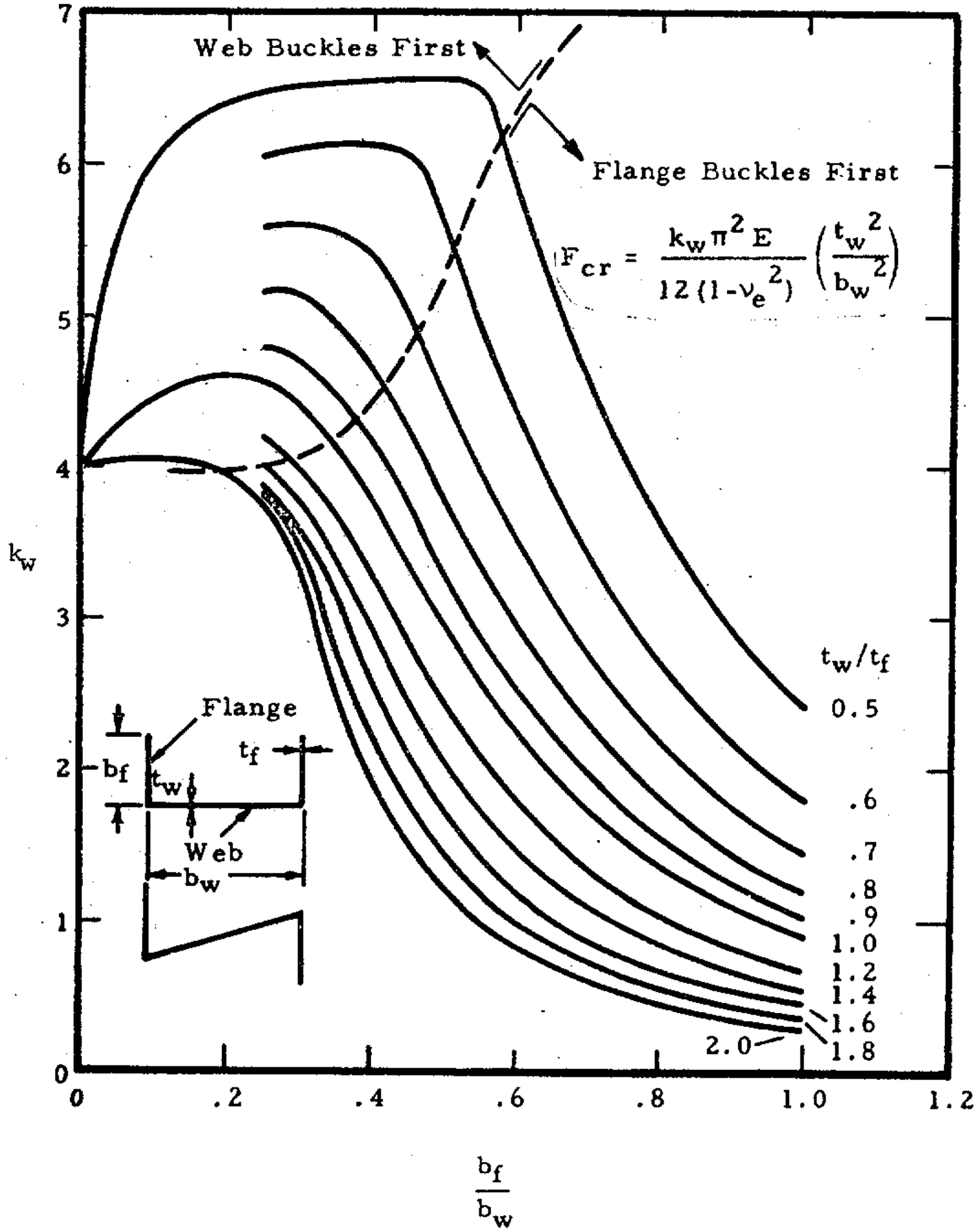
$$\eta = .95 \left(\frac{E_s}{E}\right)\left(\frac{1-\nu_e^2}{1-\nu^2}\right) \quad (7\text{-}8)$$

For other structural elements such as hat and rectangular sections, no specific plasticity correction factor has been established. However, Reference 7 recommends using the correction for a long clamped flange.

Valuesfor the buckling coefficient, k_c , for axially stiffened, infinitely wide plates are given in Figure 7-13.

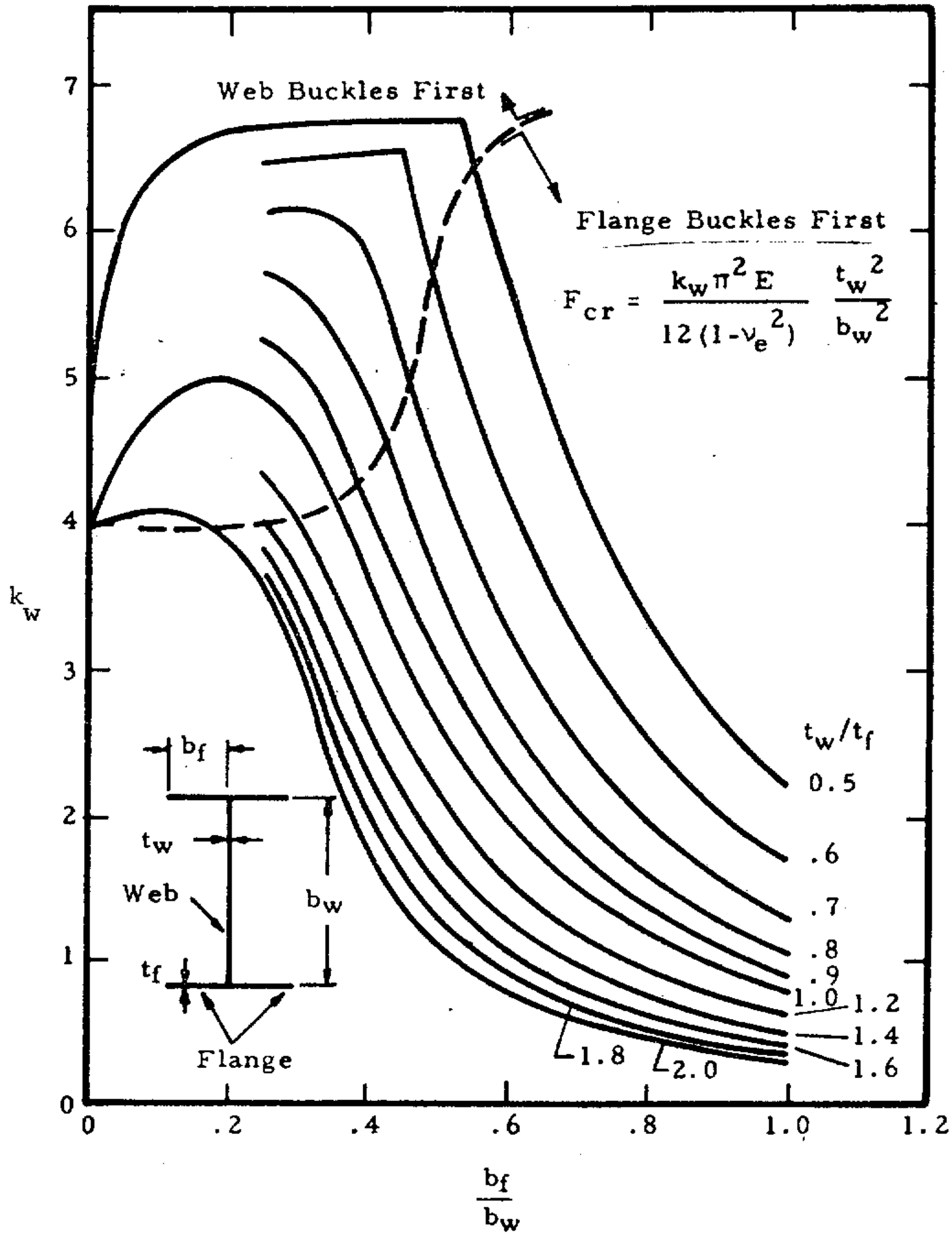
Figure 7-14 presents curves for finding a value for k for plates with transverse stiffeners. It is noted that in these plots, the torsional rigidity, GJ, of the stiffener itself is used, whereas in most data for longitudinal stiffeners, no torsional properties of the stiffeners are included.

In this brief section on buckling, an attempt has been made to present data that is most often used for routine analysis. Should the user require a more comprehensive treatment of buckling, References 2, 11, and 12 are excellent sources of additional data.



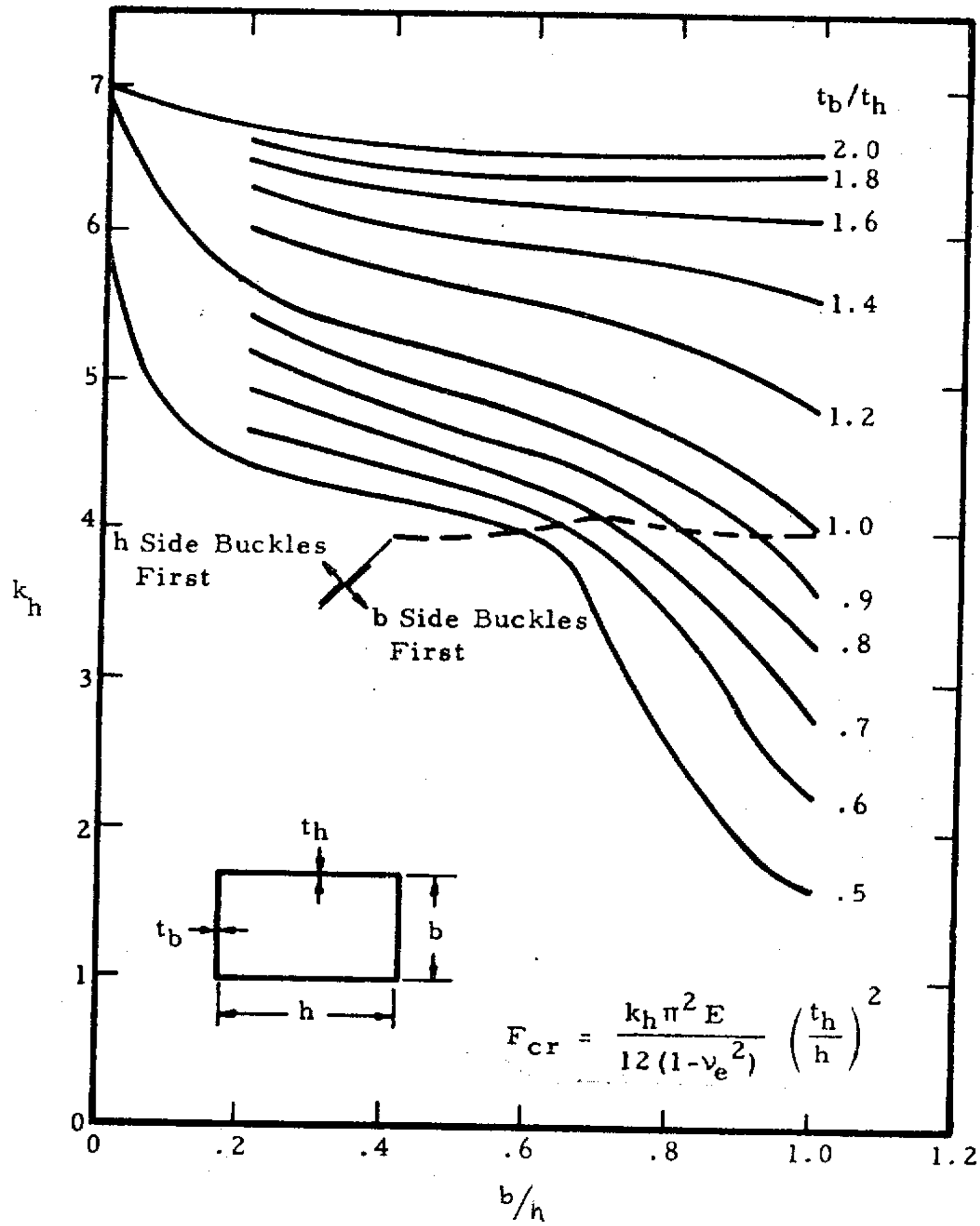
(a) Channel and Z Section Stiffeners

FIGURE 7.11 BUCKLING COEFFICIENTS FOR STIFFENERS



(b) H Section Stiffeners

FIGURE 7.11 (CONT'D) BUCKLING COEFFICIENTS FOR STIFFENERS



(c) Rectangular Tube Section Stiffeners

FIGURE 7.11 (CONT'D) BUCKLING COEFFICIENTS FOR STIFFENERS

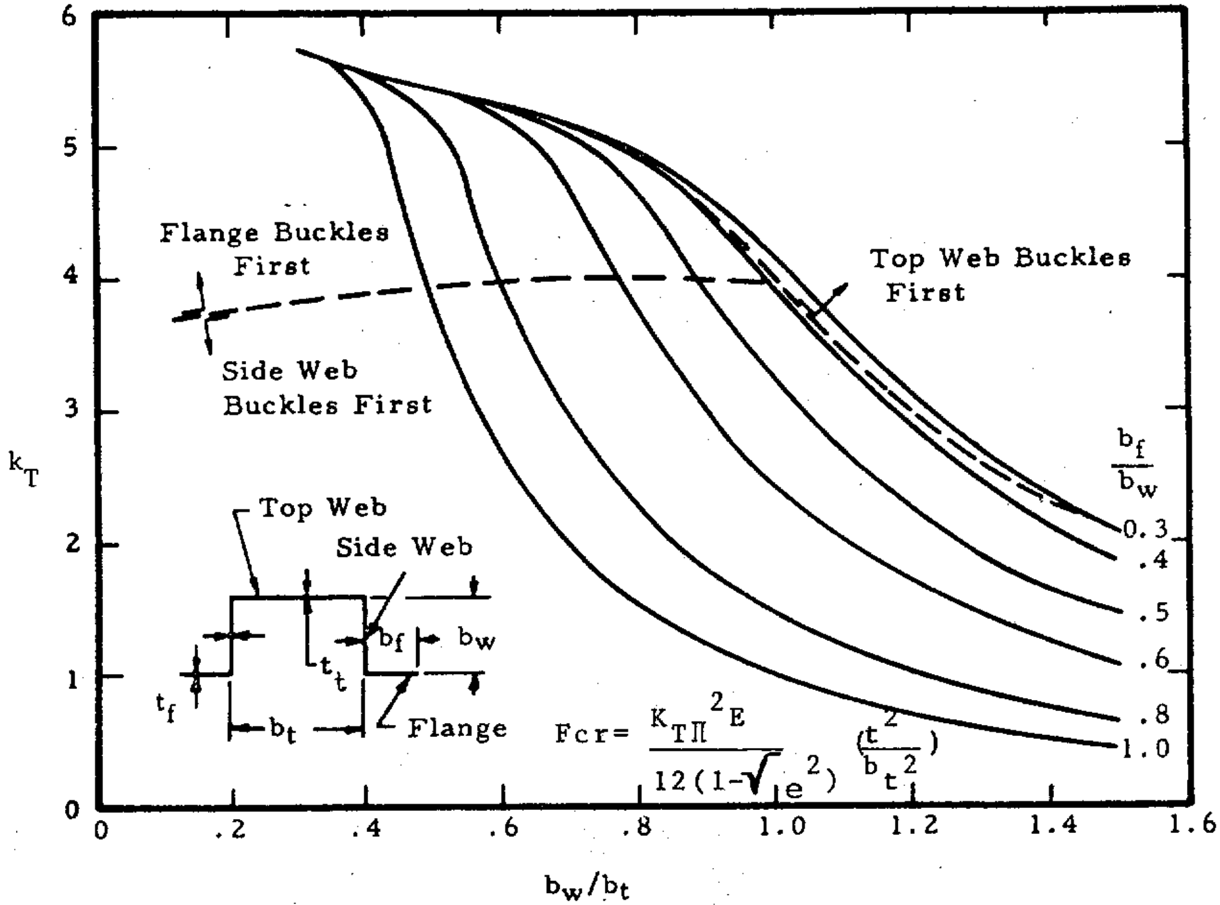
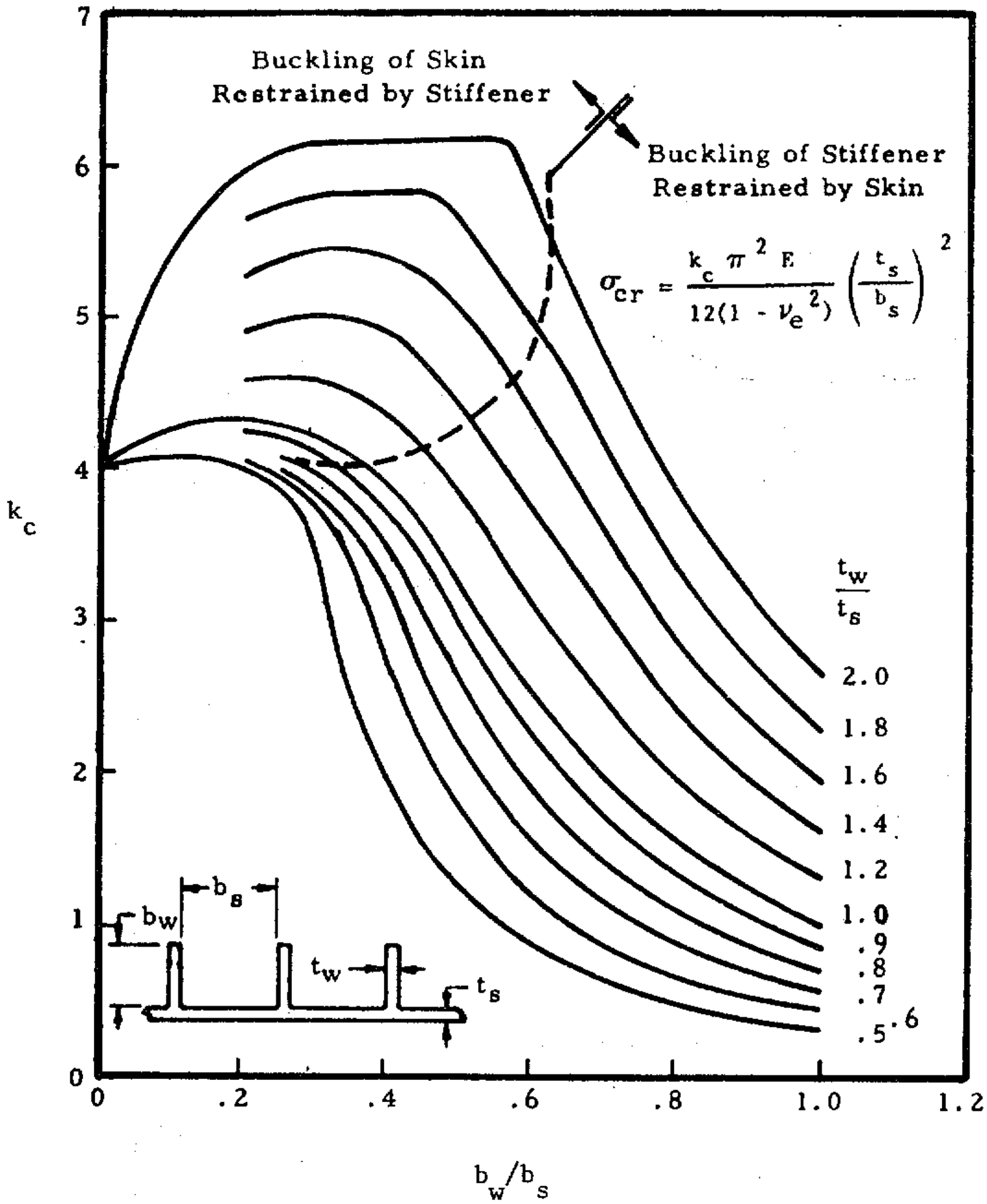
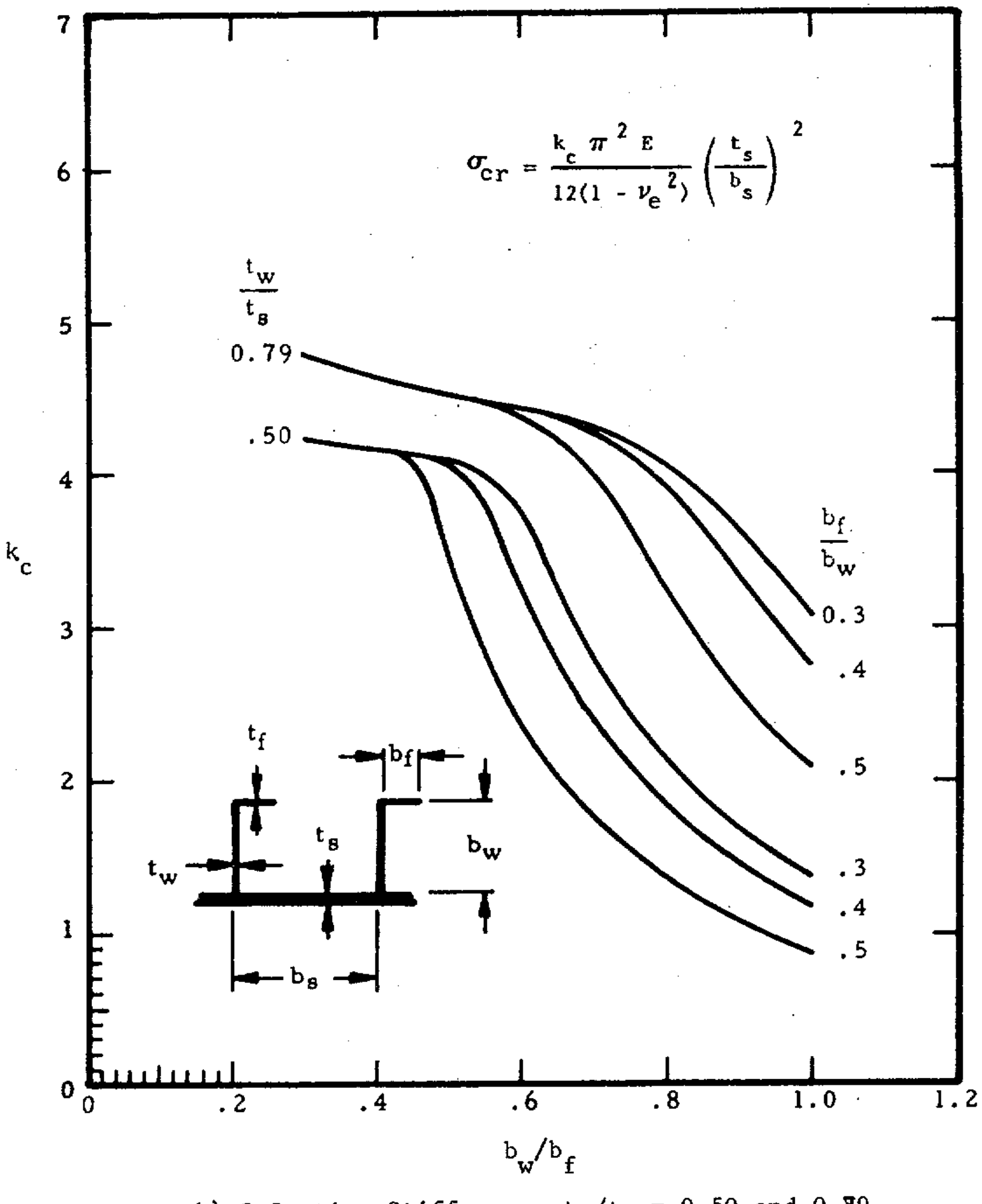


FIGURE 7.12 BUCKLING STRESS FOR HAT SECTION STIFFENERS ($t=t_f=t_w=t_t$)



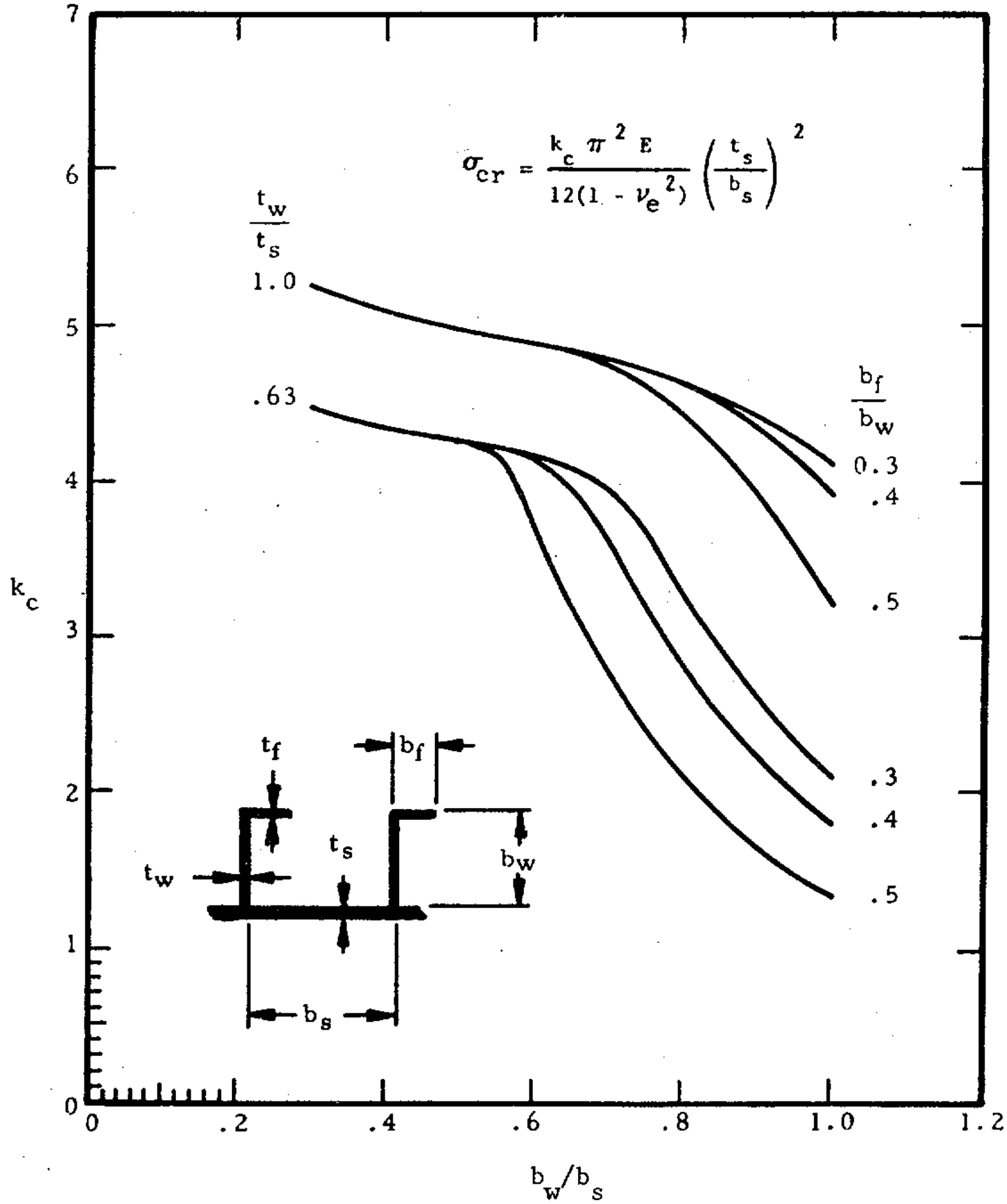
a) Web Stiffeners. $0.5 < t_w/t_s < 2.0$

FIGURE 7.13 COMPRESSIVE LOCAL BUCKLING COEFFICIENTS FOR INFINITELY WIDE IDEALIZED STIFFENED FLAT PLATES



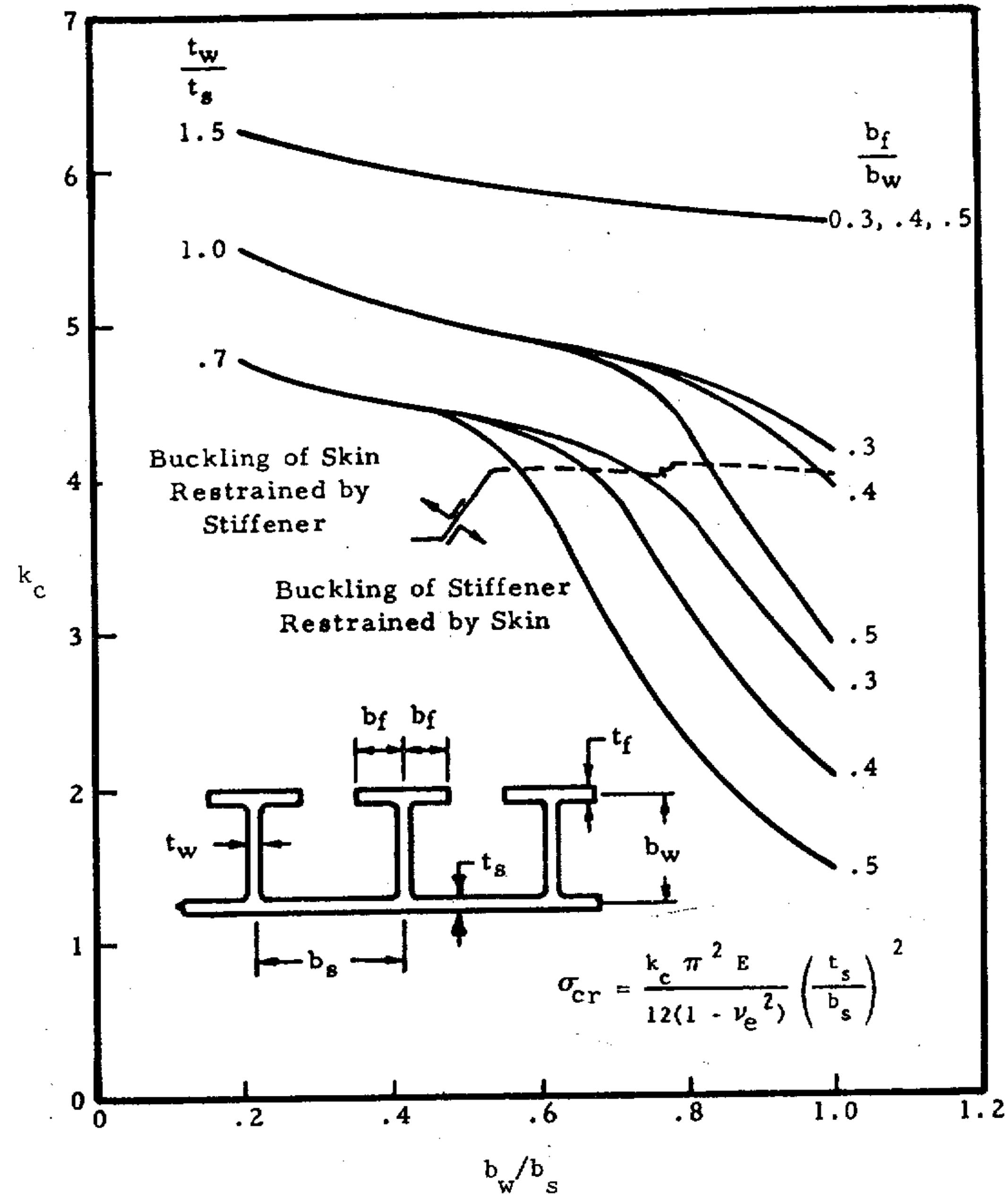
b) Z-Section Stiffeners. t_w/t_s = 0.50 and 0.79

FIGURE 7.13 (CONT'D) COMPRESSIVE LOCAL BUCKLING COEFFICIENTS FOR INFINITELY WIDE IDEALIZED STIFFENED FLAT PLATES



c) Z-Section Stiffeners. $t_w/t_s = 0.63$ and 1.0

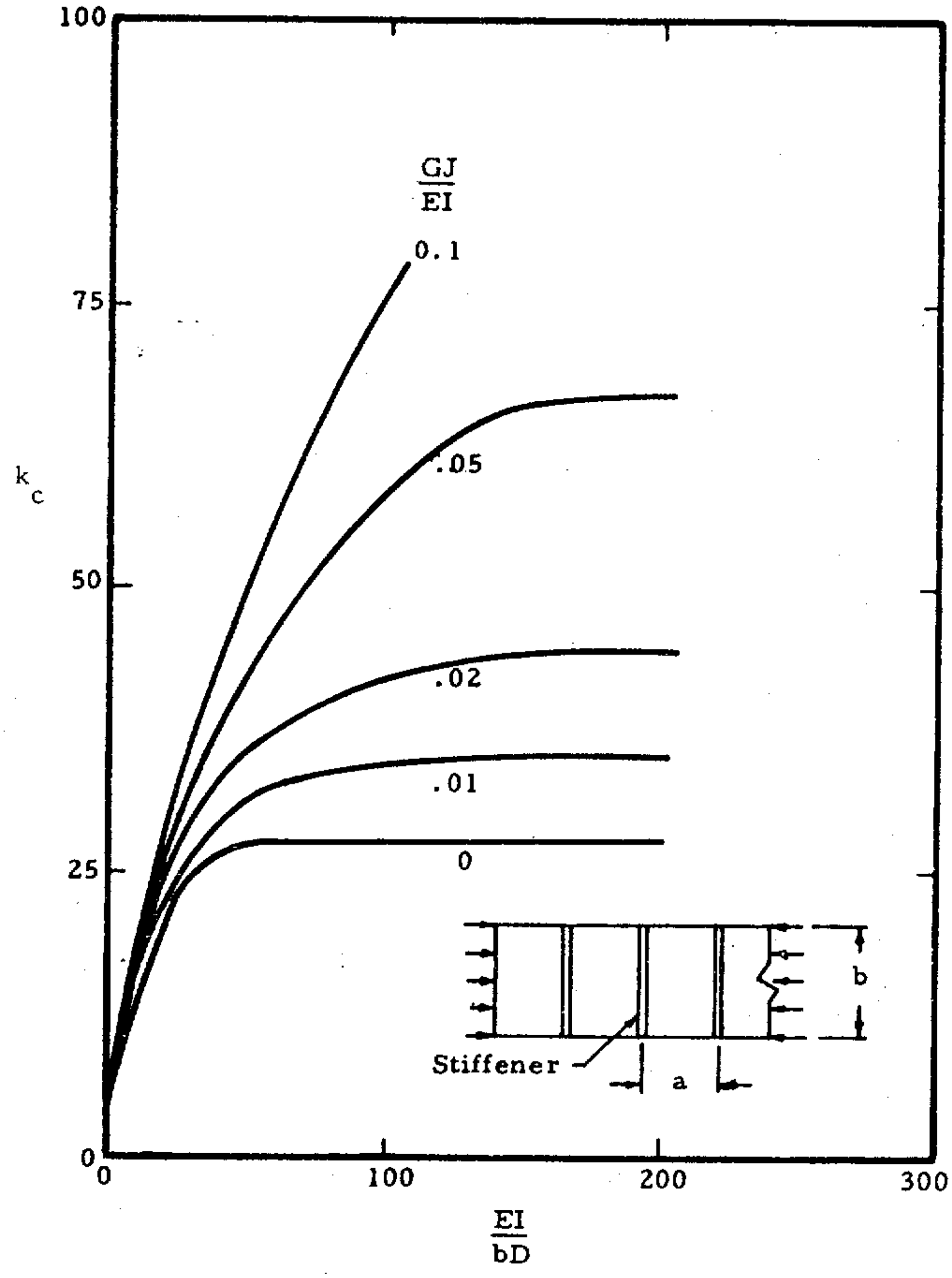
FIGURE 7.13 (CONT'D) COMPRESSIVE LOCAL BUCKLING COEFFICIENTS FOR INFINITELY WIDE IDEALIZED STIFFENED FLAT PLATES



$$\sigma_{cr} = \frac{k_c \pi^2 E}{12(1 - \nu_e^2)} \left(\frac{t_s}{b_s}\right)^2$$

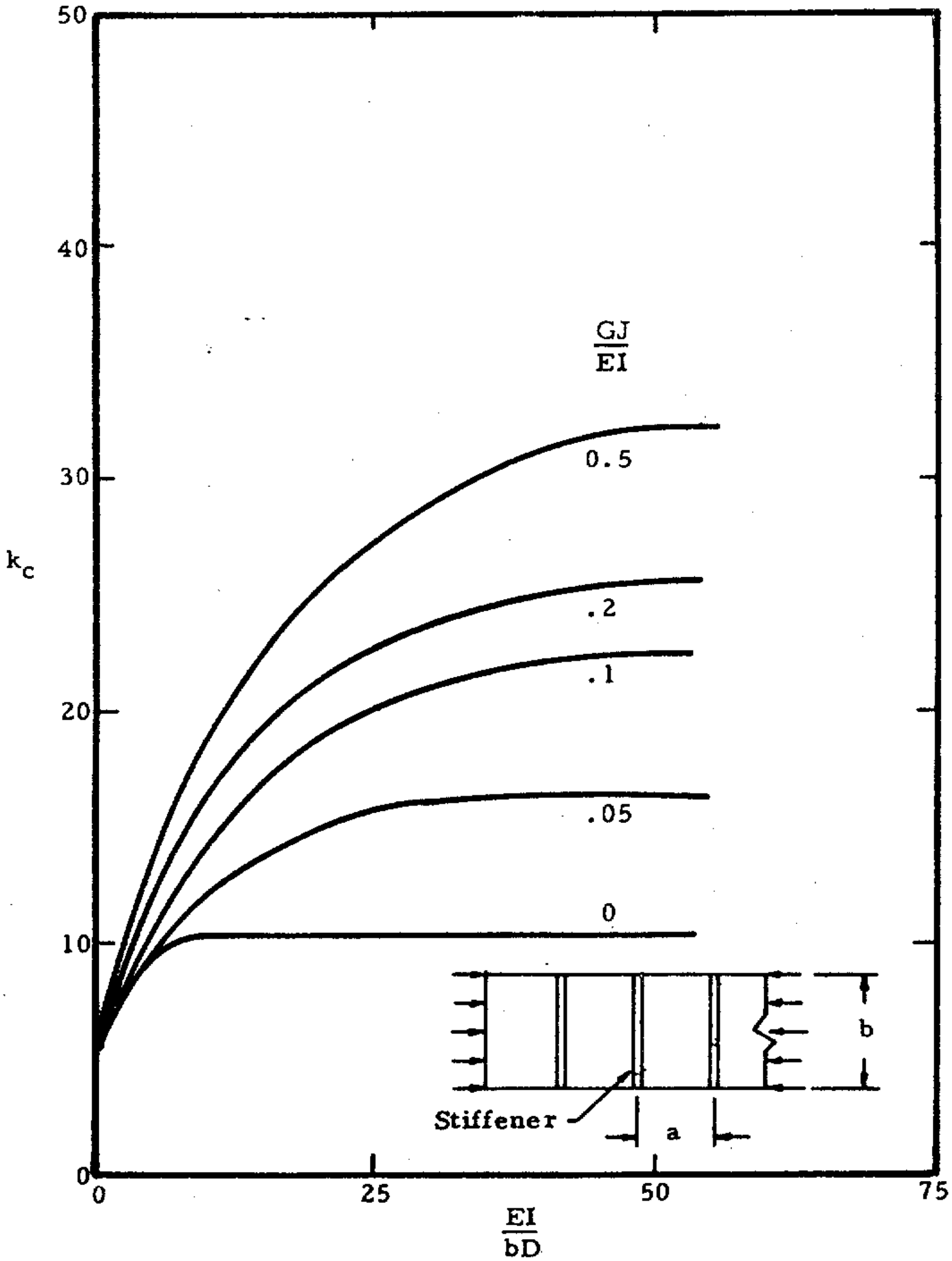
d) T-Section Stiffeners. $\frac{t_w}{t_f} = 1.0$; $\frac{b_f}{t_f} > 10$; $\frac{b_w}{b_s} > 0.25$

FIGURE 7.13 (CONT'D) COMPRESSIVE LOCAL BUCKLING COEFFICIENTS FOR INFINITELY WIDE IDEALIZED STIFFENED FLAT PLATES



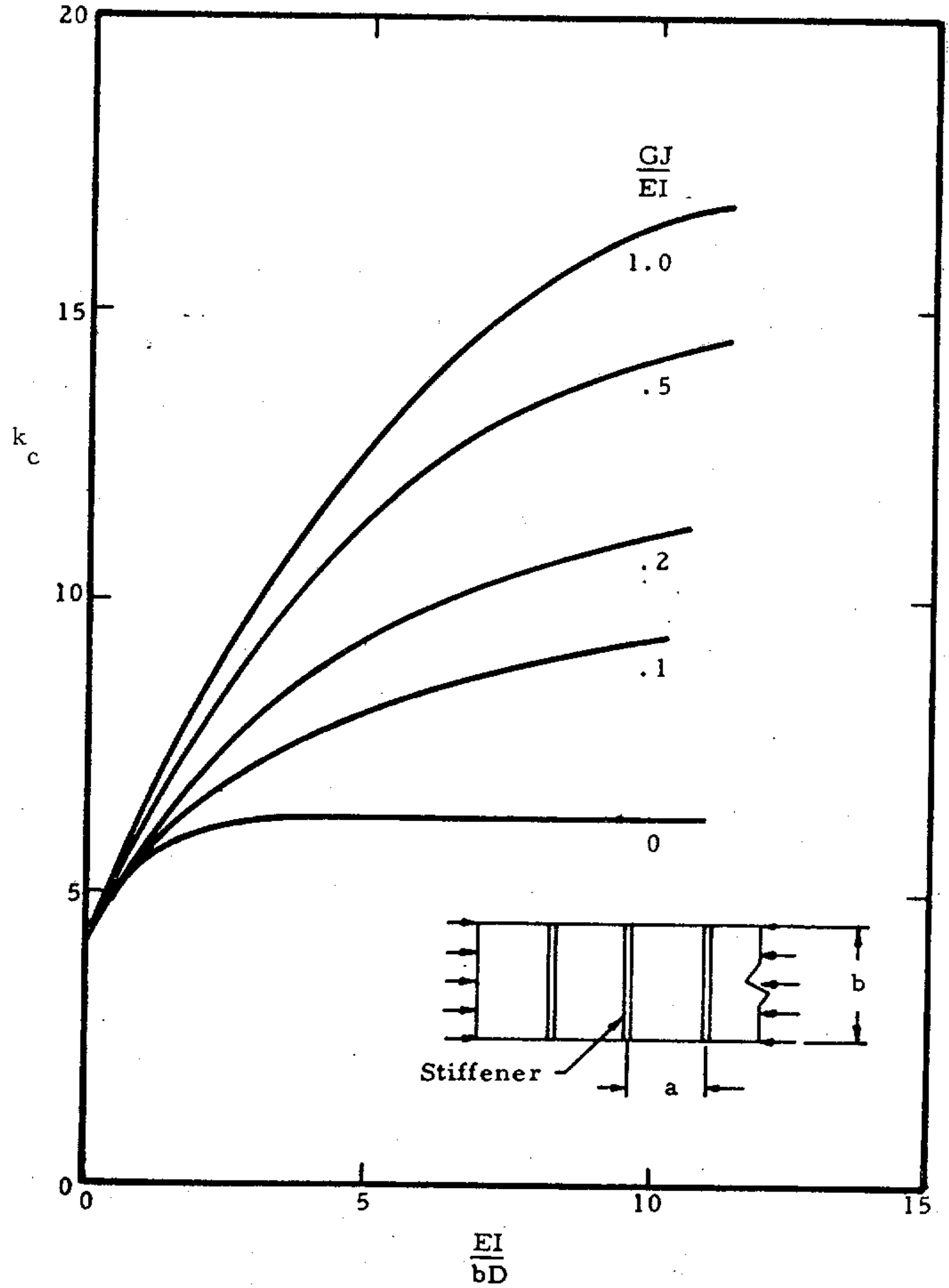
(a) a/b = 0.20

FIGURE 7.14 LONGITUDINAL COMPRESSIVE BUCKLING COEFFICIENTS FOR SIMPLY SUPPORTED PLATES WITH TRANSVERSE STIFFENERS



(b) a/b = 0.35

FIGURE 7.14 (CONT'D) LONGITUDINAL COMPRESSIVE BUCKLING COEFFICIENTS FOR SIMPLY SUPPORTED PLATES WITH TRANSVERSE STIFFENERS



(c) a/b = 0.50

FIGURE 7.14 (CONT'D) LONGITUDINAL COMPRESSIVE BUCKLING COEFFICIENTS FOR SIMPLY SUPPORTED PLATES WITH TRANSVERSE STIFFENERS

7.3.3 Crippling Failure of Flat Stiffened Plates in Compression

For stiffened plates having slenderness ratios $L'/\rho \leq 20$, considered to be short plates, the failure mode is crippling rather than buckling when loaded in compression. The crippling strength of individual stiffening elements is considered in Section 10. The crippling strength of panels stiffened by angle-type elements is given by Equation (7-9).

$$\frac{\overline{F}_f}{F_{cy}} = \beta_g \left[\frac{g t_w t_s}{A} \left(\frac{\overline{\eta} E}{F_{cy}} \right)^{\frac{1}{2}} \right]^{0.85} \qquad (7\text{-}9)$$

For more complex stiffeners such as Y sections, the relation of Equation (7-10) is used to find a weighted value of t_w.

$$\overline{t_w} = \frac{\Sigma a_i t_i}{\Sigma a_i} \qquad (7\text{-}10)$$

where a_i and t_i are the length and thickness of the cross-sectional elements of the stiffener. Figure 7-15 shows the method of determining the value of g used in Equation (7-9) based on the number of cuts and flanges of the stiffened panels. Figure 7-16 gives the coefficient β_g for various stiffening elements.

If the skin material is different from the stiffener material, a weighted value of F_{cy} given by Equation (7-11) should be used. Here $\overline{t}$ is the effective thickness of the stiffened panel.

$$\overline{F}_{cy} = \frac{F_{cy_s} + F_{cy_w}\left[(\overline{t}/t_s) - 1\right]}{(\overline{t}/t_s)} \qquad (7\text{-}11)$$

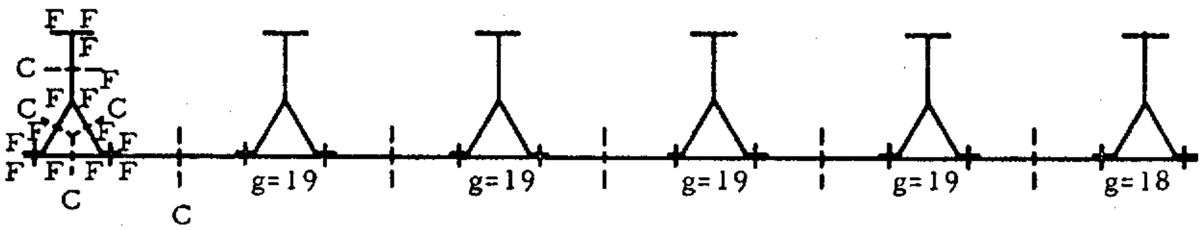
The above relations assume the stiffener-skin unit to be formed monolithically; that is, the stiffener is an integral part of the skin. For riveted construction, the failure between the rivets must be considered. The interrivet buckling stress is determined as to plate buckling stress, and is given by Equation (7-12).

$$F_i = \left(\frac{\epsilon \pi^2 \eta \overline{\eta} E}{12(1-\nu^2)} \right) \left(\frac{t_s}{p} \right)^2 \qquad (7\text{-}12)$$

Values of ϵ, the edge fixity, are given in Table 7-2.

After the interrivet buckling occurs, the resultant failure stress of the panel is given by Equation (7-13).

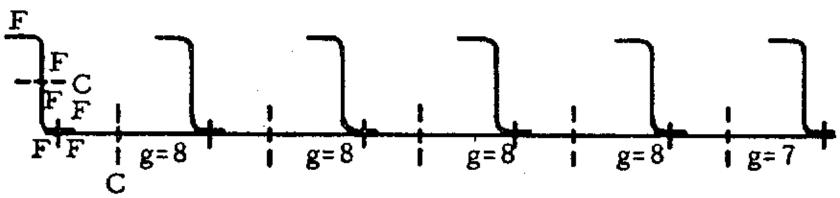
$$\overline{F}_{f_r} = \frac{F_i (2b_{ei} t_s) + \overline{F}_{fst} A_{st}}{(2b_{ei} t_s) + A_{st}} \qquad (7\text{-}13)$$



5 cuts
14 flanges
19 = g

Average g = 18.85

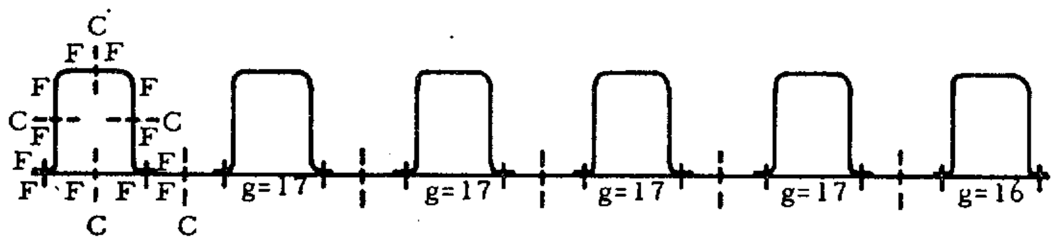
(a) Y-stiffened panel



2 cuts
6 flanges
8 = g

Average g = 7.83

(b) Z-stiffened panel



Average g = 16.83

5 cuts
12 flanges
17 = g

(c) Hat-stiffened panel

FIGURE 7.15 METHOD OF CUTTING STIFFENED PANELS TO DETERMINE g

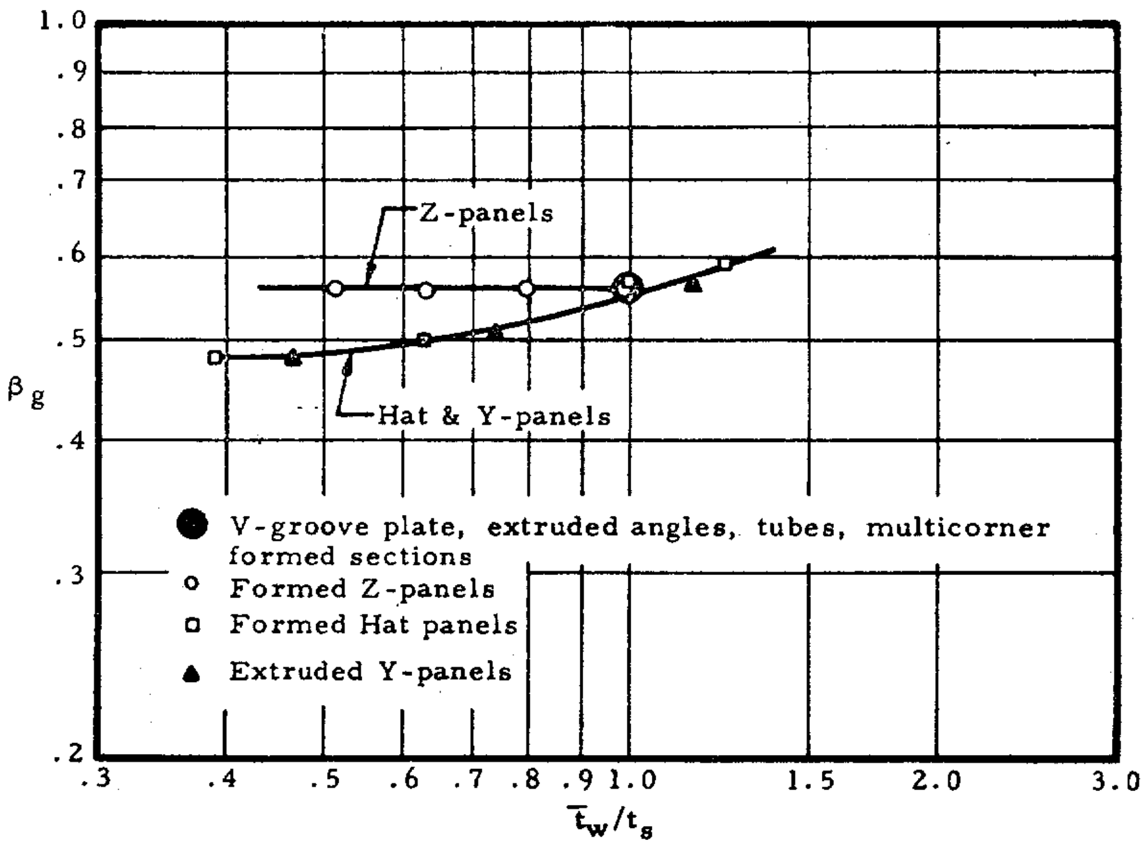


FIGURE 7.16 CRIPPLING COEFFICIENTS FOR ANGLE-TYPE ELEMENTS

Fastener Type	(Fixity-Coefficient) ϵ
Flathead rivet	4
Spotwelds	3.5
Brazier-head rivet	3
Countersunk rivet	1

TABLE 7.2 END FIXITY COEFFICIENTS FOR ANGLE-TYPE ELEMENTS

Stringer Stability	Panel Strength
$\overline{F}_{f_{st}} \geq \overline{F}_w$ – stable	$\overline{F}_{fr} = \overline{F}_w$
$\overline{F}_{f_{st}} < \overline{F}_w$ – unstable	$\overline{F}_{fr} = \dfrac{\overline{F}_w b_s t_s + \overline{F}_{f_{st}} A_{st}}{b_s t_s + A_{st}}$

TABLE 7.3 RIVETED PANEL STRENGTH DETERMINATION

Here the value b_{ei} is the effective width of skin corresponding to the interrivet buckling stress F_i . The failure stress of short riveted panels by wrinkling can be determined. The following quantities are used:

$\overline{F}_{fst}$ crippling strength of stringer alone (Compressive Crippling Section 10)

$\overline{F}_w$ wrinkling strength of the skin

$\overline{F}_f$ crippling strength of a similar monolithic panel

$\overline{F}_{fr}$ strength of the riveted panel

The wrinkling strength of the skin can be determined from Equation (7-14) and Figure 7-17. Here f is the effective rivet offset distance given in Figure 7-18. This was obtained for aluminum rivets having a diameter greater than 90% of the skin thickness.

$$F_w = \left(\frac{k_w \pi^2 \eta \overline{\eta} E}{12(1-\nu^2)} \right) \left(\frac{t_s}{b_s} \right)^2 \tag{7-14}$$

Now, based on the stringer stability, the strength of the panel can be calculated. Table 7-3 shows the various possibilities and solutions.

It is noted that in no case should $\overline{F}_{fr} > \overline{F}_f$. Thus, the lower of these two values should be used.

The use of the coefficient k_w is based upon aluminum alloy data for other materials. The procedure is to use Equation (7-15) for the panel crippling strength.

$$\frac{F_{fr}}{F_{cy}} = 17.9 \left(\frac{t_w}{f} \right)^{4/3} \left(\frac{t_w}{b_w} \right)^{1/6} \left[\frac{t_s}{b_s} \left(\frac{\eta E}{F_{cy}} \right) \right]^{\frac{1}{2}} \tag{7-15}$$

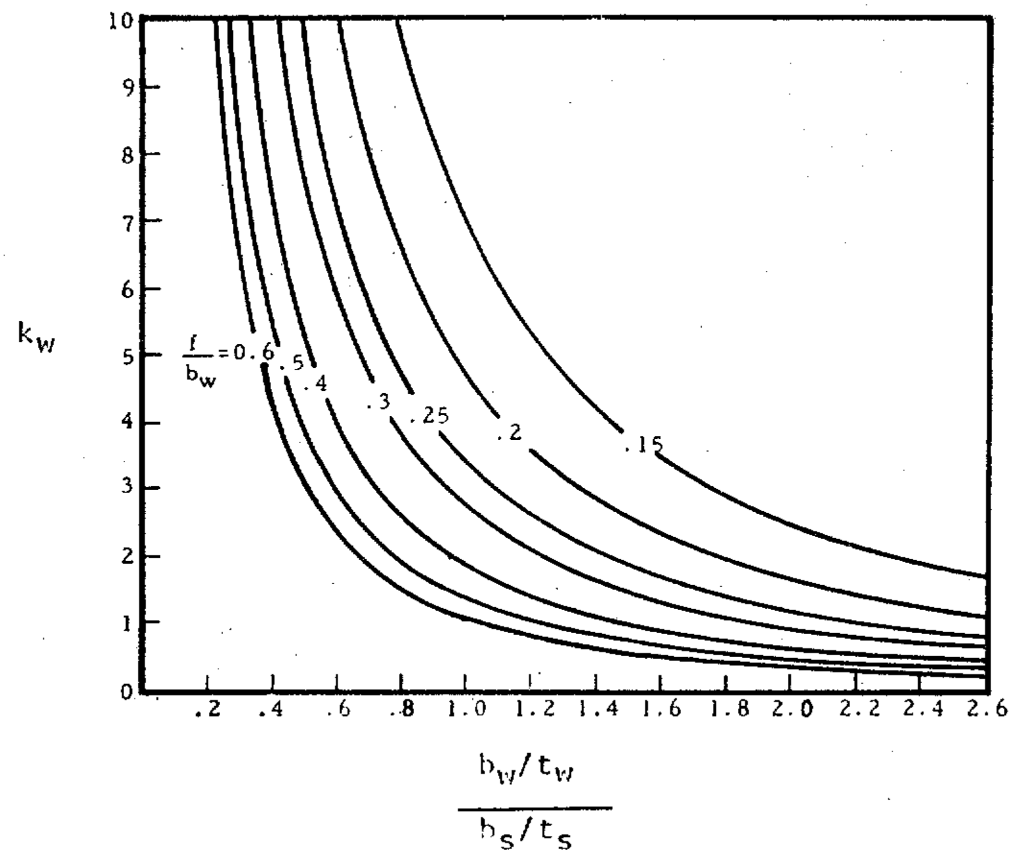


FIGURE 7.17 EXPERIMENTALLY DETERMINED COEFFICIENTS FOR FAILURE IN WRINKLING MODE

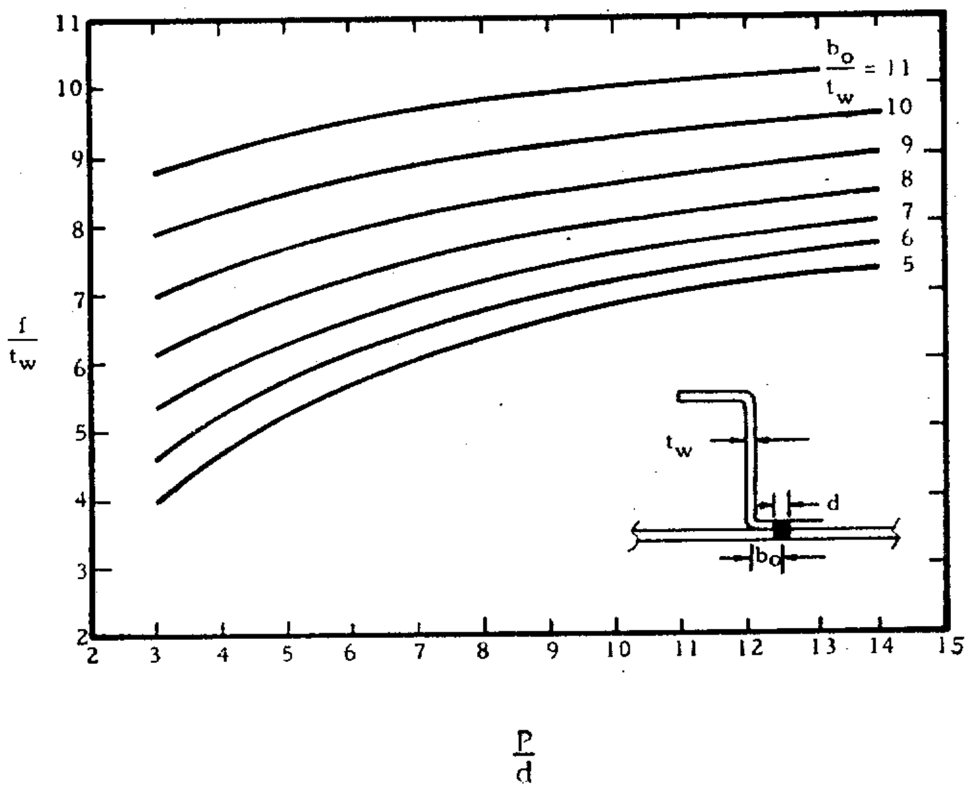


FIGURE 7.18 EXPERIMENTALLY DETERMINED VALUES OF EFFECTIVE RIVET OFFSET (P = Rivet Spacing)

7.4 Bending of Flat Plates

The bending of a flat plate can be caused by either in-plane or normal loads. In the former, the plate is subject to various buckling modes depending primarily on boundary conditions and aspect ratio. Here the critical parameter is the magnitude of stress at which buckling occurs, since redistribution of load also starts at that time. Thus, as with axial loads, it is important to know when local buckling due to bending may be expected.

An exact analysis for a flat plate loaded transversely involves a very complex mathematical treatment. A plate can be considered as a two-dimensional counterpart of a beam, except that plates bend in all planes normal to the plate, whereas a beam bends in one plane only. Also, plates exhibit varied behavior depending on thickness and have therefore been classified into four types: thick, medium-thick, thin, and membranes or diaphragms.

Since most aircraft applications of transversly loaded plates involve either medium-thick plates or membranes, only the two types are included herein. Further information is available from many sources.

7.4.1 Unstiffened Flat Plates, In-Plane Bending

The general buckling relation for plates subjected to in-plane bending is given by Equation 7-16, which has the same form as Equation 7-1. The only difference is in the coefficient, k_b.

$$F_b = \eta \, \overline{\eta} \, \frac{k_b \, \pi^2 E}{12(1-\nu_e^2)} \left(\frac{t}{b}\right)^2 \qquad (7\text{-}16)$$

Values of bending coefficient, k_b , are given in Figure 7-19 for various edge restraints and the number of buckles versus λ/b, the buckle wave length ratio, and in Figure 7-20 for various edge restraints versus the ratio a/b.

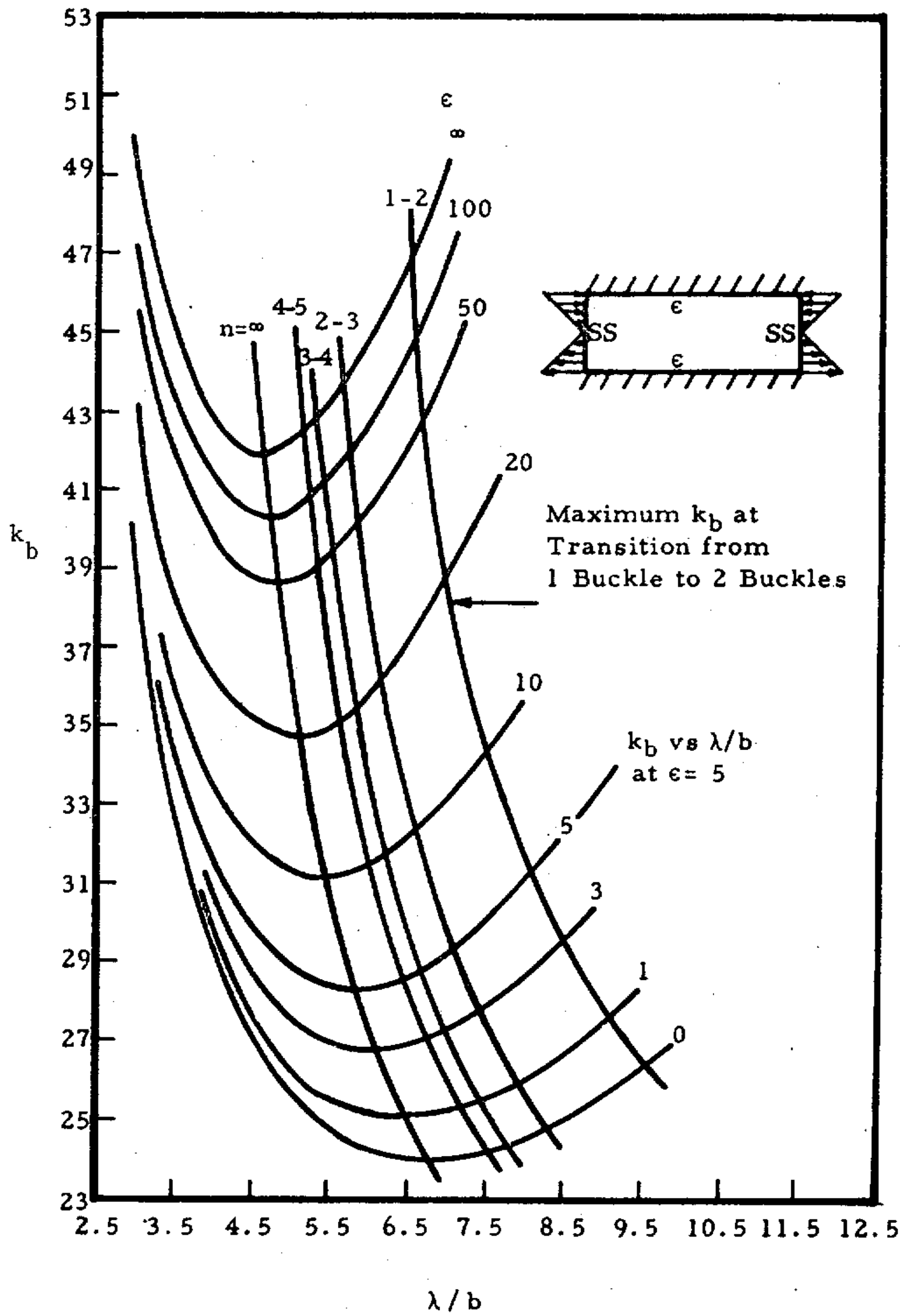


FIGURE 7.19 BENDING-BUCKLING COEFFICIENTS OF PLATES AS A FUNCTION OF λ/b FOR VARIOUS AMOUNTS OF ROTATIONAL RESTRAINT

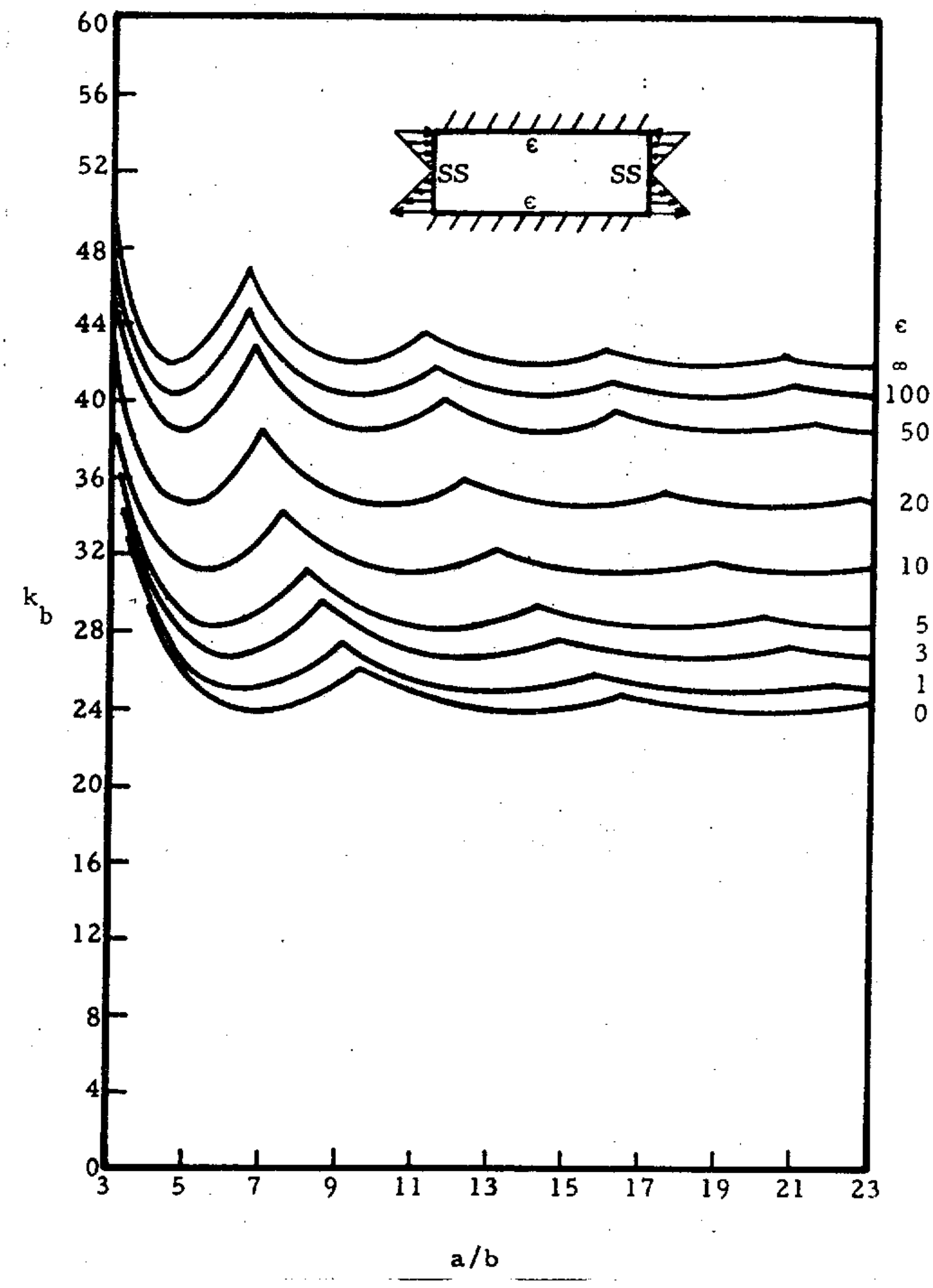


FIGURE 7.20 BENDING-BUCKLING COEFFICIENTS OF PLATES AS A FUNCTION OF a/b FOR VARIOUS AMOUNTS OF EDGE ROTATIONAL RESTRAINT

7.4.2 Unstiffened Flat Plates, Transverse Bending

The data presented in this section are predicated on the following assumptions:

1) The plates are flat, of uniform thickness, and of homogeneous isotropic material.

2) The plate width, b, is $\geq 4t$ and the deflection, y, is $\leq .5t$.

3) All forces (loads and reactions) are normal to the plane of the plate.

4) The plate is nowhere stressed beyond the elastic limit.

5) Poisson's ratio = 0.3; however, no significant error is introduced if these coefficients and formulas are used for materials with other values.

For unstiffened flat plates with various types of loading, the maximum stress and maximum deflection can be represented by simple relations by the use of a series of constants which depend upon the plate geometry and loading. Tables 7-4 through 7-9 present loading coefficients for use with Equations (7-17) through (7-22).

Equations (7-17) (a) and (b) pertain to rectangular, elliptical and triangular plates. Loading coefficients are presented in Tables 7.4, 7.7 and 7.8 respectively.

$$\text{(a)} \quad y = \frac{K w L^4}{Et^3} \qquad \text{(b)} \quad F = \frac{K_1 w L^2}{t^2} \qquad (7\text{-}17)$$

Equation (7-18) (a) and (b) pertain to corner and edge forces for simply supported rectangular plates. Loading coefficients are presented in Table 7.5.

$$\text{(a)} \quad R = K_1 wab \qquad \text{(b)} \quad V = KwL \qquad (7\text{-}18)$$

Equations (7-19) (a) and (b) pertain to partially loaded rectangular plates with supported edges. Loading coefficients are presented in Table 7-6.

$$\text{(a)} \quad y = \frac{KwL^3}{Et^3} \qquad \text{(b)} \quad F = \frac{K_1 W}{t^2} \qquad (7\text{-}19)$$

Equations (7-20), (a), (b), and (c) pertain to circular plates. Loading coefficients are presented in Table 7.9.

$$\text{(a)} \quad y = \frac{KWa^2}{Et^3} \qquad \text{(b)} \quad F = \frac{K_1 W}{t^2} \qquad \text{(c)} \quad \Theta = \frac{K_2 Wa}{Et^3} \qquad (7\text{-}20)$$

Equation (7-21) (a), (b), and (c) pertain to circular plates with end moments. Loading coefficients are presented in Table 7.9.

$$\text{(a)}\quad y = \frac{KMa^2}{Et^3} \qquad \text{(b)}\quad F = \frac{K_1 M}{t^2} \qquad \text{(c)}\quad \Theta = \frac{K_2 Ma}{Et^3} \qquad (7\text{-}21)$$

Equations (7-22), (a) and (b) apply to trunnion-loaded plates only. Loading coefficients are presented in Table 7.9.

$$\text{(a)}\quad F = \frac{K_1 M}{at^2} \qquad \text{(b)}\quad \Theta = \frac{M}{K_2\ Et^3} \qquad (7\text{-}22)$$

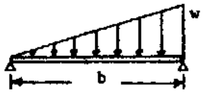
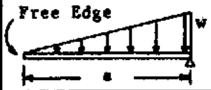
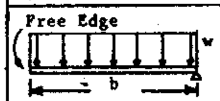
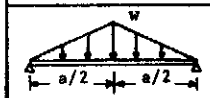
MANNER OF LOADING	a/b	1.0	1.2	1.4	1.6	1.8	2.0	2.5	3.0	3.5	4.0	Location of Maximum Stress and Deflection
All edges supported uniform load over entire surface. L = b	K_1	0.287	0.376	0.453	0.517	0.569	0.610	0.661	0.713	0.727	0.741	at the center
	K	0.044	0.062	0.077	0.091	0.102	0.111	0.122	0.134	0.137	0.140	at the center
 All edges supported, distributed load varying linearly along length. L = b	K_1	0.16	0.21	0.25	0.28	0.31	0.34	0.38	0.43	0.47	0.49	
	K	0.022	0.030	0.040	0.048	0.053	0.060	0.070	0.078	0.086	0.091	
 All edges supported, distributed load varying linearly along breadth. L = b	K_1	0.16	0.21	0.25	0.28	0.31	0.32	0.35	0.37	0.38	0.38	
	K	0.022	0.031	0.038	0.045	0.051	0.056	0.063	0.067	0.069	0.070	
All edges fixed, uniform load over entire surface. L = b	K_1	0.31	0.38	0.44	0.47	0.49	0.50	0.50	0.50	0.50	0.50	center of long edges
	K	0.014	0.019	0.023	0.025	0.027	0.028	0.028	0.028	0.028	0.028	at the center
All edges fixed, uniform load over small concentric circular area of radius r_o. $W = \pi r_o^2 w$. Max $s = K_1 \frac{W}{t^2}$, Max $y = K \frac{Wb^2}{Et^3}$	K_1	0.754	0.894	0.962	0.991	1.000	1.004	1.005	1.006	1.007	1.008	at the center
	K	0.061	0.071	0.075	0.078	0.079	0.079	0.079	0.079	0.079	0.079	at the center
Long edges fixed, short edges supported, uniform load over entire surface. L = b	K_1	0.418	0.463	0.486	0.497	0.497	0.497	0.498	0.498	0.499	0.500	center of long edges
	K	0.021	0.024	0.026	0.027	0.028	0.028	0.028	0.028	0.028	0.028	at the center
Short edges fixed, long edges supported, uniform load over entire surface. L = b	K_1	0.418	0.521	0.599	0.654	0.691	0.715	0.724	0.733	0.742	0.750	center of short edges
	K	0.021	0.035	0.050	0.066	0.0800	0.092	—	—	—	—	at the center
One long edge fixed, other free, short edges supported, uniform load over entire surface. L = b	K_1	0.714	0.973	1.232	1.482	1.693	1.914	2.285	2.568	2.780	3.000	center of fixed edge
	K	0.123	0.230	0.330	0.435	0.535	0.636	0.860	1.027	1.196	1.365	
One long edge fixed, other three edges supported, uniform load over entire surface. L = b	K_1	0.50	0.58	0.63	0.68	0.71	0.74	0.74	0.74	0.75	0.75	center of fixed edge
	K	0.030	0.038	0.042	0.047	0.050	0.054	0.056	0.057	0.058	0.058	at the center
One short edge fixed, other three edges supported, uniform load over entire surface. L = b	K_1	0.50	0.58	0.63	0.68	0.71	0.74	0.74	0.75	0.75	0.75	center of fixed edge
	K	0.030	0.050	0.068	0.080	0.090	0.100	0.122	0.132	0.137	0.139	at the center
One short edge free, other three edges supported, uniform load over entire surface. L = b	K_1	0.67	0.74	0.76	0.77	0.78	0.79	0.80	0.80	0.80	0.80	at center of free edge
	K	0.14	0.15	0.16	0.16	0.16	0.16	0.17	0.17	0.17	0.17	at the center

TABLE 7.4 LOADING COEFFICIENTS FOR RECTANGULAR FLAT PLATES UNDER VARIOUS LOADINGS

MANNER OF LOADING		a/b	1.0	1.2	1.4	1.6	1.8	2.0	2.5	3.0	3.5	4.0	Location of Maximum Stress and Deflection
	One short edge free, other three edges supported. Distributed load varying linearly along length. L = b	K_1	0.20	0.24	0.27	0.29	0.31	0.32	0.35	0.36	0.37	0.37	at center of free edge
		K	0.040	0.045	0.048	0.051	0.053	0.058	0.064	0.067	0.069	0.70	at the center
	One long edge free, other three edges supported, uniform load over entire surface. L = a	K_1	0.67	0.57	0.48	0.42	0.38	0.36	0.36	0.36	0.36	0.36	at center of free edge
		K	0.14	0.12	0.11	0.10	0.09	0.08	0.08	0.08	0.08	0.08	at the center
	One long edge free, other three edges supported, distributed load varying linearly along breadth. L = a	K_1	0.20	0.18	0.17	0.15	0.13	0.11	-	-	-	-	at center of free edge
		K	0.040	0.036	0.033	0.030	0.028	0.025	-	-	-	-	at the center
	All edges supported, distributed load in form of a triangular prism. L = b	K_1	0.204	0.262	0.311	0.352	0.385	0.411	0.450	0.476	0.476	0.500	at the center
		K	0.029	0.040	0.050	0.058	0.065	0.071	0.082	0.085	0.085	0.091	at the center

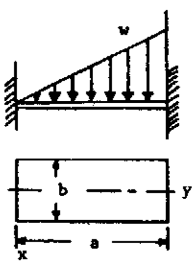
MANNER OF LOADING		b/a	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	Comments
	All edges fixed, distributed load varying linearly along length. L = a	K_1	0.1308	0.1434	0.1686	0.1800	0.1842	0.1872	0.1902	0.1908	Max F_b is at $x = \pm 0.5b$; $y = 0.55a$
		K_1	0.0636	0.0688	0.0762	0.0715	0.0612	0.0509	0.0415	0.0356	F_b at $x = 0$; $y = 0.6a$
		K_1	0.0832	0.1778	0.2365	0.2561	0.3004	0.3092	0.3100	0.3000	Max F_a is at $x = 0$; $y = a$
		K_1	0.0206	0.0497	0.0898	0.1249	0.1482	0.1615	0.1680	0.1709	F_a at $x = 0$; $y = 0$
		K_1	0.0410	0.0633	0.0869	0.1038	0.1128	0.1255	0.1157	0.1148	F_a at $x = 0$; $y = 0.6a$
		K	0.0016	0.0047	0.0074	0.0097	0.0113	0.0126	0.0133	0.0136	Max Deflection

TABLE 7.4 (CONT'D) LOADING COEFFICIENTS FOR RECTANGULAR FLAT PLATES UNDER VARIOUS LOADINGS

(a) Uniform Loading

	K		K_1
b/a	V_x(max.)	V_y(max.)	R
1	0.420	0.420	0.065
1.1	0.440	0.440	0.070
1.2	0.455	0.453	0.074
1.3	0.468	0.464	0.079
1.4	0.478	0.471	0.083
1.5	0.486	0.480	0.085
1.6	0.491	0.485	0.086
1.7	0.496	0.488	0.088
1.8	0.499	0.491	0.090
1.9	0.502	0.494	0.091
2.0	0.503	0.496	0.092
3.0	0.505	0.498	0.093
4.0	0.502	0.500	0.094
5.0	0.501	0.500	0.095
∞	0.500	0.500	0.095
Remarks L = a for V_x and V_y			

(a) Uniform Loading

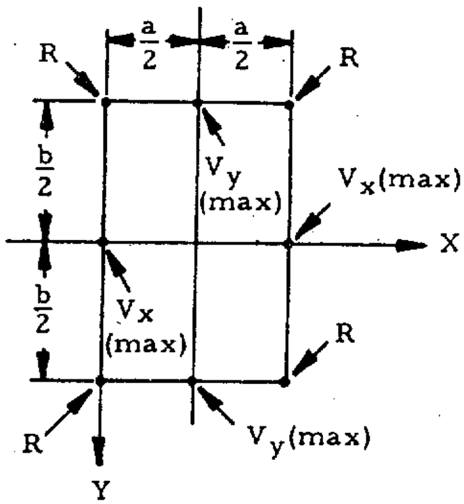
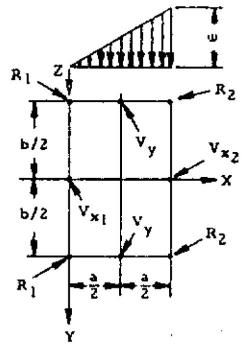


TABLE 7.5 LOADING COEFFICIENTS FOR CORNER AND EDGE FORCES FOR FLAT SIMPLY SUPPORTED RECTANGULAR PLATES UNDER VARIOUS LOADINGS

(b) Distributed Load Varying linearly Along Length (a<b)

	K			K₁		
b/a	V_{x1}	V_{x2}	V_y	R_1	R_2	Remarks
1.0	0.126	0.294	0.210	0.026	0.039	Use L = a for V_{x1}, V_{x2}
1.1	0.136	0.304	0.199	0.026	0.038	
1.2	0.144	0.312	0.189	0.026	0.037	Use L = b for V_y
1.3	0.150	0.318	0.178	0.026	0.036	Because the load is not symmetrical, the reactions R_1 are different from the reactions R_2, also V_{x1} is different than V_{x2}. The same applies to case V-3.
1.4	0.155	0.323	0.169	0.025	0.035	
1.5	0.159	0.327	0.160	0.024	0.033	
1.6	0.162	0.330	0.151	0.023	0.032	
1.7	0.164	0.332	0.144	0.022	0.030	
1.8	0.166	0.333	0.136	0.021	0.029	
1.9	0.167	0.334	0.130	0.021	0.028	
2.0	0.168	0.335	0.124	0.020	0.026	
3.0	0.169	0.336	0.083	0.014	0.018	
4.0	0.168	0.334	0.063	0.010	0.014	
5.0	0.167	0.334	0.050	0.008	0.011	
∞	0.167	0.333	--	--	--	



Notes:

1. In this case only, the formula (V) for the corner force R can be used when substituting a for b
2. V_x(max.) and V_y(max.) are at the middle of sides b and a respectively as shown in the figure for this table

(c) Distributed Load Varying Linearly Along Breadth (a>b)

	K			K₁		
a/b	V_{x1}	V_{x2}	V_y	R_1	R_2	Remarks
∞	--	--	0.250	--	--	Use L = a for V_{x1}, V_{x2}
5.0	0.008	0.092	0.250	0.002	0.017	
4.0	0.013	0.112	0.251	0.004	0.020	Use L = b for V_y
3.0	0.023	0.143	0.252	0.006	0.025	
2.0	0.050	0.197	0.251	0.013	0.033	
1.9	0.055	0.205	0.251	0.014	0.034	
1.8	0.060	0.213	0.249	0.016	0.035	
1.7	0.066	0.221	0.248	0.017	0.036	
1.6	0.073	0.230	0.245	0.018	0.037	
1.5	0.080	0.240	0.243	0.020	0.037	
1.4	0.088	0.250	0.239	0.021	0.038	
1.3	0.097	0.260	0.234	0.023	0.039	
1.2	0.106	0.271	0.227	0.024	0.039	
1.1	0.116	0.282	0.220	0.025	0.039	
1.0	0.126	0.294	0.210	0.026	0.039	

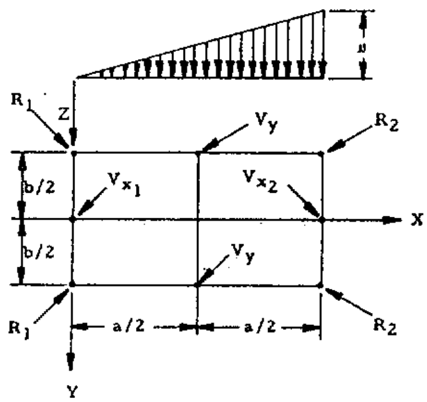
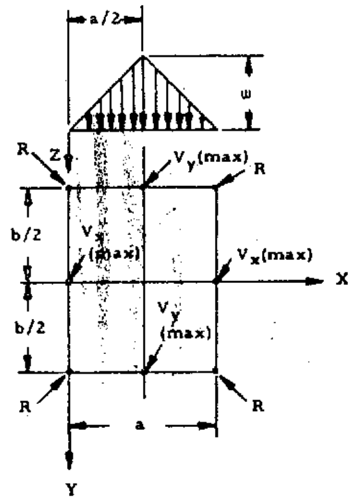


TABLE 7.5 (CONT'D) LOADING COEFFICIENTS FOR CORNER AND EDGE FORCES FOR FLAT SIMPLY SUPPORTED RECTANGULAR PLATES UNDER VARIOUS LOADINGS

(d) Load Distributed As Triangular Prism Across Breadth (a<b)

	K		K_1	
b/a	V_x(max.)	V_y(max.)	R	Remarks
1.0	0.147	0.250	0.038	Use L = a for V_x
1.1	0.161	0.232	0.038	
1.2	0.173	0.216	0.037	Use L = b for V_y
1.3	0.184	0.202	0.036	
1.4	0.193	0.189	0.035	
1.5	0.202	0.178	0.034	
1.6	0.208	0.108	0.033	
1.7	0.214	0.158	0.031	
1.8	0.220	0.150	0.030	
1.9	0.224	0.142	0.029	
2.0	0.228	0.135	0.028	
3.0	0.245	0.090	0.019	
∞	0.250	--	--	



(e) Load Distributed As Triangular Prism Along Length (a>b)

	K		K_1	
a/b	V_x(max.)	V_y(max.)	R	Remarks
∞	--	0.50	--	Use L = a for V_x
3.0	0.027	0.410	0.010	Use L = b for V_y
2.0	0.057	0.365	0.023	
1.9	0.062	0.358	0.024	
1.8	0.098	0.350	0.026	
1.7	0.074	0.342	0.028	
1.6	0.081	0.332	0.029	
1.5	0.090	0.322	0.031	
1.4	0.099	0.311	0.033	
1.3	0.109	0.298	0.035	
1.2	0.120	0.284	0.036	
1.1	0.133	0.268	0.037	
1.0	0.147	0.250	0.038	

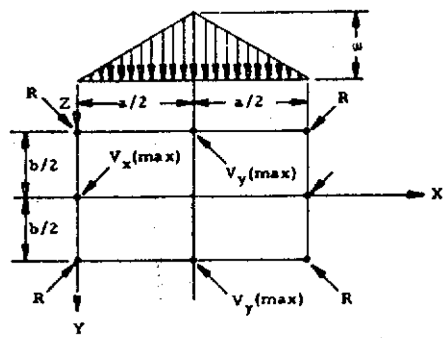
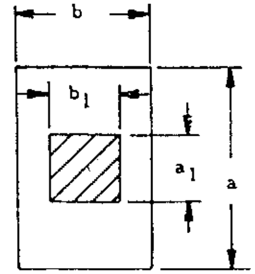


TABLE 7.5 (CONT'D) LOADING COEFFICIENTS FOR CORNER AND EDGE FORCES FOR FLAT SIMPLY SUPPORTED RECTANGULAR PLATES UNDER VARIOUS LOADINGS

K_1 factor for maximum stress at center ($F = F_b$)																		
a_1/b	a/b = 1						a/b = 1.4						a/b = 2					
b_1/b	0	0.2	0.4	0.6	0.8	1.0	0	0.2	0.4	0.8	1.2	1.4	0	0.4	0.8	1.2	1.6	2.0
0	--	1.82	1.38	1.12	0.93	0.76	--	2.0	1.55	1.12	0.84	0.75	--	1.64	1.20	0.97	0.78	0.64
0.2	1.82	1.28	1.08	0.90	0.76	0.63	1.78	1.43	1.23	0.95	0.74	0.64	1.73	1.31	1.03	0.84	0.68	0.57
0.4	1.39	1.07	0.84	0.72	0.62	0.52	1.39	1.13	1.00	0.80	0.62	0.55	1.32	1.08	0.88	0.74	0.60	0.50
0.6	1.12	0.90	0.72	0.60	0.52	0.43	1.10	0.91	0.82	0.68	0.53	0.47	1.04	0.90	0.76	0.64	0.54	0.44
0.8	0.92	0.76	0.62	0.51	0.42	0.36	0.90	0.76	0.68	0.57	0.45	0.40	0.87	0.76	0.63	0.54	0.44	0.38
1.0	0.76	0.63	0.52	0.42	0.35	0.30	0.75	0.62	0.57	0.47	0.38	0.33	0.71	0.61	0.53	0.45	0.38	0.30

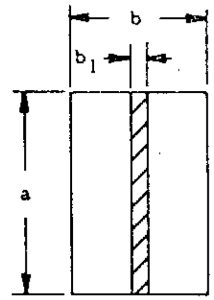
Note: Total load $W = wa_1b_1$



All edges supported
Uniform load over
central rectangular
area shown shaded

K factor for maximum deflection								
b > a	b/a	2	1.5	1.4	1.3	1.2	1.1	1.0
L a	K	0.108	0.099	0.096	0.092	0.087	0.081	0.074
b < a	a > b	1.1	1.2	1.3	1.4	1.5	2.0	∞
L b	K	0.088	0.101	0.114	0.126	0.137	0.178	0.227

Note: Use unit applied load w in this case (lb./in.)



All edges supported. Uniform load along the axis of symmetry parallel to the dimension a (b, very small)

TABLE 7.6 LOADING COEFFICIENTS FOR PARTIALLY LOADED RECTANGULAR FLAT PLATES

		Edge Supported Uniform Load Over Entire Surface		Edge Fixed Uniform Load Over Entire Surface	
Manner of Loading		K	K_1	K	K_1
a/b	1.0	0.70	1.24	0.171	0.75
	1.1	0.84	1.42	0.20	0.90
	1.2	0.95	1.57	0.25	1.04
	1.3	1.06	1.69	0.28	1.14
	1.4	1.17	1.82	0.30	1.25
	1.5	1.26	1.92	0.30	1.34
	1.6	1.34	2.04	0.33	1.41
	1.7	1.41	2.09	0.35	1.49
	1.8	1.47	2.16	0.36	1.54
	1.9	1.53	2.22	0.370	1.59
	2.0	1.58	2.26	0.379	1.63
	2.5	1.75	2.45	0.40	1.75
	3.0	1.88	2.60	0.42	1.84
	3.5	1.96	2.70	0.43	1.89
	4.0	2.02	2.78	0.43	1.9
L		b		b	
Locations of stress and deflection		F max. at center Y max. at center		F max. at end of shorter principal axis. Y max. at center	

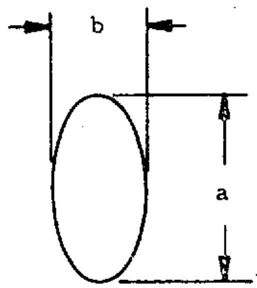


TABLE 7.7 LOADING COEFFICIENTS FOR FLAT ELLIPTICAL PLATES UNDER UNIFORM LOAD

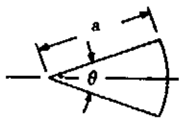
Plate Description and Type of Edge Support		θ	45°	60°	90°	180°	Locations of Stress and Deflection
 Circular Sector	All edges supported. L = a	K_1	0.102	0.147	0.240	0.522	Max radial stress, on line to midpoint of circular edge
		K_1	0.114	0.155	0.216	0.312	Max tangential stress, at midpoint of circular edge
		K	0.0054	0.0105	0.0250	0.0870	Max deflection is at midpoint of circular edge
	Straight edges supported. Circular edge fixed. L = a	K_1	0.1500	0.2040	0.2928	0.4536	Max radial stress is at curved boundary
		K	0.0035	0.0065	0.0144	0.0380	

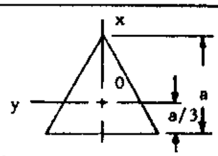
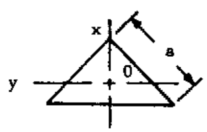
Plate Description and Type of Edge Support		Coefficients		Locations of Stress and Deflection
	Equilateral Triangle All edges supported. L = a	K_1	0.1488	Max F_x at y = 0, x = -0.062a
		K_1	0.1554	Max F_y at y = 0, x = 0.129a
		K	0.0112	Max deflection is at 0
	Right Angle Isosceles Triangle All edges supported. L = a	K_1	0.131	Max F_x, no data available on location
		K_1	0.1125	Max F_y, no data available on location
		K	0.0095	Max deflection, no data available on location

TABLE 7.8 LOADING COEFFICIENTS FOR CIRCULAR SECTOR AND TRIANGULAR FLAT PLATES UNDER UNIFORM LOAD

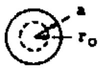
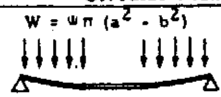
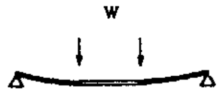
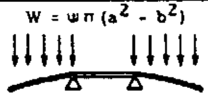
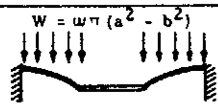
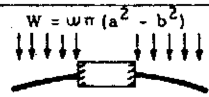
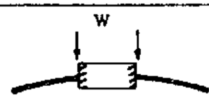
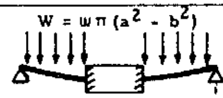
Loading	Circular Solid Plate  / Circular Plate with Concentric Hole or Circular Flange 	a/b or a/r_o: 1	1.25	1.5	2	3	4	5	Factors: K for deflection, K_1 for stress, K_2 for slope
Outer edge supported Uniform load over entire actual surface		--	0.163	0.237	0.290	0.295	0.297	0.269	K
		--	0.524	0.559	0.612	0.673	0.744	0.725	K_1 (F_t at inner edge)
		--	0.786	0.679	0.640	0.441	0.401	0.379	K_2 at outer edge
		--	0.919	0.741	0.612	0.447	0.355	0.290	K_2 at inner edge
Outer edge supported Uniform load along inner edge		--	0.341	0.519	0.672	0.734	0.724	0.704	K
		--	1.10	1.26	1.48	1.88	2.17	2.34	K_1 (F_t at inner edge)
		--	1.646	1.470	1.237	1.006	0.895	0.832	K_2 at outer edge
		--	1.758	1.650	1.475	1.238	1.082	0.932	K_2 at inner edge
Inner edge supported Uniform load over entire actual surface		--	0.179	0.281	0.383	0.437	0.444	0.434	K (at outer edge)
		--	0.584	0.682	0.867	1.196	1.458	1.688	K_1 (F_t at inner edge)
		--	0.884	0.794	0.606	0.565	0.492	0.448	K_2 at outer edge
		--	1.012	0.910	0.862	0.791	0.726	0.642	K_2 at inner edge
Outer edge fixed and supported Uniform load over entire actual surface		--	0.0018	0.0035	0.024	0.046	0.055	0.0576	K
		--	0.0072	0.025	0.069	0.145	0.192	0.222	K_1 (F_t at inner edge)
		--	0.013	0.035	0.069	0.096	0.096	0.089	K_2 at inner edge
		--	0.093	0.148	0.204	0.235	0.240	0.242	K_1 (F_r at outer edge)
Outer edge fixed and supported Uniform load along inner edge		--	0.005	0.024	0.081	0.171	0.215	0.237	K
		--	0.025	0.087	0.269	0.670	1.018	1.300	K_1 (F_t at inner edge)
		--	0.195	0.320	0.455	0.539	0.538	0.532	K_1 (F_r at outer edge)
		--	0.045	0.115	0.269	0.448	0.510	0.522	K_2 at inner edge
Inner edge fixed and supported Uniform load over entire actual surface		--	0.002	0.010	0.040	0.105	0.152	0.187	K (at outer edge)
		--	0.119	0.235	0.442	0.770	1.014	1.22	K_1 (F_r at inner edge)
		--	0.012	0.040	0.097	0.177	0.223	0.252	K_2 at outer edge
Inner edge fixed and supported Uniform load along outer edge		--	0.0051	0.025	0.088	0.209	0.293	0.350	K (at outer edge)
		--	0.227	0.428	0.753	1.205	1.514	1.745	K_1 (at inner edge)
		--	0.046	0.105	0.239	0.403	0.499	0.544	K_2 at outer edge
Outer edge supported, inner edge fixed Uniform load over entire actual surface		--	0.003	0.018	0.053	0.104	0.141	0.163	K
		--	0.108	0.192	0.314	0.433	0.491	0.526	K_1 (F_r at inner edge)
		--	0.091	0.123	0.197	0.260	0.297	0.306	K_2 at outer edge

TABLE 7.9 LOADING COEFFICIENTS FOR CIRCULAR FLAT PLATES UNDER VARIOUS LOADINGS

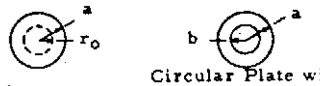
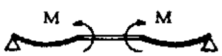
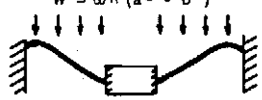
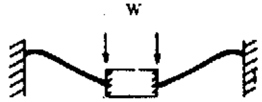
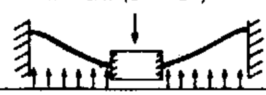
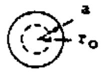
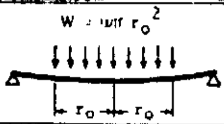
Loading	 Circular Solid Plate / Circular Plate with Concentric Hole or Circular Flange	a/b or a/r_o: 1	1.25	1.5	2	3	4	5	Factors: K for deflection, K_1 for stress, K_2 for slope
Outer edge fixed Uniform moment along inner edge		--	0.20	0.48	0.85	0.94	0.80	0.66	K max. at inner edge
		--	0.37	0.86	2.44	4.10	4.84	5.11	K_1 max. at inner edge
		--	2.20	3.16	3.88	3.31	3.12	2.72	K_2 at inner edge
Inner edge fixed Uniform moment along outer edge		--	0.23	0.66	1.49	2.55	3.10	3.41	K max. at outer edge
		--	6.87	7.50	8.14	8.71	8.94	9.04	K_1 max. at inner edge
		--	2.30	3.84	5.67	6.94	7.82	8.17	K_2 at outer edge
Inner edge supported Uniform moment along outer edge		--	10.37	9.23	7.80	6.31	5.62	5.23	K max. at outer edge
		--	33.3	21.6	16.0	13.5	12.8	12.5	K_1 max. (F_t at inner edge)
		--	51.00	28.00	16.40	11.60	10.23	9.61	K_2 at outer edge
		--	53.30	27.80	15.60	8.78	6.24	4.86	K_2 at inner edge
Outer edge supported Uniform moment along inner edge		--	8.87	6.92	4.65	2.58	1.69	1.21	K_1 max at inner edge
		--	27.36	15.60	10.0	7.50	6.80	6.50	K_1 max. F_t at inner
		--	42.70	18.75	7.81	2.93	1.56	0.94	K_2 at outer edge
		--	44.90	22.35	11.27	5.52	4.08	3.17	K_2 at inner edge
Outer edge fixed and supported, inner edge fixed Uniform load over entire actual surface		--	0.0004	0.003	0.010	0.023	0.031	0.037	K
		--	0.036	0.065	0.104	0.151	0.176	0.192	K_1 (F at inner edge)
		--	0.062	0.105	0.153	0.195	0.212	0.221	K_1 (F at outer edge)
Outer edge fixed and supported, inner edge fixed Uniform load along inner edge		--	0.0013	0.0064	0.024	0.062	0.092	0.114	K
		--	0.115	0.220	0.405	0.703	0.933	1.130	K_1 (F_r at inner edge)
		--	0.098	0.168	0.257	0.347	0.390	0.415	K_1 (F_r at inner edge)
Both edges fixed Balance loading (piston)		--	0.0007	0.003	0.014	0.039	0.061	0.077	K
		--	0.080	0.156	0.302	0.551	0.756	0.927	K_1 (F_r at inner edge)
No support Uniform edge moment		4	--	--	--	--	--	--	K max. at center
		6	--	--	--	--	--	--	K_1 ($F_1 = F_1$ at any point)
		8	--	--	--	--	--	--	At edge

TABLE 7.9 (CONT'D) LOADING COEFFICIENTS FOR CIRCULAR FLAT PLATES UNDER VARIOUS LOADINGS

Loading	Circular Solid Plate	Circular Plate with Concentric Hole or Circular Flange	a/b or a/r_o							Factors: K for deflection, K_1 for stress, K_2 for slope
			1	1.25	1.5	2	3	4	5	
Edges supported Uniform load over concentric circular area of radius r_o			0.212	0.295	0.350	0.413	0.469	0.492	0.503	K
			0.398	0.568	0.700	0.900	1.167	1.353	1.500	K_1 (F_r at center)
			0.636							K_2

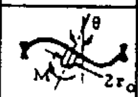
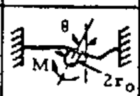
Manner of loading		r_o/a 0.1	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	K_2 slope, K_1 stress
Edges supported Central couple, trunnion loading		0.71	0.97	1.22	1.60	2.0	2.50	3.53	5.60	8.54	12.00	16.30	24.10	41.4	82.0	162.0	K_2 max. at center
		4.36	3.80	3.27	2.80	2.37	2.10	1.84	1.58	1.41	1.16	1.07	0.90	0.78	0.70	0.57	K_1 max. F at center
Edges fixed Central couple, trunnion loading		0.87	1.23	1.68	2.31	3.10	4.00	5.45	8.20	12.40	18.0	28.5	44.0	77.9	156.0	314.0	K_2 max. at center
		6.06	5.40	4.52	3.40	2.65	2.15	1.75	1.45	1.14	0.89	0.68	0.57	0.47	0.35	0.26	K_1 max. F at center

TABLE 7.9 (CONT'D) LOADING COEFFICIENTS FOR CIRCULAR FLAT PLATES UNDER VARIOUS LOADINGS

Revision A

7.5 Shear Buckling of Flat Plates

The critical shear-buckling stress of flat plates may be found from

$$F_{cr_s} = \eta \bar{\eta} \frac{K_s \pi^2 E}{12(1-\nu_e^2)} \left(\frac{t}{b}\right)^2$$

Figure 7-21 presents the shear coefficient k_s as a function of the size ratio a/b for clamped and hinged edges. For infinitely long plates, Figure 7-22 presents k_s as a function of λ/b. Figure 7-23 (a) presents $k_{s\infty}$ for long plates as a function of edge restraint, and Figure 7-23 (b) gives $k_s/k_{s\infty}$ as a function of b/a, thus allowing the determination of k_s.

The nondimensional chart in Figure 7-24 allows the calculation of inelastic shear buckling stresses if the secant yield stress, $F_{0.7}$, and n the shape parameter is known (Table 7-10).

The plasticity-reduction factor η for shear panels can be obtained from Equation 7-23.

$$\eta = \frac{E_s}{E} \left(\frac{1-\nu_e^2}{1-\nu^2}\right) \tag{7-23}$$

Cladding reduction factors, $\bar{\eta}$, are given in Table 7-1.

7.6 Axial Compression of Curved Plates

The radius of curvature of curved plates determines the method to be used to analyze their buckling stress. For large curvature ($b^2/rt < 1$), they may be analyzed as flat plates by using the relations in Section 7.3. For elastic stresses in the transition length and width ranges, Figure 7.25 may be used to find the buckling coefficient for use in Equation (7-24).

$$F_{cr} = \frac{k_c \pi^2 E}{12(1-\nu_e^2)} \left(\frac{t}{b}\right)^2 \tag{7-24}$$

For sharply curved plates, ($b^2/rt > 100$), Equations (7-25) and (7-26) can be used.

$$F_{cr} = \eta C E \left(\frac{t}{r}\right) \tag{7-25}$$

$$\eta = \frac{E_s}{E} \frac{E_t}{E_s} \frac{(1-\nu_e^2)}{(1-\nu^2)} \tag{7-26}$$

Figure 7.26 gives values of C in terms of r/t. Figure 7.27 gives η in a non-dimensional form. Here the quantity $\epsilon_{cr} = Ct/r$.

n	Material
3	One-fourth hard to full hard 18-8 stainless steel, with grain One-fourth hard 18-8 stainless steel, cross grain
5	One-half hard and three-fourths hard 18-8 stainless steel, cross grain
10	Full hard 18-8 stainless steel, cross grain 2024-T and 7075-T aluminum-alloy sheet and extrusion 2024R-T aluminum-alloy sheet
20 to 25	2024-T80, 2024-T81, and 2024-T86 aluminum-alloy sheet 2024-T aluminum-alloy extrusion SAE 4130 steel heat-treated up to 100, 000 psi ultimate stress
35 to 50	2014-T aluminum-alloy extrusions SAE 4130 steel heat-treated above 125, 000 psi ultimate stress
∞	SAE 1025 (mild) steel

TABLE 7.10 VALUES OF SHAPE PARAMETER n FOR SEVERAL ENGINEERING MATERIALS

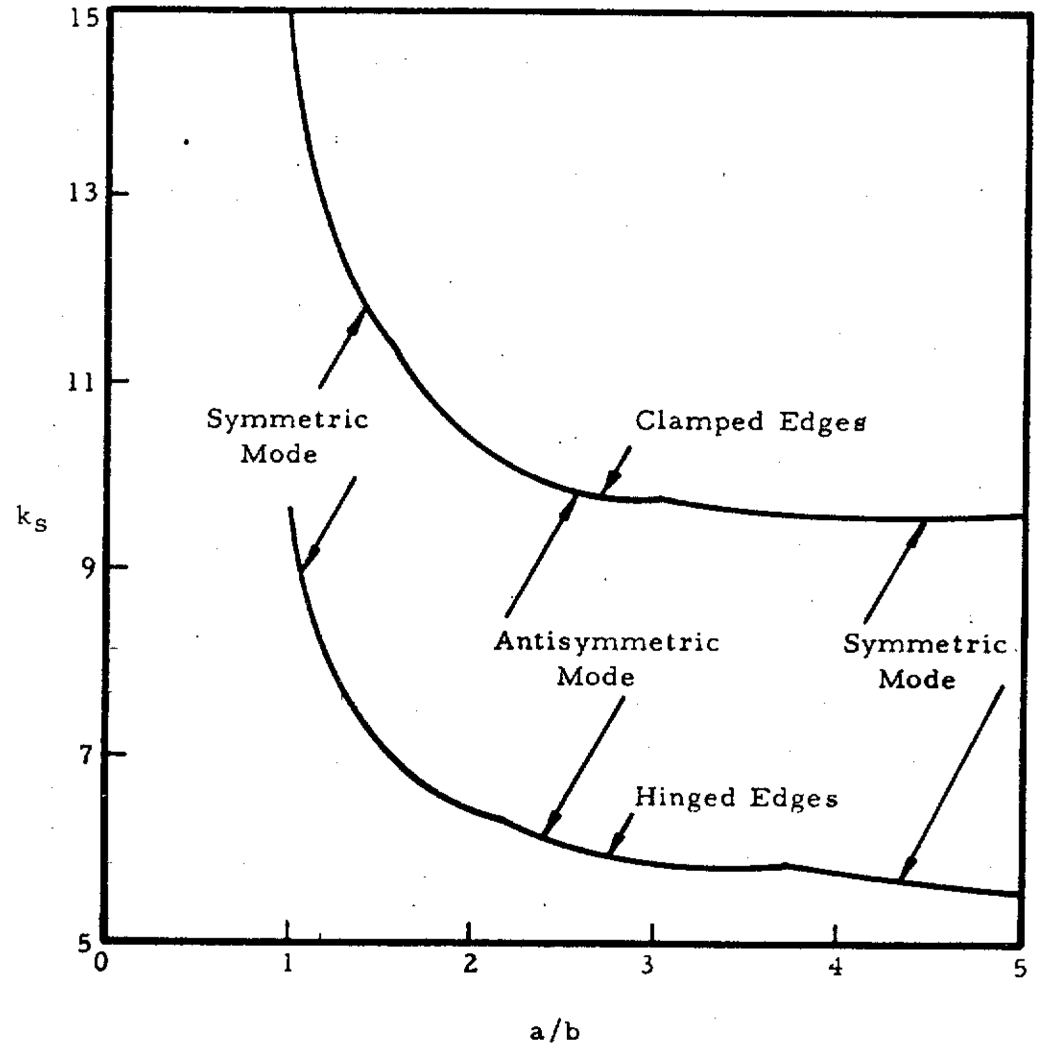


FIGURE 7.21 SHEAR-BUCKLING-STRESS COEFFICIENTS OF PLATES AS A FUNCTION OF a/b FOR CLAMPED AND HINGED EDGES

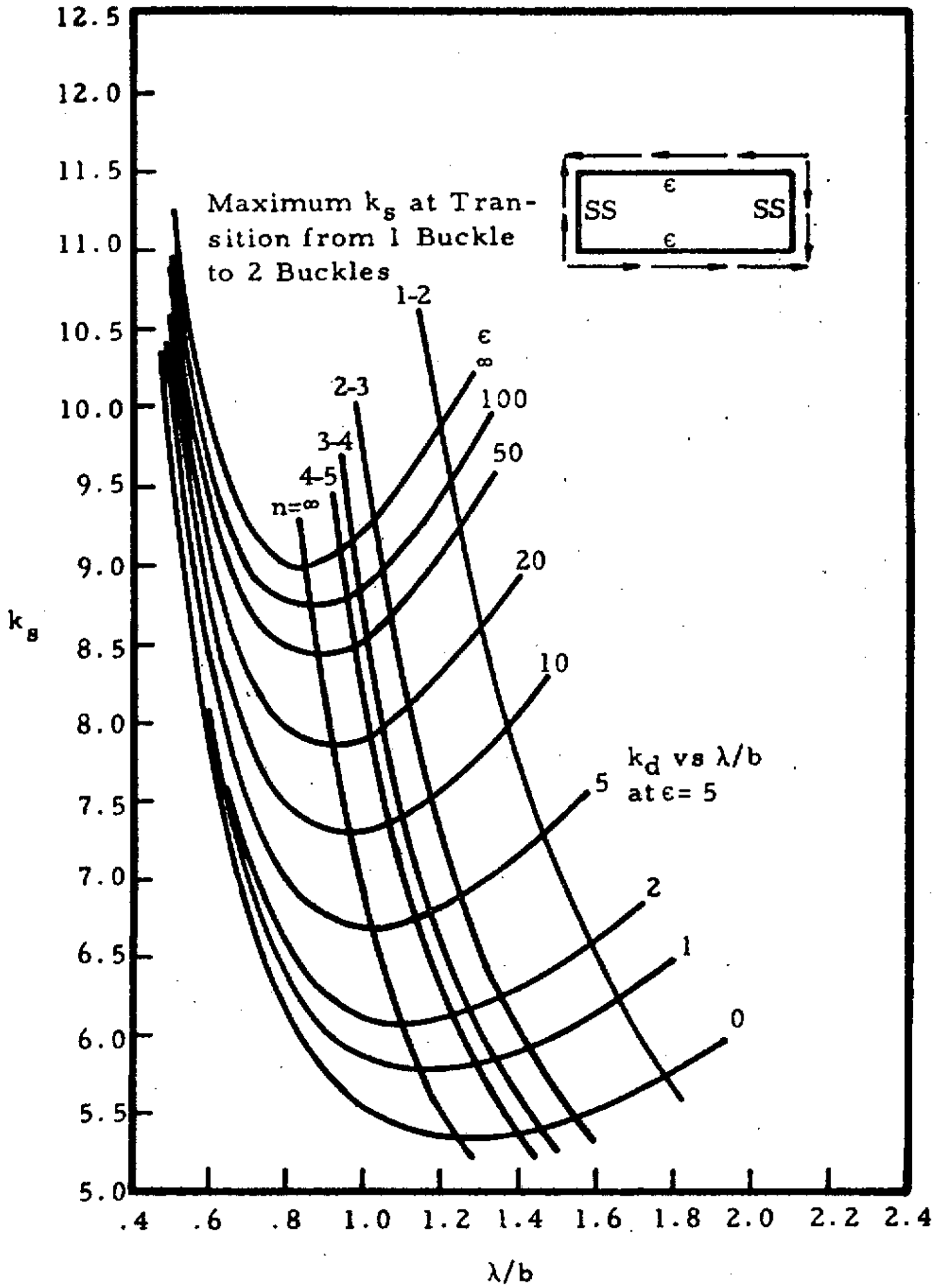
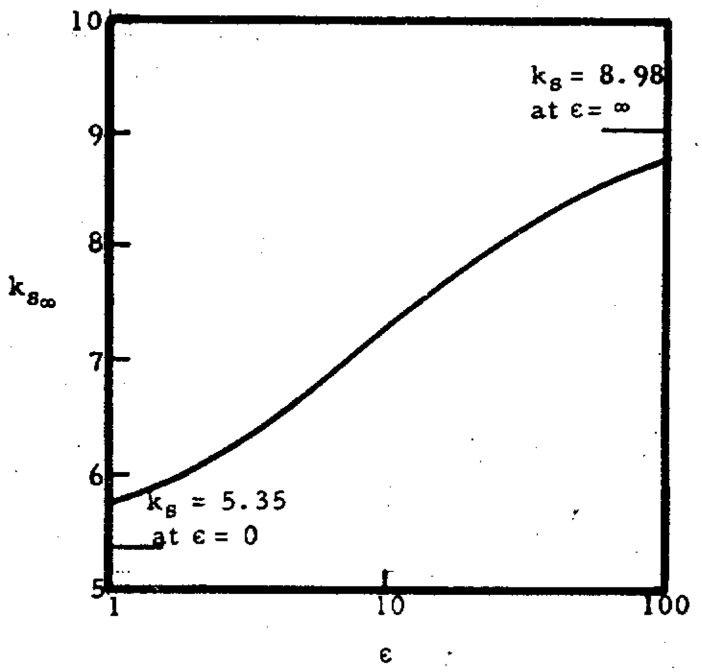
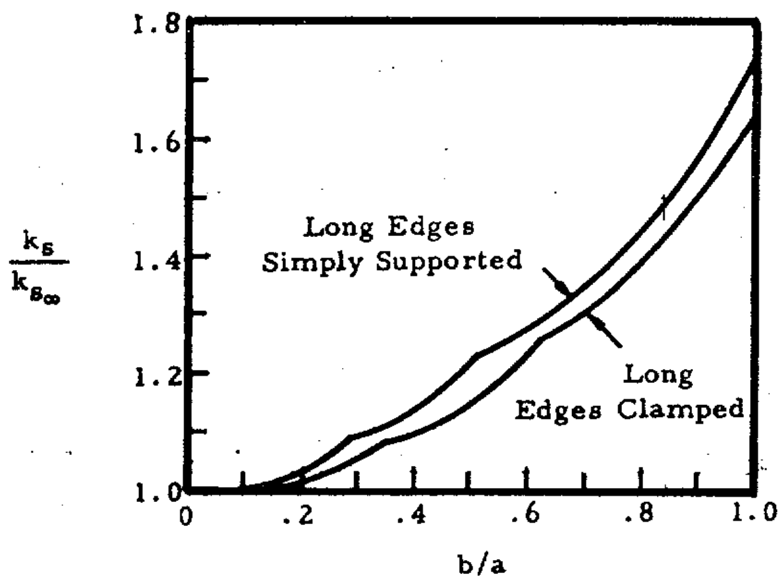


FIGURE 7.22 SHEAR-BUCKLING-STRESS COEFFICIENTS FOR PLATES OBTAINED FROM ANALYSIS OF INFINITELY LONG PLATES AS A FUNCTION OF λ/b FOR VARIOUS AMOUNTS OF EDGE ROTATIONAL RESTRAINT



(a) $K_{s\infty}$ as a function of ε



(b) $K_s/K_{s\infty}$ as a funtion of b/a

FIGURE 7.23 CURVES FOR ESTIMATION OF SHEAR-BUCKLING COEFFICIENTS OF PLATES WITH VARIOUS AMOUNTS OF EDGE ROTATIONAL RESTRAIN

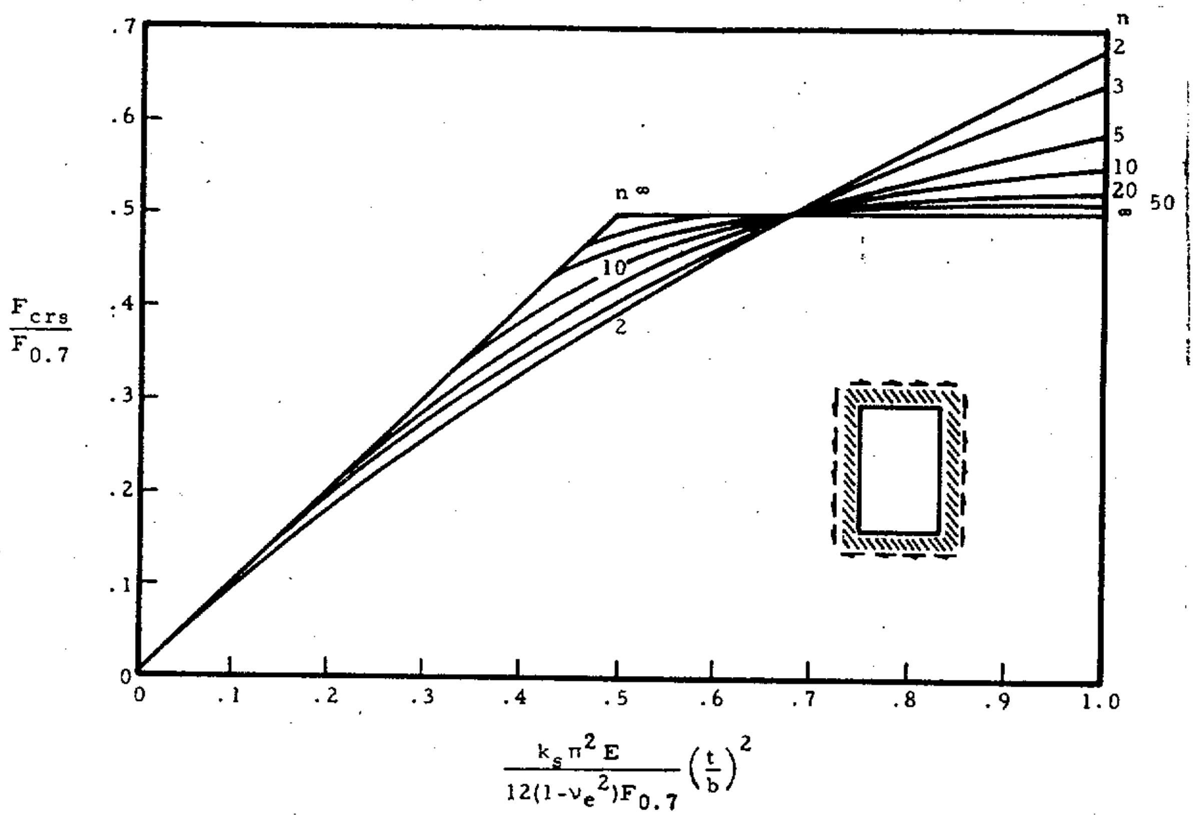
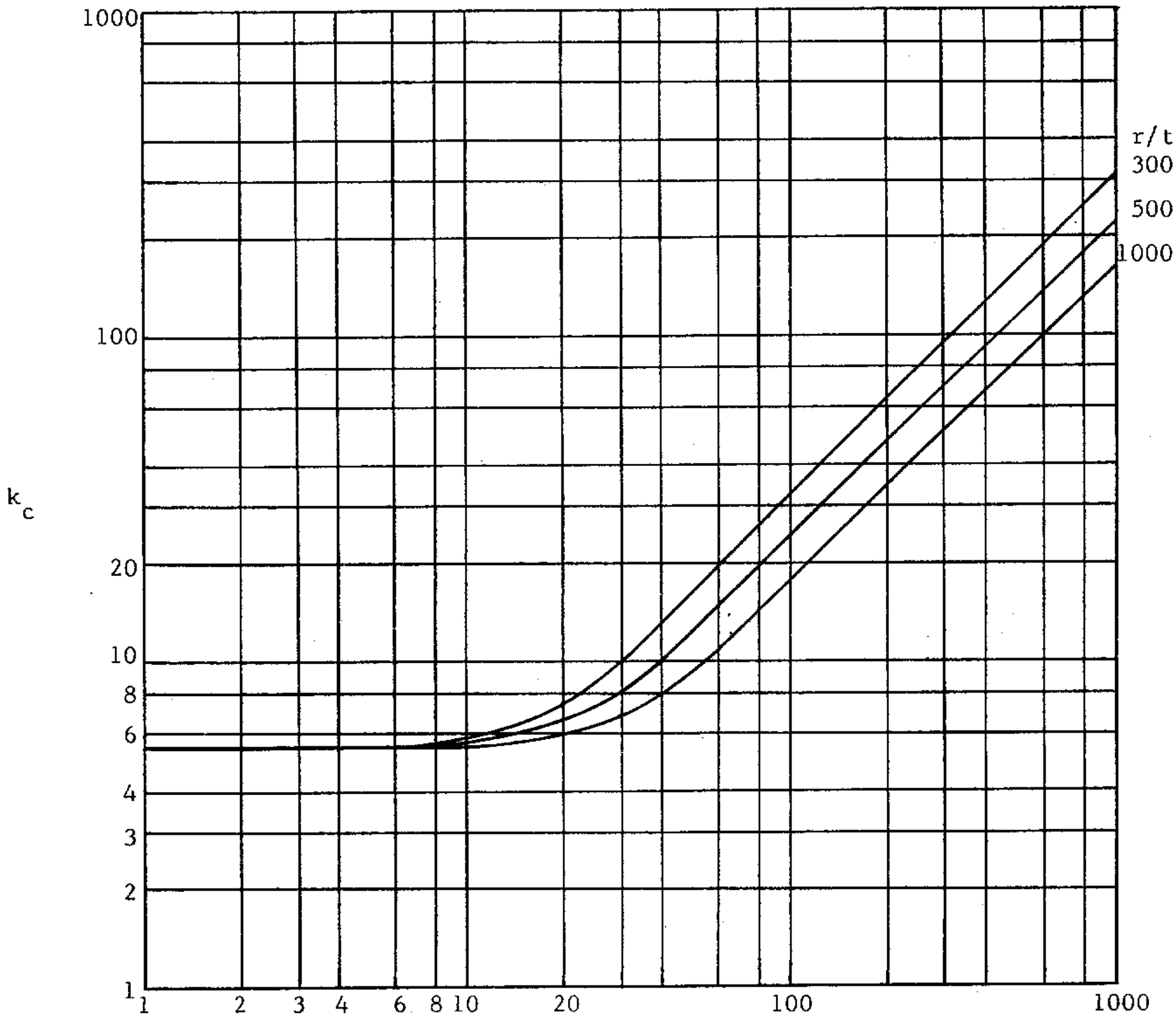


FIGURE 7.24 CHART OF NONDIMENSIONAL SHEAR BUCKLING STRESS FOR PANELS WITH EDGE ROTATIONAL RESTRAINT



$$Z_b = \frac{b^2}{rt}(1 - \nu_e^2)^{.5}$$

FIGURE 7.25 - BUCKLING COEFFICIENTS FOR CURVED PLATES

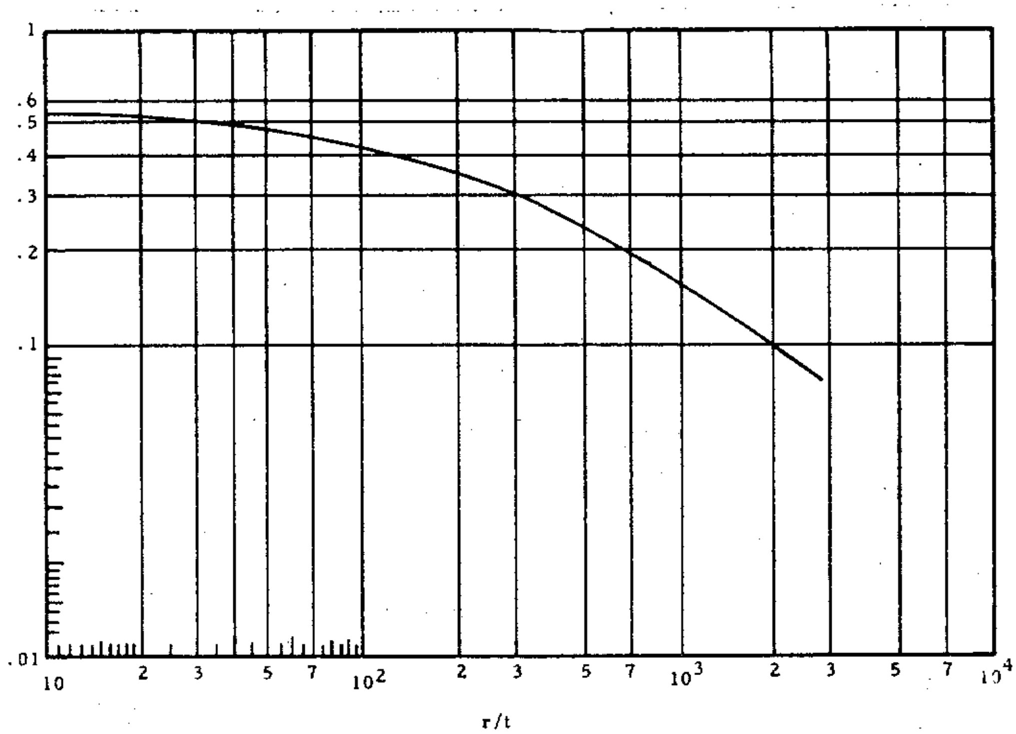


FIGURE 7.26 MODIFIED CLASSICAL BUCKLING COEFFICIENT AS A FUNCTION OF r/t FOR AXIALLY COMPRESSED CYLINDRICAL PLATES

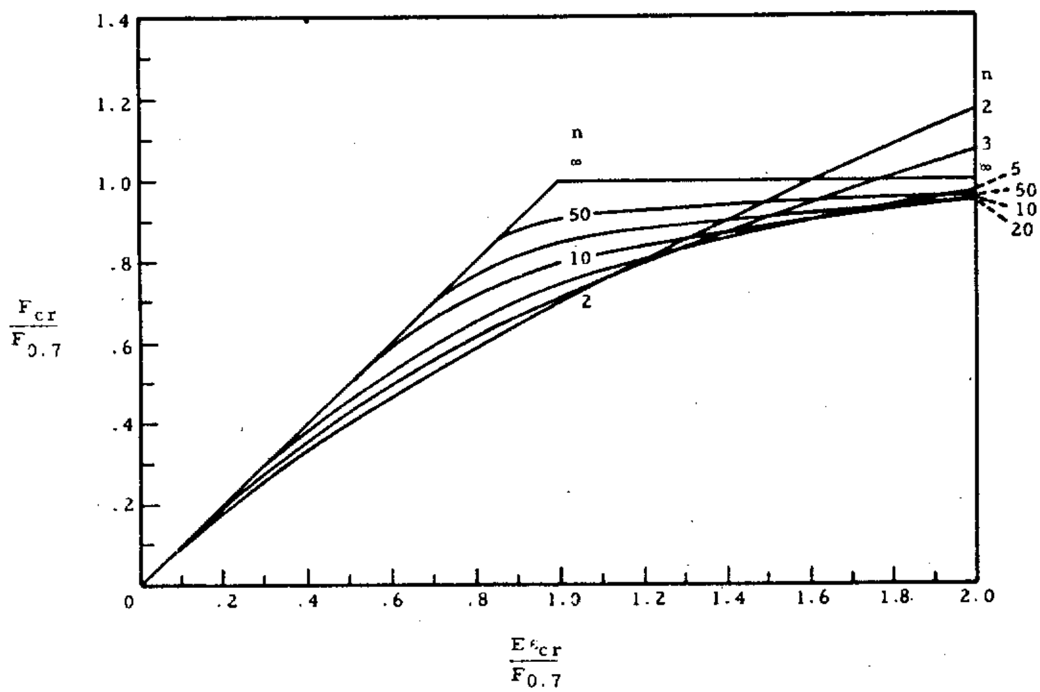


FIGURE 7.27 NONDIMENSIONAL BUCKLING CHART FOR AXIALLY COMPRESSED CURVED PLATES

$$\eta = (E_s/E)\ ((E_t/E)(1-\nu_e^{2})/(1-\nu^{2}))^{.5}$$

7.7 Shear Loading of Curved Plates

Large radius curved plates ($b^2/rt < 1$) loaded in shear may be analyzed as flat plates by the methods of Section 7.5. For transition length plates ($1 < b^2/rt < 30$), Figure 7-28 can be used to find k for use in Equation (7-27).

$$F_{cr_s} = \frac{k_s \pi^2 E}{12(1-\nu_e^2)} \left(\frac{t}{b}\right)^2 \qquad (7\text{-}27)$$

For ($b^2/rt > 30$), Equation (7-28) may be used.

$$F_{cr_s} = 0.37 (Z_b)^{\frac{1}{2}} (F_{crs})_{\text{flat plate}} \qquad (7\text{-}28)$$

Curved plates under shear loading with stiffeners can be analyzed by using Figure 7-29 for the value of the buckling coefficient k_s. Both axial stiffeners and circumferential stiffeners are treated.

7.8 Plates Under Combined Loadings

In general, the loadings on aircraft elements are a combination of two or more simple loadings. Design of such elements must consider the interaction of such loadings and a possible reduction of the allowable values of the simple stresses when combined loads are present. The method using stress ratios, R, has been used extensively in aircraft structural design. The ratio R is the ratio of the stress in the panel at buckling under combined loading to the buckling stress under the simple loading. In general, failure occurs when Equation (7-29) is satisfied. The exponents x and y must be determined experimentally and depend upon the structural element and the loading condition.

$$R_1^x + R_2^y = 1 \qquad (7\text{-}29)$$

7.8.1 Flat Plates Under Combined Loadings

Table 7-11 gives the combined loading condition for flat plates. Figures 7-30 and 7-31 give interaction curves for several loading and support conditions. It is noted that the curves present conditions of triple combinations.

7.8.2 Curved Plates Under Combined Loadings

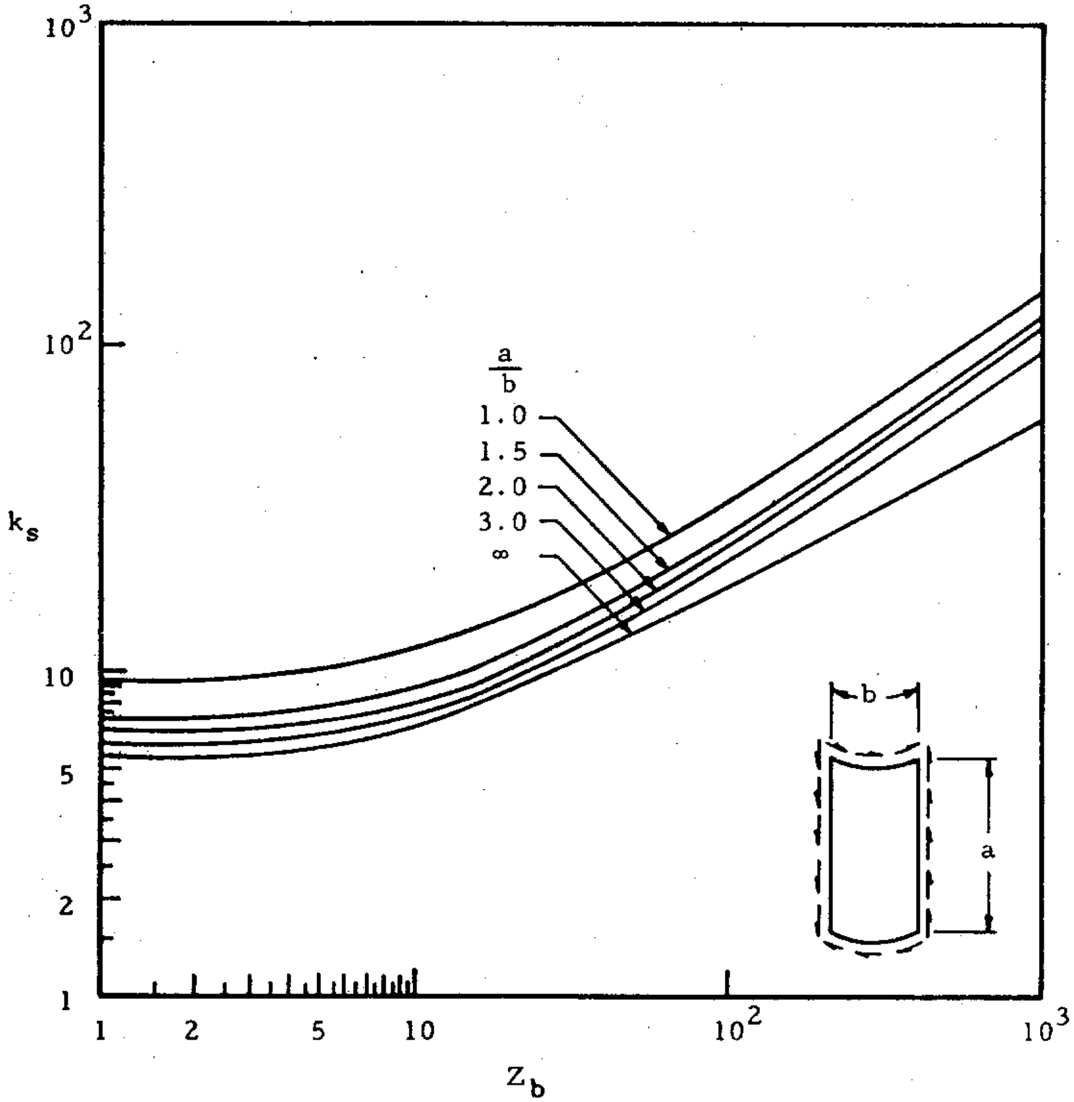
For curved plates under combined axial loading and shear with $10 < Z_b < 100$ and $1 < a/b < 3$, the interaction relation of Equation 7-30 may be used.

$$R_s^2 + R_x = 1 \qquad (7\text{-}30)$$

This may be used for either compression or tension with tension being denoted by a negative sign.

TABLE 7.11 COMBINED LOADING CONDITIONS FOR WHICH INTERACTION CURVES EXIST

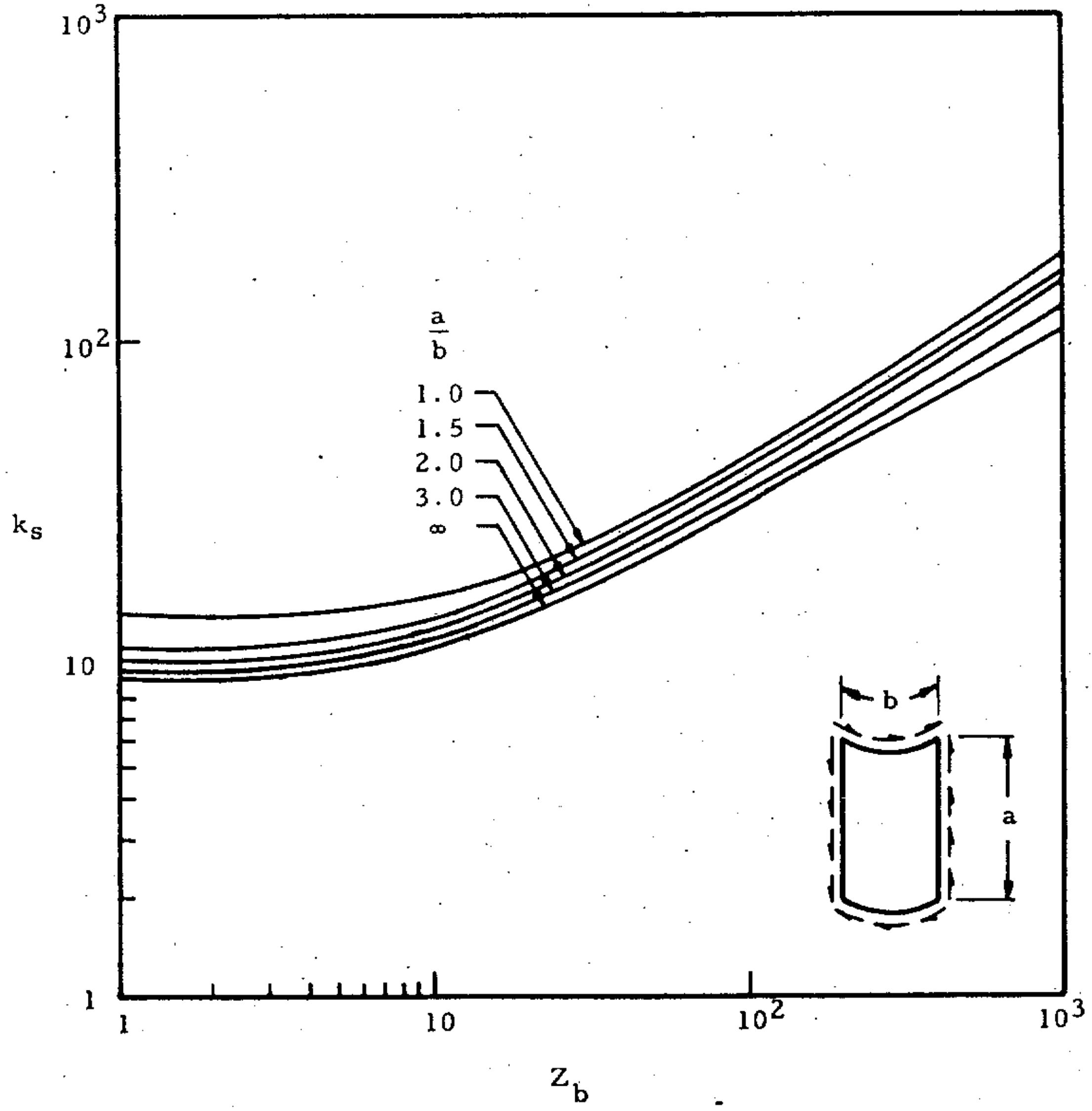
Theory	Loading Combination	Interaction Equation	Figure
Elastic	Biaxial compression	For plates that buckle in square waves, $R_x + R_y = 1$	7.31
	Longitudinal compression and shear	For long plates, $R_c + R_s^2 = 1$	7.30
	Longitudinal compression and bending	None	7.31
	Bending and shear	$R_b^2 + R_s^2 = 1$	7.30
	Bending, shear, and transverse compression	None	7.30
	Longitudinal compression and bending and transverse compression	None	7.31
Inelastic	Longitudinal compression and shear	$R_c^2 + R_s^2 = 1$	



(a) Long simply supported plates

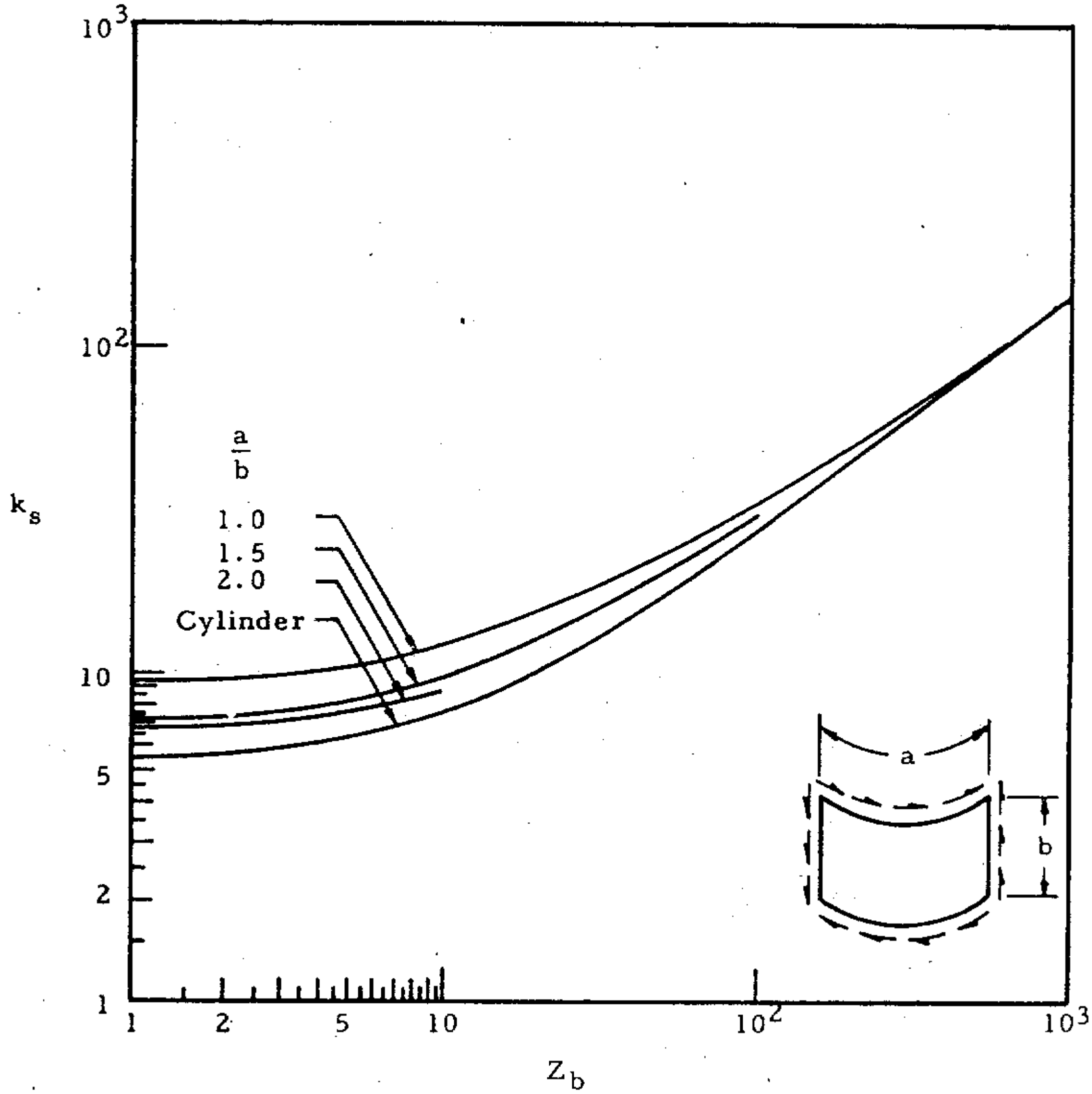
FIGURE 7.28 SHEAR BUCKLING COEFFICIENTS FOR VARIOUS CURVED PLATES

$$F_{cr} = \frac{k_s \pi^2 E}{12(1-\nu_e^2)} \left(\frac{t}{b}\right)^2 \quad ; \quad Z_b = \frac{b^2}{rt}(1-\nu_e^2)^{.5}$$



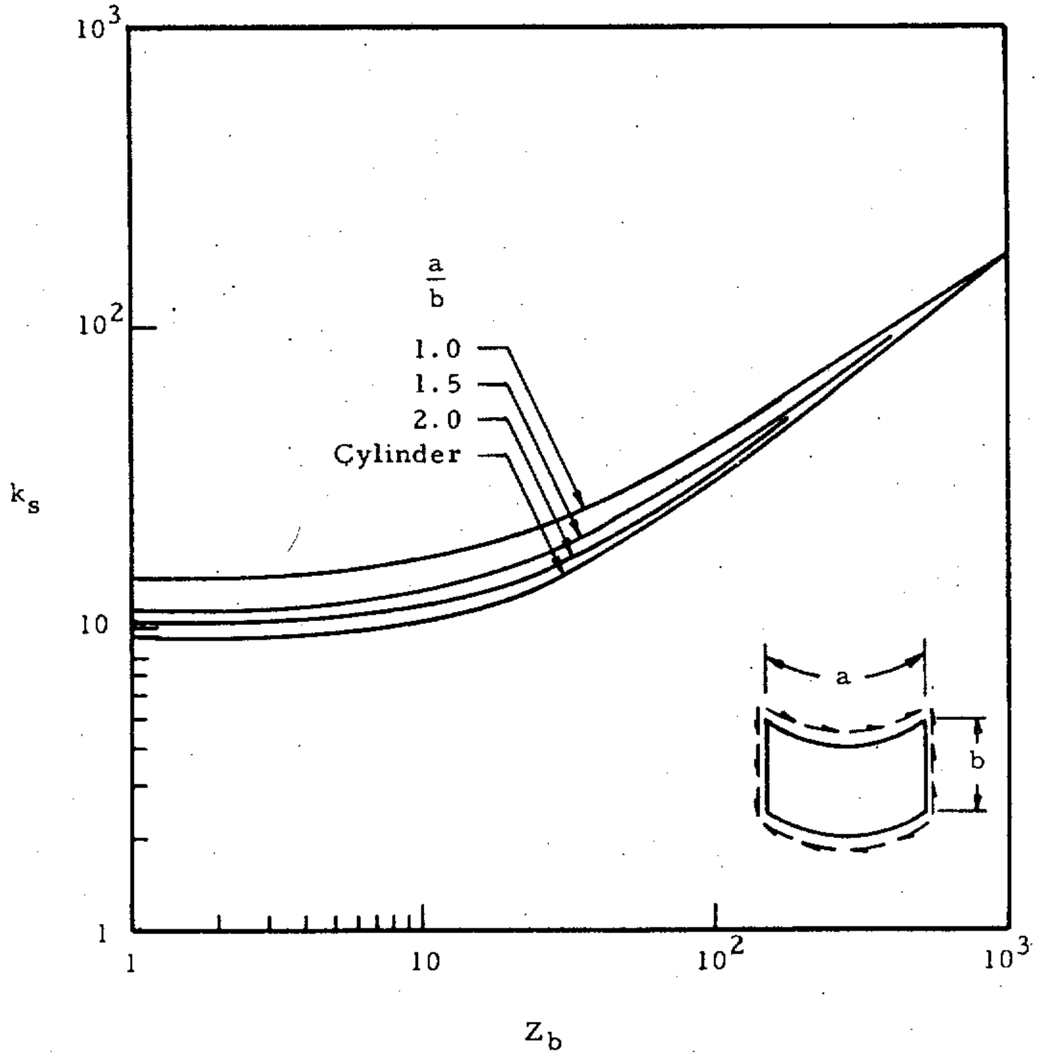
(b) Long clamped plates

FIGURE 7.28 (CONT'D) SHEAR BUCKLING COEFFICIENTS FOR VARIOUS CURVED PLATES



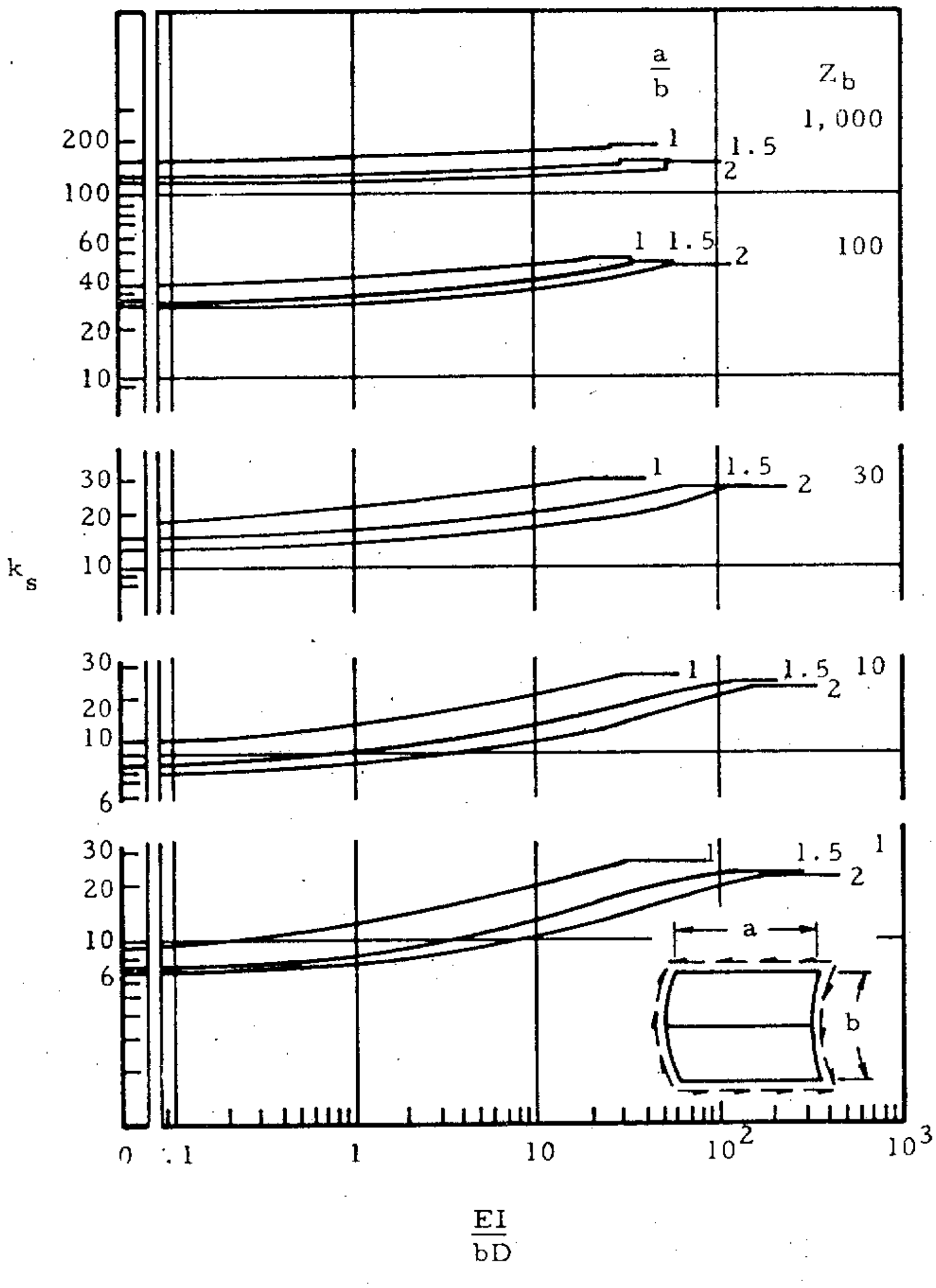
(c) Wide, simply supported plates

FIGURE 7.28 (CONT'D) SHEAR BUCKLING COEFFICIENTS FOR VARIOUS CURVED PLATES



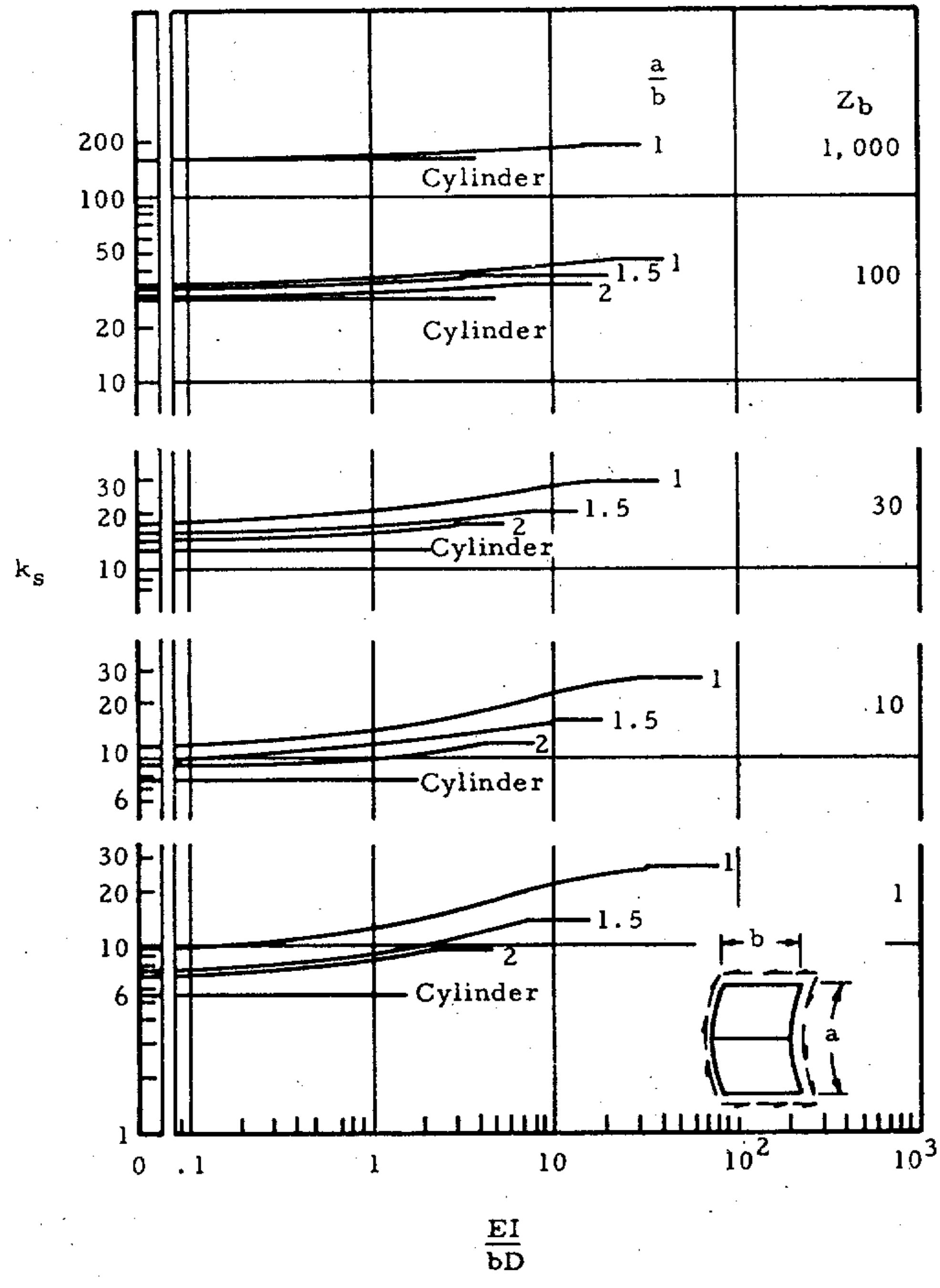
(d) Wide clamped plates

FIGURE 7.28 (CONT'D) SHEAR BUCKLING COEFFICIENTS FOR VARIOUS CURVED PLATES



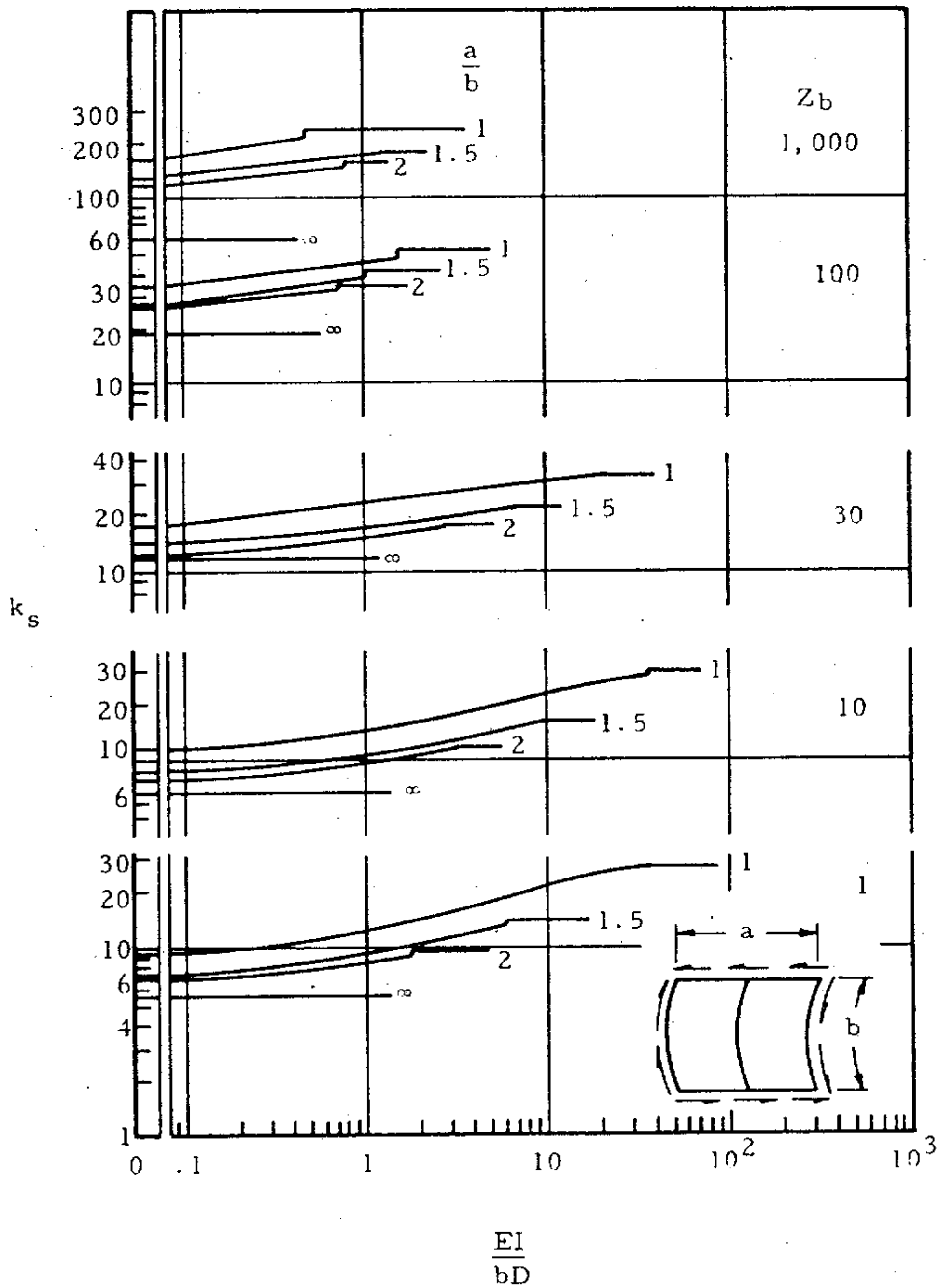
(a) Center axial stiffener; axial length greater than circumferential width

FIGURE 7.29 SHEAR-BUCKLING COEFFICIENTS FOR SIMPLY SUPPORTED CURVED PLATES WITH CENTER STIFFENER



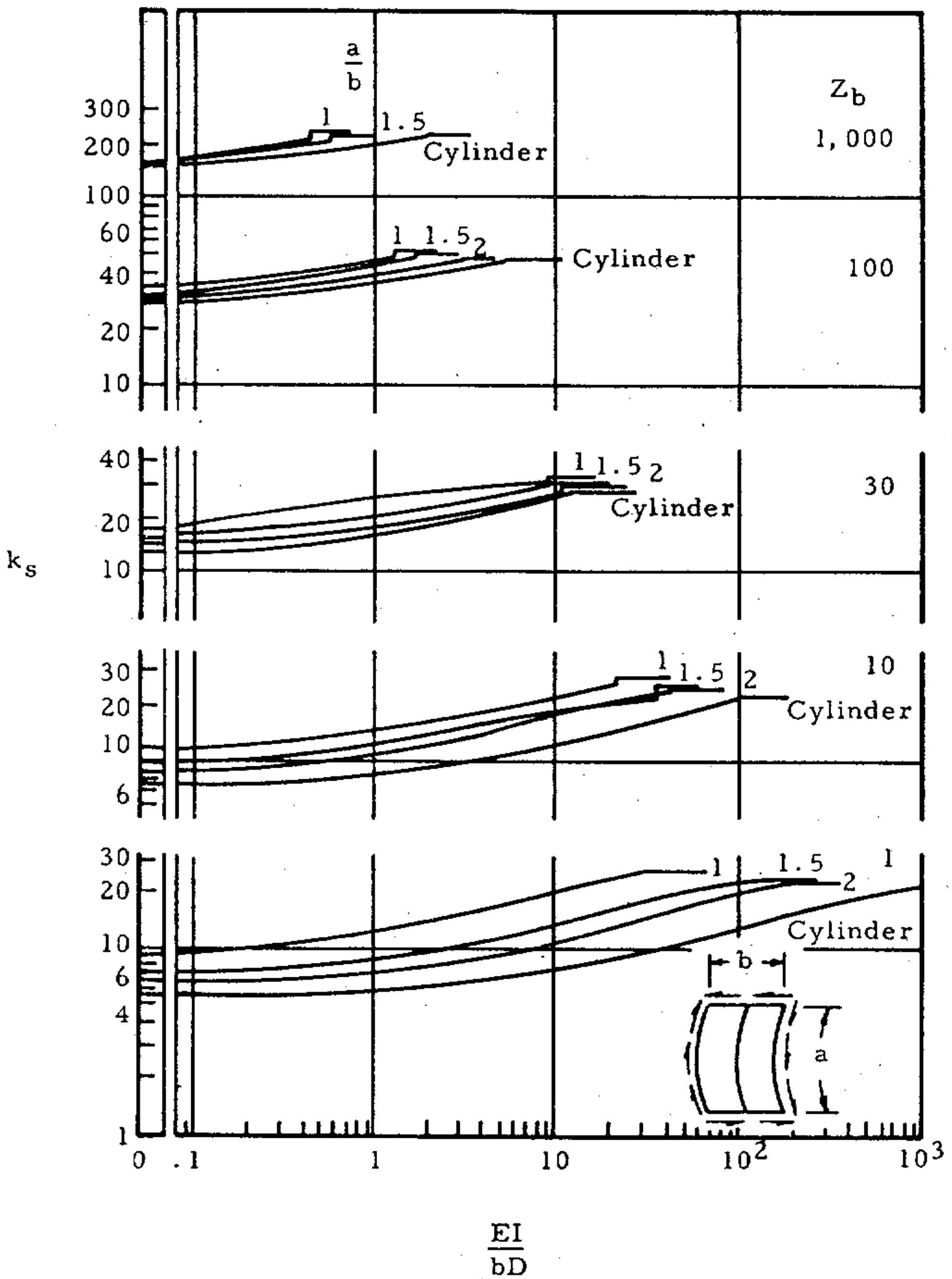
(b) Center axial stiffener; circumferential wiatn greater than axial length.

FIGURE 7.29 (CONT'D) SHEAR-BUCKLING COEFFICIENTS FOR SIMPLY SUPPORTED CURVED PLATES WITH CENTER STIFFENER



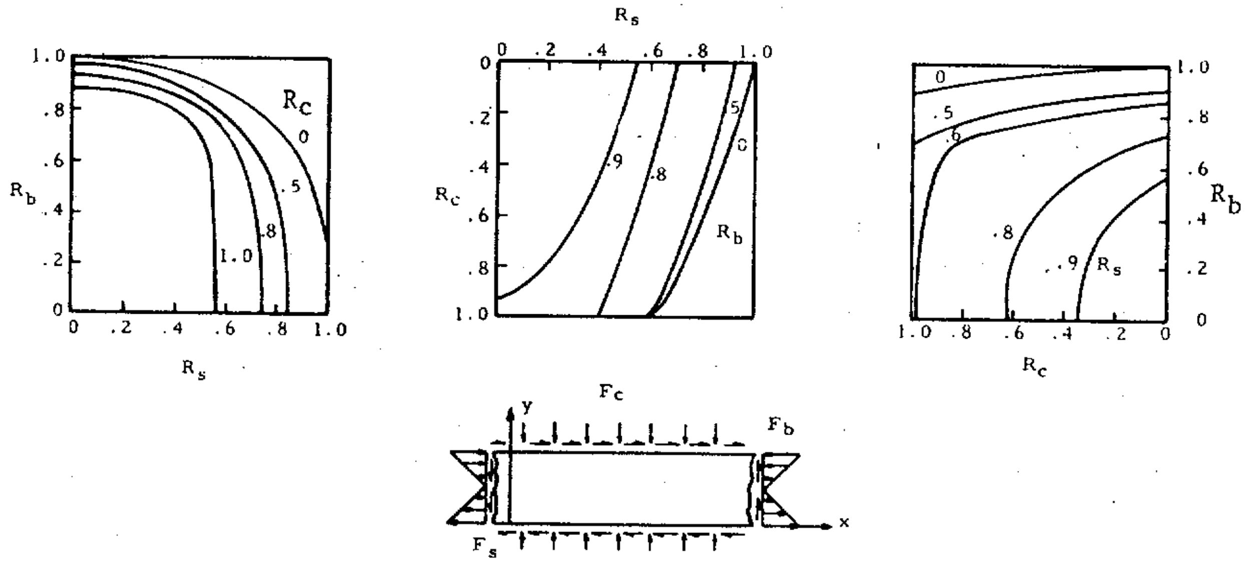
(c) Center circumferential stiffener; axial length greater than circumferential width.

FIGURE 7.29 (CONT'D) SHEAR-BUCKLING COEFFICIENTS FOR SIMPLY SUPPORTED CURVED PLATES WITH CENTER STIFFENER

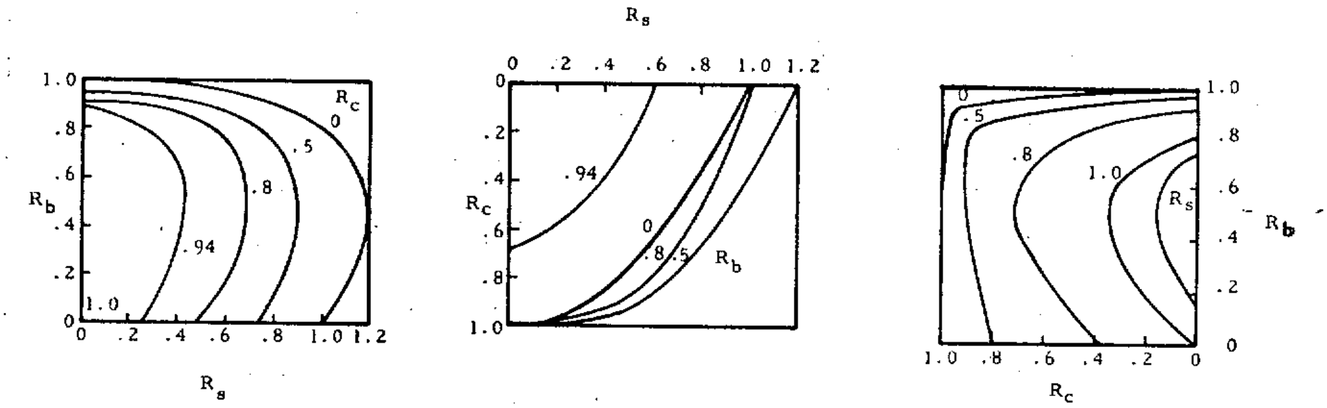


(d) Center circumferential stiffener; circumferential width greater than axial length

FIGURE 7.29 (CONT'D) SHEAR-BUCKLING COEFFICIENTS FOR SIMPLY SUPPORTED CURVED PLATES WITH CENTER STIFFENER

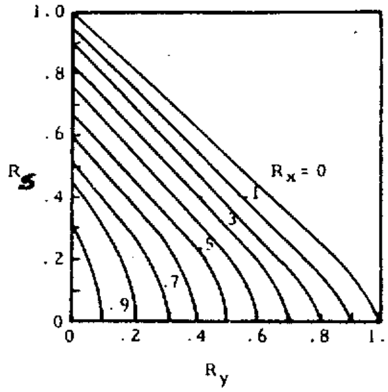


(a) Upper and lower edges simply supported

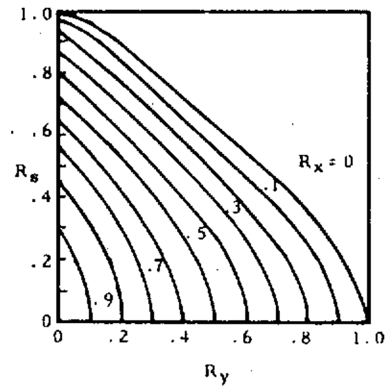


(b) Upper edges simply supported, lower edges clamped

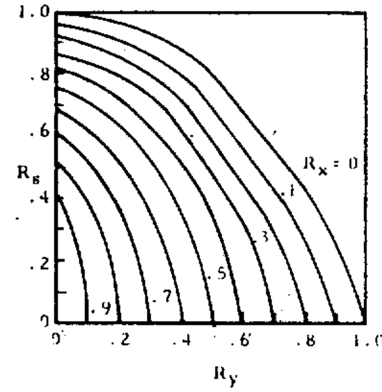
FIGURE 7.30 INTERACTION CURVES FOR LONG FLAT PLATES UNDER VARIOUS COMBINATIONS OF COMPRESSION, BENDING, AND SHEAR



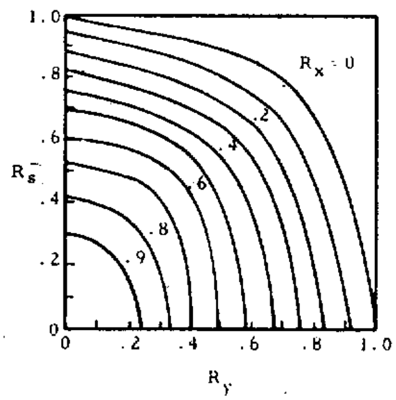
(a) a/b = 0.8



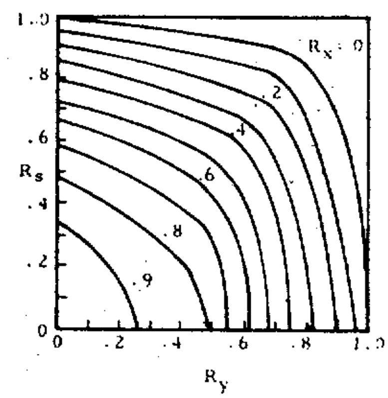
(b) a/b = 1.0



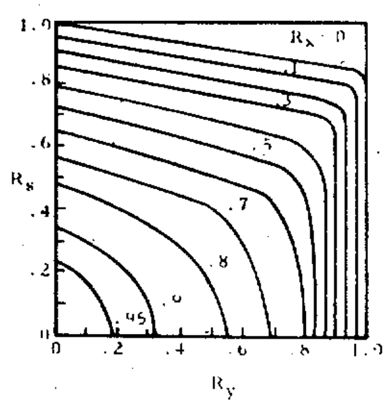
(c) a/b = 1.20



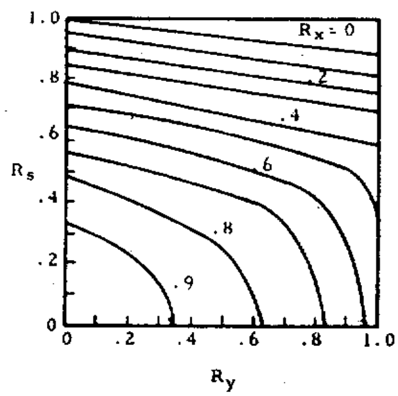
(d) a/b = 1.60



(e) a/b = 2.0



(f) a/b = 3.0



(g) a/b = ∞

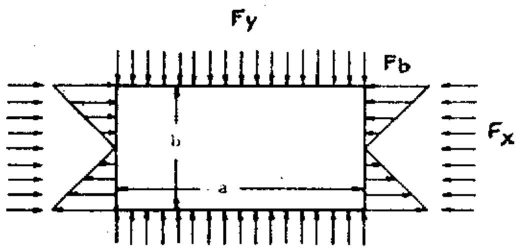


FIGURE 7.31 INTERACTION CURVES FOR FLAT RECTANGULAR PLATES UNDER COMBINED BIAXIAL-COMPRESSION AND LONGITUDINAL-BENDING LOADINGS

7.9 Triangular Flat Plates

Figure 7-32 presents buckling coefficients for isosceles triangular plates loaded under shear and compression. Equation 7-31 is the interaction equation for shear and normal stress on this type of plate.

$$\left(\frac{2F_s}{F_{crs+} + F_{crs-}} + u\right)^2 + \frac{F}{F_{cr}}(1-u^2) = 1 \qquad (7\text{-}31)$$

The + and - subscripts refer to either tension or compression along the altitude upon the hypotenuse of the triangle caused by pure shear loading. Table 7-12 contains values of k_c, k_{s+} and k_{s-} for various edge supports.

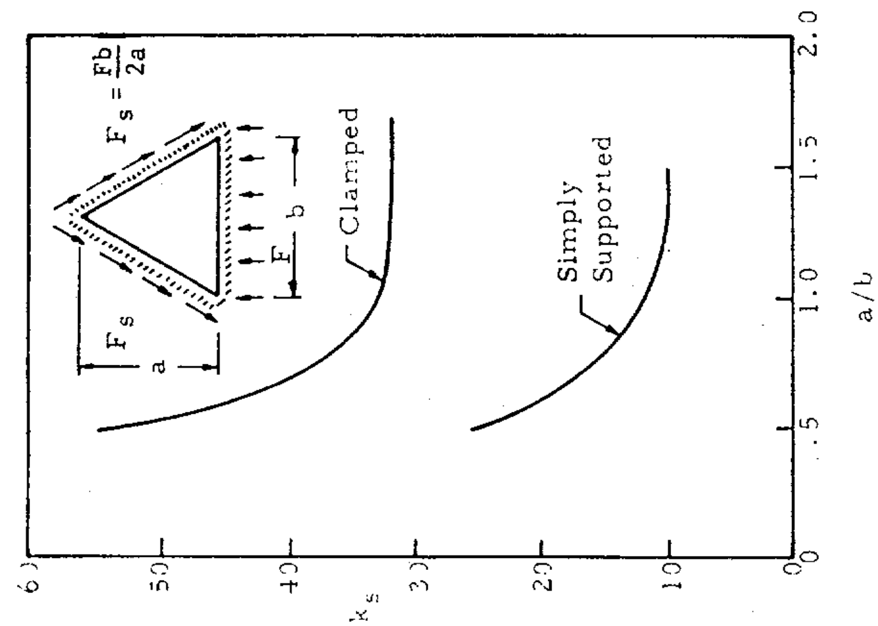
7.10 Buckling of Oblique Plates

In many instances, the use of rectangular panels is not possible. Figures 7-33 and 7-34 give buckling coefficients for panels which are oriented oblique to the loading. Figure 7-33 covers flat plates divided into oblique parallelogram panels by nondeflecting supports. Figure 7-34 covers single oblique panels.

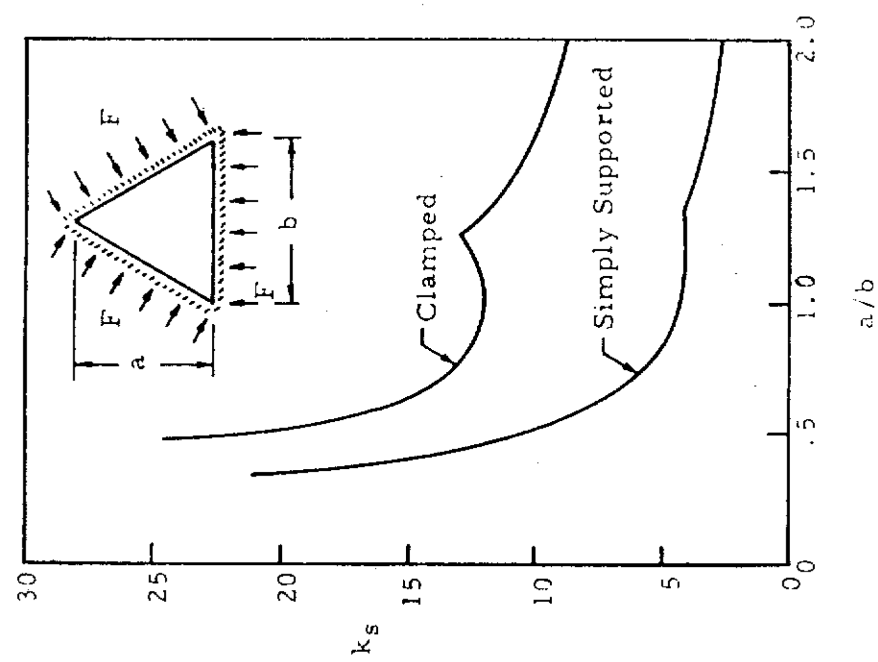
Edge Supports (a)	k_c	k_{s+}	k_{s-}
All edges simply supported	10.0	62.0	23.2
Sides simply supported, hypotenuse clamped	15.6	70.8	34.0
Sides clamped, hypotenuse simply supported	18.8	80.0	44.0

[a]Hypotenuse = b in Figure 7.32

TABLE 7.12 BUCKLING COEFFICIENTS FOR RIGHT-ANGLE ISOSCELES TRIANGULAR PLATES LOADED INDEPENDENTLY IN UNIFORM COMPRESSION, POSITIVE SHEAR, AND NEGATIVE SHEAR

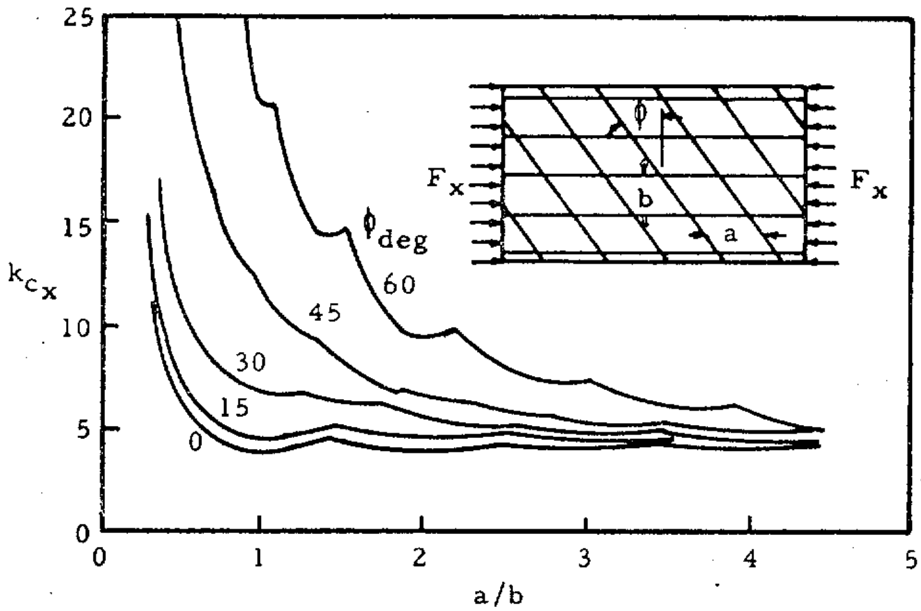


(b) Shear

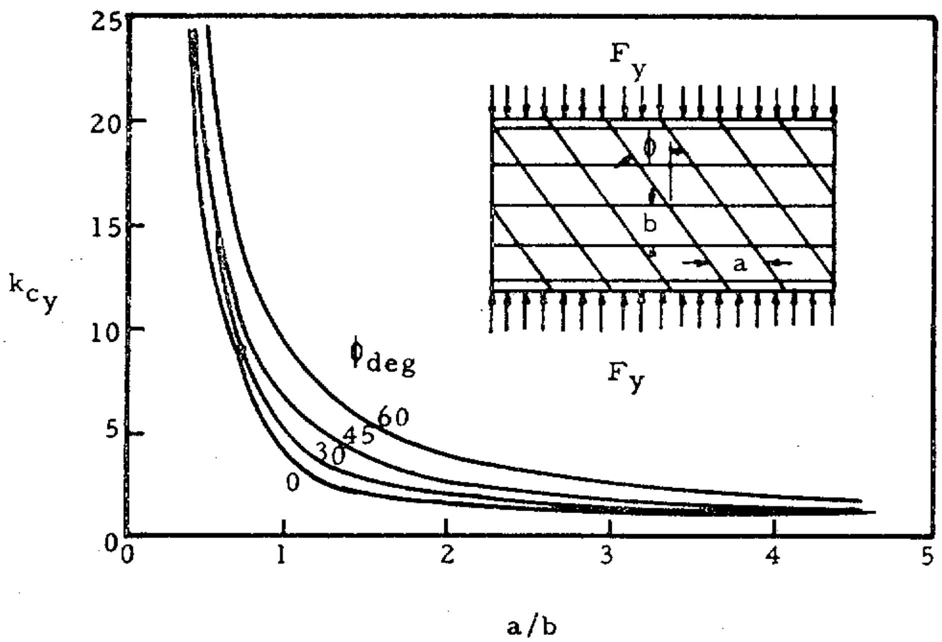


(a) Uniform Compression

FIGURE 7.32 BUCKLING COEFFICIENTS FOR ISOSCELES TRIANGULAR PLATES



(a) Loading in x-direction



(b) Loading in y-direction

FIGURE 7.33 COMPRESSIVE-BUCKLING COEFFICIENTS FOR FLAT SHEET ON NONDEFLECTING SUPPORTS DIVIDED INTO PARALLELOGRAM-SHAPED PANELS (All panel sides are equal.)

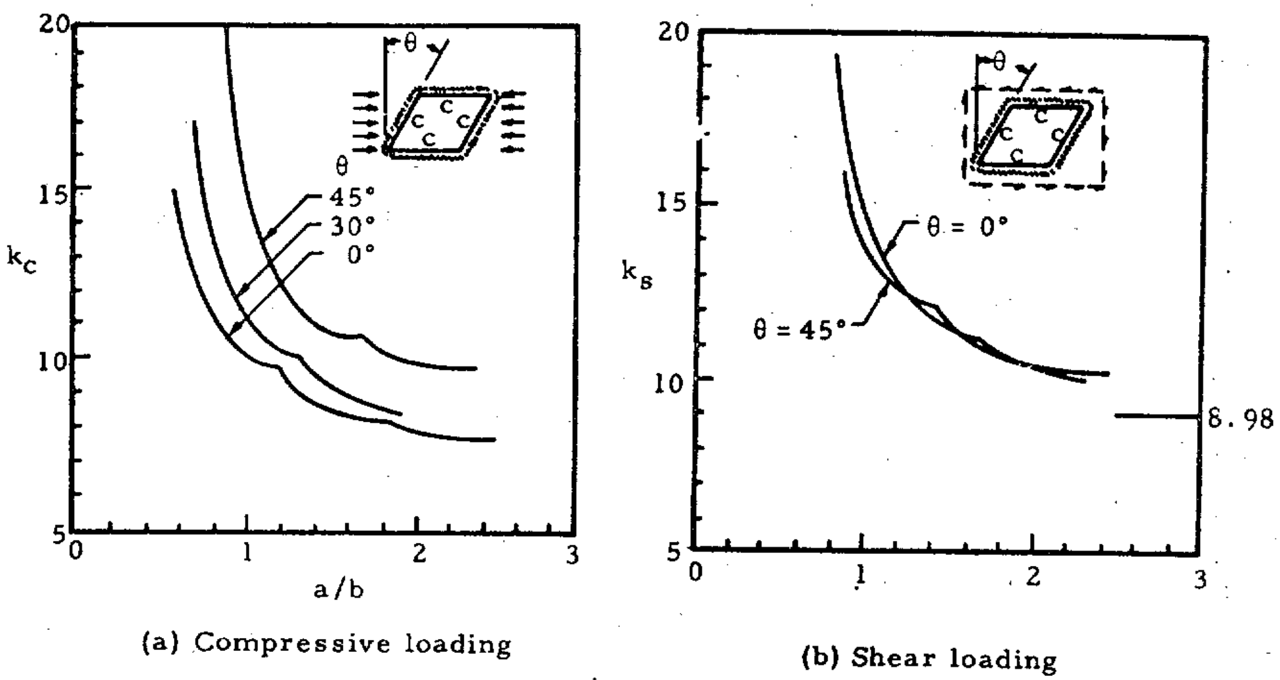


FIGURE 7.34 BUCKLING COEFFICIENT OF CLAMPED OBLIQUE FLAT PLATES

7.11 Introduction to Membranes

A membrane may be defined as a plate that is so thin that it may be considered to have no bending rigidity. The only stresses present are in the plane of the surface and are uniform throughout the thickness of the membrane. This section consists of methods of analysis of circular membranes, long rectangular membranes (a/b > 5), and short rectangular membranes (a/b < 5).

7.12 Nomenclature for Membranes

a	= length dimension of membrane
b	= width dimension of membrane
D	= diameter
E	= modulus of elasticity
f	= calculated stress
f_{max}	= calculated maximum stress
$n_1 - n_7$	= coefficients given in Figure 7.40
p	= pressure
R	= outside radius of circular membrane
r	= cylindrical coordinate
t	= thickness of membrane
x, y	= rectangular coordinates
δ	= deflection
δ_c	= center deflection of circular membrane
μ	= Poisson's ratio

7.13 Circular Membranes

Figure 7.35 shows two views of a circular membrane with the edge clamped under a uniform pressure, p.

The maximum deflection of this membrane is at the center and is given by

$$\delta_c = 0.662\, R \sqrt[3]{\frac{p\,R}{Et}} \qquad (7\text{-}32)$$

The deflection of the membrane at a distance, r, from the center is

$$\delta = \delta_c \left[1 - .09 \left(\frac{r}{R} \right)^2 - 0.1 \left(\frac{r}{R} \right)^5 \right] \qquad (7\text{-}33)$$

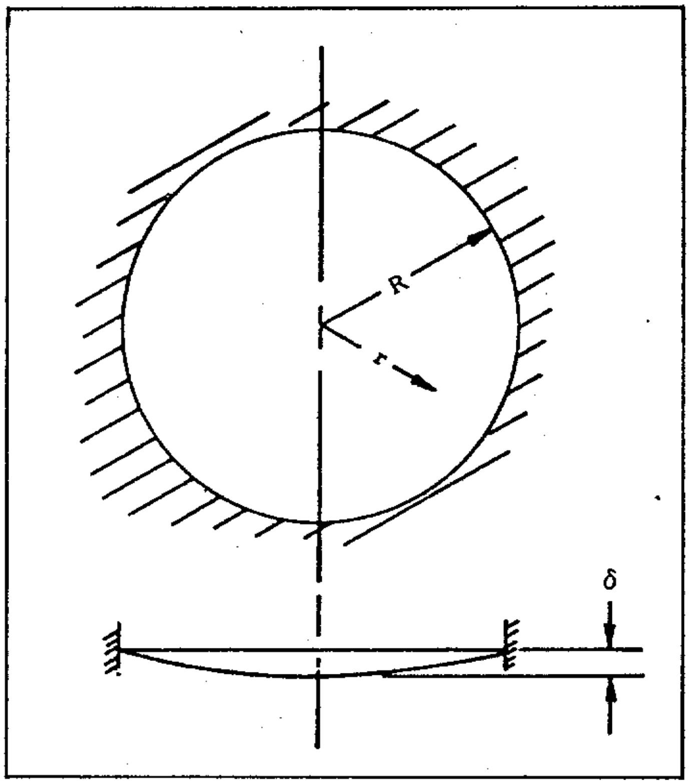


FIGURE 7.35. CIRCULAR MEMBRANE WITH CLAMPED EDGE

The stress at the center of this membrane is

$$f = 0.423 \sqrt[3]{\frac{Ep^2R^2}{t^2}} \qquad (7\text{-}34)$$

while that at the edge is

$$f = 0.328 \sqrt[3]{\frac{Ep^2R^2}{t^2}} \qquad (7\text{-}35)$$

7.14 Long Rectangular Membranes

Figure 7.36 shows a long rectangular membrane (a/b > 5) clamped along the two long sides.

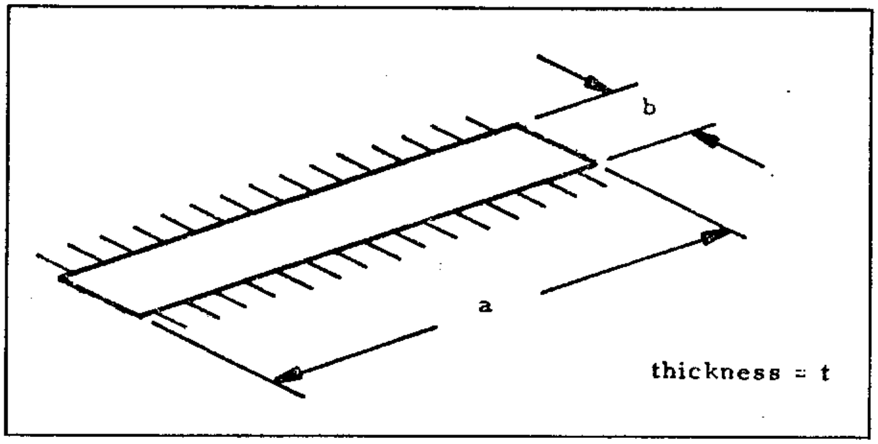


FIGURE 7.36. LONG RECTANGULAR MEMBRANE CLAMPED ON TWO LONG SIDES

The deflection and stress at the center of a long membrane clamped on all four sides are approximately the same as those in a long membrane clamped along the two long sides. The maximum stress and center deflection of the membrane in Figure 7.36 under uniform pressure p are given by Equations (7-36) and (7-37).

$$f_{max} = \left[\frac{p^2Eb^2}{24(1 - \mu^2)t^2}\right]^{1/3} \qquad (7\text{-}36)$$

$$\frac{\delta}{b} = \frac{1}{8}\left[\frac{24(1 - \mu^2)pb}{Et}\right]^{1/3} \qquad (7\text{-}37)$$

These equations are presented graphically in Figures 7.37 and 7.38 for $\mu = 0.3$.

A long rectangular plate may be considered to be a membrane if $P/E(b/t)^4$ is greater than 100.

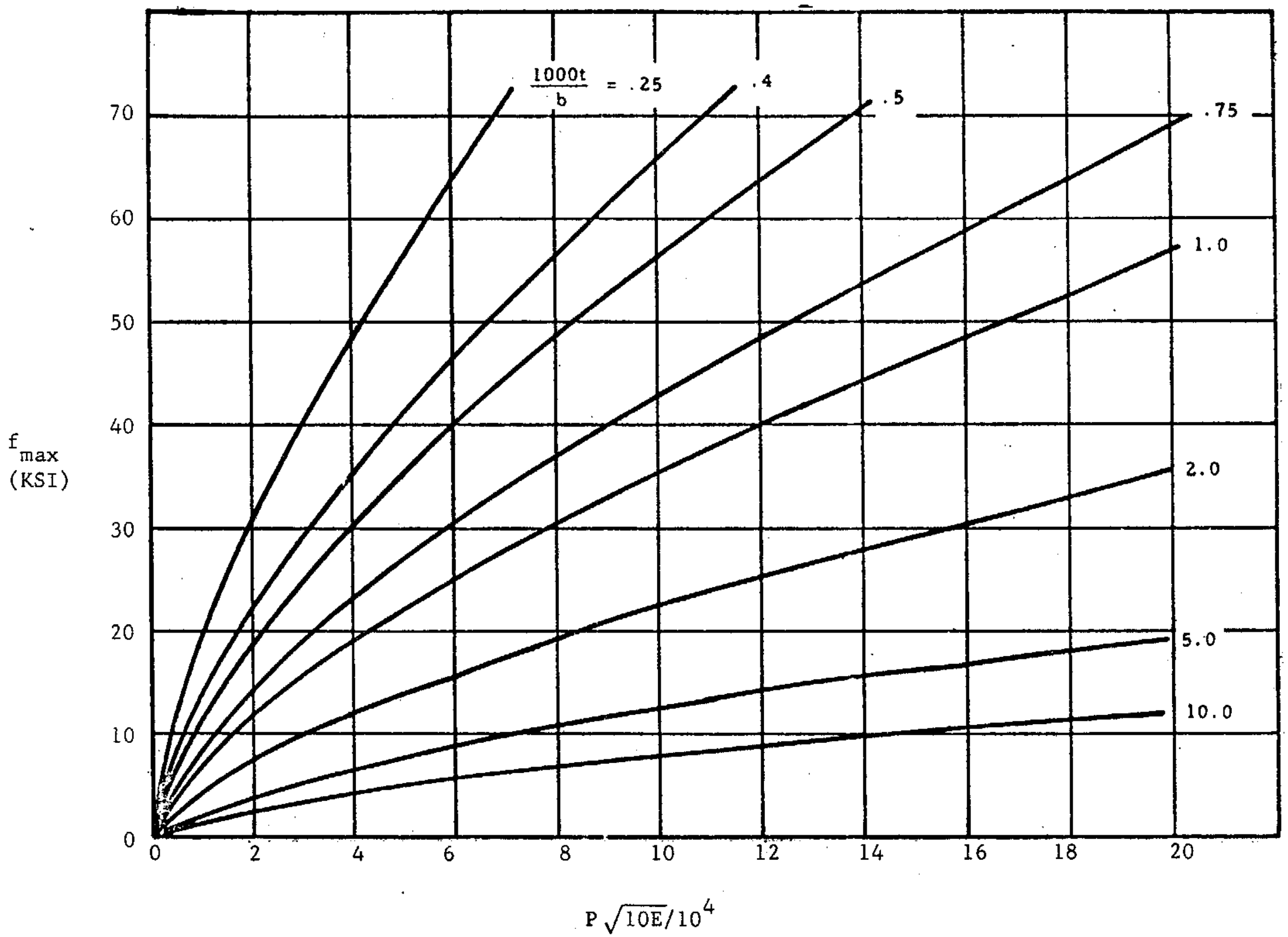


FIGURE 7.37 MAXIMUM STRESS IN LONG RECTANGULAR MEMBRANES (a/b > 5) HELD ALONG LONG SIDES (μ = 0.3)

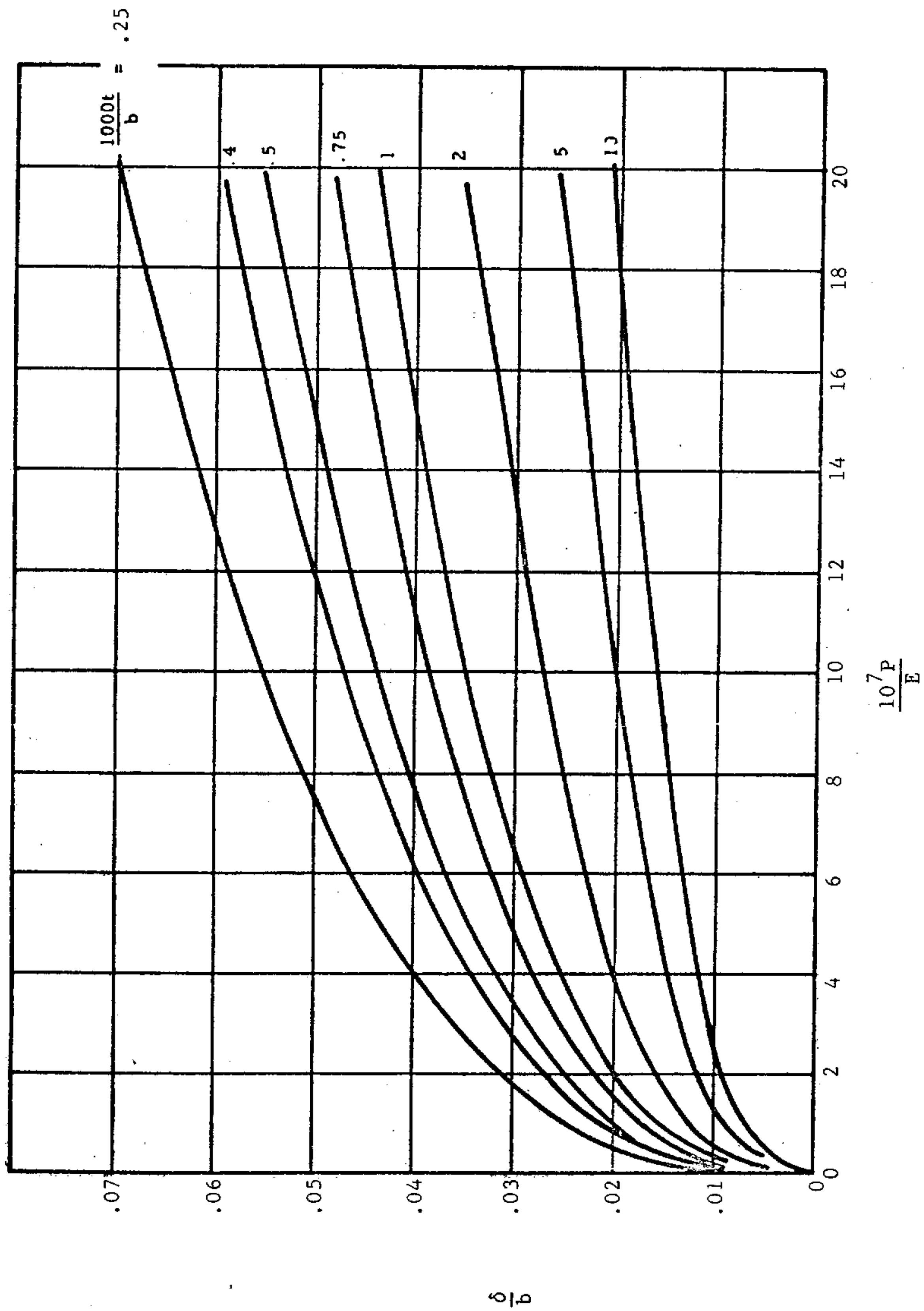


FIGURE 7.38 CENTER DEFLECTION OF LONG RECTANGULAR MEMBRANES (a/b > 5) HELD ALONG LONG SIDES (μ = 0.3)

7.15 Short Rectangular Membranes

Figure 7.39 shows a short rectangular membrane (a/b < 5) clamped on four sides under a uniform pressure p.

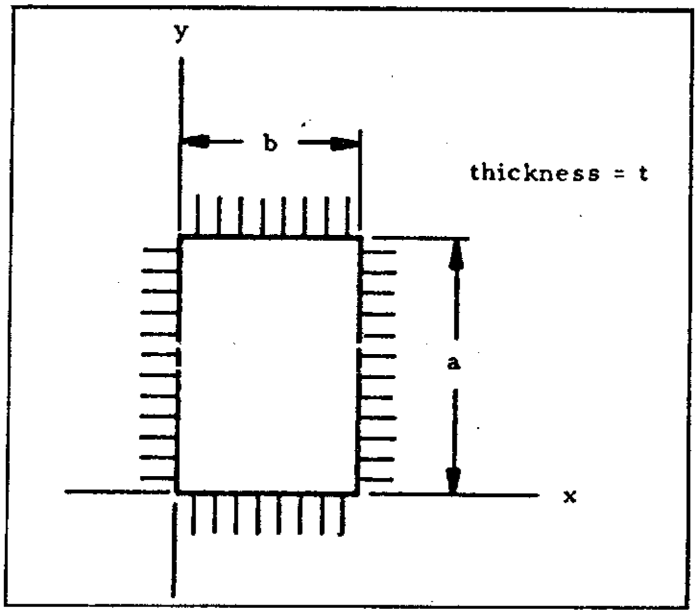


FIGURE 7.39. SHORT RECTANGULAR MEMBRANE CLAMPED ON FOUR SIDES

The deflection at the center of such a membrane is

$$\delta = n_1 \, a \sqrt[3]{\frac{pa}{Et}} \tag{7-38}$$

where n_1 is given in Figure 7.40.

The stresses at various locations on short rectangular membranes are given by the following equations for which the values of the coefficients n_2 through n_7 are given in Figure 7.40.

Center of plate (x = b/2, y = a/2)

$$f_x = n_2 \sqrt[3]{p^2 E\left(\frac{a}{t}\right)^2} \tag{7-39}$$

$$f_y = n_3 \sqrt[3]{p^2 E\left(\frac{a}{t}\right)^2} \tag{7-40}$$

Center of short side ($x = b/2$, $y = 0$)

$$f_x = n_4 \sqrt[3]{p^2 E \left(\frac{a}{t}\right)^2} \qquad (7\text{-}41)$$

$$f_y = n_5 \sqrt[3]{p^2 E \left(\frac{a}{t}\right)^2} \qquad (7\text{-}42)$$

Center of long side ($x = 0$, $y = a/2$)

$$f_x = n_6 \sqrt[3]{p^2 E \left(\frac{a}{t}\right)^2} \qquad (7\text{-}43)$$

$$f_y = n_7 \sqrt[3]{p^2 E \left(\frac{a}{t}\right)^2} \qquad (7\text{-}44)$$

It should be noted that the maximum membrane stress occurs at the center of the long side of the plate.

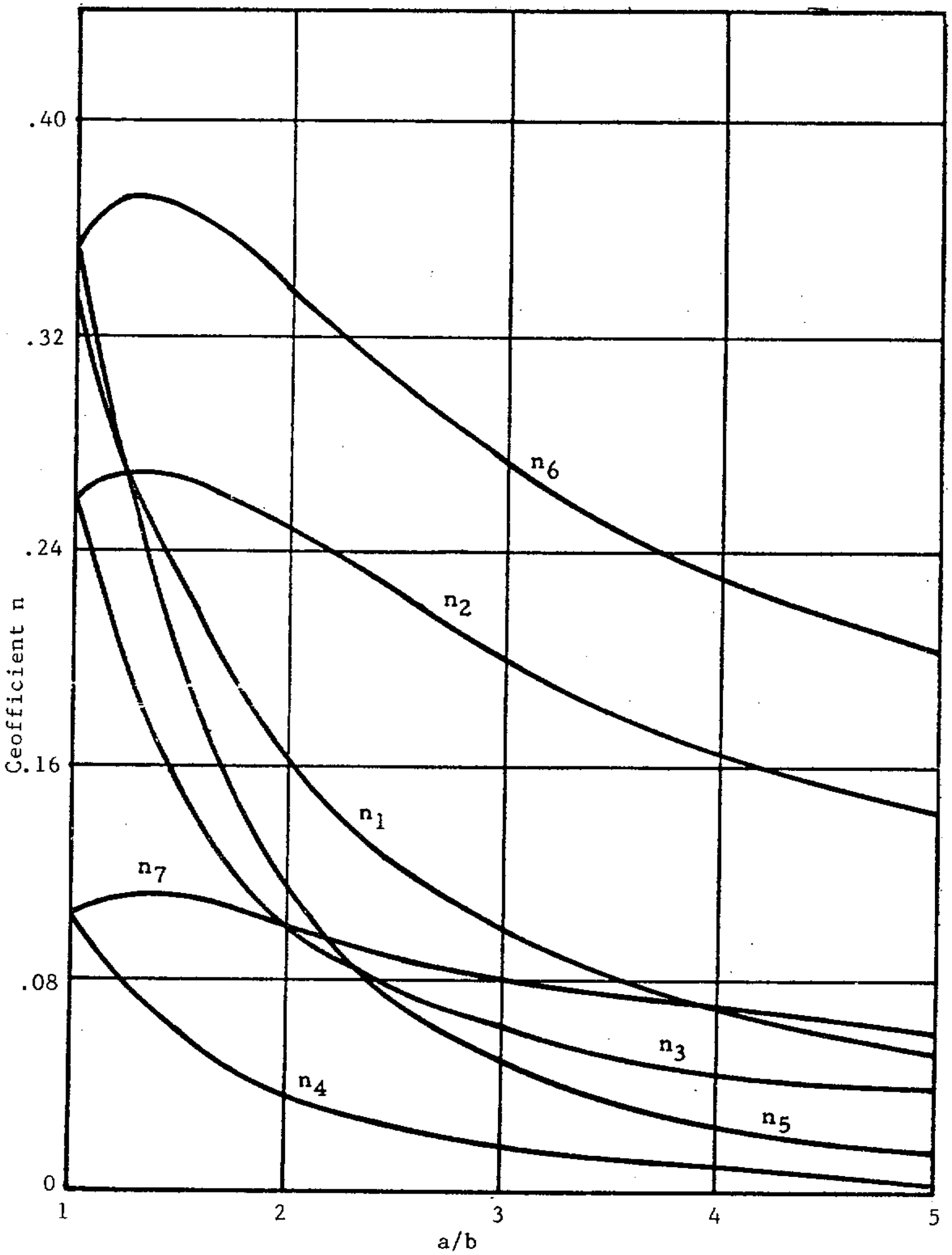


FIGURE 7.40 COEFFICIENTS FOR EQUATIONS (7-38) THROUGH (7-44)

SECTION 8

TORSION

8.0 GENERAL

This section presents the analysis methods and allowables for members torsionally loaded. Members subjected to torsion are categorized according to their cross sections for analysis purposes, i.e. (1) solid sections, (2) thin walled closed sections and (3) thin walled open sections.

8.1 Torsion of Solid Sections

The torsional stress (f_s) and resulting angle of twist (ϕ) for an applied twisting moment can be determined when the material and section properties of the bar are known.

The torsional shear stress (f_s) distribution on any cross section of a circular bar will vary linearly along any radial line emanating from the geometric centroid and will have the same distribution on all radial lines. The longitudinal shear stress (f_x) which is equal to the torsional shear stress (f_s) produces no warping of the cross section when the stress distribution is the same on adjacent radial lines. For non-circular sections the torsional shear stress distribution is nonlinear (except along lines of symmetry where the cross section contour is normal to the radial line) and will be different on adjacent radial lines. When the torsional and longitudinal shear stresses are different on adjacent radial lines, warping of the cross section will occur.

When the warping deformation induced by longitudinal shear stresses is restrained, normal stress (σ) are induced to maintain equilibrium. These normal stresses are neglected in the torsional analysis of solid sections since they are small, attenuate rapidly and have little effect on the angle of twist. Restraints to the warping deformation occur at fixed ends and at points where there is an abrupt change in the applied twisting moment.

The torsional analysis of solid cross sections is subject to the following limitations:

1) The material is homogeneous and isotropic.
2) The shear stress does not exceed the shearing proportional limit and is proportional to the shear strain (elastic analysis).
3) The stresses calculated at points of constraint and at abrupt changes of applied twisting moment are not exact.
4) The applied twisting moment cannot be an impact load.
5) If the bar has an abrupt change in cross section, stress concentrations must be used.

The basic equation for determining the torsional shear stress at some arbitrary point on an arbitrary cross section is

$$f_s = T_{(x)}/Q_{(x)} \qquad 8.1$$

where $T_{(x)}$ is the applied torque at some distance x along the beam and Q is the torsional section modulus at the same place.

The basic equation for determining the angle of twist between two points x distance apart is

$$\emptyset = 1/G \int_{x_1}^{x_2} T_{(x)}/K_{(x)} \; dx \qquad 8.2$$

where $K_{(x)}$ is the torsional constant. The area-moment technique can be used to determine the angle of twist between any two sections by plotting $T_{(x)}/GK_{(x)}$ for the beam.

Table 8.1 shows equations for calculating stress and angle of twist for some commonly used cross sections. The equations are for points of maximum torsional shear stress. Some cross sections have torsional stress equations shown for more than one section. The angle of twist equations are for a bar of length L and constant cross section.

When a circular beam of nonuniform cross section is twisted, the radii of a cross section becomes curved. Since the radii of a cross section were assumed to remain straight in the derivation of the equations for stress in uniform circular beams, these equations no longer hold if a beam is nonuniform. However, the stress at any section of a nonuniform circular beam is given with sufficient accuracy by the equations for uniform bars if the diameter changes gradually. If the change in section is abrupt, as at a shoulder with a small fillet, a stress concentration must be applied.

In nonuniform circular beams having gradual diameter changes, the angle of twist can be determined using equation 8.2. This equation is used to determine the equations for Ø in Table 8.2 for various beams of uniform taper.

8.2 Torsion of Thin-Walled Closed Sections

A closed section is any section where the center line of the wall forms a closed curve. The torsional shear stress distribution varies along any radial line emanating from the geometric centroid of the thin-walled closed section. Since the thickness of the thin walled section is small compared to the radius, the stress varies very little through the thickness of the cross section and is assumed to be constant through the thickness at that point.

The angle of twist of a thin-walled closed beam of length, L, due to an applied torque, T, is given by

$$\emptyset = \frac{TL}{4A^2G} \int \frac{du}{t} \qquad 8.3$$

$\phi = \frac{TL}{KG}$ T = Applied Torque(in-lb) G = Modulus of Rigidity(psi)

$f_s = \frac{T}{Q}$ L = Length of Beam(in) ϕ = Angle of Twist(rad)

K = Torsional Constant(in^4) f_s = Shear Stress(psi)

Q = Section Modulus(in^3)

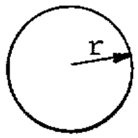
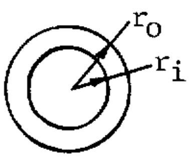
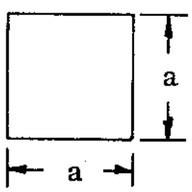
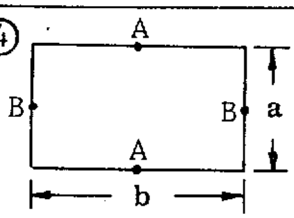
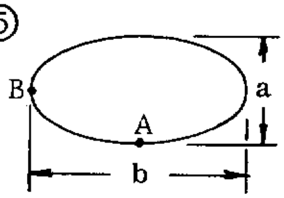
SECTION	K	Q	MAX STRESS
①  SOLID CIRCLE	$\frac{\pi r^4}{2}$	$\frac{\pi r^3}{2}$	at r_{max}
②  HOLLOW CIRCLE	$\frac{\pi}{2}(r_o^4 - r_i^4)$	$\frac{\pi}{2r_o}(r_o^4 - r_i^4)$	at r_o
③  SOLID SQUARE	$0.1406\,a^4$	$0.208\,a^3$	at midpoint of each side
④  SOLID RECTANGLE	$\beta b a^3$ $\beta = \left[.333 - \frac{.21}{(b/a)}\left(1 - \frac{0.0833}{(b/a)^4}\right)\right]$	$\alpha b a^2$ $\alpha = \frac{1}{\left[3 + \frac{1.8}{(b/a)}\right]}$	@A: $f_s = \frac{T}{Q}$ @B: $f_s = \frac{Ta}{Qb}$
⑤  SOLID ELLIPSE	$\frac{\pi b^3 a^3}{16(b^2 + a^2)}$	$\frac{\pi b a^2}{16}$	@A: $f_s = \frac{T}{Q}$ @B: $f_s = \frac{Ta}{Qb}$

TABLE 8.1 - EQUATIONS FOR STRESS AND DEFORMATION IN SOLID SECTIONS LOADED IN TORSION

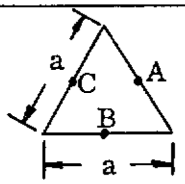
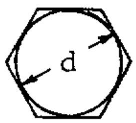
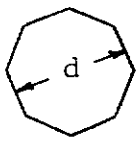
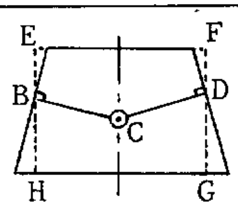
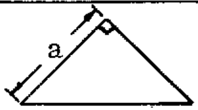
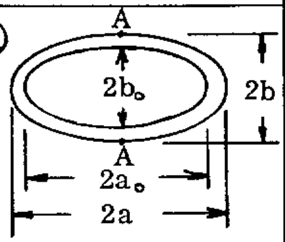
SECTION	K	Q	MAX STRESS
⑥  SOLID EQUILATERAL TRIANGLE	$\frac{a^4\sqrt{3}}{80}$	$\frac{a^3}{20}$	at A, B & C
⑦  SOLID HEXAGON	$0.1045d^4$	$0.1704d^3$	at midpoint of each side
⑧  SOLID OCTAGON	$0.1021d^4$	$0.1751d^3$	at midpoint of each side
⑨  SOLID ISOSCELES TRAPEZOID	Form equivalent rectangle through points B and D. Then use equations for rectangle to determine stress and twist. To locate B and D, construct perpendiculars from centroid (c) to each side (B and D).		
⑩  SOLID RIGHT ISOSCELES TRIANGLE	$0.0261a^4$	$0.0554a^3$	at center of long side
⑪  HOLLOW ELLIPSE	$\frac{\pi a^3 b^3 (1-q^4)}{a^2+b^2}$ $q = \frac{a_o}{a} = \frac{b_o}{b}$	$\frac{\pi a b^2 (1-q^4)}{2}$	at A

TABLE 8.1 (CONT'D) - EQUATIONS FOR STRESS AND DEFORMATION IN SOLID SECTIONS LOADED IN TORSION

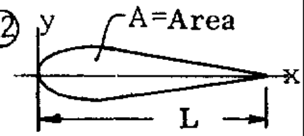
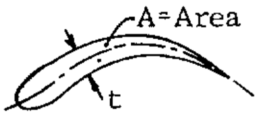
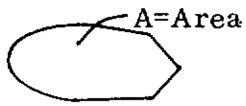
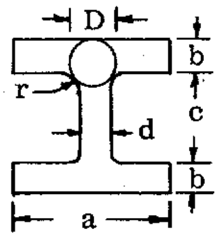
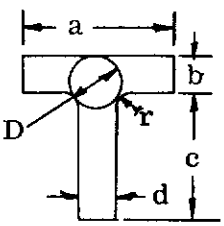
SECTION	K	Q	MAX STRESS
(12) 	$\frac{4I_x}{\left(1+\frac{16I_x}{AL^2}\right)}$	For cases 12 through 18, f_{max} occurs at or very near one of the points where the largest inscribed circle touches the boundary unless there is a sharp re-entrant at some other point on the boundary causing high local stresses. Of the points where the largest inscribed circle touches the boundary, f_{max} occurs at the one where the boundary curvature is algebraically least. Convexity represents positive, concavity negative, curvature of the boundary. At a point where the curvature is positive (boundary of section straight or convex) the maximum stress is given approximately by: $f_{s_{max}} = G\phi C/L$ or $f_{s_{max}} = TC/K$ $C = \frac{D}{1+\frac{\pi^2 D^4}{16A^2}}\left[1 + 0.15\left(\frac{\pi^2 D^4}{16A^2} - \frac{D}{2r}\right)\right]$ D = dia. of largest inscribed circle r = radius of curvature of boundary at the point (convex) A = area of section At a point where the curvature is negative (boundary of section concave or reentrant) the maximum stress is given approximately by: $f_{s_{max}} = G\phi C/L$ or $f_{s_{max}} = TC/K$ $C = \frac{D}{1+\frac{\pi^2 D^4}{16A^2}}\left[1 + \left\{0.118\ln\left(1 - \frac{D}{2r}\right) - 0.238D/2r\right\}\tanh\frac{2\phi}{\pi}\right]$	
(13)  dL = element length along median	$\frac{.333F}{1+\frac{4F}{3AL^2}}$ $F=\int_0^L t^3 dL$		
(14)  SOLID FAIRLY COMPACT SECTION WITHOUT REENTRANT ANGLES	$\frac{A^4}{40J}$ J = Polar Moment of Inertia		
(15)  D = dia inscribed cir $t=b$ if $b<d$ $t=d$ if $d<b$ $t_1=b$ if $b>d$ $t_1=d$ if $d>b$	$2K_1+K_2+2\alpha D^4$ $K_1=ab^3\left[\frac{1}{3}-\frac{0.21b}{a}\left(1-\frac{b^4}{12a^4}\right)\right]$ $K_2=\frac{cd^3}{3}$ $\alpha=\frac{t}{t_1}\left(0.15+0.1\frac{r}{b}\right)$ $D = b + \frac{(2r+d)^2}{4(2r+b)}$		
(16)  r, D, t, t_1 same as case (15)	$K_1+K_2+\alpha D^4$ $K_1=ab^3\left[\frac{1}{3}-\frac{0.21b}{a}\left(1-\frac{b^4}{12a^4}\right)\right]$ $K_2=cd^3\left[\frac{1}{3}-\frac{0.105d}{c}\left(1-\frac{d^4}{192c^4}\right)\right]$ $\alpha=\frac{t}{t_1}\left(0.15+0.1\frac{r}{b}\right)$		

TABLE 8.1 (CONT'D) - EQUATIONS FOR STRESS AND DEFORMATION IN SOLID SECTIONS LOADED IN TORSION

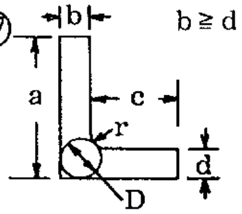
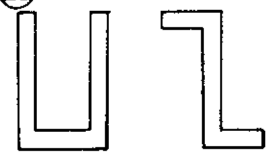
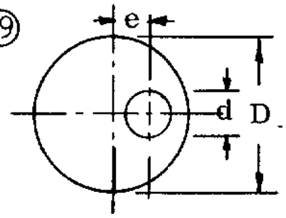
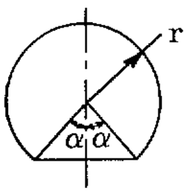
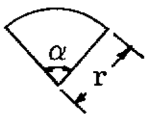
<table>
<tr><th>SECTION</th><th>K</th><th>Q</th><th>MAX STRESS</th></tr>
<tr><td>(17) </td><td>$K_1+K_2+\alpha D^4$
K_1 & K_2 per (16)
$\alpha=\frac{d}{b}\left(0.07+0.076\frac{r}{b}\right)$
$D = 2(3r + b + d) - [8(2r+b)(2r+d)]^{1/2}$</td><td colspan="2" rowspan="2">The angle Ø is the angle through which a tangent to the boundary rotates in turning or traveling around the reentrant portion, measured in radians.</td></tr>
<tr><td>(18) </td><td>Sum of K's of constituent 'L' sections computed per (17)</td></tr>
<tr><td>(19) $e/d=\lambda$
$d/D=n$</td><td>$\pi(D^4-d^4)/32Q$
$Q=1+\left[\frac{16n^2}{(1-n^2)(1-n^4)}\right]\lambda^2+\left[\frac{384n^4}{(1-n^2)^2(1-n^4)^4}\right]\lambda^4$</td><td colspan="2">$f_{s_{max}}=16TDF/\pi(D^4-d^4)$
$F=1+\left[\frac{4n^2}{1-n^2}\right]\lambda+\left[\frac{32n^2}{(1-n^2)(1-n^4)}\right]\lambda^2+\left[\frac{48n^2(1+2n^2+3n^4+2n^6)}{(1-n^2)(1-n^4)(1-n^6)}\right]\lambda^3+\left[\frac{64n^2(2+12n^2+19n^4+28n^6+18n^8+14n^{10}+3n^{12})}{(1-n^2)(1-n^4)(1-n^6)(1-n^8)}\right]\lambda^4$</td></tr>
<tr><td>(20) </td><td>Cr^4<table><tr><td>α</td><td>0</td><td>30°</td><td>60°</td><td>80°</td><td>90°</td></tr><tr><td>C</td><td>1.57</td><td>1.47</td><td>.91</td><td>.48</td><td>.296</td></tr></table></td><td colspan="2">$f_{s_{max}}=T/Q$; $Q=Cr^3$<table><tr><td>α</td><td>0</td><td>30°</td><td>60°</td><td>80°</td><td>90°</td></tr><tr><td>C</td><td>1.57</td><td>1.25</td><td>.80</td><td>.49</td><td>.35</td></tr></table></td></tr>
<tr><td>(21) </td><td>Cr^4<table><tr><td>α</td><td>45°</td><td>60°</td><td>90°</td><td>120°</td></tr><tr><td>C</td><td>.0181</td><td>.0349</td><td>.0825</td><td>.148</td></tr><tr><td>α</td><td>180°</td><td>270°</td><td>300°</td><td>360°</td></tr><tr><td>C</td><td>.296</td><td>.528</td><td>.686</td><td>.878</td></tr></table></td><td colspan="2">$f_{s_{max}}=T/Q$; $Q=Cr^3$<table><tr><td>α</td><td>60°</td><td>120°</td><td>180°</td></tr><tr><td>C</td><td>.0712</td><td>.227</td><td>.35</td></tr></table></td></tr>
</table>

TABLE 8.1 (CONT'D) - EQUATIONS FOR STRESS AND DEFORMATION IN SOLID SECTIONS LOADED IN TORSION

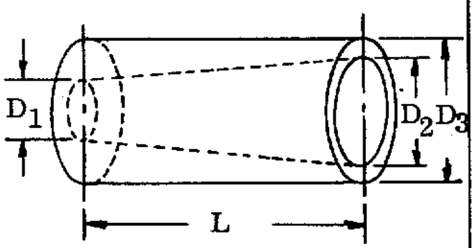
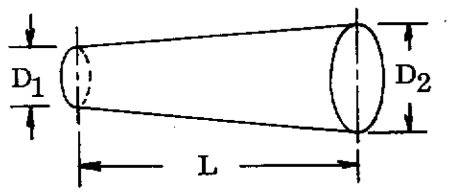
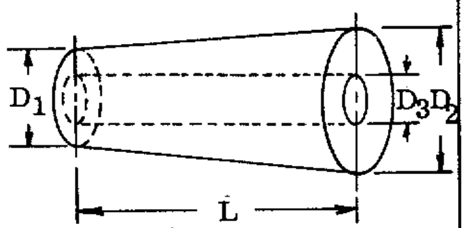
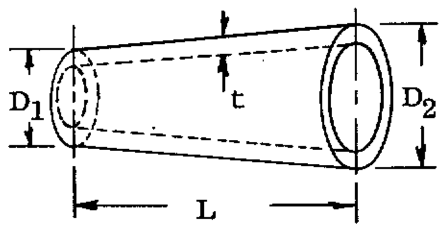
TYPE OF BEAM	ANGLE OF TWIST
 INSIDE TAPERED, OUTSIDE CONSTANT	$\Theta = \frac{8TL}{\pi G(D_2 - D_1)D_3^3}\left\{2 \arctan\left(\frac{D_2}{D_3}\right) - 2 \arctan\left(\frac{D_1}{D_3}\right) + \mathrm{Ln}\left[\left(\frac{D_3+D_2}{D_3-D_2}\right)\left(\frac{D_3-D_1}{D_3+D_1}\right)\right]\right\}$
 SOLID BEAM, OUTSIDE TAPERED	$\Theta = \frac{32TL}{3\pi G D_1 D_2}\left(\frac{1}{D_1^2} + \frac{1}{D_1 D_2} + \frac{1}{D_2^2}\right)$
 INSIDE UNIFORM OUTSIDE TAPERED	$\Theta = \frac{8TL}{\pi G(D_2 \; D_1)D_3^3}\left\{2 \arctan\left(\frac{D_2}{D_3}\right) - 2 \arctan\left(\frac{D_1}{D_3}\right) + \mathrm{Ln}\left[\left(\frac{D_3+D_2}{D_3-D_2}\right)\left(\frac{D_3-D_1}{D_3+D_1}\right)\right]\right\}$
 THIN TAPERED TUBE WITH CONSTANT WALL THICKNESS	$\Theta = \frac{2TL(D_1+D_2)}{\pi G t D_1^2 D_2^2}$

TABLE 8.2 - EQUATIONS FOR ANGLE OF TWIST FOR NONUNIFORM CIRCULAR BEAMS IN TORSION

where A is the area enclosed by the median line of the thickness, t, and u is the length along the median. The shear flow is constant around the tube and is

$$q = \frac{T}{2A} \tag{8.4}$$

The shear stress is assumed to be constant through the thickness and is

$$f_s = q/t = T/2At \tag{8.5}$$

If the cross section is of nonuniform thickness, the shear stress will be maximum where the thickness is minimum.

Table 8.3 shows the angle of twist and shear stress for thin-walled closed sections subject to an applied twist, T.

8.3 Torsion of Thin-Walled Open Sections

An open section is one in which the centerline of the wall does not form a closed curve. Channels, angles, I-beams, Tees and wide flanges are among structural shapes characterized by a combination of thin-walled rectangular shapes. Additionally many thin-walled open sections are curved. This section presents the means to calculate the stress and twisting angle for these sections.

For a bar of rectangular cross section of width b and thickness t the equations for maximum shearing stress and the angle of twist are

$$f_{s_{max}} = T/\alpha bt^2 \tag{8.6}$$

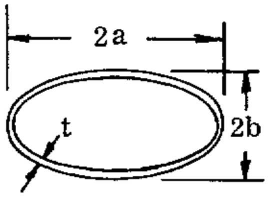
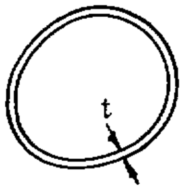
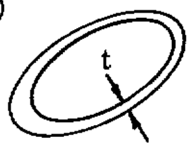
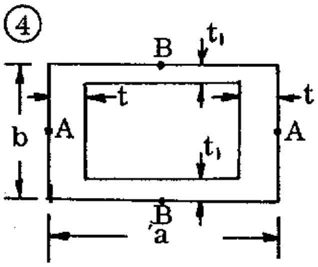
$$\phi = TL/\beta bt^3G \tag{8.7}$$

where α and β are defined in Table 8.1, Case 4 . When the ratio b/t becomes very large, α and β become 0.333. Equations 8.6 and 8.7 become

$$f_{s_{max}} = 3T/bt^2 \tag{8.8}$$

$$\phi = 3TL/bt^3G \tag{8.9}$$

These equations, 8.8 and 8.9, are applicable for narrow rectangles. They also apply to an approximate analysis of shapes made up of thin rectangular members such as those shown in Figure 8.1

SECTION	K	Q	MAX STRESS
① 	$\frac{4\pi^2 t\left[(a-.5t)^2(b-.5t)^2\right]}{U}$ U=length of median $U=\pi(a+b-t)\left[1+\frac{.27(a-b)^2}{(a+b)^2}\right]$	$2\pi t(a-.5t)(b-.5t)$	constant if t is small
② 	$\frac{4A^2 t}{U}$ U=length of median A=mean of areas enclosed by two boundaries	2tA	constant if t is small
③ 	$\frac{4A^2}{\int\frac{dU}{t}}$ U & A same as ②	2tA	at t_{min}
④ 	$\frac{2tt_1(a-t)^2(b-t_1)^2}{at+bt_1-t^2-t_1^2}$	@A: $2t(a-t)(b-t_1)$ @B: $2t_1(a-t)(b-t_1)$	There will be higher stresses at inner corners unless fillets of fairly large radius are used

Equations for twist and stress are shown in Table 8.1

TABLE 8.3 - EQUATIONS FOR STRESS AND DEFORMATION IN HOLLOW CLOSED SECTIONS LOADED IN TORSION

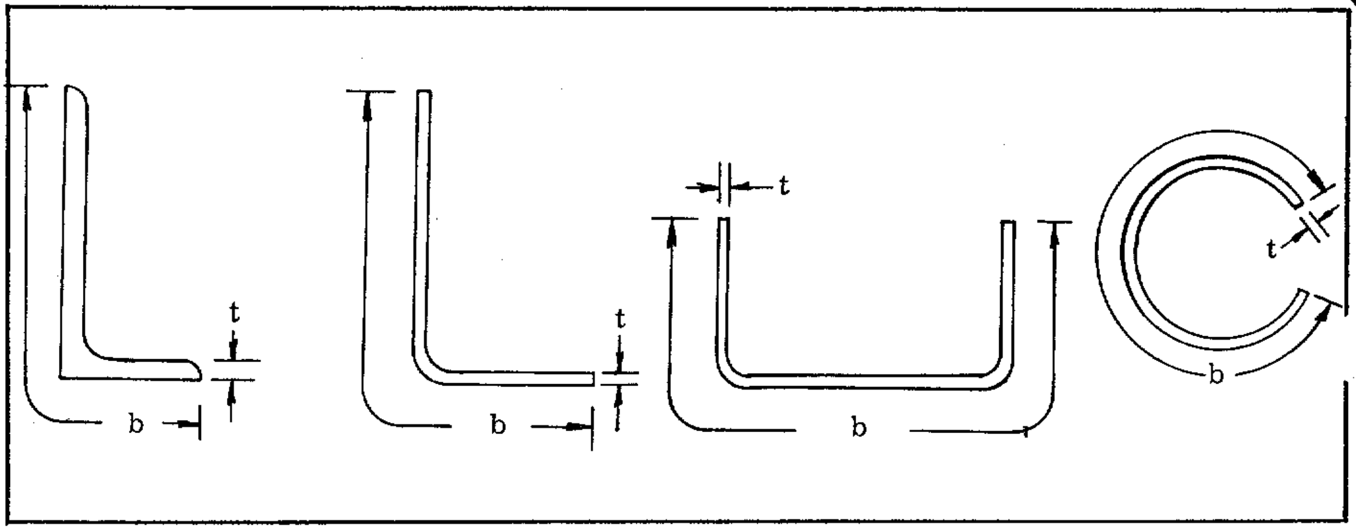


FIGURE 8.1 - BEAMS WITH THIN RECTANGULAR MEMBERS
WITH CONTINUOUS CENTERLINES

For the members shown in Figure 8.1, b can be taken as the continuous centerline of the member and equations 8.8 and 8.9 used to determine stress and angle of twist.

Shapes with a member of thin rectangular members such as T and H sections shown in Figure 8.2

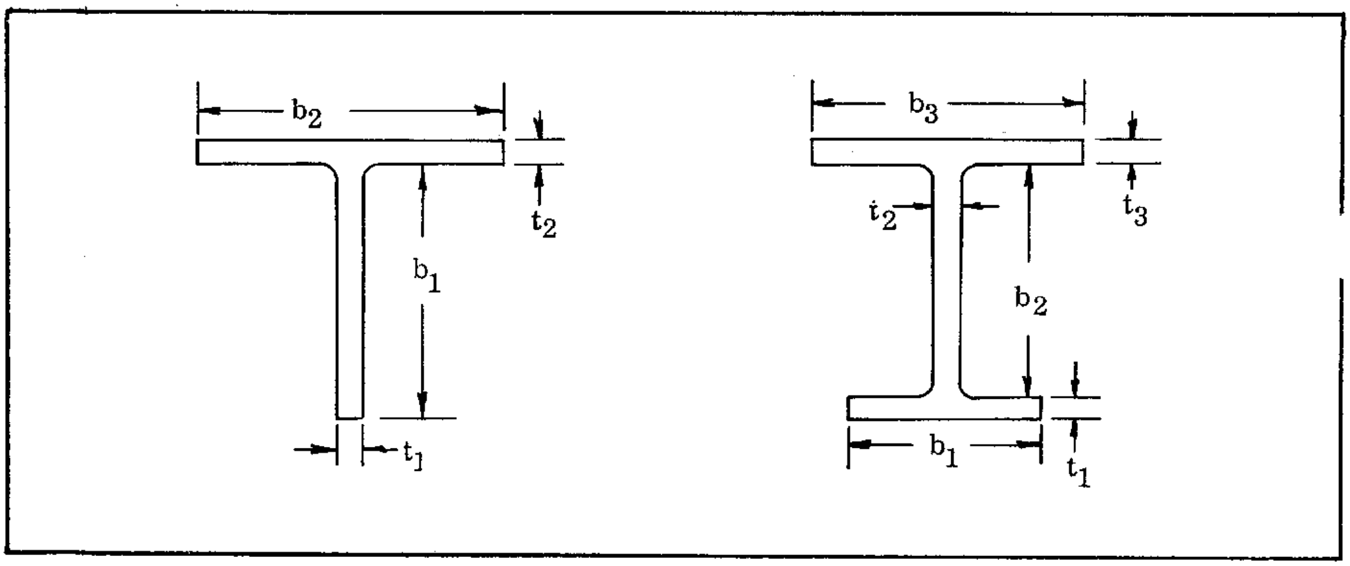


FIGURE 8.2 - BEAMS WITH THIN RECTANGULAR
MEMBERS OF COMPOSITE SHAPES

can be analyzed using equations 8.10 and 8.11

$$f_{s_{max_n}} = 3Tt_n / \Sigma bt^3 \qquad 8.10$$

$$\phi = 3TL/G\,\Sigma bt^3 \qquad 8.11$$

For the T section in Figure 8.2 the angle of twist is

$$\phi = 3TL/G(b_1 t_1^3 + b_2 t_2^3) \qquad 8.12$$

and the shear stress is

$$f_{s_1} = 3Tt_1/(b_1 t_1^3 + b_2 t_2^3) \qquad 8.13$$

$$f_{s_2} = 3Tt_2/(b_1 t_1^3 + b_2 t_2^3) \qquad 8.14$$

The same procedure is applicable for any type of shape; however, the accuracy is considerably improved when sharp corners are avoided by the use of liberal radii.

8.4 Multicell Closed Beams in Torsion

Figure 8.3 shows a multicell tube with an externally applied torsion. The torsion is reacted in the tube by internal shear flows acting around each cell.

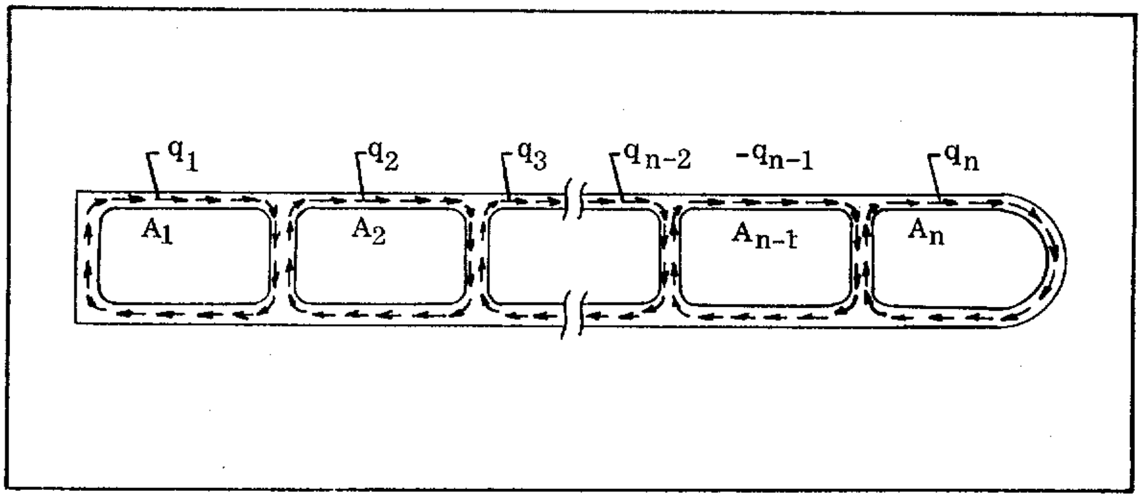


FIGURE 8.3 - MULTICELL TUBE IN TORSION

The tube has n cells with a pure torsion, T, applied. The torsion applied externally must be reacted internally. This is expressed by

$$T = 2q_1A_1 = 2q_2A_2 + \dots + 2q_nA_n \qquad 8.15$$

where A_1 through A_n are the areas enclosed by the median lines of cells 1 through n. q_1 through q_n are the reaction shear flows acting on cells 1 through n.

For elastic continuity, the twist of each cell must be equal, or

$$\phi_1 = \phi_2 = \dots = \phi_n \qquad 8.16$$

The angular twist of a cell is

$$\phi = q/2AG \oint ds/t \qquad 8.17$$

or

$$2G\phi = q/A \oint ds/t \qquad 8.18$$

Thus for each cell of multicell structure an expression for $q/A \oint ds/t$ can be written and equated to $2G\phi$. The line integral $\oint ds/t$ is represented by a. Then a_{KL} is the value of the integral along the wall between cells K and L where the area outside the tube is designated as cell 0. The shear flows acting in a clockwise direction are assumed to be positive. Using this rotation, equation 8.13 can be applied to each cell resulting in the following:

$$\text{cell (1)}: 1/A_1[q_1a_{10} + (q_1 - q_2)a_{12}] = 2G\phi \qquad 8.19$$

$$\text{cell (2)}: 1/A_2[(q_2 - q_1)a_{12} + q_2a_{20} + (q_2 - q_3)a_{23}] = 2G\phi \qquad 8.20$$

$$\text{cell (3)}: 1/A_3[(q_3 - q_2)a_{23} + q_3a_{30} + (q_3 - q_4)a_{34}] = 2G\phi \qquad 8.21$$

$$\text{cell(n-1)}: 1/A_{n-1}[(q_{n-1} - q_{n-2})a_{n-2,n-1} + q_{n-1} + q_{n-1}a_{n-1,0} + (q_{n-1} - q_n)a_{n-1,n}] = 2G\phi \qquad 8.22$$

$$\text{cell (n)}: 1/A_n\ [(q_n - q_{n-1})a_{n-1,n} + q_na_{no}] = 2G\phi \qquad 8.23$$

The shear flows, q_1 through q_n, may be found by solving equations 8.15 and 8.19 through 8.23 simultaneously. From these shear flows, the shear stresses may be found using $f_s = q/t$.

As an example, consider the multicell beam shown in Figure 8.4.

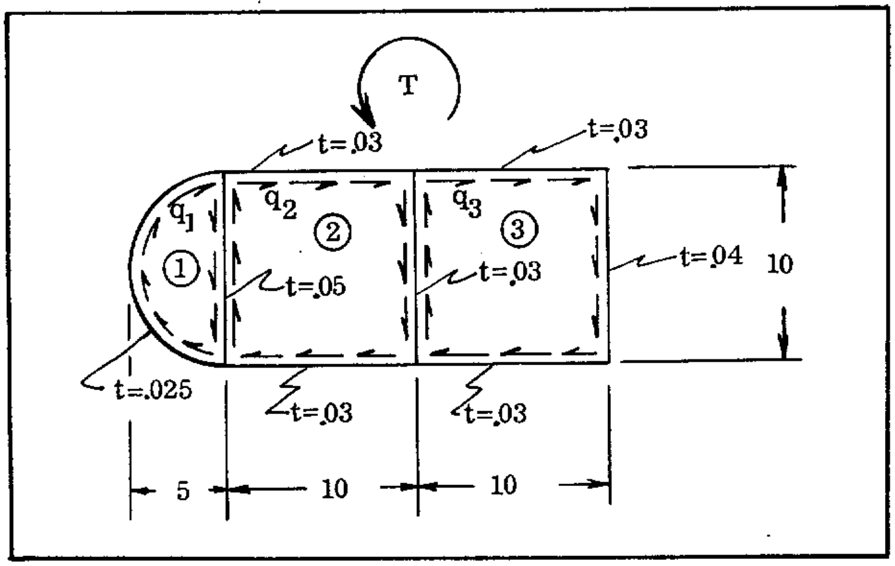


FIGURE 8.4 - EXAMPLE OF MULTICELL BEAM IN TORSION

Cell Areas:

$A_1 = 39.3 \qquad A_2 = 100 \qquad A_3 = 100$

Line integrals for each cell:

$a_{10} = 1/2(\pi)(10)/.025 = 628$

$a_{12} = 10/.05 = 200$

$a_{20} = 2(10)/.03 = 667$

$a_{23} = 10/.03 = 333$

$a_{30} = 2(10)/.03 + 10/.04 = 917$

Equate external torque to internal reactions:

$T = 2q_1A_1 + 2q_2A_2 + 2q_3A_3$, Equation 8.15

$100000 = 2(39.3)q_1 + 2(100)q_2 + 2(100)q_3$

$100000 = 78.6q_1 + 200q_2 + 200q_3$

Write the expression for angular twist of each cell:

Cell (1): $1/39.3\ [628q_1 + 200(q_1 - q_2)] = 2G\theta = 21.07q_1 - 5.09q_2$

Cell (2): $1/100\ [200(q_2 - q_1) + 667q_2 + 333(q_2 - q_3] = 2G\theta = -2q_1 + 12q_2 - 3.33q_3$

Cell (3): $1/100\ [333(q_3 - q_2) + 917q_3] = 2G\theta = -3.33q_2 + 12.5q_3$

Solving the equations simultaneously:

q_1 = 144 #/in, q_2 = 234 #/in, q_3 = 209 #/in, $\emptyset$ = .0002288 rad

8.5 Plastic Torsion

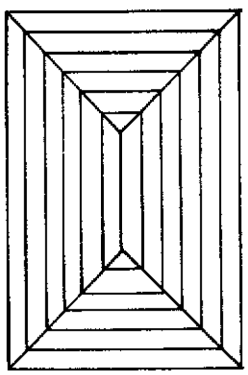
The previous methods of analysis are based on stress levels in the elastic range. These stress levels are based on limit loads. For ultimate loads, it is often desirable to allow the section to operate in the plastic region. All types of cross sections not subject to local crippling can be analyzed for allowable torsion by use of the plastic torsion theory at ultimate load. The method of analysis is called the "sand heap" analogy.

If the maximum amount of dry sand is heaped on a level platform having the same shape as the cross section of the beam in torsion, the slope of the heap represents the shear stress. The shear stress for this condition has the same magnitude over the entire cross section. The torsional moment, T, is related to the volume of the heap, V, by

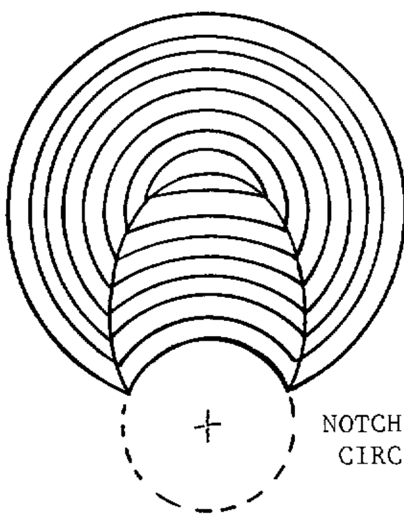
$$T = 2VF_{su} \qquad 8.24$$

where F_{su} is the ultimate allowable shear stress. The difficulty of the sand heap analogy is determining the volume of the heap. This is simplified somewhat by constructing contour lines. A contour line defines the contour of the heap at some constant elevation. It is a plane passed through the heap parallel to the torsional section. Also, it is assumed that the maximum possible slope of the heap is achieved, i.e. slope is equal unity.

It is easy to construct a contour map of the sand heap surface. Contour lines intersect normals through the section boundary at right angles and at a distance from the boundary equal to the elevation of the contour line.

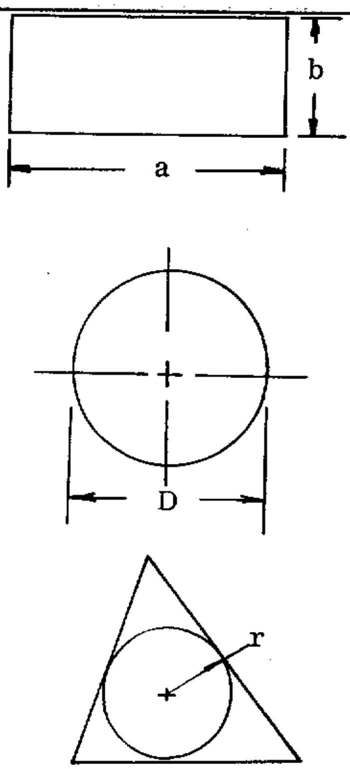


RECTANGLE



NOTCHED CIRCLE

It is possible to determine the volume of the sand heap for any cross section by integration. Figure 8.5 shows equations for sand heap volumes with various bases. For a surface with a hole, subtract the volume of sand that could be heaped on the hole alone.



$$V = \frac{b^2(3a-b)}{12} \qquad a \geq b$$

$$V = \frac{\pi D^3}{24}$$

$$V \doteq \frac{Ar}{3}$$

A = Area of triangle

FIGURE 8.5 - SAND HEAP VOLUMES

8.6 Allowable Stresses

For limit load conditions, the applied stresses should be kept below the ultimate shear stress, F_{su}. These are defined for various materials in MIL-HDBK-5.

The torsional failure of tubes may be due to plastic failure of the material, instability of the walls, or an intermediate condition. Pure shear failure will not usually occur within the range of wall thicknesses commonly used for aircraft tubing. Torsional allowable stresses are shown in Figure 8.6 through 8.22. These curves take into account the parameter L/D and are in good agreement with experimental results.

Interaction data of Section 4 should be used when other stresses are combined with torsion.

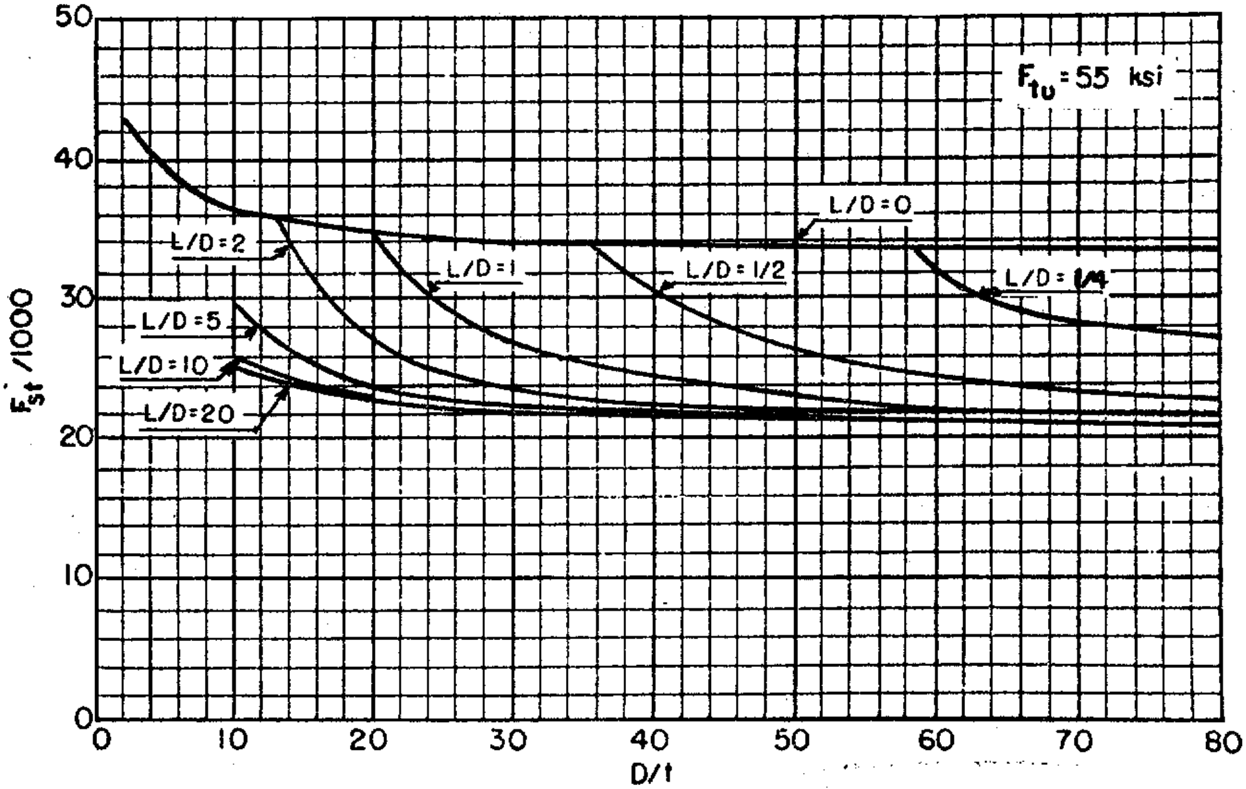


FIGURE 8.6 - TORSIONAL MODULUS OF RUPTURE - PLAIN CARBON STEELS
F_{tu} = 55 ksi

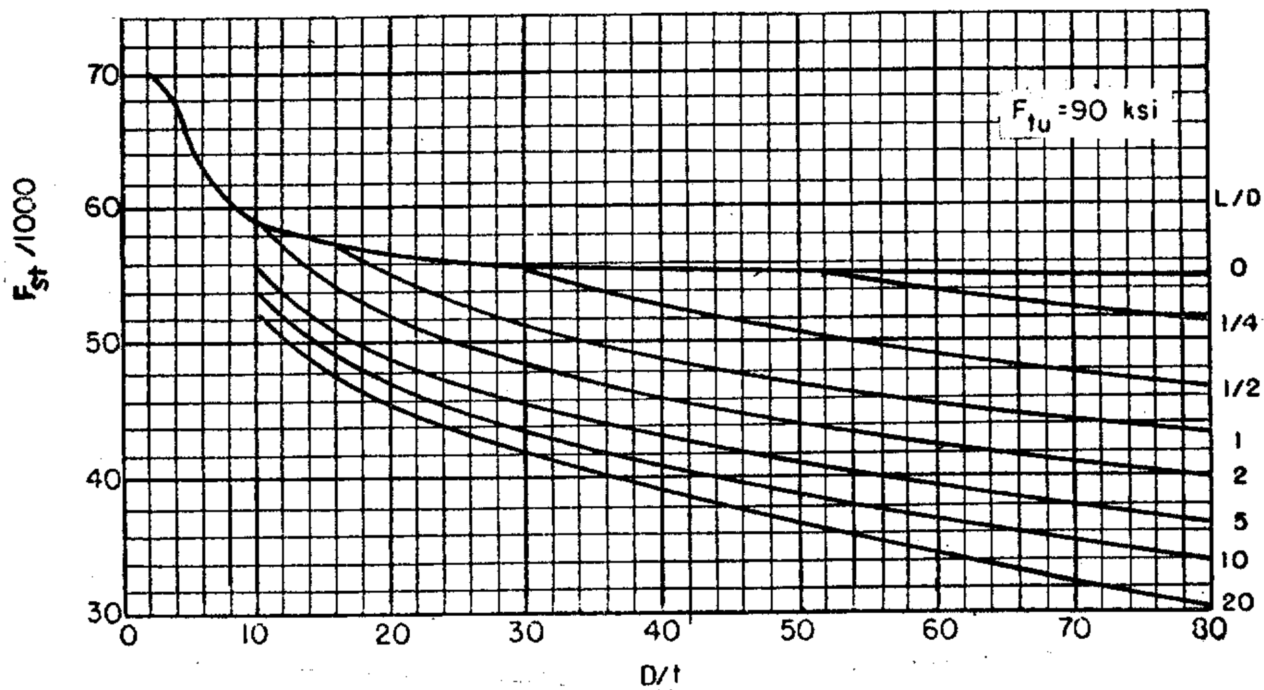


FIGURE 8.7 - TORSIONAL MODULUS OF RUPTURE - ALLOY STEELS HEAT TREATED
TO F_{tu} = 90 ksi

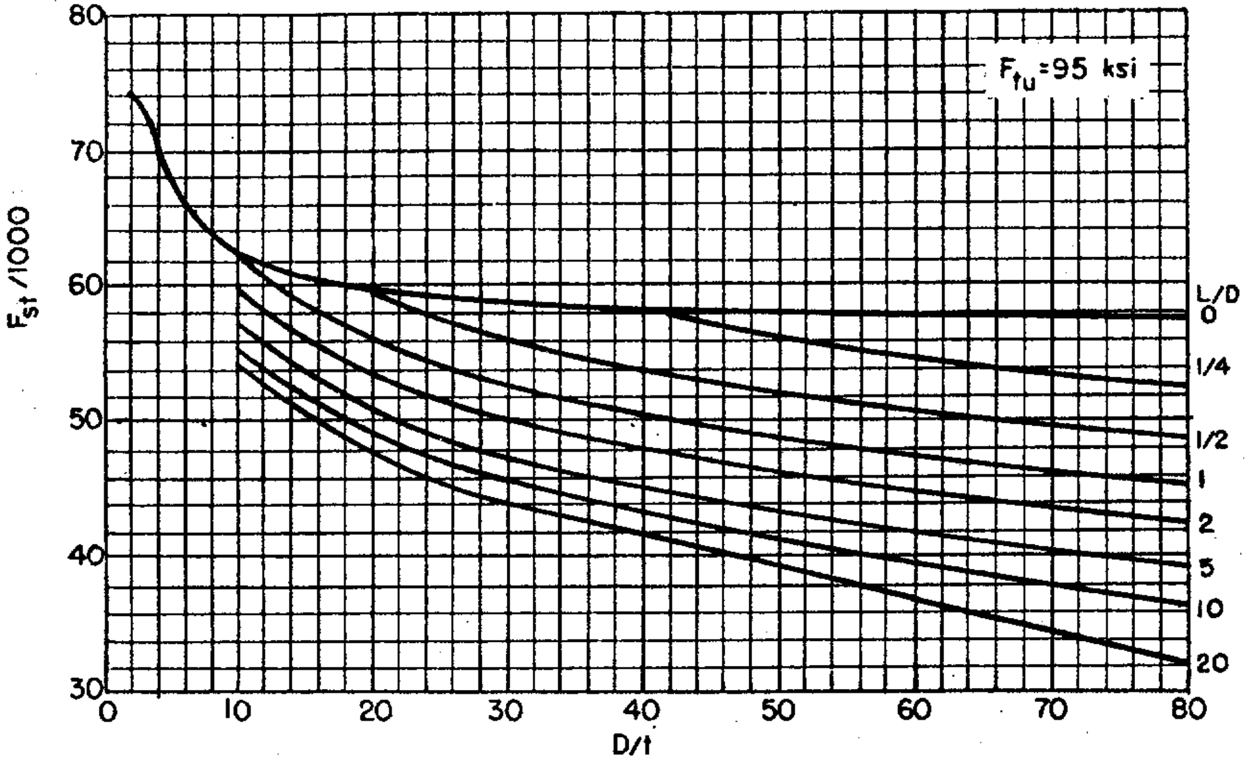


FIGURE 8.8 - TORSIONAL MODULUS OF RUPTURE - ALLOY STEELS HEAT TREATED TO F_{tu} = 95 ksi

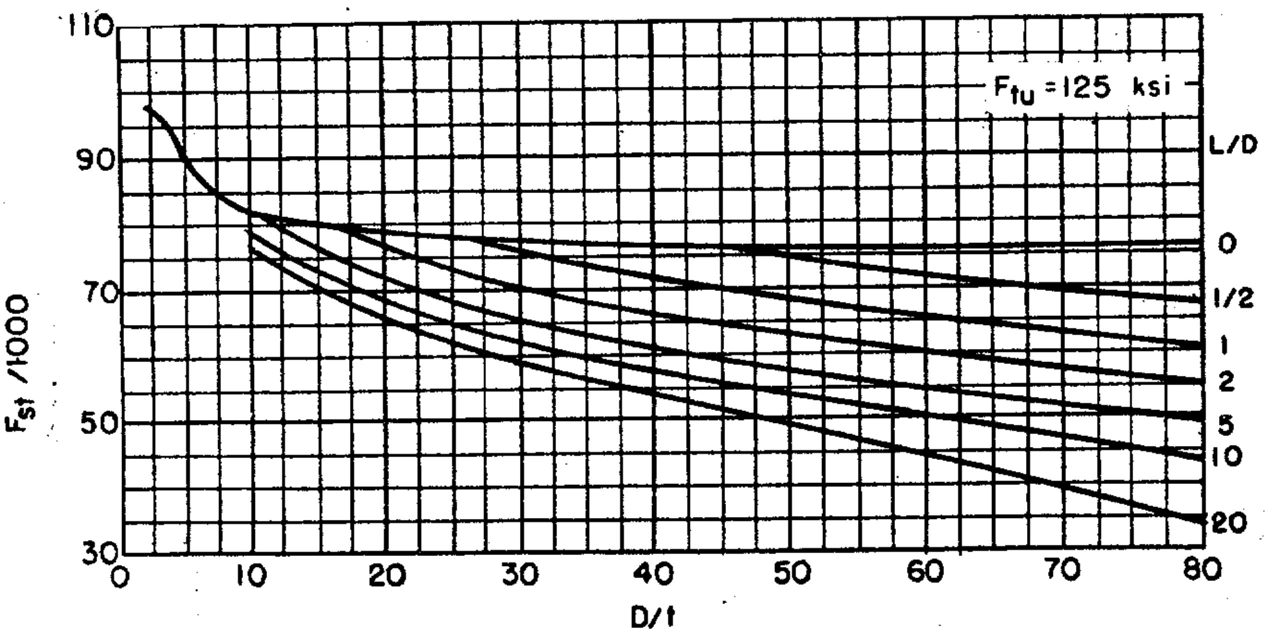


FIGURE 8.9 - TORSIONAL MODULUS OF RUPTURE - ALLOY STEELS HEAT TREATED TO F_{tu} = 125 ksi

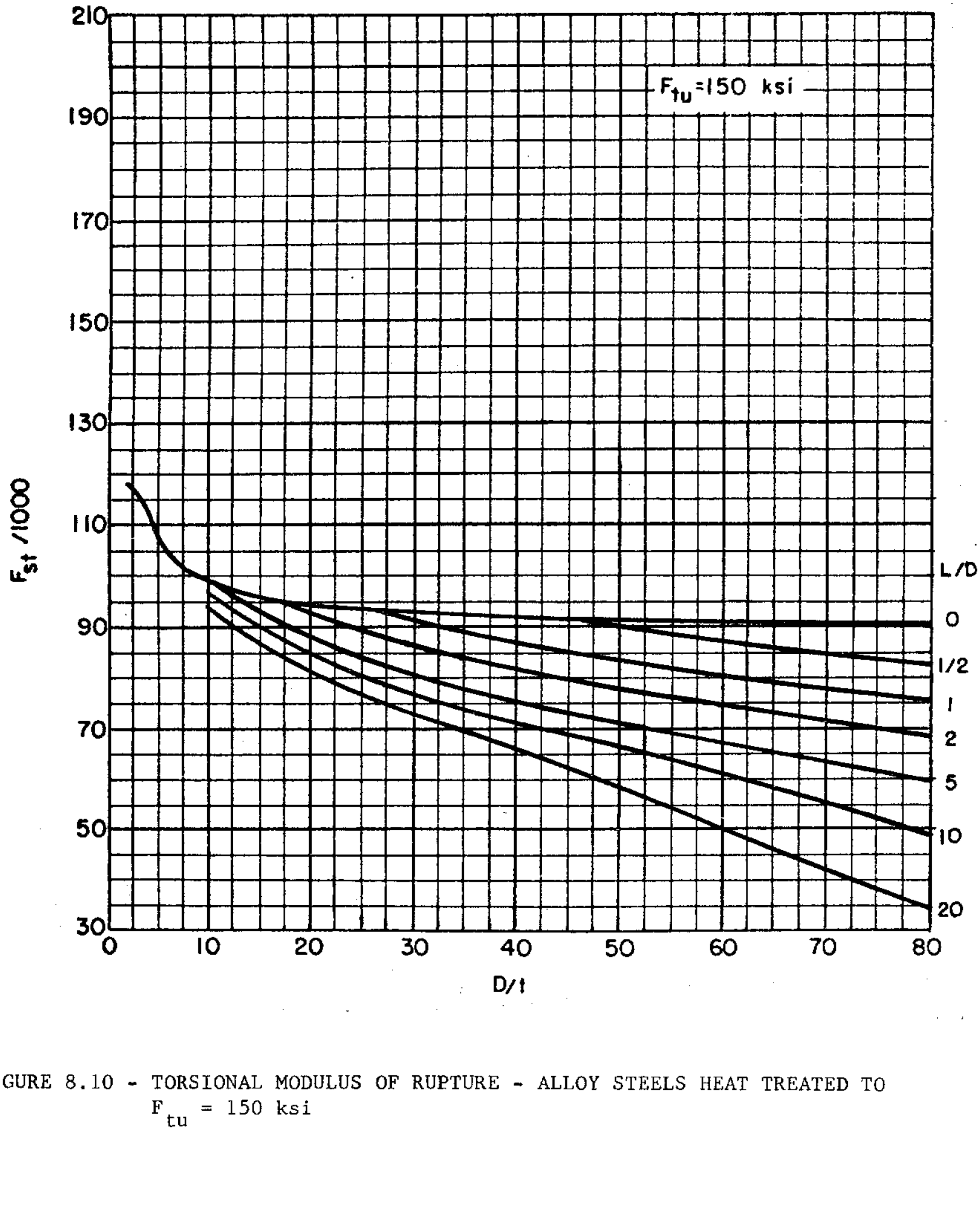


FIGURE 8.10 - TORSIONAL MODULUS OF RUPTURE - ALLOY STEELS HEAT TREATED TO F_{tu} = 150 ksi

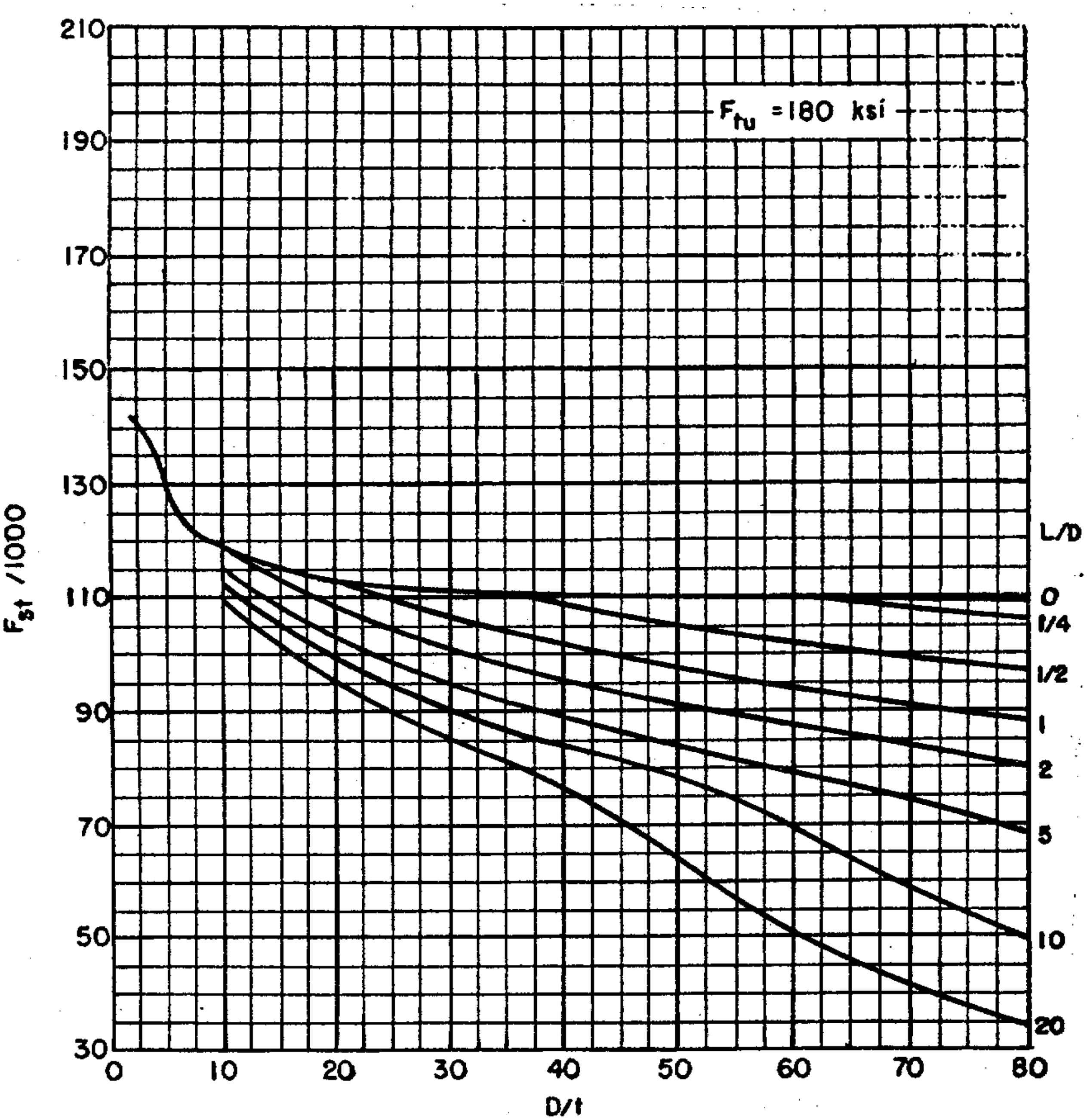


FIGURE 8.11 - TORSIONAL MODULUS OF RUPTURE - ALLOY STEEL HEAT TREATED TO F_{tu} = 180 ksi

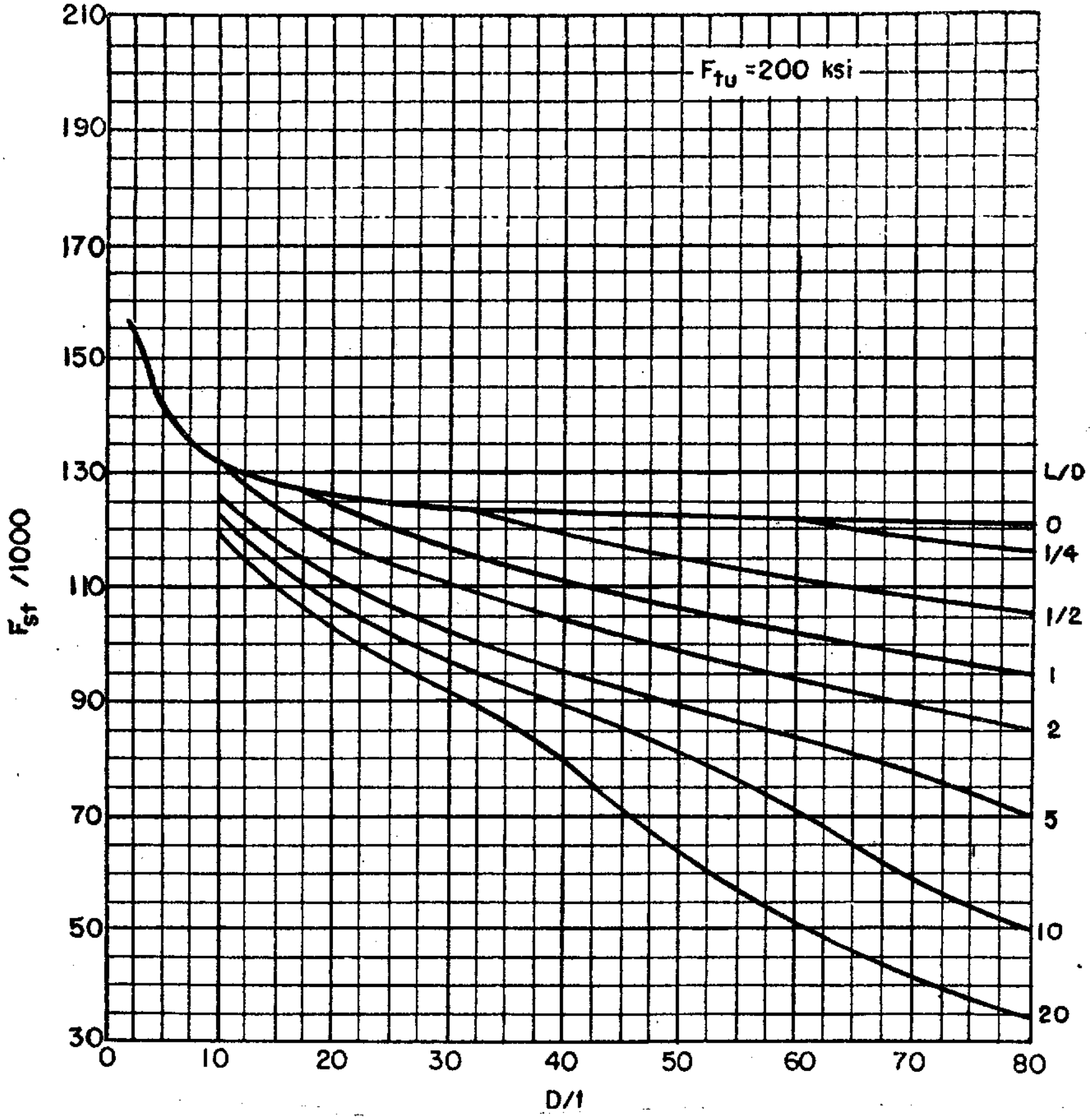


FIGURE 8.12 - TORSIONAL MODULUS OF RUPTURE - ALLOY STEELS HEAT TREATED TO F_{tu} = 200 ksi

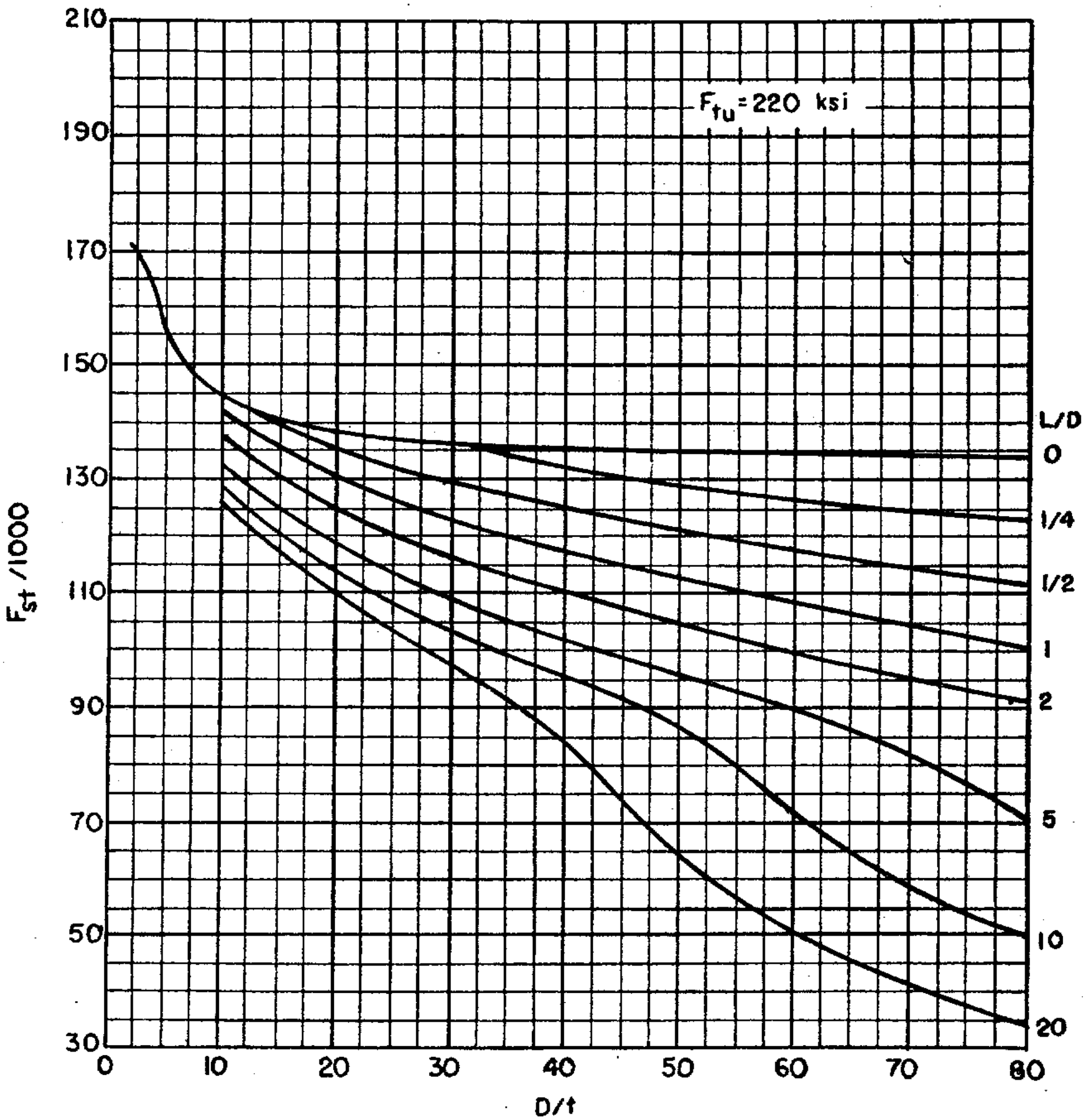


FIGURE 8.13 - TORSIONAL MODULUS OF RUPTURE - ALLOY STEELS HEAT TREATED TO F_{tu} = 220 ksi

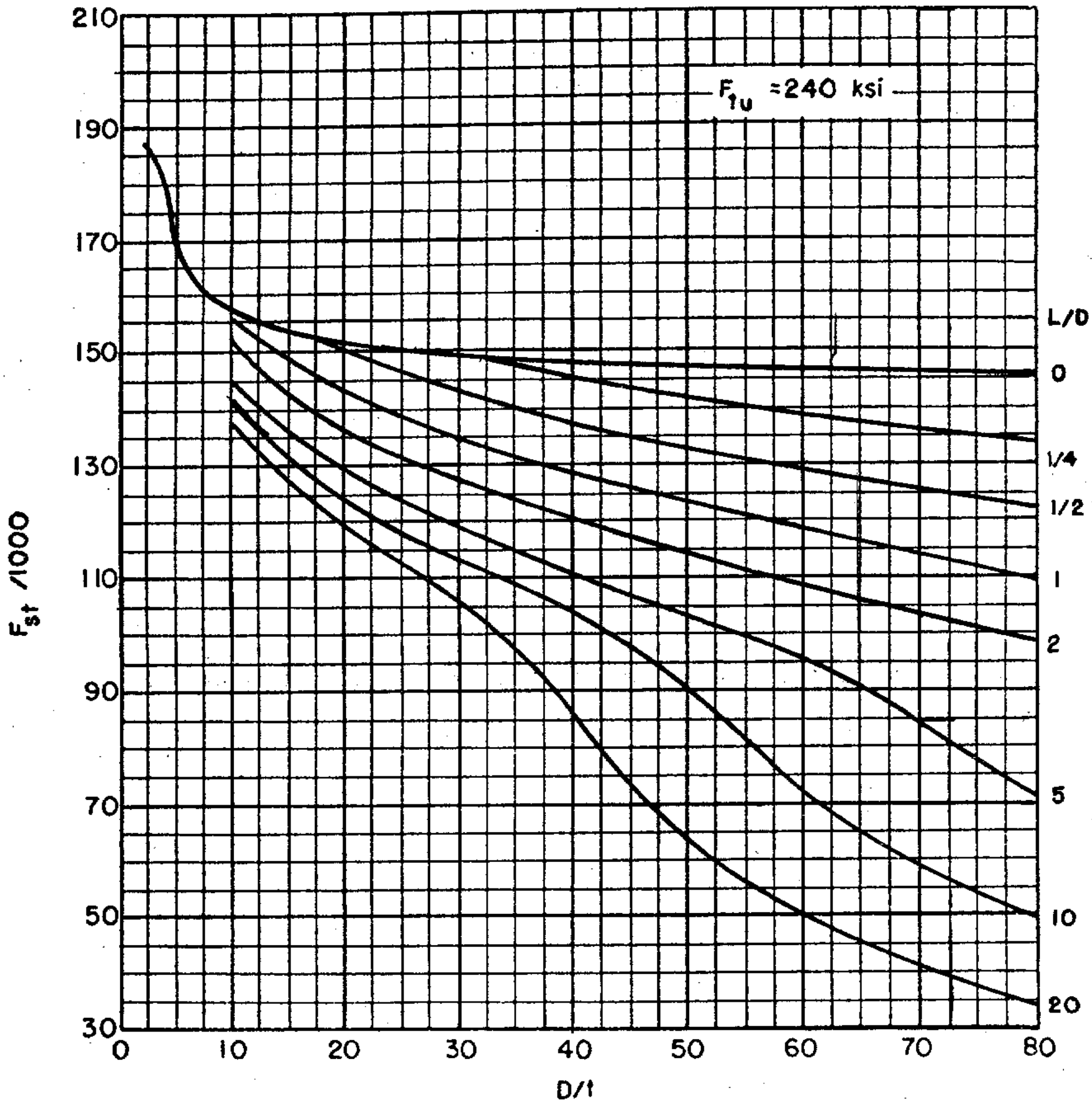


FIGURE 8.14 - TORSIONAL MODULUS OF RUPTURE - ALLOY STEELS HEAT TREATED TO F_{tu} = 240 ksi

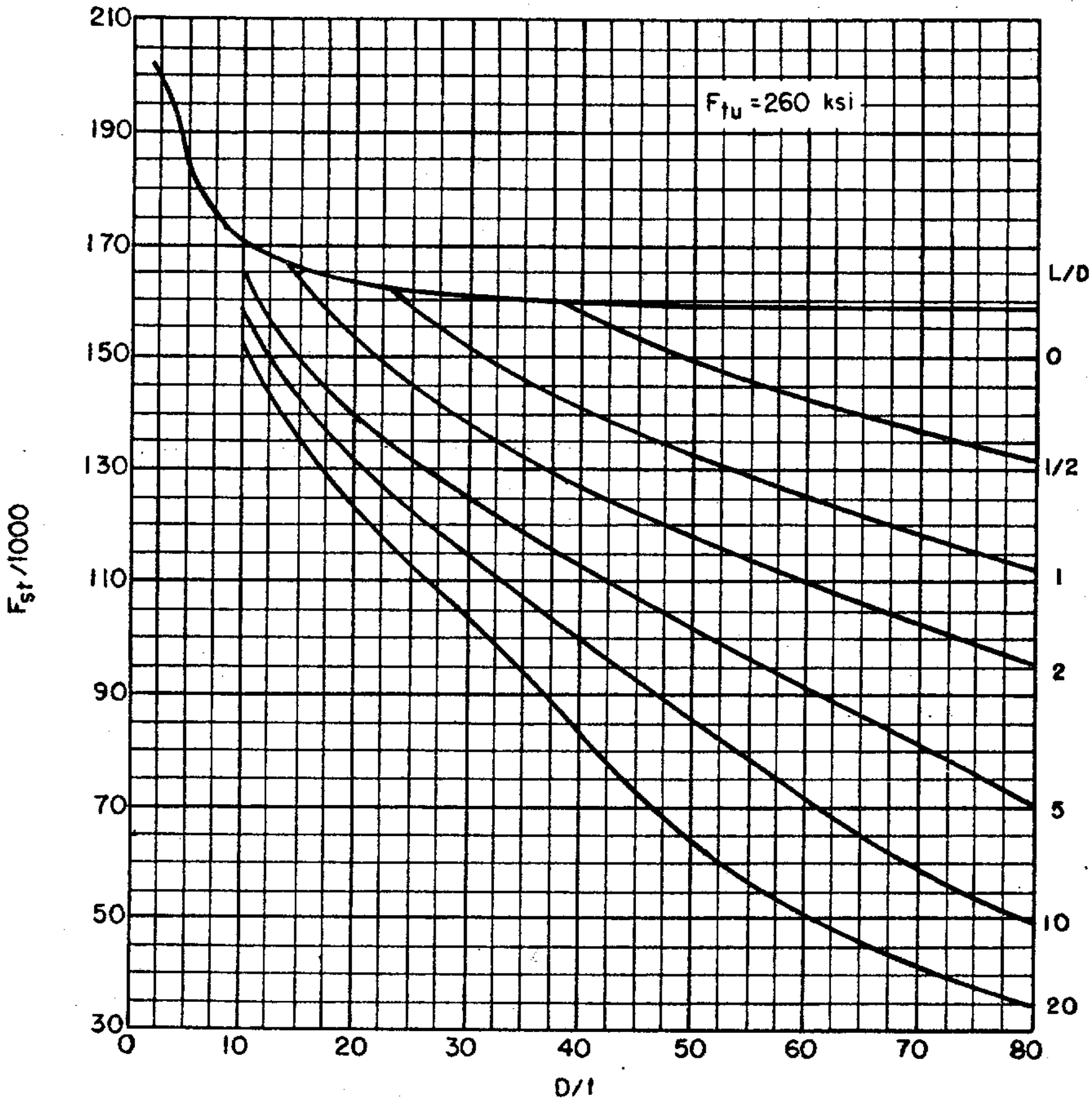


FIGURE 8.15 - TORSIONAL MODULUS OF RUPTURE - ALLOY STEELS HEAT TREATED TO F_{tu} = 260 ksi

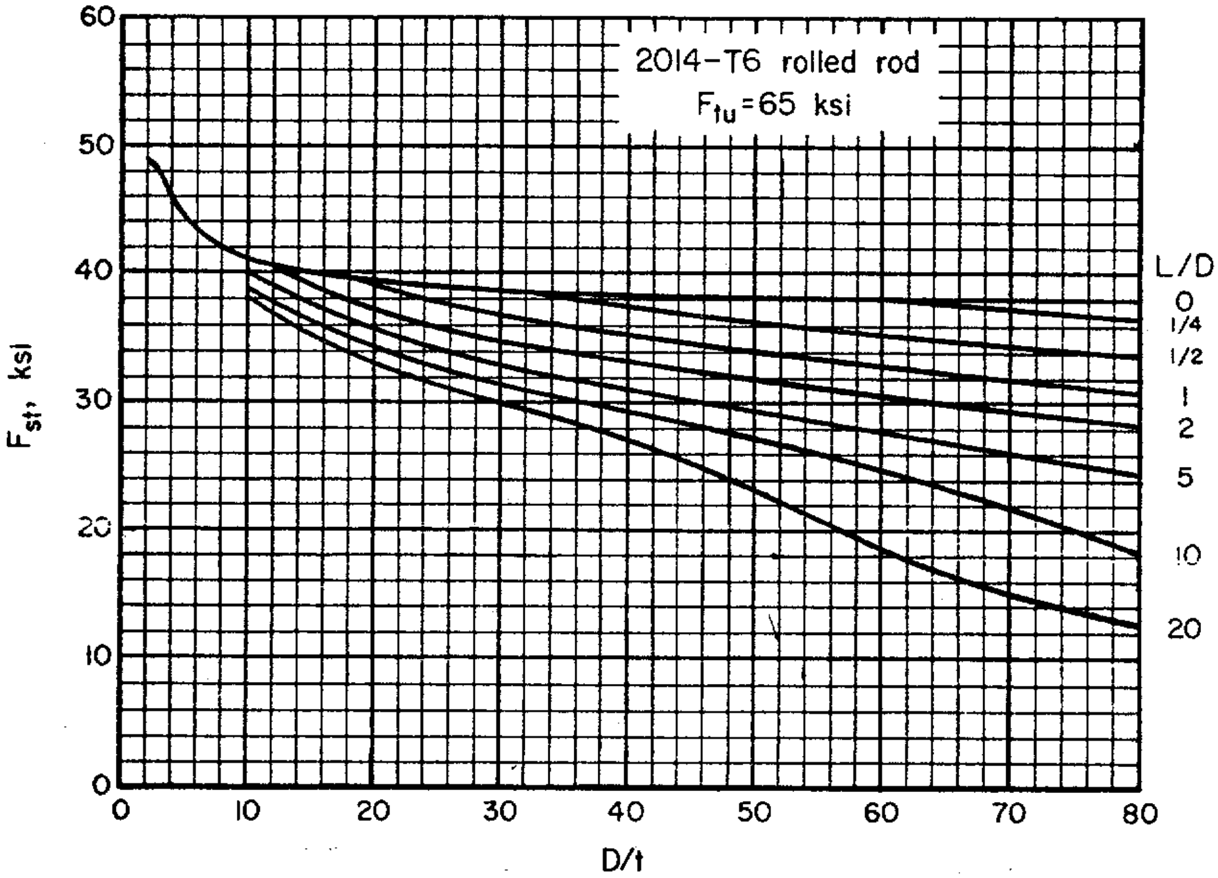


FIGURE 8.16 - TORSIONAL MODULUS OF RUPTURE - 2014-T6 ALUMINUM ALLOY ROLLED ROD

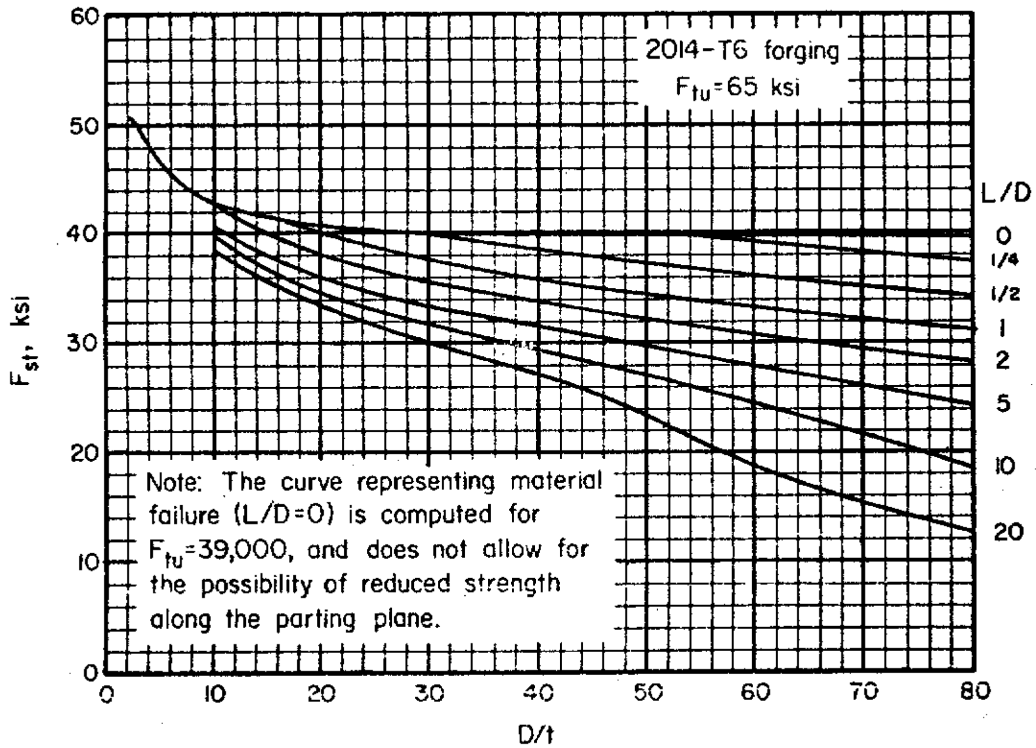


FIGURE 8.17 - TORSIONAL MODULUS OF RUPTURE - 2014-T6 ALUMINUM ALLOY FORGING

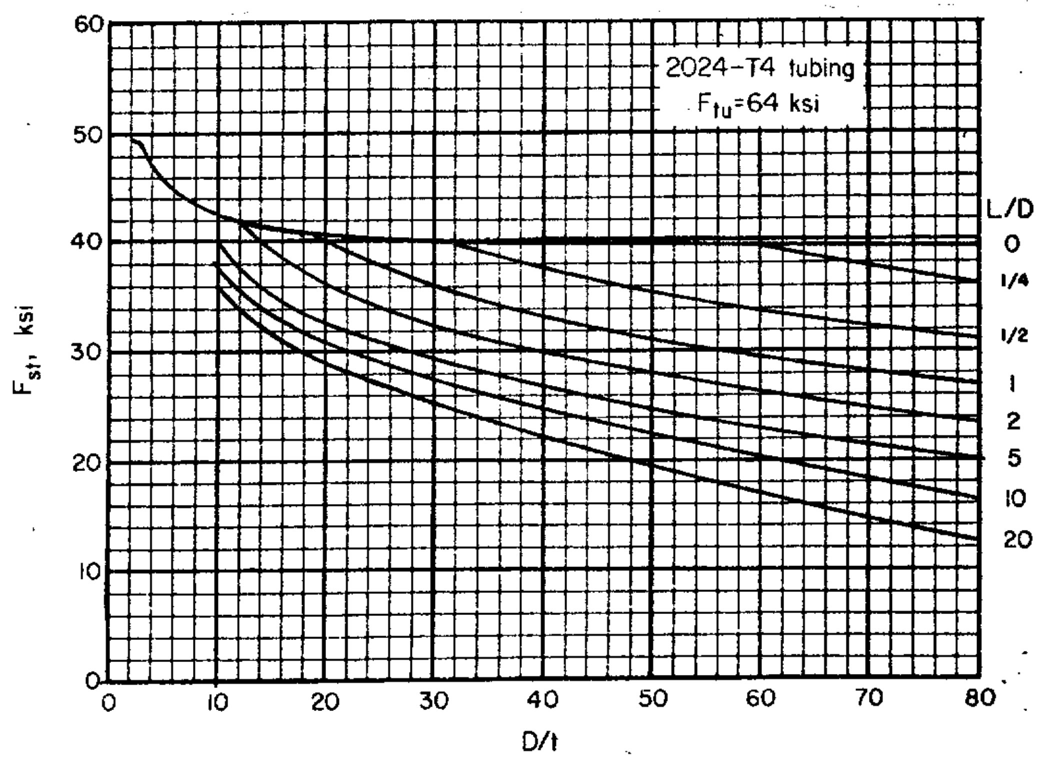


FIGURE 8.18 - TORSIONAL MODULUS OF RUPTURE - 2024-T3 ALUMINUM ALLOY TUBING

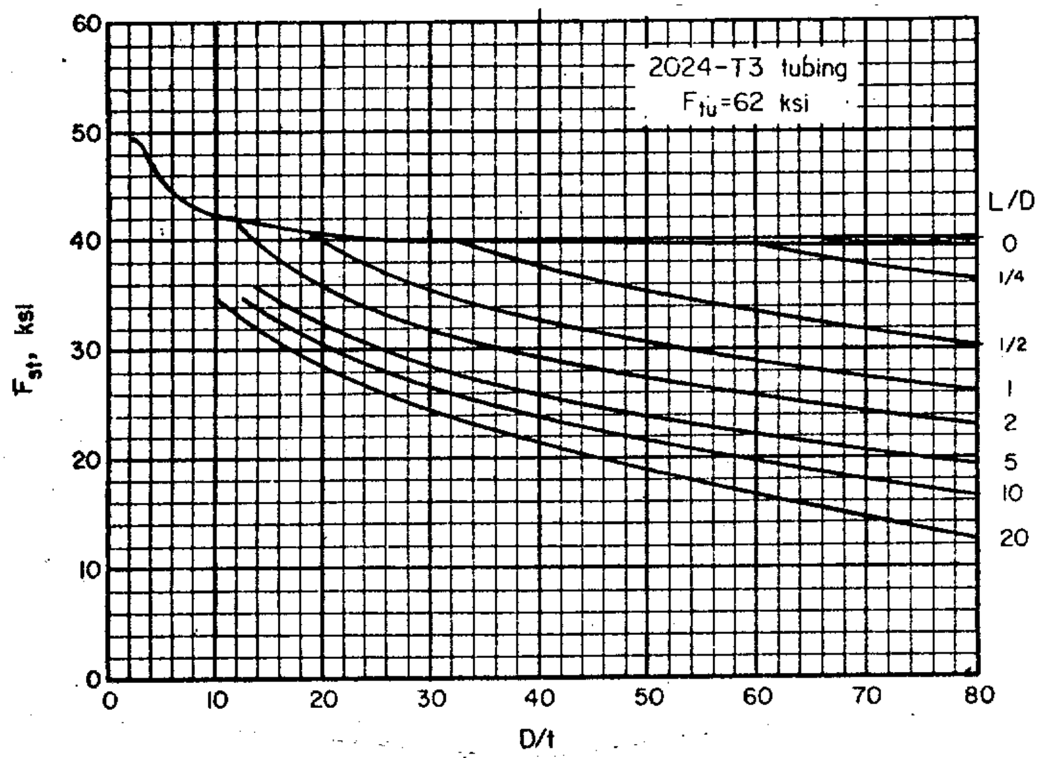


FIGURE 8.19 - TORSIONAL MODULUS OF RUPTURE - 2024-T4 ALUMINUM ALLOY TUBING

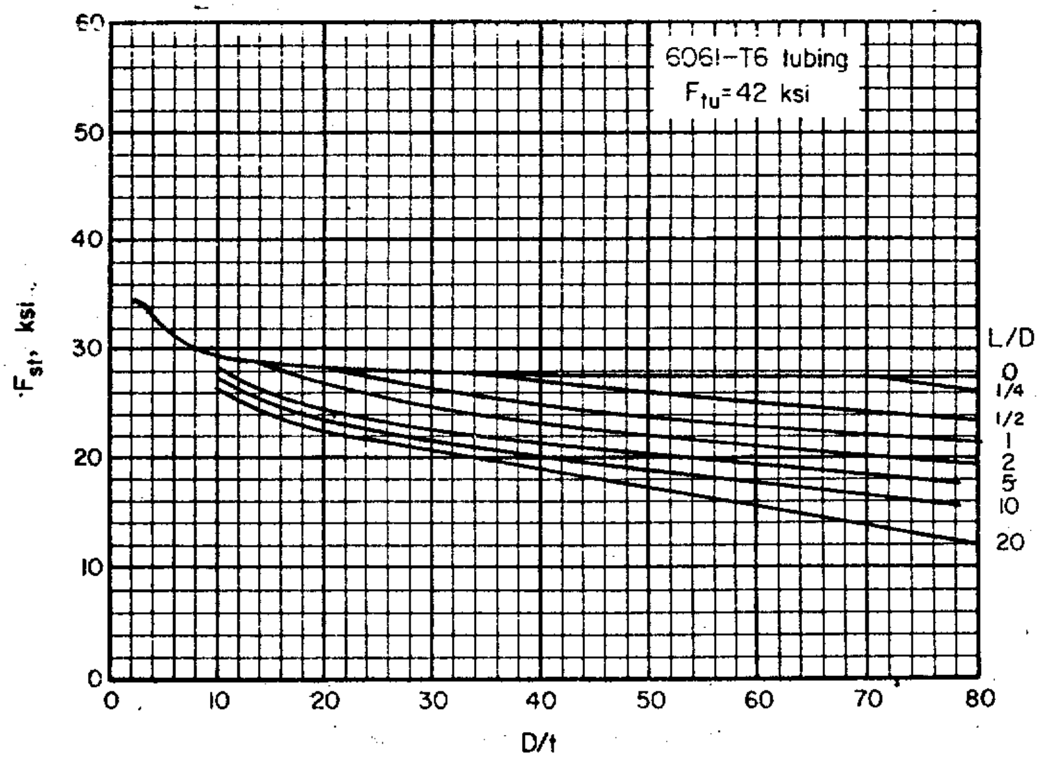


FIGURE 8.20 - TORSIONAL MODULUS OF RUPTURE - 6061-T6 ALUMINUM ALLOY TUBING

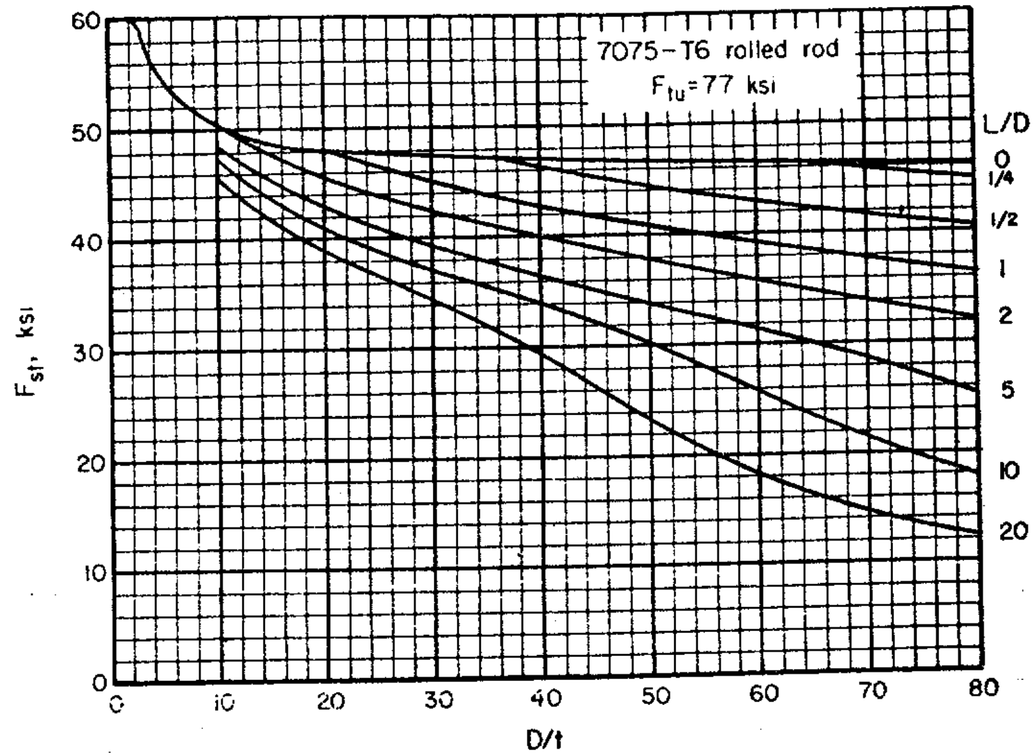


FIGURE 8.21 - TORSIONAL MODULUS OF RUPTURE - 7075-T6 ALUMINUM ALLOY ROLLED ROD

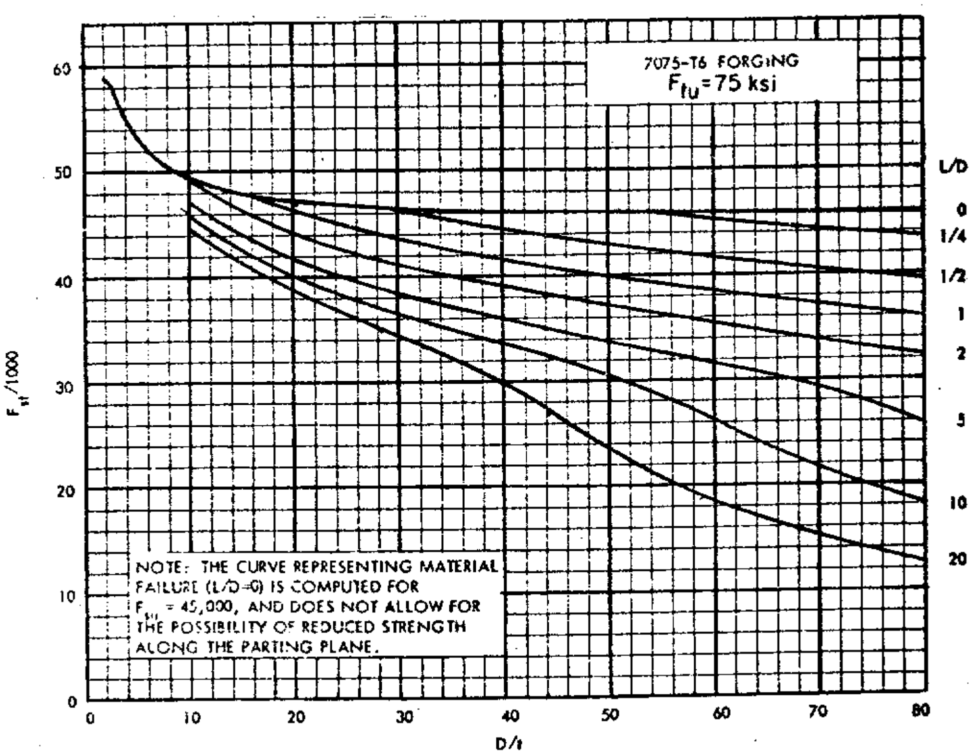


FIGURE 8.22 - TORSIONAL MODULUS OF RUPTURE - 7075-T6 ALUMINUM ALLOY FORGING

SECTION 9

BENDING

9.1 GENERAL

This section presents methods of analysis of beams in bending. Formulas for shear, moment, and deflection are shown in simple beam analysis. Beam column data for use in the three-moment procedure are included in this section, as are strain energy methods, plastic bending analysis, curved beam correction factors, and a bending analysis of bolt-spacer combinations. Allowable bending moments for tubes and channels subject to local crippling are also presented in this section.

9.2 Simple Beams

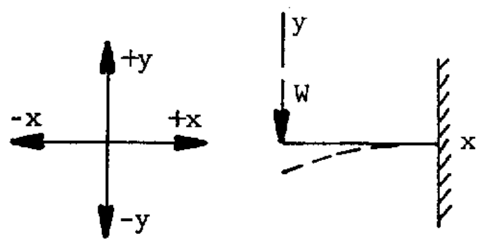
9.2.1 Shear, Moment, and Deflection

The general equations relating load, shear, bending moment, and deflection are given in Table 9.1. These equations are given in terms of deflection and bending moments.

Title	Y	M
Deflection	$\Delta = y$	$\Delta = \iint \frac{M}{EI} dx\, dx$
Slope	$\theta = dy/dx$	$\theta = \int \frac{M}{EI} dx$
Bending Moment	$M = EI\ d^2y/dx^2$	M
Shear	$V = EI\ d^3y/dx^3$	$V = dM/dx$
Load	$W = EI\ d^4y/dx^4$	$W = dV/dx = d^2M/dx^2$

TABLE 9.1 SHEAR, MOMENT, DEFLECTION EQUATIONS

Sign Convention



a) x is positive to the right.
b) y is positive upward
c) M is positive when the compressed fibers are at the top.
d) W is positive in the direction of negative y.
e) V is positive when the part of the beam to the left of the section tends to move upward under the action of the resultant of the vertical forces.

The limiting assumptions are:

a) The material follows Hooke's Law.
b) Plane cross sections remain plane.
c) Shear deflections are negligible.
d) The deflections are small.

The deflection of short, deep beams due to vertical shear may need to be considered. The differential equations of the deflection curve including the effects of shearing deformation is:

$$y = \int\int \frac{M\ dxdx}{EI} + \int \frac{KV}{AG}\ dx \qquad 9.1$$

(K) is the ratio of the maximum shearing stress on the cross section to the average shearing stress. The value of (K) is given by the equation:

$$K = \frac{A}{I\ \ b} \int_{o}^{a} b'ydy$$

(I) is the moment of inertia of the cross-section with respect to the centroidal axis and (a), (b), (b'), and (y) are the dimensions shown in Fig. 9.1
(A) is the area of the cross-section

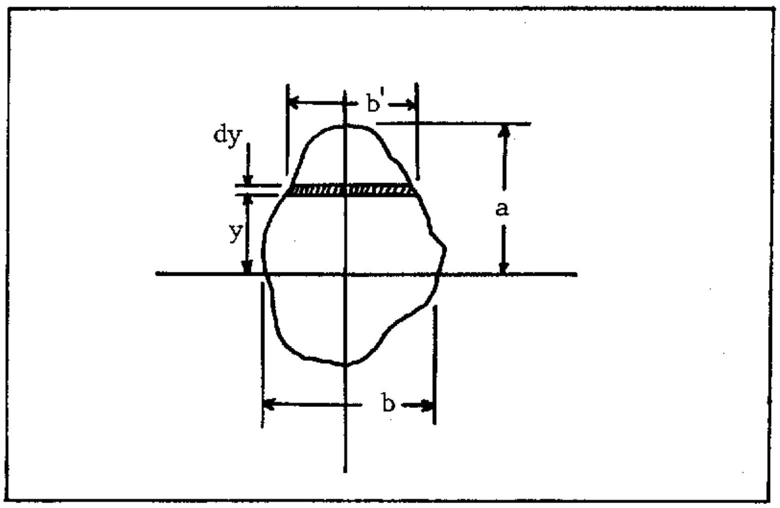


FIGURE 9.1 DEFINITION OF VARIABLES FOR DETERMINING K

Table 9.2 presents beam formulas for several different types of load cases.

Notation: W = load (lb); w = unit load (lb. per linear in.); M is positive when clockwise; V is positive when acting upward; y is positive when upward. Constraining moments, applied couples, loads; and reactions are positive when acting as shown. All forces are in pounds, all moments in inch-pounds; all deflections and dimensions in inches. θ is in radians and $\tan \theta = \theta$.

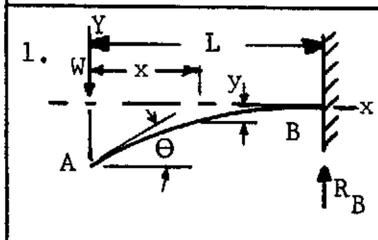
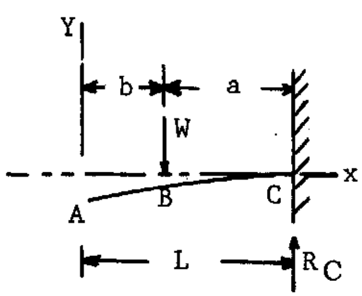
Cantilever Beams	
Type of loading and Case number	Reactions, Vertical Shear, Bending Moments, Deflection y, and Slope
1. 	$R_B = +W$; $V = -W$ $M_x = -Wx$; Max $M = -WL$ at B $y = -\frac{1}{6}\frac{W}{EI}(x^3-3L^2x+2L^3)$; Max $y = -\frac{1}{3}\frac{WL^3}{EI}$ at A; $\theta = \frac{1}{2}\frac{WL^2}{EI}$ at A
2. 	$R_C = +W$; (A to B) $V = 0$; (B to C) $V = -W$ (A to B) $M = 0$; (B to C) $M = -W(x-b)$; Max $M = -Wa$ at C (A to B)$y = -\frac{1}{6}\frac{W}{EI}(-a^3+3a^2L-3a^2x)$: (B to C)$y = -\frac{1}{6}\frac{W}{EI}\left[(x-b)^3 - 3a^2(x-b) + 2a^3\right]$; Max $y = -\frac{1}{6}\frac{W}{EI}(3a^2L-a^3)$; $\theta = \frac{1}{2}\frac{Wa^2}{EI}$ (A to B)

TABLE 9.2 BEAM FORMULAS

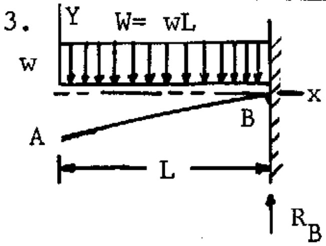
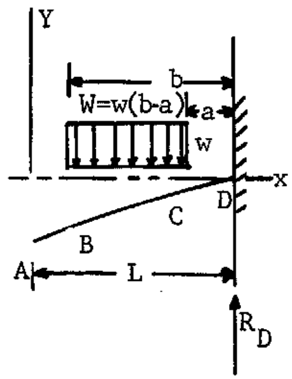
Cantilever Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
3. 	$R_B = +W;\ V = -\frac{W}{L}x$ $M = -\frac{1}{2}\frac{W}{L}x^2;$ Max $M = -\frac{1}{2}WL$ at B $y = -\frac{1}{24}\frac{W}{EIL}(x^4-4L^3x+3L^4);$ Max $y = -\frac{1}{8}\frac{WL^3}{EI}$ $\theta = +\frac{1}{6}\frac{WL^2}{EI}$ at A
4. 	$R_D = +W;$ (A to B)$V = 0$: (B to C)$V = \frac{-W}{b-a}(x-L+b)$; (C to D)$V = -W$ (A to B)$M = 0$; (B to C)$M = -\frac{1}{2}\frac{W}{b-a}(x-L+b)^2$; (C to D)$M = -\frac{1}{2}W(2x-2L+a+b)$; Max $M = -\frac{1}{2}W(a+b)$ at D (A to B)$y = -\frac{1}{24}\frac{W}{EI}\left[4(a^2+ab+b^2)(L-x)-a^3-ab^2-a^2b-b^3\right]$ (B to C)$y = -\frac{1}{24}\frac{W}{EI}\left[6(a+b)(L-x)^2-4(L-x)^3+\frac{(L-x-a)^4}{b-a}\right]$; (C to D)$y = -\frac{1}{12}\frac{W}{EI}\left[3(a+b)(L-x)^2-2(L-x)^3\right]$ Max $y = -\frac{1}{24}\frac{W}{EI}\left[4(a^2+ab+b^2)L-a^3-ab^2-a^2b-b^3\right]$ at A; $\theta = +\frac{1}{6}\frac{W}{EI}\left(a^2+ab+b^2\right)$ (A to B)

TABLE 9.2 (CONT'D) BEAM FORMULAS

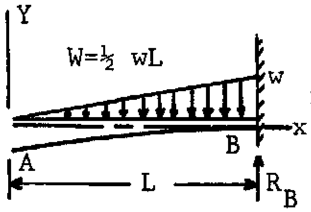
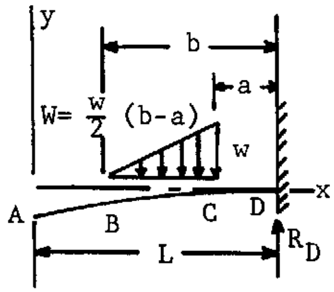
Cantilever Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
5.  	$R_B = +W;\ V = -\frac{W}{L^2}x^2$ $M = -\frac{1}{3}\frac{W}{L^2}x^3$; Max $M = -\frac{1}{3}WL$ at B $y = -\frac{1}{60}\frac{W}{EIL^2}(x^5-5L^4x+4L^5)$; Max $y = -\frac{1}{15}\frac{WL^3}{EI}$ at A; $\theta = +\frac{1}{12}\frac{WL^2}{EI}$ at A
6.  	$R_D = +W$; (A to B)$V = 0$; (B to C)$V = -\frac{W(x-L+b)^2}{(b-a)^2}$; (C to D)$V = -W$ (A to B) $M = 0$; (B to C) $M = -\frac{1}{3}\frac{W(x-L+b)^3}{(b-a)^2}$ (C to D) $M = -\frac{1}{3}W(3x-3L+b+2a)$; Max $M = -\frac{1}{3}W(b+2a)$ at D (A to B) $y = -\frac{1}{60}\frac{W}{EI}\left[(5b^2+10ba+15a^2)(L-x) - 4a^3 - 2ab^2 - 3a^2b - b^3\right]$ (B to C) $y = -\frac{1}{60}\frac{W}{EI}\left[(20a+10b)(L-x)^2 - 10(L-x)^3 + 5\frac{(L-x-a)^4}{b-a} - \frac{(L-x-a)^5}{(b-a)^2}\right]$; (C to D) $y = -\frac{1}{6}\frac{W}{EI}\left[(2a+b)(L-x)^2 - (L-x)^3\right]$; Max $Y = -\frac{1}{60}\frac{W}{EI}\left[(5b^2+10ba+15a^2)L - 4a^3 - 2ab^2 - 3a^2b - b^3\right]$ at A $\theta = +\frac{1}{12}\frac{W}{EI}(3a^2+2ab+b^2)$ (A to B)

TABLE 9.2 (CONT'D) BEAM FORMULAS

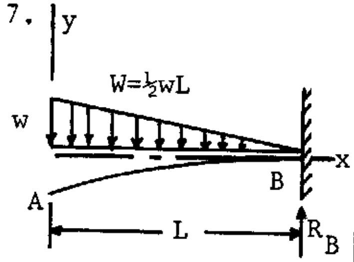
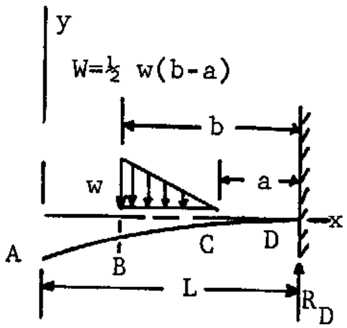
Cantilever Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
7.  	$R_B = +W; \; V = -W\left(\frac{2Lx-x^2}{L^2}\right)$ $M = -\frac{1}{3}\frac{W}{L^2}(3Lx^2-x^3)$; Max $M = -\frac{2}{3}WL$ at B $y = -\frac{1}{60}\frac{W}{EIL^2}(-x^5-15L^4x+5LX^4+11L^5)$; Max $Y = -\frac{11}{60}\frac{WL^3}{EI}$ at A $\theta = +\frac{1}{4}\frac{W}{EI}L^2$ at A
8.  	$R_D = +W$; (A to B) $V = 0$; (B to C) $V = -W\left[1 - \frac{(L-a-x)^2}{(b-a)^2}\right]$; (C to D) $V = -W$ (A to B) $M = 0$; (B to C) $M = -\frac{1}{3}W\left[\frac{3(x-L+b)^2}{b-a} - \frac{(x-L+b)^3}{(b-a)^2}\right]$ (C to D) $M = -\frac{1}{3}W(-3L+3x+2b+a)$; Max $M = -\frac{1}{3}W(2b+a)$ at D (A to B) $y = -\frac{1}{60}\frac{W}{EI}\left[(5a^2+10ab+15b^2)(L-x)-a^3-2a^2b-3ab^2-4b^3\right]$; (B to C) $y = -\frac{1}{60}\frac{W}{EI}\left[\frac{(L-x-a)^5}{(b-a)^2} -10(L-x)^3+(10a+20b)(L-x)^2\right]$ (C to D) $y = -\frac{1}{6}\frac{W}{EI}\left[(a+2b)(L-x)^2 - (L-x)^3\right]$ Max $Y = -\frac{1}{60}\frac{W}{EI}\left[(5a^2+10ab+15b^2)L-a^3-2a^2b-3ab^2-4b^3\right]$ at A $\theta = +\frac{1}{12}\frac{W}{EI}(a^2+2ab+3b^2)$ (A to B)

TABLE 9.2 (CONT'D) BEAM FORMULAS

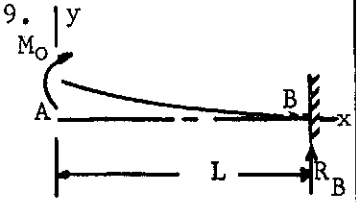
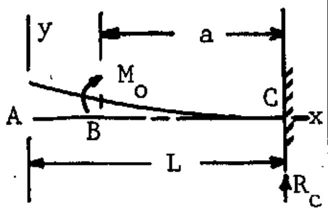
Cantilever Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
9. 	$R_B = 0;\ V = o$ $M = M_o;$ Max $M = M_o$ (A to B) $y = \frac{1}{2}\frac{M_o}{EI}(L^2 - 2Lx + x^2);$ Max $y = \frac{1}{2}\frac{M_o L^2}{EI}$ at A; $\theta = -\frac{M_o L}{EI}$ at A
10. 	$R_C = 0;\ V = 0$ (A to B) $M = 0;$ (B to C) $M = M_o;$ Max $M = M_o$ (B to C) (A to B) $y = \frac{M_o a}{EI}\left(L - \frac{1}{2}a - x\right);$ (B to C) $y = \frac{1}{2}\frac{M_o}{EI}\left[(x-L+a)^2 - 2a(x-L+a) + a^2\right]$ Max $y = \frac{M_o a}{EI}\left(L - \frac{1}{2}a\right)$ at A; $\theta = -\frac{M_o a}{EI}$ (A to B)

TABLE 9.2 (CONT'D) BEAM FORMULAS

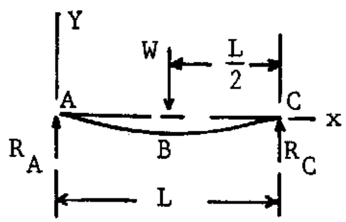
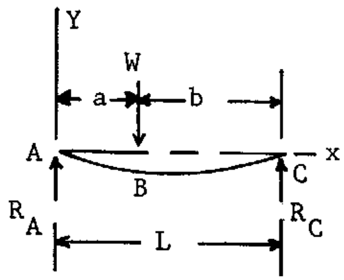
Simply Supported Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
11. 	$R_A = \frac{1}{2}W$; $R_C = \frac{1}{2}W$; (A to B) $V = \frac{1}{2}W$; (B to C) $V = -\frac{1}{2}W$ (A to B) $M = \frac{1}{2}Wx$; (B to C) $M = \frac{1}{2}W(L-x)$; Max $M = \frac{1}{4}WL$ at B (A to B) $y = \frac{1}{48}\frac{W}{EI}(3L^2x-4x^3)$; Max $y = -\frac{1}{48}\frac{WL^3}{EI}$ at B; $\theta = -\frac{1}{16}\frac{WL^2}{EI}$ at A; $\theta = +\frac{1}{16}\frac{WL^2}{EI}$ at C
12. 	$R_A = +\frac{Wb}{L}$; $R_C = +\frac{Wa}{L}$; (A to B) $V = +\frac{Wb}{L}$; (B to C) $V = -\frac{Wa}{L}$ (A to B) $M = +\frac{Wb}{L}x$; (B to C) $M = +\frac{Wa}{L}(L-x)$; Max $M = +\frac{Wab}{L}$ at B (A to B) $y = -\frac{Wbx}{6EIL}\left[2L(L-x) - b^2 - (L-x)^2\right]$; (B to C) $y = -\frac{Wa(L-x)}{6\,EIL}\left[2\,Lb - b^2 - (L-x)^2\right]$; Max $y = -\frac{Wab}{27EIL}(a+2b)\sqrt{3a(a+2b)}$ at $x = \sqrt{\frac{a}{3}(a+2b)}$ When $a > b$; $\theta = -\frac{1}{6}\frac{W}{EI}\left(bL - \frac{b^3}{L}\right)$ at A; $\theta = +\frac{1}{6}\frac{W}{EI}\left(2bL + \frac{b^3}{L} - 3b^2\right)$ at C

TABLE 9.2 (CONT'D) BEAM FORMULAS

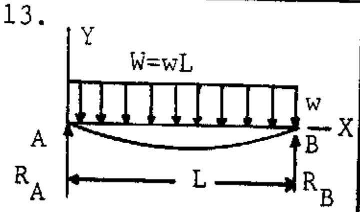
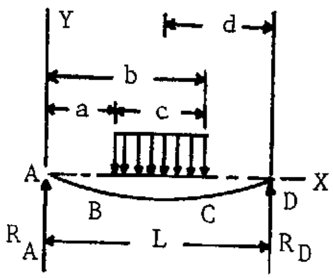
Simply Supported Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
13. 	$R_A = \frac{W}{2}$; $R_B = \frac{W}{2}$; $V = +\frac{W}{2}\left(1 - \frac{2x}{L}\right)$; $M = +\frac{W}{2}\left(x - \frac{x^2}{L}\right)$ Max. $M = +\frac{WL}{8}$ at $x = \frac{L}{2}$; $y = -\frac{Wx}{24EIL}(L^3 - 2Lx^2 + x^3)$; Max. $y = -\frac{5WL^3}{384EI}$ at $x = \frac{L}{2}$; $\theta = -\frac{WL^2}{24EI}$ at A; $\theta = +\frac{WL^2}{24EI}$ at B
14.  W= wc d= L-½b-½a	$R_A = \frac{Wd}{L}$; $R_D = \frac{W}{L}\left(a + \frac{c}{2}\right)$; (A to B) $V = R_A$: (B to C) $V = R_A - \frac{W(x-a)}{c}$ (C to D) $V = R_A - W$; (A to B) $M = R_A x$; (B to C) $M = R_A x - \frac{W(x-a)^2}{2c}$ (C to D) $M = R_A x - W\left(x - \frac{a}{2} - \frac{b}{2}\right)$; Max. $M = \frac{Wd}{L}\left(a + \frac{cd}{2L}\right)$ at $x = a + \frac{cd}{L}$ (A to B) $y = \frac{1}{48EI}\left\{8R_A(x^3 - L^2x) + Wx\left[\frac{8d^3}{L} - \frac{2bc^2}{L} + \frac{c^3}{L} + 2c^2\right]\right\}$ (B to C) $y = \frac{1}{48EI}\left\{8R_A(x^3 - L^2x) + Wx\left[\frac{8d^3}{L} - \frac{2bc^2}{L} + \frac{c^3}{L} + 2c^2\right] - \frac{2W(x-a)^4}{c}\right\}$ (C to D) $y = \frac{1}{48EI}\left\{8R_A(x^3 - L^2x) + Wx\left[\frac{8d^3}{L} - \frac{2bc^2}{L} + \frac{c^3}{L}\right]\right.$ $\left. - 8W\left(x - \frac{a}{2} - \frac{b}{2}\right)^3 + W(2bc^2 - c^3)\right\}$

TABLE 9.2 (CONT'D) BEAM FORMULAS

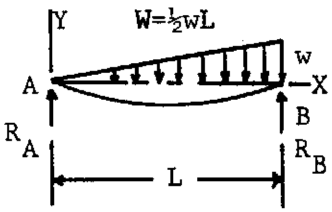
Simply Supported Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
14. (Cont'd)	$\theta = \frac{1}{48EI}\left[-8R_A L^2 + W\left(\frac{8d^3}{L} - \frac{2bc^2}{L} + \frac{c^3}{L} + 2c^2\right)\right]$ at A $\theta = \frac{1}{48EI}\left[16R_A L^2 - W\left(24d^2 - \frac{8d^3}{L} + \frac{2bc^2}{L} - \frac{c^3}{L}\right)\right]$ at B
15.  	$R_A = \frac{W}{3}$; $R_B = \frac{2W}{3}$; $V = W\left(\frac{1}{3} - \frac{x^2}{L^2}\right)$; $M = \frac{W}{3}\left(x - \frac{x^3}{L^2}\right)$; Max. M = 0.128 WL at $x = \frac{L\sqrt{3}}{3}$; $y = \frac{-Wx(3x^4 - 10L^2x^2 + 7L^4)}{180EIL^2}$ Max. $y = -\frac{0.01304\ WL^3}{EI}$ at x = 0.519 L $\theta = -\frac{7WL^2}{180EI}$ at A ; $\theta = \frac{8WL^2}{180EI}$ at B

TABLE 9.2 (CONT'D) BEAM FORMULAS

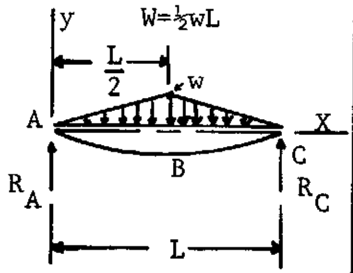
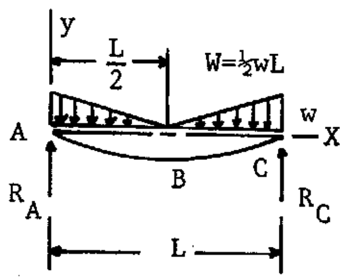
Simply Supported Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
16.  	$R_A = \frac{W}{2}$; $R_C = \frac{W}{2}$; (A to B) $V = \frac{W}{2}\left(1 - \frac{4x^2}{L^2}\right)$; (B to C) $V = -\frac{W}{2}\left[1 - \frac{4(L-x)^2}{L^2}\right]$; (A to B) $M = \frac{W}{6}\left(3x - \frac{4x^3}{L^2}\right)$ (B to C) $M = \frac{W}{6}\left[3(L-x) - \frac{4(L-x)^3}{L^2}\right]$; Max. $M = \frac{WL}{6}$ at B (A to B) $y = \frac{Wx}{6EIL^2}\left(\frac{L^2x^2}{2} - \frac{x^4}{5} - \frac{5L^4}{16}\right)$; Max. $y = -\frac{WL^3}{60EI}$ at B $\theta = -\frac{5WL^2}{96EI}$ at A ; $\theta = +\frac{5WL^2}{96EI}$ at C
17.  	$R_A = \frac{W}{2}$; $R_B = \frac{W}{2}$; (A to B) $V = \frac{W}{2}\left(\frac{L-2x}{L}\right)^2$ (B to C) $V = -\frac{W}{2}\left(\frac{2x-L}{L}\right)^2$; (A to B) $M = \frac{W}{2}\left(x - \frac{2x^2}{L} + \frac{4x^3}{3L^2}\right)$ (B to C) $M = \frac{W}{2}\left[(L-x) - \frac{2(L-x)^2}{L} + \frac{4(L-x)^3}{3L^2}\right]$; Max. $M = \frac{WL}{12}$ at B (A to B) $y = \frac{W}{12EI}\left(x^3 - \frac{x^4}{L} + \frac{2x^5}{5L^2} - \frac{3L^2x}{8}\right)$; Max. $y = -\frac{3WL^3}{320EI}$ at B $\theta = -\frac{WL^2}{32EI}$ at A; $\theta = \frac{WL^2}{32EI}$ at C

TABLE 9.2 (CONT'D) BEAM FORMULAS

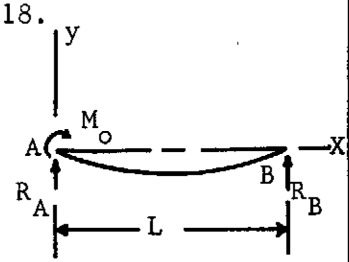
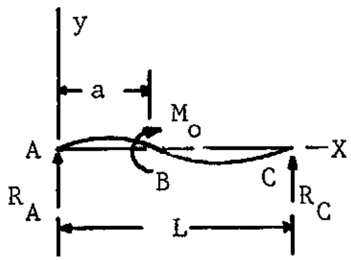
Simply Supported Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
18. 	$R_A = -\frac{M_o}{L}$; $R_B = \frac{M_o}{L}$; $V = R_A$: $M = M_o + R_A x$; Max. $M = M_o$ at A $y = \frac{M_o}{6EI}\left(3x^2 - \frac{x^3}{L} - 2Lx\right)$; Max. $y = -0.0642\frac{M_oL^2}{EI}$ at $x = 0.422L$ $\theta = -\frac{M_oL}{3EI}$ at A; $\theta \frac{M_oL}{6EI}$ at B
19. 	$R_A = -\frac{M_o}{L}$; $R_C = \frac{M_o}{L}$; (A to C) $V = +R_A$; (A to B) $M = +R_A x$ (B to C) $M = +R_A x + M_o$; Max. $(-M) = +R_A a$ just left of B Max. $(+M) = +R_A a + M_o$ just right of B (A to B) $y = \frac{M_o}{6EI}\left[\left(6a - \frac{3a^2}{L} - 2L\right)x - \frac{x^3}{L}\right]$ (B to C) $y = \frac{M_o}{6EI}\left[3a^2 + 3x^2 - \frac{x^3}{L} - \left(2L + \frac{3a^2}{L}\right)x\right]$ $\theta = -\frac{M_o}{6EI}\left(2L - 6a + \frac{3a^2}{L}\right)$ at A ; $\theta = \frac{M_o}{6EI}\left(L - \frac{3a^2}{L}\right)$ at C $\theta = \frac{M_o}{EI}\left(a - \frac{a^2}{L} - \frac{L}{3}\right)$ at B

TABLE 9.2 (CONT'D) BEAM FORMULAS

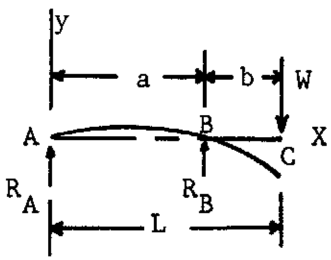
Simply Supported Beams	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
20. 	$R_A = -\frac{Wb}{a}$; $R_B = \frac{WL}{a}$; (A to B) $V = +R_A$; (B to C) $V = +W$; (A to B) $M = +R_A x$; (B to C) $M = +R_A a + W(x-a)$; Max. $M = +R_A a$ at B (A to B) $y = -\frac{Wbx}{6aEI}(x^2-a^2)$; (B to C) $y = -\frac{W}{6EI}\left[(L-x)^3 - b(L-x)(2L-b) + 2b^2L\right]$ Max. $y = -\frac{Wb^2L}{3EI}$ at C; $\theta = \frac{Wab}{6EI}$ at A; $\theta = -\frac{Wab}{3EI}$ at B $\theta = -\frac{Wb}{6EI}(2L+b)$ at C

TABLE 9.2 (CONT'D) BEAM FORMULAS

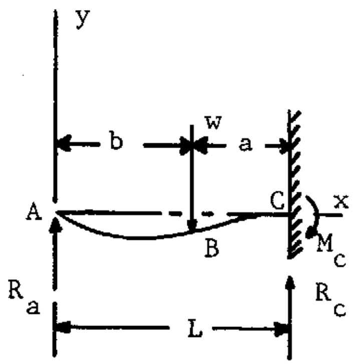
Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
21. 	$R_A = \frac{W}{2}\left(\frac{3a^2L-a^3}{L^3}\right)$; $R_C = W - R_A$; $M_c = \frac{W}{2}\left(\frac{a^3+2aL^2-3a^2L}{L^2}\right)$ (A to B) $V = R_A$; (B to C) $V = R_A - W$; (A to B) $M = R_A x$ (B to C) $M = R_A x - W(x-L+a)$; Max. $(+M) = R_A(L-a)$ at B max. possible value = 0.174 WL when a = 0.634 L Max. $(-M) = -M_c$ at C max. possible value = -0.1927 WL when a = 0.4227 L (A to B) $y = \frac{1}{6EI}\left[R_A(x^3-3L^2x) + 3Wa^2x\right]$ (B to C) $y = \frac{1}{6EI}\left\{R_A(x^3-3L^2x) + W\left[3a^2x - (x-b)^3\right]\right\}$ If $a < 0.586$ L, max. y is between A and B at $x = L\sqrt{1 - \frac{2L}{3L-a}}$ If $a > 0.586$ L, max. y is at $x = \frac{L(L^2+b^2)}{3L^2-b^2}$ If a = 0.586 L, max. y is at B and $= -0.0098\frac{WL^3}{EI}$, max. possible deflection $\theta = \frac{W}{4EI}\left(\frac{a^3}{L} - a^2\right)$ at A

TABLE 9.2 (CONT'D) BEAM FORMULAS

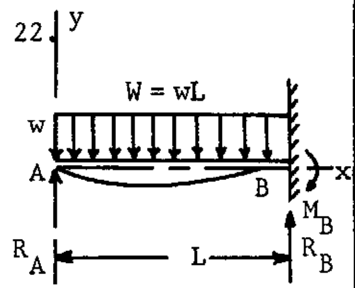
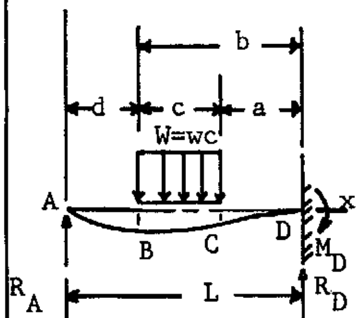
Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
22. 	$R_A = \frac{3W}{8}$; $R_B = \frac{5W}{8}$; $M_B = \frac{WL}{8}$; $V = W\left(\frac{3}{8} - \frac{x}{L}\right)$; $M = W\left(\frac{3x}{8} - \frac{x^2}{2L}\right)$ Max. (+M) = $\frac{9WL}{128}$ at $x = \frac{3L}{8}$; Max. (-M) = $-\frac{WL}{8}$ at B $y = \frac{W}{48EIL}(3Lx^3-2x^4-L^3x)$; Max. $y = -0.0054\frac{WL^3}{EI}$ at $x = 0.4215 L$ $\theta = -\frac{WL^2}{48EI}$ at A
23. 	$R_A = \frac{W}{8L^3}\left[4L(a^2+ab+b^2)-a^3-ab^2-a^2b-b^3\right]$; $R_D = W - R_A$; $M_D = -R_AL + \frac{W}{2}(a+b)$; (A to B) $V = R_A$; (B to C) $V = R_A - W\left(\frac{x-d}{c}\right)$ (C to D) $V = R_A - W$; (A to B) $M = R_Ax$; (B to C) $M = R_Ax - \frac{W(x-d)^2}{2c}$ (C to D) $M = R_Ax - W\left(x - d - \frac{c}{2}\right)$; Max. (+M) = $R_A\left(d + \frac{R_Ac}{2W}\right)$ at $x = \left(d + \frac{R_Ac}{W}\right)$; Max. (-M) = $-M_D$ (A to B) $y = \frac{1}{EI}\left[R_A\left(\frac{x^3}{6} - \frac{L^2x}{2}\right) + Wx\left(\frac{a^2}{2} + \frac{ac}{2} + \frac{c^2}{6}\right)\right]$ (B to C) $y = \frac{1}{EI}\left[R_A\left(\frac{x^3}{6} - \frac{L^2x}{2}\right) + Wx\left(\frac{a^2}{2} + \frac{ac}{2} + \frac{c^2}{6}\right) - \frac{W(x-d)^4}{24c}\right]$ (C to D) $y = \frac{1}{EI}\left\{R_A\left(\frac{x^3}{6} - \frac{L^2x}{2} + \frac{L^3}{3}\right) + W\left[\frac{1}{6}\left(a + \frac{c}{2}\right)^3 - \frac{1}{2}\left(a + \frac{c}{2}\right)^2 L - \frac{1}{6}\left(x - d - \frac{c}{2}\right)^3 + \frac{1}{2}\left(a + \frac{c}{2}\right)^2 x\right]\right\}$ $\theta = -\frac{1}{EI}\left[\frac{R_AL^2}{2} - W\left(\frac{a^2}{2} + \frac{ac}{2} + \frac{c^2}{6}\right)\right]$ at A

TABLE 9.2 (CONT'D) BEAM FORMULAS

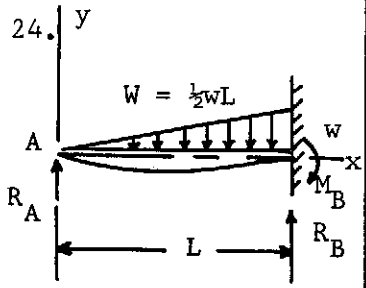
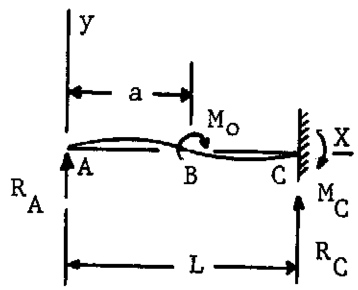
Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
24. 	$R_A = \frac{W}{5}$; $R_B = \frac{4W}{5}$; $M_B = \frac{2WL}{15}$; $V = W\left(\frac{1}{5} - \frac{x^2}{L^2}\right)$; $M = W\left(\frac{x}{5} - \frac{x^3}{3L^2}\right)$ Max. (+M) = 0.06 WL at x = 0.4474 L; Max. (-M) = $-M_B$ $y = \frac{W}{60EIL}\left(2Lx^3 - L^3x - \frac{x^5}{L}\right)$; Max. $y = -0.00477 \frac{WL^3}{EI}$ at $x = L\sqrt{\frac{1}{5}}$ $\theta = -\frac{WL^2}{60EI}$ at A
25. 	$R_A = -\frac{3M_o}{2L}\left(\frac{L^2-a^2}{L^2}\right)$; $R_C = \frac{3M_o}{2L}\left(\frac{L^2-a^2}{L^2}\right)$; $M_c = \frac{M_o}{2}\left(1 - \frac{3a^2}{L^2}\right)$ (A to B) $V = +R_A$; (B to C) $V = +R_A$; (A to B) $M = +R_A x$ (B to C) $M = +R_A x + M_o$; Max. (+M) $= M_o\left(1 - \frac{3a(L^2-a^2)}{2L^3}\right)$ just to the right of B Max. (-M) = $-M_c$ at C when a < 0.275 L Max. (-M) = $R_A a$ to the left of B when a > 0.275 L (A to B) $y = \frac{M_o}{EI}\left(\frac{L^2-a^2}{4L^3}(3L^2x-x^3) - (L-a)x\right)$ (B to C) $y = \frac{M_o}{EI}\left(\frac{L^2-a^2}{4L^3}(3L^2x-x^3) - Lx + \frac{(x^2+a^2)}{2}\right)$ $\theta = \frac{M_o}{EI}\left(a - \frac{L}{4} - \frac{3a^2}{4L}\right)$ at A

TABLE 9.2 (CONT'D) BEAM FORMULAS

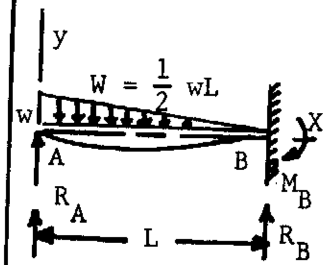
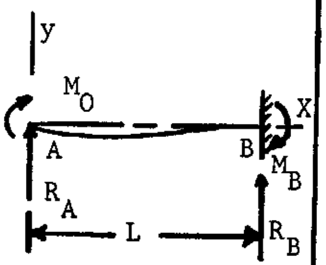
Type of loading and case number	Statically Indeterminate Cases — Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
26. 	$R_A = \frac{11W}{20}$; $R_B = \frac{9W}{20}$; $M_B = \frac{7WL}{60}$; $V = W\left(\frac{11}{20} - \frac{2x}{L} + \frac{x^2}{L^2}\right)$; $M = W\left(\frac{11x}{20} - \frac{x^2}{L} + \frac{x^3}{3L^2}\right)$ Max. (+M) = 0.0846 WL at x = 0.329 L; Max. (-M) = $-\frac{7WL}{60}$ at B $y = \frac{W}{120EIL}\left(11Lx^3 - 3L^3x - 10x^4 + \frac{2x^5}{L}\right)$ Max. y = $-0.00609\,\frac{WL^3}{EI}$ at x = 0.402 L; $\theta = \frac{WL^2}{40EI}$ at A
27. 	$R_A = -\frac{3M_o}{2L}$; $R_B = \frac{3M_o}{2L}$; $M_B = \frac{M_o}{2}$; $V = -\frac{3M_o}{2L}$; $M = \frac{M_o}{2}\left(2 - \frac{3x}{L}\right)$ Max. (+M) = M_o at A; Max. (-M) = $-\frac{M_o}{2}$ at B $y = \frac{M_o}{4EI}\left(2x^2 - \frac{x^3}{L} - Lx\right)$; Max. y = $-\frac{M_oL^2}{27EI}$ at $x = \frac{L}{3}$ $\theta = -\frac{M_oL}{4EI}$ at A

TABLE 9.2 (CONT'D) BEAM FORMULAS

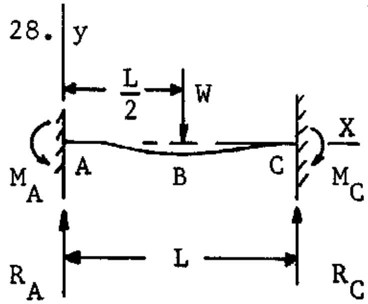
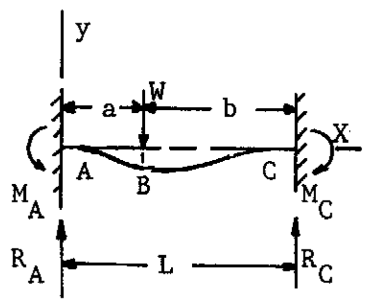
Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
28. 	$R_A = \frac{W}{2}$; $R_C = \frac{W}{2}$; $M_A = \frac{WL}{8}$; $M_C = \frac{WL}{8}$; (A to B) $V = \frac{W}{2}$; (B to C) $V = -\frac{W}{2}$ (A to B) $M = \frac{W(4x-L)}{8}$; (B to C) $M = \frac{W(3L-4x)}{8}$; Max. $(+M) = \frac{WL}{8}$ at B Max. $(-M) = -\frac{WL}{8}$ at A and C; (A to B) $y = -\frac{W}{48EI}(3Lx^2-4x^3)$ Max. $y = -\frac{WL^3}{192EI}$ at B
29. 	$R_A = \frac{Wb^2}{L^3}(3a+b)$; $R_C = \frac{Wa^2}{L^3}(3b+a)$; $M_a = \frac{Wab^2}{L^2}$; $M_c = \frac{Wba^2}{L^2}$ (A to B) $V = R_A$; (B to C) $V = R_A - W$; (A to B) $M = -\frac{Wab^2}{L^2} + R_A x$ (B to C) $M = -\frac{Wab^2}{L^2} + R_A x - W(x-a)$; Max. $(+M) = -\frac{Wab^2}{L^2} + R_A a$ at B, max. value $= \frac{WL}{8}$ when $a = \frac{L}{2}$ Max. $(-M) = -M_A$ when $a < b$, max. possible value $= -0.1481$ WL when $a = \frac{L}{3}$ Max. $(-M) = -M_c$ when $a > b$, max. possible value $= -0.1481$ WL when $a = \frac{2L}{3}$

TABLE 9.2 (CONT'D) BEAM FORMULAS

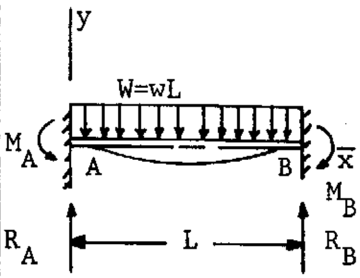
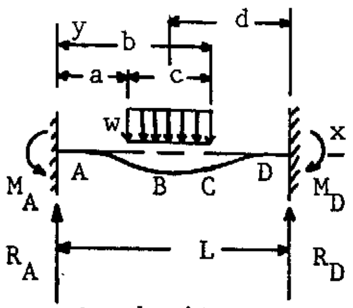
Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
29. (Cont'd)	(A to B) $y = \frac{Wb^2x^2}{6EIL^3}(3ax + bx - 3aL)$ (B to C) $y = \frac{Wa^2(L-x)^2}{6EIL^3}\left[(3b+a)(L-x) - 3bL\right]$ Max. $y = -\frac{2Wa^3b^2}{3EI(3a+b)^2}$ at $x = \frac{2aL}{3a+b}$ if $a > b$ Max. $y = -\frac{2Wa^2b^3}{3EI(3b+a)^2}$ at $x = L - \frac{2bL}{3b+a}$ if $a < b$
30. 	$R_A = \frac{W}{2}$; $R_B = \frac{W}{2}$; $M_A = \frac{WL}{12}$; $M_B = \frac{WL}{12}$; $V = \frac{W}{2}\left(1 - \frac{2x}{L}\right)$ $M = \frac{W}{2}\left(x - \frac{x^2}{L} - \frac{L}{6}\right)$; Max. $(+M) = \frac{WL}{24}$ at $x = \frac{L}{2}$ Max. $(-M) = -\frac{WL}{12}$ at A and B; $y = \frac{Wx^2}{24EIL}(2Lx - L^2 - x^2)$ Max. $y = -\frac{WL^3}{384EI}$ at $x = \frac{L}{2}$

TABLE 9.2 (CONT'D) BEAM FORMULAS

<table>
<tr><th colspan="2">Statically Indeterminate Cases</th></tr>
<tr><th>Types of loading and case number</th><th>Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope</th></tr>
<tr><td>31.

d=L-½a-½b
W=wc</td>
<td>

$R_A = \frac{W}{4L^2}\left(12d^2 - \frac{8d^3}{L} + \frac{2bc^2}{L} - \frac{c^3}{L} - c^2\right)$; $R_D = W - R_A$

$M_A = -\frac{W}{24L}\left(\frac{24d^3}{L} - \frac{6bc^2}{L} + \frac{3c^3}{L} + 4c^2 - 24d^2\right)$

$M_D = \frac{W}{24L}\left(\frac{24d^3}{L} - \frac{6bc^2}{L} + \frac{3c^3}{L} + 2c^2 - 48d^2 + 24dL\right)$; (A to B) $V = R_A$

(B to C) $V = R_A - W\left(\frac{x-a}{c}\right)$; (C to D) $V = R_A - W$; (A to B) $M = -M_A + R_A x$

(B to C) $M = -M_A + R_A x - W\frac{(x-a)^2}{2c}$; (C to D) $M = -M_a + R_A x - W(x-L+d)$

Max. (+M) is between B and C at $x = a + \frac{R_A c}{W}$;

Max. (-M) = $-M_A$ when $a < (L-b)$; Max. (-M) = $-M_D$ when $a > (L-b)$

(A to B) $y = \frac{1}{6EI}(R_A x^3 - 3M_A x^2)$

(B to C) $y = \frac{1}{6EI}\left(R_A x^3 - 3M_A x^2 - \frac{W(x-a)^4}{4c}\right)$

(C to D) $y = \frac{1}{6EI}\left[R_D(L-x)^3 - 3M_D(L-x)^2\right]$

</td></tr>
</table>

TABLE 9.2 (CONT'D) BEAM FORMULAS

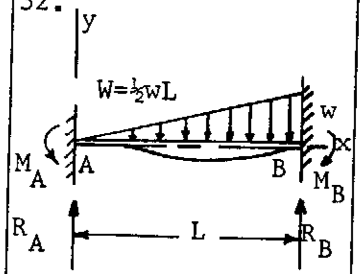
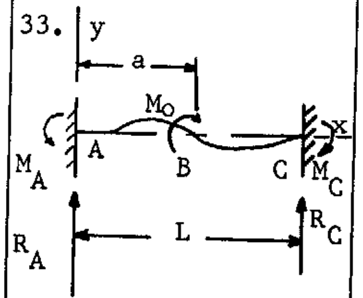
Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
32. 	$R_A = \frac{3W}{10}$; $R_B = \frac{7W}{10}$; $M_A = \frac{WL}{15}$; $M_B = \frac{WL}{10}$; $V = W\left(\frac{3}{10} - \frac{x^2}{L^2}\right)$ $M = W\left(\frac{3x}{10} - \frac{L}{15} - \frac{x^3}{3L^2}\right)$ Max. (+M) = 0.043 WL at x = 0.548 L; Max. (-M) = $-\frac{WL}{10}$ at B $y = \frac{W}{60EI}\left(3x^3 - 2Lx^2 - \frac{x^5}{L^2}\right)$; Max. y = $-0.002617\,\frac{WL^3}{EI}$ at x = 0.525 L
33. 	$R_A = -\frac{6M_o}{L^3}(aL-a^2)$; $R_C = \frac{6M_o}{L^3}(aL-a^2)$; $M_A = -\frac{M_o}{L^2}(4La-3a^2-L^2)$ $M_C = \frac{M_o}{L^2}(2La-3a^2)$; $V = +R_A$; (A to B) $M = -M_A + R_Ax$ (B to C) $M = -M_A + R_Ax + M_o$ Max. (+M) = $M_o\left(\frac{4a}{L} - \frac{9a^2}{L^2} + \frac{6a^3}{L^3}\right)$ just to the right of B Max. (-M) = $M_o\left(\frac{4a}{L} - \frac{9a^2}{L^2} + \frac{6a^3}{L^3} - 1\right)$ just to the left of B (A to B) $y = -\frac{1}{6EI}(3M_Ax^2 - R_Ax^3)$

TABLE 9.2 (CONT'D) BEAM FORMULAS

Statically Indeterminate Cases	
Type of loading and case number	Reactions, Vertical Shear, Bending Moments, Deflection Y, and Slope
33. (Cont'd)	(B to C) $y = \frac{1}{6EI}\left[(M_o - M_A)(3x^2 - 6Lx + 3L^2) - R_A(3L^2x - x^3 - 2L^3)\right]$ Max (-y) at $x = L - \frac{2M_C}{R_C}$ if $a < \frac{2L}{3}$ Max (+y) at $x = \frac{2M_A}{R_A}$ if $a > \frac{L}{3}$

TABLE 9.2 (CONT'd) BEAM FORMULAS

9.2.2 Stress Analysis of Symmetrical Sections

The bending stress in a beam with at least one axis of symmetry in the cross section is

$$f = Mc/I$$

where M is the bending moment at the cross section in question, c is the distance from the neutral axis of the cross section to the fiber in question and I is the moment of inertia of the cross section. This equation is valid when the following assumptions are satisfied:

a) plane sections remain plane
b) the material follows Hooke's law

This equation is called the flexure formula.

Beams are rarely loaded in pure bending. Generally the bending moment is a result of a shear transfer. This means that the bending moment varies along the beam. This variation in bending moment produces a shear normal to a beam cross section, i.e., normal to the section resisting the bending moment.

The flexural shear stress normal to the cross section is

$$f_s = \frac{V}{It} \int_{y=0}^{c} y\,dA$$

In its integrated form it is

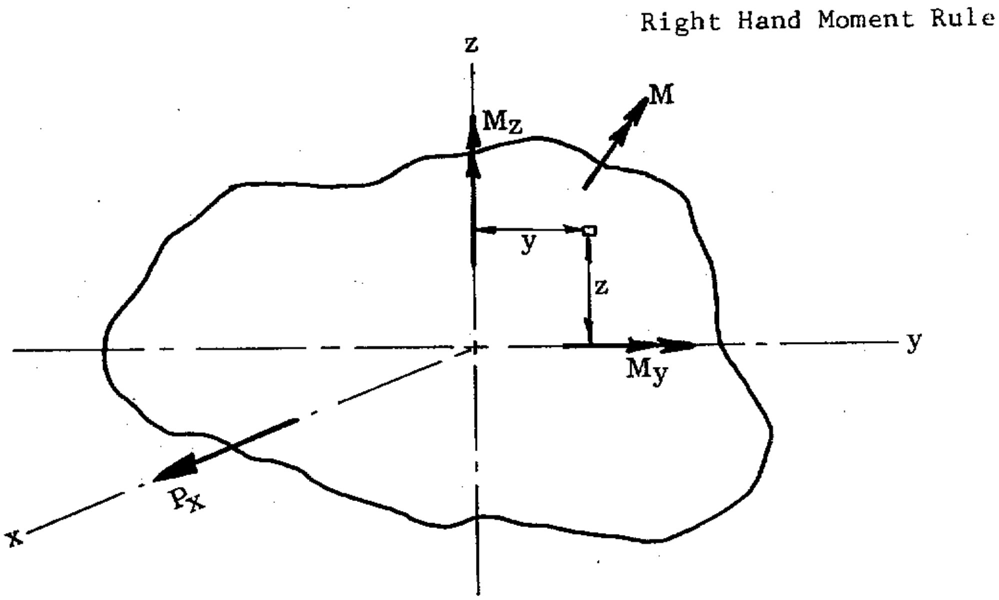
$$f_s = VQ/It$$

where V is the shear parallel to the cross section, Q is the area moment of the cross section, I is the moment of inertia and t is the section thickness.

Revision A

9.2.3 Stress Analysis of Unsymmetrical Sections

The assumption for the flexural formula, f = Mc/I, is that at least one axis of symmetry passes through the section centroid. This condition is not always possible. The section shown below is a general section



with no axis of symmetry. The equation for the stress in a homogeneous beam with axial and bending loads is

$$f_x = \frac{P_x}{A} - \frac{\left(M_y I_{yz} + M_z I_y\right) y}{I_y I_z - I_{yz}^2} + \frac{\left(M_z I_{yz} + M_y I_z\right) z}{I_y I_z - I_{yz}^2} \qquad 9.2$$

9.3 Strain Energy Methods

9.3.1 Castigliano's Theorem

Castigliano's theorem is used for determining deflections and rotations in structures. It is useful also in the analysis of statically indeterminate structures.

Castigliano's theorem states that if external forces act on a member or structure and cause small elastic deflections, the deflection of the point of application of the force in the direction of the force is equal to the partial derivative of the total internal strain energy U with respect to the force. That is

$$\delta_Q = \partial U / \partial Q \qquad 9.3$$

where δ_Q is the deflection in the direction of Q of the point of application of Q. The deflection of any point due to any system of loads may be found by introducing a fictitious load at the point in question and writing the expression for the derivative of the strain energy with respect to the fictitious load at this point.

The method of Castigliano requires that the internal strain energy U be expressed in terms of the loads in each member. Therefore, the strain energy expression for each type of loading is

$$U = \int_o^L P^2 dx/2AE; \text{ Axial} \qquad 9.4$$

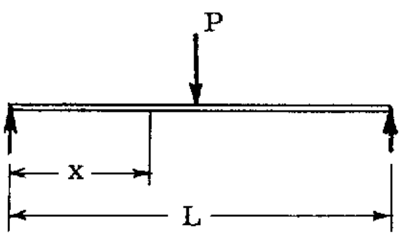
$$U = \int_o^L M^2 dx/2EI; \text{ Bending} \qquad 9.5$$

$$U = \int_o^L V^2 dx/2AG; \text{ Shear} \qquad 9.6$$

where A and I are the area and inertia of the element and E and G are the modulus of elasticity and rigidity. P, V and M are the axial force, shear force and bending moment in terms of the applied load to an element.

9.3.2 Structural Deformations Using Strain Energy

The deflection at a point can be obtained by the application of Castigliano's theorem. Consider the example shown below, a



simple beam of span L and constant cross section subjected to a concentrated load P at the center of the span. The moment at any point of the beam is

$M = Px/2$, for $x = 0$ to $x = L/2$
$M = Px/2 - P(x-L/2)$, for $x = L/2$ to $x = L$
From Equation 9.5

$$U = 1/2EI \int M^2 dx = 1/2\,EI \int_0^{L/2} (Px/2)^2 dx + 1/2EI \int_{L/2}^{L} [P/2(L-x)]^2 dx$$

$$U = P^2L^3/96EI$$

$$\delta = \partial U/\partial P = \partial/\partial P(P^2L^3/96EI) = 2PL^3/96EI = PL^3/48EI$$

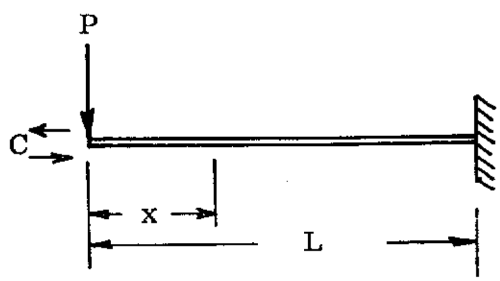
The example is about as simple as possible, but the theory is applicable to problems of any complexity.

The rotation of a point can be determined by employing Castigliano's theorem which becomes

$$\theta = \partial U/\partial C \qquad 9.7$$

where C is an applied or fictitious couple.

Consider the following example. It is



required to find the rotation of the free end of a cantilever beam. Since no couple acts at the end of the beam, one must be applied. It is denoted as C and since it is fictitious will be set equal to zero after the differentiation with respect to C. Then the example is solved as follows.

$$\theta = \frac{\partial U}{\partial C} = \frac{\partial}{\partial C} \int \frac{M^2 dx}{2EI} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial C} dx$$

$$M = C + Px$$

$$\partial M/\partial C = 1$$

$$\theta = \int_0^L \frac{(C + P_x) dx (1) dx}{EI} = \frac{CL + (PL^2/2)}{EI}$$

Now set $C = 0$ since it is fictitious, then

$$\theta = \frac{PL^2}{2EI}$$

This procedure is applicable to structures with any type of loading.

9.3.3 Deflection By The Dummy Load Method

The unit load or dummy load method may be used to determine deflection in elastic or inelastic members. Deflection of inelastic members by this method is given in section 9.5.2. The theorem as applied to elastic beams is written in integral form as

$$\delta = \int_0^L \frac{Mm}{EI}\, dx \qquad \text{9.8}$$

$$\theta = \int_0^L \frac{Mm'}{EI}\, dx \qquad \text{9.9}$$

Where (δ) is the deflection at the unit load and (θ) is the rotation at the unit moment. The moment (M) is the bending moment at any section caused by the actual loads. (m) is the bending moment at any section of the beam caused by a dummy load of unity acting at the point whose deflection is to be found and in the direction of the desired deflection. The bending moment (m') is the bending moment at any section of the beam caused by a dummy couple of unity applied at the section where the change in slope is desired. It is noted that although (m') may be thought of as a bending moment, it is evident from the expression $m' = \frac{\partial M}{\partial M_a}$ that it is actually dimensionless.

EXAMPLE: Find the elastic vertical deflection of point A (Fig. 9.2a) of the simply supported beam subjected to two concentrated loads.

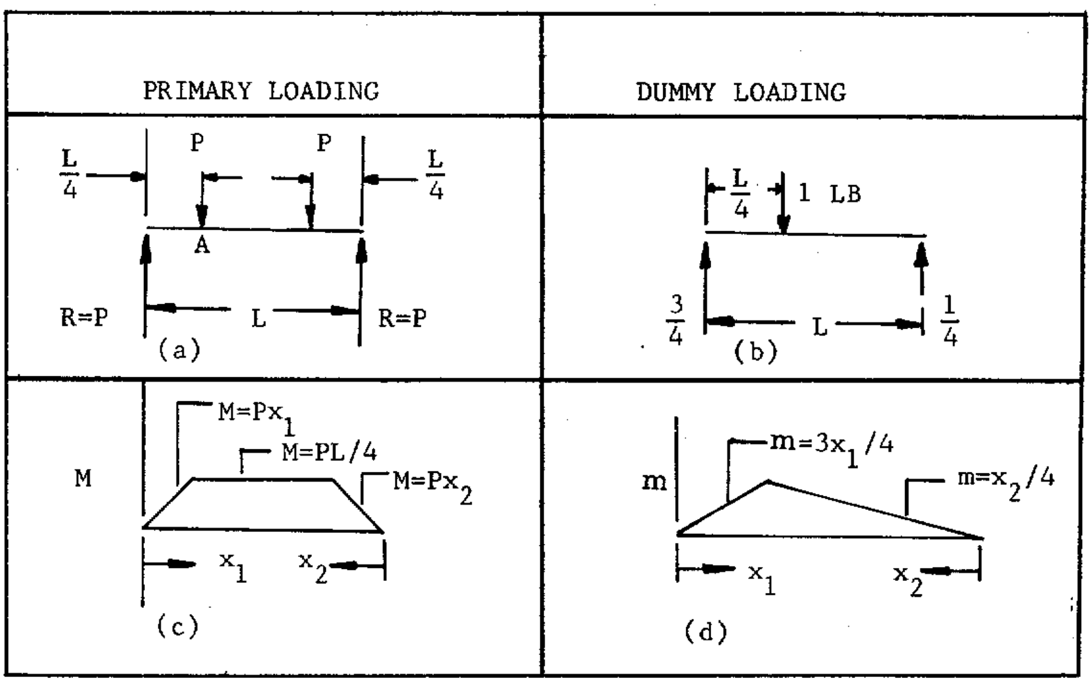


FIGURE 9.2 EXAMPLE OF DUMMY LOAD METHOD

Solution: The actual loading is shown in Fig. 9.2a and the dummy loading is shown in Fig. 9.2b. The moment for the actual loading is shown in Fig. 9.2c and the corresponding moment diagram for the dummy loading is shown in Fig. 9.2d.

The deflection by use of equation (9.8) noting that x_1 starts at the left and x_2 starts at the right is

$$\delta = \int_0^L \frac{Mm}{EI}\,dx = \int_0^{\frac{L}{4}} \frac{Px_1}{EI}\left(\frac{3x_1}{4}\right) dx_1 + \int_0^{\frac{L}{4}} \frac{Px_2}{EI}\left(\frac{x_2}{4}\right) dx_2$$

$$+ \int_{\frac{L}{4}}^{\frac{3L}{4}} \frac{PL}{4EI}\left(\frac{x_2}{4}\right) dx_2 = \frac{PL^3}{48EI}$$

The problem of Figure 9.2 can also be solved using pictorial integration. To find any deformation in a structure due to any external loading, proceed as follows:

(1) Draw the moment diagram due to the actual loading. Denote these moments as M_o.

(2) Remove the actual loading and apply a fictitious unit load where the deformation is desired. This unit load must be of such a type and applied in such a manner that (load) x (deformation) = (work); i.e., if the deformation to be found is a rotation, the unit load must be a moment. Draw the moment diagram due to this unit load and denote these moments by M_a.

(3) Compute the deformation from

$$\delta_{oa} = \frac{1}{EI}\int M_o M_a\, ds = \frac{1}{EI}\,\bar{\delta}_{oa} \qquad 9.10$$

where $\bar{\delta}_{oa}$ is given in TABLE 9.4 for various combinations of moment diagrams M_o and M_a. (TABLE 9.4 is not applicable for curved members.)

Table 9.3 shows the solution of the problem in Figure 9.2 by pictorial integration. The moment diagrams of the actual and unit loads are shown in Table 9.3a. $\bar{\delta}_{oa}$ is obtained from Table 9.3b, c, and d as,

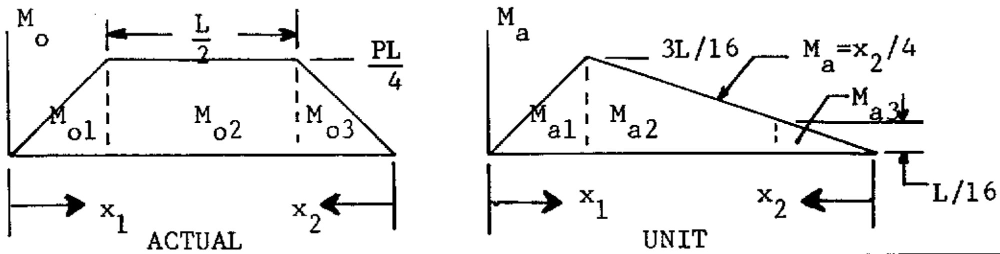
$$\bar{\delta}_{oa} = \bar{\delta}_{11} + \bar{\delta}_{22} + \bar{\delta}_{33} = PL^3/256 + PL^3/64 + PL^3/768 = PL^3/48$$

The deflection at point A is:

$$\delta_{oa} = 1/EI\ \bar{\delta}_{oa} = PL^3/48EI$$

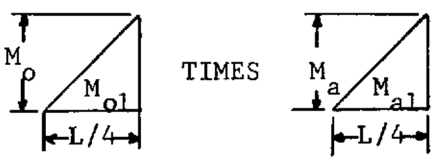
which agrees with the previous solution.

a.



MOMENT DIAGRAMS FROM FIGURES 9.2(a) and 9.2(b)

b.

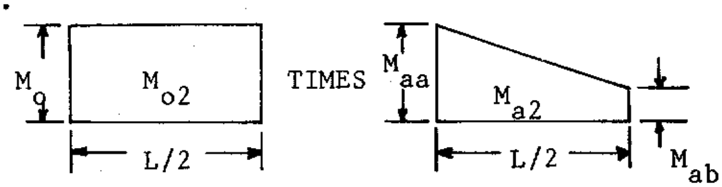


$\overline{\delta}_{11} = k(M_o)(M_a)(L)$

$k = .333$: (TABLE 9.4)

$\overline{\delta}_{11} = (.333)(PL/4)(3L/16)(L/4) = PL^3/256$

c.



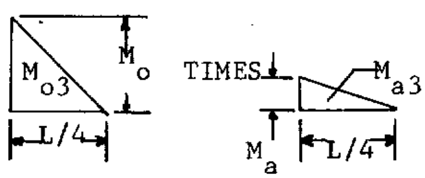
$\overline{\delta}_{22} = k(M_o)(M_{aa})(L)$

$k = .50(1.0 + M_{ab}/M_{aa})$: (TABLE 9.4)

$k = .667$

$\overline{\delta}_{22} = (2/3)(PL/4)(3L/16)(L/2) = PL^3/64$

d.



$\overline{\delta}_{33} = k(M_o)(M_a)(L)$

$k = .333$: (TABLE 9.4)

$\overline{\delta}_{33} = (.333)(PL/4)(L/16)(L/4) = PL^3/768$

TABLE 9.3 - SOLUTION OF FIGURE 9.2 BY PICTORIAL INTEGRATION

A. CORRECTION COEFFICIENTS FOR EQUATION $EI\delta = k(M_{max})^2L$

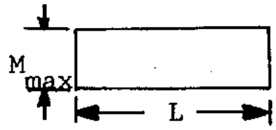
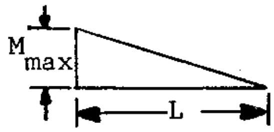
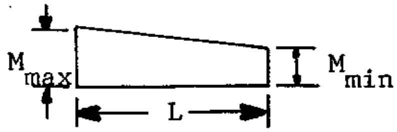
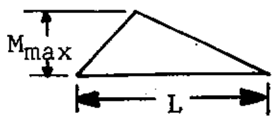
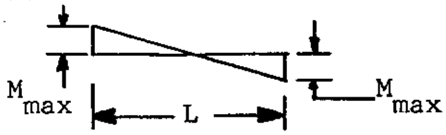
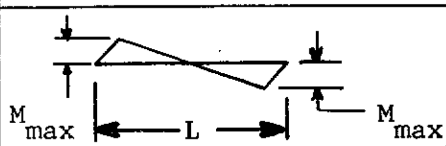
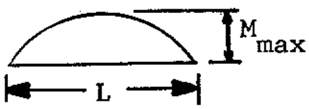
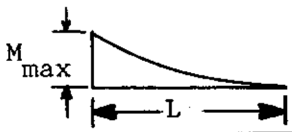
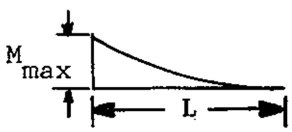
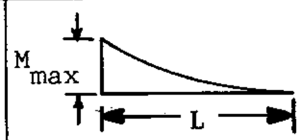
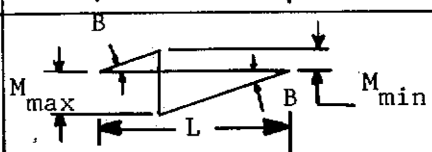
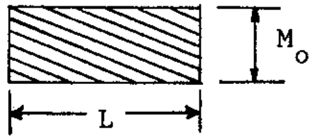
CASE	DEFINITION OF SITUATION	k
I.		1.000
II.		.333
III.	 $A = \frac{M_{min}}{M_{max}}$	$.333(1.0 + A + A^2)$
IV.		.333
V.		.333
VI.		.333
VII.	 2ND DEGREE	.533 $(= \frac{8}{15})$
VIII.	 2ND DEGREE	.200
IX.	 3RD DEGREE	.143 $(= \frac{1}{7})$
X.	 4TH DEGREE	.111 $(= \frac{1}{9})$
XI.	 $A = \frac{M_{min}}{M_{max}}$	$.333(1.0 + A + A^2)$

TABLE 9.4 CORRECTION COEFFICIENTS

B. CORRECTION COEFFICIENTS FOR EQUATION $EI\delta = k(M_o)(M_a)\,L$



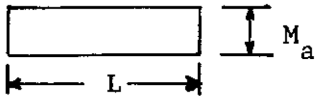
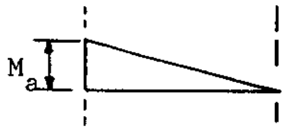
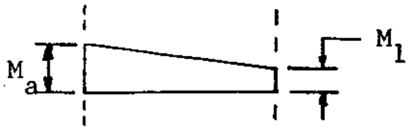
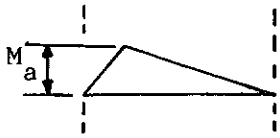
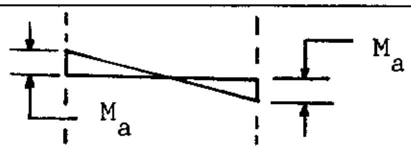
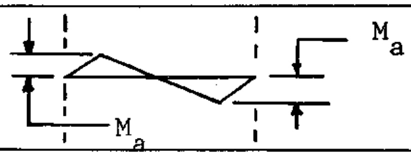
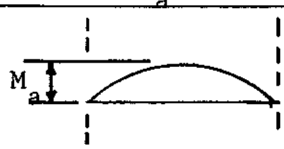
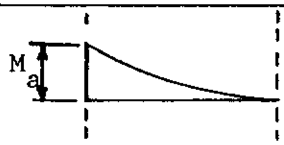
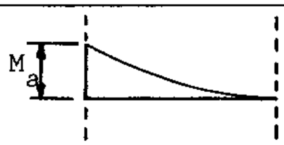
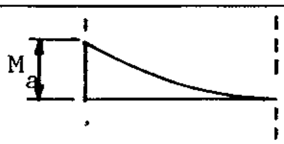
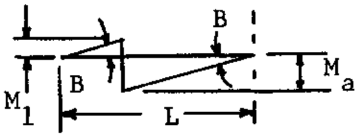
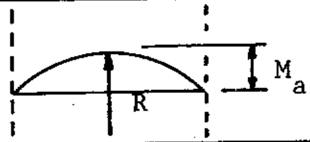
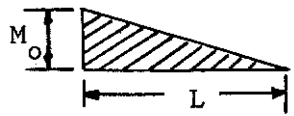
CASE	DEFINITION OF SITUATION	k
I.		1.000
II.		.500
III.	 $A = \frac{M_1}{M_a}$	.50 (1.0 + A)
IV.		.500
V.		.000
VI.		.000
VII.	 2ND DEGREE	.667
VIII.	 2ND DEGREE	.333
IX.	 3RD DEGREE	.250
X.	 4TH DEGREE	.200

TABLE 9.4 (Cont'd) CORRECTION COEFFICIENTS

(B Cont'd) XI.	 $A = \frac{M_1}{M_a}$, $B = \frac{a}{L}$	$.5\left(B(1-A)+A\right)$
XII.	 $A = \frac{R}{M_a}$ $K = \frac{\text{arc length}}{L}$	$.5\left(A\ (K-1.)+1.\right)$

C. CORRECTION COEFFICIENTS FOR EQUATION $EI\delta = k\ (M_o)(M_a)\ L$



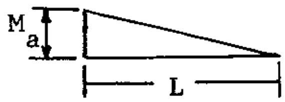
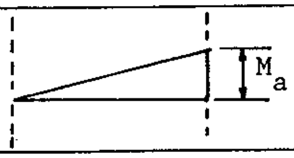
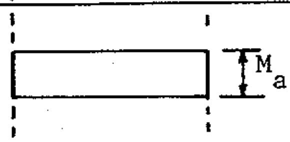
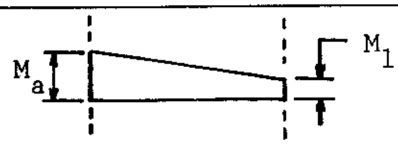
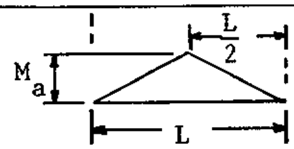
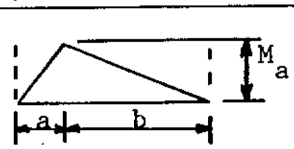
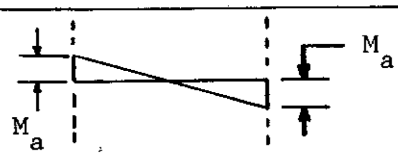
CASE	DEFINITION OF SITUATION	k
I.		.333
II.		.167 $(=\frac{1}{6})$
III.		.500
IV.	 $A = \frac{M_1}{M_a}$	.167 (2.00 + A)
V.		.250
VI.	 $A = \frac{b}{L}$ NOTE: b is dimension farthest from M_o	.167 (1.00 + A)
VII.		.167 $(=\frac{1}{6})$

TABLE 9.4 (Cont'd) CORRECTION COEFFICIENTS

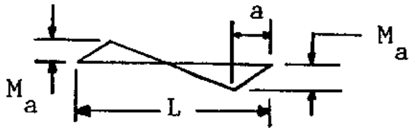
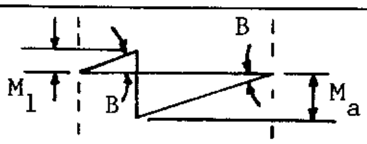
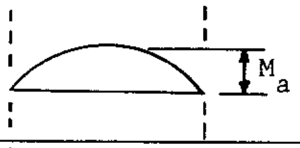
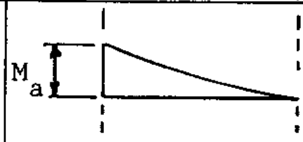
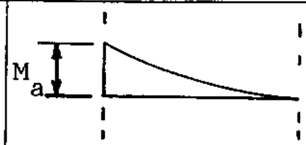
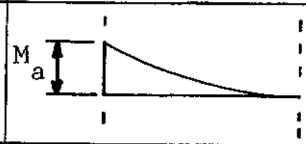
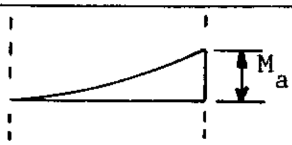
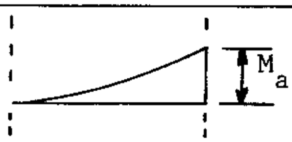
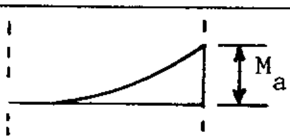
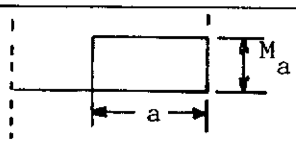
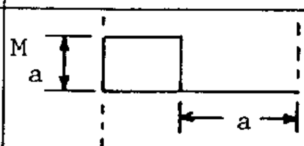
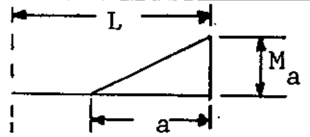
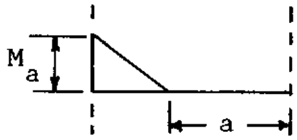
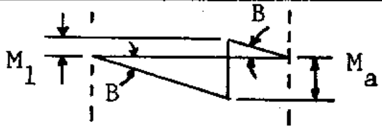
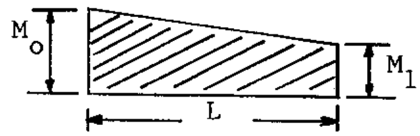
CASE (C Cont'd)	DEFINITION OF SITUATION		k
VIII.		$A = \frac{a}{L}$	.167 (1.00 - A)
IX.		$A = \frac{M_1}{M_a}$	$.167 \left(A^2(A-3) + 2\right)\left(\frac{1}{A-1}\right)^2$
X.		2ND DEGREE	.333
XI.		2ND DEGREE	.250
XII.		3RD DEGREE	.200
XIII.		4TH DEGREE	$.167\ (= \frac{1}{6})$
XIV.		2ND DEGREE	$.083\ (= \frac{1}{12})$
XV.		3RD DEGREE	$.050\ (= \frac{1}{20})$
XVI.		4TH DEGREE	$.033\ (= \frac{1}{30})$
XVII.		$A = \frac{a}{L}$	$.500\ A^2$
XVIII.		$A = \frac{a}{L}$	$.50\ (1.0 - A^2)$

TABLE 9.4 (CONT'D) CORRECTION COEFFICIENTS

CASE	DEFINITION OF SITUATION	k
(C cont'd) XIX.	 $A = \frac{a}{L}$	$.167\ A^2$
XX.	 $A = \frac{a}{L}$	$.167\ (2.0 - A - A^2)$
XXI.	 $A = \frac{M_1}{M_a}$	$.167\left(A(2A^2-3)+1\right)\ \left(\frac{1}{A-1}\right)^2$

D. CORRECTION COEFFICIENTS FOR EQUATION $EI\delta = k(M_o)(M_a)\ L$

$$A = \frac{M_1}{M_o}$$



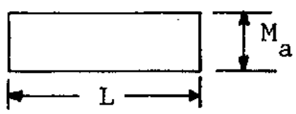
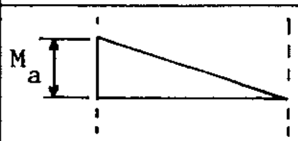
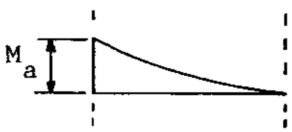
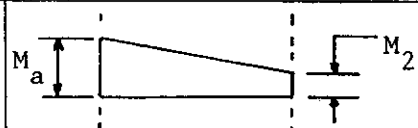
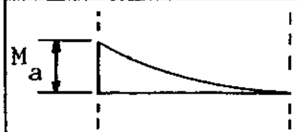
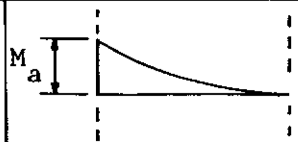
CASE	DEFINITION OF SITUATION	k
I.		$.50\ (1.0 + A)$
II.		$.167\ (2.00 + A)$
III.	 2ND DEGREE	$.083\ (3.00 + A)$
IV.	 $B = \frac{M_2}{M_a}$	$.333\ (1.0 + AB + 0.5\ (A+B))$
V.	 3RD DEGREE	$.05\ (4.0 + A)$
VI.	 4TH DEGREE	$.033\ (5.0 + A)$

TABLE 9.4 (Cont'd) CORRECTION COEFFICIENTS

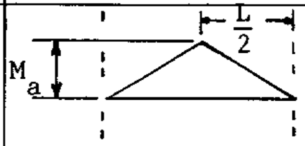
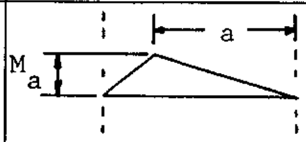
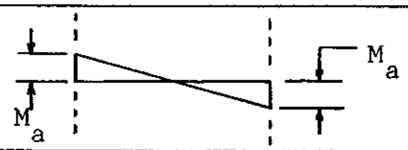
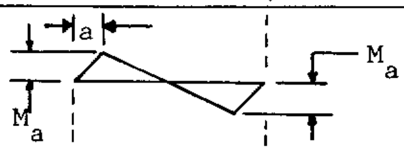
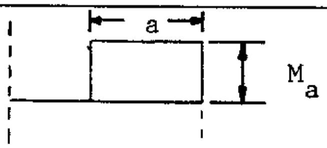
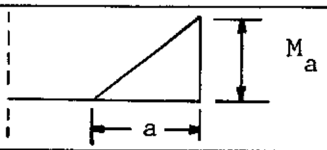
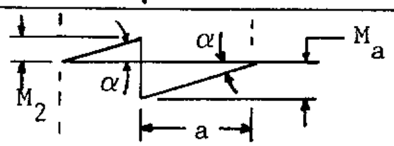
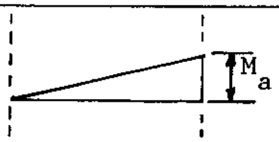
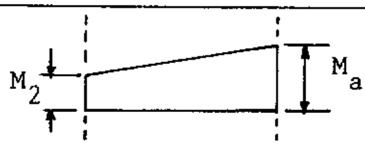
CASE	DEFINITION OF SITUATION	k
(D cont'd) VII.	 2ND DEGREE	.333 (1.0 + A)
VIII.		.250 (1.0 + A)
IX.		$.167\ (1+2A+\frac{a}{L}\ (1 - A))$
X.		.167 (1.00 - A)
XI.	 $B = \frac{a}{L}$	.167 (1 - A)(1 - B)
XII.	 $B = \frac{a}{L}$	$.50((1-A)\ B^2 + 2\ AB)$
XIII.	 $B = \frac{a}{L}$	$.167\ ((1-A)\ B^2 + 3AB)$
XIV.	 $B = \frac{a}{L}$	$.167(\frac{1}{B})(A(3B^2-6B+2)$ $+ 1 - 3B^2)$
XV.		.167 (1 + 2A)
XVI.	 $B = \frac{M_2}{M_a}$	.333 (.50(AB+1)+A+B)

TABLE 9.4 (Cont'd) CORRECTION COEFFICIENTS

9.3.4 Analysis of Redundant Structures

The theories of strain energy and Castigliano can be used to determine the reactions on a statically indeterminate structure. A statically-indeterminate structure is one where there are more supports or more members present in the structure than are necessary to maintain equilibrium of the structure. The structure will still resist the loads if one or more supports or members are omitted.

The total strain energy in a structure must be determined, i.e., the summation of the strain energy contributed by axial, shear, bending and sometimes torsion. Equations 9.4 through 9.6 are used to determine the energy in each element of the structure. Sometimes certain energy terms can be ignored because their contribution is small compared to other forms of loading. Such situation occurs in a shallow beam which is subjected to shear and bending. The shear energy can be ignored because bending energy is so much greater.

To determine the redundant forces and moments in a structure for which it may be assumed that only bending deformations affect the magnitudes of the redundants, proceed as follows:

(1) Cut the structure at convenient points to make it statically determinate. Denote the redundant forces or moments as x_a, x_b, x_c, etc. The cuts will generally be at reaction points but do not have to be there. For instance, because of symmetry, the structure could be cut at the line of symmetry.

(2) Draw the moment diagram M_o for the actual applied loads acting on the statically determinate structure.

(3) Draw the moment diagrams for unit redundant forces or moments (1 lb or 1 in-lb) acting on the statically determinate structure with the applied loads removed. Designate these moments as M_a, M_b, M_c, etc.

(4) The expression for energy is

$$U = \int M^2 \, dx/2\,EI, \text{ (equation 9.5)}$$

Write the moment equations for the structure

One Redundant: $M = M_o + x_a M_a$

Two Redundants: $M = M_o + x_a M_a + x_b M_b$

Three Redundants: $M = M_o + x_a M_a + x_b M_b + x_c M_c$

(5) Evaluate the deformations for each redundant using Catigliano's Theorem, i.e.,

$$\delta_a = \partial U/\partial x_a$$

The resulting equations will take the form

$$x_a \sum \frac{M_a^2 dx}{EI} + x_b \sum \frac{M_a M_b dx}{EI} + x_c \sum \frac{M_a M_c dx}{EI} + \sum M_a M_o dx = 0$$

$$x_a \sum \frac{M_a M_b dx}{EI} + x_b \sum \frac{M_b^2 dx}{EI} + x_c \sum \frac{M_b M_c dx}{EI} + \sum M_b M_o dx = 0$$

$$x_a \sum \frac{M_a M_c dx}{EI} + x_b \sum \frac{M_b M_c dx}{EI} + x_c \sum \frac{M_c^2 dx}{EI} + \sum M_c M_o dx = 0$$

There will be one equation for each redundant. The coefficients of x_a, x_b, and x_c can be determined using pictorial integration as explained in 9.3.3.

(6) Solve for the redundants, x_a, x_b, x_c, etc.

(7) Calculate the actual moments using the moment equations developed in step 4 if they are needed.

Examples are shown in Figures 9.3 and 9.4 with problem solutions.

9.3.4.1 Example Problem - Beam With Single Redundant

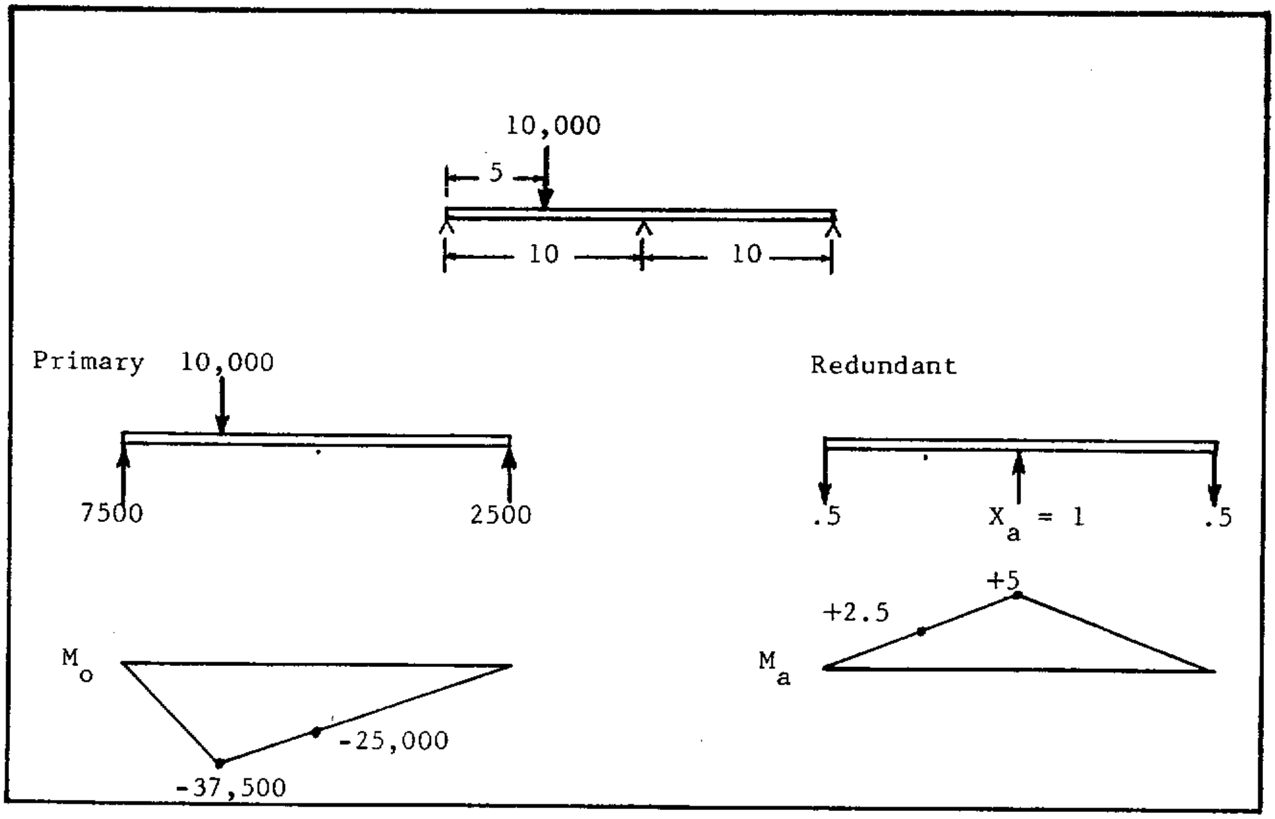


FIGURE 9.3 - BEAM WITH SINGLE REDUNDANT

$$M = M_o + x_a M_a$$

$$M^2 = M_o^2 + 2 x_a M_a M_o + x_a^2 M_a^2$$

$$U = \int M_o^2 \, dx/2\,EI + \int 2 x_a M_a M_o \, dx/2EI + \int x_a^2 M_a^2 \, dx/2\,EI$$

$$\frac{\partial U}{\partial x_a} = 0 = \int \frac{M_a M_o dx}{EI} + \int \frac{x_a M_a^2 dx}{EI}$$

$$x_a \sum k M_a^2 \, dx/EI + \sum k M_a M_o \, dx/EI = 0$$

where k is the constant of integration from Table 9.4.

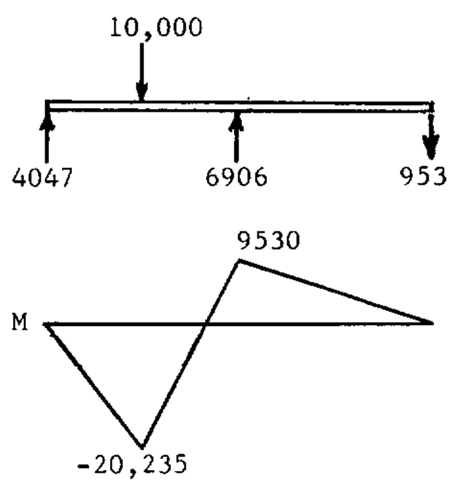
$$x_a (.333)(5)^2 (20) + (.333)(2.5)(-37500)(5) + (.611)(5)(-37500)(5)$$
$$+ (.333)(5)(-25000)(10) = 0$$

$$x_a = (156094 + 577500 + 416250)/166.5 = 6906$$

The maximum moment is then

$$M = M_o + x_a M_a = -37500 + (6906)(2.5) = -20235$$

The beam balance and bending diagram is



9.3.4.2 Example Problem - Beam With Two Redundants

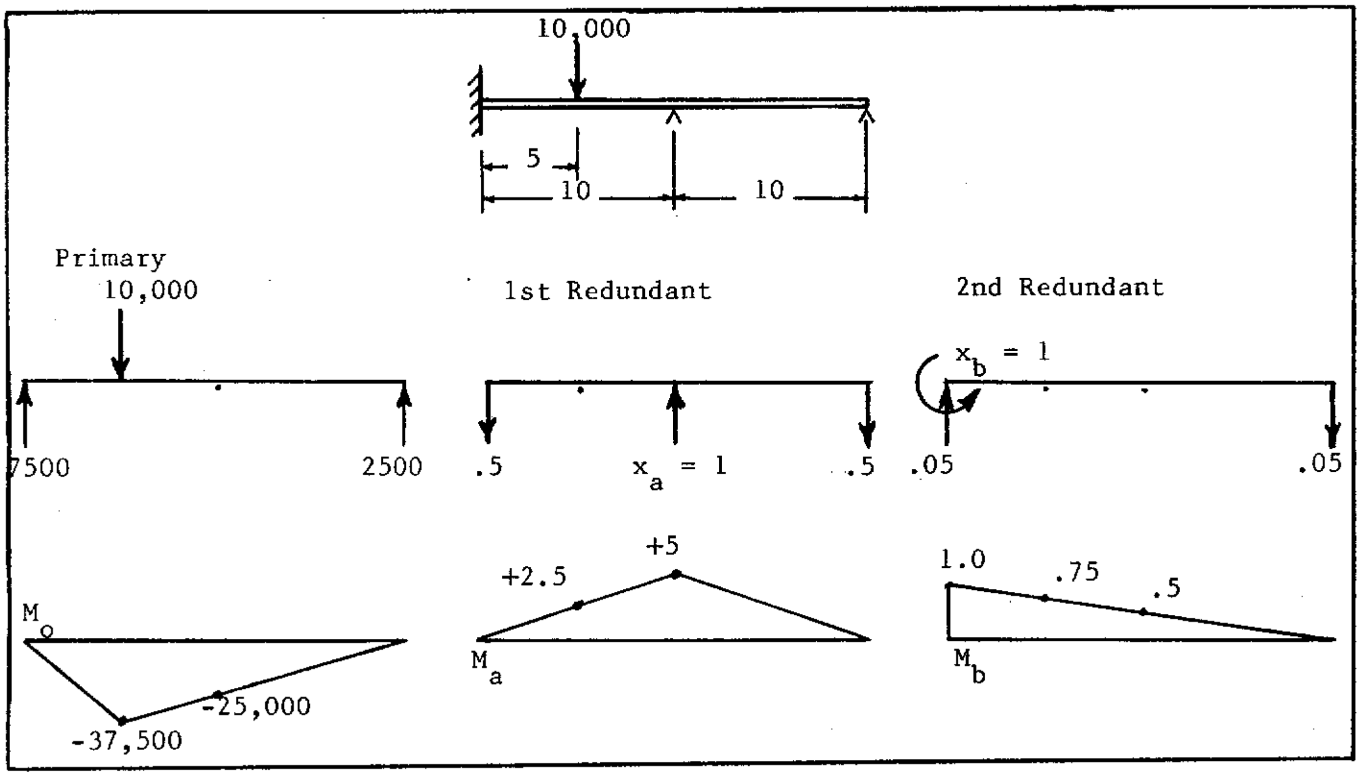


FIGURE 9.4 - BEAM WITH TWO REDUNDANTS

$$x_a \sum k\, M_a^2\, dx + x_b \sum k\, M_a\, M_b\, dx + \sum k\, M_a\, M_o\, dx = 0$$

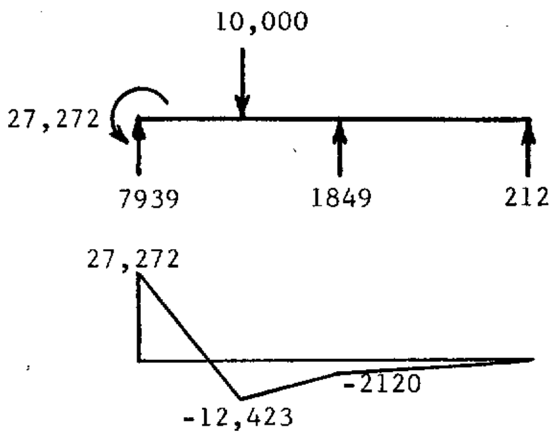
$$x_a \sum k\, M_a M_b\, dx + x_b \sum k\, M_b^2\, dx + \sum k\, M_b\, M_o\, dx = 0$$

$$166.531\, x_a + 24.994\, x_b - 989{,}219 = 0$$

$$24.994\, x_a + 6.333\, x_b - 218{,}672 = 0$$

$x_a = 1849$

$x_b = 27272$



9.3.5 Analysis of Redundant Built-Up Sheet Metal Structures

Section 9.3.4 deals with structures in which bending is the primary energy contributor. In this section, commonly-used airframe structures are discussed. Such structures are composed of thin webs and axial members. No bending members are present since all of the loads can be transferred by shear, tension and compression.

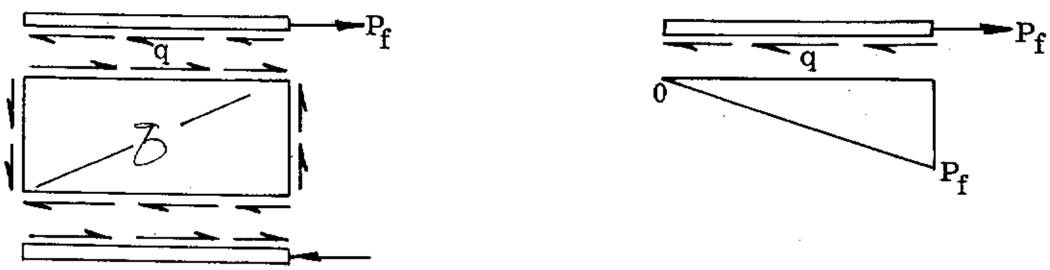
A new energy term is necessary since shear flow, q, is now present. Equation 9.6 can be written in terms of shear flow.

It then becomes

$$U = \int_0^L q^2 \ a \ dx/2Gt \qquad 9.11$$

where a is the panel dimension normal to the x direction and t is the web thickness.

In most airframe structures the shear flow, q, can be assumed to be constant over a given length. Since axial members are attached to the webs and must react the shear flow with a concentrated load the axial load will vary in an axial member. Such a situation is shown below.



The axial load in the upper member varies from zero to P_f over the length L. The load at any point x in a member is $P_x = P_f(x/L)$.
Equation 9.4 is then written as

$$U = \int_0^L P_f^2 \ x^2 \ dx/2 \ L^2 \ AE \qquad 9.12$$

Pictorial integration of the above equation is possible. Figure 9.5 shows the integration constant for various axial loading conditions.

A redundant analysis of a sheet metal beam is shown in Figure 9.6 and the following example problem.

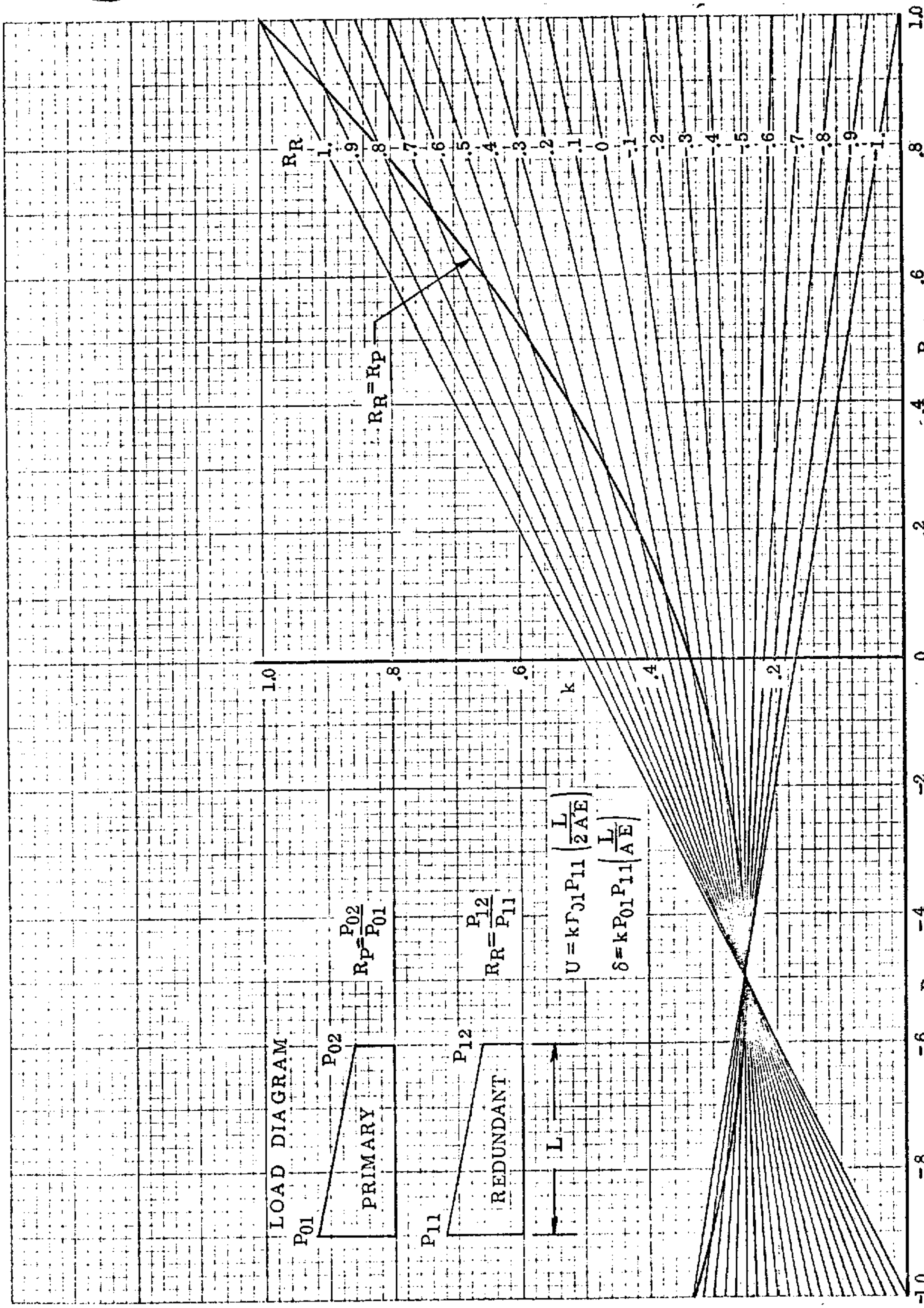


FIGURE 9.5 - INTEGRATION CONSTANTS FOR AXIAL LOADING

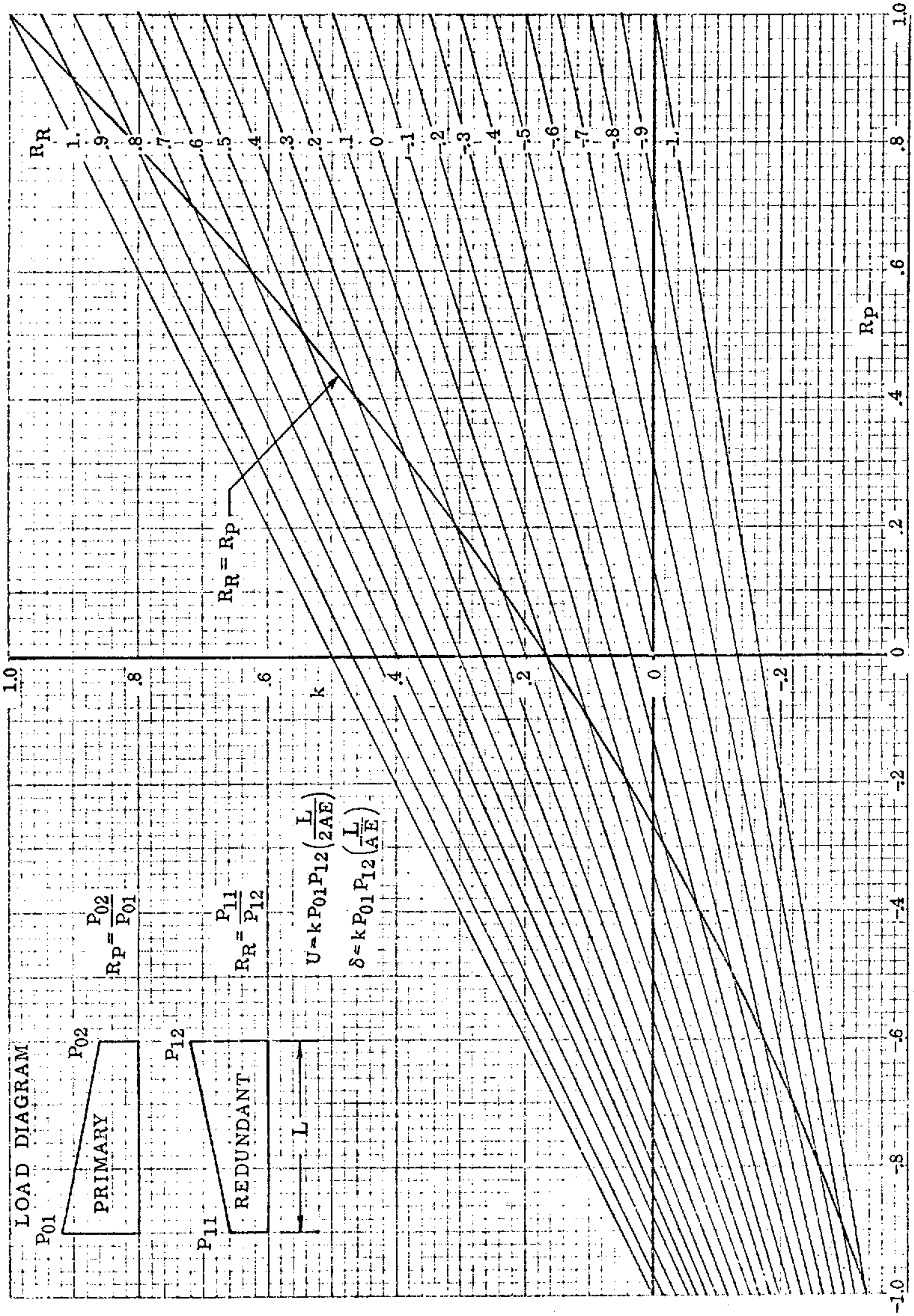


FIGURE 9.5 (Cont'd) - INTEGRATION CONSTANTS FOR AXIAL LOADING

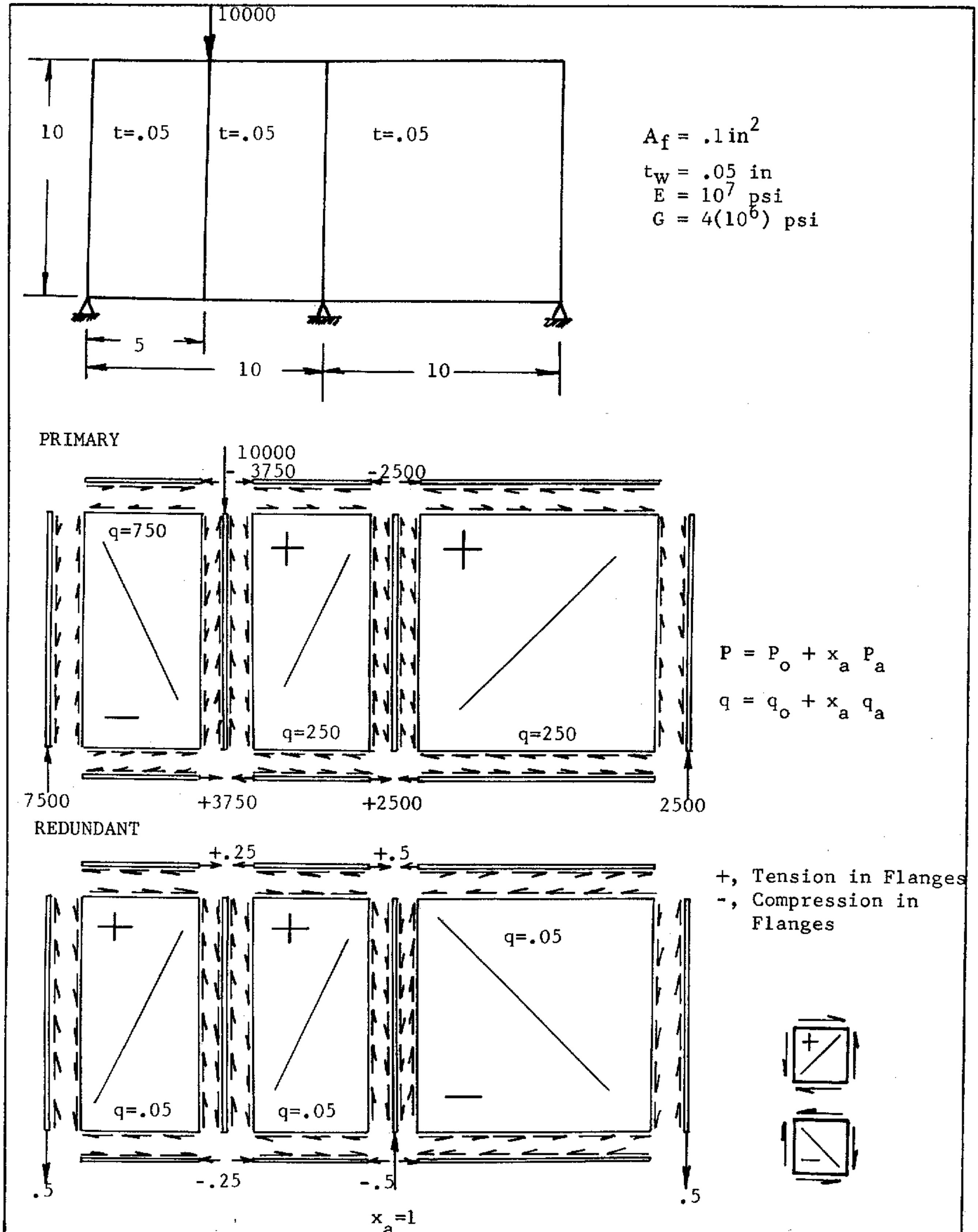


FIGURE 9.6 - EXAMPLE OF REDUNDANT SHEET METAL BEAM

Example Problem:

$$U_{TOTAL} = U_{AXIAL} + U_{SHEAR} = \int_o^L P^2\ dx/2AE + \int_o^L q^2\ adx/2Gt$$

$$U_T = \Sigma k\ (P_o + x_a\ P_a)^2\ L/2AE + \Sigma(q_o + x_a\ q_a)^2\ ab/2Gt$$

$$U_T = \Sigma k\ P_o^{\ 2}\ L/2AE + x_a \Sigma k\ P_o\ P_a\ L/AE + x_a^{\ 2}\ \Sigma k\ P_a^{\ 2}\ L/2AE$$

$$+ \Sigma q_o^{\ 2}\ ab/2Gt + x_a\ \Sigma q_o q_a\ ab/Gt + x_a^{\ 2}\ \Sigma q_a^{\ 2}\ ab/2Gt$$

$$\partial U_T/\partial x_a = \Sigma k\ P_o\ P_a\ L/AE + x_a \Sigma k\ P_a^{\ 2}\ L/AE$$

$$+ \Sigma q_o\ q_a\ ab/Gt + x_a \Sigma q_a^{\ 2}\ ab/Gt$$

$$\partial U_T/\partial x_a = \delta_{xa} = 0$$

$$x_a = \frac{-\Sigma k\ P_o\ P_a\ L/AE - \Sigma q_o\ q_a\ ab/Gt}{\Sigma k\ P_a^{\ 2}\ L/AE + \Sigma q_a^{\ 2}\ ab/Gt}$$

k = constant of integration from Figure 9.5.

FLANGE	P_o	P_a	L	AE $x(10^{-6})$	k	$\frac{k P_o P_a L}{AE} x(10^{-6})$	P_a^2	k	$\frac{k P_a^2 L}{AE} x(10^{-6})$
ab	0 / +3750	0 / - .25	5	1.0	.333	-1560	.0156	.333	.026
bc	+3750 / +2500	- .25 / - .5	5	1.0	.61	-5719	.0156	.7	.055
cd	+2500 / 0	- .5 / 0	10	1.0	.333	-4163	.25	.333	.833
hg	0 / -3750	0 / + .25	5	1.0	.333	-1560	.0156	.333	.026
gf	-3750 / -2500	+ .25 / + .5	5	1.0	.61	-5719	.25	.7	.875
fe	-2500 / 0	+ .5 / 0	10	1.0	.333	-4163	.25	.333	.833
ah	-7500 / 0	+ .5 / 0	10	1.0	.333	-12488	.25	.333	.833
bg	0 / -10000	0 / 0	10	1.0	0	0	0	.333	0
cf	0 / 0	-1.0 / 0	10	1.0	0	0	1.0	0	0
de	0 / -2500	+ .5 / 0	10	1.0	.333	-4163	.25	.333	.833
					$\sum =$	- 38135		$\sum =$	4.314

WEB	q_o	q_a	ab	Gt $x(10^{-6})$	$\frac{q_o q_a ab}{Gt} x(10^6)$	q_a^2	$\frac{q_a^2 ab}{Gt} x(10^6)$
1	-750	.05	50	.2	-9375	.0025	.625
2	250	.05	50	.2	3125	.0025	.625
3	250	-.05	100	.2	-6250	.0025	1.250
				$\sum =$	-12500	$\sum =$	2.5

$$x_a = \frac{-[-38135(10^{-6})] - [-12500(10^{-6})]}{4.314(10^{-6}) + 2.5\ (10^{-6})} = \frac{50635}{6.814} = 7431$$

9.3.6 Analysis of Structures with Elastic Supports

The previous example, Figure 9.6, could also be on spring supports instead of the rigid ones assumed. The additional energy of the springs must be added to the total strain energy of the system. The energy in a spring is

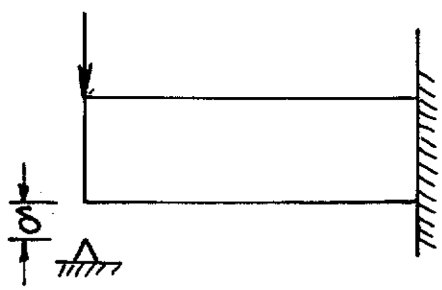
$$U_{spring} = \sum F^2/2K$$

where K is the spring rate in lbs/in. Figure 9.7 shows an example.

9.3.7 Analysis of Structures with Free Motion

In the previous sections the partial derivative of the energy with respect to the redundant ($\partial U/\partial x_a$) was set equal to zero. This the theory of least work which states the redundant does no work. This is a condition of minimum strain energy and no relative motion exists between the beam and the support. If, however, a definite free motion exists at some support the partial derivative is set equal to the free motion.

The cantilever beam shown below illustrates



a beam with free motion. The beam is statically determinant until the deflection at the end of the beam is equal the free motion. Then the beam is statically indeterminant. The total energy in the beam prior to contact of the support is

$$U_t = \sum P^2 L/2AE + \sum q^2 ab/2Gt$$

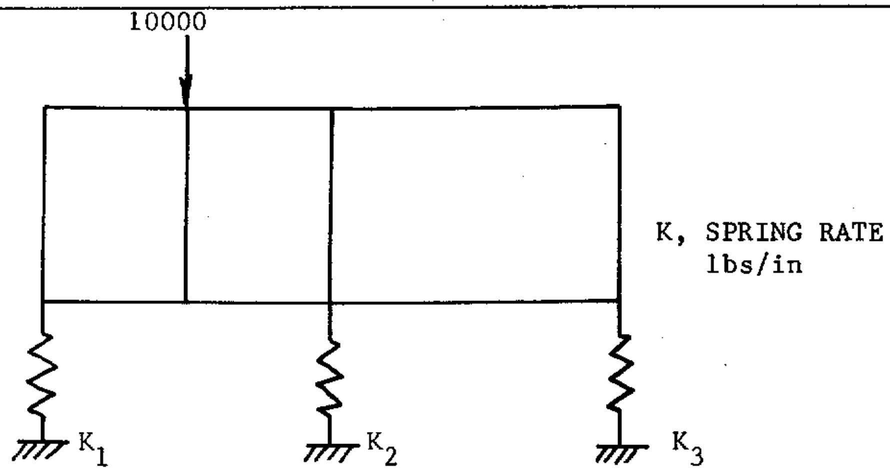
Selecting the end support as the redundant, the total axial load in any member is

$$P = P_o + x_a P_a$$

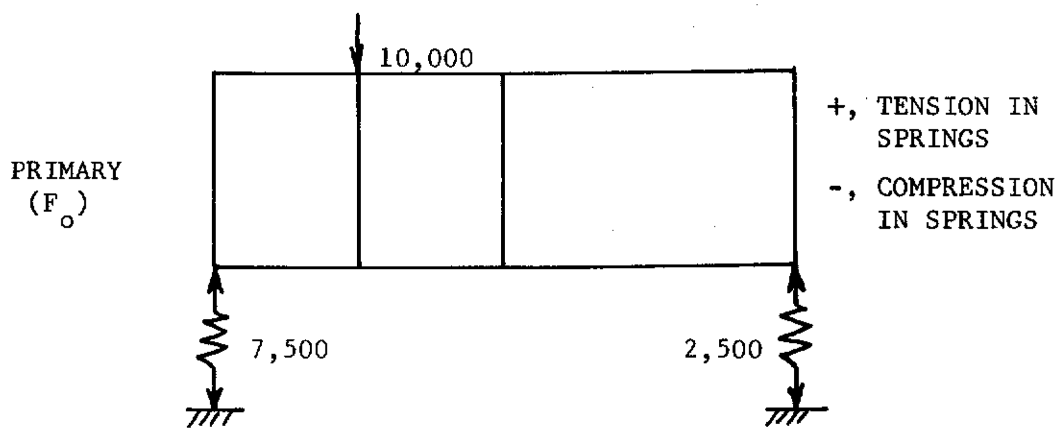
where P_o is the primary load in the member, x_a is the unknown redundant and P_a is the member load due to a unit redundant load. The shear flow in the web is

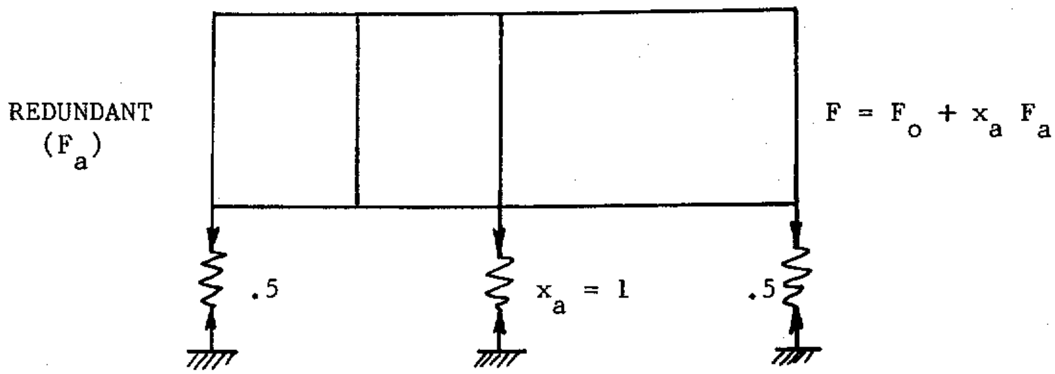
$$q = q_o + x_a q_a$$

where the terms are similar to the previous equation. The total energy in the beam is



$$U_T = U_{AXIAL} + U_{SHEAR} + U_{SPRING}$$





$$U_{SPRING} = \sum \frac{k\,(F_o + x_a F_a)^2}{2K}$$

$$x_a = \frac{-\Sigma k\ P_o\ P_a\ L/AE - \Sigma q_o\ q_a\ ab/Gt - \Sigma F_o\ F_a/K}{\Sigma k\ P_a^{2}\ L/AE + \Sigma q_a^{2}\ ab/Gt + \Sigma F_a^{2}/K}$$

FIGURE 9.7 - BEAM ON ELASTIC SUPPORTS

$$U_t = \sum k\,(P_o + x_a P_a)^2 L/2AE + \sum (q_o + x_a q_a)^2\, ab/2Gt$$

Expanding and taking $\partial U_t/\partial x_a$ gives an equation of deflection which is set equal δ_a.

$$\partial U/\partial x_a = \sum k\,P_o P_a L/AE + x_a \sum k\,P_a^2 L/AE +$$
$$\sum q_o q_1\, ab/Gt + x_a \sum q_a^2\, ab/Gt = \delta_a$$

Solving for x_a gives

$$x_a = \frac{\delta_a - \sum k\,P_o P_a L/AE - \sum q_o q_1\, ab/Gt}{\sum k\,P_a^2 L/AE + \sum q_a^2\, ab/Gt}$$

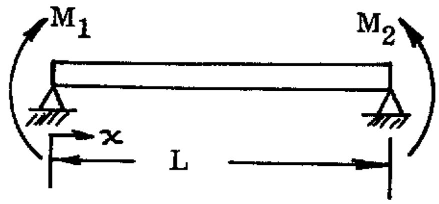
This equation is applicable only if x_a is applied in the direction to close the gap. If $x_a = 0$, the condition exists where the applied load deflects the beam to the support but no further. For this case

$$\delta_a = \sum k\,P_o P_a L/AE + \sum q_o q_a\, ab/Gt$$

which is the same as the virtual load equation.

9.4 Continuous Beams by Three-Moment Equation

The process of three moments can be applied to a beam with redundant supports and any type of loading. The procedure is to equate the slopes of the spans at a support. An equation for any type of loading can be derived and by superposition the slope of the span can be determined. The slope of either span at a support is the sum of the slope produced on a simply-supported span by the given loading (found from Table 9.2). Consider the beam shown below.



At x = 0,

$$\theta = 1/EI\ (-M_1\ L/3 - M_2\ L/6) \qquad 9.13$$

At x = L,

$$\theta = 1/EI\ (M_1\ L/6 + M_2\ L/3) \qquad 9.14$$

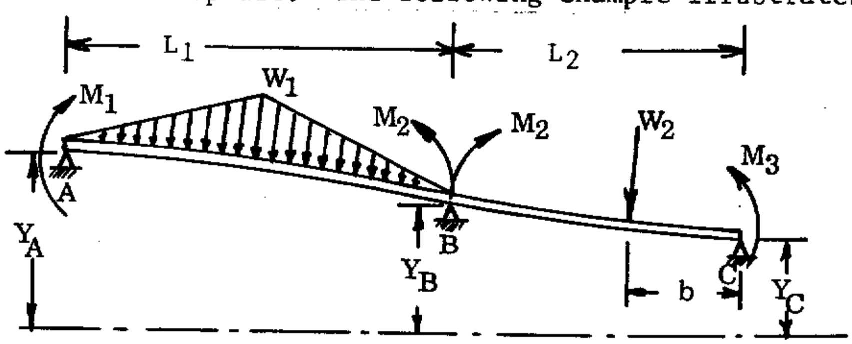
At x,

$$y = 1/EI\left[M_1\ (x^2 - Lx)/2 + (M_2 - M_1)(x^3/L - x\,L)/6\right] \qquad 9.15$$

If the end supports settle or deflect unequal amounts of y_1 and y_2, the increment of slope produced is the same at both ends and is

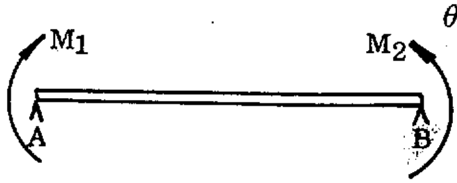
$$\theta = 1/L\,(y_2 - y_1)$$

where y is positive when upward. The following example illustrates the procedure:

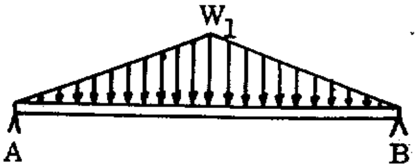


The slopes at support B for each of the adjacent spans are

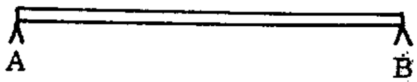
Span 1:



$$\theta_B = 1/E_1I_1\ (M_1L_1/6 + M_2L_1/3)$$

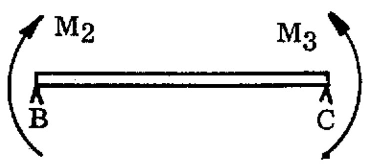


$$\theta_B = 5\ W_1L_1{}^2/96\ E_1I_1 \qquad \text{(Table 9.2)}$$



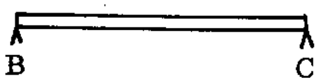
$$\theta_B = (Y_B - Y_A)/L_1$$

Span 2:



$$\theta_B = 1/E_2I_2(-M_2L_2/3 - M_3L_2/6) \quad \text{(Table 9.2)}$$

$$\theta_B = -W_2(bL_2 - b^3/L_2)/6E_2I_2$$



$$\theta_B = (Y_C - Y_B)/L_2$$

The slopes for span 1 are set equal to the slopes for span 2. If M_1 and M_3 are determinate (ends simply supported or overhanging) the equation can be solved at once for M_2 and the reactions then found by statics. If the ends are fixed, the slopes at those points can be set equal to zero; this provides two additional equations, and the three unknowns M_1, M_2 and M_3 can be found. If the two spans are parts of a continuous beam, a similar equation can be written for each successive pair of contiguous spans and these equations solved simultaneously for the unknown moments.

9.5 Lateral Buckling of Beams

Beams in bending under certain conditions of loading and restraint can fail by lateral buckling in a manner similar to that of columns loaded in axial compression. However, it is conservative to obtain the buckling load by considering the compression side of the beam as a column since this approach neglects the torsional rigidity of the beam.

In general, the critical bending moment for the lateral instability of the deep beam, such as that shown in Figure 9.8 may be expressed as

$$M_{cr} = \frac{K\sqrt{EI_y \; GJ}}{L}$$

where J is the torsion constant of the beam and K is a constant dependent on the type of loading and end restraint. Thus, the critical compressive stress is given by

$$F_{cr} = \frac{M_{cr} c}{I_x}$$

where c is the distance from the centroidal axis to the extreme compression fibers. If this compressive stress falls in the plastic range, an equivalent slenderness ratio may be calculated as

$$\frac{L'}{\rho} = \pi\sqrt{\frac{E}{F_{cr}}}$$

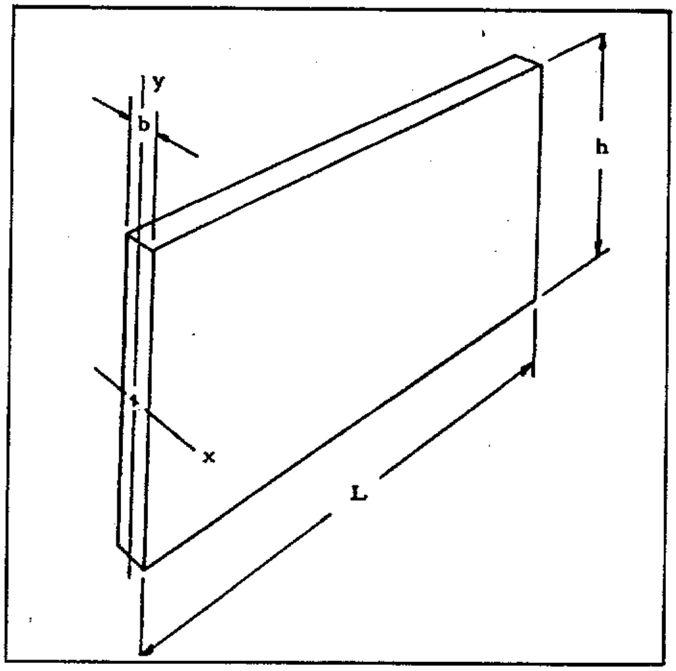


FIGURE 9.8 - DEEP RECTANGULAR BEAM

The actual critical stress may then be found by entering the column curves of Section 11 at this value of (L'/ρ). This value of stress is not the true compressive stress in the beam, but is sufficiently accurate to permit its use as a design guide.

9.5.1 Lateral Buckling of Deep Rectangular Beams

The critical moment for deep rectangular beams loaded in the elastic range loaded along the centroidal axis is given by

$$M_{cr} = 0.0985\ K_m E\ (\frac{b^3 h}{L})$$

where K_m is presented in Table 9.5 and b, h, and L are as shown in Figure 9.8. The critical stress for such a beam is

$$F_{cr} = K_f E (\frac{b^2}{Lh})$$

where K_f is presented in Table 9.5.

If the beam is not loaded along the centroidal axis, as shown in Figure 9.9, a corrected value K_f' is used in place of K_f. This factor is expressed as

$$K_f' = K_f\ (1 - n\ (\frac{s}{L}))$$

where n is a constant defined below:

(1) For simply supported beams with a concentrated load at midspan, n = 2.84.

(2) For cantilever beams with a concentrated end load, n = 0.816.

(3) For simply supported beams under a uniform load, n = 2.52.

(4) For cantilever beams under a uniform load, n = 0.725.

Note: s is negative if the point of application of the load is below the centroidal axis.

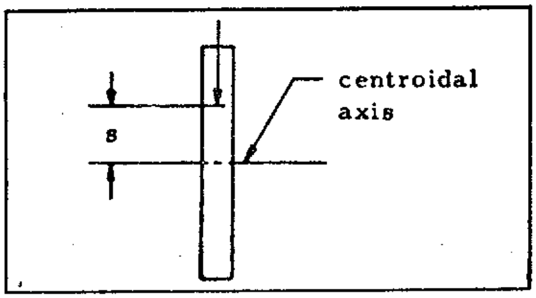


FIGURE 9.9 - DEEP RECTANGULAR BEAM LOADED AT A POINT REMOVED FROM THE CENTROIDAL AXIS

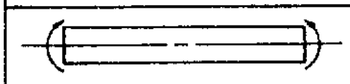
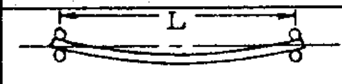
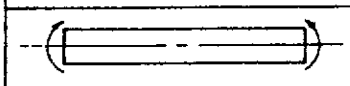
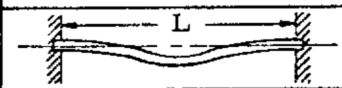
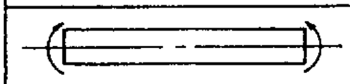
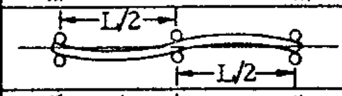
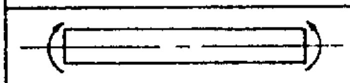
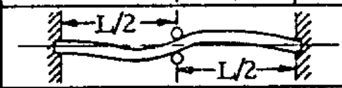
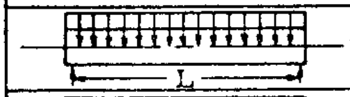
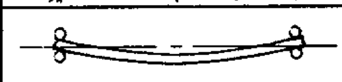
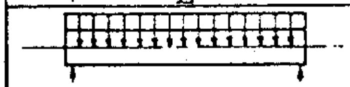
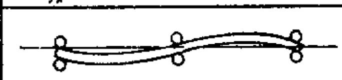
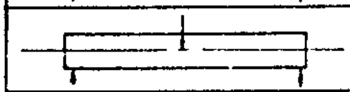
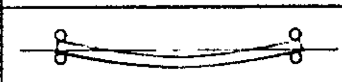
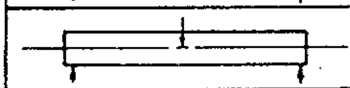
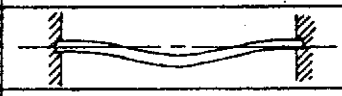
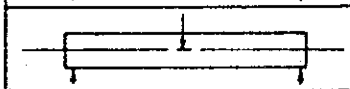
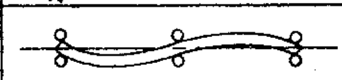
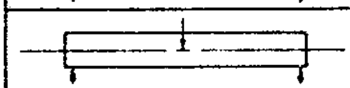
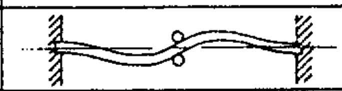
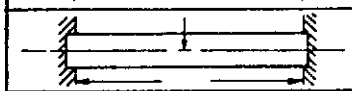
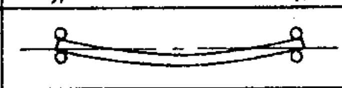
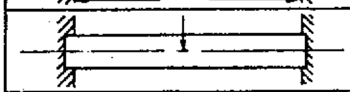
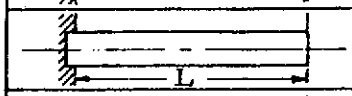
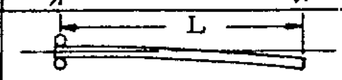
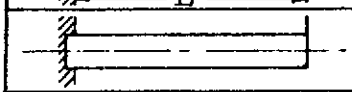
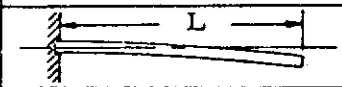
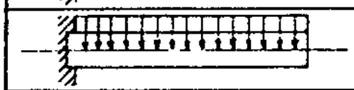
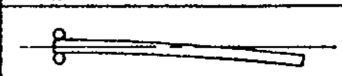
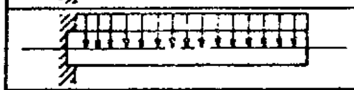
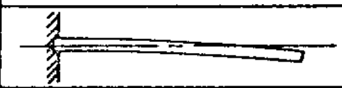
Type of Loading and Constraint		K_f	K_m
Side View	Top View		
		1.86	3.14
		3.71	6.28
		3.71	6.28
		5.45	9.22
		2.09	3.54
		3.61	6.10
		4.87	8.24
		2.50	4.235
		3.82	6.47
		6.57	11.12
		7.74	13.1
		3.13	5.29
		3.48	5.88
		2.37	4.01
		2.37	4.01
		3.80	6.43
		3.80	6.43

TABLE 9.5 - LATERAL BUCKLING CONSTANTS FOR DEEP RECTANGULAR BEAMS

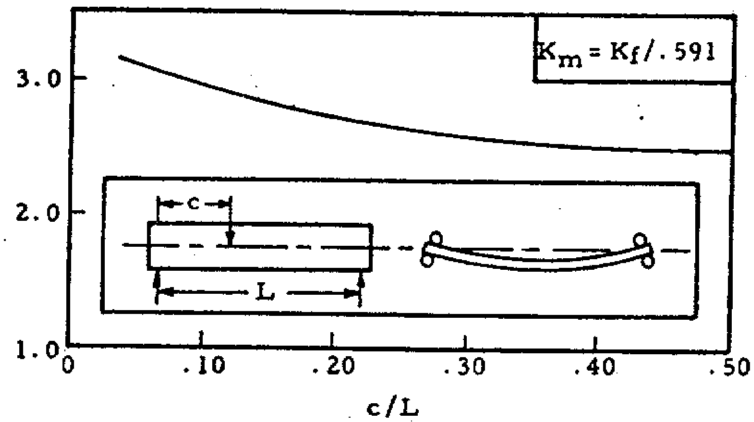


TABLE 9.5 (CONT'D) - LATERAL BUCKLING CONSTANTS FOR DEEP RECTANGULAR BEAMS

9.5.2 Lateral Buckling of Deep I Beams

Figure 9.10 shows a deep I beam.

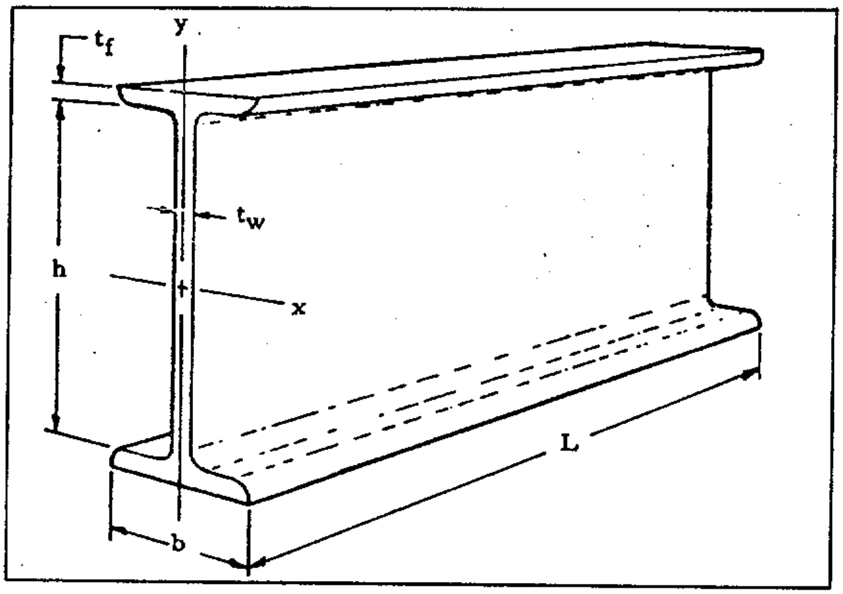


FIGURE 9.10 - DEEP I BEAM

The critical stress of such a beam in the elastic range is given by

$$F_{cr} = K_I \left(\frac{L}{a}\right) \left(\frac{h}{L}\right)^2 \frac{I_y}{I_x}$$

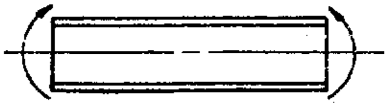
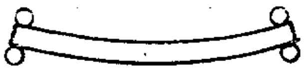
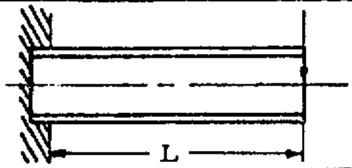
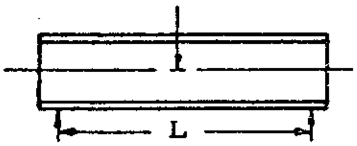
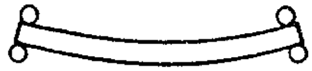
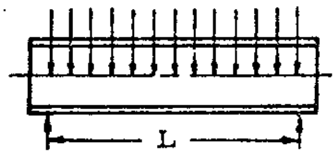
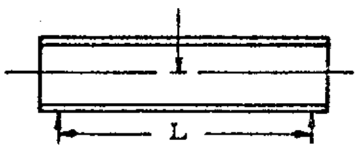
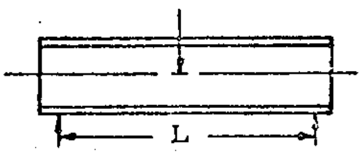
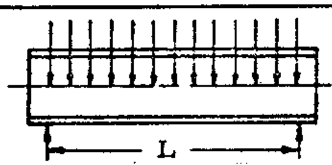
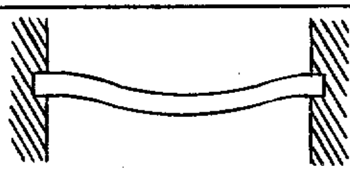
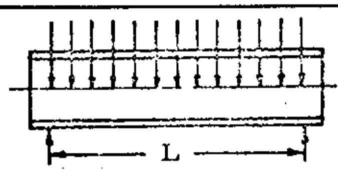
where K_I may be obtained from Table 9.6 and a is given by

$$a = \sqrt{EI_y h^2/4\ GJ}$$

where J is the torsion constant of the I beam. This constant may be approximated by

$$J = 1/3\ (2b\ t_f^{\ 3} + h\ t_w^{\ 3})$$

This method can be applied only if the load is applied at the centroidal axis.

Type of Loading and Constraint		K_I*
Side View	Top View	
		$\frac{m_1}{4}$ (E)
		$\frac{m_2}{4}$ (E)
		$\frac{m_3}{16}$ (E)
		$\frac{m_4}{32}$ (E)
		$\frac{m_5}{16}$ (E)
		$\frac{m_6}{16}$ (E)
		$\frac{m_7}{32}$ (E)
		$\frac{m_8}{32}$ (E)

* Use Figure 9.11 to obtain m

TABLE 9.6 - LATERAL BUCKLING CONSTANTS FOR DEEP I BEAMS

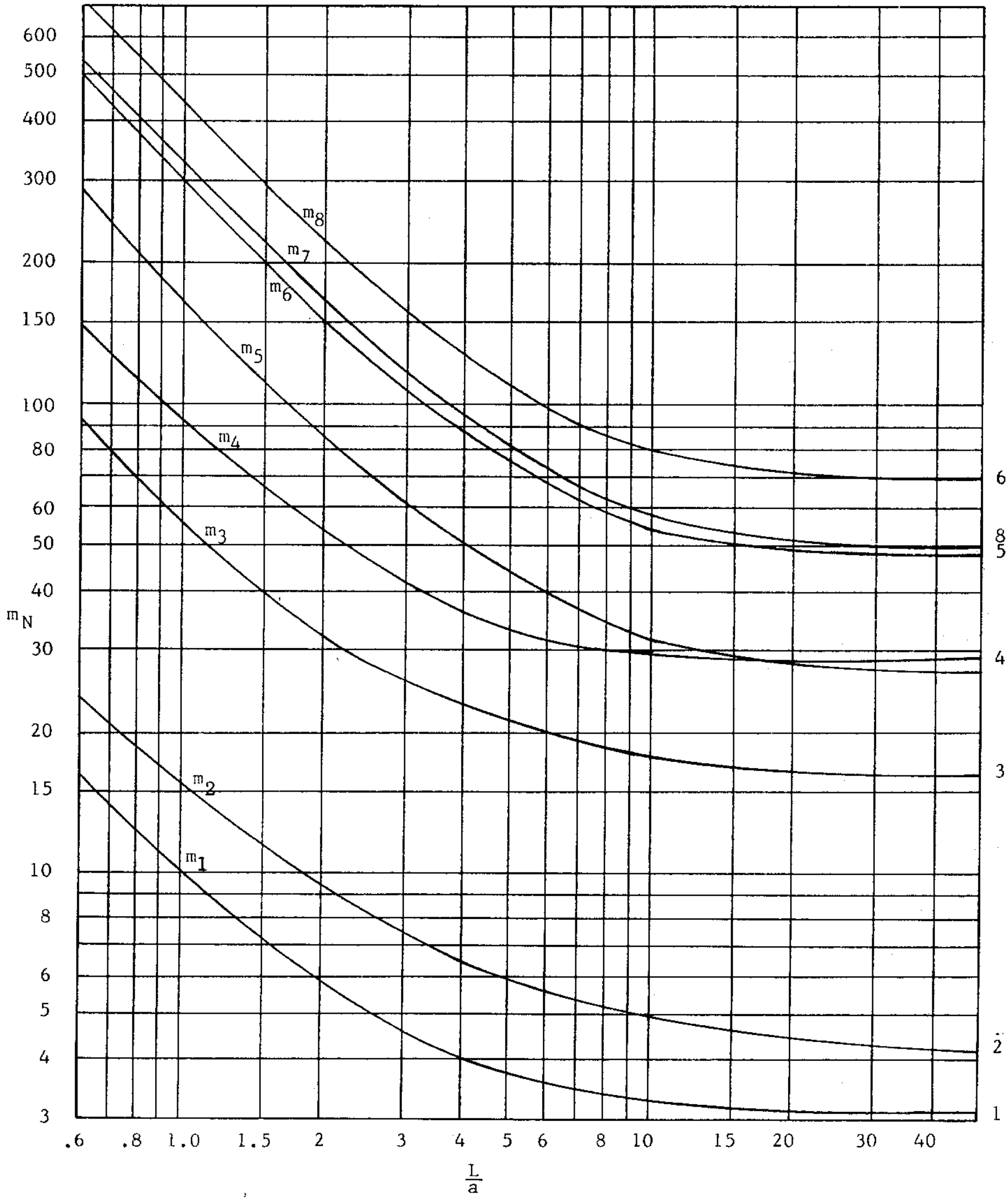


FIGURE 9.11 - VALUES OF m FOR TABLE 9.6

9.6 PLASTIC ANALYSIS OF BEAMS

Revision C

9.6.1 Bending About an Axis of Symmetry

The basic assumption of the classic theory of pure bending in the elastic range is that a plane cross-section normal to the longitudinal axis of the beam remains plane under bending deflections. This assumption is also valid for the plastic range. The strain is directly proportional to the distance from the neutral axis. The stress distribution across the plane of the cross section in a direction perpendicular to the neutral axis has the same shape as the material stress-strain curve. Such a distribution is shown in Figure 9.12. If a stress-strain

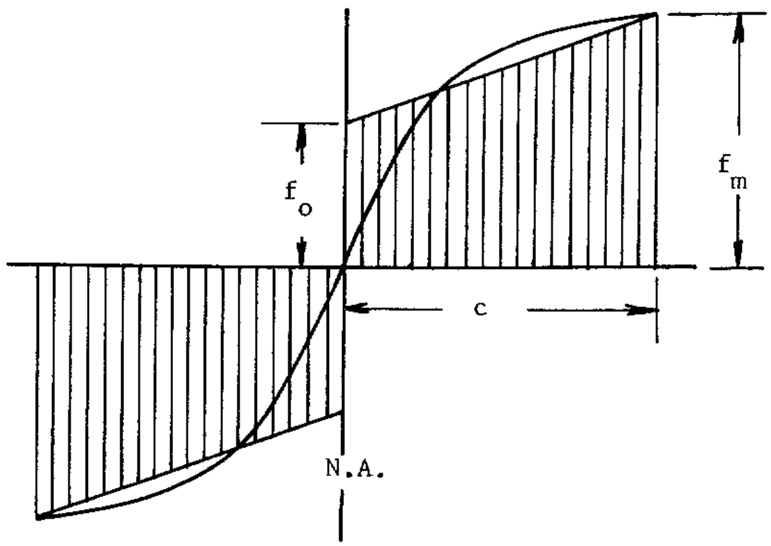


FIGURE 9.12 PLASTIC BENDING STRESS DISTRIBUTION

curve is available from a tension test and if it is assumed that the stress-strain curve in compression is the same as tension, it is possible to determine the moment carried by a given section at a specified extreme fiber stress. In Figure 9.12 the actual stress-strain distribution across a symmetrical section in bending is shown where f_m is the extreme fiber stress. The stress f_m is equal to or greater than F_{ty} and less than or equal to F_{tu}. Superimposed onto the actual distribution is a trapezoidal distribution which passes through f_m and has an intercept stress of f_o. The intercept stress f_o is defined as the stress required for the trapezoidal stress distribution to produce the same moment about the neutral axis as the actual stress distribution. This theory was introduced by Frank P. Cozzone and the following discussions are based on this paper in the May, 1943, Journal of the Aeronautical Sciences.

This stress f_o is a fictitious stress which is assumed to exist at the neutral axis or at zero strain. The value of f_o is determined by making the requirement that the internal moment of the true stress system must equal the moment which results from the assumed trapezoidal stress-strain system. Figure 9.13 shows the trapezoidal stress distribution consisting of separate distributions, one rectangular and one triangular. The total moment M_b can be defined as

$$M_b = M_R + M_T$$

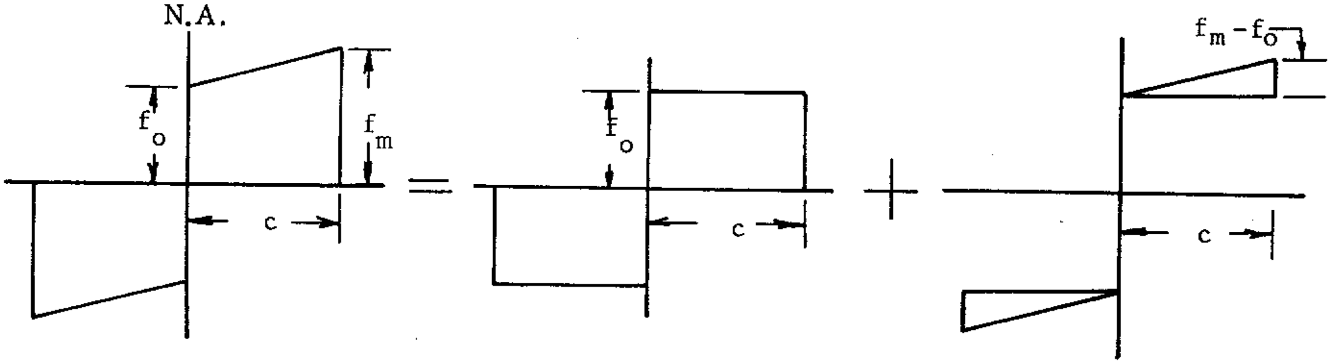


FIGURE 9.13 - ASSUMED STRESS DISTRIBUTION

where M_R = moment produced by the rectangular distribution

M_T = moment produced by the triangular distribution

The triangular distribution is equivalent to an elastic distribution and can be defined as

$$M_T\, c/I = f_m - f_o$$

$$M_T = (f_m - f_o)\, I/c$$

For a symmetrical section, the moment produced by the rectangular distribution will equal f_o times twice the area above the neutral axis times the distance from the neutral axis to the centroid of this area, or

$$M_R = 2\ f_o\ (A/2)\ \bar{y}$$

where A = total cross sectional area

A/2 = the area above or below the N.A.

$\bar{y} = c/2$ = distance from N.A. to the centroid of the area above or below N/A

But $A\bar{y}/2 = Q_m$ = the static moment of the area above or below the N.A. about the N.A.

Then substituting Q_m

$$M_R = 2\ f_o\ Q_m$$

and

$$M_b = 2\ f_o\ Q_m + (f_m - f_o)\ I/c$$

$$M_b c/I = 2\ f_o\ Q_m\, c/I + (f_m - f_o)$$

If $F_b = M_b\ c/I$ and $k = 2\ Q_m\ c/I$ then

$$F_b = f_m + f_o\ (k - 1) \qquad 9.16$$

F_b is a fictitious M c/I stress or the modulus of rupture for a particular cross section at a given maximum stress level. Equation 9.16 is applicable only to sections symmetric about the neutral axis.

The values of k vary between 1 and 2.0. If the calculated value of k is greater than 2, use 2. Figure 9.14 shows the value of k for several typical shapes. k can also

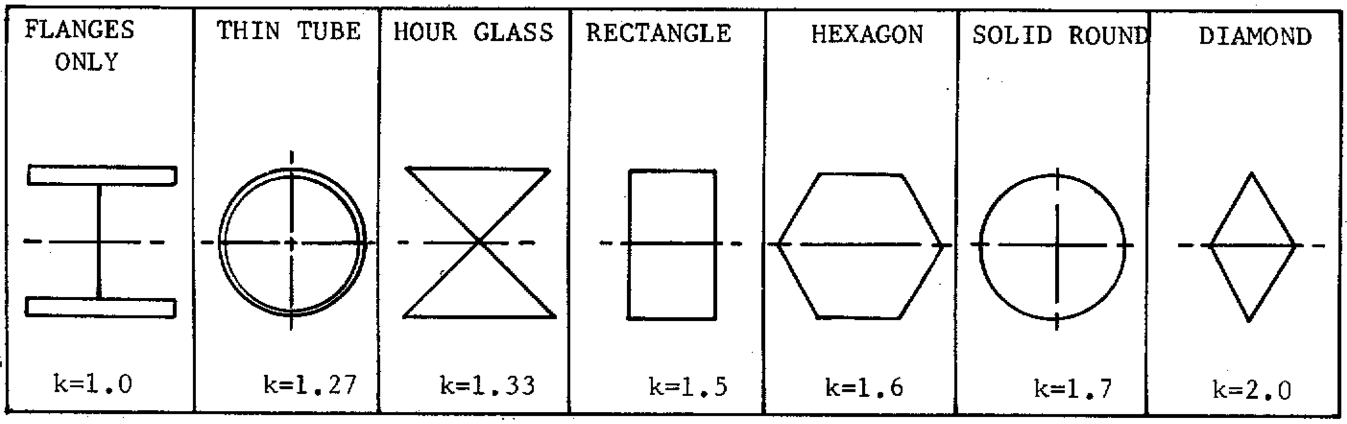
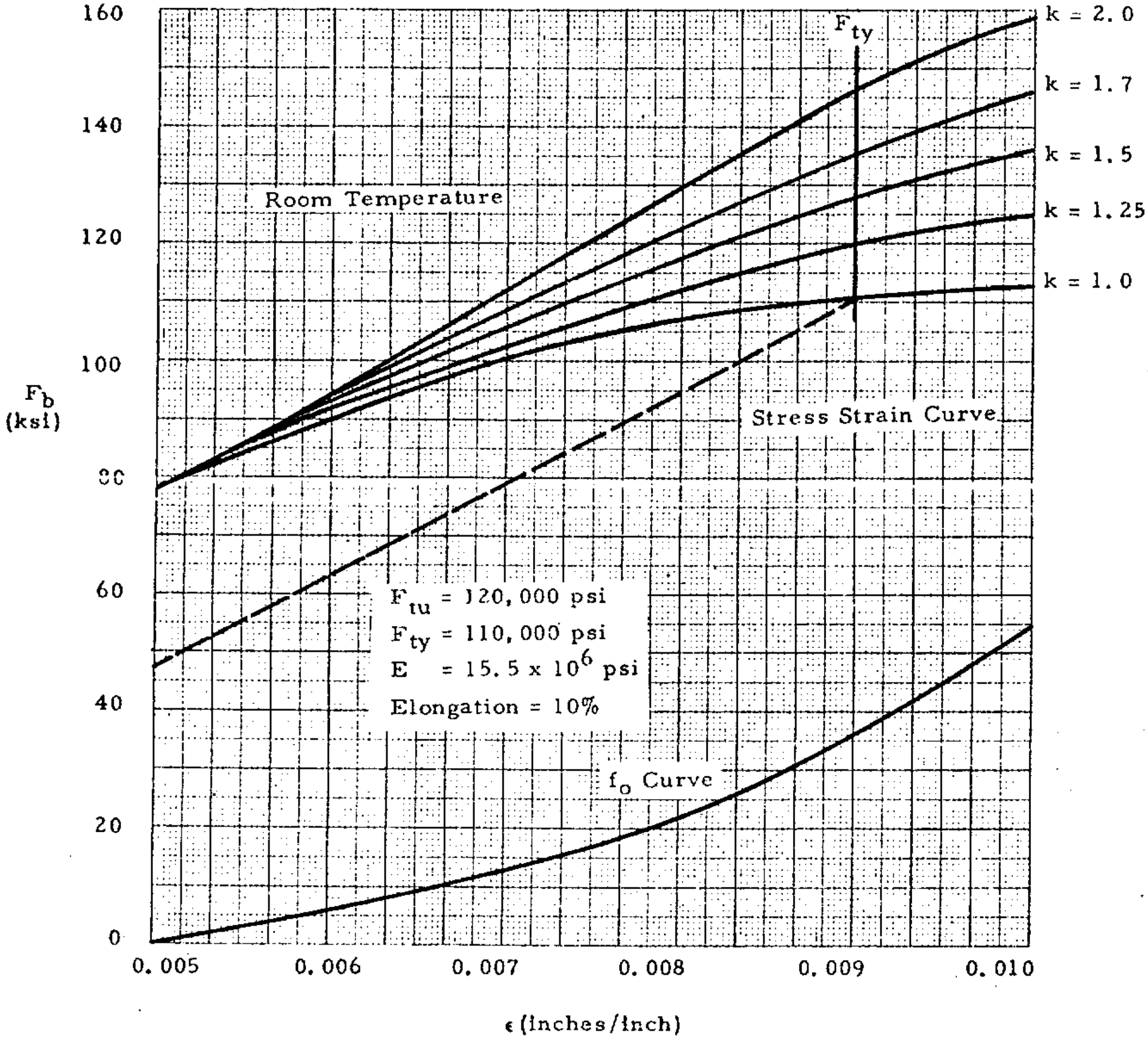


FIGURE 9.14 - SHAPE FACTORS FOR TYPICAL SECTIONS

be calculated from

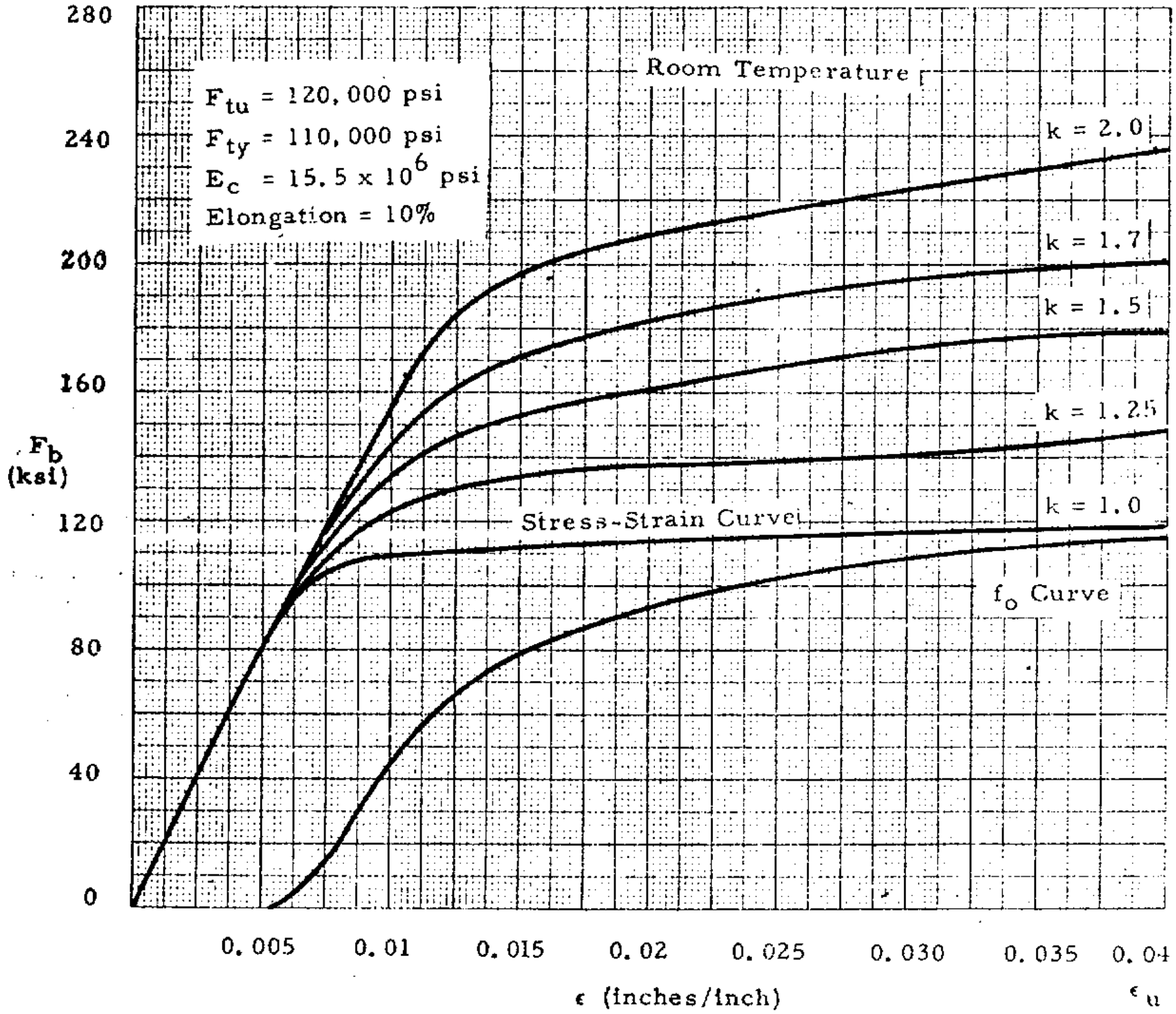
$$k = \frac{2\ Qc}{I} = \frac{2\ c \int_o^c y\ d\ A}{I} \qquad 9.17$$

The modulus of rupture F_b may be yield modulus or ultimate modulus. For yield modulus of rupture, the value of f_m in equation 9.16 is F_{ty} of the material. If ultimate modulus of rupture is desired, substitute F_{tu} of the material for f_m in equation 9.16 . The modulus of rupture may be limited to some stress between yield and ultimate stress of the material because of local crippling or by excessive distortion. Regardless of what value is used for f_m in equation 9.16, the corresponding value of f_o must be known before a value of F_b can be determined. Figures 9.15 through 9.18 give curves for F_b and f_o versus k and strain for various materials.



a. Ti-8Mn Titanium Alloy

FIGURE 9.15 - MINIMUM PLASTIC BENDING CURVES FOR TITANIUM



b. Ti-8Mn Titanium Alloy

FIGURE 9.15 - MINIMUM PLASTIC BENDING CURVES FOR TITANIUM

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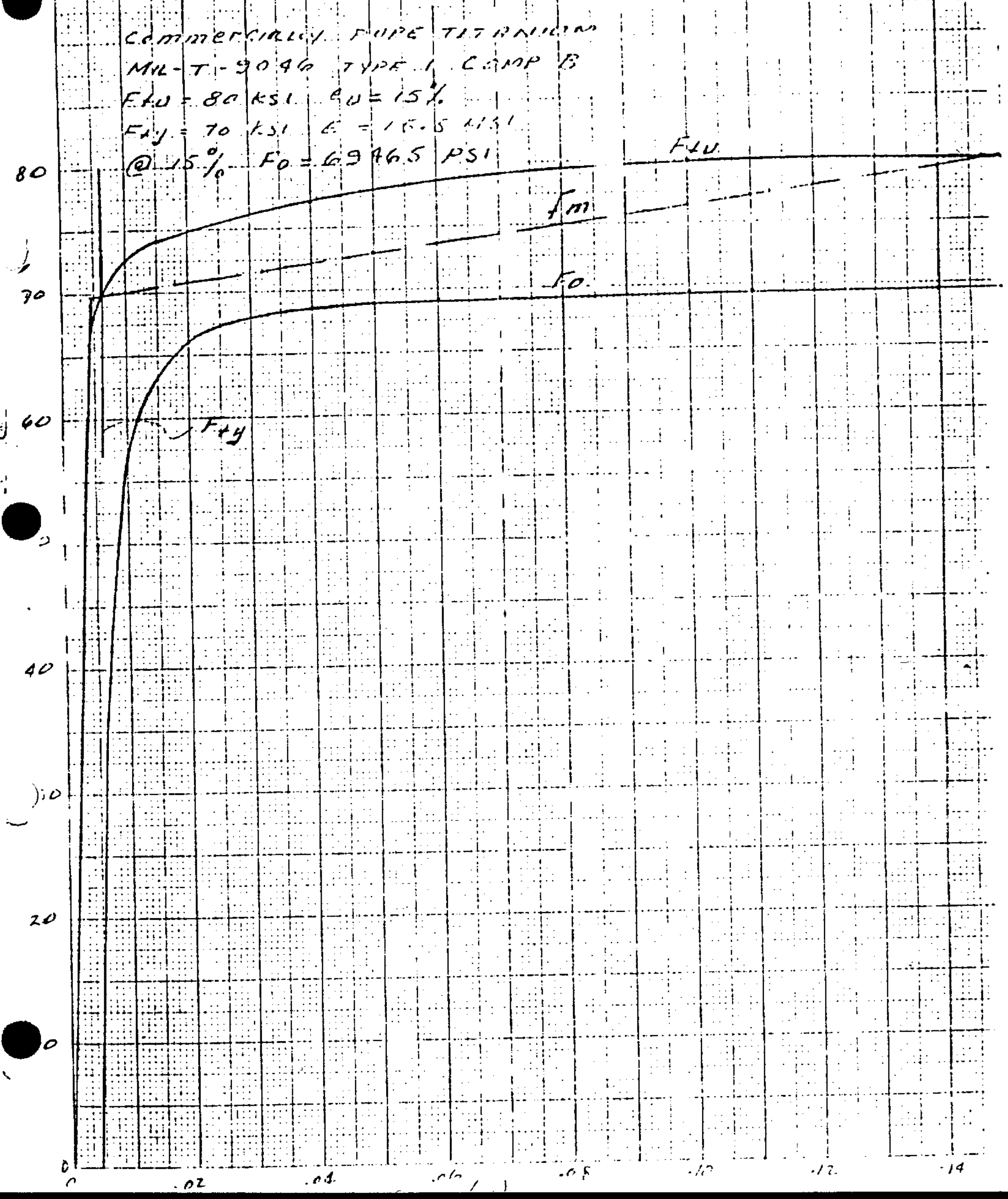
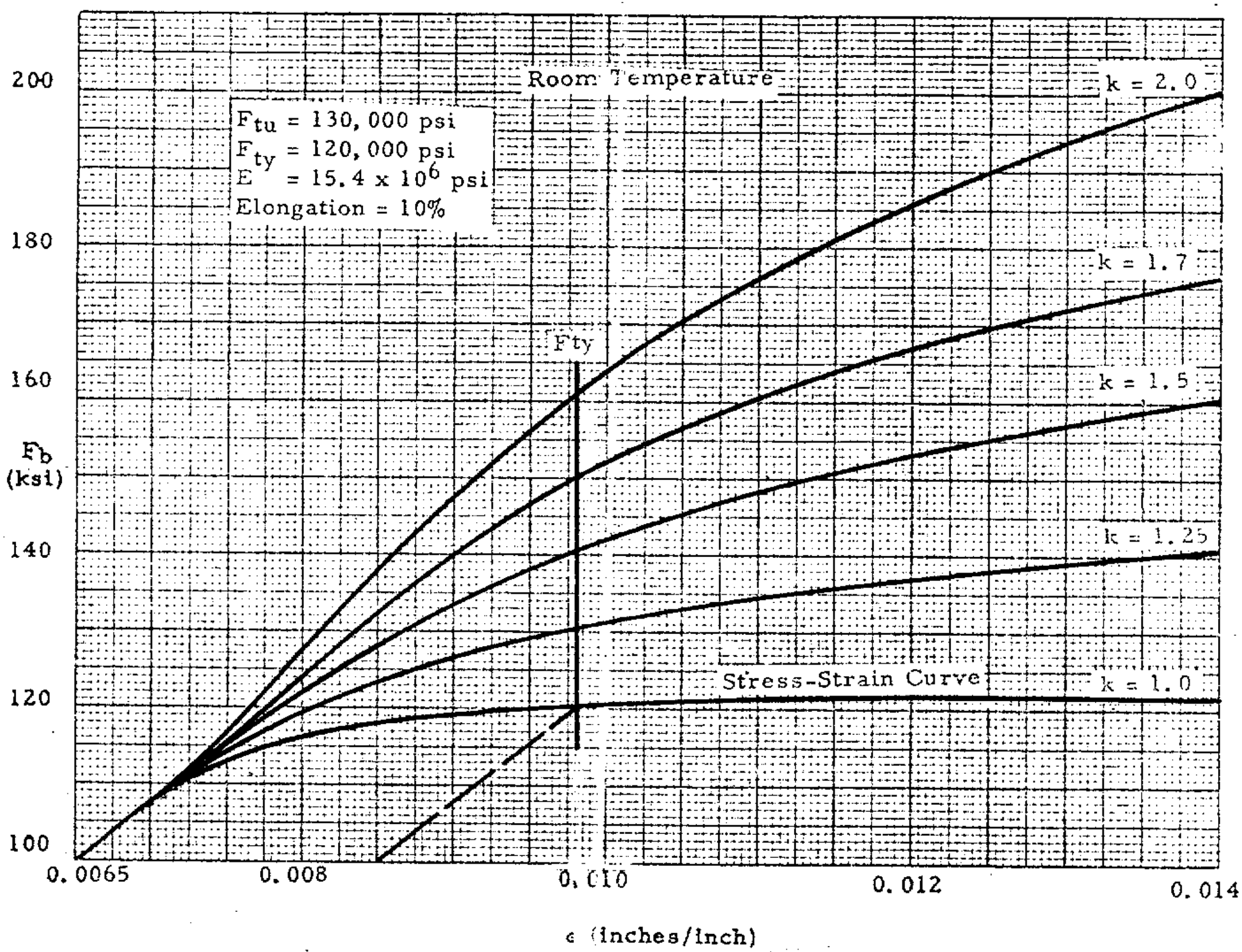
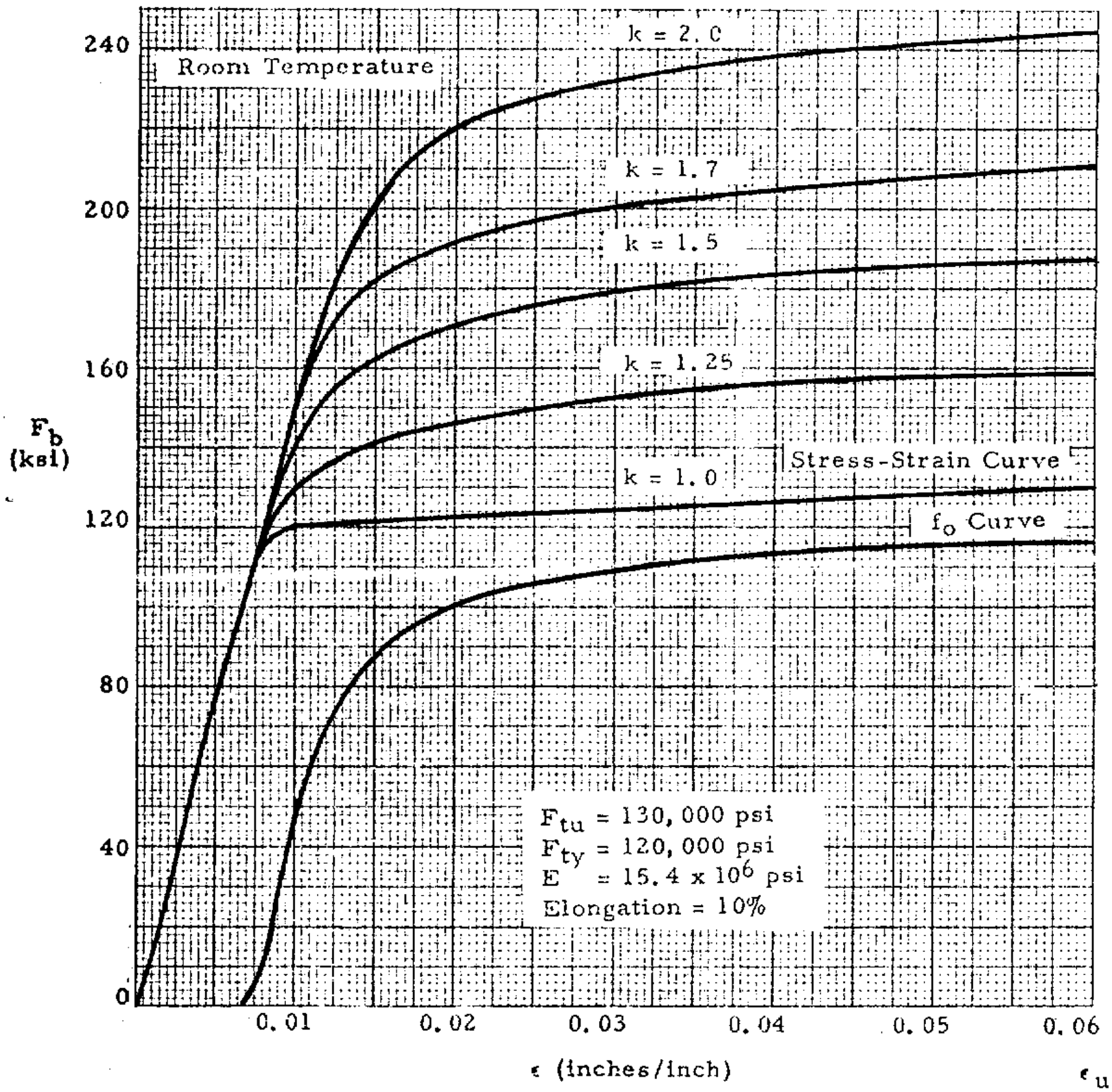




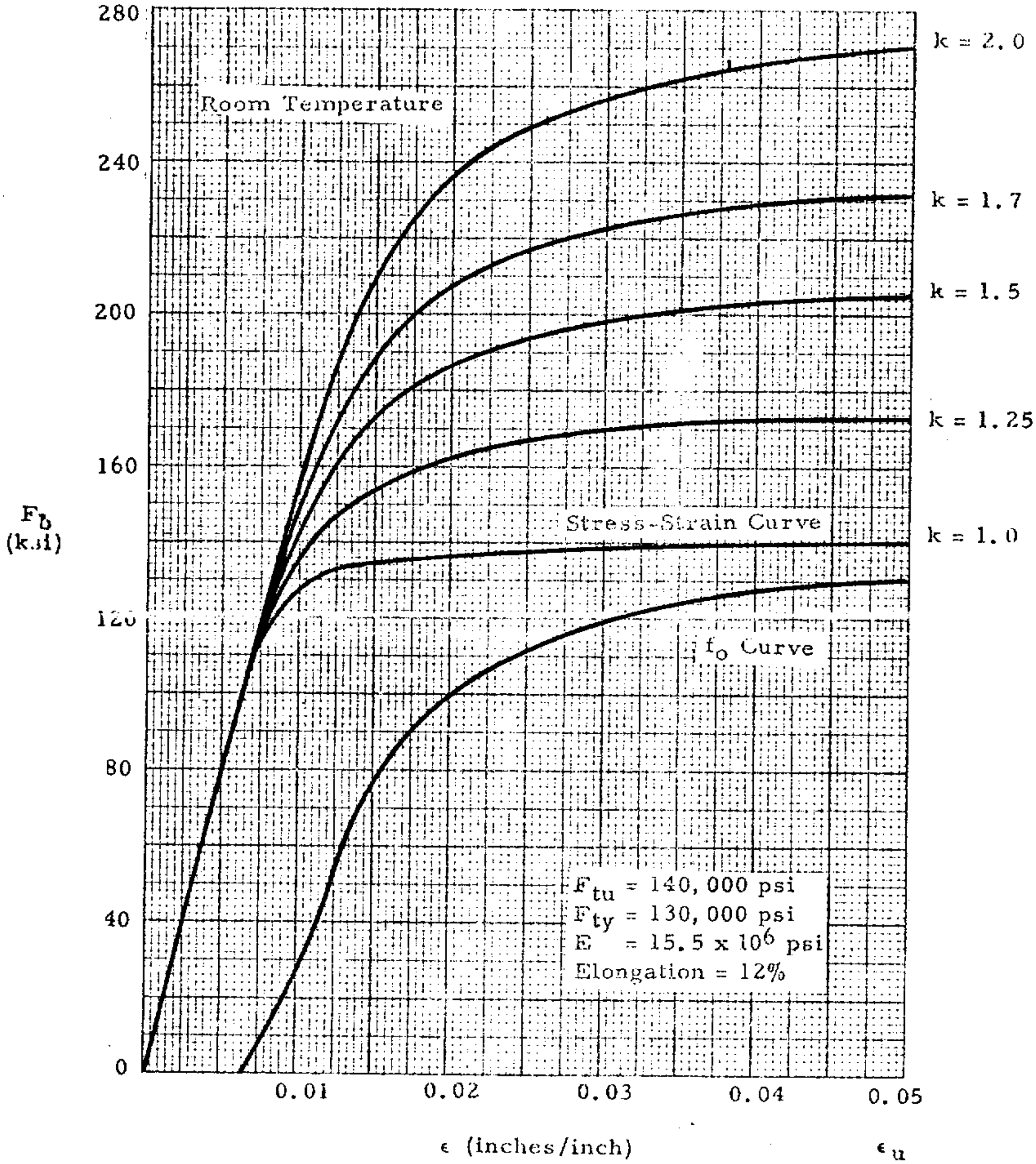
c. Ti-6Al-4V Titanium Alloy

FIGURE 9.15 - MINIMUM PLASTIC BENDING CURVES FOR TITANIUM



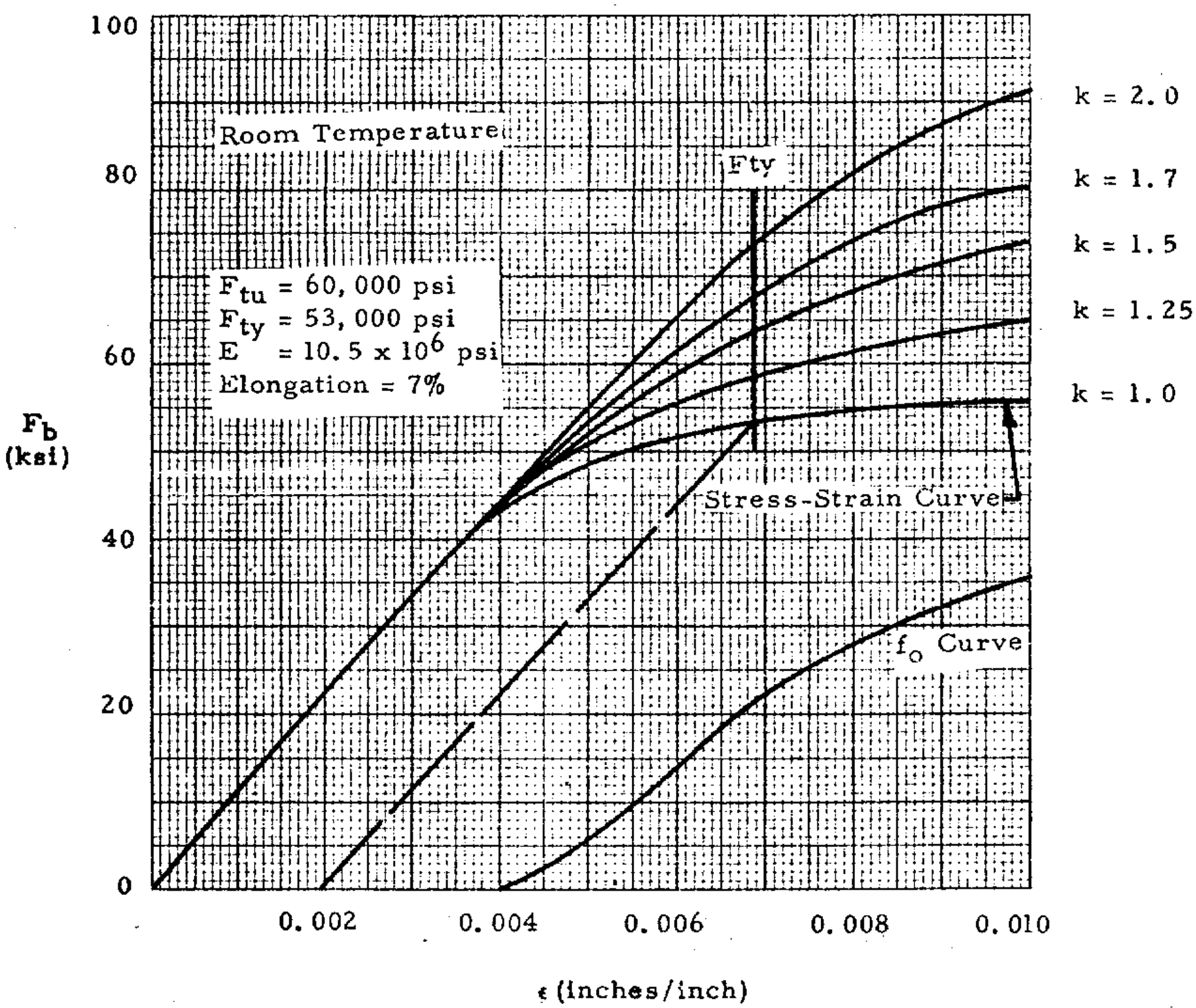
d. Ti-6Al-4V Titanium Alloy

FIGURE 9.15 - MINIMUM PLASTIC BENDING CURVES FOR TITANIUM



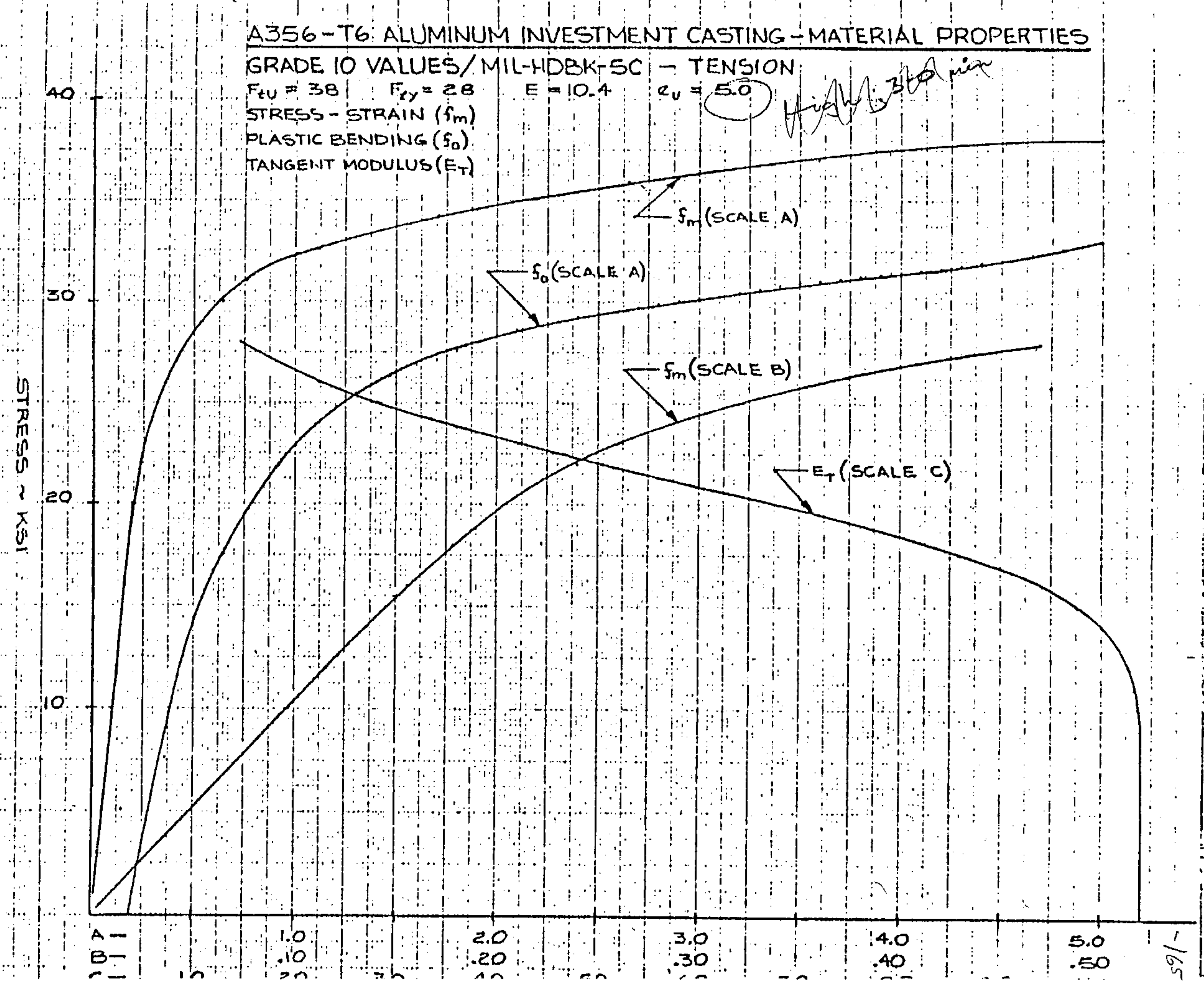
e. Ti-4Mn-4Al Titanium Alloy

FIGURE 9.15 - MINIMUM PLASTIC BENDING CURVES FOR TITANIUM



a. 2014-T6 Aluminum Alloy Extrusions
- Thickness ≦ 0.499 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



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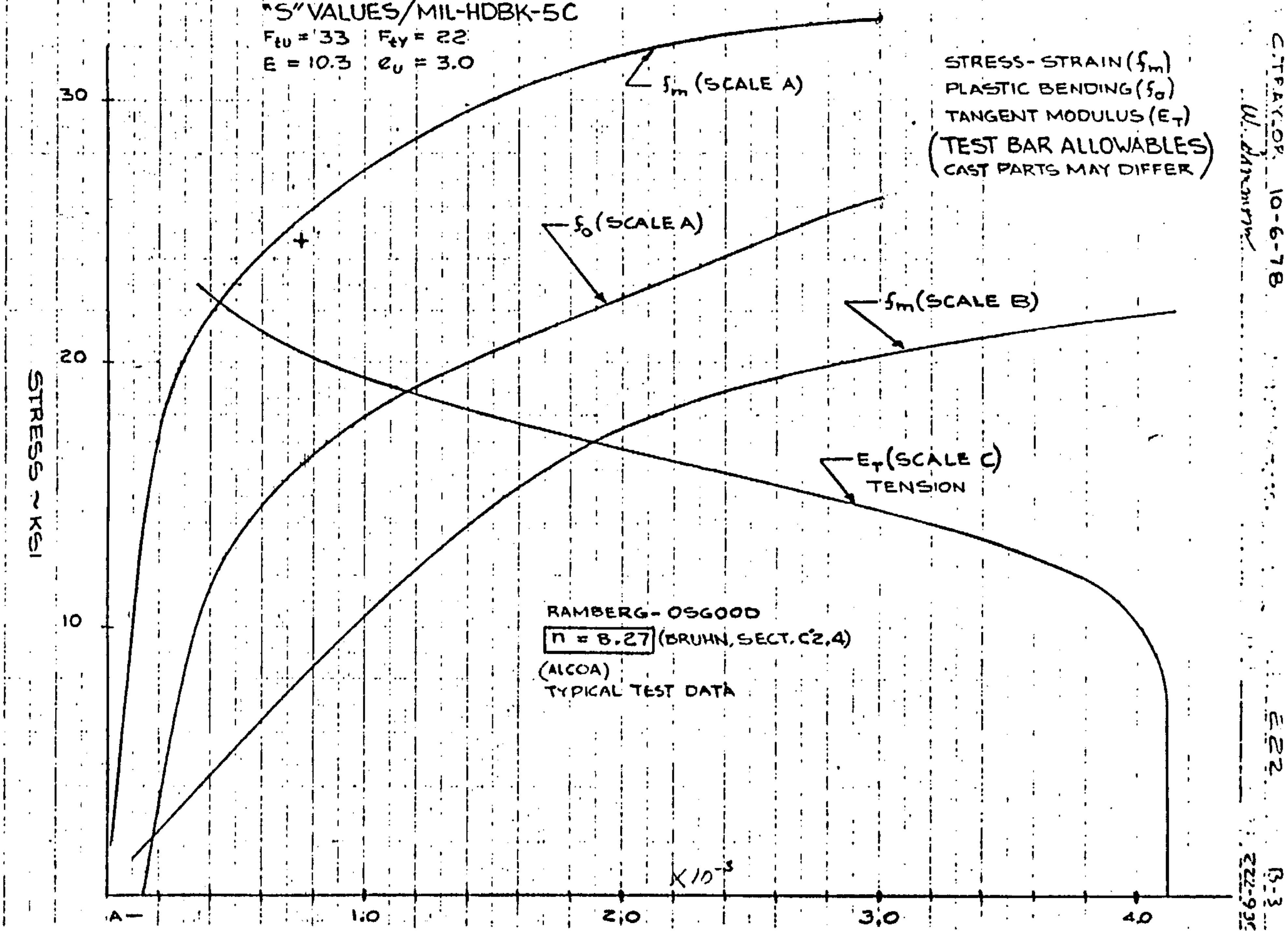
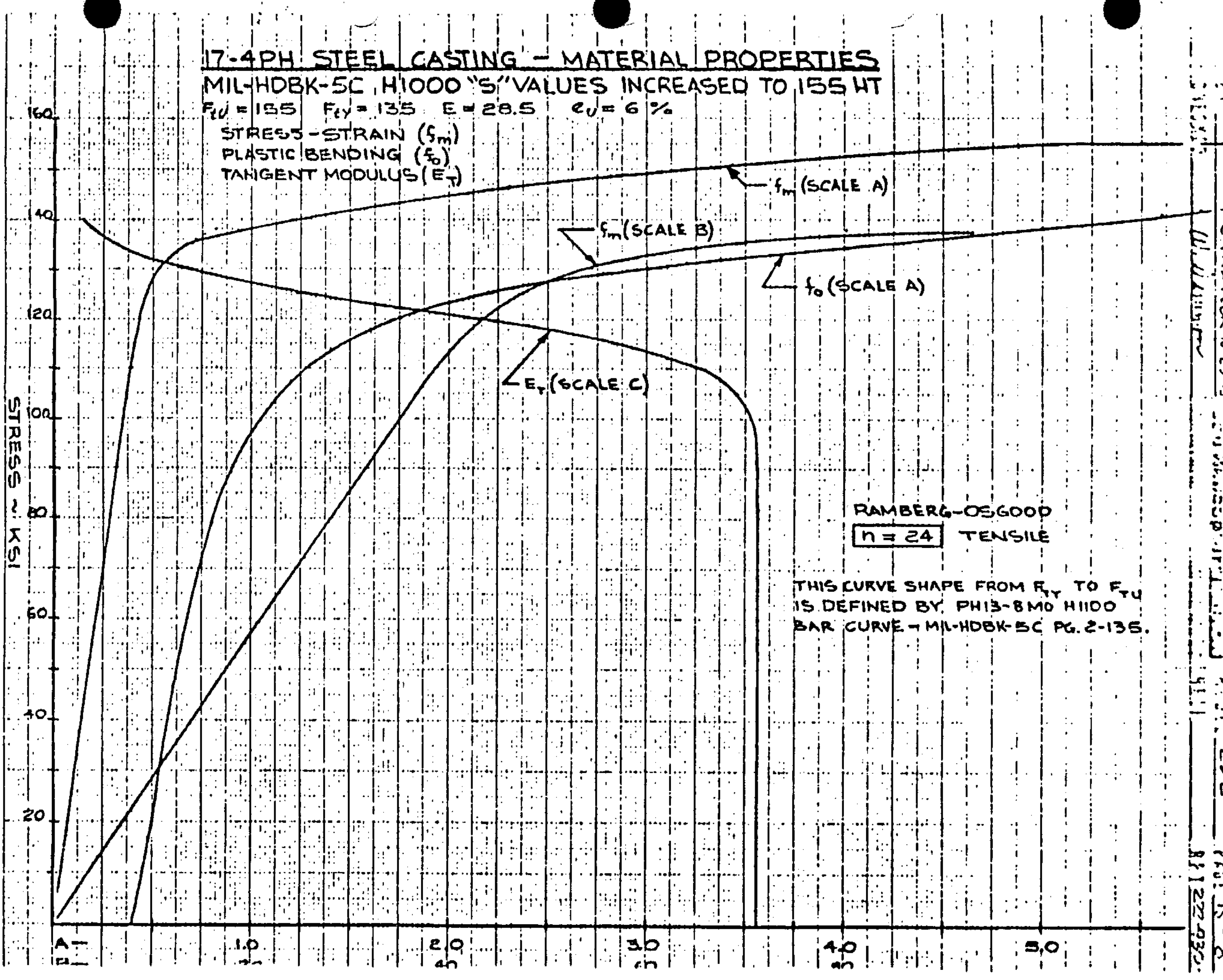
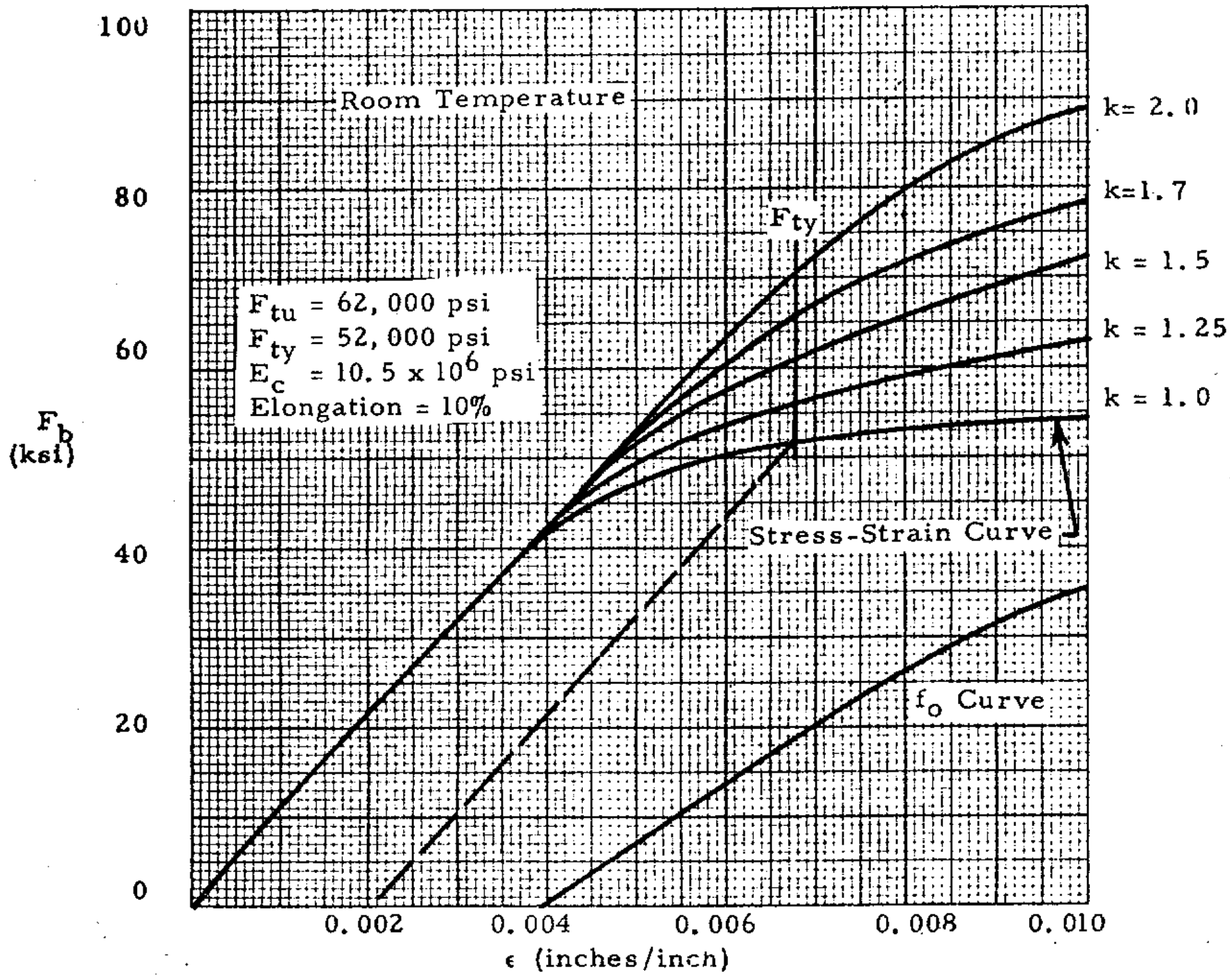
356-T6 ALUMINUM INVESTMENT CASTING - MATERIAL PROPERTIES

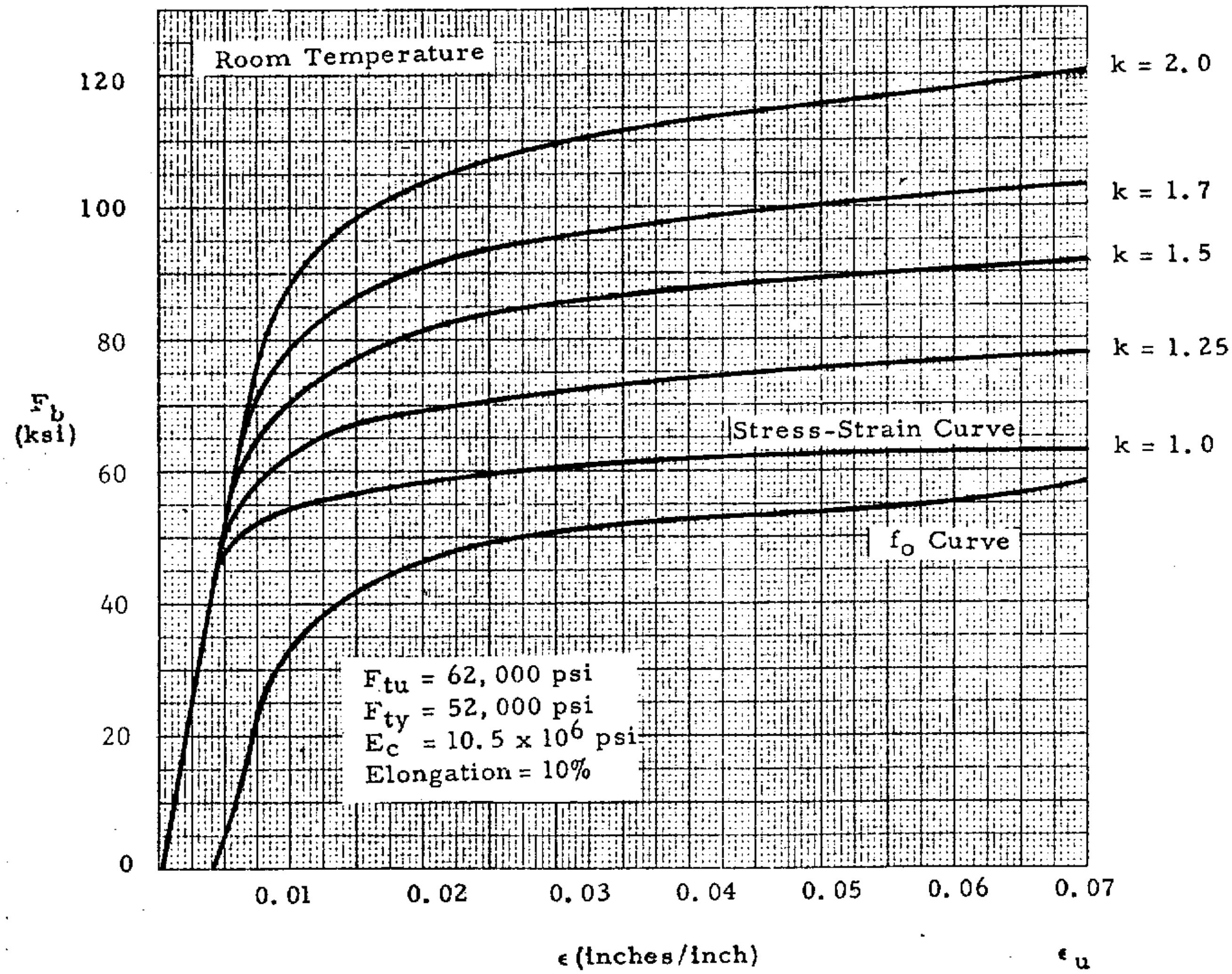



17-4PH STEEL CASTING – MATERIAL PROPERTIES
MIL-HDBK-5C H1000 "S" VALUES INCREASED TO 155 HT
Ftu = 155   Fty = 135   E = 28.5   eu = 6 %
STRESS-STRAIN (fm)
PLASTIC BENDING (fo)
TANGENT MODULUS (ET)
fm (SCALE A)
fm (SCALE B)
fo (SCALE A)
ET (SCALE C)
RAMBERG-OSGOOD
n = 24   TENSILE
THIS CURVE SHAPE FROM Fty TO Ftu
IS DEFINED BY PH13-8MO H1100
BAR CURVE – MIL-HDBK-5C PG. 2-135.
STRESS ~ KSI
160
140
120
100
80
60
40
20
A –
1.0
2.0
3.0
4.0
5.0



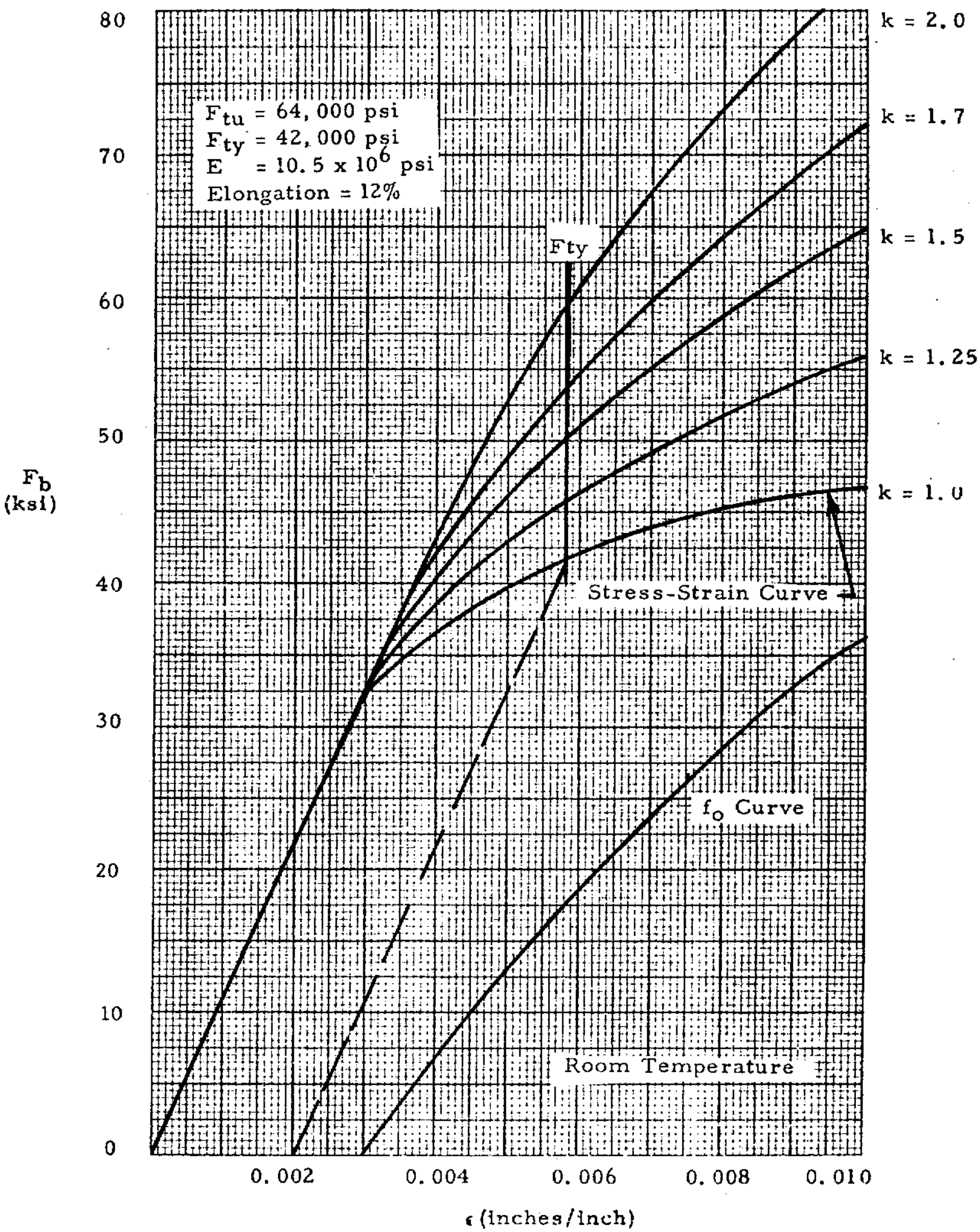
b. 2014-T6 Aluminum Alloy Die Forgings - Thickness $\leqq$ 4 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



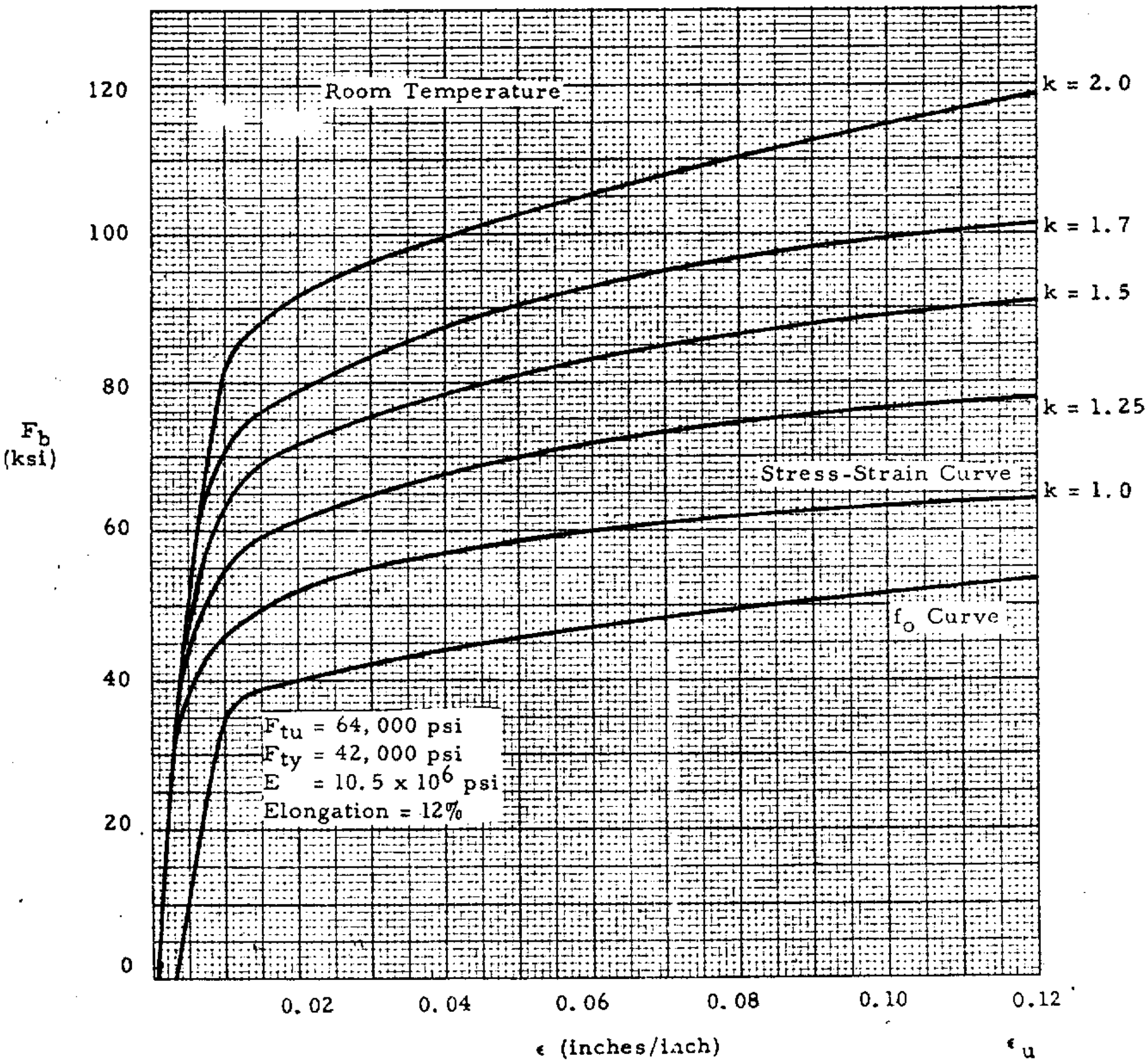
c. 2014-T6 Aluminum Alloy Die Forgings - Thickness ≦ 4 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



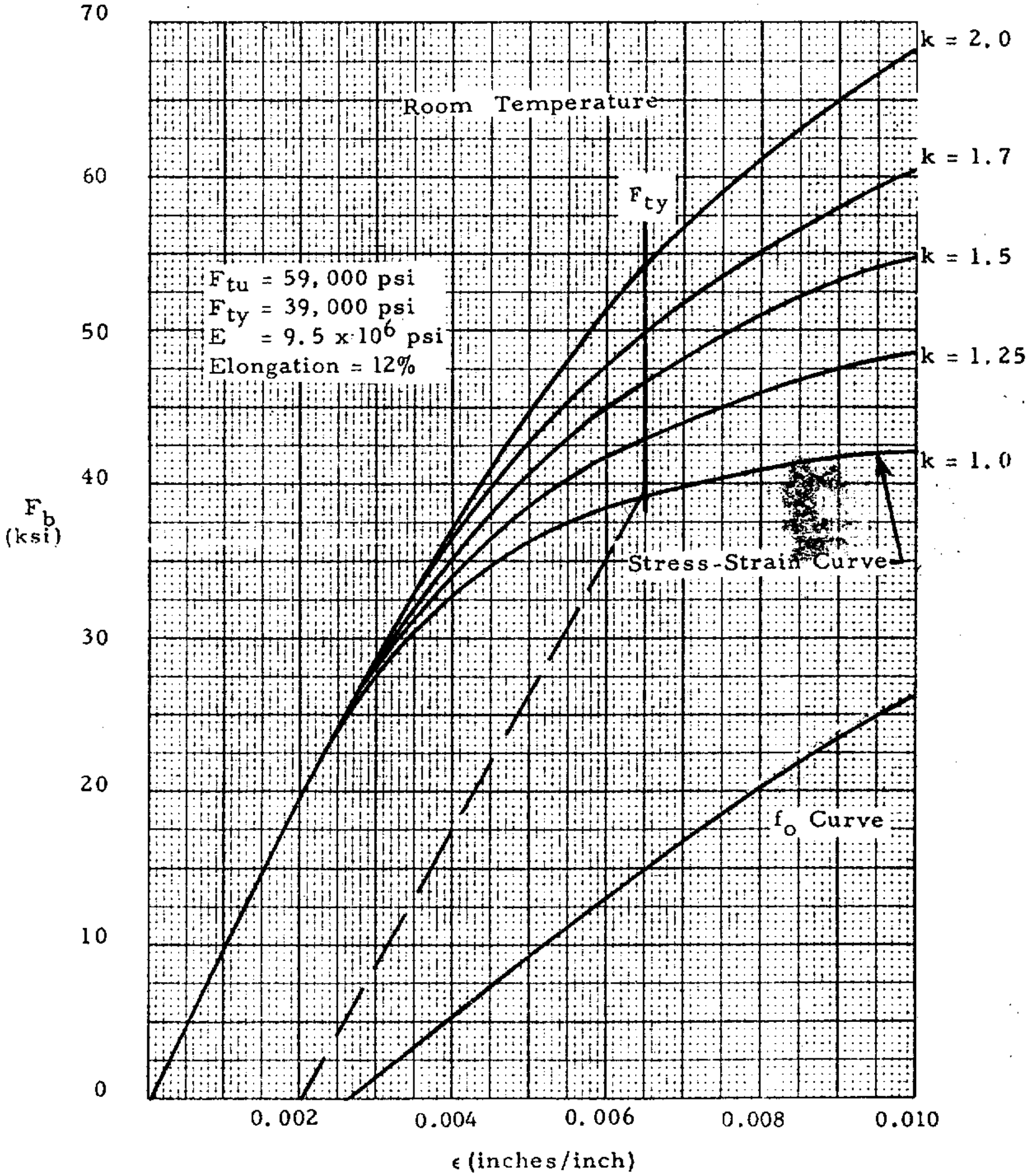
d. 2024-T3 Aluminum Alloy Sheet and Plate
- Heat Treated - Thickness ≦ 0.250 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



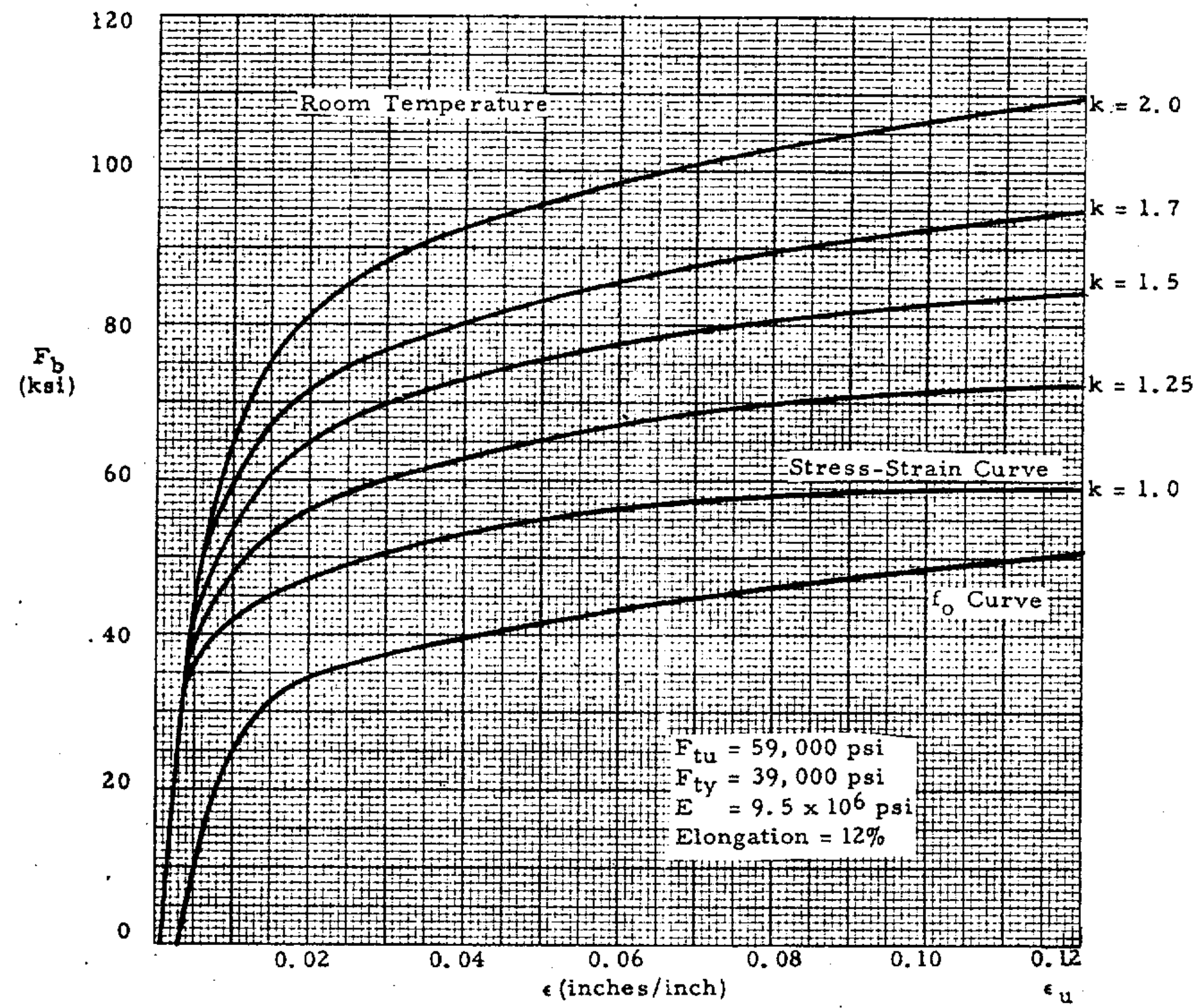
e. 2024-T3 Aluminum Alloy Sheet and Plate - Heat Threated - Thickness ≦ 0.250 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



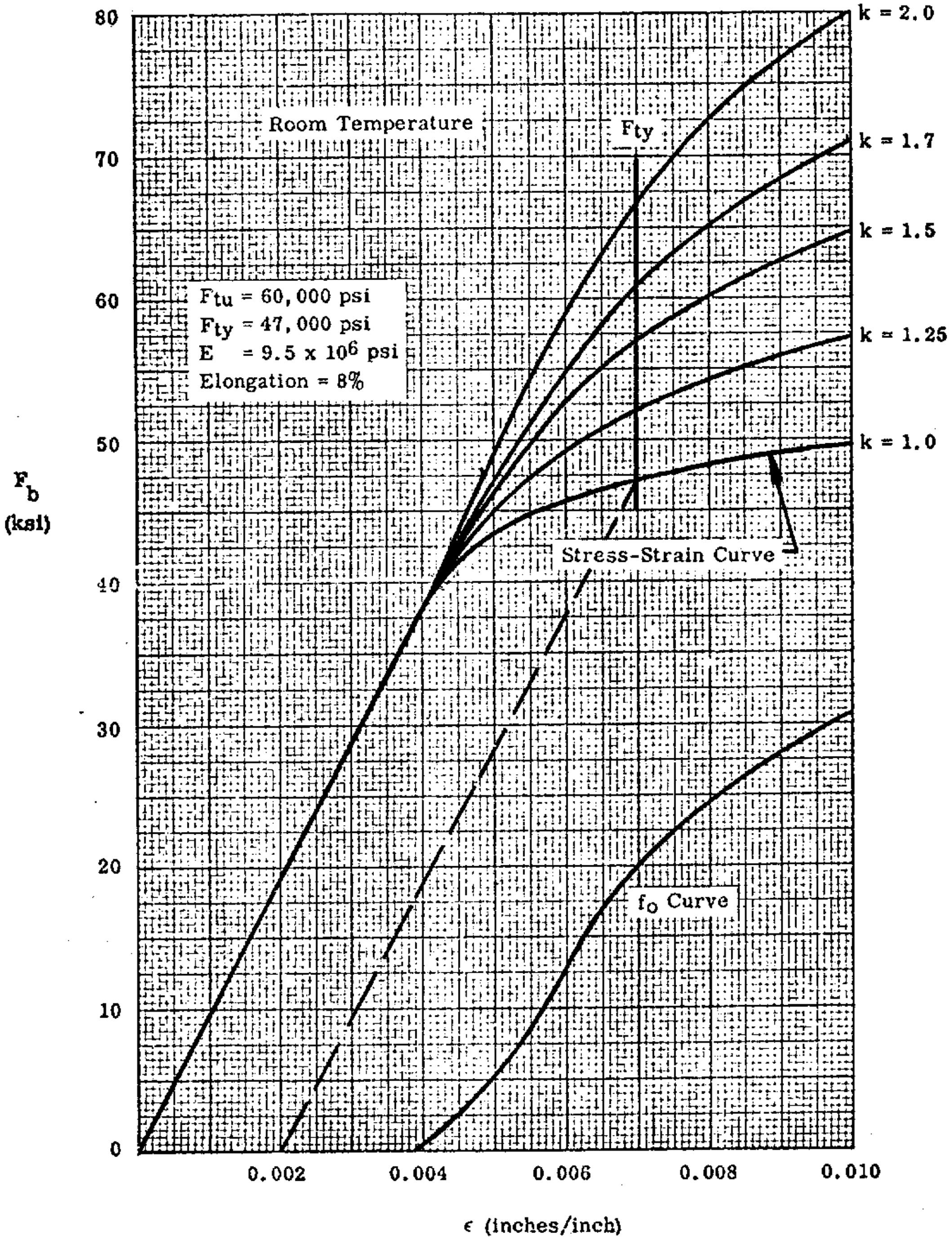
f. 2024-T3 Aluminum Alloy Clad Sheet and Plate - Heat Treated - Thickness 0.010 to 0.062 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



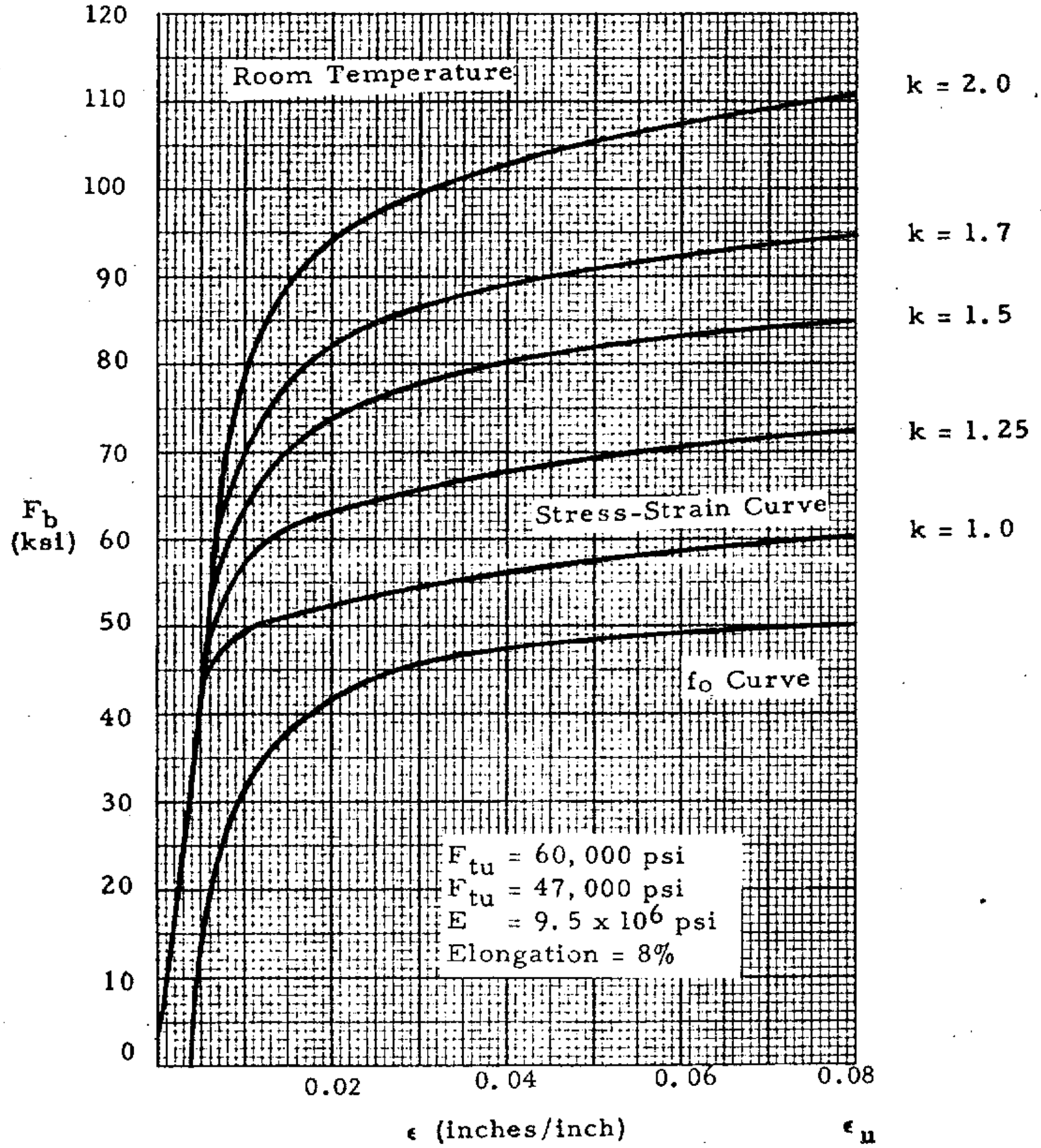
g. 2024-T3 Aluminum Alloy Clad Sheet and Plate - Heat Treated - Thickness 0.010 to 0.062 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



h. 2024-T6 Aluminum Alloy Clad Sheet - Heat Treated and Aged - Thickness < 0.064 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



i. 2024-T6 Aluminum Alloy Clad Sheet - Heat Treated and Aged - Thickness < 0.064 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM

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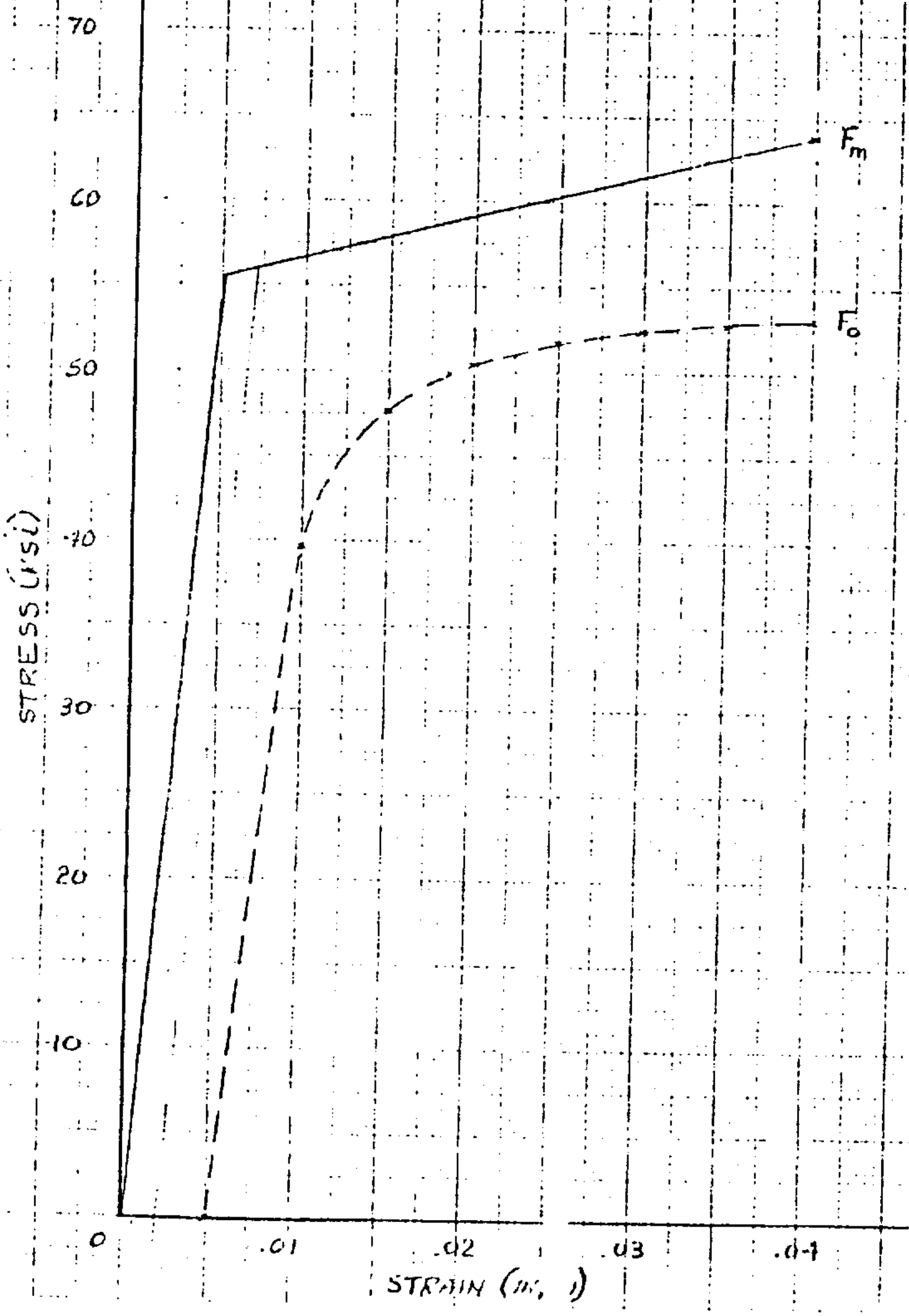
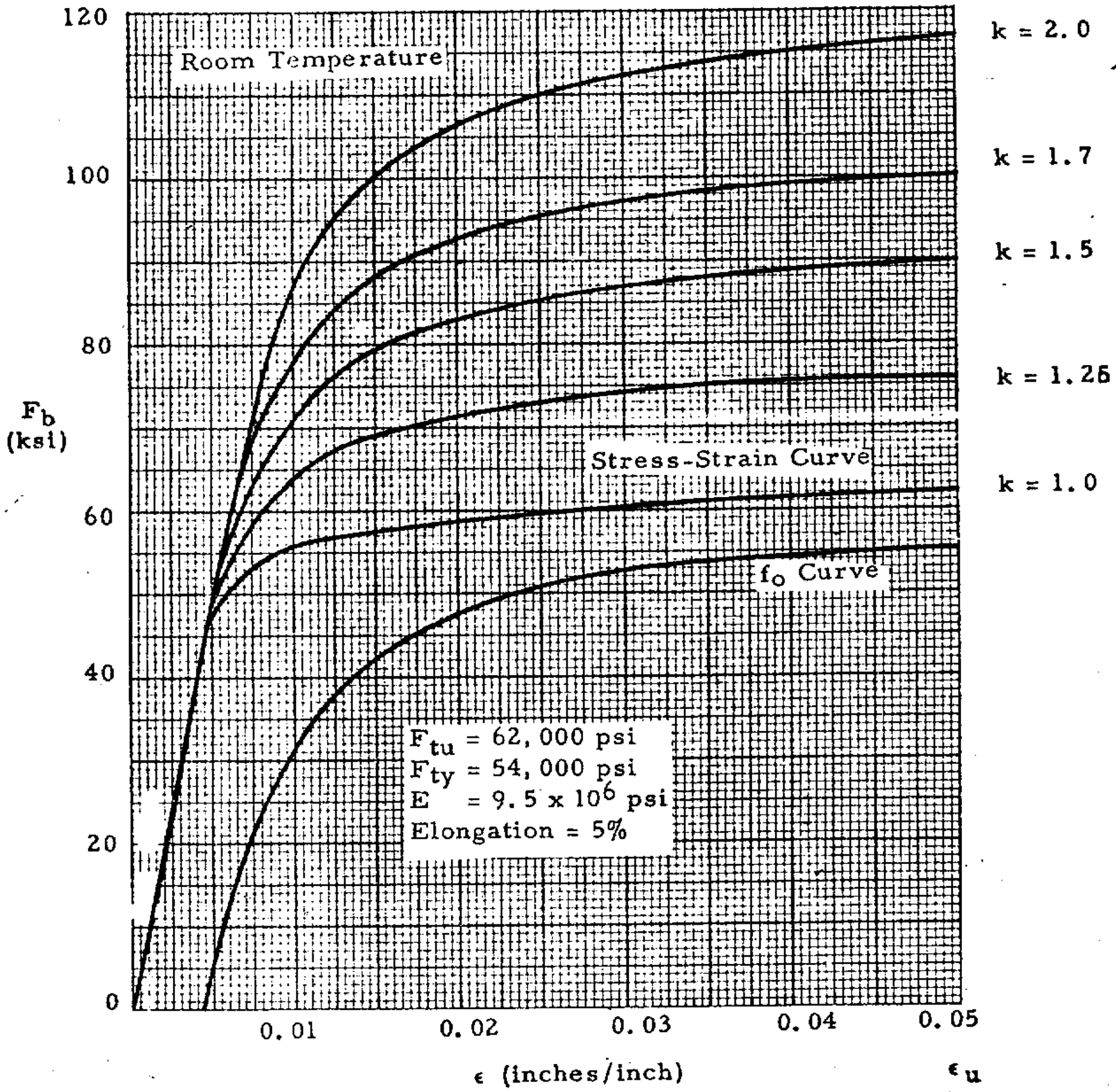
PLASTIC BENDING DATA FOR 2024-T81, T8510 + T8511
ALUMINUM ALLOY EXTRUSIONS $t = .050$ to $.249$

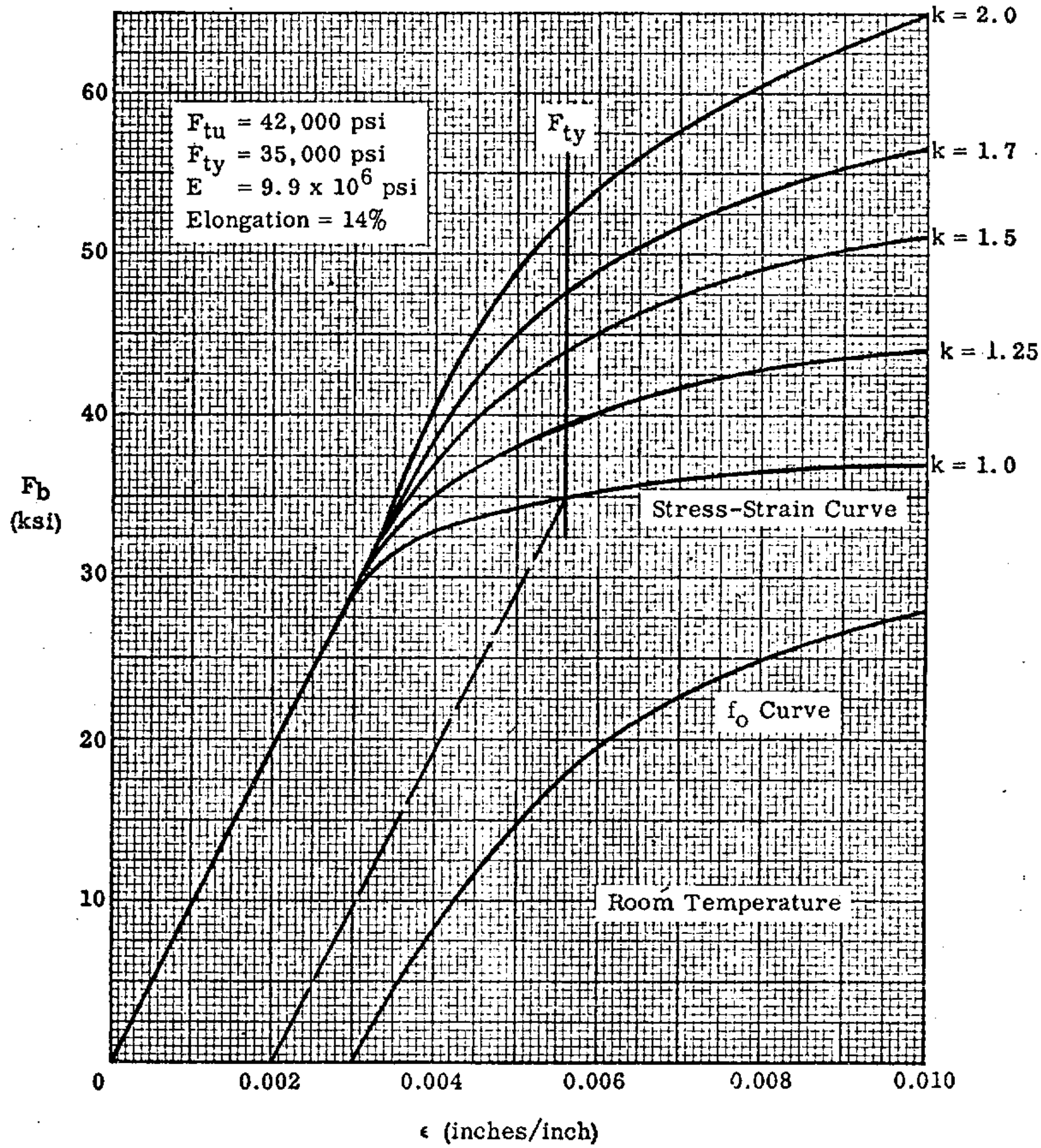
REF MIL-HDBK-5C $F_{tu} = 64$ ksi $F_{ty} = 56$ ksi $E = 10.8 \times 10^6$ psi
$e = 4.0\%$





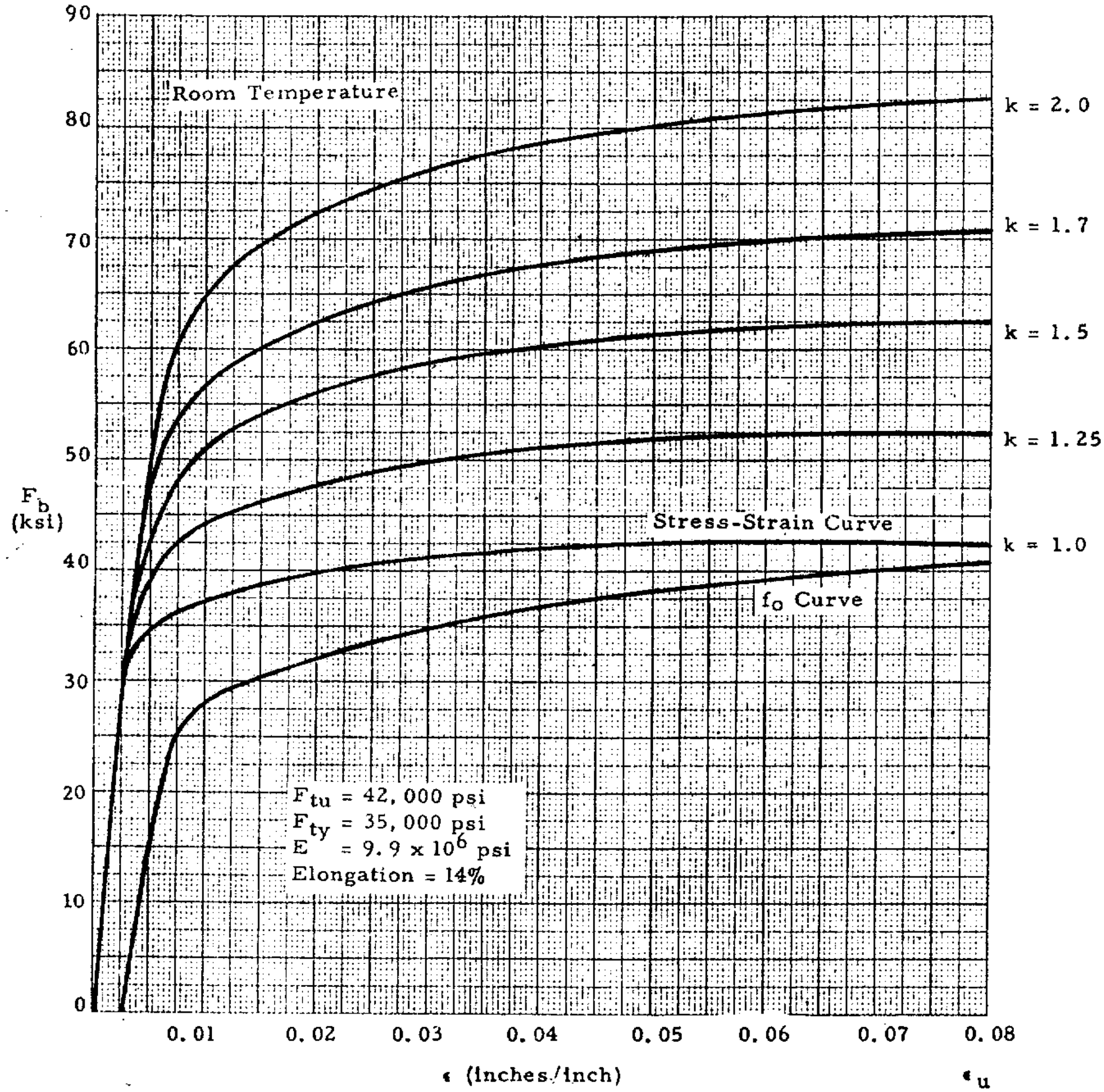
j. 2024-T81 Aluminum Alloy Clad Sheet - Heat Treated, Cold Worked and Aged - Thickness < 0.064 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



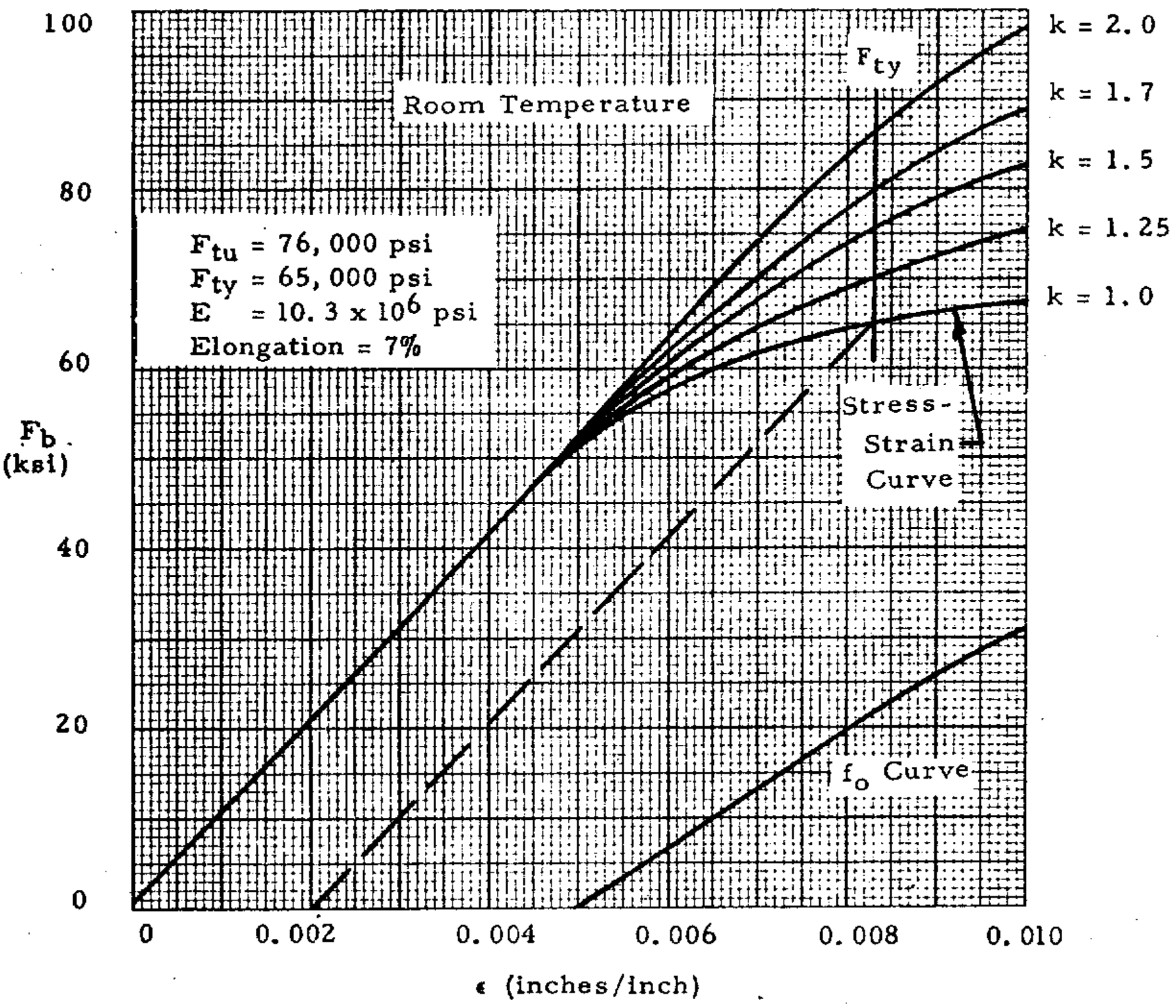
k. 6061-T6 Aluminum Alloy Sheet - Heat Treated and Aged - Thickness ≧ 0.020 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



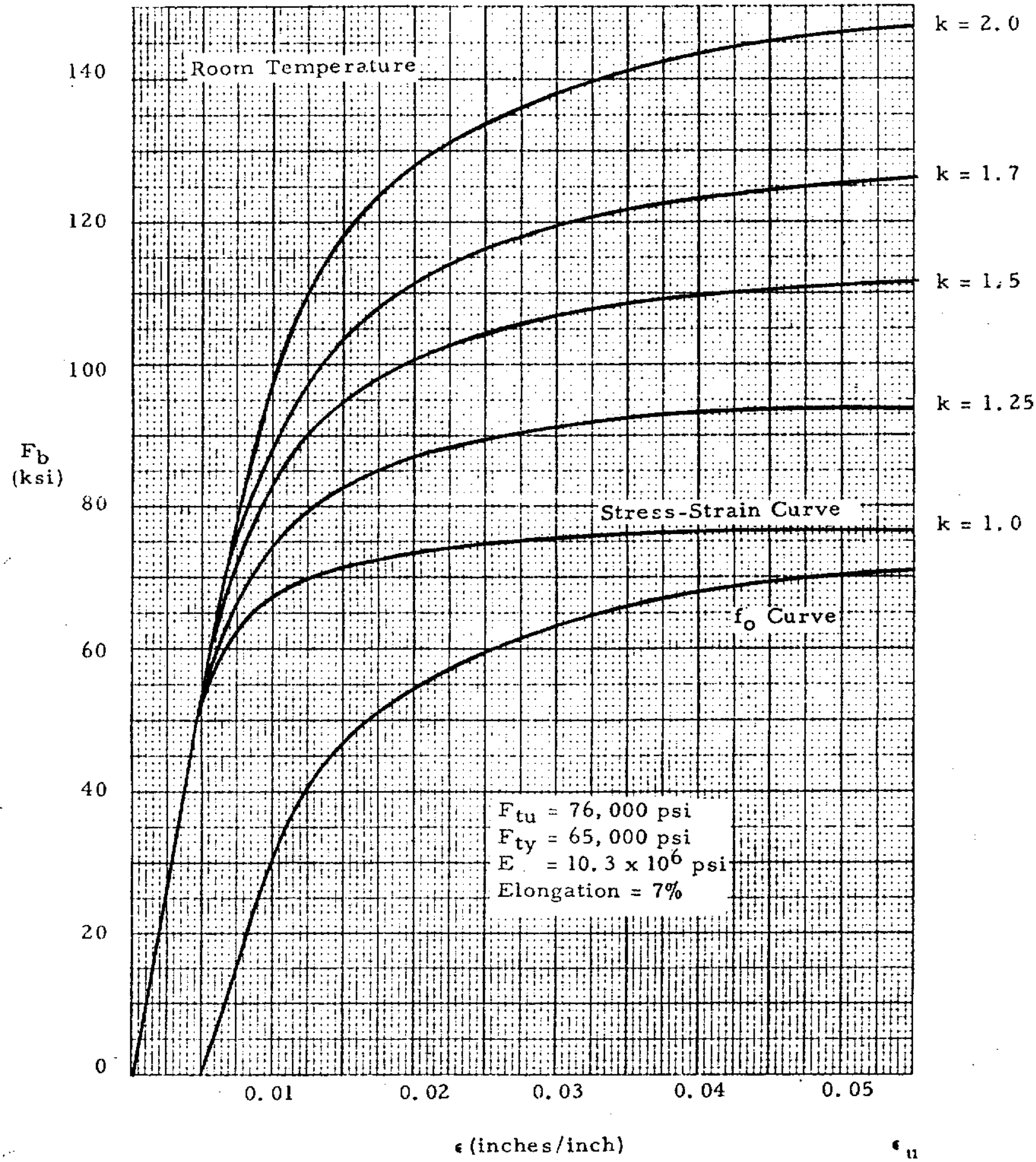
1. 6061-T6 Aluminum Alloy Sheet - Heat Treated and Aged - Thickness ≥ 0.020 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



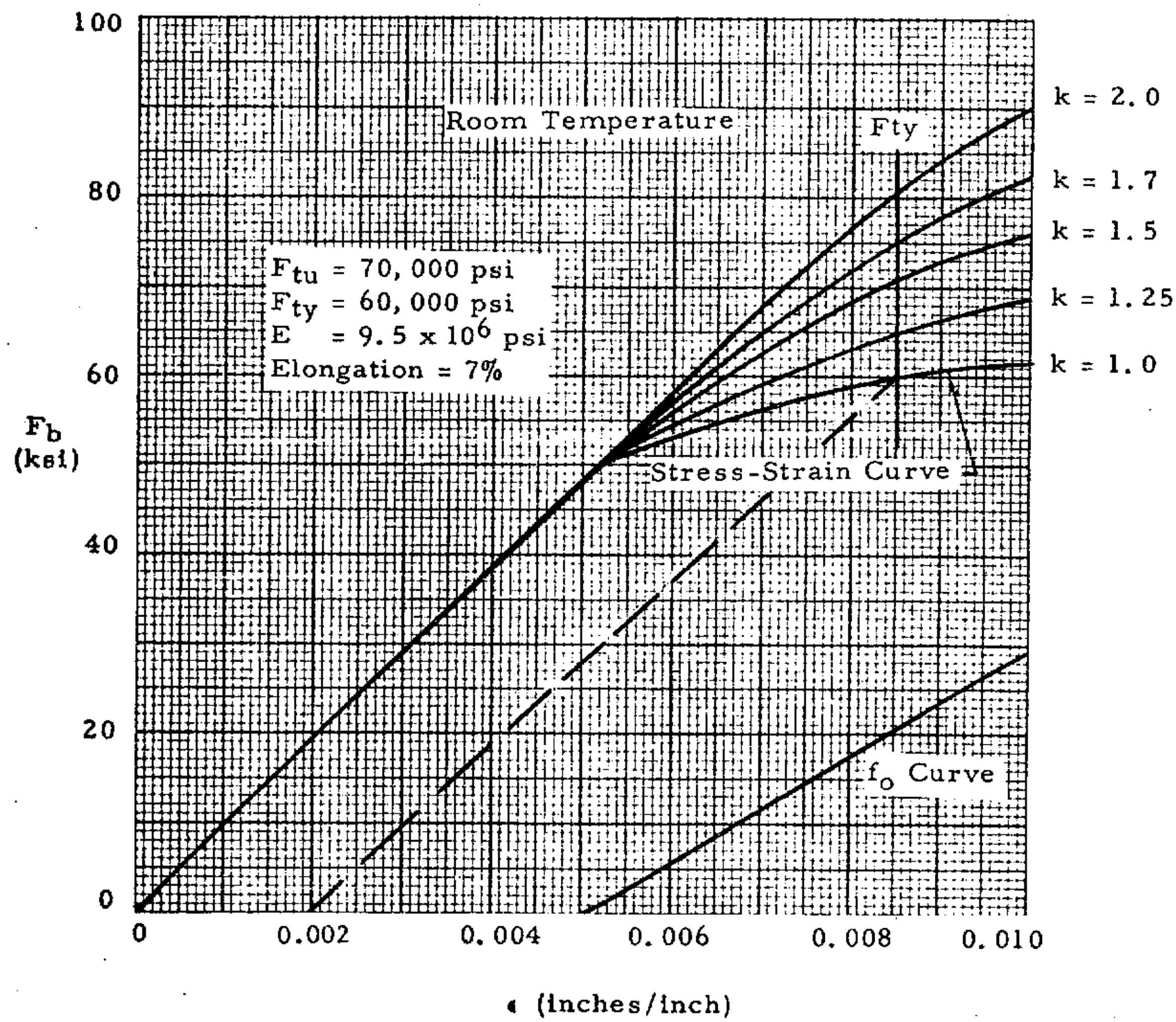
m. 7075-T6 Aluminum Alloy Bare Sheet and Plate - Thickness ≦ 0.039 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



n. 7075-T6 Aluminum Alloy Bare Sheet and Plate - Thickness ≤ 0.039 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



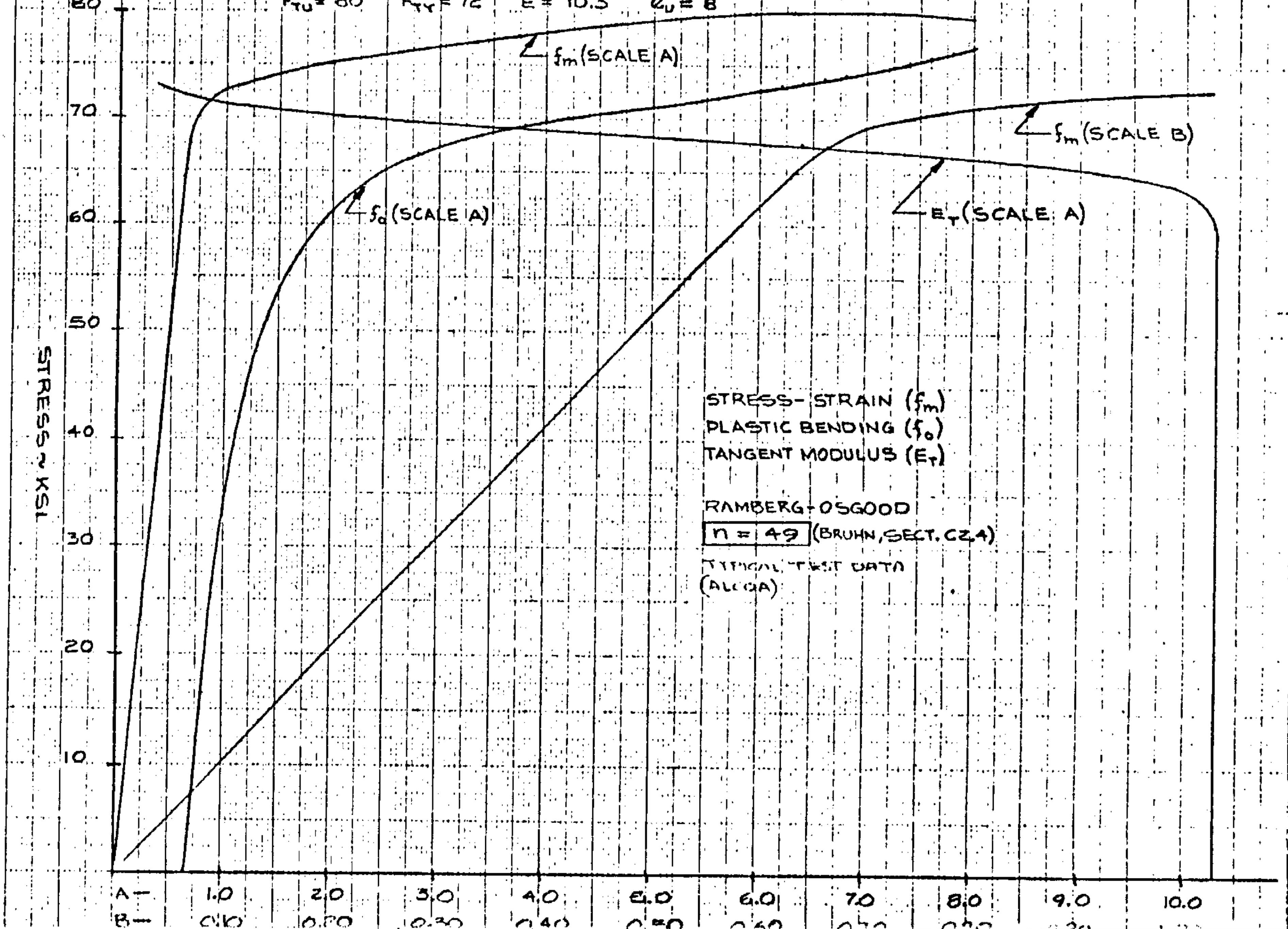
o. 7075-T6 Aluminum Alloy Clad Sheet and Plate - Thickness ≤ 0.039 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM

7075-T6 ALUMINUM BARE SHEET - MATERIAL PROPERTIES

"B" VALUES/MIL-HDBK-5C, T < .250, L-TENS.

F_{TU} = 80 F_{TY} = 72 E = 10.3 e_u = 8



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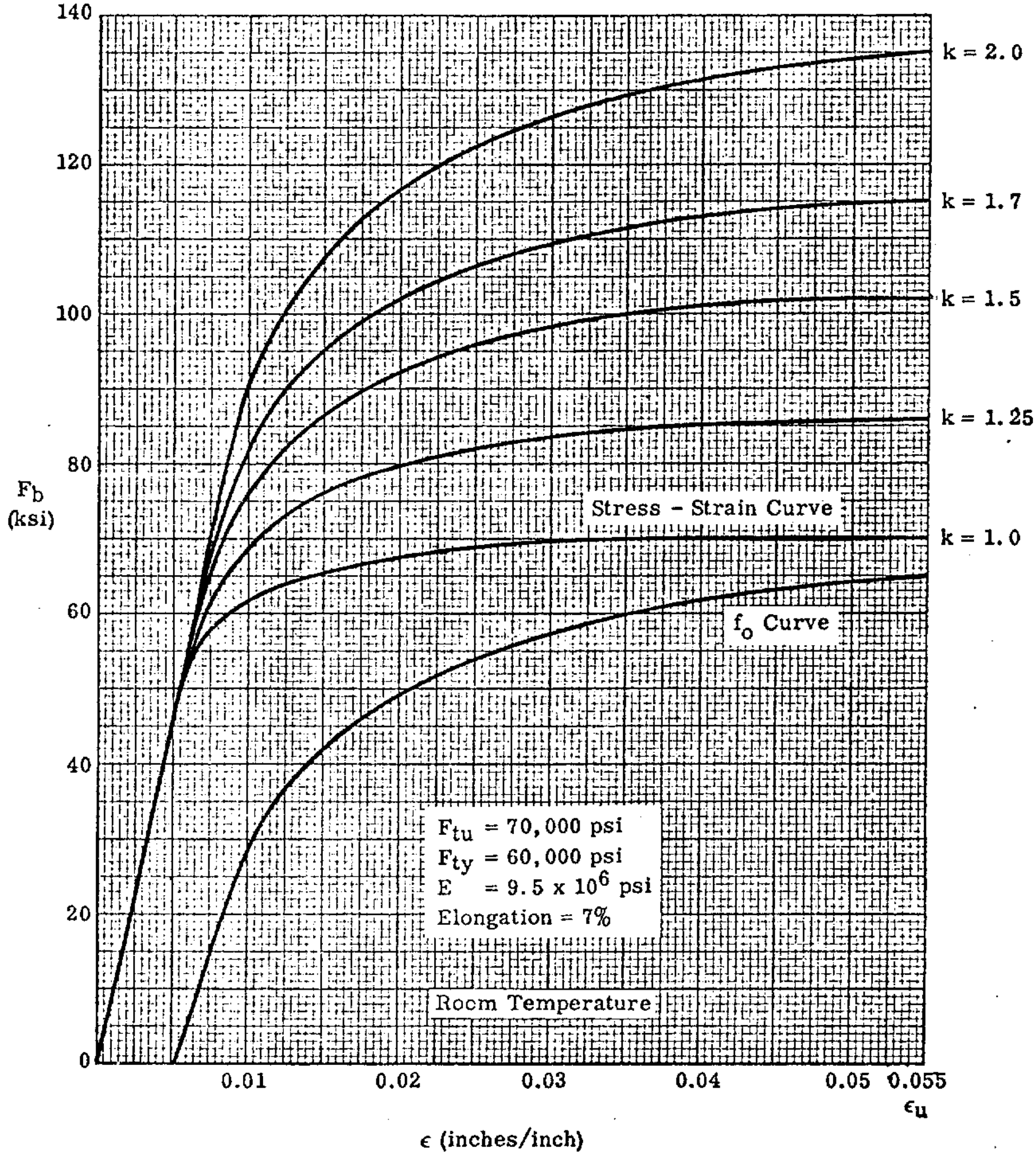
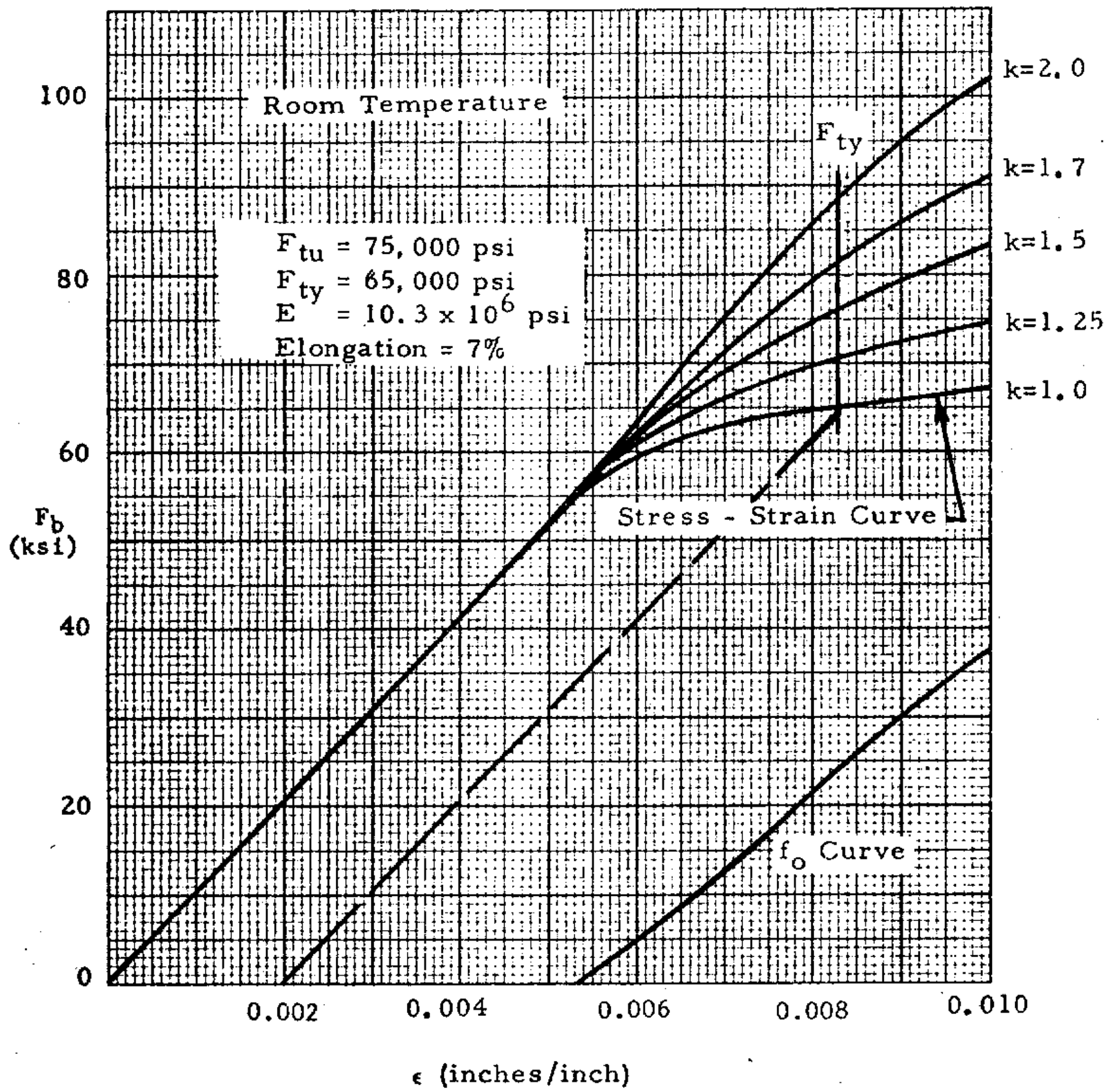


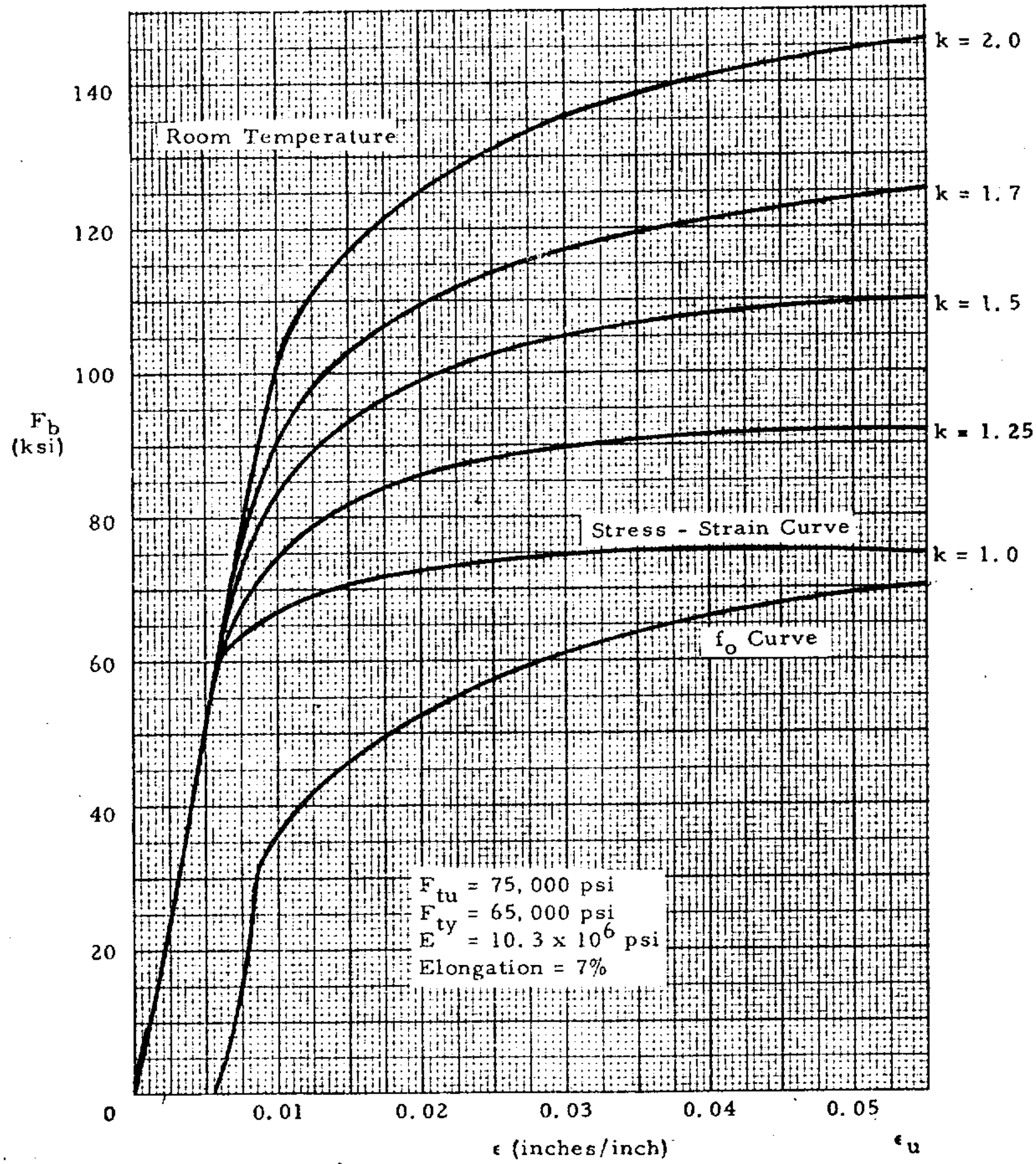
p. 7075-T6 Aluminum Alloy Clad Sheet and Plate - Thickness ≤ 0.039 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



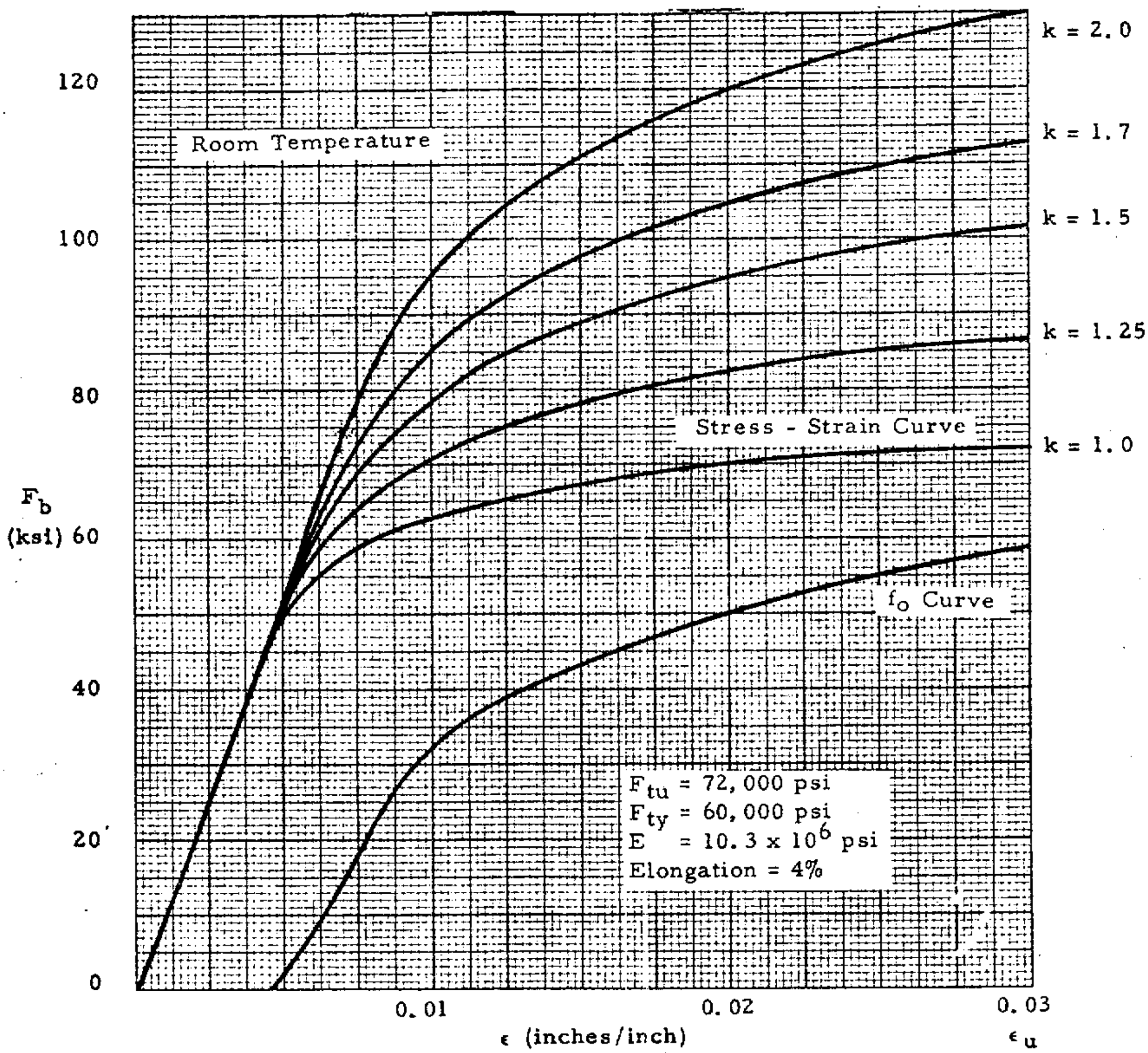
q. 7075-T6 Aluminum Alloy Extrusions - Thickness ≦ 0.25 In.

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



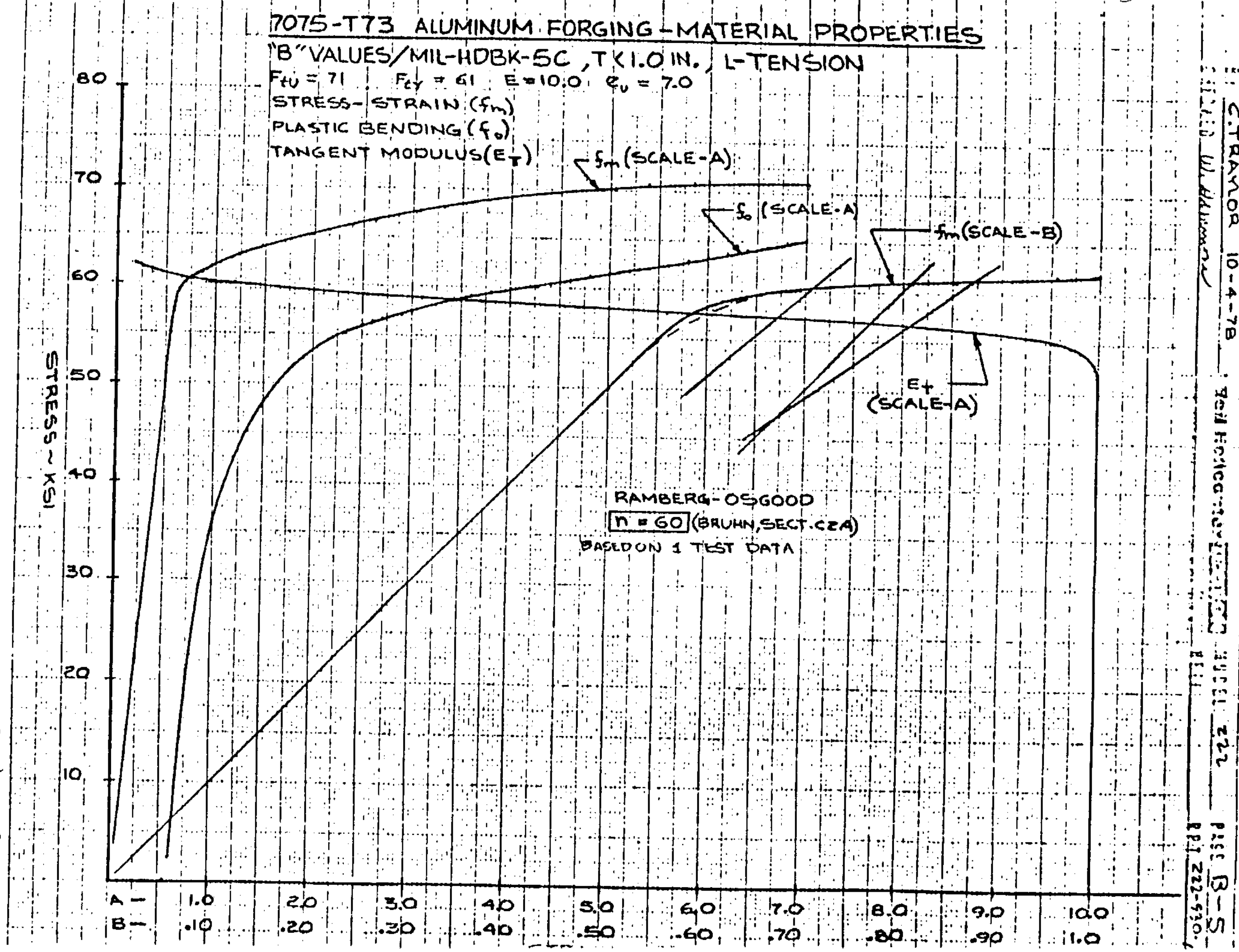
r. 7075-T6 Aluminum Alloy Extrusions
- Thickness ≦ 0.25 In.

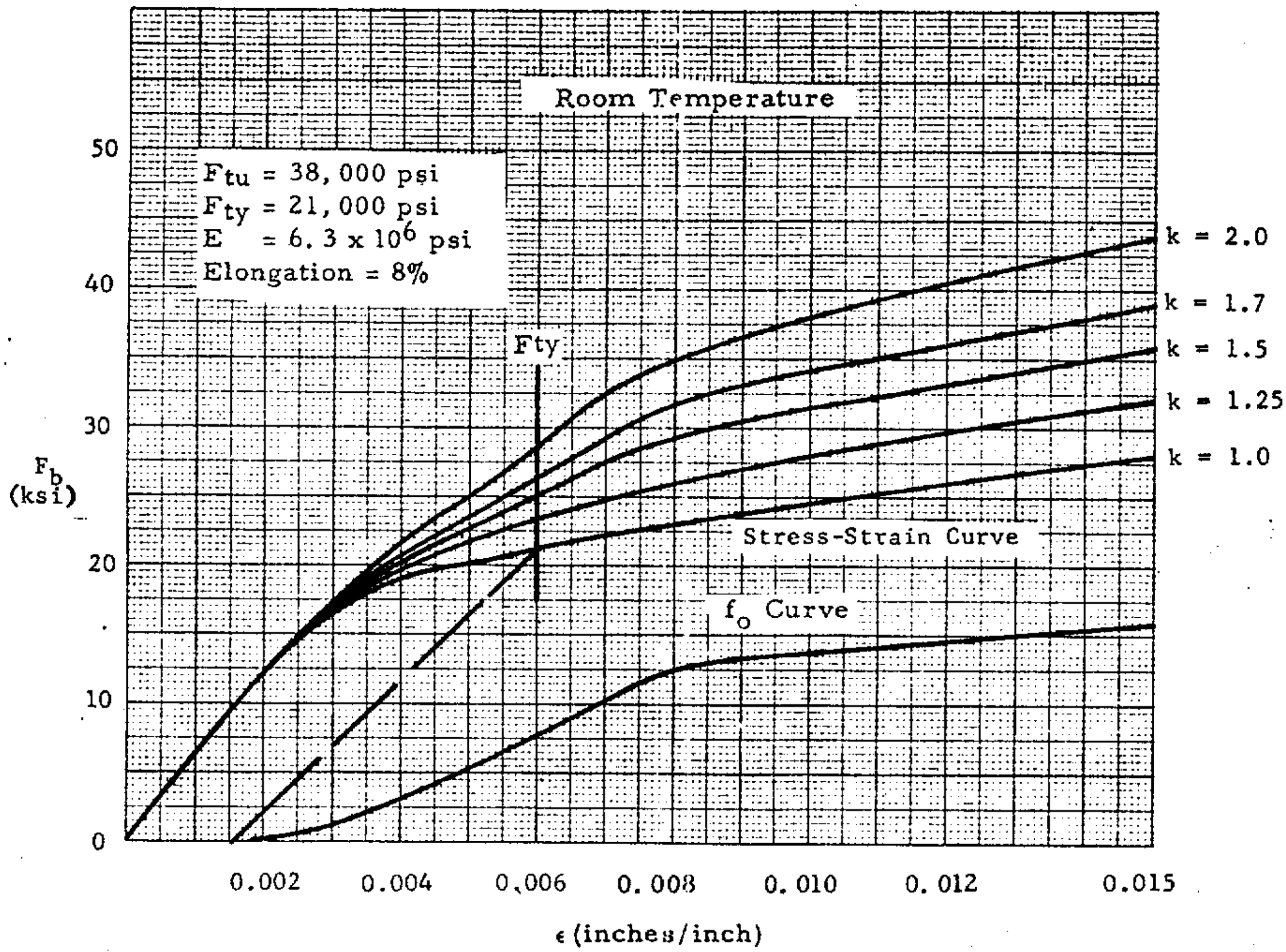
FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM



s. 7075-T6 Aluminum Alloy Hand Forgings
- Area ≦ 16 In.^{2}

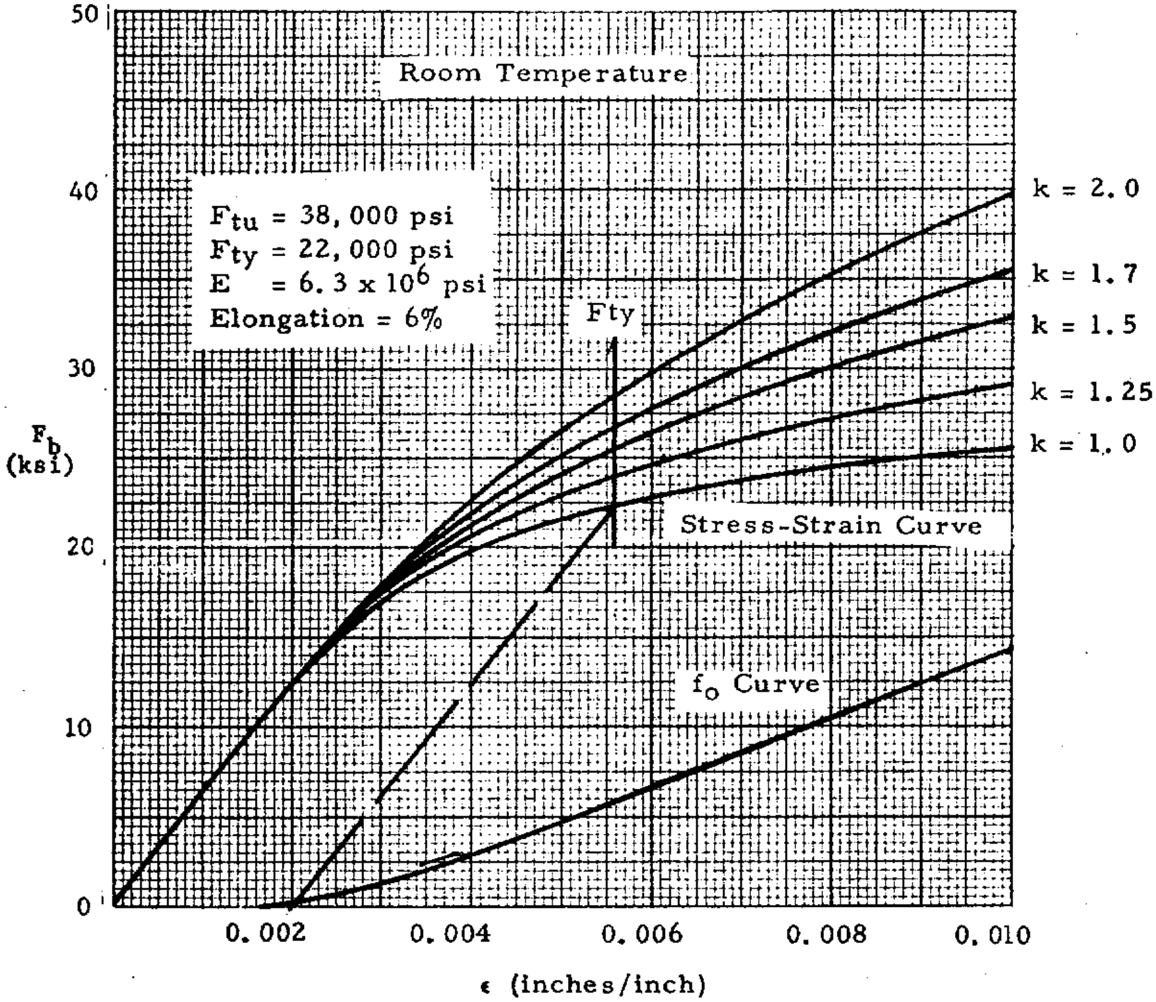
FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR ALUMINUM


7075-T73 ALUMINUM FORGING - MATERIAL PROPERTIES
"B" VALUES/MIL-HDBK-5C, T<1.0 IN., L-TENSION
Fty = 71   Fcy = 61   E = 10.0   eu = 7.0
STRESS-STRAIN (fm)
PLASTIC BENDING (fo)
TANGENT MODULUS (ET)
fm (SCALE-A)
fo (SCALE-A)
fm (SCALE-B)
Et (SCALE-A)
RAMBERG-OSGOOD
n = 60 (BRUHN, SECT. C2A)
BASED ON 1 TEST DATA
STRESS ~ KSI
80
70
60
50
40
30
20
10
A — 1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0
B — .10 .20 .30 .40 .50 .60 .70 .80 .90 1.0
C. TRAYNOR 10-4-78
PAGE B-5
9-84A



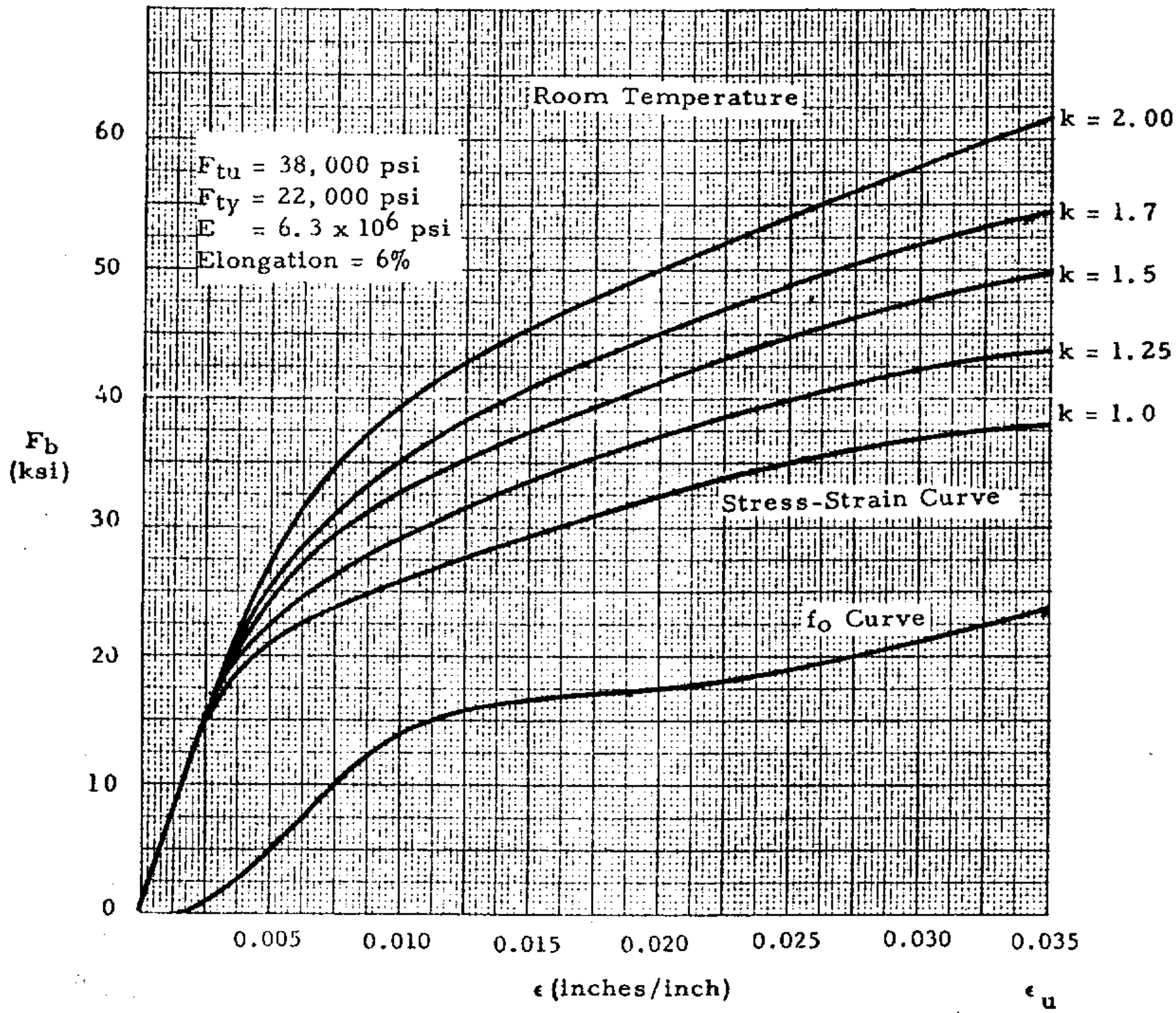
a. AZ61A Magnesium Alloy Extrusions (Longitudinal) - Thickness ≤ 0.249 In.

FIGURE 9.17 - MINIMUM PLASTIC BENDING CURVES FOR MAGNESIUM



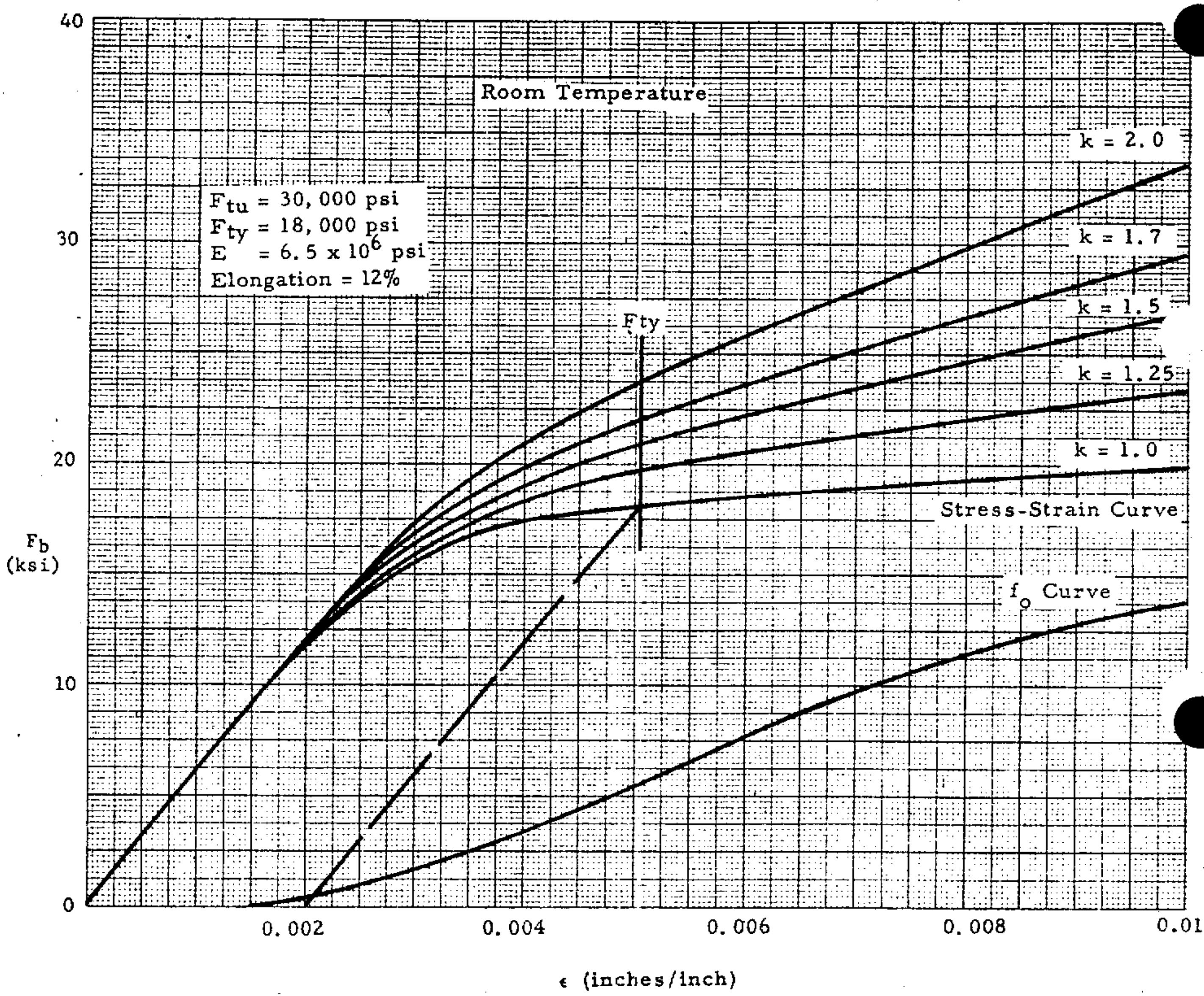
b. AZ61A Magnesium Alloy Forgings (Longitudinal)

FIGURE 9.17 - MINIMUM PLASTIC BENDING CURVES FOR MAGNESIUM



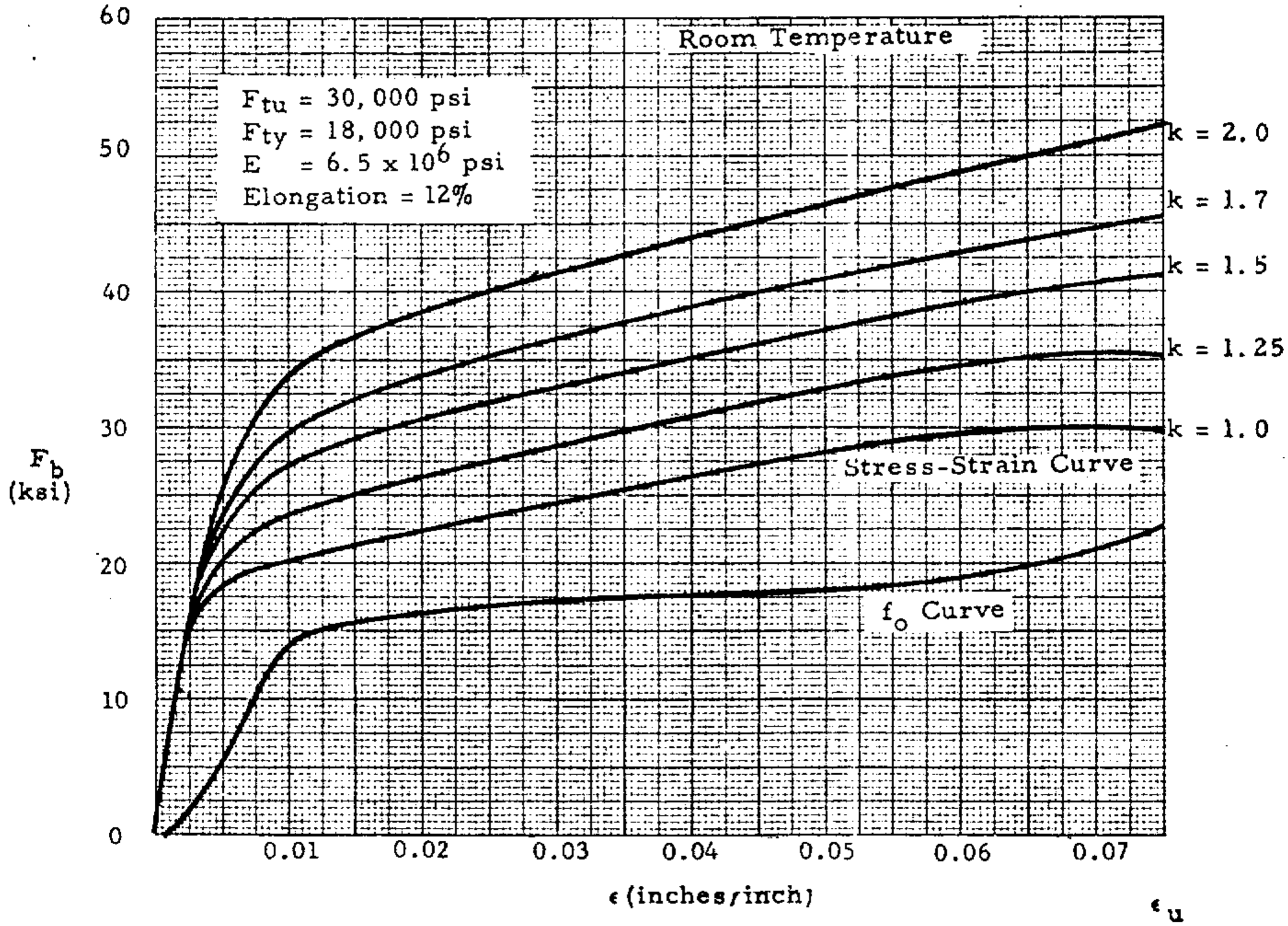
c. AZ61A Magnesium Alloy Forgings (Longitudinal)

FIGURE 9.17 - MINIMUM PLASTIC BENDING CURVES FOR MAGNESIUM



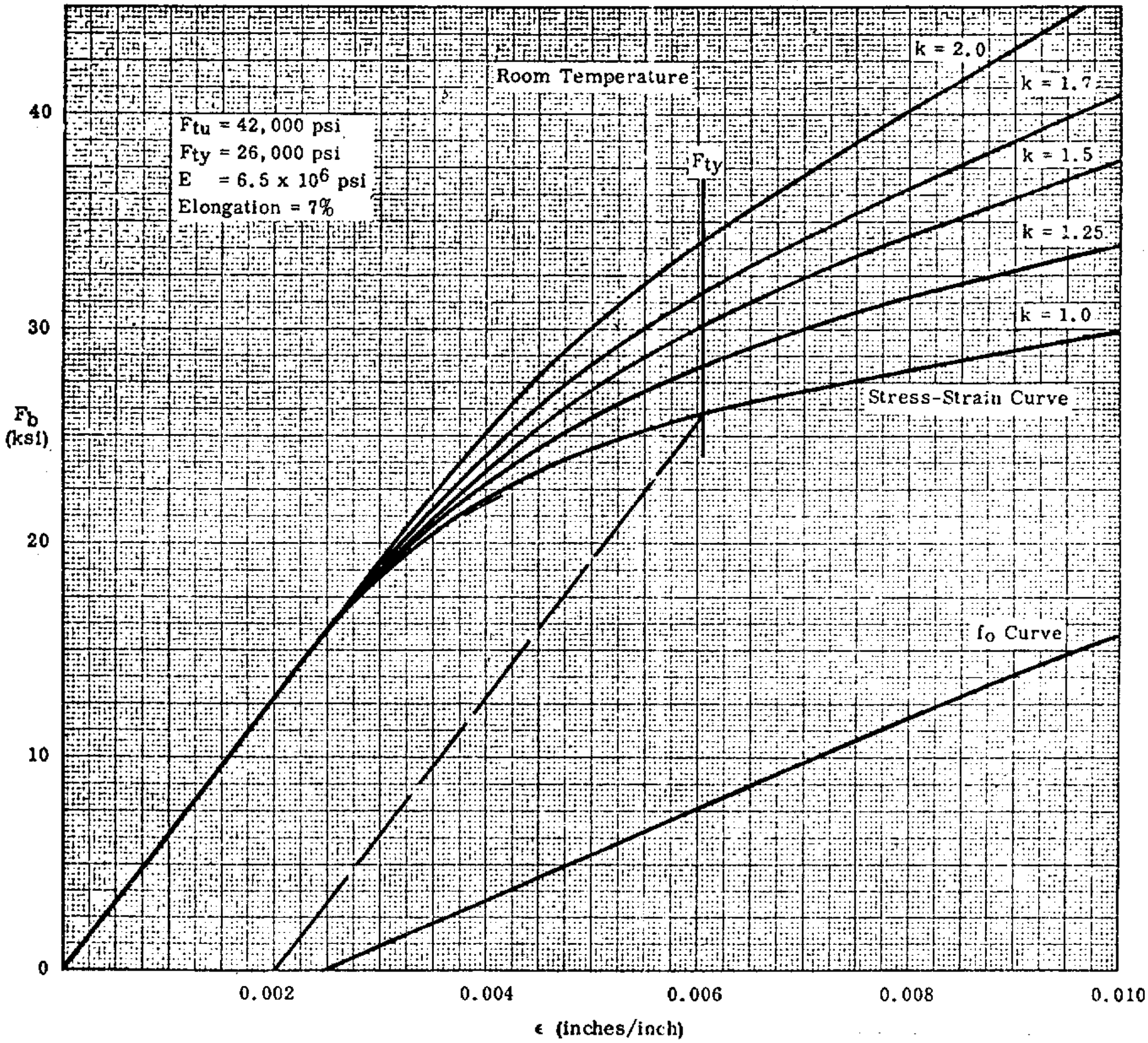
d. HK31A - O Magnesium Alloy Sheet - Thickness ≤ 0.250 In.

FIGURE 9.17 - MINIMUM PLASTIC BENDING CURVES FOR MAGNESIUM



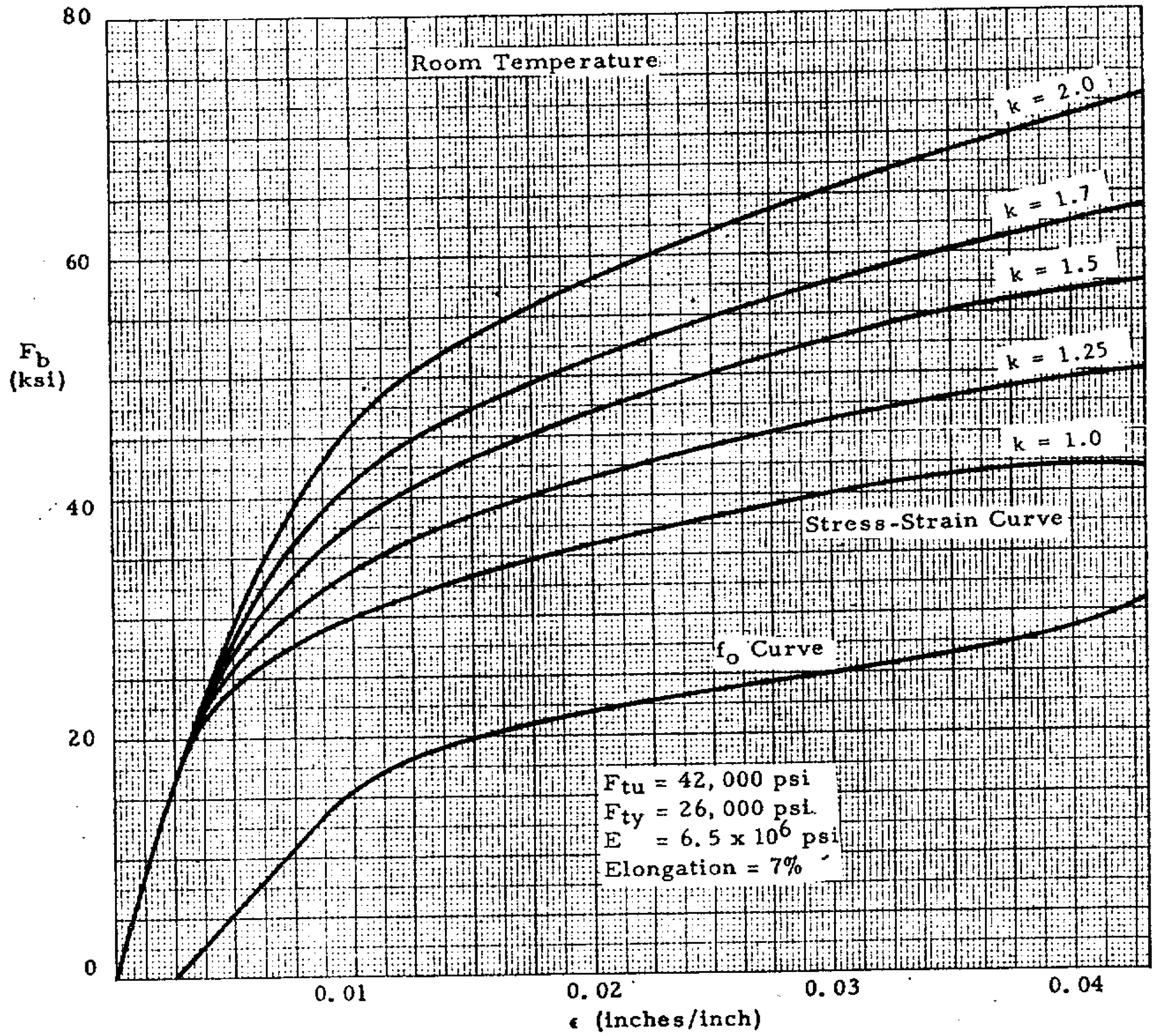
e. HK31A - O Magnesium Alloy Sheet
- Thickness ≦ 0.250

FIGURE 9.17 - MINIMUM PLASTIC BENDING CURVES FOR MAGNESIUM



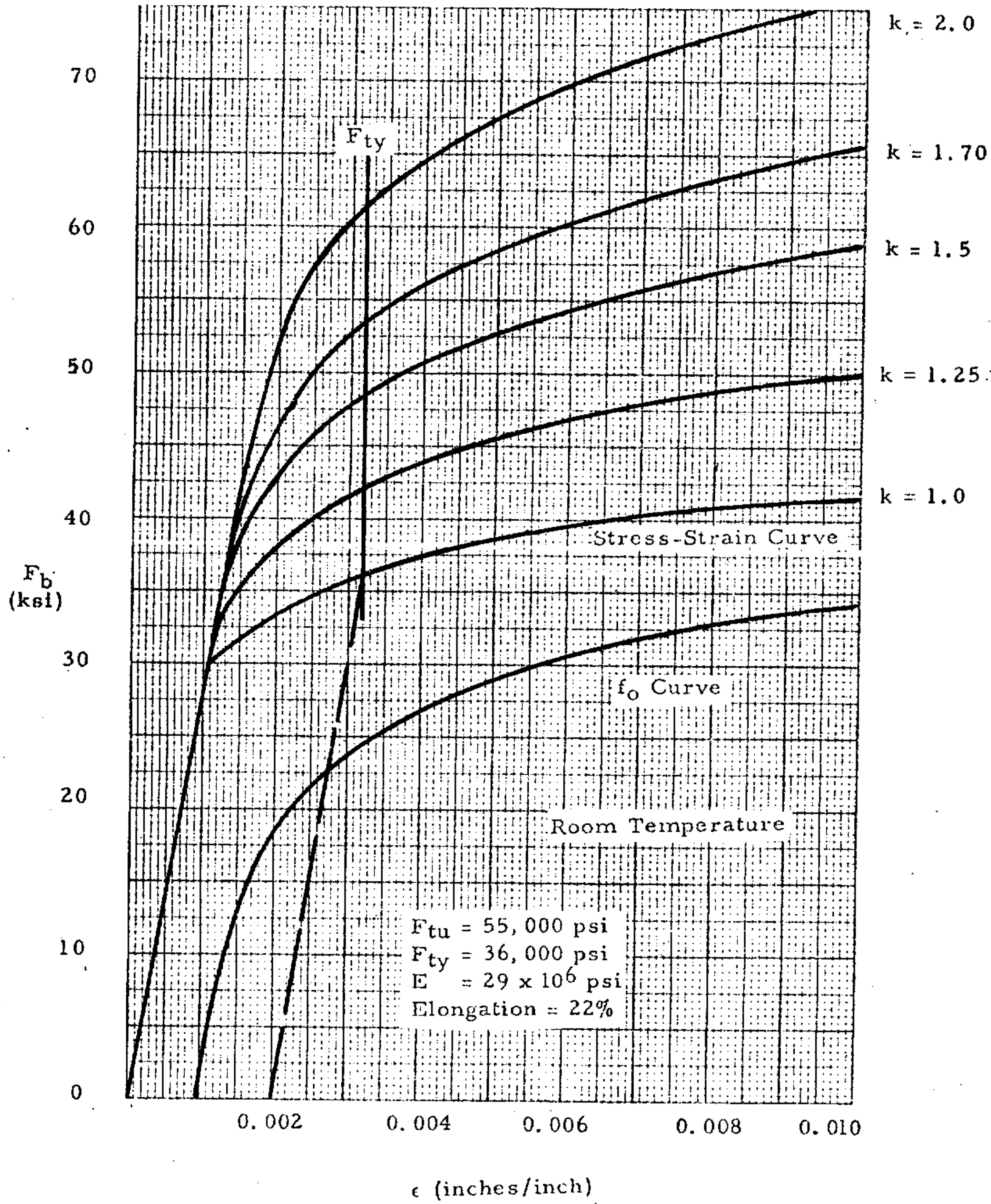
f. ZK60A Magnesium Alloy Forgings (Longitudinal)

FIGURE 9.17 - MINIMUM PLASTIC BENDING CURVES FOR MAGNESIUM



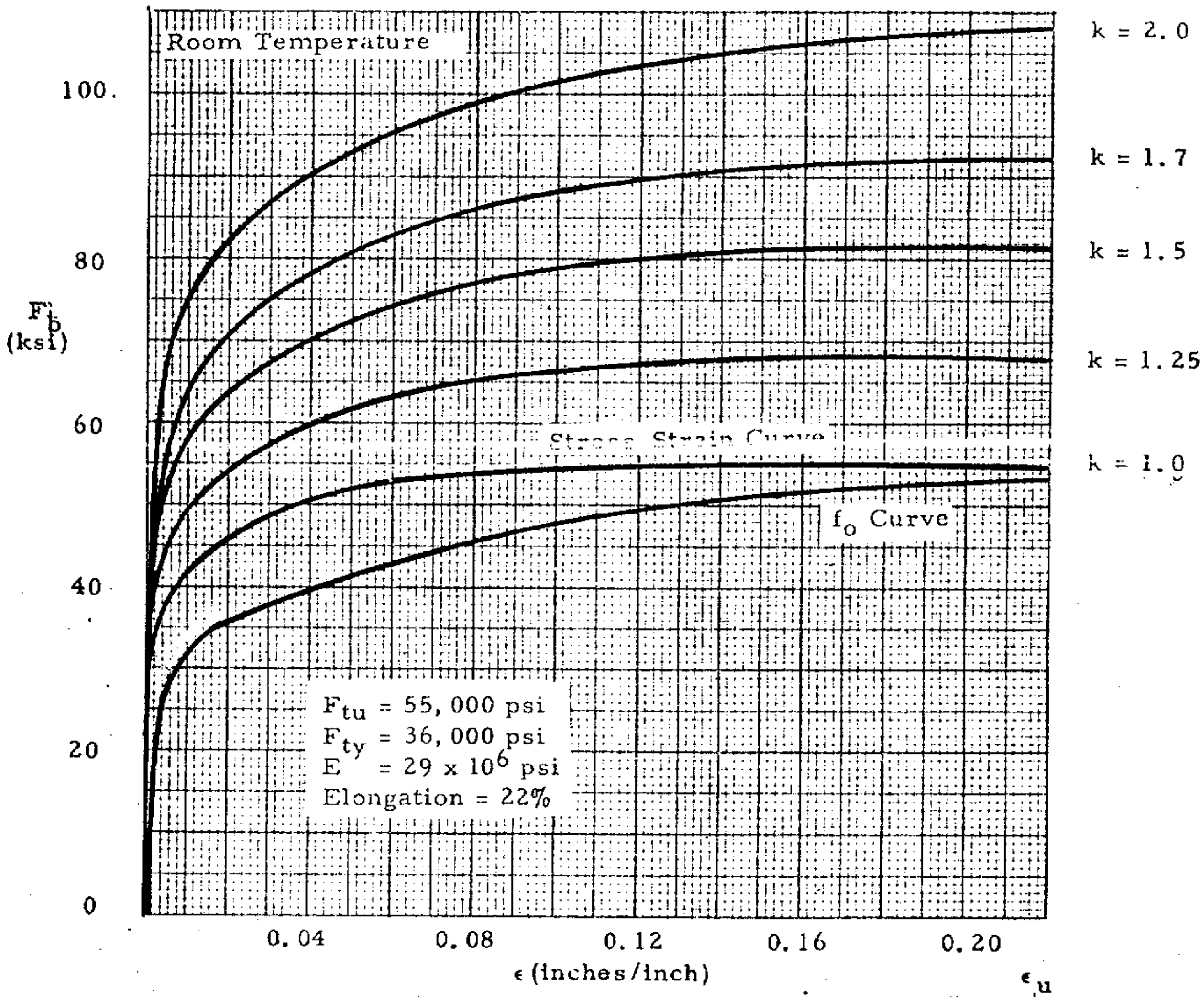
g. ZK60A Magensium Alloy Forgings (Longitudinal)

FIGURE 9.17 - MINIMUM PLASTIC BENDING CURVES FOR MAGNESIUM



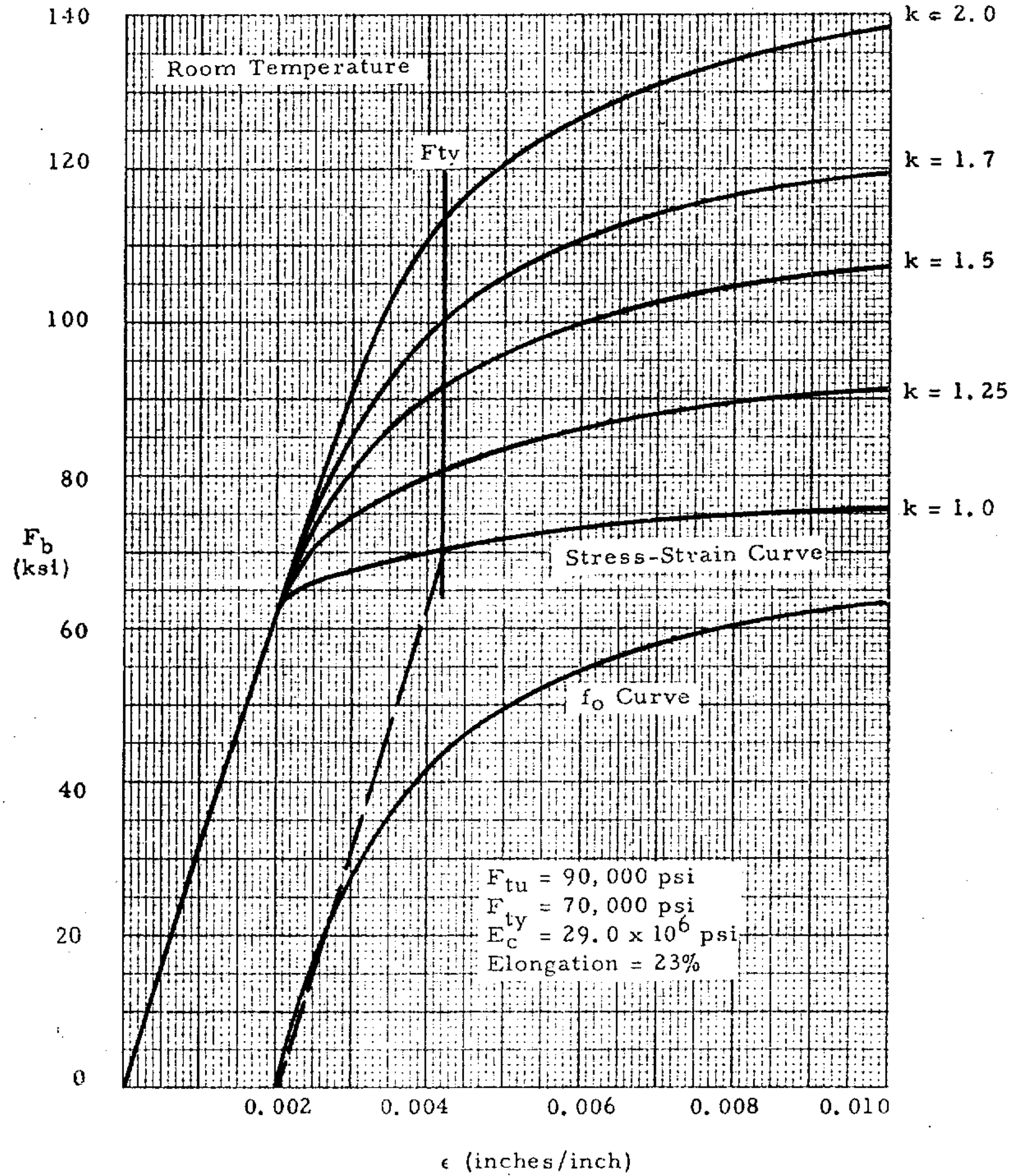
a. Carbon Steel AISI 1023-1025

FIGURE 9.18 - MINIMUM PLASTIC BENDING CURVES FOR LOW CARBON AND ALLOY STEELS



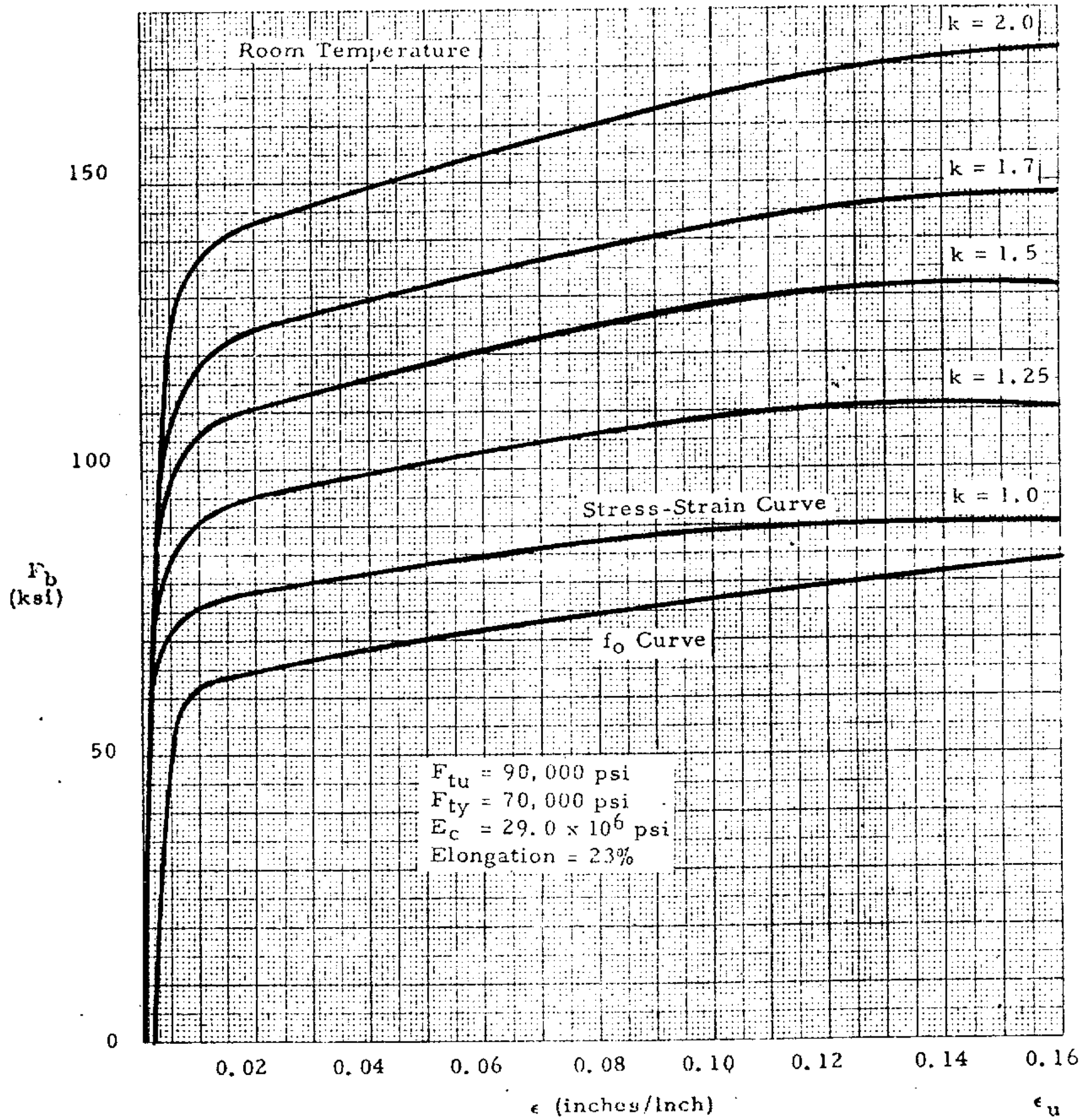
b. Carbon Steel AISI 1023-1025

FIGURE 9.18 - MINIMUM PLASTIC BENDING CURVES FOR LOW CARBON AND ALLOY STEELS



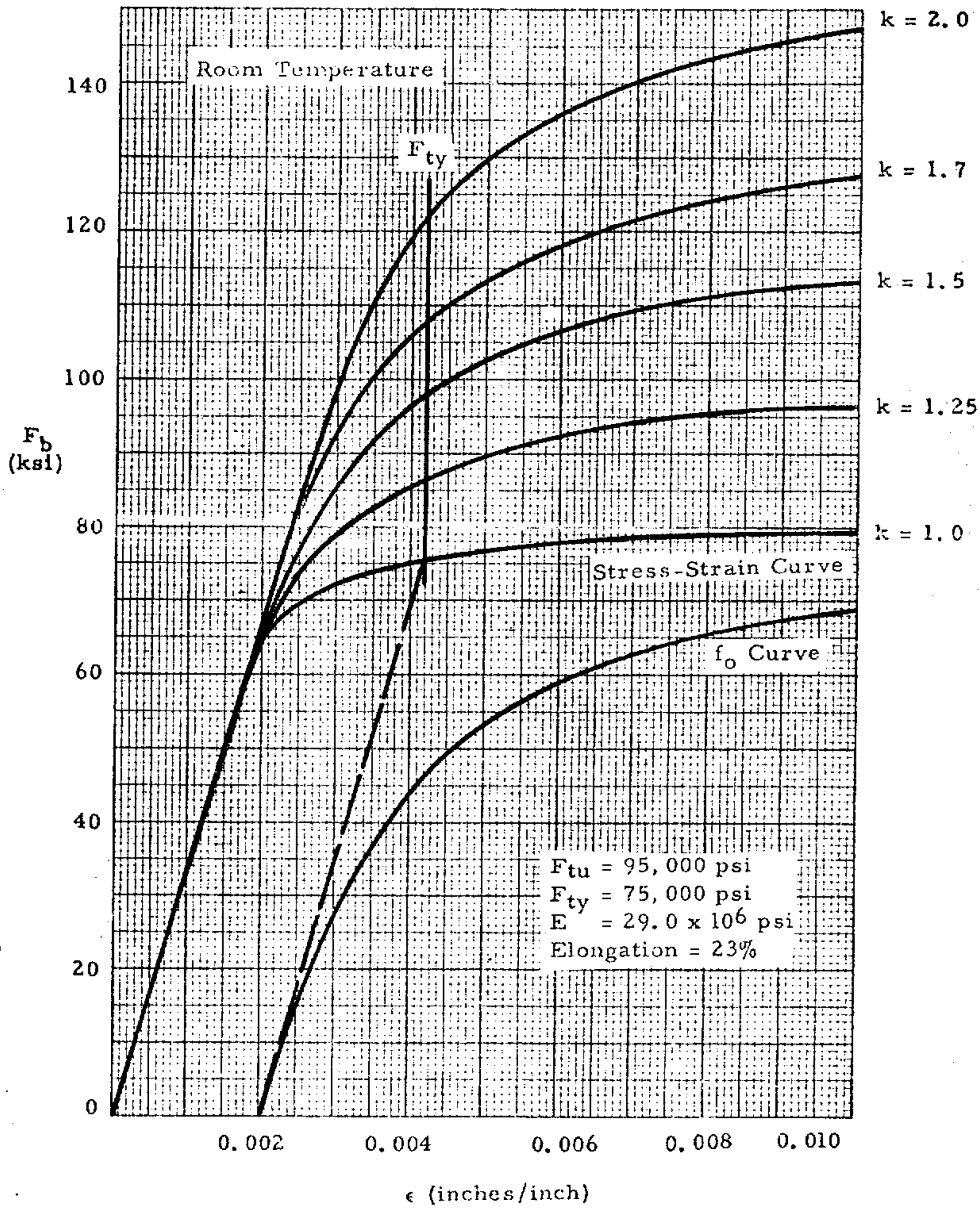
c. AISI Alloy Steel, Normalized - Thickness > 0.188 In.

FIGURE 9.18 - MINIMUM PLASTIC BENDING CURVES FOR LOW CARBON AND ALLOY STEELS



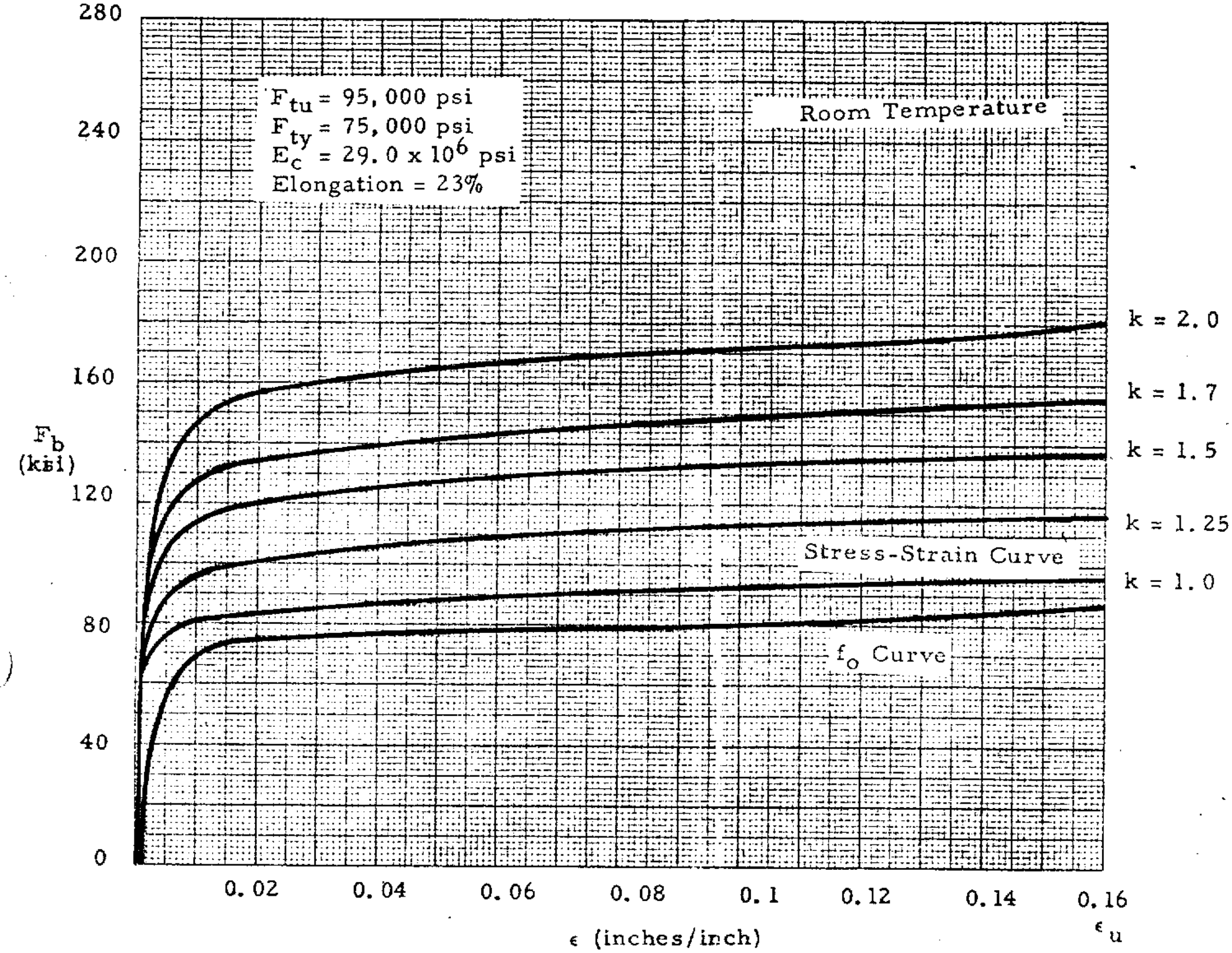
d. AISI Alloy Steel, Normalized - Thickness > 0.188 In.

FIGURE 9.18 - MINIMUM PLASTIC BENDING CURVES FOR LOW CARBON AND ALLOY STEELS



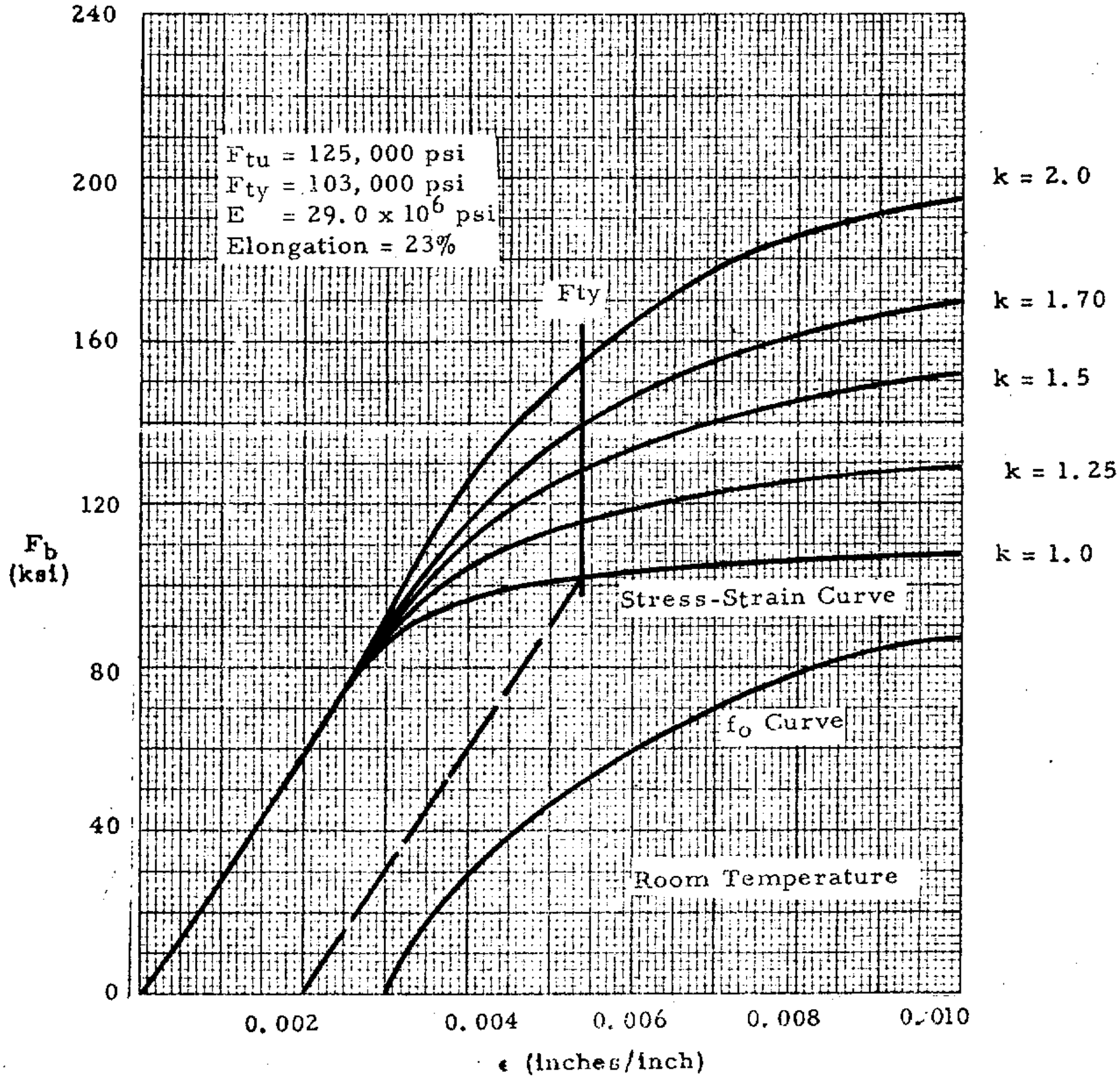
e. AISI Alloy Steel, Normalized - Thickness ≦ 0.188 In.

FIGURE 9.18 - MINIMUM PLASTIC BENDING CURVES FOR LOW CARBON AND ALLOY STEELS



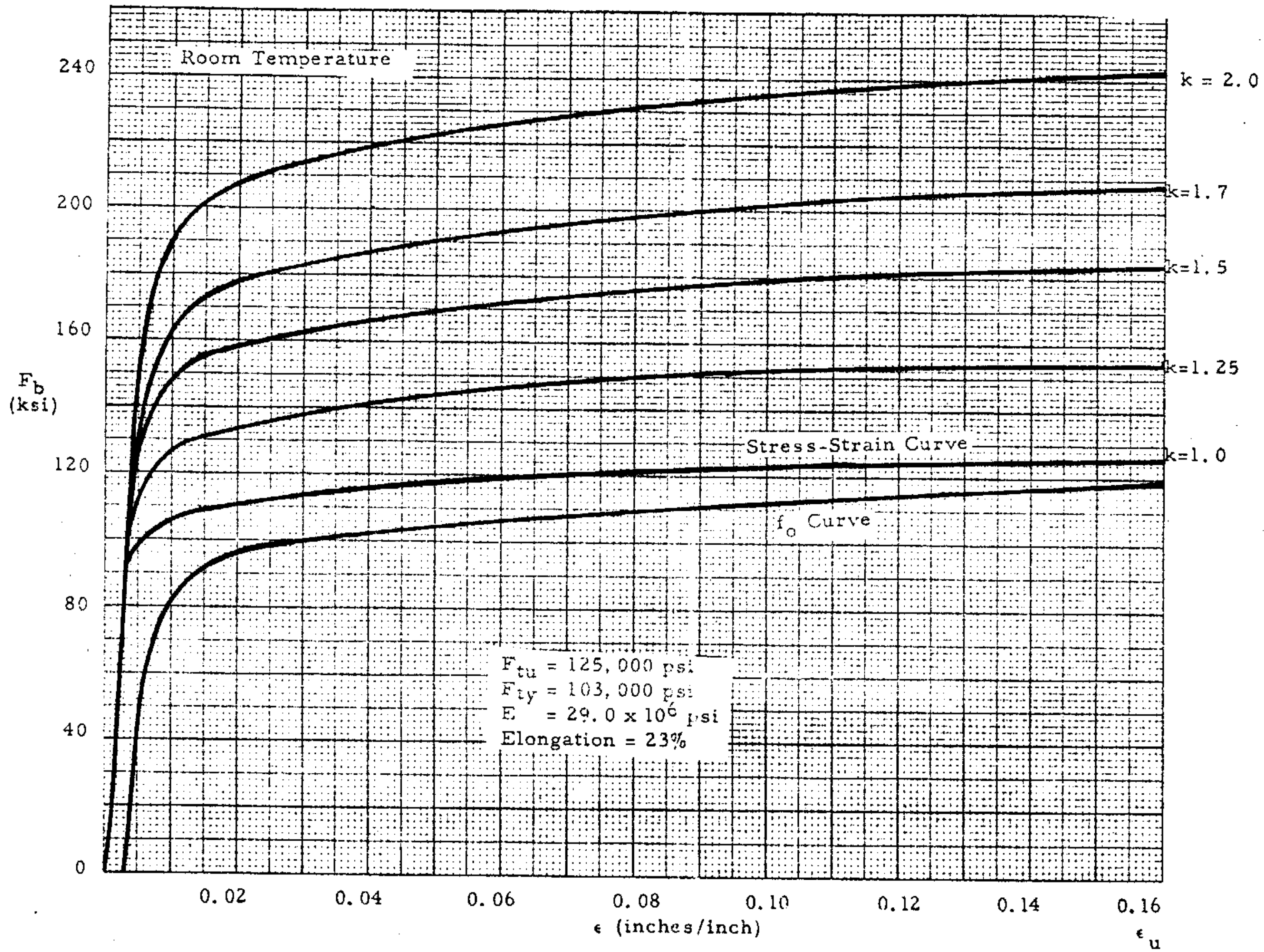
f. AISI Alloy Steel, Normalized - Thickness ≤ 0.188 In.

FIGURE 9.18 - MINIMUM PLASTIC BENDING CURVES FOR LOW CARBON AND ALLOY STEELS



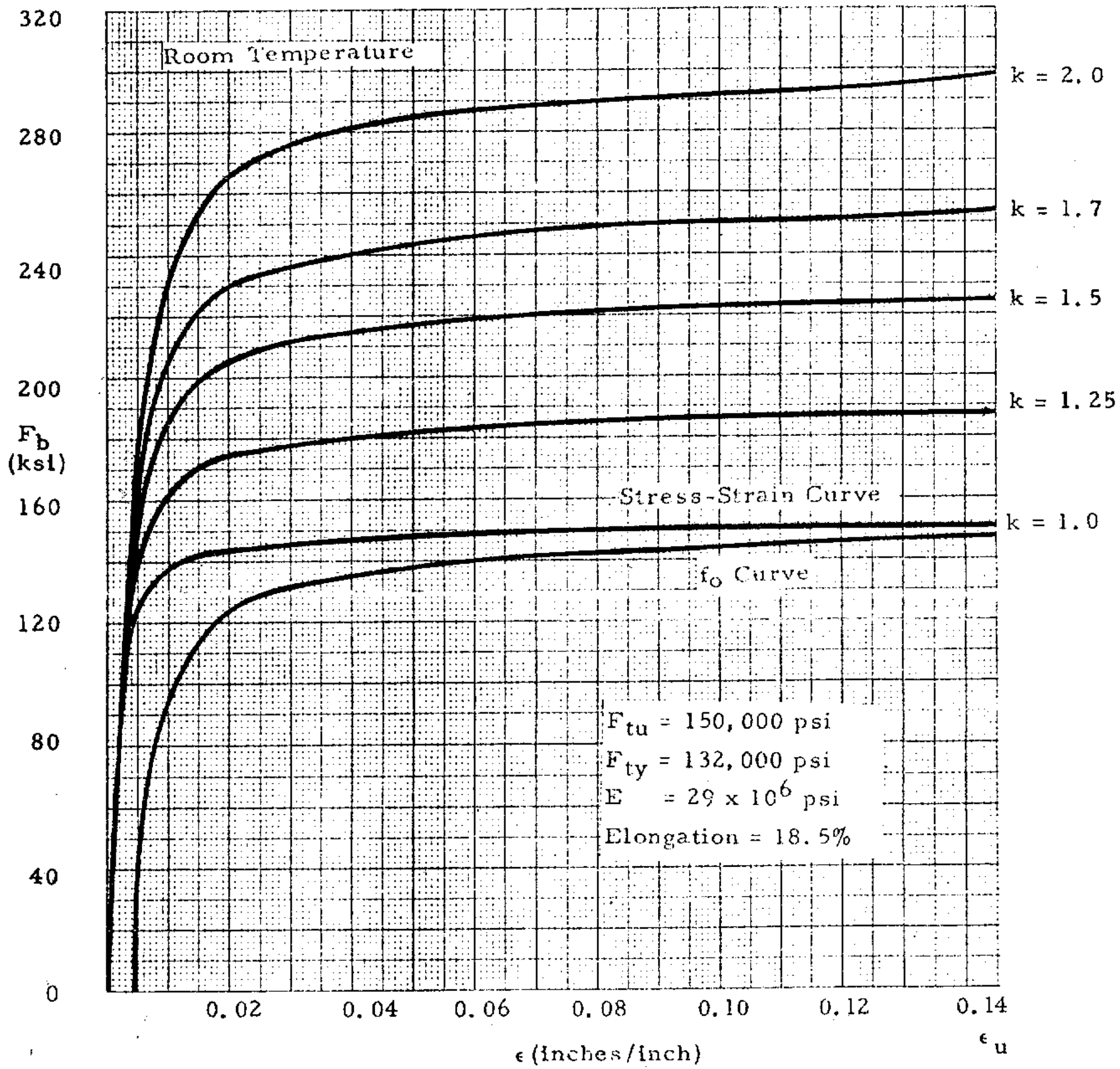
g. AISI Alloy Steel, Heat Treated

FIGURE 9.18 - MINIMUM PLASTIC BENDING CURVES FOR LOW CARBON AND ALLOY STEELS



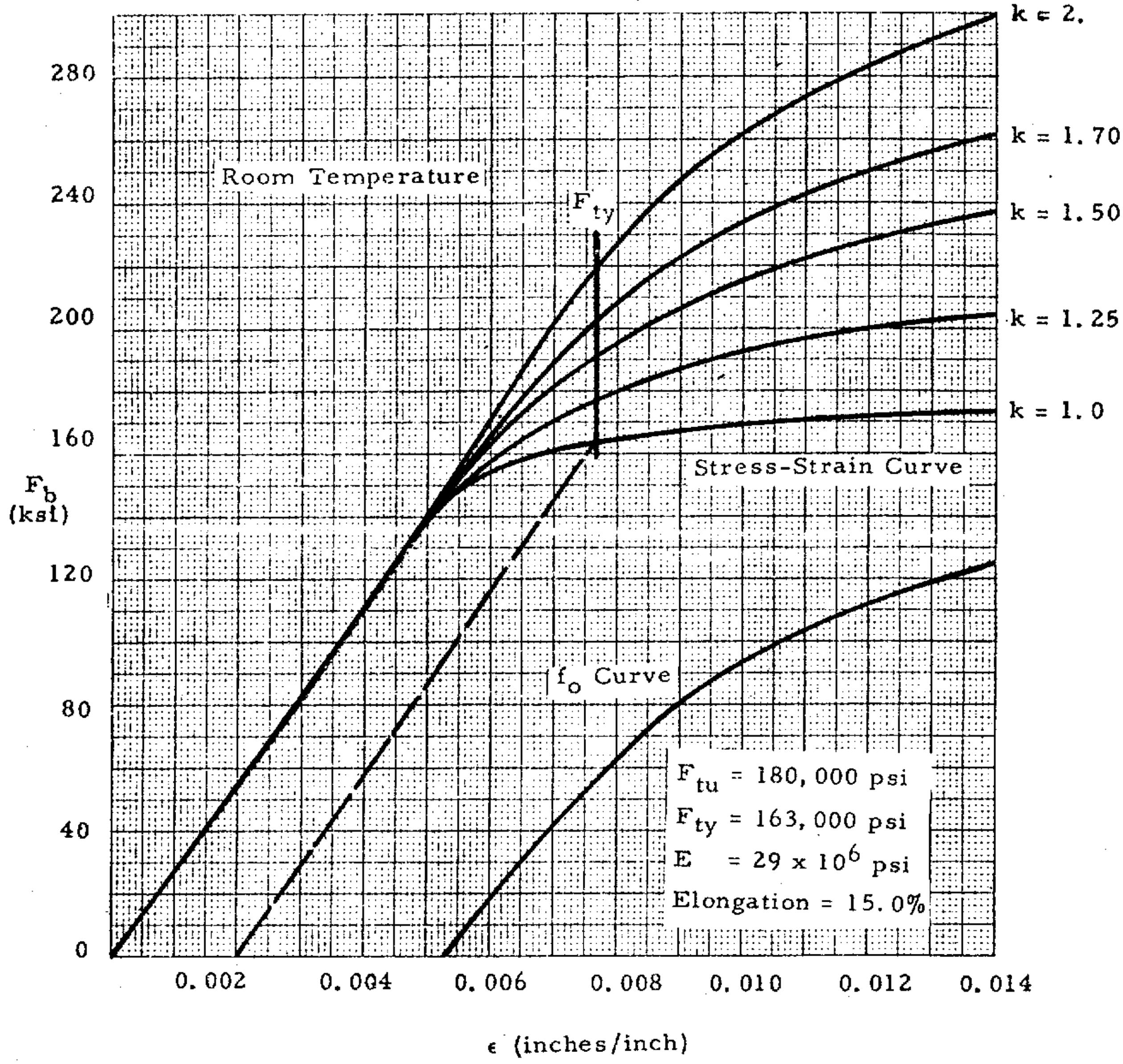
h. AISI Alloy Steel, Heat Treated

FIGURE 9.18 - MINIMUM PLASTIC BENDING CURVES FOR LOW CARBON AND ALLOY STEELS



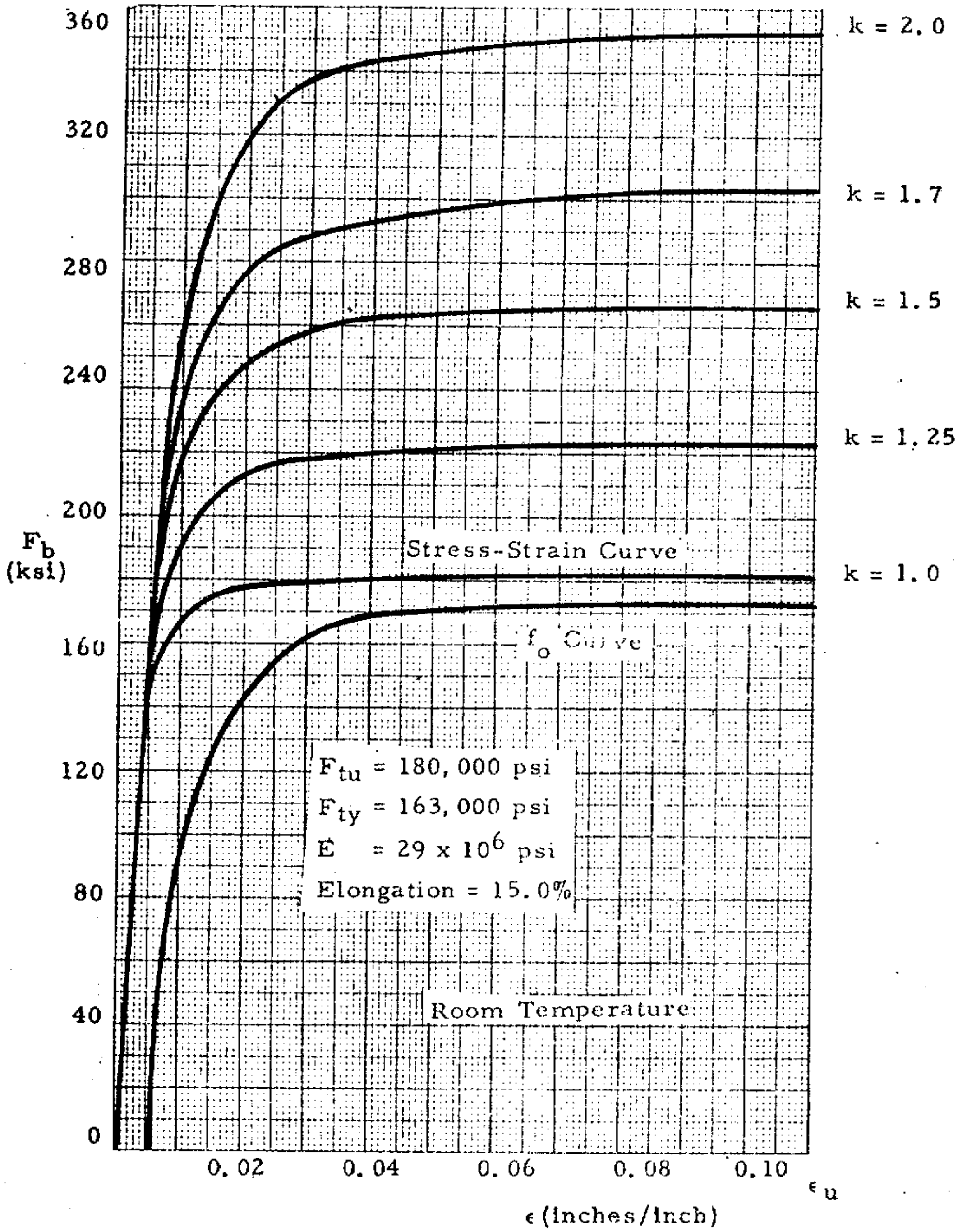
i. AISI Alloy Steel, Heat Treated

FIGURE 9.18 - MINIMUM PLASTIC BENDING CURVES FOR LOW CARBON AND ALLOY STEELS



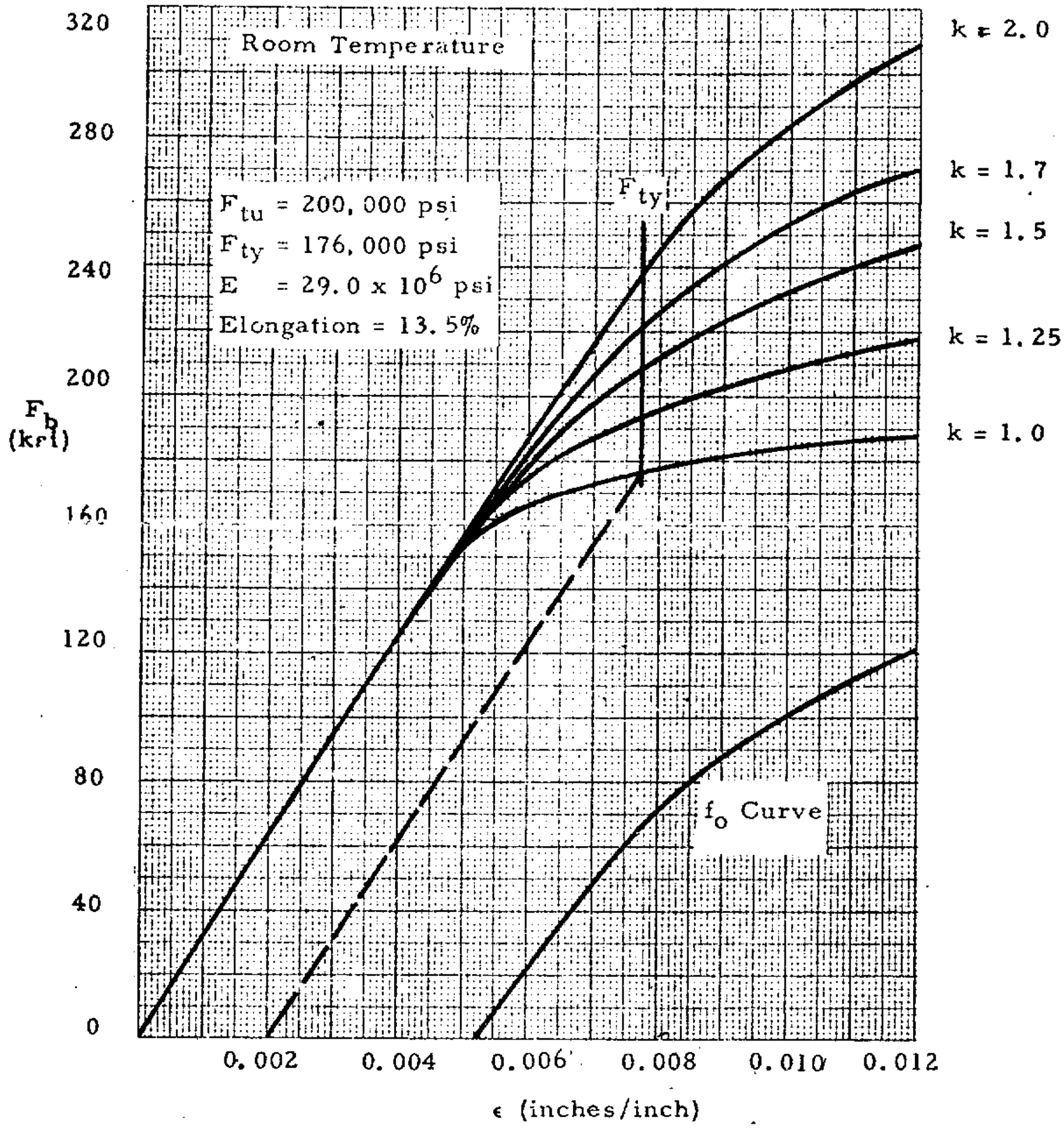
j. AISI Alloy Steel, Heat Treated

FIGURE 9.16 - MINIMUM PLASTIC BENDING CURVES FOR LOW CARBON AND ALLOY STEELS



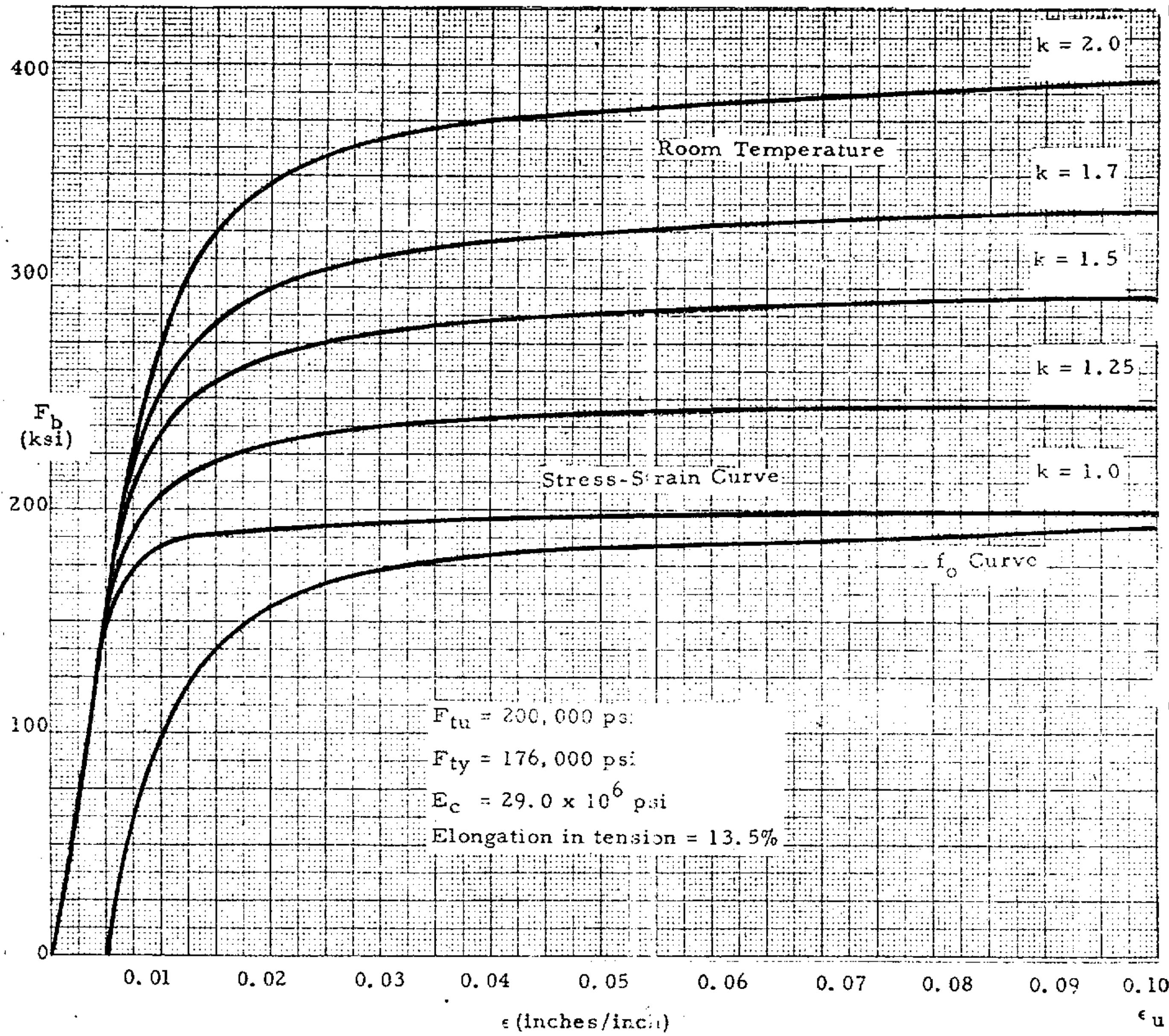
k. AISI Alloy Steel, Heat Treated

FIGURE 9.18 - MINIMUM PLASTIC BENDING CURVES FOR LOW CARBON AND ALLOY STEELS



1. AISI Alloy Steel, Heat Treated

FIGURE 9.18 - MINIMUM PLASTIC BENDING CURVES FOR LOW CARBON AND ALLOY STEELS

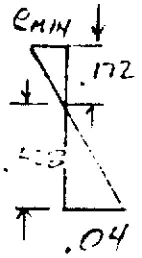


m. AISI Alloy Steel, Heat Treated

FIGURE 9.18 - MINIMUM PLASTIC BENDING CURVES FOR LOW CARBON AND ALLOY STEELS

CRITICAL AREAS A 356 INVEST. CASTIN
(SAME AS MODEL 406)

$F_{TU} = 38$ KSI
$F_{TY} = 26$ KSI
$e_{MAX} = .04$ IN
$E = 10400$ KSI



$e_{MIN} = \frac{.172}{.428}(.04) = .01607$

$e_{MAX} = .04$

$$e_u' = .04 - \frac{38}{10,400} = .036346$$

$$m = \frac{\log \frac{.036346}{.002}}{\log \frac{38}{26}} = 7.6417$$

$$K = \frac{38}{.04(10400)} = .09135$$

$$f_o/f = \frac{(m-1)}{(m+.5)}(1-K)\left[1+K\left(\frac{m-1}{m+2}\right)\right]$$

$$= \frac{(6.6417)}{(8.1417)}(1-.09135)\left[1+.09135\frac{(6.6417)}{(9.6417)}\right] = .7879$$

$$f_o = .7879(38) = 29.940$$

$e_{MIN} = .01607$

$$e_u' = .01607 - \frac{38}{10,400} = .012416$$

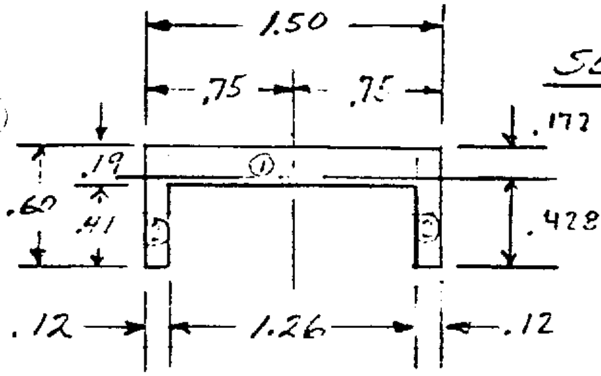
$$m = \frac{\log \frac{.012416}{.002}}{\log \frac{38}{26}} = 4.8113$$

$$K = \frac{38}{.01607(10,400)} = .22737$$

$$f_o/f = \left(\frac{3.8113}{5.3113}\right)(1-.22737)\left[1+.22737\left(\frac{3.8113}{6.8113}\right)\right] = .6250$$

$$f_o = .6250(38) = 23.750$$

SECTION @ FWD. BOLTS A-B.

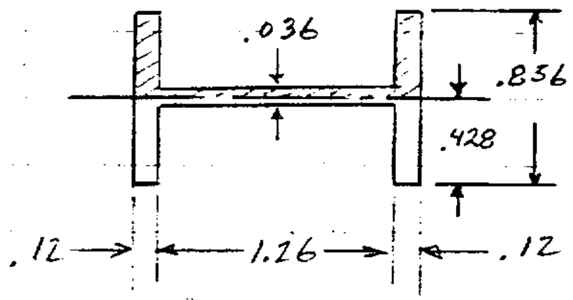


	A	Y	AY	AY²	I_o
① 1.26 × .19 =	.2394	.095	.022743	.002161	.000720
② 2 × .12 × .60 =	.1440	.300	.043200	.012960	.004320
	.3834		.065943	.015121	.005040

$\bar{Y} = \frac{.065943}{.3834} = .17200$

.600
.172
.428

.005040
.015121
.020161
$-(.3834)(.17200)^2 = -$ .011343
.008818



$I = \frac{1}{12}(.24)(.856)^3 + \frac{1}{12}(1.26)(.036)^3$

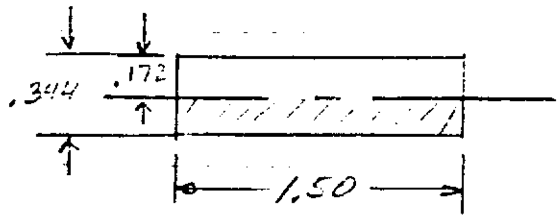
$= .012544 + .000005 = .012549$

$I/c = \frac{.012549}{.428} = .02932$

$Q = (.428)(.24)(\frac{.428}{2}) + (1.26)(.018)(\frac{.018}{2})$

$= .021982 + .000204 = .022186$

$k = 2Q\frac{c}{I} = \frac{(2)(.022186)}{.02932} = 1.513$

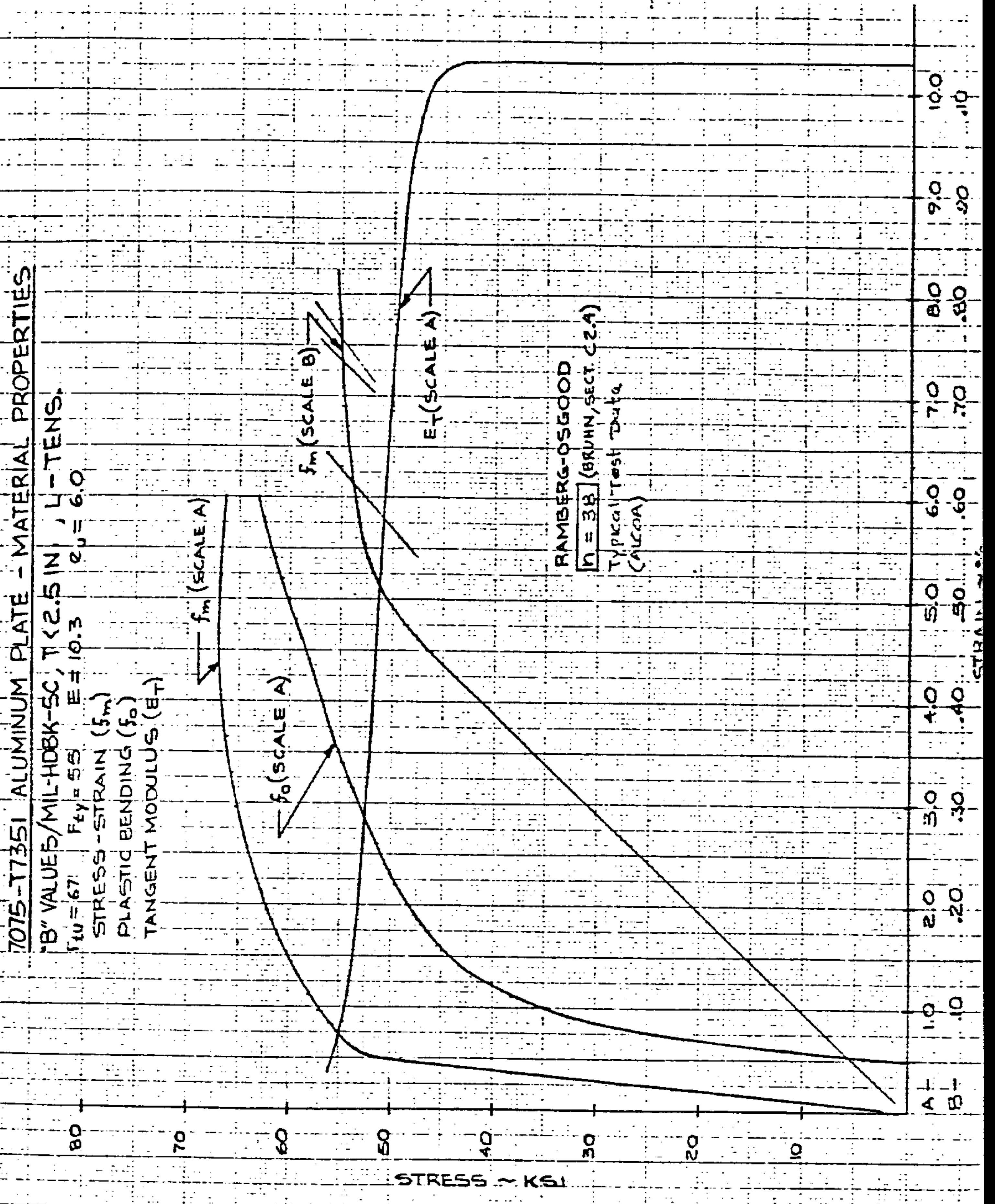


$I = \frac{1}{12}(1.50)(.344)^3 = .005088$

$I/c = \frac{.005088}{.172} = .02958$

$Q = (1.50)(.172)(\frac{.172}{2}) = .022188$

$k = \frac{(2)(.022188)}{.02958} = 1.500$


7075-T7351 ALUMINUM PLATE - MATERIAL PROPERTIES
"B" VALUES/MIL-HDBK-5C, T<2.5 IN, L-TENS.
Ftu = 67 Fty = 55 E = 10.3 eu = 6.0
STRESS-STRAIN (fm)
PLASTIC BENDING (fo)
TANGENT MODULUS (ET)
fm (SCALE A)
fo (SCALE A)
fm (SCALE B)
ET (SCALE A)
RAMBERG-OSGOOD
n = 38 (BRUHN, SECT. C2.4)
Typical Test Data (ALCOA)
STRESS ~ KSI
STRAIN ~ %
A
B
80
70
60
50
40
30
20
10
1.0
.10
2.0
.20
3.0
.30
4.0
.40
5.0
.50
6.0
.60
7.0
.70
8.0
.80
9.0
.90
10.0
1.0

9.6.2 Bending in A Plane of Symmetry

Equation 9.16 is applicable to sections with bending about an axis of symmetry. For bending in a plane of symmetry, the neutral axis is located as a line perpendicular to this plane which divides the total area into two equal areas. The static moments, Q_1 and Q_2 of the two equal areas of the cross section with respect to the neutral axis are determined.

The value of k for unsymmetric sections is

$$k = (Q_1 + Q_2)\, c/I \qquad 9.18$$

where c is the largest value of c. I and c are computed with respect to the centroidal axis.

9.6.3 Complex Bending

First consider the complex bending problem encountered when a section has two axes of symmetry, but the bending moment vector is not parallel to either. Denoting the axes of symmetry by x and y, the allowable moments M_x and M_y are determined by the methods described in Section 9.6.1. Using equation 9.16 to determine F_b about both the x and y axes of the section

$$M_x = F_{bx}\, I_x/c_y \qquad 9.19$$

$$M_y = F_{by}\, I_y/c_x \qquad 9.20$$

The external bending moment is resolved into components m_x and m_y about the two axes. The condition of failure is defined by the interaction equation

$$R_x^{\,n} + R_y^{\,n} = 1 \qquad 9.21$$

where

R_x = moment ratio for x axis = m_x/M_x

R_y = moment ratio for y axis = m_y/M_y

n = 2.0 - for circular sections

n = 1.7 - for rectangular sections

n = 1.5 - for all other sections

This interaction equation is plotted for all three values in Figure 4.10 in Section 4. The margin of safety is found by

1. Plot the point R_x, R_y on Figure 4.10
2. Draw a line through the point R_x, R_y to the proper interaction curve
3. Determine the abscissa R_{xa} of the intersection with the interaction curve

4. Compute the margin of safety

$$MS = R_{xa}/R_x - 1 \qquad 9.22$$

For complex bending of sections with a single axis of symmetry the procedure is identical to that described once the reference axes have been located. If the x-axis is the axis of symmetry, then the y axis is located as described in Section 9.6.2. M_x is then determined from equation 9.16 and becomes the same as equation 9.19.

M_y is determined from equation 9.16 using the k value determined from equation 9.18. The applied moment, m, is resolved into m_x and m_y components. The M.S. is determined as previously described, using equation 9.21 (n = 1.5) and equation 9.22.

For sections with no axis of symmetry, a different process is necessary. Any convenient set of orthogonal reference axes x and y is chosen. Various positions of the neutral axis are determined to satisfy the requirement that this axis must divide the cross section into two equal parts . The correct position for the neutral axis is determined by the requirement that the ratio of the allowable moment about the x-axis to that about the y-axis must be the same as the corresponding ratio for the components of the external moment with respect to these axes. This requirement is defined in equation 9.23.

$$\frac{m_x}{m_y} = \frac{M_x}{M_y} = \frac{F_{bx} I_x c_x}{F_{by} I_y c_y} = \frac{\overline{dx}}{\overline{dy}} \qquad 9.23$$

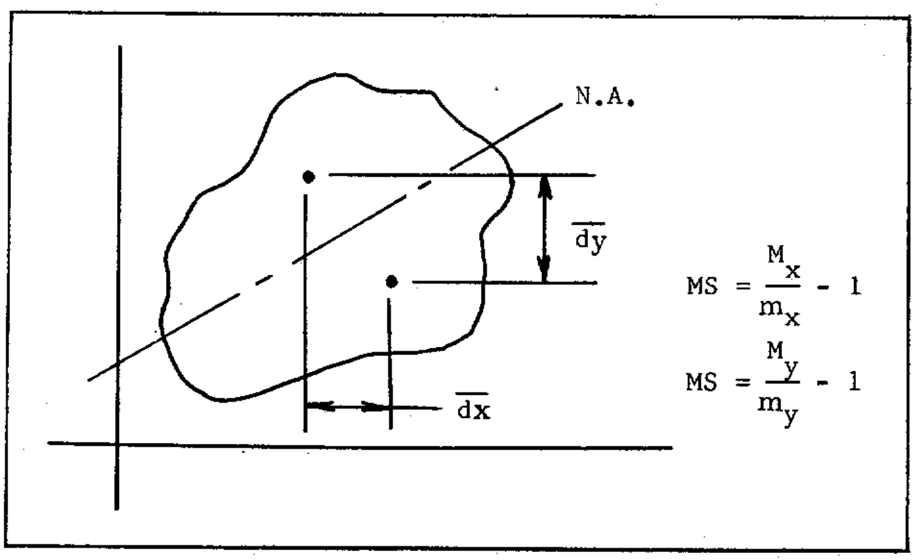


FIGURE 9.19 NEUTRAL AXIS LOCATION - COMPLEX BENDING

9.6.4 Evaluation of Intercept Stress, f_o

The preceding method of plastic bending analysis has made use of the intercept stress, f_o. It has been shown that this stress is determined from the material stress-strain curve. Often stress-strain curves are not available for the newer materials, so two methods will be shown for the determination of f_o, one when the material stress-strain curve is known and one for when it is not known, both by the method developed by G. L. Hunt and C. A. Traylor.

To begin, assume a stress-strain curve is not known. Assume that the material ultimate and yield strengths (F_{tu} and F_{ty}), ultimate strain (e_u) and elastic modulus of elasticity (E) are all that is known of the material. Now assume that the stress-strain curve of the material can be represented by a trapezoid using the four known quantities. Figure 9.20(a) shows this type of stress-strain curve.

The point F_i is the intersection of the two lines formed by the four known quantities. Using the terms defined in Figure 9.20(a)

$$F_{ty}/(e_y - .002) = E$$

$$e_y = F_{ty}/E + .002$$

$$e_i = F_i/E \qquad 9.24$$

$$m_u = (F_{tu} - F_i)/(e_u - e_i)$$

$$F_i = - m_u (e_u - e_i) + F_{tu}$$

$$F_i = - m_u (e_u - F_i/E) + F_{tu}$$

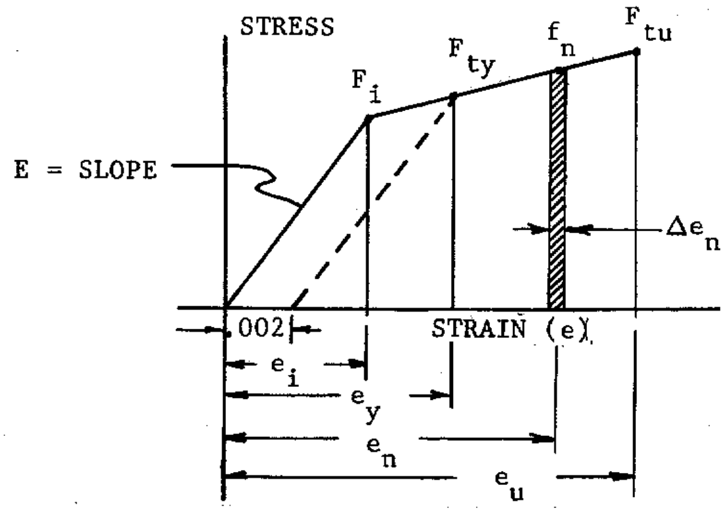
$$F_i = - m_u e_u + m_u F_i/E + F_{tu}$$

$$F_i - m_u F_i/E = -m_u e_u + F_{tu}$$

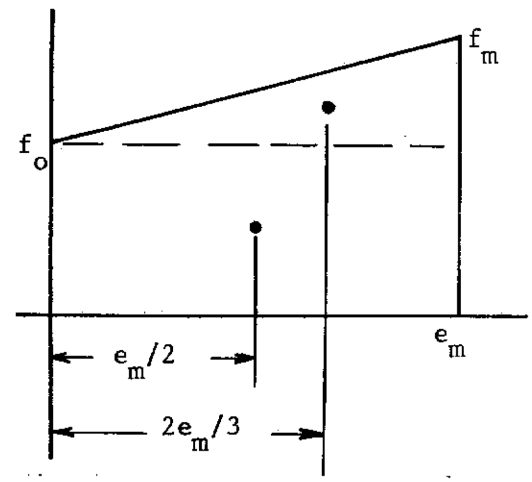
$$F_i = (F_{tu} - m_u e_u)/(1 - m_u/E)$$

but $$m_u = (F_{tu} - F_{ty})/(e_u - e_y)$$

then $$F_i = \frac{E(e_u F_{ty} - e_y F_{tu})}{E(e_u - e_y) - F_{tu} + F_{ty}} \qquad 9.25$$



(a) STRESS-STRAIN CURVE



(b) ASSUMED STRESS DISTRIBUTION

FIGURE 9.20 - STRESS DISTRIBUTIONS

Now, having equations 9.24 and 9.25 (e_i and F_i) the area under the assumed stress-strain curve can be evaluated. Proceed by dividing the assumed curve into any number of segments (a minimum of 10 is recommended). The area under the curve is

$$A = \sum_{o}^{n} f_n \Delta e_n \qquad 9.26$$

where the distance to any element, n, is e_n.

The intercept stress, f_o, may be determined by making the internal moment of the stress-strain distribution equal the moment of the assumed stress distribution. This is shown in Figure 9.8 and in Figure 9.20(b). Referring to Figure 9.20(b) the following relationships can be established:

$$A = f_o\, e_m + (f_m - f_o)\, e_m/2 \qquad 9.27$$

Now equating the moment of the equivalent stress distribution to the moment of the stress-strain distribution, gives

$$\sum_{o}^{n} f_n\, \Delta e_n\, (e_n) = f_o\, e_m\, (e_m/2) + (f_m - f_o)(e_m/2)(2\, e_m/3)$$

$$= f_o e_m^2/2 + 2\, f_m\, e_m^2/6 - 2\, f_o\, e_m^2/6$$

$$= f_o e_m^2/6 + 2\, f_m\, e_m^2/6$$

$$= (e_m^2/6)(2\, f_m + f_o)$$

Solving this equation for f_o yields:

$$2\, f_m + f_o = (6/e_m^2)\left[\sum_{o}^{n} f_n \Delta e_n\, (e_n)\right]$$

$$f_o = (6/e_m^2)\left[\sum_{o}^{n} f_n \Delta e_n\, (e_n)\right] - 2\, f_m \qquad 9.28$$

Equation 9.28 can be used to determine the intercept stress, f_o, from any stress-strain curve regardless of its shape. Therefore, f_o can be determined when no stress-strain curve is known from test results, provided an assumed stress-strain curve can be established as was done in Figure 9.20 (a).

Figure 9.21 shows an example of the determination of f_o. It should be noted that in the preceding procedure the value of e_m for the first segment should be the strain corresponding to the material proportional limit. This will result in $f_o = 0$ for the first point if f_n is the average stress.

7075-T73 Aluminum Alloy Hand Forging, MIL-A-22771

$t \leq 3.00$ in.

$F_{tu} = 66$ KSI

$F_{ty} = 56$ KSI

$E = 10^4$ KSI

$e_u = 7\%$

$$e_y = F_{ty}/E + .002 = \frac{56000}{10^7} + .002 = .0076$$

$$m_u = \frac{F_{tu} - F_{ty}}{e_u - e_y} = \frac{66000 - 56000}{.070 - .0076} = 160256 \text{ psi}$$

$$F_i = \frac{F_{tu} - m_u e_y}{1 - m_u/E} = \frac{66000 - 160256\ (.070)}{1 - 160256/10^7} = 55700 \text{ psi}$$

$$e_i = F_i/E = \frac{55700}{10^7} = .0056$$

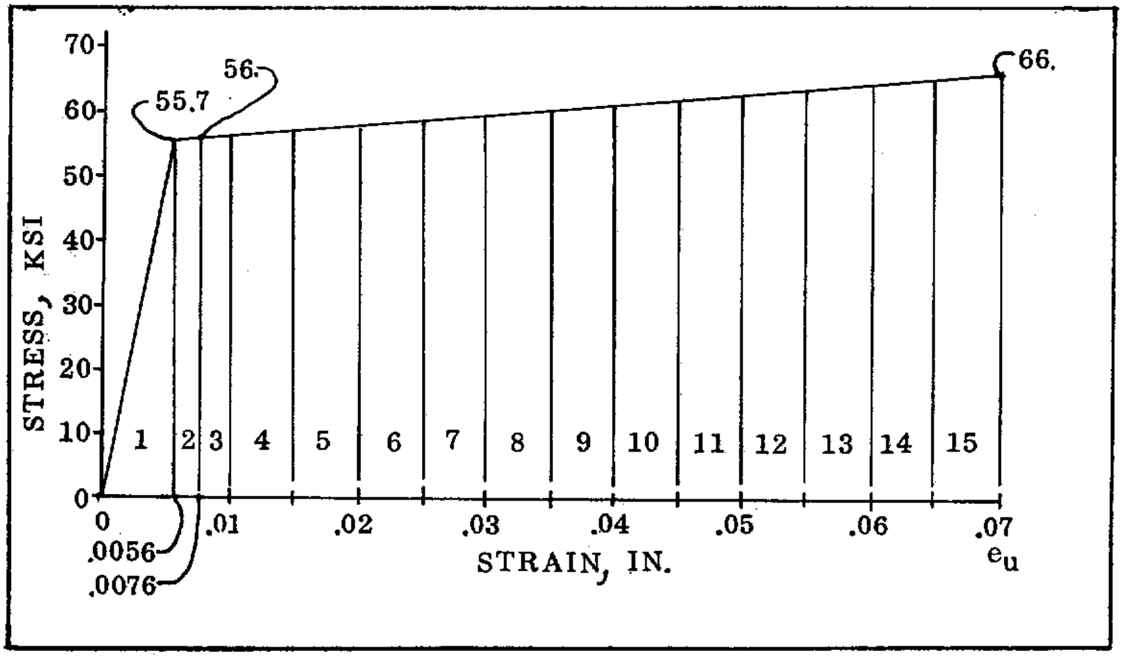


FIGURE 9.21 - SAMPLE CALCULATION OF INTERCEPT STRESS, f_o.

	①	②	③	④	⑤	⑥	⑦	⑧	⑨
n	e_n	Δe_n	f_n $x(10^3)$	$f_n e_n \Delta e_n$	6Σ④	e_m	⑤/e_m^2 $x(10^3)$	f_m $x(10^3)$	f_o $x(10^3)$
1	.0037*	.0056	27.85	.58	3.5	.0056	111.4	55.70	0
2	.0066	.0020	55.86	.74	7.9	.0076	137.1	56.02	25.06
3	.0088	.0024	56.21	1.19	15.0	.010	150.4	56.41	37.58
4	.0125	.005	56.81	3.55	36.3	.015	161.5	57.21	47.11
5	.0175	.005	57.61	5.04	66.6	.020	166.5	58.00	50.47
6	.0225	.005	58.41	6.57	106.0	.025	169.6	58.81	52.01
7	.0275	.005	59.21	8.14	154.9	.030	172.1	59.61	52.85
8	.0325	.005	60.01	9.75	213.4	.035	174.2	60.41	53.36
9	.0375	.005	60.81	11.40	281.8	.040	176.1	61.21	53.70
10	.0425	.005	61.61	13.09	360.3	.045	177.9	62.01	53.92
11	.0475	.005	62.41	14.82	449.3	.050	179.7	62.82	54.07
12	.0525	.005	63.22	16.60	548.8	.055	181.4	63.62	54.20
13	.0575	.005	64.02	18.41	659.3	.060	183.1	64.42	54.29
14	.0625	.005	64.82	20.26	780.8	.065	184.8	65.22	54.37
15	.0675	.005	65.62	22.15	913.7	.070	186.5	66.02	54.43

* e_1 = .0056 (2/3) = .0037333 . . .

FIGURE 9.21 (Cont'd) - SAMPLE CALCULATION OF INTERCEPT STRESS, f_o.

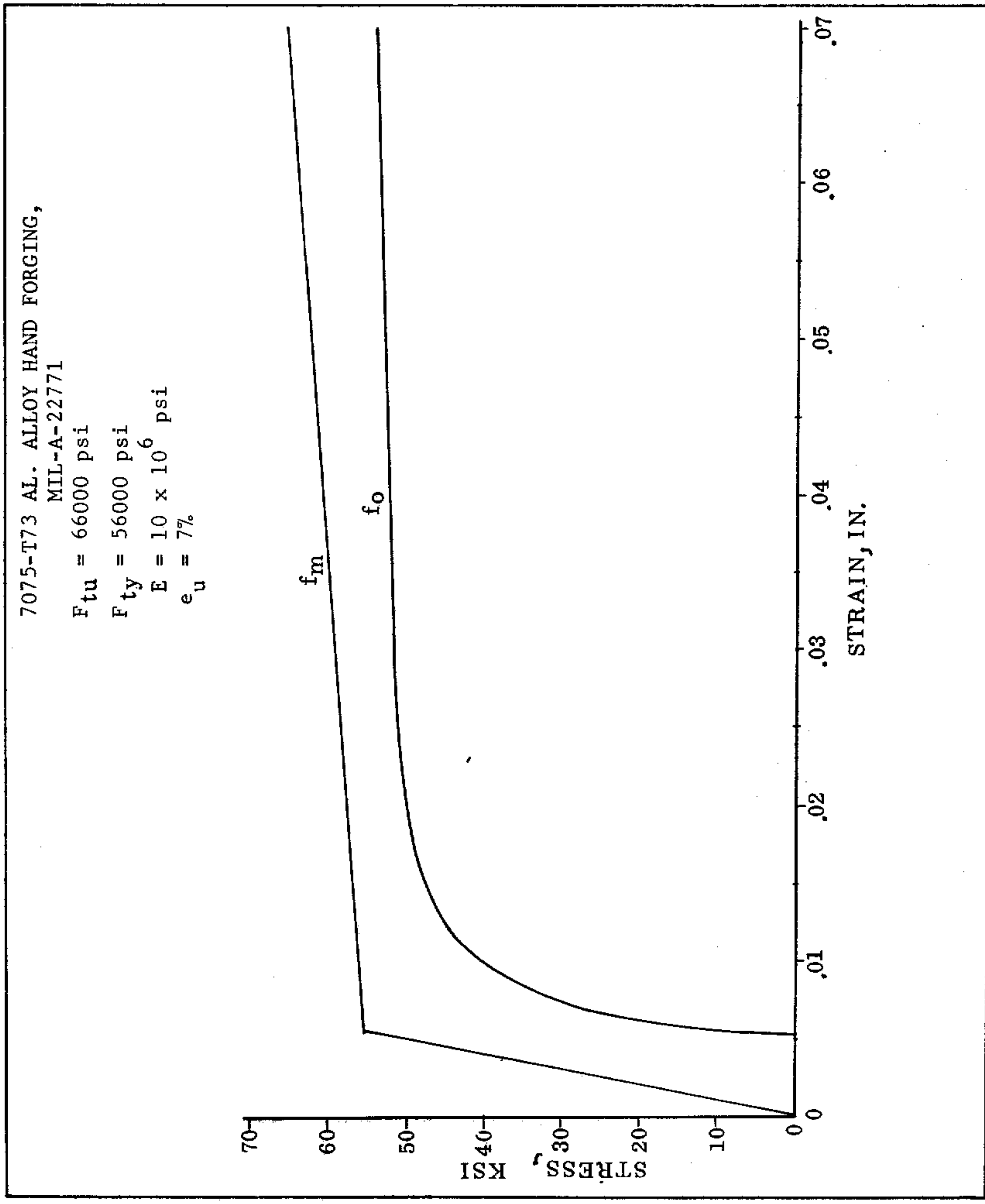


FIGURE 9.21 (Cont'd) - SAMPLE CALCULATIONS OF INTERCEPT STRESS, f_o.

9.6.5 Plastic Bending Modulus, F_b

Figures 9.15 through 9.18 show curves for various materials. The curves are plotted as $k = 2\ Qc/I$ versus F_b and strain. The strain versus F_b curves show f_o and F_b versus strain. The f_o curve is at k=1. The rest of the curves employ equation 9.16 to obtain F_b^o at various strains.

9.6.6 Application of Plastic Bending

Consider a rectangular beam section which is .25-inch thick and 1.5-inch deep. It is made of 7075-T6 extrusion and it is desired to find the yield and ultimate bending moment for the section.

$$F_b = f_m + f_o\ (k-1),\ \text{equation 9.16}$$

$$k = 2Qc/I = 2\big((.25)(.75)(.375)\big)(.75)/\big((.25)(1.5)^3/12\big) = 1.5$$

The value of k can also be found in Figure 9 14.

$$F_{tu} = 75000\ \text{psi},\ F_{ty} = 65000\ \text{psi}$$

Find the yield bending strength: The value of f_m in equation 9.16, the maximum stress permitted on the most remote fiber, is 65000 psi, the yield stress of the material. To find f_o, go to Figure 9.16 (r) to find the point on the stress-strain curve (k=1) that corresponds to a stress of 65000 psi. This point is projected downward to the f_o curve where a stress of 26000 psi is read. Then

$$F_b = 65000 + 26000\ (1.5 - 1) = 78{,}000\ \text{psi}$$

This same stress can be obtained by projecting up from the stress-strain curve in Figure 9.16 (r) to the curve labeled k=1.5 and reading F_b directly.

The yield moment is then found to be

$$M_y = F_b\ I/c = 78000\ (.0703)/.75 = 7312.5$$

The ultimate moment is found the same way.

$$F_b = 75000 + 70500\ (1.5 - 1) = 110{,}250\ \text{psi}$$

$$M_u = 110{,}250\ (.0703)/.75 = 10334$$

The previous example is for a section which is stable in compression and symmetrical about two axes. Consider now a section which is symmetrical about one axis and probably partially unstable. The Tee shown in Figure 9.22(a) is a

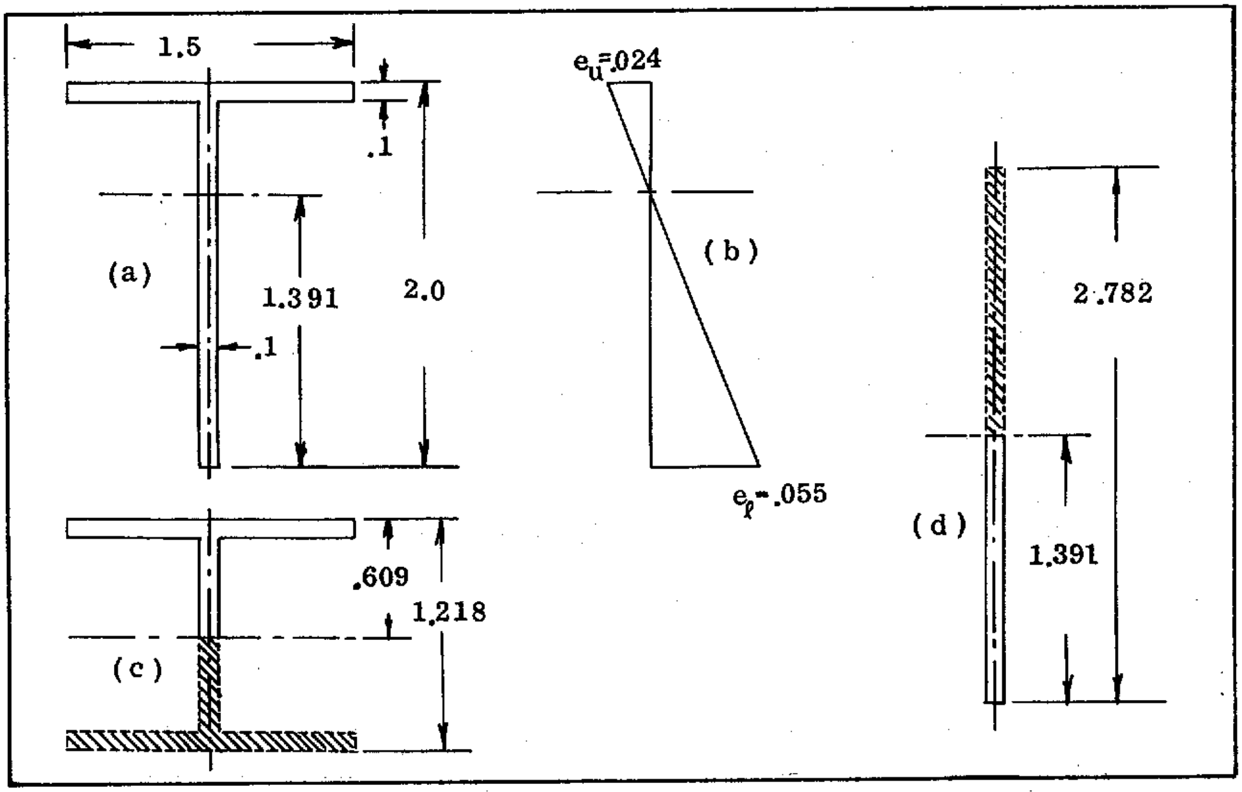


FIGURE 9.22 - UNSYMMETRICAL EXAMPLE

7075-T6 extrusion. Again, the properties of Figure 9.16(r) are used.

First consider the maximum strain, e_u, in Figure 9.16(r), e_u = .055in/in. It is apparent that the lower leg of the Tee will strain higher than the cap when the Tee is bent about the x axis, so set the lower extreme fiber strain equal 0.055. By ratioing the lower strain by the distances from the N.A. the strain in the upper extreme fiber is $e_u = (.609/1.391)(.055) = .024$ in/in.

Equation 9.16 was derived for symmetrical sections about a neutral axis. The Tee can be made into two sections which are symmetrical about their neutral axis. These are shown in Figure 9.22(c) and (d).

Figure 9.22(d) shows how the lower portion is made symmetrical about the neutral axis by adding the shaded portion above. The internal bending resistance is found for the entire section in 9.22(d). One-half of this amount will be the true moment developed by the lower portion.

$I = (.1)(2.782)^3/12 = .179$

$I/c = .179/1.391 = .129$
$k = 2\,Qc/I = 2(1.391)(.1)(.6955)/.129 = 1.5$

From Figure 9.16(r) at $e = .055$, $F_b = 110{,}000$ psi
$M = F_b\,(I/c)(1/2) = 7095$ in-lb

The 1/2 is because only one-half the beam section is used in Figure 9.22(a).

Figure 9.22(c) shows the upper half of the beam made symmetrical about the neutral axis by adding the shaded section.

$$I = (1/12)\,(1.5)\,(1.218)^3 - (1/12)(1.4)(1.018)^3 = .226 - .123 = .103$$

$$I/c = .103/.609 = .169$$

$$k = 2\,Q\,c/I = 2\,\left[(.509)(.1)(.2545) + (1.5)(.1)(.559)\right]/.169 = 1.146$$

From Figure 9.16 (r) at e = .024, F_b = 82,000 psi

$$M = F_b\,(I/c)(1/2) = 6929$$

The total ultimate resisting moment is the summation of the two moments:

$$M_{TOT} = 7095 + 6929 = 14024$$

It might also be desirable to limit some portion of the section because of crippling or stability. That would be done after calculating F_b for the section. If F_b was higher than the critical crippling or stability stress, then F_b would be set equal the lower stress. This generally occurs when the yield modulus is being calculated.

9.7 CURVED BEAMS CORRECTION FACTORS FOR USE IN STRAIGHT-BEAM FORMULA

When a curved beam is bent in the plane of initial curvature, plane sections remain plane, but the strains of the fibers are not proportional to the distance from the neutral axis because the fibers are not at equal length. If (K) denotes a correction factor, the stress at the extreme fiber of a curved beam is given by

$$f = K \frac{Mc}{I}$$

in which

$$K = \frac{\frac{M}{AR}\left[1 + \frac{c}{Z(R + c)}\right]}{\frac{Mc}{I}}$$

where M is the bending moment
A is the cross-sectional area
R is the radius of curvature to the centroidal axis
c is the distance from the centroidal axis to the extreme outer fiber
I is the moment of inertia

$$Z = -\frac{1}{A}\int \frac{y}{R+y}\, dA$$

Values for K for different sections are given in Table 9.7

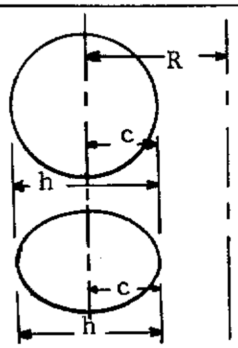
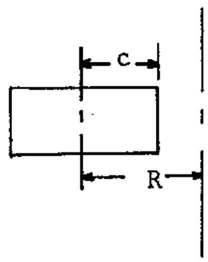
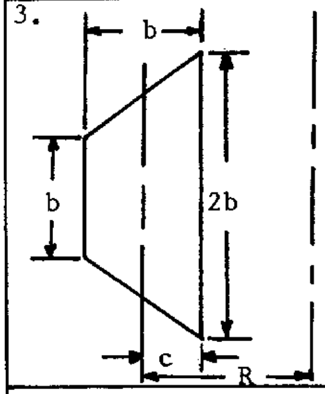
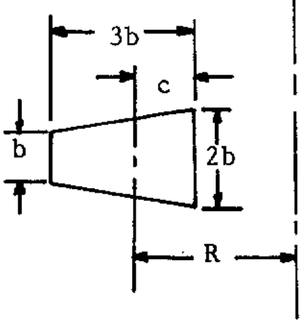
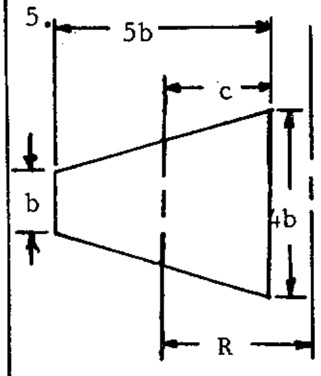
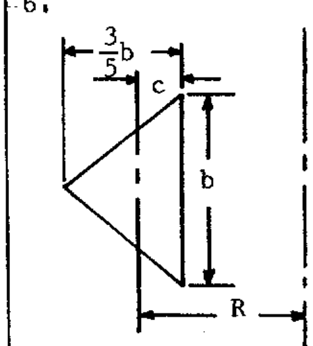
Section	R/c	Factor K Inside Fiber	Factor K Outside Fiber	Section	R/c	Factor K Inside Fiber	Factor K Outside Fiber
1. 	1.2	3.41	0.54	2. 	1.2	2.89	0.57
	1.4	2.40	0.60		1.4	2.13	0.63
	1.6	1.96	0.65		1.6	1.79	0.67
	1.8	1.75	0.68		1.8	1.63	0.70
	2.0	1.62	0.71		2.0	1.52	0.73
	3.0	1.33	0.79		3.0	1.30	0.81
	4.0	1.23	0.84		4.0	1.20	0.85
	6.0	1.14	0.89		6.0	1.12	0.90
	8.0	1.10	0.91		8.0	1.09	0.92
	10.0	1.08	0.93		10.0	1.07	0.94
K is the same for circular and elliptical sections; independent of dimensions.				K is independent of section dimensions.			

Table 9.7 - K VALUES FOR DIFFERENT SECTIONS AND RADII OF CURVATURE

Section	R/c	Factor K Inside Fiber	Factor K Outside Fiber
3. 	1.2	3.01	0.54
	1.4	2.18	0.60
	1.6	1.87	0.65
	1.8	1.69	0.68
	2.0	1.58	0.71
	3.0	1.33	0.80
	4.0	1.23	0.84
	6.0	1.13	0.88
	8.0	1.10	0.91
	10.0	1.08	0.93
4. 	1.2	3.09	0.56
	1.4	2.25	0.62
	1.6	1.91	0.66
	1.8	1.73	0.70
	2.0	1.61	0.73
	3.0	1.37	0.81
	4.0	1.26	0.86
	6.0	1.17	0.91
	8.0	1.13	0.94
	10.0	1.11	0.95
5. 	1.2	3.14	0.52
	1.4	2.29	0.54
	1.6	1.93	0.62
	1.8	1.74	0.65
	2.0	1.61	0.68
	3.0	1.34	0.76
	4.0	1.24	0.82
	6.0	1.15	0.87
	8.0	1.12	0.91
	10.0	1.10	0.93
6. 	1.2	3.26	0.44
	1.4	2.39	0.50
	1.6	1.99	0.54
	1.8	1.78	0.57
	2.0	1.66	0.60
	3.0	1.37	0.70
	4.0	1.27	0.75
	6.0	1.16	0.82
	8.0	1.12	0.86
	10.0	1.09	0.88

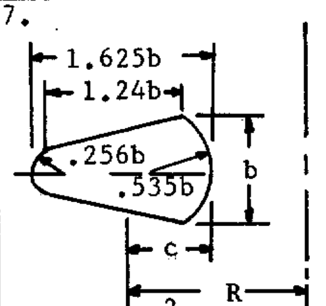
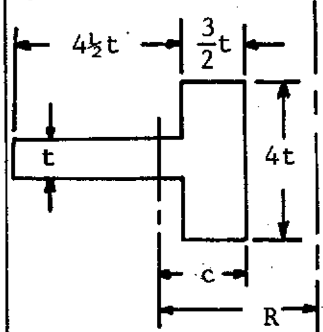
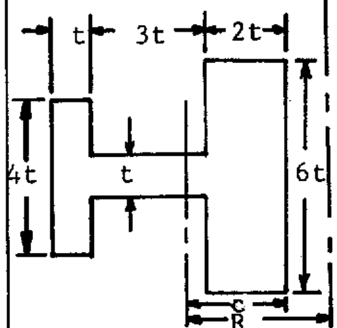
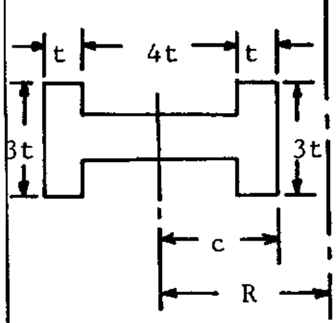
Section	R/c	Factor K Inside Fiber	Factor K Outside Fiber
7. 	1.2	3.65	0.53
	1.4	2.50	0.59
	1.6	2.08	0.63
	1.8	1.85	0.66
	2.0	1.69	0.69
	2.5	1.49	0.74
	3.0	1.38	0.78
	4.0	1.27	0.83
	6.0	1.19	0.90
$A = 1.05b^2$	8.0	1.14	0.93
$I = 0.18b^4$ $c = 0.70b$	10.0	1.12	0.96
8. 	1.2	3.63	0.58
	1.4	2.54	0.65
	1.6	2.14	0.67
	1.8	1.89	0.70
	2.0	1.73	0.72
	3.0	1.41	0.79
	4.0	1.29	0.83
	6.0	1.18	0.88
	8.0	1.13	0.91
	10.0	1.10	0.92
9. 	1.2	3.55	0.67
	1.4	2.48	0.72
	1.6	2.07	0.76
	1.8	1.83	0.78
	2.0	1.69	0.80
	3.0	1.38	0.86
	4.0	1.26	0.89
	6.0	1.15	0.92
	8.0	1.10	0.94
	10.0	1.08	0.95
10. 	1.2	2.52	0.67
	1.4	1.90	0.71
	1.6	1.63	0.75
	1.8	1.50	0.77
	2.0	1.41	0.79
	3.0	1.23	0.86
	4.0	1.16	0.89
	6.0	1.10	0.92
	8.0	1.07	0.94
	10.0	1.05	0.95

Table 9.7 (Cont'd) K VALUES FOR DIFFERENT SECTIONS AND RADII OF CURVATURE

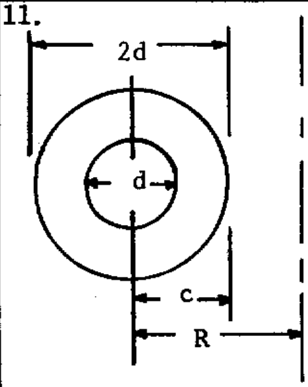
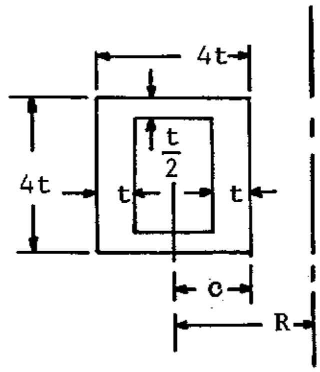
Section	R/c	Factor K Inside Fiber	Factor K Outside Fiber	Section	R/c	Factor K Inside Fiber	Factor K Outside Fiber
11. 	1.2	3.28	0.58		1.2	2.63	0.68
	1.4	2.31	0.64		1.4	1.97	0.73
	1.6	1.89	0.68		1.6	1.66	0.76
	1.8	1.70	0.71		1.8	1.51	0.78
	2.0	1.57	0.73		2.0	1.43	0.80
	3.0	1.31	0.81		3.0	1.23	0.86
	4.0	1.21	0.85		4.0	1.15	0.89
	6.0	1.13	0.90		6.0	1.09	0.92
	8.0	1.10	0.92		8.0	1.07	0.94
	10.0	1.07	0.93		10.0	1.06	0.95

Table 9.7 (CONT'D) K VALUES FOR DIFFERENT SECTIONS AND RADII OF CURVATURE

9.8 BOLT-SPACER COMBINATIONS SUBJECTED TO BENDING

When bolts and spacers are subjected to bending, the allowable may be calculated in the usual manner. However, consideration should be given to the preload induced by the nut. Figure 9.23 shows the neutral axis location of a bolt-spacer combination subjected to bending. Figure 9.24 shows the effect of preload on a bolt-spacer combination.

9.9 STANDARD BENDING SHAPES

Standard bending shapes, tubes and channels, which are subject to local crippling or crushing are presented in Figures 9.25 through 9.28 . These figures present the allowable bending moment for various materials and cross sections.

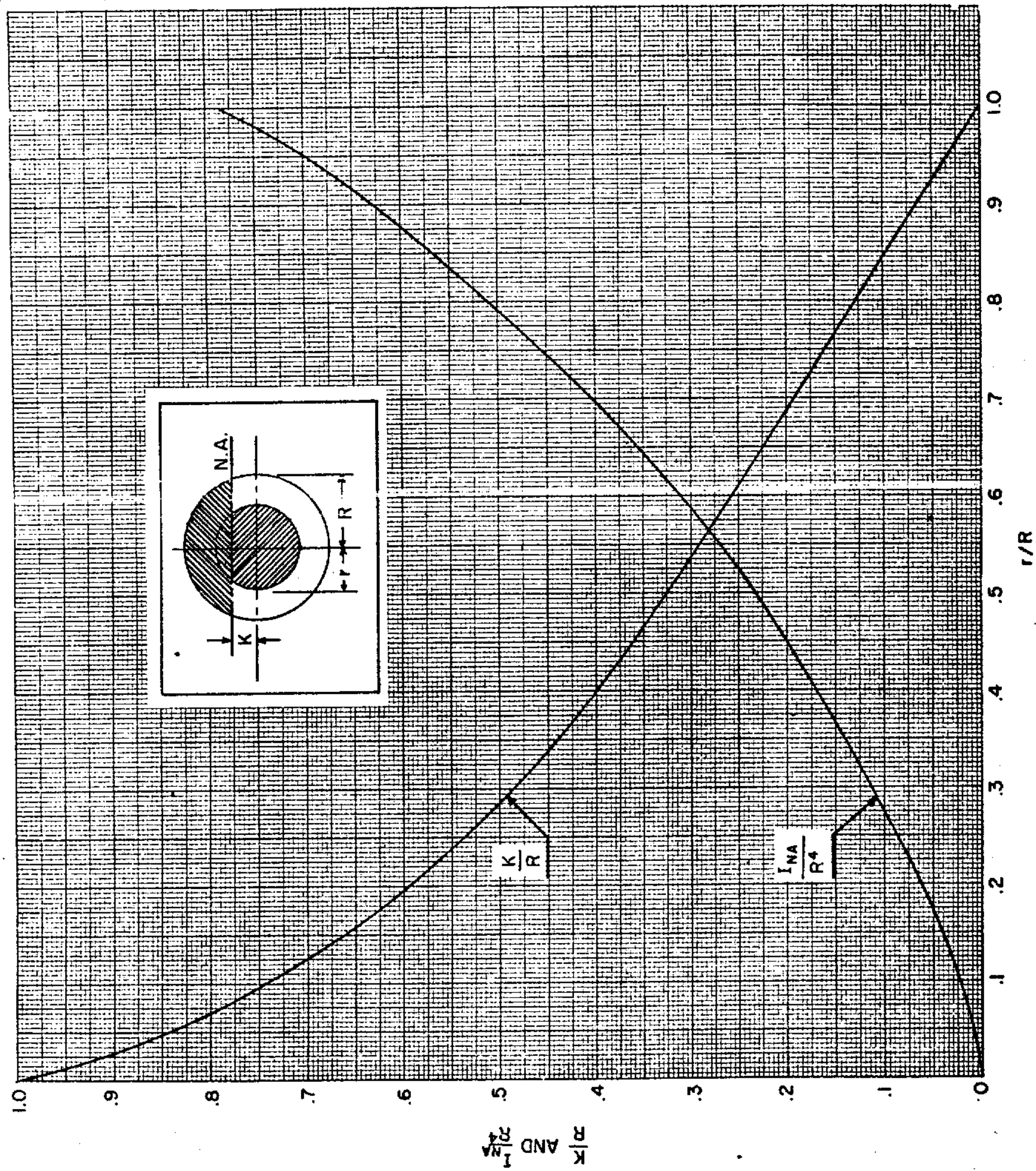


Figure 9.23 - Neutral Axis Location and Moment of Inertia of A Bolt-Spacer Combination Subjected to Bending.

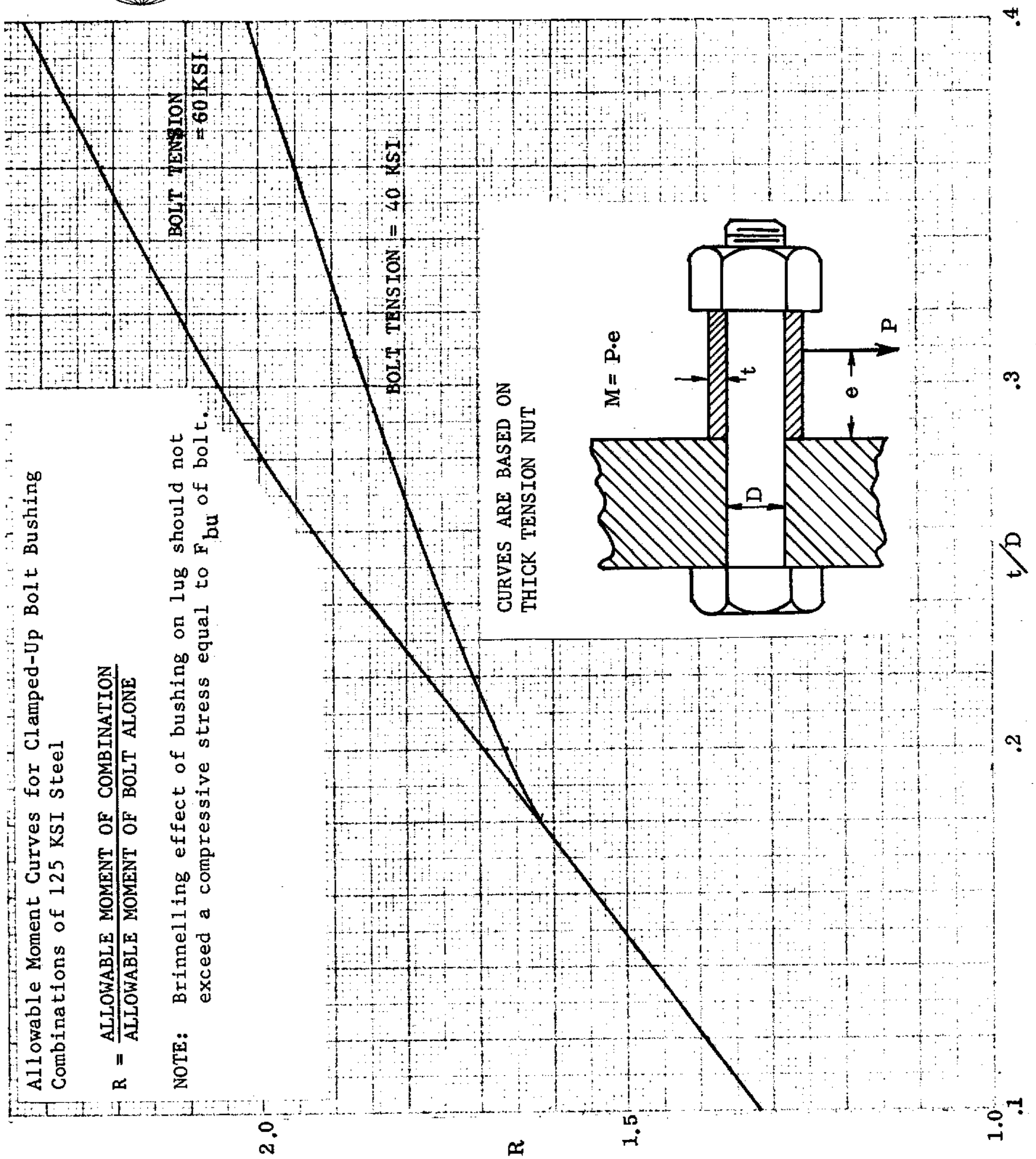
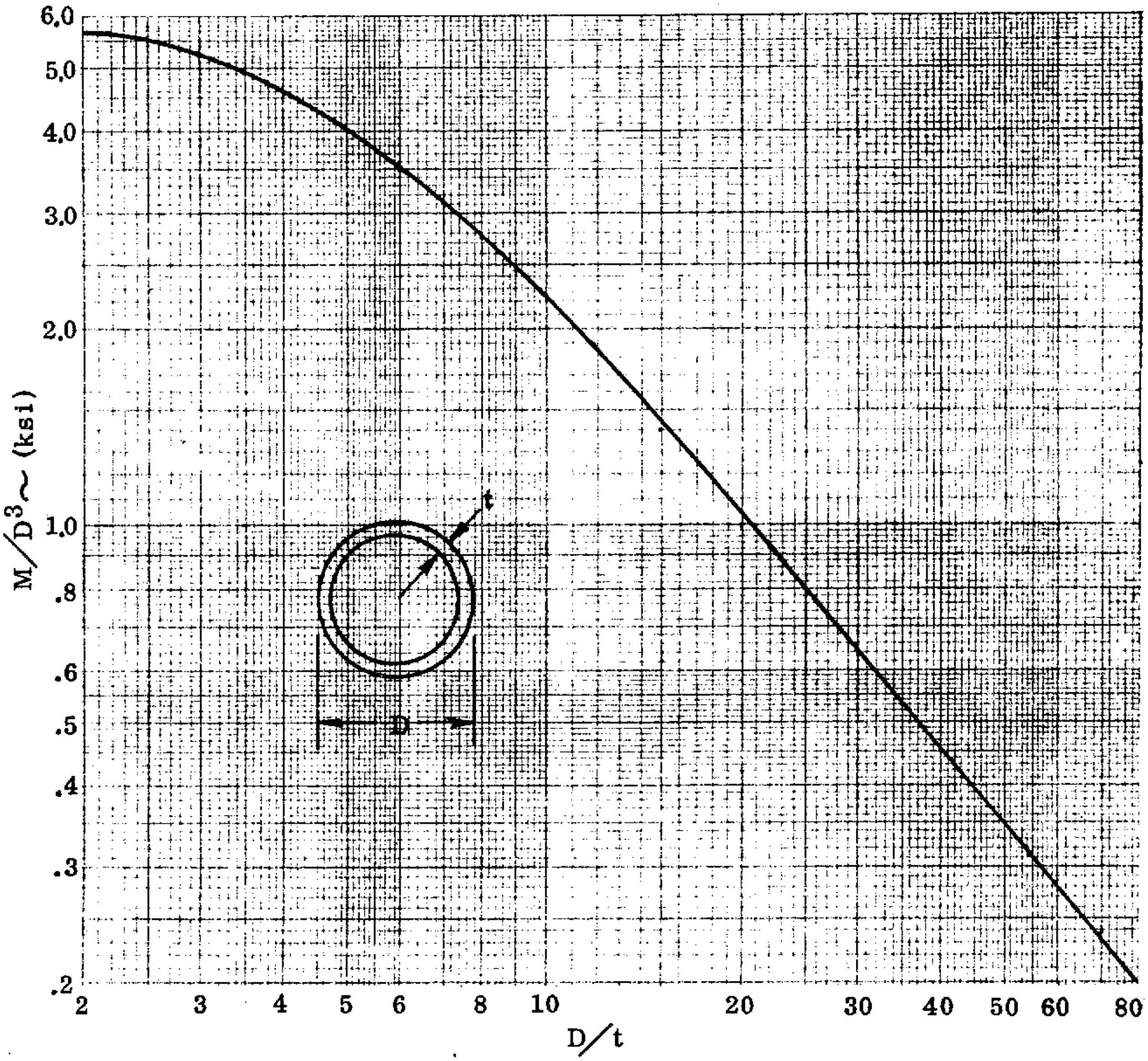
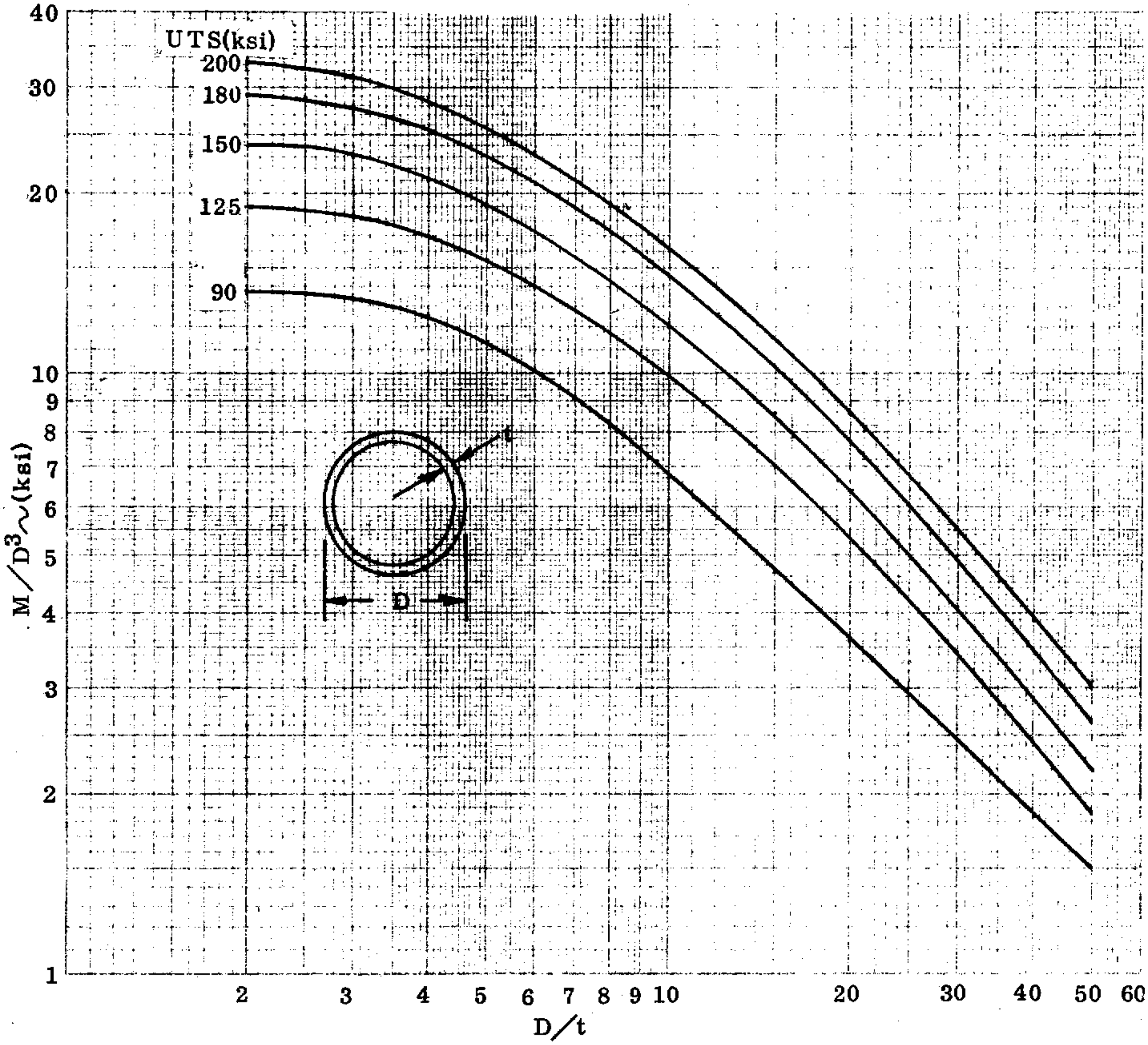


Figure 9.24 - Allowable Moment Curves for Clamped-Up Bolt-Bushing Combinations



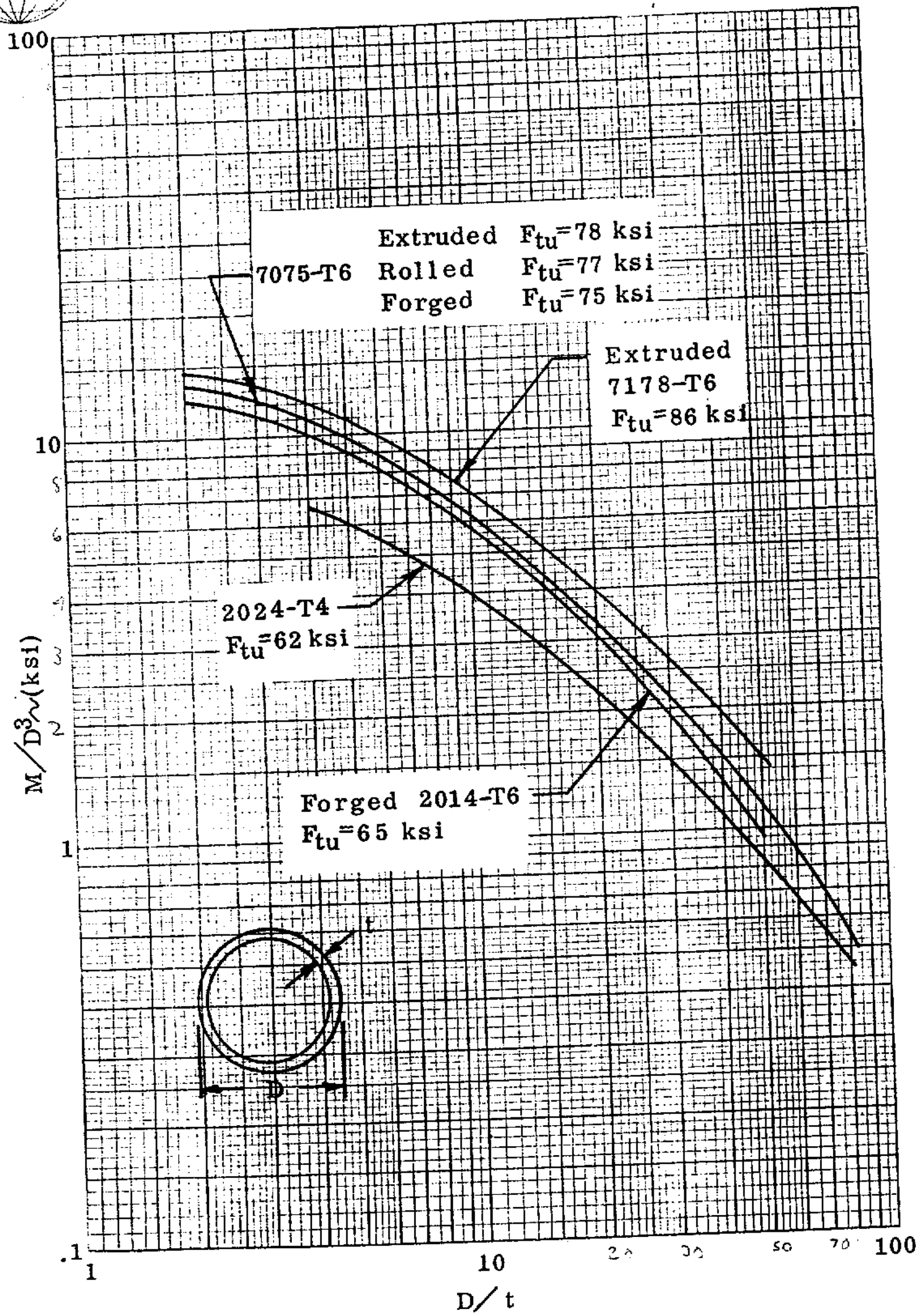
M = BENDING MOMENT (IN-LBS)

FIGURE 9.25 - BENDING ALLOWABLE FOR MAGNESIUM ALLOY TUBING, FS-1a.



M = BENDING MOMENT, IN-LBS

FIGURE 9.26 - BENDING ALLOWABLES FOR STEEL ALLOY TUBING .



M = BENDING MOMENT, IN-LBS

FIGURE 9.27 - BENDING ALLOWABLES FOR VARIOUS ALUMINUM ALLOY TUBING.

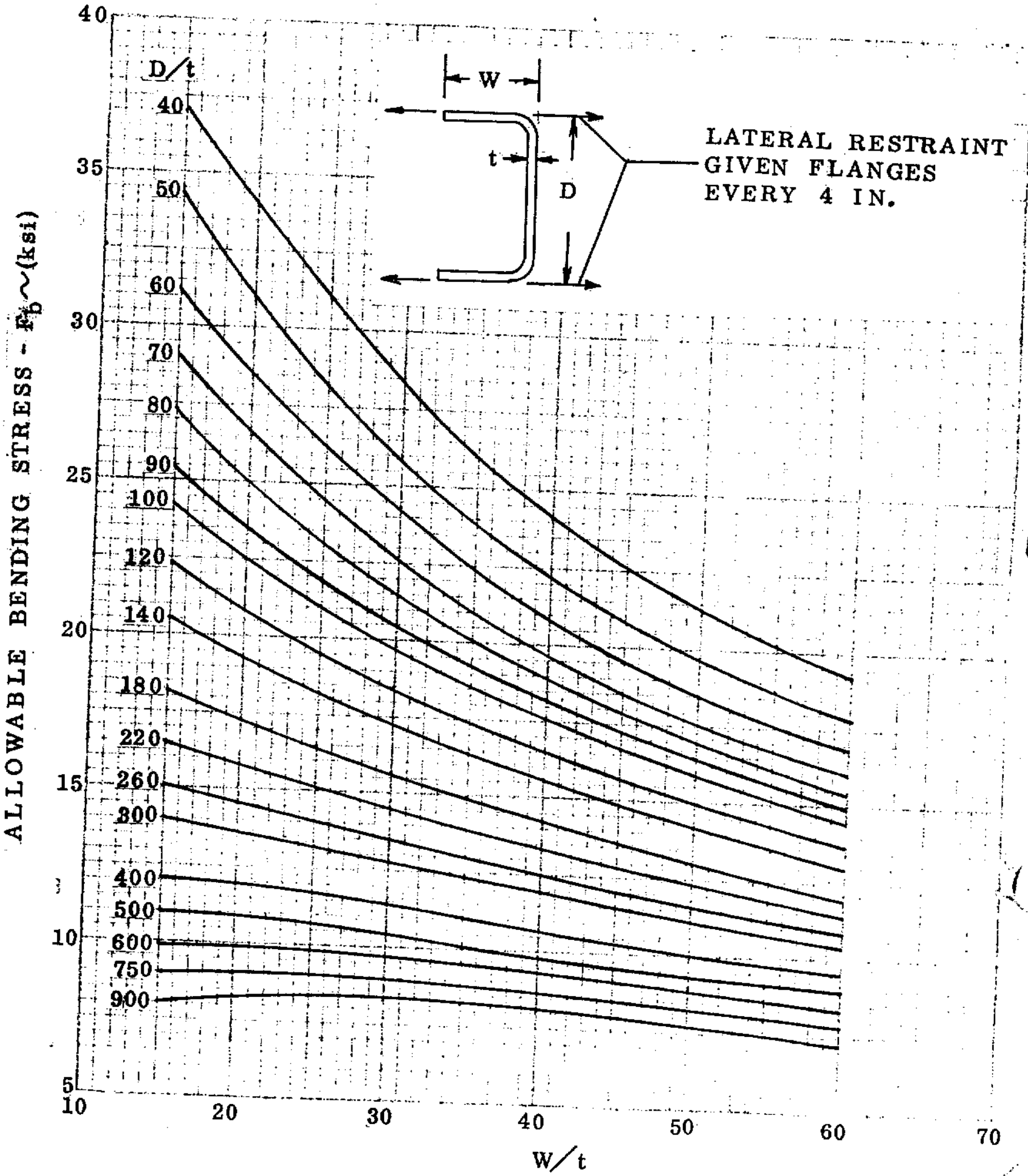


FIGURE 9.28 BENDING ALLOWABLES FOR CLAD 2024 CHANNEL SECTIONS.

7075-T73 ALUMINUM DIE FORGING-MATERIAL PROPERTIES

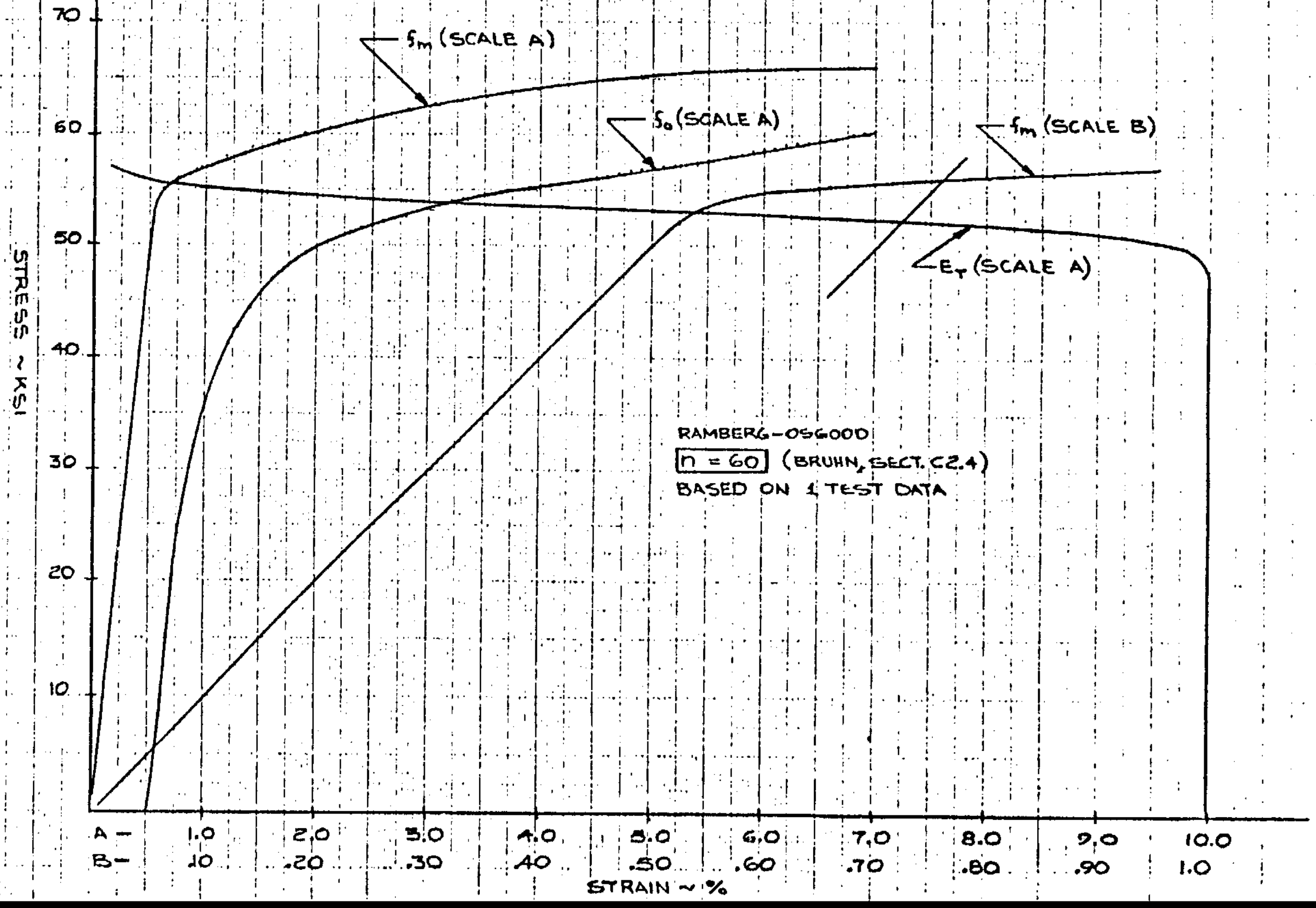
"A" VALUES/MIL-HDBK-5C, T ≤ 3.0 IN., L-TENSION

F_{tu} = 66 F_{ty} = 56 E = 10 e_u = 7.0

STRESS-STRAIN (f_m)

PLASTIC BENDING (f_o)

TANGENT MODULUS (E_T)



$F_{b\,max} = F_{TU} + f_0(k-1)$
$= 38 + 2875(1.513-1) = 53.359\ KSI$

$M = (F_{b\,max})(I/c)(1/2)$
$= (53.359)(.02932)(1/2) = 782$

$F_{b\,MIN} = F_{TU} + f_0(k-1)$
$= 38 + 23.75(1.5-1) = 49.875\ KSI$

$M = (F_{b\,min})(I/c)(1/2)$
$= (49.875(.02958)(1/2) = 738$

$M_{TOTAL} = 782 + 738 = 1520\ IN\text{-}LBS$

$$MS = \frac{1520}{951(1.5)^*} - 1 = \underline{\underline{+.07}}$$

* 1.5 CASTING FACTOR

Should 1.15 F.F. be used with bolt-out loads?

TO INCLUDE 1.15 F.F.
+ .10 WELD REPAIR ALLOWANCE

ALSO USE 356 CASTING, NOT A356

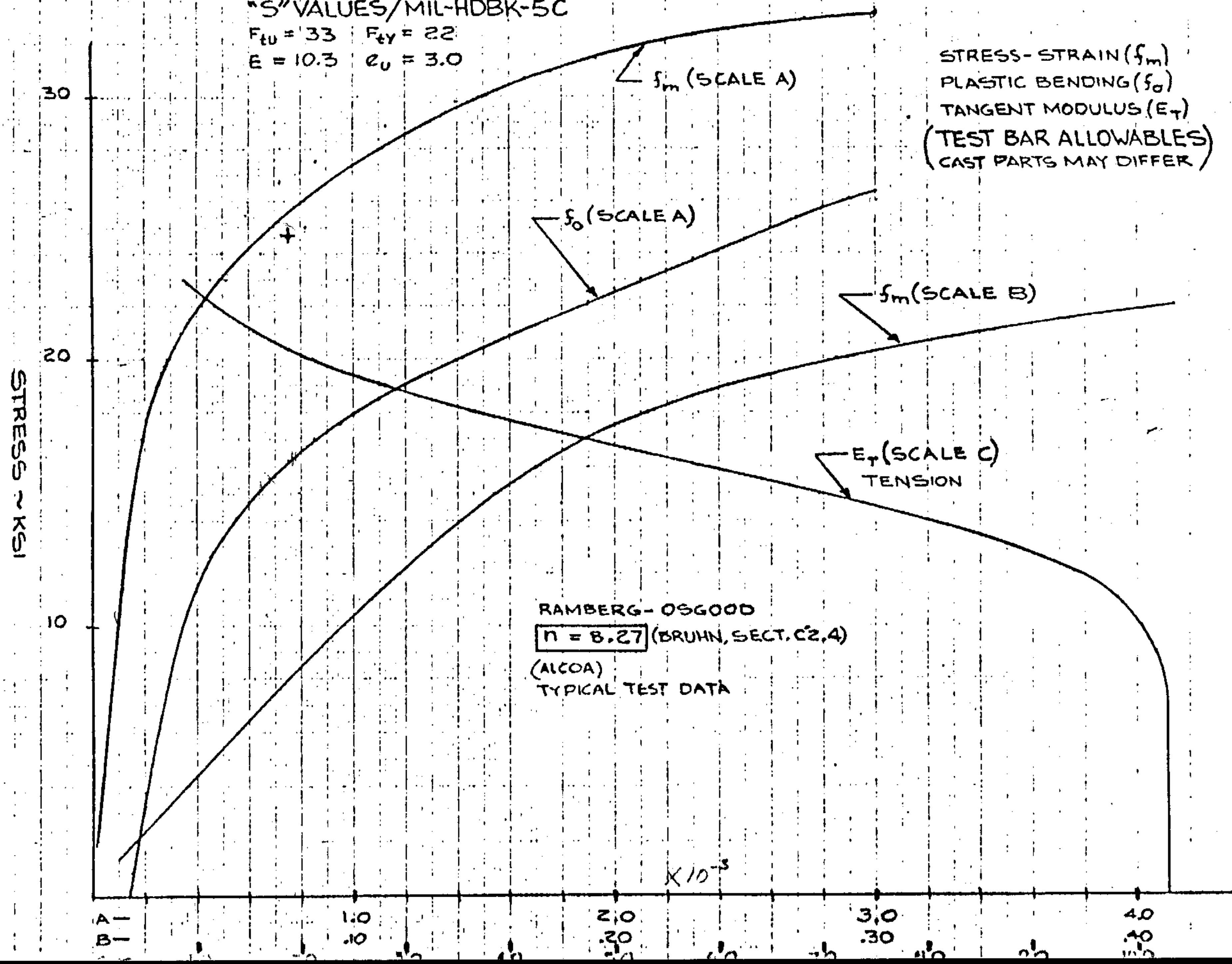
NO

356-T6 ALUMINUM INVESTMENT CASTING - MATERIAL PROPERTIES

"S" VALUES/MIL-HDBK-5C

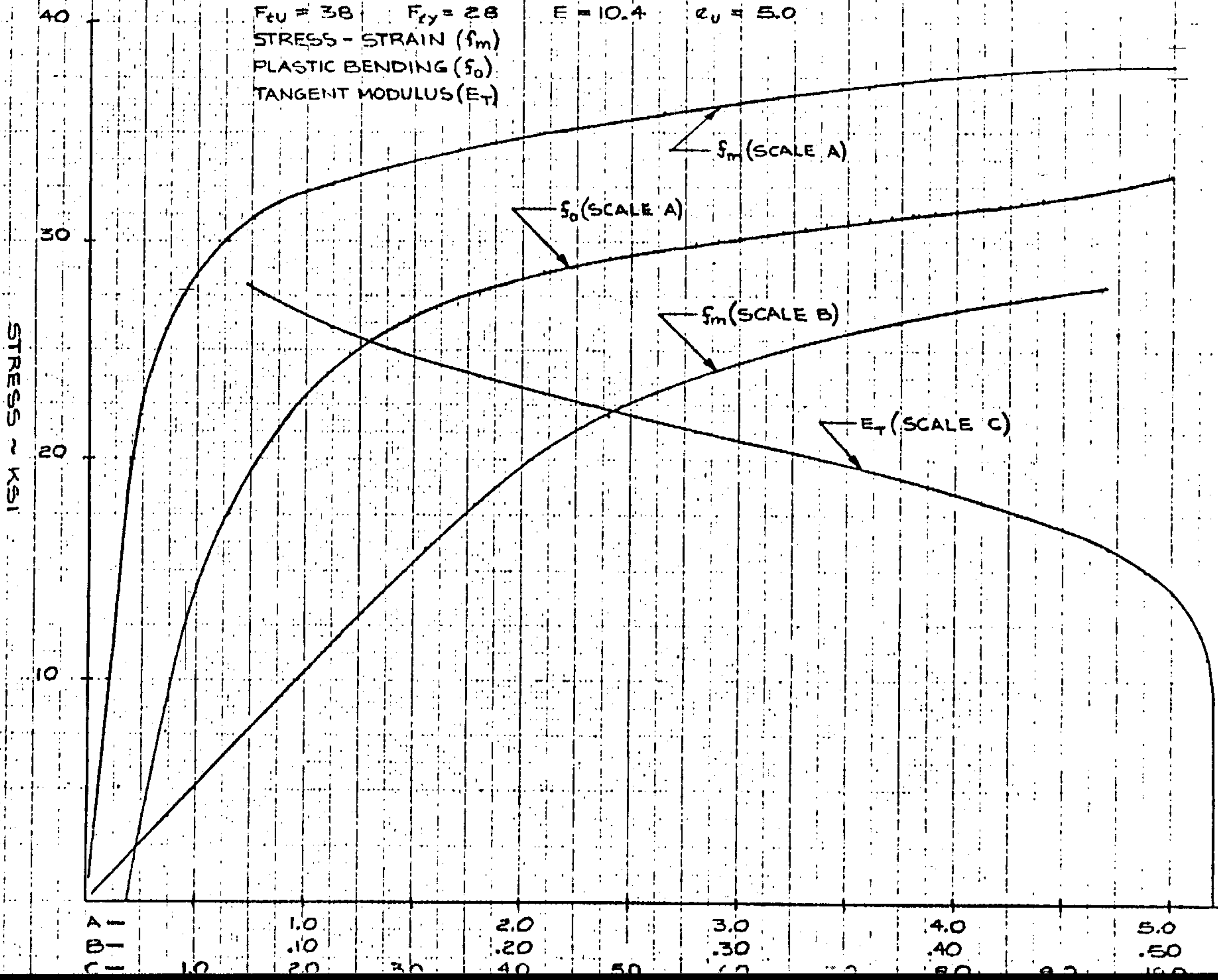
F_{tu} = 33 F_{ty} = 22

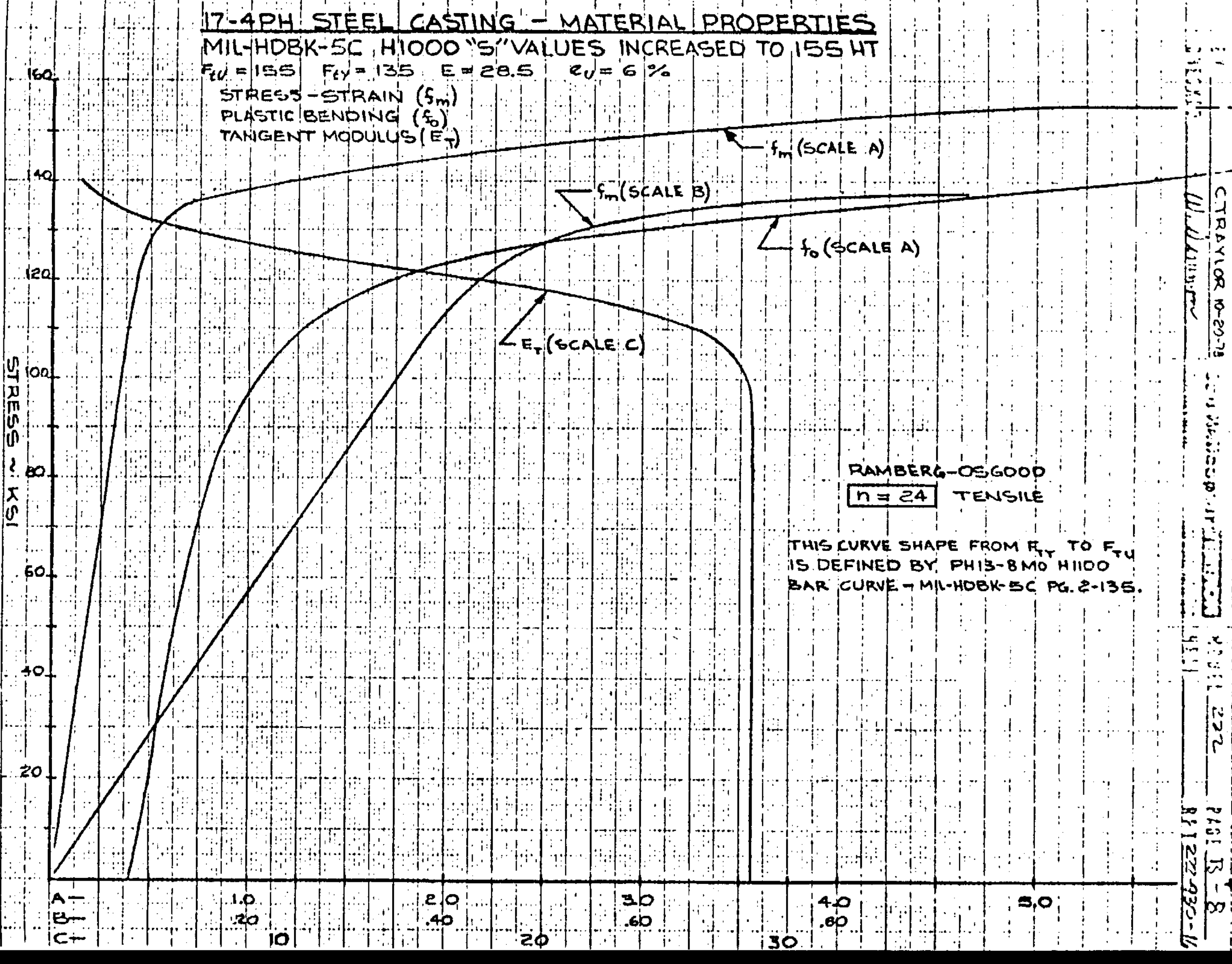
E = 10.3 e_u = 3.0



A356-T6 ALUMINUM INVESTMENT CASTING – MATERIAL PROPERTIES

GRADE 10 VALUES/ MIL-HDBK-5C – TENSION




17-4PH STEEL CASTING – MATERIAL PROPERTIES
MIL-HDBK-5C H1000 "S" VALUES INCREASED TO 155 HT
F_{tu} = 155 F_{ty} = 135 E = 28.5 e_u = 6 %
STRESS-STRAIN (ϵ_m)
PLASTIC BENDING (f_o)
TANGENT MODULUS (E_T)
f_m (SCALE A)
ϵ_m (SCALE B)
f_o (SCALE A)
E_T (SCALE C)
RAMBERG-OSGOOD
n = 24 TENSILE
THIS CURVE SHAPE FROM F_{TY} TO F_{TU}
IS DEFINED BY PH13-8Mo H1100
BAR CURVE – MIL-HDBK-5C PG. 2-135.
STRESS ~ KSI
160
140
120
100
80
60
40
20
A
B
C
1.0
2.0
3.0
4.0
5.0
.20
.40
.60
.80
10
20
30

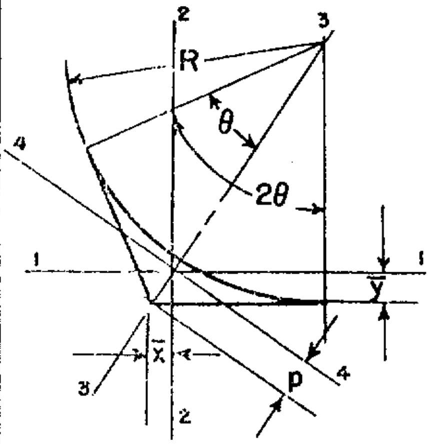
Revision D

STRUCTURAL DESIGN MANUAL

VOLUME I

Bell Helicopter TEXTRON

OBLIQUE FILLET



$$A = (\tan\theta - \theta)R^2$$

$$\bar{y} = \left[1 - \frac{\sin^2\theta \tan\theta}{3(\tan\theta - \theta)}\right]R$$

$$\bar{x} = \left[\tan\theta - \frac{\sin^2\theta \tan^2\theta}{3(\tan\theta - \theta)}\right]R$$

$$p = \left[\sec\theta - \frac{\sin^2\theta \tan^2\theta}{3(\tan\theta - \theta)}\right]R$$

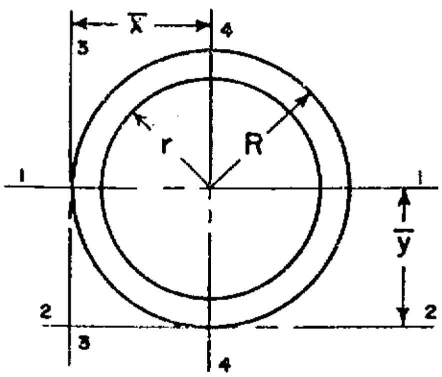
$$I_{1-1} = \left[\frac{\sin^4\theta \tan\theta}{3} + \frac{\sin\theta\cos\theta(1 + 2\sin^2\theta) - \theta}{4} - \frac{\sin^4\theta \tan^2\theta}{9(\tan\theta - \theta)}\right]R^4$$

$$I_{2-2} = \left[\frac{\tan\theta(\tan^2\theta + \sin^2\theta)}{6} - \frac{3\theta - \sin\theta\cos\theta(3 - 2\sin^2\theta)}{12} - \frac{\sin^4\theta \tan^4\theta}{9(\tan\theta - \theta)}\right]R^4$$

$$I_{3-3} = \left[\frac{\sin^3\theta}{6\cos\theta} - \frac{\theta - \sin\theta\cos\theta}{4}\right]R^4$$

$$I_{4-4} = \left[\frac{\sec^3\theta - \cos^3\theta}{6\sin\theta} - \frac{\theta + \sin\theta\cos\theta}{4} - \frac{\sin^2\theta \tan^4\theta}{9(\tan\theta - \theta)}\right]R^4$$

HOLLOW CIRCLE



$$A = 3.1416(R^2 - r^2)$$

$$\bar{x} = \bar{y} = R$$

$$I_{1-1} = I_{4-4} = 0.7854(R^4 - r^4)$$

$$I_{2-2} = I_{3-3} = 0.7854(5R^4 - 4R^2r^2 - r^4)$$

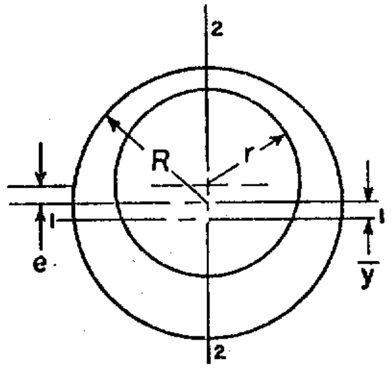
$$\rho_{1-1} = \rho_{4-4} = 0.5(R^2 + r^2)^{1/2}$$

$$\rho_{2-2} = \rho_{3-3} = \sqrt{\frac{5R^2 + r^2}{2}}$$

$$\frac{I_{1-1}}{\bar{y}} = \frac{I_{4-4}}{\bar{x}} = \frac{.07854(R^4 - r^4)}{R}$$

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

ECCENTRIC HOLLOW CIRCLE



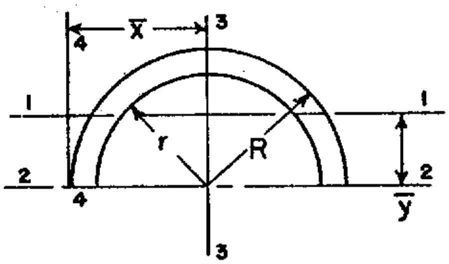
$$A = \pi \, (R^2 - r^2)$$

$$\bar{y} = \frac{e r^2}{R^2 - r^2}$$

$$I_{1-1} = .7854\,(R^4 - r^4) - \frac{\pi e^2 R^2 r^2}{(R^2 - r^2)}$$

$$I_{2-2} = .7854\,(R^4 - r^4)$$

HOLLOW SEMI-CIRCLE



$$A = 1.5708\,(R^2 - r^2)$$

$$\bar{x} = R$$

$$\bar{y} = 0.4244\,\left(R + \frac{r^2}{R + r}\right)$$

$$I_{1-1} = 0.3927\,(R^4 - r^4) - 1.5708\,(R^2 - r^2)\,\bar{y}^2$$

$$I_{2-2} = 0.3927\,(R^4 - r^4)$$

$$I_{3-3} = 0.3927\,(R^4 - r^4)$$

$$I_{4-4} = 0.3927\,(R^2 - r^2)\,(5R^2 + r^2)$$

$$\rho_{1-1} = \sqrt{\frac{2 I_{1-1}}{\pi (R^2 - r^2)}}$$

$$\rho_{2-2} = 0.5\sqrt{R^2 + r^2}$$

$$\rho_{3-3} = 0.5\sqrt{R^2 + r^2}$$

$$\rho_{4-4} = \sqrt{\frac{2 I_{4-4}}{\pi (R^2 - r^2)}}$$

TABLE 3.1 (CONT'D) PROPERTIES OF COMMON SECTIONS

(3) The margin of safety for the loading represented by point "a" can be found in three ways

a. MS = od/oa - 1
b. MS = bh/ba - 1
c. MS = cg/ca - 1

Values of od, bh and cg are referred to as allowables (load or stress) and oa, ba and ca are applied load or stress. Using this procedure and equation 4.6 procedures for two loads acting and three loads acting can be determined.

4.4.1 Procedure for Margin of Safety for Two Loads Acting

(1) Using buckling, yield or ultimate criteria and equation 4.6, calculate the stress ratio for each load acting alone.

(2) Using the calculated stress ratios locate point "a" on the proper interaction curve (using Figure 4.6 as an example).

(3) Draw a straight line from the origin "o" through point "a" and intersect the interaction curve at point "d". Read the stress ratios R_{1a}(ed) and R_{2a}(fd).

(4) Compute the applied stress ratios R_1(ba) and R_2(ca).

(5) Compute the margin of safety

$$MS = R_{1a}/R_1 - 1 = R_{2a}/R_2 - 1 \qquad 4.8$$

4.4.2 Procedure for Margin of Safety for Three Loads Acting

(1) Using buckling, yield or ultimate criteria and equation 4.6, calculate the stress ratios for each load acting above.

(2) Using the appropriate interaction family of curves locate point "a" corresponding to the calculated stress ratios R_1 and R_2 as shown in Figure 4.7.

(3) Draw a straight line from the origin "o" through point "a".

(4) Extend this line to locate the allowable point "x" which must satisfy the following relationships:

$$R_1/R_{1a} = R_2/R_{2a} = R_3/R_{3a} \qquad 4.9$$

or

$$R_{3a} = \left(R_3/R_1\right) R_{1a} \qquad 4.10$$

Point "x" is obtained by trial and error in the following manner:

(a) Select an arbitrary value of R_{1a}.

(b) Calculate R_{3a} from equation 4.10 using the known value of R_1 and R_3 and the arbitrary value of R_{1a}.

(c) Locate point "x" on the line "oa" using the calculated R_{3a} from step (b) and compare the corresponding R_{1a} with the assumed R_{1a}.

(d) Repeat steps (a) through (c) until the assumed R_{1a} and the "x" value of R_{1a} converge. At convergence, R_{1a}, R_{2a} and R_{3a} will be at a common point on line "oa".

(5) Compute the margin of safety

$$MS = R_{1a}/R_1 - 1 = R_{2a}/R_2 - 1 = R_{3a}/R_3 - 1 \qquad 4.11$$

4.5 Compact Structures

A compact structure is one in which failure does not occur by crippling or buckling. This section presents interaction criteria for compact structures with biaxial stress in a rectangular volume such as in plates, membranes and shells and with uniaxial stress in a plane such as in beams, round bars and bolts.

4.5.1 Biaxial Stress Interaction Relationships

Tests have been conducted to determine the failure theories of biaxially loaded isotropic ductile materials. The maximum shear stress theory and the octahedral shear stress theory adequately predict the yield and ultimate strengths. There are a few cases where convenient margin of safety calculations are possible. These are shown in Table 4.3. A general interaction method is required. It is shown in Figure 4.8. The method is applicable to stress conditions which combine in a two-dimensional manner like that shown in Figure 4.8. This condition exists in a rectangular volume and not on a single plane. Tension is positive, compression is negative. The interaction equations and curves are applicable for ultimate and yield by use of the parameters given in Table 4.1.

The interaction equations contain certain factors which relate one stress to the other. They are defined as follows:

The constant relating interaction in terms of tension or shear strength allowables:

$$K = F_{su}/F_{tu} \qquad 4.12$$

Tests show this value to vary from 0.5 to 0.75.

The transverse shear and torsional stress ratios combine as

$$R_s = R_{ss} + R_{st} \qquad 4.13$$

The directional tension and bending stress ratios combine as

$$R_x = R_{tx} + R_{bx} \qquad 4.14$$

$$R_y = R_{ty} + R_{by} \qquad 4.15$$

The directional compression and bending stress ratios combine as

$$R_x = R_{cx} + R_{bx} \qquad 4.16$$

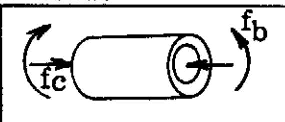
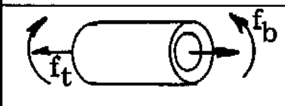
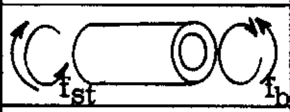
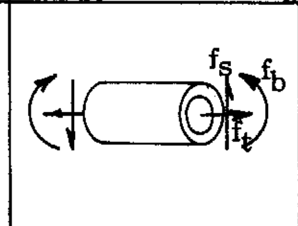
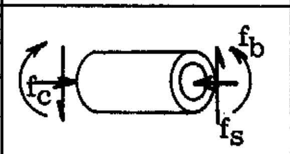
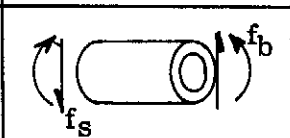
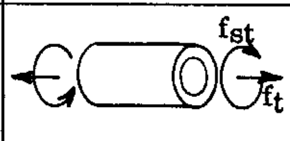
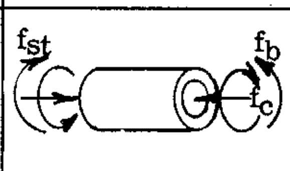
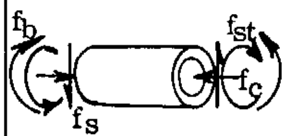
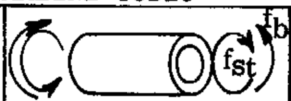
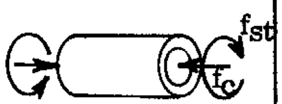
CASE	LOADING PICTURE	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY EQUATION	REMARKS
ROUND TUBES						
1		Compression + Bending	$R_c + R_b = 1$	4.10	$\frac{1}{R_c + R_b} - 1$	
2		Tension + Bending	$R_t^{1.5} + R_b = 1$	4.23		Let $R_t = R_1$, $R_b = R_2$
3		Bending + Torsion	$R_b^2 + R_{st}^2 = 1$	4.10	$\frac{1}{\sqrt{R_b^2 + R_{st}^2}} - 1$	
4		Tension + Bending + Shear	$R_{bt}^2 + R_s^2 = 1$	4.10	$\frac{1}{\sqrt{R_{bt}^2 + R_s^2}} - 1$	$R_{bt} = R_t/R_{t_a} = R_b/R_{b_a}$, where R_{t_a} and R_{b_a} are obtained from figure 4.9 ($R_t = R_1$, $R_b = R_2$, n = 1.5) by the two-loads-acting procedure as outlined in section 4.4.1
5		Compression + Bending + Shear	$(R_c + R_b)^2 + R_s^2$	4.10	$\frac{1}{\sqrt{(R_c + R_b)^2 + R_s^2}} - 1$	Let $R_c + R_b = R_1$, $R_s = R_2$
6		Bending + Shear	$R_b^2 + R_s^2 = 1$	4.10	$\frac{1}{\sqrt{R_b^2 + R_s^2}} - 1$	
7		Tension + Torsion	$R_t^2 + R_{st}^2 = 1$	4.10	$\frac{1}{\sqrt{R_t^2 + R_{st}^2}} - 1$	
8		Compression + Bending + Torsion	$R_b^2 + R_{st}^2 = (1-R_c)^2$	4.11	$\frac{1}{R_c + \sqrt{R_b^2 + R_{st}^2}} - 1$	In using figure 4.11 follow two-loads-acting procedures as outlined in section 4.9.1.

TABLE 4.5 THICK-WALLED TUBULAR STRUCTURES - INTERACTION CRITERIA-YIELD AND ULTIMATE CONDITIONS OF STRENGTH, INCLUDING THE EFFECTS OF COLUMN STABILITY

TABLE 4.5 THICK-WALLED TUBULAR STRUCTURES-INTERACTION CRITERIA-YIELD AND ULTIMATE CONDITIONS OF STRENGTH, INCLUDING THE EFFECTS OF COLUMN STABILITY (CONCLUDED)

CASE	LOADING PICTURE	LOADING DESCRIPTION	INTERACTION CURVE EQUATION	INTERACTION CURVE FIGURE	MARGIN-OF-SAFETY EQUATION	REMARKS
9		Compression + Bending + Torsion + Shear	$R_c + R_{st} + R_b^2 + R_s^2 = 1$	4.12		In using figure 4.12, follow two-loads-acting procedure as outlined in section 4.4.1
10		Tension + Torsion + Internal Pressure	$R_t^2 + R_{st}^2 + R_p^2 = 1$	4.13	$\frac{1}{\sqrt{R_t^2 + R_{st}^2 + R_p^2}} - 1$	
STREAMLINE TUBES						
11		Bending + Torsion	$R_b + R_{st} = 1$	4.10	$\frac{1}{\sqrt{R_b + R_{st}}} - 1$	
SQUARE TUBES						
12		Compression + Torsion		4.14		Let $R_c = R_1$, $R_s = R_2$

NOTES:

(1) R_c must be based on the tube column allowable.

(2) R_t must be based on the material strength allowable.

(3) R_b and R_{st} must be based on tube strength allowables.

(4) R_b must include the effects of secondary bending.

(5) For shear-bending analysis use $f_s = f_{smax}$ and $f_b = f_{bmax}$ even though the locations of the two maxima do not coincide. The allowable transverse shear stress is equal to the lower of 1.20 times the allowable torsional shear stress and the material allowable **shear stress.**

(6) $R_p = pd/2t\,F_{tu}$, d = tube mean diameter, t = wall thickness.

Alloy	Product	Temper	Product Thickness Range, Inch	Plane-Strain Fracture Toughness, K_{Ic}, ksi $\sqrt{in.}$ (L)(T) No. of Lots	(L)(T) Thick. Inch	(L)(T) Min	(L)(T) Avg	(L)(T) Max	(T)(L) No. of Lots	(T)(L) Thick. Inch	(T)(L) Min	(T)(L) Avg	(T)(L) Max	(ST)(L) No. of Lots	(ST)(L) Thick. Inch	(ST)(L) Min	(ST)(L) Avg	(ST)(L) Max
2014	Plate	T651	1 - 2	3	1	22	23	24	4	1	19	21	23	1	1/2	---	18	---
	Forgings	T652	2 - 6	4	3/4	25	29	34	4	3/4	19	23	30	2	1/2	18	19	19
2024	Plate	T351	1-1/2	---	1-1/2	---	---	---	---		---	---	---	2	1	20	23	26
	Extruded Shapes	T3510,1	1-1/2- 2	2	1-1/2	---	46	---	---		---	---	---	---		---	---	---
	Plate	T851	1 - 2	4	1-3/8	22	23	25	2	1-3/8	20	20	20	1	1	---	17	---
	Extruded Shapes	T8510,1	3/4- 4	5	3/4	22	28	32	4	3/4	16	17	18	---		---	---	---
	Forgings	T852	2 - 6	5	3/4	23	26	30	4	3/4	17	18	20	3	1/4	16	16	17
2219	Plate	T851	1 - 2	2	1-3/8	31	33	36	2	1	29	30	30	1	3/4	---	20	---
		T87	3/4- 1	3	3/4	26	27	28	---		---	---	---	1	1/2	---	20	---
7075	Plate	T651	1/2- 2	6	1/2	25	26	27	4	1/2	20	22	23	2	1/2	15	16	18
	Extruded Shapes	T6510,1	1/2- 4	10	1/2	26	28	32	10	1/2	19	22	26	4	1/4	18	19	22
	Forgings	T652	2 - 6	2	1/2	24	26	28	1	1/2	---	23	---	1	1/2	---	17	---
	Plate	T7351	1-3/8	1	1-3/8	---	30	---	2	1	25	29	33	2	1/2	19	20	21
	Extruded Shapes	T73510,1	1/2- 4	2	5/8	31	33	34	5	1/2	22	24	28	1	1	---	20	---
	Forgings	T7352	1 - 5	5	3/4	27	31	35	3	3/4	23	25	26	3	1/2	19	21	25
7079	Plate	T651	1 - 3	3	1	27	29	30	2	1	24	24	24	1	1/2	---	16	---
	Forgings	T652	2 - 6	2	3/4	28	30	31	3	3/4	21	23	25	3	1/4	17	18	18
7178	Plate	T651	1/2- 2	3	1	22	23	26	4	1/2	19	21	23	1	1/2	---	15	---
	Extruded Shapes	T6510,1	1/2- 1-1/2	1	1	---	25	11	4	1/2	16	19	20	1	1	---	14	---
	Plate	T7671	1/2- 2	3	1/2	26	29	30	2	1/2	22	22	22	1	1/2	---	17	---
	Extruded Shapes	T7650,1	1/2- 2	5	1/2	26	29	31	3	5/8	18	22	28	1	1/2	---	16	---

TABLE 5.3 - TYPICAL VALUES OF ROOM TEMPERATURE PLANE-STRAIN FRACTURE TOUGHNESS OF ALUMINUM ALLOYS (REF 1)

Alloy and Type of Temper	Estimate of Highest Sustained Tension Stress (ksi) at Which Test Specimens of Different Orientations to the Grain Structure Would Not Fail in the 3½% NaCl Alternate Immersion Test in 81 Days					
	Test Direction	Plate	Rolled Bar and Bar	Extruded Shapes Section Thickness, Inch 0.25-1	Extruded Shapes Section Thickness, Inch 1-2	Hand Forgings
2014-T6	L	45	45	50	45	30
	LT	30	..	27	22	25
	ST	8	15	..	8	8
2219-T8	L	40	..	35	35	38
	LT	38	..	35	35	38
	ST	38	..	..	35	38
2024-T3, T4	L	35	30	50	50	..
	LT	20	..	37	18	..
	ST	8	10	..	8	..
2024-T8	L	50	47	60	60	43
	LT	50	..	50	50	43
	ST	30	43	..	45	15
7075-T6	L	50	50	60	60	35
	LT	45	..	50	32	25
	ST	8	15	..	8	8
7075-T76	L	59	..	52	..	..
	LT	49	..	49	..	..
	ST	25	..	25	..	..
7075-T73	L	50	50	54	53	50
	LT	48	48	48	48	48
	ST	43	43	46	46	43
7079-T6	L	55	..	60	60	50
	LT	40	..	50	35	30
	ST	8	..	..	8	8
7178-T6	L	55	..	6	65	..
	LT	38	..	45	25	..
	ST	8	..	..	8	..
7178-T76	L	52	..	55	..	..
	LT	52	..	52	..	..
	ST	25	..	25	..	..

TABLE 5.4--COMPARISON OF THE RESISTANCE TO STRESS CORROSION OF VARIOUS ALUMINUM ALLOYS (REF. 1)

TYPE IV			
BOLT	NUT		
AN3-AN20	AN256	MS21061	NAS577
AN42-AN49	AN310	MS21062	NAS1291
AN173-AN186	AN315	MS21069	NAS1329
AN525	MS9358	MS21070	NAS1330
MS20033-MS20046	MS20365	MS21071	NAS1473
MS20073-MS20081	MS20500	MS21072	NAS1474
MS24694	MS21042	MS21073	80-004
MS27039	MS21043	MS21074	80-005
NAS333-NAS340	MS21044	MS21075	80-006
NAS517	MS21045	MS21076	80-007
NAS623	MS21047	MS21083	80-013
NAS1003-NAS1020	MS21048	MS21086	90-002
NAS1202-NAS1210	MS21049	MS21208	90-003
NAS1297	MS21051	MS21209	110-061
NAS1352 (NON-LOCKING)	MS21052	MS21991	110-062
	MS21053	MS122076	
ALL THREADED STUDS	MS21054	thru	
	MS21055	MS122275	
	MS21056	MS124651	
	MS21058	thru	
	MS21059	MS124850	
	MS21060	NAS509	

TYPE IV CONSISTS OF ANY COMBINATION OF NUT AND BOLT SHOWN

REFERENCE BELL STD 160-007

TABLE 6.5 - TYPE IV FASTENERS

NUT AND BOLT THREAD SIZE	Torque, In-Lbs			
	TYPE III		TYPE IV	
	SHEAR		TENSION	
	Recommended Installation Torque Range (a)	Max Allowable Tightening Torque (b)	Recommended Installation Torque Range (c)	Max Allowable Tightening Torque (d)
10-32	12-15	25	20-25	40
1/4-28	30-40	60	50-70	100
5/16-24	60-85	140	100-140	225
3/8-24	95-110	240	160-190	390
7/16-20	270-300	500	440-500	840
1/2-20	288-408	660	480-700	1100
9/16-18	480-600	960	800-1000	1600
5/8-18	660-780	1400	1100-1300	2400
3/4-16	1300-1500	3000	2300-2500	5000
7/8-14	1500-1800	4200	2500-3000	7000
1 - 12	2200-3300	6000	3700-5500	10000
1 1/8-12	3000-4200	9000	5000-7000	15000
1 1/4-12	5400-6600	15000	9000-11000	25000

(a) TYPE III RECOMMENDED TORQUE RANGE IS BASED ON A NOMINAL STRESS OF 24 KSI IN THE BOLT.

(b) TYPE III MAX ALLOWABLE TORQUE IS BASED ON A STRESS OF 54 KSI IN THE BOLT.

(c) TYPE IV RECOMMENDED TORQUE RANGE IS BASED ON A NOMINAL STRESS OF 40 KSI IN THE BOLT.

(d) TYPE IV MAX ALLOWABLE TORQUE IS BASED ON A STRESS OF 90 KSI IN THE BOLT.

REFERENCE BELL STD 160-007

TABLE 6.6 - TORQUE VALUES FOR THREADED FASTENERS AND FITTINGS

An example problem best illustrates the procedure. Figure 6.21 shows a bimetallic splice, titanium and aluminum sheets joined by six steel bolts. The titanium and aluminum have uniform temperature rises of 300°F and 70°F respectively.

The results of the example show that the maximum load occurs in the first attachment and that the two end attachments carry more than half of the total applied mechanical load. When plastic deformations occur in the vicinity of the bolt holes, the bolts tend to carry equal loads.

Example Problem:

$f = .900 \times 10^{-6}$ in/lb (Section 6.4.3)

$(L/AE)_T = 1/(1.5)(.125)(15)(10^6) = .356\ (10^{-6})$ in/lb

$(L/AE)_B = 1/(1.5)(.250)(10)(10^6) = .267\ (10^{-6})$ in/lb

$\Delta\phi = (\alpha\ \Delta T)_T - (\alpha\,\Delta T)_B$

$L = \Big((6.5)(300) - 12(70)\Big)\ (1)(10^{-6}) = 1110(10^{-6})$ in.

Substituting into equation (7)

$P_{jn} = \Big(A_{jn} + B_{jn}\ (.356/.900)\Big)\ (20{,}000) + B_{JN}\ (1110/.900)$

$P_{jn} = 20{,}000\ A_{jn} + 9135\ B_{jn}$

The coefficients A_{jn} and B_{jn} are now determined from Figure 6.7 through Figure 6.20.

$$Z = \left[\left(\frac{L}{Ae}\right)_T + \left(\frac{L}{Ae}\right)_B\right]\left(\frac{1}{f}\right) = (.356 + .267)\ (1/.900) = .692$$

N = 6

A_{16} = .0140; Figure 6.7 , B16 = .8000; Figure 6.8

A_{26} = .0239; Figure 6.9 , B26 = .3200; Figure 6.11

A_{36} = .0500; Figure 6.12, B36 = .0880; Figure 6.14

A_{46} = .1090; Figure 6.15, B46 = -.0860; Figure 6.16

A_{56} = .2450; Figure 6.18, B56 = -.3200; Figure 6.19

The curves give values of A_{jn} and B_{jn} up to j = 5, but the splice under consideration has 6 fasteners. In order to obtain the coefficients for the last attachment, the designation of the top and bottom plates must be interchanged as shown.

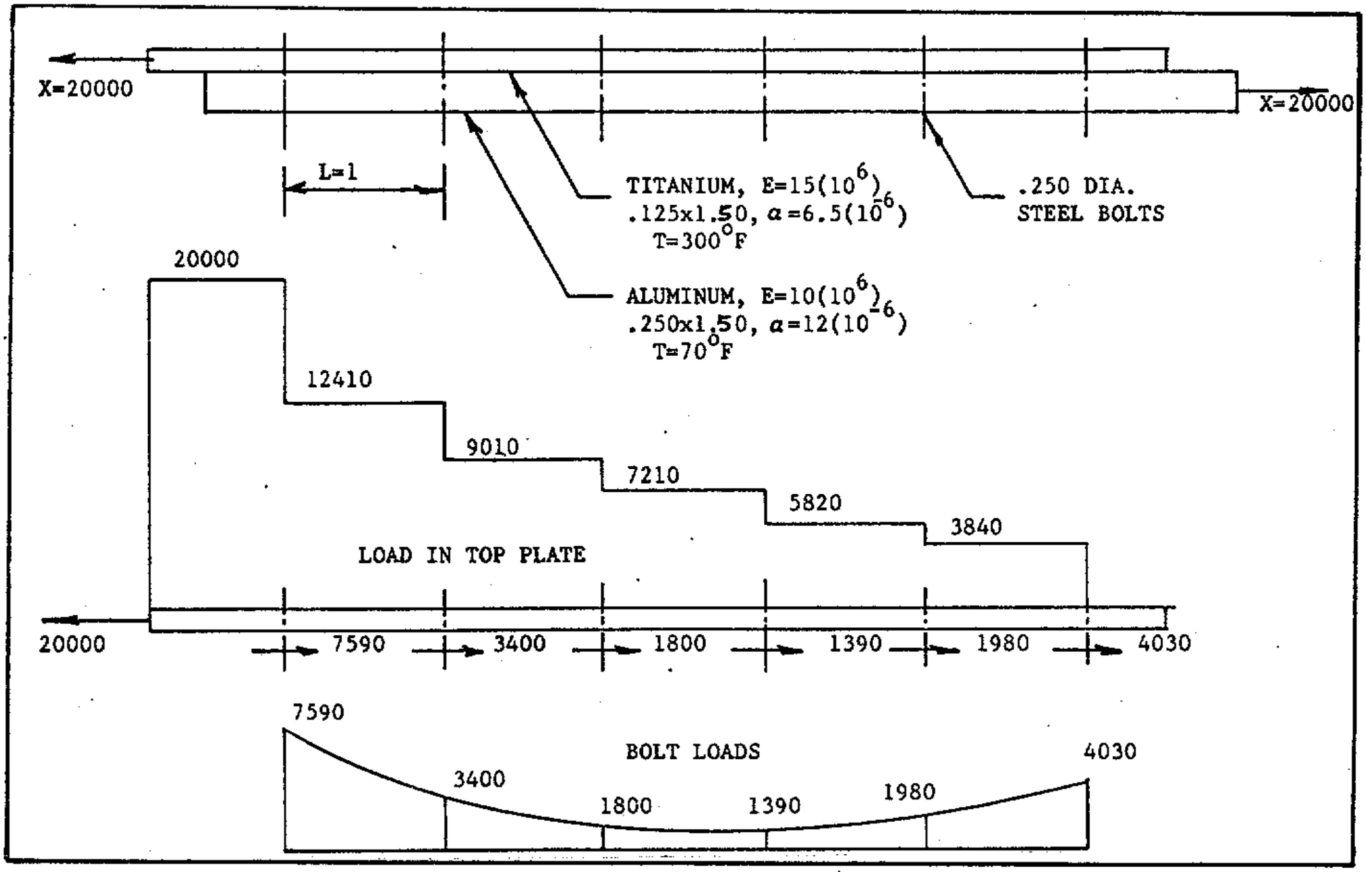


FIGURE 6.21 - EXAMPLE PROBLEM, COMPATIBILITY

As shown above, the last attachment (j = 6) in the original designation becomes the first attachment (j = 1) in the interchanged position.

$f' = f = .900\ (10^{-6})$ in/lb

$(L/AE)'_T = (L/AE)_B = .267\ (10^{-6})$ in/lb

$(L/AE)'_B = (L/AE)_T = .356\ (10^{-6})$ in/lb

$Z' = Z = .692 \qquad \Delta\phi' = -\ \Delta\phi = -1110(10^{-6})$

from equation (7)

$P'_{jn} = (A'_{jn} + B'_{jn}\ (.267/.900))(20{,}000) = B'_{jn}\ (1110/.900)$

$P'_{jn} = 20{,}000\ A'_{jn} + 4693\ B'_{jn}$

from Figures 6.7 and 6.8

$A'_{16} = A_{16} = .0140 \quad B'_{16} = B_{16} = .8000$

Unequal Areas

Add a fastener at a time as described previously. At any stage where the centroid of n bolts has been found and is joined to the (n+1) fastener, the fractional part of the connecting line measured from the previous centroid is

$$\frac{A_{n+1}}{A_1 + A_2 + \cdots + A_n + A_{n+1}}$$

6.4.2.2 Load Determination

Figure 6.26 shows a typical joint with an applied load P and three fasteners A_1, A_2 and A_3. Draw the joint to scale and locate the center of resistance G. Extend the line of action of the applied load P, and from this line erect a perpendicular that passes through the centroid G and extends a distance GQ away from P, so that

$$GQ = \frac{\Sigma Ar^2}{e \Sigma A}$$

where

A = area of fastener in shear or bearing
r = radial distance from G to fastener
e = distance from G to line of action of P

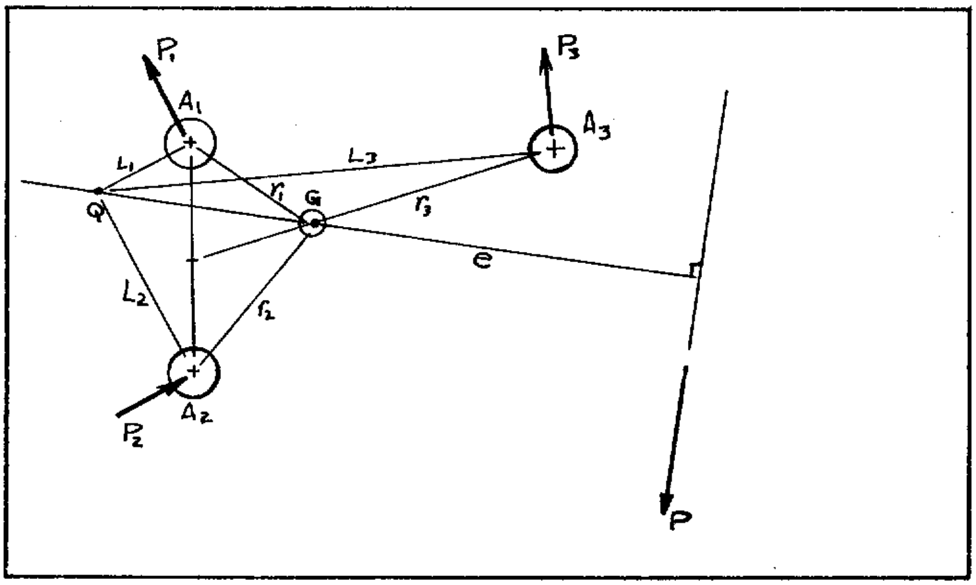


FIGURE 6.26 - TYPICAL JOINT

Next determine the radial distance L_1 of the number one fastener from Q. The load P on that bolt is

$$P_1 = \frac{PeA_1L_1}{\Sigma_{Ar}{}^2}$$

and is directed perpendicular to radial line L_1.

Repeat this procedure until the loads for all fasteners are determined.

6.4.3 Attachment Flexibility

The flexibility of an attachment/sheet combination should be determined experimentally. If load-deflection curves for a particular fastener/sheet combination are available, the flexibility is the slope of the curve at the estimated load level.

If load-deflection test data is not available for the exact fastener/sheet combination, two methods can be used to determine a spring rate.

6.4.3.1 Method I - Generalized Test Data

Some test data is available to develop generalized stiffness curves. Figure 6.27 shows a curve of t/D versus K for a single shear joint with a steel fastener. The procedure for determining joint stiffness is as follows:

DIA	1/8	5/32	3/16	1/4	5/16	3/8	7/16	1/2	9/16	5/8
	$SRx10^{-6}$									
ALUM	.163	.203	.244	.325	.406	.487	.563	.650	.732	.813
STEEL	3.62	4.53	5.44	7.25	9.06	10.9	12.6	14.5	15.5	18.1
TITAN	1.93	2.42	2.90	3.87	4.83	5.81	6.72	7.73	8.27	9.65
OTHER	(Eother/Esteel)xSRsteel									
SHEET SPRING RATE = K x SR JOINT SPRING RATE = 1/(1/SRu + 1/SR1)										

TABLE 6.9 - BASIC SPRING RATES

1. Calculate t/D for upper sheet

2. Calculate t/D for lower sheet

3. From Figure 6.27 determine K for upper sheet

4. From Figure 6.27 determine K for lower sheet

The average load is then

$$P_{avg} = (6200 + 4875)/2 = 5590 \text{ lbs.}$$

The flexibility is calculated for a deformation of 2 percent of the hole diameter per Reference 1.

$$f_{avg} = \Delta/P_{avg} = (.02)(.250)/5590 \approx 900(10^{-9}) \text{in./lb}$$

6.4.4 Lug Design

This section presents a basic method of analysis and procedure for the design of lug-pin combinations loaded axially, obliquely or transversely.

An accurate analysis of a lug-pin combination under load is difficult because the actual distributions of stresses in the lug and pin involve a combination of shear, bending and tension of varying amounts, which are a function of the ratio of lug edge distance and thickness to pin diameter, shape of lug, number of lugs in a joint, material properties, stress concentrations, rigidity of adjacent structure, etc.

The various modes of failure for a lug are:

1. Bearing of pin, lug or bushing

2. Tension across minimum net section. The full P/A_{net} stress cannot be carried because of the stress concentration around the hole.

3. Hoop tension failure of the lug across the section in line with the load.

4. Shear tearout failure of the lug.

5. Shear and bending of the pin.

Shear tearout and bearing are closely related and are covered by shear-bearing calculations based on empirical data. Also, the shear-bearing criteria precludes hoop tension failures.

Yielding of the lug is also a consideration. It is considered excessive at a permanent set of 0.02 times the pin diameter. This condition must always be checked as it is frequently reached at a lower load than would be anticipated from the ratio of the yield stress, F_{ty}, to the ultimate stress, F_{tu}, for the material.

Since lugs are elements having severe stress concentrations, the ductility and/or impact strength of the material is of importance. For this reason, attention should be paid to the longitudinal, long transverse and short transverse material properties.

Lugs are a small weight portion of a structure and are prone to fabrication errors and service damage. Since their weight is usually insignificant relative to their importance, the following criteria should be used.

1. Design lugs for a minimum margin of safety of 0.15 in both yield and ultimate.

2. If no bushing is included in the original design, design the lug so that one can be inserted in the future; however, express margins of safety with no bushings.

6.4.4.1 Nomenclature

F_{tu} = Ultimate tensile strength; F_{tuw} with grain, F_{tux} cross grain. When the plane of the lug contains both long and short transverse grain directions, F_{tux} is the smaller of the two.

F_{ty} = Tensile yield strength; F_{tyw} with grain, F_{tyx} cross grain. When the plane of the lug contains both long and short transverse grain directions, F_{tyx} is the smaller of the two.

F_{cy} = Comparison yield strength

P_u = Ultimate load

P_y = Yield load

M_{max} = Maximum bending moment on pin

P'_u = Allowable ultimate load

P'_{bru} = Allowable ultimate shear-bearing load

P'_{bry} = Allowable yield bearing load on bushing

P'_{tu} = Allowable ultimate tensile load

P'_{tru} = Allowable ultimate transverse load

P'_y = Allowable yield load of lug

A = Area; A_{br} projected bearing area, A_t minimum net section for tension, A_{av} weighted average area for transverse load.

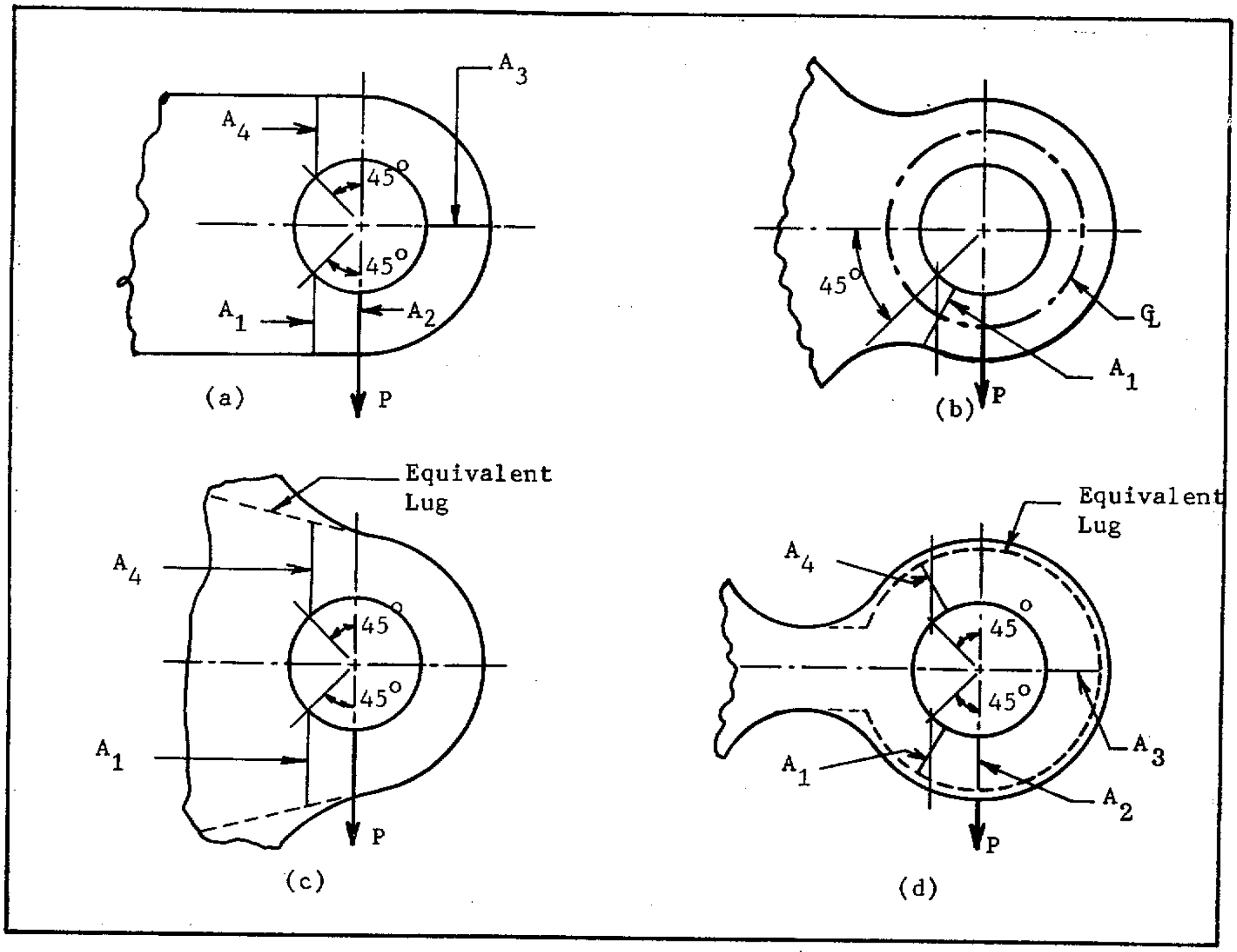


FIGURE 6.29 - TRANSVERSELY LOADED LUGS

(2) A_3 is the least area on any radial section around the hole.

(3) A_1, A_2, A_3 and A_4 should adequately reflect the strength of the lug. For lugs of unusual shape, such as severe necking or other sudden changes in cross section, an equivalent lug should be used such as shown in Figure 6.29(c) and (d).

B. P'_{tru} = Allowable ultimate load for lug failure

1. Enter Figure 6.33 with A_{av}/A_{br} and obtain K_{tru}.
2. $P'_{tru} = K_{tru} A_{br} F_{tux}$

C. P'_y = Allowable yield load of lug

1. Enter Figure 6.33 with A_{av}/A_{br} and obtain K_{try}.
2. $P'_y = K_{try} A_{br} F_{tyx}$

D. Check bushing yield per 6.4.4.2(E).

E. Margins of Safety

1. Minimum M.S. = .15 for ultimate transverse load
2. Minimum M.S. = 0 for yield of the lug and bushing

6.4.4.4 Analysis of Lugs with Oblique Loads ($0<\alpha<90^{\circ}$)

In analyzing lugs with oblique loading it is necessary to resolve the loading into axial and transverse components (denoted by the subscripts "a" and "tr" respectively), analyze the two cases separately and then combine the results using the interaction equation. The interaction equation:

$$R_a^{1.6} + R_{tr}^{1.6} = 1$$

where, for ultimate load,

$$R_a = \frac{\text{Axial component of applied ultimate load}}{\text{Smaller of } P'_{bru} \text{ or } P'_{tu} \text{ (6.4.4.2 B or C)}}$$

$$R_{tr} = \frac{\text{Transverse component of applied ultimate load}}{P'_{tru} \text{ (6.4.4.3.B)}}$$

and for yield load

$$R_a = \frac{\text{Axial component of applied yield load}}{P'_y \text{ (6.4.4.2D)}}$$

$$R_{tr} = \frac{\text{Transverse component of applied yield load}}{P'_y \text{ (6.4.4.3C)}}$$

The margin of safety should be 0.15 minimum and is calculated using the following equation:

$$MS = \frac{1}{\left(R_a^{1.6} + R_{tr}^{1.6}\right)^{0.625}} - 1$$

6.4.4.5 Analysis of Pins

The ultimate strength for a pin in a single lug/clevis joint as shown in Figure 6.34 will be analyzed first.

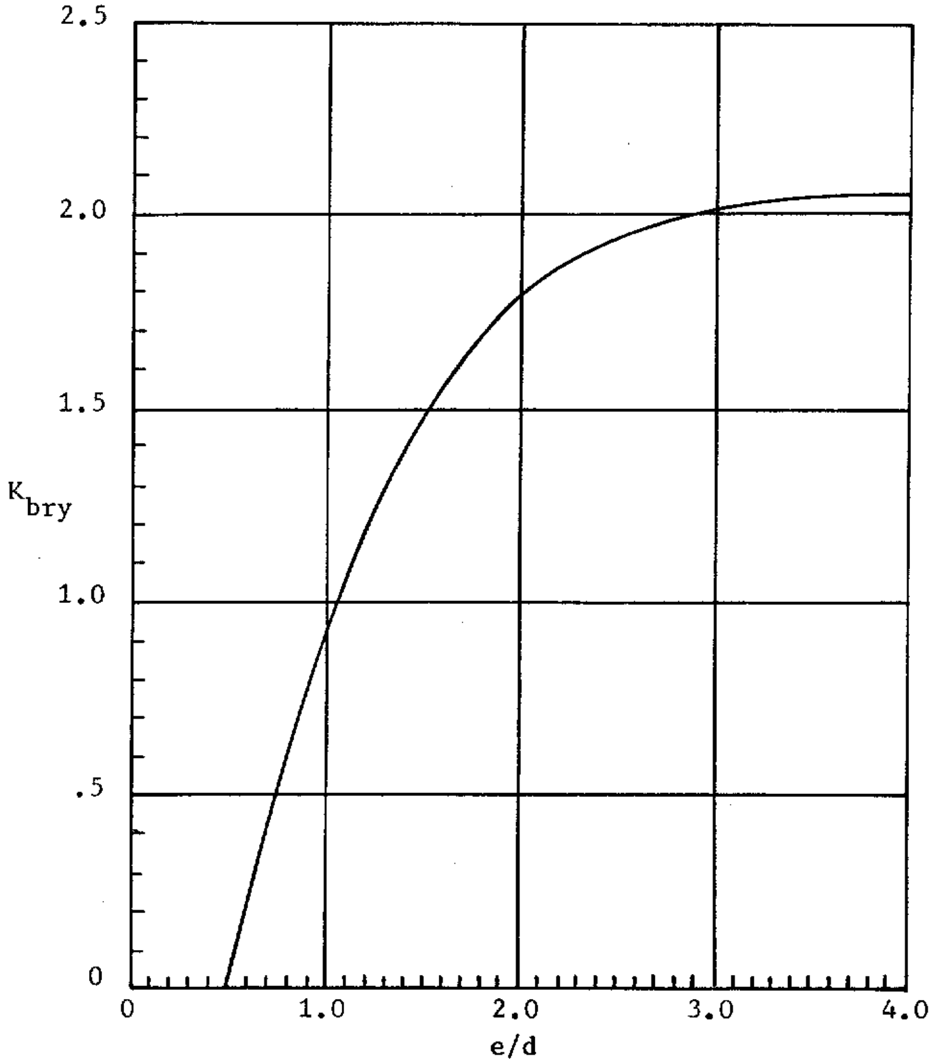


FIGURE 6.31 - BEARING YIELD EFFICIENCY FACTORS FOR AXIALLY LOADED LUGS

Revision B

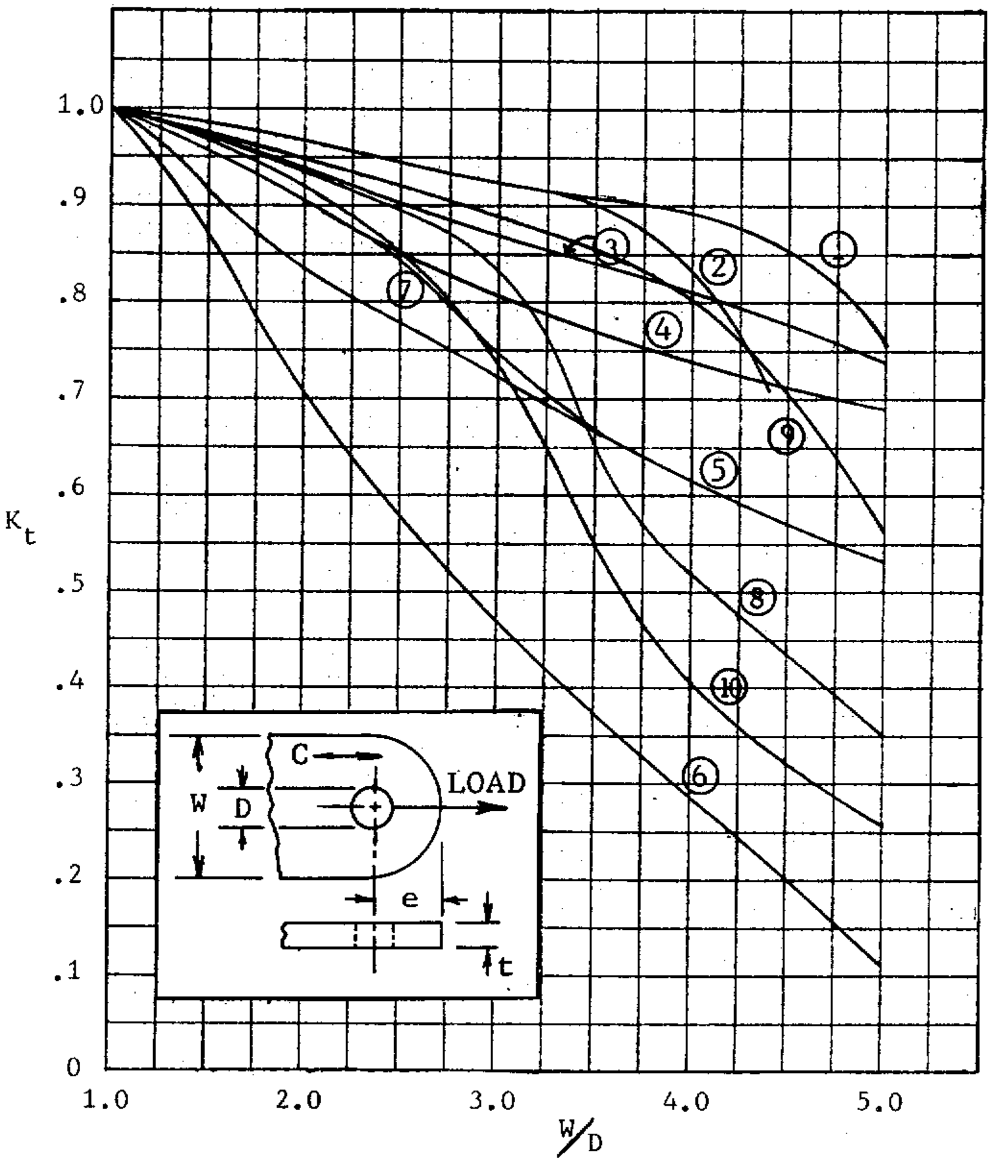


FIGURE 6.32a - TENSION EFFICIENCY FACTORS FOR AXIALLY LOADED ALUMINUM AND STEEL LUGS

Revision B

L, T and ST indicate grain in the "C" direction

Material	Curve
Ti-6Al-4V Ann. Cond. A Die Forging (T) $t \leq 5.0$	1
Ti-6Al-4V Ann. Cond. A Hand Forging (T) $A \leq 16$	1
Ti-6Al-4V Ann. Cond. A Hand Forging (T) $A > 16$	2
Ti-6Al-4V STA Die Forging (L) $t \leq 5.0$	2
Ti-6Al-4V STA Die Forging (T) $t \leq 1.0$	2
Ti-6Al-4V STA Die Forging (T) $1.0 < t \leq 3.0$	3
Ti-6Al-4V STA Hand Forging (L,T) $t \leq 2.0$	1
Ti-6Al-4V STA Hand Forging (T) $2.0 < t \leq 3.0$	2
Ti-6Al-6V-2Sn Ann. Plate (T) $t \leq 2.0$	4
Ti-6Al-6V-2Sn Ann. Die Frg. (ST) $t \leq 2.0$	4
Ti-6Al-6V-2Sn Ann. Hand Frg. (T) $t \leq 2.0$	4
Ti-6Al-6V-2Sn Ann. Plate (T) $2.0 < t \leq 4.0$	5
Ti-6Al-6V-2Sn Ann. Die Frg. (ST) $2.0 < t \leq 4.0$	5
Ti-6Al-6V-2Sn Ann. Hand Frg. (T) $2.0 < t \leq 4.0$	5
Ti-6Al-6V-2Sn STA Die Forg. (L) All	6
Ti-6Al-6V-2Sn STA Die Forging (T) All	7
Ti-6Al-6V-2Sn STA Hand Forging (L,T) $t \leq 4.0$	6
Ti-6Al-6V-2Sn STA Hand Forging (T) $t > 4.0$	7

In no case should the ultimate transverse load be taken as less than that which could be carried by cantilever beam action of the portion of the lug under the load. The load that can be carried by cantilever beam action is indicated approximately by Curve A. Should Ktru be below Curve A, separate calculation as a cantilever beam is necessary.

FIGURE 6.33b (CONT'D) - EFFICIENCY FACTORS FOR TRANSVERSELY LOADED TITANIUM LUGS

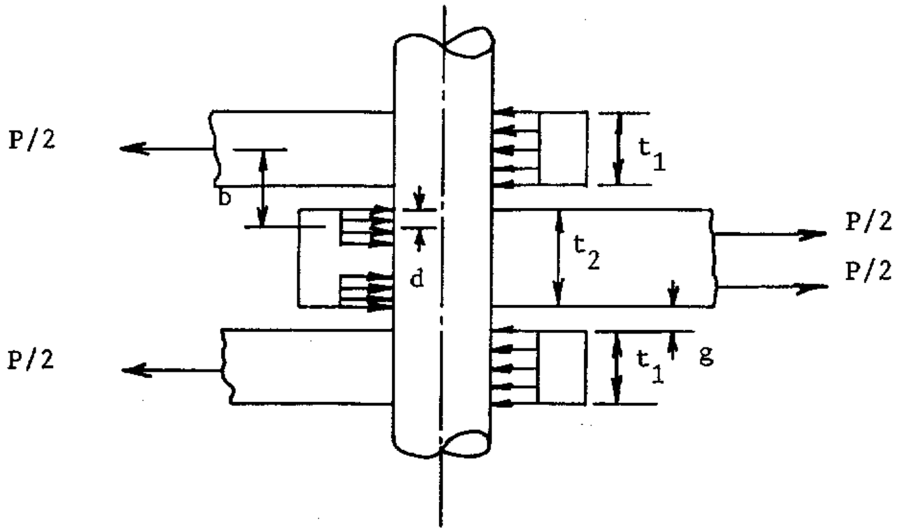


FIGURE 6.34 - SINGLE LUG/CLEVIS JOINT

A. Obtain moment arm "b". For the inner lug of Figure 6.34 calculate $r = [(e/D) - \frac{1}{2}] D/t_2$. Take the smaller of P'_{bru} and P'_{tu} for the inner lug as (P'_u) min and compute (P'_u) min$/A_{br} F_{tux}$. Enter Figure 6.36 and obtain the reduction factor "γ" which compensates for the "peaking" of the distributed pin bearing load near the shear plane. Calculate

$$b = (t_1/2) + g + \gamma(t_2/4)$$

where "g" is the gap between lugs as shown in Figure 6.34 and may be zero. Note that the peaking reduction factor applies only to the inner lugs.

B. Calculate maximum pin bending moment, "M", from the equation

$$M = P(b/2)$$

C. Calculate bending stress assuming a M_c/I distribution.

D. Obtain the ultimate strength of the pin in bending by use of Section 9.4. If the analysis should show inadequate pin bending strength it may be possible to take advantage of any excess lug strength as follows.

E. Consider a portion of the lugs to be inactive as indicated by the shaded area of Figure 6.35. The portion of the thickness to be considered active may have any desired value sufficient to carry the load and should be chosen by trial and error to give approximately equal margins of safety for the lugs and pin.

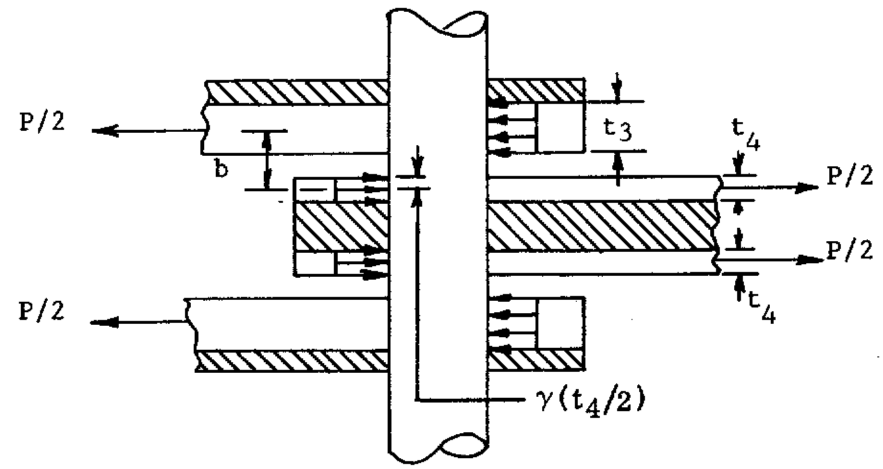


FIGURE 6.35 - ACTIVE LUG THICKNESS

F. Recalculate all lug margins of safety with allowable loads reduced in the ratio of active thickness to actual thickness.

G. Recalculate pin bending moment, $M = P(b/2)$ and margin of safety using value of "b" which is obtained as follows:

$$r = [(e/D) - \tfrac{1}{2}]\, D/2t_4.$$

Take the smaller of P'_{bru} and P'_{tu} for the inner lug, based upon the active thickness, as (P'_u) min and compute (P'_u) min/$A_{bru}\, F_{tux}$ where $A_{br} = 2t_4D$. Enter Figure 6.36 and obtain "γ". Then

$$b = t_3/2 + g + \gamma(t_4/2).$$

This reduced value of "b" should not be used if the resulting eccentricity of load on the outer lugs introduce excessive bending stresses in the adjacent structure. In such cases pins must be strong enough to distribute the load uniformly across the entire lug.

Lug-pin combinations having multiple shear connections such as those shown in Figure 6.37 are analyzed as follows.

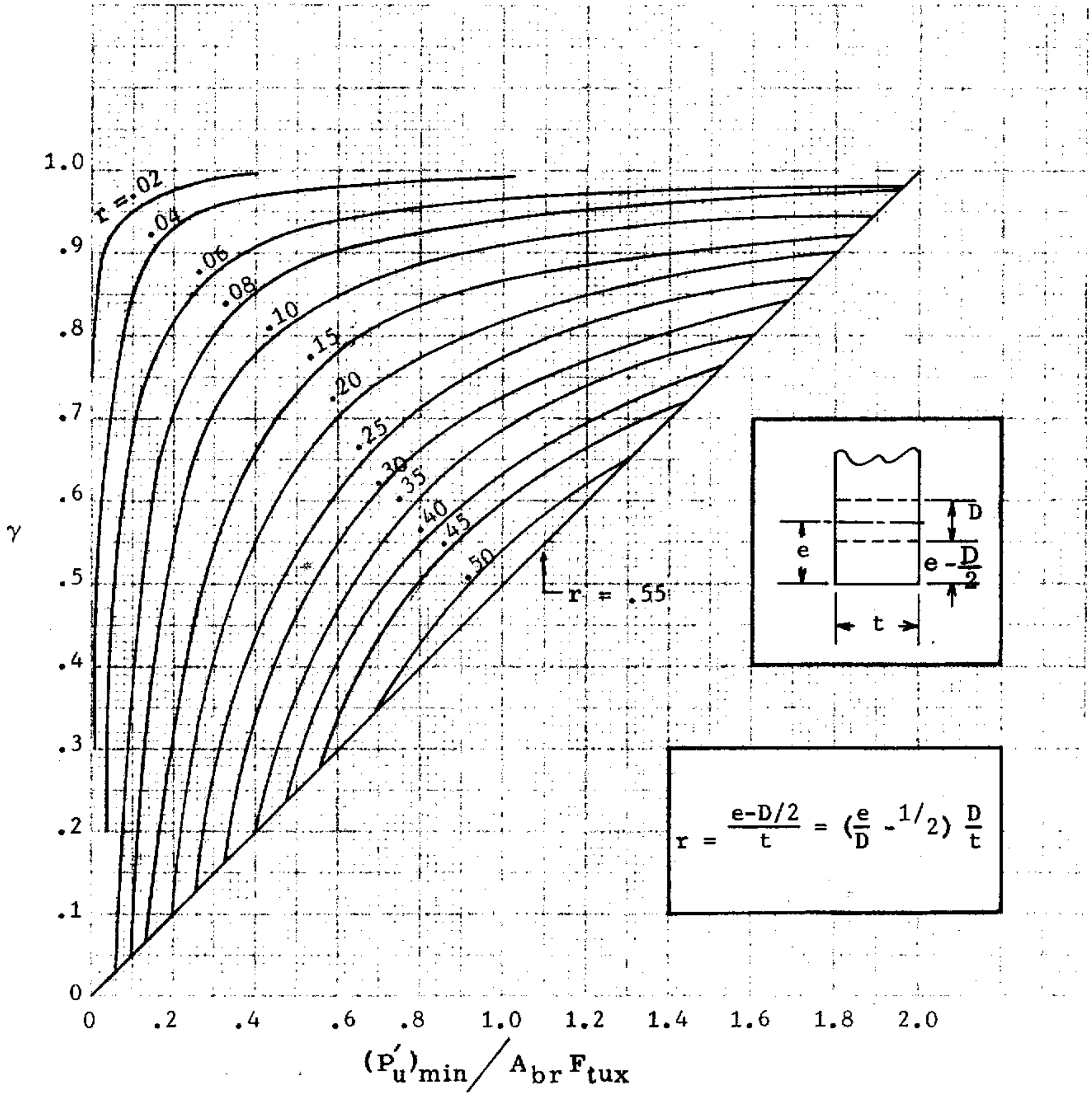


FIGURE 6.36 - REDUCTION FACTOR FOR PEAKING OF BEARING LOADS ON PINS

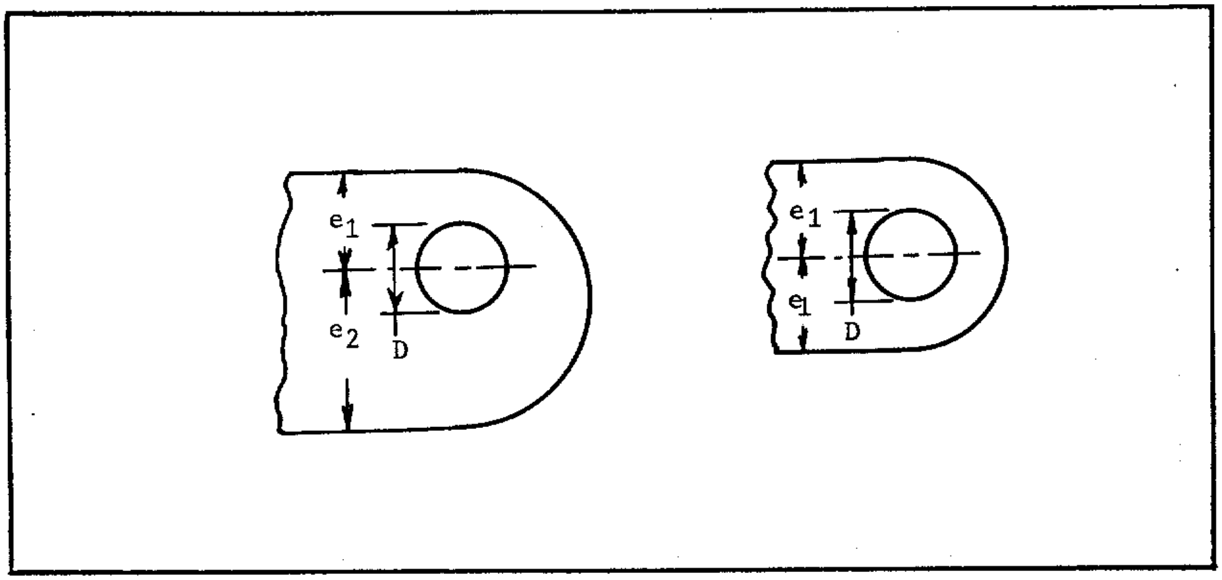


FIGURE 6.38 - LUGS WITH ECCENTRICALLY LOCATED HOLES

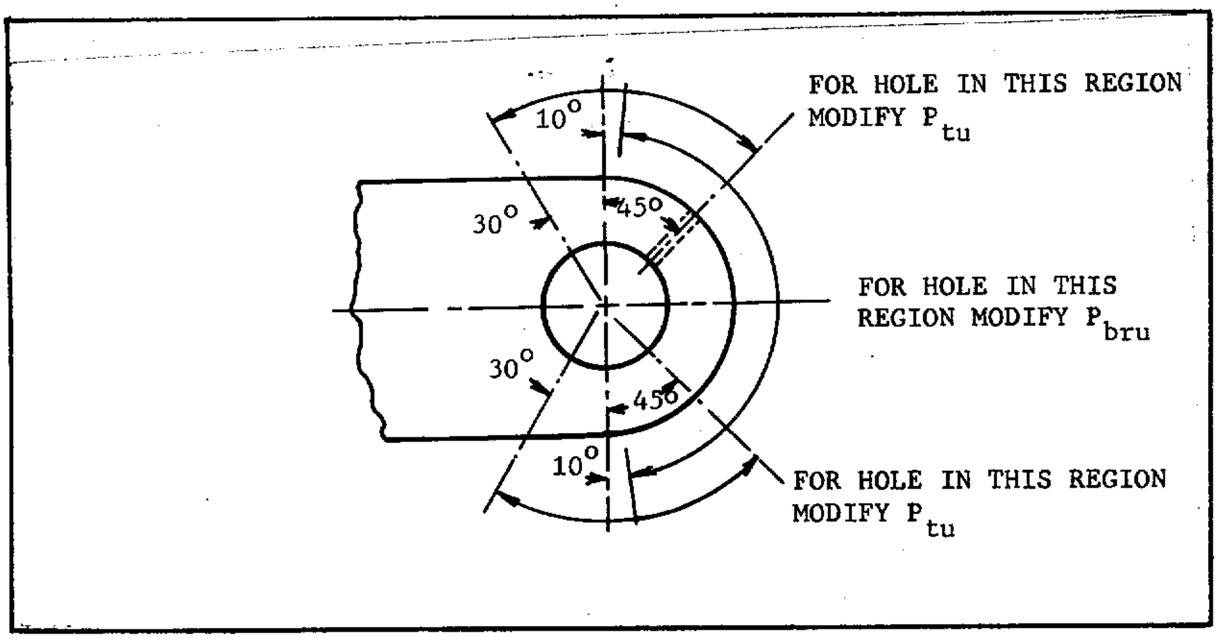


FIGURE 6.39 - LUBRICATION HOLES IN LUGS

B. Transversely loaded lugs. Obtain P'_{tru} neglecting lube hole and multiply by $0.9 \left(1 - \frac{\text{lube hole diameter}}{t}\right)$.

C. Obliquely loaded lugs. Obtain P'_{tu}, P'_{bru}, and P'_{tru} according to A and B above. Then proceed according to Section 6.4.4.4.

6.4.5 Stresses Due to Press Fit Bushings

Pressure between a lug and bushing assembly having negative clearance can be determined from consideration of the radial displacements. After assembly, the increase in inner radius of the ring (lug) plus the decrease in outer radius of the bushing equals the difference between the radii of the bushing and ring before assembly.

$$\delta = u_{ring} - u_{bushing}$$

where

δ = difference between outer radius of bushing and inner radius of the ring

u = radial displacement, positive away from the axis of ring or bushing.

Radial displacement at the inner surface of a ring subjected to internal pressure p is

$$u = \frac{D_p}{E_{ring}} \left[\frac{C^2 + D^2}{C^2 - D^2} + \mu_{ring}\right]$$

Radial displacement at the outer surface of a bushing subjected to external pressure p is

$$u = -\frac{B_p}{E_{bush}} \left[\frac{B^2 + A^2}{B^2 - A^2} - \mu_{bush}\right]$$

where:

A = inner radius of bushing
B = outer radius of bushing
C = outer radius of ring (lug)
D = inner radius of ring (lug)
E = modulus of elasticity
μ = Poisson's ratio

Substitution of the previous two equations into the first yields:

The ultimate tensile stress in the outer fibers in the lug net section is approximately

$$f_t = P/(W - D)t + 6M/k_b(W - D)t^2$$

where k_b is the plastic bending coefficient for the lug net section. The allowable ultimate is found by the methods defined in Section 6.4.4 for axial tension.

The bearing stress distribution between bushing and pin is assumed to be similar to that between the lug and bushing. At ultimate bushing load the maximum bushing bearing stress is approximated by

$$f_{br} = P/Dpt + 6M/k_{br}\ Dpt^2$$

where k_{br}, the plastic bearing coefficient, is assumed the same as the plastic bending coefficient for a rectangular section. The allowable ultimate value is F_{cy} for the bushing material.

The maximum value of pin shear can occur either within the lug or at the common shear face of the two lugs, depending upon the value of M/Pt. At the lug ultimate load,the maximum pin shear stress (fs) is approximated by

$$fs = 1.273\ P/Dp^2;\ (M/Pt \leq 2/3)$$

$$fs = \frac{1.273\ P}{Dp^2}\ \frac{\left(\sqrt{(2M/Pt)^2 + 1} - 1\right)}{\left((2M/Pt) + 1 - \sqrt{(2M/Pt)^2 + 1}\right)};\ (M/Pt > 2/3)$$

The first equation above defines the case where the maximum pin shear is obtained at the common shear face of the lugs. The second equation defines the case where the maximum pin shear occurs away from the shear face. The allowable ultimate is F_{su} of the pin material.

The maximum pin bending moment can occur within the lug or at the common shear faces of the two lugs, depending on the value of M/Pt. At the lug ultimate load,the maximum pin bending stress (f_{bu}) is approximated by

$$f_{bu} = \frac{10.19\ M}{k_b\ Dp^3}\left(\frac{Pt}{2M} - 1\right);\ (M/Pt \leq 3/8)$$

$$f_{bu} = \frac{10.19\ M}{k_b\ Dp^3}\ \frac{\left(\sqrt{(2M/Pt)^2 + 1} - 1\right)}{2M/Pt};\ (M/Pt > 3/8)$$

where k_b is the plastic bending coefficient for the pin.

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The equation for ($M/Pt \leq 3/8$) defines the case where the maximum pin bending moment is obtained at the common shear face of the lugs and the equation for ($M/Pt > 3/8$) defines the case where the maximum pin bending moment occurs away from the shear face, where the pin shear is zero. The allowable ultimate value is F_{bu} for the pin or if deflection or fatigue is critical F_{tu} should be used.

6.4.8 Socket Analysis

The method presented here applies to sockets or sleeves made of aluminum or steel alloys. It is based on the assumption that the socket or sleeve walls (section cut by a plane parallel to the beam or pin centerline) are rectangular or nearly rectangular.

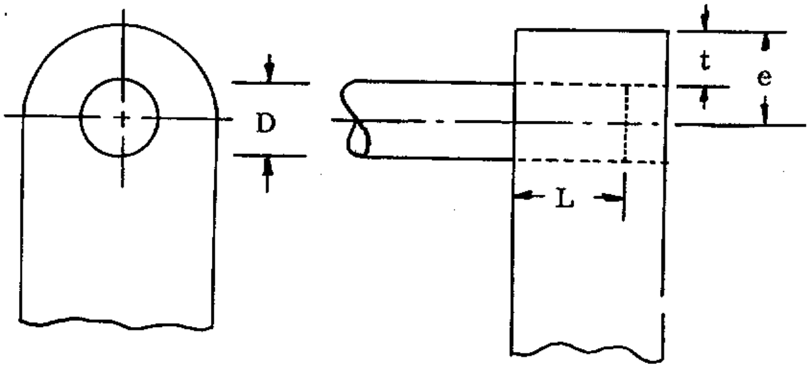
The method for obtaining bearing pressures within the socket or sleeve is also applicable to sockets or sleeves whose wall cross-sections vary appreciably from rectangular. An analysis suitable to the wall configuration must be used for the determination of the wall strengths.

This method may also be used for the analysis of single shear lug joints by considering the lug as a socket and the bolt as the beam.

The maximum wall strengths of sockets or sleeves having rectangular or nearly rectangular wall cross sections (section cut by a plane parallel to the beam or pin center-line) may be determined from the following equations.

$$P_{all} = K_{bru}\left[4(e/D)^2 - 1\right] D K_t F_{tu}\,; \quad (D/t \leq 10)$$

$$P_{all} = K_{bru}\left[16(e/D)\right]\left[\frac{2(e/D)-1}{[2(e/D)+1]^2}\right] D K_t F_{tu}\,; \quad (D/t > 10)$$



the above result in pounds per inch

e = edge distance of socket, inches
D = diameter of beam or bolt, inches
K_t = tension efficiency factor, Figure 6.32
K_{bru} = bearing rupture factor, Figure 6.45
F_{tu} = ultimate tensile strength, psi
t = wall thickness of socket, inch

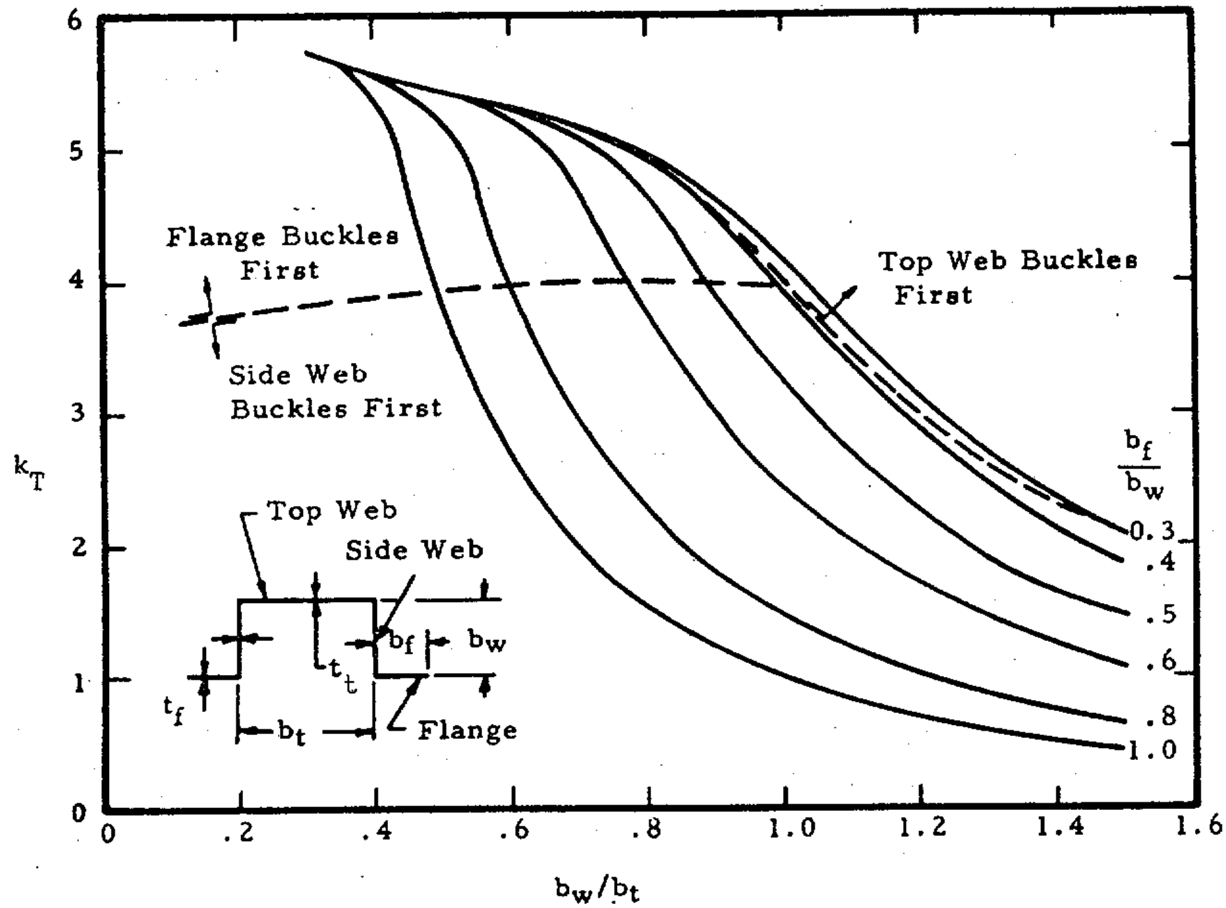
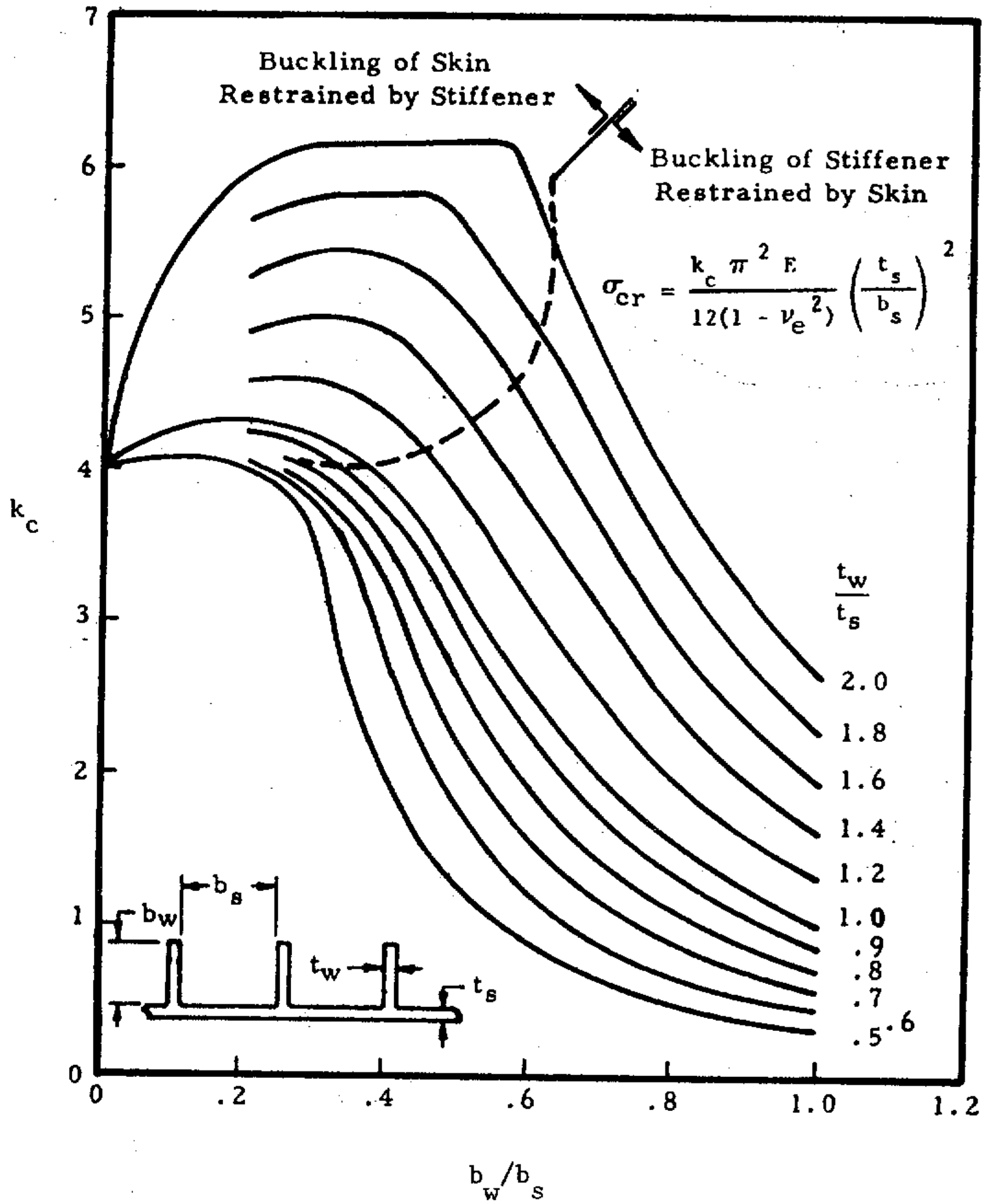


FIGURE 7.12 BUCKLING STRESS FOR HAT SECTION STIFFENERS ($t=t_f=t_w=t_t$)



a) Web Stiffeners. $0.5 < t_w/t_s < 2.0$

FIGURE 7.13 COMPRESSIVE LOCAL BUCKLING COEFFICIENTS FOR INFINITELY WIDE IDEALIZED STIFFENED FLAT PLATES

7.3.3 Crippling Failure of Flat Stiffened Plates in Compression

For stiffened plates having slenderness ratios $L'/\rho \leq 20$, considered to be short plates, the failure mode is crippling rather than buckling when loaded in compression. The crippling strength of individual stiffening elements is considered in Section 11. The crippling strength of panels stiffened by angle-type elements is given by Equation (7-9).

$$\frac{\overline{F}_f}{F_{cy}} = \beta_g \left[\frac{g t_w t_s}{A} \left(\frac{\overline{\eta} \; E}{F_{cy}} \right)^{\frac{1}{2}} \right]^{0.85} \qquad (7\text{-}9)$$

For more complex stiffeners such as Y sections, the relation of Equation (7-10) is used to find a weighted value of t_w.

$$\overline{t_w} = \frac{\Sigma \; a_i \, t_i}{\Sigma \; a_i} \qquad (7\text{-}10)$$

where a_i and t_i are the length and thickness of the cross-sectional elements of the stiffener. Figure 7-15 shows the method of determining the value of g used in Equation (7-9) based on the number of cuts and flanges of the stiffened panels. Figure 7-16 gives the coefficient β_g for various stiffening elements.

If the skin material is different from the stiffener material, a weighted value of F_{cy} given by Equation (7-11) should be used. Here $\overline{t}$ is the effective thickness of the stiffened panel.

$$\overline{F}_{cy} = \frac{F_{cy_s} + F_{cy_w} \left[(\overline{t}/t_s) - 1 \right]}{(\overline{t}/t_s)} \qquad (7\text{-}11)$$

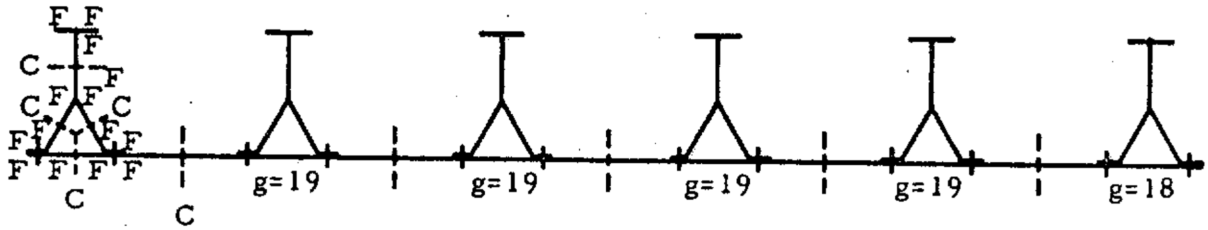
The above relations assume the stiffener-skin unit to be formed monolithically; that is, the stiffener is an integral part of the skin. For riveted construction, the failure between the rivets must be considered. The interrivet buckling stress is determined as to plate buckling stress, and is given by Equation (7-12).

$$F_i = \left(\frac{\epsilon \, \pi^2 \, \eta \, \overline{\eta} \, E}{12(1-\nu^2)} \right) \left(\frac{t_s}{p} \right)^2 \qquad (7\text{-}12)$$

Values of ϵ, the edge fixity, are given in Table 7-2.

After the interrivet buckling occurs, the resultant failure stress of the panel is given by Equation (7-13).

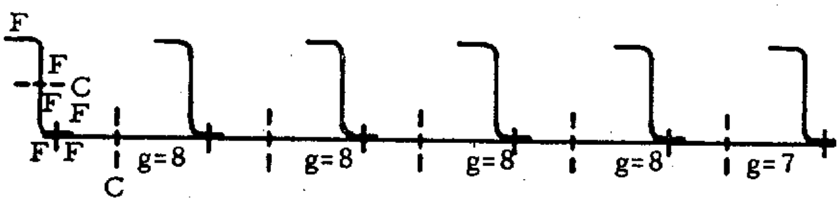
$$\overline{F}_{f_r} = \frac{F_i \, (2b_{ei} \, t_s) + \overline{F}_{fst} \, A_{st}}{(2b_{ei} \, t_s) + A_{st}} \qquad (7\text{-}13)$$



5 cuts
14 flanges
19 = g

Average g = 18.85

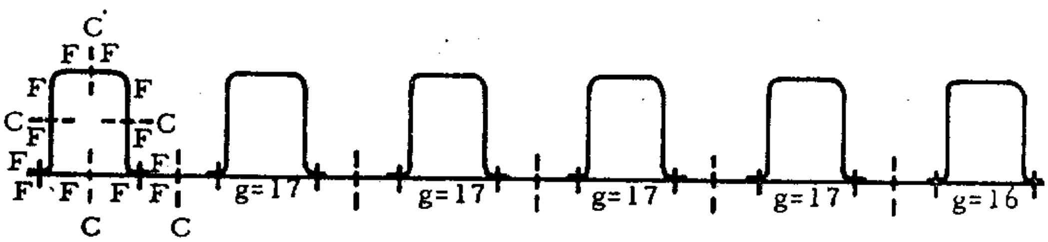
(a) Y-stiffened panel



2 cuts
6 flanges
8 = g

Average g = 7.83

(b) Z-stiffened panel



Average g = 16.83

5 cuts
12 flanges
17 = g

(c) Hat-stiffened panel

FIGURE 7.15 METHOD OF CUTTING STIFFENED PANELS TO DETERMINE g

Here the value b_{ei} is the effective width of skin corresponding to the interrivet buckling stress F_i . The failure stress of short riveted panels by wrinkling can be determined. The following quantities are used:

$\overline{F}_{fst}$ crippling strength of stringer alone (see Section 11, Column Analysis)

$\overline{F}_w$ wrinkling strength of the skin

$\overline{F}_f$ crippling strength of a similar monolithic panel

$\overline{F}_{fr}$ strength of the riveted panel

The wrinkling strength of the skin can be determined from Equation (7-14) and Figure 7-17. Here f is the effective rivet offset distance given in Figure 7-18. This was obtained for aluminum rivets having a diameter greater than 90% of the skin thickness.

$$F_w = \left(\frac{k_w \pi^2 \eta \overline{\eta} E}{12(1-\nu^2)} \right) \cdot \left(\frac{t_s}{b_s} \right)^2 \qquad (7\text{-}14)$$

Now, based on the stringer stability, the strength of the panel can be calculated. Table 7-3 shows the various possibilities and solutions.

It is noted that in no case should $\overline{F}_{fr} > \overline{F}_f$. Thus, the lower of these two values should be used.

The use of the coefficient k_w is based upon aluminum alloy data for other materials. The procedure is to use Equation (7-15) for the panel crippling strength.

$$\frac{F_{fr}}{F_{cy}} = 17.9 \left(\frac{t_w}{f} \right)^{4/3} \left(\frac{t_w}{b_w} \right)^{1/6} \left[\frac{t_s}{b_s} \left(\frac{\eta E}{F_{cy}} \right) \right]^{\frac{1}{2}}$$

(7-15)

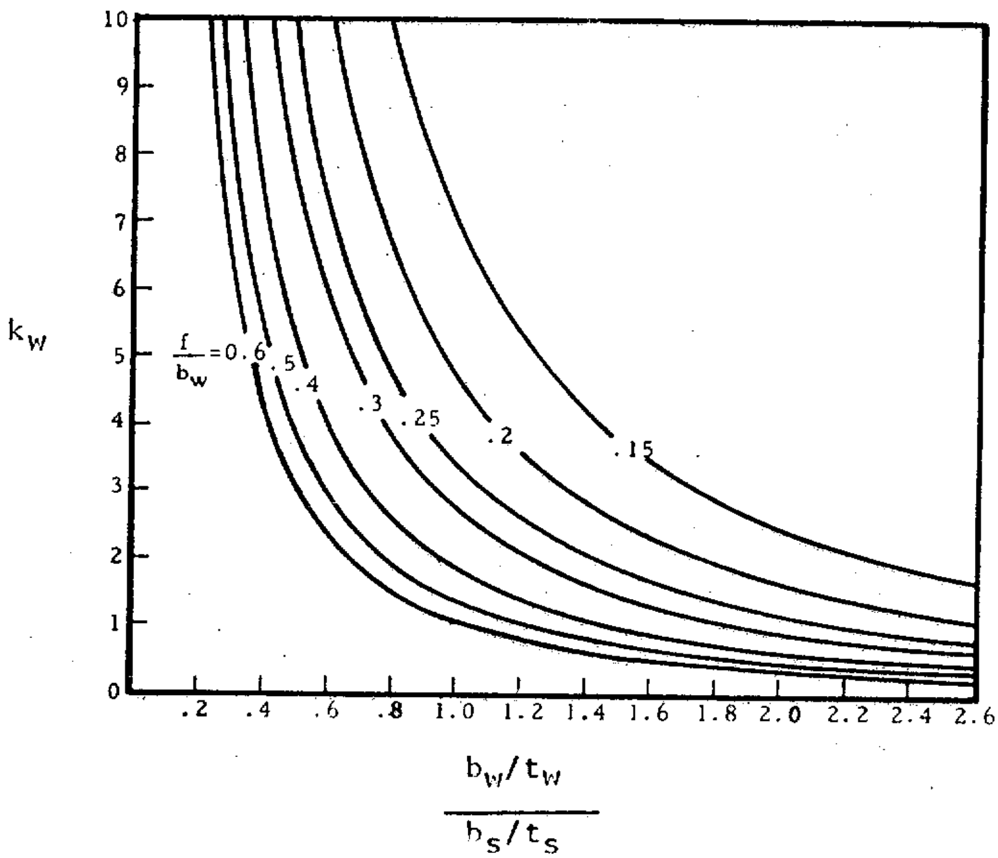


FIGURE 7.17 EXPERIMENTALLY DETERMINED COEFFICIENTS FOR FAILURE IN WRINKLING MODE

$\phi = \frac{TL}{KG}$	T = Applied Torque (in-lb)	G = Modulus of Rigidity (psi)
$f_s = \frac{T}{Q}$	L = Length of Beam (in)	ϕ = Angle of Twist (rad)
	K = Torsional Constant (in^4)	f_s = Shear Stress (psi)
	Q = Section Modulus (in^3)	

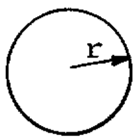
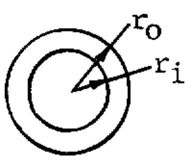
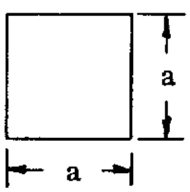
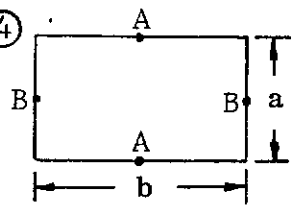
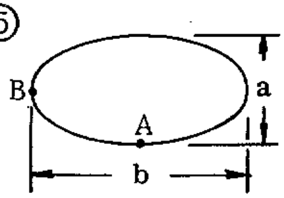
SECTION	K	Q	MAX STRESS
①  SOLID CIRCLE	$\frac{\pi r^4}{2}$	$\frac{\pi r^3}{2}$	at r_{max}
②  HOLLOW CIRCLE	$\frac{\pi}{2}(r_o^4 - r_i^4)$	$\frac{\pi}{2}(r_o^3 - r_i^3)$	at r_o
③  SOLID SQUARE	$0.1406a^4$	$0.208a^3$	at midpoint of each side
④  SOLID RECTANGLE	$\beta b a^3$ $\beta = \left[.333 - \frac{.21}{(b/a)}\left(1 - \frac{0.0833}{(b/a)^4}\right)\right]$	$\alpha b a^2$ $\alpha = \frac{1}{\left[3 + \frac{1.8}{(b/a)}\right]}$	@A: $f_s = \frac{T}{Q}$ @B: $f_s = \frac{Ta}{Qb}$
⑤  SOLID ELLIPSE	$\frac{\pi b^3 a^3}{16(b^2 + a^2)}$	$\frac{\pi b a^2}{16}$	@A: $f_s = \frac{T}{Q}$ @B: $f_s = \frac{Ta}{Qb}$

TABLE 8.1 - EQUATIONS FOR STRESS AND DEFORMATION IN SOLID SECTIONS LOADED IN TORSION

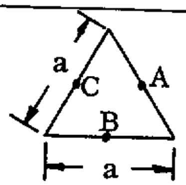
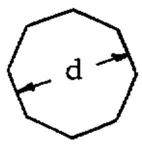
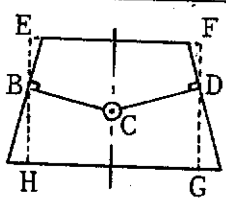
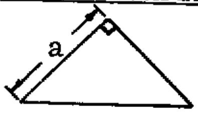
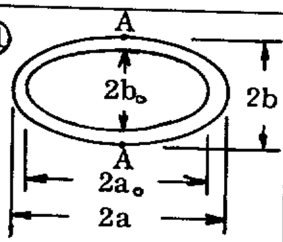
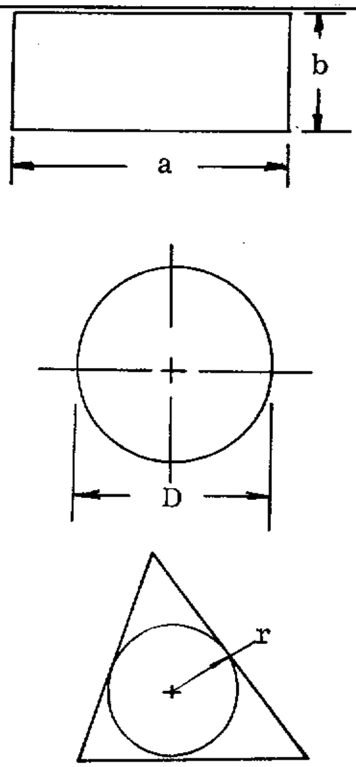
SECTION	K	Q	MAX STRESS
⑥  SOLID EQUILATERAL TRIANGLE	$\frac{a^4\sqrt{3}}{80}$	$\frac{a^3}{20}$	at A,B & C
⑦  SOLID HEXAGON	$0.1045\,d^4$	$0.1704\,d^3$	at midpoint of each side
⑧  SOLID OCTAGON	$0.1021\,d^4$	$0.1751\,d^3$	at midpoint of each side
⑨  SOLID ISOSCELES TRAPEZOID	Form equivalent rectangle through points B and D. Then use equations for rectangle to determine stress and twist. To locate B and D, construct perpendiculars from centroid (c) to each side (B and D).		
⑩  SOLID RIGHT ISOSCELES TRIANGLE	$0.0261\,a^4$	$0.0554\,a^3$	at center of long side
⑪  HOLLOW ELLIPSE	$\frac{\pi a^3 b^3 (1 - q^4)}{a^2 + b^2}$ $q = \frac{a_o}{a} = \frac{b_o}{b}$	$\frac{\pi a b^2 (1 - q^4)}{2}$	at A

TABLE 8.1 (CONT'D) - EQUATIONS FOR STRESS AND DEFORMATION IN SOLID SECTIONS LOADED IN TORSION

It is possible to determine the volume of the sand heap for any cross section by integration. Figure 8.5 shows equations for sand heap volumes with various bases.



$$V = \frac{b^2(3a-b)}{12} \qquad a \geq b$$

$$V = \frac{\pi D^3}{24}$$

$$V = \frac{Ar}{3}$$

A = Area of triangle

FIGURE 8.5 - SAND HEAP VOLUMES

8.6 Allowable Stresses

For limit load conditions, the applied stresses should be kept below the ultimate shear stress, F_{su}. These are defined for various materials in MIL-HDBK-5.

The torsional failure of tubes may be due to plastic failure of the material, instability of the walls, or an intermediate condition. Pure shear failure will not usually occur within the range of wall thicknesses commonly used for aircraft tubing. Torsional allowable stresses are shown in Figure 8.6 through 8.22. These curves take into account the parameter L/D and are in good agreement with experimental results.

Interaction data of Section 4 should be used when other stresses are combined with torsion.

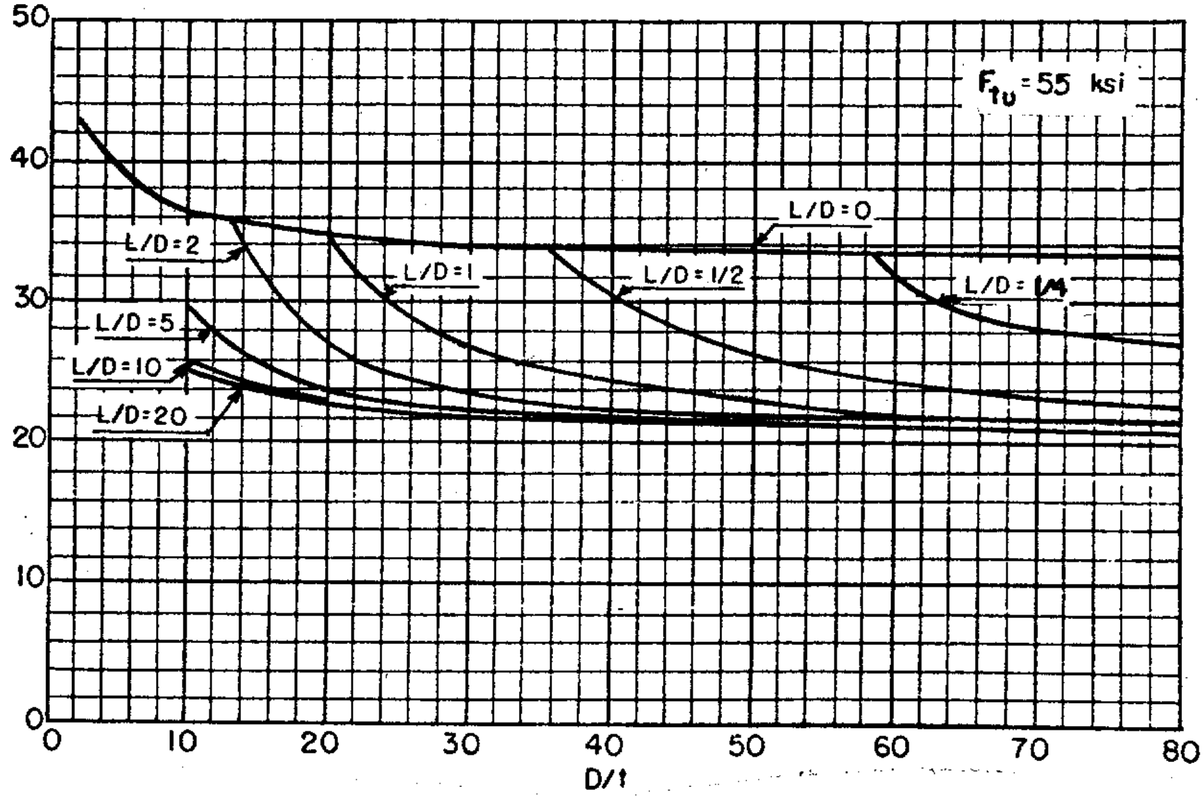


FIGURE 8.6 - TORSIONAL MODULUS OF RUPTURE - PLAIN CARBON STEELS
F_{tu} = 55 ksi

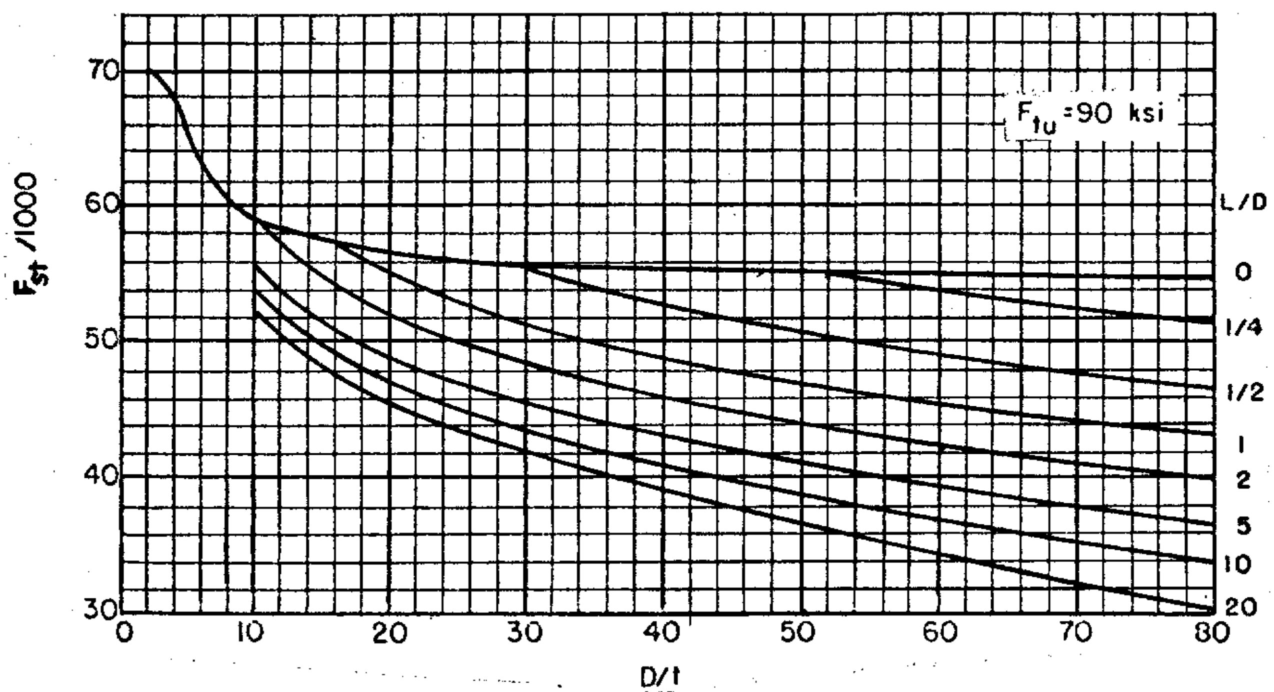


FIGURE 8.7 - TORSIONAL MODULUS OF RUPTURE - ALLOY STEELS HEAT TREATED
TO F_{tu} = 90 ksi

9.5 Lateral Buckling of Beams

Beams in bending under certain conditions of loading and restraint can fail by lateral buckling in a manner similar to that of columns loaded in axial compression. However, it is conservative to obtain the buckling load by considering the compression side of the beam as a column since this approach neglects the torsional rigidity of the beam.

In general, the critical bending moment for the lateral instability of the deep beam, such as that shown in Figure 9.8 may be expressed as

$$M_{cr} = \frac{K\sqrt{EI_y\ GJ}}{L}$$

where J is the torsion constant of the beam and K is a constant dependent on the type of loading and end restraint. Thus, the critical compressive stress is given by

$$F_{cr} = \frac{M_{cr}c}{I_x}$$

where c is the distance from the centroidal axis to the extreme compression fibers. If this compressive stress falls in the plastic range, an equivalent slenderness ratio may be calculated as

$$\frac{L'}{\rho} = \pi\sqrt{\frac{E}{F_{cr}}}$$

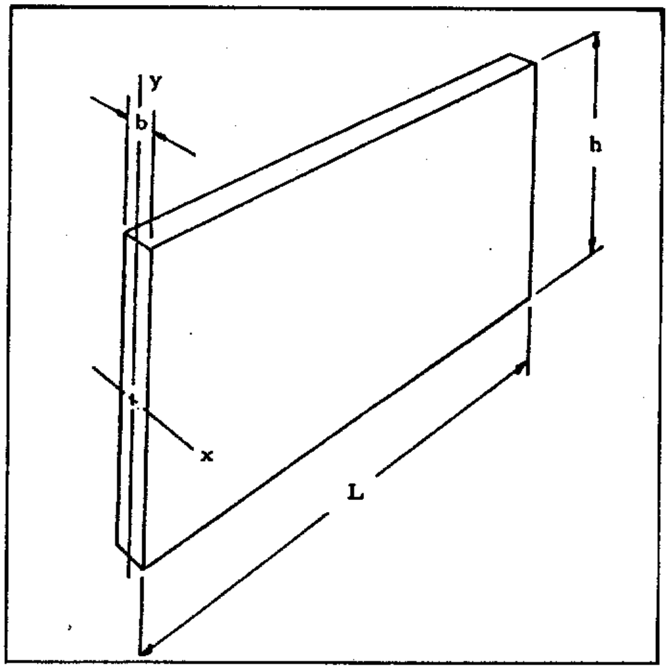


FIGURE 9.8 - DEEP RECTANGULAR BEAM

The actual critical stress may then be found by entering the column curves of Section 11 at this value of (L'/ρ). This value of stress is not the true compressive stress in the beam, but is sufficiently accurate to permit its use as a design guide.

9.5.1 Lateral Buckling of Deep Rectangular Beams

The critical moment for deep rectangular beams loaded in the elastic range loaded along the centroidal axis is given by

$$M_{cr} = 0.0985\ K_m E\ (\frac{b^3 h}{L})$$

where K_m is presented in Table 9.5 and b, h, and L are as shown in Figure 9.8. The critical stress for such a beam is

$$F_{cr} = K_f E(\frac{b^2}{Lh})$$

where K_f is presented in Table 9.5.

If the beam is not loaded along the centroidal axis, as shown in Figure 9.9, a corrected value K_f' is used in place of K_f. This factor is expressed as

$$K_f' = K_f\ (1 - n)\ (\frac{s}{L})$$

where n is a constant defined below:

(1) For simply supported beams with a concentrated load at midspan, n = 2.84.

(2) For cantilever beams with a concentrated end load, n = 0.816.

(3) For simply supported beams under a uniform load, n = 2.52.

(4) For cantilever beams under a uniform load, n = 0.725.

Note: s is negative if the point of application of the load is below the centroidal axis.

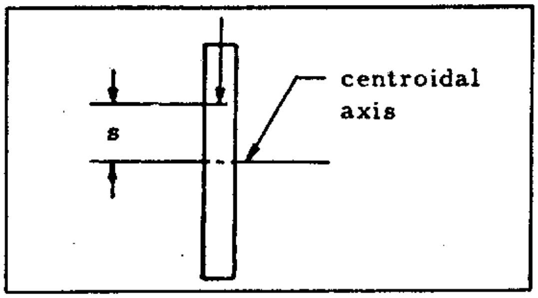


FIGURE 9.9 - DEEP RECTANGULAR BEAM LOADED AT A POINT REMOVED FROM THE CENTROIDAL AXIS

9.6.5 Plastic Bending Modulus, F_b

Figures 9.15 through 9.18 show curves for various materials. The curves are plotted as k = 2 Qc/I versus F_b and strain. The strain versus F_b curves show f_o and F_b versus strain. The f_o curve is at k=1. The rest of the curves employ equation 9.16 to obtain F_b^o at various strains.

9.6.6 Application of Plastic Bending

Consider a rectangular beam section which is .25-inch thick and 1.5-inch deep. It is made of 7075-T6 extrusion and it is desired to find the yield and ultimate bending moment for the section.

$$F_b = f_m + f_o\ (k-1),\ \text{equation 9.16}$$

$$k = 2Qc/I = 2\left((.25)(.75)(.375)\right)(.75)/\left((.25)(1.5)^3/12\right) = 1.5$$

The value of k can also be found in Figure 9.10.

$$F_{tu} = 75000\ \text{psi},\ F_{ty} = 65000\ \text{psi}$$

Find the yield bending strength: The value of f_m in equation 9.16, the maximum stress permitted on the most remote fiber, is 65000 psi, the yield stress of the material. To find f_o, go to Figure 9.16 (r) to find the point on the stress-strain curve (k=1) that corresponds to a stress of 65000 psi. This point is projected downward to the f_o curve where a stress of 26000 psi is read. Then

$$F_b = 65000 + 26000\ (1.5 - 1) = 78{,}000\ \text{psi}$$

This same stress can be obtained by projecting up from the stress-strain curve in Figure 9.16 (r) to the curve labeled k=1.5 and reading F_b directly.

The yield moment is then found to be

$$M_y = F_b\ I/c = 78000\ (.0703)/.75 = 7312.5$$

The ultimate moment is found the same way.

$$F_b = 75000 + 70500\ (1.5 - 1) = 110{,}250\ \text{psi}$$

$$M_u = 110{,}250\ (.0703)/.75 = 10334$$

The previous example is for a section which is stable in compression and symmetrical about two axes. Consider now a section which is symmetrical about one axis and probably partially unstable. The Tee shown in Figure 9.22(a) is a

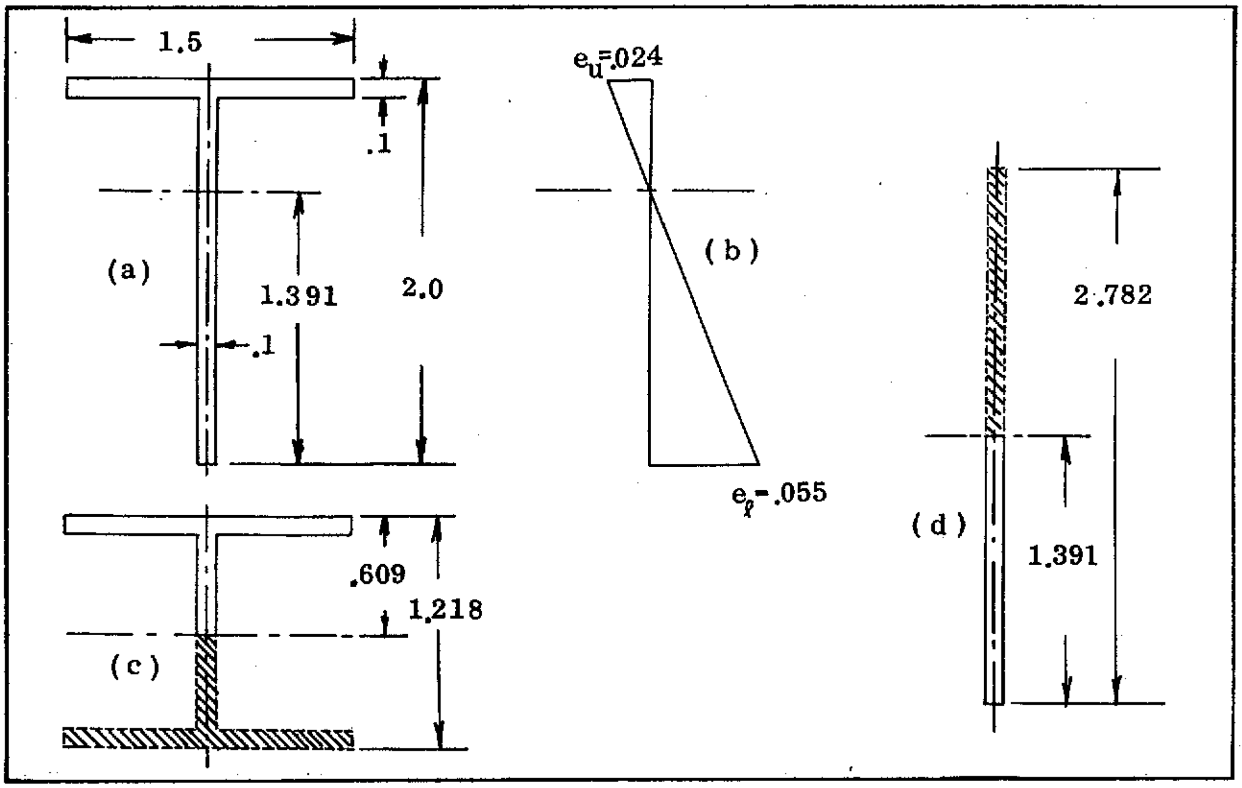


FIGURE 9.22 - UNSYMMETRICAL EXAMPLE

7075-T6 extrusion. Again, the properties of Figure 9.16(r) are used.

First consider the maximum strain, e_u, in Figure 9.16(r), e_u = .055 in/in. It is apparent that the lower leg of the Tee will strain higher than the cap when the Tee is bent about the x axis, so set the lower extreme fiber strain equal 0.055. By ratioing the lower strain by the distances from the N.A. the strain in the upper extreme fiber is $e_u = (.609/1.391)(.055) = .024$ in/in.

Equation 9.16 was derived for symmetrical sections about a neutral axis. The Tee can be made into two sections which are symmetrical about their neutral axis. These are shown in Figure 9.22(c) and (d).

Figure 9.22(d) shows how the lower portion is made symmetrical about the neutral axis by adding the shaded portion above. The internal bending resistance is found for the entire section in 9.22(d). One-half of this amount will be the true moment developed by the lower portion.

$I = (.1)(2.782)^3/12 = .179$

$I/c = .179/1.391 = .129$

$k = 2\,Qc/I = 2(1.391)(.1)(.6955)/.129 = 1.5$

From Figure 9.16(r) at $e = .055$, F_b = 110,000 psi

$M = F_b\,(I/c)(1/2) = 7095$ in-lb

The 1/2 is because only one-half the beam section is used in Figure 9.22(a).

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SECTION 10

BUCKLING

10.1 SHEAR RESISTANT BEAMS

10.1.1 Introduction

For shear resistant beams in bending, the simplifying assumption that all the mass is concentrated at the centroids of the flanges may be made if the web is sufficiently thin. The simple beam formulas may then be reduced to

$$f_b = \frac{M}{A_f h} \quad \text{for bending, and} \qquad 10.1$$

$$f_s = \frac{V}{ht} = \frac{q}{t} \text{ for shear.} \qquad 10.2$$

The bending is, therefore, resisted by the flanges, and the shear is resisted by the webs. Figure 10.1 may be used to determine if the panel in question is shear resistant or a state of incomplete diagonal tension is developed. An estimate of efficient stiffener size and minimum stiffener moment of inertia are presented in Figures 10.2 and 10.3.

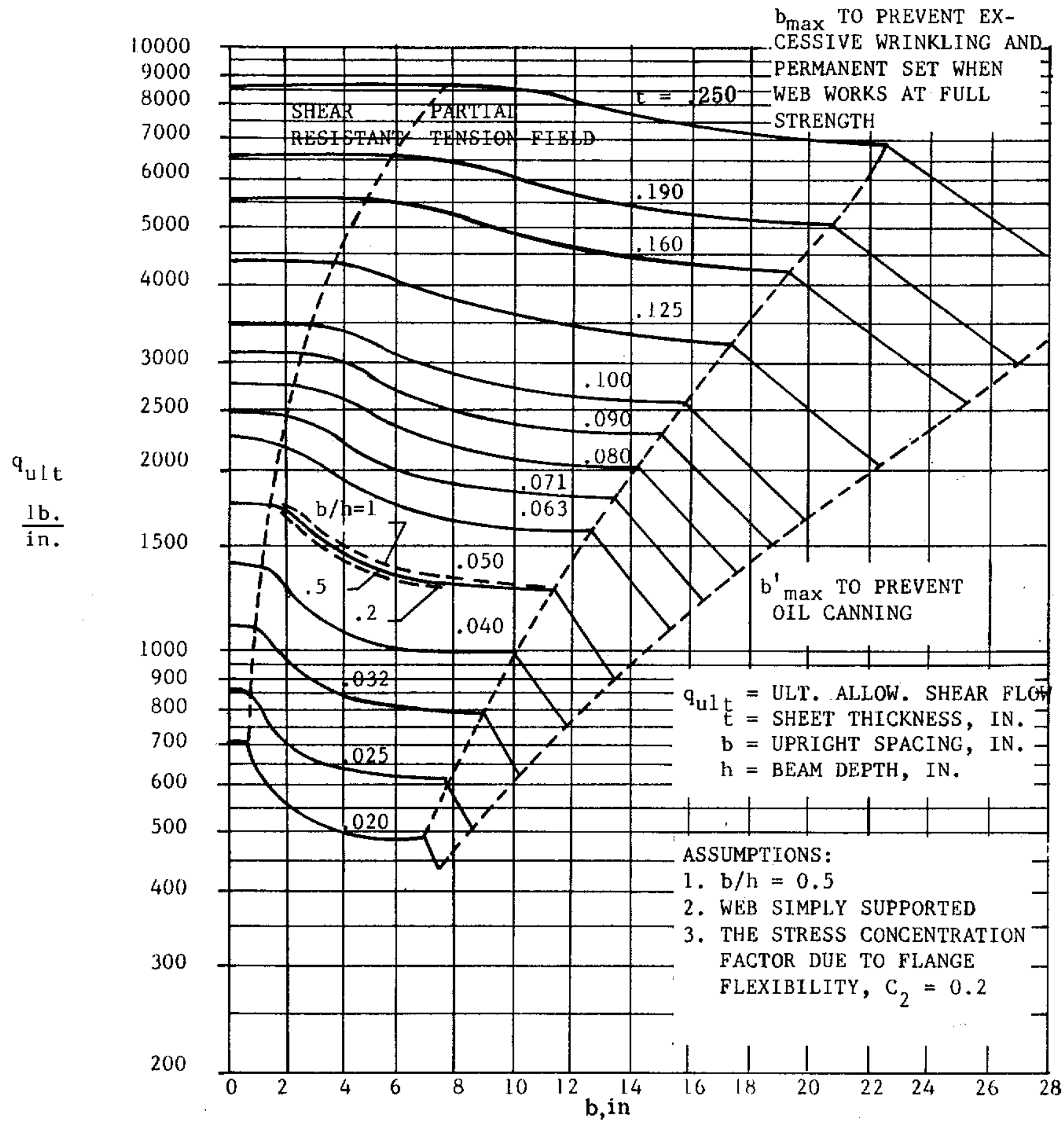
10.1.2 Unstiffened Shear Resistant Beams

Failure checks must be made for both the web and flanges of the beam. The flange is usually considered to have failed if its bending stress exceeds the yield stress of the material, unless some permanent set is allowed. The allowable average stress at the ultimate load F_s is either 85% of the ultimate strength in shear or 125% of the yield strength in shear, if the web is not subject to collapse. For thin webs ($h/t > 60$), initial buckling does not cause collapse. The collapsing stress for two aluminum alloys is given in Figure 10.4.

The required web thickness is

$$t = \frac{V}{hF_s} \quad \text{or} \quad t = \frac{V}{hF_{scoll}},$$

whichever is larger.



NOTE: FOR BARE 7075-T6, MULTIPLY BY 1.07

FIGURE 10.1 - ULTIMATE ALLOWABLE SHEAR FLOW FOR ALCLAD 7075-T6 SHEET

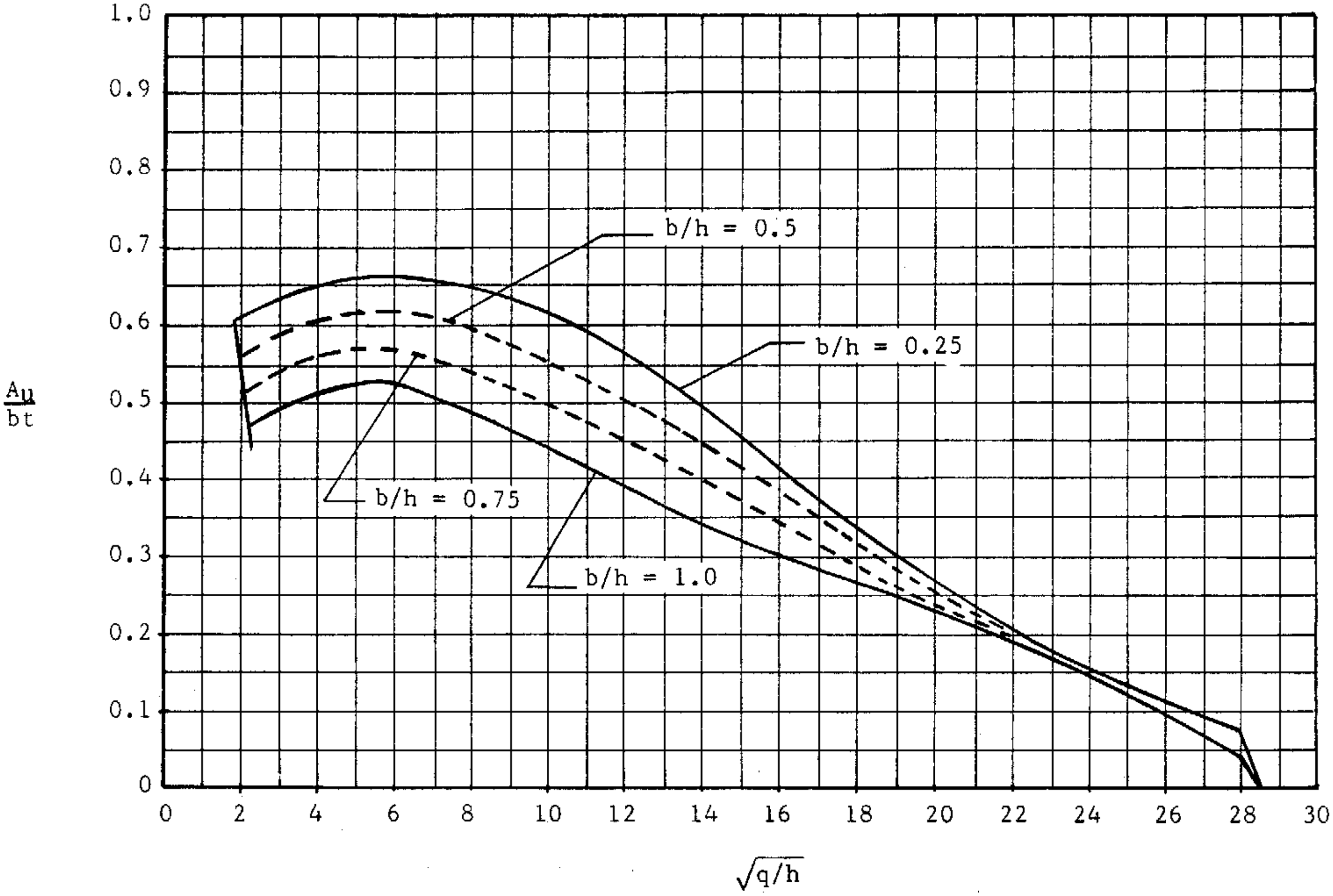


FIGURE 10.2 - CHART FOR ESTIMATING EFFICIENT STIFFENER SIZE, 7075-T6 WEB WITH 7075-T6 SINGLE ANGLE STIFFENER

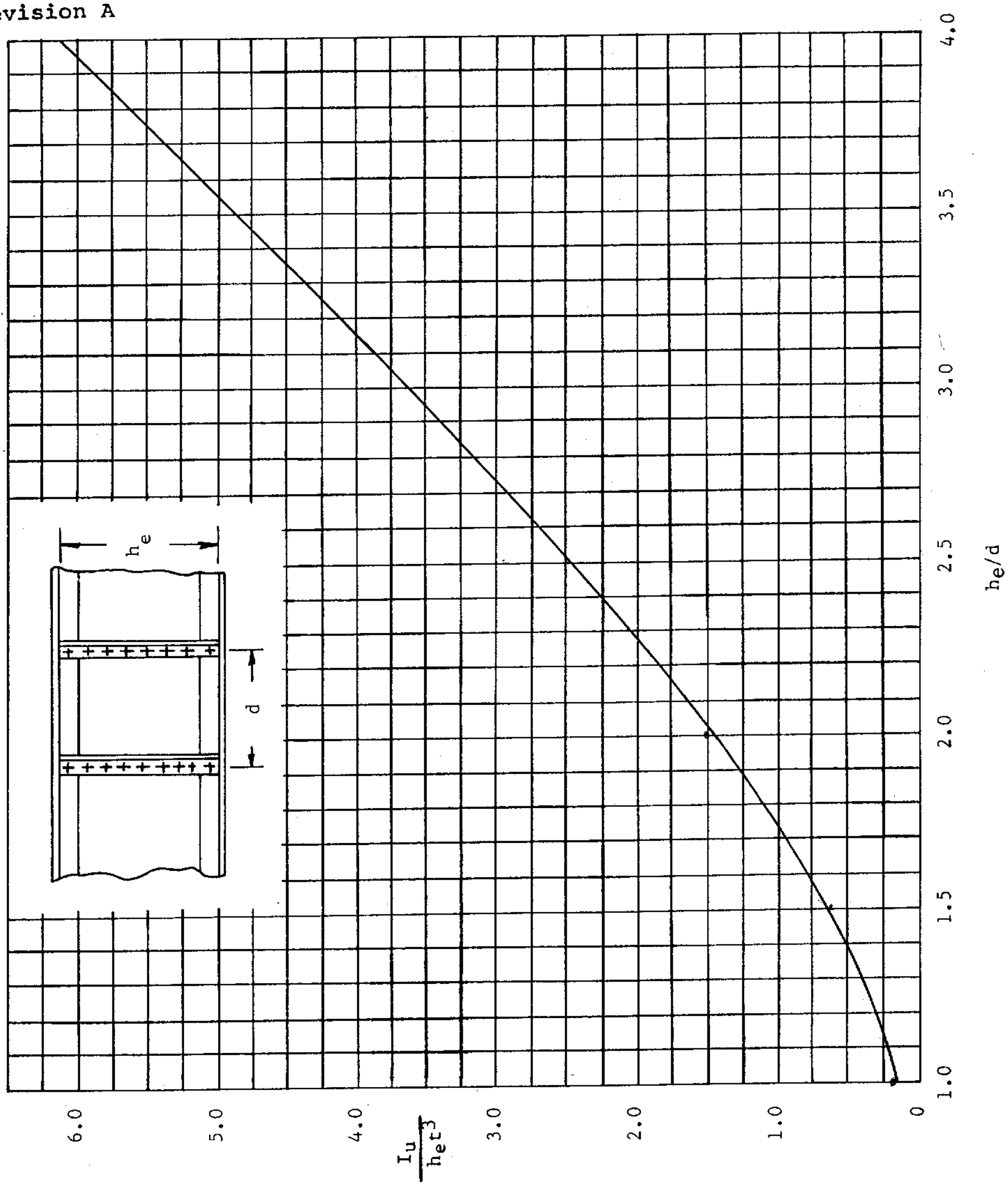


FIGURE 10.3 - MINIMUM STIFFENER MOMENT OF INERTIA FOR SHEAR RESISTANT WEBS

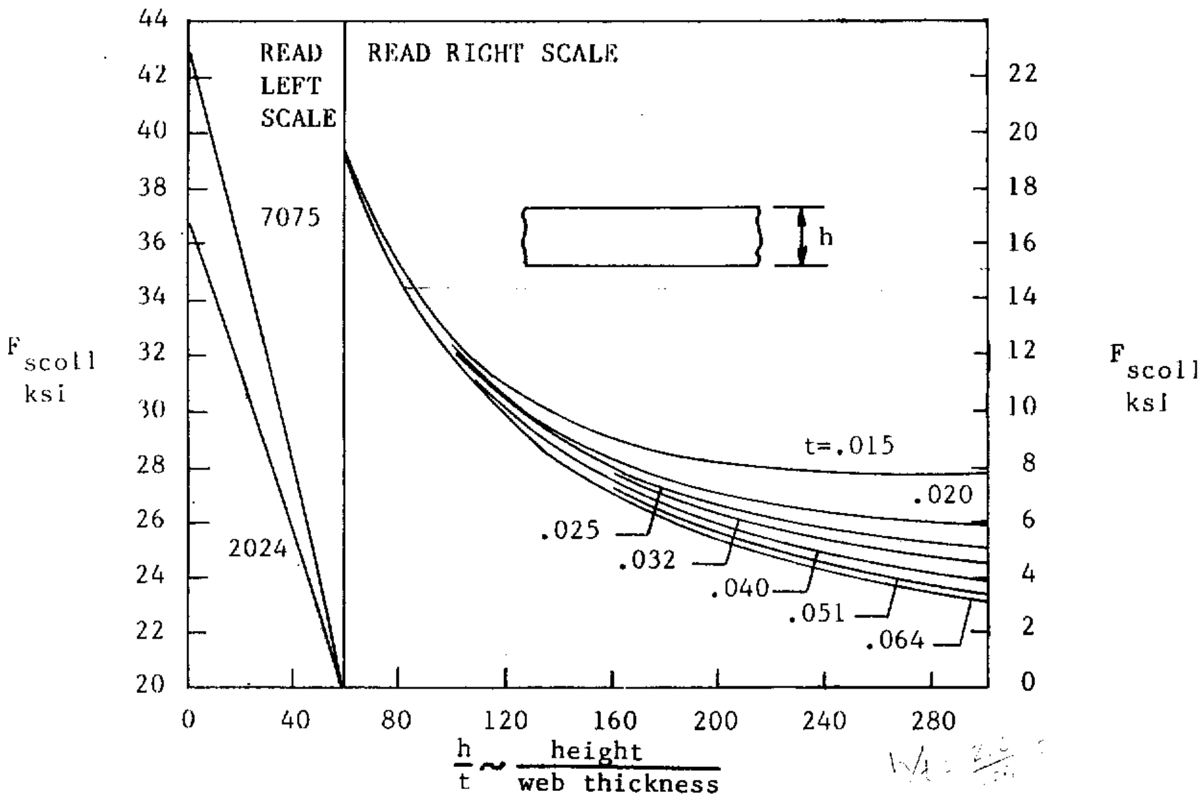


FIGURE 10.4 - COLLAPSING SHEAR STRESS, F_{scoll}, FOR SOLID WEBS

10.1.3 Stiffened Shear Resistant Beams

The vertical stiffeners of a shear resistant beam increase the web buckling stress. They resist no compressive load, but divide the web into smaller unsupported rectangles. An analysis of the flange, web and rivets is required.

The yielding or ultimate strength of the flanges must be checked by using equation 10.1.

In addition to the strength of the web panel, stability must also be checked. The strength of the web may be checked by using equation 10.2. Stability of the web may be checked by using the below equation in conjunction with Figures 10.5 through 10.11.

$$\frac{F_{scr}}{\eta} = K_s E \left(\frac{t}{d}\right)^2$$

where F_{scr} = critical buckling stress of the web

η = plasticity coefficient

and K_s = critical shear stress coefficient
$K_s = f(d/h, \text{edge restraint})$

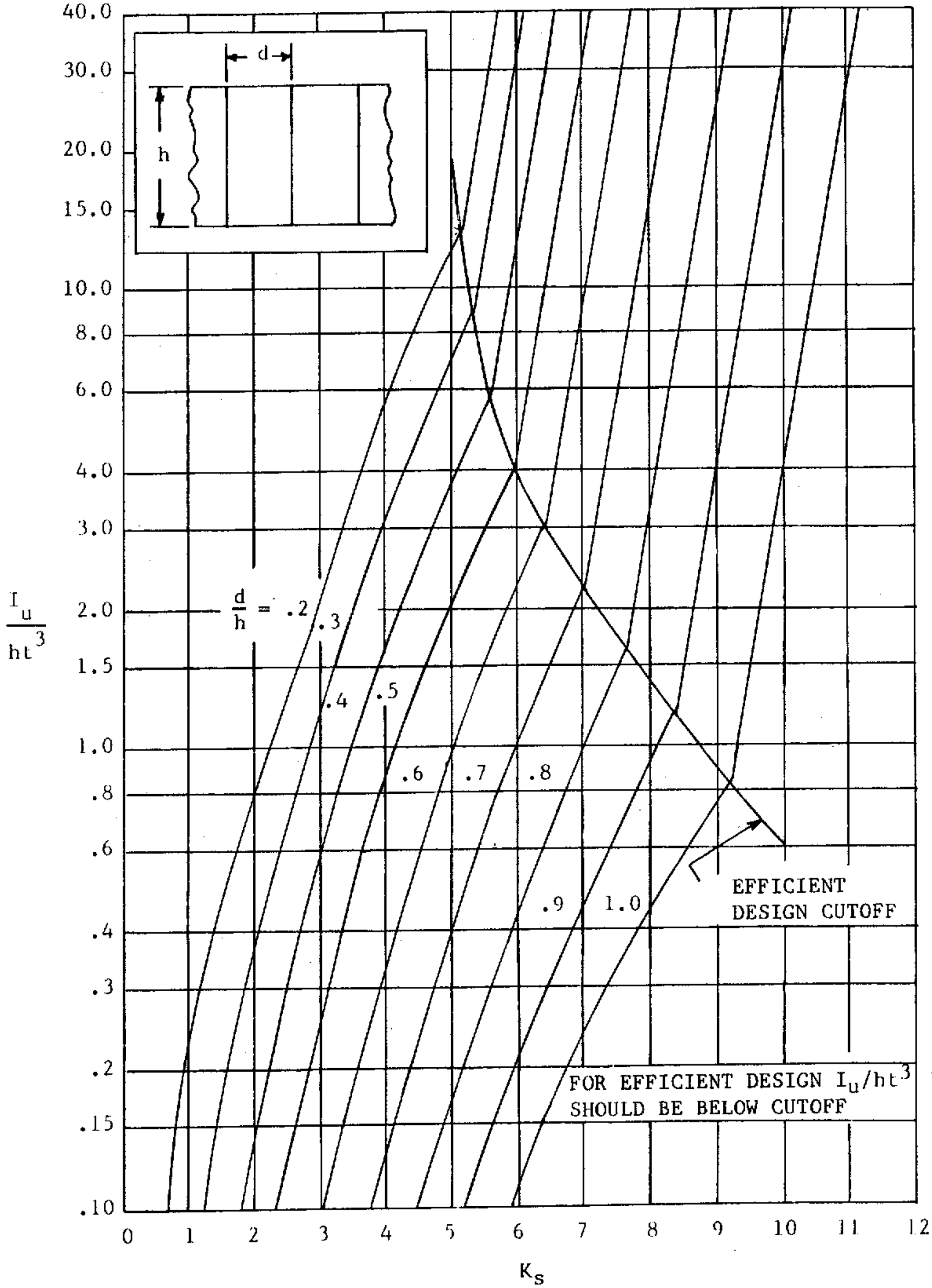


FIGURE 10.5 - CRITICAL SHEAR STRESS COEFFICIENT, K_s

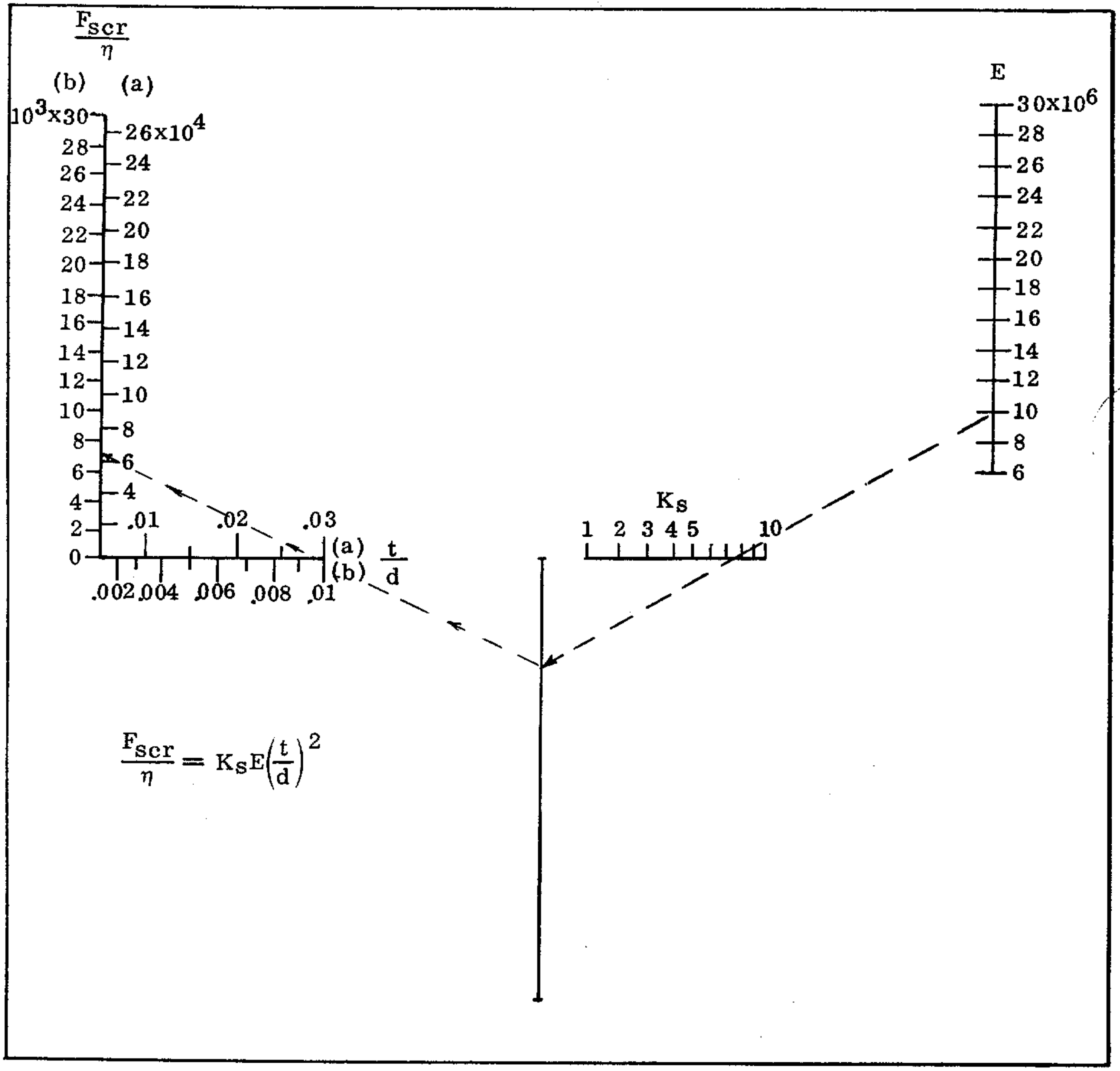


FIGURE 10.6 - NOMOGRAPH FOR CRITICAL BUCKLING STRESS

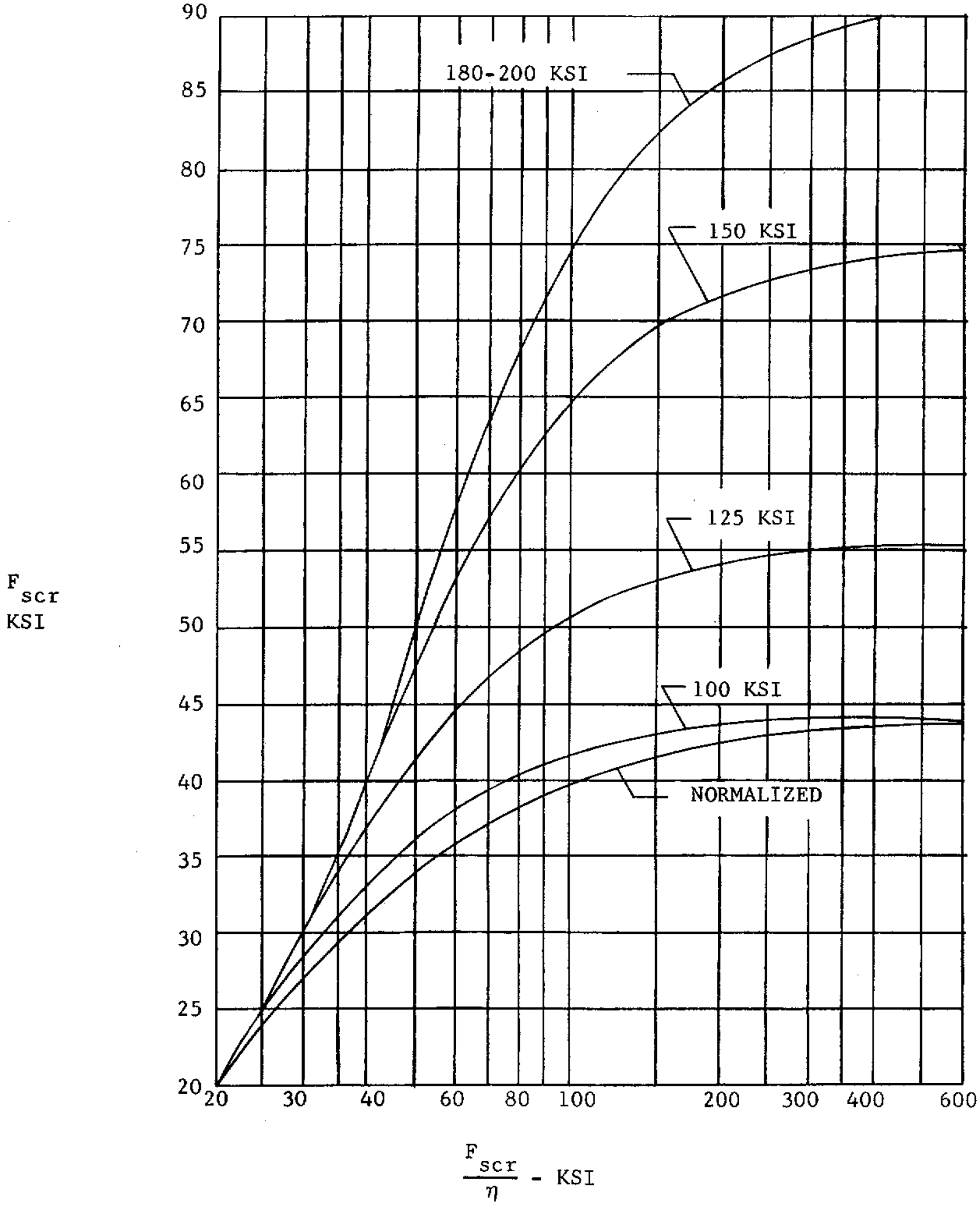


FIGURE 10.7 - F_{scr} VS. F_{scr}/η FOR ALLOY STEEL

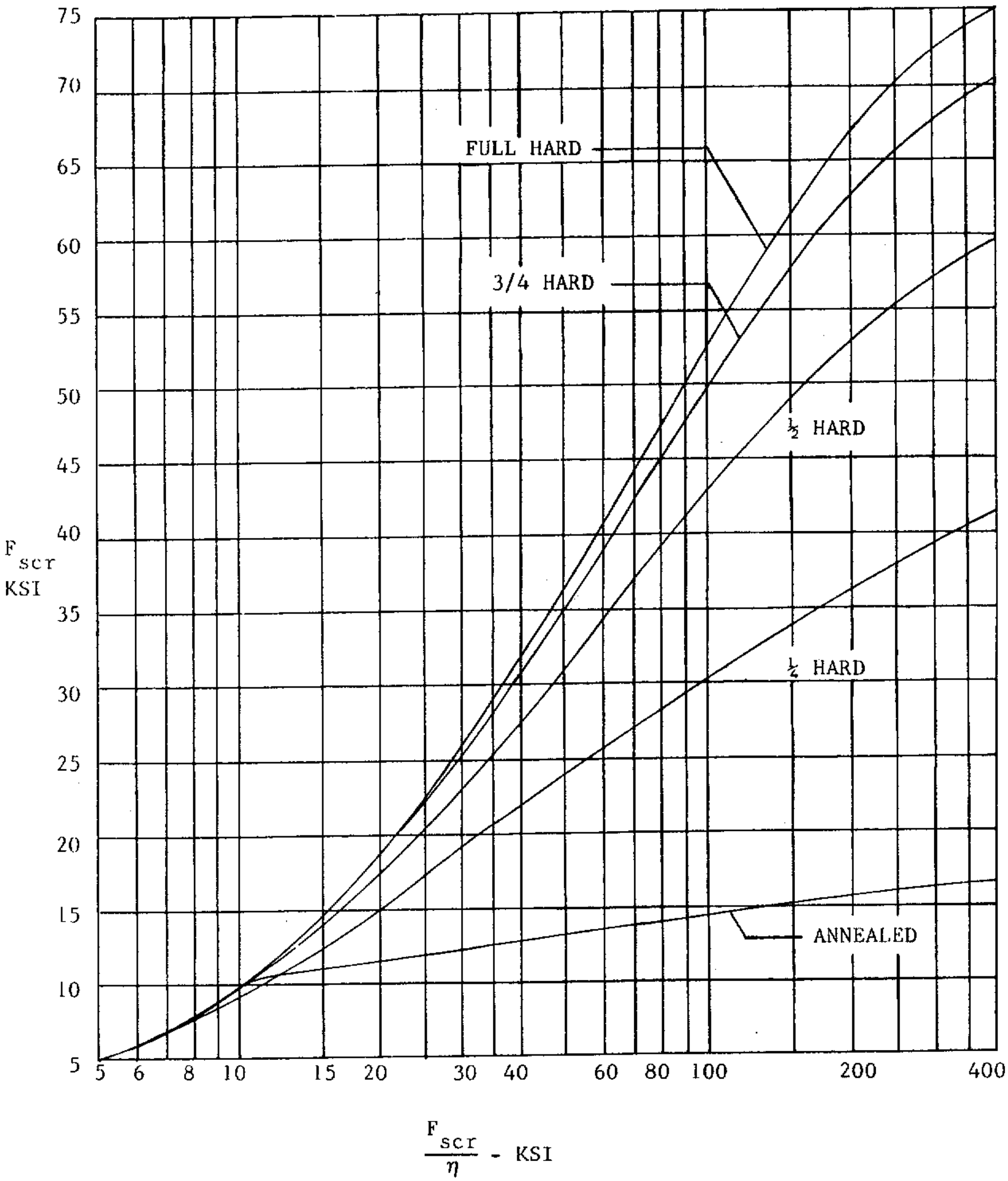
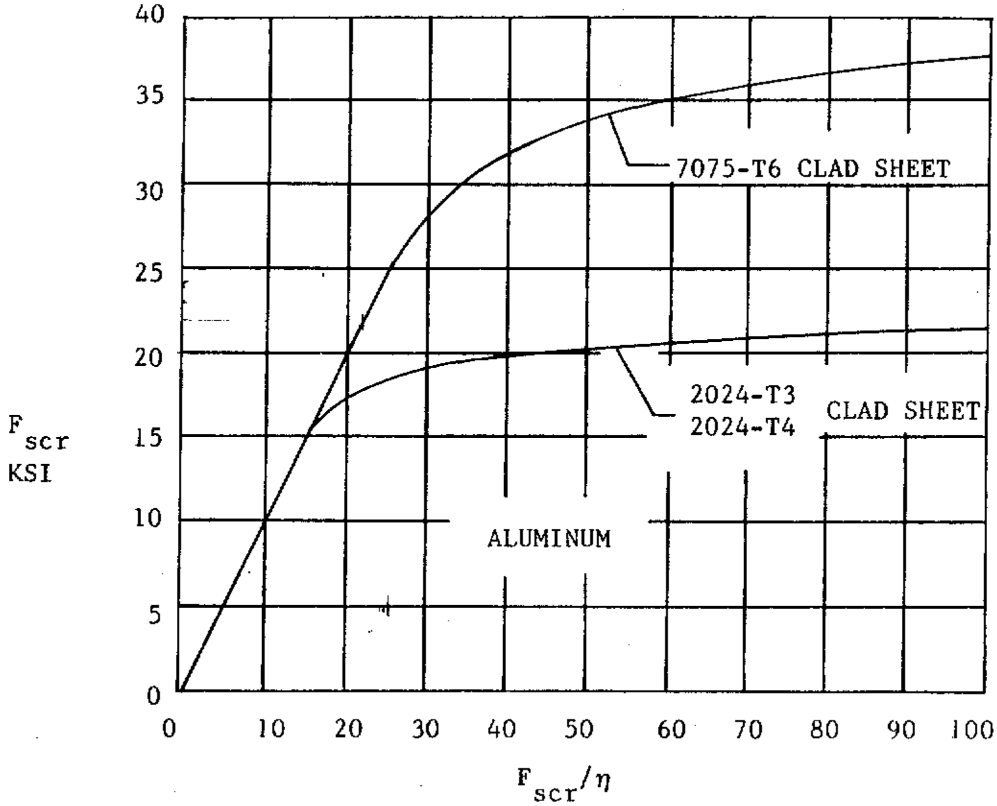


FIGURE 10.8 - F_{scr} VS. F_{scr}/η FOR STAINLESS STEEL



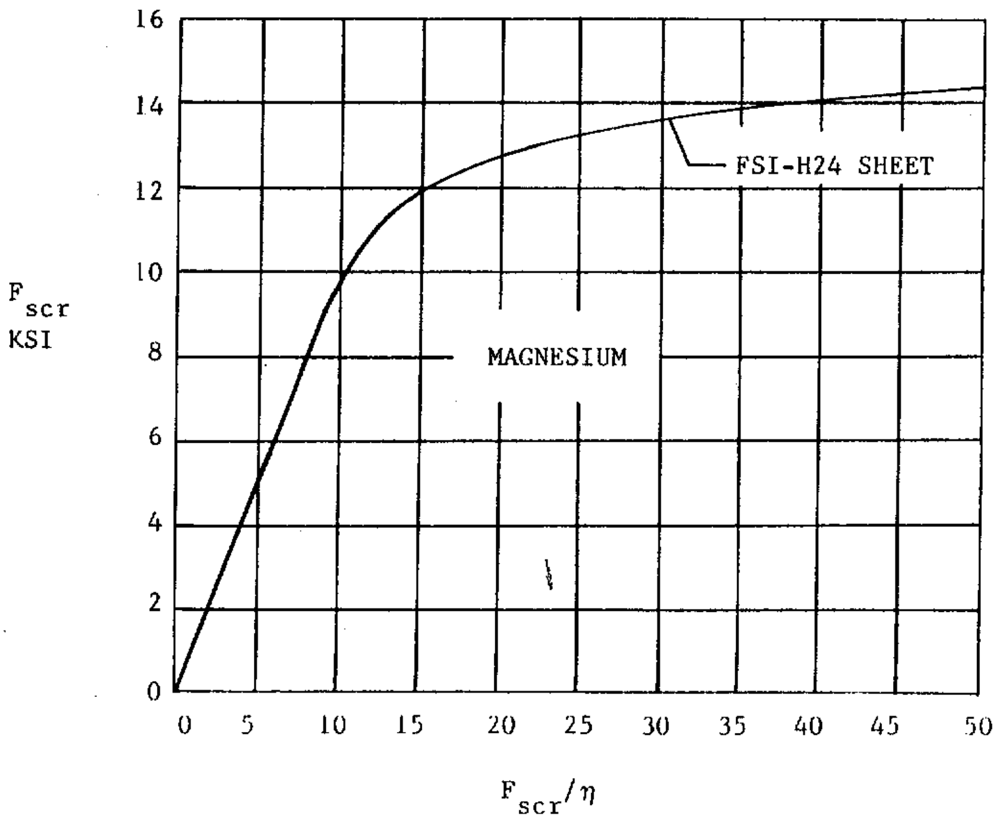


FIGURE 10.9 - F_{scr} VS. F_{scr}/η FOR ALUMINUM AND MAGNESIUM

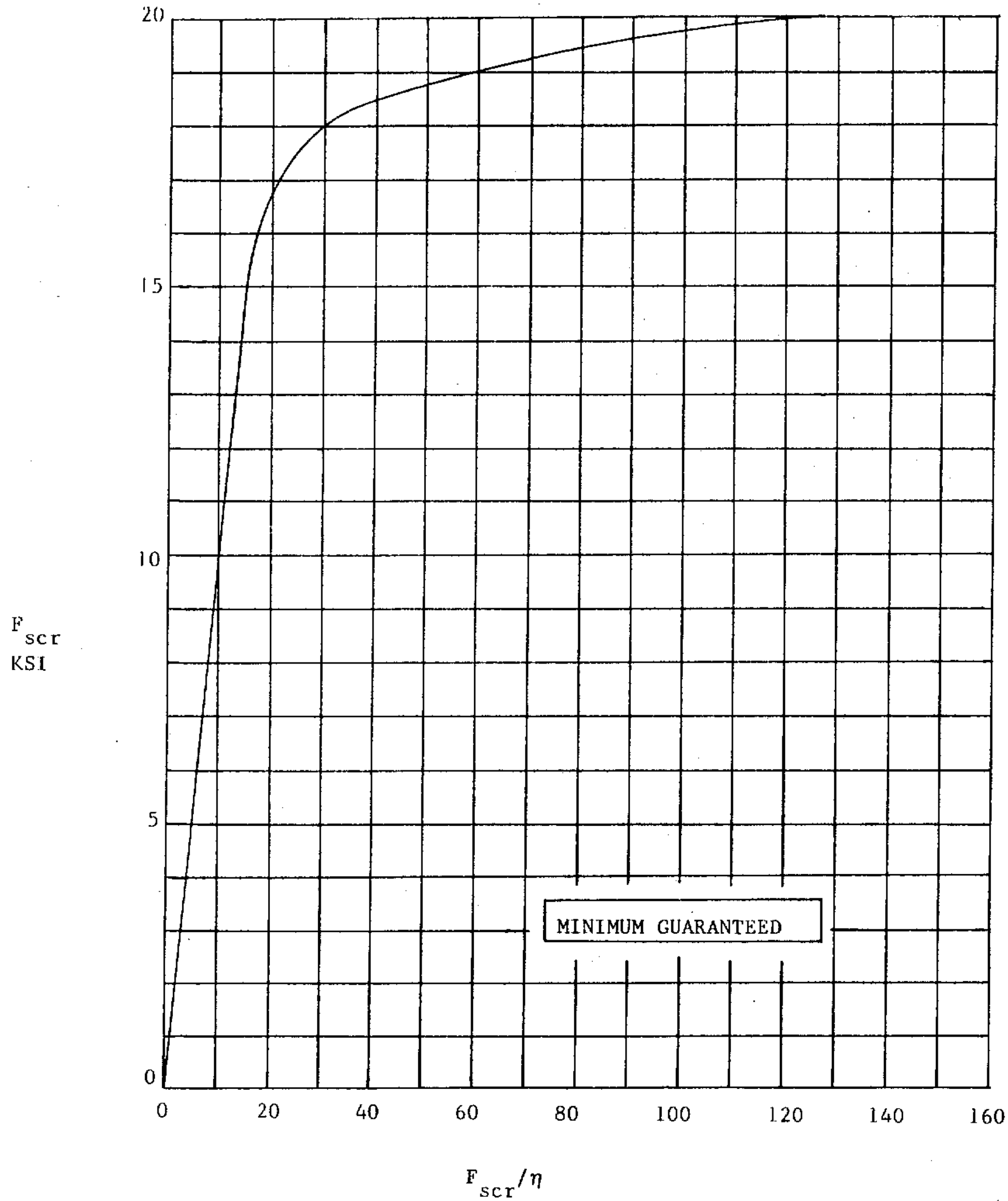


FIGURE 10.10 - F_{scr} VS. F_{scr}/η FOR 6061-T6 SHEET AND PLATE

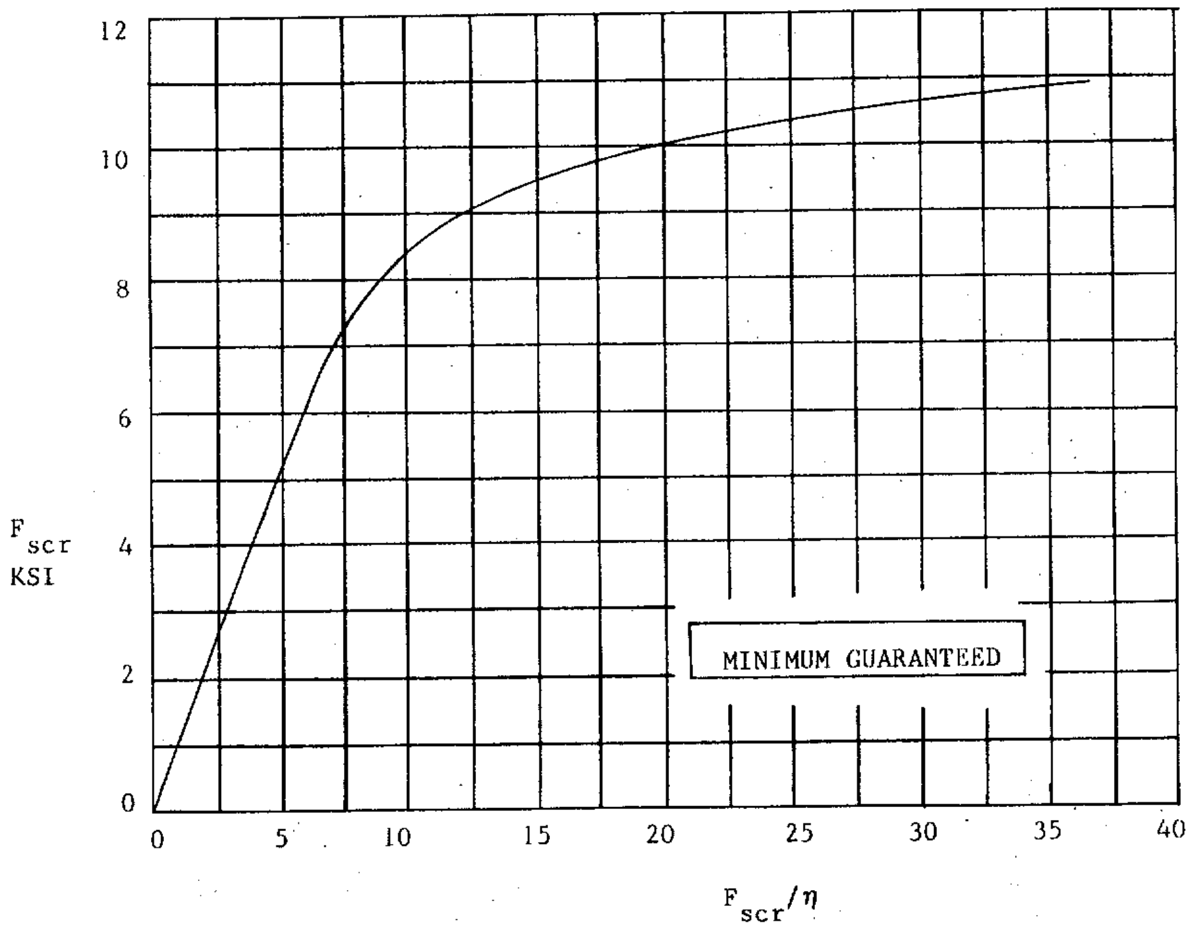


FIGURE 10.11 - F_{scr} VS. F_{scr}/η FOR 356-T6 SAND CASTINGS

K_s is related to $\frac{d}{h}$ and $\frac{I_u}{ht^3}$ in Figure 10.5.

The moment of inertia of the upright, I_u, should be calculated about the base of the stiffener (stiffener-web connection). Knowing K_s, $\frac{F_{scr}}{\eta}$ may be found from Figure 10.6. The critical buckling stress of the web F_{scr} is then obtained from Figures 10.7 through 10.11.

10.2 SHEAR WEB REINFORCEMENT FOR ROUND HOLES

10.2.1 Doubler Reinforcement

The thickness of the reinforcing doubler may be obtained through the use of the equations below. The figure below defines the variables used in the equations.

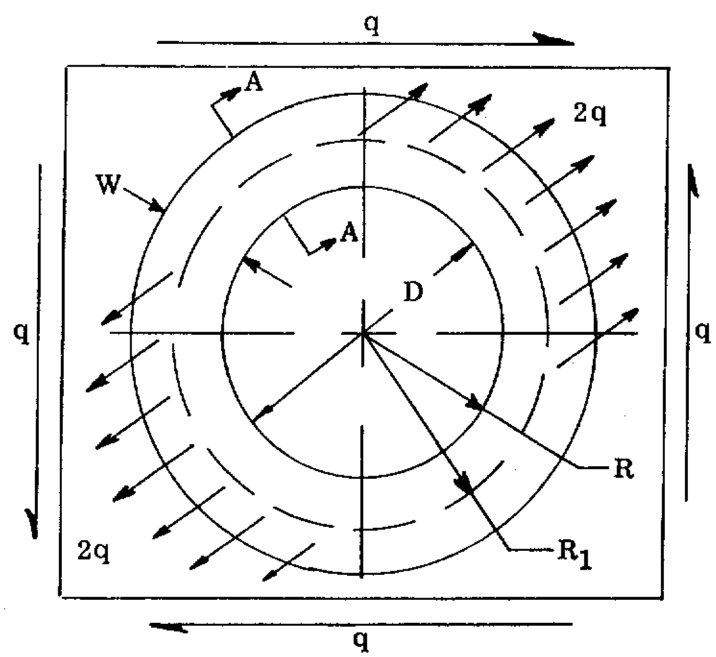


FIGURE 10.12 - SHEAR WEB REINFORCEMENT

$R_1 = R + \frac{W}{2}$	$D = 2R$
q = Web shear flow	F_{tu} = Ultimate tensile strength
f_b = Bending stress	f_t = Tensile stress
W = Doubler width	t_d = Doubler thickness

The stresses at section A-A are:

$$f_b = \frac{\text{Moment}}{\text{Section Modulus}} = \frac{2q(.25R)(R_1)}{t\left(\frac{W^2}{6}\right)}$$

$$f_t = \frac{2qR}{Wt_d} = \frac{Dq}{Wt_d}$$

The stress interaction, assumed at failure, is:

$$F_{tu} = f_b + f_t = \frac{.75qD(D + W)}{t_d W^2} + \frac{qD}{Wt_d}$$

Therefore,

$$t_d = \frac{.75q\left(\frac{D}{W}\right)^2 + 1.75q\left(\frac{D}{W}\right)}{F_{tu}}$$

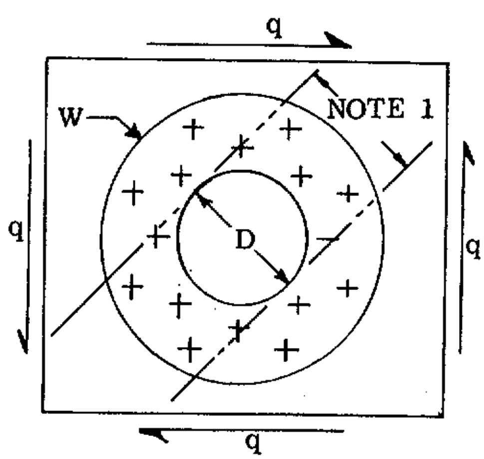


FIGURE 10.13 - RIVET PATTERN LOAD

For flanged doublers the total thickness, $t_{web} + t_d$, may be obtained from Figure 10.14.

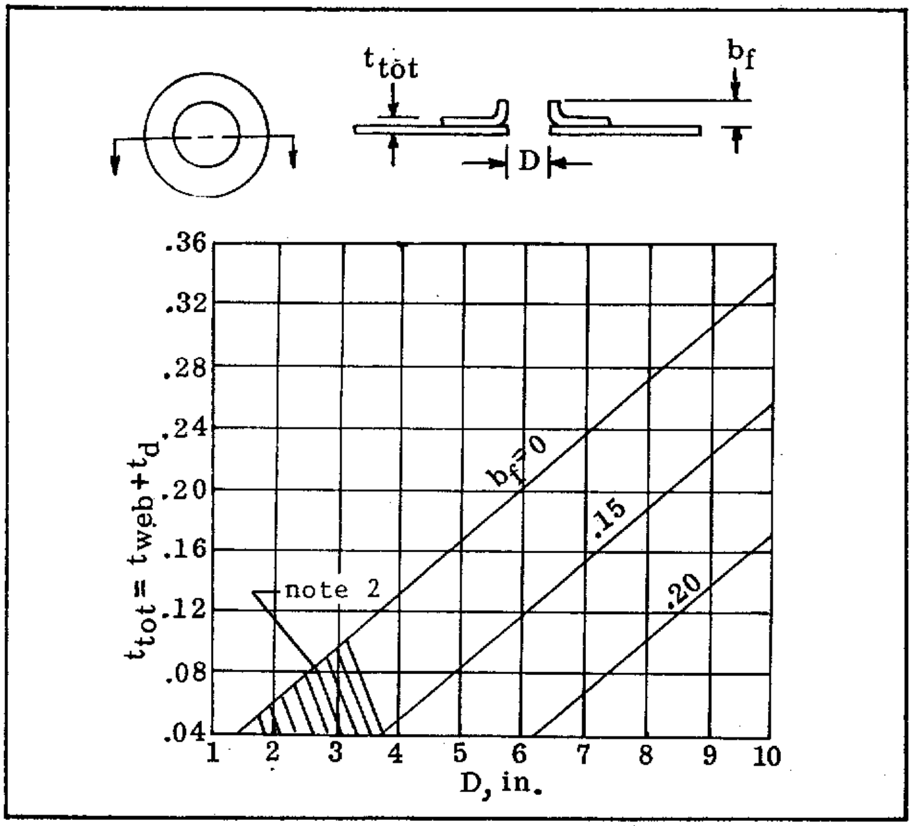
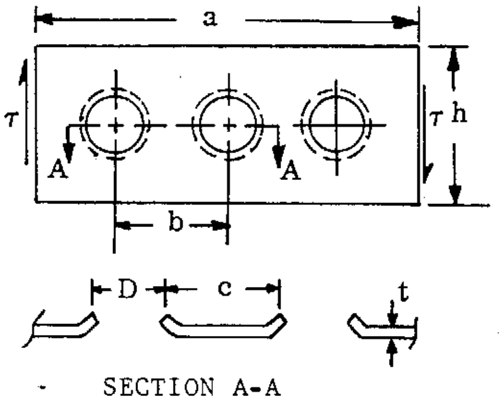


FIGURE 10.14 - TOTAL THICKNESS FOR FLANGED DOUBLERS

Note 1: The rivet pattern is to be uniform, and develop a running load strength per inch (between the tangent lines) of

$$2q\left[\frac{t_d}{t_d + 0.8\ t_{web}}\right]$$

Note 2: In this region, increase t_{tot} to correspond to $b_f = 0$ for the same D and omit the flange.

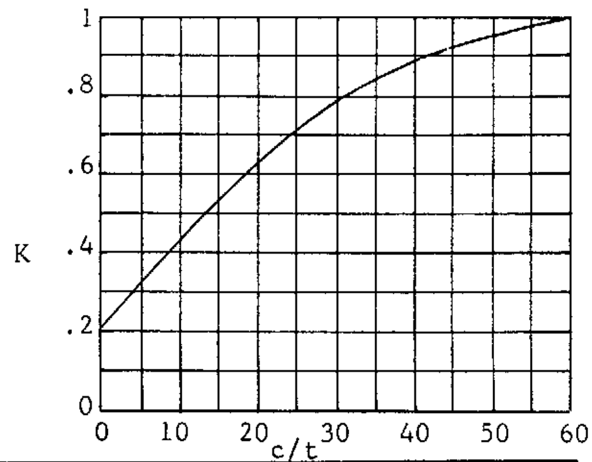


NOTE: The limits of the curve for τ_{all}:

$50 < h/t < 300$
$.15 < D/h < .75$
$60 < c/t < 300$

For $c/t < 60$, use correction factor, K, at right

$\tau_{cor} = K\,\tau_{all}$



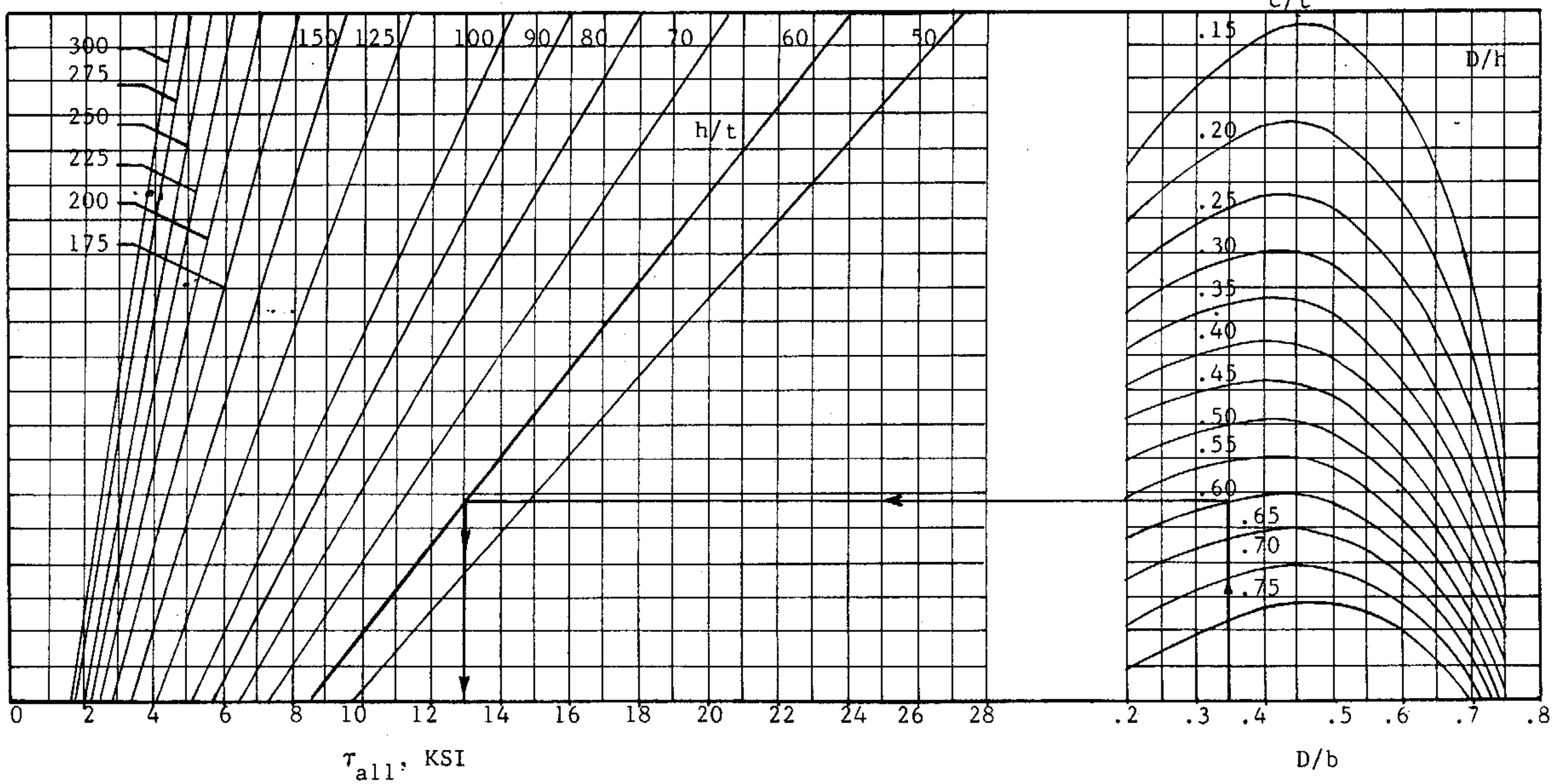


FIGURE 10.15 - ALLOWABLE SHEAR STRESS FOR 2024 WEBS WITH CIRCULAR HOLES HAVING 45° FLANGES

10.2.2 45° Flange Reinforcement

Allowables for panels loaded by pure shear (no addition bending forces) are given in Figure 10.15. Limited available data indicates that beaded lightening panels are more efficient than flanged panels. (Reference NACA RB No. 4B23, "Tests of Beams with Large Circular Lightening Holes".)

10.3 SHEAR WEBS WITH BEADS

Beaded panels are one type of non-buckling shear webs. Stiffeners must be added at load points to prevent premature collapse. Since the collapsing stress is only slightly higher than the buckling stress, the buckling stress is considered the ultimate allowable. The critical shear stress τ_{cr} for a beaded web can be expressed as:

$$\tau_{cr} = K_s K_1 E \left(\frac{t}{h}\right)^2 \left(\frac{\pi^2}{12(1-\mu^2)}\right)$$

where

K_s = Simply supported flat sheet, shear buckling constant based on a/b from Figure 10.18.

K_1 = Beaded-web shear buckling coefficient obtained from Figure 10.17.

Figure 10.17 is based on test results obtained from 2024-T4 clad panels with a bead spacing of 2 to 5 inches, panel heights of 7 to 12 inches, and gages of 0.032 to 0.064 inches. It is suggested that above the proportional limit τ_{cr} be reduced by the factor G_t/G.

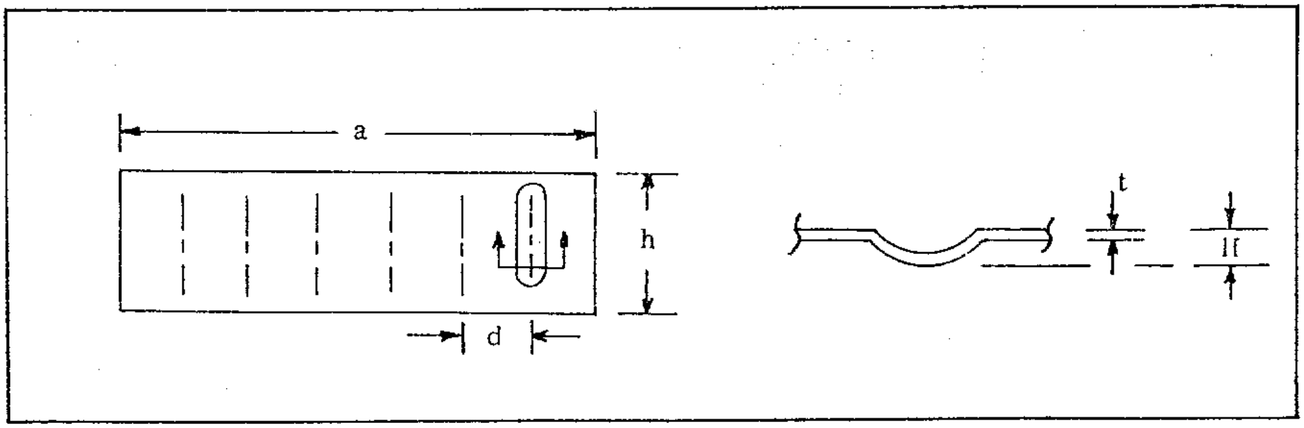


FIGURE 10.16 - GEOMETRY OF BEADED WEBS

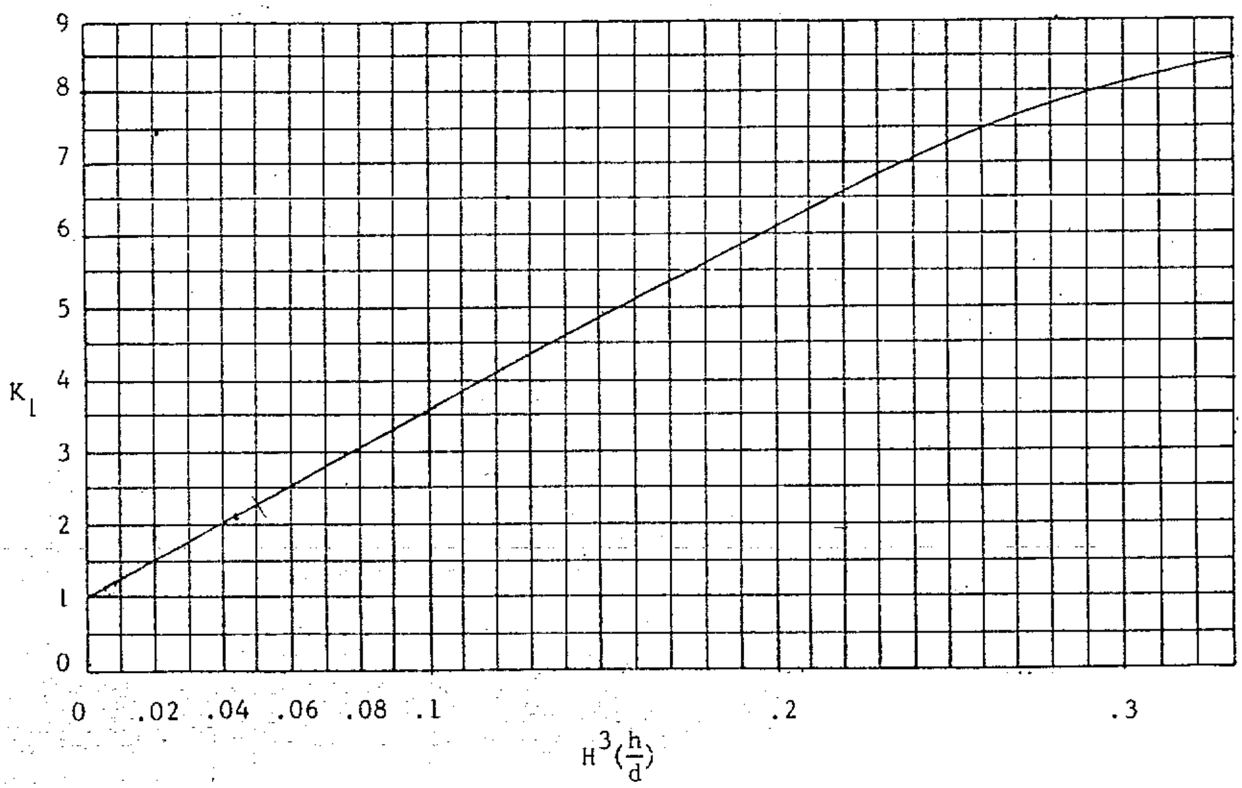


FIGURE 10.17 - BEADED WEB SHEAR BUCKLING CONSTANT

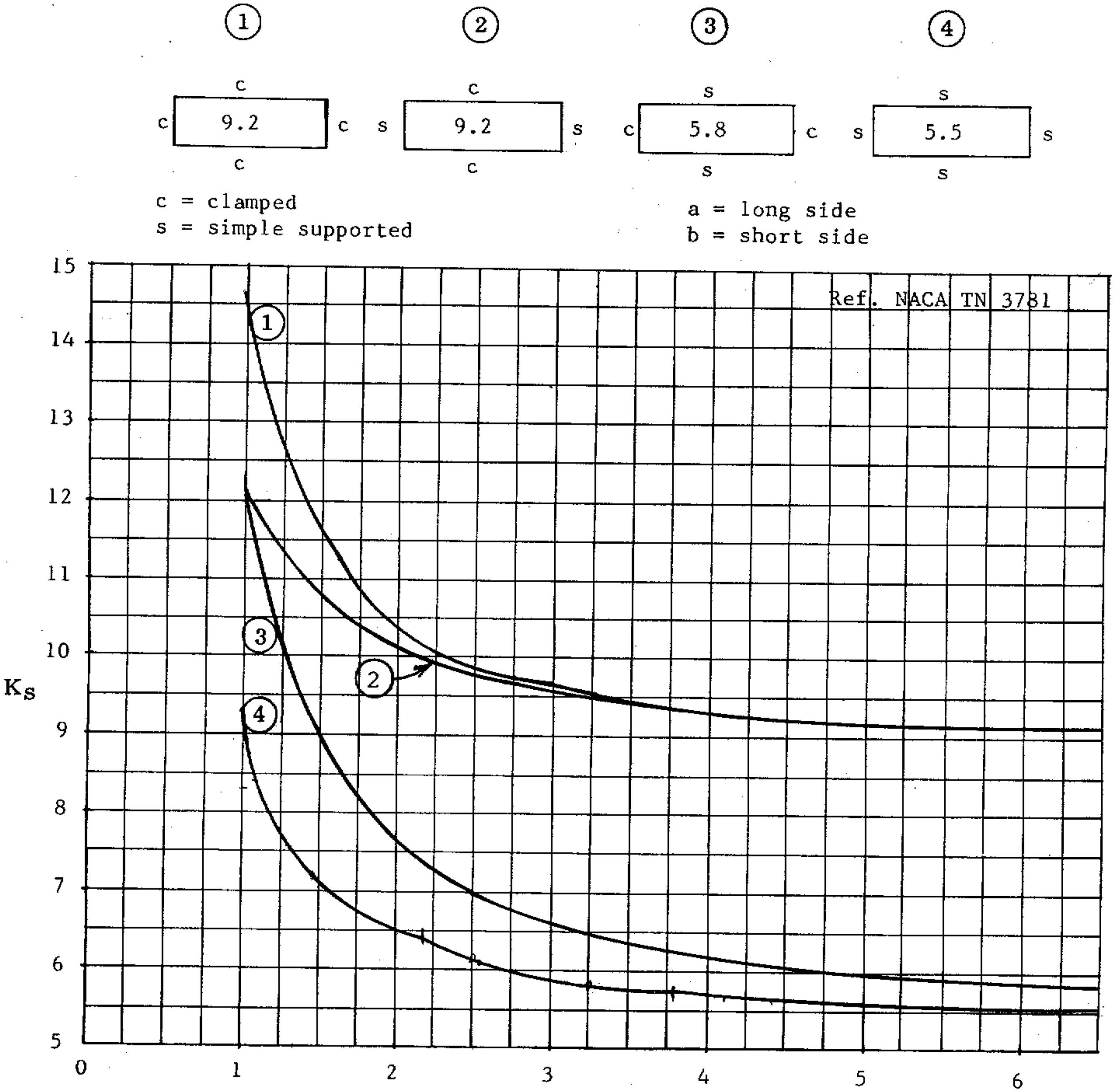


FIGURE 10.18 - SHEAR BUCKLING COEFFICIENT, K_s

Revision A

10.4 SHEAR BUCKLING

The critical shear stress at which a plate first buckles is given by the equation:

$$\tau_{cr} = \frac{K_s \pi^2 \eta E}{12(1-\mu^2)} \left(\frac{t}{b}\right)^2$$

where K_s (Fig. 10.18) is the non-dimensional shear buckling coefficient and is a function of the plate geometry and edge restraints. The values of K_s and μ are always the elastic values since the plasticity correction factor, η, contains all changes in those terms resulting from inelastic behavior. The term b is the smaller dimension of the panel.

A great deal of work has been done relative to the value of the plasticity correction factor. The expression for η must involve a measure of the stiffness of the material in the elastic and inelastic ranges. A simple means of obtaining a value of η is to take the ratio of the shear secant modulus to the shear modulus.

$$\eta = \frac{G_s}{G} = \frac{\text{shear secant modulus}}{\text{shear modulus}}$$

10.4.1 CRITICAL BUCKLING STRESS WITH AXIAL LOADS

When axial loads are present the actual shear buckling stress defined in paragraph 10.4 will be different. The presence of compressive stresses together with shear stresses causes the panel to buckle at a lower value of shear than if no compression were present. Tension causes the panel to buckle at a higher shear stress.

When shear and compression are present the panel buckles according to the interaction

$$f_c/F_{c_{cr}} + (f_s/F_{s_{cr}})^2 = 1.0$$

where $F_{c_{cr}}$ and $F_{s_{cr}}$ are the critical panel buckling stresses for pure compression and pure shear. From chapter 7, section 7.3 the buckling stress for a panel under compression is

$$F_{c_{cr}} = \frac{\pi^2 \eta k_c E}{12(1-\mu^2)} \left(\frac{t}{b}\right)^2$$

For any particular panel

$$F_{c_{cr}}/F_{s_{cr}} = A, \text{ (a constant)}$$

From conventional means the applied compressive stress, f_c, and the applied shear stress, f_s, can be calculated. These stresses will have a constant relationship with each other until the panel buckles, after which the compressive stress no longer increases. Thus

$$f_c/f_s = B$$

Now the interaction equation can be rewritten as

$$\frac{Bf_s}{AF_{s_{cr}}} + \left(\frac{f_s}{F_{s_{cr}}}\right)^2 = 1$$

$$f_s = F_{s_{cr}} \left[\frac{-B/A + \sqrt{(B/A)^2 + 4}}{2} \right]$$

where f_s is the actual shear stress at which the panel buckles due to the presence of compression stresses.

The expression in the brackets can be called R_c and the equation rewritten as

$$f_s = R_c F_{s_{cr}}$$

where R_c is always less than 1.0 when compression stresses are present.

When shear and tension are present the panel buckles according to the interaction

$$f_s/F_{s_{cr}} - f_t/2F_{c_{cr}} = 1.0$$

where $F_{c_{cr}}$ and $F_{s_{cr}}$ are as before. The shear stress, f_s, at which the panel buckles with tension, f_t, present is

$$f_s = F_{s_{cr}} (1.0 + f_t/2F_{c_{cr}})$$

and can be rewritten, substituting R_t for the term in parenthesis, as

$$f_s = R_t F_{s_{cr}}$$

Since R_t is always greater than 1.0 when tension is present, the actual shear buckling stress will always be greater than $F_{s_{cr}}$, the buckling stress for shear only.

10.5 INCOMPLETE DIAGONAL TENSION

The incomplete diagonal tension theory is a usable engineering theory which is a combination of shear-resistant beam theory, the pure diagonal tension theory, and empirical results of tests. BHT computer program SSCS01 performs a computer analysis of incomplete diagonal tension. (See Section 2.6.) A physical description of what occurs during incomplete diagonal tension is given below. To a beam with a plane web, stiffened by uprights and free from large imperfections, apply a gradually increasing load. For low loads the beam behaves in accordance with the shear-resistant beam theory. The web remains plane and no stresses are developed in the uprights. At a certain critical load the web will begin to buckle. Incomplete diagonal tension has begun at this point. As the load is increased, the buckles become more distinct and the pure diagonal tension theory is approached. The state of pure diagonal tension is a theoretical limiting case, which can never be reached because some failure will occur preceding the limit. As the process of buckle formation progresses, axial stresses in the uprights develop.

The portion of the total shear, V, carried by diagonal tension, V_{DT}, is found by using the diagonal tension factor, k.

$$V_{DT} = kV$$

$$V_S = (1-k)(V)$$

$$V = V_{DT} + V_S$$

The shear stresses are calculated from the shear flow equation:

$$f_s = \frac{V}{(h_c)(t)} = q/t = f_{s_{DT}} + f_{s_s}$$

$$f_{s_{DT}} = (k)(f_s); \; f_{s_s} = (1-k)(f_s)$$

As the load increases beyond the initial buckling load, a higher percentage of the total shear is carried by tension field. This causes the ratio $f_s/f_{s_{cr}}$ to become an important parameter.

Methods of analysis for three specific types of tension field beams are given:

1. Flat tension field beams with single uprights.
2. Flat tension field beams with single uprights and access holes.
3. Curved tension field beams.

The curves given for use in these analyses yield results with a reasonable assurance of conservative strength predictions, provided that normal design practices and proportions are used.

10.5.1 Effective Area of the Uprights

In order to make the design curves apply to both single and double uprights, it is necessary to define an effective upright area A_{ue}.

For double uprights, which are symmetric with respect to the web:

$A_{ue} = A_u$ = total cross-sectional area of the uprights.

For single uprights:

$$A_{ue} = \frac{A_u}{1 + \left(\frac{e}{\rho}\right)^2}$$

where ρ = radius of gyration of the stiffener and
e = distance from the centroid of the stiffener to the center of the web.

If the upright has a very deep web, A_{ue} should be taken to be the sum of the cross-sectional area of the attached leg and an area $12\ t_u{}^2$, where t_u is the upright thickness.

10.5.2 Moment of Inertia of the Uprights

The uprights must have a sufficient moment of inertia to prevent buckling of the web system as a whole before formation of the tension field, in addition to preventing column failure due to the loads imposed upon the upright by the tension field. Forced crippling failure, caused by the waves of the buckled web and possibly most critical, must also be prevented by the upright. The required moment of inertia of the upright may be determined by iterating through the appropriate Table 10.1, 10.2, 10.3, or 10.4.

Revision B

10.5.3 Effective Column Length

The effective column (upright) length is calculated by the equations:

If $d_c < 1.5\ h_u$, $$L_e = \frac{h}{\sqrt{1 + k^2\ (3 - \frac{2d}{h})}}$$

If $d_c > 1.5\ h_u$, $$L_e = h$$

10.5.4 Discussion of the End Panel of a Beam

The following analyses are concerned with the "interior" bays of a beam. The uprights in these areas are subjected, primarily, only to axial compressive loads. The end panel, however, is a special case. Since the diagonal tension effect results in an inward pull on the end upright, bending, in addition to the usual compressive axial load, is also produced. There are three general ways of dealing with the edge member subjected to bending.

1. Sufficiently strengthen the edge member so it can carry all of the loads (which is inefficient, weight-wise, for long unsupported lengths).

2. Increase the thickness of the end panel either to make it non-buckling or to reduce k, which would reduce the running load producing bending in the edge member. (This is usually inefficient for large panels.)

3. Additional uprights may be provided to support the edge member and thus reduce its bending moment.

10.5.5 Analysis of a Flat Tension Field Beam with Single Uprights

Table 10.1 is a step-by-step procedure which yields the stresses in the flanges, webs, rivets, and uprights of a flat tension field beam with single uprights (Figure 10.19).

Table 10.1 is based on a single web with parallel flanges and parallel uprights. Most beams consist of more than one web. At various locations in the following table adjacent panels must be considered. Such a situation occurs for rivet load, stringer axial stress, upright stress and moment in stringer.

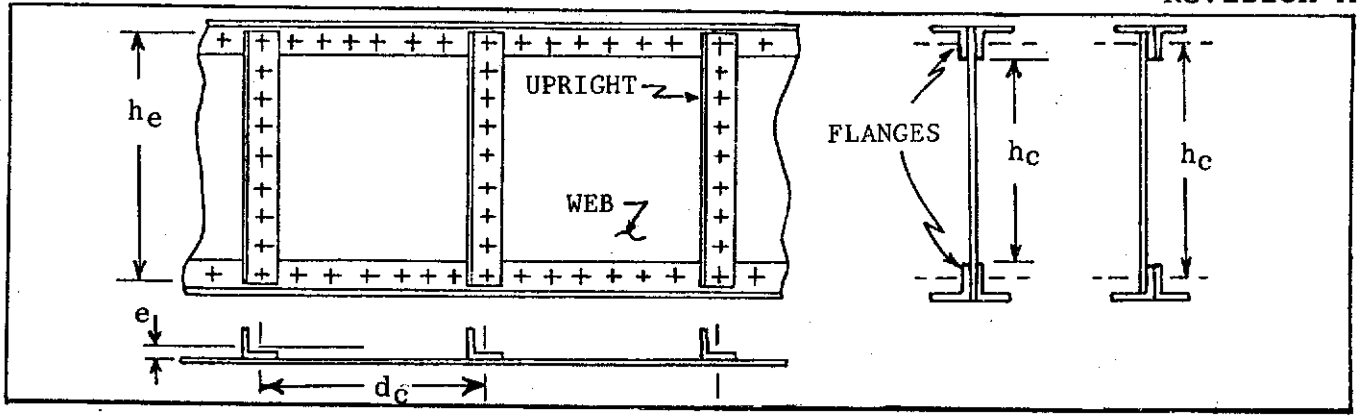


FIGURE 10.19 - FLAT TENSION FIELD BEAM WITH SINGLE UPRIGHTS

Description	Variable and Equation	Numerical Value
(1) Elastic modulus	E_c	
(2) Upright spacing, (NA to NA)	d	
(3) Clear web between uprights (rivet to rivet)	d_c	
(4) Distance from median plane of web to centroid of upright	e	
(5) Clear web between flanges (rivet to rivet)	h_c	
(6) Distance between flange centroids	h_e	
(7) Length of upright between upright to flange rivets	h_u	
(8) Web thickness	t	
(9) Upright thickness	t_u	
(10) Flange thickness	t_f	
(11) Upright area	A_u	
(12) Flange area	A_f	
(13) Radius of gyration of upright	ρ	
(14) Moment of inertia of upright	I_u	
(15) Moment of inertia of flange	I_F	
(16) Applied load - upright	P_u	
(17) Applied load - flange	P_f	
(18) Applied web shear flow	q	
(19) Web shear stress	$\tau = q/t =$ (18) / (8)	

TABLE 10.1 - ANALYSIS OF FLAT TENSION FIELD BEAM WITH SINGLE UPRIGHT

(20) Effective area of upright	$A_{ue} = (11) / 1+((4)^2/(13)^2)$	
(21) Parameter	$A_{ue}/d_c t = (20) / (3)(8)$	
(22) Parameter	$h_e t = (6)(8)$	
(23) Parameter	$d_c/h_u = (3) / (7)$	
(24) Parameter	$t_f/t = (10) / (8)$	
(25) Parameter	$t_u/t = (9) / (8)$	
(26) Parameter	$h_c/d_c = (5) / (3)$	
(27) Parameter	$d_c/h_c = 1 / (26)$	
(28) Parameter	$t/d_c = (8) / (3)$	
(29) Parameter	$t/h_c = (8) / (5)$	
(30) Upright restraint coefficient	R_h, Figure 10.20(b)	
(31) Flange restraint coefficient	R_d, Figure 10.20(b)	
(32) Theoretical buckling coefficient	k_{ss}, Figure 10.20(a)	
(33) Elastic buckling stress: $d_c < h_c$	$\tau_{cr_e} = (32)(1)(28)^2 \left\| (30) + \frac{1}{2}((31) - (30))(27)^3 \right\|$	
$d_c > h_c$	$\tau_{cr_e} = (32)(1)(29)^2 \left\| (31) + \frac{1}{2}((30) - (31))(26)^3 \right\|$	
(34) Initial buckling stress	τ_{cr}, Figure 10.21 (See Note 2)	
(35) Stress ratio	$\tau/\tau_{cr} = (19) / (34)$	
(36) Diagonal tension factor	k, Figure 10.22 @ $300td_c/12h_c = 0$	
(37) Parameter	$\frac{A_{ue}}{d_c t} + \frac{1}{2}(1-k) = (21) + \frac{(1-(36))}{2}$	
(38) Ratio of upright stresses	$\sigma_{u_{max}}/\sigma_u$, Figure 10.23	
(39) Ratio of upright to shear stresses	σ_u/τ, Figure 10.24	
(40) Diagonal tension angle	$\tan\alpha$, Figure 10.25(a)	
(41) Stress in median plane upright/web	$\sigma_u = -(36)(19)(40) / (37)$	
(42) Upright average stress	$\sigma_{u_{avg}} = (41)(20) / (11)$	
(43) Upright maximum stress	$\sigma_{u_{max}} = (41)(38)$	
(44) Effective column length: If $(23) < 1.5$	$L_e = (7) / \left\| 1 + (36)^2 (3 - 2(23)) \right\|^{\frac{1}{2}}$	
If $(23) > 1.5$	$L_e = h_u = (7)$	
(45) Slenderness ratio	$L_e/2\rho = (44) / 2(13)$	
(46) Column allowable	$\sigma_{co} = \pi^2 (1) / (45)^2$ or Section 11	
(47) Proportional limit	F_{pl}, Section 5 ($F_{pl} = F_{tp}$)	
(48) Strain, if $(41) > (47)$	$\sigma_u/E = (41) / (1)$	

TABLE 10.1 (cont'd) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH SINGLE UPRIGHT

(49) From stress strain curve	F_c, use (48) to determine all.	
(50) Margin of Safety: column yield (41) > (47) (41) < (47)	 MS = (49) / (41) - 1 MS = (47) / (41) - 1	
(51) MS - Column	MS = (46) / (42) -1	
(52) Parameter	$k^{2/3}(t_u/t)^{1/3}$ = (36) $^{2/3}$ (25) $^{1/3}$	
(53) Upright allowable (forced crippling)	σ_o, Figure 10.26	
(54) MS - Forced crippling	MS = (53) / (43) -1	
(55) Parameter	wd_c = .7 (3) ((8) /2 (15) (6)) $^{1/4}$	
(56) Parameter	C_1, Figure 10.27	
(57) Parameter	C_2, Figure 10.28	
(58) Maximum web stress	τ'_{max} = (19) (1 + (36) ^{2} (56))(1 + (36) (57))	
(59) Web allowable	τ^*_{all}, Figure 10.29 @ α_{PDT} = 45°	
(60) MS - Web	MS = (59) / (58) -1	
(61) Parameter	C_3, Figure 10.28	
(62) Secondary bending in flange	M_{SB} = (1/12)((36) (61) (18) (3) ^{2} (40))	
(63) Distance from NA to extreme fiber of flange	C_f	
(64) Distance - NA to near fiber of flange	D_f	
(65) Flange applied stress	σ_a = (17) / (12)	
(66) Diagonal tension stress-flange (comp)	σ_{DT} = -((36) (19) / (40)) / [2 (12) / (22) + .5(1 - (36))	
(67) Secondary bending stress-flange (comp)	σ_{SB} = - (62) (63) / (15)	
(68) Secondary bending stress-flange (tension)	σ_{SB} = (67) (64) / (63)	
(69) Flange stress-inside fiber	σ_{tot} = (65) + (66) + (67)	
(70) Flange stress-extreme fiber	σ_{tot} = (65) + (66) + (68)	
(71) Allowable crippling stress-flange	F_{cc}	
(72) Allowable tension stress-flange	F_{tu} or F_{ty}	
(73) MS - Flange (tension)	MS = (72) / (70) -1	
(74) MS - Flange (compression)	MS = (71) / (69) -1	
(75) Rivet factor	R = 1 + 0.414 (36)	
(76) Rivet load-web to flange	R" = qR = (18) (75)	

TABLE 10.1 (cont'd) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH SINGLE UPRIGHT

(77) Allowable rivet shear load	P_{af}	
(78) MS - Flange rivets	MS = (77) / (76) -1	
(79) Rivet load-upright to flange	P_u = (41) (20)	
(80) Allowable rivet load-upright	P_{au}	
(81) MS - Upright rivets	MS = (80) / (79) -1	
(82) Interrivet buckling allowable	F_{ir}, Section 10.6	
(83) MS - Interrivet buckling	MS = (82) / (43) -1	
(84) Ultimate tensile stress of web	F_{tu}, Section 5	
(85) Rivet tensile strength upright/web per inch*	σ_R = .22 (8) (84)	
(86) Rivet allowable tensile load per inch	F_{RT}, Section 6	
(87) MS - Rivet tension	MS = (86) / (85) -1	

NOTES:

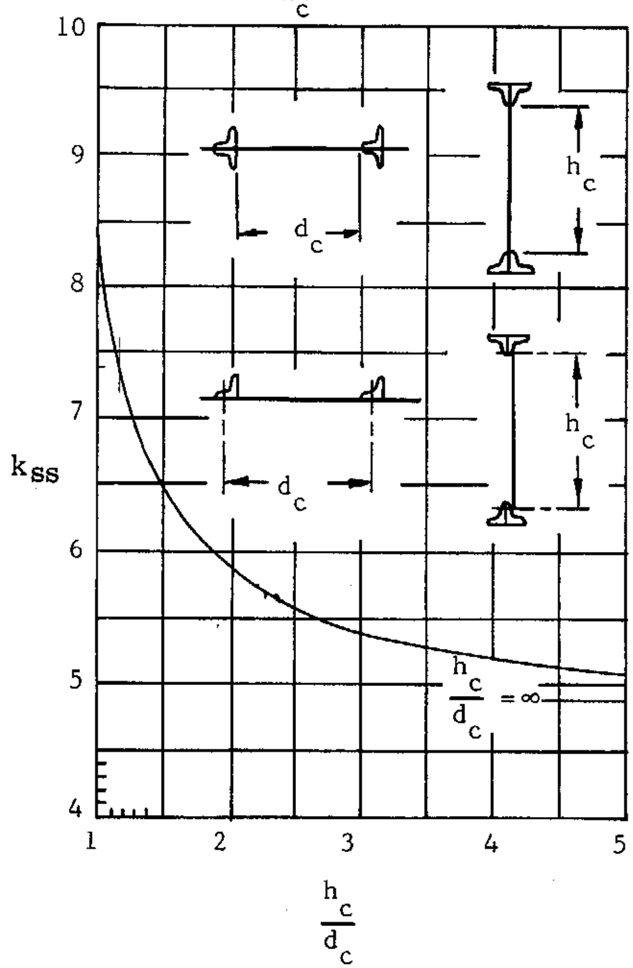
(1) If any of the margins of safety are less than zero, the design is inadequate. The deficient area must be corrected and this table repeated.

(2) If the web is subjected to tension or compression as well as shear, the initial buckling stress of the web must be modified according to the method described in Section 10.4.1.

* See NACA TN 2661, "A Summary of Diagonal Tension", 1952, page 49 for explanation.

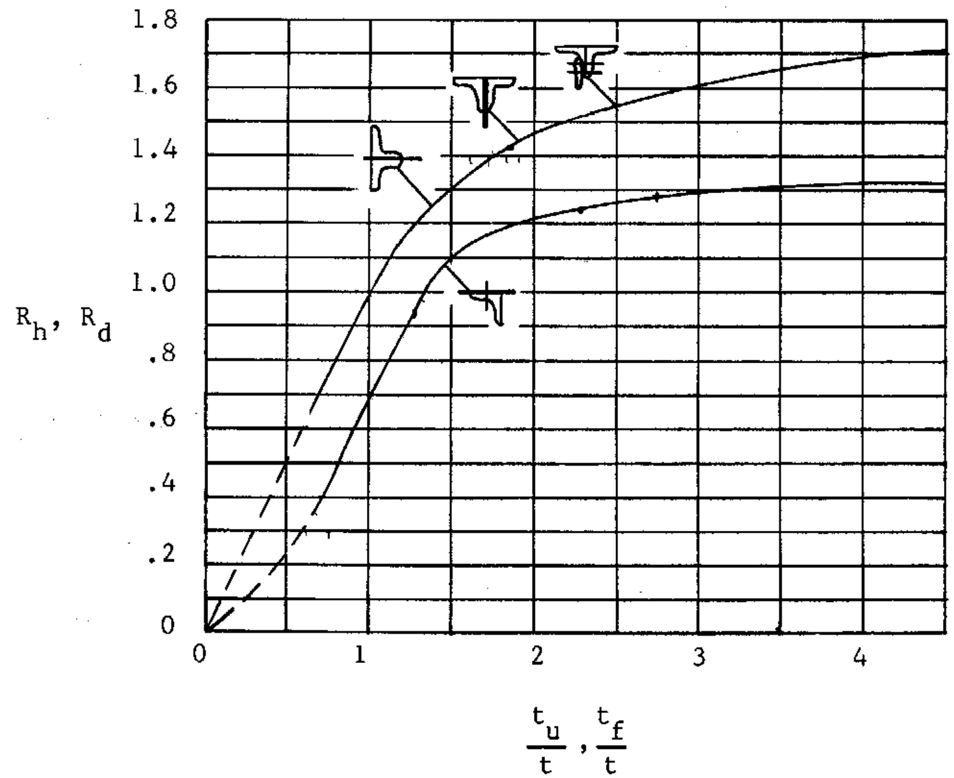
TABLE 10.1 (cont'd) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH SINGLE UPRIGHT

THIS CURVE IS SHOWN FOR $d_c < h_c$.
IF $d_c > h_c$ THE ABSCISSA SHOULD BE READ AS $\frac{d_c}{h_c}$



(A) THEORETICAL COEFFICIENTS FOR PLATES WITH SIMPLY SUPPORTED EDGES.

R_h = UPRIGHT COEFFICIENT
R_d = FLANGE COEFFICIENT



(B) EMPIRICAL RESTRAINT COEFFICIENTS

FIGURE 10.20 - GRAPHS FOR CALCULATING BUCKLING STRESS OF WEBS

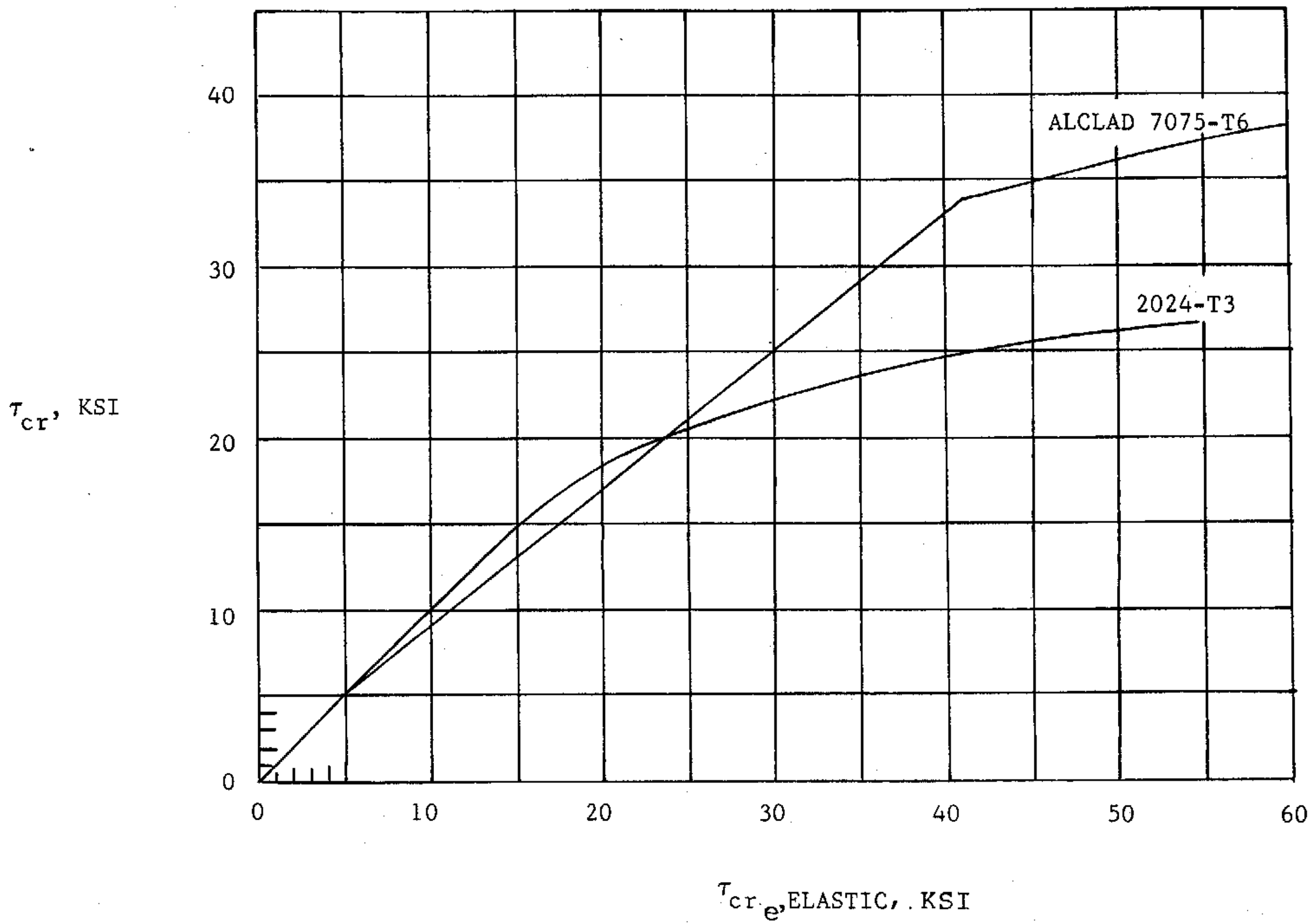
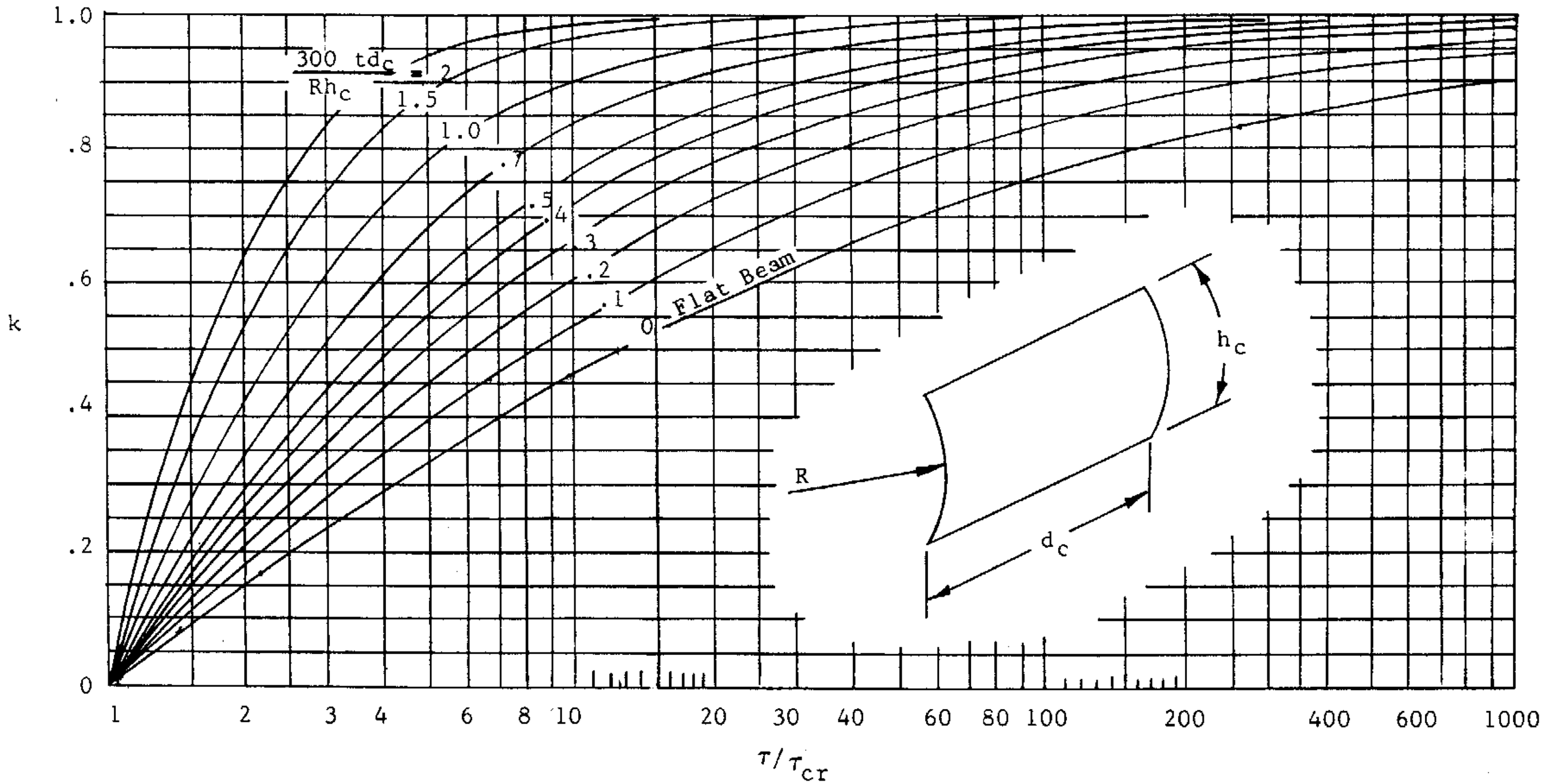


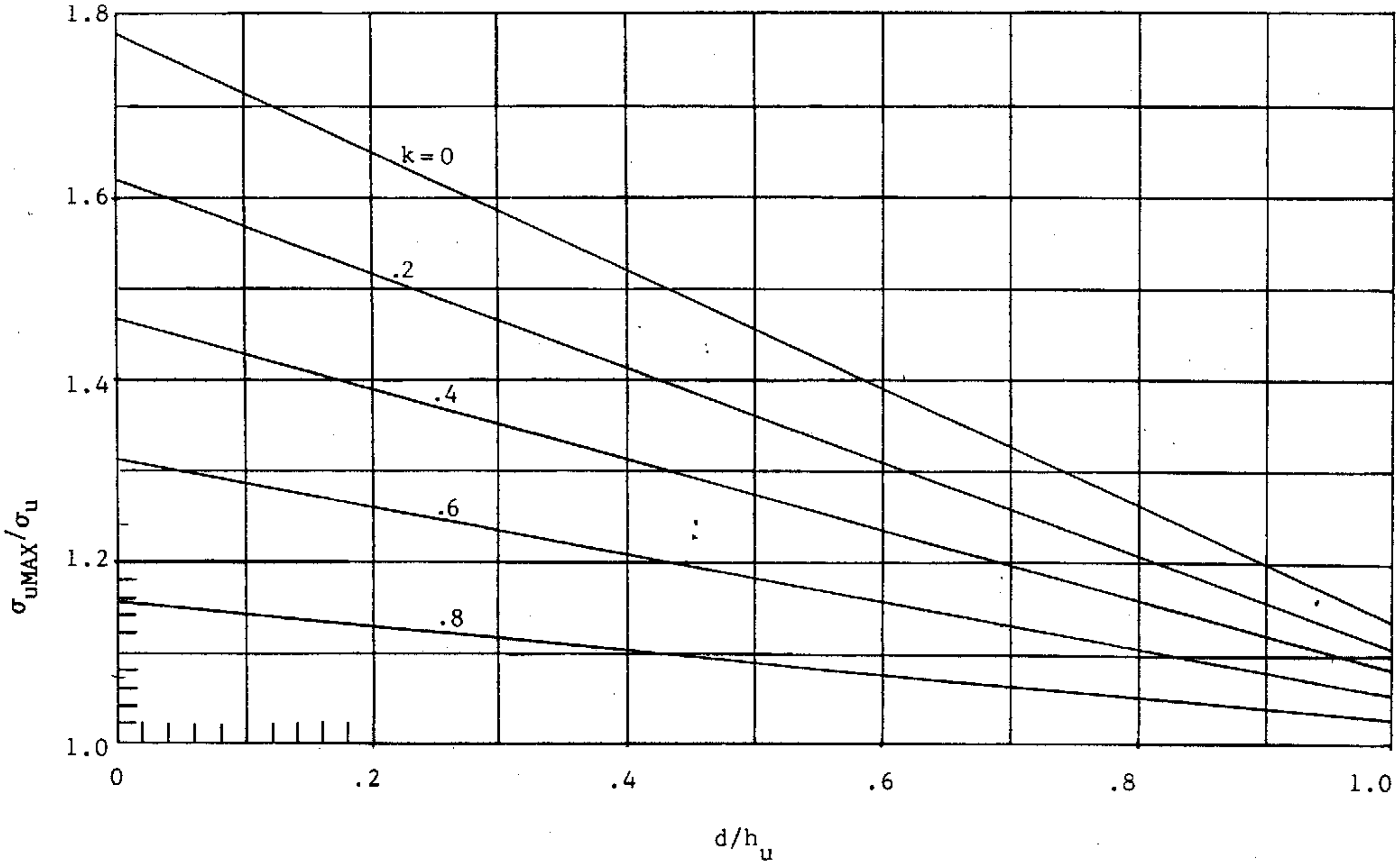
FIGURE 10.21 - GRAPHS FOR CALCULATING BUCKLING STRESS OF WEBS



If $h_c > d_c$, REPLACE td_c/Rh_c BY th_c/Rd_c

If d/h (or h/d) > 2, use 2

FIGURE 10.22 - DIAGONAL TENSION FACTOR, k



FOR CURVED WEBS, READ ABSCISSA AS d/h FOR RINGS AND READ ABSCISSA AS h/d FOR STRINGERS

FIGURE 10.23 - RATIO OF MAXIMUM STRESS TO AVERAGE STRESS IN WEB STIFFENER

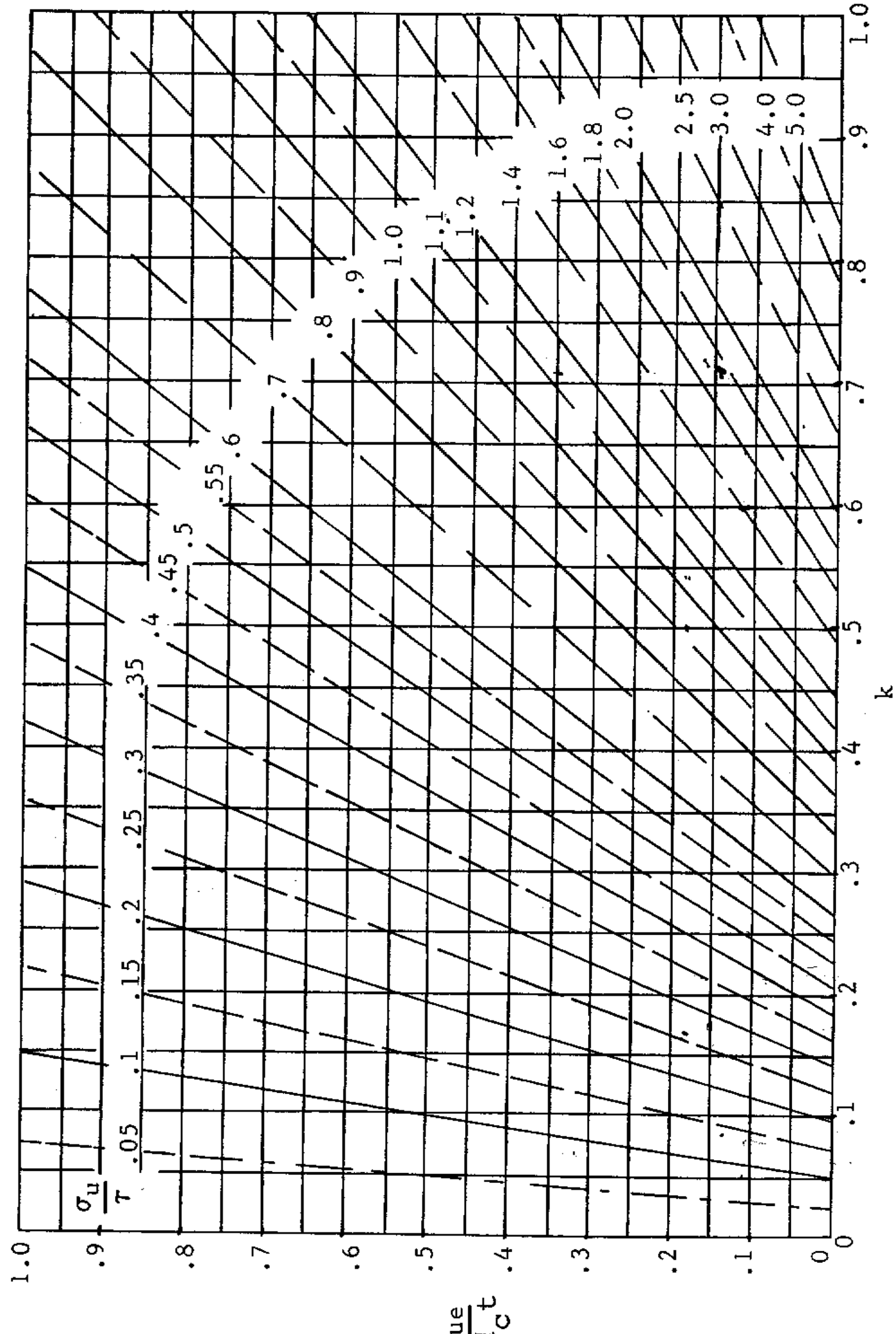
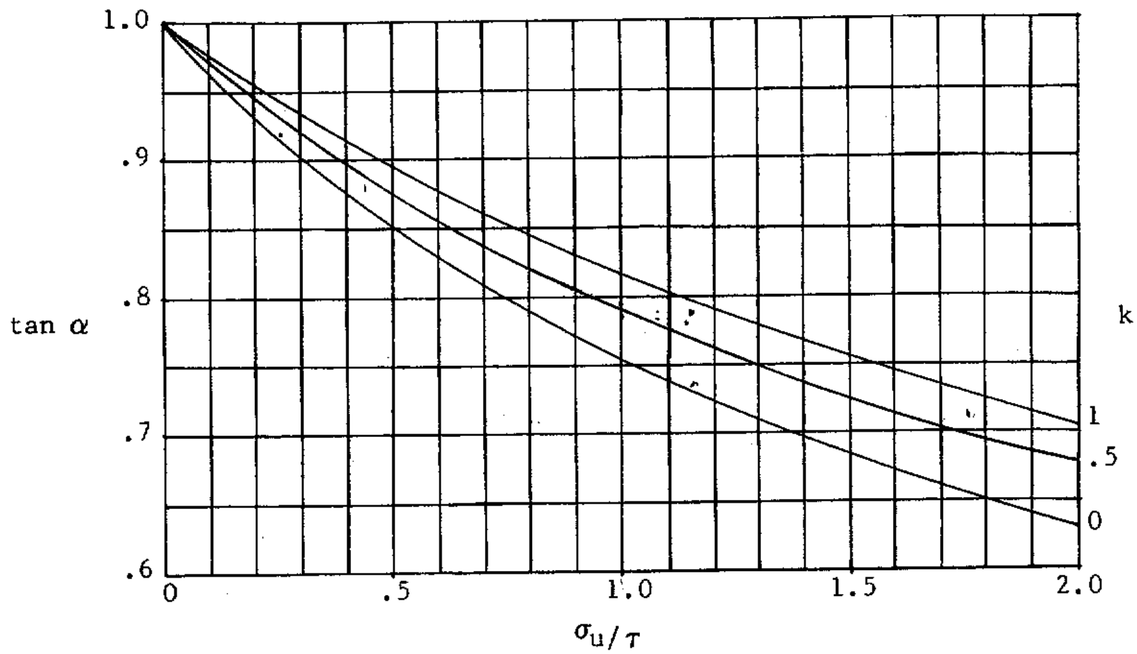
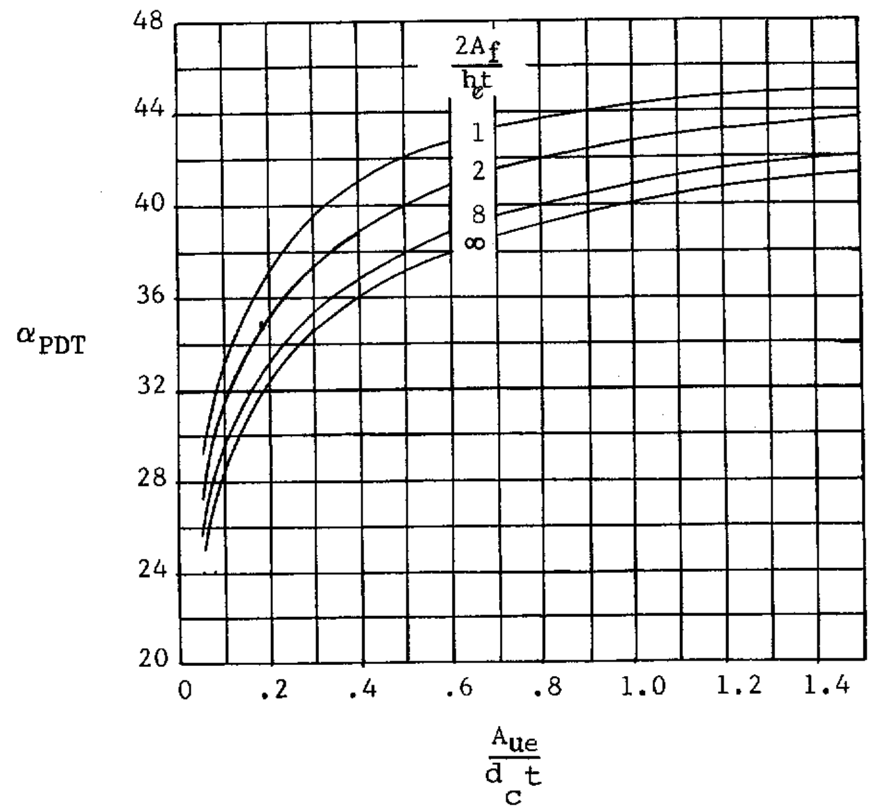


FIGURE 10.24 - DIAGONAL TENSION ANALYSIS

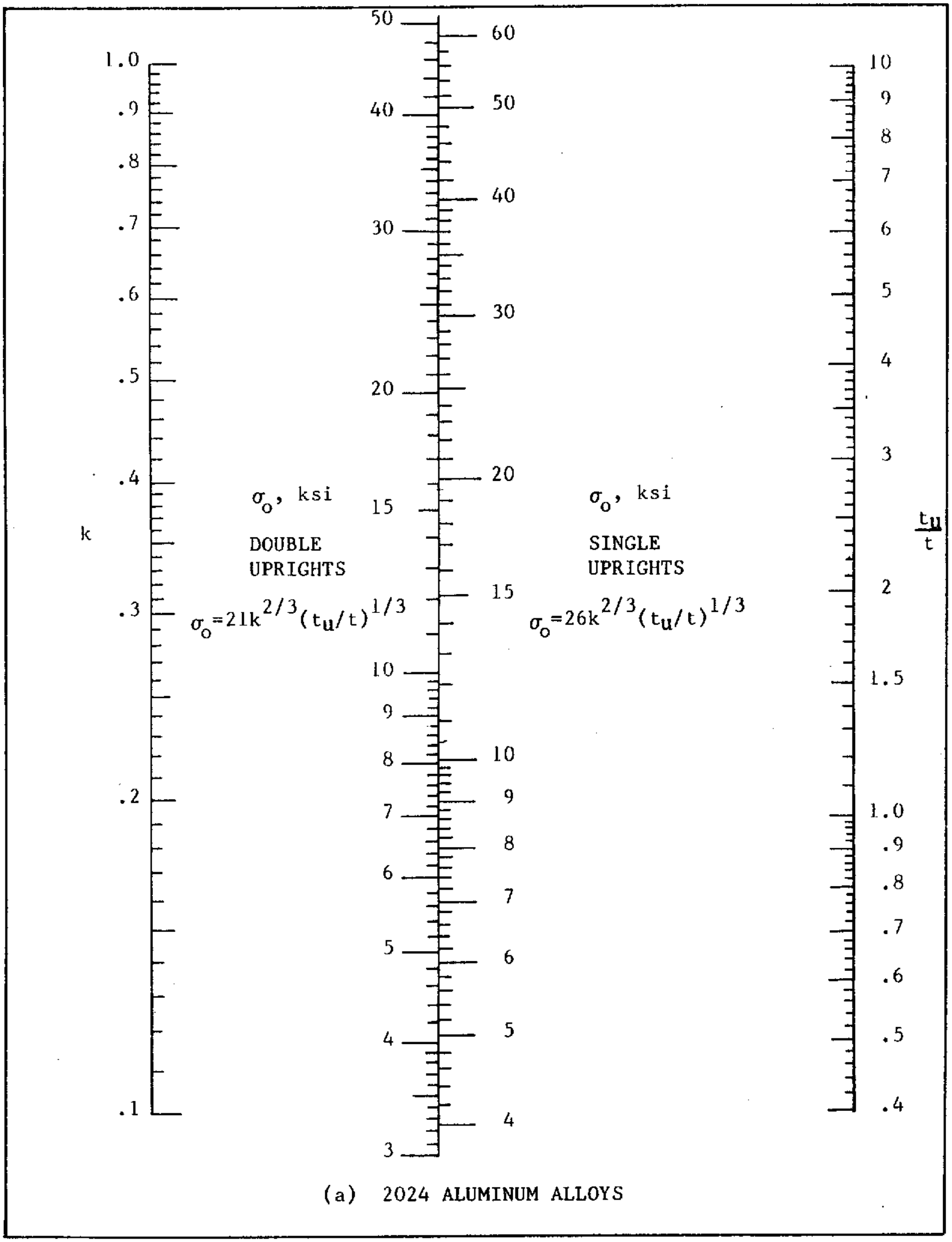


(a) INCOMPLETE DIAGONAL TENSION



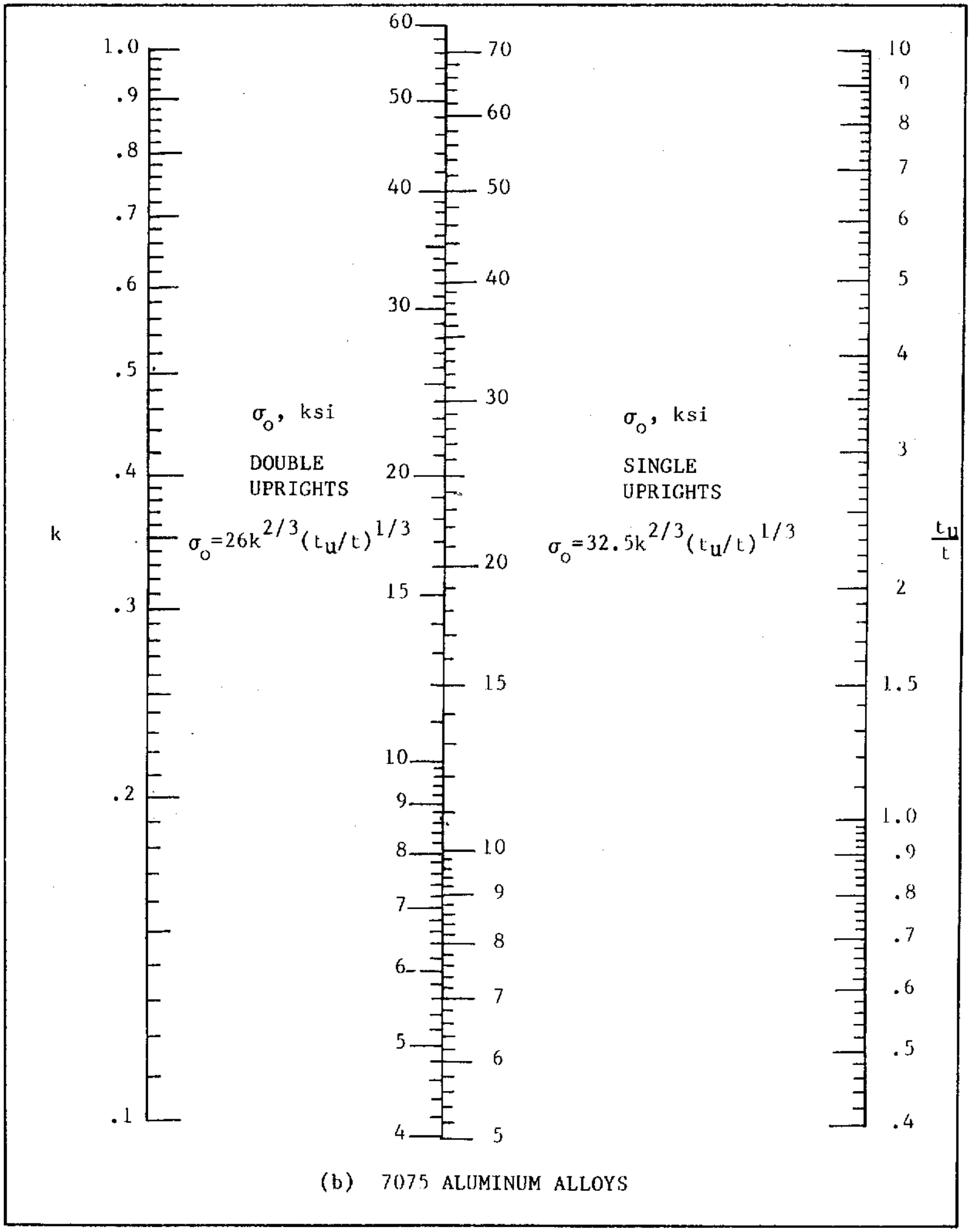
(b) PURE DIAGONAL TENSION

FIGURE 10.25 - ANGLE OF DIAGONAL TENSION



(a) 2024 ALUMINUM ALLOYS

FIGURE 10.26 - NOMOGRAPH FOR ALLOWABLE UPRIGHT STRESS (FORCED CRIPPLING)



(b) 7075 ALUMINUM ALLOYS

FIGURE 10.26 (cont'd) - NOMOGRAPH FOR ALLOWABLE UPRIGHT STRESS (FORCED CRIPPLING)

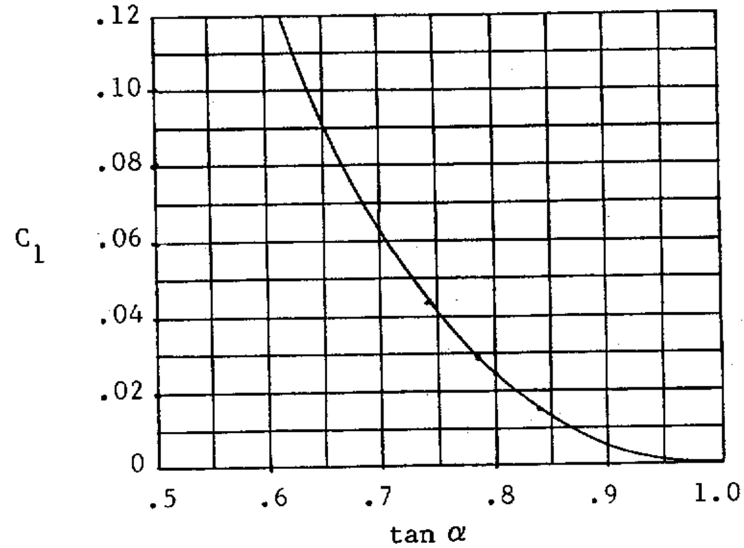


FIGURE 10.27 - ANGLE FACTOR C_1

$$wd_c = 0.7 d_c \left(\frac{t}{(I_C + I_T) h e} \right)^{1/4}$$

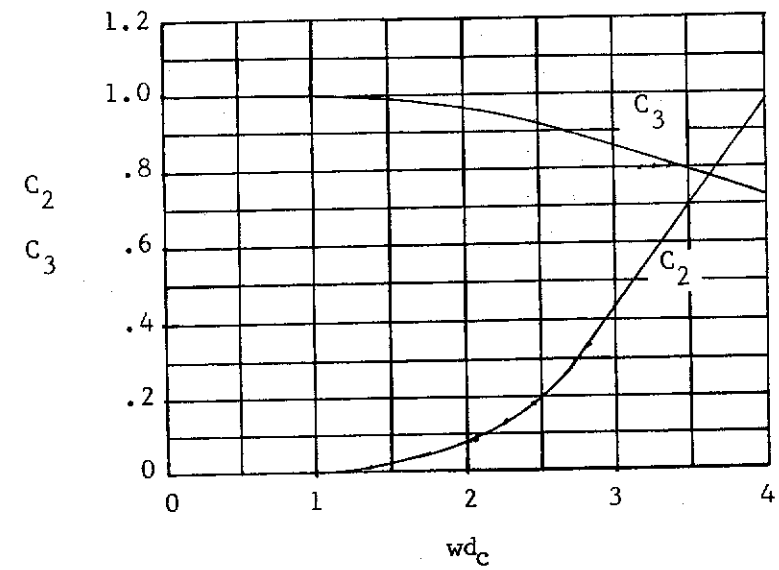
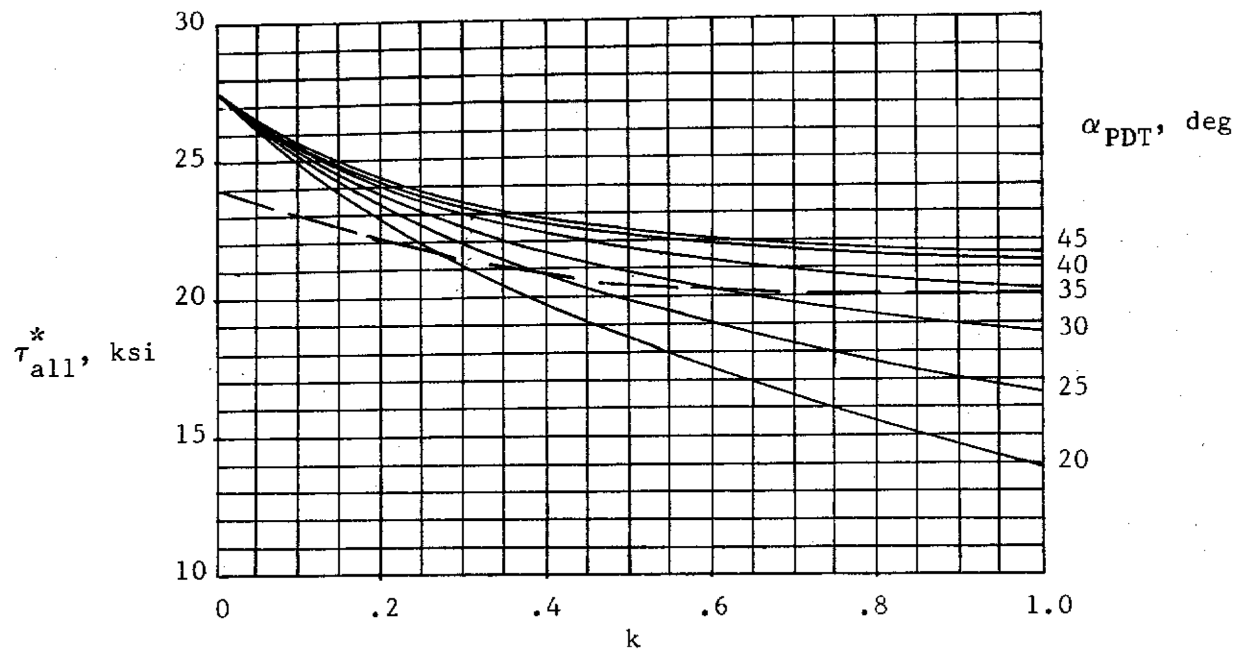
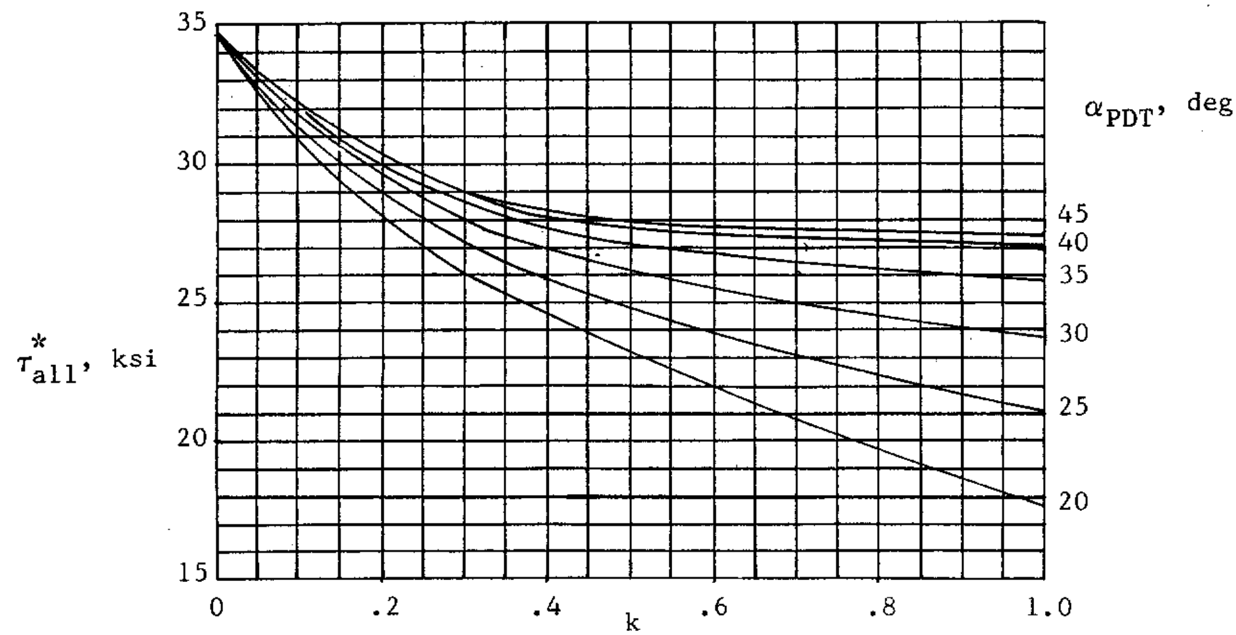


FIGURE 10.28 - STRESS CONCENTRATION FACTORS, C_2 AND C_3



2024 ALUMINUM ALLOY. F_{tu} = 62000 psi
DASHED LINE IS ALLOWABLE YIELD STRESS



7075 ALUMINUM ALLOY. F_{tu} = 72000 psi

FIGURE 10.29 - BASIC ALLOWABLE VALUES OF τ_{MAX}

10.5.6 Analysis of Flat Tension Field Beams with Double Uprights

Table 10.2 is a step-by-step procedure which yields the stresses in the flanges, webs, rivets and uprights of a flat tension field beam with double uprights as shown in Figure 10.30.

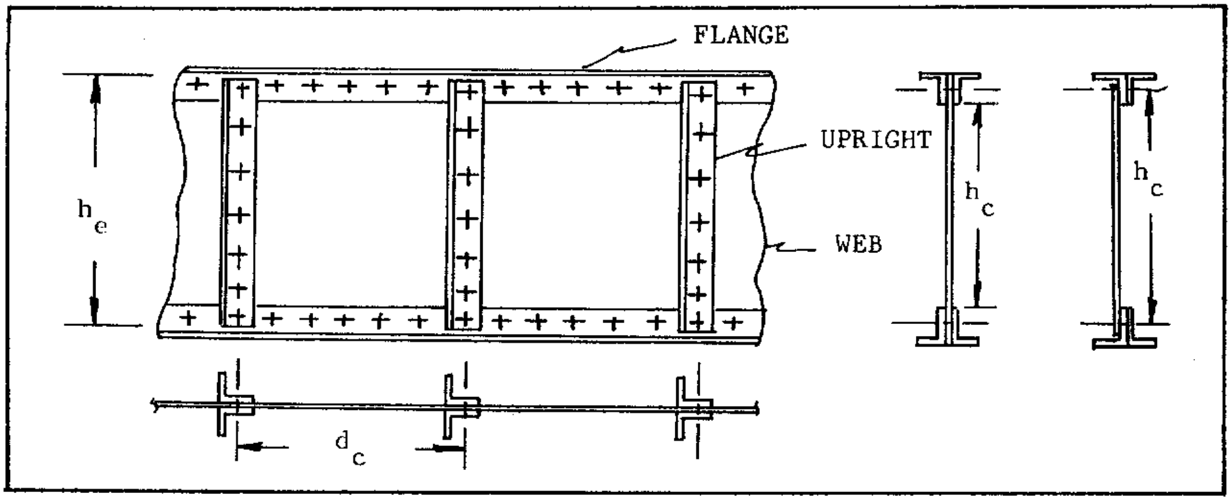
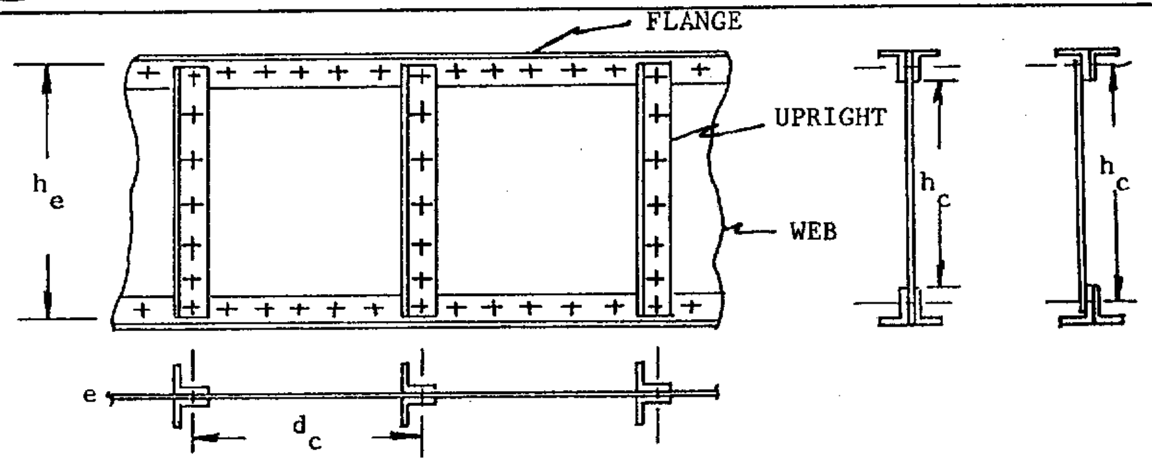


FIGURE 10.30 - FLAT TENSION FIELD BEAM WITH DOUBLE UPRIGHTS

Table 10.2 is based on a single web with parallel flanges and parallel uprights. Most beams consist of more than one web. At various locations in the following table adjacent panels must be considered. Such a situation occurs for rivet load, stringer axial stress, upright stress and moment in stringer.



DESCRIPTION	VARIABLE AND EQUATION	NUMERICAL VALUE
(1) ELASTIC MODULUS	E_c	
(2) UPRIGHT SPACING, (N.A. TO N.A.)	d	
(3) CLEAR WEB BETWEEN UPRIGHTS (RIVET TO RIVET)	d_c	
(4) DISTANCE FROM MEDIAN PLANE OF WEB TO CENTROID OF UPRIGHT	e	
(5) CLEAR WEB BETWEEN FLANGES (RIVET TO RIVET)	h_c	
(6) DISTANCE BETWEEN FLANGE CENTROIDS	h_e	
(7) LENGTH OF UPRIGHT BETWEEN UPRIGHT TO FLANGE RIVETS	h_u	
(8) WEB THICKNESS	t	
(9) UPRIGHT THICKNESS	t_u	
(10) FLANGE THICKNESS	t_f	
(11) UPRIGHT AREA	A_u	
(12) FLANGE AREA	A_f	
(13) RADIUS OF GYRATION OF UPRIGHT	ρ	
(14) MOMENT OF INERTIA OF UPRIGHT	I_u	
(15) MOMENT OF INERTIA OF FLANGE	I_F	
(16) APPLIED LOAD - UPRIGHT	P_u	
(17) APPLIED LOAD - FLANGE	P_f	
(18) APPLIED WEB SHEAR FLOW	q	
(19) WEB SHEAR STRESS	$\tau = q/t =$ (18) / (8)	

TABLE 10.2 - ANALYSIS OF FLAT TENSION FIELD BEAM WITH DOUBLE UPRIGHTS

(20) EFFECTIVE AREA OF UPRIGHT	$A_{ue} = A_u$	
(21) PARAMETER	$A_{ue}/d_c t$ = (20) / (3)(8)	
(22) PARAMETER	$h_e t$ = (6)(8)	
(23) PARAMETER	d_c/h_u = (3) / (7)	
(24) PARAMETER	t_f/t = (10) / (8)	
(25) PARAMETER	t_u/t = (9) / (8)	
(26) PARAMETER	h_c/d_c = (5) / (3)	
(27) PARAMETER	d_c/h_c = 1 / (26)	
(28) PARAMETER	t/d_c = (8) / (3)	
(29) PARAMETER	t/h_c = (8) / (5)	
(30) UPRIGHT RESTRAINT COEFFICIENT	R_h, FIGURE 10.20 (b)	
(31) FLANGE RESTRAINT COEFFICIENT	R_d, FIGURE 10.20 (b)	
(32) THEORETICAL BUCKLING COEFFICIENT	k_{ss}, FIGURE 10.20 (a)	
(33) ELASTIC BUCKLING STRESS: $d_c < h_c$ $d_c > h_c$	τ_{cr_e} = (32)(1)(28)^2 [(30) + ½ ((31) - (30))(27)^3] τ_{cr_e} = (32)(1)(29)^2 [(31) + ½ ((30) - (31))(26)^3]	
(34) INITIAL BUCKLING STRESS	τ_{cr}, FIGURE 10.21 (See Note 2)	
(35) STRESS RATIO	τ/τ_{cr} = (19) / (34)	
(36) DIAGONAL TENSION FACTOR	k, FIGURE 10.22 @ $300td_c/12h_c = 0$	
(37) PARAMETER	$\frac{A_{ue}}{d_c t} + \frac{1}{2}(1-k)$ = (21) + $\frac{(1-(36))}{2}$	
(38) RATIO OF UPRIGHT STRESSES	σ_{uMAX}/σ_u, FIGURE 10.23	
(39) RATIO OF UPRIGHT TO SHEAR STRESSES	σ_u/τ, FIGURE 10.24	
(40) DIAGONAL TENSION ANGLE	TANα, FIGURE 10.25 (a)	
(41) STRESS IN MEDIAN PLANE UPRIGHT/WEB	σ_u = - (36)(19)(40) / (37)	
(42) UPRIGHT AVERAGE STRESS	σ_{uAVG} = (41)(20) / (11)	
(43) UPRIGHT MAXIMUM STRESS	σ_{uMAX} = (41)(38)	
(44) EFFECTIVE COLUMN LENGTH: IF (23) <1.5 (23) >1.5	L_e = (7) / [1 + (36)^2(3 - 2 (23))]$^{1/2}$ $L_e = h_u$ = (7)	

TABLE 10.2.(CONT'D) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH DOUBLE UPRIGHTS

(45)	SLENDERNESS RATIO	L_e/ρ = (44) / (13)	
(46)	COLUMN ALLOWABLE	$\sigma_{co} = \pi^2$ (1) / (45)^2 or SECTION 11	
(47)	PROPORTIONAL LIMIT	F_{pl}, SECTION 5 $(F_{pl} = F_{tp})$	
(48)	STRAIN, IF (41) > (47)	σ_u/E = (41) / (1)	
(49)	FROM STRESS STRAIN CURVE	F_c, USE (48) TO DETERMINE ALLOWABLE	
(50)	MARGIN OF SAFETY: (41) > (47) COLUMN YIELD (41) < (47)	MS = (49) / (41) - 1 MS = (47) / (41) - 1	
(51)	MS - COLUMN	MS = (46) / (42) - 1	
(52)	PARAMETER	$k^{2/3}(t_u/t)^{1/3}$ = (36)$^{2/3}$ (25)$^{1/3}$	
(53)	UPRIGHT ALLOWABLE (FORCED CRIPPLING)	σ_o, FIGURE 10.26	
(54)	PLASTICITY CORRECTION: IF (53) > (47)	σ_o, = $(E_{SEC}/$(1)$)$ (53)	
(55)	MS - FORCED CRIPPLING	MS = (54) / (43) - 1	
(56)	PARAMETER	wd_c = .7 (3) $[$(8) / 2 (15) (6)$]^{1/4}$	
(57)	PARAMETER	C_1, FIGURE 10.27	
(58)	PARAMETER	C_2, FIGURE 10.28	
(59)	MAXIMUM WEB STRESS	τ'_{MAX} = (19) (1 + (36)^2 (57)) (1 + (36) (58))	
(60)	WEB ALLOWABLE	τ^*_{all}, FIGURE 10.29 @ $\alpha_{PDT} = 45^o$	
(61)	MS - WEB	MS = (60) / (59) -1	
(62)	PARAMETER	C_3, FIGURE 10.28	
(63)	SECONDARY BENDING IN FLANGE	M_{SB} = (1/12)((36) (62) (18) (3)^2 (40))	
(64)	DISTANCE FROM N.A. TO EXTREME FIBER OF FLANGE	C_f	
(65)	DISTANCE - N.A. TO NEAR FIBER OF FLANGE	D_f	
(66)	FLANGE APPLIED STRESS	σ_a = (17) / (12)	
(67)	DIAGONAL TENSION STRESS - FLANGE (COMP)	σ_{DT} = -((36) (19) / (40)) / $[$2 (12) / (22) + .5(1-(36))$]$	
(68)	SECONDARY BENDING STRESS - FLANGE (COMP)	σ_{SB} = -(63) (64) / (15)	
(69)	SECONDARY BENDING STRESS - FLANGE (TENSION)	σ_{SB} = (68) (65) / (64)	

TABLE 10.2 (CONT'D) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH DOUBLE UPRIGHTS

(70)	FLANGE STRESS - INSIDE FIBER	σ_{tot} = (66) + (67) + (68)	
(71)	FLANGE STRESS - EXTREME FIBER	σ_{tot} = (66) + (67) + (69)	
(72)	ALLOWABLE CRIPPLING STRESS - FLANGE	F_{cc}	
(73)	ALLOWABLE TENSION STRESS - FLANGE	F_{tu} or F_{ty}	
(74)	MS - FLANGE (TENSION)	MS = (73) / (71) - 1	
(75)	MS - FLANGE (COMP)	MS = (72) / (70) - 1	
(76)	RIVET FACTOR	R = 1 + 0.414 (36)	
(77)	RIVET LOAD - WEB TO FLANGE	R'' = qR = (18) (76)	
(78)	ALLOWABLE RIVET SHEAR LOAD	P_{af}	
(79)	MS - FLANGE RIVETS	MS = (78) / (77) - 1	
(80)	RIVET LOAD - UPRIGHT TO FLANGE	P_u = (41) (20)	
(81)	ALLOWABLE RIVET LOAD	P_{au}	
(82)	MS - UPRIGHT RIVETS	MS = (81) / (80) - 1	
(83)	STATIC MOMENT OF CROSS SECT. OF ONE UPRIGHT ABOUT MEDIAN PLANE OF WEB	Q	
(84)	WIDTH OF OUTSTANDING LEG OF UPRIGHT	b	
(85)	UPRIGHT COLUMN YIELD STRESS	F_{coy}, SECTION 11	
(86)	RIVET LOAD - UPRIGHT TO WEB	R_R = 2 (83) (85) (7) / (84) (44)	
(87)	RIVET ALLOWABLE LOAD	P_{ar}	
(88)	MS - RIVET, UPRIGHT TO WEB	MS = (87) / (86) - 1	
(89)	ULTIMATE TENSILE STRESS OF WEB	F_{tu}, SECTION 5	
(90)	RIVET TENSILE STRENGTH UPRIGHT/WEB PER INCH *	σ_R = .15 (8) (89)	*See NACA TN 2661, "A Summary of Diagonal Tension", 1952, page 49 for explanation.
(91)	RIVET ALLOWABLE TENSILE LOAD PER INCH	F_{RT}, SECTION 6	
(92)	MS - RIVET TENSION	MS = (91) / (90) - 1	

NOTES:

(1) If any of the margins of safety are less than zero, the design is inadequate. The deficient area must be corrected and this table repeated.

(2) If the web is subjected to tension or compression as well as shear, the initial buckling stress of the web must be modified according to the method described in Section 10.4.1.

TABLE 10.2 (CONT'D) - ANALYSIS OF FLAT TENSION FIELD BEAMS WITH DOUBLE UPRIGHTS

Revision A

10.5.7 Analysis of a Flat Tension Field Beam with Single Uprights and Access Holes

The following step-by-step procedure (in Table 10.3) is an analysis of a flat tension field beam with single uprights and access holes (Figure 10.31).

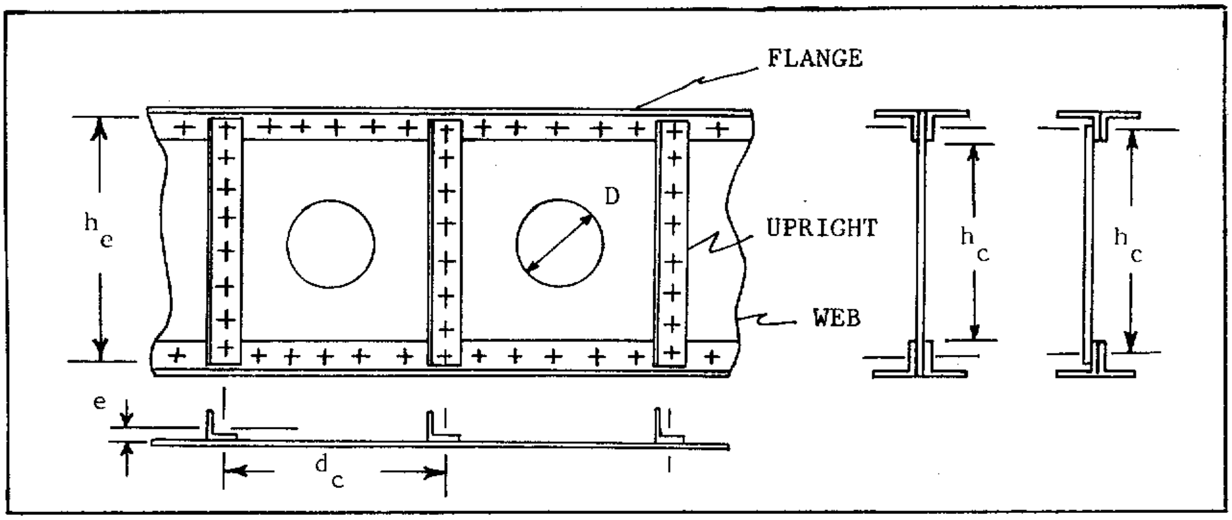
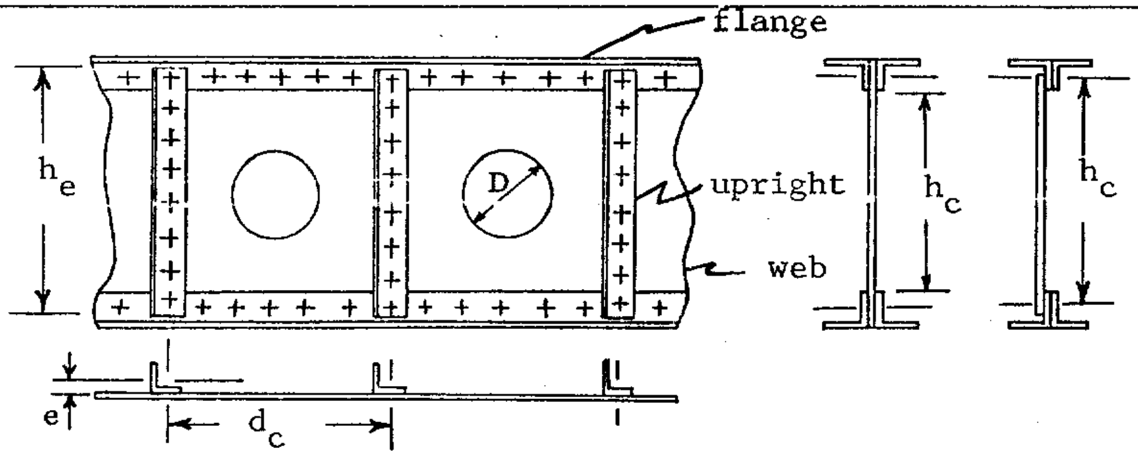


FIGURE 10.31 - FLAT TENSION FIELD BEAM WITH SINGLE UPRIGHTS AND ACCESS HOLES

Table 10.3 is based on a single web with parallel flanges and parallel uprights. Most beams consist of more than one web. At various locations in the following table adjacent panels must be considered. Such a situation occurs for rivet load, stringer axial stress, upright stress and moment in stringer.



DESCRIPTION	VARIABLE AND EQUATION	NUMERICAL VALUE
① Elastic Modulus	E_c	
② Upright Spacing (N.A. to N.A.)	d	
③ Clear Web Between Uprights (Rivet to Rivet)	d_c	
④ Distance from Median Plane of Web to Centroid of Upright	e	
⑤ Clear Web Between Flanges	h_c	
⑥ Distance Between Flange Centroids	h_e	
⑦ Length of Upright Between Upright to Flange Rivets	h_u	
⑧ Access Hole Diameter	D	
⑨ Web Thickness	t	
⑩ Upright Thickness	t_u	
⑪ Flange Thickness	t_f	
⑫ Upright Area	A_u	
⑬ Flange Area	A_f	
⑭ Radius of Gyration of Upright	ρ	
⑮ Moment of Inertia of Upright	I_u	

TABLE 10.3 - ANALYSIS OF FLAT TENSION FIELD BEAMS WITH ACCESS HOLE AND SINGLE UPRIGHTS

(16)	Moment of Inertia of Flange	I_f	
(17)	Flange Applied Load	P_f	
(18)	Upright Applied Load	P_u	
(19)	Applied Web Shear Flow	q	
(20)	Web Shear Stress	$\tau = (19)/(9)$	
(21)	Effective Area of Upright	$A_{ue} = (12)/\left\|1 + (4)^2/(14)^2\right\|$	
(22)	Parameter	$A_{ue}/d_c t = (21)/(3)(9)$	
(23)	Parameter	$h_e t = (6)(9)$	
(24)	Parameter	$d_c/h_u = (3)/(7)$	
(25)	Parameter	$t_f/t = (11)/(9)$	
(26)	Parameter	$t_u/t = (10)/(9)$	
(27)	Parameter	$h_c/d_c = (5)/(3)$	
(28)	Parameter	$d_c/h_c = 1/(27)$	
(29)	Parameter	$t/d_c = (9)/(3)$	
(30)	Parameter	$t/h_c = (9)/(5)$	
(31)	Parameter	$D/h_c = (8)/(5)$	
(32)	Parameter	$D/d_c = (8)/(3)$	
(33)	Parameter	$1 + D/d_c = 1 + (32)$	
(34)	Parameter	$A_w = t(d_c - D) = (9)((3) - (8))$	
(35)	Parameter	$(26)/(28)$	
(36)	Upright Restraint Coefficient	R_h, Figure 10.20 (b)	
(37)	Flange Restraint Coefficient	R_d, Figure 10.20 (b)	
(38)	Theoretical Buckling Coefficient	k_{ss}, Figure 10.20 (a)	
(39)	Elastic Buckling Stress: $d_c < h_c$ $d_c > h_c$	 $\tau_{cr_e} = (38)(1)(29)^2\left[(36) + \frac{1}{2}((37) - (36))(28)^3\right]$ $\tau_{cr_e} = (38)(1)(30)^2\left[(37) + \frac{1}{2}((36) - (37))(27)^3\right]$	
(40)	Initial Buckling Stress	τ_{cr}, Figure 10.21 (See Note 2)	
(41)	Stress Ratio	$\tau/\tau_{cr} = (20)/(40)$	

TABLE 10.3 (CONT'D) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH ACCESS HOLE AND SINGLE UPRIGHTS

(42)	Diagonal Tension Coefficient	k, Figure 10.22 @ $300\, td_c/12 h_c = 0$	
(43)	Parameter	(22) + ½(1 - (42))	
(44)	Ratio of Upright Stresses	σ_{umax}/σ_u, Figure 10.23	
(45)	Ratio of Upright to Shear Stress	σ_u/τ, Figure 10.24	
(46)	Diagonal Tension Angle	tan α, Figure 10.25 (a)	
(47)	Stress in Median Plane-Upright to Web	σ_u = - (42) (20) (46) / (43)	
(48)	Upright Average Stress	σ_{uavg} = (47) (21) / (12)	
(49)	Upright Maximum Stress	σ_{umax} = (47) (44)	
(50)	Effective Column Length: If (24) < 1.5	L_e = (7) / [1 + (42)^2 (3 - 2 (24))]$^{1/2}$	
	If (24) > 1.5	$L_e = h_u$ = (7)	
(51)	Slenderness Ratio	$L_e/2\rho$ = (50)/2 (14)	
(52)	Column Allowable	$\sigma_{co} = \pi^2$ (1)/(51)^2 or Section 11	
(53)	Proportional Limit	F_{pl}, Section 5 ($F_{pl} = F_{tp}$)	
(54)	Strain, If (47) > (53)	σ_u/E = (47) / (1)	
(55)	From Stress-Strain Curve	F_c, use (54) to determine allowable	
(56)	Margin of Safety: Column Yield (47)>(53)	MS = (55) / (47) - 1	
	(47)<(53)	MS = (53) / (47) - 1	
(57)	MS - Column	MS = (52) / (48) - 1	
(58)	Parameter	$k^{2/3}(t_u/t)^{1/3}$ = (42)$^{2/3}$ (26)$^{1/3}$	
(59)	Parameter	C_4, Figure 10.32	
(60)	Parameter	C_5, Figure 10.32	
(61)	Parameter	(12) (32) (59)	
(62)	Parameter	(34) + (61)	
(63)	Parameter	(62) / (3)(9)	
(64)	Upright Allowable (Without Access Hole)	σ_o, Figure 10.26	

TABLE 10.3 (CONT'D) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH ACCESS HOLE AND SINGLE UPRIGHTS

(65) Upright Allowable (With Access Hole)	$\sigma_o' = (64) / (33)$	
(66) MS Forced Crippling	$MS = (65) / (49) - 1$	
(67) Parameter	$wd_c = .7 (3) [(9)/2 (16)(6)]^{1/4}$	
(68) Parameter	C_1, Figure 10.27	
(69) Parameter	C_2, Figure 10.28	
(70) Maximum Web Stress	$\tau'_{max} = (20) [(1 + (42)^2 (68)) (1 + (42)(69))]$	
(71) Web Allowable (Without Access Hole)	τ^*_{all}, Figure 10.29 @ $\alpha_{PDT} = 45°$	
(72) Web Allowable (With Access Hole)	$\tau_s = (71)(63) / (60)$	
(73) MS - Web	$MS = (72) / (70) - 1$	
(74) Parameter	C_3, Figure 10.28	
(75) Secondary Bending in Flange	$M_{SB} = (1/12) ((42)(74)(19)(3)^2 (46))$	
(76) Distance From N.A. to Extreme Fiber of Flange	C_f	
(77) Distance - N.A. to Near Fiber of Flange	D_f	
(78) Flange Applied Stress	$\sigma_a = (17) / (13)$	
(79) Diagonal Tension Stress Flange (Comp)	$\sigma_{DT} = -((42)(20) / (46)) / [2 (13) / (23) + .5 (1 - (42))]$	
(80) Secondary Bending Stress - Flange (Comp)	$\sigma_{SB} = - (75)(76) / (16)$	
(81) Secondary Bending Stress - Flange (Tension)	$\sigma_{SB} = (80)(77) / (76)$	
(82) Flange Stress - Inside Fibers	$\sigma_{tot} = (78) + (79) + (80)$	
(83) Flange Stress - Extreme Fibers	$\sigma_{tot} = (78) + (79) + (81)$	
(84) Allowable Crippling Stress - Flange	F_{cc}	
(85) Allowable Tensile Stress - Flange	F_{tu} or F_{ty}	

TABLE 10.3 (CONT'D) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH ACCESS HOLE AND SINGLE UPRIGHTS

(86)	MS - Flange (Tension)	MS = (85) / (83) - 1	
(87)	MS - Flange (Comp)	MS = (84) / (82) - 1	
(88)	Rivet Factor	R = 1 + 0.414 (42)	
(89)	Rivet Load - Web to Flange	R" = qR = (19) (88)	
(90)	Allowable Rivet Shear Load	P_{af}	
(91)	MS - Flange Rivets	MS = (90) / (89) - 1	
(92)	Rivet Load - Upright to Flange	P_u = (47) (21)	
(93)	Allowable Rivet Upright Load	P_{au}	
(94)	MS - Upright Rivets	MS = (93) / (92) - 1	
(95)	Inter Rivet Buckling Allowable	F_{IR}, Section 10.6	
(96)	MS - Inter Rivet Buckling	MS = (95) / (49) - 1	
(97)	Ultimate Tensile Stress of Web	F_{tu}, Section 5	
(98)	Rivet Tensile Strength-Upright to Web Per Inch *	σ_R = .22 (9) (97)	
(99)	Rivet Allowable Tensile Load per inch	F_{RT}, Section 6	
(100)	MS - Rivet Tension	MS = (99) / (98) - 1	

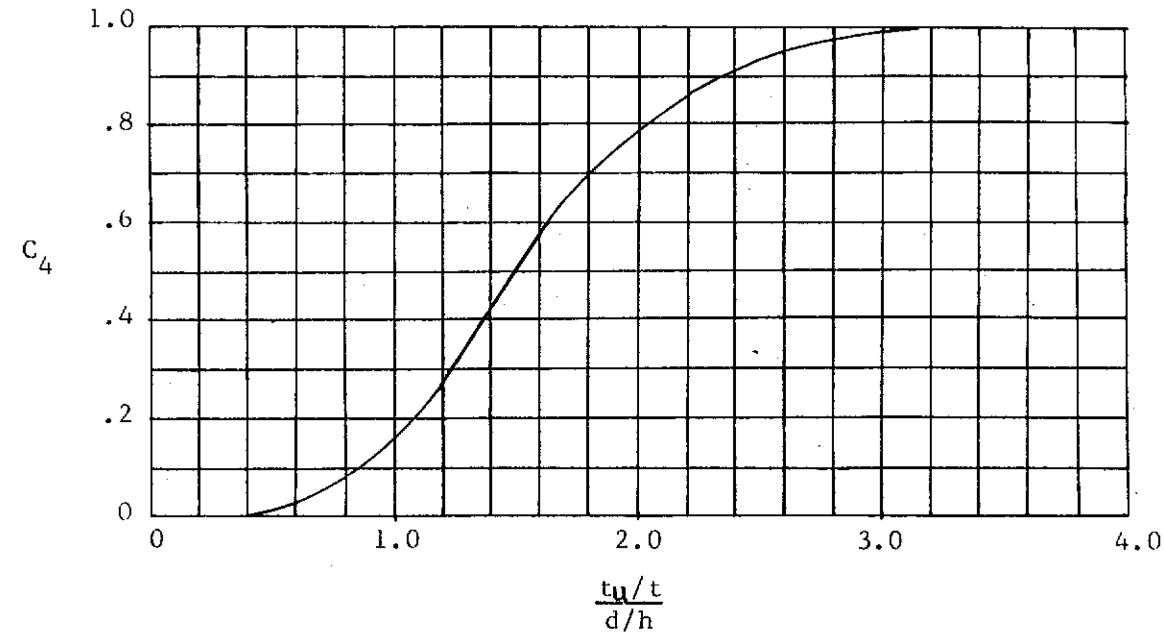
NOTES:

(1) If any of the margins of safety are less than zero, the design is inadequate. The deficient area must be corrected and this table repeated.

(2) If the web is subjected to tension or compression as well as shear, the initial buckling stress of the web must be modified according to the method described in Section 10.4.1.

* See NACA TN 2661, "A Summary of Diagonal Tension", 1952, page 49 for explanation.

TABLE 10.3 (CONT'D) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH ACCESS HOLE AND SINGLE UPRIGHTS



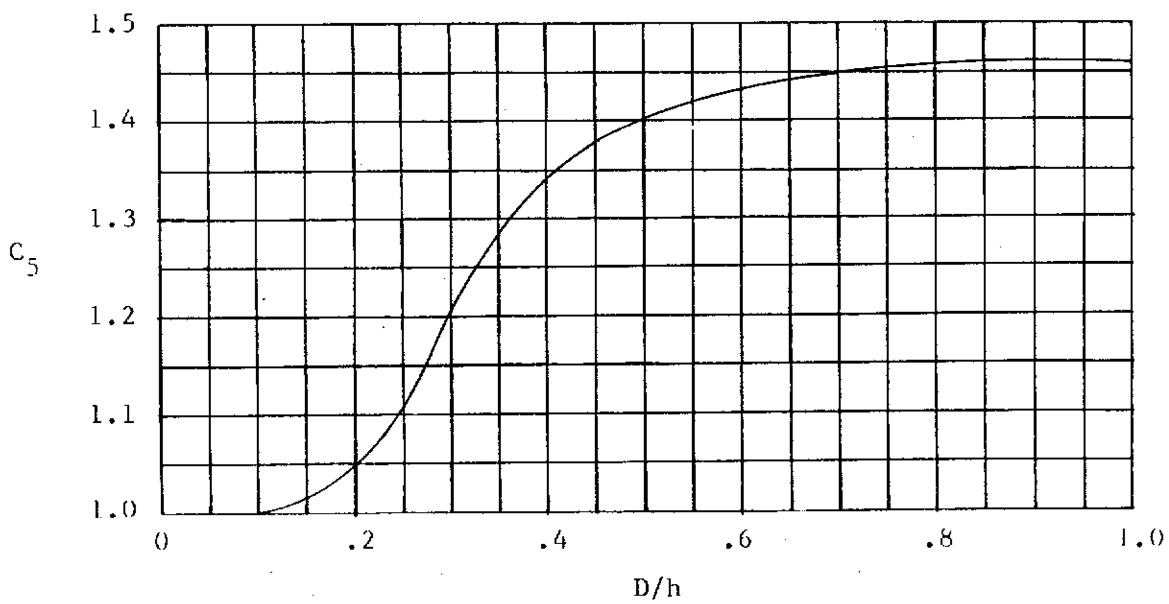


FIGURE 10.32 - ACCESS HOLE REDUCTION FACTORS

10.5.8 Analysis of a Tension Field Beam with Curved Panels

Table 10.4 presents an analysis procedure for a tension field beam with curved panels. The basic geometry of a sheet/stringer beam with curved panels is given in Figure 10.33. The applied loads are the shear flows (positive acting clockwise) on the edges of the bays and the axial stresses at each end of the stringers (tension is positive). Section properties and allowable stresses are required for each stringer and web.

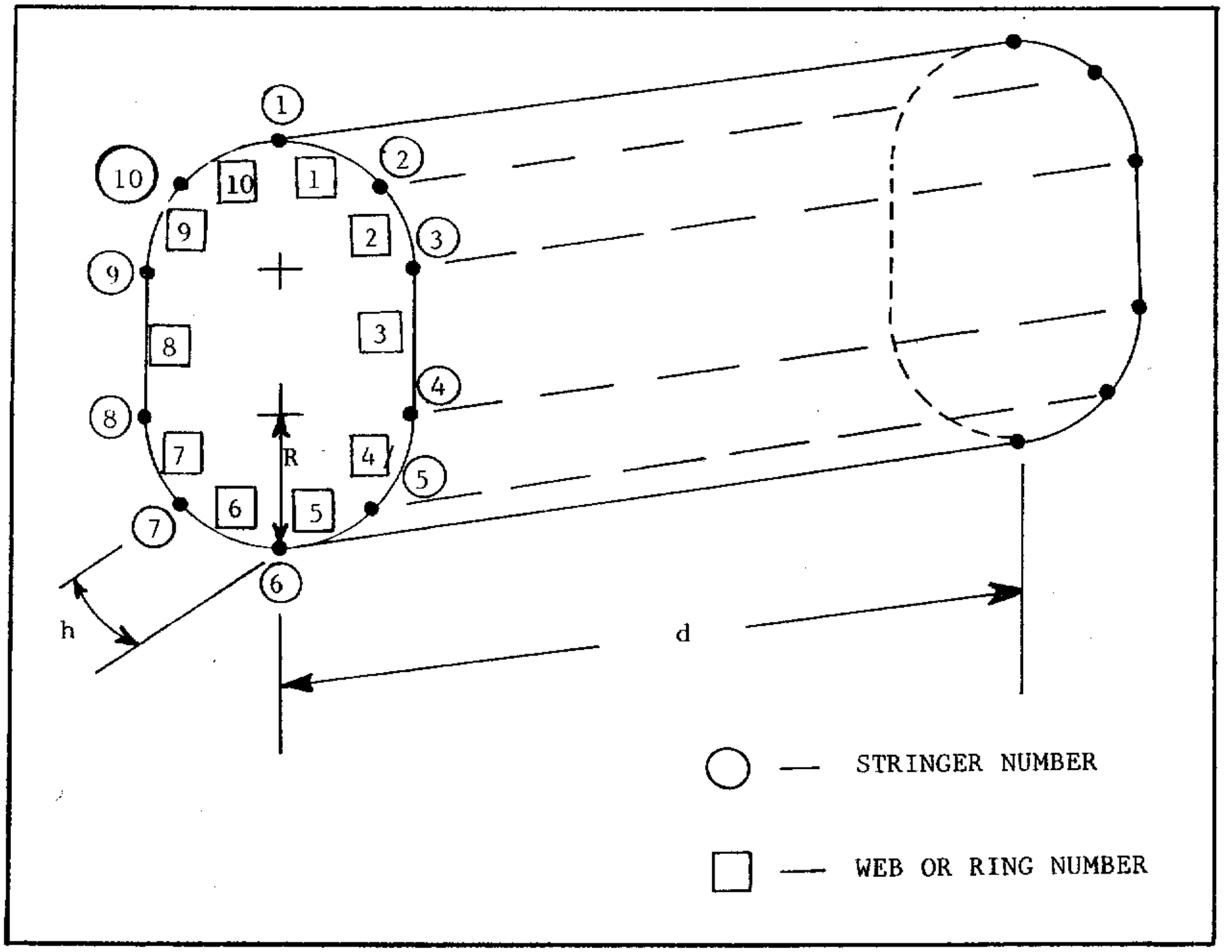


FIGURE 10.33 - GEOMETRY OF BAY WITH CURVED PANELS

Table 10.4 is based on a single web with parallel flanges and parallel uprights. Most beams consist of more than one web. At various locations in the following table adjacent panels must be considered. Such a situation occurs for rivet load, stringer axial stress, upright stress and moment in stringer.

DESCRITPION	VARIABLE AND EQUATION	NUMERICAL VALUE
① Element Number	EL	
② Skin Thickness	t	
③ Stringer Thickness	t_{st}	
④ Frame Spacing (Panel Length)	d	
⑤ Panel Height	h	
⑥ Aspect Ratio	h/d = ⑤ / ④	
⑦ Aspect Ratio	d/h = ④ / ⑤	
⑧ Stringer Area	A_{ST}	
⑨ Average Stringer Area	A_{AVG} = (⑧$_n$ + ⑧$_{n+1}$) /2, Note 1	
⑩ Area of Frame or Ring	A_{RG}	
⑪ Radius of Curvature	R	
⑫ Poisson's Ratio	μ	
⑬ Parameter	Z = ⑤^2(1 - ⑫^2)$^{1/2}$/⑪②	
⑭ Shear Stress Coefficient	k_s, Figure 10.34 or 10.35	
⑮ Elastic Modulus	E	
⑯ Elastic Buckling Stress of Web (See Note 5)	τ_{cr} = π^2 ⑭ ⑮ ⑤^2/12 ⑪^2 ⑬^2	
⑰ Parameter	300 ②④/ ⑪⑤	
⑱ Parameter	R_S = ⑤②/ ⑨	
⑲ Parameter	R_R = ④②/ ⑩	
⑳ Parameter	(1 + ⑱)/(1 + ⑲)	
㉑ Ultimate Applied Shear Flow	q_{ULT}	
㉒ Web Shear Stress	τ_{ULT} = ㉑ / ②	
㉓ Shear Stress Ratio	τ_{ULT}/τ_{CR} = ㉒ / ⑯	
㉔ Diagonal Tension Factor	k, Figure 10.22	
㉕ Parameter	(⑤/ ⑪) (⑮ / ㉒)$^{1/2}$ / (1 + ⑲)$^{1/2}$	
㉖ Pure Diagonal Tension Angle	α_{PDT}, Figure 10.36	

TABLE 10.4 - ANALYSIS OF CURVED TENSION FIELD BEAM

㉗ Web Allowable	τ^*_{all}, Figure 10.29	
㉘ Parameter	$1/R_R$ = 1 / ⑲	
㉙ Parameter	$1/R_S$ = 1 / ⑱	
㉚ Correction Factor	Δ, Figure 10.38	
㉛ Ultimate Allowable Shear Stress	τ_{all} = ㉗ (.65 + ㉚)	
㉜ Margin of Safety - Web	MS = ㉛ / ㉒ - 1	
㉝ Parameter	α/α_{PDT}, Figure 10.37	
㉞ Tension Field Angle	α = ㉝ ㉖	
㉟ Rivet Shear Flow	q_{RIVET}, See Note 2	
㊱ Required Rivet Shear Strength	R" = ㉟ [1 + ㉔(1/cos ㉞ - 1)]	
㊲ Allowable Rivet Shear/ Inch	q_{all}, Section 6	
㊳ MS - Rivets	MS = ㊲ / ㊱ - 1	
㊴ Stringer Stress at the Ring	σ_{ST} = [-㉔ ㉒ (cot ㉞)] / [(⑧/② ⑤) + .5(1 - ㉔)]	
㊵ Stress Ratio	$\sigma_{STMAX}/\sigma_{ST}$, Figure 10.23	
㊶ Maximum Stringer Stress	σ_{STMAX} = ㊵ ㊴	
㊷ Average Maximum Stringer Stress	$\sigma_{STMAXAVG}$ = (㊶$_n$ + ㊶$_{n+1}$)/2, Note 1	
㊸ Thickness Ratio	③ / ②	
㊹ Avg. Diagonal Tension Factor	k_{AVG} = (㉔$_n$ + ㉔$_{n+1}$), Note 3	
㊺ Forced Crippling Allowable	σ_o, Figure 10.26	
㊻ Moment in Stringer	M_{ST} = [㉔ ㉒ ② ⑤ ④^2 tan ㉞] / [24 ⑪]	
㊼ Allowable Moment (Outside of Stringer)	MACO	
㊽ Allowable Moment (Inside of Stringer)	MACI	
㊾ Local Crippling Stress	σ_{cr}, Note 4	
㊿ Bending Stress @ Center of Bay (From External Bending)	f_{bc}	

TABLE 10.4 (CONT'D) - ANALYSIS OF CURVED TENSION FIELD BEAM

(51) Bending Stress at Ring (From External Bending)	f_{br}	
(52) Stringer Tension Allowable	F_{tu}	
(53) Local Crippling Allowable	F_{cc}, Section 10.7	
(54) Effective Length of Stringer	ℓ'	
(55) Radius of Gyration of Stringer	ρ	
(56) Slenderness Ratio	(54) / (55)	
(57) Johnson-Euler Buckling Coefficient	C, Section 11	
(58) Johnson-Euler Column Allowable	F_c, Section 11	
(59) MS - Stringer at Center of Bay	Figure 10.39	
(60) MS - Stringer at Ring	Figure 10.40	

NOTES:

(1) Average over stringer length, d

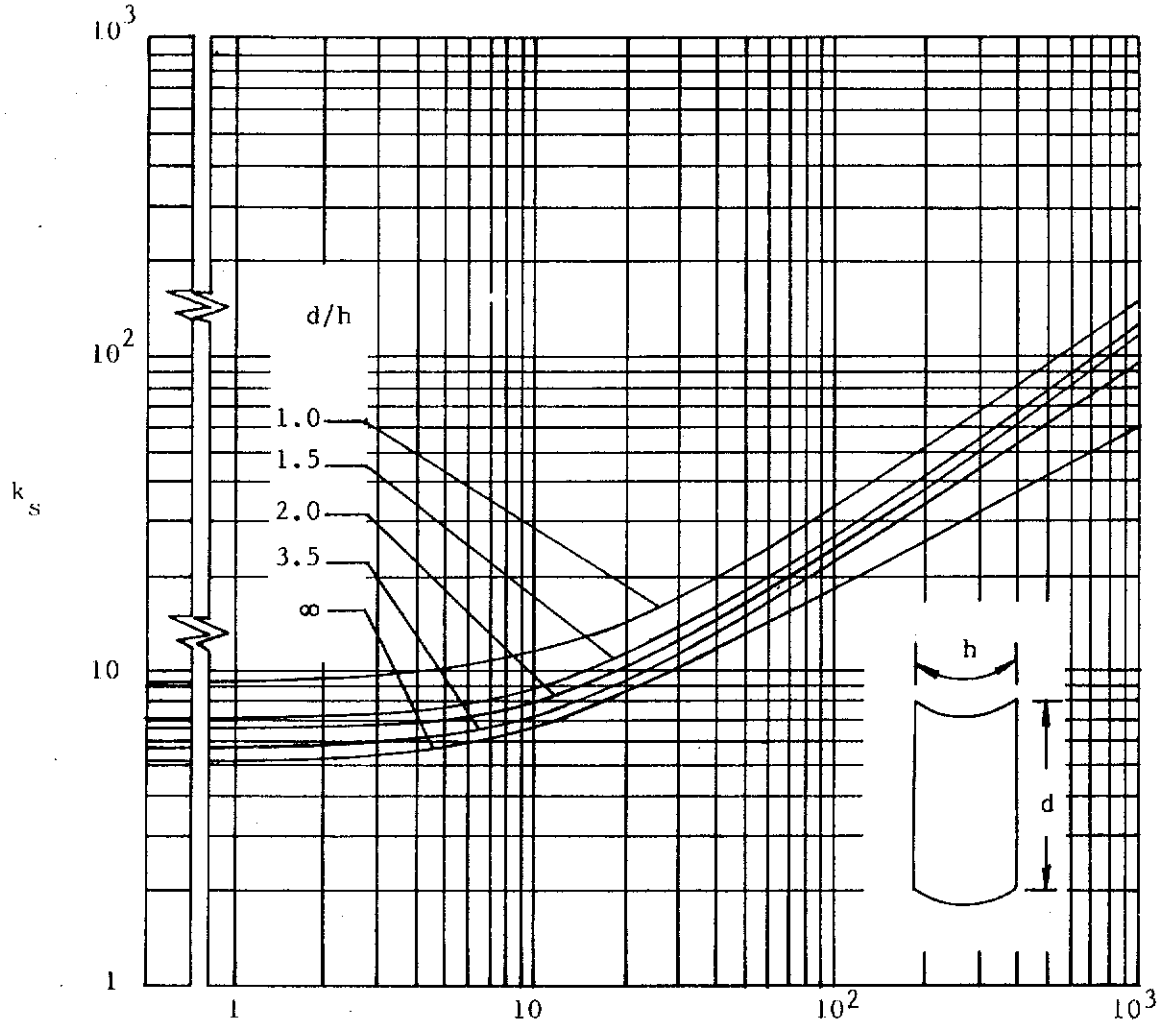
(2) q_{RIVET} along stringer is the difference in shear flow between adjacent panels or the shear flow in the outside skin of the lap splice if one exists, whichever is greater.

(3) Average of adjacent panels

(4) Portion of element adjacent to skin

(5) If the web is subjected to tension or compression as well as shear, the initial buckling stress of the web must be modified according to the method described in Section 10.4.1.

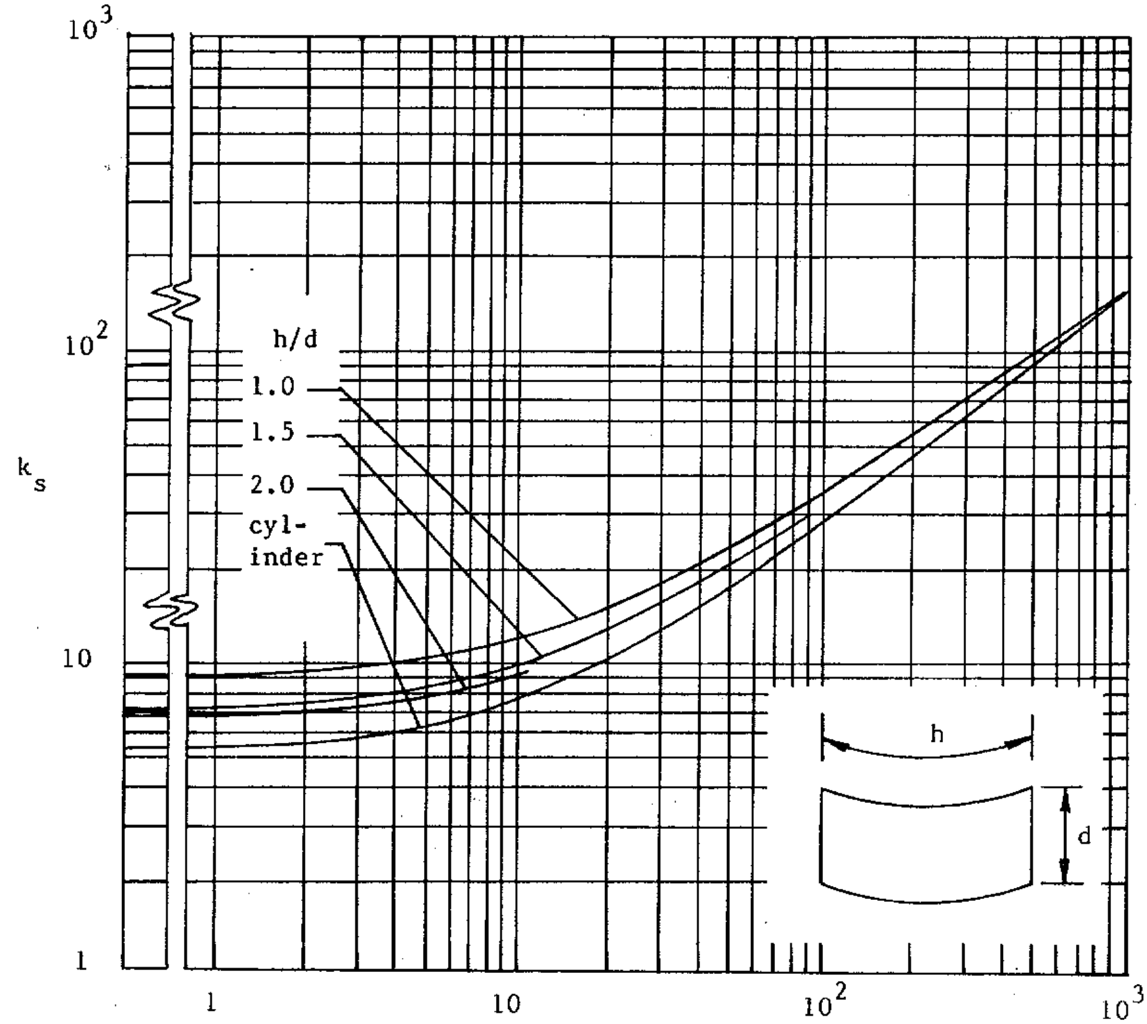
TABLE 10.4 (CONT'D) - ANALYSIS OF CURVED TENSION FIELD BEAM



$$Z = \frac{h^2}{Rt}(1-\mu^2)^{1/2} = (13)$$

PLATES LONG AXIALLY ($d \geq h$), τ_{cr}, elastic $= k_s \dfrac{\pi^2 E h^2}{12 R^2 Z^2}$

FIGURE 10.34 - CRITICAL SHEAR STRESS COEFFICIENTS FOR SIMPLY SUPPORTED CURVED PLATES

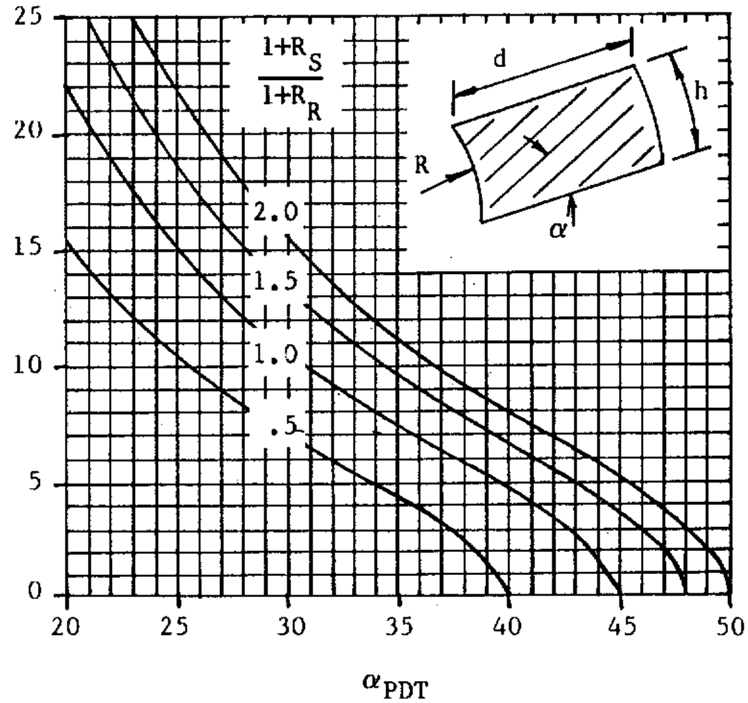


$$Z = \frac{d^2}{Rt}\,(1-\mu^2)^{1/2} = ⑬$$

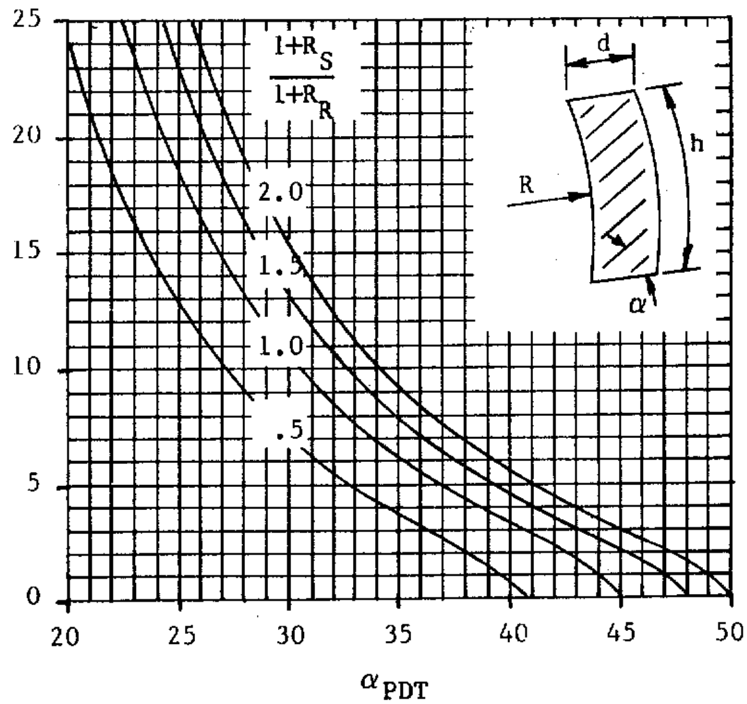
PLATES LONG CIRCUMFERENTIALLY ($h \geq d$), τ_{cr}, elastic $= k_s \dfrac{\pi^2 E d^2}{12R^2 Z^2}$

FIGURE 10.35 - CRITICAL SHEAR STRESS COEFFICIENTS FOR SIMPLY SUPPORTED CURVED PLATES

$$25 = \frac{\frac{h}{R}\left(\frac{E}{\tau}\right)^{1/2}}{(1+R_R)^{1/2}}$$



$$25 = \frac{\frac{d}{R}\left(\frac{E}{\tau}\right)^{1/2}}{(1+R_R)^{1/2}}$$



$$R_R = \frac{dt}{A_{RG}}, \quad R_S = \frac{ht}{A_{ST}}$$

FIGURE 10.36 - ANGLE OF PURE DIAGONAL TENSION

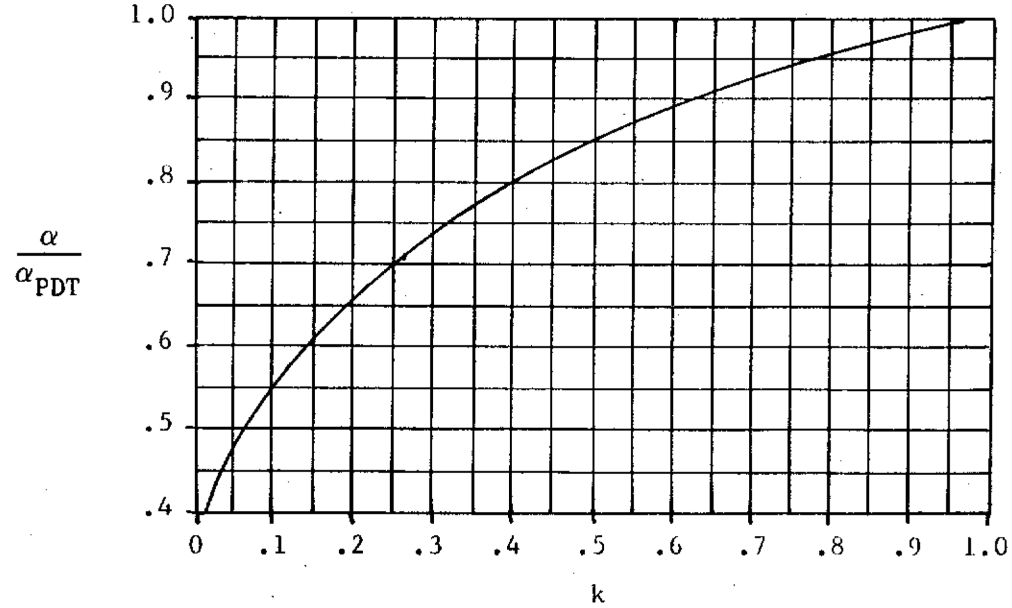


FIGURE 10.37 - CORRECTION FACTOR FOR ANGLE OF DIAGONAL TENSION ($1/R_S = 1/R_R = 1.0$)

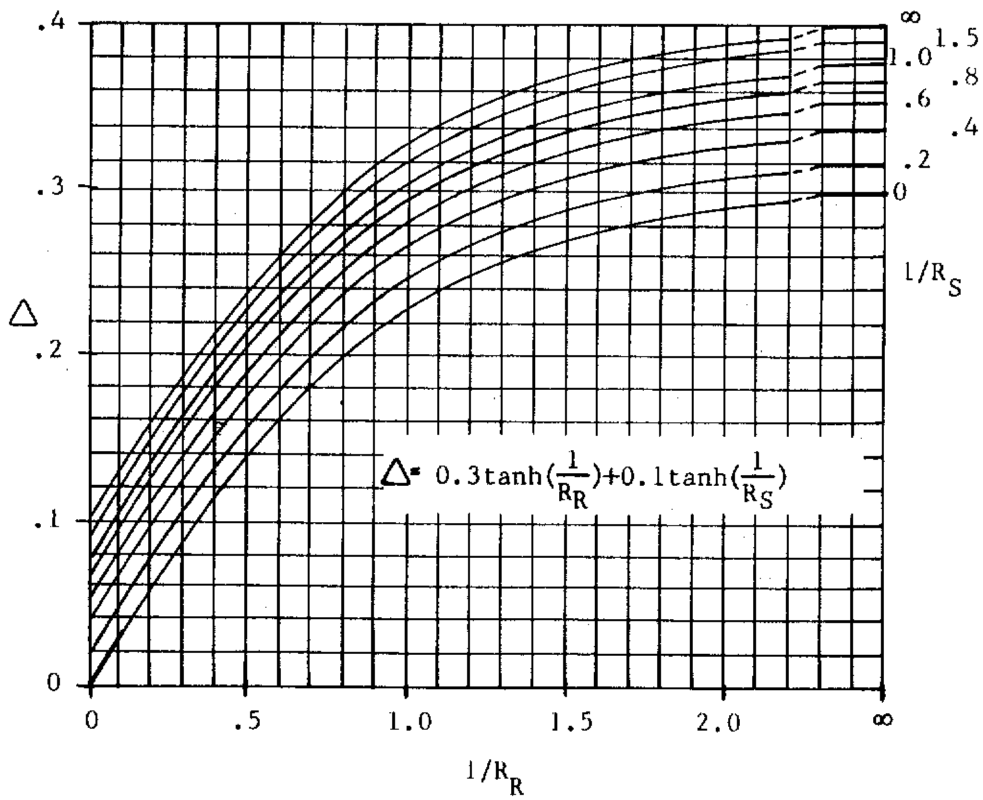


FIGURE 10.38 - CORRECTION FOR ALLOWABLE ULTIMATE SHEAR STRESS IN CURVED WEBS

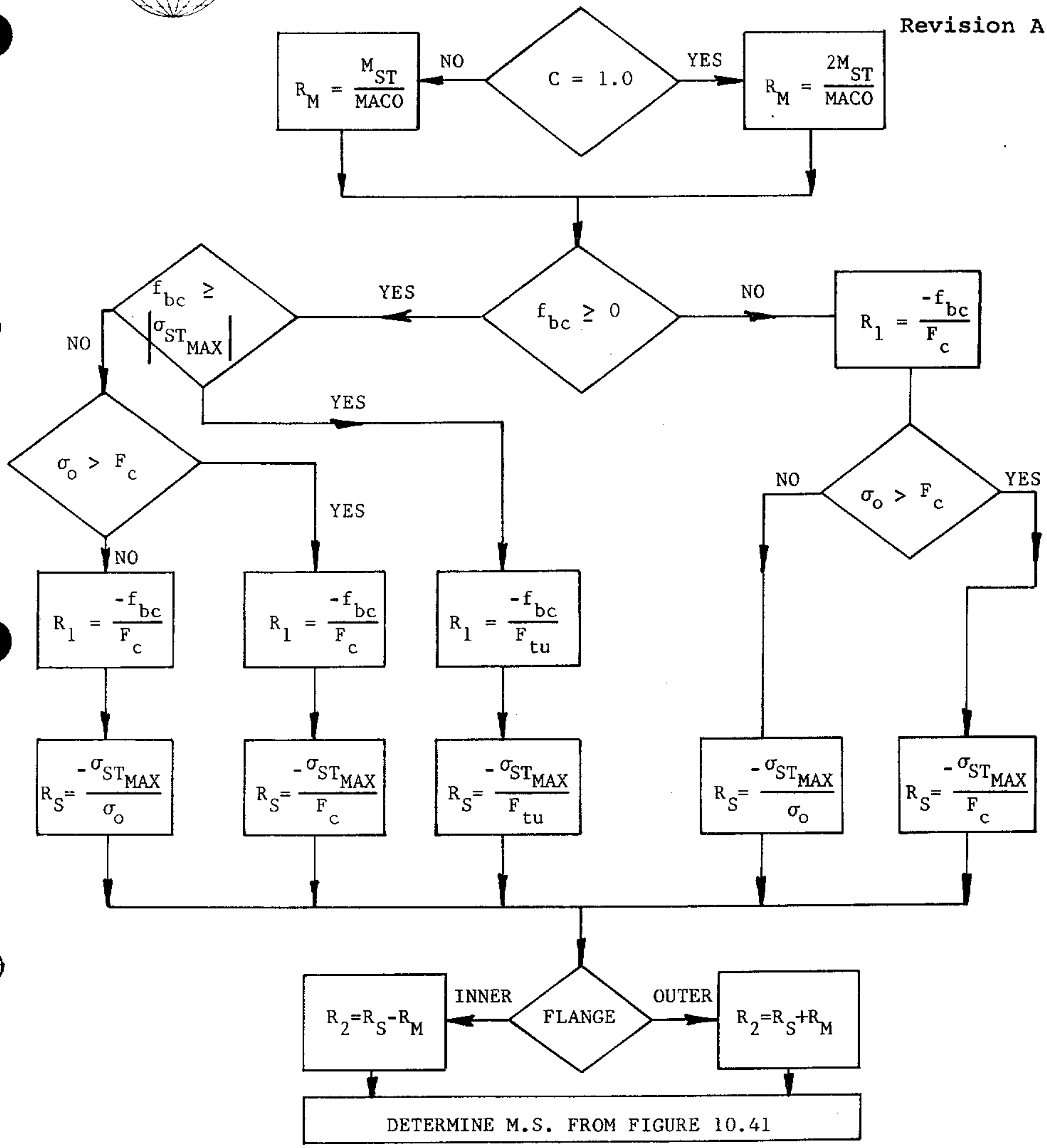


FIGURE 10.39 - STRINGER MARGIN OF SAFETY AT CENTER OF BAY

Revision B

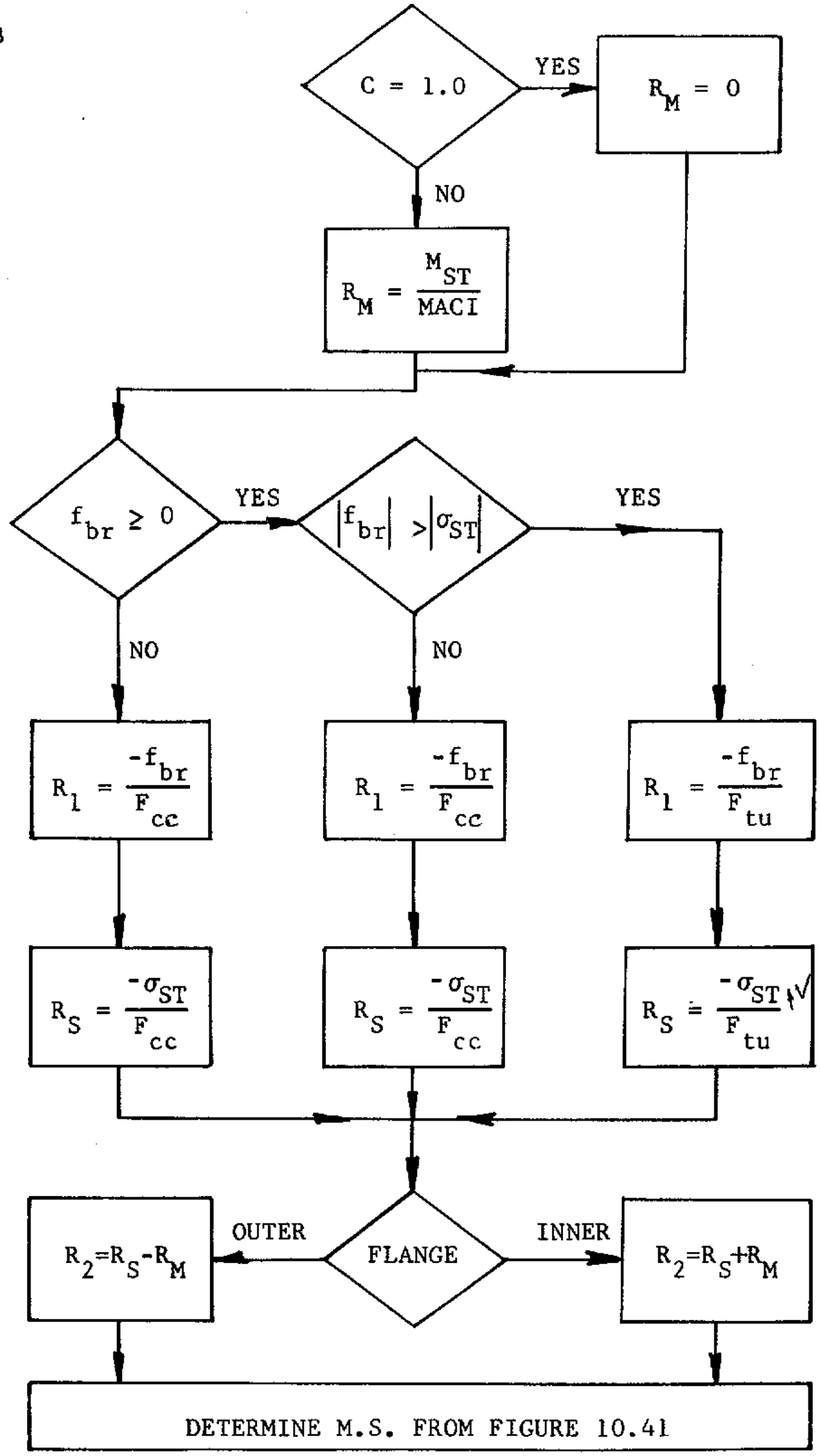
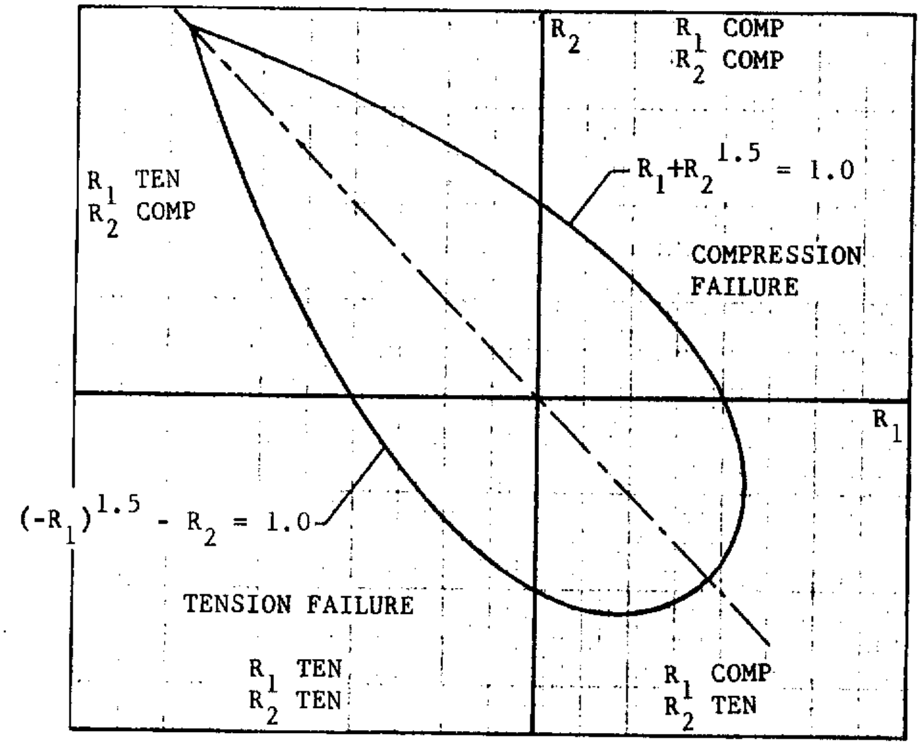


FIGURE 10.40 - STRINGER MARGIN OF SAFETY AT RING



OUTER FLANGE: $R_2 = R_S + R_M$

INNER FLANGE: $R_2 = R_S - R_M$

FIGURE 10.41 - STRINGER MARGIN OF SAFETY INTERACTION DIAGRAM

10.6 INTER-RIVET BUCKLING

The effective sheet area and stiffener are considered to act monolithically in most designs. If, however, the rivet spacing is too large, the sheet will buckle between the rivets before the crippling stress of the stiffener is reached. Thus, the sheet is less effective in aiding the stiffener in carrying compressive loads. It is, therefore, necessary to calculate the inter-rivet buckling stress to ensure that inter-rivet buckling does not occur.

In calculating the inter-rivet buckling stress, it is assumed that the sheet between adjacent rivets acts as a column with fixed ends. The general column equation is

$$F_c = C\,\pi^2\,E_t/(L/\rho)^2,$$

where C is the end fixity coefficient and is equal to 4 for fixed end supports. Since the effective column length $L' = L/\sqrt{C}$:

$$F_c = \pi^2\,E_t/(L'/\rho)^2$$

The radius of gyration ρ for a unit width of sheet is 0.29 t. Letting the rivet spacing p replace the column length L, the above equation becomes:

$$F_{ir} = \frac{\pi^2 E_t}{\left(\frac{p}{\sqrt{C}}/0.29t\right)^2}$$

The fixity coefficient C = 4 is used for flat head rivets or bolts. For spotwelds it should be decreased to C = 3.5. For Brazier head rivets or screws use C = 3 and for counter-sunk or dimpled installations C = 1. The inter-rivet buckling allowable stresses based on the column allowables of Section 11 and using the previous equation are shown in Figure 10.42.

FIGURE 10.42 - INTER-RIVET BUCKLING ALLOWABLES FOR ALUMINUM

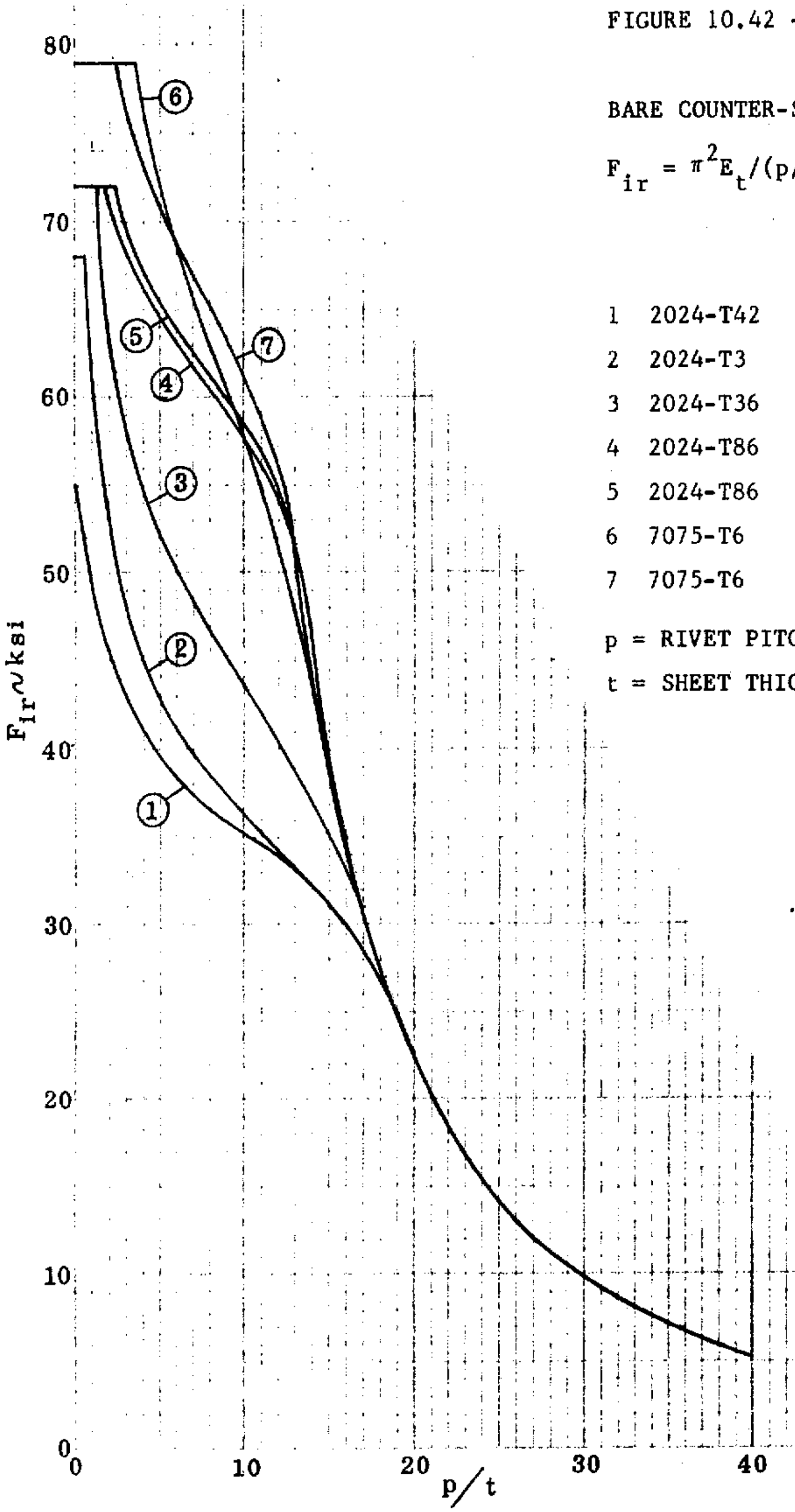
BARE COUNTER-SUNK OR DIMPLED (C=1)

$F_{ir} = \pi^2 E_t/(p/0.29t)^2$

		t	Basis
1	2024-T42	< .25	A
2	2024-T3	< .25	B
3	2024-T36	< .50	B
4	2024-T86	< .063	A
5	2024-T86	> .063 < .5	A
6	7075-T6	.016 - .039	B
7	7075-T6	.04 - .249	B

p = RIVET PITCH

t = SHEET THICKNESS



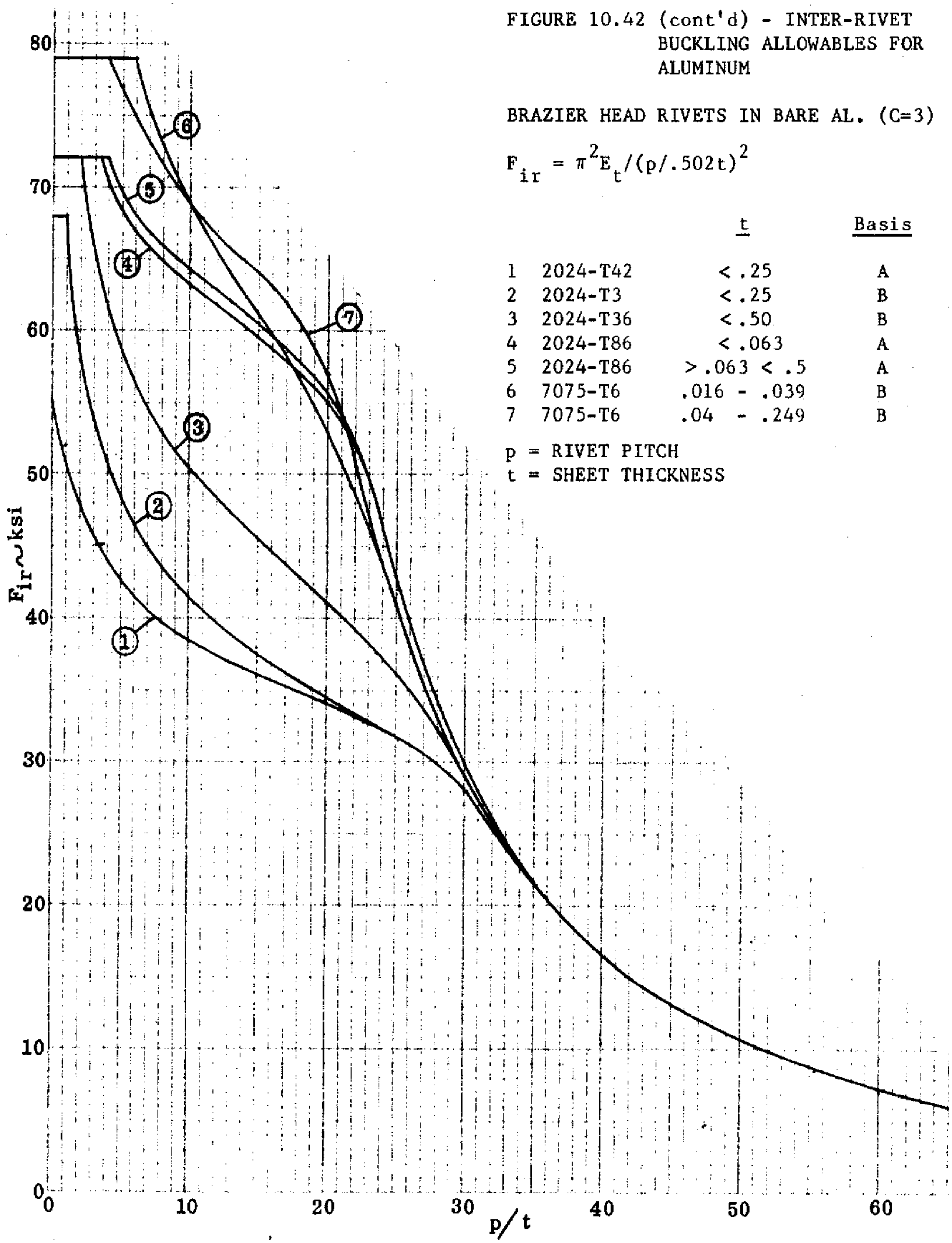


FIGURE 10.42 (cont'd) - INTER-RIVET BUCKLING ALLOWABLES FOR ALUMINUM

BRAZIER HEAD RIVETS IN BARE AL. (C=3)

$$F_{ir} = \pi^2 E_t / (p/.502t)^2$$

		t	Basis
1	2024-T42	< .25	A
2	2024-T3	< .25	B
3	2024-T36	< .50	B
4	2024-T86	< .063	A
5	2024-T86	> .063 < .5	A
6	7075-T6	.016 - .039	B
7	7075-T6	.04 - .249	B

p = RIVET PITCH
t = SHEET THICKNESS

FIGURE 10.42 (cont'd) - INTER-RIVET BUCKLING ALLOWABLES FOR ALUMINUM

SPOTWELDS IN BARE ALUMINUM (C=3.5)

$$F_{ir} = \pi^2 E_t / (p/.543t)^2$$

		t	Basis
1	2024-T42	< .25	A
2	2024-T3	< .25	B
3	2024-T36	< .50	B
4	2024-T86	< .063	A
5	2024-T86	> .063 < .5	A
6	7075-T6	.016 - .039	B
7	7075-T6	.04 - .249	B

p = RIVET PITCH
t = SHEET THICKNESS

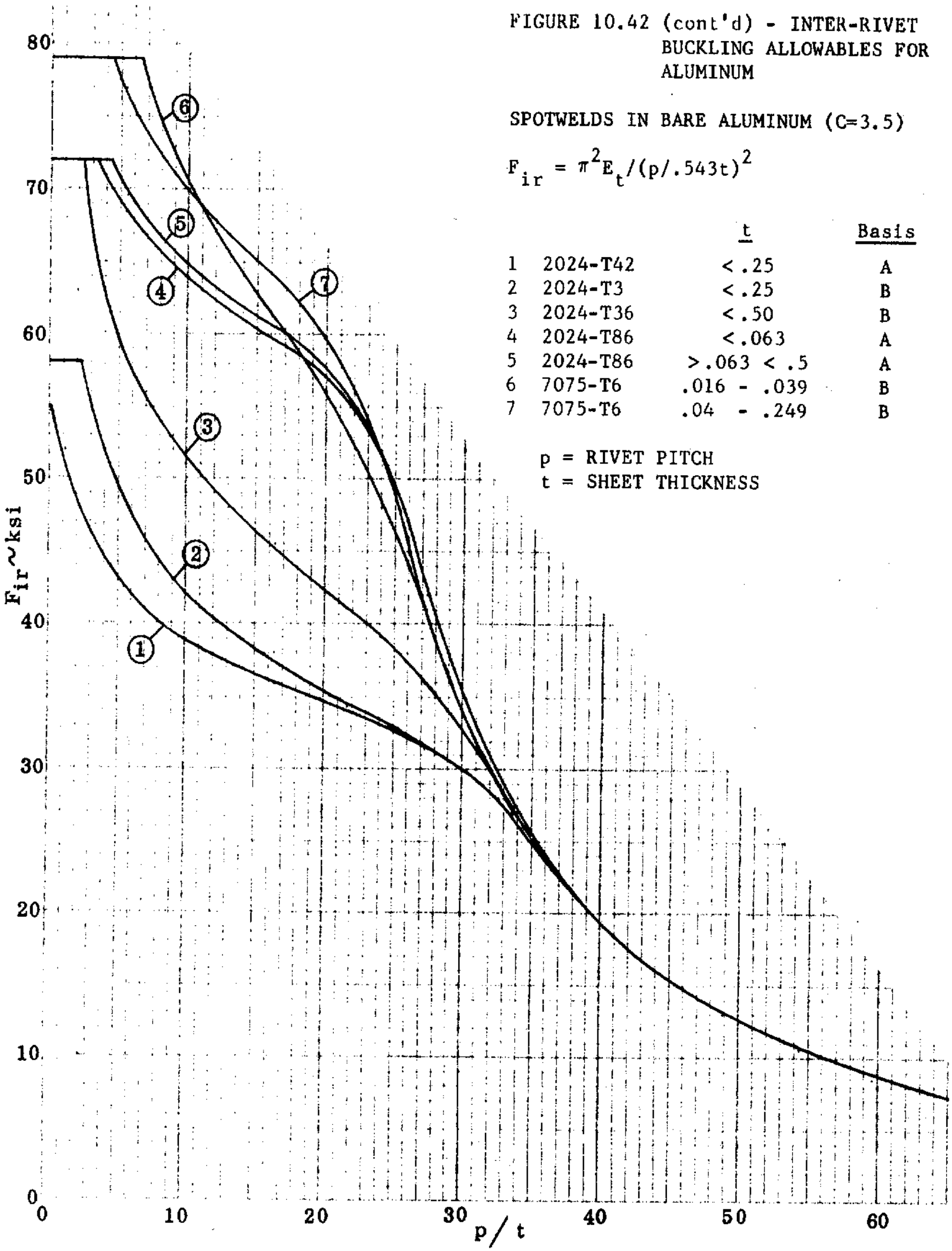


FIGURE 10.42 (cont'd) - INTER-RIVET BUCKLING ALLOWABLES FOR ALUMINUM

FLAT HEAD RIVETS AND BOLTS IN BARE AL. (C=4)

$F_{ir} = \pi^2 E_t/(p/.58t)^2$

		t	Basis
1	2024-T42	< .25	A
2	2024-T3	< .25	B
3	2024-T36	< .50	B
4	2024-T86	< .063	A
5	2024-T86	> .063 < .5	A
6	7075-T6	.016 - .039	B
7	7075-T6	.04 - .249	B

p = RIVET PITCH

t = SHEET THICKNESS

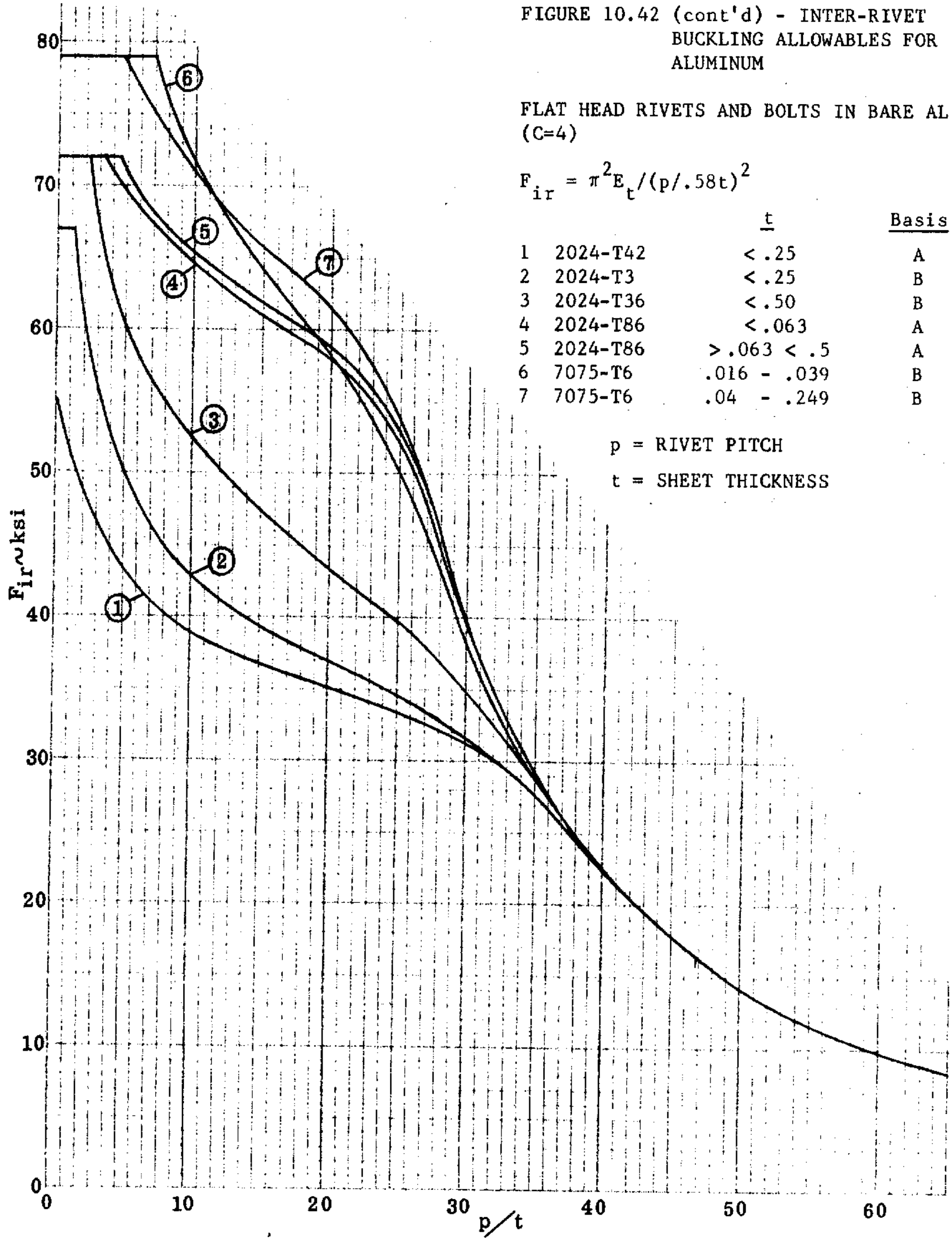


FIGURE 10.42 (cont'd) - INTER-RIVET BUCKLING ALLOWABLES FOR ALUMINUM

COUNTER-SUNK OR DIMPLED IN CLAD AL. (C=1)

$F_{ir} = \pi^2 E_t/(p/.29t)^2$

		t	Basis
1	2024-T42	< .063	B
2	2024-T3	< .063	B
3	2024-T36	< .063	B
4	2024-T6	< .063	A
5	2024-T81	< .063	A
6	2024-T86	< .063	B
7	7075-T6	.016 - .039	B
8	7075-T6	.04 - .062	B
9	7075-T6	.063 - .187	B
10	7075-T6	.188 - .249	B

p = RIVET PITCH

t = SHEET THICKNESS

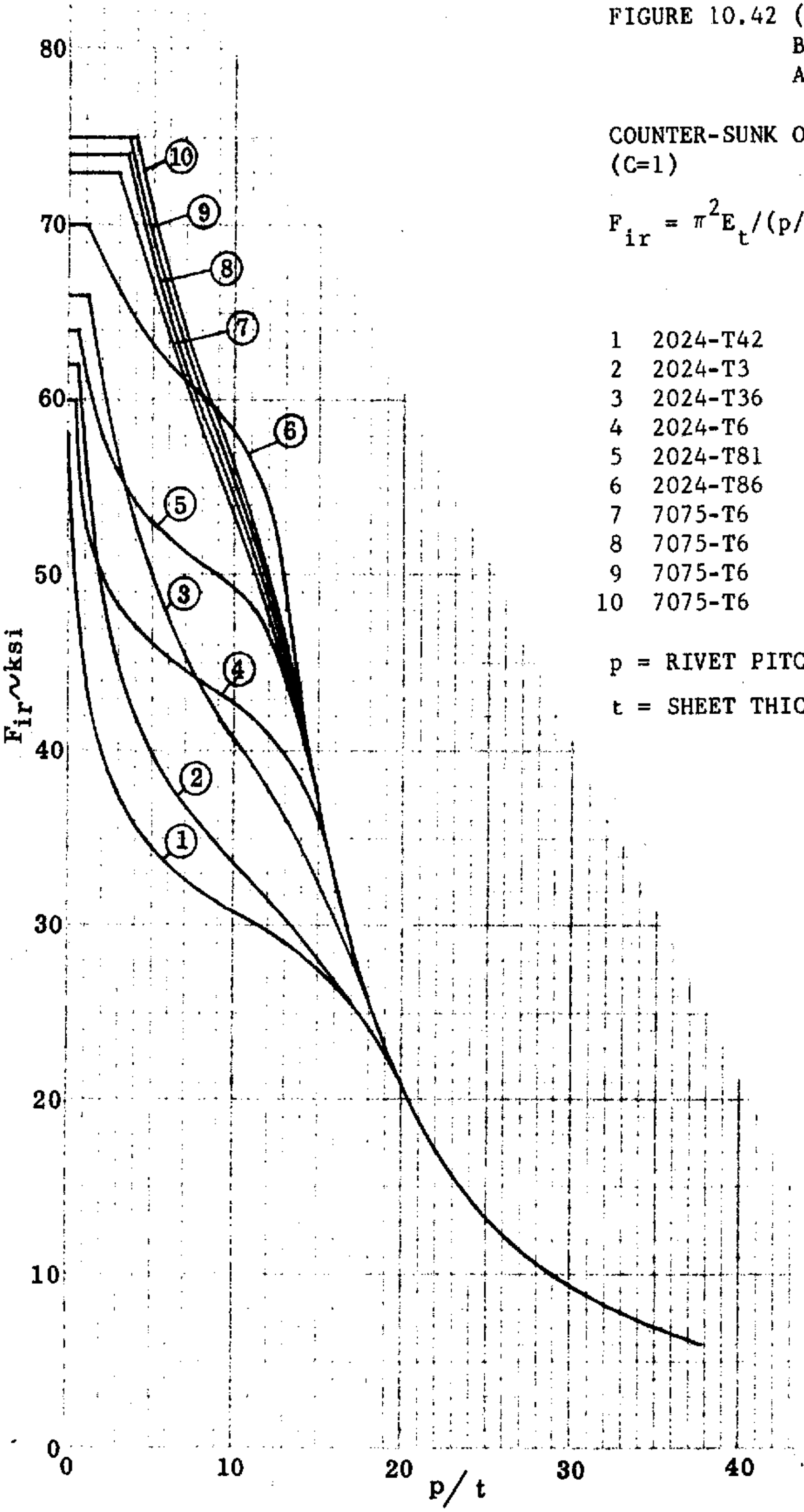


FIGURE 10.42 (cont'd) - INTER-RIVET BUCKLING ALLOWABLES FOR ALUMINUM

BRAZIER HEAD RIVETS IN CLAD ALUMINUM (C=3)

$F_{ir} = \pi^2 E_t/(p/.502t)^2$

		t	Basis
1	2024-T42	< .063	B
2	2024-T3	< .063	B
3	2024-T36	< .063	B
4	2024-T6	< .063	A
5	2024-T61	< .063	A
6	2024-T86	< .063	B
7	7075-T6	.016 - .039	B
8	7075-T6	.04 - .062	B
9	7075-T6	.063 - .187	B
10	7075-T6	.188 - .249	B

p = RIVET PITCH

t = SHEET THICKNESS

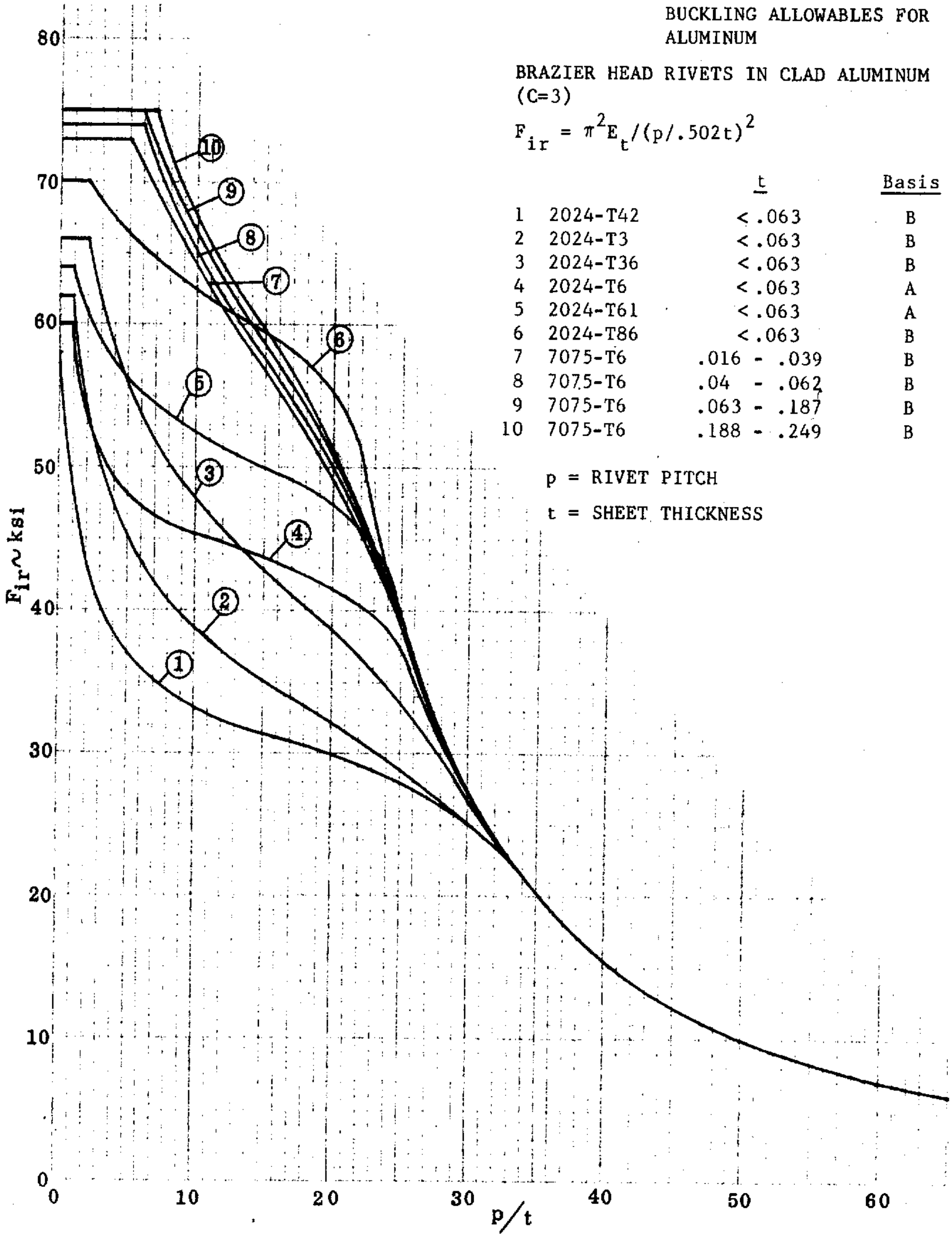


FIGURE 10.42 (cont'd) - INTER-RIVET BUCKLING ALLOWABLES FOR ALUMINUM

SPOTWELDS IN CLAD ALUMINUM (C=3.5

$F_{ir} = \pi^2 E_t/(p/.543t)^2$

		t	Basis
1	2024-T42	< .063	B
2	2024-T3	< .063	B
3	2024-T36	< .063	B
4	2024-T6	< .063	A
5	2024-T61	< .063	A
6	2024-T86	< .063	B
7	7075-T6	.016 - .039	B
8	7075-T6	.04 - .062	B
9	7075-T6	.063 - .187	B
10	7075-T6	.188 - .249	B

p = RIVET PITCH
t = SHEET THICKNESS

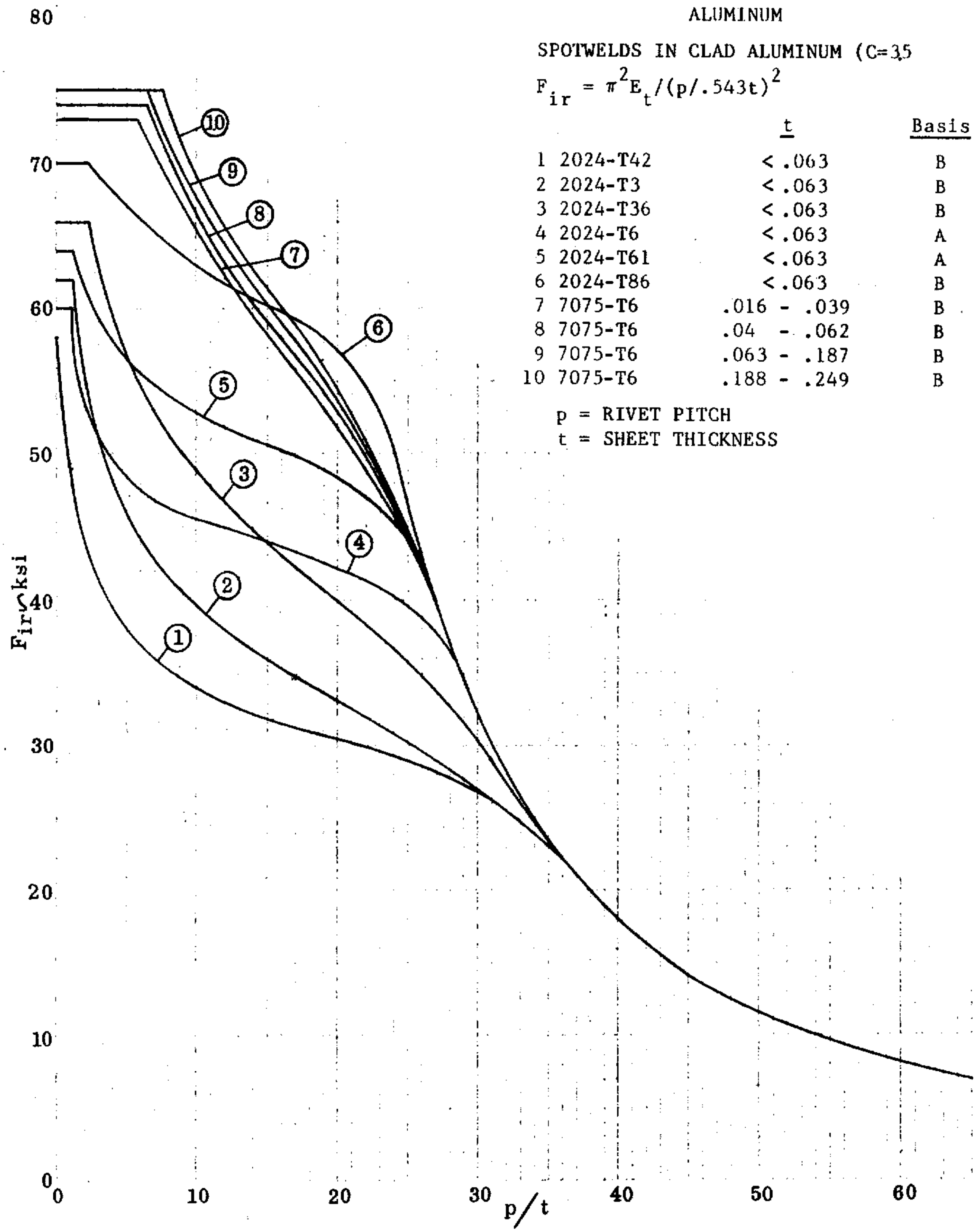


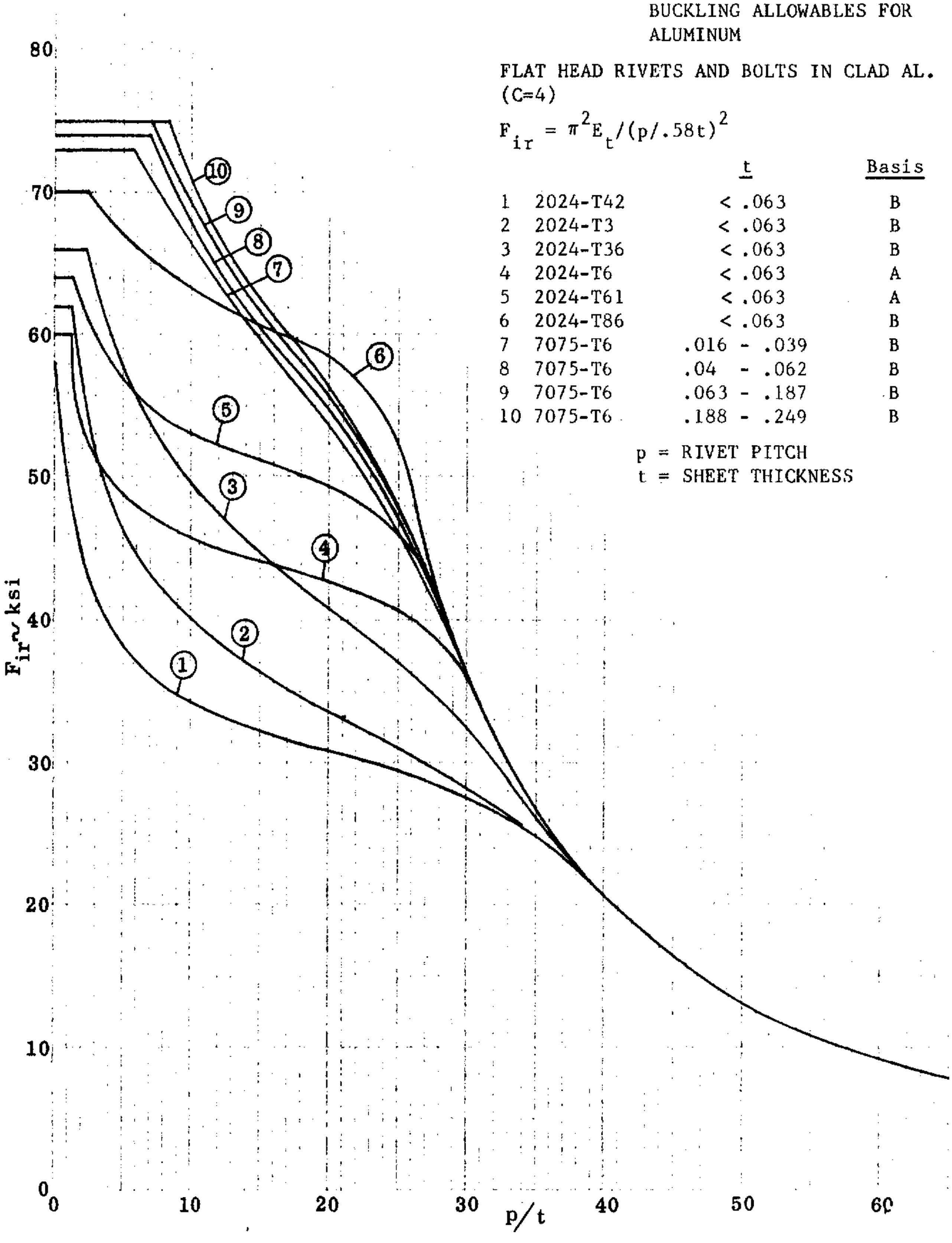
FIGURE 10.42 (cont'd) - INTER-RIVET BUCKLING ALLOWABLES FOR ALUMINUM

FLAT HEAD RIVETS AND BOLTS IN CLAD AL. (C=4)

$F_{ir} = \pi^2 E_t / (p/.58t)^2$

		t	Basis
1	2024-T42	< .063	B
2	2024-T3	< .063	B
3	2024-T36	< .063	B
4	2024-T6	< .063	A
5	2024-T61	< .063	A
6	2024-T86	< .063	B
7	7075-T6	.016 - .039	B
8	7075-T6	.04 - .062	B
9	7075-T6	.063 - .187	B
10	7075-T6	.188 - .249	B

p = RIVET PITCH
t = SHEET THICKNESS



10.7 COMPRESSIVE CRIPPLING

Introduction

Compressive crippling or local buckling is defined as an inelastic distortion of the cross-section of a structural element in its own plane (rather than along the longitudinal axis, as in column buckling). The crippling stress, which is the maximum average stress developed by a structural shape, is a function of the cross-sectional area rather than the length. The crippling stress for a given cross-section is calculated by assuming that a uniform stress is acting over the entire section, $P_{cc} = F_{cc} \cdot A$. In reality, however, the stress is not uniform over the entire cross-section. Parts of the section will buckle at a stress below the gross area crippling stress, while the more stable areas, such as intersections and corners, reach a higher stress than the buckled elements.

Method of Analysis

The allowable crippling stress may be obtained from the procedure outlined below.

1. Divide the section into individual segments as shown in Figures 10.43 and 10.44. Define for each segment a width b and a thickness t. Each segment will have either zero or one edge free.

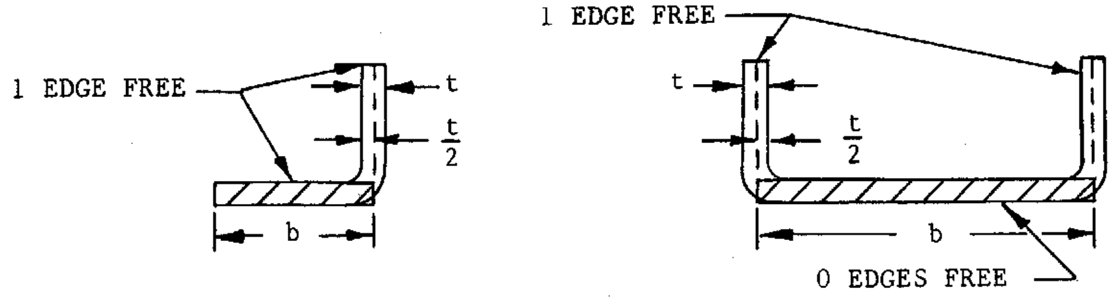
2. The allowable crippling stress, F_{cc}, for each segment is obtained from the compressive crippling curves of Figures 10.43 or 10.44.

3. The allowable crippling stress for the entire section is found by taking a weighted average of the allowable stresses for each segment:

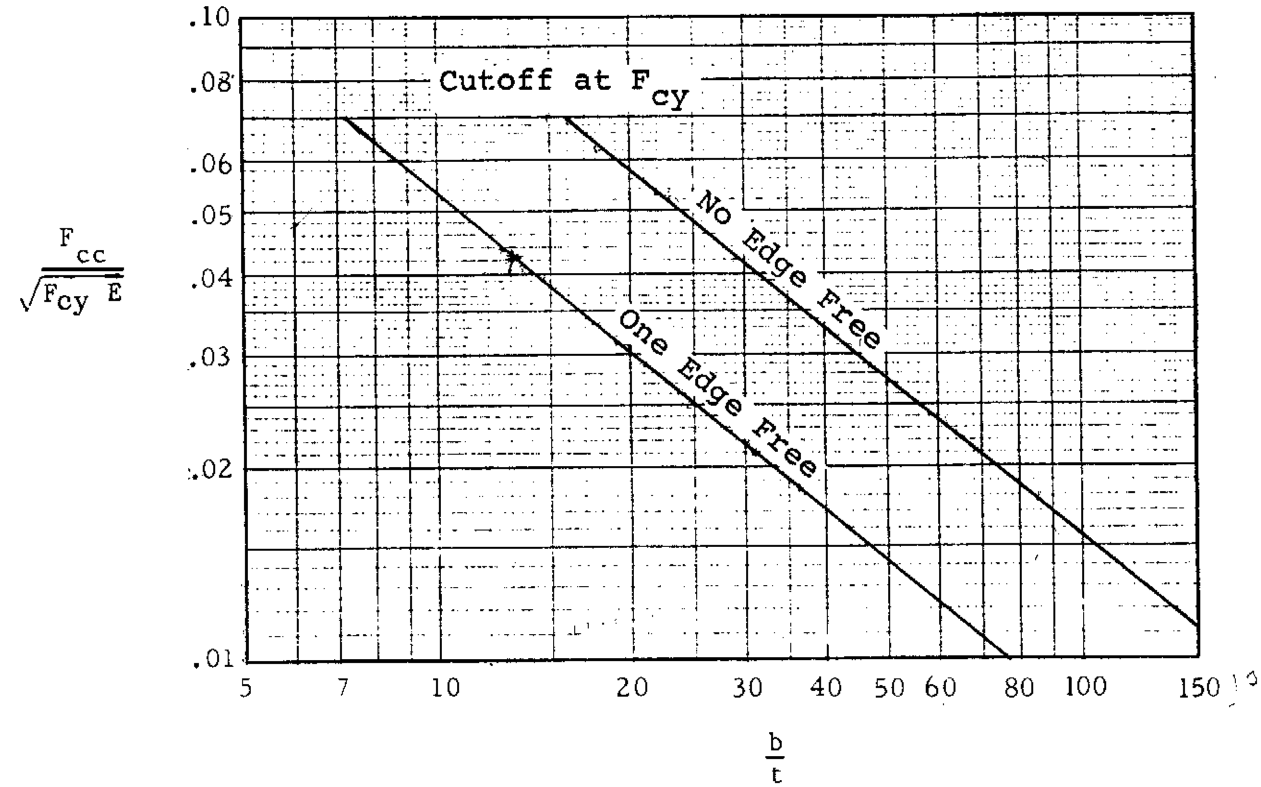
$$F_{cc} = \frac{b_1 t_1 F_{cc1} + b_2 t_2 F_{cc2} + \ldots}{b_1 t_1 + b_2 t_2 + \ldots} = \frac{\Sigma b_n t_n F_{ccn}}{\Sigma b_n t_n}$$

The same procedure is used to analyze formed and extruded sections. Care must be taken in segmenting an unbalanced extruded section. When the thicknesses of the segments differ by a factor of 3.0 or more, the excess thickness should be discounted in calculating both the crippling stress of the segment and the effective load carrying area of the section. Also note that the bend radii of formed sections are ignored. For formed sections with lips, Figure 10.45 may be used to determine whether the lip provides sufficient stability to the adjacent segment.

FIGURE 10.43 - COMPRESSIVE CRIPPLING FORMED SECTIONS, GENERAL SOLUTION



$$F_{cc} = \frac{\Sigma\, b_N\, t_N\, F_{ccN}}{\Sigma\, b_N\, t_N}$$



$$\frac{F_{cc}}{\sqrt{F_{cy}\,E}} = C_o\,(b/t)^{A_o}$$

No edge free

C_o = .67882

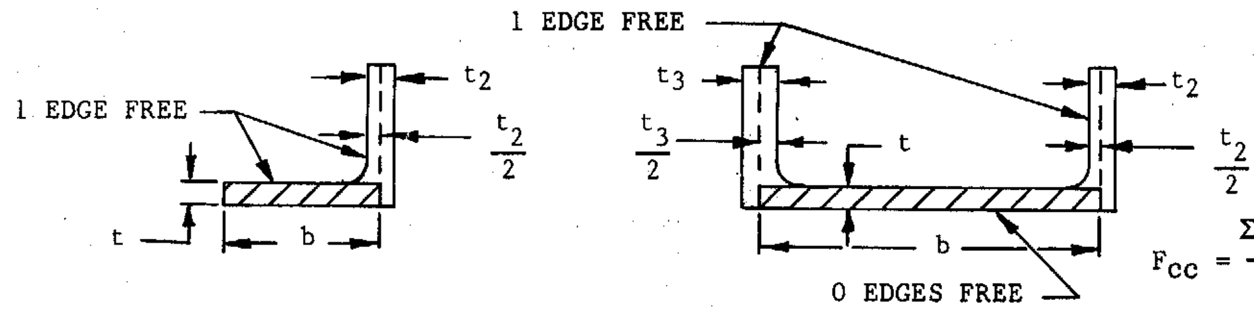
A_o = -.81940

One edge free

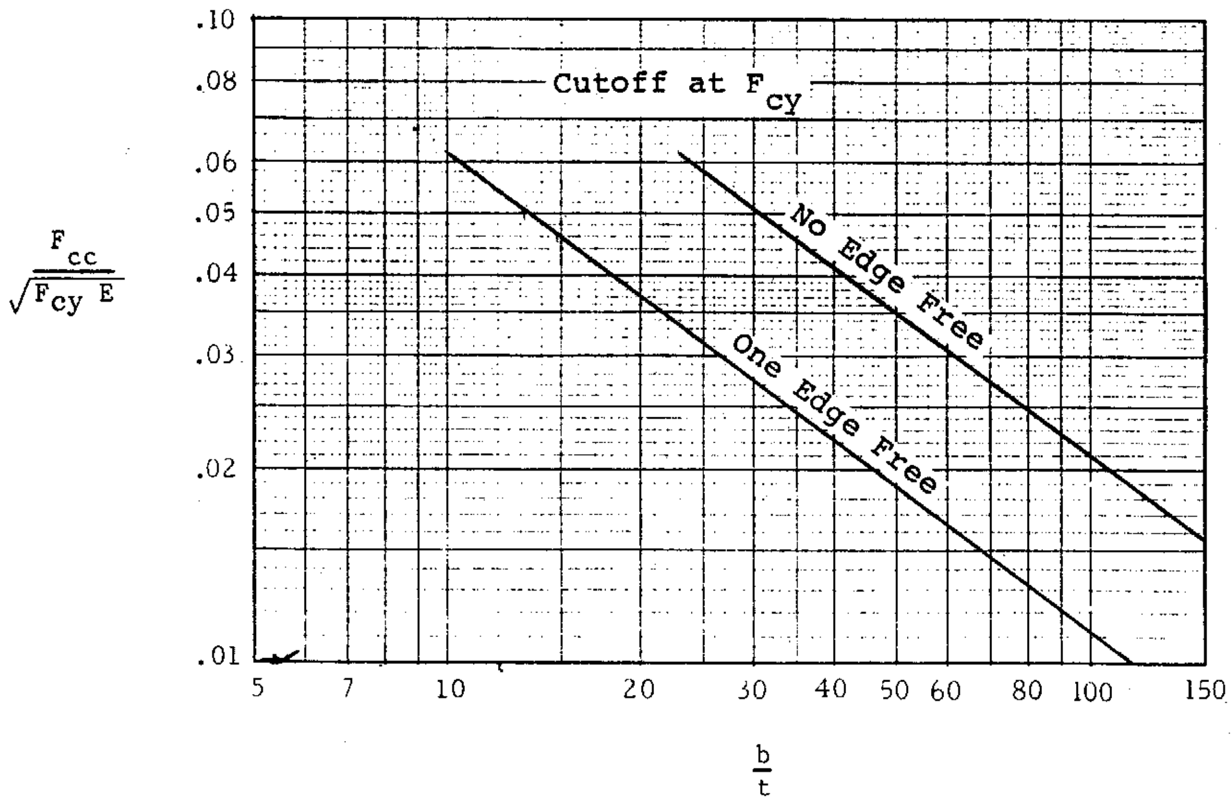
C_o = .35728

A_o = -.82571

FIGURE 10.44 - COMPRESSIVE CRIPPLING OF EXTRUSIONS, GENERAL SOLUTION



$$F_{cc} = \frac{\Sigma b_N \; t_N \; F_{cc_N}}{\Sigma b_N \; t_N}$$



$$\frac{F_{cc}}{\sqrt{F_{cy} \; E}} = C_o \; (^b/_t)^{A_o}$$

No edge free

C_o = .62153

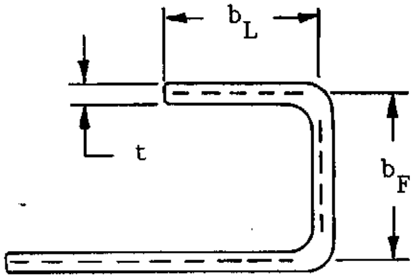
A_o = −.73562

One edge free

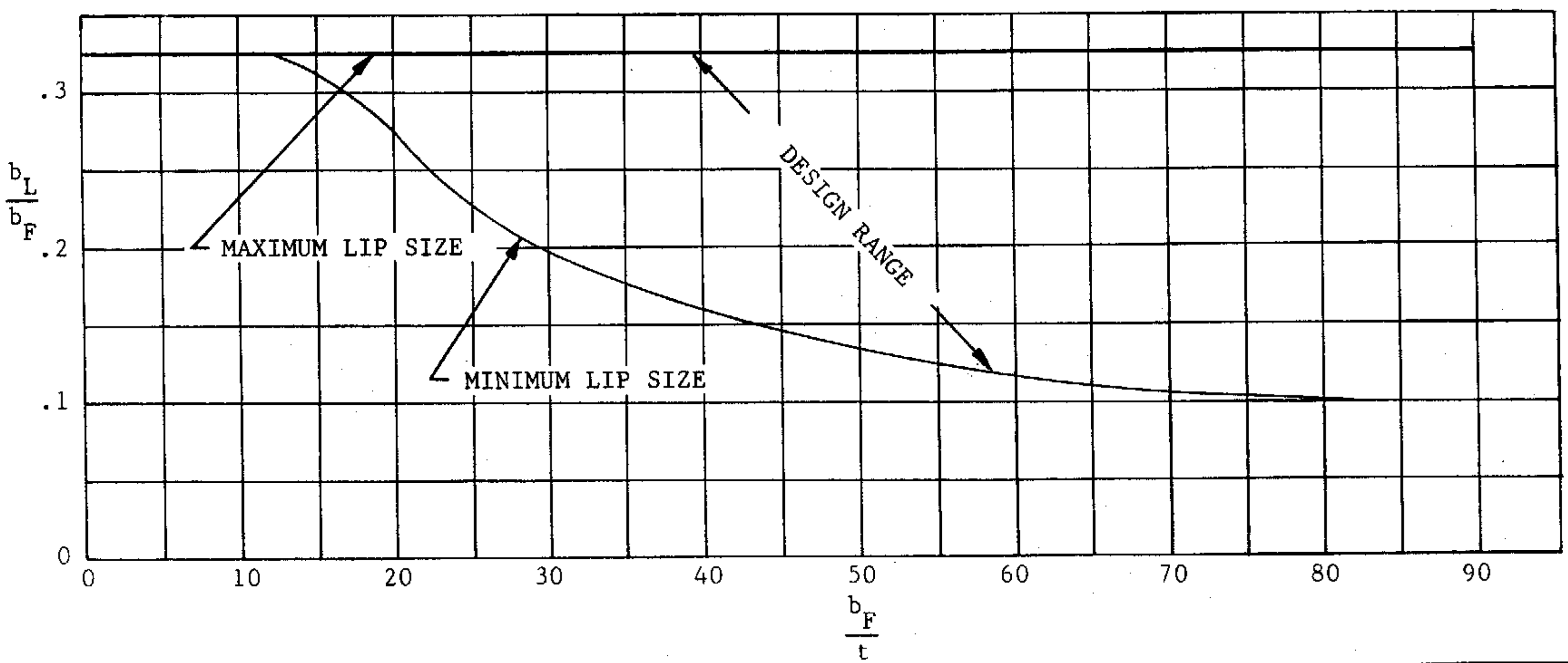
C_o = .34881

A_o = −.74858

FIGURE 10.45 - DESIGN RANGE FOR LIPS ON FORMED SECTIONS



ABOVE DESIGN RANGE: LIP BUCKLES BEFORE FLANGE
BELOW DESIGN RANGE: LIP TOO SMALL TO PROVIDE SIMPLE SUPPORT TO FLANGE
WITHIN DESIGN RANGE: LIP PROVIDES SIMPLE SUPPORT TO ADJACENT ELEMENT AND IS TREATED AS A FLANGE IN CRIPPLING ANALYSIS



10.8 EFFECTIVE SKIN WIDTH

The effective skin width is used to calculate the amount of skin that contributes to the stiffness of the attached flange. Figure 10.46 shows several types of skin-flange attachments and the corresponding effective skin widths. The skin width equations are based on the buckling compressive stress equation for sheet panels:

$$F_{cr} = \frac{k_c \pi^2 E}{12(1 - \mu^2)} \left(\frac{t}{b}\right)^2$$, where b is the stiffener spacing.

If the stiffener provides a boundary restraint equal to a simple support, then $k_c = 4.0$.

Assuming $\mu = .3$,

$$F_{cr} = 3.6\ E\left(\frac{t}{b}\right)^2.$$

When F_{cr} is equal or less than the yield stress of the material, the ultimate strength of a simply supported sheet is independent of the width of the sheet. The term b may then be replaced with an effective width term, w.

$$F_c = 3.6\ E\left(\frac{t}{w}\right)^2$$

Solving for w -

$$w = 1.9\ t\left(\frac{E}{F_c}\right)^{.5} \qquad 10.4$$

The constant 1.9 in the preceeding equation is valid for heavy stiffeners. For relatively light stiffeners a constant of 1.7 is suggested. The radius of gyration of the stiffener should include the effective skin area.

For skin-stiffener attachments that develop a fixed or clamped condition -

$$w = 2.52\ t(E/F_c)^{.5} \qquad 10.5$$

Note A - (Fig. 10.46-b) Staggered Rivet Rows
In calculating the crippling stress of the stiffener, use a stiffener flange thickness of three-fourths the sum of the flange thickness plus the sheet thickness.

Note B - (Fig. 10.46-c) $t_s \leq t_f < 2t_s$
Find the crippling stress for the tee section, assuming the vertical member of the tee has both ends simply supported. For t (equ. 10.4) use

$$(t_s + t_f)/2.$$

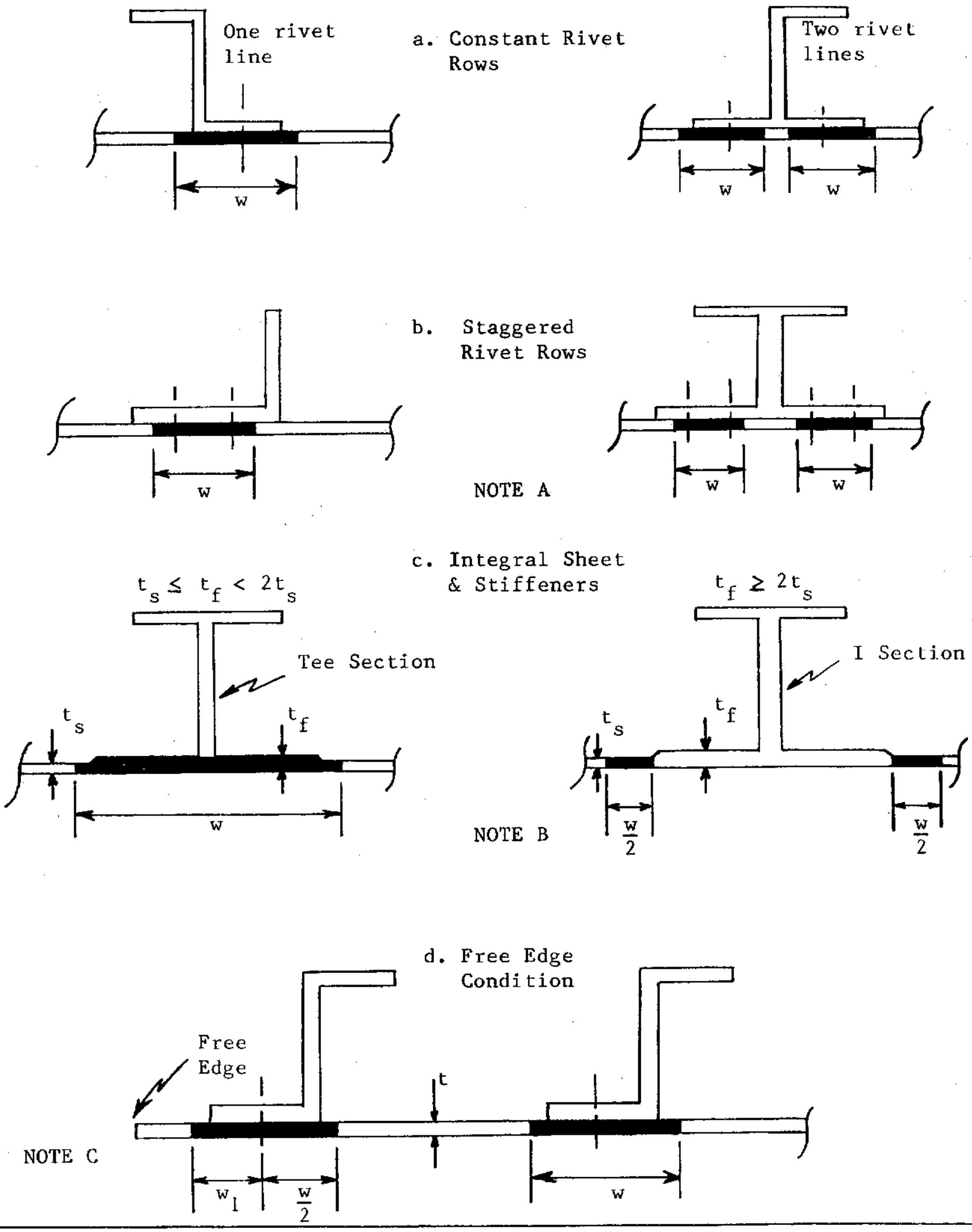


FIGURE 10.46 - EFFECTIVE SKIN WIDTHS

$t_f \geq 2t_s$ - Find the crippling stress for the I section. The properties should include the I section plus effective skin.

Note C - (Fig. 10.46-d) For a sheet with one edge free, the buckling coefficient is 0.43. The effective width w_1 on the free-edge side of the attachment is:

$$w_1 = 0.62\ t(E/F_c)^{.5}$$

10.9 JOGGLED ANGLES

Figure 10.47 shows crippling efficiency factors of each joggled leg of aluminum angles with joggle depth (D) relative to thickness (t) of 0 to 3, and joggle length-to-depth ratios (L/D) of 4, 6, and 8.

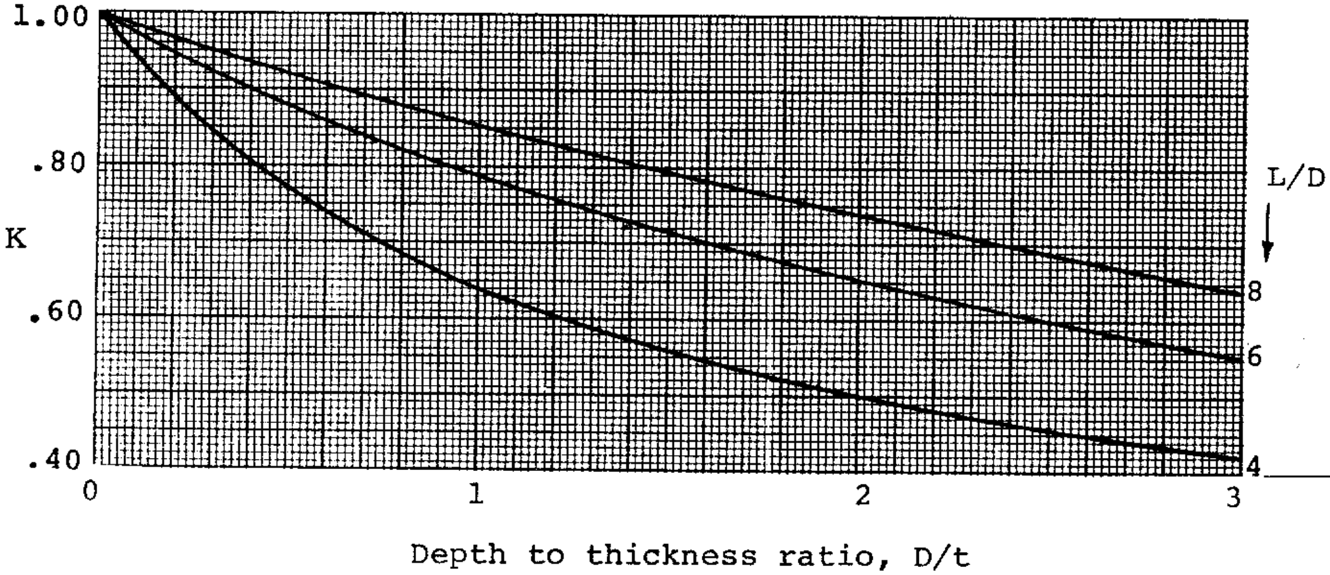


FIGURE 10.47 - CRIPPLING EFFICIENCY FACTORS OF JOGGLED LEG OF ALUMINUM ANGLES

The allowable crippling load is $P_{cc} = Kbt\ F_{cc}$ for each leg, and $P_{cc} = K_1 b_1 t_1 F_{cc_1} + K_2 b_2 t_2 F_{cc2}$ for the joggled angle.

The allowable crippling stress equation on page 10-69 then becomes

$$F_{cc} = \frac{K_1 b_1 t_1 F_{cc1} + K_2 b_2 t_2 F_{cc2}}{b_1 t_1 + b_2 t_2}$$

for joggled aluminum angles.

SECTION 11

COLUMNS AND BEAM COLUMNS

11.0 GENERAL

The stresses that a structural element can sustain in compression are functions of several parameters. These parameters are:

(1) The length of the element along its loading axis,
(2) The moment of inertia of the element normal to its loading axis,
(3) The cross-sectional variation of the element with length,
(4) The eccentricity of the applied load,
(5) The continuity of the integral parts of the element,
(6) The cross-sectional characteristics of the elements,
(7) The homogeneity of the element material,
(8) The straightness of the element, and
(9) The end fixity of the element.

The effects of these parameters can be categorized by first establishing certain necessary assumptions. For the following analysis, it is assumed that the material is homogeneous and isotropic. It is further assumed that the element is initially straight and, if it is composed of several attached parts, that the parts act as integral components of the total structural configuration.

The remainder of the previously mentioned parameters dictate more general classifications of compression elements. If a compression element is of uniform cross section and satisfies the previously mentioned assumptions, it is referred to as a simple column and is treated in Section 11.1. On the other hand columns with non-rigid end supports, intermediate string supports or eccentrically loaded are called complex columns and are treated later in this section.

11.1 Simple Columns

In general, failures in simple columns may be classed under two headings:

(1) Primary failure (general instability)
(2) Secondary failure (local instability)

Primary or general instability failure is any type of column failure, whether elastic or inelastic, in which the cross sections are translated and/or rotated but not distorted in their own planes. Secondary or local instability failure of a column is defined as any type of failure in which cross sections are distorted in their own planes but not translated or rotated. However, the distinction between primary and secondary failure is largely theoretical because most column failures are a combination of the two types.

Figure 11.1 illustrates the curves for several types of column failure.

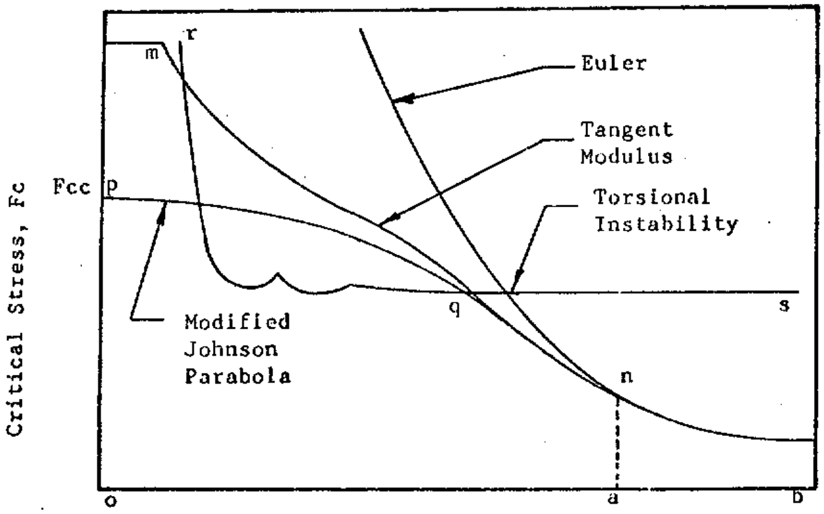


Figure 11.1 - Types of Column Failure

L' represents the effective length of the column and is dependent upon the manner in which the column is constrained, and ρ is the minimum radius of gyration of the cross sectional area of the column.

For a value of L'/ρ in the range "a" to "b", the column buckles in the classical Euler manner. If the slenderness ratio, L'/ρ, is in the range of "0" to "a", a column may fail in one of the three following ways:

(1) Inelastic Bending Failure. This is a primary failure described by the Tangent Modulus equation, curve mn. This type of failure depends only on the mechanical properties of the material.

(2) Combined Inelastic Bending and Local Instability. The elements of a column section may buckle, but the column can continue to carry load until complete failure occurs. This failure is predicted by a modified Johnson Parabola, "pq", a curve defined by the crippling strength of the section. At low values of L'/ρ the tendency to cripple predominates; while at L'/ρ approaching the point "q", the failure is primarily inelastic bending. Geometry of the section, as well as material properties, influences this combined type of failure.

(3) Torsional Instability. This failure is characterized by twisting of the column and depends on both material and section properties. The curve "rs" is superimposed on Figure 11.1 for illustration. Torsional instability is presented in Section 11.4.

11.1.1 Long Elastic Columns

A column with a slenderness ratio (L'/ρ) greater than the critical slenderness ratio (point "a" in Figure 11.1) is called long column. This type of column fails through lack of stiffness instead of a lack of strength.

The critical slenderness ratio is given by:

$$\left(L'/\rho\right)_{crit} = \left[\sqrt{\frac{2E}{F_{cc}}}\right]\pi \qquad (11.1)$$

The critical stress in the column that produces Euler buckling is

$$F_{crit} = \frac{\pi^2 E}{(L'/\rho)^2} \qquad (11.2)$$

Values for F_{crit} for various materials are shown in Section 11.1.4.

The value for L' depends on the column's end constraints. The effective length is

$$L' = L/\sqrt{C} \qquad (11.3)$$

The values of C for different end constraints are given in Figure 11.2.

11.1.2 Short Columns

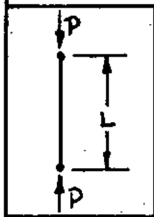
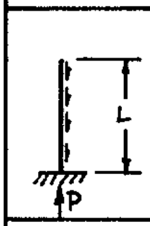
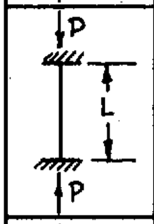
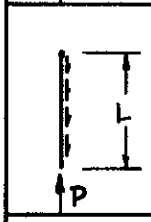
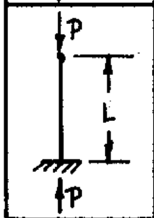
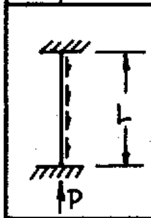
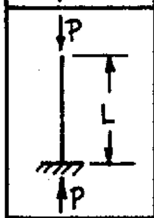
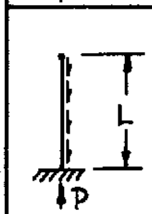
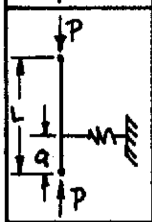
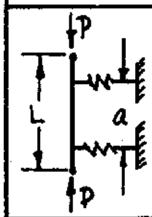
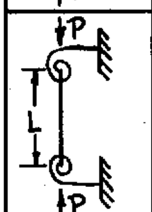
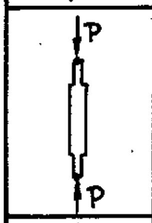
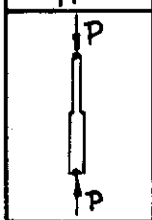
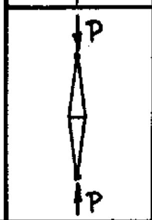
A column with a slenderness ratio (L'/ρ) less than the critical slenderness ratio is called a short column. This distinction is made on the basis that column behavior departs from that described by the classical Euler equation, Eq. 11.2.

Elements of a short column may buckle, but the column can continue to carry load until complete failure occurs. This failure is predicted by a modified Johnson Parabola, "pq", in Figure 11.1, a curve defined by the crippling strength of the section.

The Johnson Parabola defines a column allowable stress as:

$$F_{crit} = F_{cc}\left[1 - F_{cc}\left(\frac{(L'/\rho)^2}{4\pi^2 E}\right)\right] \qquad (11.4)$$

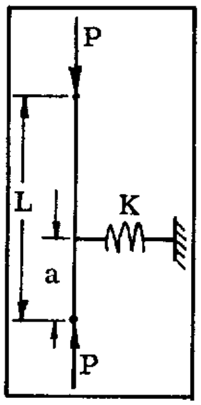
Values of F_{crit} for various materials are shown in Section 11.1.4.

COLUMN LOADING AND END CONSTRAINTS		END CONSTRAINT COEFFICIENT	COLUMN LOADING AND END CONSTRAINTS		END CONSTRAINT COEFFICIENT
	UNIFORM COLUMN AXIALLY LOADED PINNED ENDS	$C = 1$ $\frac{1}{\sqrt{C}} = 1$		UNIFORM COLUMN DISTRIBUTED AXIAL LOAD, ONE END FIXED, ONE END FREE	$C = .794$ $\frac{1}{\sqrt{C}} = 1.12$
	UNIFORM COLUMN AXIALLY LOADED FIXED ENDS	$C = 4$ $\frac{1}{\sqrt{C}} = .5$		UNIFORM COLUMN, DISTRIBUTED AXIAL LOAD, PINNED ENDS	$C = 1.87$ $\frac{1}{\sqrt{C}} = .732$
	UNIFORM COLUMN AXIALLY LOADED ONE END PINNED ONE END FIXED	$C = 2.05$ $\frac{1}{\sqrt{C}} = 0.7$		UNIFORM COLUMN DISTRIBUTED AXIAL LOAD, FIXED ENDS	$C = 7.5$ $\frac{1}{\sqrt{C}} = .365$
	UNIFORM COLUMN AXIALLY LOADED ONE END FREE ONE END FIXED	$C = 0.25$ $\frac{1}{\sqrt{C}} = 2$		UNIFORM COLUMN DISTRIBUTED AXIAL ONE END FIXED, ONE END PINNED	$C = 6.08$ (Approx.) $\frac{1}{\sqrt{C}} = .406$
	UNIFORM COLUMN AXIALLY LOADED PINNED ENDS ONE INTERMEDIATE SPRING SUPPORT	SEE FIGURE 11.3		UNIFORM COLUMN AXIALLY LOADED PINNED ENDS TWO INTERMEDIATE SPRING SUPPORTS	SEE FIGURE 11.5
	UNIFORM COLUMN AXIALLY LOADED, ELASTICALLY RESTRAINED ENDS	SEE FIGURE 11.4		STEPPED COLUMN AXIALLY LOADED PINNED ENDS	SEE FIGURE 11.10
	STEPPED COLUMN AXIALLY LOADED PINNED ENDS	SEE FIGURE 11.10		TAPERED COLUMN AXIALLY LOADED PINNED ENDS	SEE FIGURE 11.10

(See Figure 11.6 for more loading cases.)

FIGURE 11.2 - BUCKLING CONSTRAINT COEFFICIENTS

FIGURE 11.3 - END CONSTRAINT COEFFICIENTS - UNIFORM SECTION COLUMNS WITH PINNED ENDS AND INTERMEDIATE SPRING SUPPORT.



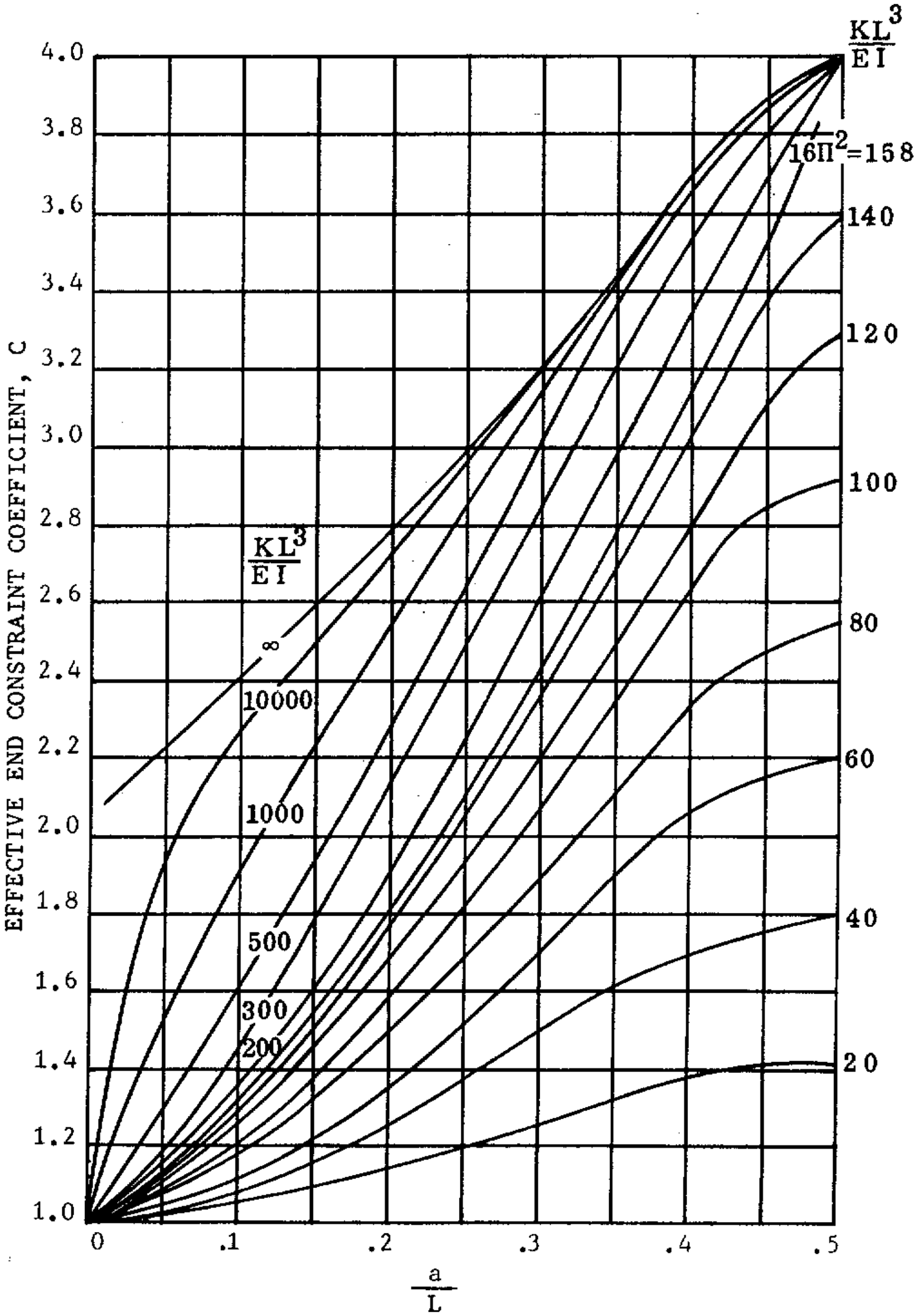
K is the spring constant with units of

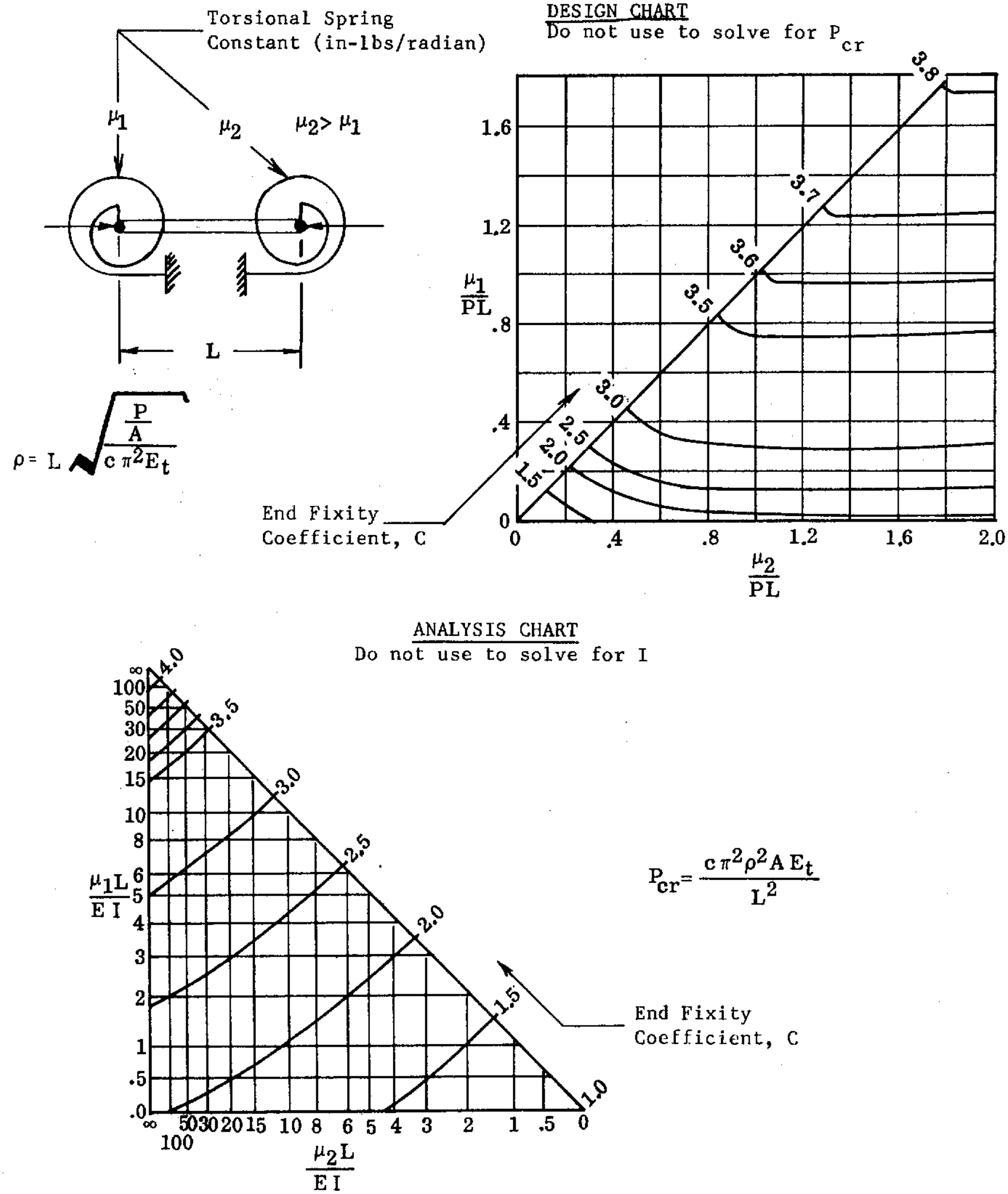
$$\frac{lb}{in}$$

A center support is effectively rigid if

$$\frac{KL^3}{EI} = 16\ \pi^2$$

DO NOT use this chart to solve for Required I.



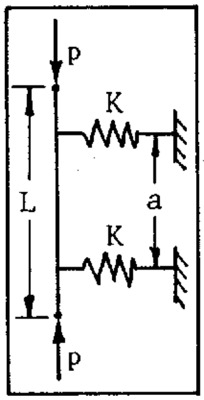


$$\rho = L\sqrt{\frac{P/A}{c\,\pi^2 E_t}}$$

$$P_{cr} = \frac{c\,\pi^2 \rho^2 A\, E_t}{L^2}$$

Figure 11.4 - Fixity Coefficients-Single Span Columns With Elastic Restraints

FIGURE 11.5 - END CONSTRAINT COEFFICIENTS - COLUMNS WITH PINNED ENDS AND TWO SYMMETRICALLY PLACED SPRING SUPPORTS



K is the spring constant with units of lb/in.

DO NOT use this curve to solve for required I

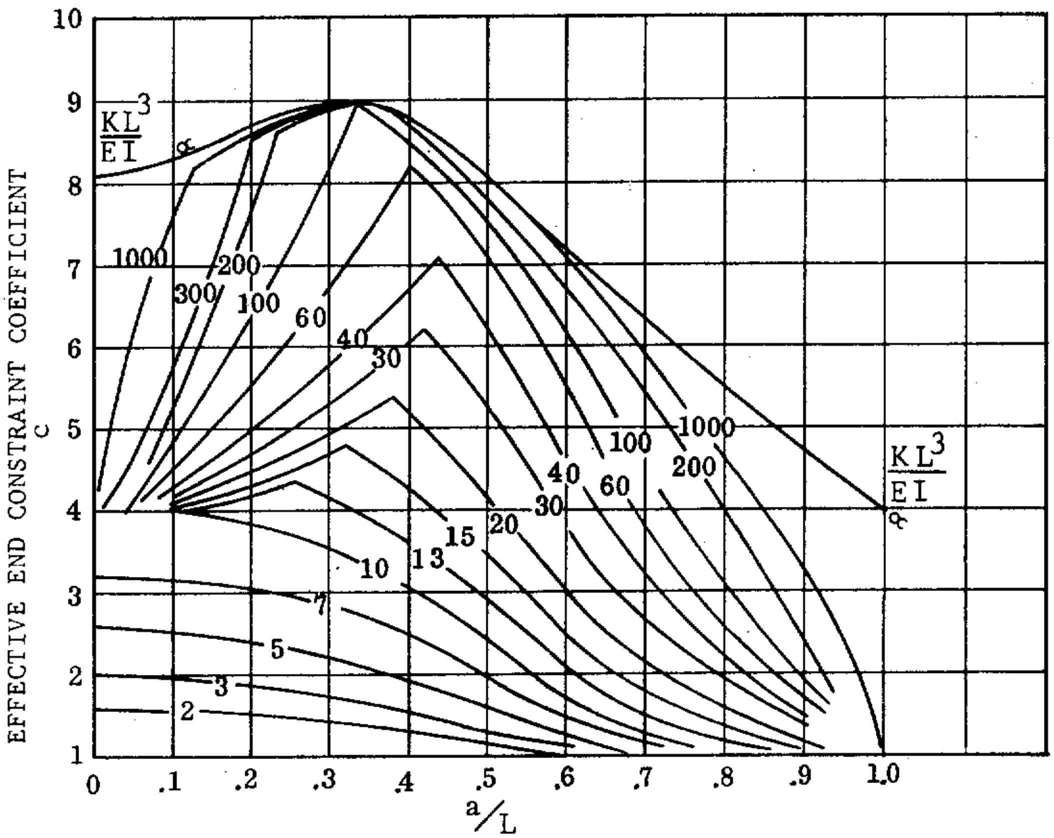
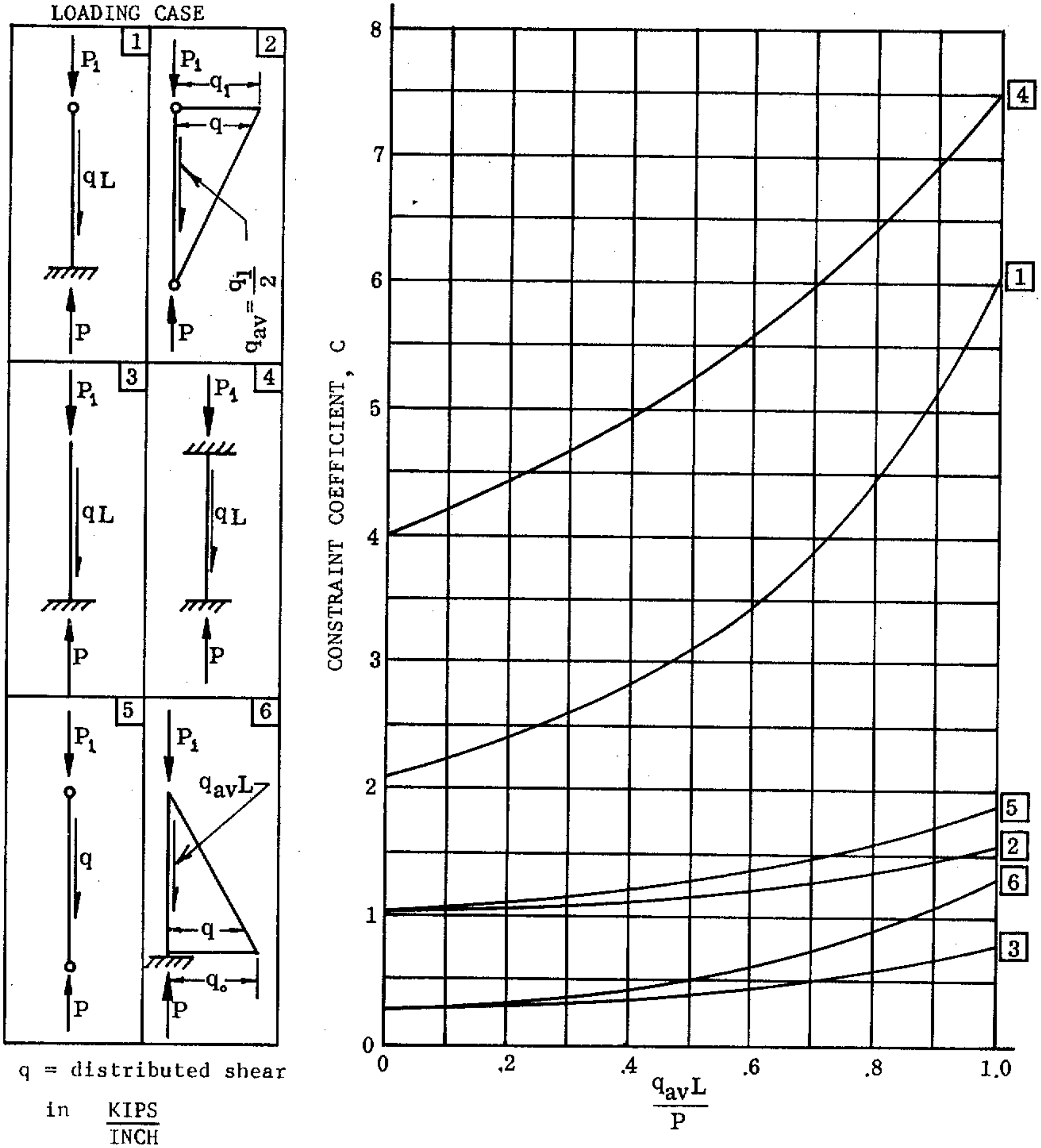


FIGURE 11.6 - END CONSTRAINT COEFFICIENTS - UNIFORM SECTION COLUMNS SUBJECTED TO CONCENTRATED AXIAL LOAD AND DISTRIBUTED SHEAR LOAD

NOTE: Pcr found by using these coefficients is P, NOT P_1, shown in loading case diagrams.



q = distributed shear in $\frac{\text{KIPS}}{\text{INCH}}$

In the short column range failure is often due to plastic crushing of the column. In other words, the column is too short to bow or buckle under end load but crushes under the high stresses. This column range of stresses is usually referred to as the block compression strength.

The influence of end supports on plastic buckling is the same as it is for elastic buckling. The allowable compressive stress is given by:

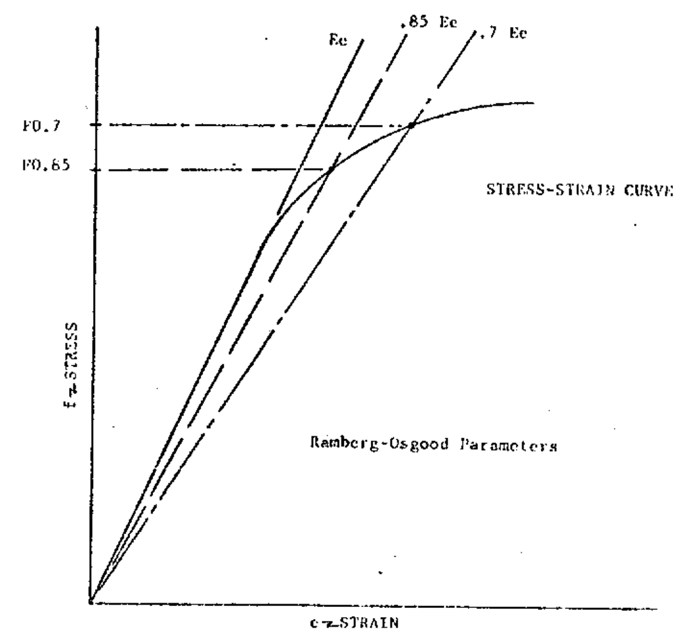
$$F_{crit} = \frac{\pi^2 E_t}{(L'/\rho)^2} \tag{11.5}$$

This equation is very simular to the Euler allowable except the Tangent Modulus is used in place of the Modulus of Elasticity. The values for L' are calculated using Equation 11.3 and values for C are determined from Figure 11.2.

The ratio of Tangent Modulus to Modulus of Elasticity (E_t/E) is given by the Ramberg-Osgood relationship :

$$\frac{E_t}{E} = \frac{1}{1 + \frac{3}{7} n \left(\frac{F}{F_{0.7}}\right)^{n-1}} \tag{11.6}$$

The values for E, n, and $F_{0.7}$ are material dependent. $F_{0.7}$ is a secant yield stress which is determined by the intersection of the stress strain curve and a secant modulus curve for $E_{0.7} = .7E$. This is shown below. Another secant stress ($F_{0.85}$) is needed to determine the constant n.



The value for n is given by :

$$n = 1 + \left|\log_e (17/7)/\log_e(F_{0.7}/F_{0.85})\right| \tag{11.7}$$

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Equation 11.6 is plotted in Figure 11.7. For a given material, n, $F_{0.7}$ and E must be known. Values of n, $F_{0.7}$ and E may be obtained from Table 11.1 for several materials. Then assuming values of F, E_t can be calculated.

To use this approach an initial F_{crit} is calculated and E_t is determined from Figure 11.7. This E_t is used in Equation 11.5 to determine a new F_{crit} which is used to obtain a new E_t. This proceedure is repeated until the critical stress does not change after each iteration.

The Euler column equation can be rewritten as

$$\frac{E_t}{F_{crit}} = \frac{(L'/\rho)^2}{\pi^2} \tag{11.8}$$

The problem therefore resolves itself to obtaining an expression for E_t/F_{crit} from the nondimensional equation 11.6. Both sides of Equation 11.6 are multiplied by $(F_{0.7}/F_{crit})$. This yields

$$\left(\frac{E_t}{E}\right)\left(\frac{F_{0.7}}{F_{crit}}\right) = \frac{1}{\frac{F_{crit}}{F_{0.7}} + \frac{3}{7}\, n\left(\frac{F_{crit}}{F_{0.7}}\right)^{n}} = B^2 \tag{11.9}$$

Rearranging and substituting equation 11.8 into 11.9 yields:

$$\frac{(L'/\rho)^2}{\pi^2}\ \frac{F_{0.7}}{E} = \frac{1}{\frac{F_{crit}}{F_{0.7}} + \frac{3}{7}\, n\left(\frac{F_{crit}}{F_{0.7}}\right)^{n}} = B^2 \tag{11.10}$$

or

$$B = \frac{L'/\rho}{\pi}\sqrt{\frac{F_{0.7}}{E}} \tag{11.11}$$

Figure 11.8 shows plots of this equation, $F_{crit}/F_{0.7}$ versus B for various values of n.

Thus for given values of E, n, and $F_{0.7}$; $\frac{F_{crit}}{F_{0.7}}$ can be determined from Fig. 11.8. F_{crit} can be calculated directly by

$$F_{crit} = F_{0.7}\left(\frac{F_{crit}}{F_{0.7}}\right) \tag{11.12}$$

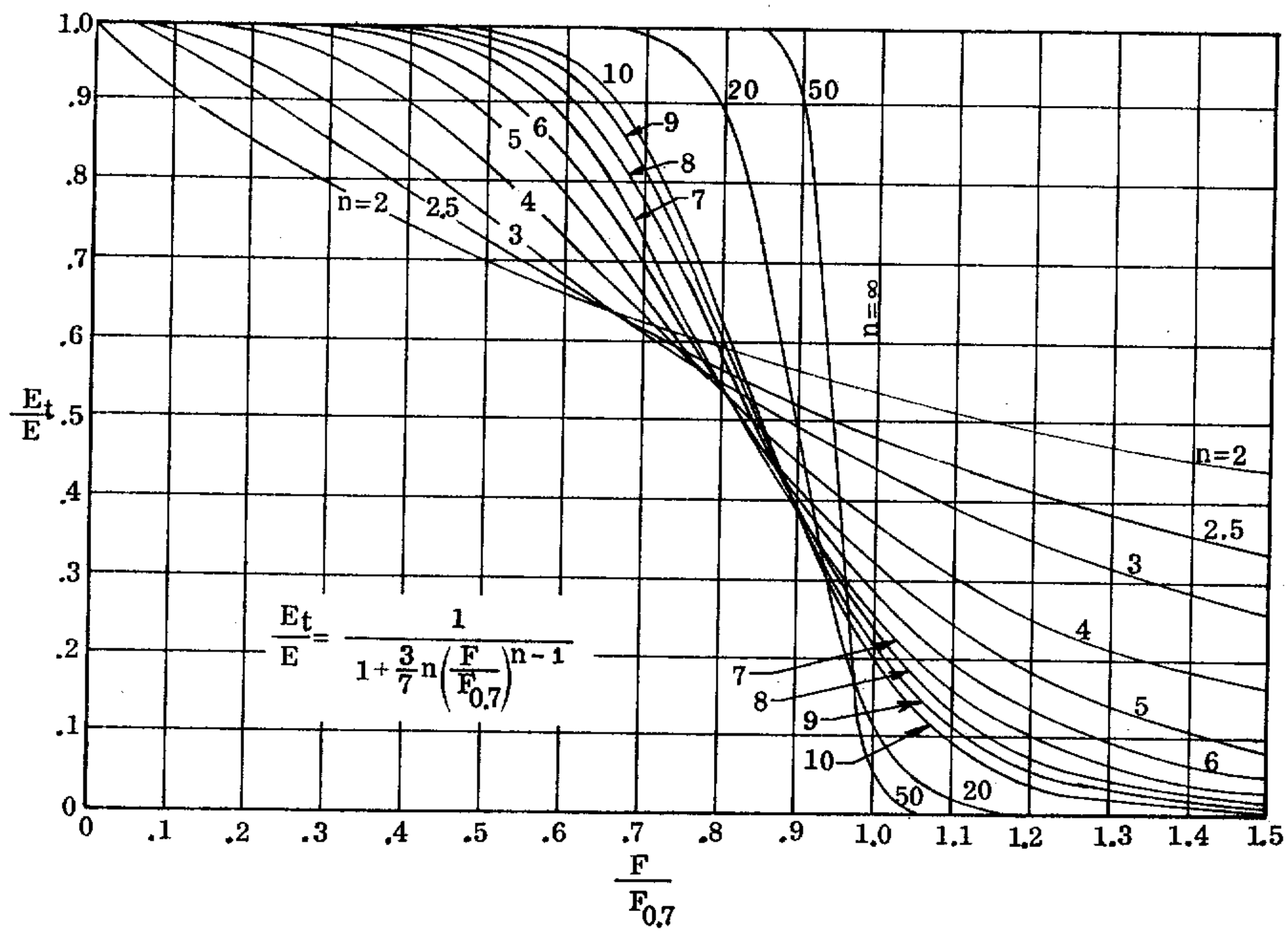

FIGURE 11.7 - DIMENSIONLESS TANGENT MODULUS STRESS CURVES

Material	Temp. Exp. Hr.	Temp. °F	e, %	F_{tu}, ksi	F_{cy}, ksi	E_c 10^6psi	$F_{0.7}$ ksi	$F_{0.85}$ ksi	n
Stainless Steel									
AISI 301 1/4 Hard Sheet									
Transverse Comp.	1/2	RT	25	125	80	27.0	73	63	6.9
Longitudinal Comp.	1/2	RT	25	125	43	26.0	28.2	23	5.2
AISI 301 1/2 Hard Sheet	1/2	RT	15	150	118	27.0	116.5	105	9.2
Transverse Comp.	1/2	400		118	108.5	23.2	108.5	97	8.6
	1/2	600		110	107.5	20.9	108.5	96.5	8.2
	1/2	1000		86	86	16.2	94.5	83.5	8.0
Longitudinal Comp.	1/2	RT	15	150	58	26.0	48	37	4.4
	1/2	400		118	53.3	22.4	45.5	36	4.7
	1/2	600		110	52.8	20.1	44	31	3.5
	1/2	1000		86	45.2	15.6	40	30.5	4.3
AISI 301 3/4 Hard Sheet	1/2	RT	12	175	160	27.0	163.5	151.5	13.2
Transverse Comp.	1/2	400		148	148	24.1	153	142.5	13.2
	1/2	600		138	138	22.4	152	140	11.2
	1/2	1000		112	112	18.9	127	121	19.2
Longitudinal Comp.	1/2	RT	12	175	76	26.0	70	61.5	7.6
	1/2	400		148	71	23.3	65	56	6.8
	1/2	600		138	70.3	21.6	65.5	56.5	6.8
	1/2	1000		112	59.3	18.2	55	46	5.9
AISI 301 Full Hard Sheet	1/2	RT	8	185	179	27.0	183	172	16
Transverse Comp.	1/2	400		168	168	25.1	174	164	16
	1/2	600		159	159	23.8	172	162	16
	1/2	1000		131	130	21.6	141.5	135.5	21.5
Longitudinal Comp.	1/2	RT	8	185	85	26.0	77.5	63	5.2
	1/2	400		168	80.8	24.2	74	59.5	5
	1/2	600		159	79.9	22.9	74	58	4.6
	1/2	1000		131	66.3	20.8	58	42.5	3.9
17-4 PH Bar & Forgings	1/2	RT	6	180	165	27.5	166	160	24
	1/2	400		162	135	25.3	137	129	16
	1/2	700		146	105.5	23.1	106	97	11
	1/2	1000		88	62.6	21.2	60	52	7.1
17-7 PH(TH1050) Sheet, Strip & Plate, t = .010 to .125 in.	1/2	RT		180	162	29.0	166	145	7.4
	1/2	400		169	144	27.8	146	126	6.8
	1/2	700		144	118	24.9	117	104	8.4
	1/2	1000		88	61.5	20.3	56	47	6
17-7 PH(RH950) Sheet, Strip & Plate, t = .010 to .125 in.	1/2	RT		210	205	29.0	208	196	16.4
19-9DL(AMS5526) & 19-9DX (AMS5538), Sheet, Strip & Plate	1/2	RT	30	95	45	29.0	36.5	32	7.6
19-9DL(AMS5527) & 19-9DX (AMS5539) Sheet, Strip & Plate	1/2	RT	12	125	90	29.0	85	74	7.2

TABLE 11.1 VALUES OF RAMBERG-OSGOOD PARAMETERS

Material	Temp. Exp. Hr.	Temp. °F	e, %	F_{tu}, ksi	F_{cy}, ksi	E_c 10^6 psi	$F_{0.7}$ ksi	$F_{0.85}$ ksi	n
PH15-7Mo(TH1050) Sheet & Strip, t = .020 to .187 in.	1/2	RT	5	190	170	28.0	171	164	22.5
PH15-7Mo(RH950) Sheet & Strip, t = .020 to .187 in.	1/2	RT	4	225	200	28.0	218	189	7.3
Low Carbon & Alloy Steels									
AISI 1023 & 1025 Tube, Sheet & Bar, Cold Finished		RT	22	55	36	29.0	32.7	31.5	24
AISI 4130 Normalized, t>.188 in.	1/2	RT	23	90	70	29.0	61.5	53	6.8
	1/2	500		81	61.5	27.3	55	48	7.3
	1/2	800		68	46.2	23.8	40	32.5	5.2
	1/2	1000		46	30.8	20.6	28	22	4.7
AISI 4130, 4140, 4340 Heat Treated	1/2	RT	23	125	113	29.0	111	102	10.9
	1/2	500		113	98.3	27.3	96	88	10.9
	1/2	850		88	68.9	23.2	66.5	61.5	12
	1/2	1000		64	49.7	20.6	45.5	41	9.2
AISI 4130, 4140, 4340 Heat Treated	1/2	RT	18.5	150	145	29.0	145	140	25
	1/2	500		135	126	27.3	126	122	29
	1/2	850		105	88.5	23.2	88	83.5	18.5
	1/2	1000		76	63.8	20.6	62	57	10.9
AISI 4130, 4140, 4340 Heat Treated	1/2	RT	15	180	179	29.0	179	176	50
	1/2	500		162	156	27.3	156	153	46
	1/2	850		126	109.3	23.2	109.4	105	22
	1/2	1000		92	77	20.6	75	68	9.8
AISI 4130, 4140, 4340 Heat Treated	1/2	RT	13.5	200	198	29.0	198	196	90
	1/2	500		180	170	27.3	172.5	169	46
	1/2	850		140	121	23.2	121.5	117	25
	1/2	1000		104	87.1	20.6	87	83	19
Heat Resistant Alloys									
A-286(AMS5725A) Sheet, Plate & Strip	1/2	RT	15	140	95	29.0	93	87	14
	1/2	600		129	88.4	24.4	87	81	13.5
	1/2	1000		115	81.7	19.8	81	75	12.5
	1/2	1400		52	50.3	14.2	50	47	15.3
K-Monel Sheet, Age Hardened	1/2	RT	15	125	90	26.0	88	82	13.5
Monel Sheet, Cold Rolled & Annealed	1/2	RT	35	70	28	26.0	20	17	6.4

TABLE 11.1 (CONT'D) - VALUES OF RAMBERG-OSGOOD PARAMETERS

Material	Temp. Exp. Hr.	Temp. ^oF	e, %	F_{tu}, ksi	F_{cy}, ksi	E_c 10^6psi	$F_{0.7}$ ksi	$F_{0.85}$ ksi	n
Inconel-X	1/2	RT	20	155	105	31.0	104	100	23.5
	1/2	400		152	95.6	28.9	94	89	17
	1/2	800		141	90.2	26.4	88.6	84	18.5
	1/2	1200		104	83	23.2	82	78.6	21
Titanium Alloys									
Ti-8Mn Annealed Sheet, Plate & Strip	1000	RT	10	120	110	15.5	119.5	102	13.7
Ti-6Al-4V Annealed Bar & Sheet, t≤.187 in.	1/2	RT	10	130	126	16.0	127	124.5	43
	1/2	400		105	96	14.1	97	93	22
	1/2	600		99	84.5	13.0	85.5	82	22
	1/2	800		87	79.4	11.8	80.5	77	21.5
	1/2	1000		70	60.6	7.7	61	59.5	36
Aluminum Alloys									
2014-T6 Extrusions t≤0.499 in.	2	RT	7	60	53	10.7	53	50.3	18.5
	2	300		51	42.5	10.2	41.5	40	24
	2	450		28	21	9.2	20.5	19.5	25
	2	600		10	8.0	7.4	5.5	4.5	5.4
	1/2	300		51	43.5	10.2	44.0	42.5	25
	1/2	450		31	26	9.2	26	25.2	29
2014-T6 Forgings t≤4 in.	2	RT	7	62	52	10.7	52.3	50	20
	2	300		53	41	10.2	40.5	38.5	19
	2	450		29	22	9.2	21.5	20	12.6
	2	600		10	7.5	7.4	4.5	3.0	3.2
	1/2	300		53	43	10.2	42.5	40	15.8
	1/2	450		32	25.5	9.2	25.0	23.5	15.6
2024-T3 Sheet & Plate, Heat Treated, t≤.250 in	2	RT	12	65	40	10.7	39	36	11.5
	2	300			37	10.3	35.7	33.5	15
	2	500			26	8.4	24.8	22.8	10.9
	2	700			7.5	6.4	6.2	5.5	8.2
2024-T4 Sheet & Plate, Heat Treated, t≤0.50 in	2	RT	12	65	38	10.7	36.7	34.5	15.6
	2	300			34	10.3	32.5	30.5	14.6
	2	500			24	8.4	23	21	10.2
	2	700			7	6.4	60	5.7	18.5
2024-T3 Clad Sheet & Plate, Heat Treated, t = .020 to .062 in.	2	RT	12	60	37	10.7	35.7	33	12
	2	300			34	10.3	33	30.3	11
	2	500			24.5	8.4	22.7	20	7.9
	2	700			6.5	6.4	5.8	5.5	18.5
2024-T6 Clad Sheet & Plate, Heat Treated, t≥0.063 in.	2	RT	8	62	49	10.7	49	45	11
	2	300			45	10.3	44.3	40.7	11
	2	500			22	8.4	31.5	28	8.3
	2	700			6	6.4	7.0	6.0	6.6

TABLE 11.1 (CONT'D) - VALUES OF RAMBERG-OSGOOD PARAMETERS

Material	Temp. Exp. Hr.	Temp. ^oF	e, %	F_{tu}, ksi	F_{cy}, ksi	E_c 10^6psi	$F_{0.7}$ ksi	$F_{0.85}$ ksi	n
2024-T6 Clad Sheet & Plate, Heat Treated, t<0.063 in.	2	RT	8	60	47	10.7	47	43	10.6
	2	300			43.2	10.3	42.3	38.7	10.8
	2	500			21	8.4	29.5	26	7.8
	2	700			6	6.4	5.0	4.0	4.9
2024-T81 Clad Sheet, Heat Treated, t<0.063 in.	2	RT	5	62	55	10.7	56	51.6	11.2
	2	300			50.5	10.3	51.2	46.5	10
	2								
	2								
6061-T6 Sheet, Heat Treated & Aged, t < 0.25 in.	1/2	RT	10	42	35	10.1	35	34	31
	1/2	300			29.5	9.5	29	28	26
	1/2	450			20.5	8.5	19.3	17.7	10.9
	1/2	600			7.5	7.0	6.6	6.2	15.2
7075-T6 Bare Sheet & Plate, t≤0.50 in.	2	RT	7	76	67	10.5	70	63	9.2
	2	300			54	9.4	55.8	52.5	15.6
	2	425			25.5	8.1	25.4	23.5	12.1
	2	600			8	5.3	7.2	5.2	3.7
	1/2	425			30	8.1	34.5	32.5	16
7075-T6 Extrusions, t≤0.25 in.	2	RT	7	75	70	10.5	72	68	16.6
	2	300			54	9.4	58.5	54.5	13.4
	2	450			22.5	7.8	21.3	18.5	7.2
	2	600			8	5.3	6.5	4.3	3.2
	1/2	450			25	7.8	29	26	8.8
7075-T6 Die Forgings, t≤2 in.	2	RT	7	71	58	10.5	58.5	55.1	15.2
	2	300			47.6	9.4	47.8	45	15.6
	2	450			18.5	7.8	17.3	16	12
	2	600			7.0	5.3	5.0	3.7	3.9
	1/2	450			23	7.8	24	22	10.9
7075-T6Hand Forgings, Area≤16 sq. in.	2	RT	4	72	63	10.5	63.8	61.5	25
	2	300			51.6	9.4	52.2	50	21.5
	2	450			20.2	7.8	20.3	19	13.7
	2	600			7.6	5.3	6.0	5.0	5.8
	1/2	450			24	7.8	26.5	25.3	19.5
7075-T6 Clad Sheet & Plate, t≤0.50 in.	2	RT	8	70	64	10.5	64.5	61.6	19.5
	2	300			50	9.4	54	51.7	20
	2	450			20.5	7.8	19.7	17.5	4.6
	2	600			7.7	5.3	7.7	5.5	3.6
	1/2	450			23	7.8	27.2	25.3	12.4
7079-T6 Hand Forgings, t≤6.0 in.	1/2	RT	4	67	59	10.5	59.5	57.5	26
	1/2	300			47	9.4	46.5	45	29
	1/2	450			21	7.8	20	18.5	12
	1/2	600			7.0	5.3	5.5	3.5	3.0

TABLE 11.1 (CONT'D) - VALUES OF RAMBERG-OSGOOD PARAMETERS

Material	Temp. Exp. Hr.	Temp. ^{o}F	e, %	F_{tu}, ksi	F_{cy}, ksi	E_c 10^6psi	$F_{0.7}$ ksi	$F_{0.85}$ ksi	n
Magnesium Alloys									
AZ61A Extrusions,									
t≦0.249 in.		RT	8	38	14	6.3	12.9	12.3	19
HK31A-0 Sheet	1/2	RT	12	30	12	6.5	10	8.4	6
t = 0.016 to 0.250 in.	1/2	300		20	11.1	6.16	8.9	6.9	4.5
	1/2	500		15	9.3	4.94	7.5	5.6	4.2
	1/2	600		10	4.9	3.77	3.3	1.6	2.2
HK31A-H24 Sheet,	1/2	RT	4	34	19	6.5	17.3	14.6	6.2
t<.25	1/2	300		22	17.7	6.2	15.6	12.6	5.1
	1/2	500		17	14.8	4.9	13.1	10.5	4.9
	1/2	600		11	7.8	3.8	6.7	5.2	4.5

TABLE 11.1 (CONT'D) - VALUES OF RAMBERG-OSGOOD PARAMETERS

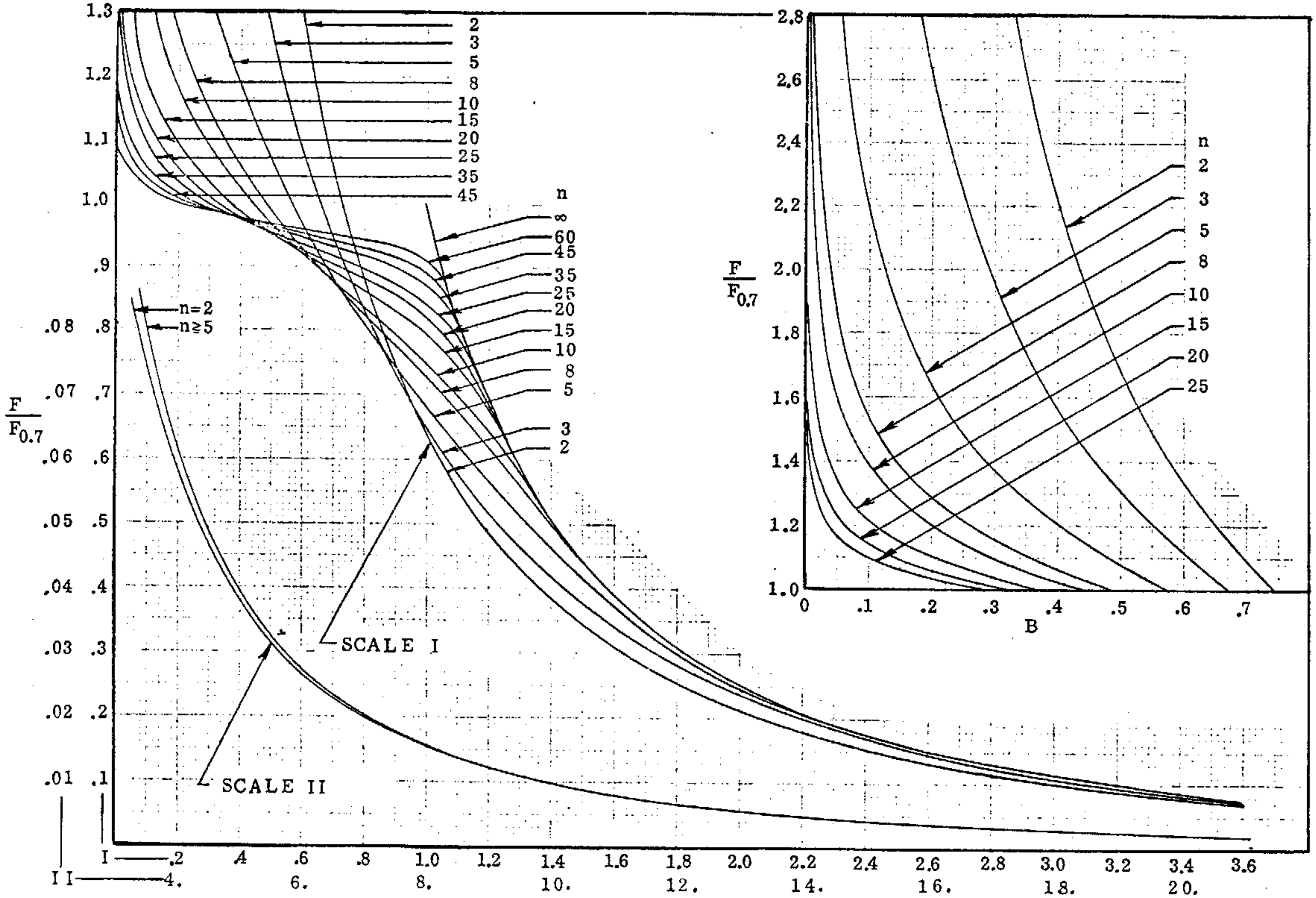

FIGURE 11.8 - COLUMN BUCKLING

$$B = \frac{L'/\rho}{\Pi}\sqrt{\frac{F_{0.7}}{E}}$$

11.1.3 Columns With Varying Cross Section

The conventional Euler critical column stress equation :

$$F_{crit} = \frac{\pi^2 E}{\left(L'/\rho\right)^2}$$

is only valid for a straight column under compression with constant bending rigidity (EI) and a constant area along its length. When the bending rigidity varies along the length of the column, determination of the Euler load becomes more difficult. In this section column buckling coefficient charts and the appropriate formulas for the Euler loads are given for numerous columns of varying cross section. CPS Program SC5001 is a computer analysis of stepped columns.

The critical buckling load for variable section columns in the elastic range is given by general equations of the form

$$P_{cr} = m EI/L^2 \qquad (11.13)$$

where m is the column buckling coefficient and is a function of the column geometry, bending rigidity and end restraint. Values of the column buckling coefficient, m, for various stepped columns shown in Figure 11.10 are given in Figures 11.11 through 11.29.

For tapered columns with the moment of inertia varying at the ends according to

$$I_x = I_2 (x/b)^n \qquad (11.14)$$

where b, x, Ix and I_2 are defined in Figure 11.9, the values of the coefficient, m, to be used in Equation 11.13 are obtained from Figures 11.11 through 11.29 for the cases given in Figure 11.10.

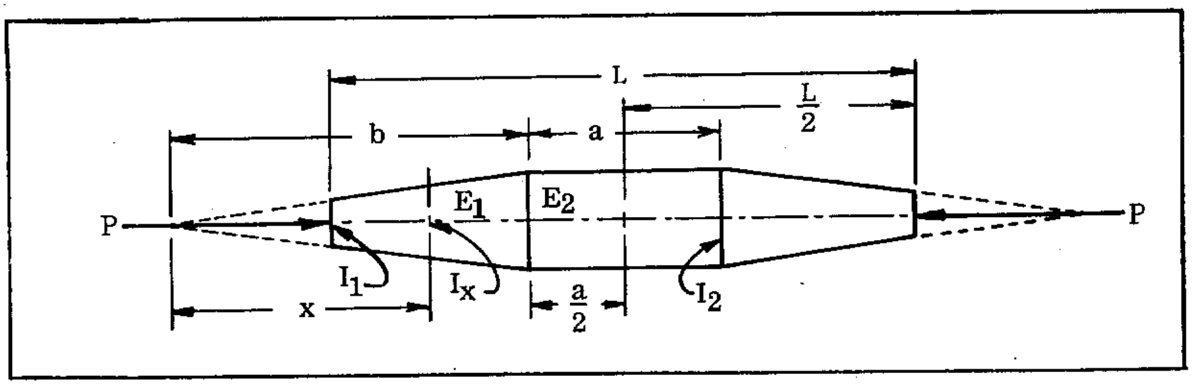

Figure 11.9 - Column with Varying Cross-Sections

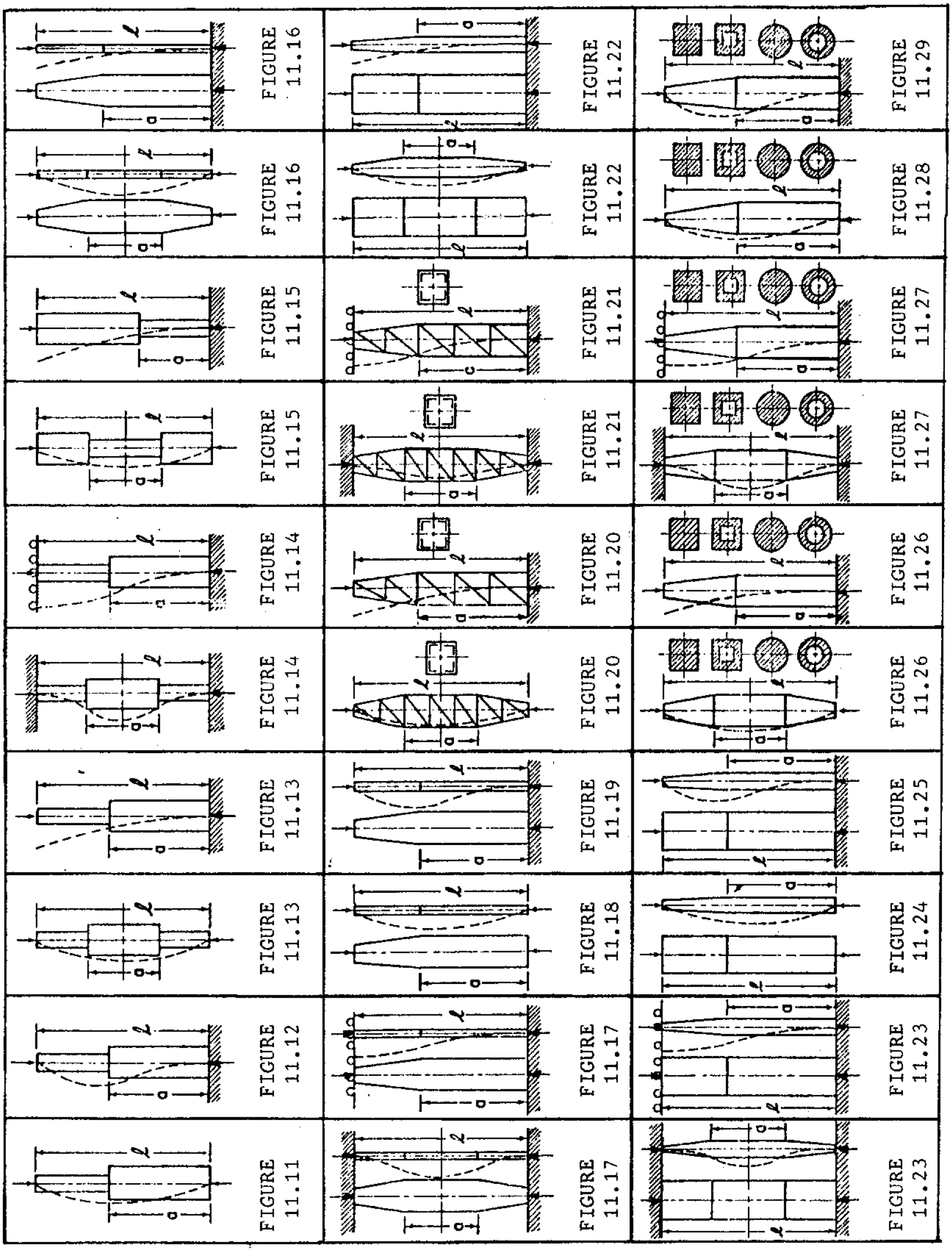

FIGURE 11.10 - LIST OF CASES OF COLUMNS OF VARIABLE CROSS-SECTION

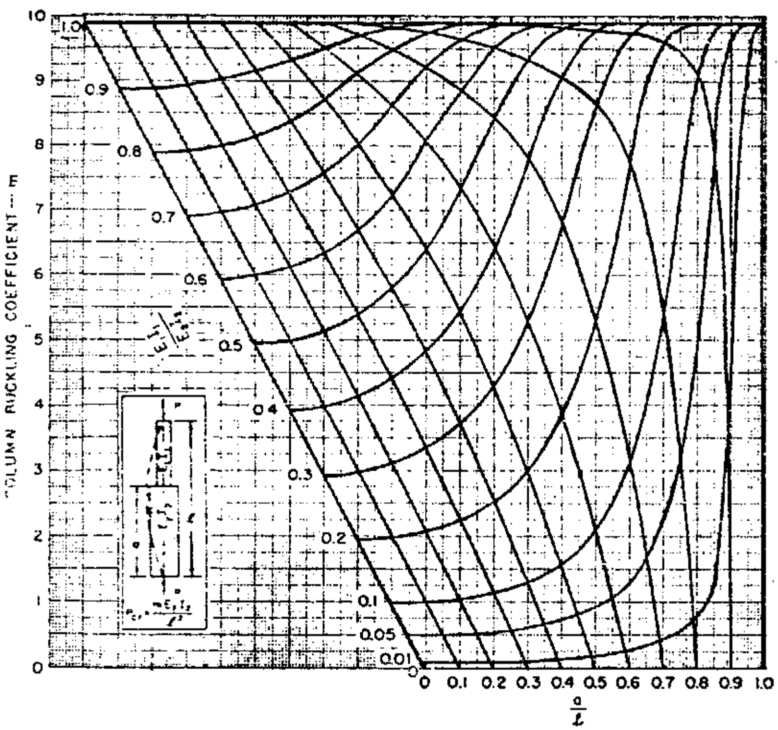

FIGURE 11.11 - BUCKLING OF VARIABLE SECTION COLUMNS: STEPPED COLUMN WITH BOTH ENDS PINNED AND NO TRANSVERSE AXIS OF SYMMETRY.

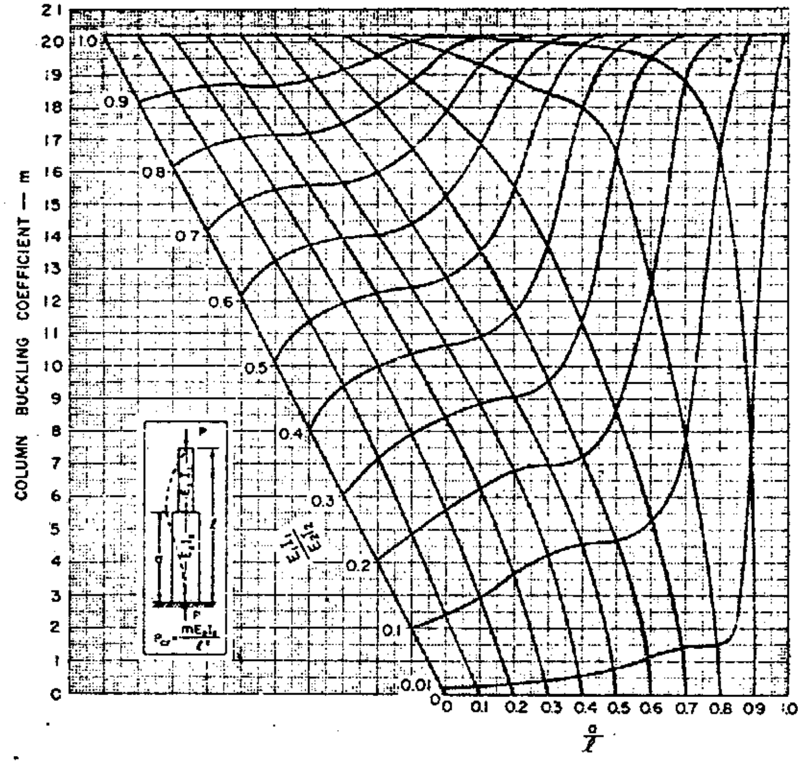

FIGURE 11.12 - BUCKLING OF VARIABLE SECTION COLUMNS: STEPPED COLUMN WITH ONE END PINNED AND THE OTHER FIXED AND NO TRANSVERSE AXIS OF SYMMETRY.

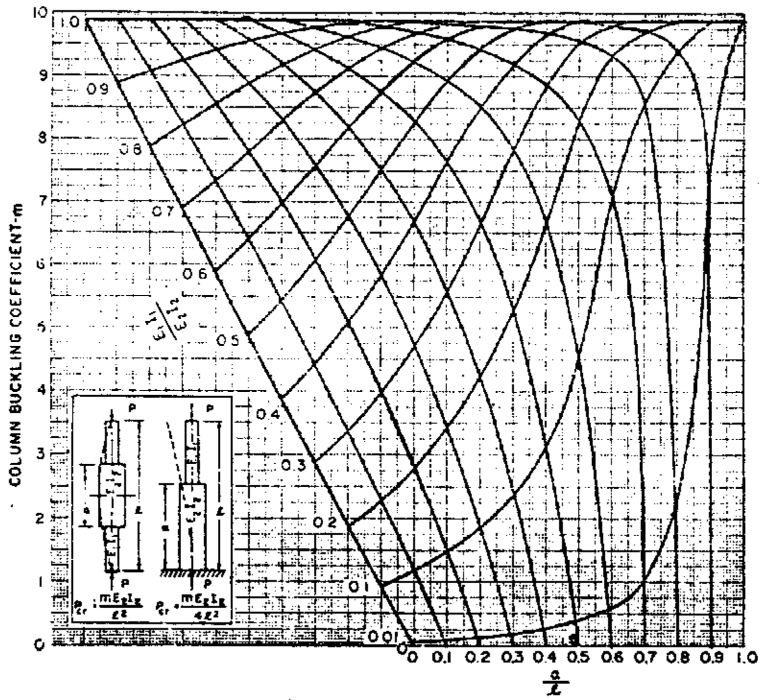

FIGURE 11.13 - BUCKLING OF VARIABLE SECTION COLUMNS: STEPPED COLUMN WITH SMALLER MOMENT OF INERTIA AT ENDS, BOTH ENDS PINNED, AND A TRANSVERSE AXIS OF SYMMETRY.

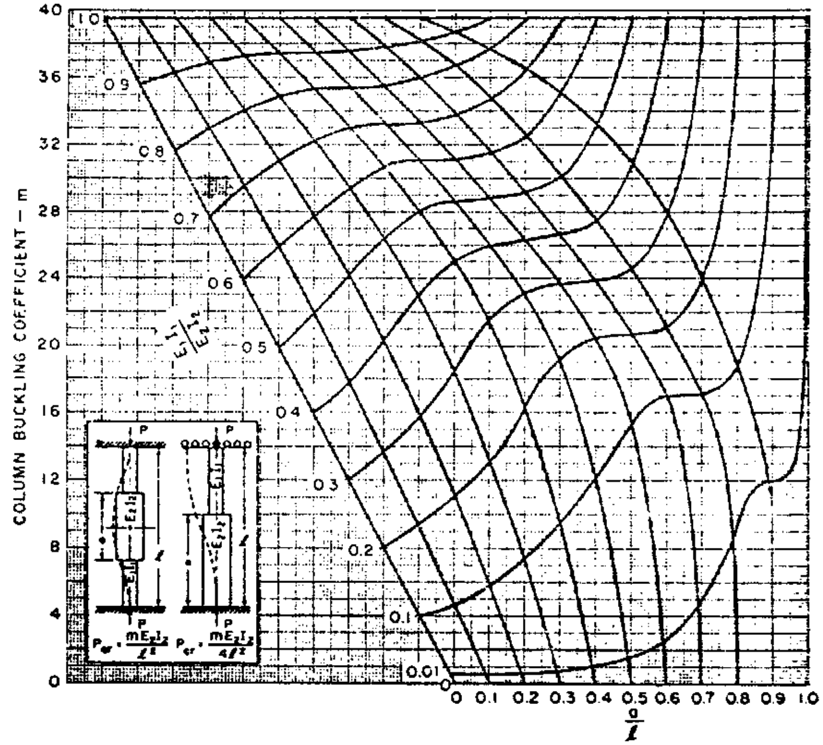

FIGURE 11.14 - BUCKLING OF VARIABLE SECTION COLUMNS: STEPPED COLUMN WITH SMALLER MOMENT OF INERTIA AT ENDS, BOTH ENDS FIXED AND A TRANSVERSE AXIS OF SYMMETRY.

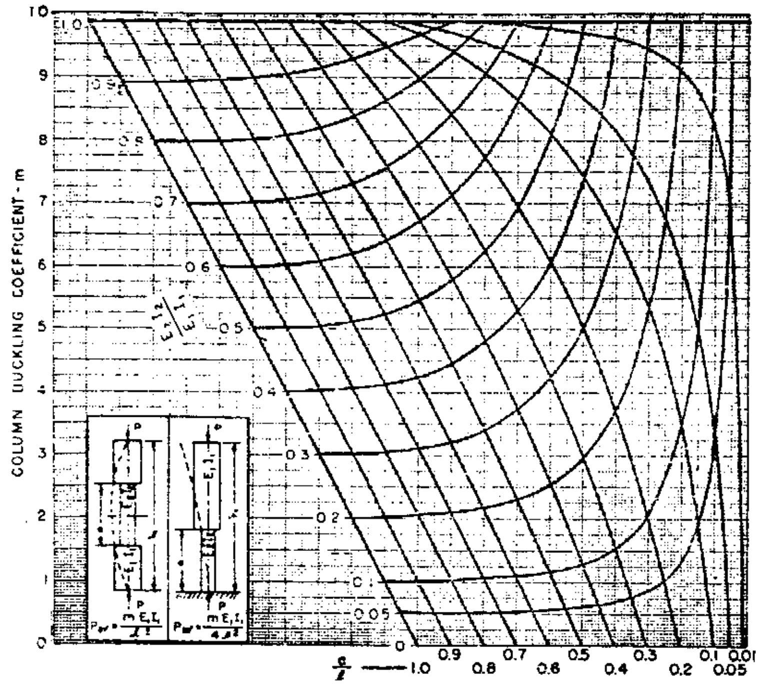

FIGURE 11.15 - BUCKLING OF VARIABLE SECTION COLUMNS: STEPPED COLUMN WITH LARGER MOMENT OF INERTIA AT ENDS, BOTH ENDS PINNED AND A TRANSVERSE AXIS OF SYMMETRY.

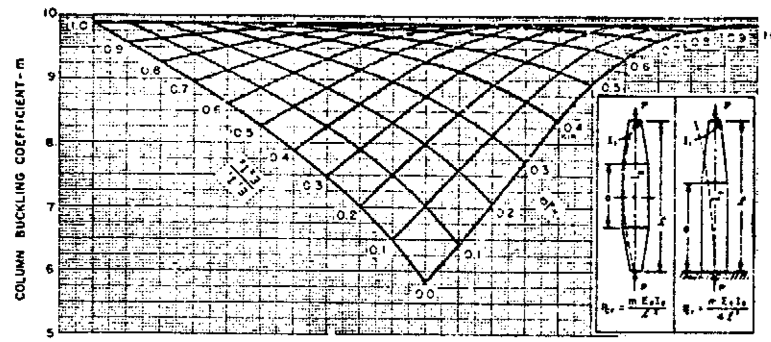

FIGURE 11.16 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ENDS VARYING AS THE FIRST POWER (n=1) WITH BOTH ENDS PINNED AND A TRANSVERSE AXIS OF SYMMETRY.

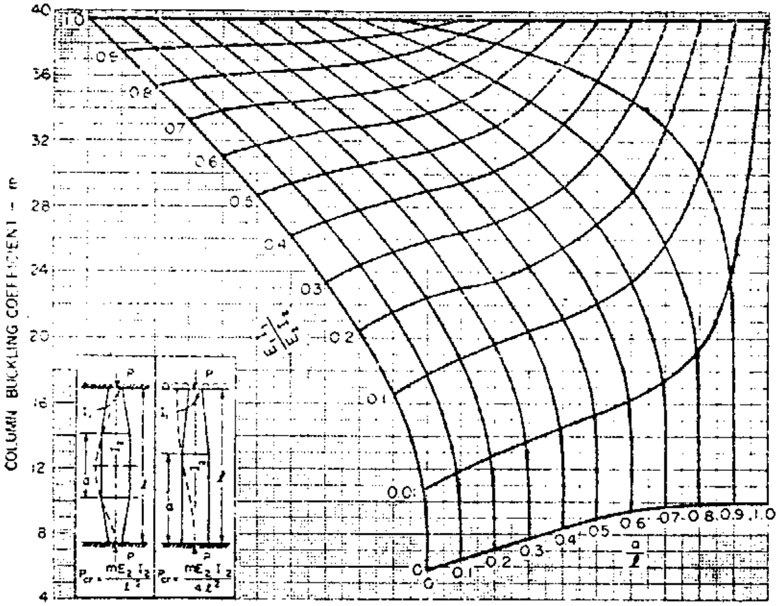

FIGURE 11.17 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ENDS VARYING AS THE FIRST POWER (n = 1) WITH BOTH ENDS FIXED AND A TRANSVERSE AXIS OF SYMMETRY.

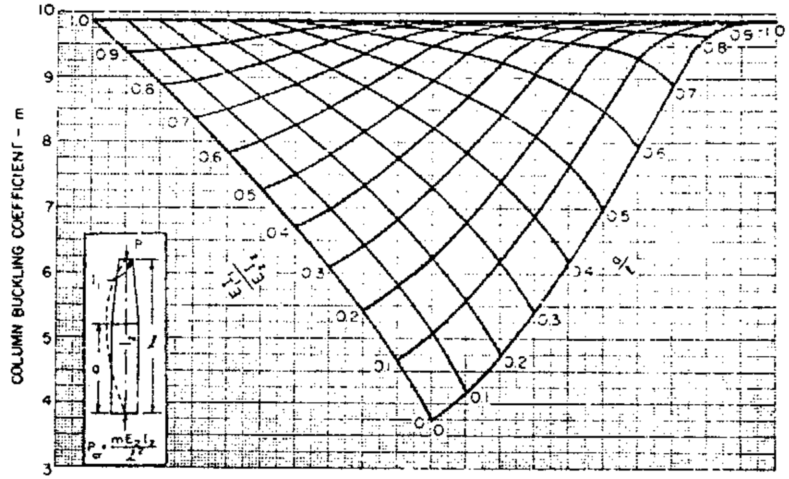

FIGURE 11.18 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ONE END VARYING AS THE FIRST POWER (n = 1) WITH BOTH ENDS PINNED AND NO TRANSVERSE AXIS OF SYMMETRY.

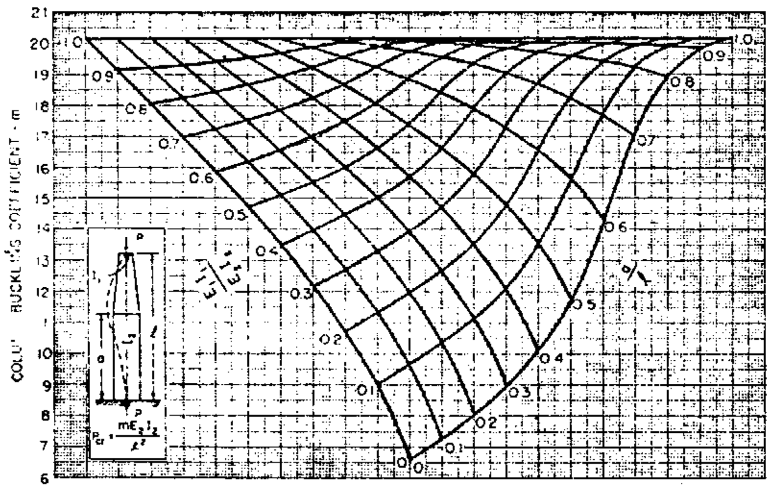

FIGURE 11.19 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ONE END VARYING AS THE FIRST POWER (n = 1) WITH ONE END PINNED AND THE OTHER FIXED AND NO TRANSVERSE AXIS OF SYMMETRY.

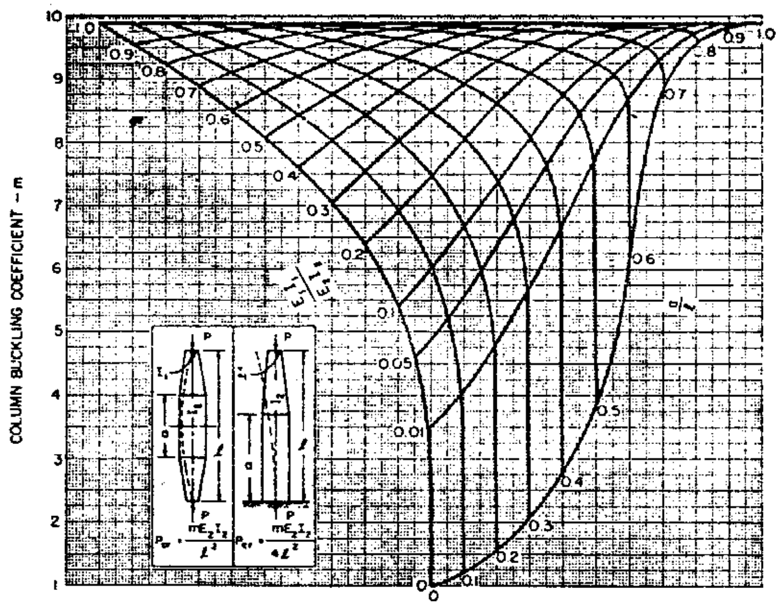

FIGURE 11.20 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ENDS VARYING AS THE SECOND POWER (n = 2) WITH BOTH ENDS PINNED AND A TRANSVERSE AXIS OF SYMMETRY.

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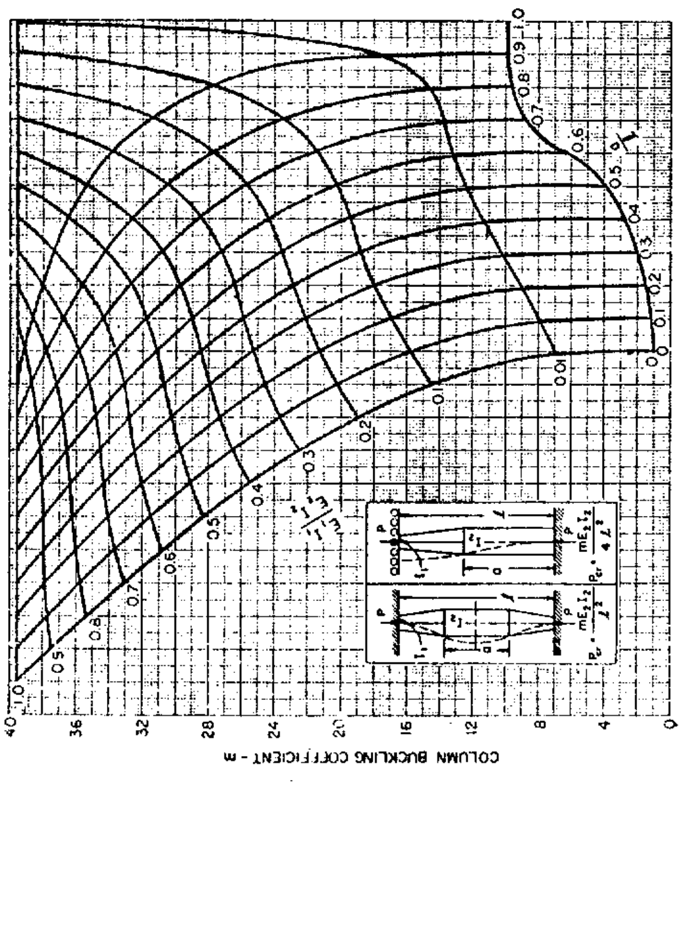

FIGURE 11.21 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ENDS VARYING AS THE SECOND POWER (n = 2) WITH BOTH ENDS FIXED AND A TRANSVERSE AXIS OF SYMMETRY.

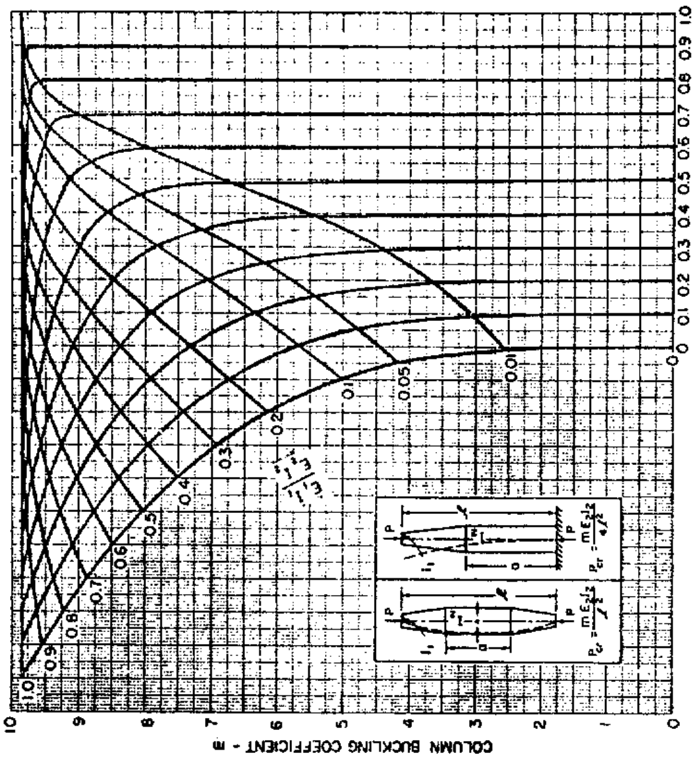

FIGURE 11.22 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ENDS VARYING AS THE THIRD POWER (n = 3) WITH BOTH ENDS PINNED AND A TRANSVERSE AXIS OF SYMMETRY.

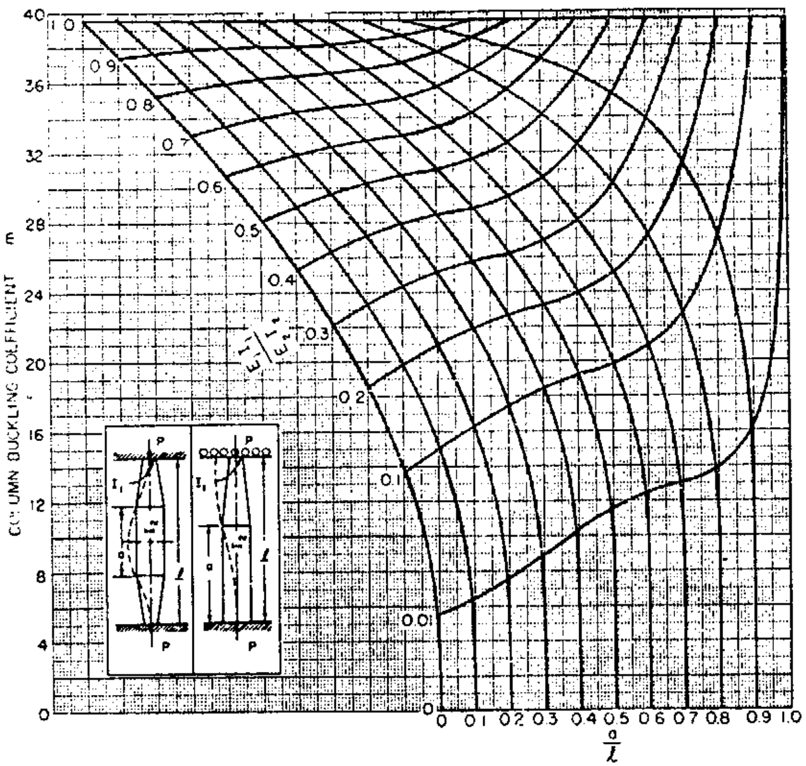

FIGURE 11.23 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ENDS VARYING AS THE THIRD POWER (n = 3) WITH BOTH ENDS FIXED AND A TRANSVERSE AXIS OF SYMMETRY.

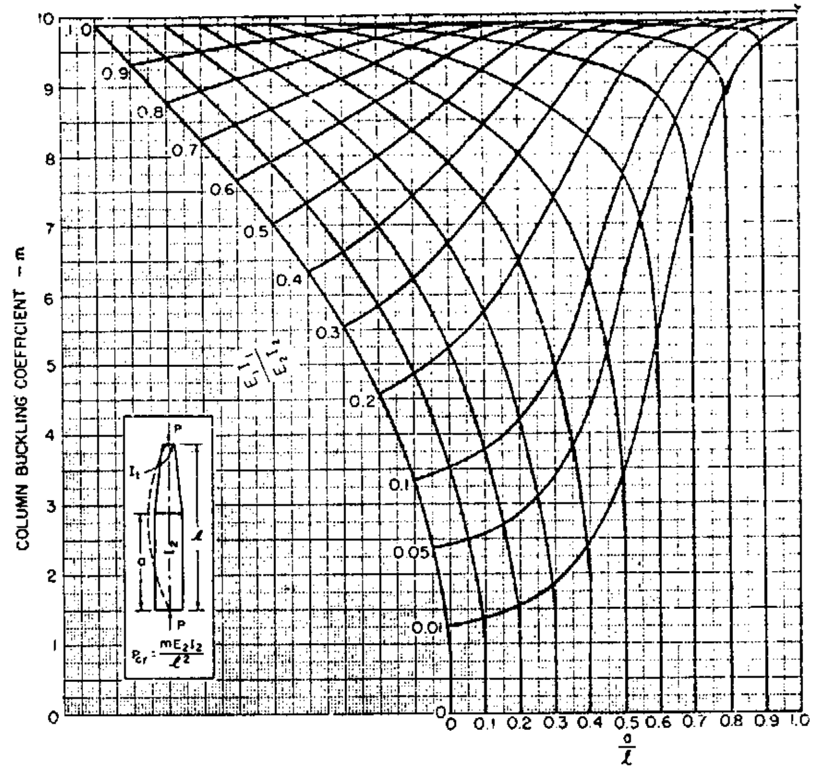

FIGURE 11.24 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ONE END VARYING AS THE THIRD POWER (n = 3) WITH BOTH ENDS PINNED AND NO TRANSVERSE AXIS OF SYMMETRY.

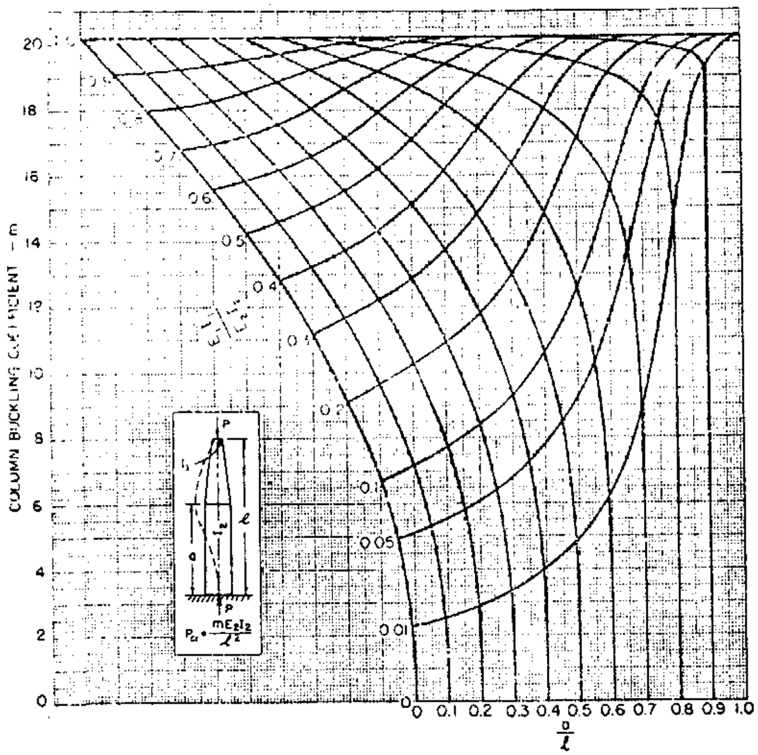

FIGURE 11.25 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ONE END VARYING AS THE THIRD POWER (n = 3) WITH ONE END PINNED AND THE OTHER FIXED AND NO TRANSVERSE AXIS OF SYMMETRY.

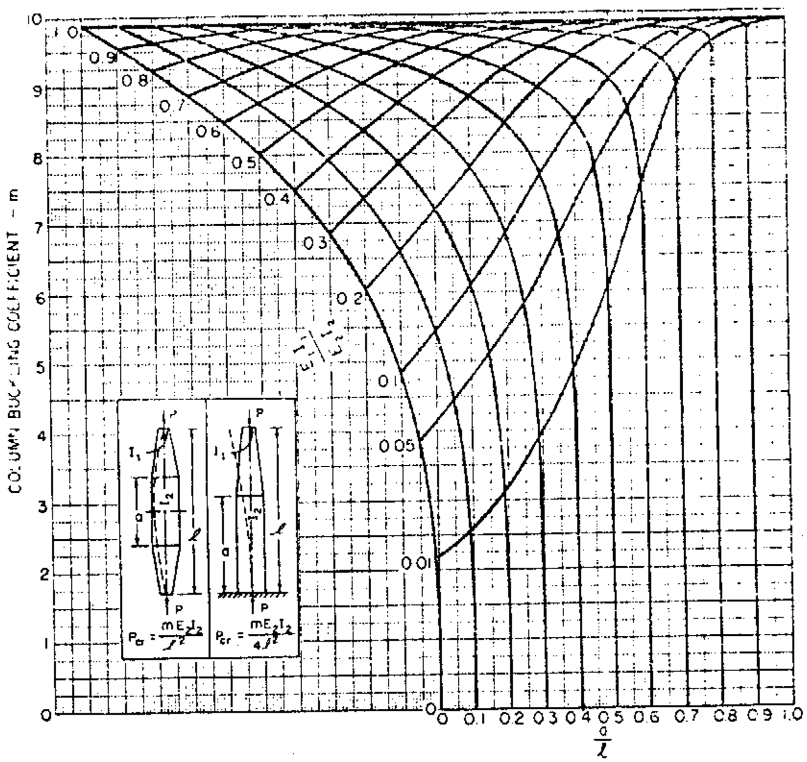

FIGURE 11.26 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ENDS VARYING AS THE FOURTH POWER (n = 4) WITH BOTH ENDS PINNED AND A TRANSVERSE AXIS OF SYMMETRY.

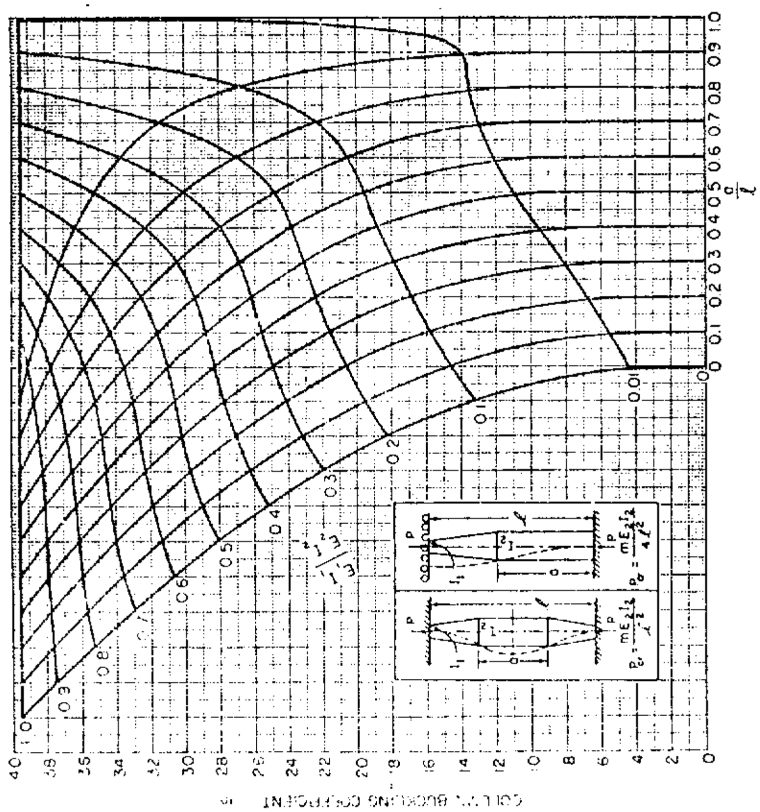

FIGURE 11.27 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ENDS VARYING AS THE FOURTH POWER (n = 4) WITH BOTH ENDS FIXED AND A TRANSVERSE AXIS OF SYMMETRY.

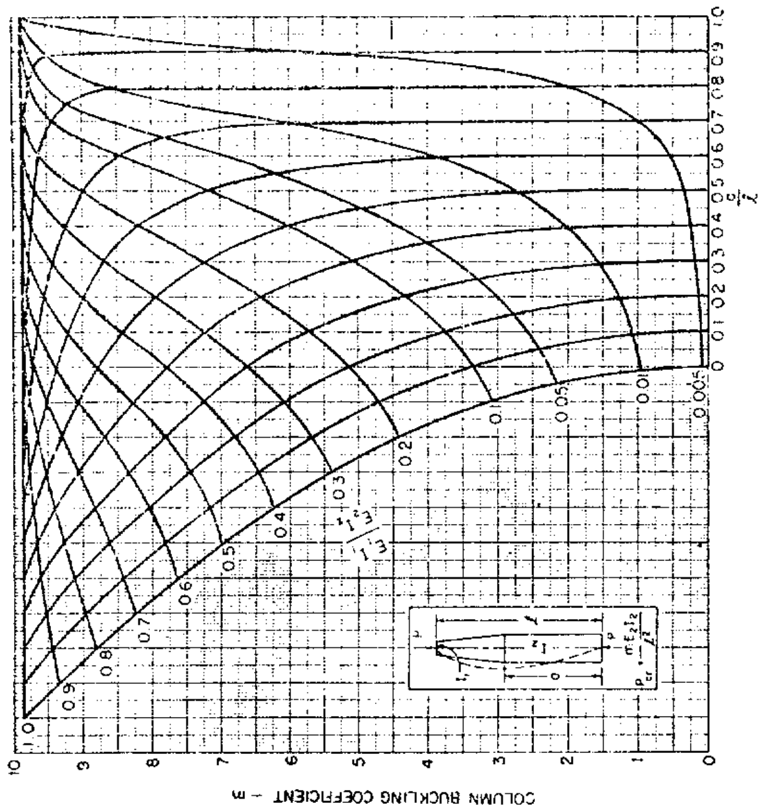

FIGURE 11.28 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ONE END VARYING AS THE FOURTH POWER (n = 4) WITH BOTH ENDS PINNED AND A TRANSVERSE AXIS OF SYMMETRY.

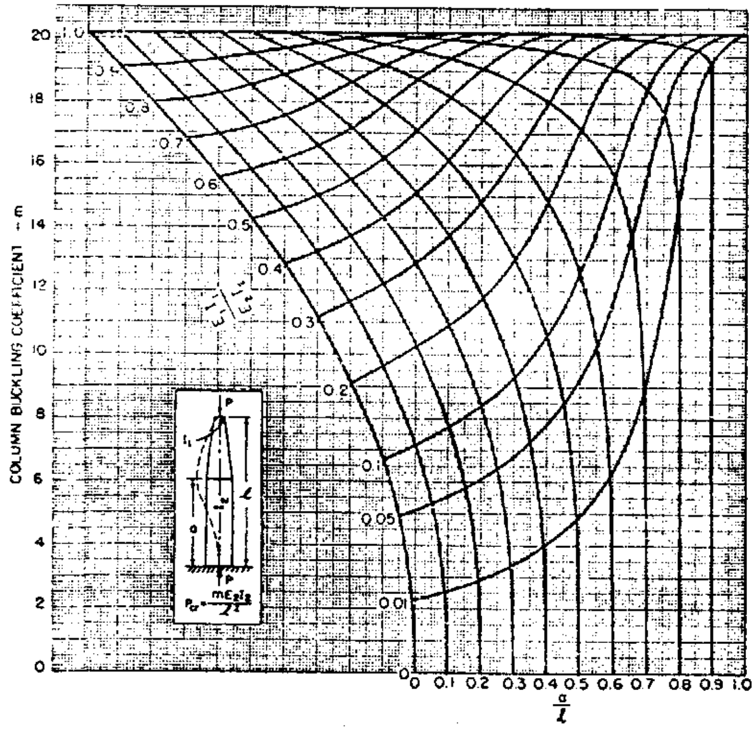

FIGURE 11.25 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ONE END VARYING AS THE THIRD POWER (n = 3) WITH ONE END PINNED AND THE OTHER FIXED AND NO TRANSVERSE AXIS OF SYMMETRY.

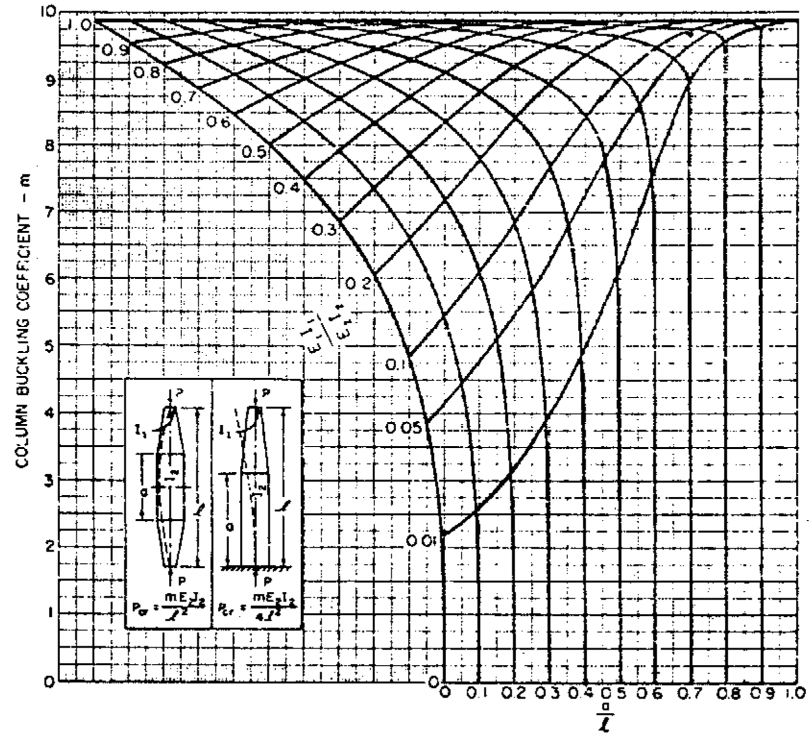

FIGURE 11.26 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ENDS VARYING AS THE FOURTH POWER (n = 4) WITH BOTH ENDS PINNED AND A TRANSVERSE AXIS OF SYMMETRY.

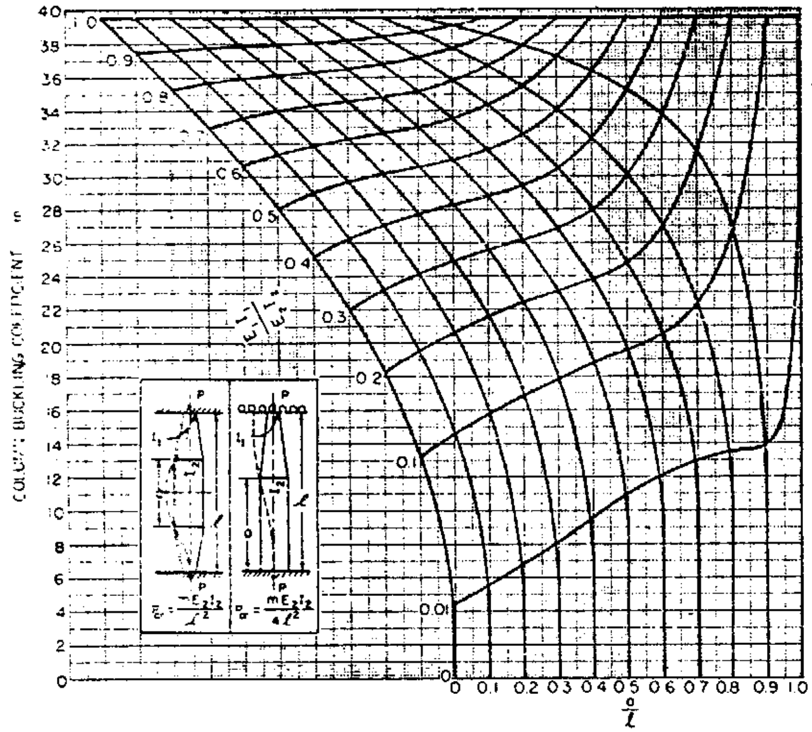

FIGURE 11.27 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ENDS VARYING AS THE FOURTH POWER (n = 4) WITH BOTH ENDS FIXED AND A TRANSVERSE AXIS OF SYMMETRY.

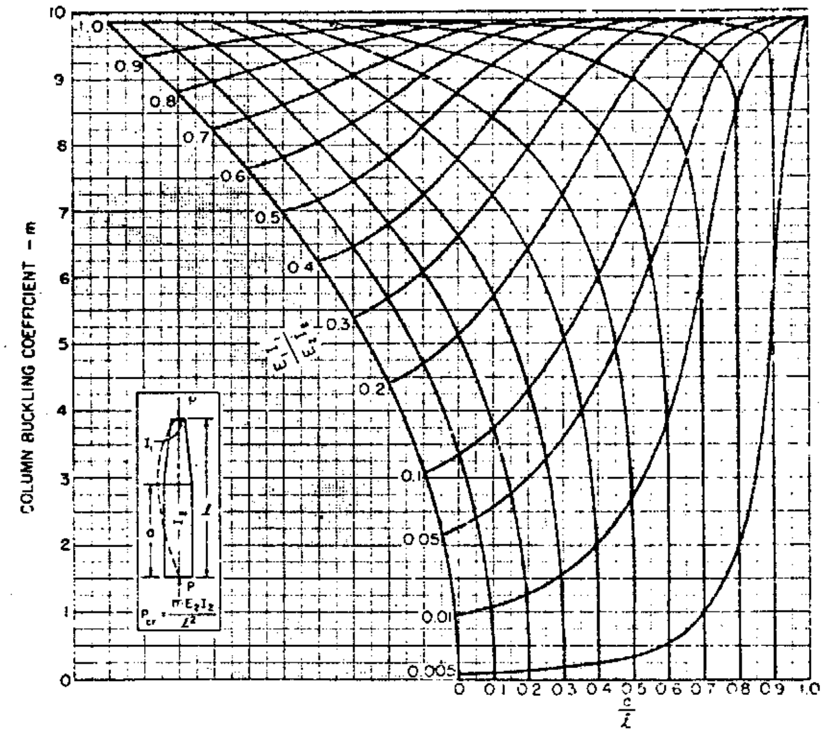

FIGURE 11.28 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ONE END VARYING AS THE FOURTH POWER (n = 4) WITH BOTH ENDS FIXED AND A TRANSVERSE AXIS OF SYMMETRY.

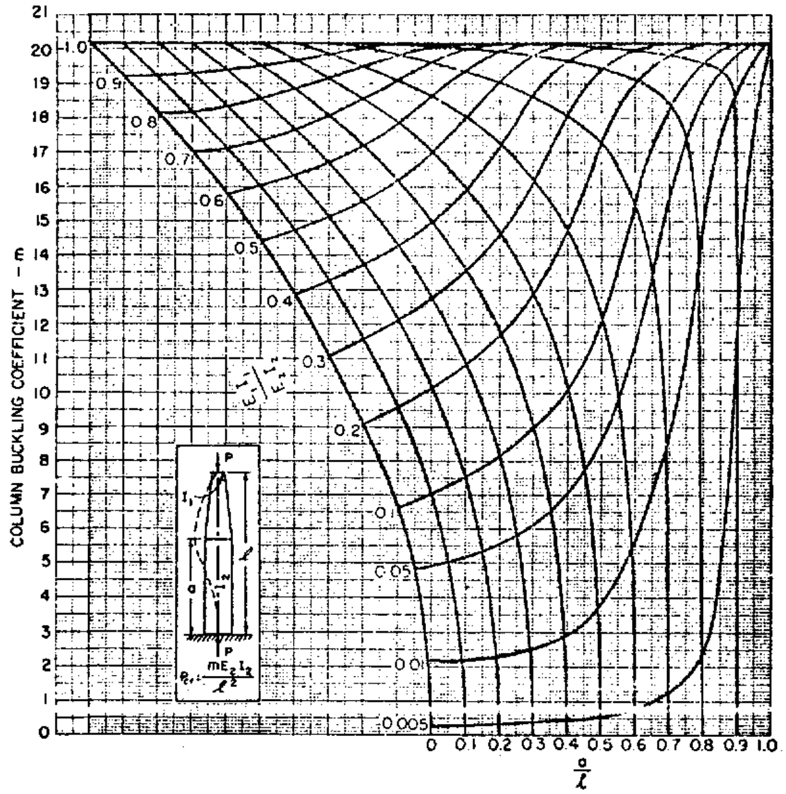

FIGURE 11.29 - BUCKLING OF VARIABLE SECTION COLUMNS: MOMENT OF INERTIA FOR ONE END VARYING AS THE FOURTH POWER (n = 4) WITH ONE END PINNED AND THE OTHER FIXED AND NO TRANSVERSE AXIS OF SYMMETRY.

The significance of the value of the exponent n is as follows:

$n = 1$

For this condition, the column is of constant thickness with a linear taper in width. The thickness must always be smaller than the width at any section, so that actually this represents a thin, wide rectangular cross-section.

$n = 2$

For this case we obtain a braced polygonal or pyramidical column. Thus the moment of inertia for such a configuration is approximately proportioned to the square of the distance to the centroids of the concentrated outer areas. Often the moment of inertia of the individual sections about their own axes can be neglected. Also, when $n = 2$, the edge of the column is a parabola, the axis of which is perpendicular to the centerline of the column. The thickness is constant and smaller than the width at any section.

$n = 3$

For this exponent, the resulting column is a column of constant width and linearly varying thickness. In this case, although the thickness tapers, its dimension is always smaller than the width. Thus a column of constant width is the result.

$n = 4$

A solid or hollowed cone, square or pyramid results for this exponent. A column tapering linearly in width and depth, although not of the same taper ratio, falls in this class if extensions of the taper lines meet at a common apex.

Equation 11.13 is valid only in the elastic range and must be modified for stresses in the plastic range. The following procedure is used where Equation 11.13 is in the plastic range.

A) Determine the buckling coefficient, m, from Figures 11.11 through 11.29.
B) Calculate the equivalent slenderness ratio, $(L/\rho\sqrt{c})_e$, for each section corresponding to the smallest cross-sectional area. If the column is composed of one material, then the equivalent slenderness ratio corresponding to the smallest column cross sectional area is all that is necessary. The equivalent slenderness ratio is obtained from Table 11.2 for the particular column.
C) From the appropriate column curves of Section 11.1.4 and the equivalent slenderness ratios determine the critical compressive stress, $(\sigma_{cr})_1$ and $(\sigma_{cr})_2$.
D) Calculate the critical buckling loads for each section

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$$P_{cr} = \frac{m E_2 I_2}{L^2}$$	
$$\left[\frac{L}{\rho\sqrt{C}}\right]_{e_1} = \frac{\pi L}{\rho_1 \sqrt{\frac{m E_2 I_2}{E_1 I_1}}}$$	$$\left[\frac{L}{\rho\sqrt{C}}\right]_{e_2} = \frac{\pi L}{\rho_2 \sqrt{m}}$$
$$P_{cr} = \frac{m E_1 I_1}{L^2}$$	
$$\left[\frac{L}{\rho\sqrt{C}}\right]_{e_1} = \frac{\pi L}{\rho_1 \sqrt{m}}$$	$$\left[\frac{L}{\rho\sqrt{C}}\right]_{e_2} = \frac{\pi L}{\rho_2 \sqrt{\frac{m E_1 I_1}{E_2 I_2}}}$$
$$P_{cr} = \frac{m E_2 I_2}{4L^2}$$	
$$\left[\frac{L}{\rho\sqrt{C}}\right]_{e_1} = \frac{2\pi L}{\rho_1 \sqrt{\frac{m E_2 I_2}{E_1 I_1}}}$$	$$\left[\frac{L}{\rho\sqrt{C}}\right]_{e_2} = \frac{2\pi L}{\rho_2 \sqrt{m}}$$
$$P_{cr} = \frac{m E_1 I_1}{4L^2}$$	
$$\left[\frac{L}{\rho\sqrt{C}}\right]_{e_1} = \frac{2\pi L}{\rho_1 \sqrt{m}}$$	$$\left[\frac{L}{\rho\sqrt{C}}\right]_{e_2} = \frac{2\pi L}{\rho_2 \sqrt{\frac{m E_1 I_1}{E_2 I_2}}}$$

TABLE 11.2 - EQUIVALENT SLENDERNESS RATIOS

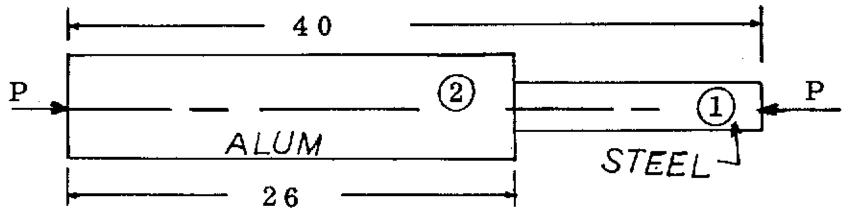
E) The smallest critical load is taken as the critical load of the column.

It should be noted that in the elastic range use of the equivalent slenderness ratios and the column curves will give the same buckling load as that calculated by Equation 11.13.

To illustrate the procedure consider the following example:

Example Problem

Determine the critical buckling load for the following stepped column where the larger section is made of 2024 aluminum and the smaller section alloy steel heat treated to 125000 psi.



A) Section Properties

Section 1 - Steel	Section 2 - Aluminum
$I_1 = 0.2665 \text{ in}^4$	$I_2 = 1.549 \text{ in}^4$
$A_1 = 0.4462 \text{ in}^2$	$A_2 = 1.964 \text{ in}^2$
$\rho_1 = 0.774$ in	$\rho_2 = 0.888$ in
$E_1 = 29 \times 10^6$ psi	$E_2 = 10.5 \times 10^6$ psi
$F_{cy} = 100{,}000$ psi	$F_{cy} = 42{,}000$ psi

B) Column Buckling Coefficient - m

$E_1 I_1/E_2 I_2 = (29)(10^6)(0.2665)/(10.5)(10^6)(1.549) = 0.475$

$a/L = 26/40 = 0.65$

Using Table 11.2 and Figure 11.10, $m = 7.75$, then

$P_{cr} = m(E_2 I_2/L^2) = 7.75\ (10.5)(10^6)(1.549)/(40)^2 = 78{,}800$ lbs.

C) Stresses in Each Section at Buckling

$(\sigma_{cr})_1 = (P_{cr})_1/A_1 = 78800/0.4462 = 176{,}600$ psi

$(\sigma_{cr})_2 = (P_{cr})_2/A_2 = 78800/1.964 = 40{,}100$ psi

These stresses are in the plastic region so the Euler load must be corrected to include the effects of plasticity.

Revision C

D) Equivalent Slenderness Ratio

For Section 1 of the column, from Table 11.2

$$(L/\rho\sqrt{c})_{e_1} = \pi L/\rho_1\sqrt{\pi E_2 I_2/E_1 I_1} = 40\pi/0.774\sqrt{7.75/0.476} = 40.2$$

For Section 2 of the column

$$(L/\rho\sqrt{c})_{e_2} = \pi L/\rho_2\sqrt{m} = 40\pi/0.888\sqrt{7.75} = 50.8$$

E) For Section 1, from column curves in Section 11.1.4, using the 125000 psi steel curve and $(L/\rho\sqrt{c}) = 40.2$ the column stress is $(\sigma_{cr})_1 = 107{,}000$.

For Section 2, from the column curves in Section 11.1.4, using the 2024 curve and $(L/\rho\sqrt{c}) = 50.8$, the column stress is $(\sigma_{cr})_2 = 31{,}000$ psi.

F) Critical Column Load

$P_{cr1} = (107{,}000)(0.4462) = 47{,}750$ lbs.

$P_{cr2} = (31{,}000)(1.964) = 60{,}900$ lbs.

∴ The critical compression load of the column is $P_{cr} = 47{,}750$ lbs.

11.1.4 Column Data for Both Long and Short Columns (FIGURES 11.30-11.65)

Critical buckling stresses for different materials and geometry are given. Given the slenderness ratio (L'/ρ), and the material, the critical stress is determined. Both short and long columns are accounted for.

INDEX OF COLUMN ALLOWABLE CURVES

Index of Column Allowable Curves (Continued)

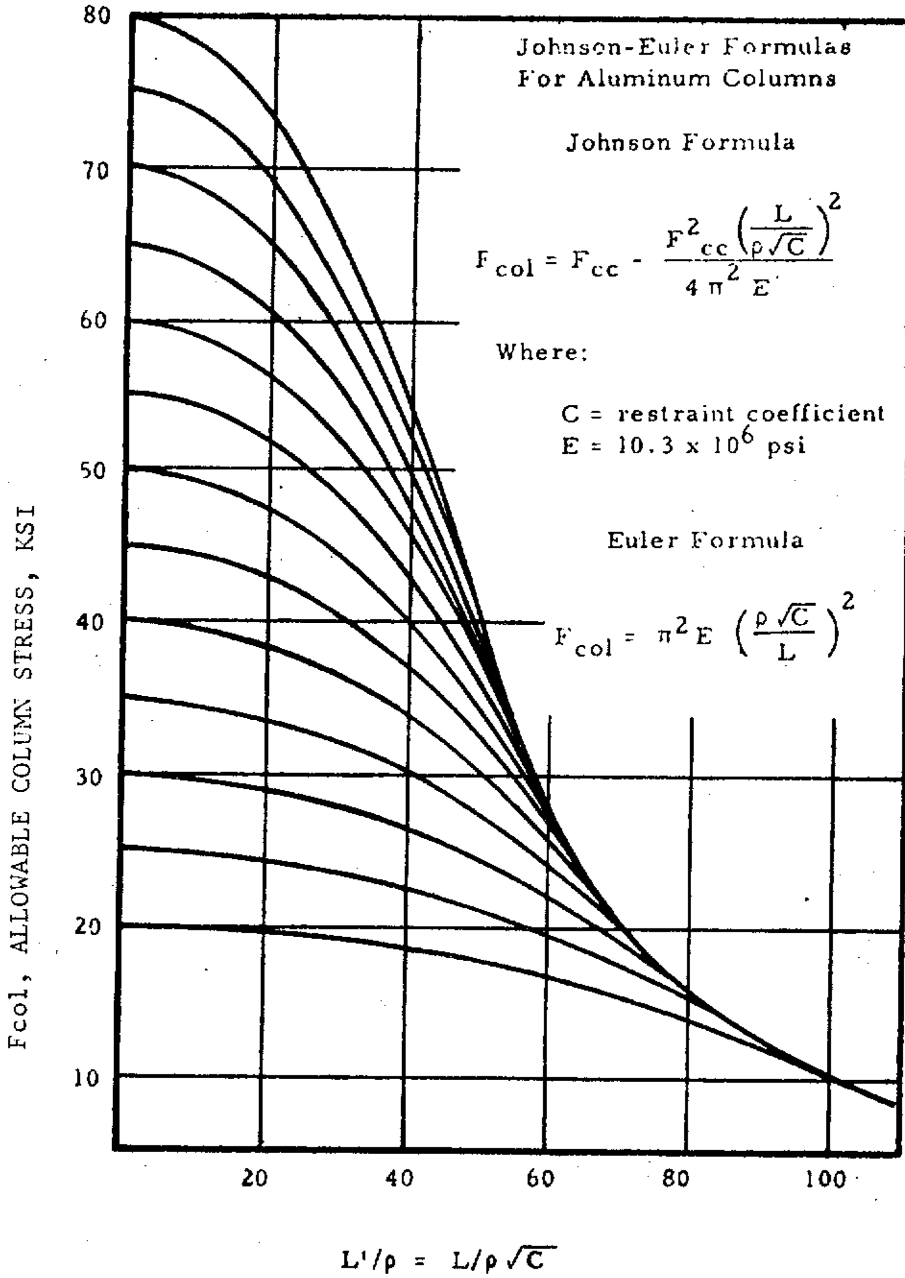


FIGURE 11.30 - JOHNSON-EULER COLUMN CURVES FOR ALUMINUM ALLOYS

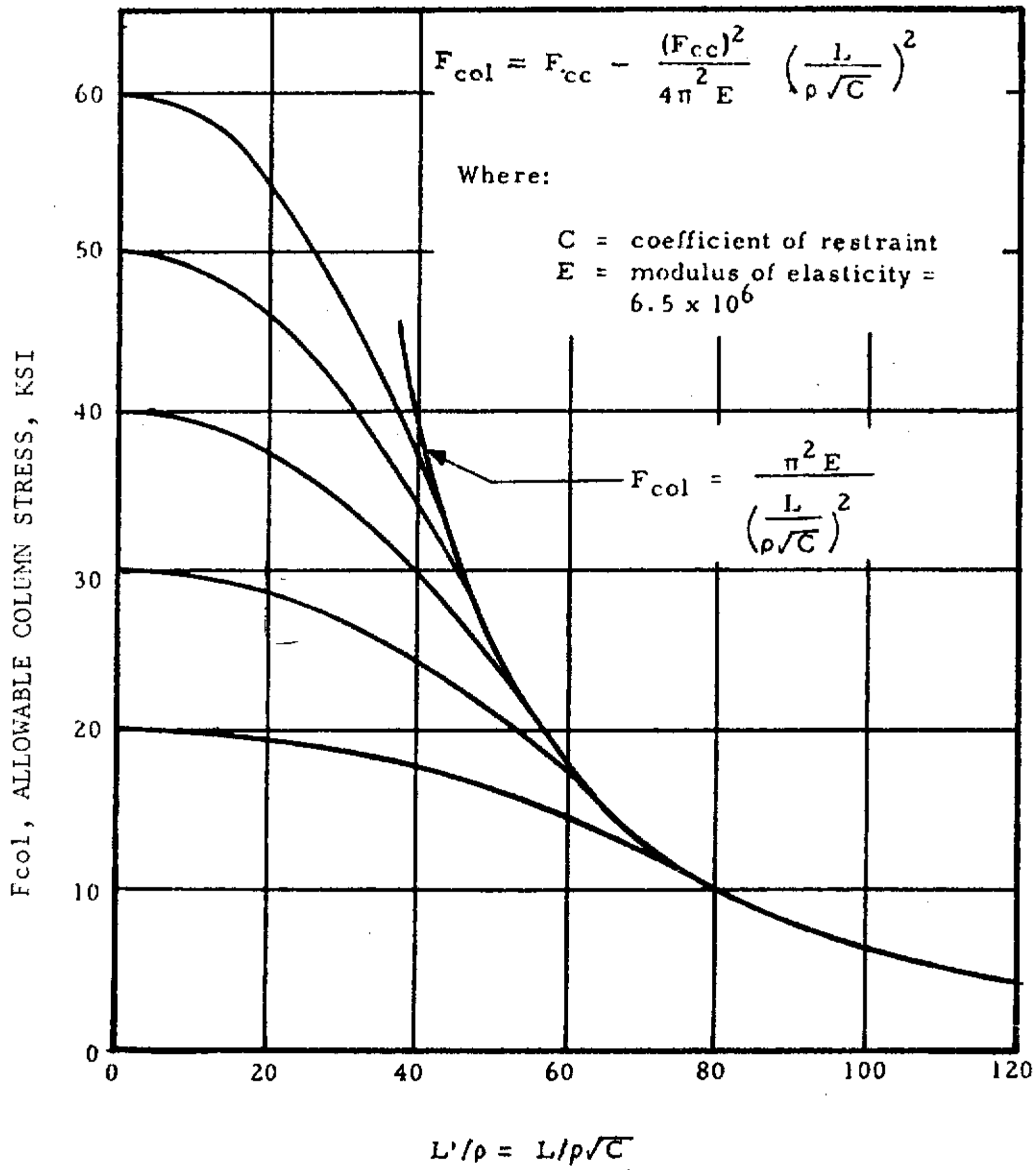


FIGURE 11.31 - JOHNSON-EULER COLUMN CURVES FOR MAGNESIUM ALLOYS

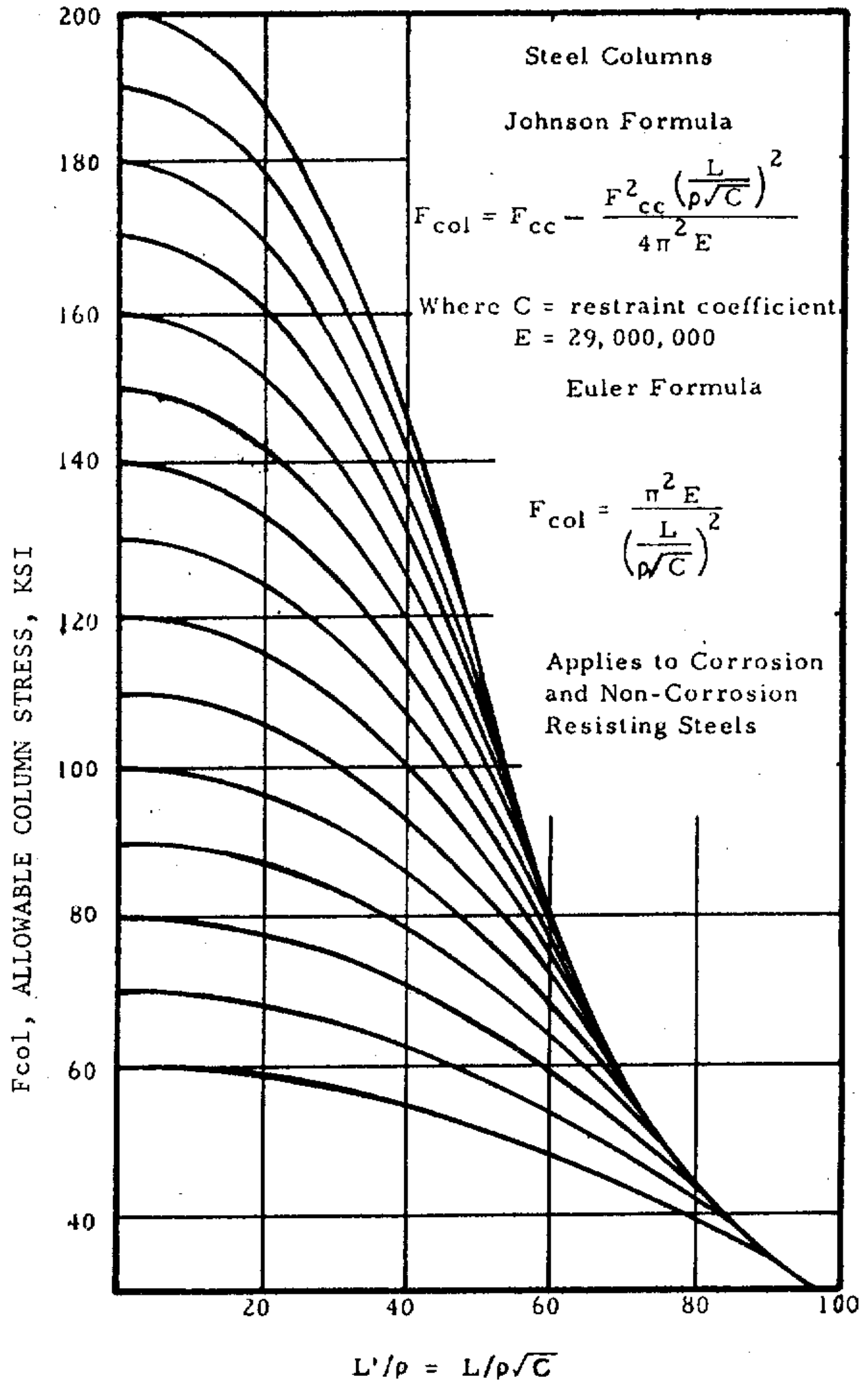


FIGURE 11.32 - JOHNSON-EULER COLUMN CURVES FOR STEEL ALLOYS

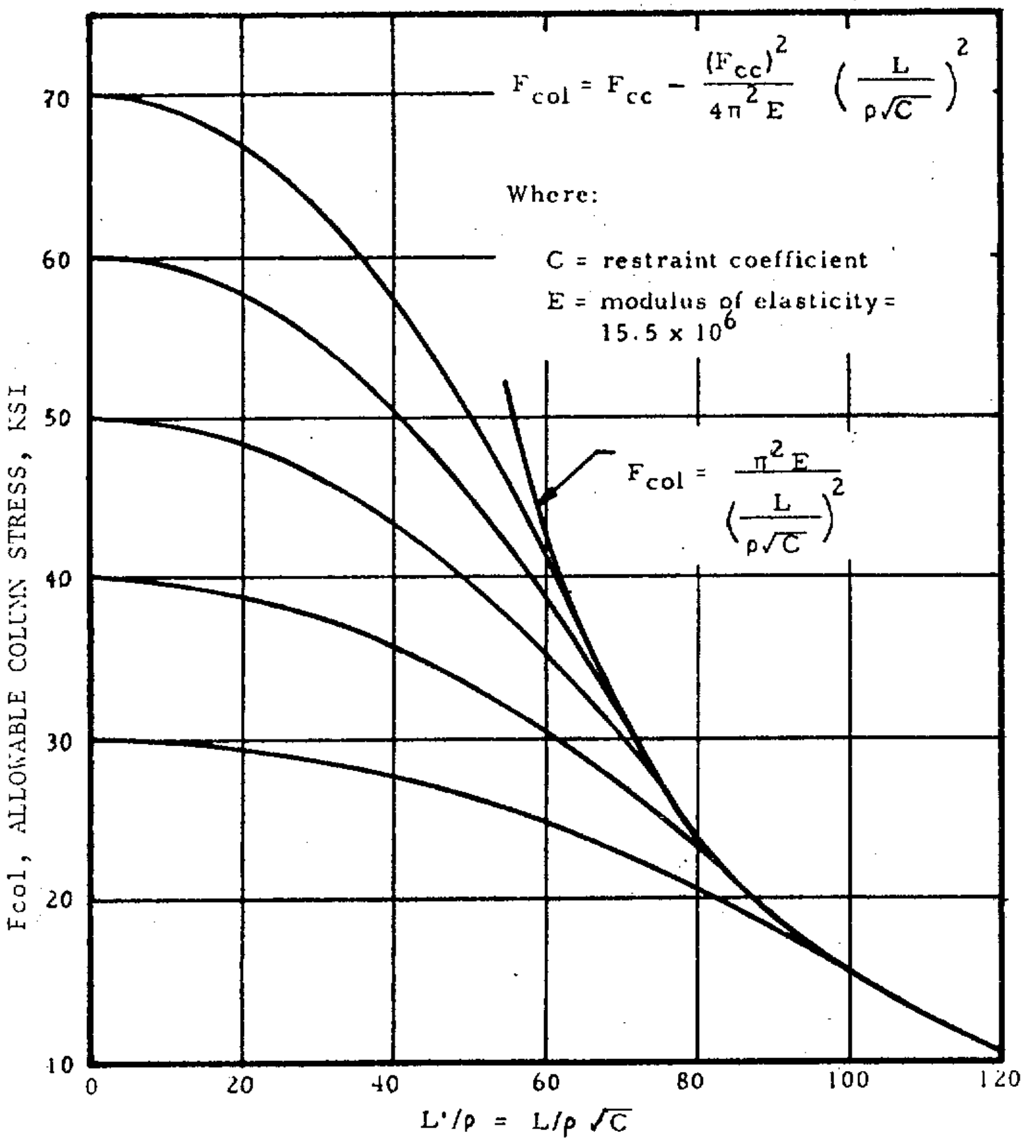


FIGURE 11.33 - JOHNSON-EULER COLUMN CURVES FOR COMMERCIALLY PURE TITANIUM

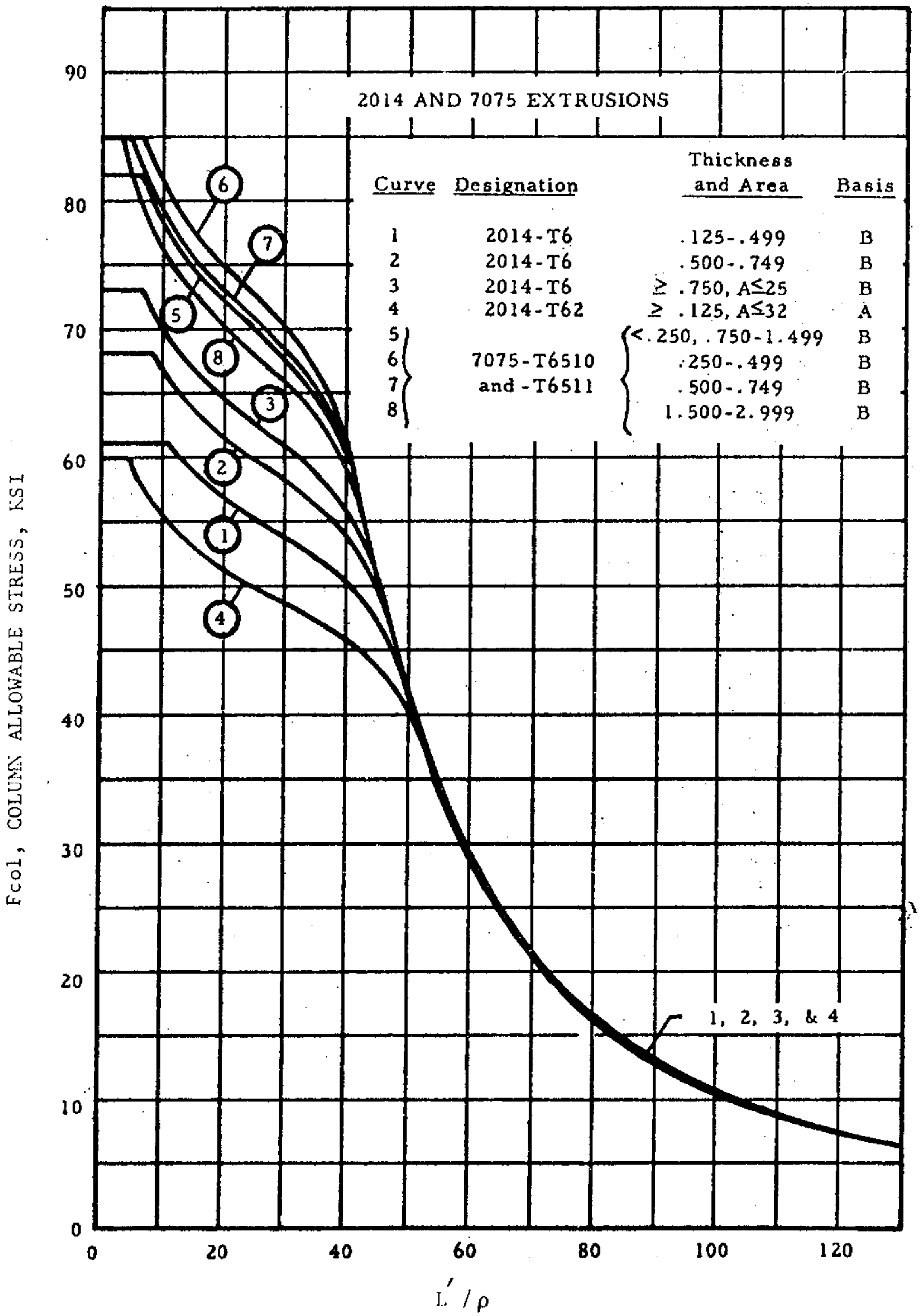
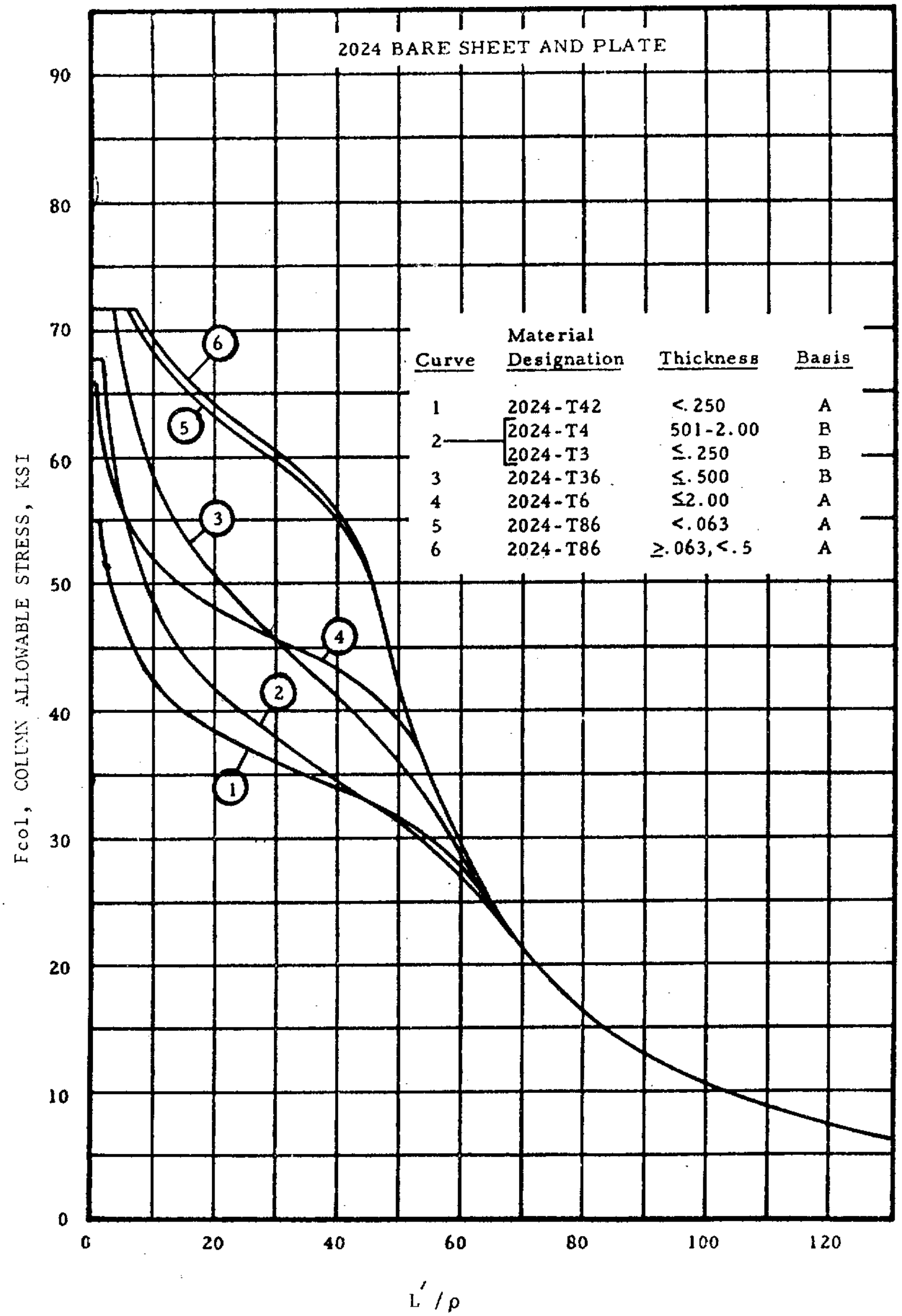


FIGURE 11.34 - COLUMN ALLOWABLE CURVES



Curve	Material Designation	Thickness	Basis
1	2024-T42	<.250	A
2	2024-T4	501-2.00	B
2	2024-T3	≤.250	B
3	2024-T36	≤.500	B
4	2024-T6	≤2.00	A
5	2024-T86	<.063	A
6	2024-T86	≥.063,<.5	A

FIGURE 11.35 - COLUMN ALLOWABLE CURVES

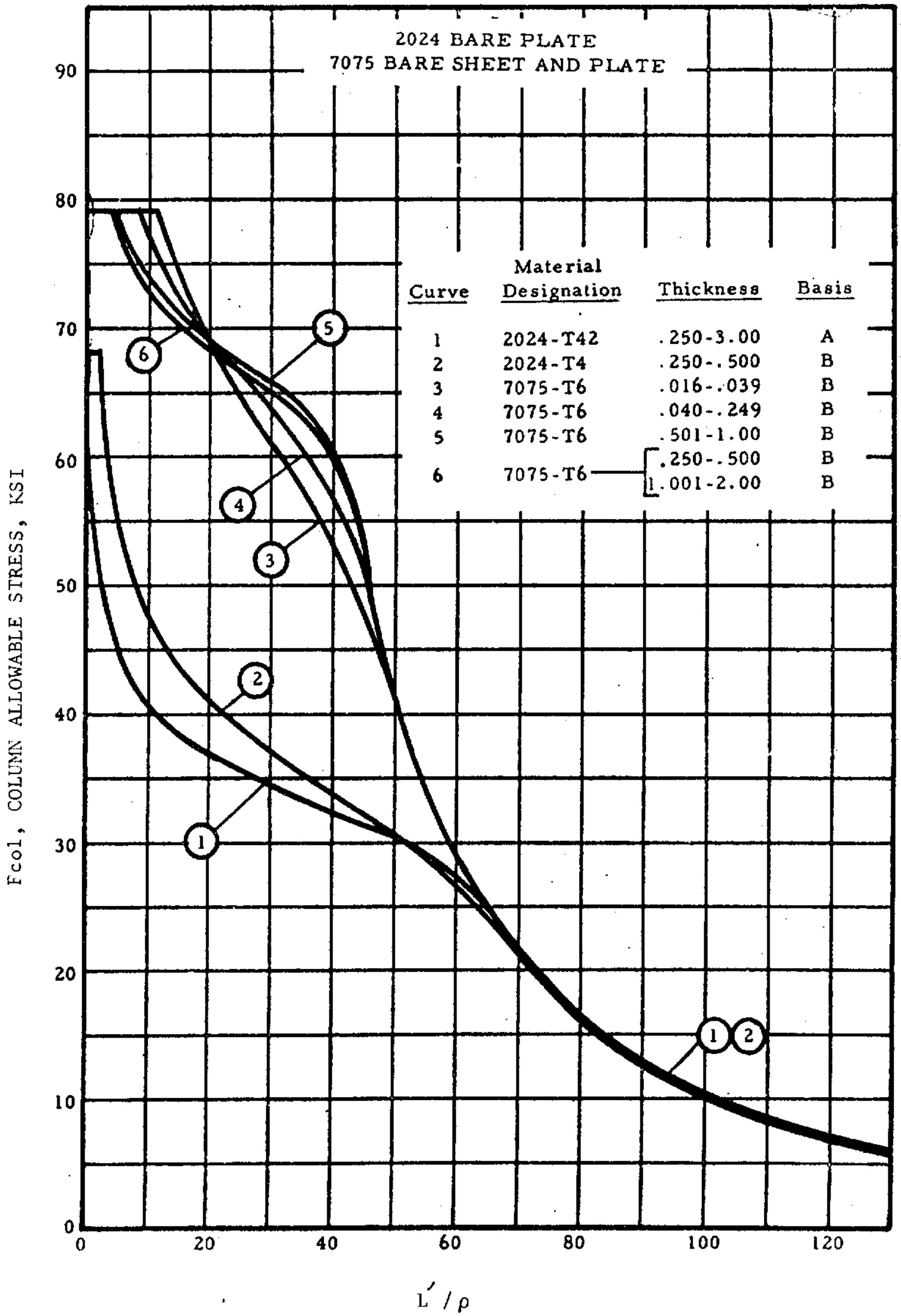


FIGURE 11.36 - COLUMN ALLOWABLE CURVES

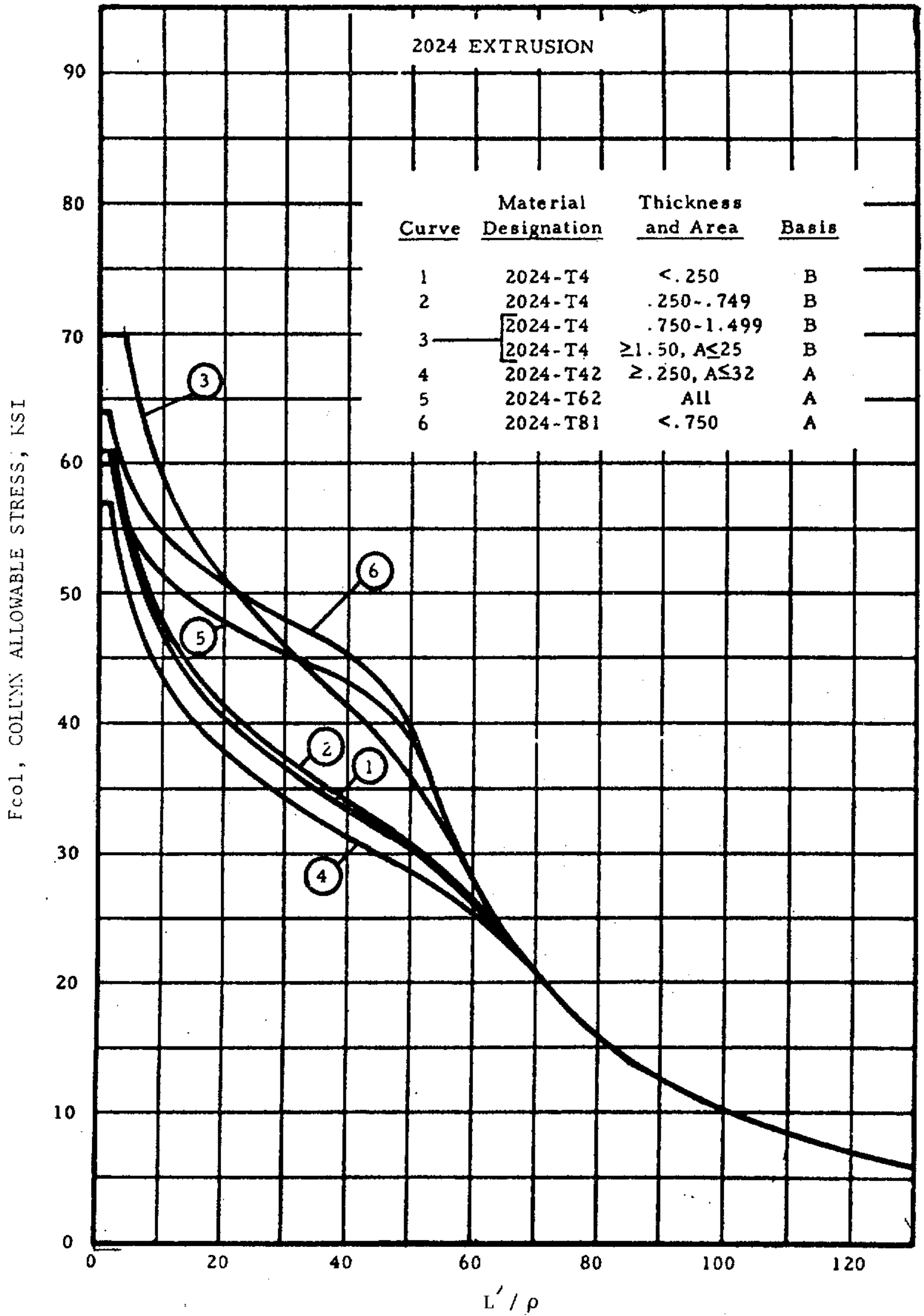


FIGURE 11.37 - COLUMN ALLOWABLE CURVES

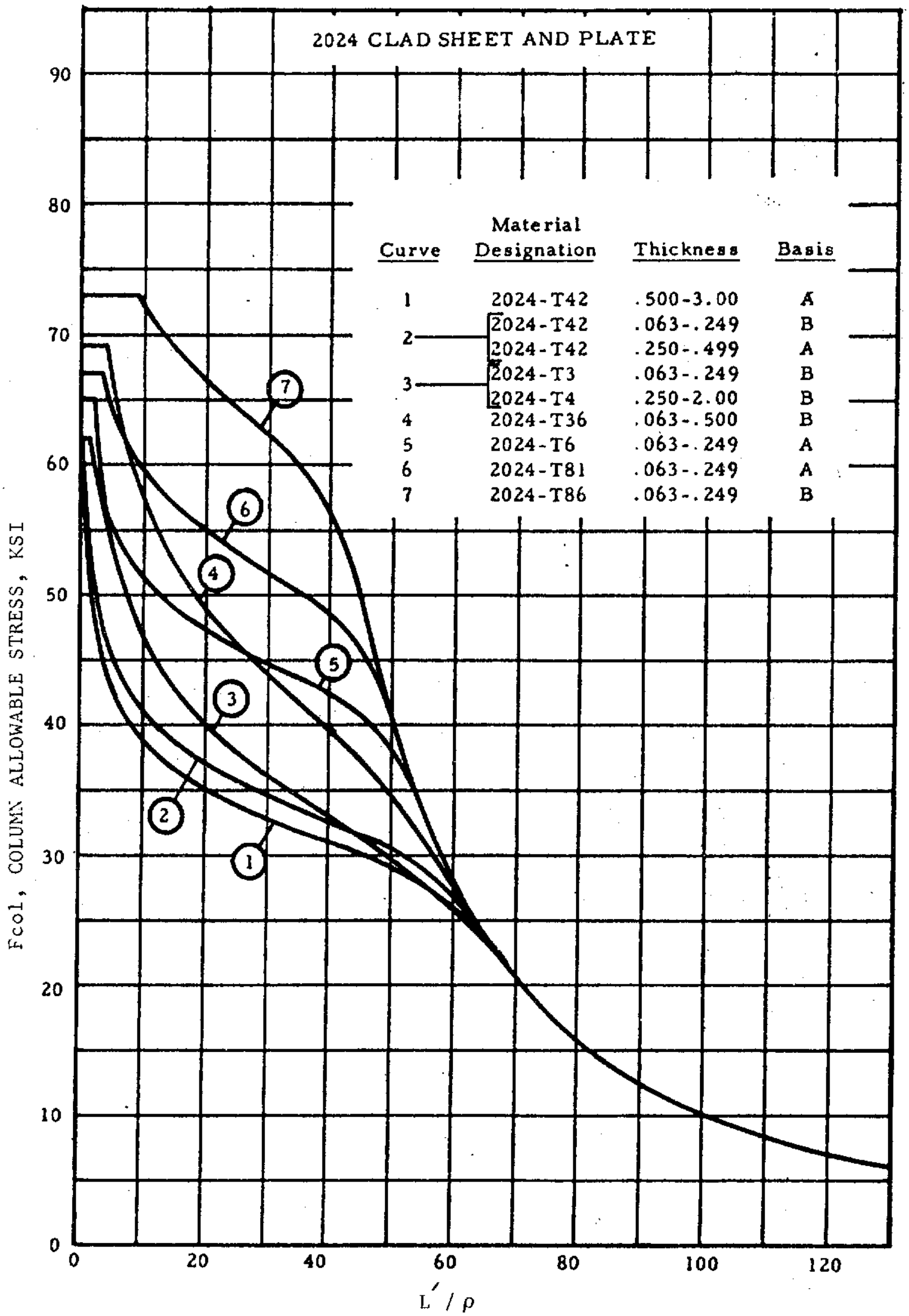


FIGURE 11.38 - COLUMN ALLOWABLE CURVES

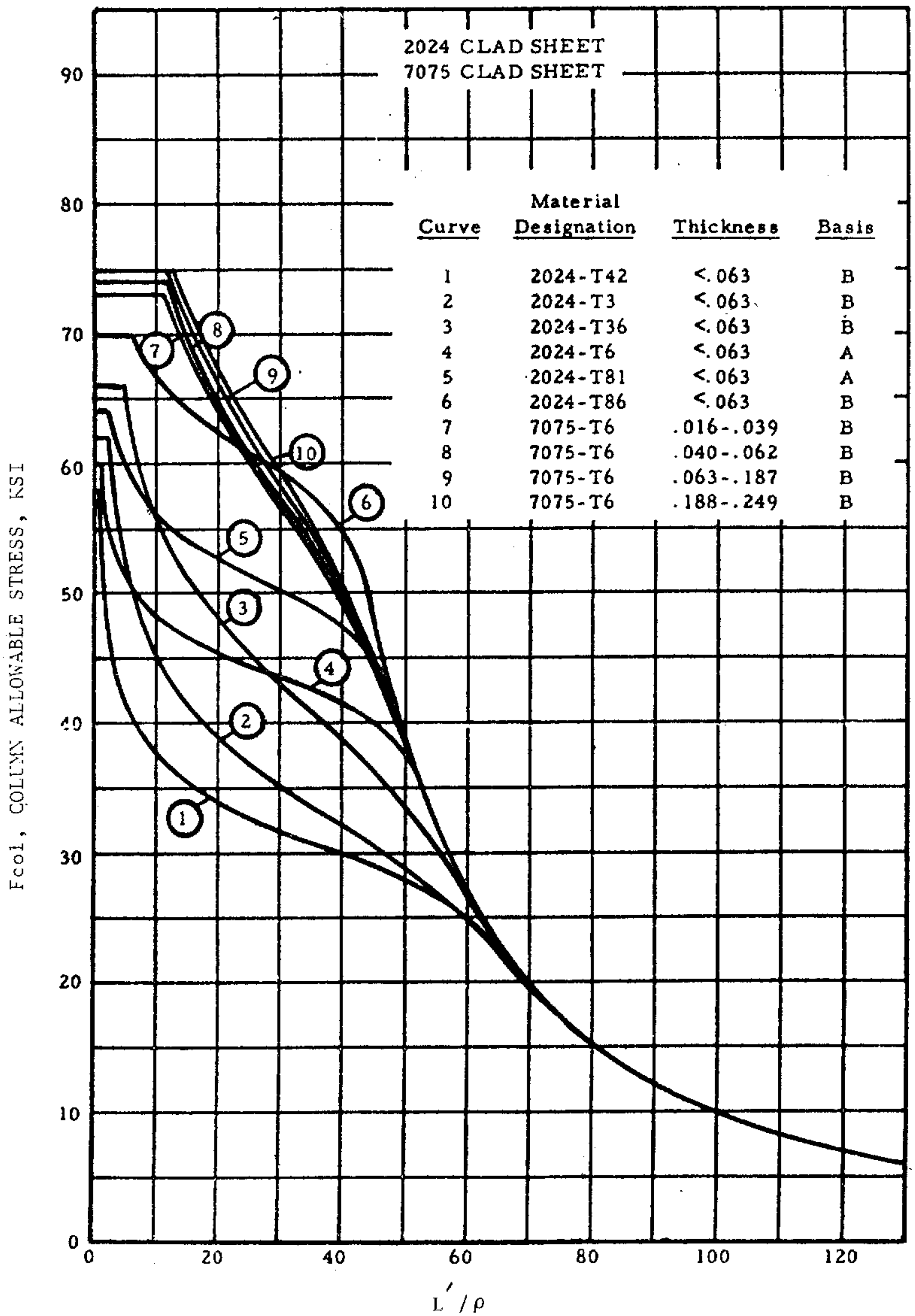


FIGURE 11.39 - COLUMN ALLOWABLE CURVES

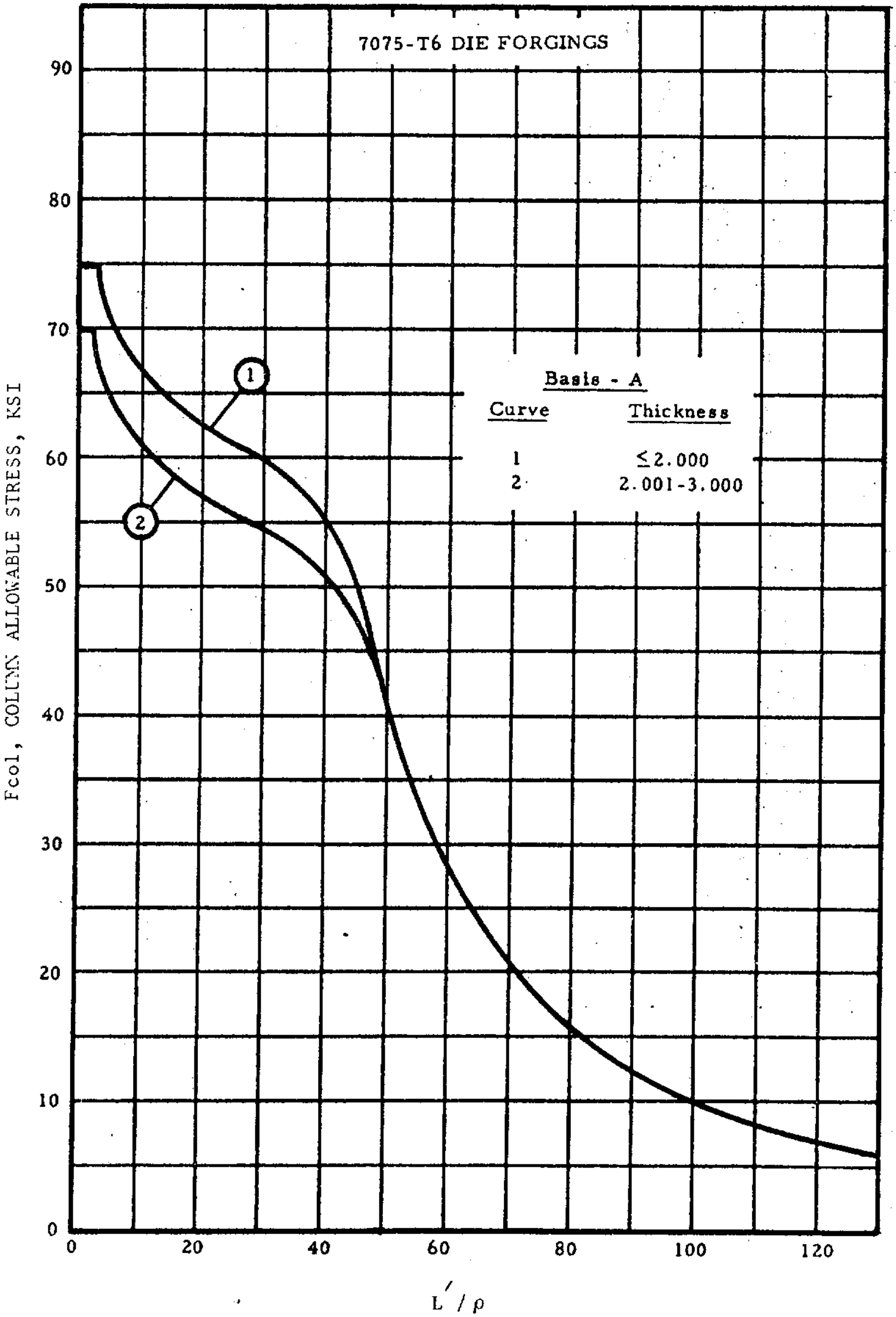


FIGURE 11.40 - COLUMN ALLOWABLE CURVES

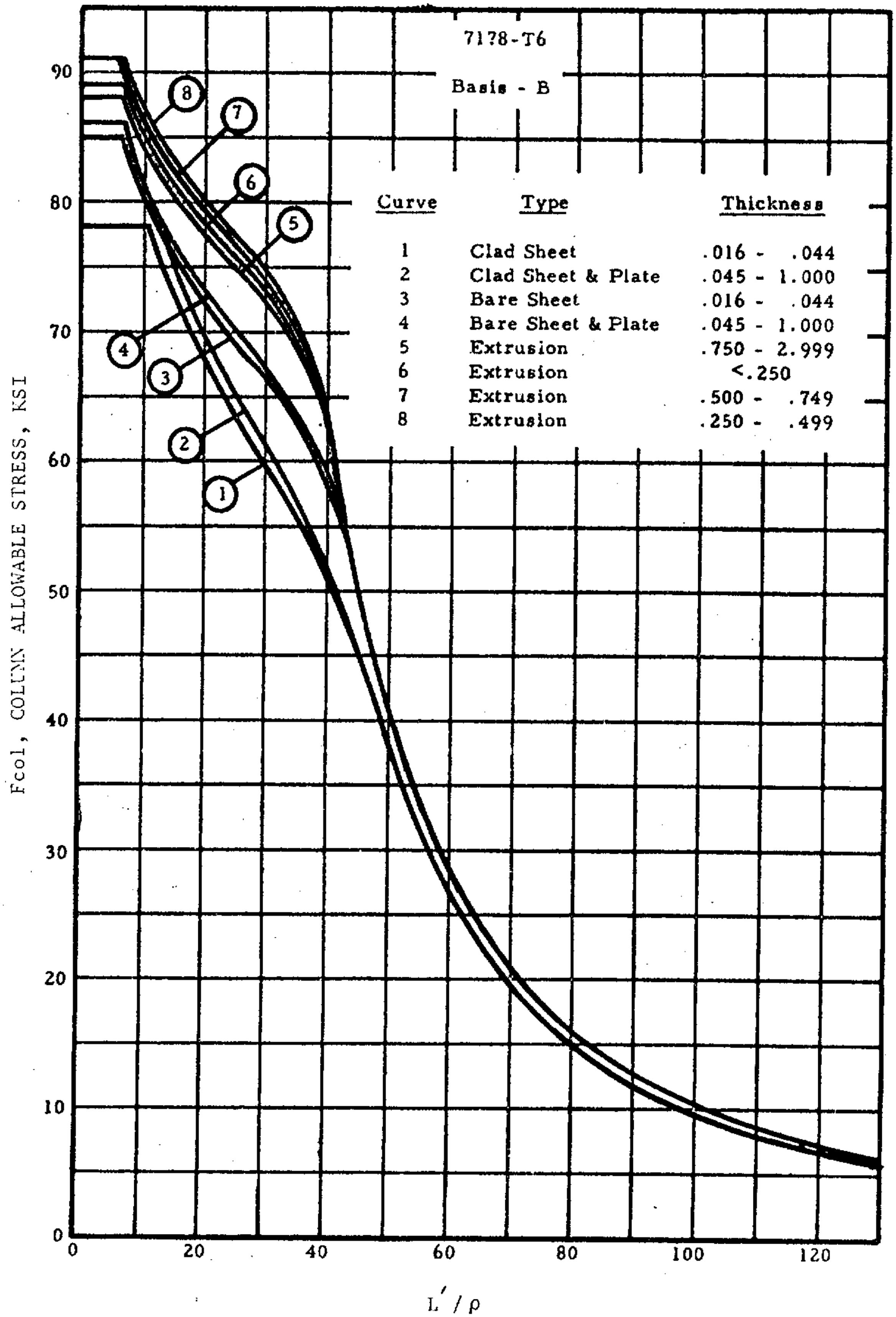


FIGURE 11.41 - COLUMN ALLOWABLE CURVES

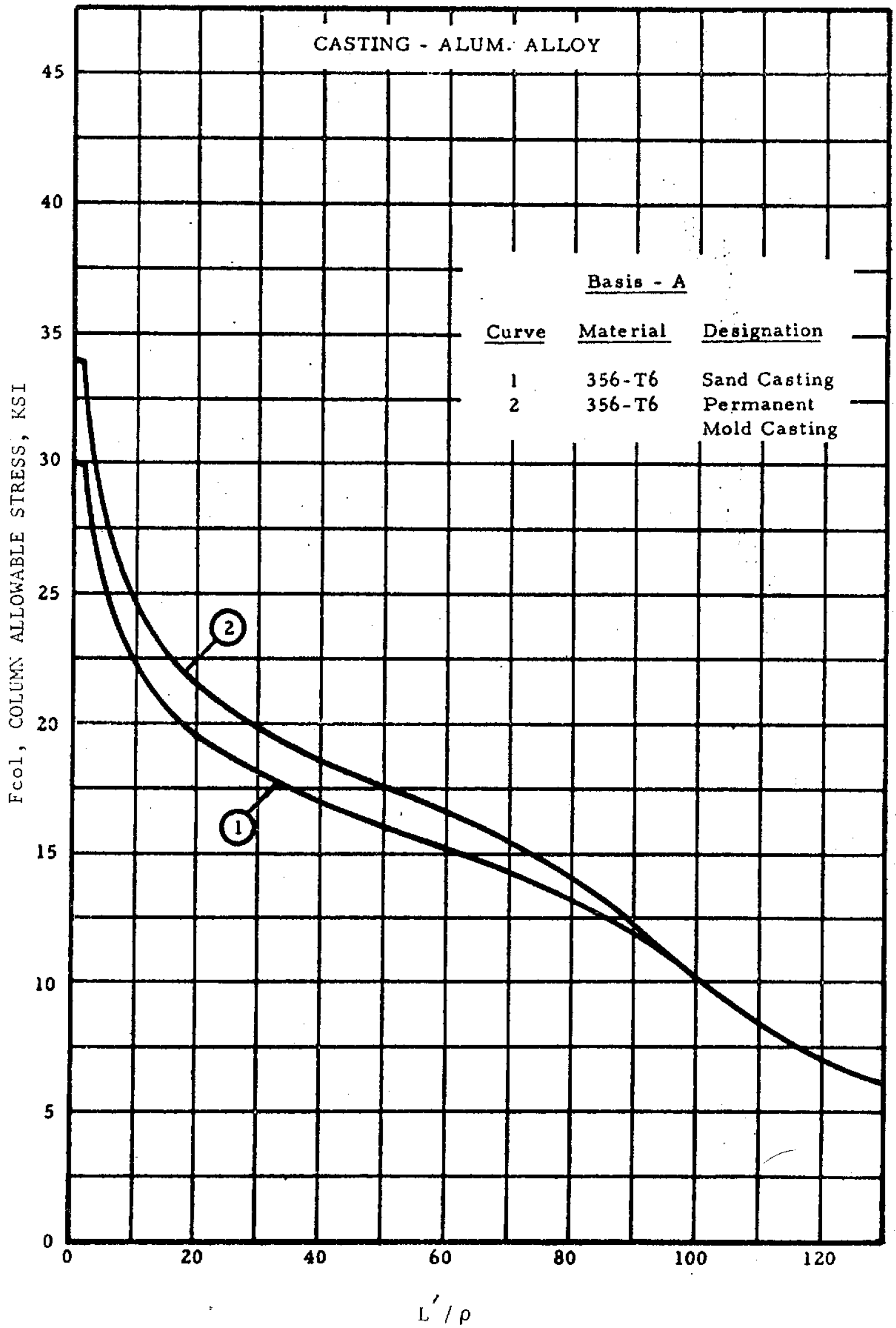


FIGURE 11.42 - COLUMN ALLOWABLE CURVES

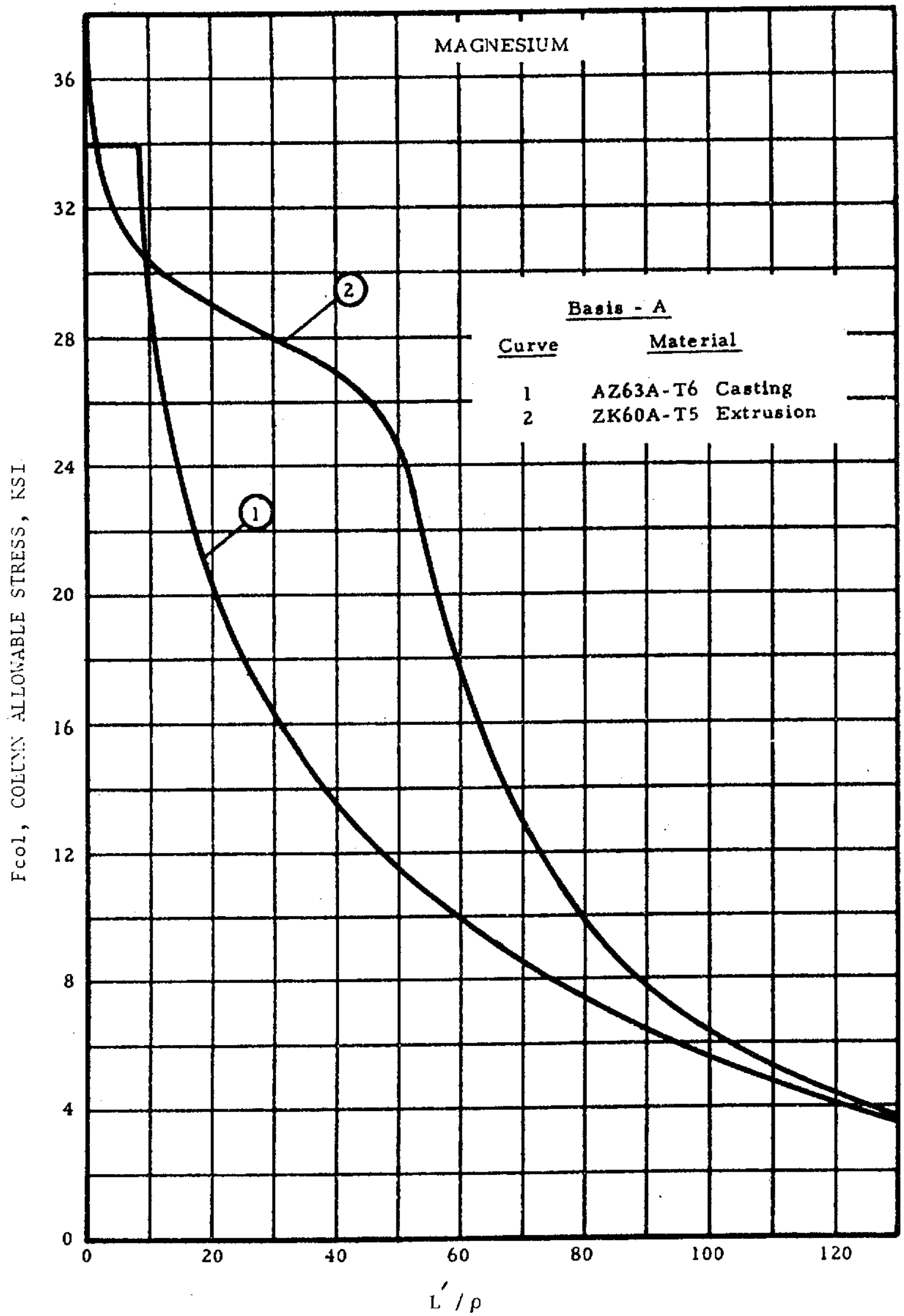


FIGURE 11.43 - COLUMN ALLOWABLE CURVES

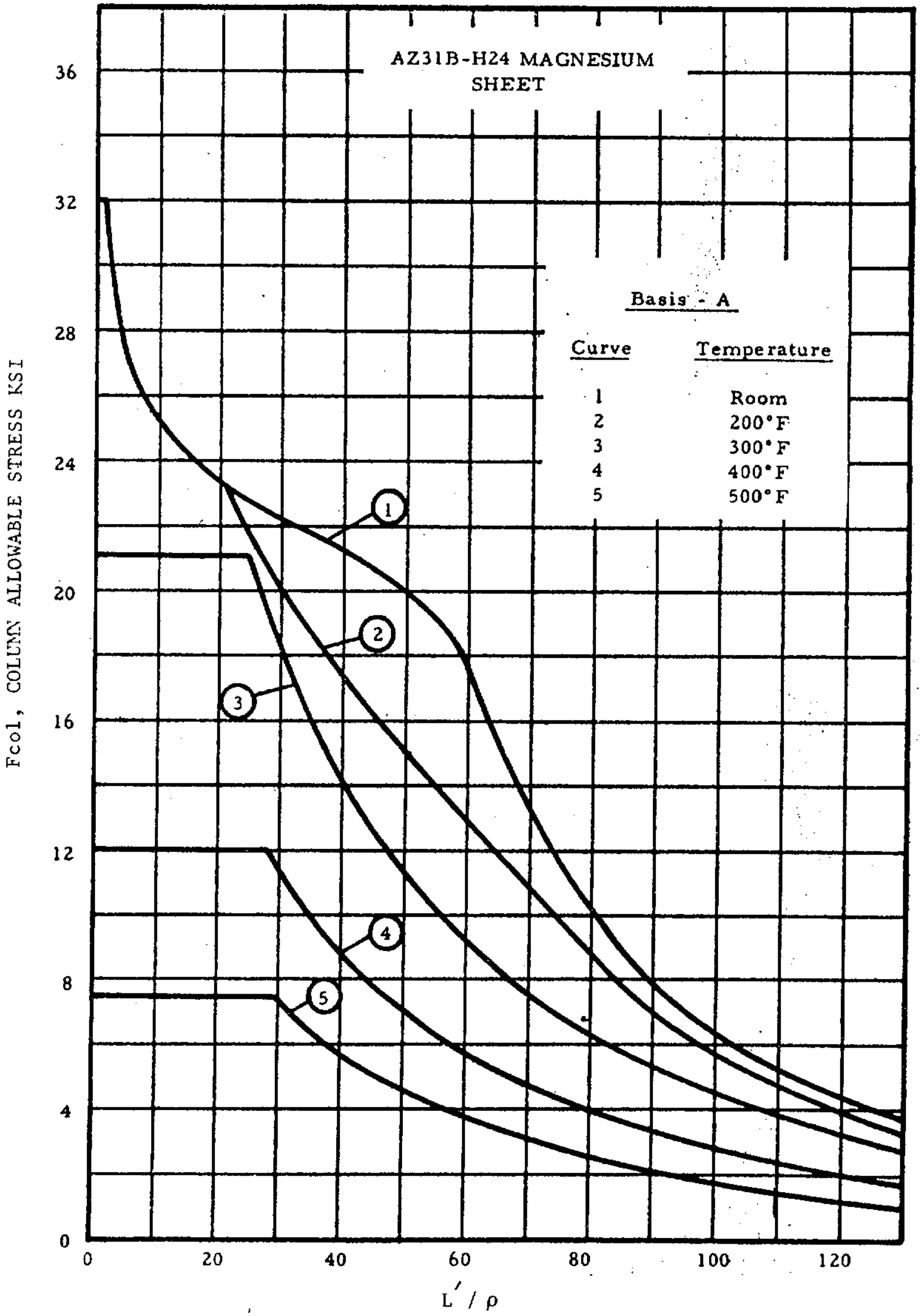


FIGURE 11.44 - COLUMN ALLOWABLE CURVES

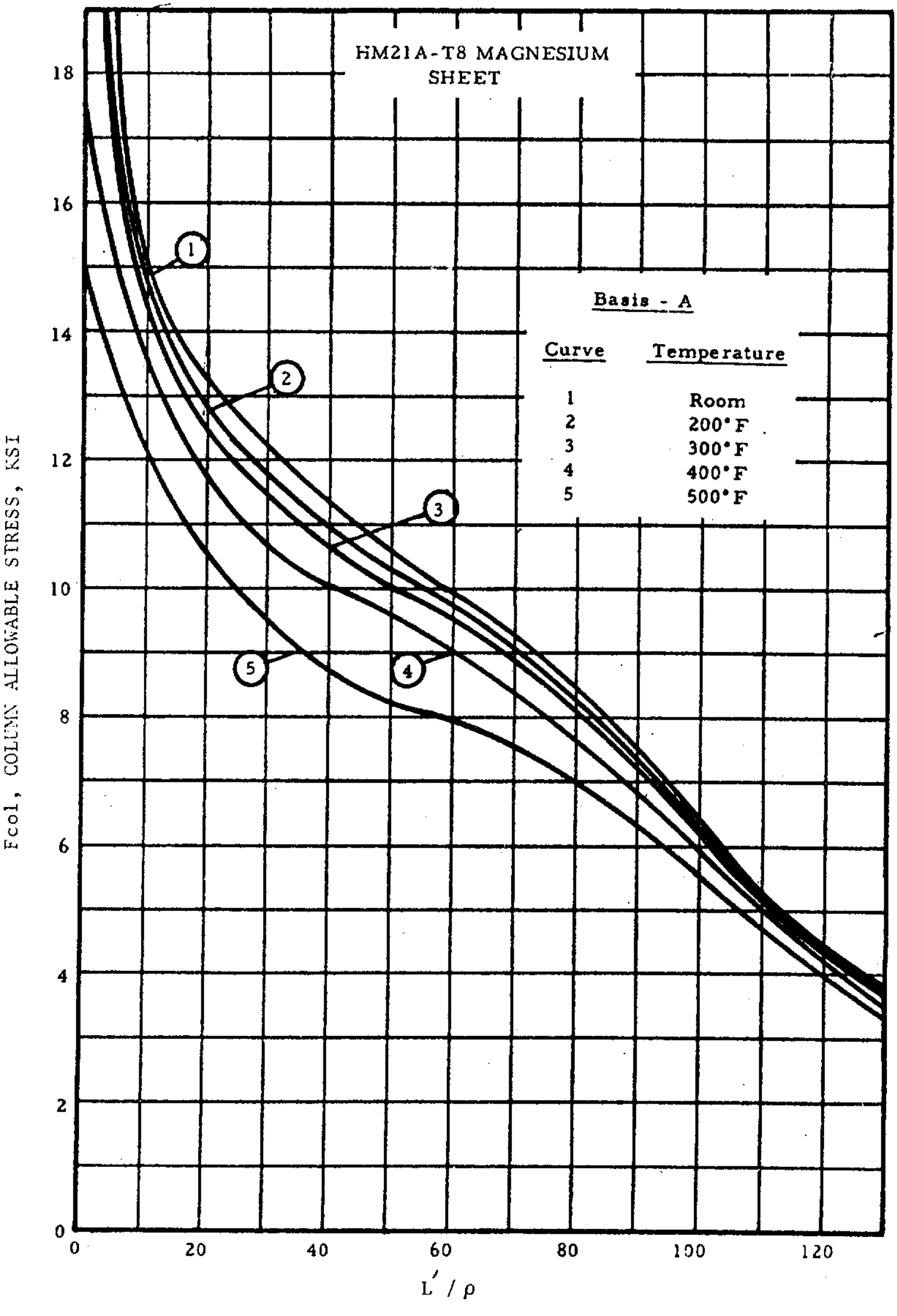


FIGURE 11.45 - COLUMN ALLOWABLE CURVES

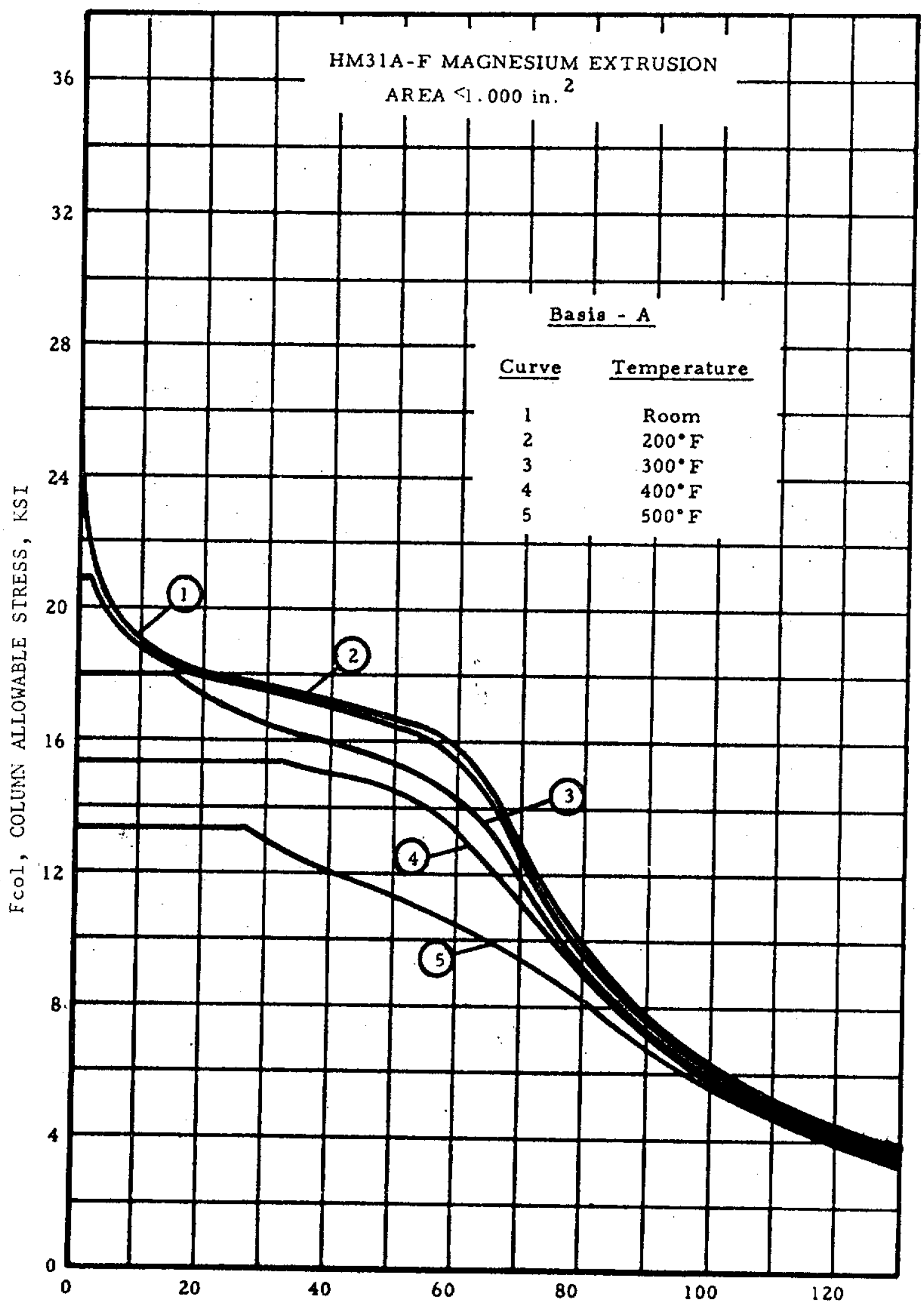


FIGURE 11.46 - COLUMN ALLOWABLE CURVES

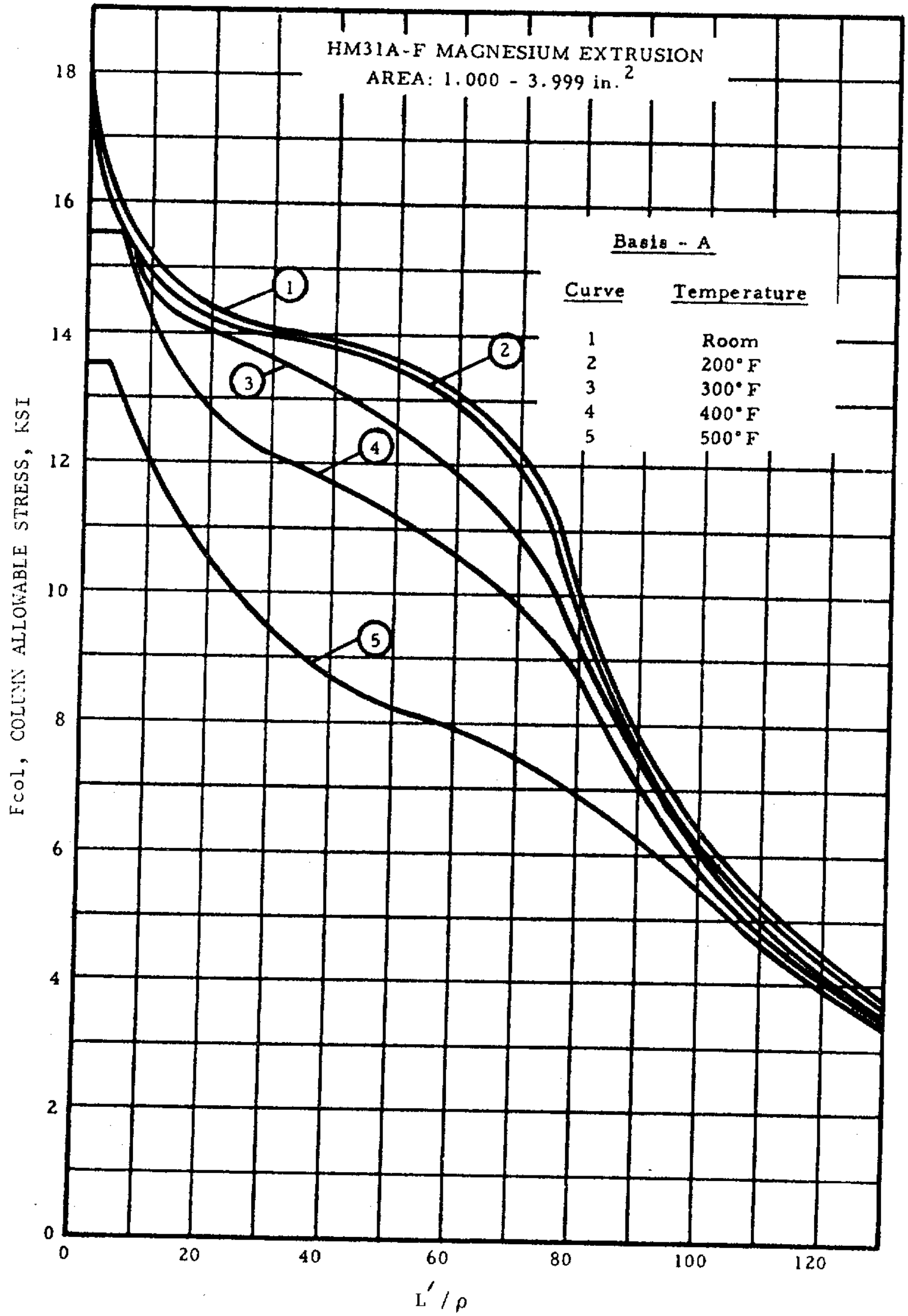


FIGURE 11.47 - COLUMN ALLOWABLE CURVES

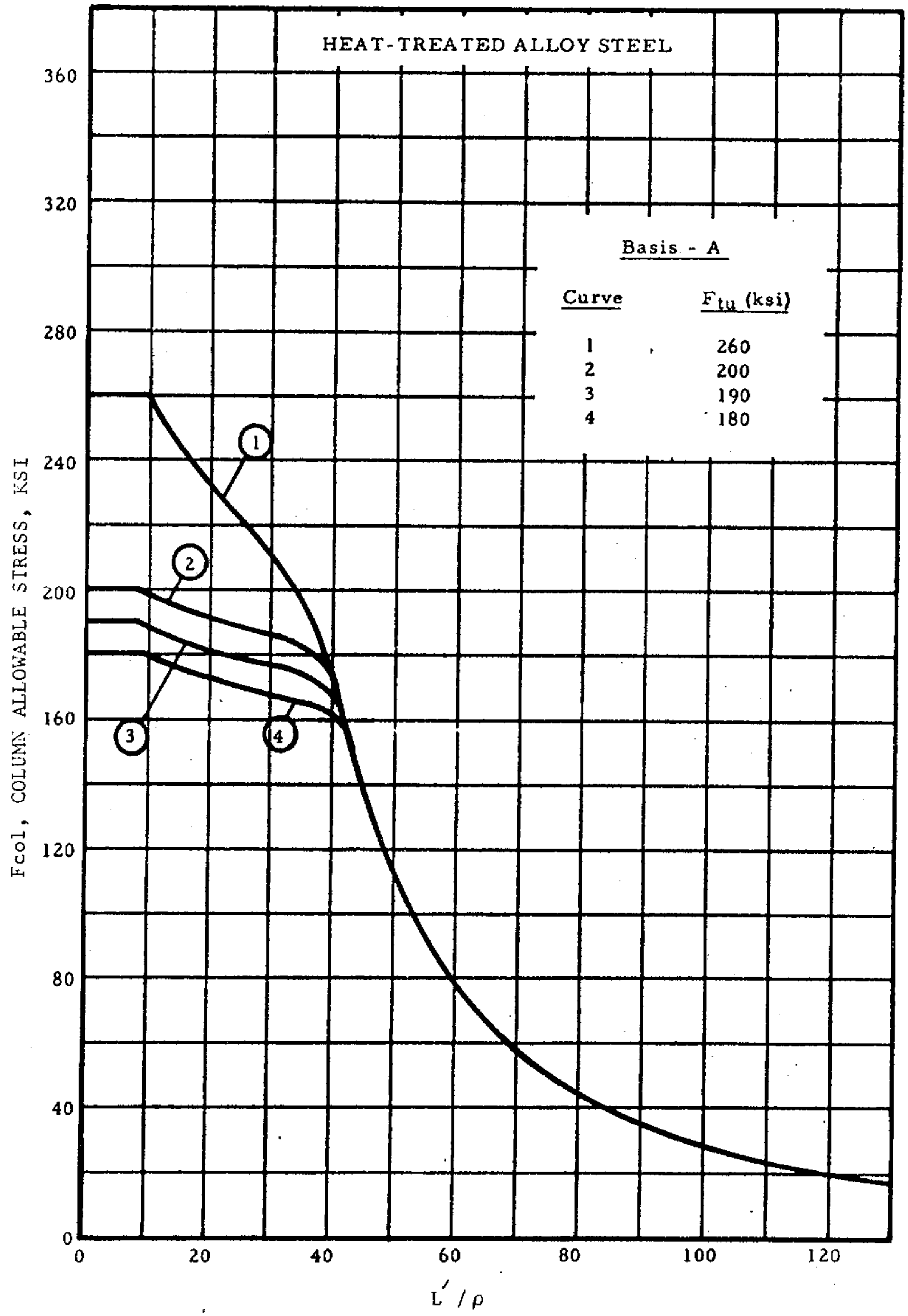


FIGURE 11.48 - COLUMN ALLOWABLE CURVES

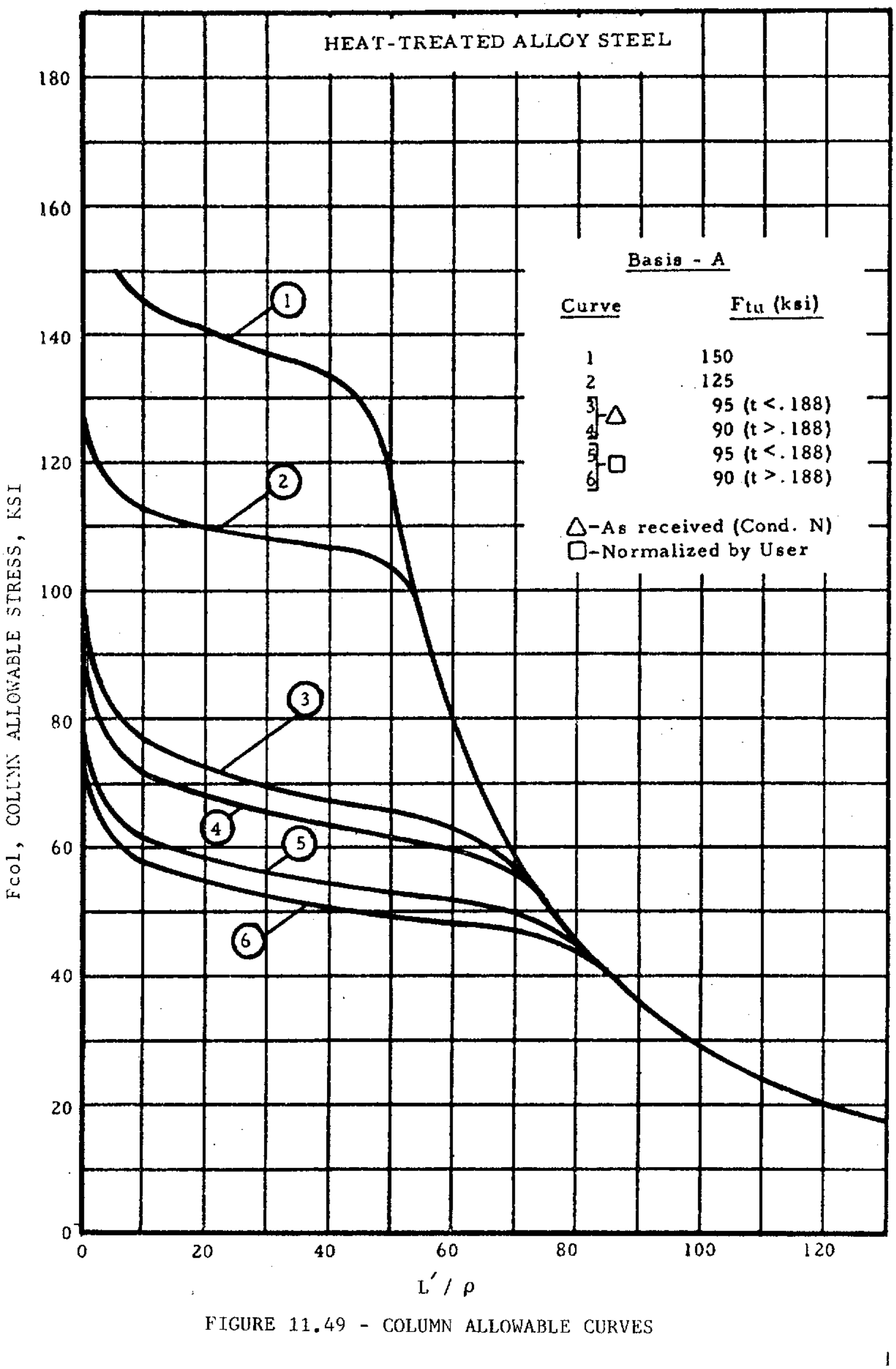


FIGURE 11.49 - COLUMN ALLOWABLE CURVES

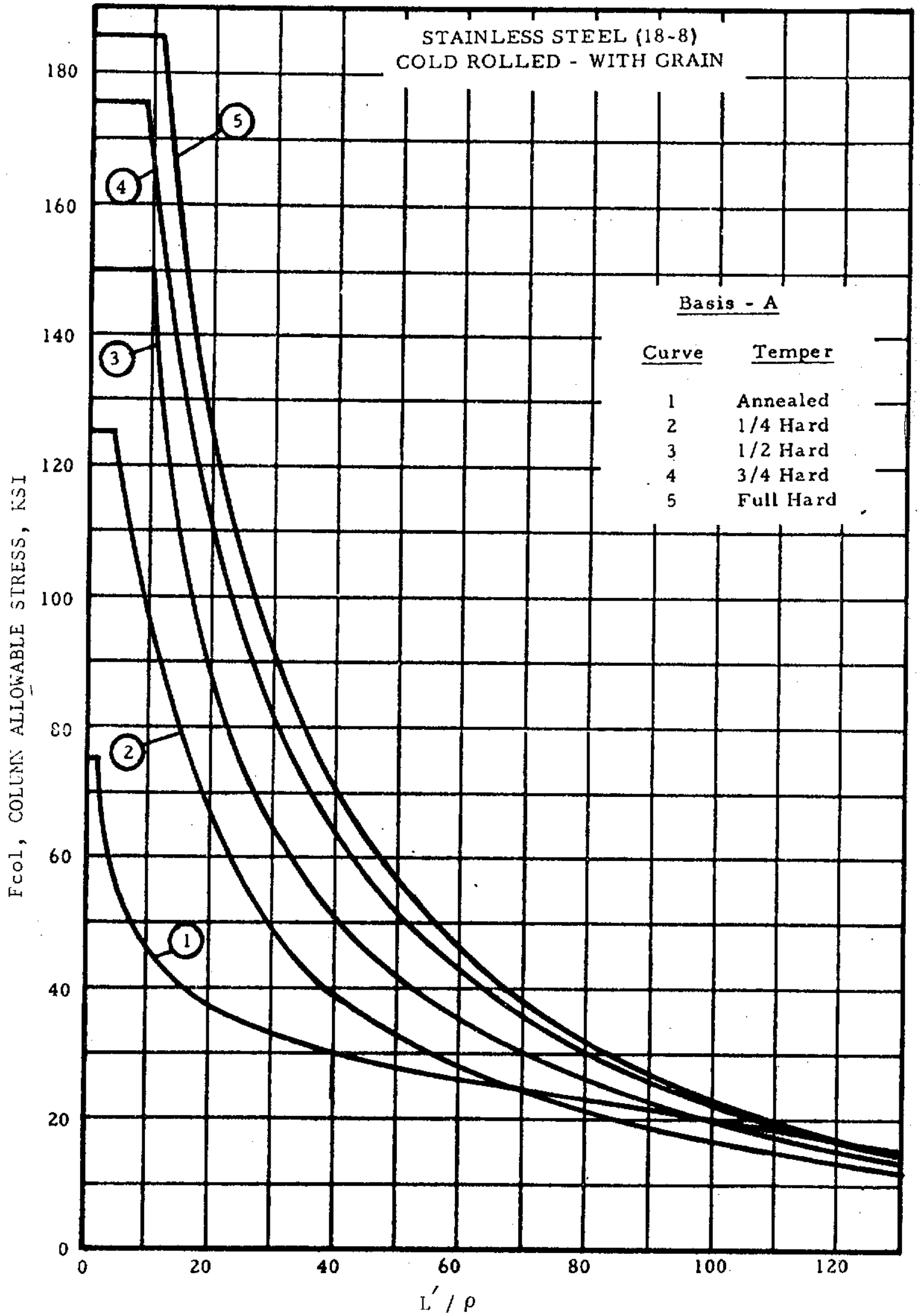


Figure 11.50 - COLUMN ALLOWABLE CURVES

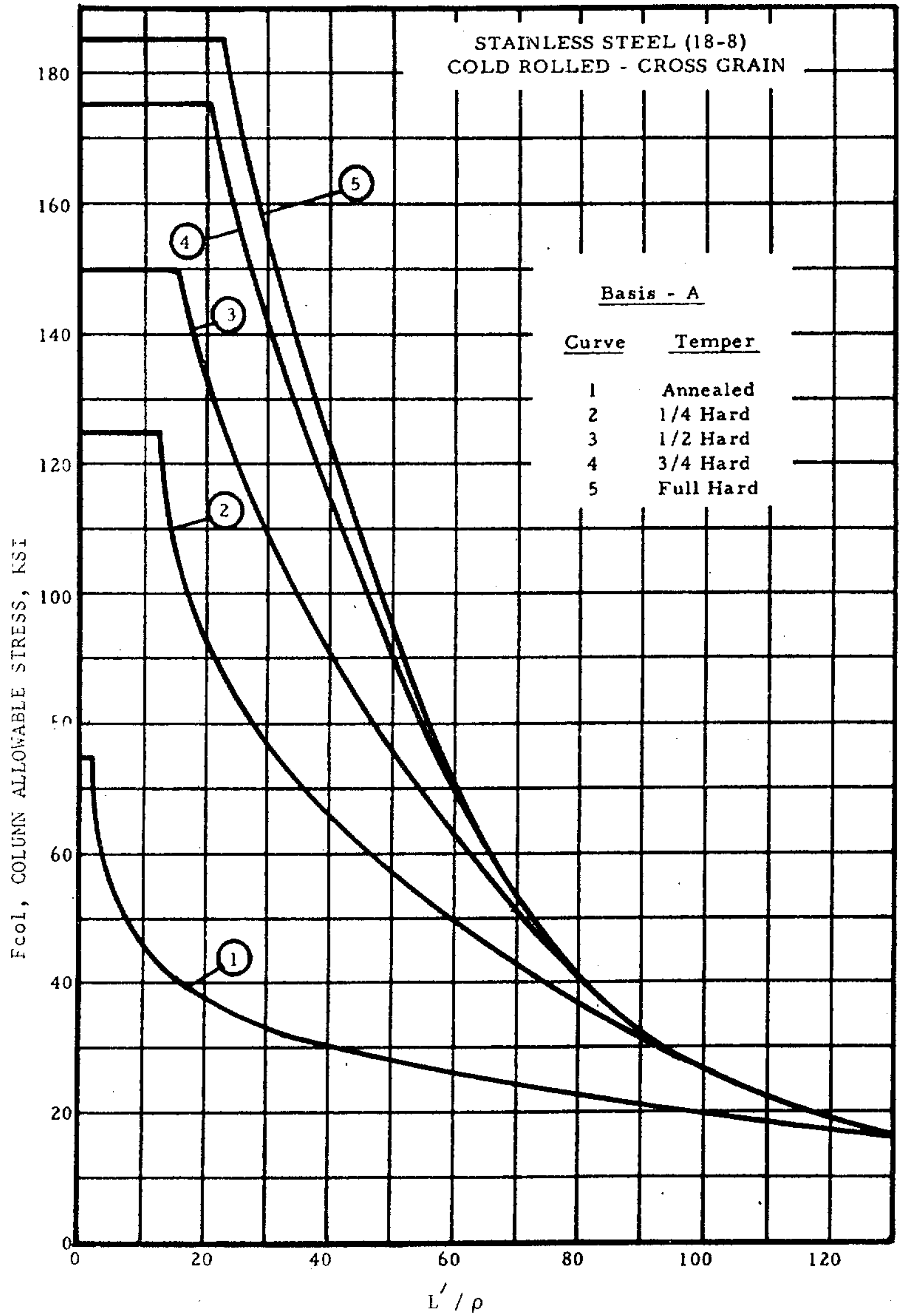


FIGURE 11.51 - COLUMN ALLOWABLE CURVES

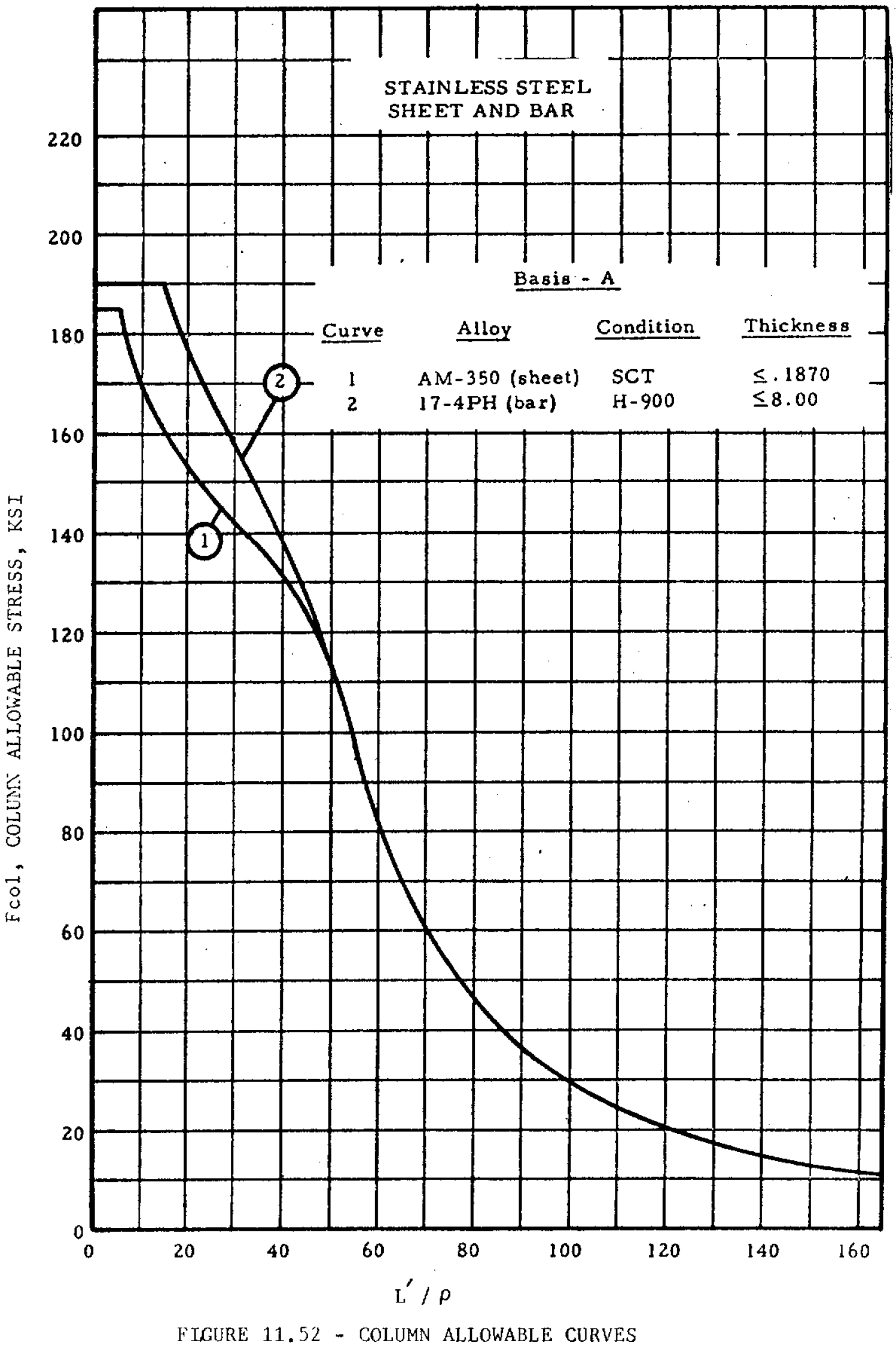


FIGURE 11.52 - COLUMN ALLOWABLE CURVES

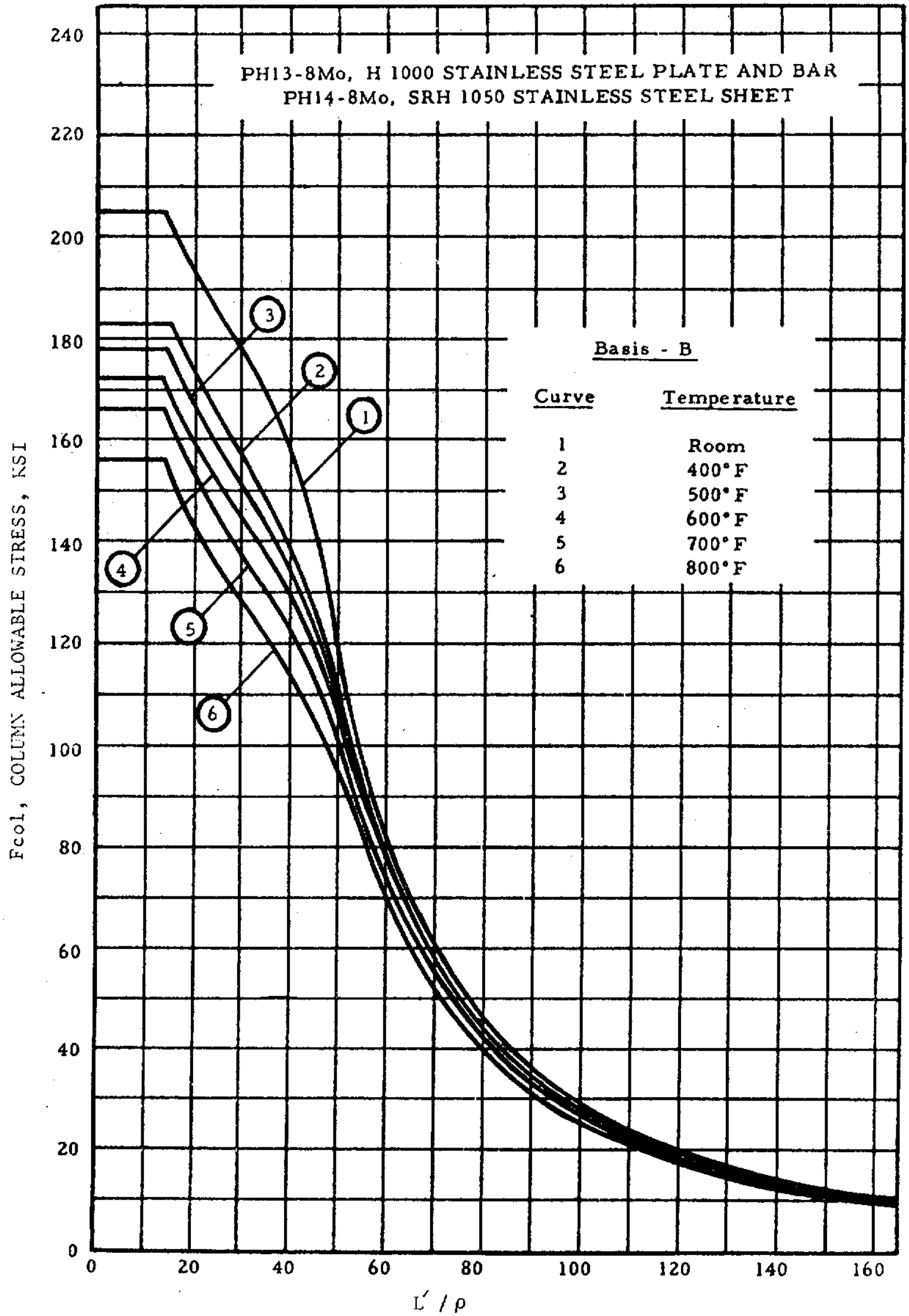


FIGURE 11.53 - COLUMN ALLOWABLE CURVES

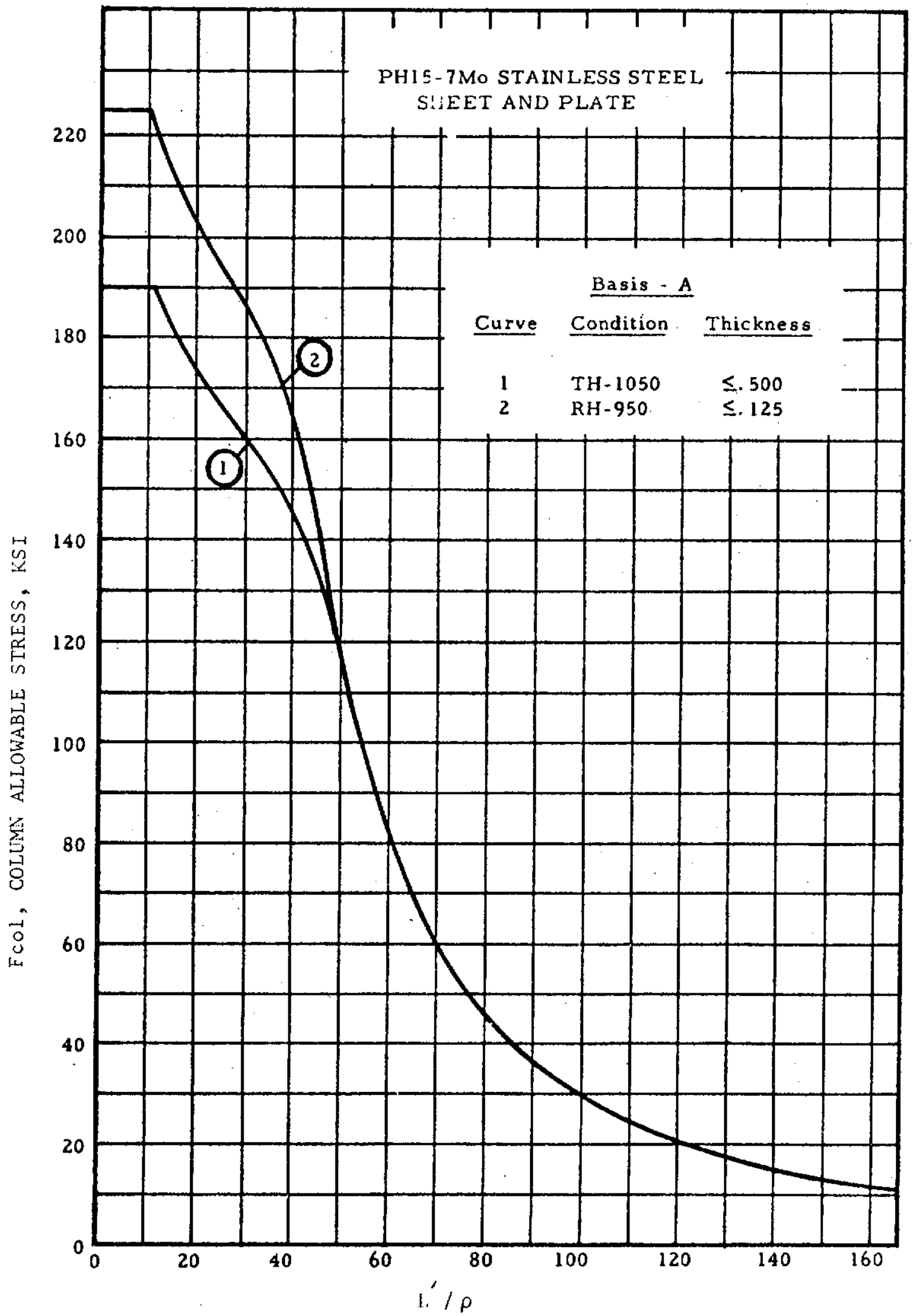


FIGURE 11.54 - COLUMN ALLOWABLE CURVES

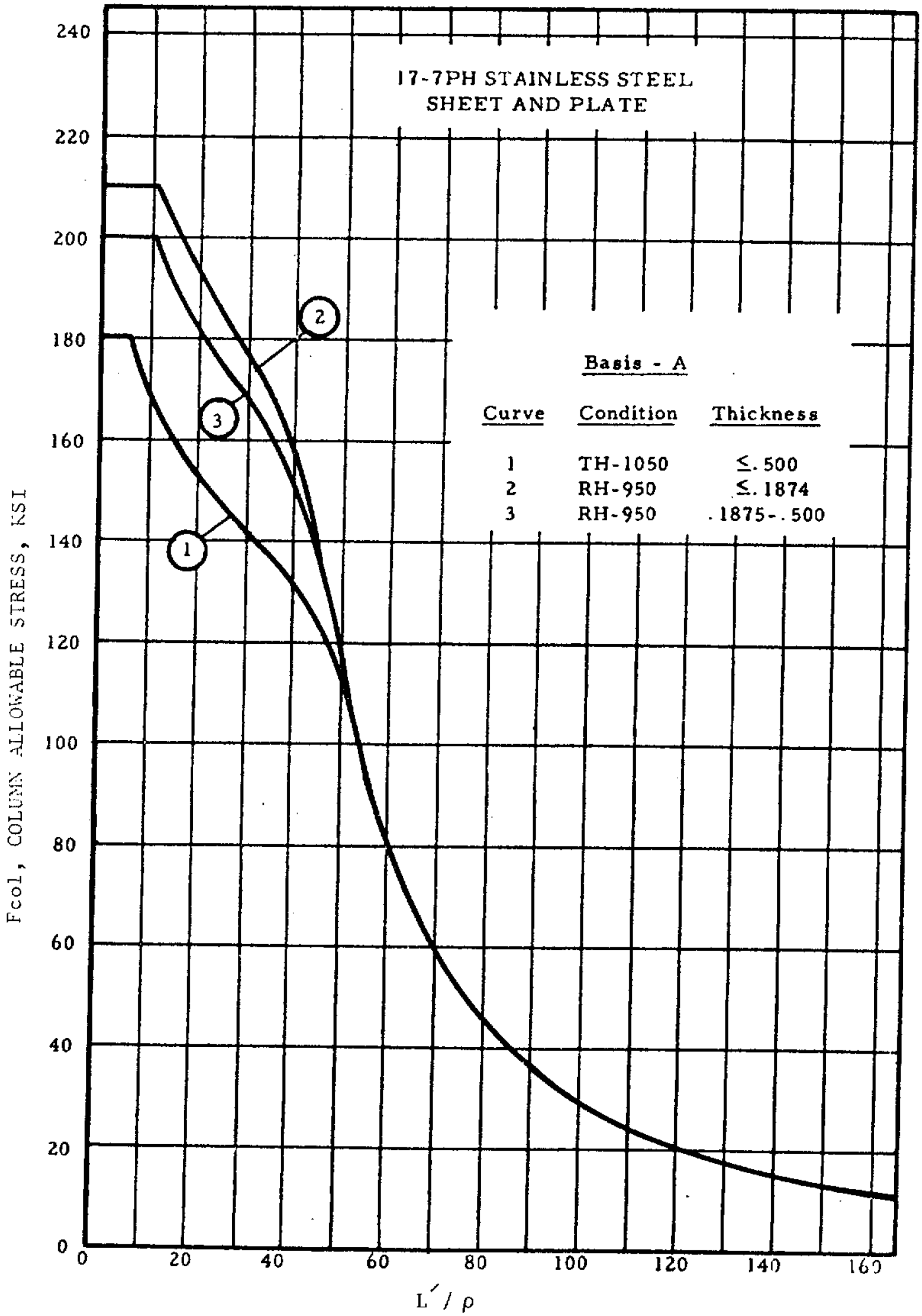


FIGURE 11.55 - COLUMN ALLOWABLE CURVES

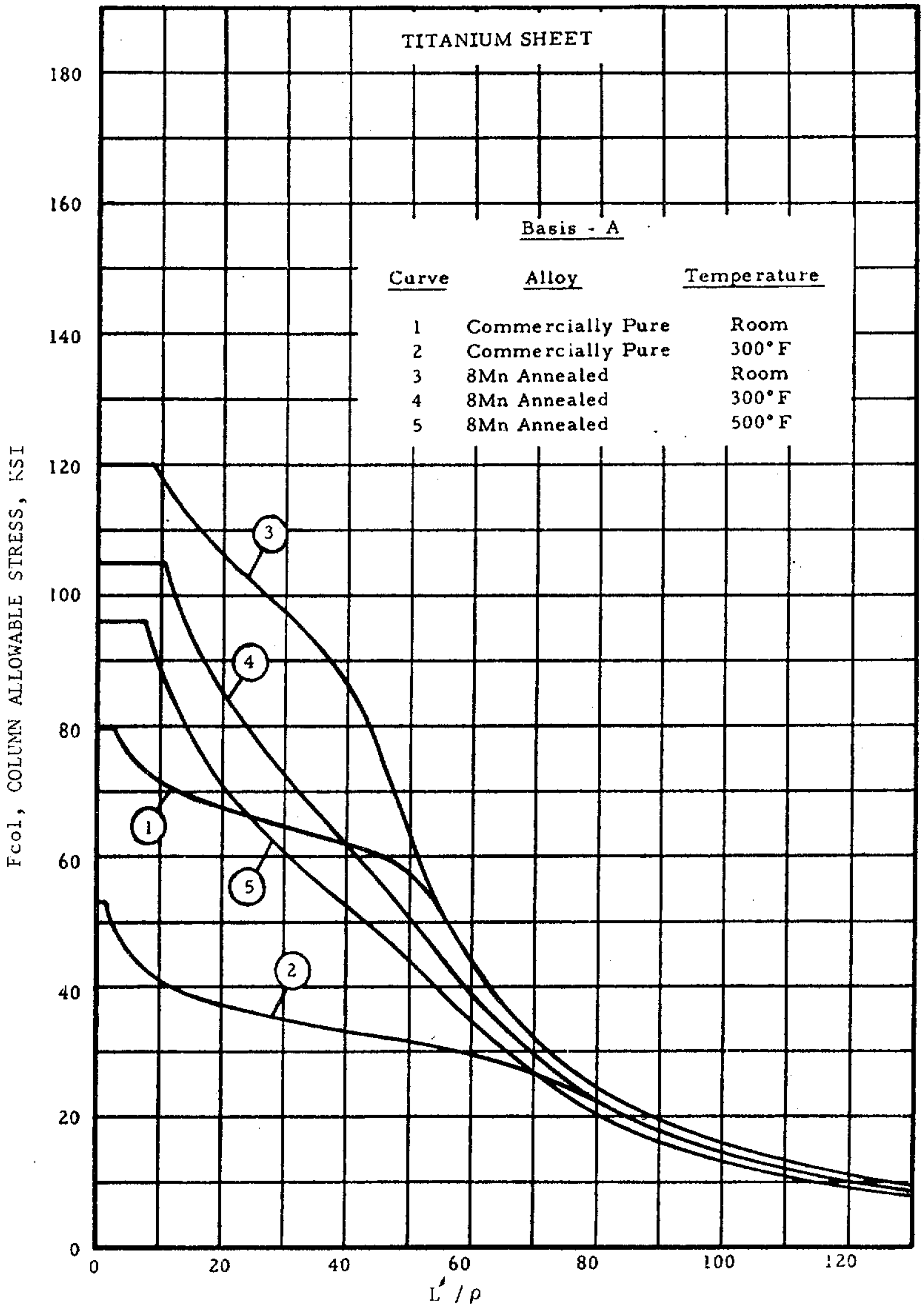


FIGURE 11.56 - COLUMN ALLOWABLE CURVES

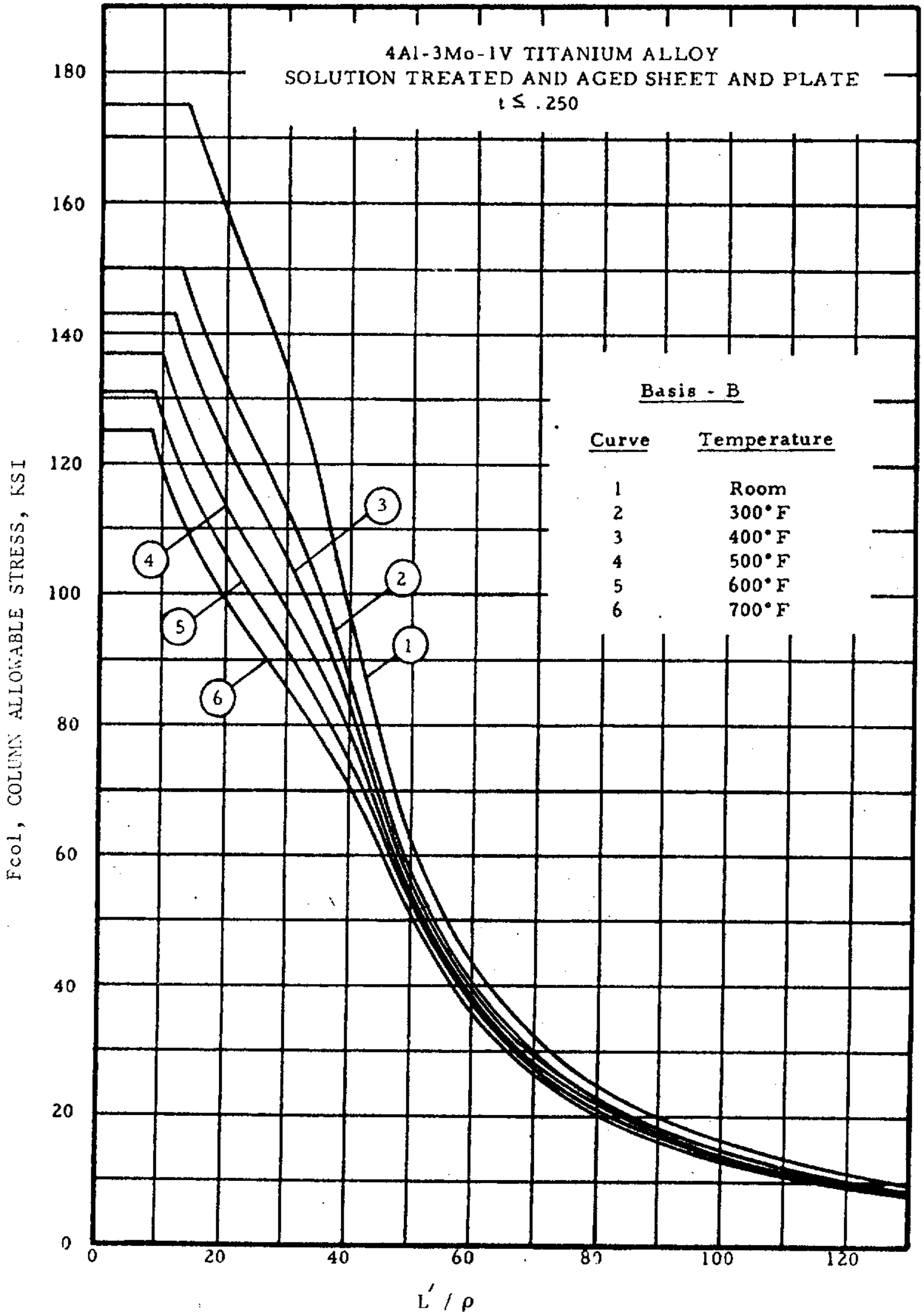


FIGURE 11.57 - COLUMN ALLOWABLE CURVES

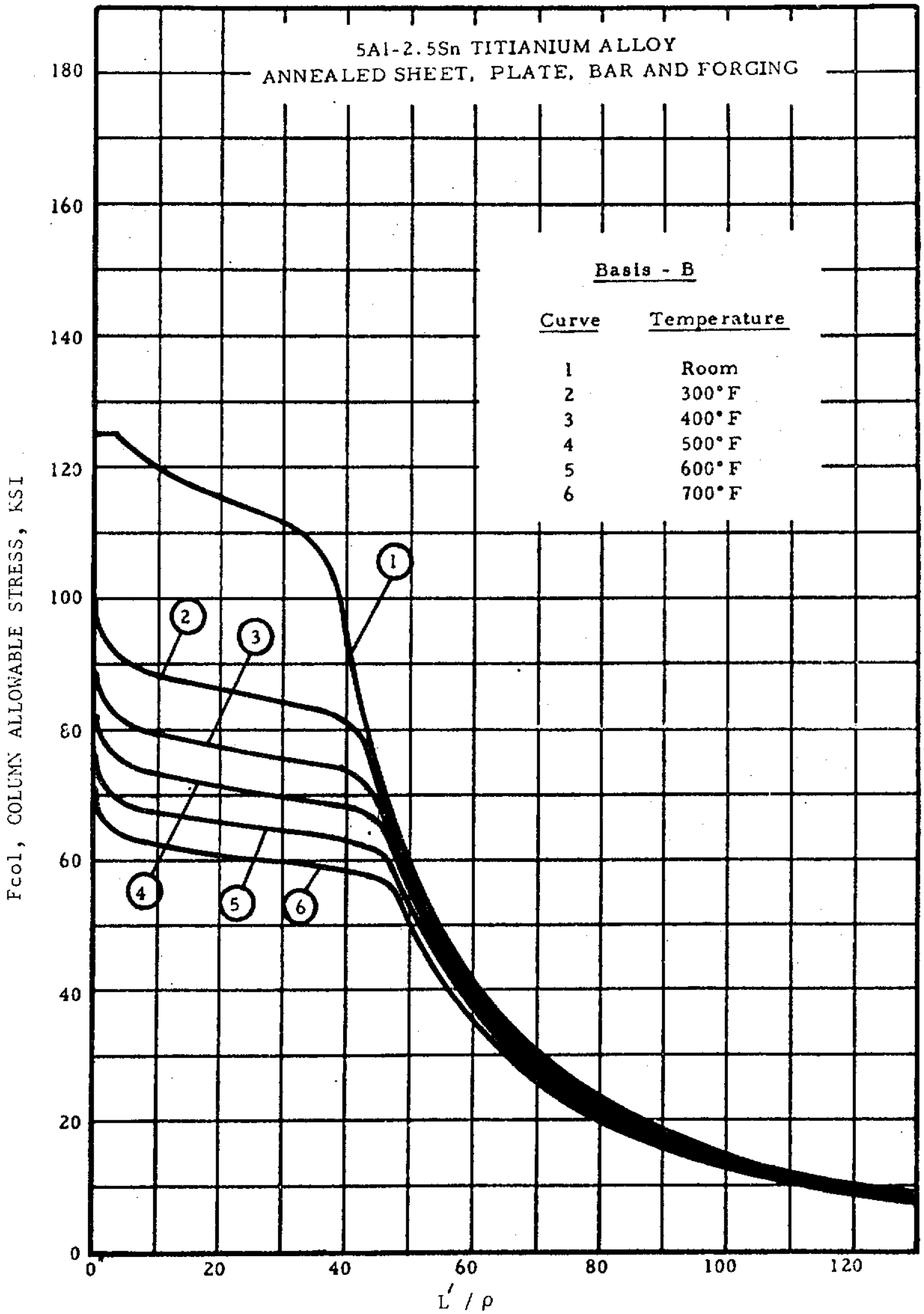


FIGURE 11.58 - COLUMN ALLOWABLE CURVES

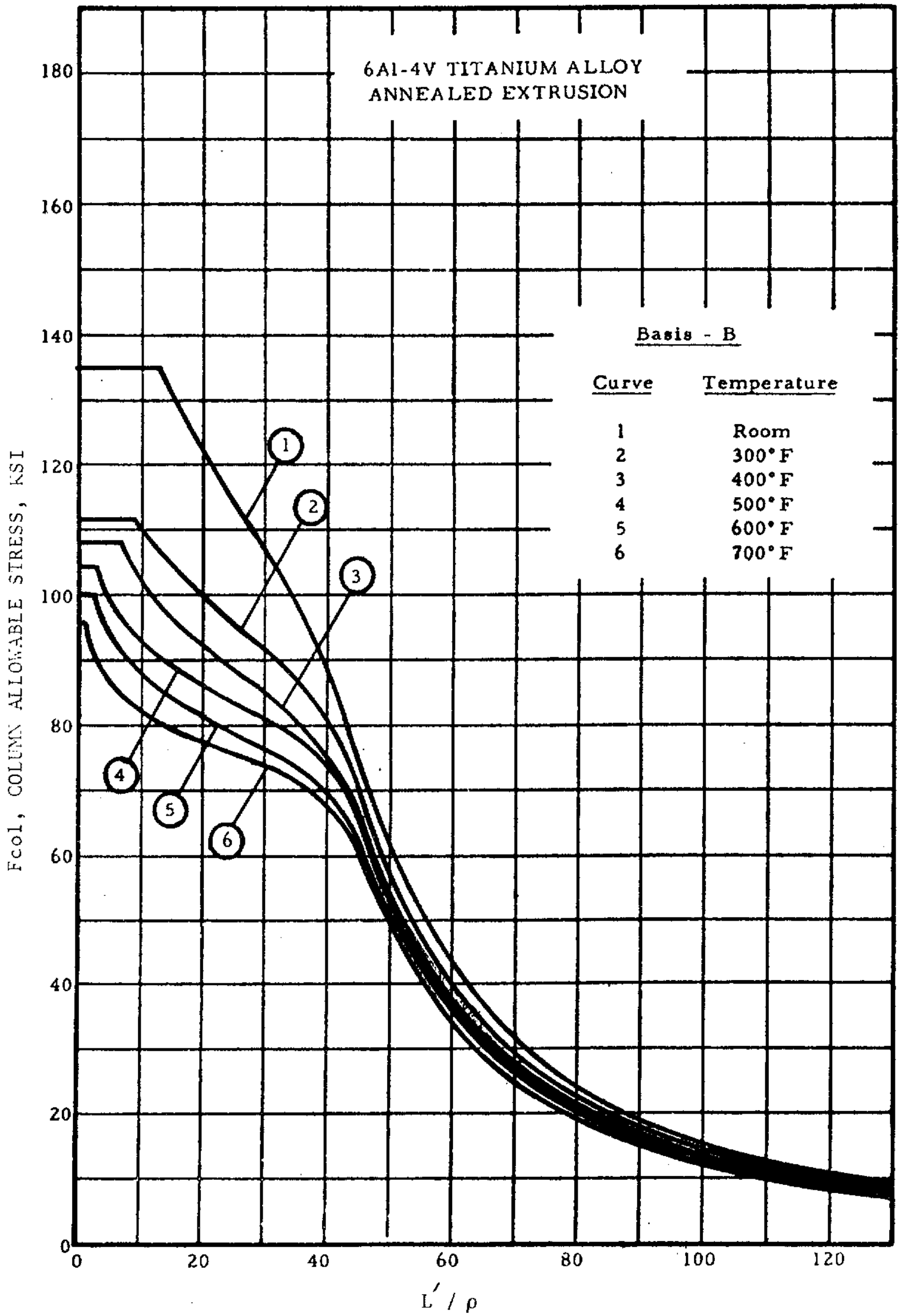


FIGURE 11.59 - COLUMN ALLOWABLE CURVES

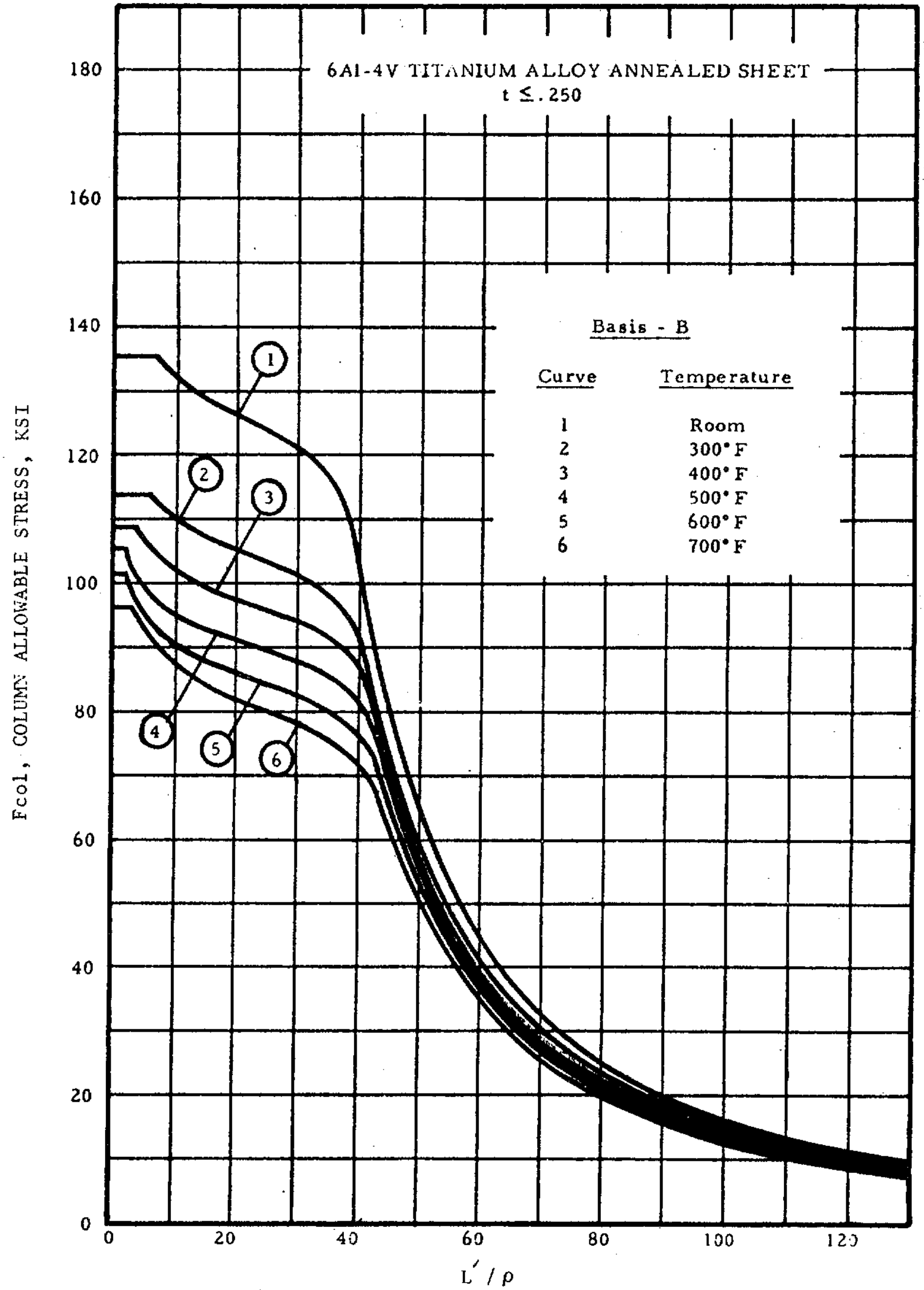


FIGURE 11.60 - COLUMN ALLOWABLE CURVES

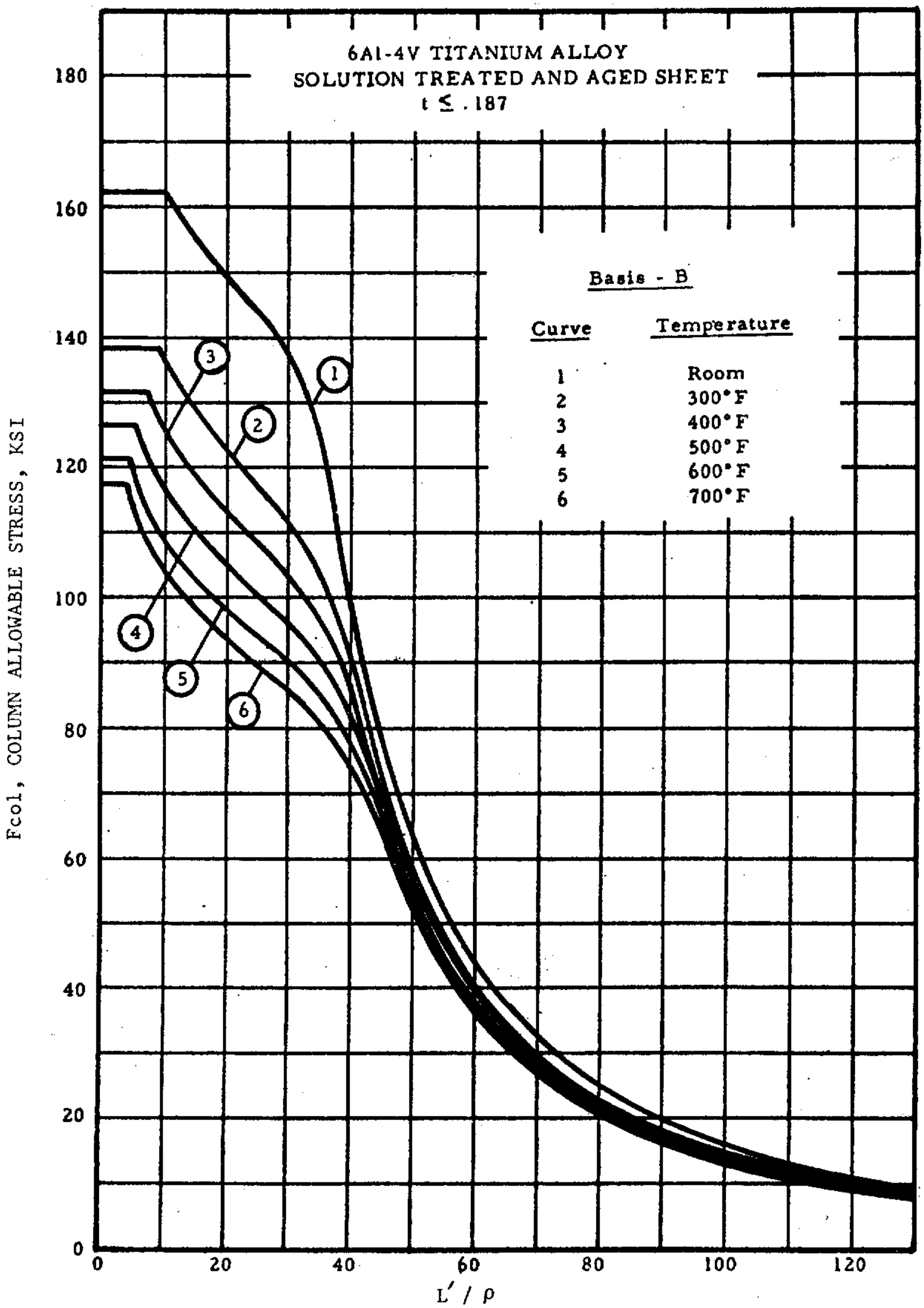


FIGURE 11.61 - COLUMN ALLOWABLE CURVES

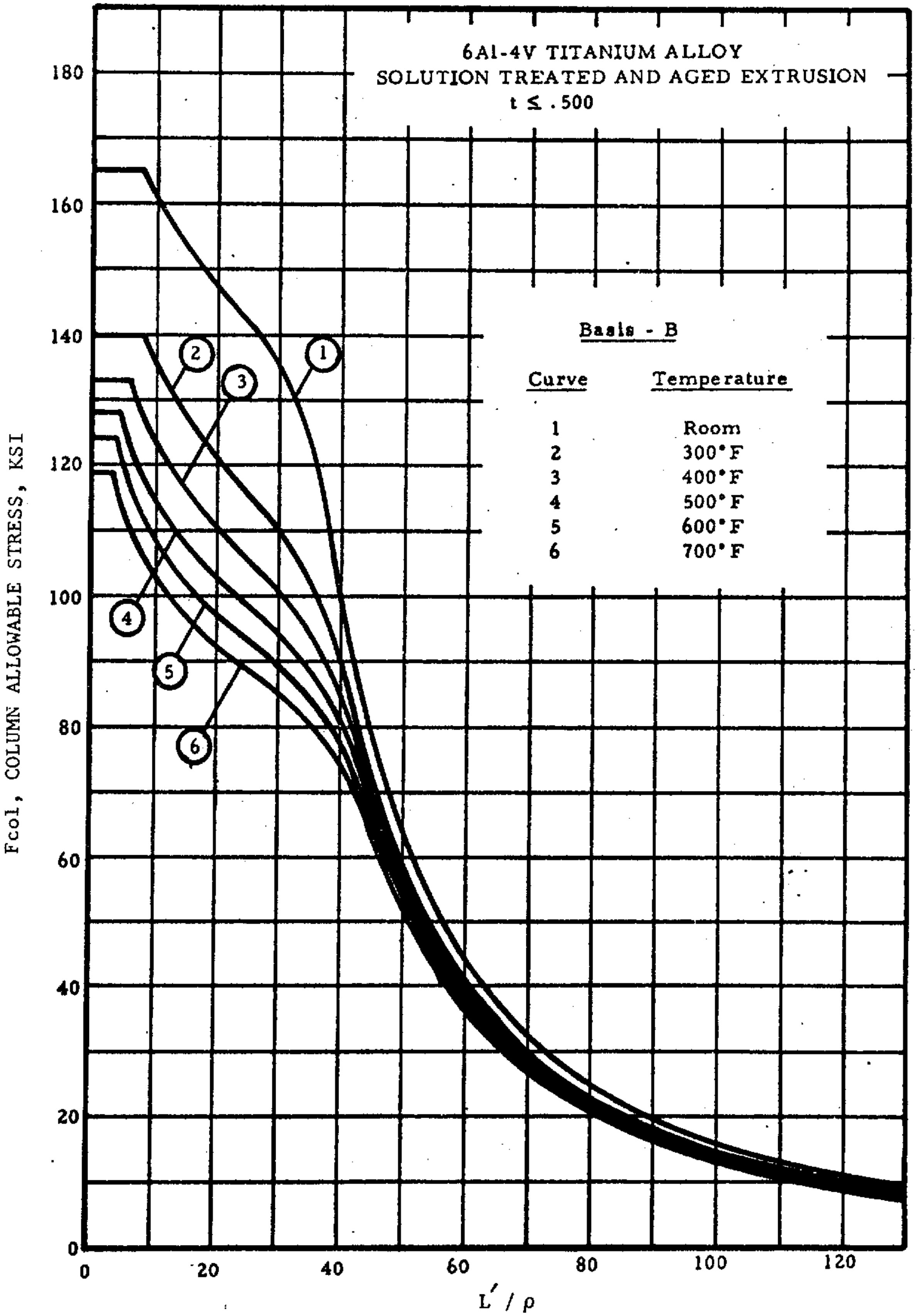


FIGURE 11.62 - COLUMN ALLOWABLE CURVES

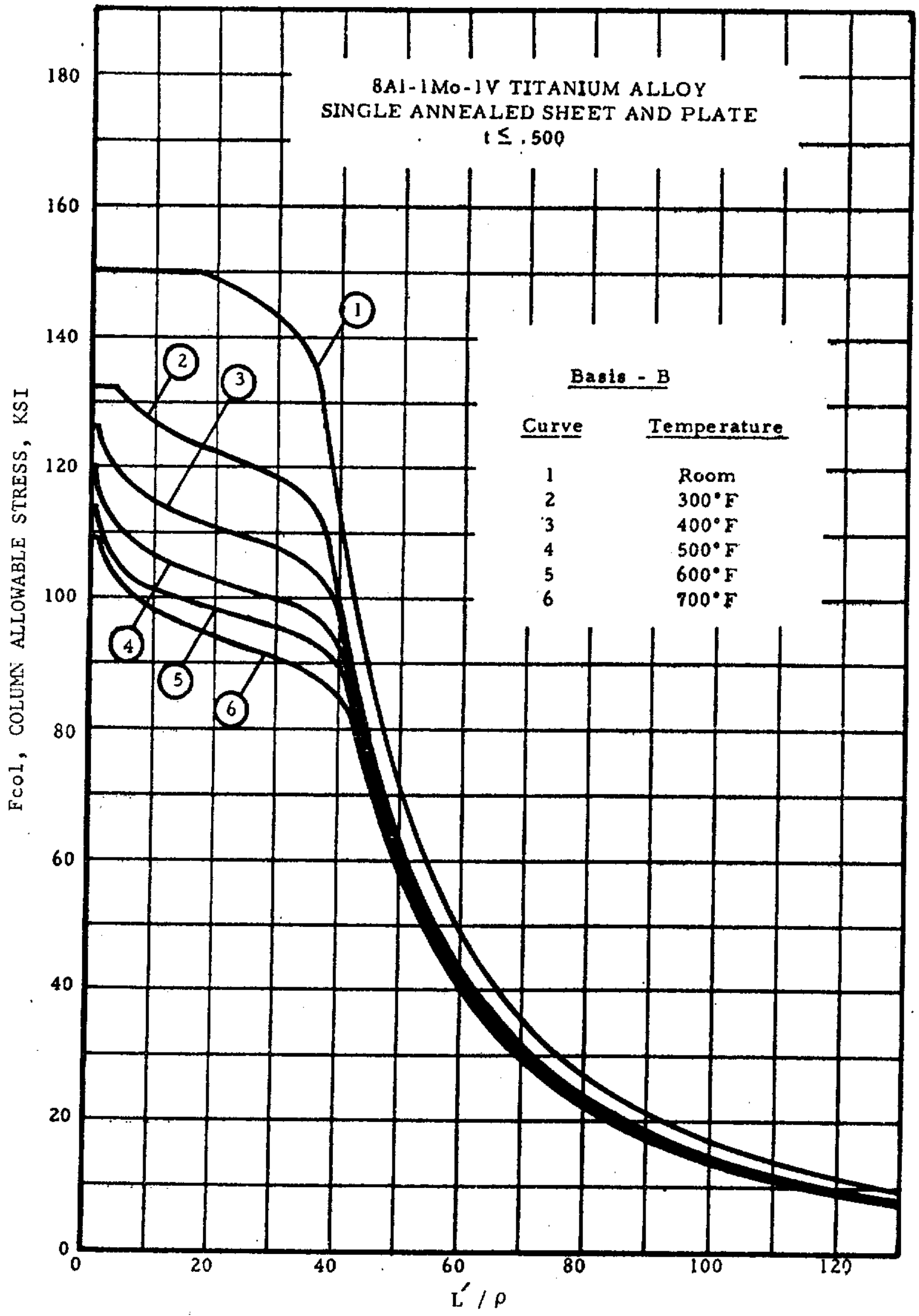


FIGURE 11.63 - COLUMN ALLOWABLE CURVES

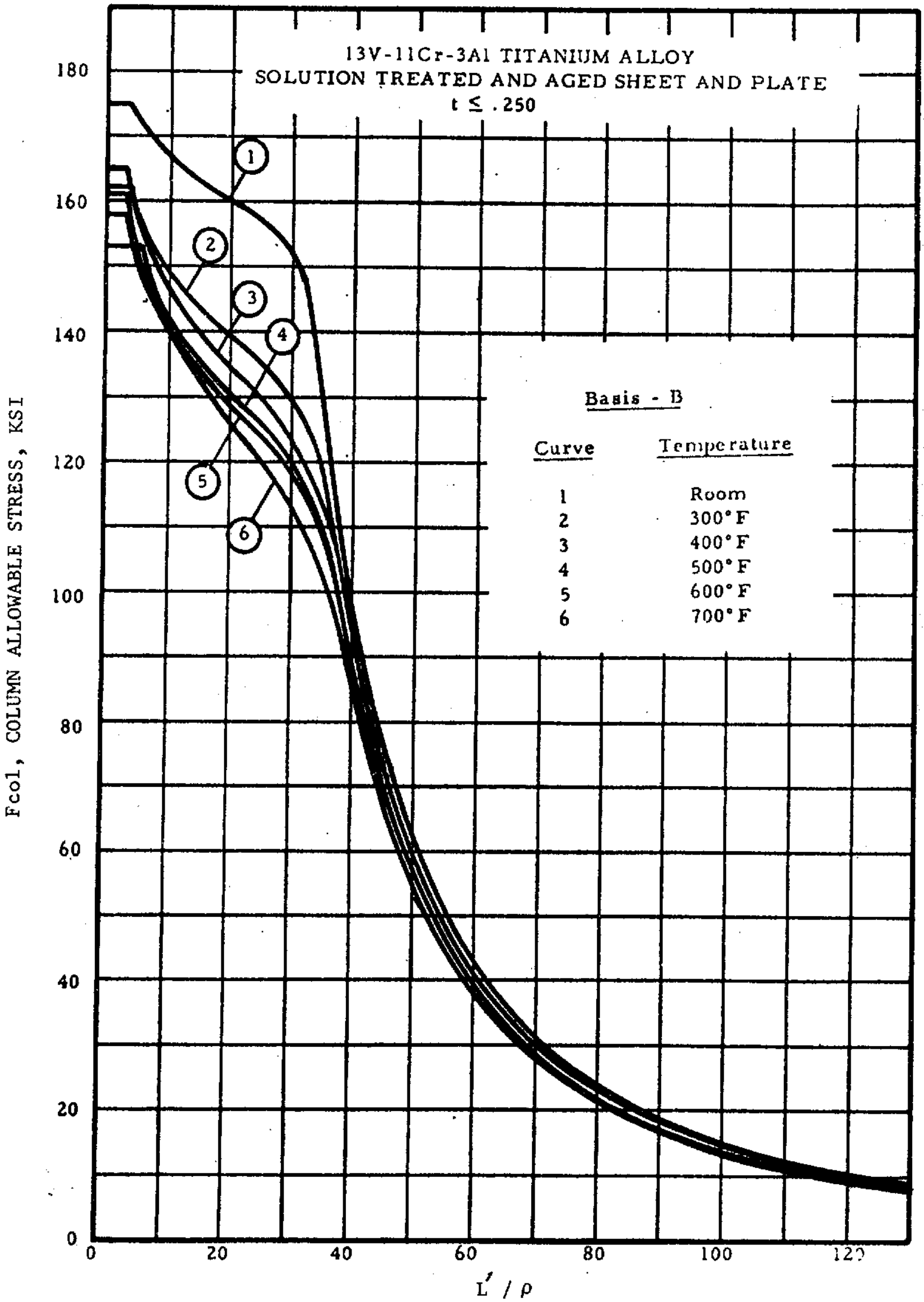


FIGURE 11.64 - COLUMN ALLOWABLE CURVES

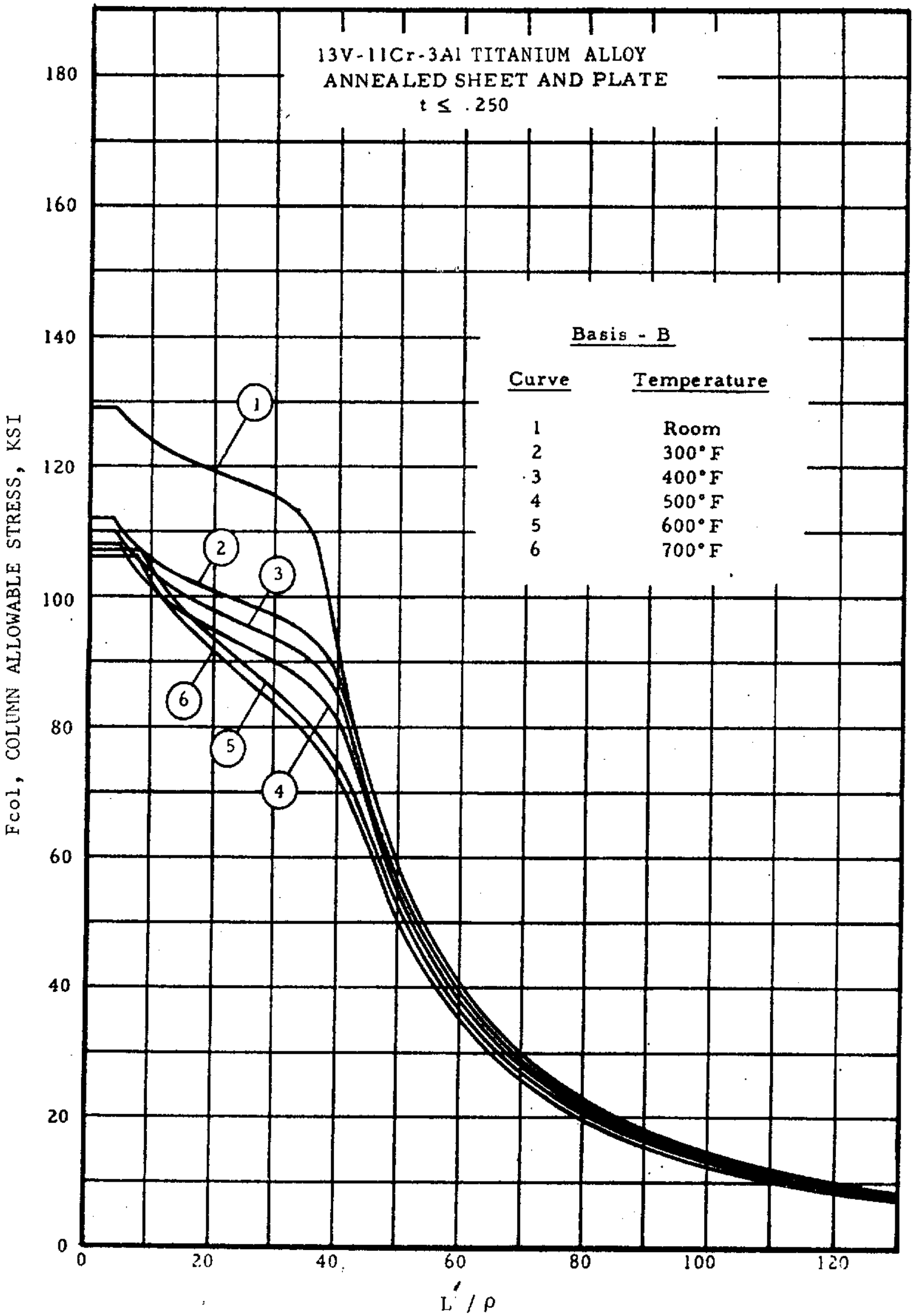


FIGURE 11.65 - COLUMN ALLOWABLE CURVES

11.2 Beam Columns

A beam column is a member subjected to transverse loads or end moments plus axial loads. The beam can be straight or have an initial curvature. Its cross sectional dimensions are small with respect to its length. The axial loads (either tension or compression) produce secondary bending moments because of the lateral deflection caused by the transverse loads or bending moments. In the case of the axial compression load, the primary transverse bending moments will be increased, while the axial tension load will decrease them.

11.2.1 Beam Columns With Axial Compression Loads

Beam columns under axial compressive loads are far more critical than those with tension loads. Axial compressive loads increase the bending moment and increase the possibility of instability or buckling failure. The critical column load is defined in Sections 11.1 and is independent of the magnitude or distribution of the transverse loading. The critical load is the load under which the member would be unstable if there were no side load. A beam column must satisfy both the criteria of a column and a beam.

Beam columns may be designed to function in the elastic and plastic range of the material but normal structural design procedures must be followed, i.e., no yield at limit load and no failure at ultimate.

The following equations are used to calculate the combined effects on a single span beam column:

Bending Moment: $$M = C_1 \sin x/j + C_2 \cos x/j + f(w) \tag{11.15}$$

Shear: $$V = C_1/j \cos x/j - C_2/j \sin x/j + f'(w) \tag{11.16}$$

where $f'(w)$ is 1st derivative of $f(w)$

Deflection: $$\delta = (M_o - M)/P \tag{11.17}$$

Slope: $$\theta = (V_o - V)/P \tag{11.18}$$

In the above equations $j = \sqrt{EI/P}$ and M_o and V_o are the primary bending moment and shear, i.e., the bending moment and shear that would be produced by the transverse loads and end moments acting without the axial loads. The constants C_1 and C_2 and the expression $f(w)$ depend on the type of transverse load, that is, distributed, point, moment, etc. The moment M is positive when compression is produced in the upper fibers and W or w is positive when upward. The load P and the distance X are as shown in Table 11.3.

The results of this method of beam column analysis are inaccurate when M/Mo becomes less than 1.1. It is recommended that at least four significant figures be used in all computations.

Table 11.3 shows the constants for use with Equations 11.15 and 11.16. Not all combinations of loads and moments are possible to present. The method of superposition can be applied provided each transverse loading is used with the total axial load for the systems which are being combined. The principle of superposition does not apply to a beam column when used in the conventional manner. The sum of the bending moments due to the transverse loads and the axial loads acting separately are not the same as the moments when they act simultaneously. To find the moments for several combined loadings, add the values of C_1, C_2, and f(w) for the several loadings using Table 11.3 for each individual case. Then substitute these values into Equations 11.15 and 11.16.

In a beam column, the bending moments do not vary directly as the load increases. Thus, it should be noted that direct ratioing of moments to loads should be avoided and will result in inadequate structure.

It is recommended that four significant figures be used in computations, making use of the so-called precise equations, since the results in many cases involve small differences in large numbers.

In Equations 11.15 and 11.16 the terms x/j are in radians. The values of sin x/j and cos x/j can be found in several textbooks. They are not presented here since they are easily obtained on hand calculators.

The equations presented previously assume that E is constant, that is, the stresses are in the elastic range. This is used for loads up to limit; however, for ultimate loads the stresses may be in the plastic range. The method and equations used for the elastic analysis are also used for the plastic analysis with a plastic E used where previously E was elastic. A good approximation of the plastic E is as follows:

1) Compute $F_c = P/A$
2) Enter the basic column curve for the material at F_c and find $L'/\rho\sqrt{C}$ corresponding to F_c. C = 1 for this step.
3) Using these values of F_c and L'/ρ compute $E' = \frac{F_c}{\pi^2}\left(\frac{L'}{\rho}\right)^2$
4) Then $j = \left(\frac{E'I}{P}\right)^{\frac{1}{2}}$

Then proceed as explained previously.

TABLE 11.3 - BEAM COLUMNS WITH AXIAL COMPRESSION

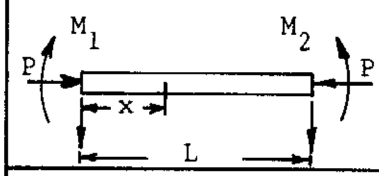
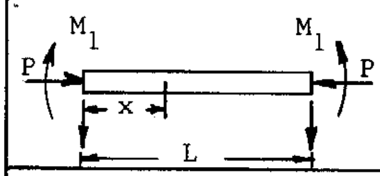
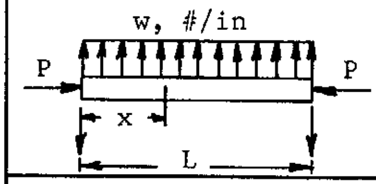
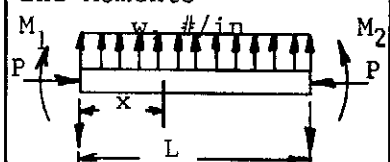
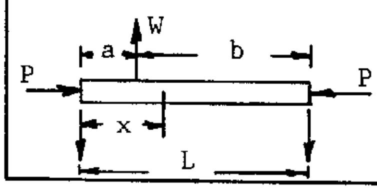
LOADING	C_1	C_2	f(w)	MAXIMUM MOMENT
Unequal End Moments 	$\frac{M_2 - M_1 \text{Cos}(L/j)}{\text{Sin}(L/j)}$	M_1	0	$M_{max} = \frac{M_1}{\text{Cos}(x/j)}$ where $\text{Tan}(x/j) = \frac{M_2 - M_1 \text{Cos}(L/j)}{M_1 \text{Sin}(L/j)}$
Equal End Moments 	$M_1 \text{Tan}(L/2j)$	M_1	0	$M_{max} = \frac{M_1}{\text{Cos}(L/2j)}$ At Midspan
Uniform 	$\frac{wj^2 [\text{Cos}(L/j) - 1]}{\text{Sin}(L/j)}$	$-wj^2$	wj^2	$M_{max} = wj^2[1 - \text{Sec}(L/2j)]$ At Midspan
Uniform Load Plus End Moments 	$\frac{D_2 - D_1 \text{Cos}(L/j)}{\text{Sin}(L/j)}$ See Note 4	D_1 See Note 4	wj^2	$M_{max} = \frac{D_1}{\text{Cos}(x/j)} + wj^2$ where $\text{Tan}(x/j) = \frac{D_2 - D_1 \text{Cos}(L/j)}{D_1 \text{Sin}(L/j)}$
Concentrated Load 	x < a: $\frac{-Wj\text{Sin}(b/j)}{\text{Sin}(L/j)}$ x > a: $\frac{Wj\text{Sin}(a/j)}{\text{Tan}(L/j)}$	0 $-Wj\text{Sin}(a/j)$	0 0	$M_{max} = \frac{C_1^2 + C_2^2}{C_2} \text{Cos}(x/j)$ where $\text{Tan}(x/j) = \frac{C_1}{C_2}$

TABLE 11.3 (CONT'D) - BEAM COLUMNS WITH AXIAL COMPRESSION

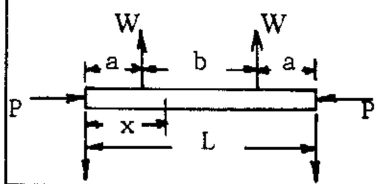
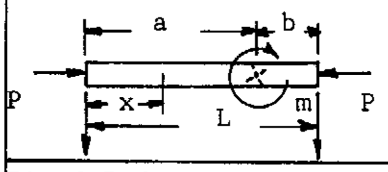
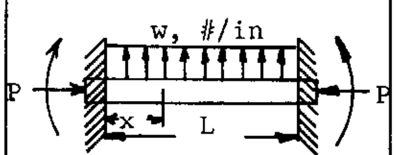
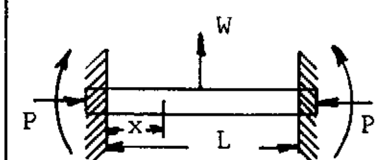
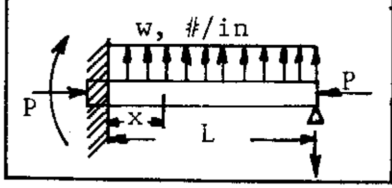
LOADING	C_1	C_2	f(w)	MAXIMUM MOMENT
Two Symmetrical Concentrated Loads 	$x < a$: $-Wj\frac{Cos(b/2j)}{Cos(L/2j)}$ $a < x < L-a$: $-WjSin(a/j)Tan(L/2j)$ $L-a < x < L$: $Wj\frac{Cos(L/j)Cos(b/2j)}{Cos(L/2j)}$	0 $-WjSin(a/j)$ $-Wj\frac{Sin(L/j)Cos(b/2j)}{Cos(L/2j)}$	0 0 0	$-Wj\frac{Sin(a/j)}{Cos(L/2j)}$ at midspan
Concentrated Moment 	$x < a$: $-\frac{mCos(b/j)}{Sin(L/j)}$ $x > a$: $-\frac{mCos(a/j)}{Tan(L/j)}$	0 $mCos(a/j)$	0 0	See Note (6)
Fixed End Beam - Uniform Load 	$\frac{-wjL}{2}$	$\frac{-wLj}{2Tan(L/2j)}$	wj^2	At $x = 0$: $wj^2\left[1 - \frac{L/2j}{Tan(L/2j)}\right]$ At $x = L/2$ $-wj^2\left(\frac{L/2j}{Sin\,L/2j} - 1\right)$
Fixed End Beam - Concentrated Load at Center 	$x < L/2$: $\frac{-Wj}{2}$ $x > L/2$: $\frac{Wj}{2}\left(2Cos(L/2j) - 1\right)$	$\frac{Wj}{2}\left(\frac{1 - Cos(L/2j)}{Sin(L/2j)}\right)$ $\frac{Wj}{2}\left(\frac{Cos(L/j) - Cos(L/2j)}{Sin(L/2j)}\right)$	0 0	$\frac{Wj}{2}\left(\frac{1 - Cos(L/2j)}{Sin(L/2j)}\right)$
Fixed Simple Beam Uniform Load 	$x < L/2$: $-\frac{Tan(L/2j) - L/2j}{Tan(L/j) - L/j}\ -wLj$ $x > L/2$: $-\frac{1 - Cos(L/j)}{Sin(L/j)}\ -wj^2$	$\frac{Tan(L/2j) - (L/2j)}{Tan(L/j) - (L/j)}$ $(Tan(L/j)(wLj)) - wj^2$	0 wj^2	$wLj\left(\frac{Tan(L/2j) - (L/2j)}{Tan(L/j) - (L/j)}\right)$ $x(Tan\ (L/j))$ at $x = 0$

TABLE 11.3 (CONT'D) - BEAM COLUMNS WITH AXIAL COMPRESSION

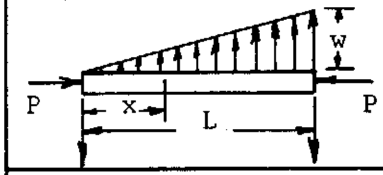
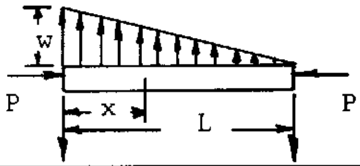
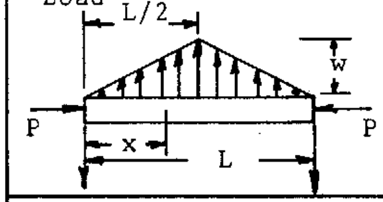
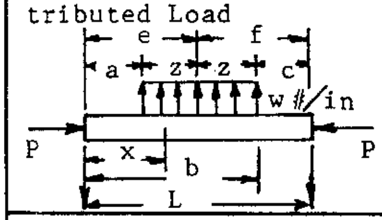
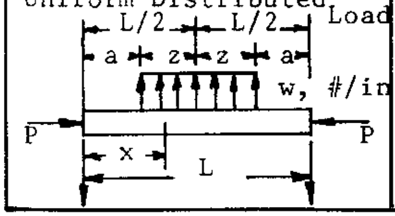
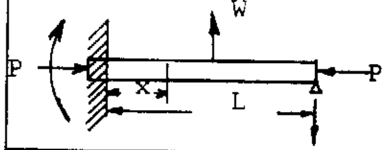
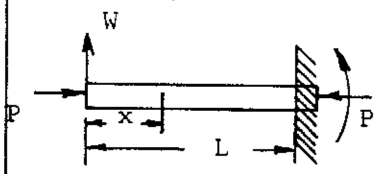
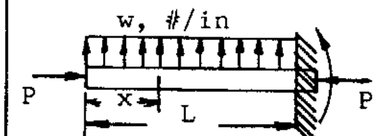
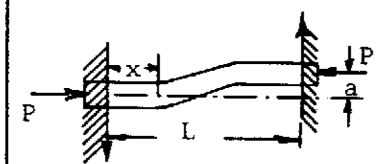
LOADING	C_1	C_2	f(w)	MAXIMUM MOMENT
Uniformly Increasing Load 	$\frac{-wj^2}{Sin(L/j)}$	0	$\frac{wj^2 x}{L}$	Occurs at $Cos(x/j)=(j/L)Sin(L/j)$ Solve for x/j and x Substitute into Eq. 11.15.
Uniformly Decreasing Load 	$\frac{wj^2}{Tan(L/j)}$	$-wj^2$	$wj^2(1-x/L)$	Occurs at $Cos(\frac{L-x}{j}) = (j/L)Sin(L/j)$ Solve for x/j and x Substitute into Eg. 11.15.
Symmetrical Triangle Load 	$x < L/2$: $\frac{-2wj^3}{L\,Cos(L/2j)}$ $x > L/2$: $\frac{2wj^3 Cos(L/j)}{L\,Cos(L/2j)}$	0 $\frac{-4wj^3 Sin(L/2j)}{L}$	$\frac{2wj^2 x}{L}$ $2wj^2(1-x/L)$	$\frac{-2wj^3 Tan(L/2j)}{L} + wj^2$ at midspan
Partial Uniformly Distributed Load 	$x < a$: $\frac{-2wj^2 Sin(z/j)Sin(f/j)}{Sin(L/j)}$ $a < x < b$: $\frac{2wj^2 Sin(z/j)Sin(e/f)}{Tan(L/j)} - wj^2 Sin(b/j)$ $b < x < L$: $\frac{2wj^2 Sin(z/j)Sin(e/f)}{Tan(L/j)}$	0 $-wj^2 Cos(a/j)$ $-2wj^2 Sin(z/j)Sin(e/j)$	0 wj^2 0	See Note (6)
Symmetrical Partially Uniform Distributed Load 	$x < a$: $-wj^2 Sin(z/j)Sec(L/2j)$ $a < x < L-a$: $-wj^2 Tan(L/2j)Cos(a/j)$ $L-a < x < L$: $wj^2 Sin(z/j)Sec(L/2j)Cos(L/j)$	0 $-wj^2 Cos(a/j)$ $-2wj^2 Sin(z/j)Sin(L/2j)$	0 wj^2 0	$wj^2 \{ 1 - \frac{Cos(a/j)}{Cos(L/2j)} \}$ at midspan

TABLE 11.3 (CONT'D) - BEAM COLUMNS WITH AXIAL COMPRESSION

LOADING	C_1	C_2	f(w)	MAXIMUM MOMENT
Fixed Simple Beam Concentrated Load 	x < L/2: $\frac{-Wj}{2}\left(\frac{jTan(L/j)Sec(L/2j)-1}{jTan(L/j) - L}\right)$ x > L/2: $\frac{Wj}{2}\left(\frac{L+2jSin(L/2j)-2LCos(L/2j)}{jTan(L/j)-L}\right)$	$\frac{WL}{2}\left[\frac{jTan(L/j)(Sec(L/2j)-1)}{jTan(L/j)-L}\right]$ $\frac{Wj}{2}\left[\frac{LTan(L/j)Sec(L/2j)-1}{jTan(L/j)-L} - 2Sin(L/2j)\right]$	0 0	$\frac{WL}{2}\left[\frac{jTan(L/j)Sec(L/2j)-1}{jTan(L/j)-L}\right]$ at x = 0
Cantilever - Concentrated Load 				WjTan(L/j) at x = L
Cantilever - Uniform Load 				Wj{j(1 - SecL/j) + LTan L/j} at x = L
				$\frac{aPTan(L/2j)}{2(Tan(L/2j) - L/2j)}$

NOTES:

(1) W or w is positive when upward.

(2) M is positive when producing compression in upper fibers.

(3) $j = \sqrt{\frac{EI}{P}}$ with a dimension of length.

(4) $D_1 = M_1 - wj^2$; $D_2 = M_2 - wj^2$

(5) All angles for trigonometric functions are in radians.

(6) When the formula for the maximum moment is not provided in the table, methods of differential calculus may be employed, if applicable, to find the location of maximum moment; or moments at several points in a span may be computed and a smooth curve then drawn through the plotted results. The same principle applies in the case of a complicated combination of loadings.

(7) All points where concentrated loads or moments are acting should also be checked for maximum possible bending moments.

(8) Before the total stress can reach the yield point, a compression beam column may fail due to buckling. This instability failure is independent of lateral loads and the maximum P that the structure can sustain may be computed pertaining to the boundary condition without regard to lateral loads. A check using ultimate loads should always be made to insure that P is not beyond the critical value.

TABLE 11.3 (Continued) BEAM COLUMNS WITH AXIAL COMPRESSION

11.2.2 Beam Columns With Axial Tension Loads

Axial tension loads usually decrease the bending moment in the beam column. Beam columns may be designed neglecting the axial loads; however, this will give conservative results. More precise results can be obtained with the following procedure:

Bending Moment: $M = C_1 \sinh x/j + C_2 \cosh x/j + f(w)$ (11.19)

Shear: $V = C_1/j \cosh x/j + C_2/j \sinh x/j + f'(w)$ (11.20)

where $f'(w)$ is 1st derivative of $f(w)$

Deflection: $\delta = (M-M_o)/P$ (11.21)

Slope: $\theta = (V-V_o)/P$ (11.22)

In the above equations $j = \sqrt{EI/P}$ and M_o and V_o are primary bending moment and shear, i.e., the bending moment and shear that would be produced by the transverse loads and end moments acting without axial loads. The constants C_1 and C_2 and the expression $f(w)$ depend on the type of transverse load, that is, distributed, point, moment, etc. The moment M is positive when compression is produced in the upper fibers and W or w is positive when upward. The load P and the distance x are as shown in Table 11.4.

The results of this method of beam column analysis are inaccurate when M/Mo becomes greater than 0.9. To maintain accuracy in the analysis four significant figures should be used in the calculations.

Various loading conditions are shown in Table 11.4. The method of analysis for other types of loading is the same as for compression loaded members described in Section 11.2.1.

11.2.3 Multi-Span Columns and Beam Columns

Multi-span beams are those with three or more supports and in general it is not possible to develop simple equations as previously described.

The determination of moments is more involved for multi-span columns and the method of "moment distribution" is used to determine the beam moments at each support. These moments are then used as previously described in Sections 11.2.1 and 11.2.2 in the single span equations to determine bending moments between supports.

The moment distribution method is sometimes called the "Hardy Cross" method after the man who originated it. The method is simple and useful for the solution of continuous structures, i.e., multi-span beams. This method starts by assuming

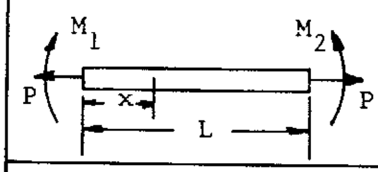
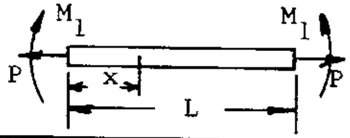
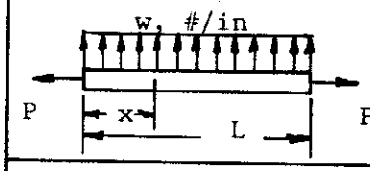
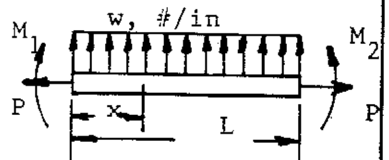
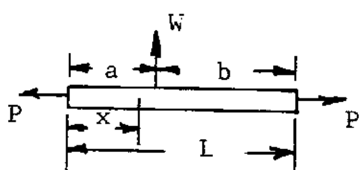
LOADING	C_1	C_2	f(w)	MAXIMIM MOMENT
Unequal End Moments 	$\frac{M_2 - M_1 Cosh(L/j)}{Sinh(L/j)}$	M_1	0	$\frac{M_1}{Cosh(x/j)}$ where $Tanh(x/j) = \frac{M_2 - M_1 Cosh(L/j)}{-M_1 Sinh(L/j)}$
Equal End Moments 	$-M_1 Tanh(L/2j)$	M_1	0	$\frac{M_1}{Cosh(L/2j)}$
Uniform Load 	$\frac{wj^2[1 - Cosh(L/j)]}{Sinh(L/j)}$	wj^2	$-wj^2$	$wj^2[Sech(L/2j) -1]$
Uniform Load with End Moments 	$\frac{D_2 - D_1 Cosh(L/j)}{Sinh(L/j)}$	D_1 See Note 4	$-wj^2$	$\frac{D_1}{Cosh(x/j)} -wj^2$ where $Tanh(x/j) = -\frac{D_2 - D_1 Cosh(L/j)}{D_1 Sinh(L/j)}$ See Note 4
Concentrated Load 	$x < a$: $\frac{-WjSinh(b/j)}{Sinh(L/j)}$ $x > a$: $\frac{WjSinh(a/j)}{Tanh(L/j)}$	0 $-WjSinh(a/j)$	0 0	$\frac{C_2^2 - C_1^2}{C_2} Cosh(x/j)$ where $Tanh(x/j) = -\frac{C_1}{C_2}$

TABLE 11.4 - BEAM COLUMNS WITH AXIAL TENSION

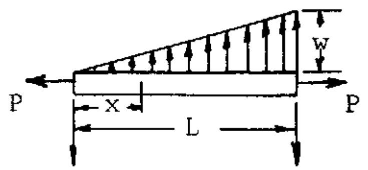
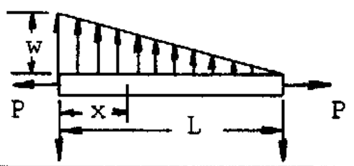
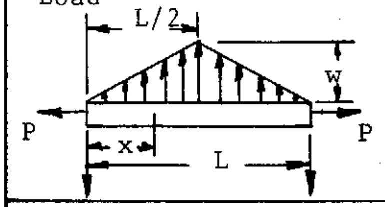
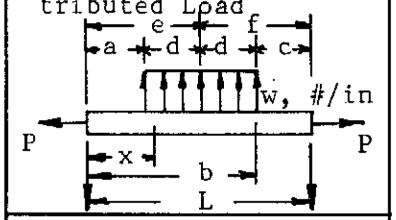
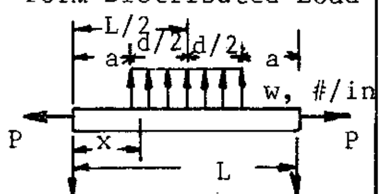
LOADING	C_1	C_2	f(w)	MAXIMUM MOMENT
Uniform Increasing Load 	$\frac{wj^2}{Sinh(L/j)}$	0	$\frac{-wj^2x}{L}$	Occurs at: $Cosh(x/j)=(j/L)Sinh(L/j)$ Solve for x/j and x, Substitute into Equation 11.19
Uniform Decreasing Load 	$\frac{-wj^2}{Tanh(L/j)}$	wj^2	$-wj^2(1-x/L)$	Occurs at: $Cosh(\frac{L-x}{j})=(j/L)Sinh(L/j)$ Solve for x/j and x, Substitute into Equation 11.19
Symmetrical Triangular Load 	$x < L/2$: $\frac{2wj^3}{LCosh(L/2j)}$ $x > L/2$: $\frac{-2wj^3Cosh(L/j)}{LCosh(L/2j)}$	0 $\frac{4wj^3}{L}Sinh(L/2j)$	$\frac{-2wj^2x}{L}$ $-2wj^2(1-x/L)$	$\frac{2wj^3}{L}Tanh(\frac{L}{2j}) - wj^2$
Partial Uniformly Distributed Load 	$x < a$: $\frac{-2wj^2Sinh(d/2j)Sinh(f/j)}{Sinh(L/j)}$ $a < x < b$: $\frac{2wj^2Sinh(d/2j)Sinh(e/j)}{Tanh(L/j)} - wj^2Sinh(b/j)$ $b < x < L$: $\frac{2wj^2Sinh(d/2j)Sinh(e/f)}{Tanh(L/j)}$	0 $wj^2Cosh(a/j)$ $-2wj^2Sinh(d/j)Sinh(e/f)$	0 $-wj^2$ 0	See Note 6
Symmetrical Partial Uniform Distributed Load 	$x < a$: $\frac{-wj^2Sinh(d/2j)}{Cosh(L/2j)}$ $a < x < L-a$: $-wj^2Cosh(a/j)Tanh(L/2j)$ $L-a < x < L$: $\frac{wj^2Sinh(d/2j)Cosh(L/j)}{Cosh(L/2j)}$	0 $-wj^2Cosh(a/j)$ $-2wj^2Sinh(d/2j)Sinh(L/2j)$	0 $-wj^2$ 0	$wj^2\left[\frac{Cosh(a/j)}{Cosh(L/2j)} - 1\right]$

TABLE 11.4 (CONT'D) - BEAM COLUMNS WITH AXIAL TENSION

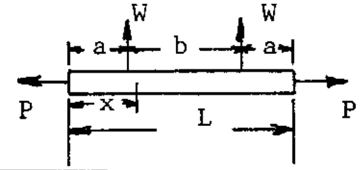
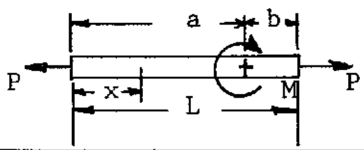
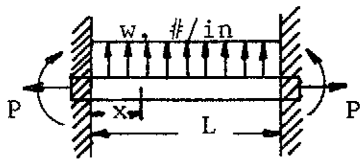
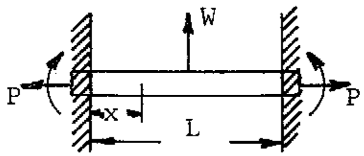
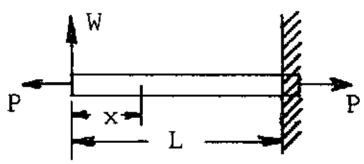
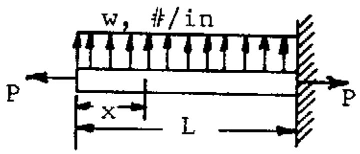
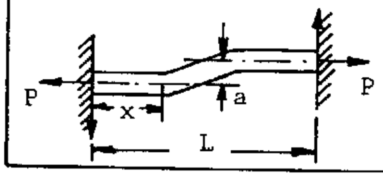
LOADING	C_1	C_2	f(w)	MAXIMUM MOMENT
Two Symmetrical Concentrated Loads 	x<a: $\frac{-WjCosh(b/2j)}{Cosh(L/2j)}$ a<x<L-a: $WjSinh(a/j)Tanh(L/2j)$ L-a<x<L: $\frac{WjCosh(L/j)Cosh(b/2j)}{Cosh(L/2j)}$	0 $-WjSinh(a/j)$ $\frac{-WjSinh(L/j)Cosh(b/2j)}{Cosh(L/2j)}$	0 0 0	$-Wj\frac{Sinh(a/j)}{Cosh(L/2j)}$ at midspan
Concentrated Moment 	x<a: $\frac{-MCosh(b/j)}{Sinh(L/j)}$ x>a: $\frac{-MCosh(a/j)}{Tanh(L/j)}$	0 $MCosh(a/j)$	0 0	See Note 6
Fixed End Beam - Uniform Load 	$\frac{-wLj}{2}$	$\frac{wLj}{2Tanh(L/2j)}$	$-wj^2$	At x = 0 $wj^2\left[\frac{L/2j}{Tanh(L/2j)} -1\right]$
Fixed End Beam-Concentrated Load at Center 	x < L/2: $\frac{-Wj}{2}$ x > L/2: $\frac{Wj}{2}[2Cosh(L/2j) -1]$	$\frac{Wj}{2}\left\|\frac{Cosh(L/2j) -1}{Sinh(L/2j)}\right\|$ $\frac{Wj}{2}\left\|\frac{Cosh(L/2j)-Cosh(L/j)}{Sinh(L/2j)}\right\|$	0 0	$\frac{Wj}{2}\left\|\frac{1 - Cosh(L/2j)}{Sinh(L/2j)}\right\|$
Cantilever - Concentrated End Load 				$WjTanh(L/j)$

TABLE 11.4 (CONT'D) - BEAM COLUMNS WITH AXIAL TENSION

LOADING	C_1	C_2	f(w)	MAXIMUM MOMENT
Cantilever - Uniform Load 				$wj[LTanh(L/j) - j(1-Sech(L/j)]$
Fixed End Beam - Lateral Displacement 				$\frac{aPTanh(L/2j)}{2[L/2j - Tanh(L/2j)]}$

Notes:

(1) W or w is positive upward.

(2) M is positive when producing compression in upper fibers.

(3) $j = \sqrt{EI/P}$ with a dimension of "in".

(4) $D_1 = M_1 + wj^2$; $D_2 = M_2 + wj^2$

(5) All angles for hyperbolic functions are in radians.

(6) When formula for max. moment is not given in the Table, methods of differential calculus may be employed, if applicable, to find the location of the maximum moment; or moments at several points in the span may be computed and a curve drawn thru the results. The same applies to a complicated combination of loadings.

(7) Values given in Table 11.4 were obtained from Table 11.3 by the following substitution:
SinL/j = iSinhL/j: CosL/j = CoshL/j;
Sinx/j = iSinhx/j; Cosx/j = Coshx/j and
j = ij where $i = \sqrt{-1}$

(8) All points where concentrated loads or moments are acting should also be checked for possible bending moments.

(9) Axial tension helps to stabilize the structure. Usually instability need not be considered unless the beam is very thin for which bending buckling should be checked.

TABLE 11.4 (CONT'D) - BEAM COLUMNS WITH AXIAL TENSION

an arbitrary restrained state for the beam and then gradually releases these restraints according to definite laws of continuity and statics until every part of the structure rests in its true state of equilibrium.

Certain terms are used in the Hardy Cross method and they are defined following:

Fixed End Moments - The moment which would exist at the ends of a member if these ends were fixed against rotation. The effect of a compressive axial load is to increase the fixed end moments (F.E.M.) while tensile axial loads will decrease the F.E.M. Values of F.E.M. for various loadings are shown in Figures 11.66 through 11.72.

Stiffness Factor - The stiffness factor (S.F.) is taken as the resistance to rotation at a joint between the subject joint and the adjacent joint. This is the ratio of the change in slope at joint A to the applied moment at joint A for the beam between joint A and joint B if these are adjacent joints. The S.F. of a beam at a joint can be thought of simply as its effective torsional spring constant at the joint. If any value of M_A were applied and the resulting value of θ computed (by using the equations of Tables 11.3 and 11.4) the value of S.F. could be calculated using:

$$S.F. = M_A / \theta_A$$

This is the method for spans without constant EI. For beams with constant EI over a span:

$$S.F. = S.C. \, (4EI/L) \qquad (11.23)$$

where S.C. is the "Stiffness Coefficient" for the span and is obtained from Tables 11.5 and 11.6. There are two cases in Tables 11.5 and 11.6, one for "full fixity" at the far end and the other for "pinned" at the far end.

Joint Stiffness - When one or more beams meet at a joint or support, the total stiffness factor is the sum of the S.F.'s of each member at the joint.

$$\text{Joint Stiffness} = \sum S.F. \qquad (11.24)$$

The total joint stiffness must be positive or the structure is unstable. Instability can exist even though the joint stiffness is positive. This occurs when members are not fully fixed at the far end but are somewhere between pinned and fixed. Sometimes good engineering judgment is necessary to obtain a solution more exact than is possible if pinned and fixed are assumed.

Distribution Factor - The distribution factor (D.F.) is a measure of the amount of the moment at a joint that is resisted by each of the members meeting at the joint. It is expressed as:

$$D.F. = S.F. / \sum S.F. \qquad (11.25)$$

Carry Over Factor - The carry over factor (C.O.F.) is the ratio of the moment (M_B) generated at the "far end" of a span when a moment (M_A) is applied at the near end. It is expressed as

$$COF_{A-B} = M_B/M_A \tag{11.26}$$

Values of COF are given in Tables 11.5 and 11.6. The sign of the carry over factor is given in the tables based on COF = $-M_B/M_A$ where positive M(+M) produces compression in the upper fibers of the beam. The values of COF in Tables 11.5 and 11.6 are for spans with constant E.I. If the span does not have a constant E.I. the COF's in Tables 11.5 and 11.6 cannot be used. A unit moment is applied at end A and resultant moment at end B is calculated; then COF = M_B/M_A.

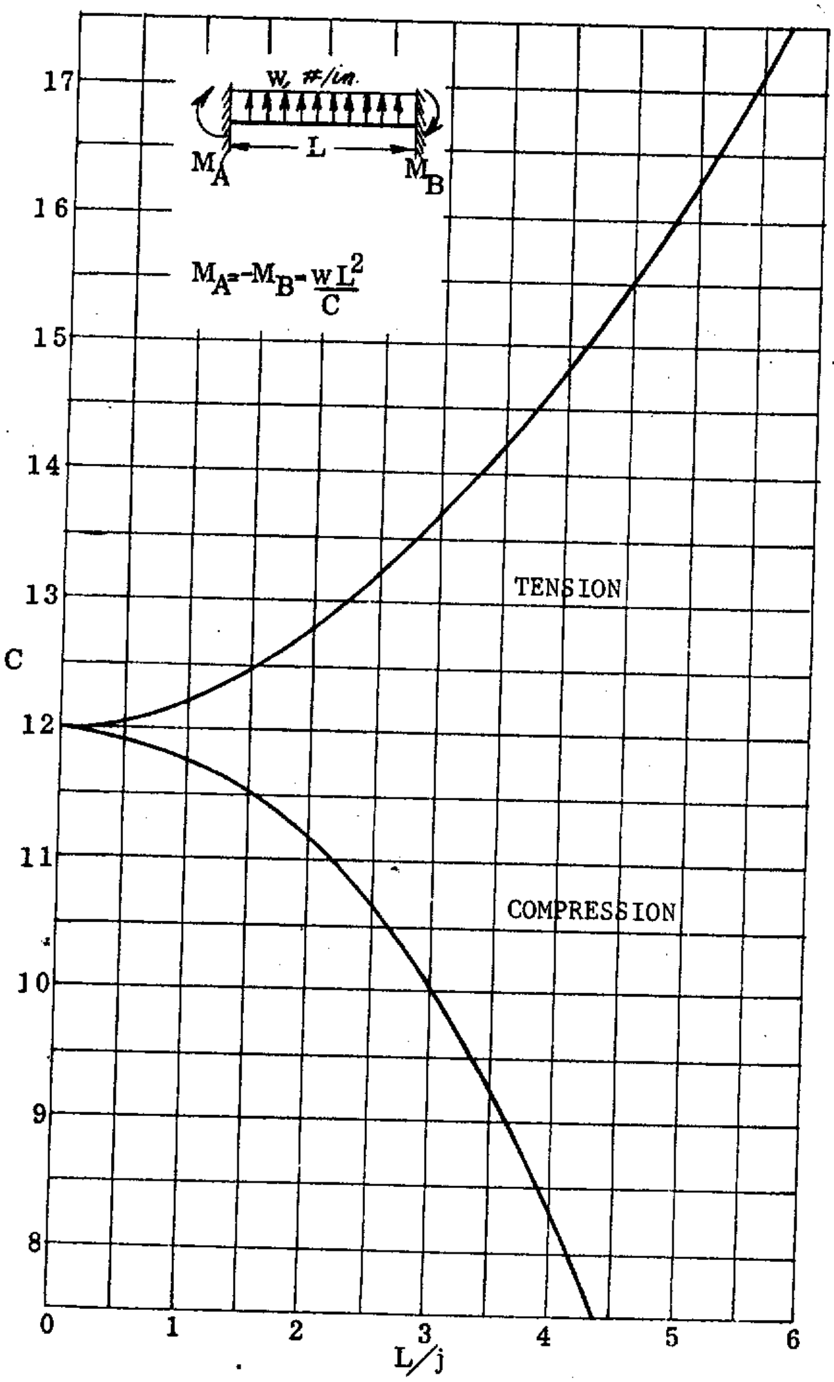


FIGURE 11.66 - FIXED END MOMENTS FOR UNIFORM LOAD WITH TENSION OR COMPRESSION

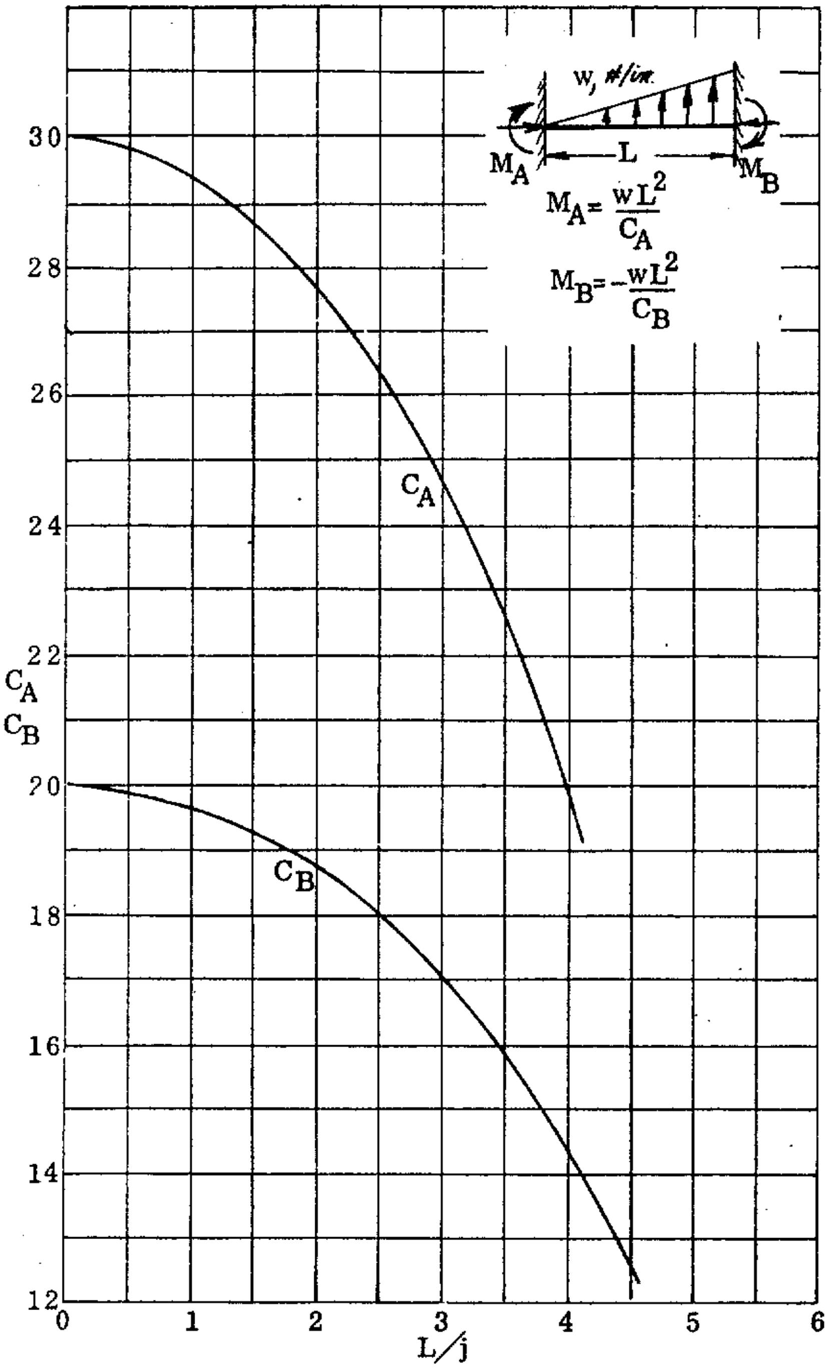


FIGURE 11.67 - FIXED END MOMENTS FOR UNIFORMLY VARYING LOAD WITH AXIAL COMPRESSION

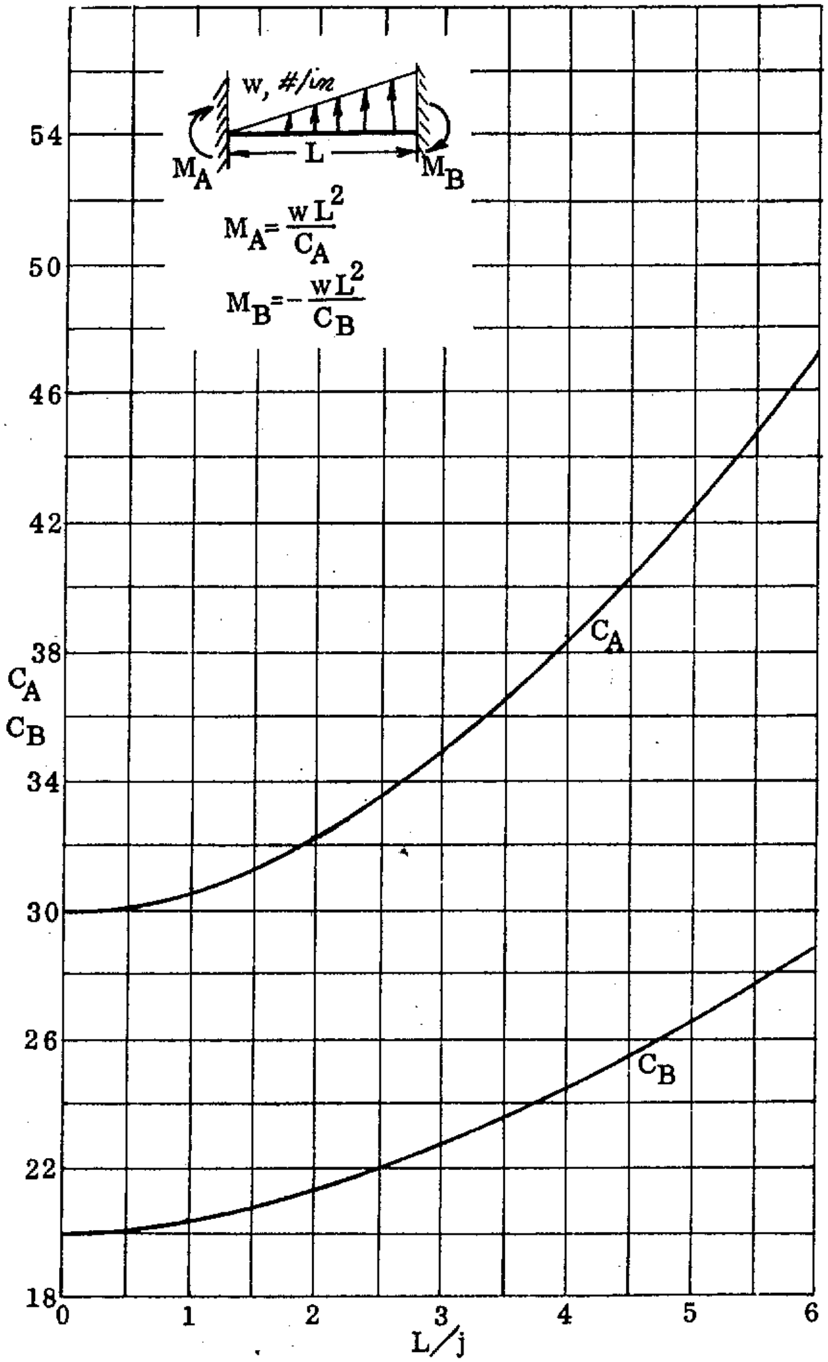


FIGURE 11.68 - FIXED END MOMENTS FOR UNIFORMLY VARYING LOAD WITH AXIAL TENSION

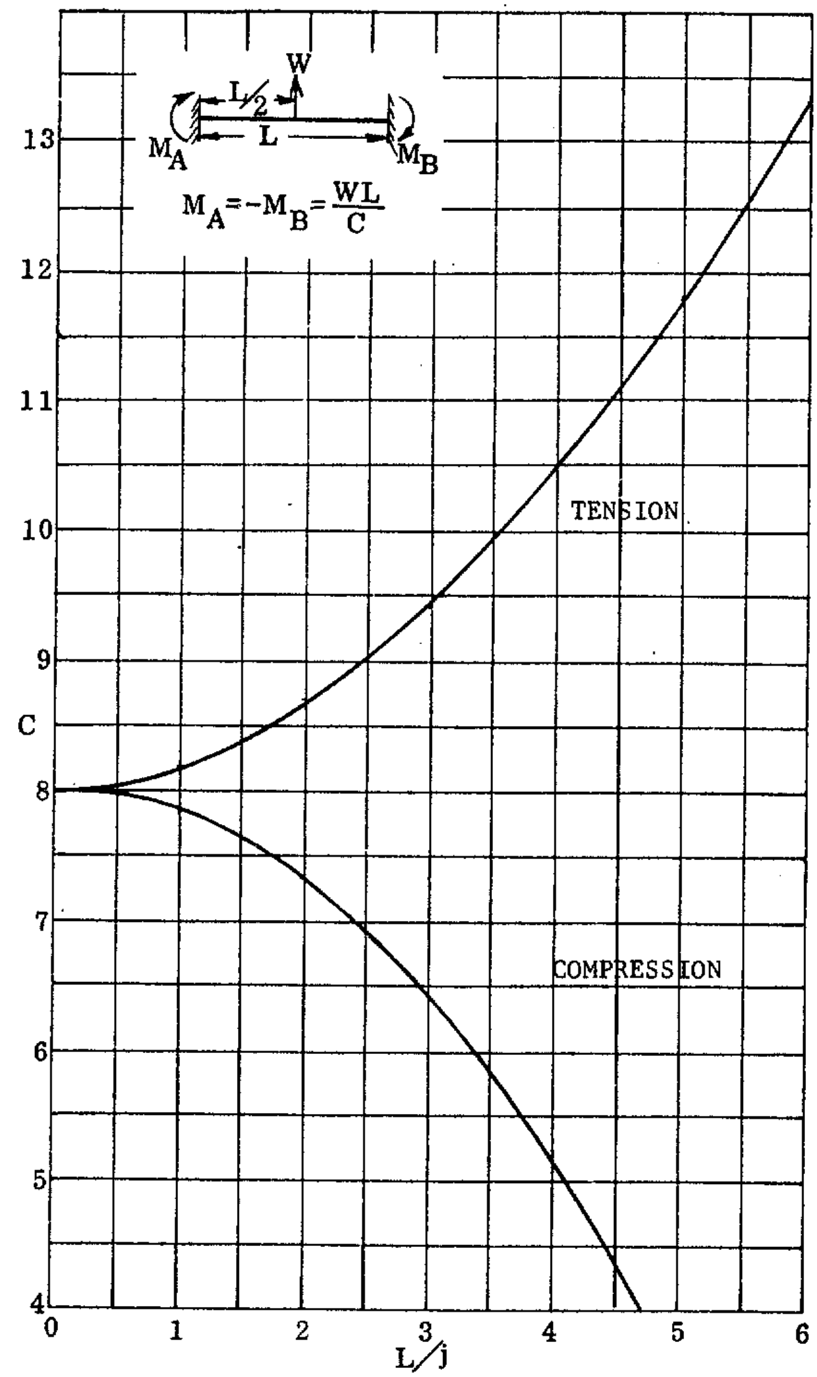


FIGURE 11.69 - FIXED END MOMENTS FOR CONCENTRATED LOAD AT CENTER WITH TENSION OR COMPRESSION

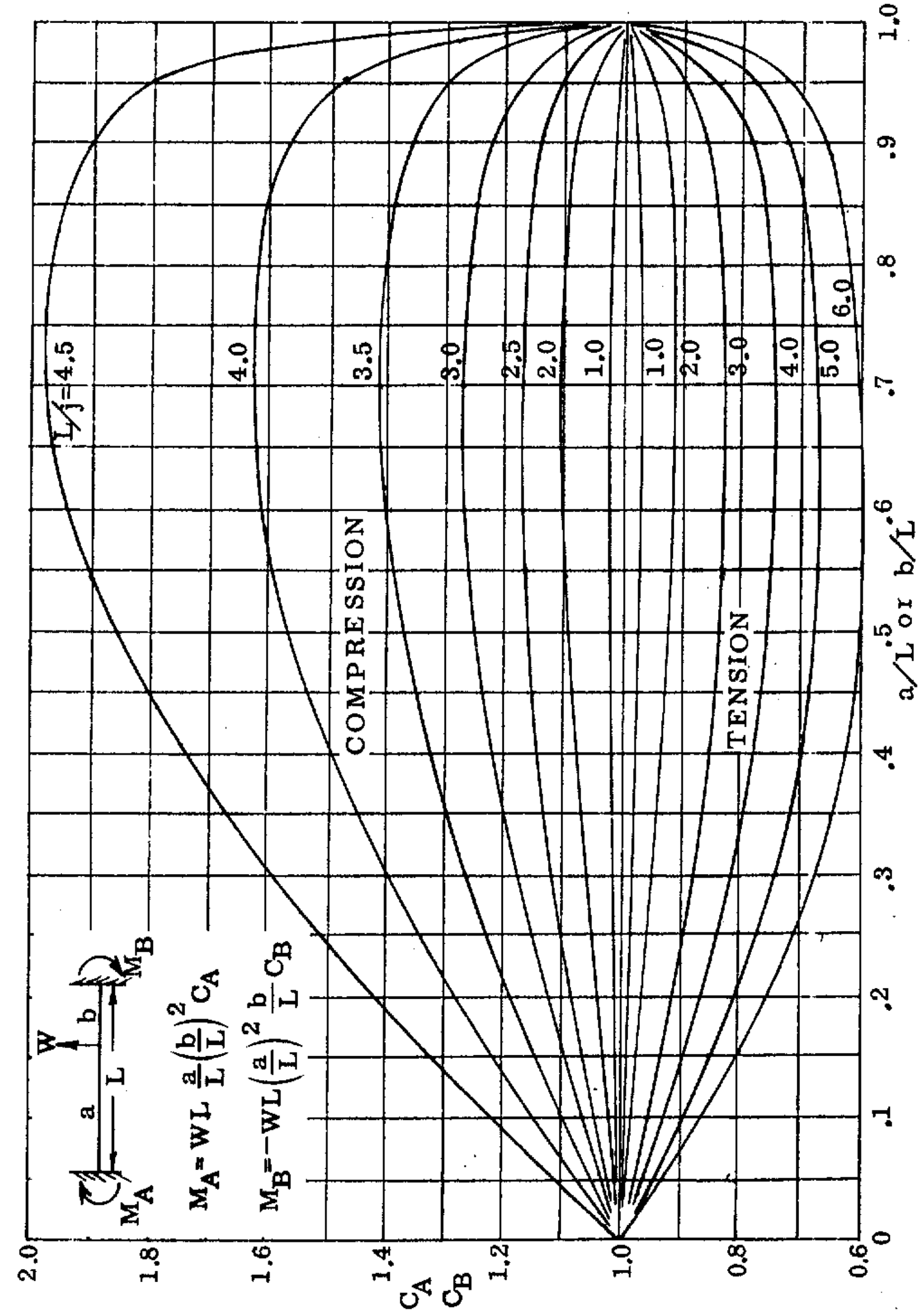


FIGURE 11.70 - FIXED END MOMENTS FOR CONCENTRATED LOAD WITH TENSION OR COMPRESSION

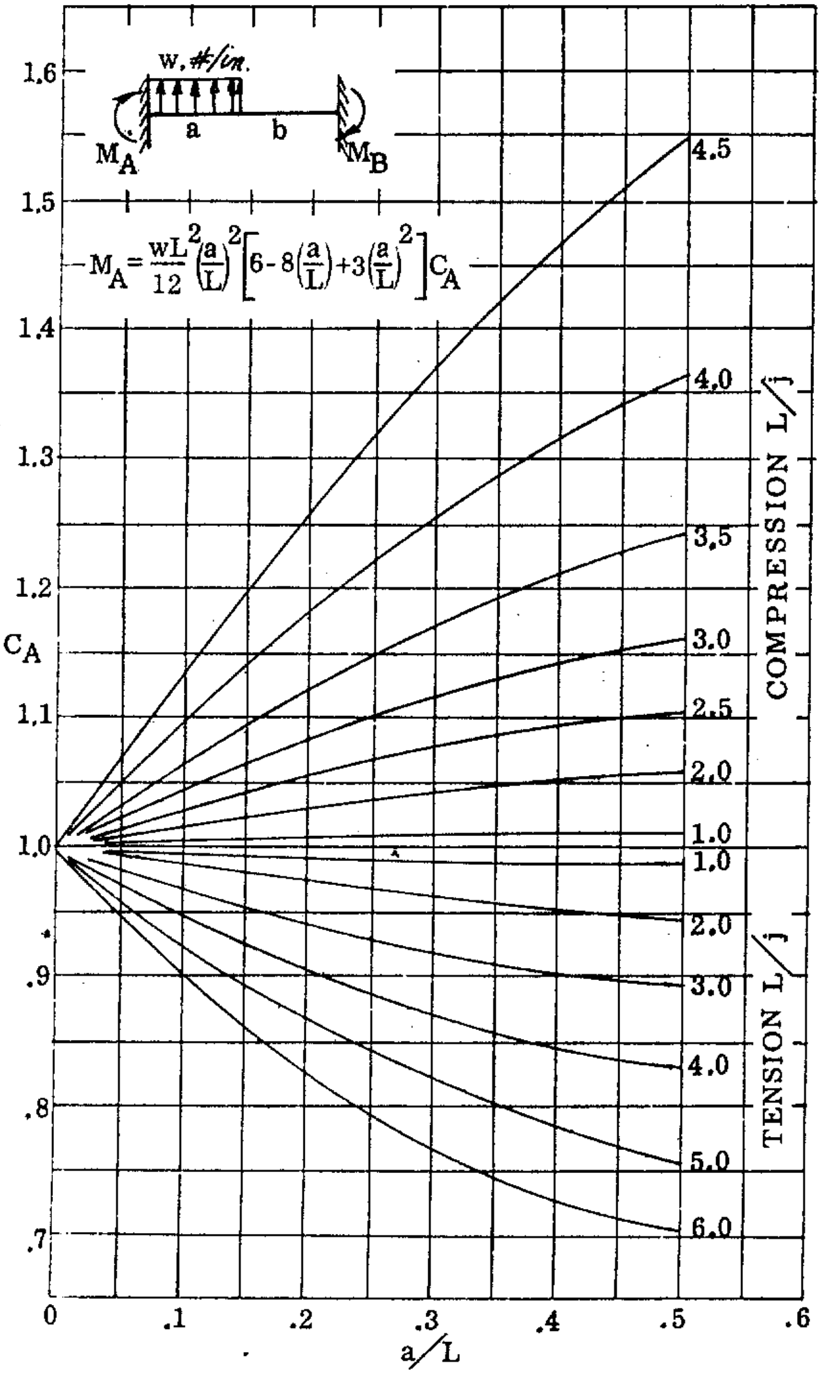


FIGURE 11.71 - FIXED END MOMENTS FOR Ma WITH UNIFORM LOAD OVER PART OF SPAN

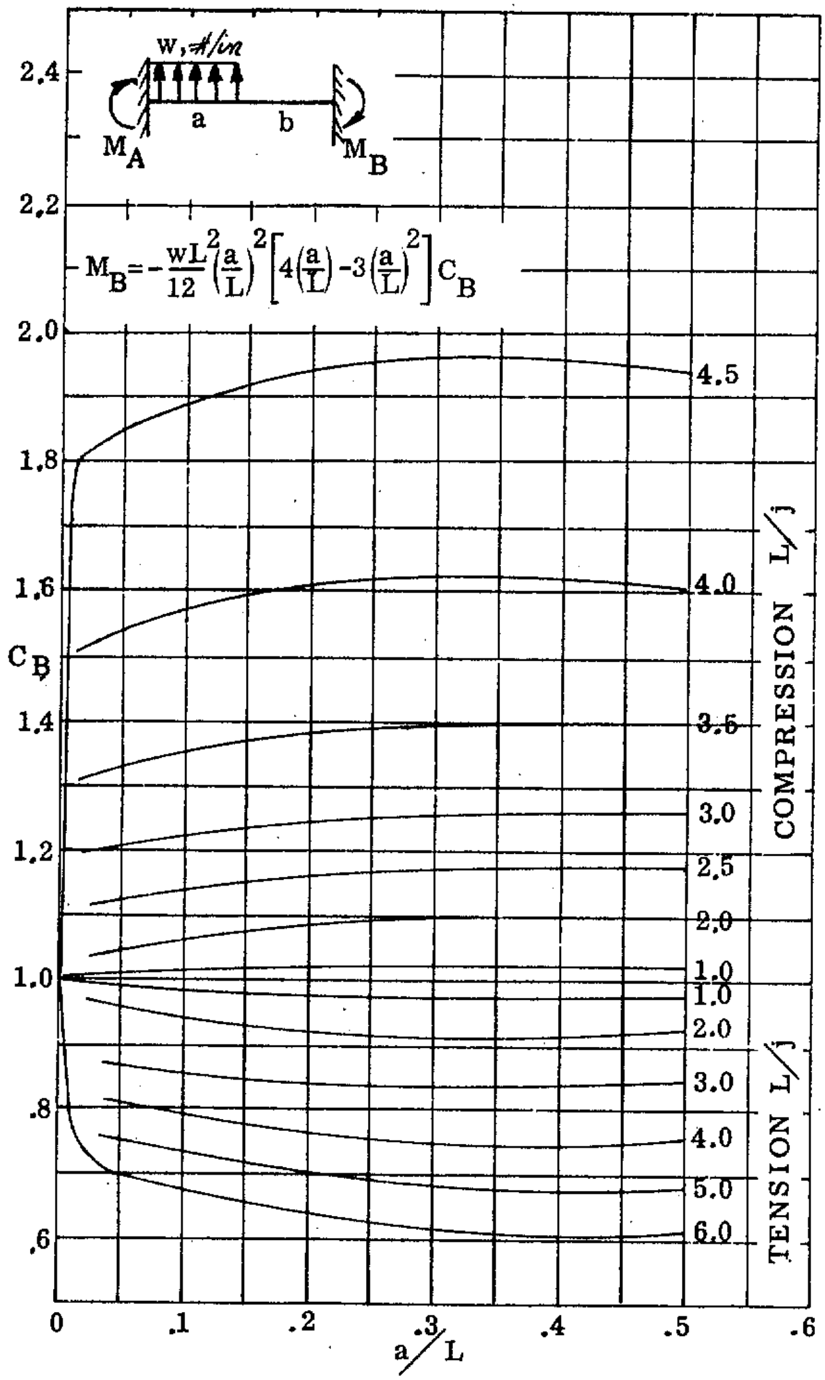


FIGURE 11.72 - FIXED END MOMENTS FOR Mb WITH UNIFORM LOAD OVER PART OF SPAN

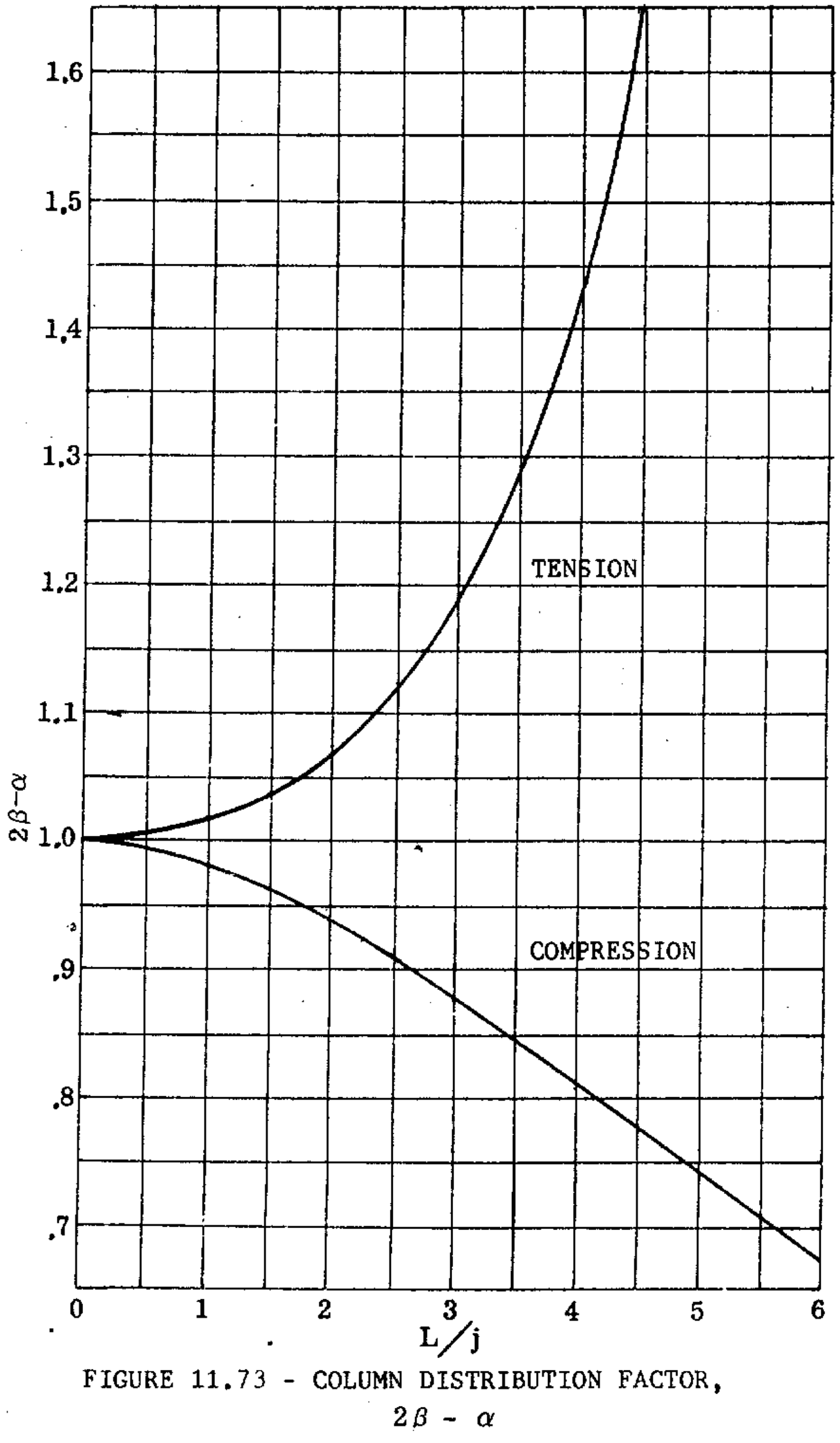


FIGURE 11.73 - COLUMN DISTRIBUTION FACTOR, $2\beta - \alpha$

L/j	C.O.F.	S.C. Far End Pinned	S.C. Far End Fixed
0.0	0.500000	0.750000	1.000000
.1	.500243	.749505	.999664
.2	.501001	.747996	.998663
.3	.502260	.746408	.996996
.4	.504034	.741963	.994656
.5	.506333	.737410	.991639
.6	.509173	.731812	.987943
.7	.512572	.725149	.983561
.8	.516588	.717398	.978486
.9	.521146	.708528	.972709
1.0	.526380	.698505	.966221
1.05	.529249	.693049	.962707
1.10	.532298	.687289	.959011
1.15	.525517	.681220	.955131
1.20	.528929	.674834	.951066
1.25	.524535	.668125	.946814
1.30	.546341	.661086	.942374
1.35	.550355	.653709	.937742
1.40	.554505	.645984	.933917
1.45	.559041	.637903	.927896
1.50	.563733	.629457	.922679
1.55	.568668	.620634	.917261
1.60	.573861	.611423	.911641
1.65	.579821	.601812	.905815
1.70	.585062	.591788	.8999781
1.75	.591098	.581337	.893536
1.80	.597444	.570444	.887077
1.85	.604116	.559093	.880400
1.90	.611131	.547266	.878502
1.95	.618508	.534944	.866379
2.00	.626268	.522107	.859028
2.05	.634433	.508733	.851444
2.10	.643028	.494798	.843624
2.15	.652078	.480276	.835563
2.20	.661613	.465139	.827255
2.25	.671664	.449357	.818697
2.30	.682268	.432895	.809090
2.35	.693454	.415718	.800809
2.40	.705272	.397785	.791468
2.45	.717765	.379058	.781854
2.50	.730983	.359473	.771961
2.55	.744980	.336996	.761782
2.60	.759820	.317560	.751311
2.65	.775563	.295101	.740541
2.70	.792302	.271547	.729462
2.75	.810106	.246820	.718069
2.80	.829074	.220831	.706351
2.85	.849313	.193479	.694300
2.90	.870941	.164654	.681906
2.95	.894096	.134228	.669160
3.00	.918930	.102060	.656050

L/j	C.O.F.	S.C. Far End Pinned	S.C. Far End Fixed
3.05	.945618	.0679879	.642565
3.10	.974360	.0318257	.628694
	1.000000	0	.616850
3.15	1.00539	-.006639	.614429
3.20	1.03897	-.047650	.599738
3.25	1.07541	-.091492	.584625
3.30	1.11506	-.138494	.569072
3.35	1.15837	-.189047	.553050
3.40	1.20503	-.243607	.536568
3.45	1.25802	-.302722	.619584
3.50	1.31569	-.367045	.502005
3.55	1.37870	-.437371	.484051
3.60	1.45111	-.514668	.465457
3.65	1.53126	-.600135	.446252
3.70	1.62179	-.695273	.426498
3.75	1.72480	-.801982	.406077
3.80	1.84302	-.922712	.384989
3.85	1.97998	-1.06067	.363202
3.90	2.14045	-1.22015	.340680
3.95	2.33090	-1.40701	.317386
4.00	2.56049	-1.62948	.293278
4.05	2.84246	-1.89953	.268312
4.10	3.19680	-2.23517	.242427
4.15	3.65515	-2.66491	.215605
4.20	4.27073	-3.23670	.187753
4.25	5.14045	-4.03787	.158820
4.30	6.46148	-5.24604	.128735
4.35	8.70595	-7.28654	.097422
4.40	13.3569	-11.4953	.064797
4.45	20.7800	-25.3636	.030766
4.48	92.9960	-83.2686	.009629
4.49	365.751	-328.980	.0002459
4.50	-188.680	170.465	-.004788
4.51	-75.1658	68.2040	-.012074
4.53	-34.0630	31.1862	-.026871
4.55	-22.0401	20.3312	-.041940
4.60	-11.7074	11.0020	-.080859
4.65	-7.97597	7.61877	-.121674
4.70	-6.05282	5.86397	-.164549
4.75	-4.88096	4.78531	-.209664
4.80	-4.09288	4.05174	-.257227
4.85	-5.52721	3.51784	-.307471
4.90	-3.10197	3.10973	-.360666
4.95	-2.77106	2.78587	-.417117
5.00	-2.50666	2.52111	-.477180
5.05	-2.29087	2.29934	-.541267
5.10	-2.11173	2.10973	-.608855
5.15	-1.96093	1.94475	-.683511
5.20	-1.88251	1.79898	-.762898
5.25	-1.72210	1.66844	-.848813
5.30	-1.67640	1.55011	-.942210

TABLE 11.5 - CARRY OVER FACTORS AND STIFFNESS COEFFICIENTS FOR BEAM COLUMNS WITH AXIAL COMPRESSION LOAD

L/j	C.O.F.	S.C. Far End Pinned	S.C. Far End Pinned
5.35	-1.54291	1.44166	-1.04434
5.40	-1.46967	1.34126	-1.15634
5.45	-1.40514	1.24748	-1.20024
5.50	-1.34009	1.15911	-1.41816
5.55	-1.29753	1.07519	-1.57289
5.60	-1.25266	.994913	-1.74806
5.65	-1.21282	.917677	-1.94542
5.70	-1.17746	.842593	-2.18038
5.75	-1.14618	.769440	-2.45268
5.80	-1.11056	.697656	-2.77765
5.85	-1.09431	.626825	-3.17364
5.90	-1.07319	.556562	-3.66787
5.95	-1.05500	.486507	-4.30474
6.00	-1.03956	.416317	-5.15938
6.05	-1.02677	.345650	-6.37157
6.10	-1.01651	.274197	-8.23357
6.15	-1.00874	.201600	-11.4768
6.20	-1.00342	.127521	-18.5908
6.25	-1.00055	.051597	-47.0647
2 π	-1.00000	0	

TABLE 11.5 (CONT'D) - CARRY OVER FACTORS AND STIFFNESS COEFFICIENTS FOR BEAM COLUMNS WITH AXIAL COMPRESSION LOAD

L/j	C.O.F.	S.C. Far End Pinned	S.C. Far End Fixed
0.0	0.500000	0.750000	1.00000
.1	.499757	.750512	1.00036
.2	.499001	.751990	1.00133
.3	.497760	.754488	1.00300
.4	.496033	.757964	1.00532
.5	.493831	.762412	1.00831
.6	.491167	.767818	1.01194
.7	.488057	.774165	1.01623
.8	.484519	.781431	1.02116
.9	.480575	.789595	1.02672
1.0	.476246	.798632	1.03291
1.1	.471556	.808515	1.03971
1.2	.466530	.819215	1.04712
1.3	.461194	.830700	1.05513
1.4	.455575	.842949	1.06372
1.5	.449699	.855921	1.07289
1.6	.443594	.869506	1.08262
1.7	.437286	.883915	1.09290
1.8	.430802	.898873	1.10371
1.9	.424167	.914429	1.11505
2.0	.417408	.930553	1.12689
2.1	.410548	.947214	1.13923
2.2	.403610	.964380	1.15205
2.3	.396616	.982024	1.16534
2.4	.389588	1.00012	1.17906
2.5	.382544	1.01863	1.19325
2.6	.375502	1.03754	1.20785
2.7	.368480	1.05683	1.22287
2.8	.361492	1.07646	1.23827
2.9	.354553	1.09642	1.25406
3.0	.347676	1.11668	1.27022
3.1	.340871	1.13722	1.28673
3.2	.334149	1.15803	1.30358
3.3	.327520	1.17909	1.32076
3.4	.320990	1.20037	1.33826
3.5	.314566	1.22188	1.35606
3.6	.308255	1.24350	1.37416
3.7	.302063	1.26547	1.39233
3.8	.295990	1.28754	1.41117
3.9	.290042	1.30976	1.43007
4.0	.284221	1.33214	1.44921
4.1	.278529	1.35466	1.46059
4.2	.272967	1.37731	1.40020
4.3	.267535	1.40009	1.50002
4.4	.262234	1.42297	1.52505
4.5	.257068	1.44597	1.54820
4.6	.252022	1.46907	1.56870
4.7	.247110	1.49225	1.58930
4.8	.242324	1.51553	1.61005
4.9	.237664	1.53889	1.63101
5.0	.233128	1.56232	1.65211
5.1	.228713	1.58503	1.67836
5.2	.224418	1.60940	1.69476
5.3	.220239	1.62304	1.71629
5.4	.216176	1.65674	1.73793
5.5	.212224	1.68049	1.75974
5.6	.206381	1.70429	1.78166
5.7	.204645	1.72814	1.80360
5.8	.201012	1.75204	1.82502
5.9	.197481	1.77399	1.84006
6.0	.194048	1.79997	1.87040
6.1	.190711	1.82400	1.89204
6.2	.187466	1.84806	1.91837
6.3	.184312	1.87215	1.93799
6.4	.181245	1.89628	1.96069
6.5	.178264	1.92044	1.98348
6.6	.175364	1.94463	2.00638
6.7	.172544	1.96885	2.02927
6.8	.169802	1.99310	2.05227
6.9	.167134	2.01737	2.07534
7.0	.164539	2.04166	2.09847

TABLE 11.6 - CARRY OVER FACTORS AND STIFFNESS COEFFICIENTS FOR BEAM COLUMNS WITH AXIAL TENSION LOAD

L/j	C.O.F.	S.C. Far End Pinned	S.C. Far End Fixed
7.1	.162013	2.06598	2.18167
7.2	.159556	2.09032	2.14493
7.3	.157164	2.11468	2.16824
7.4	.154836	2.18906	2.19160
7.5	.152570	2.16346	2.21502
7.6	.150363	2.18788	2.23849
7.7	.148213	2.21231	2.26200
7.8	.146119	2.23676	2.28556
7.9	.144079	2.26123	2.30917
8.0	.142090	2.28571	2.33281
8.1	.140152	2.31021	2.35630
8.2	.138263	2.33472	2.38022
8.3	.136421	2.35925	2.40399
8.4	.134625	2.38378	2.42778
8.5	.132872	2.40833	2.45162
8.6	.131162	2.43289	2.47540
8.7	.129494	2.45747	2.49938
8.8	.127865	2.48205	2.52381
8.9	.126275	2.50665	2.54726
9.0	.124722	2.53128	2.57125
9.1	.123206	2.55586	2.59526
9.2	.121724	2.58049	2.61830
9.3	.120277	2.60512	2.64286
9.4	.118862	2.62876	2.66748
9.5	.117480	2.65441	2.69186
9.6	.116128	2.67707	2.71569
9.7	.114806	2.70374	2.73985
9.8	.113513	2.72841	2.76402
9.9	.112248	2.75309	2.78822
10.0	.111010	2.77778	2.81244
10.5	.105202	2.90132	2.93379
11.0	.099863	3.02500	3.05553
11.5	.095216	3 14881	3.17762
12.0	.090396	3.27273	3.29999
12.5	.086948	3.39674	3.42261
13.0	.083328	3.52083	3.54545
13.5	.079997	3.64800	3.66048
14.0	.076921	3.76923	3.79167
14.5	.074873	3.89352	3.91500
15.0	.071428	4.01786	4.03846
15.5	.068965	4.14224	4.16204
16.0	.066666	4.26667	4.28571
16.5	.064516	4.39113	4.04940
17.0	.062500	4.51562	4.53333
17.5	.060606	4.64015	4.65726
18.0	.058824	4.76471	4.78125
18.5	.057143	4.88929	4.90530
19.0	.055555	5.01389	5.02941
19.5	.054054	5.13851	5.15357
20.0	.052632	5.26316	5.27770
21.	.050000	5.51250	5.52632
22	.047619	5.76190	5.77500
23	.045455	6.01136	6.02301
24	.043470	6.26087	6.27273
25	.041667	6.51042	6.58174
26	.040000	6.76000	6.77083
27	.035462	7.00962	7.02000
28	.037037	7.25926	7.26923
29	.033714	7.50893	7.51052
30	.034433	7.75862	7.76786
31	.033823	8.00833	8.01724
32	.032356	8.25806	8.26667
33	.031250	8.50781	8.51613
34	.030303	8.78768	8.76563
35	.029412	9.00735	9.01515
36	.028571	9.25714	9.26471
37	.027778	9.50694	9.51429
38	.027027	9.75676	9.76389
39	.026313	10.0066	10.0135
40	.025641	10.2564	10.2632
41	.025000	10.5063	10.5128
42	.024390	10.7561	10.7625
43	.023810	11.0060	11.0122
44	.023256	11.2550	11.2619
45	.022727	11.5057	11.5116
46	.022222	11.7536	11.7614
47	.021749	12.0034	12.0111
48	.021277	12.2553	12.2609
49	.020833	12.5052	12.5106
50	.020408	12.7551	12.7604

TABLE 11.6 (CONT'D) - CARRY OVER FACTORS AND STIFFNESS COEFFICIENTS FOR BEAM COLUMNS WITH AXIAL TENSION LOAD

Moment distribution (Hardy Cross method) is used to determine the moment in the beam at each support point. Once determined; (1) the reactions at each support can be determined from statics for each span, and (2) the bending moment and deflection at any point in a span between supports can be obtained by considering each span to be simply supported and with the applied axial and transverse loads the internal loading can be determined.

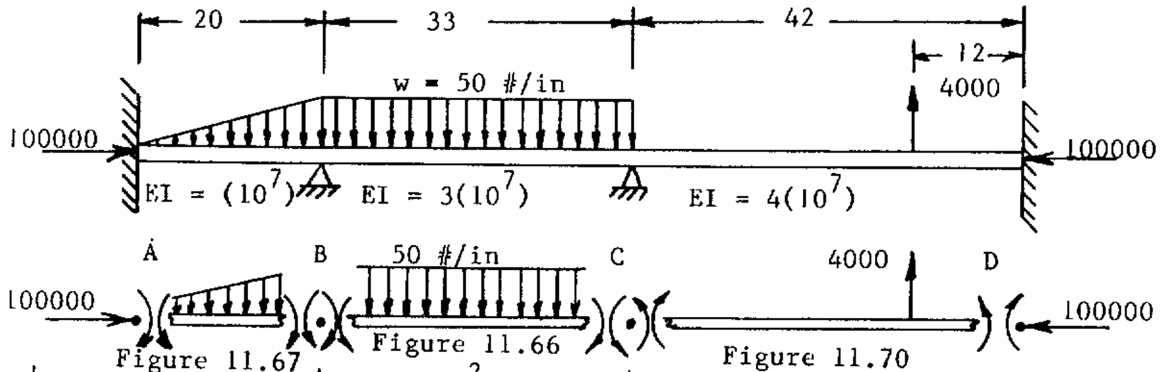
Moment distribution can be used as follows:

(1) Assume each span to be fixed against rotation at both ends.
(2) Determine the fixed end moments, FEM, for each span due to the applied transverse loads or moments using Figures 11.66 through 11.73.
(3) Determine the "net" moment at each joint. The net moment will be the difference in fixed end moments of the spans on each side of the joint plus any applied moment.
(4) Free a joint allowing it to rotate due to the net moment. Balance the joint by distributing this net moment to the members at the joint in proportion to their distribution factors (D.F.). The balancing moments are opposite in sign to the net moment. The sum of the moments at any joint is always zero after each balancing.

(5) Determine the moments at the far end of each span. This is done by multiplying each distributed moment by the members carry over factor (C.O.F.). This carry over moment is assumed to be acting on the joint at the far end when that joint is freed and balanced.
(6) Repeat for all joints.
(7) Repeat the entire process. This time only carry over moments will be balanced since there will be no fixed end moments.
(8) Repeat until the carry over moments are negligible.
(9) Add up all balancing and carry over moments at each joint to obtain the final moment at the joint.

In the moment distribution process the sign of all moments (FEM, COM and balancing moments) are defined by the direction in which they act on the joint between each span. If the moment tends to rotate the joint clockwise it is positive (+). If the moment tends to rotate the joint counterclockwise it is negative (-).

Figure 11.74 is an example of the moment distribution as applied to a beam column with a compression load. Figure 11.75 shows the same example with a tension load.



FEM (Figure 11.66-11.73)

- A: $\frac{50(20)^2}{27.7} = 722$
- B: $\frac{50(20)^2}{18.75} = -1067$
- B: $\frac{50(33)^2}{11.2} = 4862$
- C: $\frac{50(33)^2}{11.2} = 4862$
- C: $4000(\frac{30}{42})(\frac{12}{42})^2(42)(1.11) = 10847$
- : $-4000(\frac{30}{42})^2(\frac{12}{42})(42)(1.08) = -26449$

	AB	BC	CD
$j = \sqrt{EI/P}$	$\sqrt{10^7/10^5} = 10$	$\sqrt{3(10^7)/10^5} = 17.32$	$\sqrt{4(10^7)/10^5} = 20$
L/j	20/10 = 2	33/17.32 = 1.905	42/20 = 2.1

S.C.(T11.5)		.859	.859	.873	.873	.844	.844	
S.F.(E11.23)	∞	$1.72(10^6)$	$1.72(10^6)$	$3.17(10^6)$	$3.17(10^6)$	$3.22(10^6)$	$3.22(10^6)$	∞
Σ S.F.	∞		$4.89(10^6)$		$6.39(10^6)$			∞
D.F.(E11.25)	1.0	0	.351	.649	.497	.503	0	1.0
C.O.F.(T11.5)		.627	.627	.612	.612	.643	.643	
FEM	0	722	-1067	4862	-4862	-10847	26449	0
1st BAL	-722	0	-1322	-2463	7807	7902	0	-26449
C.O.	0	-835	0	4778	-1507	0	5081	0
2nd BAL	835	0	-1677	-3101	749	578	0	-5081
C.O.	0	-1051	0	458	-1898	0	487	0
3rd BAL	1051	0	-161	-297	943	955	0	-487
C.O.	0	-101	0	577	-182	0	614	0
4th BAL	101	0	-203	-374	90	92	0	-614
C.O.	0	-127	0	55	-229	0	59	0
5th BAL	127	0	-19	-36	114	115	0	-59
Final Moment	1392	-1392	-4459	4459	1025	-1025	32690	-32690

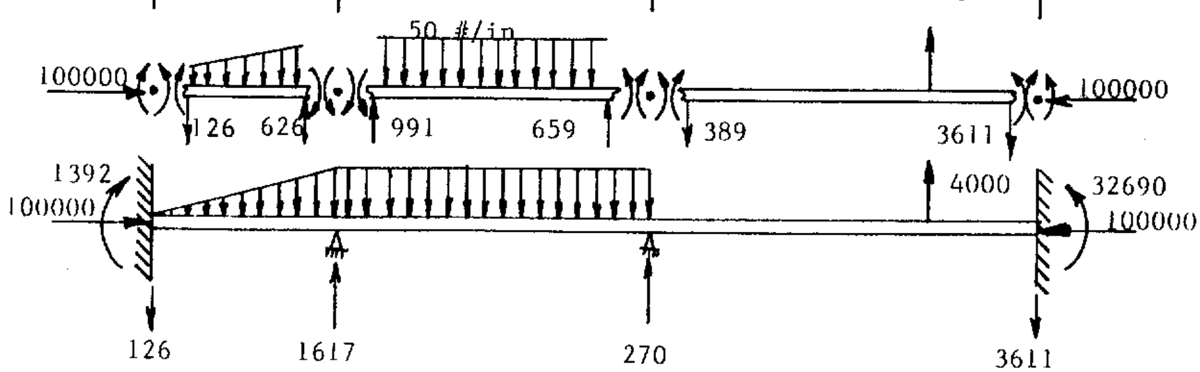


FIGURE 11.74 - EXAMPLE OF BEAM COLUMN WITH AXIAL COMPRESSION BY THE MOMENT DISTRIBUTION METHOD

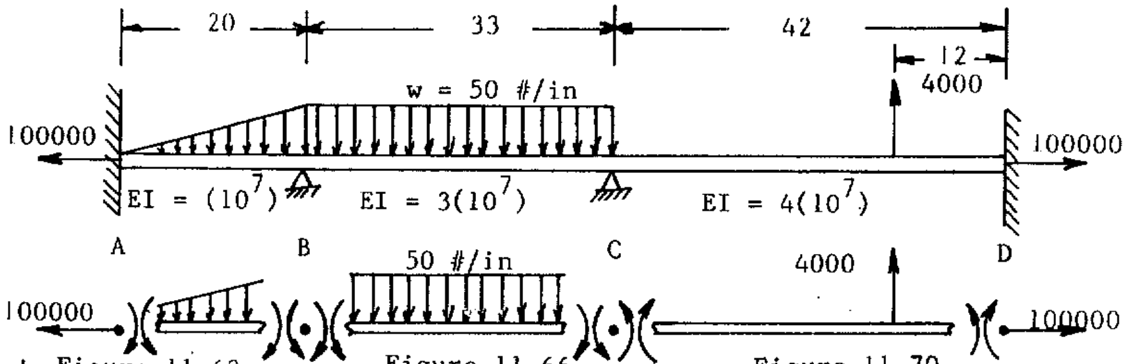


		Figure 11.68		Figure 11.66		Figure 11.70		
FEM (Figure 11.66-11.73)		A: $\frac{50(20)^2}{32.2} = 621$	B: $-\frac{50(20)^2}{21.3} = -939$	B: $\frac{50(20)^2}{12.7} = 4287$	C: $-\frac{50(20)^2}{12.7} = -4287$	C: $4000(42)(\frac{30}{42})(\frac{12}{42})^2(.91) = 8911$	D: $-4000(42)(\frac{30}{42})^2(\frac{12}{42})(.925) = -22676$	
$j = \sqrt{EI/P}$		$\sqrt{10^7/10^5} = 10$		$\sqrt{3(10^7)/10^5} = 17.32$		$\sqrt{4(10^7)/10^5} = 20$		
L/j		20/10 = 2		33/17.32 = 1.905		42/20 = 2.1		
S.C.(T11.6)		1.127	1.127	1.116	1.116	1.139	1.139	
S.F.(E11.23)	∞	$2.25(10^6)$	$2.25(10^6)$	$4.06(10^6)$	$4.06(10^6)$	$4.34(10^6)$	$4.34(10^6)$	∞
ΣS.F.		∞	$6.31(10^6)$		$8.4(10^6)$			$4.34(10^6)$
D.F.(E11.25)	1.0	0	.357	.643	.483	.517	0	1.0
C.O.F.(T11.6)		.412	.417	.424	.424	.411	411	
FEM	0	621	-939	4287	-4287	-8911	22676	0
1st BAL	-621	0	-1195	-2153	6375	6823	0	-22676
C.O.		-498	0	2703	913	0	2804	
2nd BAL	498	0	-965	-1738	-441	-472	0	-2804
C.O.		-402	0	-187	-737	0	-194	
3rd BAL	402	0	67	120	356	381	0	194
C.O.		28	0	151	51	0	157	
4th BAL	-28	0	-54	-97	-25	-26	0	-157
C.O.		-23	0	-11	-41	0	-11	
5th BAL	23		5	6	20	21	0	11
FINAL MOMENT	274	-274	-3081	3081	2184	-2184	25432	-25432

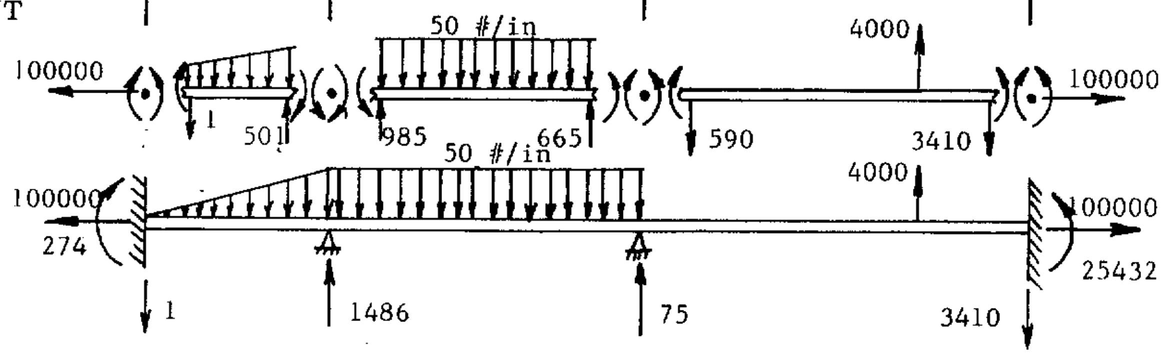


FIGURE 11.75 - EXAMPLE OF BEAM COLUMN WITH AXIAL TENSION BY THE MOMENT DISTRIBUTION METHOD

The slope and deflection at any point in the beam may be determined by considering each span separately and using the equations presented in Tables 11.3 and 11.4.

11.2.4 Control Rod Design

Control rods are characteristically long rods with swaged ends having rod ends threaded into them. Two column analyses are necessary to insure the rods are sufficient. A typical control rod is shown below.

The column must be stable as a stepped column using the analysis in Section 11.1.3. It must also satisfy the beam column analysis. Following is the procedure for control rod design:

(1) Calculate P_{cr} for the stepped column using Section 11.1.3.

(2) Determine P_{all} from

$$\left(1 - \frac{P_{all}}{P_{cr}}\right)\left(\frac{1}{P_{all}/P_{cr}} - \frac{P_{cr}}{F_{cc}A}\right) = \frac{P_{cr}\,e}{M_{all}}$$

where

P_{cr} = Critical column load, Section 11.1.3

F_{cc} = Crippling stress of column section

A = Cross sectional area

e = L/800

M_{all} = Allowable moment of section in plastic bending

(3) M.S. = $P_{all}/P - 1$

11.2.5 Beam Columns By The Three-Moment Procedure

The three-moment procedure can be applied to beams carrying axial compression or tension in addition to transverse loading. The procedure is described in Section 9.4.

11.3 Torsional Instability of Columns

The previous sections have assumed that the column was torsionally stable; i.e., the column would either fail by bending in a plane of symmetry of the cross section, by crippling or by a combination of crippling and bending. There are cases when a column will fail either by twisting or by a combination of bending and twisting. These torsional buckling failures occur when the torsional rigidity of the section is very low. Thin walled open sections, for instance, can buckle by twisting at loads well below the Euler load. Often in thin open sections the centroid and shear center do not coincide, therefore, torsion and flexure interact.

In this section, it will be assumed that the plane cross sections of the column warp, but their geometric shape does not change during buckling; that is, the theories consider primary failure of columns and not secondary failures, characterized by distortion of the cross sections. There is coupling of primary and secondary failures but no method has been developed to handle them so secondary failures will be ignored.

11.3.1 Centrally Loaded Columns

Centrally loaded columns can buckle in one of three possible modes: (1) they can bend in the plane of one of the principal axes; (2) they can twist about the shear center axis; or (3) they can bend and twist simultaneously. Bending in the plane of one of the principal axes has been discussed previously. The latter two modes will be discussed here.

11.3.1.1 Two Axes of Symmetry

When the cross section has two axes of symmetry or is point symmetric, the shear center and centroid coincide. In this case, the purely torsional buckling load about the shear center is given by

$$P_{\phi} = 1/r_o^2 \; \{ GJ + E\Gamma \pi^2/L^2 \} \qquad (11.27)$$

where:

r_o = polar radius of gyration of the section about its shear center

G = shear modulus of elasticity

J = torsion constant (Section 8.0)

E = modulus of elasticity

Γ = warping constant of the section (Figure 11.85)

L = effective length of the member

For a cross section with two axes of symmetry there are three critical values of the axial load. They are the flexural buckling loads about the principal axes, P_x and P_y and the purely torsional buckling load, P_ϕ. One of these loads will be minimum and will determine the mode of failure. In this case there is no interaction and the column fails either in pure bending or in pure twisting. Shapes in this category include I-sections, Z-sections and cruciforms.

11.3.1.2 General Cross Section

In general a thin walled open section buckling occurs by a combination of torsion and bending. Purely flexural or purely torsional failure cannot occur. Consider a general section with the x and y axes the principal centroidal axes of the cross section and x_o and y_o the coordinates of the shear center. The cross section will undergo translation and rotation during buckling. The translation is defined by the deflections of the shear center u and v in the x and y directions. During translation of the cross section, point 0 moves to 0' and point C to C' where 0 is the shear center and C is the centroid. The cross section rotates an angle ϕ about the shear center. Equilibrium of a longitudinal element yields three simultaneous equations, the solution of which results in the following cubic equation for calculating the critical value of buckling load.

$$r_o^2(P_{cr}-P_y)(P_{cr}-P_x)(P_{cr}-P_\phi) - P_{cr}^2 y_o^2(P_{cr}-P_x) - P_{cr}^2 x_o^2(P_{cr}-P_y) = 0 \qquad (11.28)$$

where

$$P_x = \pi^2 EI_x/L^2 \qquad (11.29)$$

$$P_y = \pi^2 EI_y/L^2 \qquad (11.30)$$

$$P_\phi = 1/r_o^2(GJ + E\Gamma\,\pi^2/L^2) \qquad (11.31)$$

Solution of the cubic equation, 11.28, gives three values of the critical load, P_{cr}, of which the smallest will be used. The lowest value of P_{cr} will always be less than P_x, P_y, or P_ϕ. By use of the effective length, L, various end conditions can be considered.

11.3.1.3 Cross Sections With One Axis of Symmetry

A number of singly symmetric sections are shown in Figure 11.76. If the x-axis is considered to be the axis of symmetry, the $y_o = 0$ and the equation for a general section reduces to

$$(P_{cr} - P_y)\left\{ r_o^2(P_{cr} - P_x)(P_{cr} - P_\phi) - P_{cr}^2 x_o^2 \right\} = 0 \qquad (11.32)$$

There are again three solutions, one of which is $P_{cr} = P_y$ and represents purely flexural buckling about the y-axis. The other two are the roots of the quadratic term inside the brackets equated to zero and give two torsional-flexural buckling loads. The lowest torsional-flexural load will always be below P_x and P_ϕ. It may, however, be above or below P_y. Therefore, a singly symmetrical section (such as an angle, channel or hat) can buckle in either of two modes, by bending or in torsional-flexural buckling. Which of these two actually occurs depends on the dimensions and shape of the given section.

Failure of singly symmetrical sections can occur either in pure bending or in simultaneous bending and twisting. The evaluation of the torsional-flexural buckling load can never be as simple as the determination of the Euler load, therefore, steps have been taken to categorize modes of failure. Certain combinations of dimensions will prohibit torsional-flexural failures.

For sections symmetrical about the x-axis, the critical buckling load is given by equation 11.32. The load at which the member actually buckles is either P_y or the smaller root of the quadratic equation.

The buckling domain can be visualized as being composed of three regions. These are shown in Figure 11.77 for a section whose shape is defined by the ratio, b/a. Region 1 contains all sections for which $I_y > I_x$. In this region only torsional-flexural buckling can occur. Sections for which $I_x > I_y$ falls into Region 2 or 3. In Region 2, the mode of buckling depends on the parameter tL/a^2. The $(tL/a^2)_{min}$ curve represents the boundary between the two possible modes of failure. It is a plot of the value of tL/a^2 at which the buckling mode changes from purely flexural to torsional-flexural. The boundary between Regions 2 and 3 is located at the intersection of the $(tL/a^2)_{min}$ curve with the b/a axis. Sections in Region 3 will always fail in the flexural mode regardless of the value of tL/a^2.

Figure 11.78 defines these curves for angles, channels, and hat sections. In this figure, members that plot below and to the right of the curve fail in the torsional-flexural mode, whereas those to the left and above fail in the pure bending mode. The curves in Figure 11.78 also give the location of the boundaries between the various buckling domains. Each of the curves approaches a vertical asymptote, indicated as a dashed line in the figure. The asymptote, which is the boundary between Regions 1 and 2, is located at b/a corresponding to sections for which $I_x = I_y$. Sections with b/a larger than the transition value at the asymptote will always fail in torsional-flexural buckling, regardless of their other dimensions. If b/a is smaller than the value for the asymptote, then the section fails in Region 2 and failure can be either by pure flexural buckling or in the torsional-flexural mode. In this region, the parameter, tL/a^2, will determine which of the two possible modes of failure is critical. In the case of the plain and lipped channel section, there is a lower boundary Region 2. This transition occurs where the $(tL/a^2)_{lim}$ curve intersects the b/a axis. Sections for which b/a is less than the value at this intersection are located in Region 3. These sections will always fail in the flexural mode, regardless of the value of tL/a^2. For the lipped angle and hat sections the $(tL/a^2)_{lim}$ curve does not intersect the b/a axis. Region 3, where only flexural buckling occurs, does not exist for these sections.

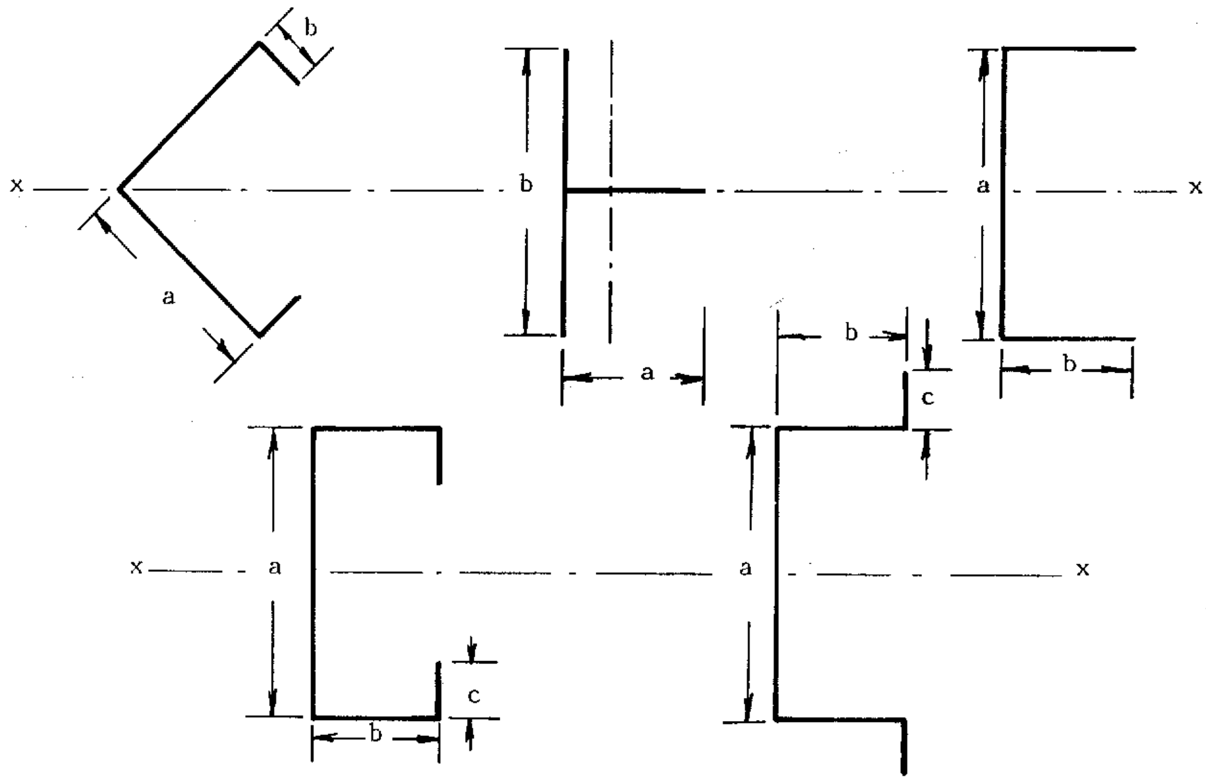


FIGURE 11.76 - SINGLY SYMMETRICAL SECTIONS

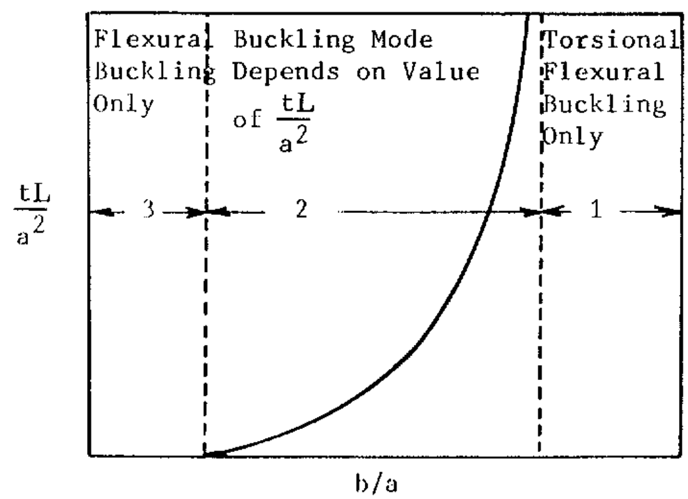


FIGURE 11.77 BUCKLING REGIONS

The critical buckling load for singly symmetrical sections (x-axis is the axis of symmetry) that buckle in the torsional-flexural mode is given by the lowest root of

$$r_o^2(P_{cr}-P_x)(P_{cr}-P_\phi) - P_{cr}^2 x_o^2 = 0. \qquad (11.33)$$

Dividing this equation by $P_x P_\phi r_o^2$, and rearranging results in the following interaction equation:

$$\frac{P}{P_\phi} + \frac{P}{P_x} - K\frac{P^2}{P_x P_\phi} = 1 \qquad (11.34)$$

in which

$$K = 1 - \left(\frac{x_o}{r_o}\right)^2 \qquad (11.35)$$

is a shape factor that depends on geometrical properties of the cross section.

Figure 11.79 is a plot of equation 11.34. This plot provides a simple method for checking the safety of a column against failure by torsional-flexural buckling.

To determine if a given member can safely carry a certain load, P, it is only necessary to compute P_x and P_ϕ for the section in question and then, knowing K, use the correct curve to check whether the point determined by the arguments P/P_x and P/P_ϕ falls below (safe) or above (unsafe) the pertinent curve. If it is desired to determine the critical load of a member instead of ascertaining whether it can safely carry a given load, use

$$P_{cr} = \frac{1}{2K}\left\{(P_\phi + P_x) - \sqrt{(P_\phi + P_x)^2 - 4KP_\phi P_x}\right\} \qquad (11.36)$$

which is another form of equation 11.34.

The interaction equation 11.34 indicates that P_{cr} depends on three factors: the loads, P_x and P_ϕ, and the shape factor, K. P_x and P_ϕ are the two factors which interact, while K determines the extent to which they interact. The reason bending and twisting interact is that the shear center and the centroid do not coincide. A decrease in x_o, the distance between these points, therefore causes a decrease in the interaction.

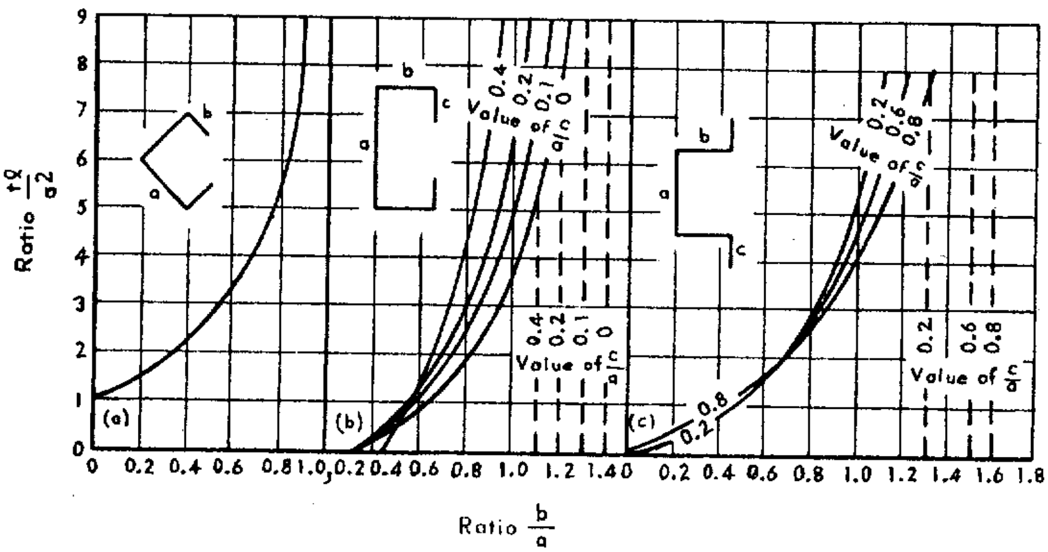


FIGURE 11.78 - BUCKLING MODE OF SINGLY SYMMETRICAL SECTIONS

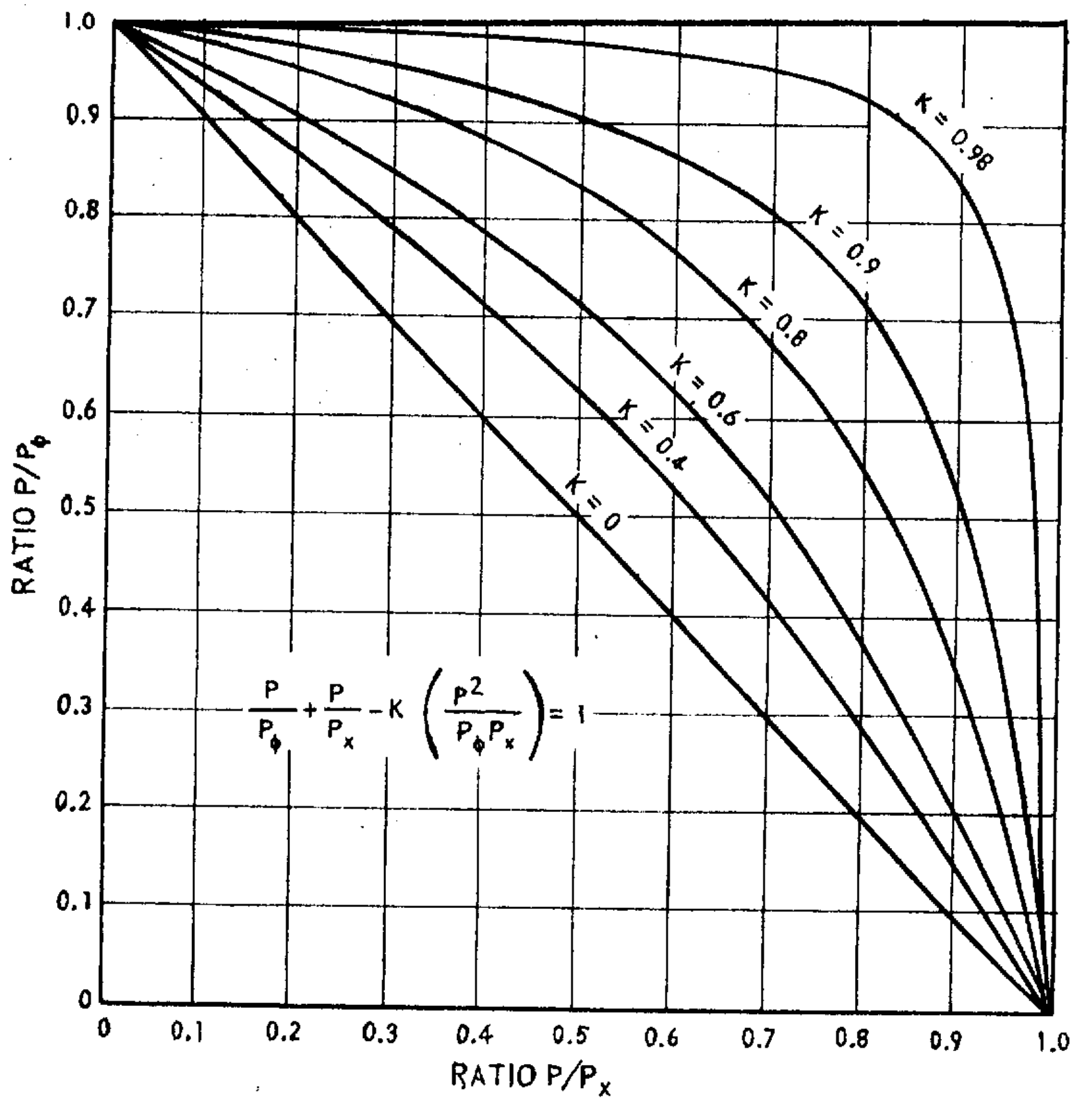


FIGURE 11.79 - INTERACTION CURVES

To evaluate the torsional-flexural buckling load by means of the interaction equation, it is necessary to know P_ϕ and K. A convenient method for determining these two parameters is therefore an assential part of the procedure.

For any given section, K is a function of certain parameters that define the shape of the section. Starting with equation 11.35 and substituting for x_o and r_o, K can be reduced to an expression in terms of one or more of these parameters. If the thickness of the member is uniform, the parameters will be of the form b/a, in which a and b are the widths of two of the flat components of the section. In the case of a tee section, for example, equation 11.35 can be reduced to:

$$K = 1 - \frac{4}{\{1 + b/a\}\{(b/a)^3 + 1\}} \qquad (11.37)$$

in which b/a is the ratio of the flange to the leg width (Fig. 11.76).

In general, the number of elements of which a section is composed and the number of width ratios required to define its shape will determine the complexity of the relation for K. Because all equal-legged angles without lips have the same shape, K is a constant for this section. For channels and lipped angles, K is a function of a single variable, b/a, while lipped channels and hat sections require two parameters, b/a and c/a, to define K (Fig. 11.76).

Curves for determination of K have been obtained for angles, channels, and hat sections. These curves are shown in Figures 11.80 and 11.81. A single curve covers all equal-legged lipped angle sections. The value of K for all plain equal-legged angles, K = 0.625, is given by the point b/a = 0 on this curve (Fig. 11.80). For hats and channels (Fig. 11.81), a series of curves is given.

The evaluation of P_ϕ follows the same scheme as that used to determine K. Starting with the equation for P_ϕ, given in equation 11.27, and substituting for r_o, J, and Γ yields:

$$P_\phi = EA\{C_1(t/a)^2 + C_2(a/l)^2\} \qquad (11.38)$$

a general relation for P_ϕ, in which, E = Young's modulus, A = cross-sectional area; t = the thickness of the section; L = effective length of the member; a = the width of one of the elements of the section; and C_1 and C_2 = functions of b/a and c/a, in which b and c are the widths of the remaining elements.

Equation 11.35 indicates the important parameters in torsional buckling and their effect on the buckling load. Similar to Euler buckling, P_ϕ varies directly with E and A. The term inside the bracket consists of two parts, the St. Venant torsional resistance and the warping resistance. In the first of these, the parameter, t/a, indicates the decrease in torsional resistance with decreasing relative wall thickness; whereas, in the second the parameter a/L shows the decrease in warping resistance with increasing slenderness.

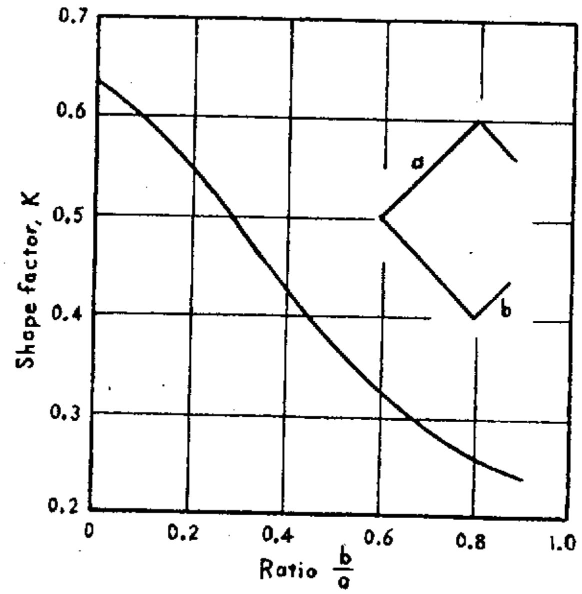


FIGURE 11.80 - SHAPE FACTOR FOR ANGLES.

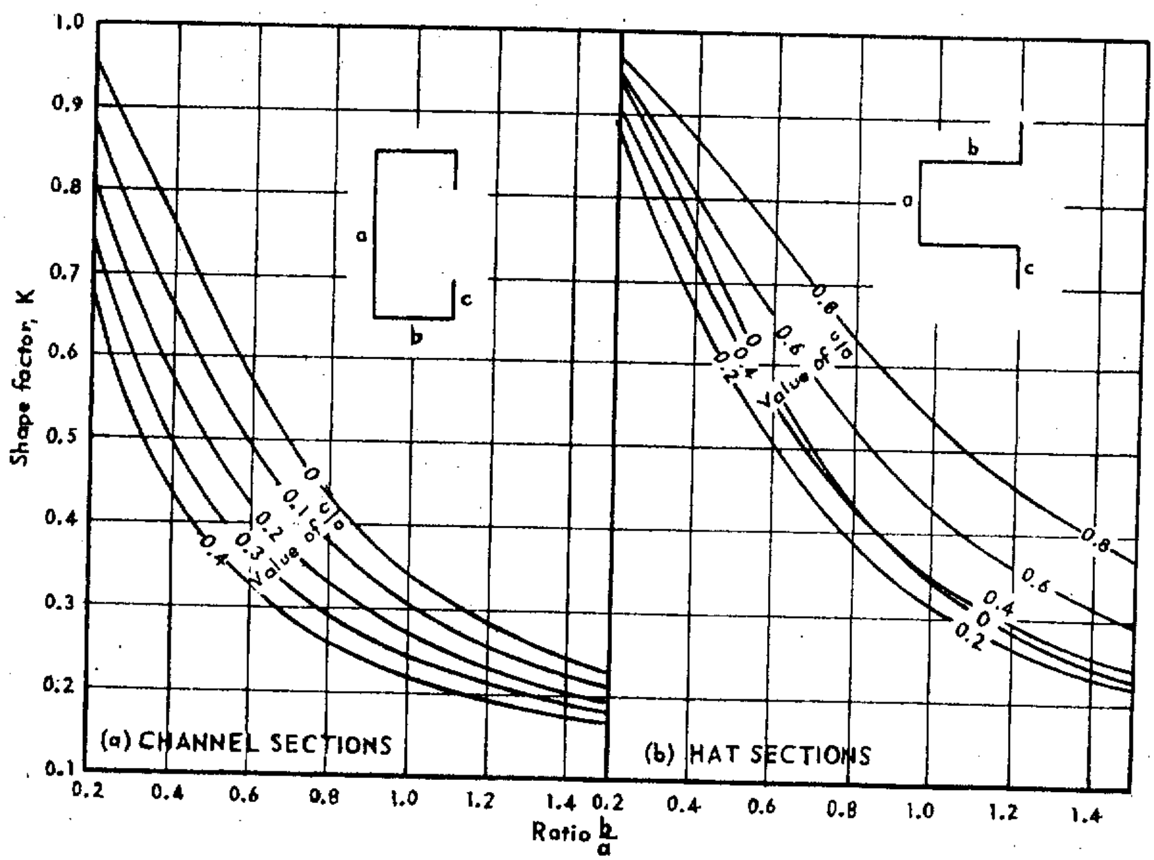


FIGURE 11.81 - SHAPE FACTORS FOR COMPLEX SECTIONS

The coefficients, C_1 and C_2, in the St. Venant and warping terms are functions of b/a and c/a, respectively. These terms therefore indicate the effect that the shape of the section has on P_ϕ.

Sections composed of thin rectangular elements whose middle lines intersect at a common point have negligible warping stiffness; i.e., $\Gamma = 0$. Because C_2 is proportional to Γ, the torsional buckling load of these sections reduces to:

$$P_\phi = EAC_1(t/a)^2 \tag{11.39}$$

For the plain equal-legged angle, which falls into this category, P_ϕ can be further reduced to:

$$P_\phi = AG(t/a)^2 \tag{11.40}$$

in which G is the shear modulus of elasticity, and a is the length of one of the legs.

In general, however, C_1 and C_2 must be evaluated. Curves for these values are given in Figures 11.82, 11.83, and 11.84 for angles, hats, and channels.

For other cross sections values of the warping constant, Γ, and location of shear center are given in Figure 11.85.

11.3.2 Eccentrically Loaded Columns

The previous section described the buckling of columns with centrally applied loads, i.e., at the centroid of the section. If the load, P, is applied eccentrically as shown in Figure 11.86 the general cubic equation for calculating P_{cr} is:

$$A_3 P_{cr}^3 + A_2 P_{cr}^2 + A_1 P_{cr} + A_o = 0 \tag{11.41}$$

Where

$$A_3 = A/I_o \{ e_x \beta_2 + e_y \beta_1 - (e_y - y_o)^2 - (e_x - x_o)^2 \} + 1$$

$$A_2 = A/I_o \{ P_x(y_o - e_y)^2 + P_y(x_o - e_x)^2 - e_x \beta_2 (P_x + P_y) - e_y \beta_1 (P_x + P_y) \} - (P_x + P_y + P_\phi)$$

$$A_1 = A/I_o \{ P_x P_y e_x \beta_2 + P_x P_y e_y \beta_1 \} + (P_x P_y + P_y P_\phi + P_x P_\phi)$$

$$A_o = -P_x P_y P_\phi \qquad I_o = I_x + I_y + A(X_o^2 + Y_o^2)$$

$P_x = EI_x \; \pi^2/L^2$ $\qquad \beta_1 = 1/I_x(\int_A Y^3 dA + \int_A X^2 Y dA) - 2Y_o$

$P_y = EI_y \; \pi^2/L^2$ $\qquad \beta_2 = 1/I_y(\int_A X^3 dA + \int_A XY^2 dA) - 2X_o$

$P_\phi = A/I_o(GJ + E\Gamma \; \pi^2/L^2)$

In the general case, buckling occurs by combined bending and torsion. In each case the three roots of the cubic can be evaluated for the lowest value.

If P acts along the shear center axis:

$e_x = X_o$

$e_y = Y_o$

and the buckling loads become independent of each other, the critical load will be the lowest of the two Euler loads, P_x, P_y and the load corresponding to purely torsional buckling which is:

$$P_\phi = (I_o/A)\; P/e_y\, \beta_1 + e_x\, \beta_2 + I_o/A \qquad (11.42)$$

When the column has one plane of symmetry and the load acts in the plane of symmetry $e_x = 0$ and buckling in this plane takes place independently and the critical load is the same as the Euler load. However, lateral buckling and torsional buckling are coupled and the critical loads are obtained from the following quadratic equation:

$$(P_y - P)\left\{(I_o/A)\; P_\phi - P(e_y\, \beta_1 + I_o/A)\right\} - P^2(Y_o - e_y)^2 = 0 \qquad (11.43)$$

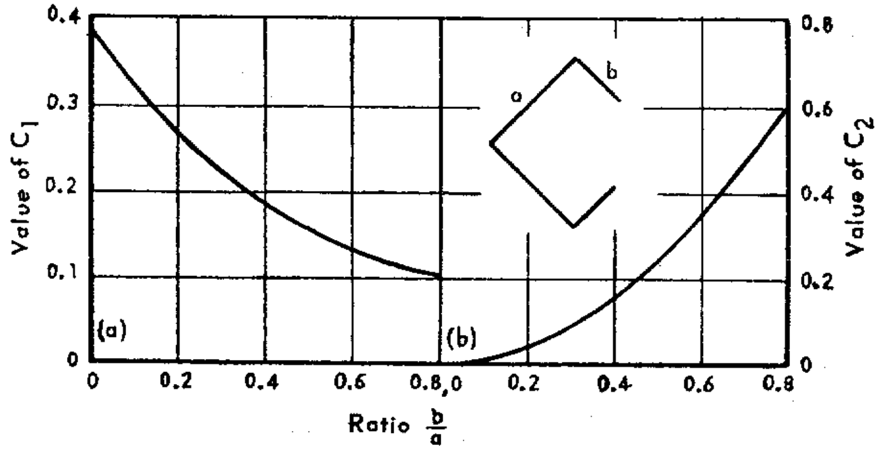


FIGURE 11.82 - TORSIONAL BUCKLING COEFFICIENTS C_1 AND C_2 FOR ANGLES

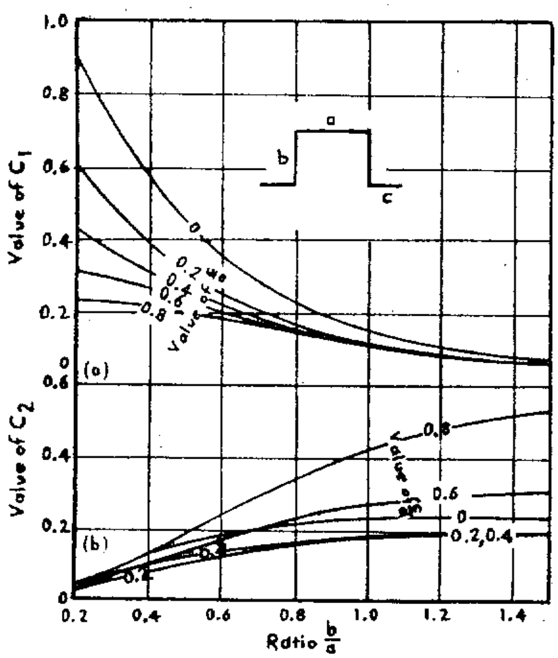


FIGURE 11.83 - TORSIONAL BUCKLING COEFFICIENTS C_1 AND C_2 FOR HAT SECTIONS.

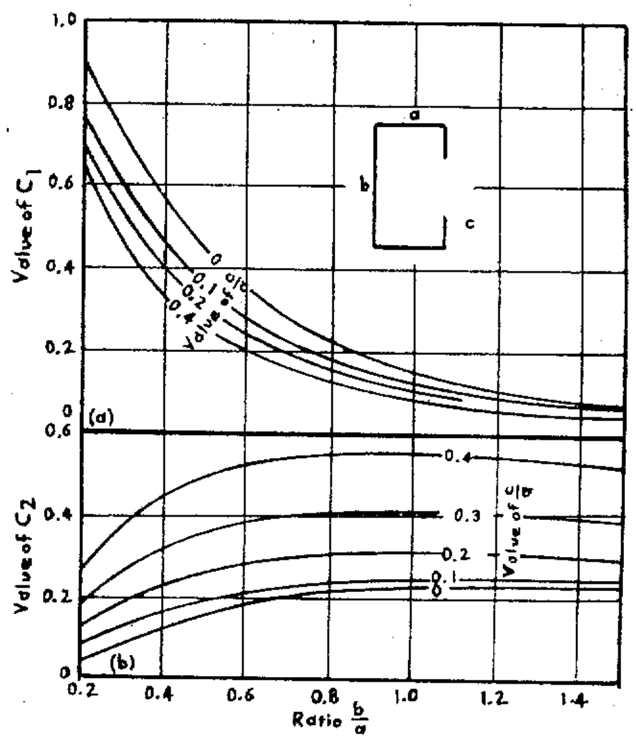


FIGURE 11.84 - TORSIONAL BUCKLING COEFFICIENTS C_1 AND C_2 FOR CHANNEL SECTIONS.

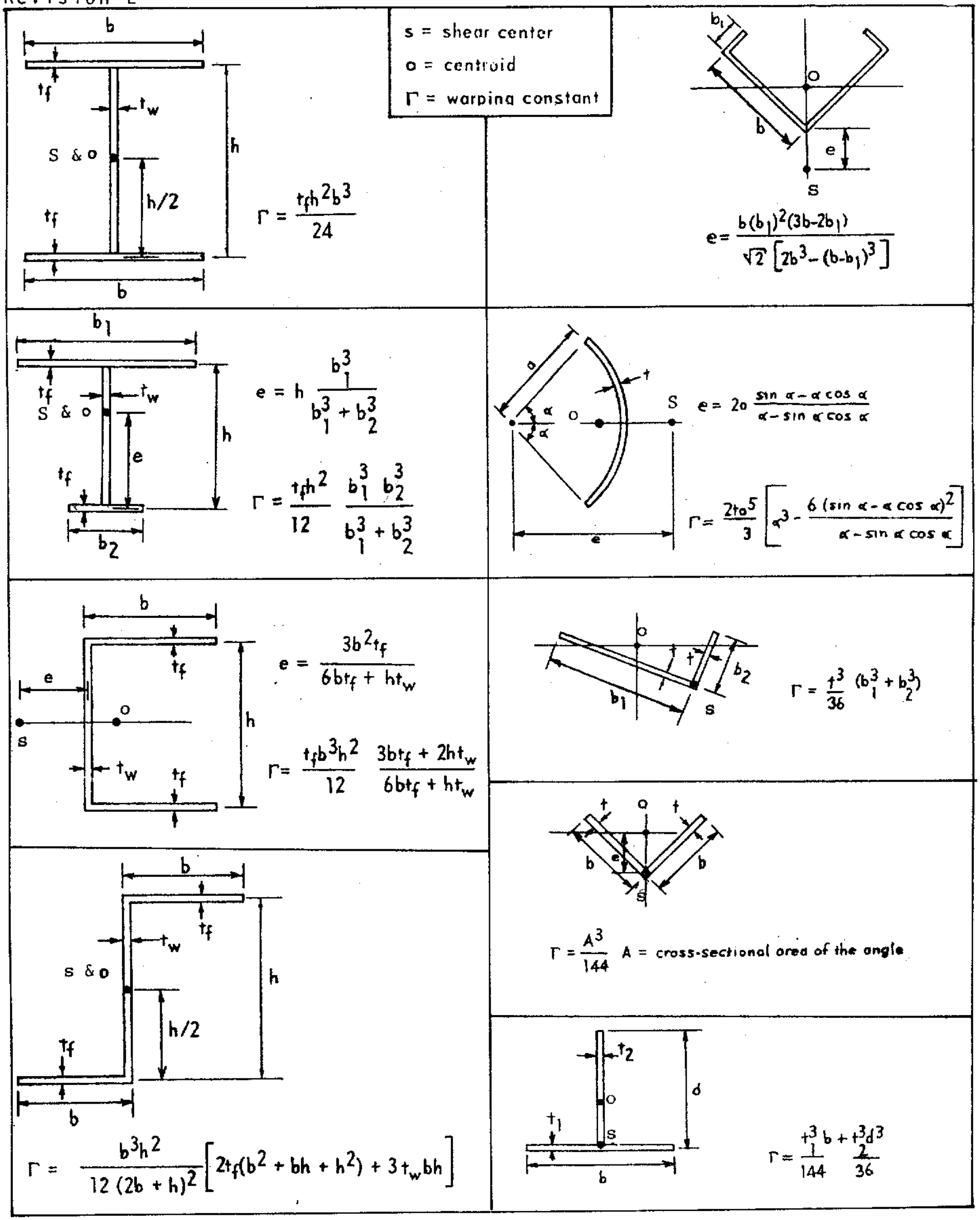
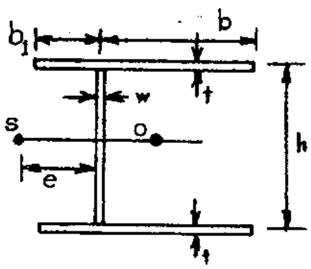
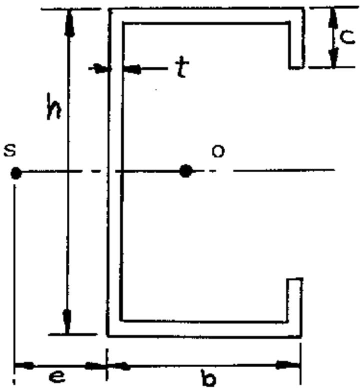


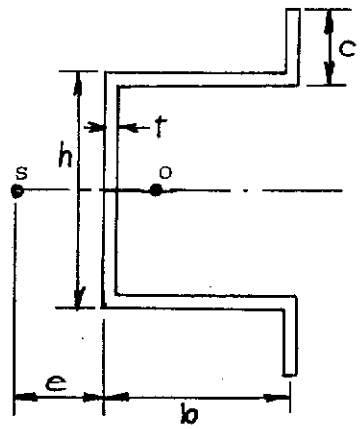
FIGURE 11.85 - SHEAR CENTER LOCATIONS AND WARPING CONSTANTS



$$e = \frac{3(b^2 - b_1^2)}{(w/t)h + 6(b + b_1)} \quad ; b_1 < b$$



Values of e/h							
c/h \ b/h	1.0	0.8	0.6	0.4	0.2		
0	.430	.330	.236	.141	.055		
0.1	.477	.280	.280	.183	.087		
0.2	.530	.425	.325	.222	.115		
0.3	.575	.470	.365	.258	.138		
0.4	.610	.503	.394	.280	.155		
0.5	.621	.517	.405	.290	.161		



Values of e/h							
c/h \ b/h	1.0	0.8	0.6	0.4	0.2		
0	.430	.330	.236	.141	.055		
0.1	.464	.367	.270	.173	.080		
0.2	.474	.377	.280	.182	.090		
0.3	.453	.358	.265	.172	.085		
0.4	.410	.320	.235	.150	.072		
0.5	.355	.275	.196	.123	.056		
0.6	.300	.225	.155	.095	.040		

Figure 11.85 (Cont'd) - Shear Center Locations and Warping Constants.

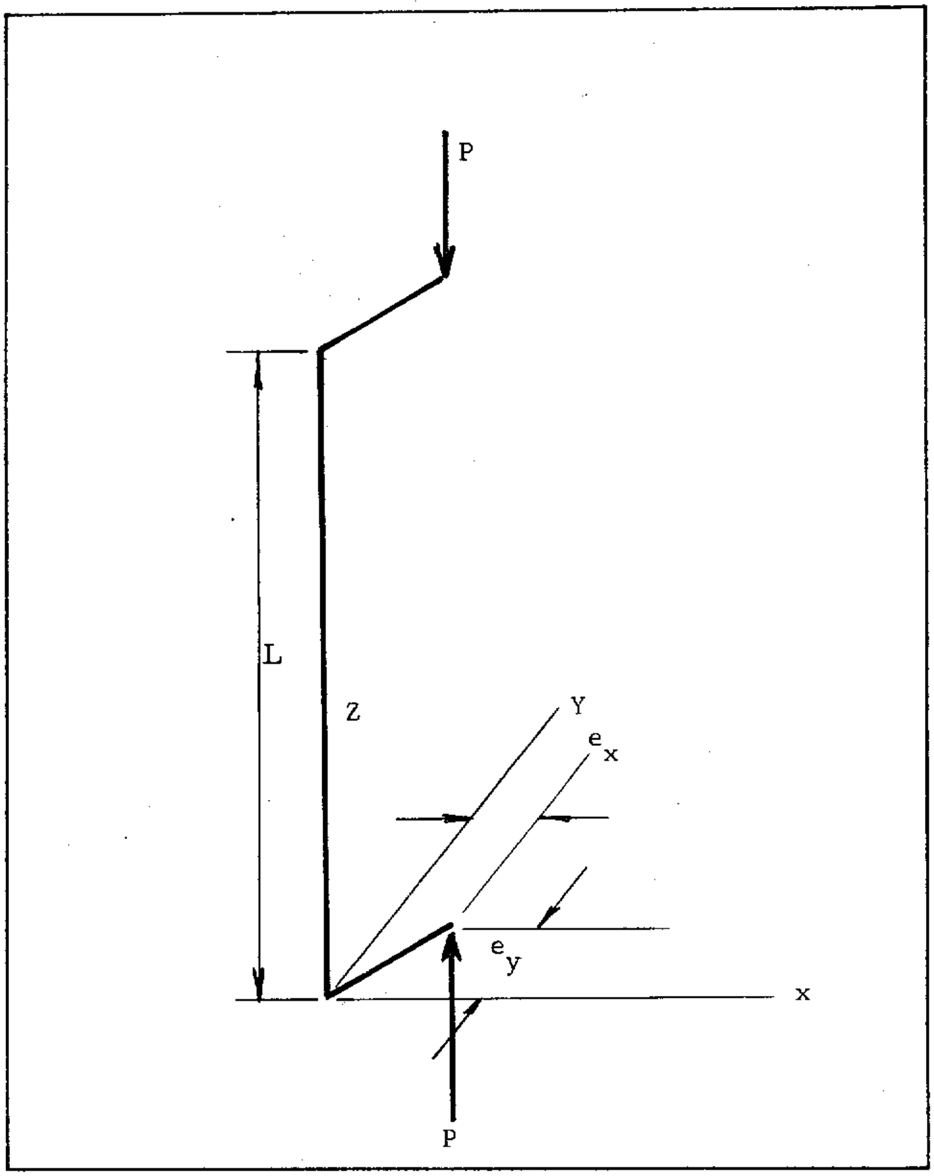


FIGURE 11.86 - ECCENTRICALLY APPLIED LOAD

SECTION 12
FRAMES AND RINGS

12.1 GENERAL

This section presents the methods of analyzing statically indeterminate rings and frames. The principal analysis method discussed in this section is that of moment distribution. Particular solutions of bents and semicircular arches under various loading cases are also given.

12.2 Analysis of Statically Indeterminate Frames by the Method of Moment Distribution

Moment distribution is a convenient method of reducing statically indeterminate structures to a problem in statics. Moment distribution does not involve the solution of simultaneous equations, but consists of a series of converging cycles that may be terminated at the degree of precision required by the problem. The theory of moment distribution is not discussed since many publications are available on the subject. Instead, a step-by-step procedure along with an example problem is shown.

12.2.1 Sign Convention

The sign convention for moments in the method of moment distribution is to consider moments acting clockwise on the ends of a member as positive. This convention is illustrated in Figure 12.1.

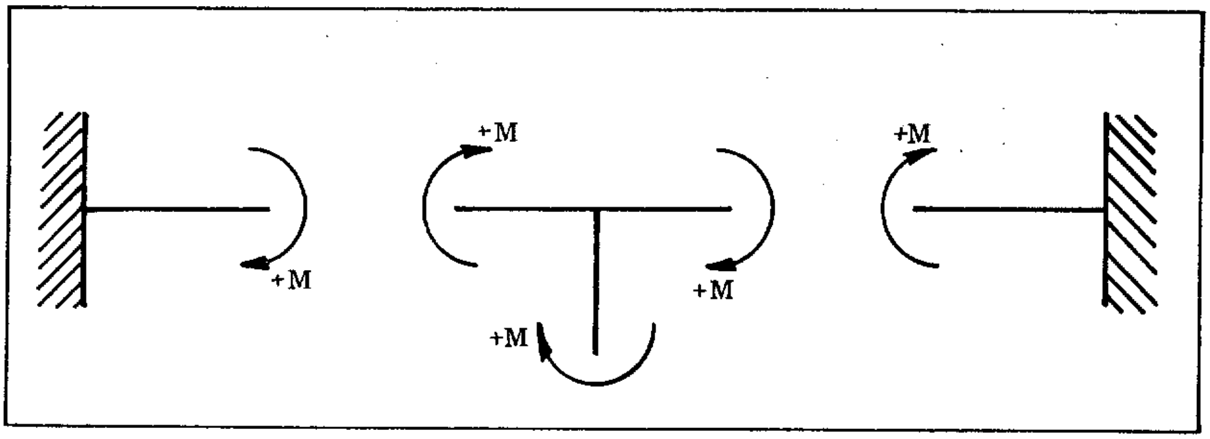


FIGURE 12.1 - SIGN CONVENTION FOR MOMENTS

12.2.2 Moment Distribution Procedure

(1) Compute the stiffness factor, K, for each member.

$K = \frac{EI}{L}$, general; $K = \frac{I}{L}$ constant E; $K = \frac{1}{L}$, constant EI.

(2) Compute the distribution factor, DF, of each member at each joint.

$DF = \frac{K}{\sum K}$, where the summation includes all members meeting at the joint.

(3) Compute the fixed-end moments, FEM, for each loaded span and record. Fixed-end moments for various types of loading are given in Table 12.1.

(4) Balance the moments at a joint by multiplying the unbalanced moment by the distribution factor, changing sign, and recording the balancing moment below the fixed-end moment. The unbalanced moment is the sum of the fixed-end moments of a joint.

(5) Draw a horizontal line below the balancing moment. The algebraic sum of all moments at any joint above the horizontal line must be zero.

(6) Record the carry-over moment at the opposite ends of the members. Carry-over moments have the same sign as the balancing moments and are half their magnitude.

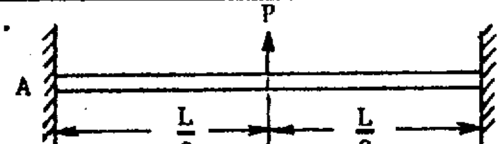
(7) Move to a new joint and repeat the process for the balance and carry-over moments for as many cycles as desired to meet the required accuracy of the problem. The unbalanced moment for each cycle will be the algebraic sum of the moments at the joint recorded below the last horizontal line.

(8) Obtain the final moment at the end of each member as the algebraic sum of all moments tabulated at this point. The total of the final moments for all members at any joint must be zero.

(9) Reactions, vertical shears, and bending moments of the members may be found through statics by utilizing the above-mentioned final moments.

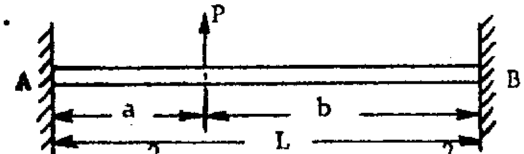
It should be noted that simpler methods may be found for the solution of rectangular, trapezoidal, and triangular frames in Section 12.3.

1.



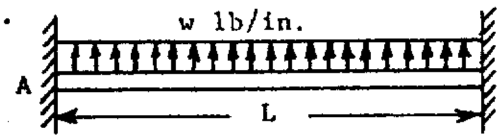
$M_A = \frac{PL}{8}$ $\quad M_B = -\frac{PL}{8}$

2.



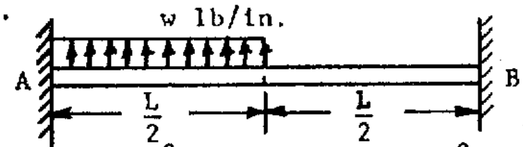
$M_A = \frac{Pab^2}{L^2}$ $\quad M_B = -\frac{Pa^2b}{L^2}$

3.



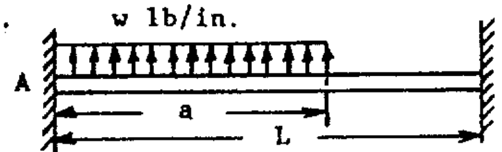
$M_A = \frac{wL^2}{12}$ $\quad M_B = -\frac{wL^2}{12}$

4.



$M_A = \frac{11wL^2}{192}$ $\quad M_B = -\frac{5wL^2}{192}$

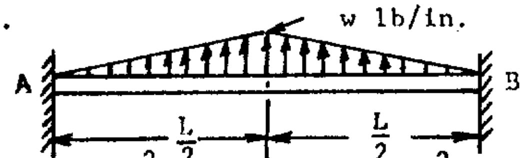
5.



$M_A = \frac{wa^2}{12L^2}(6L^2-8aL+3a^2)$

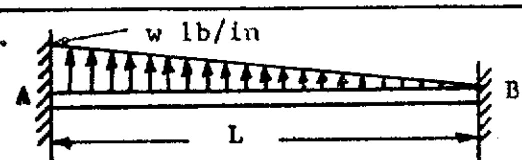
$M_B = -\frac{wa^2}{12L^2}(4aL-3a^2)$

6.



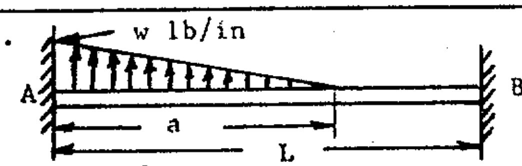
$M_A = \frac{5wL^2}{96}$ $\quad M_B = -\frac{5wL^2}{96}$

7.



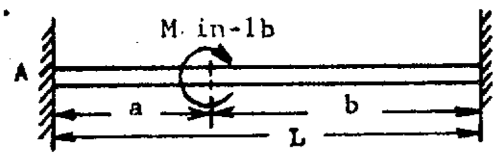
$M_A = \frac{wL^2}{20}$ $\quad M_B = -\frac{wL^2}{30}$

8.

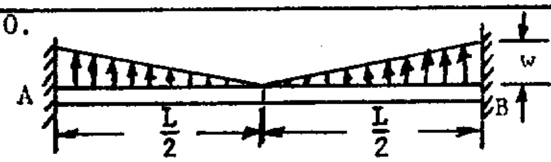


$M_A = \frac{wa^2}{60L^2}(10L^2-10aL+3a^2)$

$M_B = -\frac{wa^3}{60L^2}(5L-3a)$

9.

$M_A = \frac{Mb}{L}(3\frac{a}{L}-1)$, $M_B = \frac{Ma}{L}(3\frac{b}{L}-1)$

10.



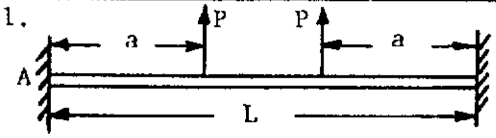
$M_A = \frac{wL^2}{32}$ $\quad M_B = -\frac{wL^2}{32}$

P = load (lb), w = unit load (lb/in)

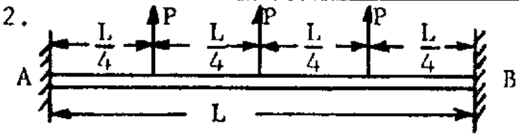
M = bending moment (in-lb), positive when clockwise

TABLE 12.1 - FIXED END MOMENTS FOR BEAMS

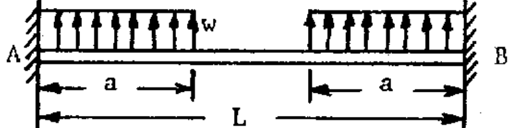
11.



$M_A = Pa(1 - \frac{a}{L})$ $\quad M_B = -M_A$

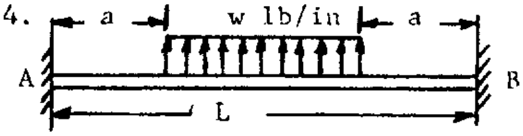
12.

$M_A = \frac{15PL}{48}$ $\quad M_B = -M_A$

13.



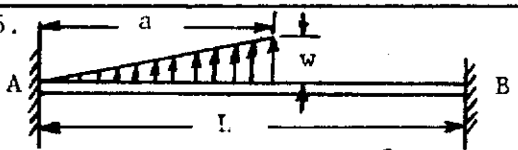
$M_A = \frac{wa^2}{6L}(3L-2a)$ $\quad M_B = -M_A$

14.



$M_A = \frac{w}{12L}(L^3-a^2L+4a^3)$ $\quad M_B = -M_A$

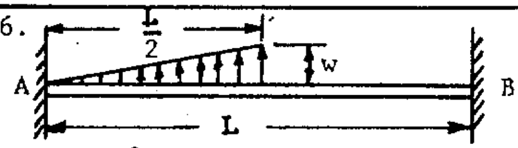
15.



$M_A = \frac{wa^2}{30}(10 - 15\frac{a}{L} + 6\frac{a^2}{L^2})$

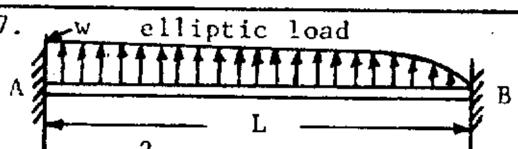
$M_B = \frac{wa^3}{20L^2}(5L-4a)$

16.



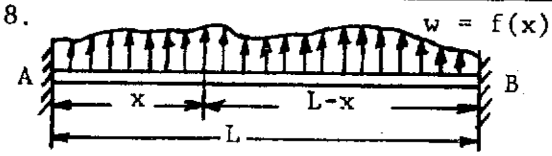
$M_A = \frac{wL^2}{30}$ $\quad M_B = -\frac{3wL^2}{160}$

17.



$M_A = \frac{wL^2}{13.52}$ $\quad M_B = -\frac{wL^2}{15.86}$

18.



$$M_A = \frac{1}{L^2}\int_0^L x(L-x)^2 f(x)dx$$

$$M_B = \frac{-1}{L^2}\int_0^L x^2(L-x)f(x)dx$$

TABLE 12.1 (CONT'D) - FIXED END MOMENTS FOR BEAMS

12.2.3 Sample Problem

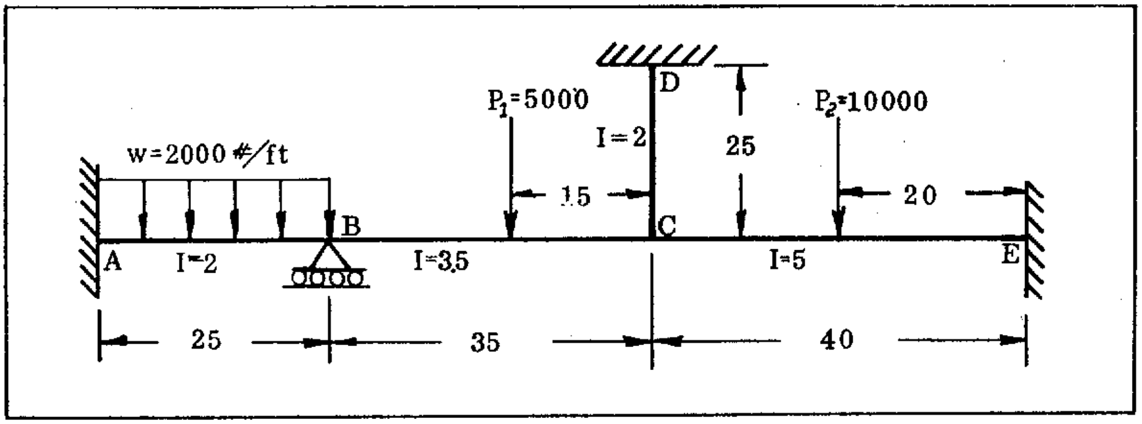


FIGURE 12.2 - SAMPLE PROBLEM BY MOMENT DISTRIBUTION

Find: The member end moments for the frame of Figure 12.2.

Solution:

(1) The stiffness factors of the members are:

$$K_{BA} = \frac{I_{BA}}{L_{BA}} = \frac{2}{25} = 0.08, \quad K_{BC} = 0.1$$

$$K_{CE} = 0.125, \quad K_{CD} = 0.08$$

(2) The distribution factors of the members are:

$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = 0.44, \quad DF_{BC} = 0.56$$

$$DF_{CB} = 0.33, \quad DF_{CE} = 0.41, \quad DF_{CD} = 0.26$$

(3) The fixed-end moments are obtained from Table 12.1. From case 3,

$$FEM_{AB} = \frac{wL^2}{12} = \frac{-2000(25)^2}{12} = -104{,}000 \text{ ft.lbs.}$$

and

$$FEM_{BA} = \frac{-wL^2}{12} = \frac{2000(25)^2}{12} = 104{,}000 \text{ ft.lbs.}$$

(FEM_{BA} = Fixed-End Moment acting on the end of member AB labeled as B)

From case 2 of Table 12.1,

$$FEM_{BC} = \frac{Pab^2}{L^2} = \frac{-5000(20)(15)^2}{(35)^2} = -18{,}000 \text{ ft.lbs.}$$

and

$$FEM_{CB} = \frac{-Pa^2b}{L^2} = \frac{5000(20)^2(15)}{(35)^2} = 25{,}000 \text{ ft.lbs.}$$

From case 1 of Table 12.1,

$$FEM_{CE} = \frac{PL}{8} = \frac{-10000(40)}{8} = -50{,}000 \text{ ft.lbs.}$$

and

$$FEM_{EC} = \frac{-PL}{8} = 50{,}000 \text{ ft.lbs.}$$

$FEM_{CD} = FEM_{DC} = 0$ since member CD is unloaded

(4) Prepare a table similar to the one shown in Table 12.2. Enter the stiffness factors, distribution factors and fixed and moments for each element of the structure. These numbers are shown in lines 1, 2 and 3 of Table 12.2.

		AB	BA	BC	DC	CB	CE	CD	EC
1	K		0.08	0.1		0.1	0.125	0.08	
2	DF		0.44	0.56		0.33	0.41	0.26	
3	FEM	-104	+104	-18	0	+25	-50	0	+50
4			-38	-48		+8	+10	+7	
5		-19		+4	+3	-24			+5
6			-2	-2		+8	+10	+6	
7		-1		+4	+3				+5
8			-2	-2					
9	Σ	-124	+62	-62	+6	+17	-30	+13	+60

TABLE 12.2 - SOLUTION OF FRAME SHOWN IN FIGURE 12.2

(5) The unbalanced moment at joint B is

$\Sigma FEM = 104 - 18 = 86.$

The moments at joint B may be balanced by multiplying this unbalanced moment by the distribution factor and changing sign.

(6) A horizontal line may be drawn below the balancing moments. The algebraic sum of all the moments at this joint above this line is zero.

(7) Record the carry-over moments at the opposite ends of the members. The carry-over moments have the same sign as the corresponding balancing moments and are half their magnitude.

(8) Steps 5, 6, and 7 may be repeated for joint C to obtain the rest of the values in rows 4 and 5 of Table 12.2. The process for the balance and carry-over moments may be repeated for as many cycles as desired to meet the required accuracy of the problem. The unbalanced moment for each cycle will be the algebraic sum of the moments at the joint recorded below the last horizontal line. Lines 6, 7, and 8 of Table 12.2 show this process.

(9) The final moment at the end of each member may be obtained as the algebraic sum of all moments tabulated in lines 3 through 7 of Table 12.2.

12.3 FORMULAS FOR SIMPLE FRAMES

This section presents formulas for determining the reaction forces and moments acting on simple frames under various simple loadings. The reaction forces and moments acting on frames under more complicated loadings may often be obtained by the superposition of several of the simple cases.

12.3.1 Rectangular Frames

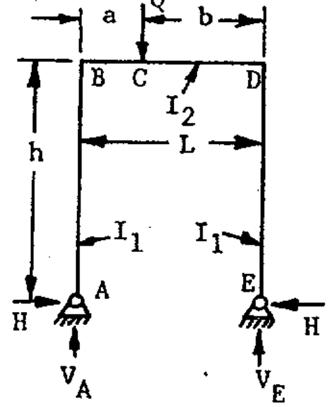
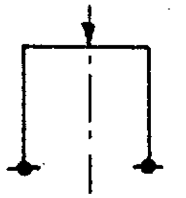
Table 12.3 shows reaction forces and moments for various loadings of rectangular frames. In all cases in Table 12.3, $K = I_2 \, h/I_1 L$.

12.3.2 Triangular Frames

Table 12.4 shows reaction forces and moments for various loadings of triangular frames. In all cases in Table 12.4, $K = I_1 \, S_2/I_2 S_1$.

12.3.3 Semicircular Frames and Arches

Table 12.5 shows reaction forces and moments for various loadings on semicircular frames.

<table>
<tr>
<td>1. VERT. CONCENTRATED LOAD

</td>
<td>$V_A = \frac{Qb}{L}$ $V_E = Q - V_A$
$H = \frac{3Qab}{2Lh(2K + 3)}$
<u>FOR SPECIAL CASE</u>: $a = b = \frac{L}{2}$
$V_A = V_E = \frac{Q}{2}$
$H = \frac{3QL}{8h(2K + 3)}$
</td>
</tr>
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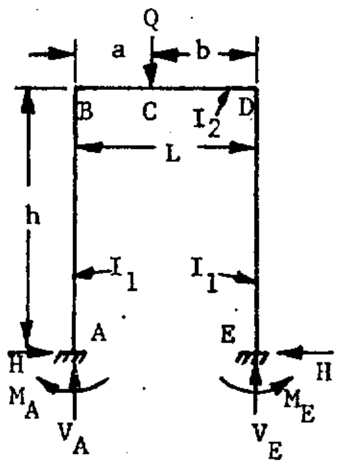
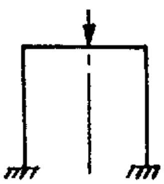
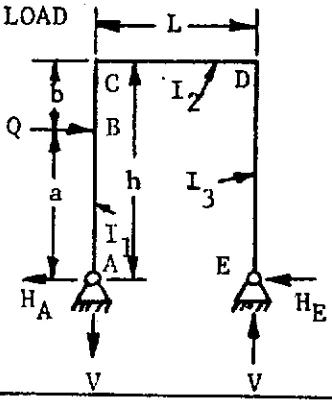
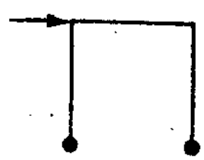
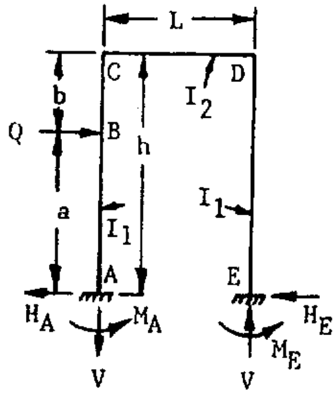
<td>2. VERT. CONCENTRATED LOAD

</td>
<td>$V_A = \frac{Qb}{L}\left[1 + \frac{a(b - a)}{L^2(6K + 1)}\right]$ $V_E = Q - V_A$
$H = \frac{3Qab}{2Lh(K + 2)}$
$M_A = \frac{Qab}{L}\left[\frac{1}{2(K + 2)} - \frac{(b - a)}{2L(6K + 1)}\right]$
$M_E = \frac{Qab}{L}\left[\frac{1}{2(K + 2)} + \frac{(b - a)}{2L(6K + 1)}\right]$
<u>FOR SPECIAL CASE</u>: $a = b = \frac{L}{2}$
$V_A = V_E = \frac{Q}{2}$
$H = \frac{3QL}{8h(K + 2)}$ $M_A = M_E = \frac{QL}{8(K + 2)}$
</td>
</tr>
<tr>
<td>3. HORIZ. CONCENTRATED LOAD

</td>
<td>$V = \frac{Qa}{L}$ $H_A = Q - H_E$
$H_E = \frac{Qa}{2h}\left[\frac{bK(a + h)}{h^2(2K + 3)} + 1\right]$
FOR SPECIAL CASE: $b = 0$, $a = h$
$V = \frac{Qh}{L}$
$H_E = H_A = \frac{Q}{2}$
</td>
</tr>
</table>

TABLE 12.3 - REACTIONS AND CONSTRAINING MOMENTS IN RECTANGULAR FRAMES.

4. HORIZ. CONCENTRATED LOAD



$$V = \frac{3Qa^2K}{Lh(6K + 1)} \qquad H_A = Q - H_E$$

$$H_E = \frac{Qab}{2h^2}\left[\frac{h}{b} - \frac{h + b + K(b - a)}{h(K + 2)}\right]$$

$$M_A = \frac{Qa}{2h}\left[\frac{b(h + b + bK)}{h(K + 2)} + h - \frac{3aK}{(6K + 1)}\right]$$

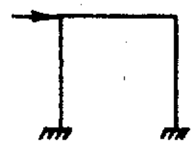
$$M_E = \frac{Qa}{2h}\left[\frac{-b(h + b + bK)}{h(K + 2)} + h - \frac{3aK}{(6K + 1)}\right]$$

FOR SPECIAL CASE: $b = 0,\ a = h$

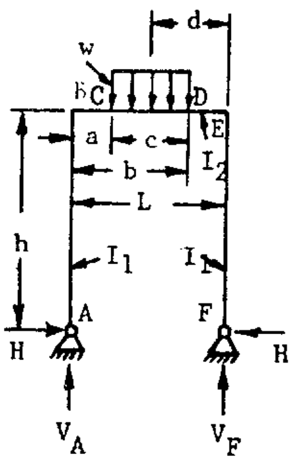
$$V = \frac{3QhK}{L(6K + 1)}$$

$$H_A = H_E = \frac{Q}{2}$$

$$M_A = M_E = \frac{Qh(3K + 1)}{2(6K + 1)}$$



5. VERT. UNIFORM RUNNING LOAD



$$d = L - \frac{a}{2} - \frac{b}{2}$$

$$V_A = \frac{wcd}{L}$$

$$V_F = wc - V_A = \frac{wc}{L}\left(a + \frac{c}{2}\right) = wc\left(1 - \frac{d}{L}\right)$$

$$H = \frac{3}{2h}\left[\frac{X_1 + X_2}{2K + 3}\right] = \frac{3wc}{24Lh(2K + 3)}\left[12dL - 12d^2 - c^2\right]$$

where:

$$X_1 = -\frac{wc}{24L}\left[24\frac{d^3}{L} - 6\frac{bc^2}{L} + 3\frac{c^2}{L} + 4c^2 - 24d^2\right]$$

$$X_2 = \frac{wc}{24L}\left[24\frac{d^3}{L} - 6\frac{bc^2}{L} + 3\frac{c^3}{L} + 2c^2 - 48d^2 + 24dL\right]$$

FOR SPECIAL CASE: $a = 0,\ c = b = L,\ d = \frac{L}{2}$

$$V_A = V_F = \frac{wL}{2}$$

$$H = \frac{wL^2}{4h(2K + 3)}$$

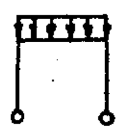
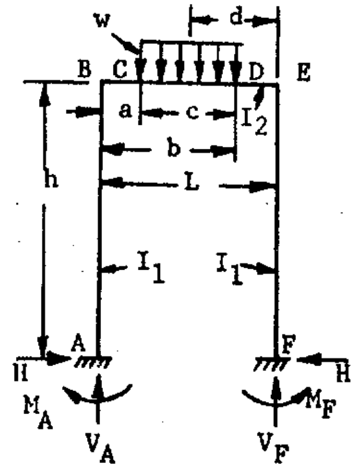


TABLE 12.3 (CONT'D) - REACTIONS AND CONSTRAINING MOMENTS IN RECTANGULAR FRAMES

6. VERT. UNIFORM RUNNING LOAD



$$d = L - \frac{a}{2} - \frac{b}{2}$$

$$V_A = \frac{wcd}{L} + \frac{X_1 - X_2}{L(6K + 1)}$$

X_1 and X_2 are given in case 5

$$V_F = wc - V_A$$

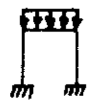
$$H = \frac{3(X_1 + X_2)}{2h(K + 2)} \qquad M_A = \frac{X_1 + X_2}{2(K + 2)} - \frac{X_1 - X_2}{2(6K + 1)}$$

$$M_F = \frac{X_1 + X_2}{2(K + 2)} + \frac{X_1 - X_2}{2(6K + 1)}$$

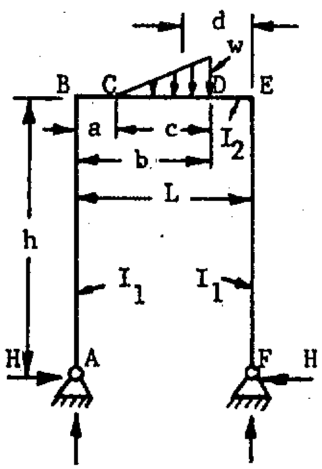
FOR SPECIAL CASE: $a = 0,\ c = b = L,\ d = \frac{L}{2}$

$$V_A = V_F = \frac{wL}{2}$$

$$H = \frac{wL^2}{4h(K + 2)} \qquad M_A = M_F = \frac{wL^2}{12(K + 2)}$$



7. VERT. TRIANGULAR RUNNING LOAD



$$d = L - \frac{a}{3} - \frac{2b}{3}$$

$$V_A = \frac{wcd}{2L}$$

$$V_F = \frac{wc}{2} - V_A = \frac{wc}{2L}\left(a + \frac{2c}{3}\right)$$

$$H = \frac{3}{2h}\left[\frac{X_3 + X_4}{2K + 3}\right] = \frac{3wc}{4Lh(2K + 3)}\left[dL - \frac{c^2}{18} - d^2\right]$$

WHERE:

$$X_3 = -\frac{wc}{2L}\left[\frac{d^3}{L} + \frac{c^2}{9} + \frac{51c^3}{810L} + \frac{c^2b}{6L} - d^2\right]$$

$$X_4 = \frac{wc}{2L}\left[\frac{d^3}{L} + \frac{c^2}{18} + \frac{51c^3}{810L} - \frac{c^2b}{6L} - 2d^2 + dL\right]$$

FOR SPECIAL CASE: $a=o,\ c=b=L,\ d = \frac{L}{3}$

$$V = \frac{wL}{6}$$

$$H = \frac{wL^2}{8h(2K + 3)}$$

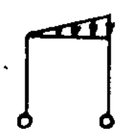
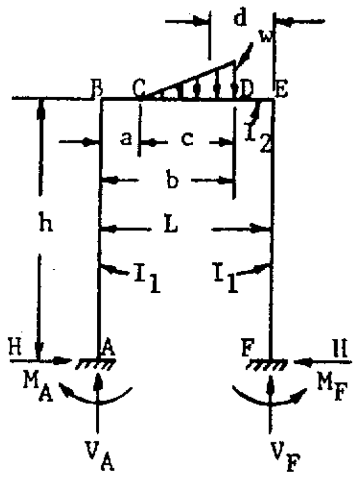


TABLE 12.3 (CONT'D) - REACTIONS AND CONSTRAINING MOMENTS IN RECTANGULAR FRAMES

8. VERT. TRIANGULAR RUNNING LOAD



$$d = L - \frac{a}{3} - \frac{2b}{3}$$

$$V_A = \frac{wcd}{2L} + \frac{X_3 - X_4}{L(6K + 1)}$$

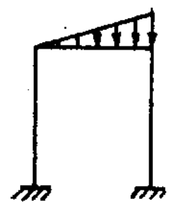
X_3 and X_4 are given in case 7

$$V_F = \frac{wc}{2} - V_A \qquad H = \frac{3(X_3 + X_4)}{2h(K + 2)}$$

$$M_A = \frac{X_3 + X_4}{2(K + 2)} - \frac{X_3 - X_4}{2(6K + 1)}$$

$$M_F = \frac{X_3 + X_4}{2(K + 2)} + \frac{X_3 - X_4}{2(6K + 1)}$$

FOR SPECIAL CASE: a=o, c=b=L, $d = \frac{L}{3}$



$$V_A = \frac{wL}{6}\left[1 - \frac{1}{10(6K + 1)}\right]$$

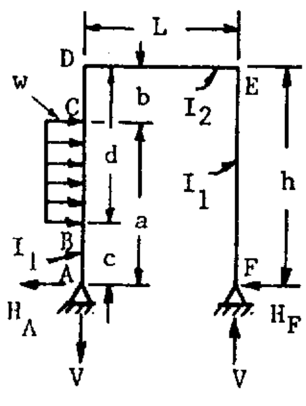
$$V_F = \frac{wL}{3}\left[1 + \frac{1}{20(6K + 1)}\right]$$

$$H = \frac{wL^2}{8h(K + 2)}$$

$$M_A = \frac{wL^2}{120}\left[\frac{5}{K + 2} + \frac{1}{6K + 1}\right]$$

$$M_F = \frac{wL^2}{120}\left[\frac{5}{K + 2} - \frac{1}{6K + 1}\right]$$

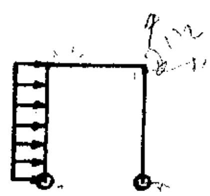
9. HORIZ. UNIFORM RUNNING LOAD



$$V = \frac{w(a^2 - c^2)}{2L} \qquad H_A = w(a - c) - H_F$$

$$H_F = \frac{w(a^2 - c^2)}{4h} + \frac{K\left[w(a^2-c^2)(2h^2-a^2-c^2)\right]}{8h^3(2K + 3)}$$

FOR SPECIAL CASE: c=o, b=o, a=d=h



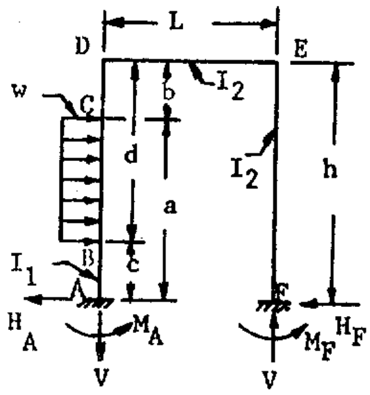
$$V = \frac{wh^2}{2L}$$

$$H_A = wh - H_F$$

$$H_F = \frac{wh}{4}\left[1 + \frac{K}{2(2K + 3)}\right]$$

TABLE 12.3 (CONT'D) - REACTIONS AND CONSTRAINING MOMENTS IN RECTANGULAR FRAMES

10. HORIZ. UNIFORM RUNNING LOAD



$$V = \frac{w(a^2 - c^2)}{2L} - \frac{M_A}{L} - \frac{M_F}{L}$$

$$H_A = w(a - c) - H_F$$

$$H_F = \frac{w(a^2 - c^2)}{4h} - \frac{X_5}{2h} + \frac{X_6(K - 1)}{2h(K + 2)}$$

WHERE:

$$X_5 = \frac{w}{12h^2}\left[d^3(4h - 3d) - b^3(4h - 3b)\right]$$

$$X_6 = \frac{w}{12h^2}\left[a^3(4h - 3a) - c^3(4h - 3c)\right]$$

$$M_A = \frac{(3K + 1)\left[\frac{w(a^2 - c^2)}{2} - X_5\right]}{2(6K + 1)} + \frac{X_6}{2}\left[\frac{1}{K + 2} + \frac{3K}{6K + 1}\right] + X_5$$

$$M_F = \frac{(3K + 1)\left[\frac{w(a^2 - c^2)}{2} - X_5\right]}{2(6K + 1)} - \frac{X_6}{2}\left[\frac{1}{K + 2} - \frac{3K}{6K + 1}\right]$$

FOR SPECIAL CASE: c=o, b=o, a=d=h:

$$V = \frac{wh^2K}{L(6K + 1)} \qquad H_A = wh - H_F$$

$$H_F = \frac{wh(2K + 3)}{8(K + 2)} \qquad M_F = \frac{wh^2}{24}\left[\frac{18K + 5}{6K + 1} - \frac{1}{K + 2}\right]$$

$$M_A = \frac{wh^2}{24}\left[\frac{30K + 7}{6K + 1} + \frac{1}{K + 2}\right]$$

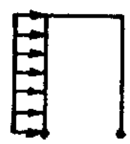
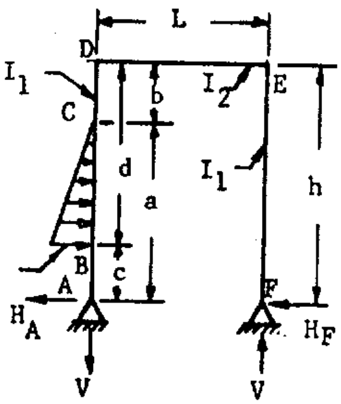


TABLE 12.3 (CONT'D) - REACTIONS AND CONSTRAINING MOMENTS IN RECTANGULAR FRAMES

11. HORIZ. TRIANGULAR RUNNING LOAD



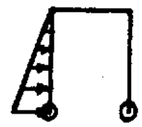
$$V = \frac{w}{6L}\left(a^2 + ac - 2c^2\right) \qquad H_A = \frac{w(a-c)}{2} - H_F$$

$$H_F = \frac{VL}{2h} + \frac{KX_7}{(2K+3)h} \qquad \text{WHERE:}$$

$$X_7 = \frac{w}{120h^2(d-b)}\left[3(4d^5+b^5) - 15h(3d^4+b^4) + 20h^2(2d^3+b^3) - 15bd^2(2h-d)^2\right]$$

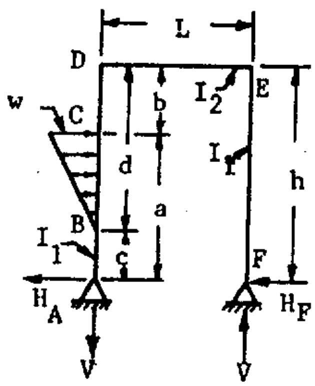
FOR SPECIAL CASE: b=c=o, a=d=h:

$$V = \frac{wh^2}{6L} \qquad H_A = \frac{wh}{2} - H_F$$



$$H_F = \frac{wh}{12}\left[1 + \frac{7K}{10(2K+3)}\right]$$

12. HORIZ. TRIANGULAR RUNNING LOAD



$$V = \frac{w}{6L}(2a + c)(a - c)$$

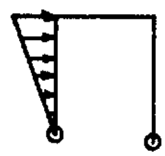
$$H_A = \frac{w(a-c)}{2} - H_F \qquad H_F = \frac{VL}{2h} + \frac{KX_{10}}{h(2K+3)}$$

WHERE:

$$X_{10} = \frac{w}{120h^2(a-c)}\left[-30h^2c\,(a^2-c^2) + 20h^2(a^3-c^3) + 15c(a^4-c^4) - 12(a^5-c^5)\right]$$

FOR SPECIAL CASE: b=c=o, a=d=h:

$$V = \frac{wh^2}{3L}$$

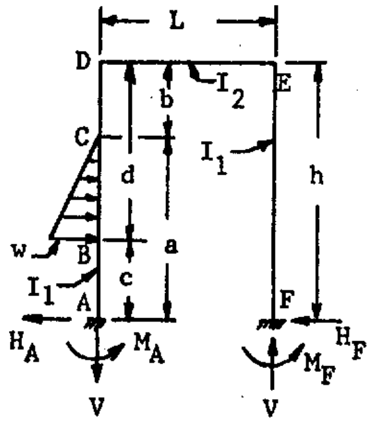


$$H_A = \frac{wh}{2} - H_F$$

$$H_F = \frac{wh}{10}\left[\frac{4K+5}{2K+3}\right]$$

TABLE 12.3 (CONT'D) - REACTIONS AND CONSTRAINING MOMENTS IN RECTANGULAR FRAMES

13. HORIZ. TRIANGULAR RUNNING LOAD



$$V = \frac{w(a^2 + ac - 2c^2)}{6L} - \frac{M_A}{L} - \frac{M_F}{L}$$

$$H_A = \frac{w(a - c)}{2} - H_F$$

$$H_F = \frac{w(a^2 + ac - 2c^2)}{12h} - \frac{X_8}{2h} + \frac{X_9(K-1)}{2h(K+2)}$$

WHERE:

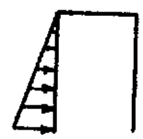
$$X_8 = \frac{w}{60h^2(d-b)}\left[15(h+b)(d^4-b^4) - 12(d^5-b^5) - 20bh(d^3-b^3)\right]$$

$$X_9 = \frac{w}{60h^2(d-b)}\left[10d^2h^2(2d-3b) + 10bh(4d^3+b^2h-b^3) - d^4(30h+15b) + 12d^5 + 3b^5\right]$$

$$M_A = \frac{(3K + 1)\left[\frac{w(a^2 + ac - 2c^2)}{6} - X_8\right]}{2(6K + 1)} + \frac{X_9}{2}\left[\frac{1}{K + 2} + \frac{3K}{6K + 1}\right] + X_8$$

$$M_F = \frac{(3K + 1)\left[\frac{w(a^2 + ac - 2c^2)}{6}\right] - X_8}{2(6K + 1)} - \frac{X_8}{2}\left[\frac{1}{K + 2} - \frac{3K}{6K + 1}\right]$$

<u>FOR SPECIAL CASE</u>: $b=c=o$, $a=d=h$



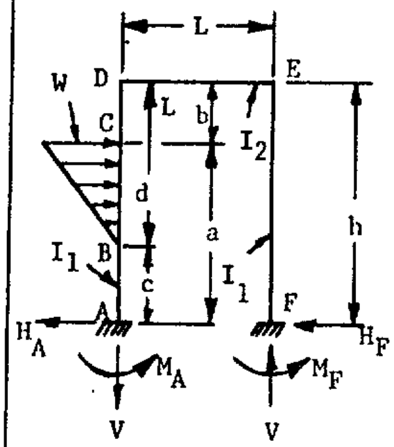
$$V = \frac{wh^2K}{4L(6K + 1)} \qquad H_A = \frac{wh}{2} - H_F$$

$$H_F = \frac{wh(3K + 4)}{40(K + 2)} \qquad M_F = \frac{wh^2}{60}\left[\frac{27K+7}{2(6K+1)} - \frac{1}{K+2}\right]$$

$$M_A = \frac{wh^2}{60}\left[\frac{27K + 7}{2(6K + 1)} + \frac{3K + 7}{K + 2}\right]$$

TABLE 12.3 (CONT'D) - REACTIONS AND CONSTRAINING MOMENTS IN RECTANGULAR FRAMES

14. HORIZ. TRIANGULAR RUNNING LOAD



$$V = \frac{w(2a+c)(a-c)}{6L} - \frac{M_A}{L} - \frac{M_F}{L}$$

$$H_A = \frac{w(a-c)}{2} - H_F$$

$$H_F = \frac{w(2a^2-ac-c^2)}{12h} - \frac{X_{11}}{2h} + \frac{X_{12}(K-1)}{2h(K+2)}$$

where:

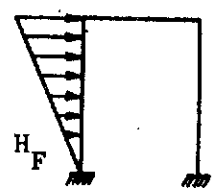
$$X_{11} = \frac{w}{60h^2(d-b)}\left[5hd^4-3d^5-20hdb^3-12b^4(d+h)\right]$$

$$X_{12} = \frac{w}{60h^2(a-c)}\left[15(h+c)(a^4-c^4) - 12(a^5-c^5) - 20ch(a^3-c^3)\right]$$

$$M_A = \frac{\left[3K+1\right]\left[\frac{w(2a^2-ac-c^2)}{6} - X_{11}\right]}{2(6K+1)} + \frac{X_{12}}{2}\left[\frac{1}{K+2} + \frac{3K}{6K+1}\right] + X_{11}$$

$$M_F = \frac{\left[3K+1\right]\left[\frac{w(2a^2-ac-c^2)}{6} - X_{11}\right]}{2(6K+1)} - \frac{X_{22}}{2}\left[\frac{1}{K+2} - \frac{3K}{6K+1}\right]$$

FOR SPECIAL CASE: b=c=o, a=d=h



$$V = \frac{3Kwh^2}{4L(6K+1)} \qquad H_A = \frac{wh}{2} - H_F$$

$$H_F = \frac{wh(7K+11)}{40(K+2)} \qquad M_A = \frac{wh^2}{120}\left[\frac{87K+22}{6K+1} + \frac{3}{K+2}\right]$$

$$M_F = \frac{wh^2}{40}\left[\frac{21K+6}{6K+1} - \frac{1}{K+2}\right]$$

TABLE 12.3 (CONT'D) - REACTIONS AND CONSTRAINING MOMENTS IN RECTANGULAR FRAMES

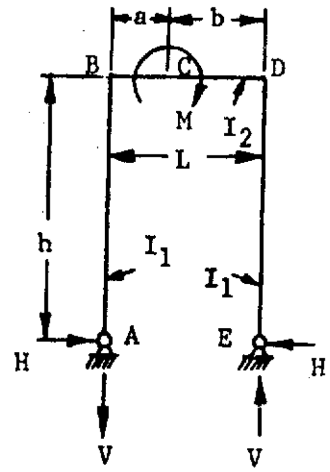
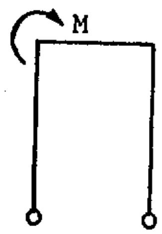
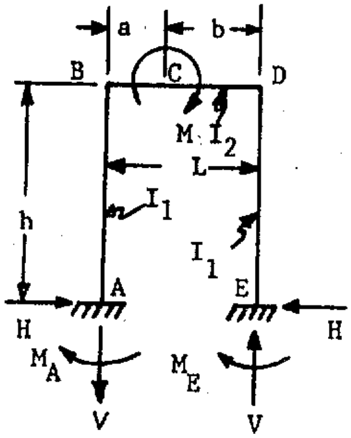
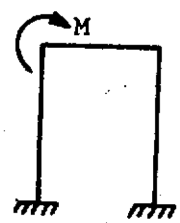
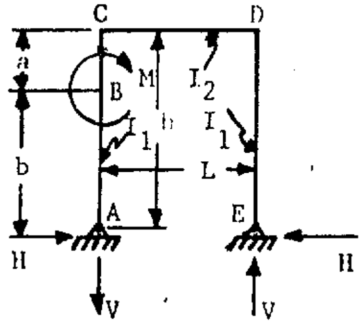
15. MOMENT ON HORIZ. SPAN 	$V = \frac{M}{L}$ $H = \frac{3(b - L/2)M}{Lh(2K + 3)}$ FOR SPECIAL CASE: a=o, b=L $V = \frac{M}{L}$ $H = \frac{3M}{2h(2K + 3)}$ 
16. MOMENT ON HORIZ. SPAN 	$V = \frac{6(ab + L^2K)M}{L^3(6K + 1)}$ $H = \frac{3(b - a)M}{2Lh(K + 2)}$ $M_A = M\left[\frac{6ab(K+2) - L\left[a(7K+3) - b(5K-1)\right]}{2L^2(K+2)(6K+1)}\right]$ $M_E = VL - M - M_A$ FOR SPECIAL CASE: a=o, b=L $V = \frac{6KM}{L(1 + 6K)}$ $H = \frac{3M}{2h(K + 2)}$ $M_A = \frac{(5K - 1)M}{2(K + 2)(6K + 1)}$ 

TABLE 12.3 (CONT'D) - REACTIONS AND CONSTRAINING MOMENTS IN RECTANGULAR FRAMES

17. MOMENT ON SIDE SPAN

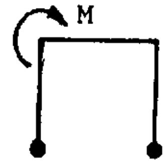


$$V = \frac{M}{L} \qquad H = \frac{3\left[K(2ab+a^2) + h^2\right]M}{2h^3(2K+3)}$$

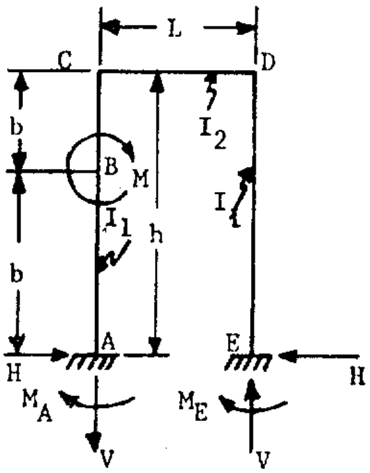
<u>FOR SPECIAL CASE:</u> a=o, b=h

$$V = \frac{M}{L}$$

$$H = \frac{3M}{2h(2K+3)}$$



18. MOMENT ON SIDE SPAN



$$V = \frac{6bKM}{hL(6K+1)} \qquad H = \frac{3bM\left[2a(K+1) + b\right]}{2h^3(K+2)}$$

$$M_A = \frac{-M}{2h^2(K+2)(6K+1)}\left[4a^2+2ab+b^2+K(26a^2-5b^2) + 6aK^2(2a-b)\right]$$

$$M_E = VL - M - M_A$$

<u>FOR SPECIAL CASE:</u> a=o; b=h

$$V = \frac{6KM}{L(6K+1)}$$

$$H = \frac{3M}{2h(K+2)}$$

$$M_A = \frac{M(5K-1)}{2(K+2)(6K+1)}$$

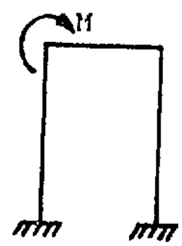
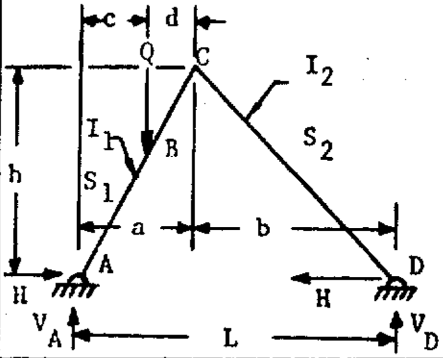


TABLE 12.3 (CONT'D) - REACTIONS AND CONSTRAINING MOMENTS IN RECTANGULAR FRAMES

1. VERT. CONCENTRATED LOAD

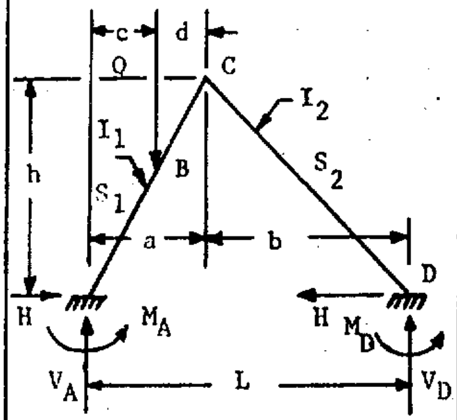


$$V_A = Q - V_D$$

$$V_D = \frac{Qc}{L}$$

$$H = \frac{Qc}{h}\left[\frac{b}{L} + \frac{d(a+c)}{2a^2(K+1)}\right]$$

2. VERT. CONCENTRATED LOAD



$$V_A = Q - V_D$$

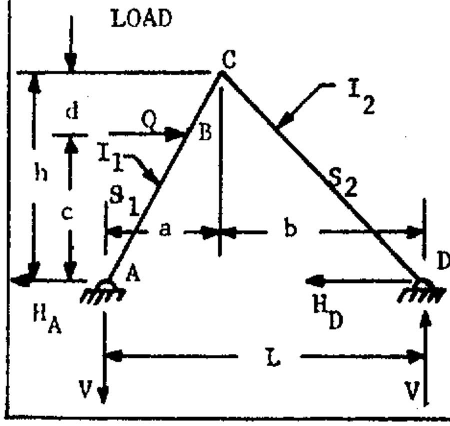
$$V_D = \frac{Qc}{L}\left[1 - \frac{d(a+d)}{2a^2}\right]$$

$$H = \frac{Qcb}{Lh} + \frac{Qcd}{6La^2h(K+1)}\left\{\left[-\,b(3K+4) - 2L\right](a+d) + 2(2L+b)(a+c) + 3ac\right\}$$

$$M_A = \frac{Qcd}{6a^2(K+1)}\left[(a+d)(3K+4) - 2(a+c)\right]$$

$$M_D = \frac{Qc^2d}{2a^2(K+1)}$$

3. HORIZ. CONCENTRATED LOAD



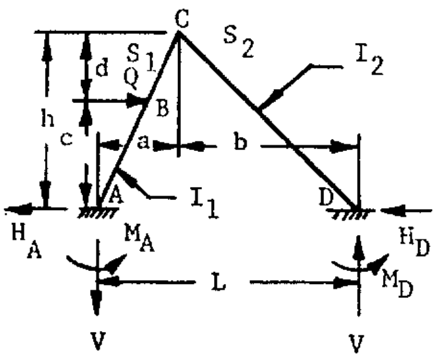
$$V = \frac{Qc}{L}$$

$$H_A = Q - H_D$$

$$H_D = \frac{Qc}{h}\left[\frac{b}{L} + \frac{d(h+c)}{2h^2(K+1)}\right]$$

TABLE 12.4 - REACTIONS AND CONSTRAINING MOMENTS IN TRIANGULAR FRAMES

4. HORIZ. CONCENTRATED LOAD



$$V = \frac{Qc}{L}\left[1 - \frac{d(h+d)}{2h^2}\right]; \qquad H_A = Q - H_D$$

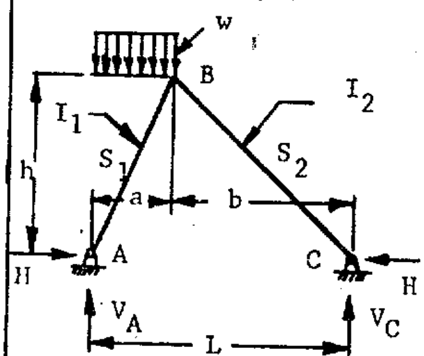
$$H_D = \frac{Qc}{Lh}\left\{b + \frac{d}{6h^2(K+1)}\left[(h+d)(-b[3K+4] - 2L) + 2(2L+b)(h+e) + 3ac\right]\right\}$$

ASSUME C

$$M_A = \frac{Qcd}{6h^2(K+1)}\left[(h+d)(3K+4) - 2(h+c)\right]$$

$$M_D = \frac{Qcd}{6h^2(K+1)}\quad(h + 2c + d)$$

5. VERTICAL UNIFORM RUNNING LOAD

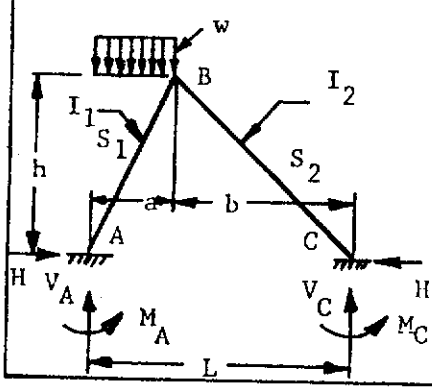


$$V_A = wa\left[1 - \frac{a}{2L}\right]$$

$$V_C = \frac{wa^2}{2L}$$

$$H = \frac{wa^2}{8h}\left[\frac{4b}{L} + \frac{1}{1+K}\right]$$

6. VERTICAL UNIFORM RUNNING LOAD



$$V_A = wa\left[1 - \frac{3a}{8L}\right] \quad : \quad V_C = \frac{3wa^2}{8L}$$

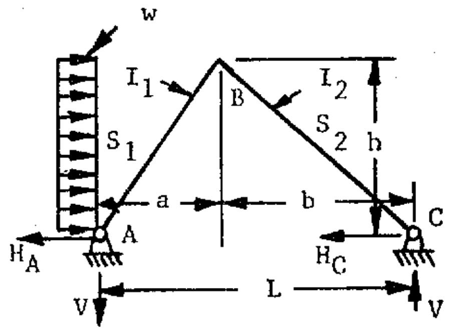
$$H = \frac{wa^2}{24Lh(K+1)}\left[b(10 + 9K) + 2L + a\right]$$

$$M_A = \frac{wa^2(3K+2)}{24(K+1)}$$

$$M_c = \frac{wa^2}{24(K+1)}$$

TABLE 12.4 (CONT'D) - REACTIONS AND CONSTRAINING MOMENTS IN TRIANGULAR FRAMES

7. HORIZ. UNIFORM RUNNING LOAD

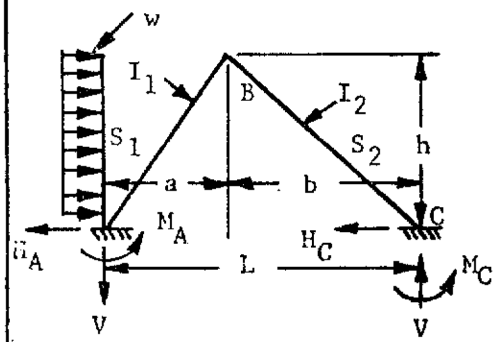


$$V = \frac{wh^2}{2L}$$

$$H_A = wh - H_c$$

$$H_C = \frac{wh}{8}\left[\frac{4b}{L} + \frac{1}{K+1}\right]$$

8. HORIZ. UNIFORM RUNNING LOAD



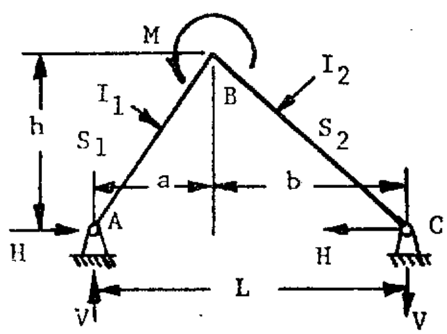
$$V = \frac{3wh^2}{8L}$$

$$H_A = wh - H_C$$

$$H_C = \frac{wh}{8L(K+1)}\left[b(3K+4) + a \right]$$

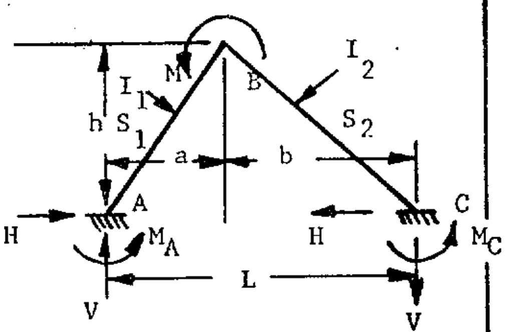
$$M_A = \frac{wh^2(3K+2)}{24(K+1)} \qquad M_C = \frac{wh^2}{24(K+1)}$$

9. APPLIED MOMENT AT APEX



$$V = \frac{M}{L}$$

$$H = \frac{M}{hL}\left[\frac{a - bK}{K+1}\right]$$



$$V = \frac{3M}{2L}$$

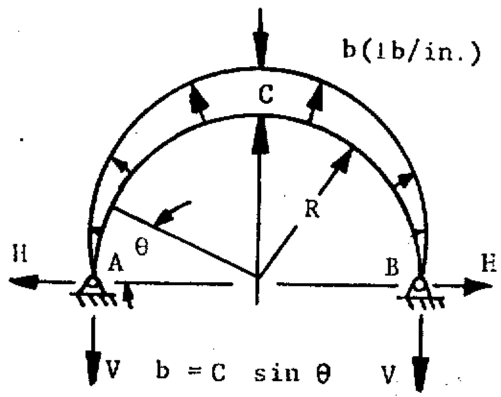
$$H = \frac{3M(a - bK)}{2hL(K+1)}$$

$$M_A = \frac{KM}{2(K+1)}$$

$$M_C = \frac{M}{2(K+1)}$$

TABLE 12.4 (CONT'D) - REACTIONS AND CONSTRAINING MOMENTS IN TRIANGULAR FRAMES

1. Sinusoidal Normal Pressure



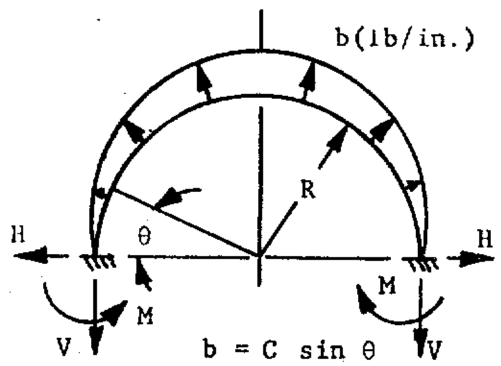
$$V = \frac{C\pi R}{4}$$

$$H = \frac{CR}{4}$$

$$M_\theta = \frac{CR^2}{4}\left[(\pi - 2\theta)\cos\theta - \pi + 3\sin\theta\right]$$

(Positive moment acts clockwise on section ahead.)

2. Sinusoidal Normal Pressure



$$V = \frac{C\pi R}{4}$$

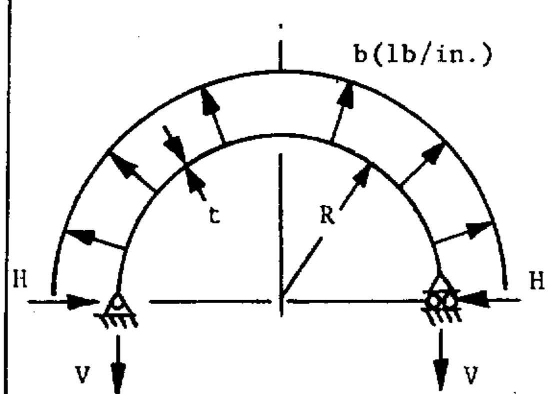
$$H = \frac{CR}{4}\left[\frac{3\pi^2 - 32}{8 - \pi^2}\right] = .31974CR$$

$$M = \frac{CR^2}{4}\left[\frac{\pi^3 - 10\pi}{8 - \pi^2}\right] = .05478CR^2$$

$$M_\theta = CR^2\left[.81974\sin\theta - .84018 + \frac{\cos\theta}{2}\left(\frac{\pi}{2} - \theta\right)\right]$$

(Positive moment acts clockwise on section ahead.)

3. Uniform Normal Pressure



M = 0 at all points since pin points permit a uniform hoop tension. T, where:

$$T = V = bR$$

$$H = 0$$

TABLE 12.5 - REACTIONS AND CONSTRAINING MOMENTS IN SEMICIRCULAR FRAMES OR ARCHES

12.4 Analysis of Rings

Tables and figures are presented for the analysis of rings and ring-supported shells. Sections 12.4.1 and 12.4.2 show analysis methods for rings which are rigid with respect to the resisting structure for out-of-plane loads. The plane of the ring remains plane and the supporting structure deforms.

Only bending is considered in the deflection curves for the in-plane load cases given in Figures 12.4 through 12.29. Refer to Figures 12.30 through 12.33 to include the effects of shear and normal forces.

Section 12.4.3 shows methods of analysis for circular cylindrical shells supported by "flexible" rings.

12.4.1 Analysis of Rigid Rings with In-Plane Loading

Coefficients to obtain slope, deflection, bending moment, shear, and axial force along with equations for these values are given for some of the frequently-used load cases. Figure 12.4 shows an index for the various load cases presented in Figures 12.5 through 12.29.

The sign convention used throughout the rigid frame analysis in-plane load cases is shown in Figure 12.3. It basically consists of: moments which produce tension

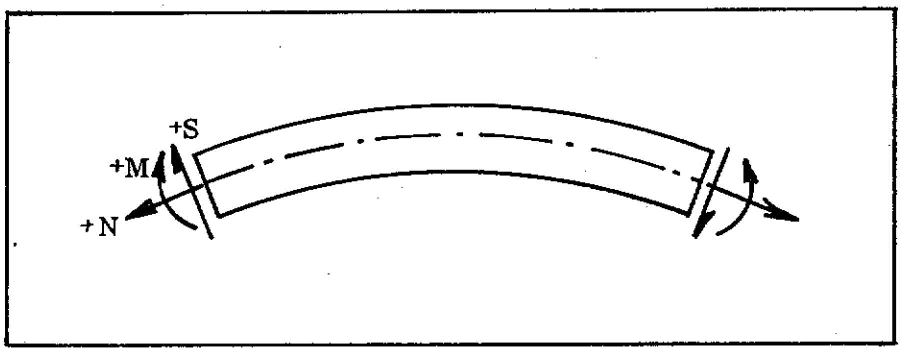


FIGURE 12.3 - SIGN CONVENTION FOR RIGID RINGS WITH IN-PLANE LOADS

on the inner fibers are positive, transverse forces which act upward to the left of the cut are positive and axial forces which produce tension in the frame are positive.

Deflections in Figures 12.5 through 12.29 are based on bending only. Deflection curves for the three basic load cases due to shear and concentrated loads are shown in Figures 12.31 through 12.33. A shape factor (β) that is to be used with the curves for shear deflection of various cross sections is shown in Figure 12.30.

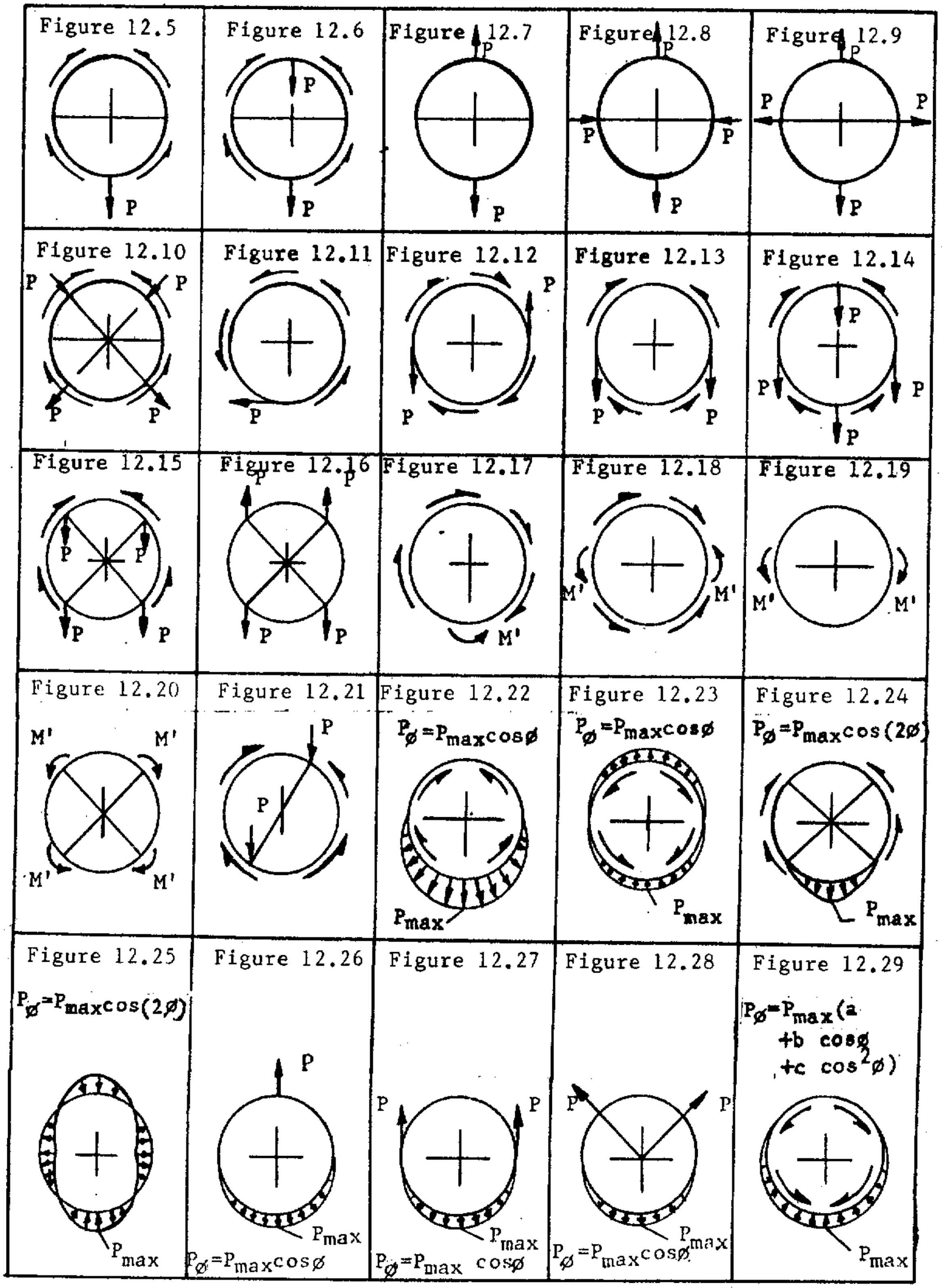
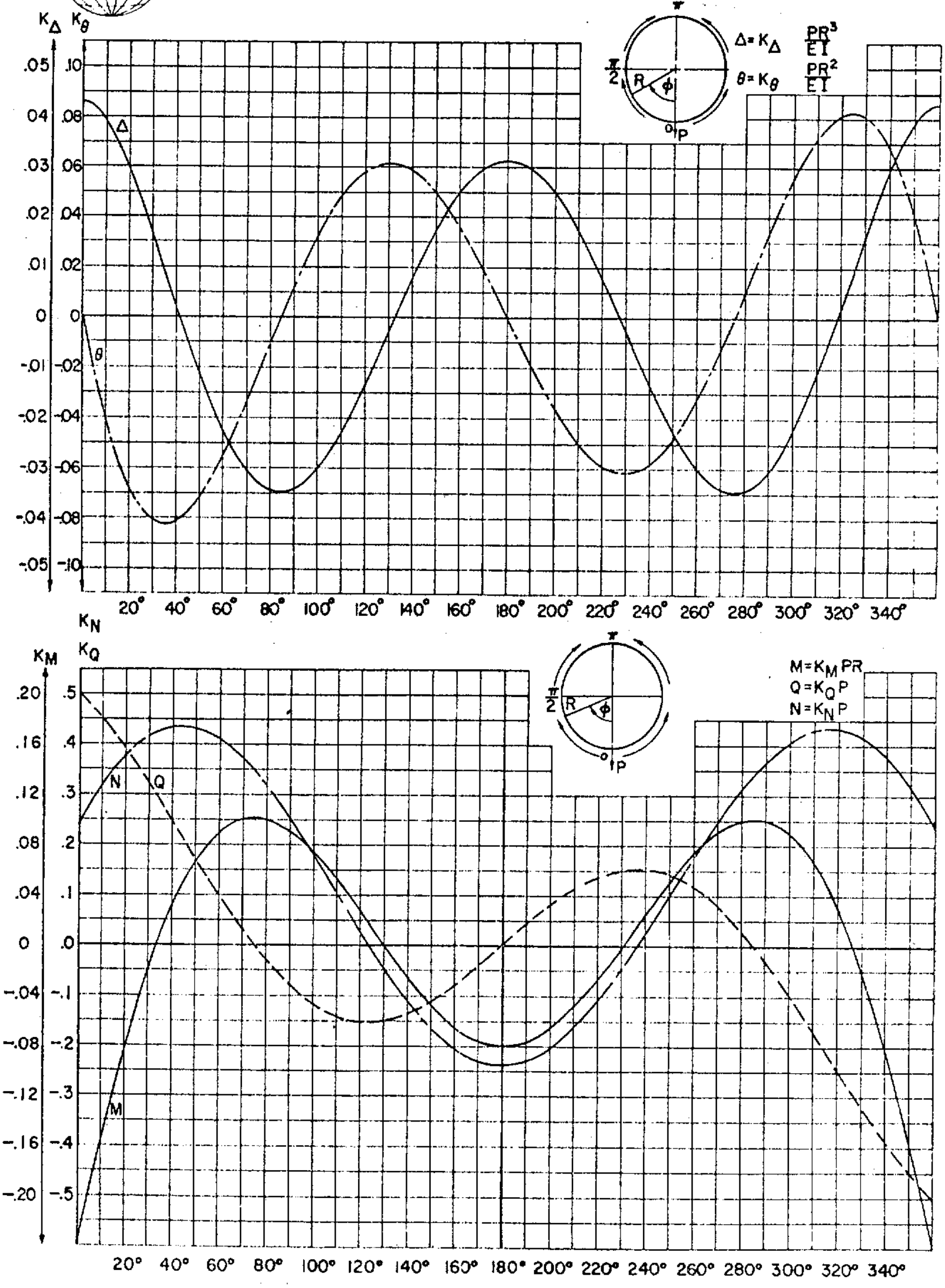
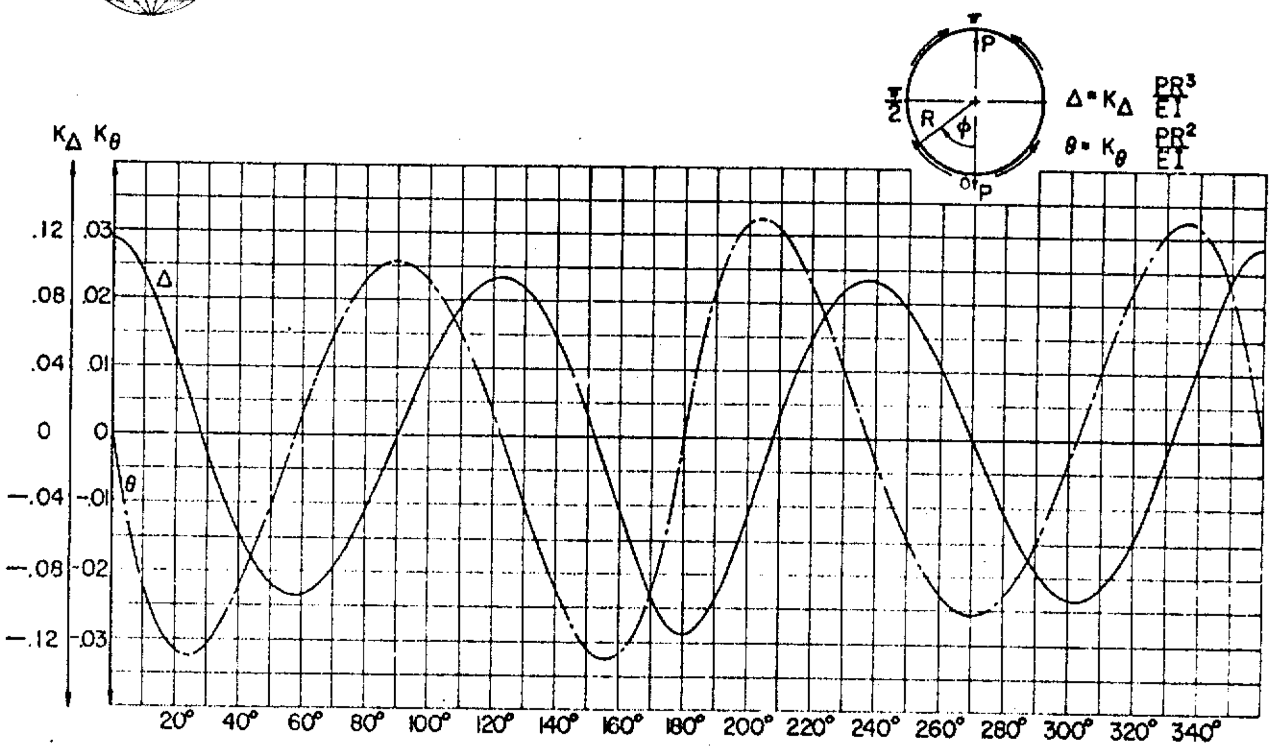


FIGURE 12.4 - INDEX OF IN-PLANE LOADS CASES FOR RIGID RINGS



K_Δ K_θ
.05 .10
.04 .08
.03 .06
.02 .04
.01 .02
0 0
-.01 -.02
-.02 -.04
-.03 -.06
-.04 -.08
-.05 -.10
Δ
θ
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°
$\Delta = K_\Delta \frac{PR^3}{EI}$
$\theta = K_\theta \frac{PR^2}{EI}$
R
φ
P
K_N
K_Q
K_M
.20 .5
.16 .4
.12 .3
.08 .2
.04 .1
0 .0
-.04 -.1
-.08 -.2
-.12 -.3
-.16 -.4
-.20 -.5
N
Q
M
$M = K_M PR$
$Q = K_Q P$
$N = K_N P$
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.5


K_Δ K_θ
$\Delta = K_\Delta \frac{PR^3}{EI}$
$\theta = K_\theta \frac{PR^2}{EI}$
P
R
φ
$\frac{\pi}{2}$
0
.12 .03
.08 .02
.04 .01
0 0
-.04 -.01
-.08 -.02
-.12 -.03
Δ
θ
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

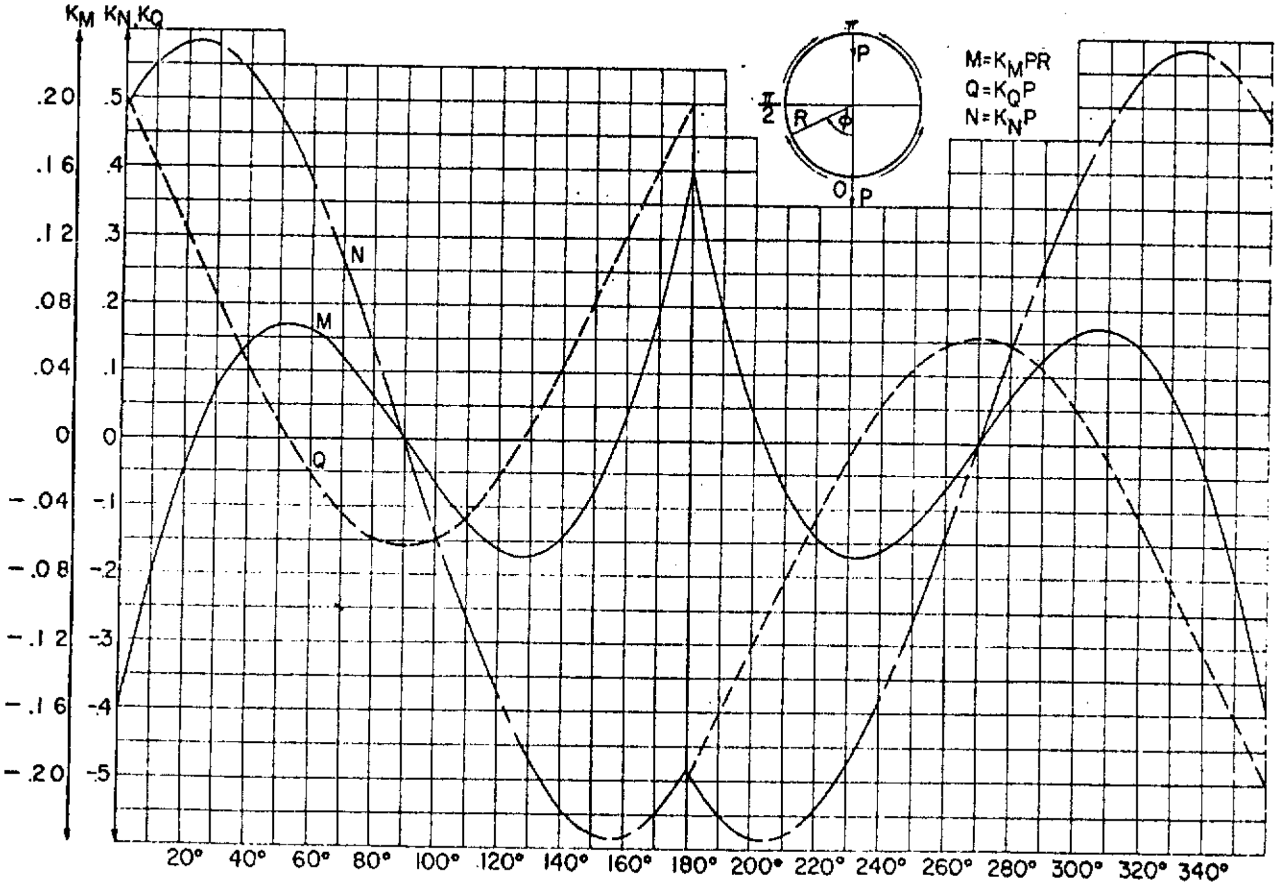
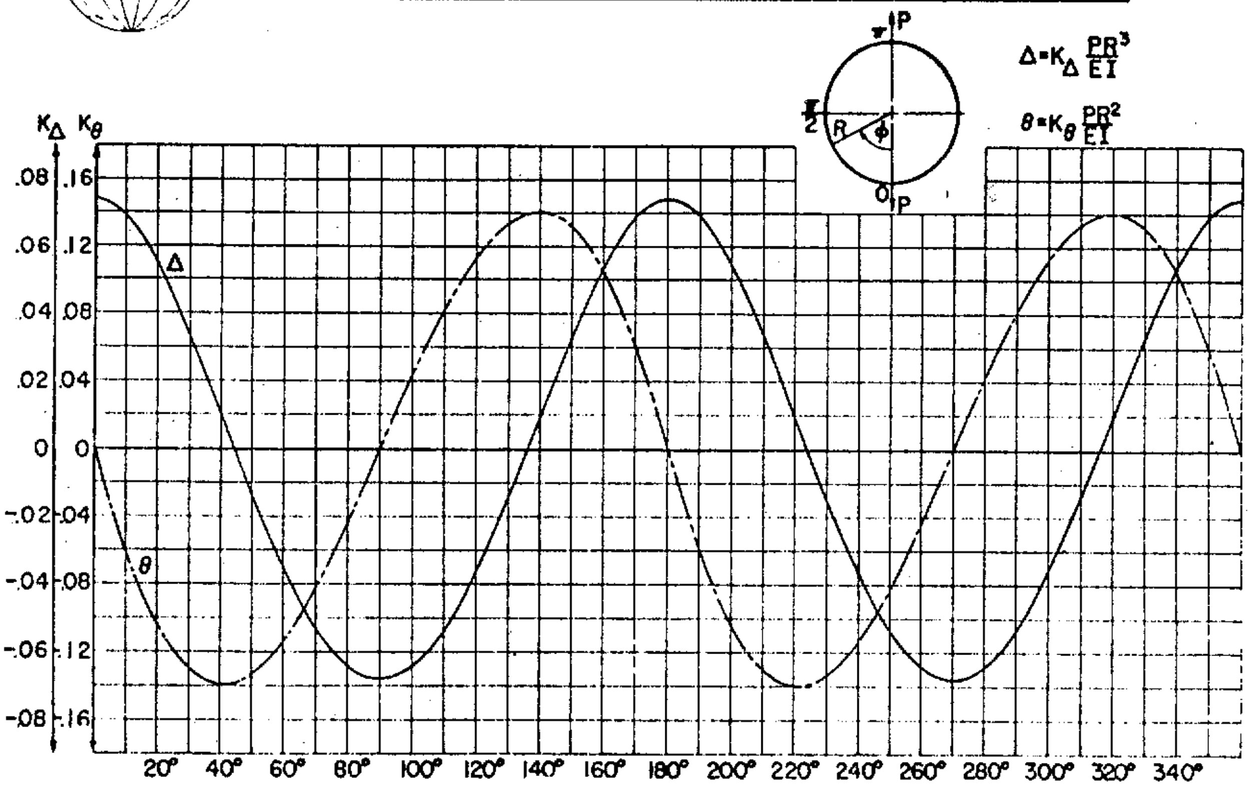
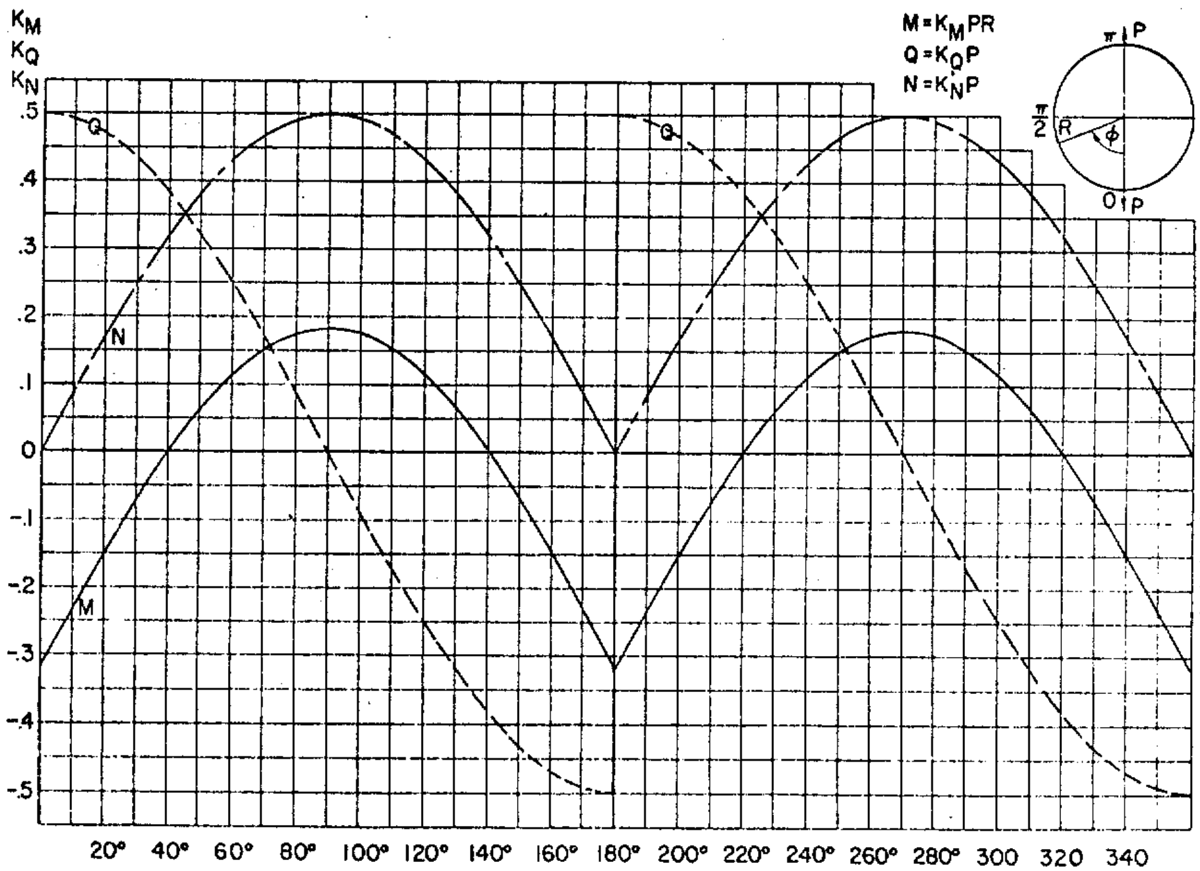

K_M K_N K_Q
$M = K_M PR$
$Q = K_Q P$
$N = K_N P$
P
R
φ
$\frac{\pi}{2}$
0
.20 .5
.16 .4
.12 .3
.08 .2
.04 .1
0 0
-.04 -.1
-.08 -.2
-.12 -.3
-.16 -.4
-.20 -.5
N
M
Q
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.6

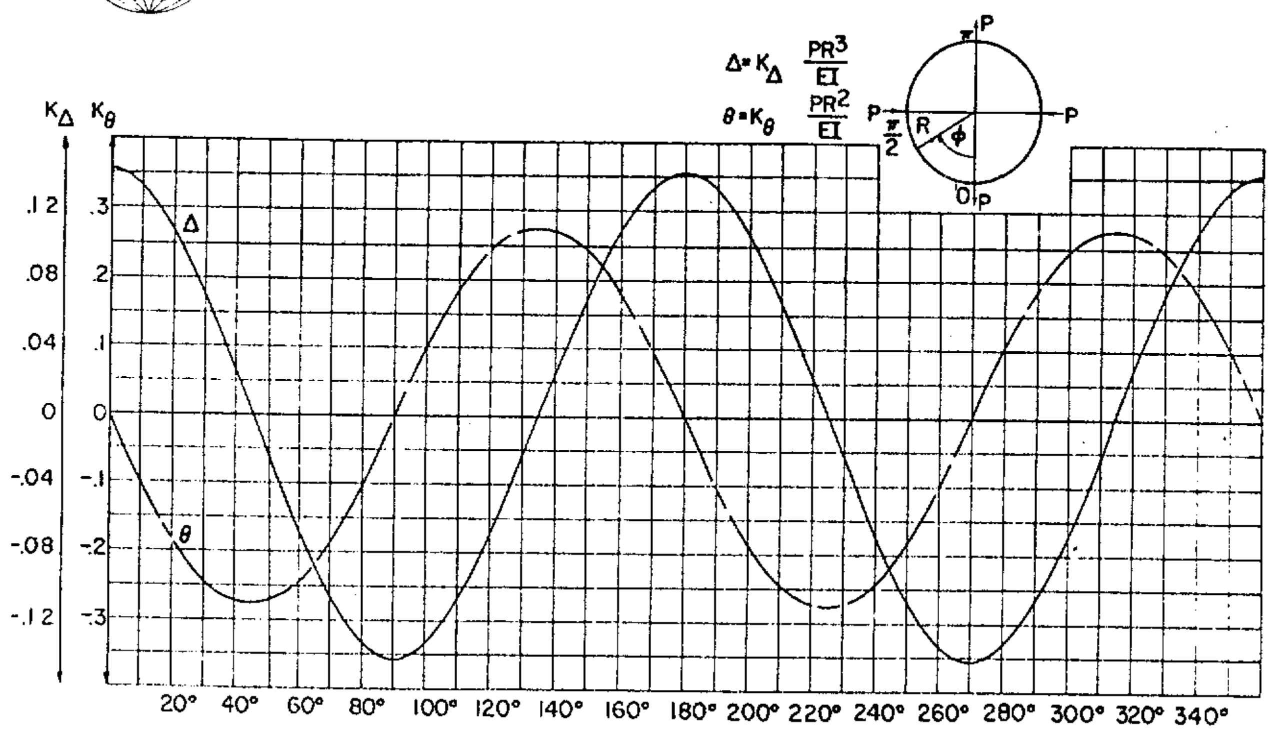


$\Delta = K_\Delta \frac{PR^3}{EI}$
$\theta = K_\theta \frac{PR^2}{EI}$
K_Δ K_θ
Δ
θ
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°



K_M
K_Q
K_N
$M = K_M PR$
$Q = K_Q P$
$N = K_N P$
Q
N
M
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300 320 340

FIGURE 12.7


$\Delta = K_\Delta \frac{PR^3}{EI}$
$\theta = K_\theta \frac{PR^2}{EI}$
K_Δ K_θ
P
π
$\frac{\pi}{2}$
R
ϕ
0
Δ
θ
.12 .3
.08 .2
.04 .1
0 0
-.04 -.1
-.08 -.2
-.12 -.3
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

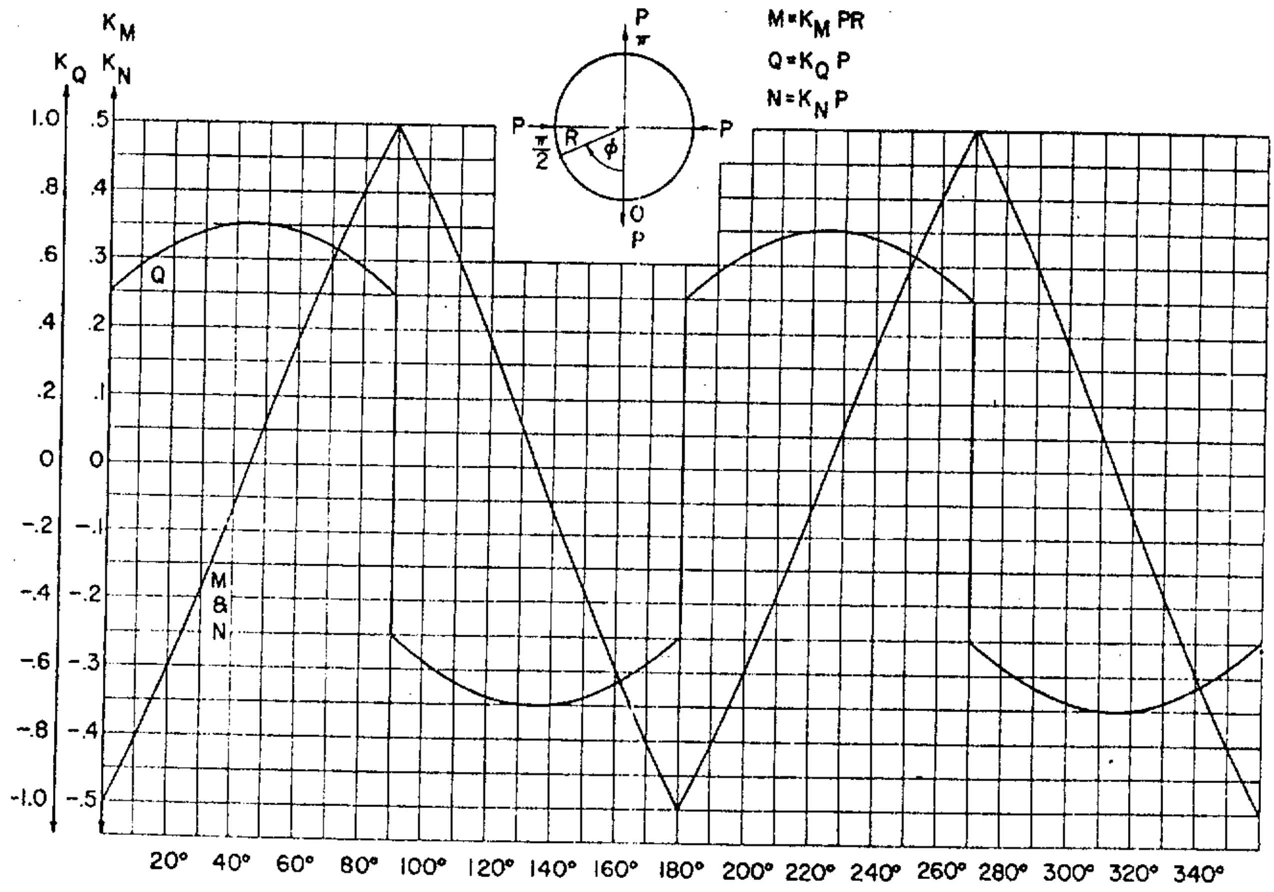
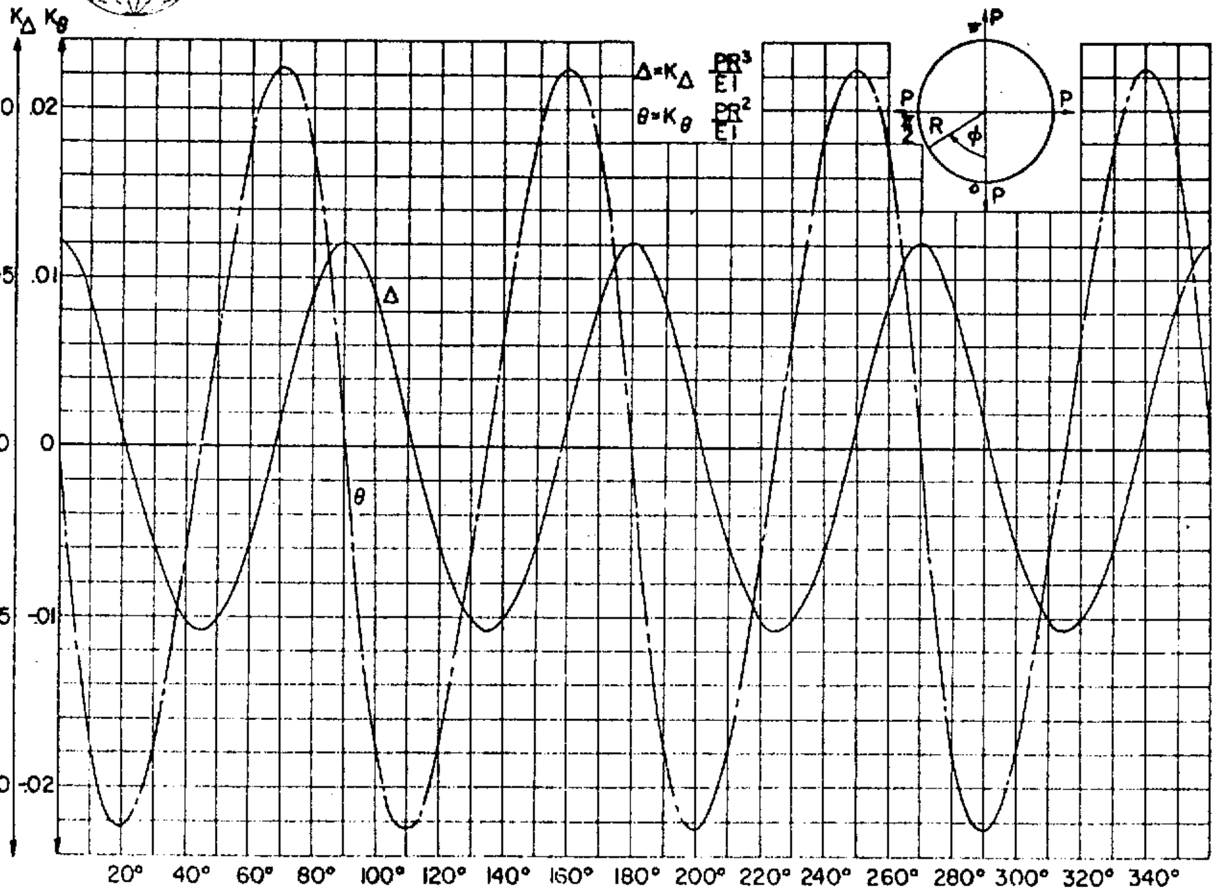

$M = K_M PR$
$Q = K_Q P$
$N = K_N P$
K_Q K_M K_N
P
π
$\frac{\pi}{2}$
R
ϕ
0
Q
M & N
1.0 .5
.8 .4
.6 .3
.4 .2
.2 .1
0 0
-.2 -.1
-.4 -.2
-.6 -.3
-.8 -.4
-1.0 -.5
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.8


K_Δ K_θ
$\Delta = K_\Delta \frac{PR^3}{EI}$
$\theta = K_\theta \frac{PR^2}{EI}$
P
P
P
P
R
ϕ
$\frac{\pi}{2}$
Δ
θ
.010 .02
.005 .01
0 0
-.005 -.01
-.010 -.02
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

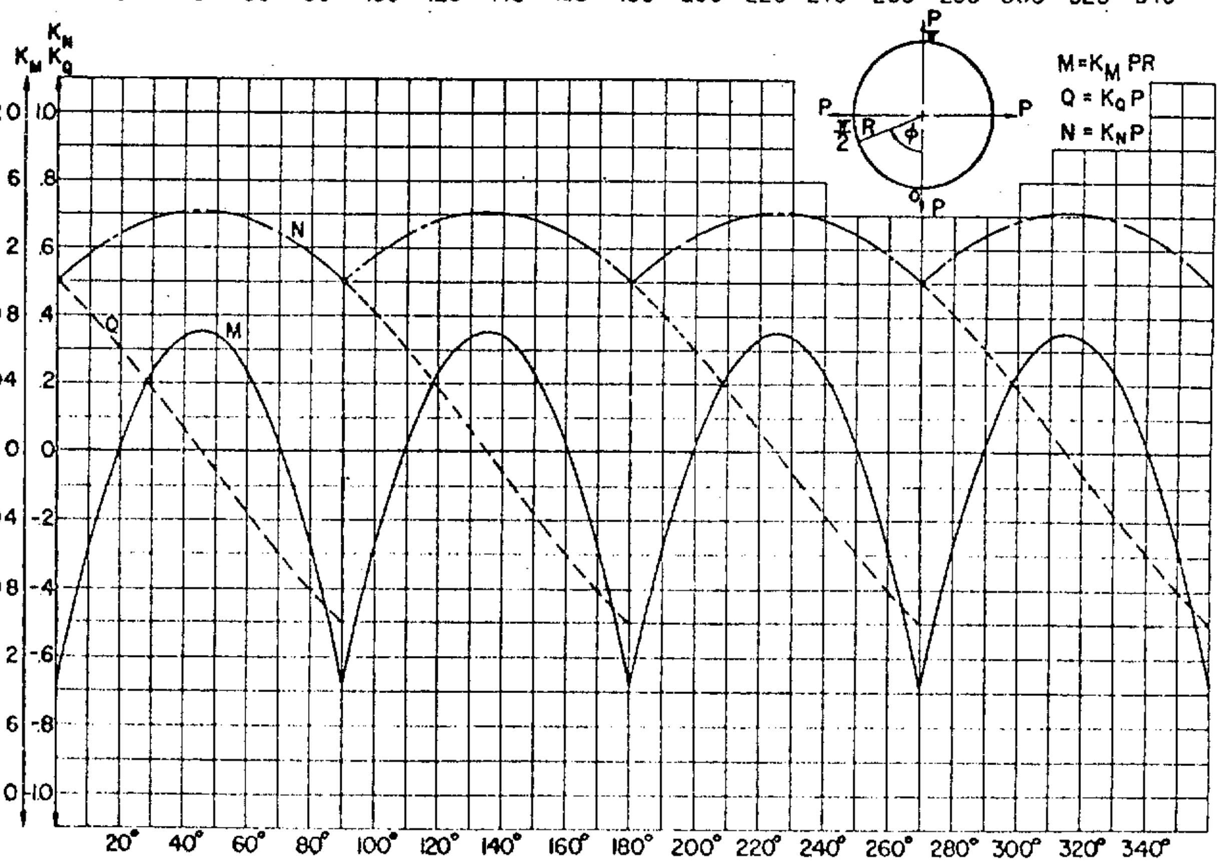
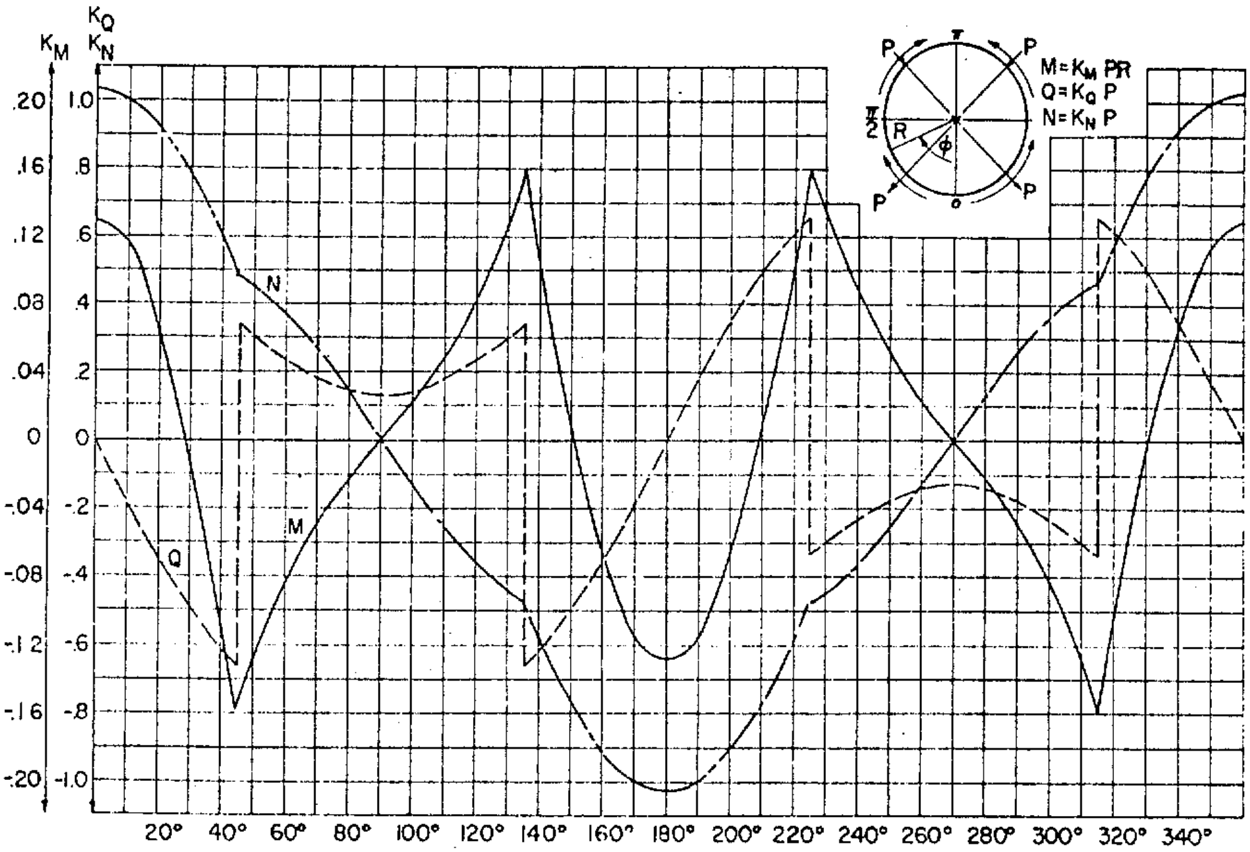
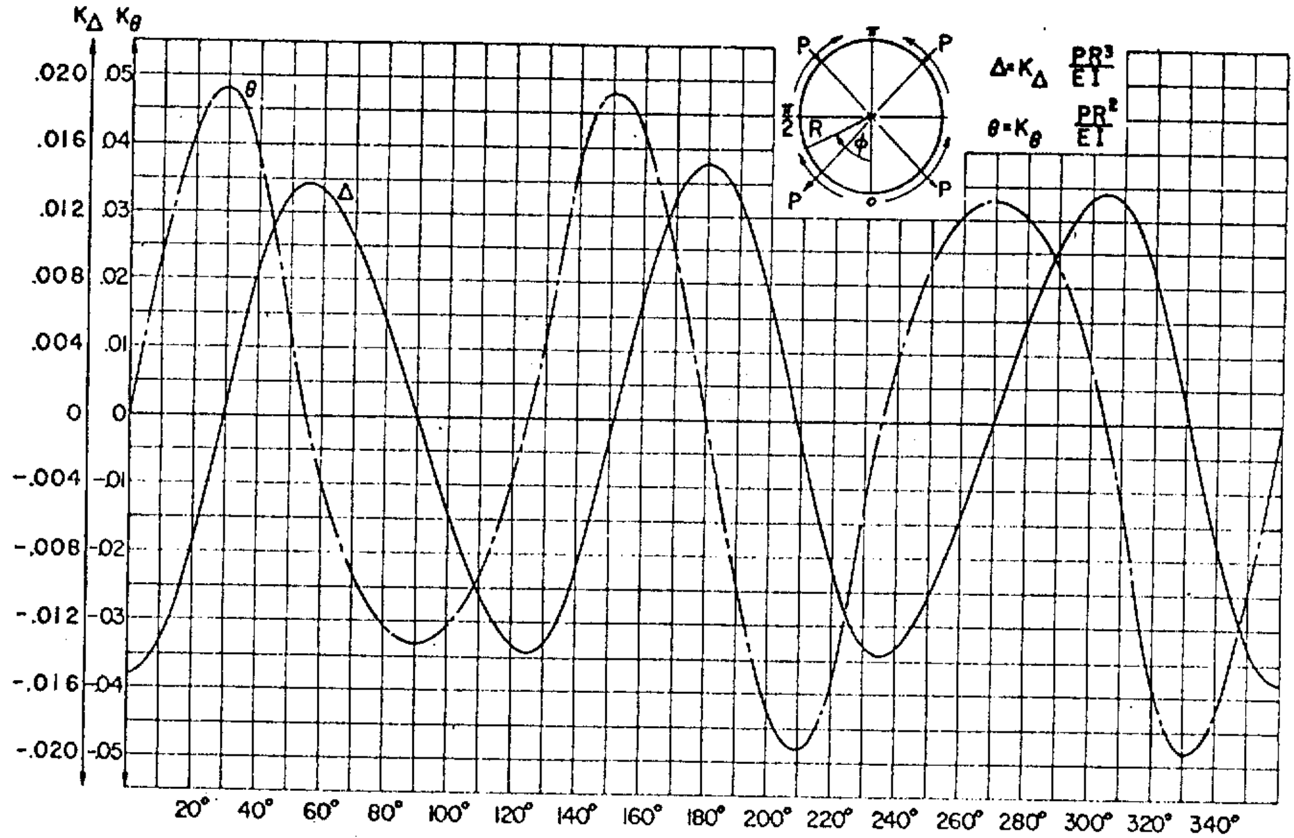

K_M K_Q K_N
$M = K_M PR$
$Q = K_Q P$
$N = K_N P$
P
P
P
P
R
ϕ
$\frac{\pi}{2}$
N
Q
M
.20 1.0
.16 .8
.12 .6
.08 .4
.04 .2
0 0
−.04 -.2
−.08 -.4
−.12 -.6
−.16 -.8
−.20 -1.0
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.9

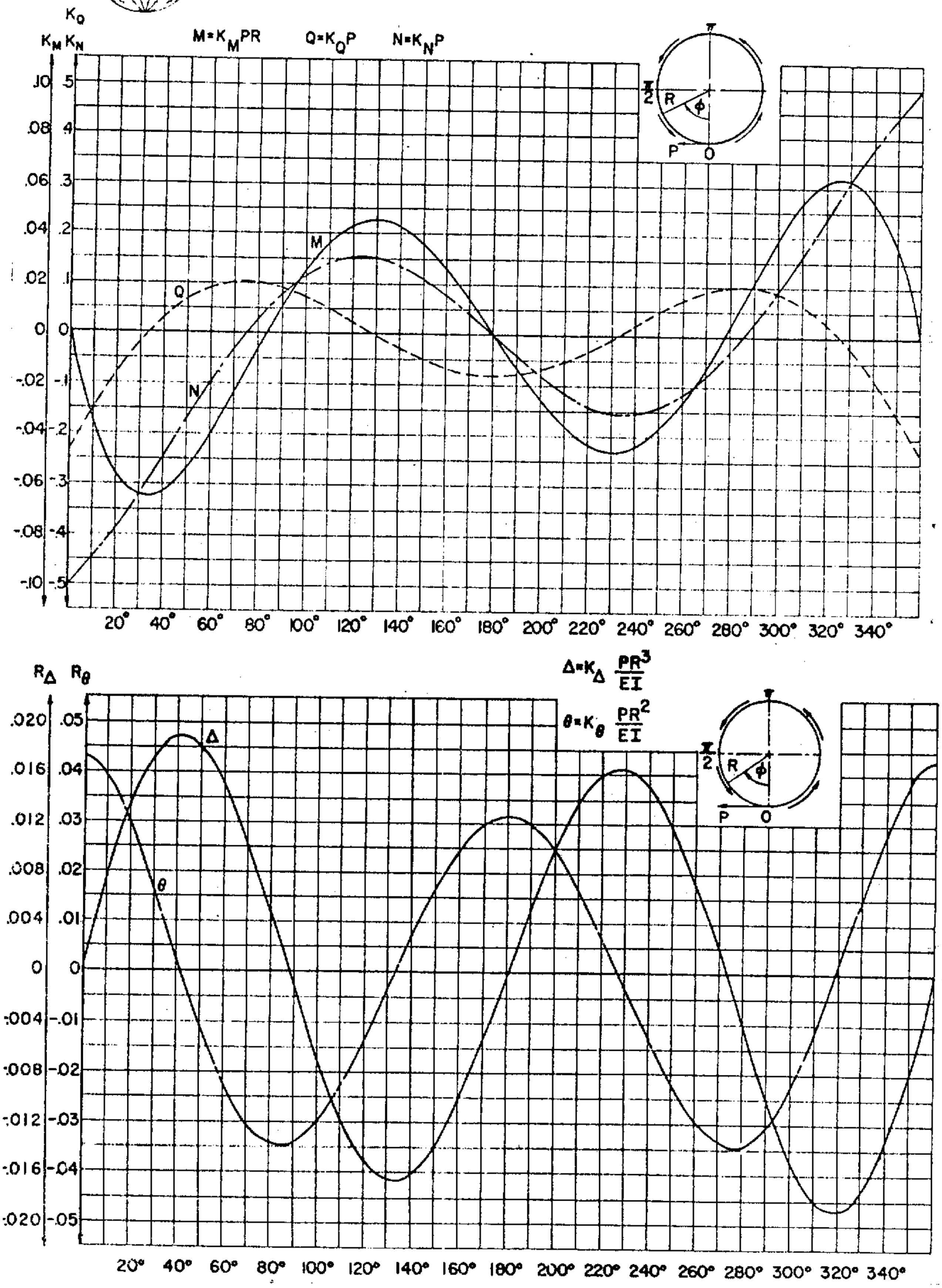


K_M
K_Q
K_N
M = K_M PR
Q = K_Q P
N = K_N P
P
R
φ
N
M
Q
.20 1.0
.16 .8
.12 .6
.08 .4
.04 .2
0 0
-.04 -.2
-.08 -.4
-.12 -.6
-.16 -.8
-.20 -1.0
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°



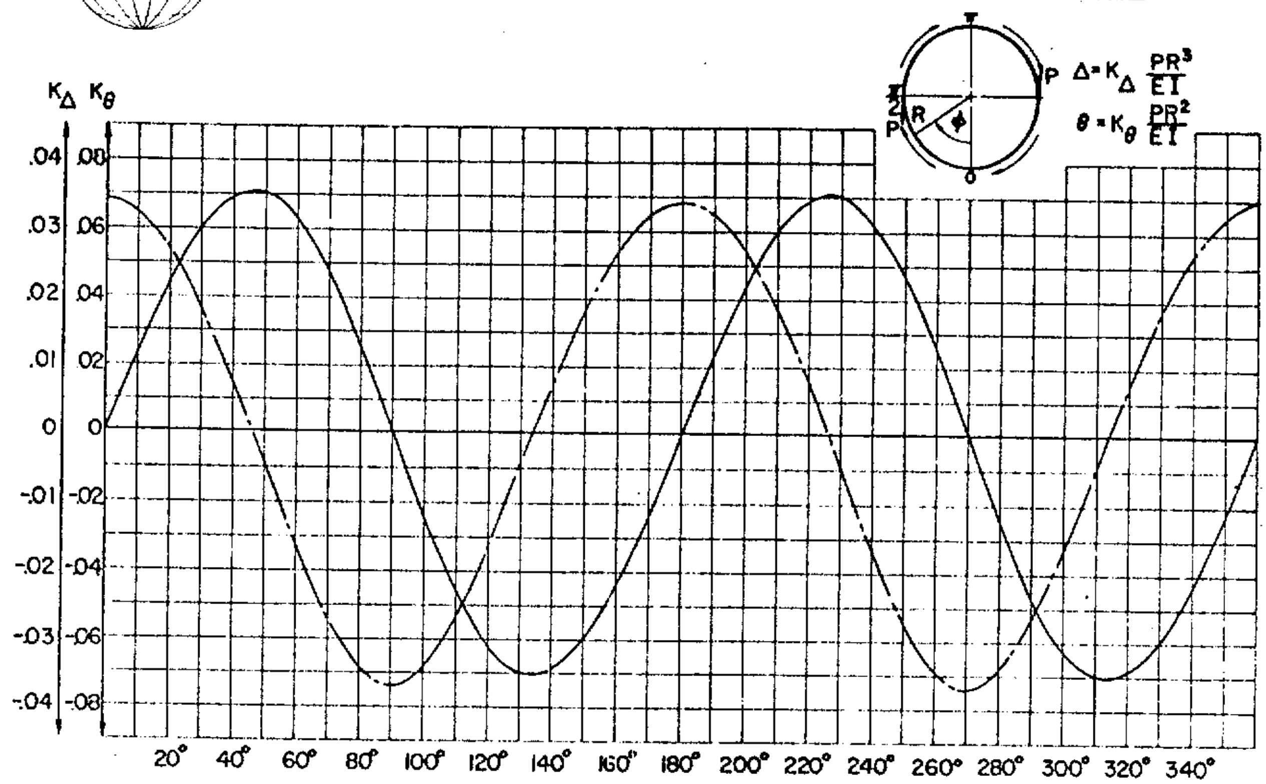
K_Δ K_θ
$\Delta = K_\Delta \frac{PR^3}{EI}$
$\theta = K_\theta \frac{PR^2}{EI}$
P
R
φ
θ
Δ
.020 .05
.016 .04
.012 .03
.008 .02
.004 .01
0 0
-.004 -.01
-.008 -.02
-.012 -.03
-.016 -.04
-.020 -.05
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12,10



K_Q
K_M K_N
M=K_MPR
Q=K_QP
N=K_NP
.10 .5
.08 .4
.06 .3
.04 .2
.02 .1
0 0
-.02 -.1
-.04 -.2
-.06 -.3
-.08 -.4
-.10 -.5
M
Q
N
R
ϕ
P
0
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°
R_Δ R_θ
$\Delta = K_\Delta \frac{PR^3}{EI}$
$\theta = K_\theta \frac{PR^2}{EI}$
.020 .05
.016 .04
.012 .03
.008 .02
.004 .01
0 0
-.004 -.01
-.008 -.02
-.012 -.03
-.016 -.04
-.020 -.05
Δ
θ
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.11


KΔ Kθ
.04 .08
.03 .06
.02 .04
.01 .02
0 0
-.01 -.02
-.02 -.04
-.03 -.06
-.04 -.08
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°
π/2
P
R
φ
0
Δ = KΔ PR³/EI
θ = Kθ PR²/EI

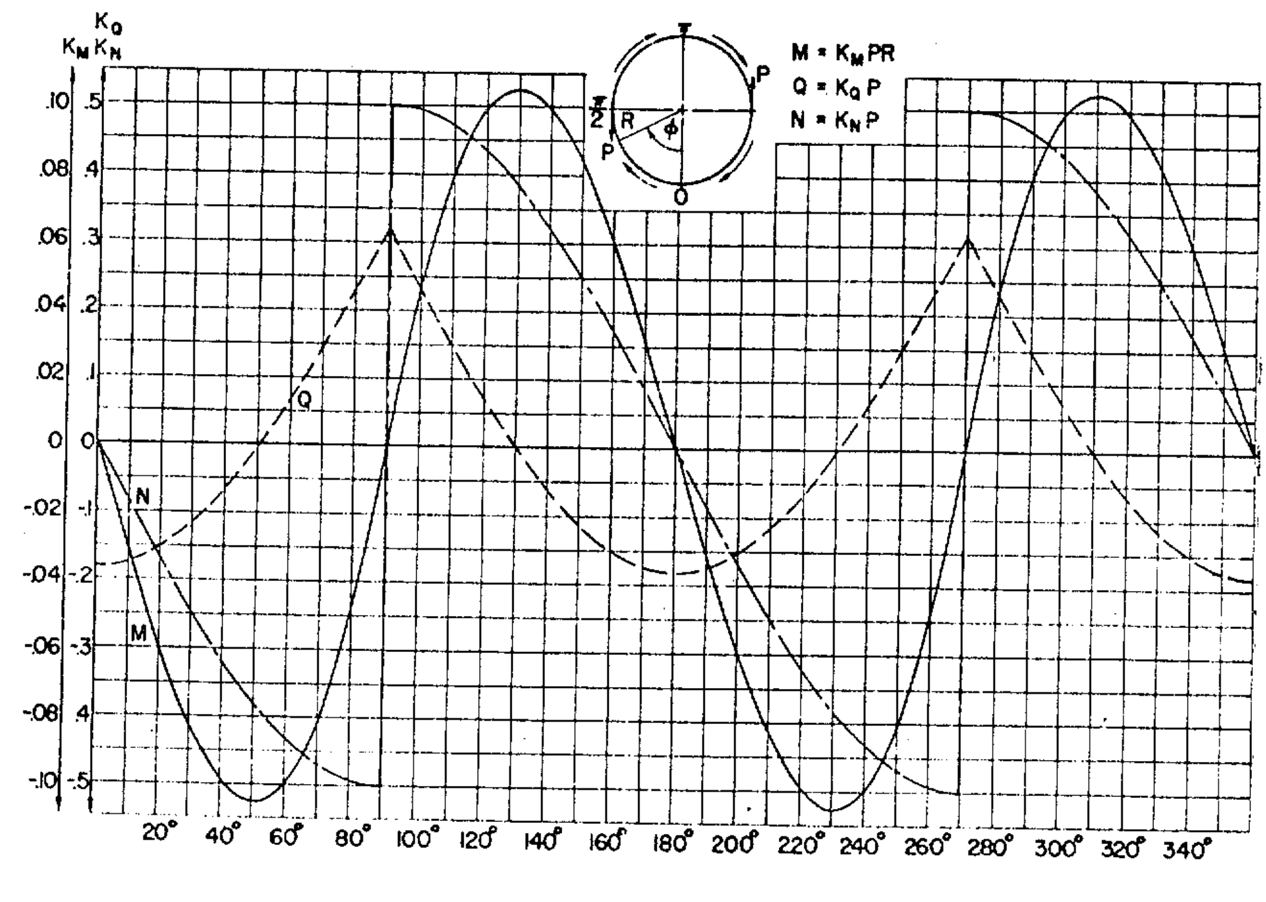
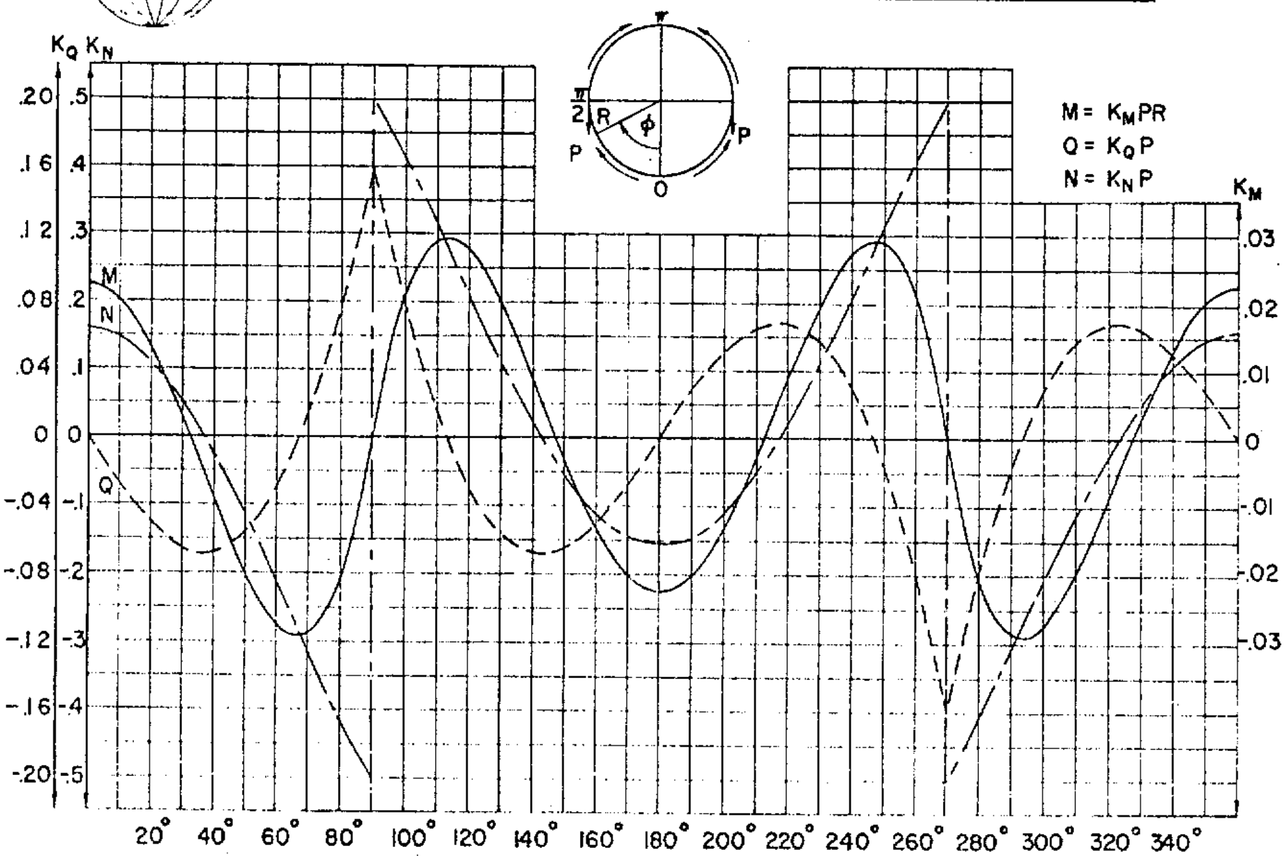

KQ
KM KN
.10 .5
.08 .4
.06 .3
.04 .2
.02 .1
0 0
-.02 -.1
-.04 -.2
-.06 -.3
-.08 -.4
-.10 -.5
Q
N
M
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°
π/2
P
R
φ
0
M = KM PR
Q = KQ P
N = KN P

FIGURE 12.12


K_Q K_N
M = K_MPR
Q = K_QP
N = K_NP
K_M
M
N
Q
π/2
R
φ
P
0
.20 .16 .12 .08 .04 0 -.04 -.08 -.12 -.16 -.20
.5 .4 .3 .2 .1 0 -.1 -.2 -.3 -.4 -.5
.03 .02 .01 0 -.01 -.02 -.03
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

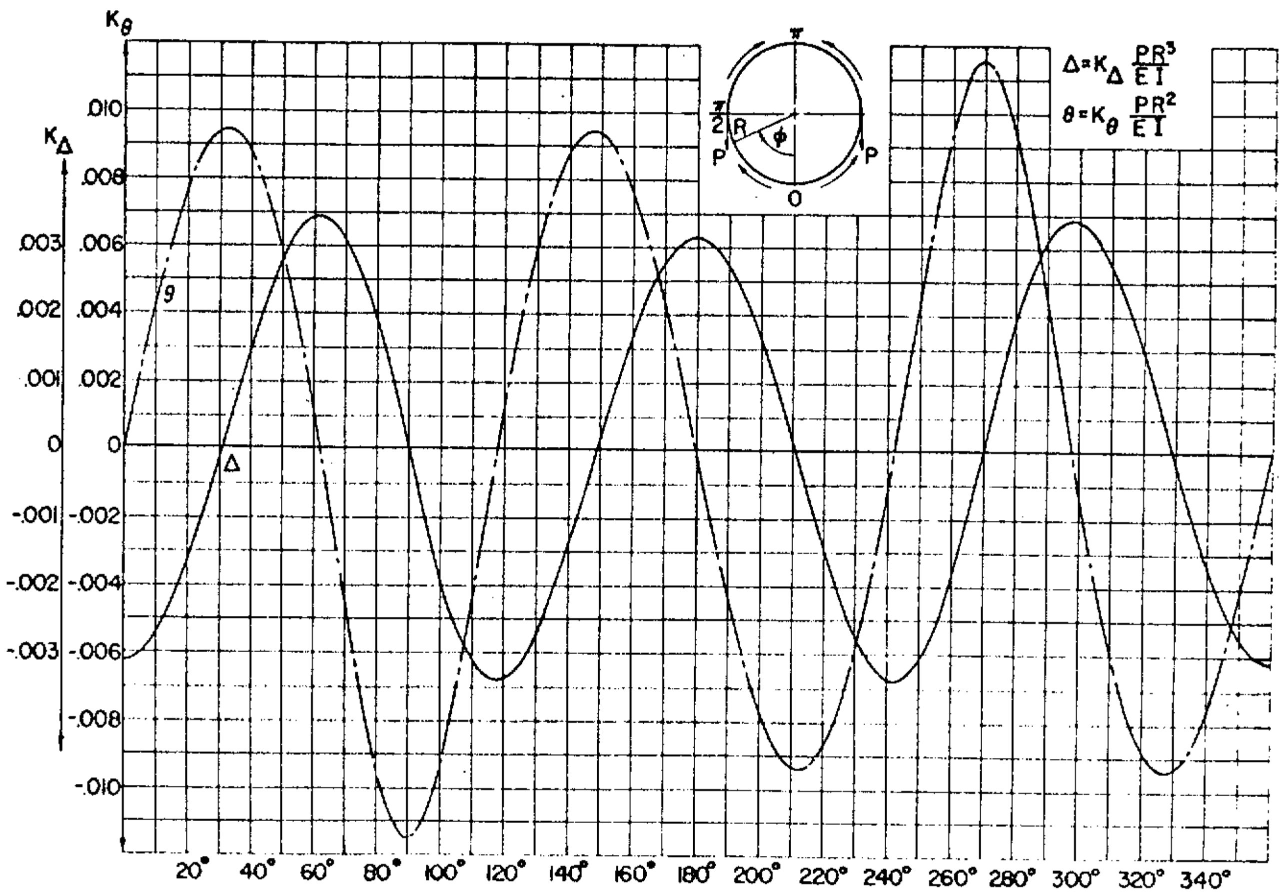
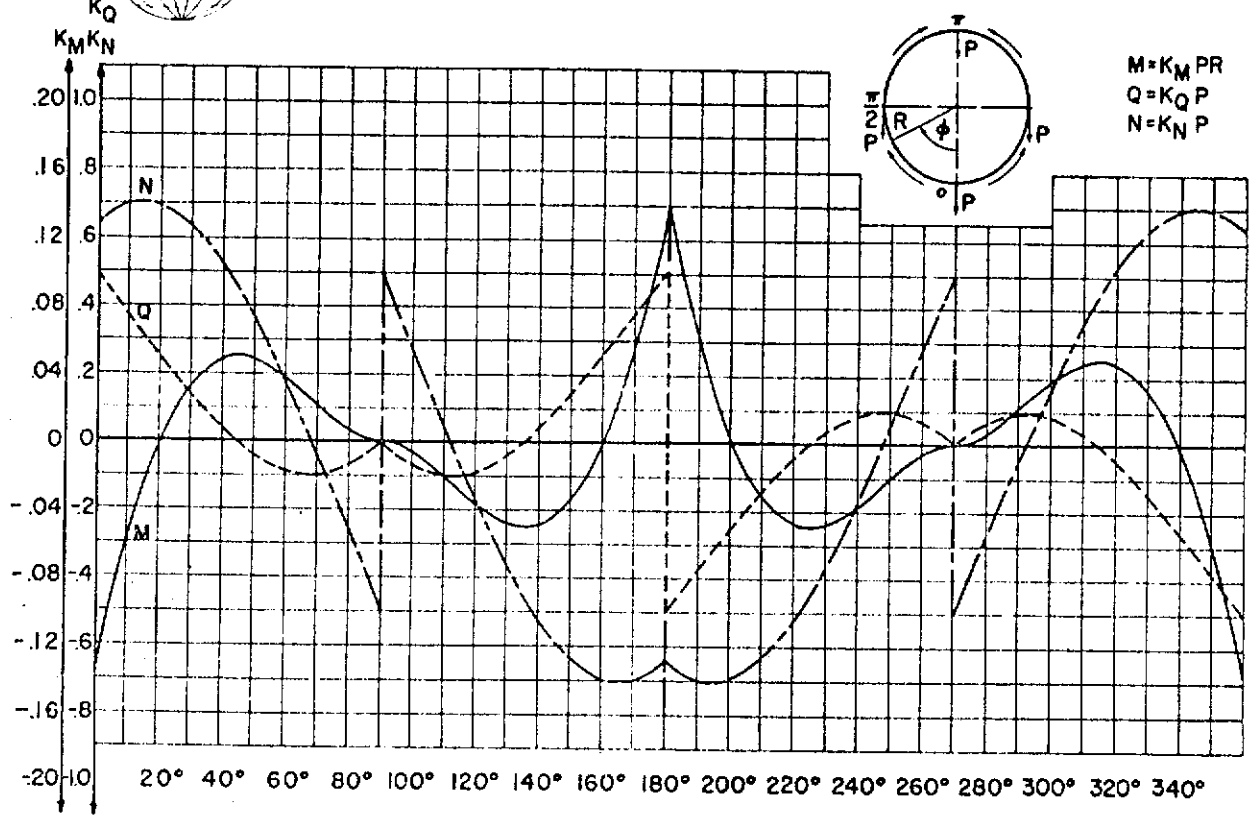
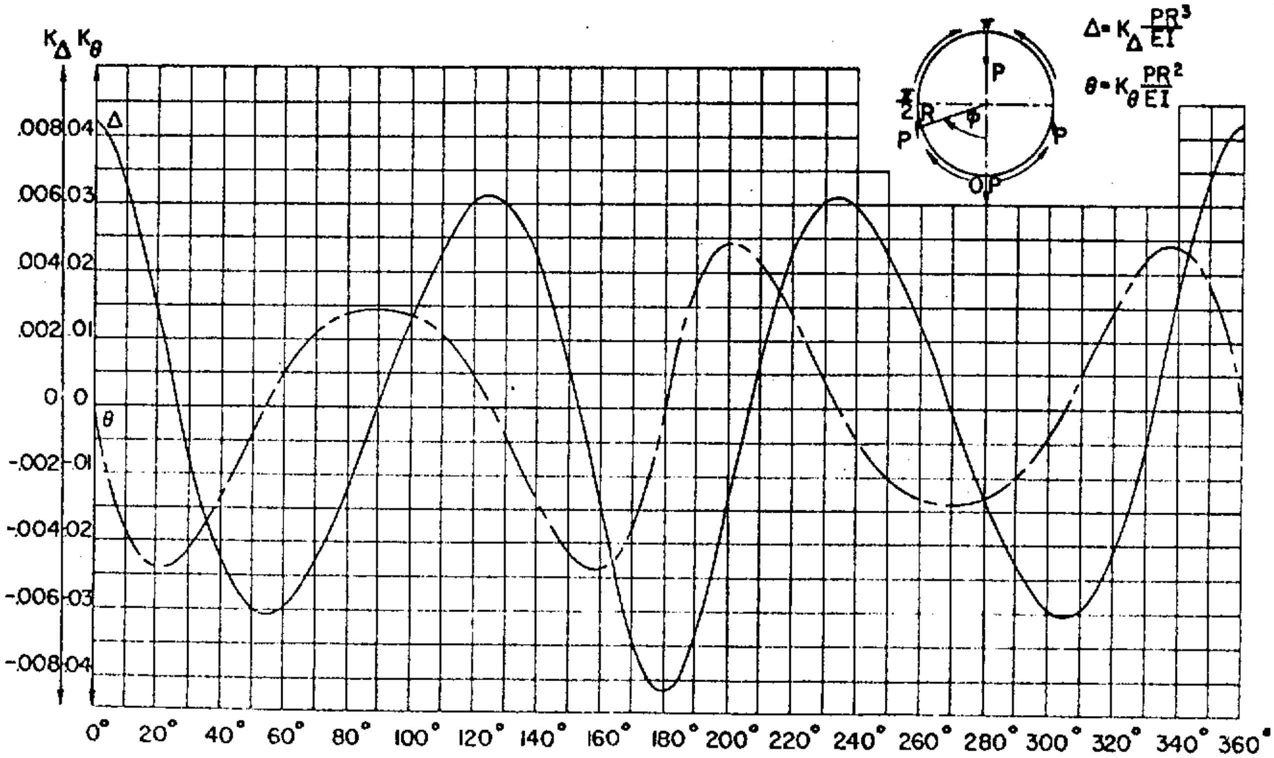

K_θ
K_Δ
Δ=K_Δ PR³/EI
θ=K_θ PR²/EI
θ
Δ
π/2
R
φ
P
0
.010 .008 .006 .004 .002 0 -.002 -.004 -.006 -.008 -.010
.003 .002 .001 0 -.001 -.002 -.003
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.13

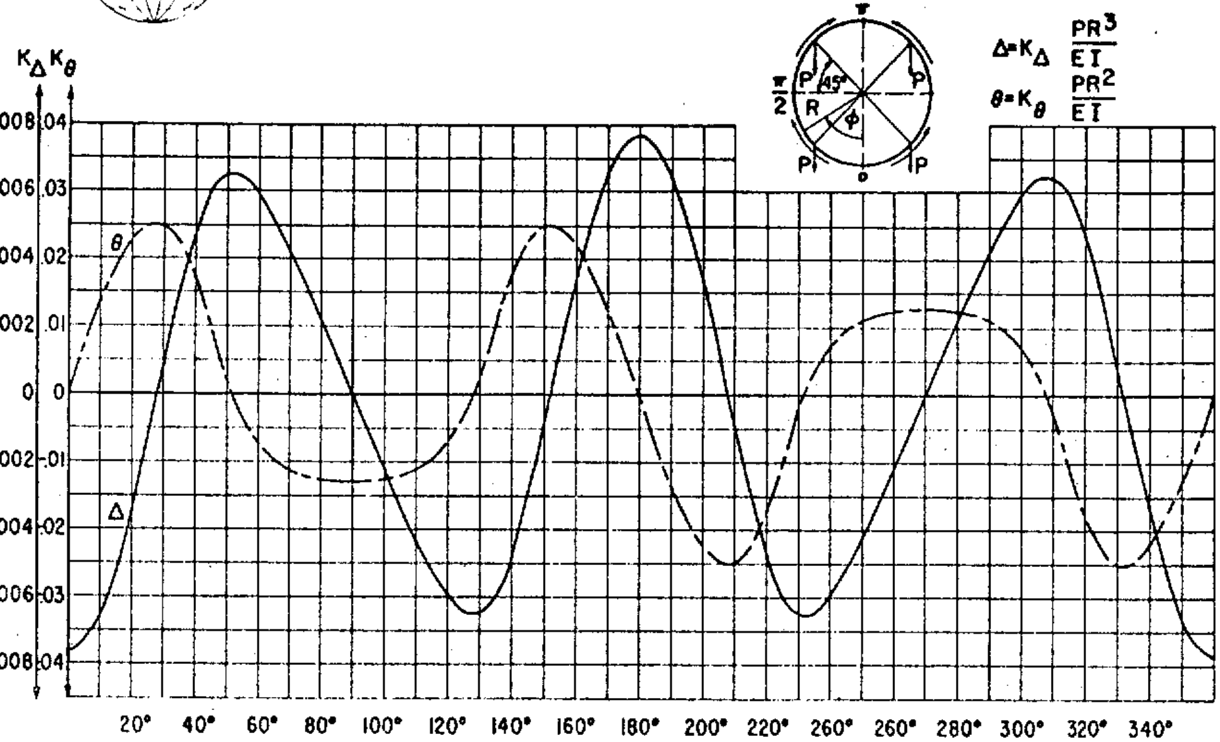


K_Q
K_M K_N
M = K_M PR
Q = K_Q P
N = K_N P
N
Q
M
.20 1.0
.16 .8
.12 .6
.08 .4
.04 .2
0 0
-.04 -.2
-.08 -.4
-.12 -.6
-.16 -.8
-.20 -1.0
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°



K_Δ K_θ
$\Delta = K_\Delta \frac{PR^3}{EI}$
$\theta = K_\theta \frac{PR^2}{EI}$
Δ
θ
.008 .04
.006 .03
.004 .02
.002 .01
0 0
-.002 -.01
-.004 -.02
-.006 -.03
-.008 -.04
0° 20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340° 360°

FIGURE 12.14


$\Delta = K_\Delta \frac{PR^3}{EI}$
$\theta = K_\theta \frac{PR^2}{EI}$
K_Δ K_θ
θ
Δ
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 260° 260° 280° 300° 320° 340°

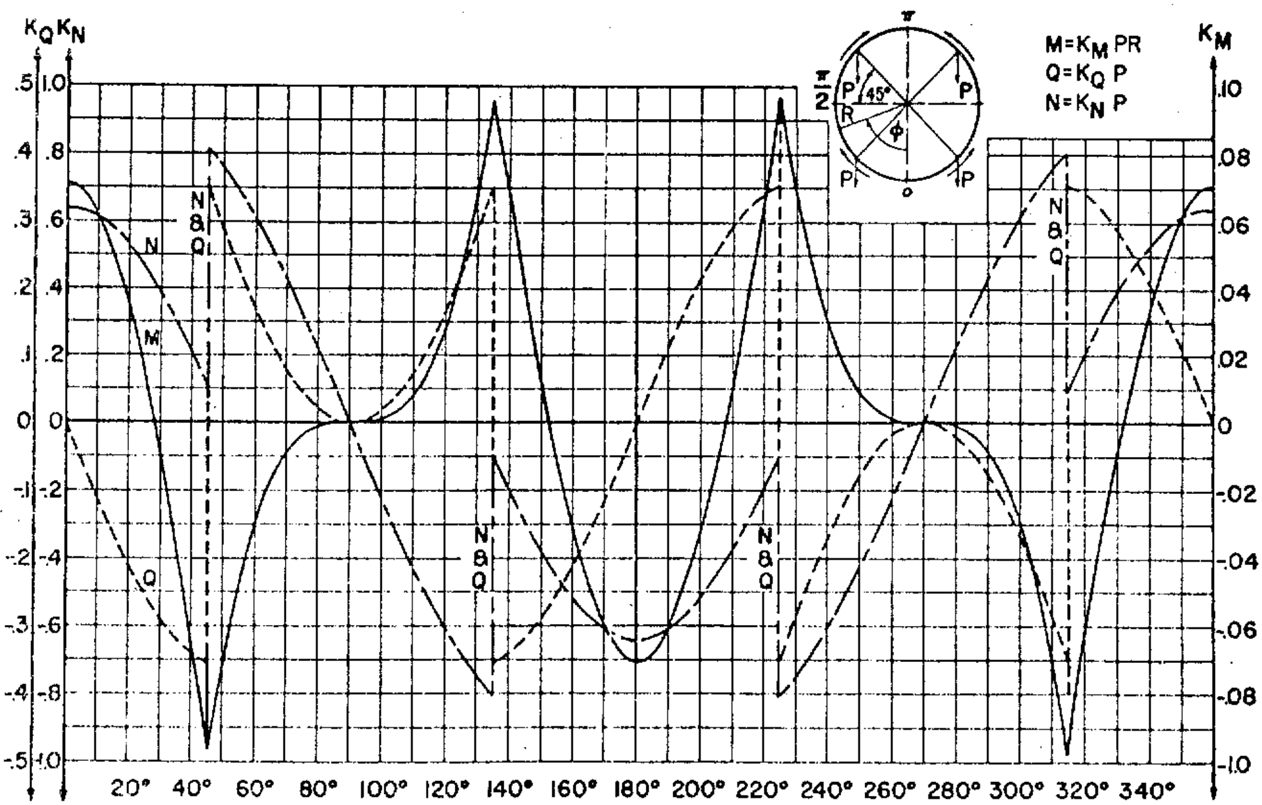
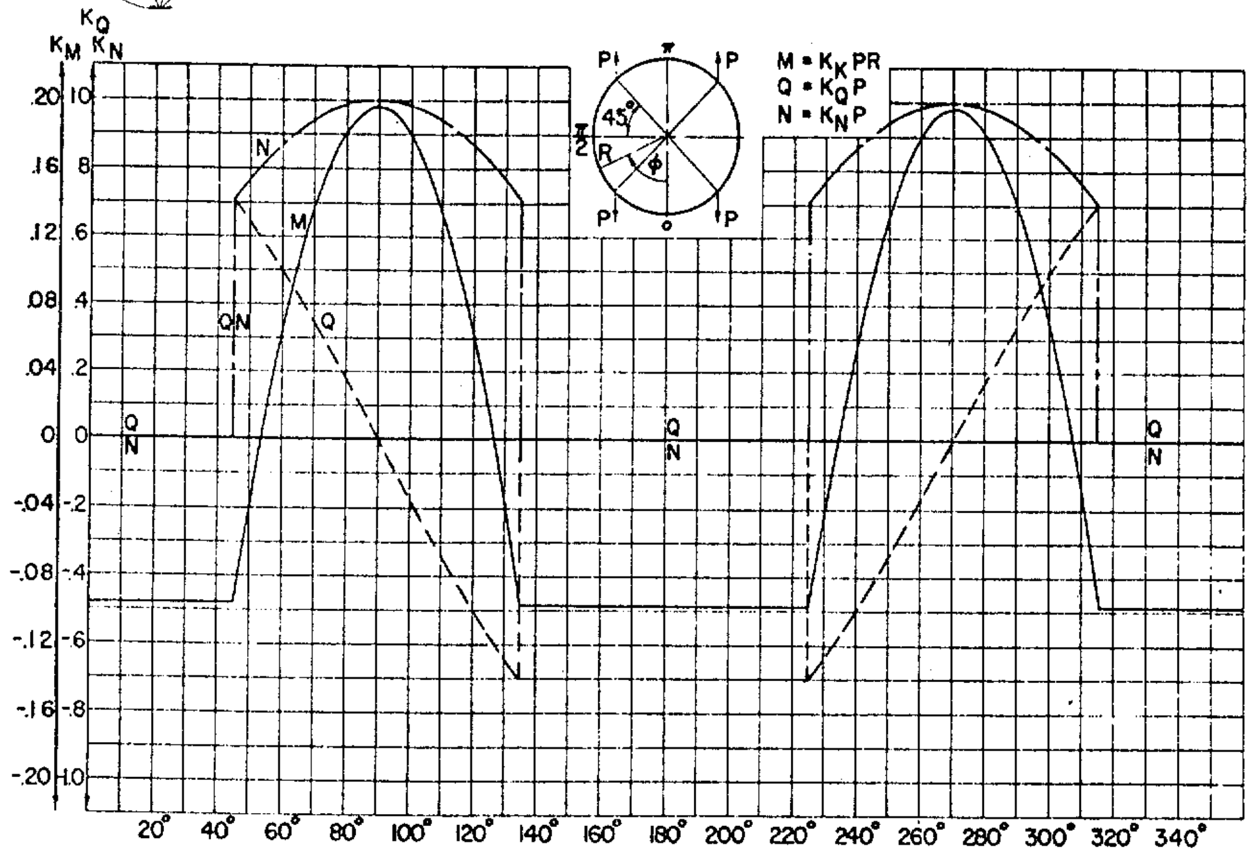
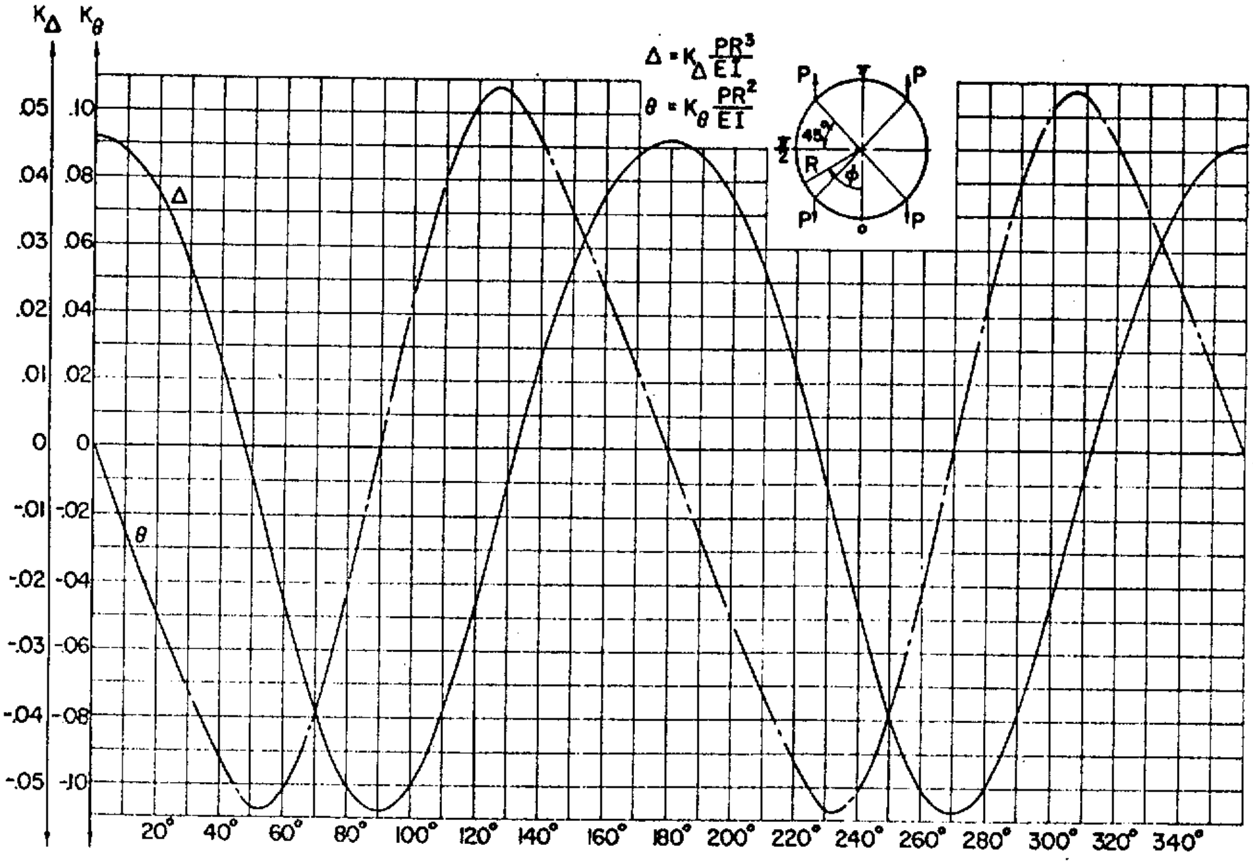

M=K_M PR
Q=K_Q P
N=K_N P
K_Q K_N
K_M
N
M
Q
N & Q
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.15

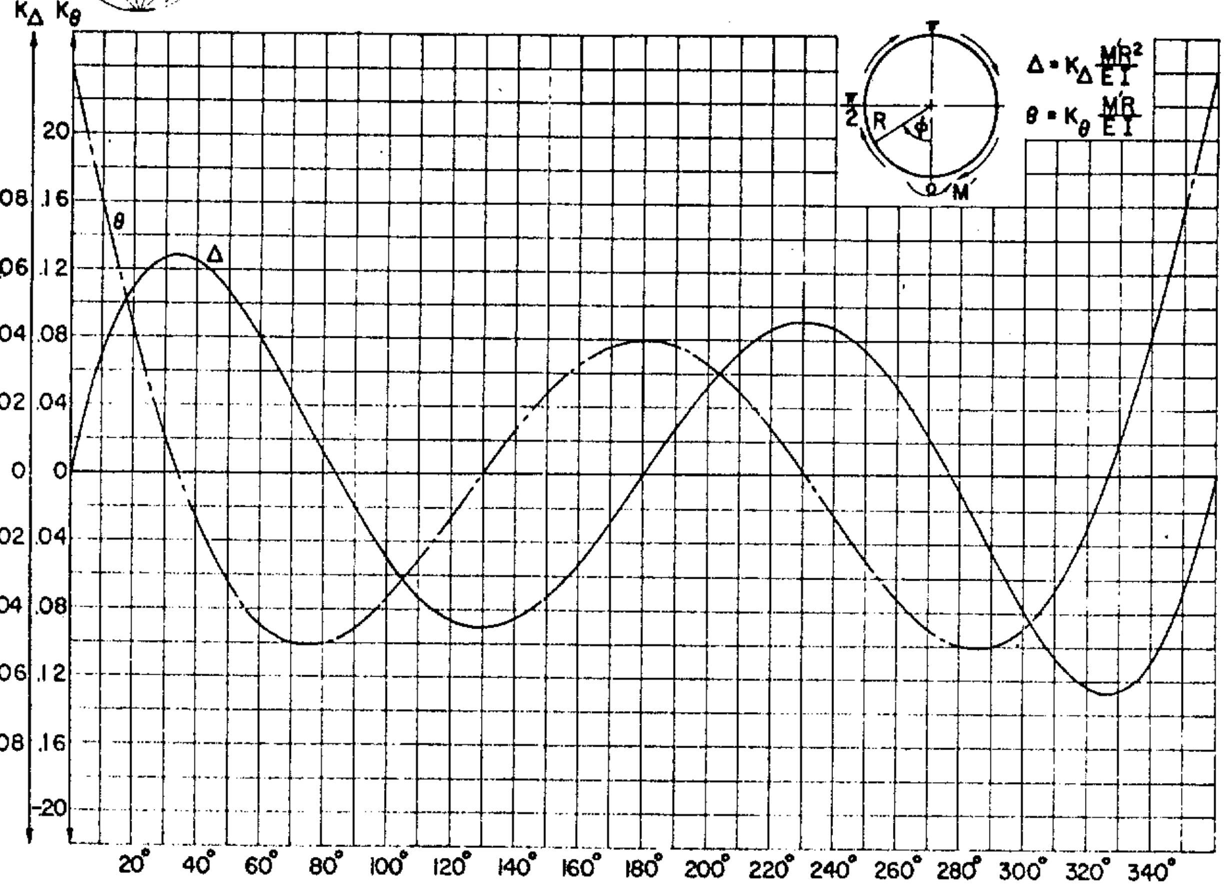


K_M K_Q K_N
M = K_K PR
Q = K_Q P
N = K_N P
P
π
45°
π/2
R
φ
o
N
M
Q
QN
Q N
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

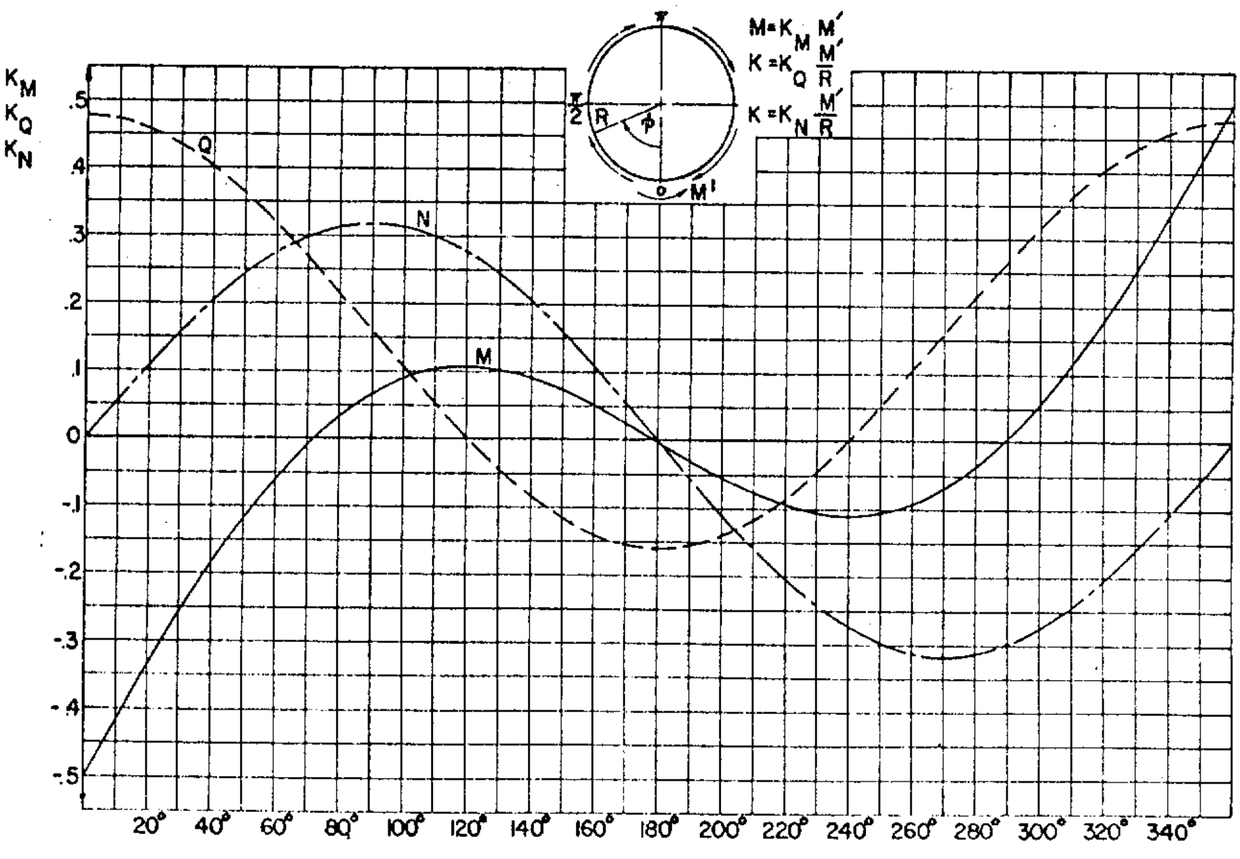


K_Δ K_θ
$\Delta = K_\Delta \frac{PR^3}{EI}$
$\theta = K_\theta \frac{PR^2}{EI}$
P
π
45°
π/2
R
φ
o
Δ
θ
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12,16

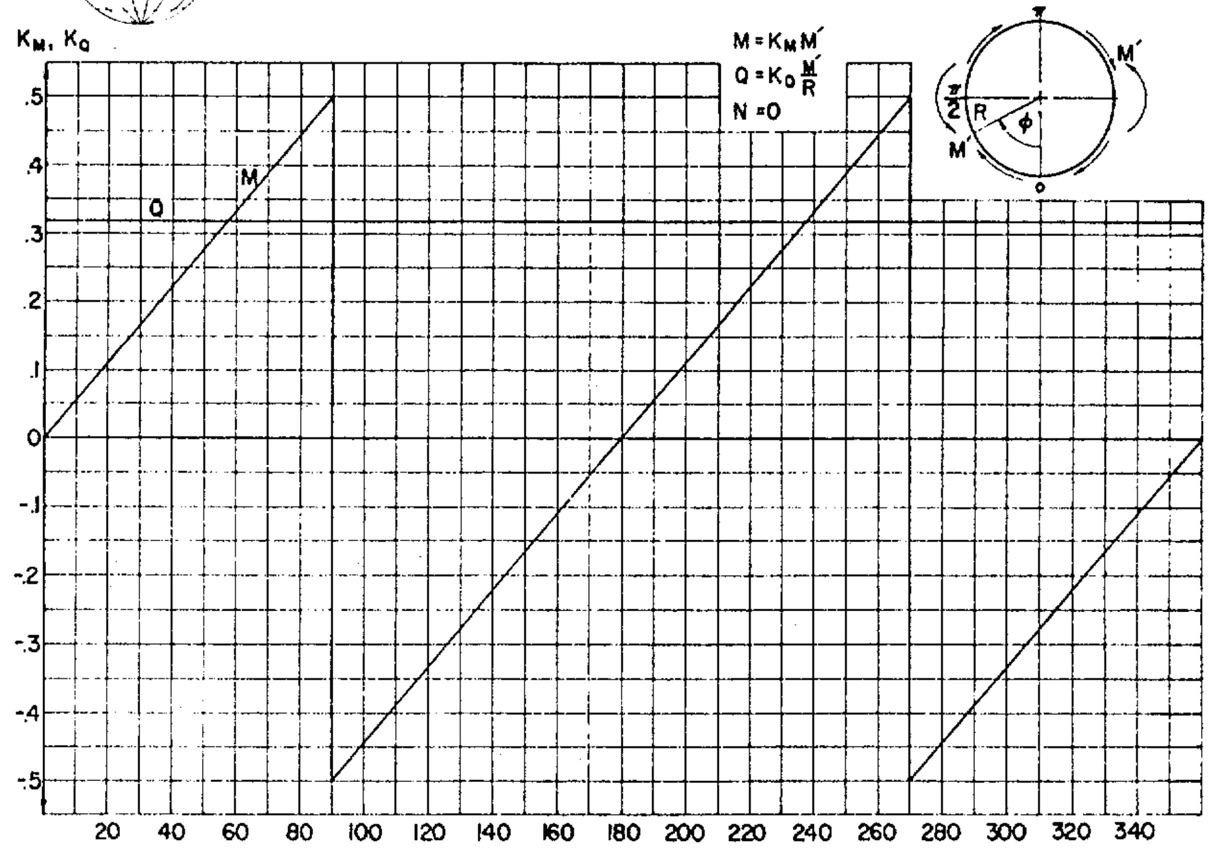


K_Δ K_θ
$\Delta = K_\Delta \frac{M'R^2}{EI}$
$\theta = K_\theta \frac{M'R}{EI}$
θ
Δ
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

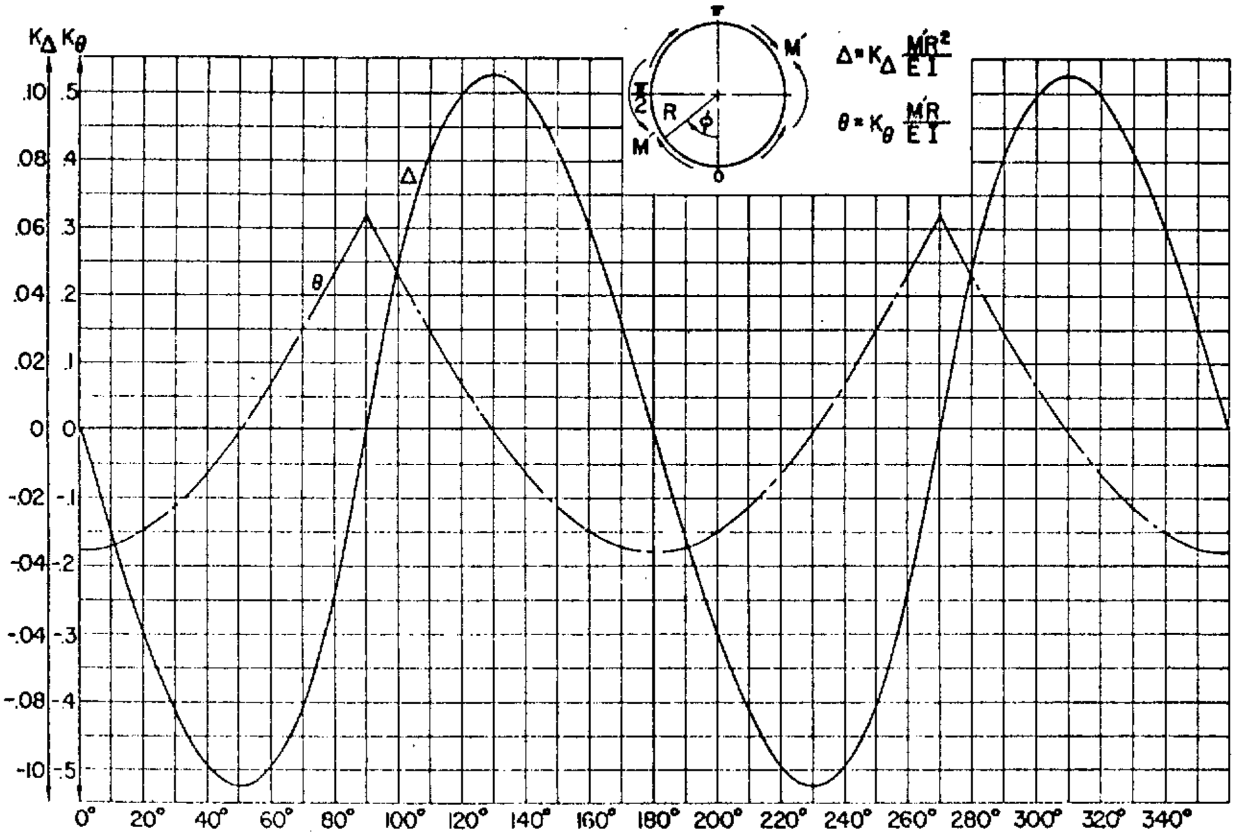


K_M
K_Q
K_N
$M = K_M M'$
$K = K_Q \frac{M'}{R}$
$K = K_N \frac{M'}{R}$
Q
N
M
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.17

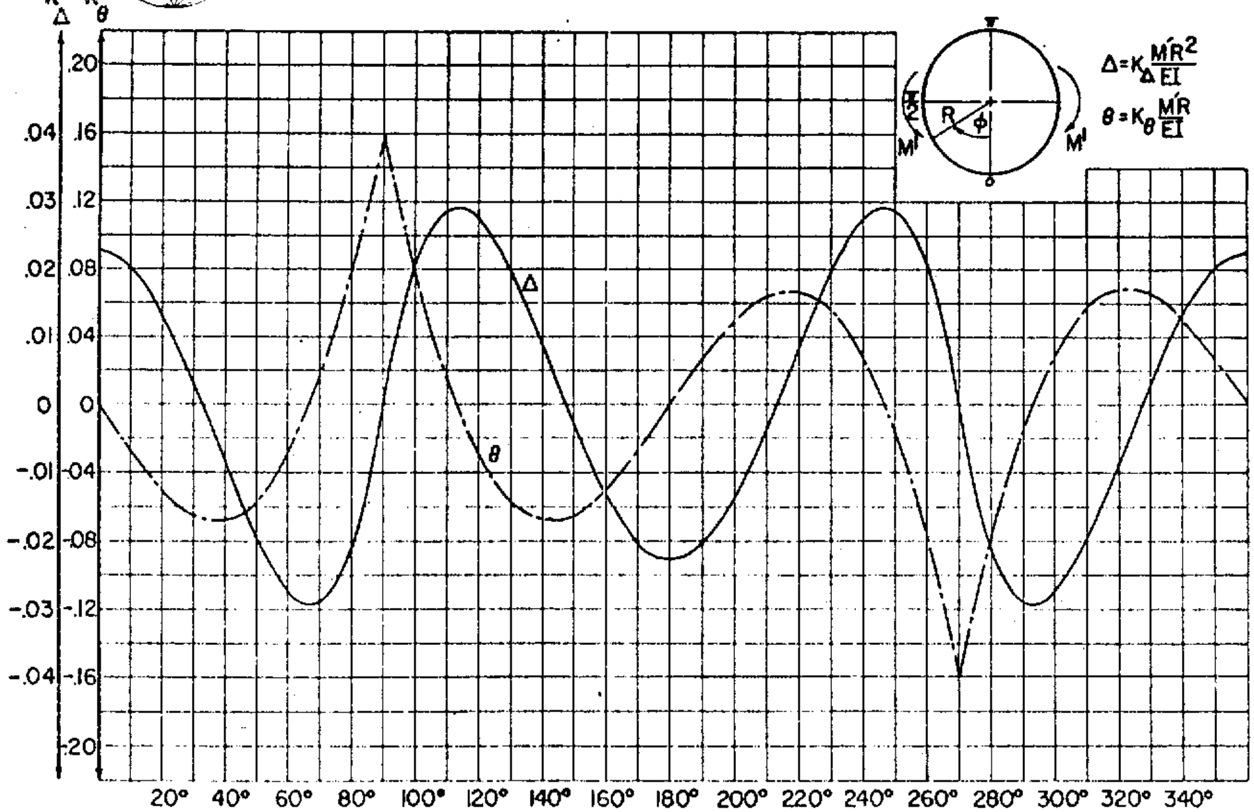


K_M, K_Q
M = K_M M′
Q = K_Q M′/R
N = 0
M′
π/2
R
φ
M
Q
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
-.5
20 40 60 80 100 120 140 160 180 200 220 240 260 280 300 320 340



K_Δ K_θ
Δ = K_Δ M′R²/EI
θ = K_θ M′R/EI
M′
π/2
R
φ
Δ
θ
.10 .5
.08 .4
.06 .3
.04 .2
.02 .1
0 0
-.02 -.1
-.04 -.2
-.04 -.3
-.08 -.4
-10 -.5
0° 20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.18


K_Δ K_θ
.20
.04 .16
.03 .12
.02 .08
.01 .04
0 0
-.01 -.04
-.02 -.08
-.03 -.12
-.04 -.16
-.20
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°
Δ
θ
π/2
R
φ
M'
M'
$\Delta = K_\Delta \frac{M'R^2}{EI}$
$\theta = K_\theta \frac{M'R}{EI}$

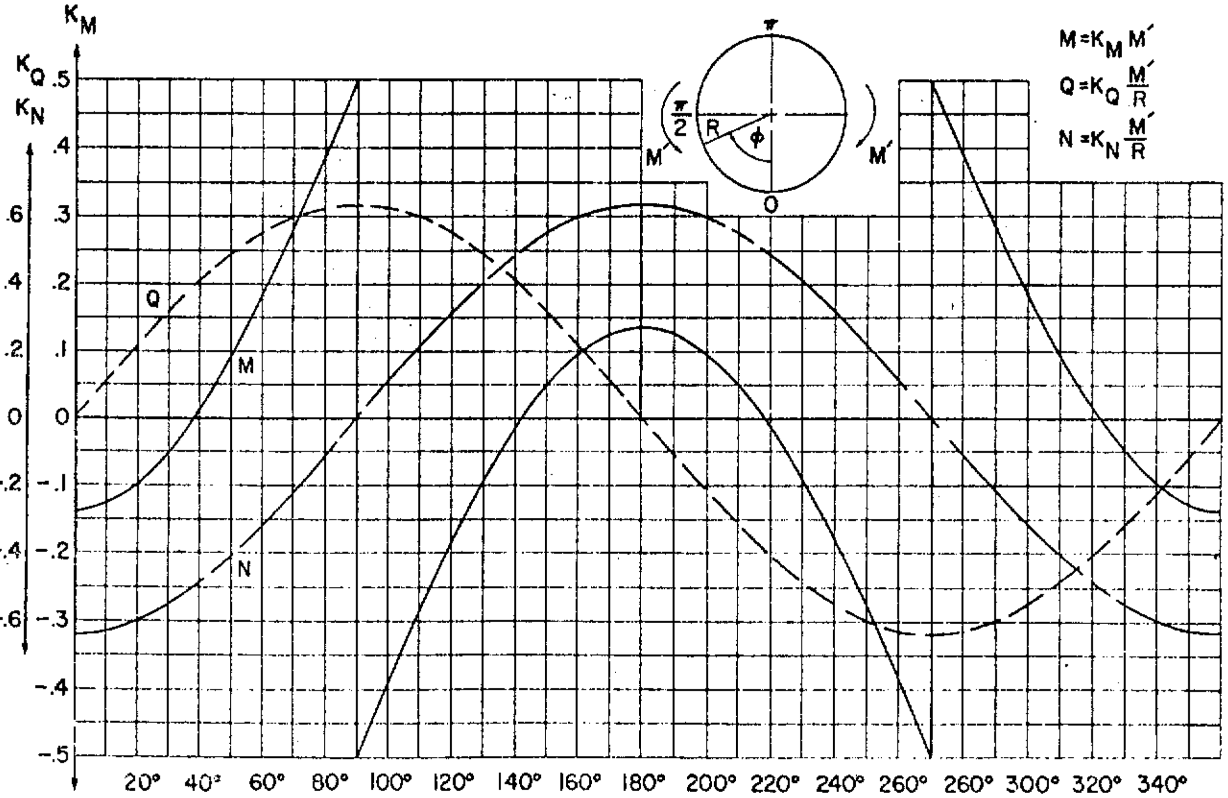
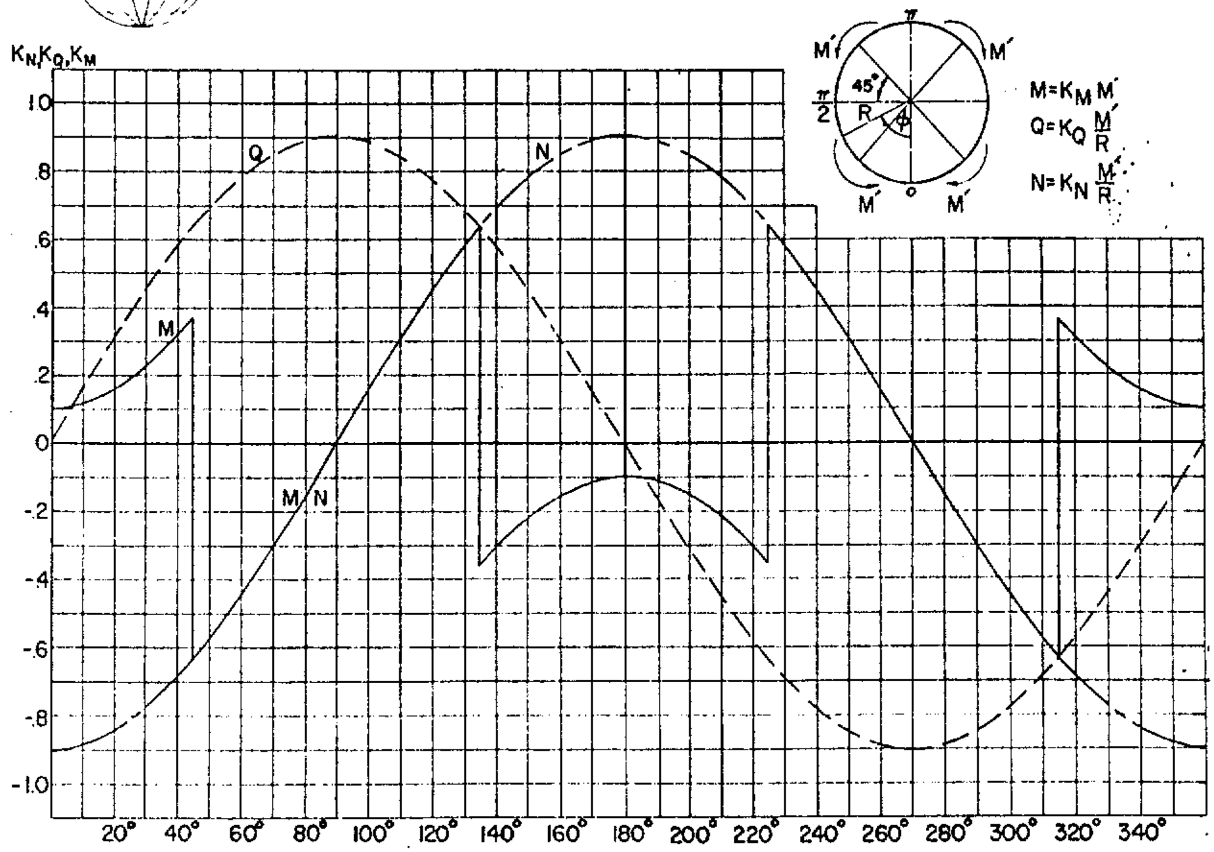

K_M
K_Q
K_N
.5
.4
.6 .3
.4 .2
.2 .1
0 0
-.2 -.1
-.4 -.2
-.6 -.3
-.4
-.5
Q
M
N
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 260° 300° 320° 340°
π/2
R
φ
O
M'
M'
$M = K_M M'$
$Q = K_Q \frac{M'}{R}$
$N = K_N \frac{M'}{R}$

FIGURE 12.19


K_N, K_Q, K_M
M=K_M M'
Q=K_Q M'/R
N=K_N M'/R
Q
N
M
M N
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

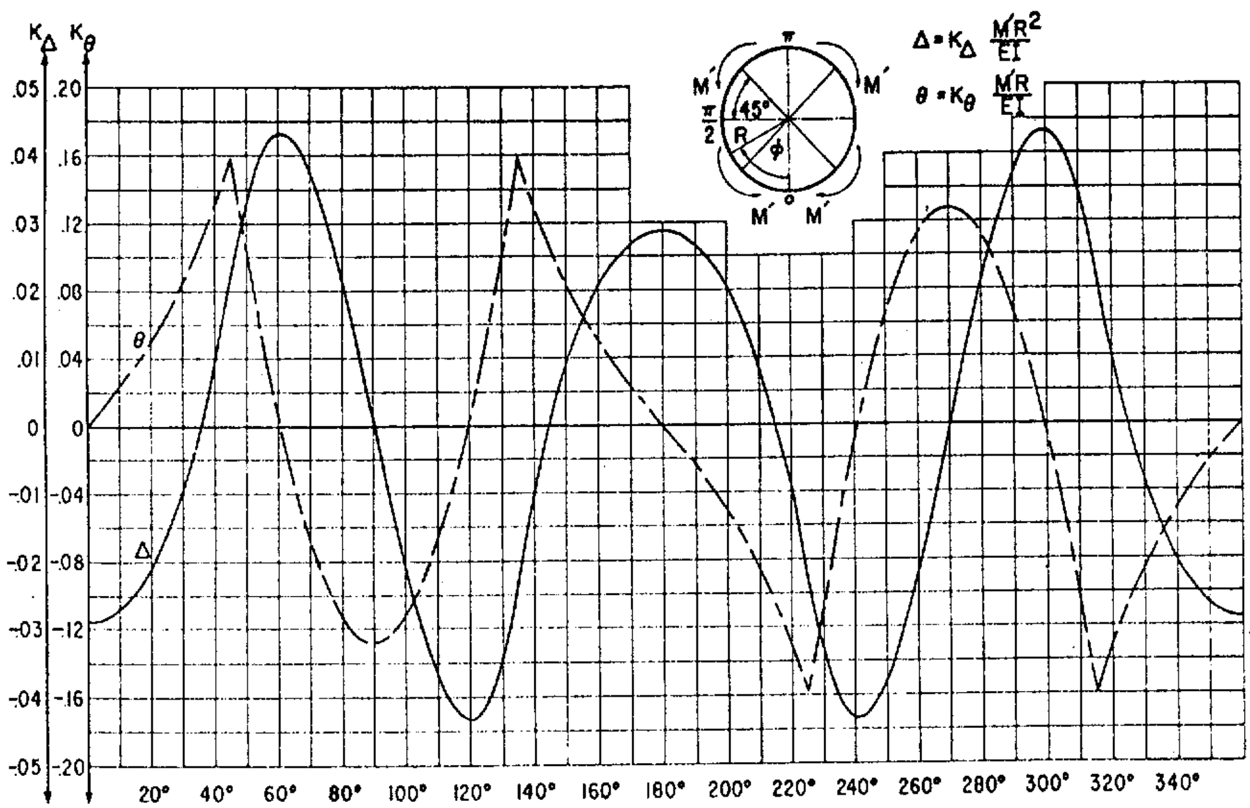
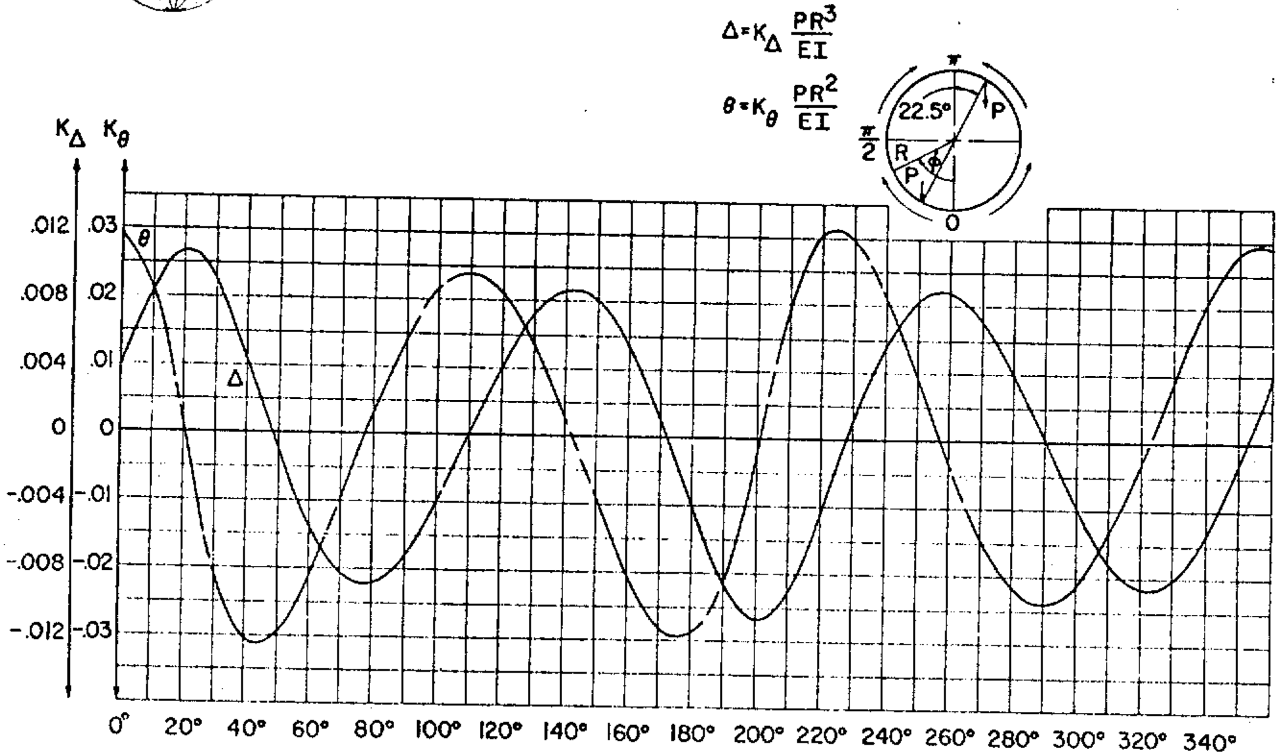
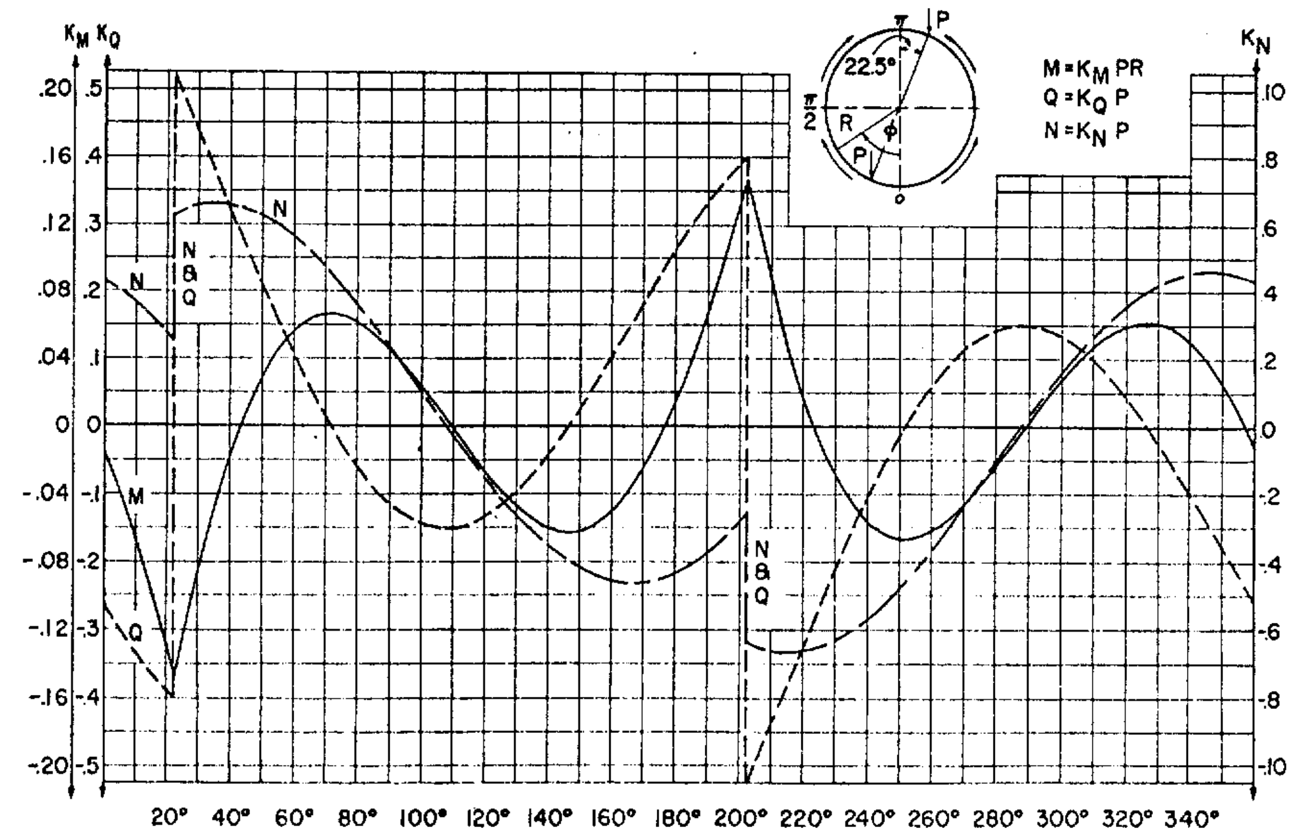

K_Δ K_θ
Δ = K_Δ M'R²/EI
θ = K_θ M'R/EI
θ
Δ
.05 .20
.04 .16
.03 .12
.02 .08
.01 .04
0 0
-.01 -.04
-.02 -.08
-.03 -.12
-.04 -.16
-.05 -.20
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.20

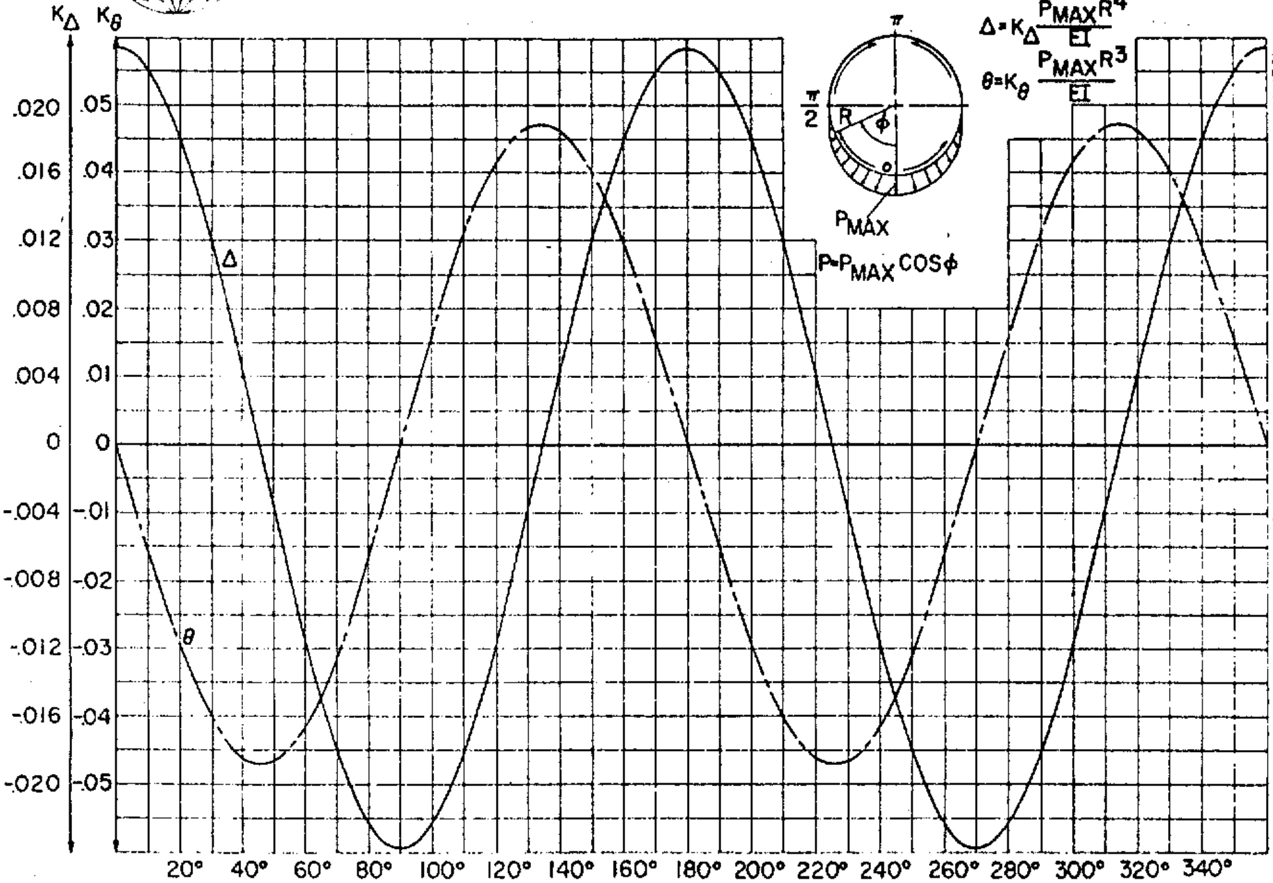


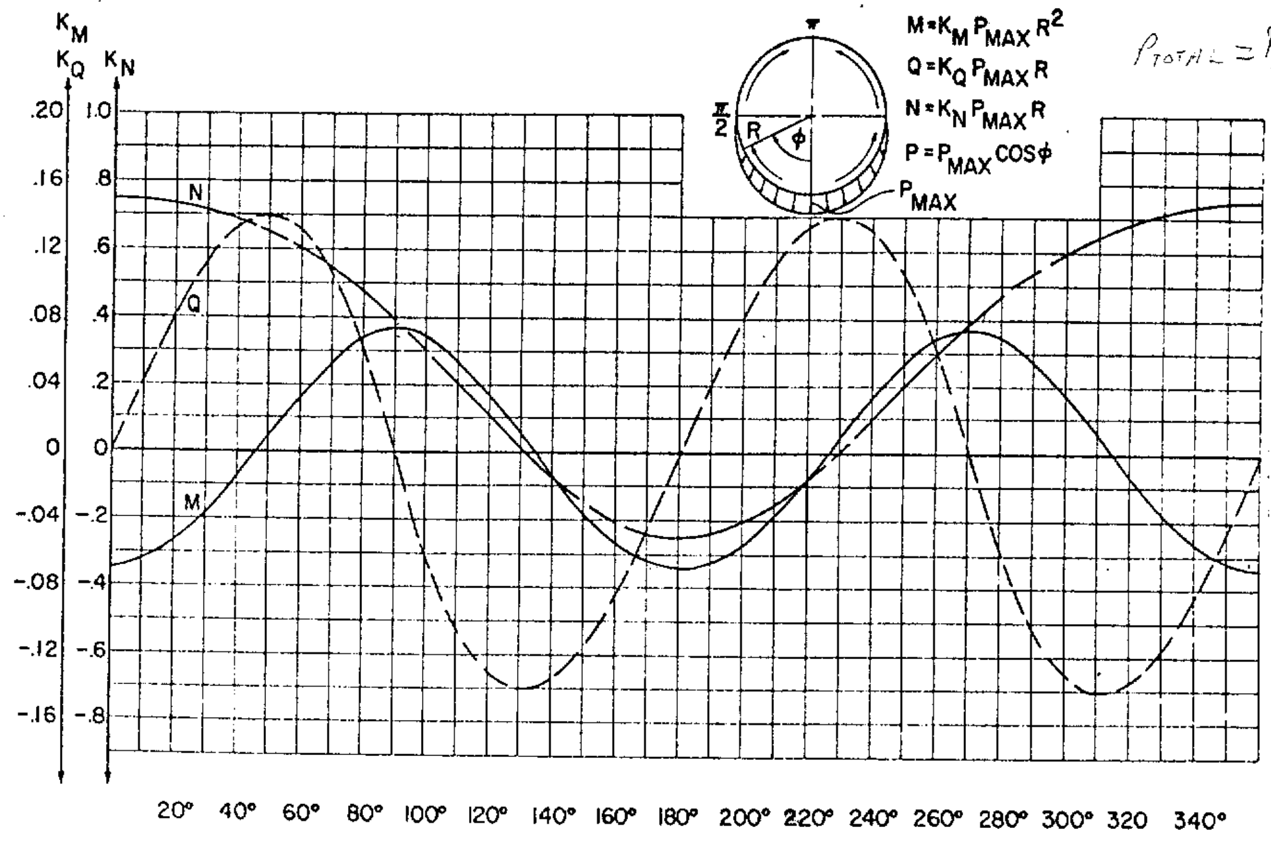
Δ=K_Δ PR³/EI
θ=K_θ PR²/EI
K_Δ K_θ
θ
Δ
22.5°
P
R
φ
π/2
π
0
.012 .03
.008 .02
.004 .01
0 0
-.004 -.01
-.008 -.02
-.012 -.03
0° 20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°



K_M K_Q
K_N
M=K_M PR
Q=K_Q P
N=K_N P
22.5°
P
R
φ
π/2
π
0
N
M
Q
N & Q
.20 .5
.16 .4
.12 .3
.08 .2
.04 .1
0 0
-.04 -.1
-.08 -.2
-.12 -.3
-.16 -.4
-.20 -.5
.10
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-10
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

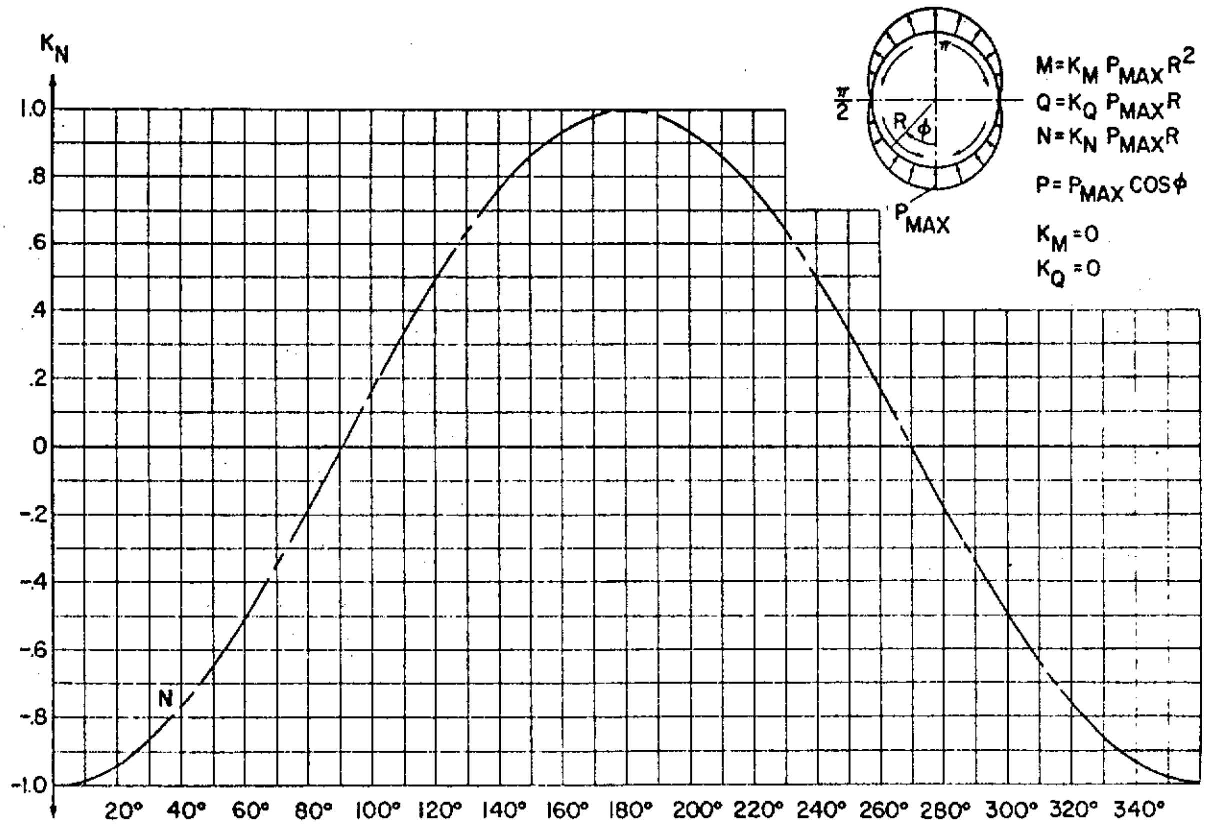
FIGURE 12.21





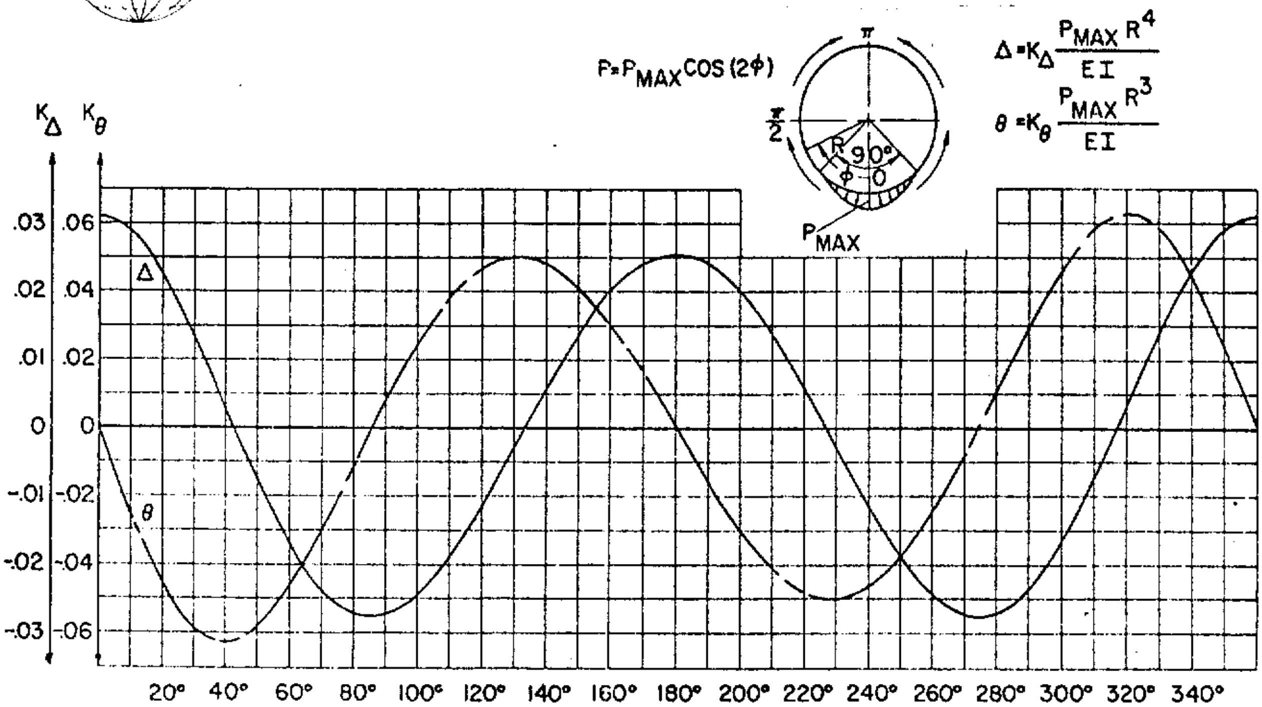
$P_{TOTAL} = P_{MAX} \frac{r\pi}{2}$

FIGURE 12.22



K_N
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°
N
R
ϕ
π
$\frac{\pi}{2}$
P_{MAX}
$M = K_M P_{MAX} R^2$
$Q = K_Q P_{MAX} R$
$N = K_N P_{MAX} R$
$P = P_{MAX} \cos\phi$
$K_M = 0$
$K_Q = 0$

FIGURE 12.23


P=PMAX COS (2φ)
Δ=KΔ PMAX R4 / EI
θ=Kθ PMAX R3 / EI
KΔ
Kθ
PMAX
Δ
θ
.03 .06
.02 .04
.01 .02
0 0
-.01 -.02
-.02 -.04
-.03 -.06
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

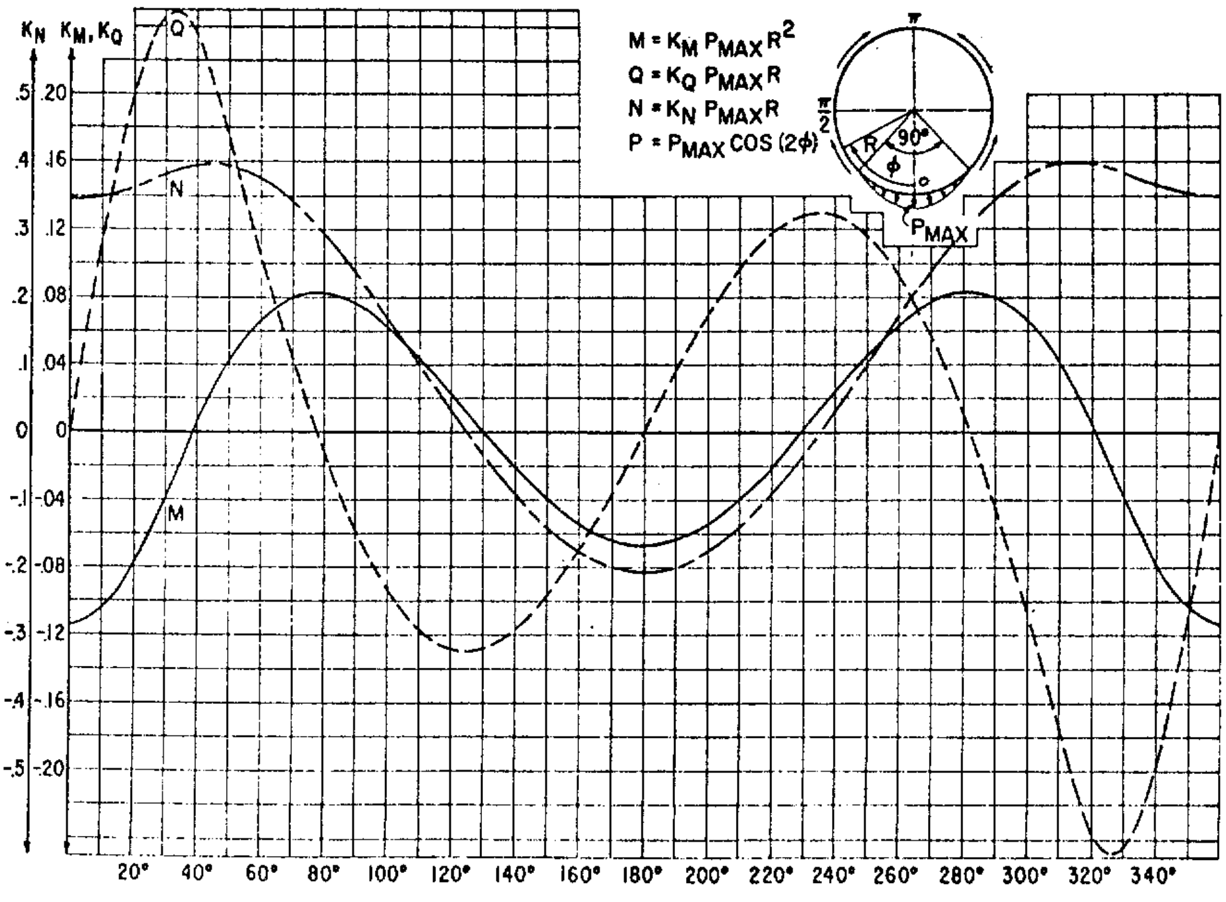
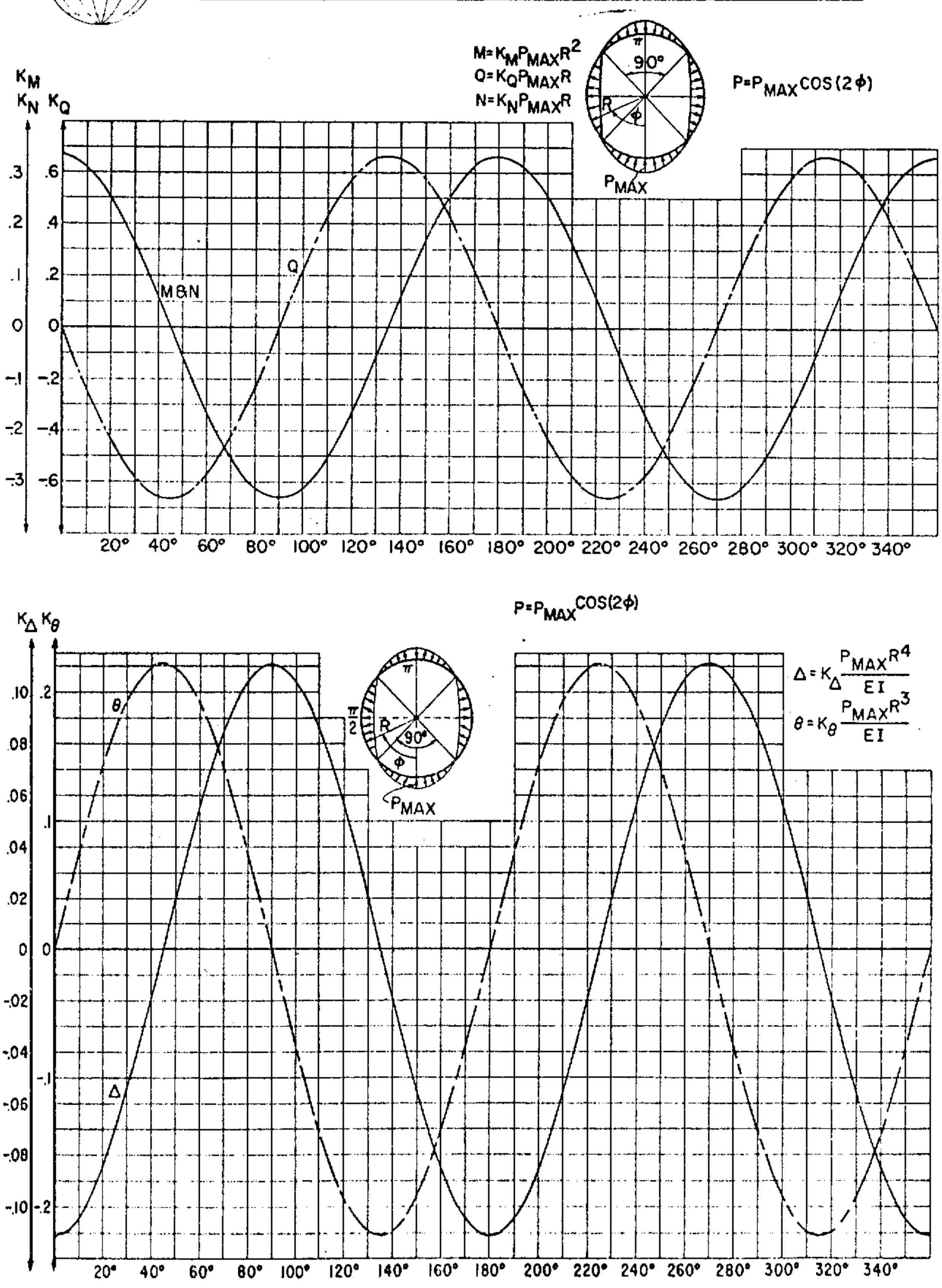

M = KM PMAX R2
Q = KQ PMAX R
N = KN PMAX R
P = PMAX COS (2φ)
KN
KM, KQ
PMAX
Q
N
M
.5 .20
.4 .16
.3 .12
.2 .08
.1 .04
0 0
-.1 -.04
-.2 -.08
-.3 -.12
-.4 -.16
-.5 -.20
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.24



M=K_M P_MAX R^2
Q=K_Q P_MAX R
N=K_N P_MAX R
P=P_MAX COS(2φ)
90°
R
φ
P_MAX
K_M K_N K_Q
M&N
Q
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°
P=P_MAX COS(2φ)
K_Δ K_θ
Δ=K_Δ P_MAX R^4 / EI
θ=K_θ P_MAX R^3 / EI
θ
Δ

FIGURE 12.25

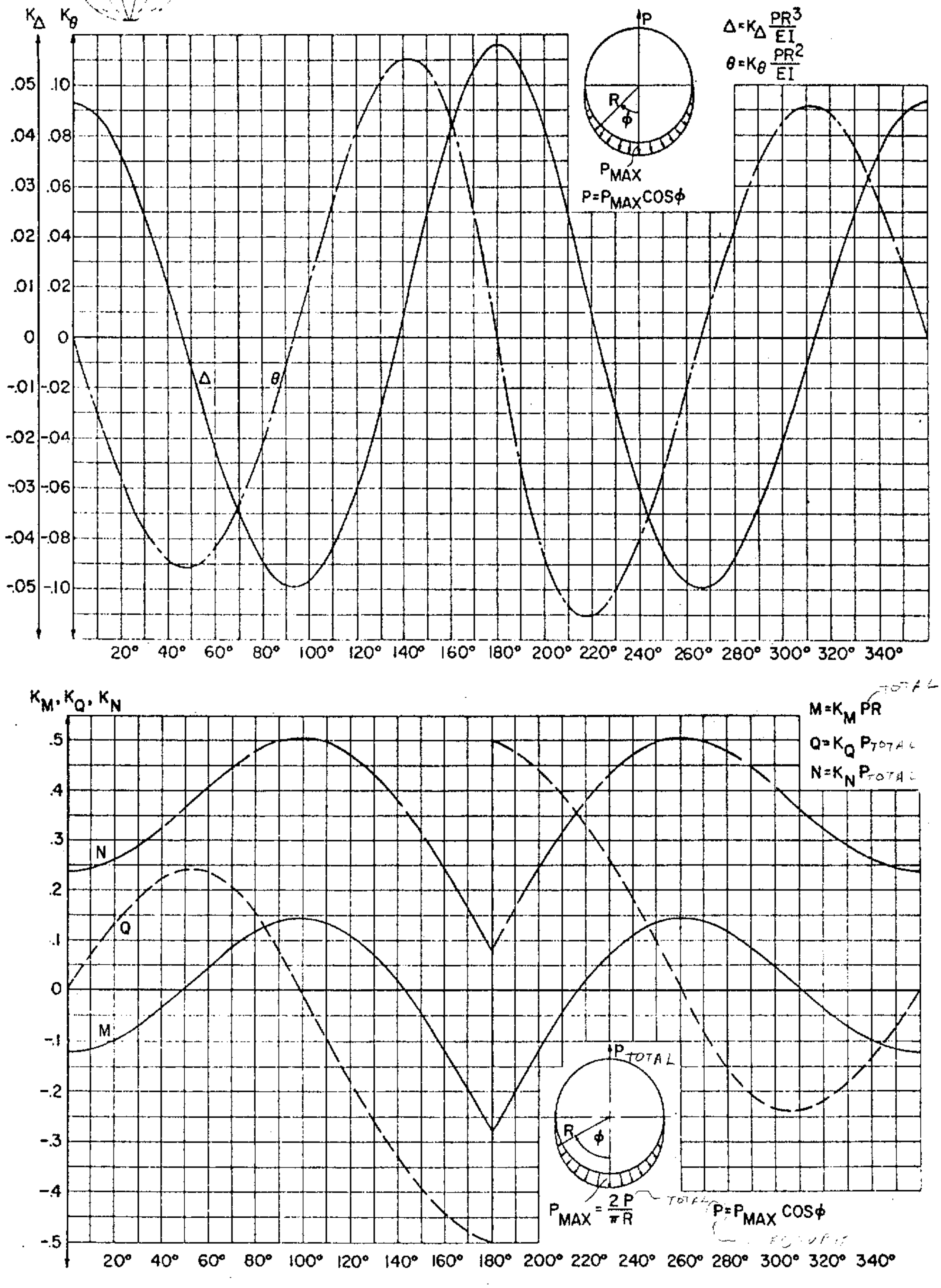

K_Δ K_θ
$\Delta = K_\Delta \frac{PR^3}{EI}$
$\theta = K_\theta \frac{PR^2}{EI}$
P
R
ϕ
P_{MAX}
$P = P_{MAX} \cos\phi$
Δ
θ
.05 .10
.04 .08
.03 .06
.02 .04
.01 .02
0 0
-.01 -.02
-.02 -.04
-.03 -.06
-.04 -.08
-.05 -.10
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°
K_M, K_Q, K_N
$M = K_M P_{TOTAL} R$
$Q = K_Q P_{TOTAL}$
$N = K_N P_{TOTAL}$
N
Q
M
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
-.5
P_{TOTAL}
R
ϕ
$P_{MAX} = \frac{2P}{\pi R}$ TOTAL
$P = P_{MAX} \cos\phi$
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.26

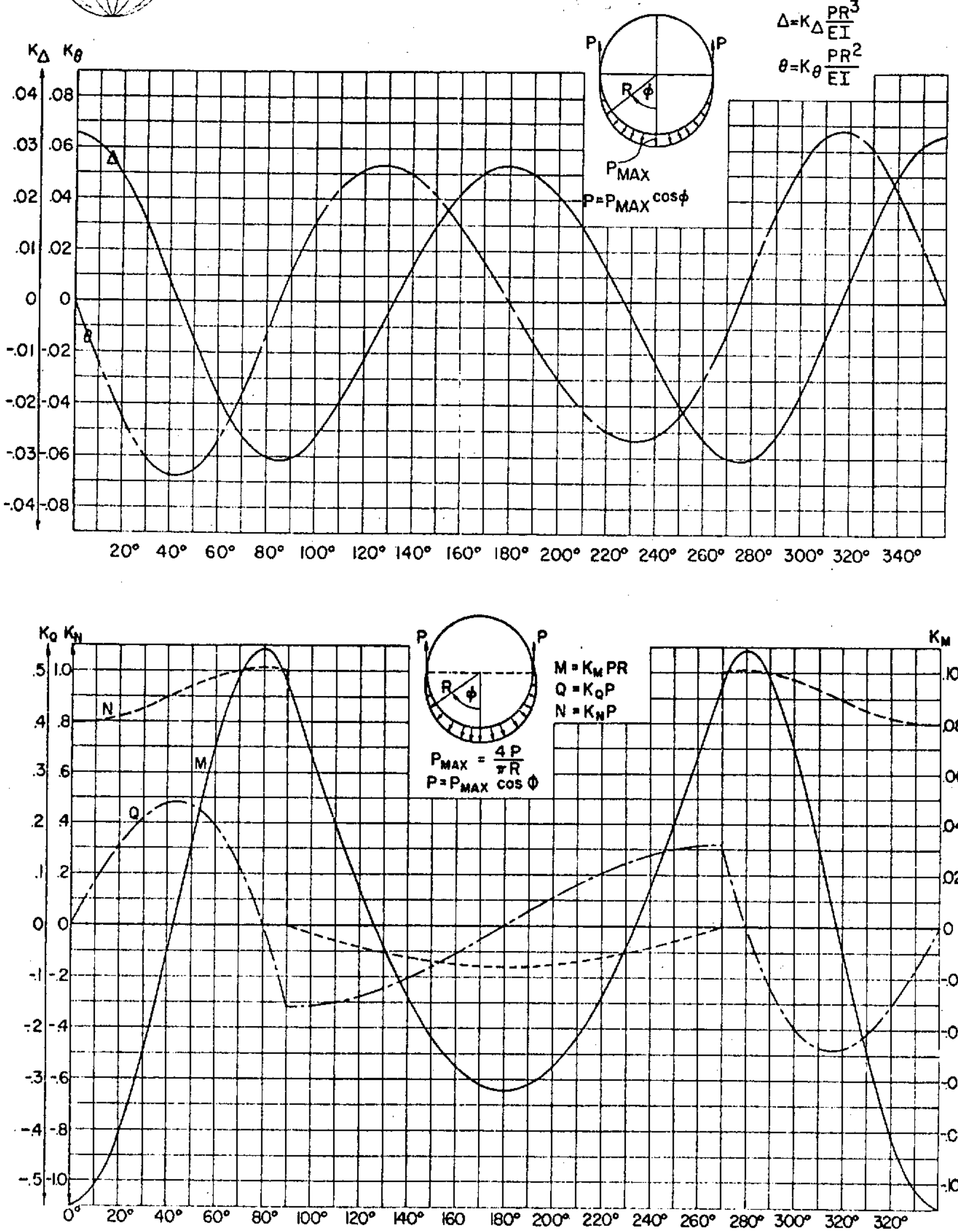
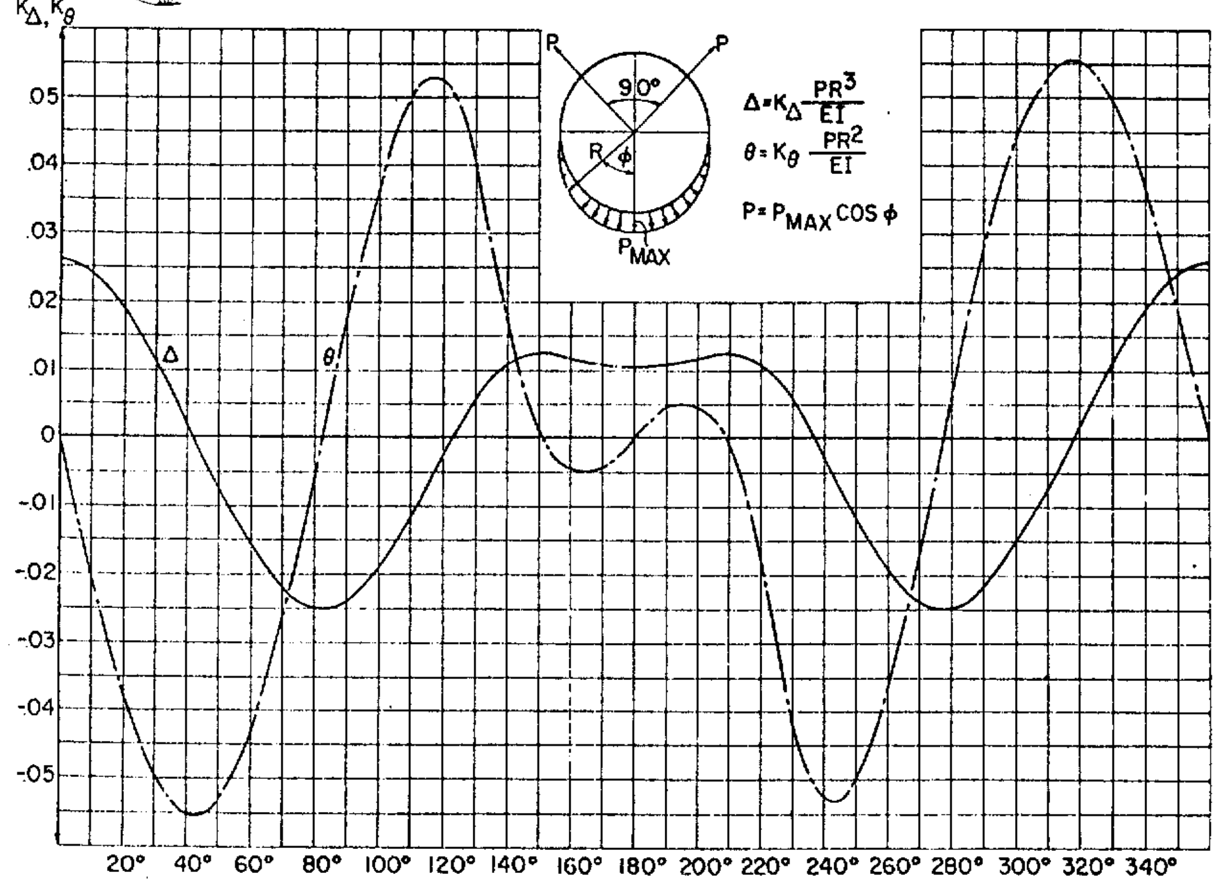
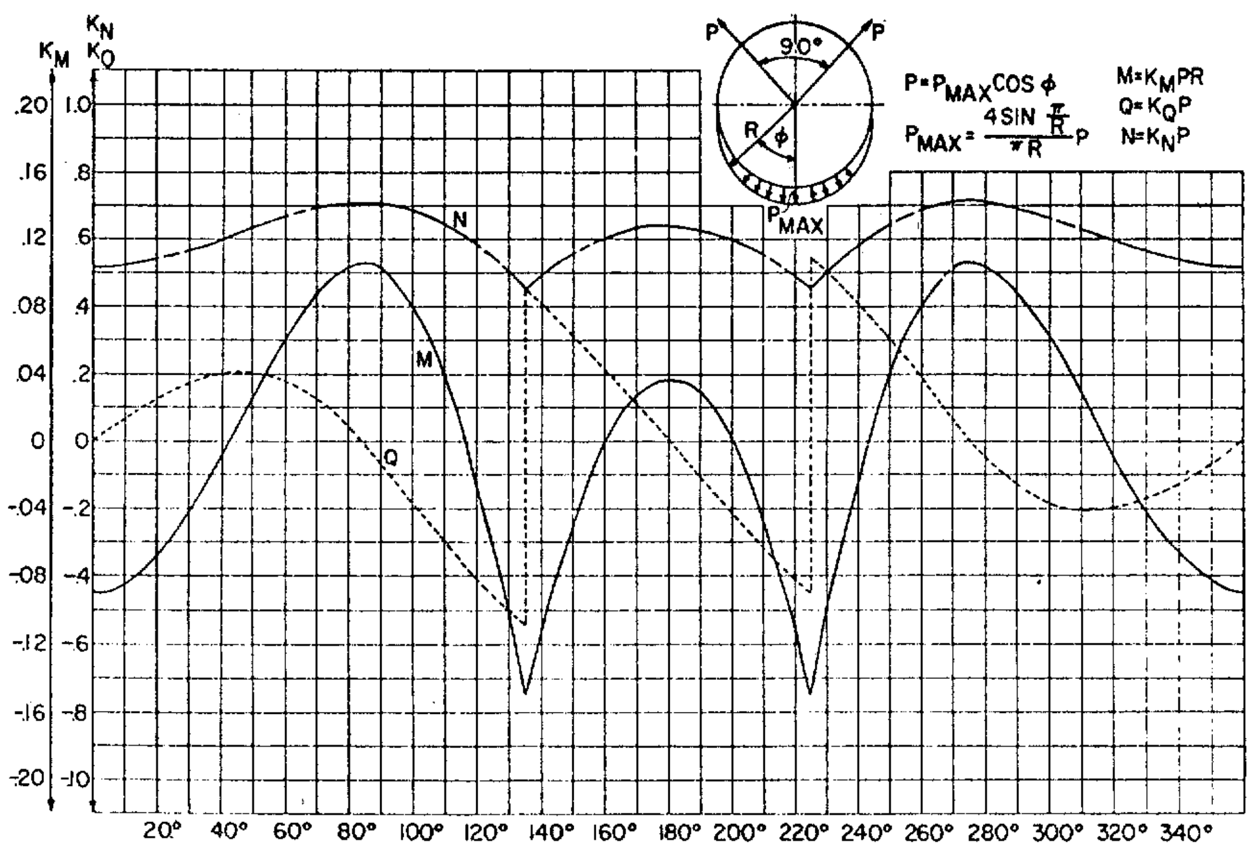

$\Delta = K_\Delta \frac{PR^3}{EI}$
$\theta = K_\theta \frac{PR^2}{EI}$
P
P
R
φ
P_{MAX}
$P = P_{MAX} \cos\phi$
K_Δ K_θ
Δ
θ
$M = K_M PR$
$Q = K_Q P$
$N = K_N P$
$P_{MAX} = \frac{4P}{\pi R}$
$P = P_{MAX} \cos\phi$
K_Q K_N
K_M
N
M
Q

FIGURE 12.27



K_Δ, K_θ
$\Delta = K_\Delta \frac{PR^3}{EI}$
$\theta = K_\theta \frac{PR^2}{EI}$
$P = P_{MAX} \cos\phi$
P_{MAX}
90°
Δ
θ
.05
.04
.03
.02
.01
0
-.01
-.02
-.03
-.04
-.05
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°



K_M
K_N
K_Q
$P = P_{MAX} \cos\phi$
$P_{MAX} = \frac{4 \sin \frac{\pi}{R}}{\pi R} P$
$M = K_M PR$
$Q = K_Q P$
$N = K_N P$
P_{MAX}
90°
N
M
Q
.20 1.0
.16 .8
.12 .6
.08 .4
.04 .2
0 0
-.04 -.2
-.08 -.4
-.12 -.6
-.16 -.8
-.20 -1.0
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.28

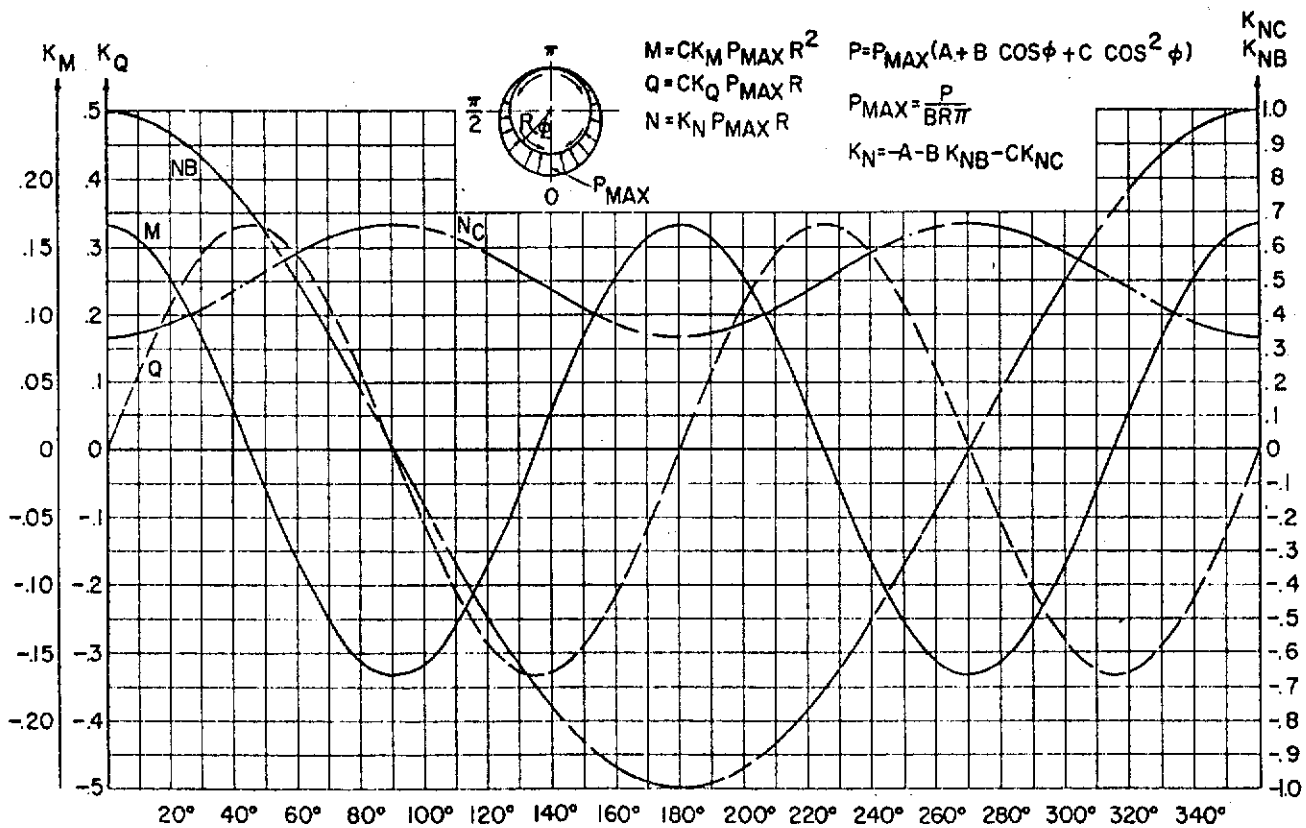

K_M K_Q
$M = CK_M P_{MAX} R^2$
$Q = CK_Q P_{MAX} R$
$N = K_N P_{MAX} R$
$P = P_{MAX}(A + B \cos\phi + C \cos^2\phi)$
$P_{MAX} = \frac{P}{BR\pi}$
$K_N = -A - B K_{NB} - CK_{NC}$
K_{NC} K_{NB}
π
$\frac{\pi}{2}$
R
ϕ
0
P_{MAX}
NB
M
N_C
Q
.20
.15
.10
.05
0
-.05
-.10
-.15
-.20
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
-.5
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
20° 40° 60° 80° 100° 120° 140° 160° 180° 200° 220° 240° 260° 280° 300° 320° 340°

FIGURE 12.29

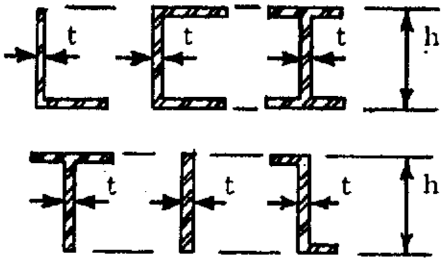
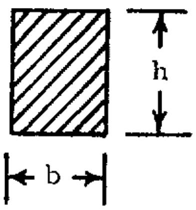
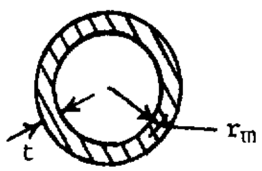
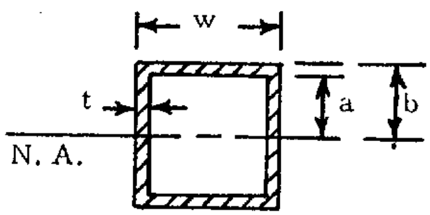
Cross-Section	Shear Area	Shape Factor, β
	Area of Web $A_Q = th$	$\beta = 1.00$
	Entire Area $A_Q = bh$	$\beta = 1.20$ for $b \geq 0.50\,h$ $\beta = 1.00$ for $b < 0.50\,h$
	Entire Area $A_Q = 2\pi r_m t$	$\beta = 2.00$
 ρ = radius of gyration with respect to the neutral axis	Entire Area $A_Q = (w)^2 - (2a)(w-2t)$	$\beta = \left[1 + \frac{3(b^2 - a^2)\,a}{2b^3}\left(\frac{w}{t} - 1\right)\right]\left[\frac{4\,b^2}{10\,\rho^2}\right]$ If the flanges are of nonuniform thickness, they may be replaced by an "equivalent" section whose flanges have the same width and area as those of the actual section.

FIGURE 12.30 - SHAPE FACTORS FOR SHEAR DEFLECTIONS FOR VARIOUS CROSS SECTIONS

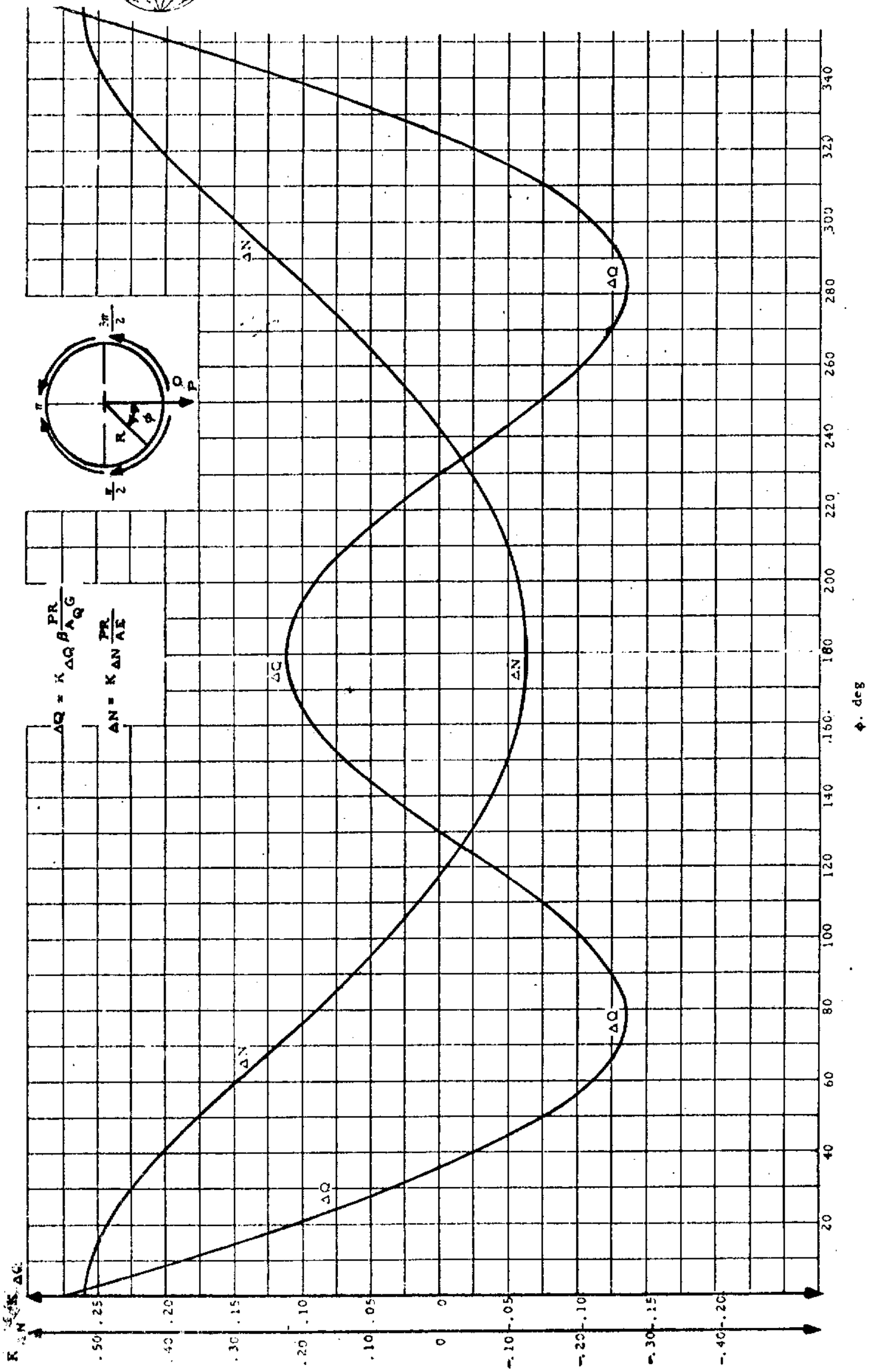


FIGURE 12.31 - COEFFICIENTS TO ACCOUNT FOR SHEAR DEFORMATION

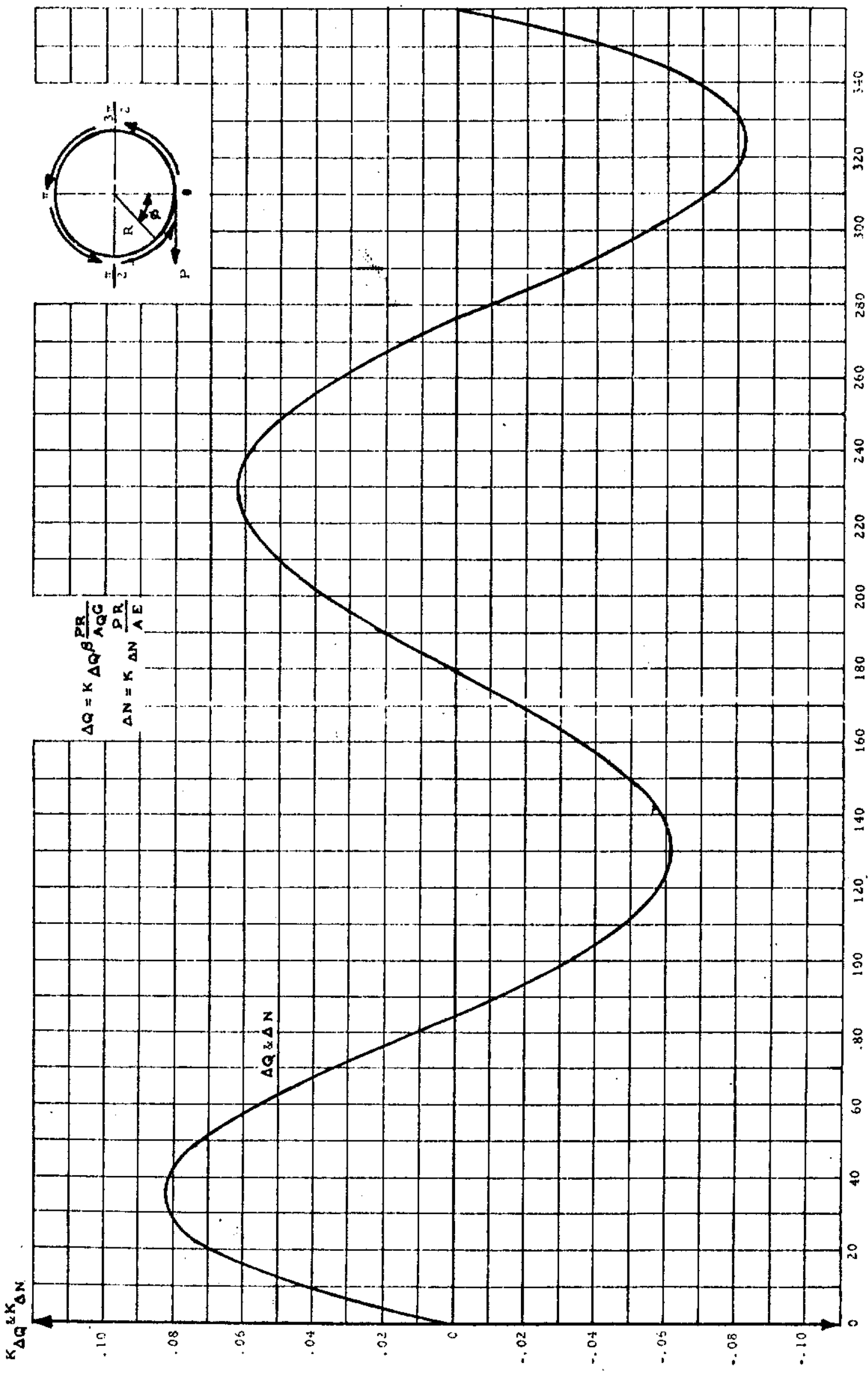


FIGURE 12.32 - COEFFICIENTS TO ACCOUNT FOR SHEAR DEFORMATION

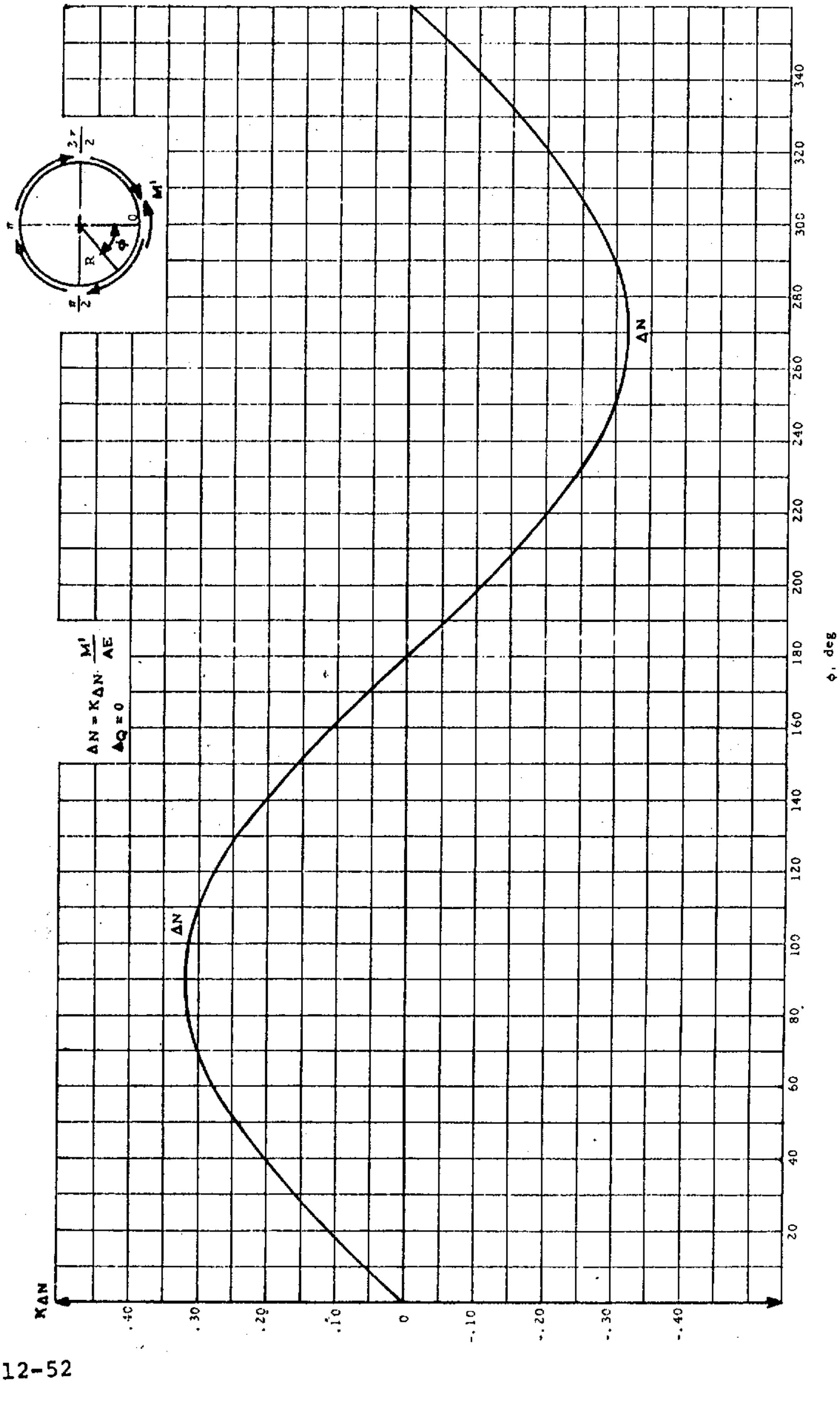


FIGURE 12.33 - COEFFICIENTS TO ACCOUNT FOR SHEAR DEFORMATION

12.4.2 Analysis of Rigid Rings with Out-of-Plane Loads

Coefficients to obtain slope, deflection, bending moment, shear and axial force along with equations for these values are given for some of the frequently used load cases. Figure 12.35 shows an index for the various load cases presented in Figures 12.36 through 12.38.

The sign convention used throughout the rigid frame analysis of out-of-plane load cases is shown in Figure 12.34. It basically consists of moments which produce tension on the inner fibers are positive, torque "T" and lateral shear "V" are positive as shown.

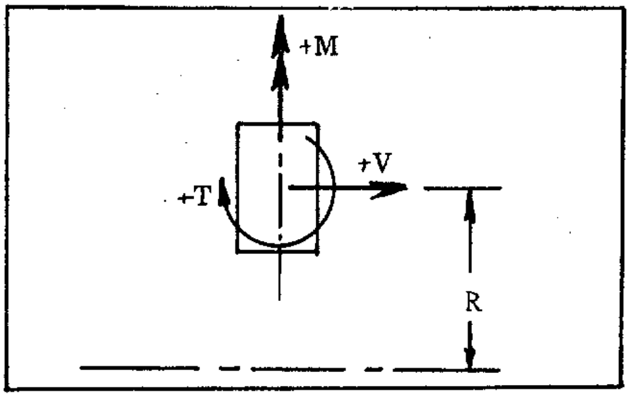


FIGURE 12.34 - SIGN CONVENTION FOR RIGID RINGS WITH OUT-OF-PLANE LOADS

12.4.3 Analysis of Frame Reinforced Cylindrical Shells

Tables are presented giving the loads and displacements in a flexible frame supported by a circular cylindrical shell and subjected to concentrated radial, tangential, and moment loads. Additional tables give the loads in the shell. The solutions are presented in terms of two basic parameters, one of which is of second-order importance. Procedure for modifying the important parameter to account for certain non-uniform properties of the structure are presented.

Notation

A $\quad \dfrac{2.25}{\gamma^4}$

B $\quad \left(\dfrac{L_r}{L_c}\right)^2 / \gamma^2$

E $\quad$ Young's modulus ~ lb/in^2

E_f $\quad$ Young's modulus of unloaded frames ~ lb/in^2

E_o $\quad$ Young's modulus of loaded frame ~ lb/in^2

Figure 12.36

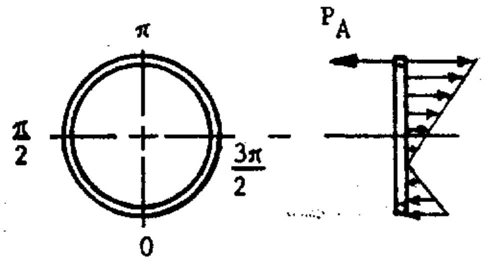


Figure 12.37

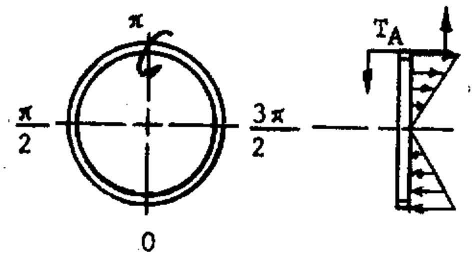


Figure 12.38

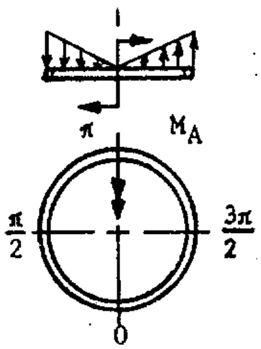


Figure 12.35 - Index for Rings with Out-of-Plane Loads

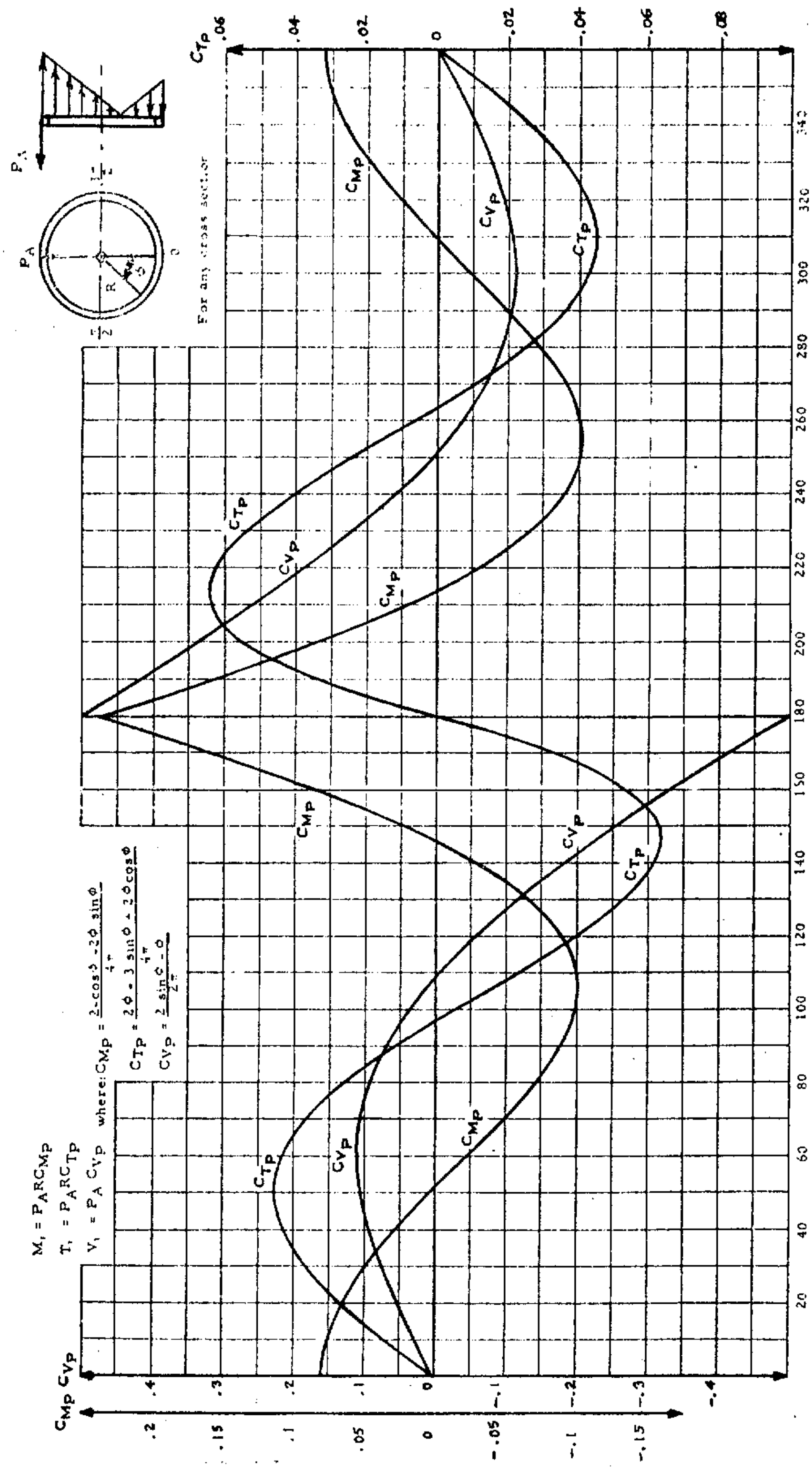


Figure 12.36 - Coefficients for Concentrated Out-of-Plane Load

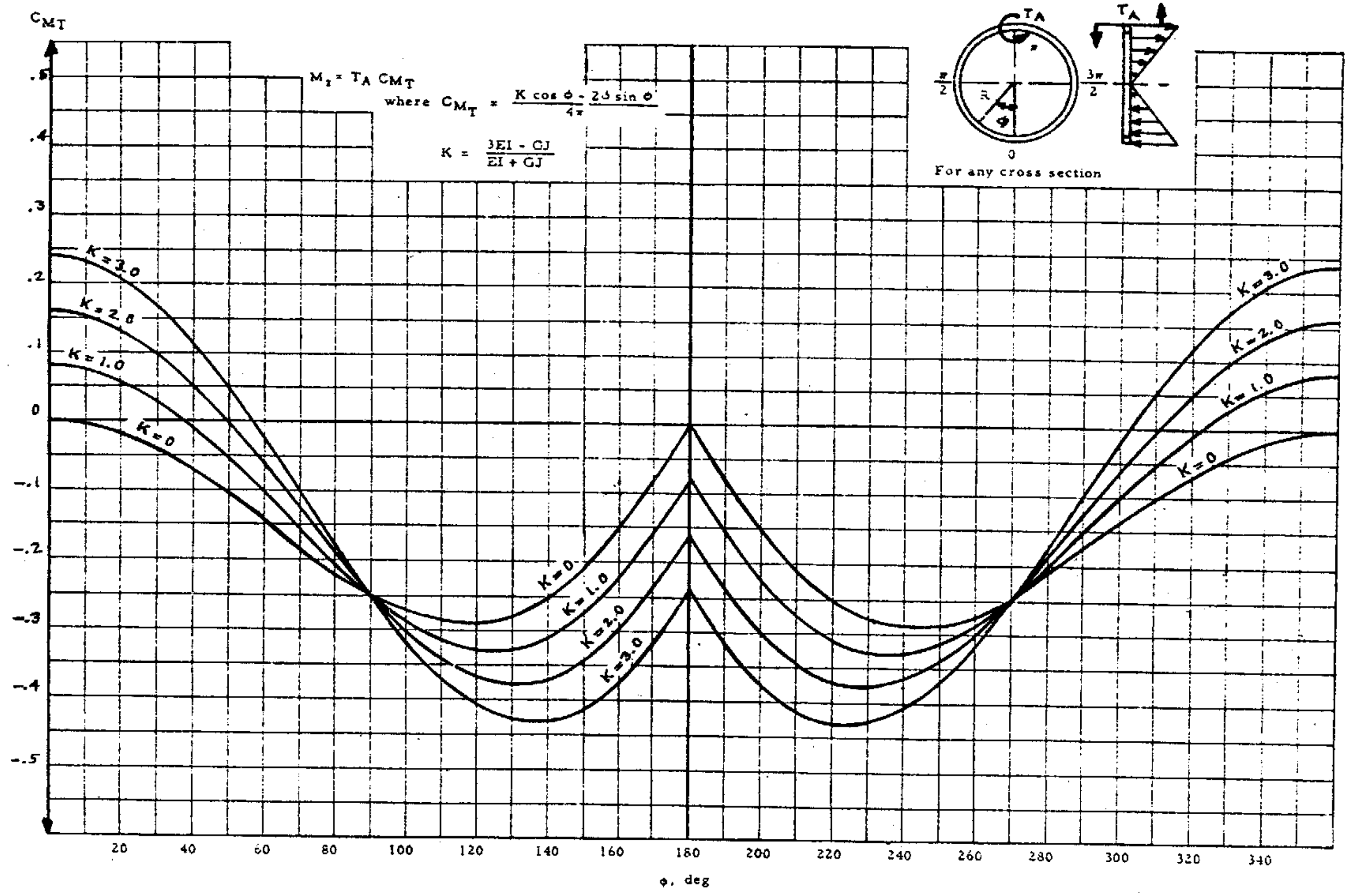


Figure 12.37 - Coefficients for Concentrated Out-of-Plane Torsion

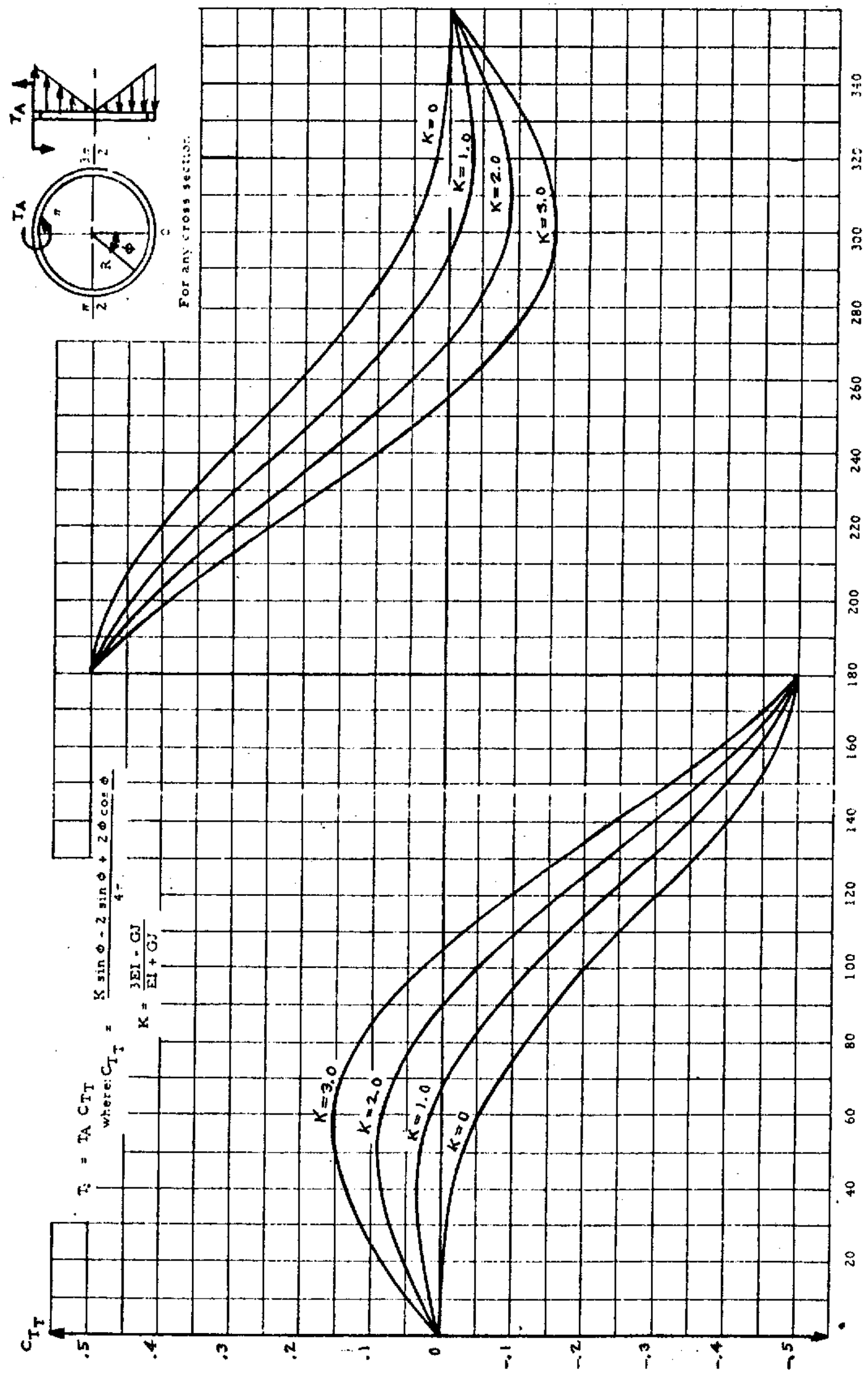


Figure 12.37 (Cont'd) - Coefficients for Concentrated Out-of-Plane Torsion

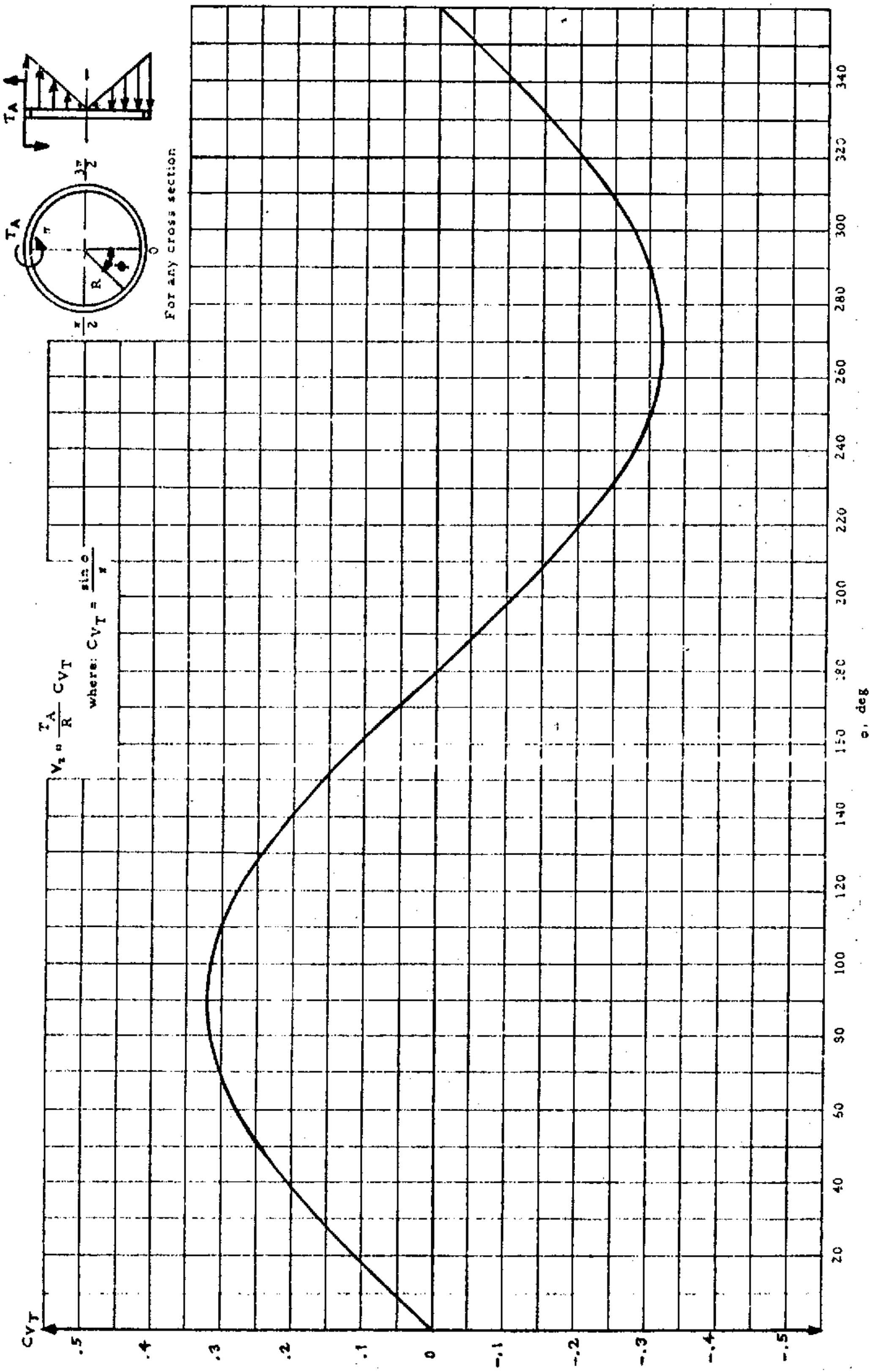


Figure 12.37 (Cont'd) - Coefficients for Concentrated Out-of-Plane Torsion

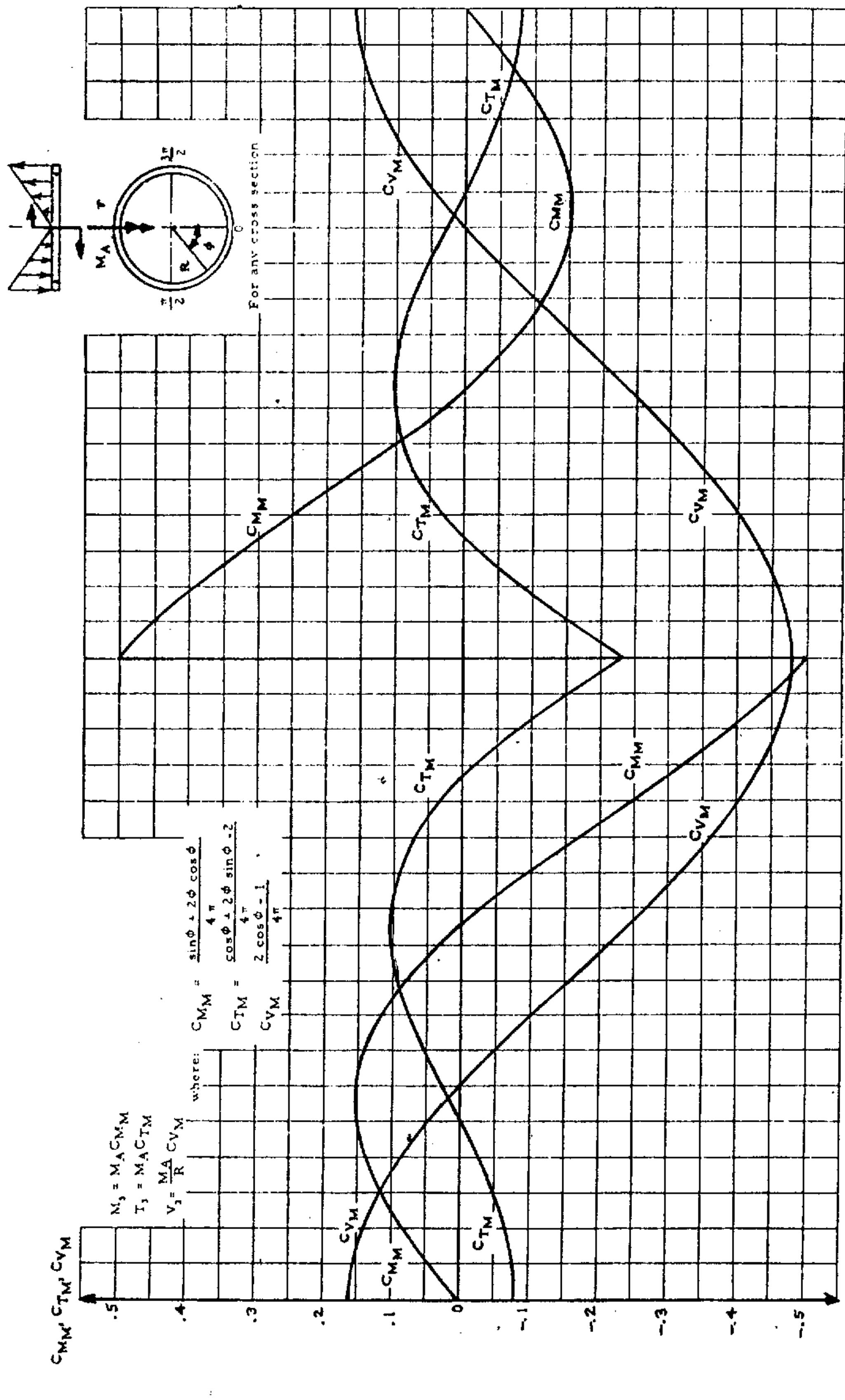


Figure 12.38 - Coefficients for Concentrated Out-of-Plane Moment

E_{sk}	Young's modulus of skin ~ lb/in^2
e	base of natural logarithms
F	axial force in loaded frame ~ lb
G	shear modulus ~ lb/in^2
I	moment of inertia of a typical unloaded frame ~ in^4
I_ℓ	moment of inertia of an unloaded frame, distance "ℓ" from the loaded frame ~ in^4
I_o	moment of inertia of the loaded frame ~ in^4
i	I/ℓ_o ~ in^3
K_n	$\dfrac{n\sqrt{n^2-1}}{2\sqrt{3}} \quad \dfrac{1 + 2\,\dfrac{n^2-1}{3}\left(\dfrac{L_r}{L_c}\right)^2}{\sqrt{1 + \dfrac{n^2-1}{3}\left(\dfrac{L_r}{L_c}\right)^2}}$
ℓ	distance from loaded frame to undistorted shell section ~ in
L_c	characteristic length (see Glossary) = $\dfrac{r}{\sqrt{6}}\left[\dfrac{t'r^2}{\ell}\right]^{1/4}$ ~ in
L_r	characteristic length (see Glossary) = $\dfrac{r}{2}\sqrt{\dfrac{Et'}{Gt}}$ ~ in
ℓ_o	frame spacing ~ in
M	bending moment in loaded frame ~ in-lb
M_o	externally applied concentrated moment ~ in-lb
P_o	externally applied radial load ~ lb
P	axial load per inch in the shell ~ lb/in
q	shear flow in shell ~ lb/in
r	radius of skin line ~ in
S	transverse shear force in loaded frame ~ lb
s	transverse shear per inch in shell ~ lb/in

T_o	externally applied tangential load ~ lb
t	skin panel thickness ~ in
t'	effective skin panel thickness for axial loads ~ in
t_e	weighted average of all the bending material (skin and stiffeners) adjacent to the loaded frame, assumed uniformly distributed around the perimeter ~ in.
u	axial displacement of shell ~ in.
v	tangential displacement of shell ~ in.
w	radial displacement of shell ~ in.
x	axial coordinate of shell ~ in.
γ	"beef up" parameter $I_o/2i\ L_c$
γ_ℓ	γ for nearby heavy frame
θ	rotational displacement ~ radians
ϕ	polar coordinate of frame and shell

In the method of attack with which this section is mainly concerned, a simplified structure, as shown in Figure 12.39, is used to obtain a solution for a uniform

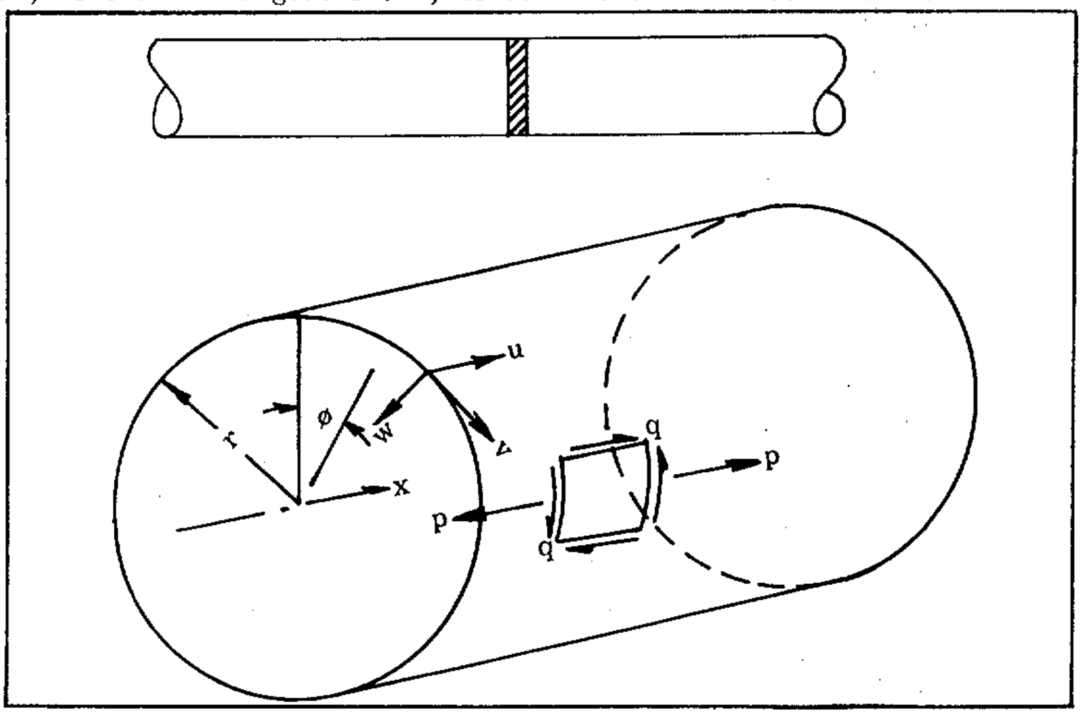


FIGURE 12.39 - SHELL WITH FLEXIBLE EXTERNALLY-LOADED FRAME

shell stretching to infinity on both sides of the loaded frame. Clearly the effects of any frame can be propagated only a finite distance along the shell. In practice, the perturbations from the "elementary beam theory" are, at worst, negligible at some characteristic length "L_c" inches away from the loaded frame. Procedures for modifying the solution to account for discontinuities and non-uniform properties are discussed in the following sections. For the structure used, the following assumptions are made:

(1) Concentrated loads are applied to the loaded frame and are reacted an infinite distance away on either one or both sides. The shell extends to infinity on both sides.

(2) The loaded frame has in-plane bending flexibility. It is free to warp out of its plane and to twist. It has no axial or shearing flexibilities. Its moment of inertia for circumferential bending is constant.

(3) The effects of the eccentricity of the skin attachment with respect to the frame neutral axis is ignored for both the loaded and unloaded frames.

(4) The shell consists of skin, longerons, and frames similar to the loaded frame, but possibly with different moments of inertia. The skin and longerons have no bending stiffness. All properties of the shell are uniform.

(5) The longerons are "smeared out" over the circumference giving an equivalent constant thickness, t', (including effective skin), for axial loads.

(6) The shell frames, but not the loaded frame, are "smeared out" in the direction of the shell axis, giving an equivalent moment of inertia per inch, "i", for circumferential bending loads.

Characteristic length - In this section there are two characteristic lengths, defined as follows: L_c is the distance required for the exponential envelope of the lowest order self-equilibrating stress system to decay to 1/e (e ~ base of natural logarithms) of its value at x = 0, provided that the skin panels are rigid in shear. L_r is the distance required for the envelope of the lowest order self-equilibrating stress system to decay to 1/e of its value at x = 0, provided that the frames are rigid in bending.

Evaluation of Parameters L_r, L_o, and γ

Case of uniform shell

In cases where the shell happens to satisfy all the assumptions listed and, in particular, if the skin thickness, stringer area, and shell-frame moment of inertia are uniform in both the axial and circumferential directions, the following formulas may be used:

$$L_c = \frac{r}{\sqrt{6}} \left[\frac{t' r^2}{i}\right]^{1/4} \qquad 12.1$$

$$L_r = \frac{r}{2} \sqrt{\frac{E\, t'}{G t}} \qquad 12.2$$

$$\gamma = \frac{I_o}{2 i L_c} \qquad 12.3$$

Young's modulus for skin, stiffeners and all frames is assumed equal. Coefficients are obtained by use of these parameters (L_c, L_r, γ) in the tables. These coefficients yield the required loads and deformations when substituted into Eqs. 12.14 through 12.21. In non-uniform shells, use the modified parameters indicated in the following equations:

Case of non-uniform shell

(a) In the case that the shell properties, i, t, and t', vary over the surface of the shell to a moderate degree, the following formulas and definitions are appropriate:

$$L_c = \frac{r}{\sqrt{6}} \left[\frac{E_{sk}\, t_e\, r^2}{E_f\, i}\right]^{1/4} \qquad 12.4$$

$$L_r = \frac{r}{2} \sqrt{\frac{E_{sk}\, t'}{G\, t}} \qquad 12.5$$

$$\gamma = \frac{E_o\, I_o}{2 E_f\, i\, L_c} \qquad 12.6$$

The stiffness factors, Gt, E_{sk}, t_e, and $E_f i$, must be averaged in the neighborhood of the loaded frame. The factors Gt and $E_{sk} i$ shall be averaged over a length of shell extending approximately one-half of a characteristic length from the loaded frame in both directions.

(b) When unloaded frames have unequal moment of inertia or are unequally spaced, the following weighting factor is used for computing $E_f i$:

$$E_f i = (E_f i)_{fwd} + (E_f i)_{aft} \qquad 12.7$$

$$(E_f i)_{fwd} = \frac{1}{L_{c_{fwd}}} \Sigma (W E_f I_f) \qquad 12.8$$

$$(E_f i)_{aft} = \frac{1}{L_{c_{aft}}} \Sigma (W E_f I_f) \qquad 12.9$$

Where

$$W = 1 - \frac{x}{L_c} \text{ for } x < L_c$$

$$= 0 \quad \text{for } x > L_c$$

(x is measured forward and aft of loaded frame)

The summations in Eqs. 12.8 and 12.9 are to be extended over all frames except the loaded frame. The method of calculation gives greater importance to frames closest to the loaded frame and less importance to those farther away. For the case of a single, particularly heavy, neighboring frame, or for other neighboring discontinuities such as rigid bulkheads, a free end, or a plane of symmetry, the correction factors to be discussed are applicable. If those corrections are applied, the heavy frame or other discontinuity must be ignored in applying Eqs. 12.7, 12.8, and 12.9 . In particular, if the loaded frame is near the end of the shell, the shell must be continued beyond the end, fictitiously, in the summations of Eqs. 12.7, 12.8 and 12.9, as though the shell were symmetric about the loaded frame and extended for a length greater than L_c on both sides of the loaded frame.

The method of calculation indicated in this subsection exaggerates the effect of frames which are heavier than average when compared with the more accurate method of correction given in the next section. Since L_c depends on $(E_f i)^{1/4}$, an initial estimate of $E_f i$ is required in order to calculate the L_c used in Eqs. 12.7, 12.8 and 12.9.

Corrections to γ, the "Beef-Up parameter

The general form of the modified "beef-up" parameter, γ^*, is:

$$\gamma^* = \gamma \cdot f_a \cdot f_b \cdot f_c, \text{ etc.} \qquad 12.10$$

where γ is computed by the methods of the preceding section, and f_a, f_b, and f_c are factors accounting for effects of nearby heavy frames etc.

Modification for different value of L_r/L_c

The value of L_r/L_c used in the graphs are 0.2, 0.4, and 1.0. To account for values of this parameter between 0.2 and 1.0, graphical interpolation should be used. Otherwise, the following formula may be applied.

$$\gamma^* = \gamma \frac{\sqrt{1 + \left[\left(\frac{L_r}{L_c}\right)^*\right]^2}}{1 + 2\left[\left(\frac{L_r}{L_c}\right)^*\right]^2} \quad \frac{1 + 2\left[\left(\frac{L_r}{L_c}\right)''\right]^2}{\sqrt{1 + \left[\left(\frac{L_r}{L_c}\right)''\right]^2}} \qquad 12.11$$

where $(L_r/L_c)''$ is the value of the parameter for the shell, and $(L_r/L_c)^*$ is the value of the parameter closest to $(L_r/L_c)''$, for which graphs are available.

Modification for finite frame spacing.

The modification for finite frame spacing is as follows:

$$\gamma * = \gamma \left\{ 1 + \frac{\ell_o}{2L_c K_2} \left(1 + \frac{1}{2\,\gamma K_2}\right) \left[4 \left(\frac{L_r}{L_c}\right)^2 + \frac{1}{1+\left(\frac{L_r}{L_c}\right)^2} \right] \right\} \quad 12.12$$

where

ℓ_o = distance from loaded frame to adjacent frames

$$K_2 = \frac{1 + 2\left(\frac{L_r}{L_c}\right)^2}{\sqrt{1 + \left(\frac{L_r}{L_c}\right)^2}}$$

Modification for nearby heavy frames and for other similar nearby discontinuities.

The corrections to "γ" in a previous section are not intended to account for discontinuities in circumferential bending stiffness. The form of the correction for these effects is:

$$\gamma * = \gamma \cdot f_{(2)} \quad \ldots\ldots\ldots\ldots \quad 12.13$$

Fig. 12.40 shows $f_{(2)}$ plotted for nearby heavy frames and for nearby rigid bulkheads. Fig. 12.41 shows $f_{(2)}$ plotted for a finite length of shell terminated in various ways on one side of the loaded frame. The validity of the correction is considered doubtful for $f_{(2)} < 0.25$, due to the importance of higher order stress systems. Figures 12.40 and 12.41 are for $L_r/L_c = 0.4$, but their variation with L_r/L_c is negligible for conventional shell-frame structures and adequate in other applications for $L_r/L_c < 0.75$. The corrections for nearby planes of symmetry and antisymmetry can be used to solve problems where two similar frames are simultaneously loaded. To illustrate the method the two following examples are given:

Example 1

A frame of moment of inertia 4.0 in^4 that is subjected to concentrated loads is supported in a uniform shell whose characteristic length, L_c, is 200 inches and moment of inertia per unit length, i, is 0.10 $in.^3$. A heavy frame having a moment of inertia 16.0 in^4 is 50 inches to one side of this frame. The loaded frame and shell loads are required.

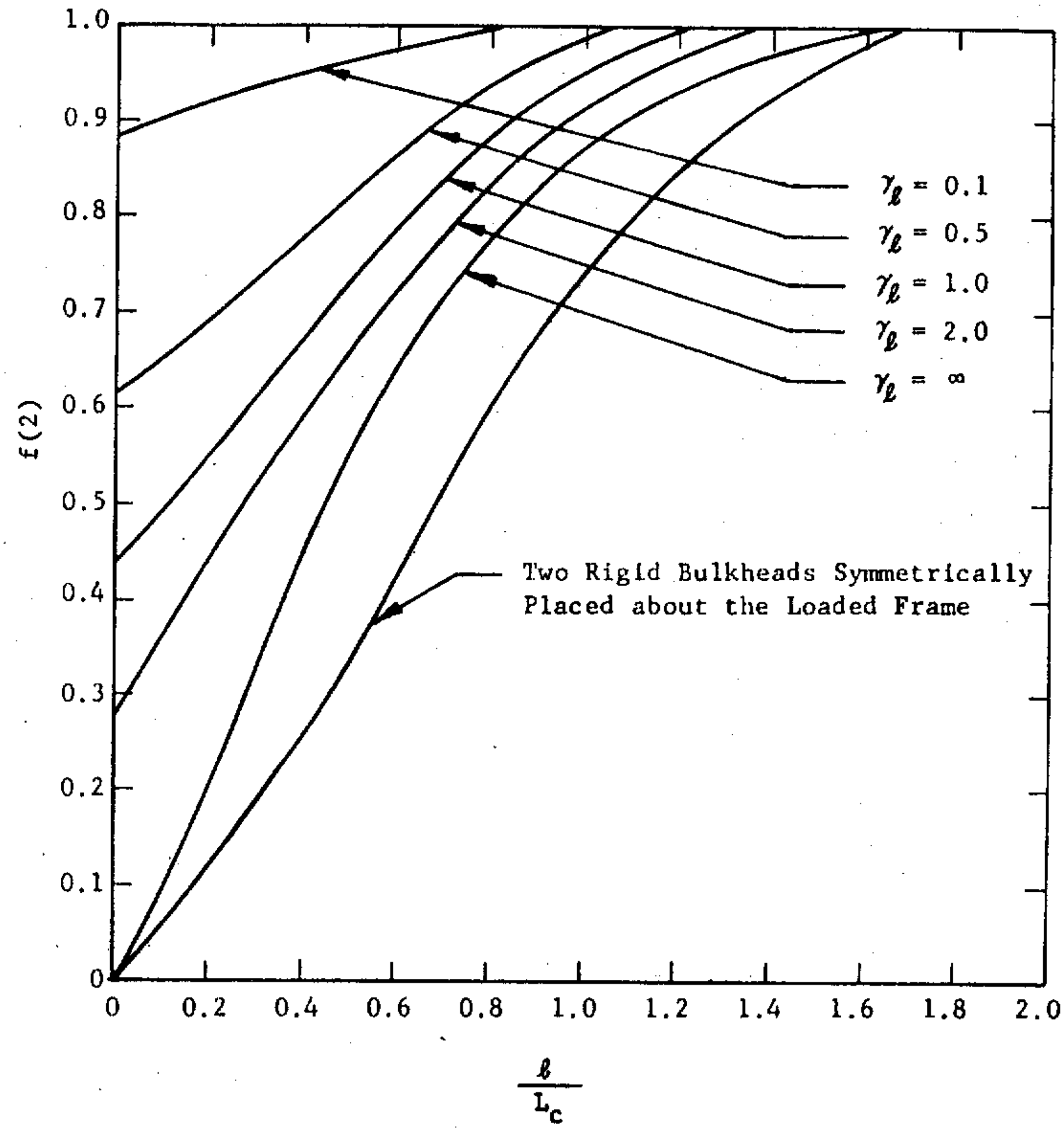


Figure 12.40 - A Single Frame on One Side of Loaded Frame or Two Rigid Bulkheads Symmetrically Placed about the Loaded Frame Curves of f(2) and f(3). $L_r/L_c = 0.4$

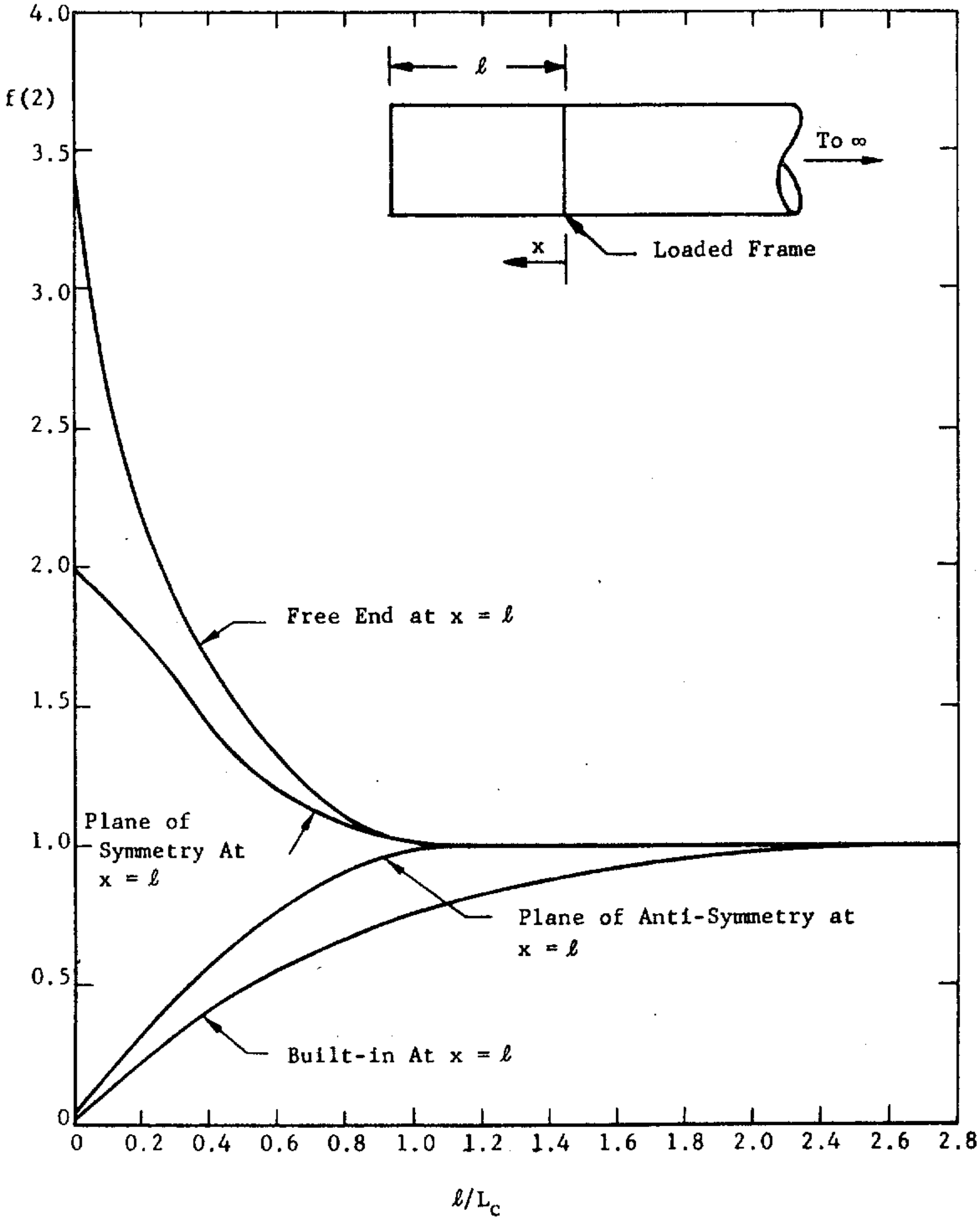


Figure 12.41 - Finite Length of Shell on One Side of Loaded Frame f(2) vs ℓ/L_c For Various Boundary Conditions at $x = \ell$, $L_r/L_c = 0.4$

The parameters needed are:

$$\gamma = \frac{4.0}{2(.1)(200)} = 0.10$$

$$\gamma_\ell = \frac{16}{2(.1)\ 200} = 0.40$$

$$\frac{\ell}{L_c} = \frac{50}{200} = 0.25$$

Using γ_ℓ and ℓ/L_c in Fig. 12.40 yields f(2) = 0.75

$\therefore\ \gamma^* = 0.75\ (0.10) = 0.075$ by Eq. 12.13

Use $\gamma = 0.075$ instead of 0.10 in the curves to account for the presence of the heavy frame on the stresses in and near the loaded frame.

Example 2:

A shell whose characteristic length, L_c, is 250 inches is supported by a large number of identical frames whose moments of inertia are 2.0 in^4, spaced 24 inches apart. A pair of frames 96 inches apart are subjected to concentrated loads at the same polar angle, ϕ. The two radial loads are of equal magnitude but opposite sign, while the tangential loads are of the same magnitude and sign. The loads in the loaded frames and shell are to be found.

$$i = \frac{I}{\ell_o} = \frac{2}{24} = .0833$$

$$\gamma = \frac{I_o}{2iL_c} = \frac{2}{2(.0833)(250)} = 0.048 \qquad \text{by Eq. 12.3}$$

$$\frac{\ell}{L_c} = \frac{48}{250} = 0.192$$

For the tangential loads there is a plane of symmetry midway between the loaded frames, while for the radial loads a plane of anti-symmetry exists at the same place. From Fig. 12.41 it is seen that for the radial load stress system, f(2) = 0.32, while for the tangential loading f(2) = 1.75. Hence, the values of γ^* to be used in the graphs are 0.015 and 0.084, respectively.

Eccentricity between skin line and neutral axis of the loaded frame:

In the three types of perturbation just discussed, it is possible to account for the effects by modifying γ only, since the "elementary-beam-theory" part of the solution is always valid. In the case when the eccentricity between skin line and neutral axis of the loaded frame exists, the "elementary-beam-theory" solution is also affected.

Eqs. 12.14 thru 12.21 are given later in this section, by which the effects of a concentrated load or moment on a shell-supported frame may be computed by using the tabulated coefficients given in Table 12.6. The method of computing γ is indicated in a previous section. These enable the shear flow and axial load at all points in the shell and the internal loads and displacements of the loaded frame to be computed.

The following parts of the overall solution are omitted in the tabulated coefficients:

(1) The "elementary-beam-theory" part of skin shear flow which is calculated from beam theory.

(2) The "elementary-beam-theory" part of the axial load intensity in longerons which should be calculated from beam theory.

(3) The rigid translations and rotation of the loaded frame.

As a consequence of items (1) and (2), shear flow and axial load intensity in the shell, as calculated from the tables, can be added directly to the results of an "engineers bending theory" calculation. The shear flow and axial load distributions given in the tables are assumed to be symmetrical with respect to the loaded frame. In a shell that is unsymmetric about the loaded frame, the shear flows and axial loads are not symmetric about the loaded frame. It is not possible to derive a simple correction for this effect, but the exact solutions indicated in reference 7 are applicable.

Distributed loads on a frame:

The effect of a distributed load on one frame may be obtained by superimposing the effects of the concentrated loads into which the distributed load can be resolved. The axial load and shear flow in the shell can be obtained for loads on several frames by a similar superposition, since "p" and "q" are tabulated in Ref. 8 NASA TN D402 as a function of x/L_c.

Frames adjacent to the loaded frame:

At the present time it is not possible by use of tables to compute the internal forces in frames adjacent to the loaded frame. It is, however, a simple matter to tabulate the frame-bending moment per inch, "m", and the other internal forces as a function of x/L_c. The bending moment in an adjacent frame, due to a force applied at the loaded frame, is then obtained by multiplying "m" at the frame station by I /i (see Appendix D of reference 6).

Effect of local reinforcement of the loaded frame:

It is not practical to attempt to cover, by a set of tables or charts, the many possible reinforcing patterns that can be used to locally strengthen frames in the region of applied concentrated loads. A solution is presented in Appendix A of reference 8, together with a simple example, to illustrate the numerical procedure. A loaded frame, whose moment of inertia varies around the circumference in any manner can be treated as a frame of constant moment of inertia that is reinforced to produce the actual inertia variation.

Tables (TABLE 12.6)

The loads and displacements of the loaded frame and loads in the shell are given in terms of the non-dimensional coefficients of the tables by the formulas below. The tables contained in this section are for M, S, F, p, and q at x = o.

Coefficients for displacements v, w, and γ are tabulated in reference 8 along with coefficients for "q" and "p" as a function of x/L_c.

$$q = C_{qp}\frac{P_o}{r} + C_{qt}\frac{T_o}{r} + C_{qm}\frac{M_o}{r^2} \quad \text{12.14}$$

$$p = C_{pp}\frac{P_o}{r}\left(\frac{L_c}{r}\right) + C_{pt}\frac{T_o}{r}\left(\frac{L_c}{r}\right) + C_{pm}\frac{M_o}{r^2}\left(\frac{L_c}{r}\right) \quad \text{12.15}$$

$$M = C_{mp}P_o r + C_{mt}T_o r + C_{mm}M_o \quad \text{12.16}$$

$$S = C_{sp}P_o + C_{st}T_o + C_{sm}\frac{M_o}{r} \quad \text{12.17}$$

$$F = C_{fp}P_o + C_{tt}T_o + C_{fm}\frac{M_o}{r} \quad \text{12.18}$$

$$v = C_{vp}P_o\frac{\gamma r^3}{EI_o} + C_{vt}T_o\frac{\gamma r^3}{EI_o} + C_{vm}M_o\frac{\gamma r^2}{EI_o} \quad \text{12.19}$$

$$W = C_{wp}P_o\frac{\gamma r^3}{EI_o} + C_{wt}T_o\frac{\gamma r^3}{EI_o} + C_{wm}M_o\frac{\gamma r^2}{EI_o} \quad \text{12.20}$$

$$\theta = C_{\theta p}P_o\frac{\gamma r^2}{EI_o} + C_{\theta t}T_o\frac{\gamma r^2}{EI_o} + C_{\theta m}M_o\frac{\gamma r}{EI_o} \quad \text{12.21}$$

The sign convention for loads, moments and displacements are positive in the loaded frame as shown in Figure 12.42.

FRAME LOADS		INDEX OF TABLES (FIGURE NO.)		
	COEFFICIENT	L_r/L_c = .200	L_r/L_c = .400	L_r/L_c = 1.000
Bending Moment, M	C_{mp}	12.7	12.11	12.15
	C_{mt}	12.19	12.23	12.27
	C_{mm}	12.31	12.35	12.39
Shear, S	C_{sp}	12.8	12.12	12.16
	C_{st}	12.20	12.24	12.28
	C_{sm}	12.32	12.36	12.40
Axial Load, F	C_{fp}	12.9	12.13	12.17
	C_{ft}	12.21	12.25	12.29
	C_{fm}	12.33	12.37	12.41
Shear Flow, q At Ring	C_{qp}	12.10	12.14	12.18
	C_{qt}	12.22	12.26	12.30
	C_{qm}	12.34	12.38	12.42

TABLE 12.6 - INDEX OF TABLES FOR CALCULATION OF FRAME LOADS

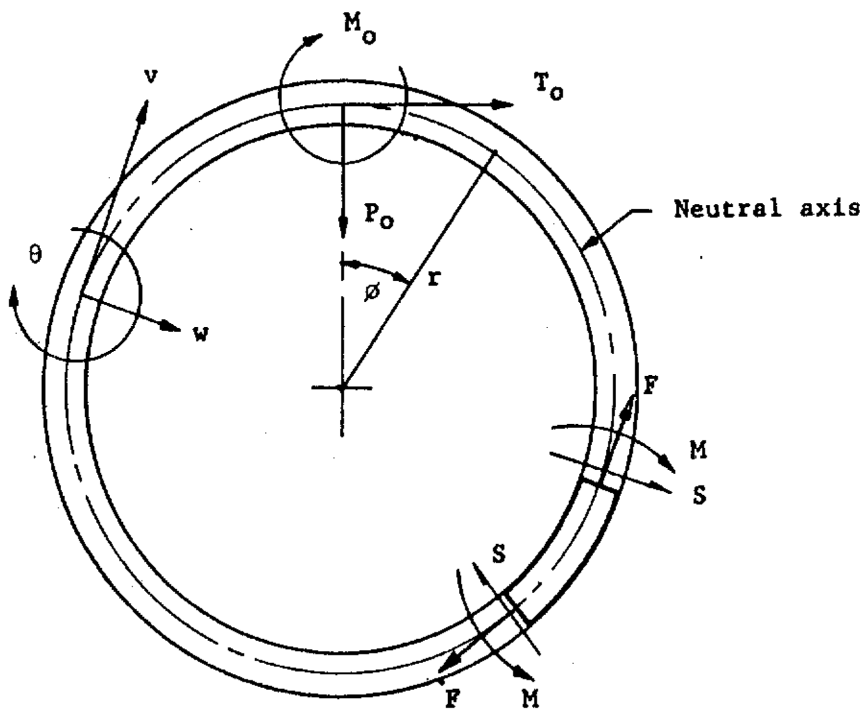


Figure 12.42 - Sign Convention for Tables 12.7 through 12.42

C_{mp}				L_r/L_c = .200			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ø°									
0	.0463	.0536	.0646	.0833	.1066	.1221	.1430	.1713	.2072
5	.0141	.0201	.0296	.0465	.0682	.0829	.1030	.1303	.1652
10	.0004	.0036	.0096	.0219	.0396	.0522	.0698	.0946	.1267
15	-.0038	-.0032	-.0007	.0064	.0189	.0286	.0429	.0638	.0918
20	-.0047	-.0054	-.0055	-.0029	.0043	.0109	.0214	.0376	.0605
25	-.0041	-.0056	-.0073	-.0081	-.0057	-.0022	.0043	.0156	.0327
30	-.0033	-.0050	-.0075	-.0108	-.0122	-.0116	-.0089	-.0028	.0083
35	-.0029	-.0045	-.0072	-.0120	-.0164	-.0182	-.0190	-.0176	-.0128
40	-.0027	-.0041	-.0069	-.0123	-.0189	-.0226	-.0264	-.0295	-.0307
45	-.0024	-.0037	-.0064	-.0122	-.0202	-.0254	-.0317	-.0367	-.0454
50	-.0022	-.0034	-.0059	-.0117	-.0206	-.0269	-.0351	-.0455	-.0572
55	-.0022	-.0033	-.0056	-.0112	-.0205	-.0274	-.0371	-.0503	-.0662
60	-.0021	-.0031	-.0053	-.0106	-.0198	-.0272	-.0379	-.0531	-.0726
65	-.0019	-.0029	-.0049	-.0098	-.0189	-.0263	-.0376	-.0543	-.0766
70	-.0018	-.0027	-.0045	-.0091	-.0177	-.0250	-.0365	-.0540	-.0783
75	-.0017	-.0025	-.0042	-.0083	-.0163	-.0234	-.0347	-.0525	-.0779
80	-.0015	-.0022	-.0037	-.0075	-.0148	-.0214	-.0323	-.0499	-.0757
85	-.0013	-.0020	-.0033	-.0066	-.0132	-.0193	-.0294	-.0464	-.0718
90	-.0012	-.0017	-.0029	-.0058	-.0115	-.0169	-.0262	-.0421	-.0665
100	-.0008	-.0012	-.0020	-.0039	-.0080	-.0119	-.0190	-.0317	-.0523
110	-.0005	-.0006	-.0011	-.0021	-.0044	-.0068	-.0113	-.0200	-.0349
120	-.0000	-.0001	-.0001	-.0004	-.0010	-.0018	-.0036	-.0078	-.0158
130	.0003	.0004	.0007	.0013	.0022	.0029	.0036	.0040	.0033
140	.0006	.0009	.0014	.0027	.0051	.0070	.0101	.0147	.0211
150	.0008	.0012	.0020	.0039	.0074	.0104	.0154	.0237	.0362
160	.0009	.0015	.0025	.0048	.0091	.0130	.0194	.0305	.0478
170	.0011	.0017	.0027	.0053	.0102	.0145	.0219	.0347	.0550
180	.0011	.0017	.0028	.0055	.0105	.0150	.0227	.0361	.0575

TABLE 12.7

C_{sp}				L_r/L_c = .200			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ø°									
0	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000
5	-.2464	-.2738	-.3066	-.3467	-.3813	-.3989	-.4182	-.4393	-.4610
10	-.0863	-.1199	-.1637	-.2230	-.2787	-.3085	-.3423	-.3803	-.4204
15	-.0223	-.0449	-.0804	-.1374	-.1988	-.2342	-.2762	-.3256	-.3796
20	.0000	-.0111	-.0342	-.0802	-.1381	-.1744	-.2197	-.2755	-.3390
25	.0098	.0049	-.0087	-.0426	-.0927	-.1269	-.1719	-.2301	-.2991
30	.0080	.0074	.0014	-.0205	-.0602	-.0901	-.1323	-.1894	-.2603
35	.0026	.0045	.0038	-.0082	-.0374	-.0623	-.0994	-.1530	-.2227
40	.0025	.0042	.0052	-.0005	-.0208	-.0405	-.0721	-.1205	-.1865
45	.0032	.0041	.0056	.0039	-.0091	-.0236	-.0494	-.0914	-.1519
50	.0007	.0020	.0042	.0056	-.0011	-.0111	-.0307	-.0657	-.1190
55	.0002	.0014	.0037	.0067	.0048	-.0013	-.0152	-.0428	-.0880
60	.0023	.0028	.0044	.0080	.0093	.0065	-.0023	-.0226	-.0590
65	.0020	.0026	.0043	.0084	.0124	.0125	.0084	-.0049	-.0320
70	.0006	.0018	.0038	.0086	.0146	.0171	.0171	.0106	-.0072
75	.0018	.0027	.0045	.0092	.0166	.0208	.0244	.0240	.0153
80	.0028	.0035	.0051	.0098	.0180	.0238	.0303	.0355	.0355
85	.0016	.0027	.0048	.0099	.0189	.0259	.0349	.0451	.0533
90	.0014	.0026	.0048	.0101	.0197	.0274	.0385	.0530	.0687
100	.0025	.0034	.0054	.0106	.0205	.0293	.0431	.0641	.0922
110	.0021	.0032	.0052	.0104	.0203	.0293	.0445	.0694	.1061
120	.0017	.0028	.0048	.0098	.0192	.0280	.0432	.0696	.1107
130	.0024	.0031	.0047	.0090	.0174	.0254	.0396	.0652	.1069
140	.0009	.0019	.0036	.0074	.0148	.0217	.0341	.0570	.0954
150	.0018	.0022	.0033	.0061	.0117	.0172	.0270	.0456	.0774
160	.0004	.0009	.0019	.0040	.0080	.0118	.0187	.0317	.0545
180	0	0	0	0	0	0	0	0	0

TABLE 12.8

C_{fp}				L_r/L_c = .200				X = 0	
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ø°									
0	-3.1241	-2.7540	-2.3243	-1.8128	-1.3837	-1.1695	-.9368	-.6856	-.4302
5	-2.4792	-2.2671	-1.9943	-1.6314	-1.2958	-1.1182	-.9180	-.6942	-.4597
10	-1.1964	-1.2557	-1.2662	-1.1891	-1.0445	-.9445	-.8155	-.6537	-.4679
15	-.3940	-.5499	-.6954	-.7949	-.7955	-.7625	-.6996	-.5999	-.4669
20	-.1715	-.2693	-.3960	-.5335	-.6033	-.6121	-.5966	-.5472	-.4617
25	-.0452	-.0990	-.1944	-.3340	-.4412	-.4787	-.4997	-.4932	-.4517
30	.0683	.0250	-.0532	-.1847	-.3089	-.3648	-.4124	-.4406	-.4382
35	.0355	.0235	-.0153	-.1084	-.2212	-.2818	-.3428	-.3944	-.4231
40	-.0207	-.0085	-.0149	-.0690	-.1610	-.2193	-.2858	-.3527	-.4062
45	.0148	.0157	.0087	-.0318	-.1104	-.1659	-.2351	-.3131	-.3869
50	.0284	.0231	.0159	-.0128	-.0764	-.1264	-.1939	-.2776	-.3661
55	-.0182	-.0109	-.0050	-.0147	-.0590	-.1007	-.1625	-.2466	-.3444
60	-.0167	-.0114	-.0064	-.0107	-.0436	-.0786	-.1348	-.2173	-.3211
65	.0197	.0127	.0074	-.0014	-.0290	-.0591	-.1101	-.1897	-.2963
70	.0022	.0005	-.0007	-.0043	-.0233	-.0473	-.0912	-.1652	-.2710
75	-.0223	-.0164	-.0114	-.0091	-.0204	-.0386	-.0756	-.1427	-.2449
80	.0040	.0018	-.0004	-.0030	-.0132	-.0283	-.0599	-.1206	-.2177
85	.0156	.0099	.0048	.0001	-.0084	-.0204	-.0467	-.1001	-.1902
90	-.0125	-.0091	-.0066	-.0051	-.0083	-.0162	-.0365	-.0816	-.1626
100	.0151	.0102	.0060	.0028	.0002	-.0039	-.0157	-.0460	-.1069
110	-.0151	-.0100	-.0056	-.0014	.0020	.0026	-.0006	-.0153	-.0528
120	.0139	.0102	.0076	.0071	.0098	.0124	.0148	.0130	-.0014
130	-.0071	-.0038	-.0003	.0047	.0118	.0176	.0260	.0365	.0450
140	.0044	.0044	.0054	.0090	.0167	.0241	.0366	.0570	.0856
150	.0052	.0052	.0064	.0107	.0199	.0288	.0446	.0730	.1185
160	-.0059	-.0022	.0022	.0094	.0209	.0314	.0500	.0845	.1428
170	.0140	.0115	.0109	.0143	.0245	.0349	.0544	.0920	.1579
180	-.0103	-.0050	.0008	.0094	.0223	.0339	.0546	.0939	.1627

TABLE 12.9

C_{gp}				L_r/L_c = .200				X = 0	
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ø°									
0	0	0	0	0	0	0	0	0	0
5	6.8999	5.3381	3.7739	2.2737	1.3216	.9481	.6150	.3339	.1206
10	6.6814	5.4895	4.1517	2.7002	1.6671	1.2306	.8224	.4601	.1711
15	2.4739	2.5437	2.3603	1.6606	1.3094	1.0231	.7232	.4273	.1668
20	.5810	.9941	1.2537	1.2436	1.0055	.8322	.6217	.3870	.1582
25	.7800	.8745	.9778	.9864	.8433	.7205	.5574	.3597	.1521
30	.1713	.2726	.4288	.5741	.5809	.5312	.4381	.3000	.1334
35	-.5612	-.3448	-.0862	.1887	.3234	.3387	.3111	.2325	.1106
40	-.1377	-.1158	-.0359	.1134	.2230	.2483	.2420	.1911	.0952
45	.1709	.0721	.0278	.0709	.1456	.1729	.1800	.1513	.0794
50	-.2963	-.2542	-.1992	-.0968	.0138	.0624	.0965	.0994	.0588
55	-.3463	-.2896	-.2340	-.1492	-.0509	-.0011	.0417	.0613	.0422
60	.0732	-.0008	-.0616	-.0832	-.0473	-.0162	.0171	.0383	.0302
65	-.0570	-.0879	-.1158	-.1236	-.0899	-.0594	-.0230	.0079	.0156
70	-.3732	-.3042	-.2486	-.1985	-.1441	-.1077	-.0645	-.0227	.0010
75	-.1369	-.1430	-.1497	-.1522	-.1324	-.1096	-.0762	-.0377	-.0084
80	.0374	-.0236	-.0761	-.1166	-.1223	-.1104	-.0857	-.0506	-.0169
85	-.2538	-.2231	-.1982	-.1793	-.1591	-.1406	-.1113	-.0706	-.0275
90	-.2942	-.2513	-.2155	-.1879	-.1666	-.1497	-.1226	-.0824	-.0350
100	-.0925	-.1122	-.1287	-.1417	-.1449	-.1397	-.1242	-.0925	-.0443
110	-.1401	-.1429	-.1449	-.1461	-.1447	-.1402	-.1277	-.0996	-.0507
120	-.2095	-.1865	-.1671	-.1515	-.1421	-.1363	-.1250	-.1000	-.0528
130	-.0055	-.0420	-.0724	-.0956	-.1065	-.1083	-.1049	-.0886	-.0492
140	-.2383	-.1952	-.1589	-.1301	-.1145	-.1080	-.0992	-.0817	-.0453
150	.0514	.0103	-.0242	-.0507	-.0636	-.0671	-.0674	-.0597	-.0348
160	-.1563	-.1241	-.0970	-.0756	-.0641	-.0597	-.0547	-.0455	-.0253
170	.0282	.0107	-.0040	-.0154	-.0210	-.0226	-.0232	-.0209	-.0125
180	0	0	0	0	0	0	0	0	0

TABLE 12.10

C_{mp}			L_r/L_c = .400				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	.0546	.0626	.0744	.0938	.1172	.1325	.1526	.1792	.2116
5	.0206	.0274	.0380	.0558	.0780	.0926	.1120	.1379	.1695
10	.0028	.0071	.0145	.0284	.0472	.0600	.0775	.1012	.1306
15	-.0048	-.0033	.0005	.0096	.0237	.0341	.0487	.0691	.0951
20	-.0073	-.0077	-.0070	-.0026	.0064	.0138	.0250	.0414	.0630
25	-.0071	-.0088	-.0103	-.0101	-.0060	-.0015	.0059	.0178	.0344
30	-.0060	-.0083	-.0112	-.0142	-.0144	-.0129	-.0091	-.0019	.0092
35	-.0049	-.0072	-.0109	-.0160	-.0199	-.0209	-.0207	-.0182	-.0127
40	-.0040	-.0062	-.0100	-.0165	-.0231	-.0264	-.0293	-.0312	-.0312
45	-.0032	-.0052	-.0089	-.0160	-.0247	-.0297	-.0354	-.0413	-.0466
50	-.0027	-.0044	-.0078	-.0151	-.0250	-.0315	-.0394	-.0489	-.0589
55	-.0024	-.0039	-.0069	-.0140	-.0246	-.0319	-.0417	-.0541	-.0683
60	-.0022	-.0034	-.0061	-.0127	-.0235	-.0315	-.0424	-.0572	-.0750
65	-.0020	-.0030	-.0054	-.0115	-.0220	-.0302	-.0420	-.0585	-.0792
70	-.0018	-.0028	-.0048	-.0103	-.0203	-.0285	-.0406	-.0582	-.0810
75	-.0017	-.0025	-.0043	-.0091	-.0184	-.0263	-.0384	-.0565	-.0807
80	-.0015	-.0022	-.0038	-.0080	-.0164	-.0238	-.0356	-.0537	-.0784
85	-.0013	-.0019	-.0033	-.0069	-.0144	-.0212	-.0323	-.0498	-.0744
90	-.0011	-.0017	-.0028	-.0058	-.0123	-.0184	-.0286	-.0451	-.0689
100	-.0007	-.0011	-.0018	-.0038	-.0082	-.0126	-.0204	-.0339	-.0542
110	-.0004	-.0006	-.0009	-.0019	-.0043	-.0069	-.0119	-.0213	-.0361
120	.0000	.0000	.0000	-.0001	-.0005	-.0014	-.0035	-.0082	-.0164
130	.0003	.0005	.0008	.0016	.0028	.0036	.0043	.0045	.0034
140	.0007	.0010	.0016	.0031	.0057	.0079	.0112	.0159	.0218
150	.0009	.0013	.0022	.0043	.0081	.0115	.0169	.0254	.0375
160	.0011	.0016	.0027	.0052	.0099	.0141	.0211	.0326	.0495
170	.0012	.0018	.0029	.0057	.0110	.0157	.0237	.0370	.0570
180	.0012	.0018	.0030	.0059	.0113	.0163	.0245	.0385	.0595

TABLE 12.11

C_{sp}			L_r/L_c = .400				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000
5	-.2862	-.3103	-.3382	-.3715	-.3998	-.4141	-.4298	-.4469	-.4644
10	-.1334	-.1664	-.2075	-.2605	-.3084	-.3336	-.3619	-.3936	-.4265
15	-.0508	-.0784	-.1175	-.1744	-.2313	-.2629	-.2997	-.3421	-.3875
20	-.0096	-.0279	-.0585	-.1101	-.1680	-.2023	-.2438	-.2933	-.3479
25	.0100	-.0003	-.0214	-.0636	-.1173	-.1514	-.1944	-.2477	-.3083
30	.0141	.0105	-.0014	-.0322	-.0782	-.1098	-.1516	-.2056	-.2693
35	.0113	.0122	.0078	-.0120	-.0486	-.0763	-.1149	-.1672	-.2311
40	.0093	.0119	.0119	.0009	-.0264	-.0495	-.0836	-.1322	-.1940
45	.0074	.0103	.0128	.0085	-.0102	-.0284	-.0572	-.1007	-.1583
50	.0042	.0074	.0113	.0121	.0012	-.0121	-.0351	-.0724	-.1243
55	.0026	.0054	.0097	.0138	.0092	.0005	-.0168	-.0472	-.0921
60	.0029	.0048	.0087	.0145	.0149	.0102	-.0015	-.0249	-.0619
65	.0023	.0039	.0075	.0142	.0185	.0174	.0109	-.0053	-.0339
70	.0014	.0029	.0063	.0136	.0208	.0227	.0210	.0117	-.0081
75	.0019	.0031	.0060	.0132	.0224	.0267	.0291	.0264	.0154
80	.0025	.0034	.0059	.0128	.0233	.0295	.0355	.0389	.0364
85	.0018	.0029	.0054	.0122	.0236	.0313	.0404	.0492	.0550
90	.0018	.0029	.0053	.0118	.0237	.0325	.0441	.0577	.0710
100	.0024	.0034	.0055	.0114	.0232	.0333	.0482	.0694	.0955
110	.0022	.0032	.0053	.0108	.0221	.0323	.0489	.0747	.1099
120	.0019	.0029	.0050	.0100	.0203	.0302	.0468	.0745	.1147
130	.0021	.0030	.0047	.0091	.0181	.0269	.0425	.0695	.1107
140	.0012	.0021	.0038	.0076	.0152	.0227	.0362	.0605	.0987
150	.0016	.0021	.0032	.0061	.0120	.0178	.0285	.0483	.0801
160	.0006	.0011	.0020	.0041	.0082	.0122	.0196	.0336	.0564
170	.0006	.0008	.0011	.0022	.0042	.0062	.0100	.0172	.0291
180	0	0	0	0	0	0	0	0	0

TABLE 12.12

C_{fp}			L_r/L_c = .400				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	-2.5833	-2.2692	-1.9132	-1.4981	-1.1540	-.9826	-.7962	-.5944	-.3897
5	-2.1796	-1.9745	-1.7225	-1.4023	-1.1162	-.9672	-.8006	-.6156	-.4238
10	-1.3098	-1.3065	-1.2558	-1.1296	-.9685	-.8695	-.7484	-.6028	-.4422
15	-.6505	-.7536	-.8308	-.8520	-.8017	-.7514	-.6773	-.5753	-.4514
20	-.3303	-.4325	-.5401	-.6297	-.6514	-.6384	-.6039	-.5418	-.4543
25	-.1256	-.2088	-.3175	-.4404	-.5120	-.5286	-.5280	-.5028	-.4513
30	.0134	-.0537	-.1529	-.2867	-.3888	-.4273	-.4541	-.4611	-.4433
35	.0334	-.0013	-.0689	-.1840	-.2930	-.3433	-.3884	-.4205	-.4319
40	.0152	.0080	-.0279	-.1150	-.2180	-.2732	-.3299	-.3811	-.4173
45	.0322	.0293	.0069	-.0607	-.1555	-.2123	-.2764	-.3423	-.3996
50	.0334	.0330	.0212	-.0271	-.1088	-.1635	-.2302	-.3057	-.3796
55	.0027	.0115	.0136	-.0130	-.0773	-.1265	-.1917	-.2719	-.3578
60	-.0017	.0062	.0119	-.0024	-.0525	-.0958	-.1577	-.2397	-.3341
65	.0138	.0143	.0161	.0066	-.0328	-.0705	-.1280	-.2092	-.3088
70	.0027	.0047	.0087	.0065	-.0213	-.0523	-.1036	-.1814	-.2824
75	-.0113	-.0062	.0004	.0038	-.0139	-.0386	-.0830	-.1555	-.2551
80	.0019	.0017	.0037	.0059	-.0065	-.0262	-.0644	-.1307	-.2268
85	.0078	.0052	.0047	.0062	-.0016	-.0167	-.0485	-.1077	-.1981
90	-.0068	-.0049	-.0021	.0024	-.0001	-.0104	-.0356	-.0867	-.1692
100	.0078	.0051	.0035	.0046	.0056	.0015	-.0127	-.0476	-.1111
110	-.0077	-.0049	-.0023	.0019	.0073	.0085	.0042	-.0141	-.0547
120	.0077	.0059	.0049	.0064	.0121	.0162	.0192	.0157	-.0014
130	-.0030	-.0010	.0015	.0058	.0141	.0211	.0305	.0406	.0469
140	.0032	.0036	.0049	.0088	.0178	.0264	.0404	.0614	.0887
150	.0037	.0042	.0058	.0105	.0205	.0304	.0479	.0780	.1227
160	-.0020	.0006	.0040	.0104	.0219	.0330	.0532	.0898	.1478
170	.0085	.0078	.0086	.0133	.0243	.0356	.0569	.0972	.1633
180	-.0042	-.0007	.0035	.0109	.0235	.0354	.0575	.0993	.1684

TABLE 12.13

C_{qp}			L_r/L_c = .400				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	0	0	0	0	0	0	0	0	0
5	4.5167	3.4237	2.3731	1.4049	.8096	.5797	.3761	.2046	.0742
10	4.8735	3.8926	2.8593	1.8075	1.0973	.8055	.5364	.2998	.1115
15	2.6158	2.4071	2.0290	1.4710	.9846	.7551	.5254	.3066	.1186
20	1.2698	1.4050	1.3858	1.1586	.8536	.6825	.4948	.3004	.1203
25	.9842	1.0669	1.0863	.9696	.7586	.6247	.4669	.2923	.1203
30	.3646	.5031	.6291	.6717	.5895	.5094	.3990	.2609	.1115
35	-.2178	-.0112	.2055	.3780	.4094	.3808	.3182	.2202	.0985
40	-.1082	-.0244	.1032	.2464	.3036	.2973	.2609	.1888	.0877
45	.0058	-.0028	.0447	.1448	.2106	.2197	.2043	.1557	.0754
50	-.2519	-.2087	-.1306	-.0042	.0948	.1257	.1361	.1152	.0599
55	-.2785	-.2426	-.1841	-.0793	.0194	.0585	.0831	.0813	.0460
60	-.0536	-.0965	-.1150	-.0804	-.0141	.0199	.0470	.0550	.0342
65	-.1146	-.1384	-.1505	-.1240	-.0638	-.0282	.0054	.0257	.0211
70	-.2757	-.2460	-.2202	-.1765	-.1128	-.0742	-.0342	-.0026	.0081
75	-.1503	-.1593	-.1687	-.1615	-.1229	.0924	-.0561	-.0218	-.0019
80	-.0576	-.0946	-.1285	-.1473	-.1291	-.1062	-.0742	-.0386	-.0111
85	-.2088	-.1952	-.1878	-.1800	-.1551	-.1313	-.0977	-.0576	-.0208
90	-.2300	-.2084	-.1939	-.1836	-.1632	-.1428	-.1118	-.0710	-.0285
100	-.1227	-.1337	-.1443	-.1554	-.1545	-.1446	-.1231	-.0869	-.0394
110	-.1445	-.1458	-.1474	-.1509	-.1509	-.1447	-.1285	-.0962	-.0465
120	-.1750	-.1625	-.1523	-.1458	-.1427	-.1380	-.1256	-.0977	-.0493
130	-.0610	-.0805	-.0962	-.1086	-.1156	-.1160	-.1100	-.0895	-.0471
140	-.1732	-.1500	-.1308	-.1160	-.1090	-.1058	-.0988	-.0806	-.0429
150	-.0111	-.0332	-.0510	-.0645	-.0716	-.0736	-.0723	-.0617	-.0340
160	-.1076	-.0903	-.0760	-.0648	-.0592	-.0572	-.0539	-.0450	-.0246
170	.0016	-.0078	-.0155	-.0212	-.0242	-.0251	-.0251	-.0218	-.0123
180	0	0	0	0	0	0	0	0	0

TABLE 12.14

C_{mp}				L_r/L_c = 1.000			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
φ°									
0	.0713	.0811	.0951	.1171	.1421	.1574	.1763	.1988	.2222
5	.0349	.0438	.0568	.0777	.1017	.1166	.1350	.1569	.1799
10	.0117	.0183	.0287	.0465	.0678	.0813	.0983	.1187	.1403
15	-.0018	.0021	.0091	.0225	.0400	.0515	.0662	.0834	.1036
20	-.0087	-.0075	-.0040	.0047	.0176	.0267	.0387	.0538	.0701
25	-.0115	-.0123	-.0119	-.0080	.0001	.0064	.0153	.0269	.0398
30	-.0117	-.0141	-.0162	-.0166	-.0133	-.0097	-.0042	.0037	.0129
35	-.0108	-.0141	-.0181	-.0220	-.0230	-.0222	-.0200	-.0160	-.0108
40	-.0094	-.0131	-.0182	-.0249	-.0298	-.0315	-.0324	-.0323	-.0311
45	-.0079	-.0116	-.0173	-.0260	-.0341	-.0380	-.0420	-.0455	-.0482
50	-.0065	-.0100	-.0159	-.0258	-.0364	-.0423	-.0488	-.0557	-.0621
55	-.0053	-.0085	-.0141	-.0246	-.0371	-.0445	-.0533	-.0632	-.0730
60	-.0043	-.0071	-.0123	-.0229	-.0365	-.0451	-.0557	-.0682	-.0809
65	-.0035	-.0059	-.0106	-.0208	-.0350	-.0444	-.0564	-.0708	-.0860
70	-.0029	-.0048	-.0090	-.0185	-.0328	-.0426	-.0555	-.0714	-.0885
75	-.0024	-.0040	-.0075	-.0162	-.0300	-.0399	-.0533	-.0702	-.0887
80	-.0020	-.0032	-.0062	-.0139	-.0269	-.0366	-.0500	-.0674	-.0866
85	-.0016	-.0026	-.0050	-.0116	-.0235	-.0328	-.0459	-.0632	-.0825
90	-.0013	-.0020	-.0039	-.0095	-.0201	-.0287	-.0411	-.0578	-.0768
100	-.0007	-.0011	-.0021	-.0055	-.0131	-.0199	-.0300	-.0443	-.0609
110	-.0002	-.0003	-.0006	-.0021	-.0065	-.0109	-.0180	-.0285	-.0411
120	.0003	.0005	.0007	.0008	-.0005	-.0024	-.0060	-.0118	-.0191
130	.0007	.0011	.0018	.0033	.0048	.0052	.0053	.0045	.0030
140	.0011	.0016	.0027	.0053	.0091	.0118	.0152	.0194	.0237
150	.0014	.0021	.0035	.0068	.0126	.0171	.0234	.0319	.0415
160	.0016	.0024	.0040	.0079	.0151	.0209	.0295	.0413	.0551
170	.0017	.0026	.0043	.0086	.0166	.0233	.0332	.0472	.0636
180	.0018	.0027	.0044	.0088	.0171	.0241	.0345	.0492	.0665

TABLE 12.15

C_{sp}				L_r/L_c = 1.000			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
φ°									
0	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000
5	-.3359	-.3556	-.3778	-.4037	-.4251	-.4357	-.4471	-.4589	-.4700
10	-.2032	-.2338	-.2700	-.3145	-.3526	-.3723	-.3935	-.4159	-.4371
15	-.1117	-.1433	-.1839	-.2374	-.2867	-.3125	-.3413	-.3723	-.4022
20	-.0517	-.0791	-.1174	-.1728	-.2275	-.2574	-.2915	-.3290	-.3657
25	-.0143	-.0352	-.0679	-.1199	-.1756	-.2074	-.2446	-.2865	-.3281
30	.0055	-.0082	-.0331	-.0782	-.1311	-.1630	-.2012	-.2454	-.2901
35	.0141	.0069	-.0099	-.0461	-.0937	-.1239	-.1615	-.2060	-.2521
40	.0174	.0151	.0052	-.0218	-.0625	-.0900	-.1255	-.1687	-.2144
45	.0173	.0183	.0142	-.0041	-.0369	-.0610	-.0931	-.1336	-.1774
50	.0148	.0182	.0186	.0084	-.0164	-.0364	-.0644	-.1010	-.1416
55	.0122	.0167	.0204	.0168	-.0002	-.0158	-.0391	-.0709	-.1071
60	.0103	.0150	.0205	.0222	.0125	.0012	-.0170	-.0433	-.0743
65	.0082	.0129	.0194	.0253	.0220	.0150	.0019	-.0184	-.0435
70	.0063	.0107	.0178	.0266	.0290	.0260	.0181	.0040	-.0148
75	.0053	.0092	.0161	.0269	.0340	.0345	.0316	.0237	.0116
80	.0047	.0080	.0145	.0264	.0372	.0410	.0427	.0409	.0355
85	.0039	.0068	.0129	.0254	.0391	.0457	.0516	.0555	.0568
90	.0034	.0059	.0115	.0241	.0399	.0489	.0585	.0678	.0753
100	.0032	.0050	.0095	.0211	.0392	.0515	.0670	.0854	.1041
110	.0028	.0044	.0079	.0181	.0364	.0504	.0696	.0943	.1216
120	.0025	.0038	.0067	.0153	.0324	.0466	.0674	.0957	.1281
130	.0024	.0035	.0058	.0127	.0277	.0410	.0614	.0902	.1244
140	.0018	.0027	.0047	.0101	.0225	.0341	.0524	.0791	.1114
150	.0016	.0023	.0037	.0077	.0171	.0263	.0412	.0634	.0908
160	.0009	.0015	.0025	.0051	.0115	.0178	.0283	.0442	.0640
170	.0006	.0008	.0013	.0026	.0058	.0090	.0144	.0227	.0331
180	0	0	0	0	0	0	0	0	0

TABLE 12.16

C_{fp}			L_r/L_c = 1.000				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	-1.9400	-1.6935	-1.4193	-1.1056	-.8503	-.7251	-.5917	-.4532	-.3238
5	-1.7475	-1.5619	-1.3438	-1.0794	-.8532	-.7392	-.6155	-.4850	-.3618
10	-1.2762	-1.2137	-1.1129	-.9562	-.7967	-.7086	-.6082	-.4978	-.3900
15	-.8448	-.8724	-.8677	-.8103	-.7190	-.6594	-.5857	-.4992	-.4107
20	-.5472	-.6122	-.6608	0.6724	-.6371	-.6034	-.5553	-.4931	-.4252
25	-.3183	-.3983	-.4781	-.5399	-.5517	-.5415	-.5179	-.4800	-.4337
30	-.1485	-.2305	-.3245	-.4190	-.4678	-.4777	-.4763	-.4616	-.4366
35	-.0607	-.1268	-.2147	-.3201	-.3922	-.4173	-.4341	-.4398	-.4347
40	-.0174	-.0631	-.1358	-.2392	-.3243	-.3604	-.3919	-.4153	-.4284
45	.0185	-.0139	-.0732	-.1704	-.2624	-.3065	-.3496	-.3882	-.4178
50	.0329	.0129	-.0317	-.1168	-.2090	-.2576	-.3090	-.3598	-.4034
55	.0248	.0180	-.0097	-.0782	-.1645	-.2146	-.2710	-.3307	-.3858
60	.0221	.0220	.0059	-.0476	-.1262	-.1757	-.2347	-.3008	-.3651
65	.0251	.0266	.0174	-.0236	-.0935	-.1410	-.2006	-.2706	-.3416
70	.0164	.0213	.0195	-.0085	-.0675	-.1115	-.1695	-.2408	-.3159
75	.0068	.0142	.0179	.0013	-.0466	-.0860	-.1408	-.2114	-.2882
80	.0090	.0143	.0190	.0095	-.0288	-.0634	-.1142	-.1823	-.2589
85	.0089	.0128	.0180	.0144	-.0147	-.0443	-.0900	-.1541	-.2283
90	.0011	.0061	.0133	.0155	-.0044	-.0285	-.0684	-.1269	-.1968
100	.0051	.0067	.0115	.0181	.0114	-.0028	-.0304	-.0753	-.1324
110	-.0019	.0008	.0062	.0160	.0198	.0147	-.0002	-.0287	-.0684
120	.0044	.0046	.0073	.0161	.0257	.0277	.0245	.0128	-.0072
130	.0001	.0017	.0050	.0143	.0284	.0363	.0436	.0483	.0489
140	.0028	.0037	.0062	.0145	.0305	.0426	.0586	.0779	.0977
150	.0032	.0043	.0067	.0145	.0318	.0469	.0696	.1011	.1377
160	.0010	.0030	.0061	.0142	.0323	.0495	.0771	.1177	.1672
170	.0054	.0060	.0082	.0152	.0332	.0514	.0817	.1279	.1854
180	.0003	.0026	.0062	.0143	.0328	.0515	.0829	.1312	.1915

TABLE 12.17

C_{qp}			L_r/L_c = 1.000				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	0	0	0	0	0	0	0	0	0
5	2.3911	1.7706	1.1976	.6909	.3905	.2770	.1778	.0954	.0340
10	2.8901	2.2299	1.5779	.9578	.5636	.4076	.2669	.1461	.0529
15	2.0653	1.7386	1.3439	.8934	.5626	.4195	.2832	.1596	.0593
20	1.4211	1.3108	1.1070	.8040	.5387	.4130	.2865	.1657	.0628
25	1.1162	1.0708	.9505	.7324	.5142	.4028	.2855	.1685	.0650
30	.6514	.7014	.6937	.5926	.4465	.3605	.2631	.1595	.0629
35	.2195	.3460	.4322	.4374	.3633	.3050	.2308	.1443	.0583
40	.1100	.2086	.2973	.3371	.3016	.2612	.2035	.1307	.0539
45	.0457	.1095	.1864	.2444	.2392	.2148	.1731	.1144	.0482
50	-.1338	-.0564	.0398	.1328	.1643	.1584	.1352	.0936	.0407
55	-.1900	-.1289	-.0439	.0540	.1043	.1107	.1015	.0740	.0334
60	-.1221	-.1085	-.0642	.0099	.0611	.0735	.0732	.0566	.0265
65	-.1581	-.1494	-.1142	-.0451	.0129	.0325	.0421	.0372	.0188
70	-.2275	-.2065	-.1675	-.0973	-.0328	-.0068	.0118	.0180	.0109
75	-.1753	-.1774	-.1631	-.1161	-.0601	-.0336	-.0112	.0022	.0042
80	-.1343	-.1524	-.1568	-.1295	-.0827	-.0568	-.0318	-.0124	-.0023
85	-.1929	-.1921	-.1857	-.1566	-.1099	-.0825	-.0536	-.0275	-.0089
90	-.1981	-.1946	-.1896	-.1676	-.1268	-.1004	-.0703	-.0399	-.0146
100	-.1472	-.1565	-.1658	-.1648	-.1416	-.1208	-.0925	-.0582	-.0235
110	-.1493	-.1527	-.1585	-.1615	-.1479	-.1317	-.1060	-.0703	-.0298
120	-.1537	-.1500	-.1497	-.1516	-.1441	-.1320	-.1099	-.0756	-.0330
130	-.0972	-.1059	-.1148	-.1249	-.1263	-.1196	-.1030	-.0731	-.0327
140	-.1315	-.1220	-.1154	-.1137	-.1122	-.1066	-.0929	-.0659	-.0303
150	-.0515	-.0607	-.0687	-.0773	-.0825	-.0809	-.0725	-.0535	-.0247
160	-.0763	-.0690	-.0634	-.0608	-.0607	-.0587	-.0523	-.0386	-.0179
170	-.0157	-.0196	-.0228	-.0261	-.0285	-.0284	-.0259	-.0195	-.0091
180	0	0	0	0	0	0	0	0	0

TABLE 12.18

C_{mt}			L_r/L_c = .200				X = 0		
γ	.20	.03	.05	.10	.20	.30	.50	1.00	3.00
φ°									
0	0	0	0	0	0	0	0	0	0
5	-.0025	-.0031	-.0040	-.0056	-.0076	-.0089	-.0107	-.0131	-.0162
10	-.0030	-.0040	-.0056	-.0085	-.0122	-.0147	-.0182	-.0229	-.0289
15	-.0028	-.0040	-.0059	-.0097	-.0147	-.0182	-.0231	-.0298	-.0385
20	-.0024	-.0036	-.0056	-.0098	-.0157	-.0199	-.0258	-.0342	-.0451
25	-.0020	-.0031	-.0051	-.0093	-.0156	-.0202	-.0269	-.0365	-.0491
30	-.0017	-.0026	-.0044	-.0084	-.0148	-.0196	-.0267	-.0370	-.0509
35	-.0015	-.0022	-.0038	-.0074	-.0135	-.0183	-.0255	-.0361	-.0506
40	-.0012	-.0019	-.0037	-.0064	-.0120	-.0165	-.0235	-.0340	-.0487
45	-.0010	-.0015	-.0026	-.0053	-.0102	-.0144	-.0209	-.0310	-.0454
50	-.0008	-.0012	-.0021	-.0043	-.0085	-.0121	-.0180	-.0273	-.0409
55	-.0006	-.0009	-.0015	-.0033	-.0067	-.0097	-.0148	-.0231	-.0355
60	-.0004	-.0006	-.0011	-.0023	-.0049	-.0074	-.0115	-.0186	-.0294
65	-.0002	-.0004	-.0006	-.0014	-.0032	-.0050	-.0082	-.0139	-.0229
70	-.0001	-.0001	-.0002	-.0006	-.0016	-.0028	-.0050	-.0092	-.0161
75	.0001	.0001	.0002	.0002	-.0001	-.0007	-.0019	-.0045	-.0093
80	.0002	.0003	.0005	.0009	.0012	.0013	.0010	-.0001	-.0026
85	.0003	.0005	.0008	.0015	.0025	.0031	.0037	.0042	.0039
90	.0005	.0007	.0011	.0020	.0035	.0047	.0062	.0080	.0099
100	.0006	.0009	.0015	.0029	.0053	.0072	.0101	.0145	.0204
110	.0007	.0011	.0018	.0034	.0063	.0088	.0128	.0190	.0280
120	.0008	.0011	.0019	.0036	.0068	.0096	.0141	.0214	.0324
130	.0007	.0011	.0018	.0035	.0067	.0095	.0141	.0218	.0335
140	.0007	.0010	.0016	.0032	.0061	.0086	.0129	.0201	.0314
150	.0005	.0008	.0013	.0026	.0050	.0071	.0106	.0167	.0263
160	.0004	.0006	.0009	.0018	.0035	.0050	.0075	.0120	.0189
170	.0002	.0003	.0005	.0010	.0018	.0026	.0039	.0062	.0099
180	0	0	0	0	0	0	0	0	0

TABLE 12.19

C_{st}			L_r/L_c = .200				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
φ°									
0	-.0463	-.0536	-.0646	-.0833	-.1066	-.1221	-.1430	-.1713	-.2072
5	-.0141	-.0201	-.0296	-.0465	-.0682	-.0829	-.1030	-.1303	-.1652
10	-.0004	-.0036	-.0096	-.0219	-.0396	-.0522	-.0698	-.0946	-.1267
15	.0038	.0032	.0007	-.0064	-.0189	-.0286	-.0429	-.0638	-.0918
20	.0047	.0054	.0055	.0029	-.0043	-.0109	-.0214	-.0376	-.0605
25	.0041	.0056	.0073	.0081	.0057	.0022	-.0043	-.0156	-.0327
30	.0033	.0050	.0075	.0108	.0122	.0116	.0089	.0027	-.0083
35	.0029	.0045	.0072	.0020	.0164	.0182	.0190	.0176	.0128
40	.0027	.0041	.0069	.0123	.0189	.0226	.0264	.0295	.0307
45	.0024	.0037	.0064	.0122	.0202	.0254	.0317	.0387	.0454
50	.0022	.0034	.0059	.0117	.0206	.0269	.0351	.0455	.0572
55	.0022	.0033	.0056	.0112	.0205	.0274	.0371	.0503	.0662
60	.0021	.0031	.0053	.0106	.0198	.0272	.0379	.0531	.0726
65	.0019	.0029	.0049	.0098	.0189	.0263	.0376	.0543	.0766
70	.0018	.0027	.0045	.0091	.0177	.0250	.0365	.0540	.0783
75	.0017	.0025	.0042	.0083	.0163	.0234	.0347	.0525	.0779
80	.0015	.0022	.0037	.0075	.0148	.0214	.0323	.0499	.0757
85	.0013	.0020	.0033	.0066	.0132	.0193	.0294	.0464	.0718
90	.0012	.0017	.0029	.0058	.0115	.0169	.0262	.0421	.0665
100	.0008	.0012	.0020	.0039	.0080	.0119	.0190	.0317	.0523
110	.0005	.0006	.0011	.0021	.0044	.0068	.0113	.0200	.0349
120	.0000	.0001	.0001	.0004	.0010	.0018	.0036	.0078	.0158
130	-.0003	-.0004	-.0007	-.0013	-.0022	-.0029	-.0036	-.0040	-.0033
140	-.0006	-.0009	-.0014	-.0027	-.0051	-.0070	-.0101	-.0147	-.0211
150	-.0008	-.0012	-.0020	-.0039	-.0074	-.0104	-.0154	-.0237	-.0362
160	-.0010	-.0015	-.0025	-.0048	-.0091	-.0130	-.0194	-.0305	-.0478
170	-.0011	-.0017	-.0027	-.0053	-.0102	-.0145	-.0219	-.0347	-.0550
180	-.0011	-.0017	-.0028	-.0055	-.0105	-.0150	-.0227	-.0361	-.0575

TABLE 12.20

C_{ft}				L_r/L_c = .200			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ø°									
0	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000
5	-.2464	-.2738	-.3066	-.3467	-.3813	-.3989	-.4182	-.4393	-.4610
10	-.0863	-.1199	-.1637	-.2230	-.2787	-.3085	-.3423	-.3803	-.4204
15	-.0223	-.0449	-.0804	-.1374	-.1988	-.2342	-.2762	-.3256	-.3796
20	.0000	-.0111	-.0342	-.0802	-.1381	-.1744	-.2197	-.2755	-.3390
25	.0098	.0049	-.0087	-.0426	-.0927	-.1269	-.1719	-.2301	-.2991
30	.0080	.0074	.0014	-.0205	-.0602	-.0903	-.1323	-.1894	-.2603
35	.0026	.0045	.0038	-.0082	-.0374	-.0623	-.0994	-.1530	-.2227
40	.0025	.0042	.0052	-.0005	-.0208	-.0405	-.0721	-.1205	-.1865
45	.0032	.0041	.0056	.0039	-.0091	-.0238	-.0494	-.0914	-.1519
50	.0007	.0020	.0042	.0056	-.0011	-.0111	-.0307	-.0657	-.1190
55	.0002	.0014	.0037	.0067	.0048	-.0013	-.0152	-.0428	-.0880
60	.0023	.0028	.0044	.0080	.0093	.0065	-.0023	-.0226	-.0590
65	.0020	.0026	.0043	.0084	.0124	.0125	.0084	-.0049	-.0320
70	.0006	.0018	.0038	.0086	.0146	.0171	.0171	.0106	-.0073
75	.0018	.0027	.0045	.0092	.0166	.0208	.0244	.0240	.0153
80	.0028	.0035	.0051	.0098	.0180	.0238	.0303	.0355	.0355
85	.0016	.0027	.0048	.0099	.0189	.0259	.0349	.0451	.0533
90	.0014	.0026	.0048	.0101	.0197	.0274	.0385	.0530	.0687
100	.0025	.0034	.0054	.0106	.0205	.0293	.0431	.0641	.0922
110	.0021	.0032	.0052	.0104	.0203	.0293	.0445	.0694	.1061
120	.0017	.0028	.0048	.0098	.0192	.0280	.0432	.0696	.1107
130	.0024	.0031	.0047	.0090	.0174	.0254	.0396	.0652	.1069
140	.0009	.0019	.0036	.0074	.0148	.0217	.0341	.0570	.0934
150	.0018	.0022	.0033	.0061	.0117	.0172	.0270	.0456	.0774
160	.0004	.0009	.0019	.0040	.0080	.0118	.0187	.0317	.0545
170	.0007	.0008	.0012	.0022	.0041	.0061	.0096	.0163	.0281
180	0	0	0	0	0	0	0	0	0

TABLE 12.21

C_{qt}				L_r/L_c = .400			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ø°									
0	1.3464	1.1650	.9557	.7093	.5064	.4071	.3012	.1897	.0799
5	1.0085	.9055	.7738	.6008	.4439	.3624	.2723	.1741	.0743
10	.3621	.3933	.4016	.3692	.3057	.2620	.2063	.1378	.0610
15	-.0382	.0401	.1140	.1674	.1739	.1623	.1379	.0986	.0461
20	-.1457	-.0972	-.0339	.0362	.0747	.0824	.0798	.0633	.0320
25	-.2033	-.1771	-.1303	-.0609	-.0061	.0145	.0282	.0306	.0184
30	-.2532	-.2324	-.1946	-.1304	-.0691	-.0408	-.0157	.0015	.0058
35	-.2291	-.2239	-.2059	-.1618	-.1076	-.0781	-.0480	-.0216	-.0048
40	-.1925	-.1993	-.1975	-.1732	-.1305	-.1032	-.0718	-.0399	-.0137
45	-.2007	-.2018	-.1997	-.1823	-.1470	-.1219	-.0904	-.0549	-.0214
50	-.1972	-.1952	-.1928	-.1814	-.1540	-.1321	-.1025	-.0659	-.0274
55	-.1629	-.1671	-.1712	-.1691	-.1516	-.1342	-.1082	-.0727	-.0318
60	-.1519	-.1550	-.1586	-.1591	-.1473	-.1335	-.1107	-.0771	-.0349
65	-.1576	-.1546	-.1530	-.1511	-.1418	-.1304	-.1106	-.0791	-.0370
70	-.1360	-.1356	-.1359	-.1364	-.1312	-.1229	-.1067	-.0784	-.0377
75	-.1105	-.1139	-.1171	-.1204	-.1188	-.1131	-.1003	-.0757	-.0373
80	-.1100	-.1092	-.1089	-.1094	-.1080	-.1038	-.0934	-.0719	-.0362
85	-.1019	-.0994	-.0975	-.0968	-.0959	-.0929	-.0848	-.0666	-.0343
90	-.0739	-.0759	-.0777	-.0799	-.0812	-.0799	-.0744	-.0598	-.0315
100	-.0599	-.0576	-.0559	-.0553	-.0560	-.0559	-.0536	-.0448	-.0247
110	-.0178	-.0204	-.0229	-.0255	-.0283	-.0299	-.0305	-.0275	-.0162
120	-.0069	-.0051	-.0039	-.0037	-.0054	-.0071	-.0092	-.0104	-.0072
130	.0264	.0248	.0232	.0210	.0180	.0154	.0115	.0065	.0019
140	.0404	.0406	.0404	.0392	.0365	.0338	.0291	.0212	.0101
150	.0561	.0563	.0561	.0549	.0520	.0491	.0437	.0336	.0171
160	.0734	.0718	.0701	.0677	.0641	.0608	.0547	.0430	.0225
170	.0708	.0722	.0731	.0727	.0700	.0670	.0609	.0485	.0257
180	.0853	.0830	.0806	.0776	.0737	.0702	.0637	.0507	.0270

TABLE 12.22

C_{mt}			L_r/L_c = .400				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	0	0	0	0	0	0	0	0	0
5	-.0031	-.0038	-.0048	-.0065	-.0085	-.0098	-.0115	-.0138	-.0166
10	-.0041	-.0052	-.0070	-.0101	-.0139	-.0164	-.0197	-.0242	-.0297
15	-.0039	-.0053	-.0076	-.0117	-.0169	-.0204	-.0252	-.0316	-.0395
20	-.0034	-.0048	-.0073	-.0119	-.0182	-.0225	-.0284	-.0364	-.0464
25	-.0027	-.0041	-.0065	-.0113	-.0182	-.0230	-.0297	-.0389	-.0506
30	-.0022	-.0033	-.0056	-.0103	-.0172	-.0223	-.0295	-.0396	-.0525
35	-.0017	-.0027	-.0046	-.0089	-.0157	-.0208	-.0282	-.0387	-.0523
40	-.0013	-.0021	-.0037	-.0075	-.0138	-.0188	-.0260	-.0365	-.0504
45	-.0010	-.0016	-.0029	-.0061	-.0117	-.0163	-.0232	-.0334	-.0469
50	-.0007	-.0012	-.0021	-.0047	-.0096	-.0136	-.0199	-.0294	-.0423
55	-.0005	-.0008	-.0015	-.0035	-.0074	-.0109	-.0163	-.0249	-.0368
60	-.0003	-.0005	-.0009	-.0023	-.0053	-.0081	-.0127	-.0200	-.0305
65	-.0001	-.0002	-.0004	-.0012	-.0033	-.0054	-.0090	-.0150	-.0237
70	.0000	.0000	.0000	-.0003	-.0015	-.0028	-.0054	-.0099	-.0167
75	.0002	.0003	.0004	.0006	.0002	-.0004	-.0019	-.0048	-.0096
80	.0003	.0005	.0008	.0013	.0018	.0018	.0013	.0000	-.0027
85	.0004	.0007	.0011	.0020	.0031	.0037	.0043	.0045	.0040
90	.0006	.0008	.0013	.0025	.0043	.0055	.0070	.0087	.0102
100	.0007	.0011	.0017	.0034	.0061	.0082	.0112	.0156	.0210
110	.0008	.0012	.0020	.0039	.0071	.0099	.0141	.0204	.0290
120	.0008	.0013	.0021	.0040	.0076	.0106	.0154	.0230	.0336
130	.0008	.0012	.0020	.0039	.0074	.0104	.0153	.0233	.0347
140	.0007	.0011	.0018	.0035	.0066	.0094	.0140	.0215	.0325
150	.0006	.0009	.0014	.0028	.0054	.0077	.0115	.0179	.0272
160	.0004	.0006	.0010	.0020	.0038	.0054	.0082	.0128	.0196
170	.0002	.0003	.0005	.0010	.0020	.0028	.0042	.0066	.0102
180	0	0	0	0	0	0	0	0	0

TABLE 12.23

C_{st}			L_r/L_c = .400				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	-.0546	-.0626	-.0744	-.0938	-.1172	-.1325	-.1526	-.1792	-.2116
5	-.0206	-.0274	-.0380	-.0558	-.0780	-.0926	-.1120	-.1379	-.1695
10	-.0028	-.0071	-.0145	-.0284	-.0472	-.0600	-.0775	-.1012	-.1306
15	.0048	.0033	-.0005	-.0096	-.0237	-.0341	-.0487	-.0691	-.0951
20	.0073	.0077	.0070	.0026	-.0064	-.0138	-.0250	-.0414	-.0630
25	.0071	.0088	.0103	.0101	.0060	.0015	-.0059	-.0178	-.0344
30	.0060	.0083	.0112	.0142	.0144	.0129	.0091	.0019	-.0092
35	.0049	.0072	.0109	.0160	.0199	.0209	.0207	.0182	.0127
40	.0040	.0062	.0100	.0165	.0231	.0264	.0293	.0312	.0312
45	.0032	.0052	.0089	.0160	.0247	.0297	.0354	.0413	.0486
50	.0027	.0044	.0078	.0151	.0250	.0315	.0394	.0489	.0589
55	.0024	.0039	.0069	.0140	.0246	.0319	.0417	.0541	.0683
60	.0022	.0034	.0061	.0127	.0235	.0315	.0424	.0572	.0750
65	.0020	.0030	.0054	.0115	.0220	.0302	.0420	.0585	.0792
70	.0018	.0028	.0048	.0103	.0203	.0285	.0406	.0582	.0810
75	.0017	.0025	.0043	.0091	.0184	.0263	.0384	.0565	.0807
80	.0015	.0022	.0038	.0080	.0164	.0238	.0356	.0537	.0784
85	.0013	.0019	.0033	.0069	.0144	.0212	.0323	.0498	.0744
90	.0011	.0017	.0028	.0058	.0123	.0184	.0286	.0451	.0689
100	.0007	.0011	.0018	.0038	.0082	.0126	.0204	.0339	.0542
110	.0004	.0006	.0009	.0019	.0043	.0069	.0119	.0213	.0361
120	.0000	.0000	.0000	.0001	.0005	.0014	.0035	.0082	.0164
130	-.0003	-.0005	-.0008	-.0016	-.0028	-.0036	-.0043	-.0045	-.0034
140	-.0007	-.0010	-.0016	-.0031	-.0057	-.0079	-.0112	-.0159	-.0218
150	-.0009	-.0013	-.0022	-.0043	-.0081	-.0115	-.0169	-.0254	-.0375
160	-.0011	-.0016	-.0027	-.0052	-.0099	-.0141	-.0211	-.0326	-.0495
170	-.0012	-.0018	-.0029	-.0057	-.0110	-.0157	-.0237	-.0370	-.0570
180	-.0012	-.0018	-.0030	-.0059	-.0113	-.0163	-.0245	-.0385	-.0595

TABLE 12.24

C_{ft}				$L_r/L_c = .400$			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
φ°									
0	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000
5	-.2862	-.3103	-.3382	-.3715	-.3998	-.4141	-.4298	-.4469	-.4644
10	-.1334	-.1664	-.2075	-.2605	-.3084	-.3336	-.3619	-.3936	-.4265
15	-.0508	-.0784	-.1175	-.1744	-.2313	-.2629	-.2997	-.3421	-.3875
20	-.0096	-.0279	-.0585	-.1101	-.1680	-.2023	-.2438	-.2933	-.3479
25	.0100	-.0003	-.0214	-.0636	-.1173	-.1514	-.1944	-.2477	-.3083
30	.0141	.0105	-.0014	-.0322	-.0782	-.1098	-.1516	-.2056	-.2693
35	.0113	.0122	.0078	-.0120	-.0486	-.0763	-.1149	-.1672	-.2311
40	.0093	.0119	.0119	.0009	-.0264	-.0495	-.0836	-.1322	-.1940
45	.0074	.0103	.0128	.0085	-.0102	-.0284	-.0572	-.1007	-.1583
50	.0042	.0074	.0113	.0121	.0012	-.0121	-.0351	-.0724	-.1243
55	.0026	.0054	.0097	.0138	.0092	.0005	-.0168	-.0472	-.0921
60	.0029	.0048	.0087	.0145	.0149	.0102	-.0015	-.0249	-.0619
65	.0023	.0039	.0075	.0142	.0185	.0174	.0109	-.0053	-.0339
70	.0014	.0029	.0063	.0136	.0208	.0227	.0210	.0117	-.0081
75	.0019	.0031	.0060	.0132	.0224	.0267	.0291	.0264	.0154
80	.0025	.0034	.0059	.0128	.0233	.0295	.0355	.0389	.0364
85	.0018	.0029	.0054	.0122	.0236	.0313	.0404	.0492	.0550
90	.0018	.0029	.0053	.0118	.0237	.0325	.0441	.0577	.0710
100	.0024	.0034	.0055	.0114	.0232	.0333	.0482	.0694	.0955
110	.0022	.0032	.0053	.0108	.0221	.0323	.0489	.0747	.1099
120	.0019	.0029	.0050	.0100	.0203	.0302	.0468	.0745	.1147
130	.0021	.0030	.0047	.0091	.0181	.0269	.0425	.0695	.1107
140	.0012	.0021	.0038	.0076	.0152	.0227	.0362	.0605	.0987
150	.0006	.0021	.0032	.0061	.0120	.0178	.0285	.0483	.0801
160	.0006	.0011	.0020	.0041	.0082	.0122	.0196	.0336	.0564
170	.0006	.0008	.0011	.0022	.0042	.0062	.0100	.0172	.0291
180	0	0	0	0	0	0	0	0	0

TABLE 12.25

C_{qt}				$L_r/L_c = .400$			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
φ°									
0	1.0802	.9271	.7551	.5572	.3969	.3188	.2357	.1481	.0619
5	.8619	.7629	.6421	.4909	.3590	.2918	.2182	.1386	.0585
10	.4200	.4205	.3988	.3427	.2715	.2285	.1766	.1157	.0501
15	.0895	.1419	.1824	.1975	.1794	.1594	.1297	.0889	.0399
20	-.0676	-.0167	.0374	.0844	.0998	.0970	.0853	.0625	.0295
25	-.1646	-.1238	-.0702	-.0087	.0292	.0397	.0432	.0365	.0190
30	-.2271	-.1947	-.1465	-.0812	-.0302	-.0102	.0051	.0122	.0088
35	-.2291	-.2129	-.1809	-.1260	-.0734	-.0488	-.0261	-.0088	-.0004
40	-.2111	-.2086	-.1925	-.1523	-.1040	-.0781	-.0512	-.0266	-.0086
45	-.2098	-.2094	-.2000	-.1698	-.1267	-.1008	-.0716	-.0417	-.0156
50	-.2000	-.2006	-.1964	-.1759	-.1400	-.1159	-.0865	-.0535	-.0215
55	-.1735	-.1785	-.1811	-.1714	-.1445	-.1236	-.0959	-.0620	-.0281
60	-.1594	-.1640	-.1681	-.1644	-.1447	-.1270	-.1015	-.0679	-.0296
65	-.1547	-.1555	-.1576	-.1559	-.1415	-.1267	-.1039	-.0715	-.0320
70	-.1363	-.1378	-.1408	-.1424	-.1335	-.1221	-.1025	-.0724	-.0333
75	-.1160	-.1189	-.1231	-.1272	-.1230	-.1146	-.0985	-.0713	-.0336
80	-.1089	-.1092	-.1109	-.1141	-.1122	-.1061	-.0928	-.0687	-.0330
85	-.0980	-.0970	-.0974	-.1000	-.0998	-.0957	-.0853	-.0645	-.0316
90	-.0767	-.0780	-.0799	-.0837	-.0857	-.0836	-.0761	-.0588	-.0294
100	-.0562	-.0551	-.0546	-.0562	-.0589	-.0590	-.0558	-.0451	-.0235
110	-.0215	-.0230	-.0244	-.0270	-.0309	-.0329	-.0332	-.0287	-.0159
120	-.0038	-.0029	-.0025	-.0032	-.0063	-.0088	-.0114	-.0119	-.0075
130	.0244	.0235	.0224	.0206	.0171	.0140	.0096	.0047	.0010
140	.0411	.0410	.0407	.0395	.0363	.0331	.0278	.0195	.0089
150	.0568	.0568	.0565	.0552	.0521	.0488	.0427	.0320	.0156
160	.0715	.0705	.0693	.0674	.0640	.0605	.0539	.0414	.0208
170	.0735	.0742	.0743	.0734	.0705	.0672	.0605	.0471	.0240
180	.0823	.0809	.0793	.0771	.0735	.0700	.0631	.0492	.0251

TABLE 12.26

C_{mt}			L_r/L_c = 1.000				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ø°									
0	0	0	0	0	0	0	0	0	0
5	-.0045	-.0054	-.0066	-.0084	-.0106	-.0119	-.0136	-.0155	-.0175
10	-.0065	-.0080	-.0102	-.0138	-.0179	-.0205	-.0237	-.0275	-.0315
15	-.0069	-.0088	-.0118	-.0168	-.0226	-.0263	-.0308	-.0363	-.0421
20	-.0064	-.0086	-.0120	-.0179	-.0251	-.0296	-.0354	-.0423	-.0497
25	-.0055	-.0077	-.0113	-.0177	-.0258	-.0311	-.0377	-.0458	-.0544
30	-.0044	-.0065	-.0100	-.0166	-.0252	-.0309	-.0382	-.0471	-.0567
35	-.0034	-.0053	-.0085	-.0149	-.0236	-.0295	-.0371	-.0466	-.0568
40	-.0026	-.0041	-.0069	-.0129	-.0213	-.0271	-.0348	-.0444	-.0550
45	-.0018	-.0030	-.0054	-.0106	-.0185	-.0241	-.0315	-.0410	-.0514
50	-.0012	-.0020	-.0039	-.0084	-.0154	-.0205	-.0275	-.0366	-.0466
55	-.0007	-.0012	-.0026	-.0062	-.0122	-.0167	-.0231	-.0314	-.0407
60	-.0002	-.0006	-.0014	-.0041	-.0090	-.0128	-.0183	-.0256	-.0339
65	.0001	.0000	-.0004	-.0022	-.0058	-.0089	-.0134	-.0196	-.0266
70	.0004	.0005	.0004	-.0005	-.0029	-.0051	-.0085	-.0133	-.0190
75	.0006	.0009	.0011	.0011	-.0001	-.0015	-.0038	-.0071	-.0113
80	.0008	.0012	.0017	.0024	.0024	.0019	.0008	-.0011	-.0036
85	.0010	.0014	.0022	.0035	.0046	.0049	.0050	.0046	.0038
90	.0011	.0016	.0026	.0044	.0065	.0076	.0088	.0099	.0108
100	.0013	.0019	.0031	.0057	.0094	.0118	.0150	.0188	.0229
110	.0013	.0021	.0033	.0064	.0111	.0145	.0192	.0252	.0318
120	.0013	.0020	.0033	.0065	.0117	.0157	.0213	.0287	.0371
130	.0012	.0019	.0031	.0061	.0113	.0154	.0213	.0293	.0385
140	.0011	.0016	.0027	.0053	.0101	.0139	.0195	.0272	.0361
150	.0009	.0013	.0022	.0043	.0082	.0114	.0161	.0227	.0304
160	.0006	.0009	.0015	.0030	.0057	.0080	.0115	.0163	.0219
170	.0003	.0005	.0008	.0015	.0030	.0042	.0060	.0085	.0114
180	0	0	0	0	0	0	0	0	0

TABLE 12.27

C_{st}			L_r/L_c = 1.000				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ø°									
0	-.0713	-.0811	-.0951	-.1171	-.1421	-.1574	-.1763	-.1988	-.2222
5	-.0349	-.0438	-.0568	-.0777	-.1017	-.1166	-.1350	-.1569	-.1799
10	-.0117	-.0183	-.0287	-.0465	-.0678	-.0813	-.0983	-.1187	-.1403
15	.0018	-.0021	-.0091	-.0225	-.0400	-.0515	-.0662	-.0843	-.1036
20	.0087	.0075	.0040	-.0047	-.0176	-.0267	-.0387	-.0538	-.0701
25	.0115	.0123	.0119	.0080	-.0001	-.0064	-.0153	-.0269	-.0398
30	.0117	.0141	.0162	.0166	.0133	.0097	.0042	-.0037	-.0129
35	.0108	.0141	.0181	.0220	.0230	.0222	.0200	.0160	.0108
40	.0094	.0131	.0182	.0249	.0298	.0315	.0324	.0323	.0341
45	.0079	.0116	.0173	.0260	.0341	.0380	.0420	.0455	.0482
50	.0065	.0010	.0159	.0258	.0364	.0423	.0488	.0557	.0621
55	.0053	.0085	.0141	.0246	.0371	.0445	.0533	.0632	[illegible]
60	.0043	.0071	.0123	.0229	.0365	.0451	.0557	.0682	.0809
65	.0035	.0059	.0106	.0208	.0350	.0444	.0564	.0708	.0860
70	.0029	.0048	.0090	.0185	.0328	.0426	.0555	.0714	.0885
75	.0024	.0040	.0075	.0162	.0300	.0399	.0533	.0702	.0887
80	.0020	.0032	.0062	.0139	.0269	.0366	.0500	.0674	.0866
85	.0016	.0026	.0050	.0116	.0235	.0328	.0459	.0632	.0825
90	.0013	.0020	.0039	.0095	.0201	.0287	.0411	.0578	.0768
100	.0007	.0011	.0021	.0055	.0131	.0199	.0300	.0443	.0609
110	.0002	.0003	.0006	.0021	.0065	.0109	.0180	.0285	.0411
120	-.0003	-.0005	-.0007	-.0008	.0005	.0024	.0060	.0118	.0191
130	-.0007	-.0011	-.0018	-.0033	-.0048	-.0052	-.0053	-.0045	-.0030
140	-.0011	-.0016	-.0027	-.0053	-.0091	-.0118	-.0152	-.0194	-.0237
150	-.0014	-.0021	-.0035	-.0068	-.0126	-.0171	-.0234	-.0319	-.0415
160	-.0016	-.0024	-.0040	-.0079	-.0151	-.0209	-.0295	-.0413	-.0551
170	-.0017	-.0026	-.0043	-.0086	-.0166	-.0233	-.0332	-.0472	-.0636
180	-.0018	-.0027	-.0044	-.0088	-.0171	-.0241	-.0345	-.0492	-.0665

TABLE 12.28

C_{ft}				L_r/L_c = 1.000			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000	-.5000
5	-.3359	-.3556	-.3778	-.4037	-.4251	-.4357	-.4471	-.4589	-.4700
10	-.2032	-.2338	-.2700	-.3145	-.3528	-.3723	-.3935	-.4159	-.4371
15	-.1117	-.1433	-.1839	-.2374	-.2867	-.3125	-.3413	-.3723	-.4022
20	-.0517	-.0791	-.1174	-.1728	-.2275	-.2574	-.2915	-.3290	-.3656
25	-.0143	-.0352	-.0679	-.1199	-.1756	-.2074	-.2446	-.2865	-.3281
30	.0055	-.0082	-.0332	-.0782	-.1311	-.1630	-.2012	-.2454	-.2901
35	.0141	.0069	-.0099	-.0461	-.0937	-.1239	-.1615	-.2060	-.2521
40	.0174	.0151	.0052	-.0218	-.0625	-.0900	-.1255	-.1687	-.2144
45	.0173	.0183	.0142	-.0041	-.0369	-.0610	-.0931	-.1336	-.1774
50	.0148	.0182	.0186	.0084	-.0164	-.0364	-.0644	-.1010	-.1416
55	.0122	.0167	.0204	.0168	-.0002	-.0158	-.0391	-.0709	-.1071
60	.0103	.0150	.0205	.0222	.0125	.0012	-.0170	-.0433	-.0743
65	.0082	.0129	.0194	.0253	.0220	.0150	.0019	-.0184	-.0435
70	.0063	.0107	.0178	.0266	.0290	.0260	.0181	.0040	-.0148
75	.0053	.0092	.0161	.0269	.0340	.0345	.0316	.0237	.0116
80	.0047	.0080	.0145	.0264	.0372	.0410	.0427	.0409	.0355
85	.0039	.0068	.0129	.0254	.0391	.0457	.0516	.0555	.0568
90	.0034	.0059	.0115	.0241	.0399	.0489	.0585	.0678	.0753
100	.0032	.0050	.0095	.0211	.0392	.0515	.0670	.0854	.1041
110	.0028	.0044	.0079	.0181	.0364	.0504	.0696	.0943	.1216
120	.0025	.0038	.0067	.0153	.0324	.0466	.0674	.0957	.1281
130	.0024	.0035	.0058	.0127	.0277	.0410	.0614	.0902	.1244
140	.0018	.0027	.0047	.0101	.0225	.0341	.0524	.0791	.1114
150	.0016	.0023	.0037	.0077	.0171	.0263	.0412	.0634	.0908
160	.0009	.0015	.0025	.0051	.0115	.0178	.0283	.0442	.0640
170	.0006	.0008	.0013	.0026	.0058	.0090	.0144	.0227	.0331
180	0	0	0	0	0	0	0	0	0

TABLE 12.29

C_{qt}				L_r/L_c = 1.000			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ o									
0	.7669	.6485	.5185	.3726	.2574	.2025	.1453	.0872	.0343
5	.6531	.5647	.4622	.3404	.2393	.1897	.1371	.0829	.0327
10	.4076	.3797	.3345	.2650	.1959	.1587	.1170	.0719	.0288
15	.1882	.2039	.2051	.1831	.1462	.1221	.0927	.0585	.0239
20	.0401	.0732	.0993	.1094	.0982	.0859	.0678	.0443	.0185
25	-.0704	-.0308	.0093	.0421	.0521	.0501	.0428	.0296	.0129
30	-.1490	-.1092	-.0633	-.0162	.0099	.0165	.0187	.0152	.0073
35	-.1855	-.1536	-.1116	-.0609	-.0254	-.0124	-.0029	.0020	.0020
40	-.1975	-.1765	-.1427	-.0943	-.0543	-.0370	-.0218	-.0100	-.0029
45	-.2053	-.1910	-.1642	-.1199	-.0780	-.0579	-.0383	-.0208	-.0073
50	-.2016	-.1933	-.1740	-.1363	-.9560	-.0742	-.0518	-.0299	-.0112
55	-.1859	-.1841	-.1731	-.1441	-.1071	-.0858	-.0620	-.0371	-.0145
60	-.1724	-.1737	-.1683	-.1468	-.1143	-.0939	-.0696	-.0428	-.0171
65	-.1611	-.1631	-.1608	-.1454	-.1176	-.0985	-.0747	-.0469	-.0191
70	-.1437	-.1471	-.1482	-.1390	-.1166	-.0996	-.0770	-.0493	-.0203
75	-.1254	-.1298	-.1335	-.1295	-.1125	-.0977	-.0770	-.0502	-.0210
80	-.1127	-.1160	-.1198	-.1189	-.1063	-.0938	-.0751	-.0497	-.0211
85	-.0987	-.1011	-.1049	-.1064	-.0979	-.0877	-.0714	-.0480	-.0206
90	-.0808	-.0837	-.0882	-.0921	-.0874	-.0797	-.0659	-.0450	-.0196
100	-.0549	-.0558	-.0587	-.0637	-.0642	-.0605	-.0517	-.0364	-.0162
110	-.0243	-.0257	-.0285	-.0342	-.0383	-.0379	-.0341	-.0250	-.0115
120	-.0020	-.0021	-.0033	-.0076	-.0131	-.0151	-.0153	-.0123	-.0060
130	.0230	.0224	.0211	.0172	.0109	.0072	.0036	-.0009	-.0002
140	.0415	.0413	.0406	.0378	.0317	.0269	.0207	.0131	.0054
150	.0573	.0572	.0567	.0544	.0487	.0434	.0352	.0237	.0102
160	.0703	.0697	.0689	.0669	.0614	.0557	.0462	.0318	.0139
170	.0753	.0755	.0752	.0739	.0689	.0631	.0529	.0368	.0163
180	.0803	.0796	.0787	.0769	.0717	.0659	.0554	.0386	.0171

TABLE 12.30

C_{mm}			L_r/L_c = .200				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
5	.2439	.2708	.3026	.3411	.3737	.3900	.4075	.4262	.4448
10	.0833	.1159	.1581	.2145	.2665	.2937	.3241	.3574	.3915
15	.0195	.0409	.0745	.1278	.1841	.2160	.2532	.2958	.3411
20	-.0024	.0075	.0285	.0704	.1224	.1545	.1939	.2414	.2940
25	-.0119	-.0080	.0036	.0334	.0771	.1067	.1450	.1937	.2500
30	-.0098	-.0100	-.0059	.0121	.0455	.0707	.1056	.1524	.2094
35	-.0040	-.0067	-.0075	.0008	.0239	.0440	.0740	.1169	.1721
40	-.0037	-.0060	-.0083	-.0059	.0089	.0240	.0486	.0864	.1378
45	-.0042	-.0056	-.0081	-.0092	-.0012	.0094	.0285	.0604	.1065
50	-.0015	-.0032	-.0062	-.0098	-.0074	-.0010	.0128	.0384	.0781
55	-.0008	-.0023	-.0052	-.0100	-.0114	-.0084	.0004	.0197	.0525
60	-.0027	-.0034	-.0055	-.0102	-.0142	-.0139	-.0093	.0040	.0296
65	-.0022	-.0030	-.0049	-.0098	-.0156	-.0175	-.0166	-.0091	.0091
70	-.0007	-.0019	-.0041	-.0092	-.0162	-.0199	-.0221	-.0198	-.0088
75	-.0017	-.0026	-.0044	-.0091	-.0167	-.0215	-.0263	-.0286	-.0245
80	-.0026	-.0032	-.0046	-.0090	-.0168	-.0225	-.0293	-.0356	-.0380
85	-.0013	-.0022	-.0040	-.0084	-.0165	-.0228	-.0312	-.0410	-.0494
90	-.0010	-.0020	-.0038	-.0080	-.0161	-.0228	-.0324	-.0450	-.0587
100	-.0019	-.0025	-.0039	-.0077	-.0152	-.0221	-.0330	-.0497	-.0718
110	-.0014	-.0021	-.0035	-.0070	-.0140	-.0205	-.0317	-.0504	-.0781
120	-.0009	-.0016	-.0030	-.0061	-.0124	-.0184	-.0291	-.0481	-.0783
130	-.0016	-.0020	-.0029	-.0055	-.0107	-.0160	-.0256	-.0435	-.0733
140	-.0002	-.0009	-.0019	-.0043	-.0087	-.0131	-.0212	-.0369	-.0640
150	-.0013	-.0014	-.0019	-.0035	-.0068	-.0101	-.0164	-.0289	-.0511
160	.0000	-.0004	-.0009	-.0022	-.0045	-.0068	-.0111	-.0198	-.0356
170	-.0005	-.0005	-.0007	-.0012	-.0023	-.0035	-.0056	-.0101	-.0182
180	0	0	0	0	0	0	0	0	0

TABLE 12.31

C_{sm}			L_r/L_c = .200				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	-3.1704	-2.8075	-2.3889	-1.8961	-1.4903	-1.2916	-1.0799	-.8569	-.6374
5	-2.4933	-2.2873	-2.0239	-1.6779	-1.3640	-1.2011	-1.0209	-.8245	-.6249
10	-1.1968	-1.2592	-1.2758	-1.2111	-1.0840	-.9967	-.8853	-.7483	-.5946
15	-.3902	-.5468	-.6947	-.8013	-.8144	-.7911	-.7425	-.6638	-.5588
20	-.1669	-.2639	-.3905	-.5306	-.6077	-.6230	-.6179	-.5848	-.5222
25	-.0411	-.0934	-.1871	-.3259	-.4355	-.4766	-.5041	-.5088	-.4844
30	.0716	.0299	-.0457	-.1740	-.2967	-.3532	-.4035	-.4379	-.4464
35	.0383	.0279	-.0081	-.0964	-.2048	-.2636	-.3239	-.3768	-.4103
40	-.0180	-.0044	-.0080	-.0566	-.1421	-.1967	-.2594	-.3232	-.3755
45	.0172	.0194	.0150	-.0196	-.0902	-.1405	-.2034	-.2744	-.3414
50	.0306	.0266	.0218	-.0010	-.0557	-.0995	-.1588	-.2320	-.3089
55	-.0160	-.0076	.0006	-.0035	-.0386	-.0733	-.1254	-.1963	-.2782
60	-.0146	-.0082	-.0011	-.0002	-.0237	-.0514	-.0969	-.1642	-.2484
65	.0216	.0156	.0123	.0085	-.0102	-.0328	-.0725	-.1354	-.2198
70	.0040	.0032	.0038	.0048	-.0056	-.0222	-.0548	-.1112	-.1927
75	-.0206	-.0138	-.0073	-.0008	-.0041	-.0153	-.0409	-.0902	-.1669
80	.0055	.0040	.0034	.0045	.0016	-.0069	-.0277	-.0707	-.1420
85	.0169	.0119	.0081	.0067	.0048	-.0011	-.0173	-.0537	-.1184
90	-.0113	-.0073	-.0037	.0007	.0032	.0007	-.0103	-.0395	-.0961
100	.0159	.0114	.0080	.0067	.0082	.0080	.0033	-.0142	-.0546
110	-.0147	-.0094	-.0045	.0007	.0064	.0094	.0108	.0047	-.0180
120	.0139	.0103	.0078	.0074	.0107	.0141	.0185	.0208	.0144
130	-.0074	-.0042	-.0009	.0034	.0095	.0147	.0224	.0325	.0417
140	.0038	.0035	.0040	.0063	.0117	.0171	.0265	.0422	.0645
150	.0044	.0040	.0044	.0068	.0125	.0184	.0292	.0493	.0823
160	-.0069	-.0037	-.0003	.0046	.0118	.0185	.0306	.0540	.0950
170	.0128	.0099	.0081	.0090	.0143	.0204	.0325	.0573	.1028
180	-.0114	-.0067	-.0020	.0039	.0119	.0189	.0319	.0577	.1052

TABLE 12.32

C_{fm}				L_r/L_c = .200			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	0	0	0	0	0	0	0	0	0
5	-13.8275	-10.7039	-7.5756	-4.5751	-2.6709	-1.9239	-1.2578	-.6955	-.2691
10	-13.4182	-11.0342	-8.3587	-5.4556	-3.3895	-2.5165	-1.7000	-.9755	-.3975
15	-5.0301	-5.1699	-4.8030	-3.8036	-2.7011	-2.1287	-1.5289	-.9369	-.4161
20	-1.2709	-2.0970	-2.6162	-2.5961	-2.1198	-1.7732	-1.3523	-.8830	-.4253
25	-1.6945	-1.8835	-2.0901	-2.1072	-1.8211	-1.5756	-1.2492	-.8539	-.4387
30	-.5018	-.7044	-1.0167	-1.3074	-1.3210	-1.2216	-1.0354	-.7591	-.4259
35	.9398	.5070	-.0102	-.5600	-.8293	-.8600	-.8049	-.6476	-.4038
40	.0709	.0269	-.1329	-.4314	-.6506	-.7012	-.6886	-.5867	-.3951
45	-.5668	-.3693	-.2806	-.3669	-.5163	-.5709	-.5851	-.5276	-.3839
50	.3487	.2645	.1546	-.0503	-.2714	-.3686	-.4368	-.4426	-.3613
55	.4319	.3185	.2072	.0377	-.1590	-.2586	-.3441	-.3834	-.3451
60	-.4221	-.2741	-.1525	-.1092	-.1811	-.2434	-.3098	-.3523	-.3361
65	-.1745	-.1127	-.0568	-.0413	-.1087	-.1697	-.2425	-.3042	-.3197
70	.4473	.3093	.1982	.0979	-.0110	-.0837	-.1702	-.2537	-.3010
75	-.0336	-.0216	-.0080	-.0030	-.0427	-.0884	-.1550	-.2321	-.2906
80	-.3882	-.2663	-.1613	-.0803	-.0688	-.0927	-.1421	-.2123	-.2796
85	.1905	.1292	.0794	.0414	.0012	-.0358	-.0946	-.1759	-.2621
90	.2702	.1844	.1126	.0575	.0149	-.0188	-.0732	-.1534	-.2483
100	-.1285	-.0891	-.0561	-.0301	-.0236	-.0341	-.0650	-.1284	-.2249
110	-.0190	-.0134	-.0094	-.0070	-.0097	-.0188	-.0437	-.0999	-.1978
120	.1433	.0974	.0585	.0272	.0086	-.0030	-.0257	-.0756	-.1700
130	-.2329	-.1599	-.0990	-.0526	-.0309	-.0272	-.0340	-.0667	-.1455
140	.2719	.1858	.1131	.0556	.0244	.0114	-.0061	-.0412	-.1141
150	-.2619	-.1797	-.1108	-.0578	-.0320	-.0250	-.0242	-.0398	-.0895
160	.2038	.1394	.0851	.0423	.0194	.0106	.0005	-.0178	-.0573
170	-.1116	-.0766	-.0472	-.0245	-.0134	-.0101	-.0090	-.0134	-.0303
180	0	0	0	0	0	0	0	0	0

TABLE 12.33

C_{qm}				L_r/L_c = .200			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	-97.7290	-74.4364	-51.6332	-30.3642	-17.2652	-12.2413	-7.8356	-4.1889	-1.4893
5	-41.7926	-34.0444	-25.4518	-16.2837	-9.8887	-7.2309	-4.7779	-2.6374	-.9661
10	40.2106	26.5155	14.7833	5.9408	1.9823	.9005	.2249	-.0799	-.0984
15	42.0065	31.0848	20.2264	10.4161	4.9552	3.1130	1.6913	.7197	.1872
20	2.8684	5.1413	5.4306	3.9087	2.2096	1.4673	.8248	.3489	.0828
25	-.5785	2.1281	3.5464	3.3068	2.2199	1.6092	1.0069	.4945	.1494
30	11.7559	9.6871	7.7161	5.3971	3.4144	2.4917	1.6068	.8383	.2800
35	1.2469	1.7724	2.3110	2.4197	1.9284	1.5390	1.0773	.6052	.2152
40	-7.9488	-5.0669	-2.3947	-.2853	.4982	.5901	.5280	.3514	.1408
45	2.0650	1.4856	1.2581	1.2770	1.1823	1.0331	.7994	.4997	.1979
50	5.0499	3.4313	2.2455	1.5479	1.2076	1.0236	.7873	.4990	.2028
55	-4.3005	-3.0045	-1.8179	-.6911	-.0276	.1653	.2573	.2332	.1173
60	-2.9426	-2.0933	-1.3235	-.5619	-.0555	.1126	.2073	.2037	.1088
65	4.5897	3.0818	1.8342	.9697	.6422	.5471	.4486	.3173	.1466
70	.4406	.2655	.1117	.0524	.1224	.1709	.2022	.1849	.1012
75	-4.4378	-3.0739	-1.9354	-1.0138	-.4589	-.2426	-.0658	.0406	.0508
80	1.2554	.8245	.4588	.1911	.1172	.1218	.1356	.1309	.0780
85	3.2939	2.2316	1.3312	.6320	.3181	.2394	.1905	.1479	.0802
90	-2.6982	-1.8694	-1.1800	-.6454	-.3409	-.2145	-.0965	-.0060	.0251
100	3.1751	2.1480	1.2849	.6090	.2751	.1787	.1178	.0826	.0464
110	-3.3286	-2.3047	-1.4447	-.7765	-.4300	-.3035	-.1879	-.0864	-.0184
120	2.6050	1.7598	1.0489	.4929	.2079	.1170	.0543	.0220	.0098
130	-1.7893	-1.2520	-.7998	-.4472	-.2669	-.2037	-.1458	-.0883	-.0341
140	.4320	.2718	.1362	.0294	-.0266	-.0447	-.0547	-.0504	-.0276
150	.6222	.3999	.2137	.0684	-.0071	-.0325	-.0504	-.0537	-.0335
160	-1.8151	-1.2679	-.8087	-.4512	-.2690	-.2081	-.1577	-.1113	-.0562
170	2.3464	1.5815	.9388	.4373	.1790	.0912	.0214	-.0237	-.0292
180	-2.7488	-1.9078	-1.2014	-.6510	-.3698	-.2758	-.1996	-.1351	-.0671

TABLE 12.34

C_{mm}			L_T/L_c = .400				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ°									
0	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
5	.2831	.3065	.3334	.3651	.3914	.4043	.4183	.4331	.4478
10	.1294	.1612	.2005	.2504	.2946	.3172	.3422	.3694	.3968
15	.0469	.0731	.1099	.1627	.2144	.2425	.2745	.3105	.3480
20	.0063	.0231	.0512	.0982	.1498	.1798	.2154	.2569	.3015
25	-.0127	-.0038	.0149	.0522	.0991	.1284	.1647	.2088	.2577
30	-.0162	-.0138	-.0042	.0219	.0609	.0875	.1220	.1660	.2168
35	-.0130	-.0149	-.0124	.0031	.0329	.0555	.0867	.1285	.1788
40	-.0107	-.0140	-.0156	-.0084	.0125	.0307	.0576	.0957	.1436
45	-.0084	-.0119	-.0156	-.0145	-.0015	.0121	.0340	.0673	.1114
50	-.0050	-.0085	-.0134	-.0168	-.0108	-.0016	.0152	.0430	.0820
55	-.0032	-.0062	-.0112	-.0172	-.0166	-.0114	.0004	.0223	.0554
60	-.0032	-.0053	-.0096	-.0167	-.0202	-.0183	-.0111	.0049	.0315
65	-.0024	-.0041	-.0079	-.0155	-.0218	-.0228	-.0199	-.0096	.0101
70	-.0013	-.0029	-.0063	-.0139	-.0223	-.0255	-.0263	-.0216	-.0087
75	-.0017	-.0028	-.0056	-.0126	-.0221	-.0271	-.0310	-.0312	-.0250
80	-.0021	-.0029	-.0051	-.0115	-.0215	-.0277	-.0342	-.0339	-.0391
85	-.0014	-.0023	-.0043	-.0102	-.0205	-.0276	-.0361	-.0448	-.0510
90	-.0012	-.0021	-.0040	-.0093	-.0194	-.0270	-.0371	-.0491	-.0608
100	-.0016	-.0023	-.0037	-.0080	-.0172	-.0251	-.0370	-.0538	-.0744
110	-.0014	-.0020	-.0033	-.0069	-.0149	-.0225	-.0349	-.0543	-.0809
120	-.0010	-.0017	-.0029	-.0060	-.0128	-.0196	-.0314	-.0515	-.0812
130	-.0013	-.0018	-.0027	-.0052	-.0108	-.0165	-.0272	-.0463	-.0760
140	-.0005	-.0011	-.0020	-.0042	-.0087	-.0133	-.0223	-.0391	-.0663
150	-.0010	-.0012	-.0018	-.0033	-.0066	-.0101	-.0170	-.0305	-.0529
160	-.0002	-.0005	-.0010	-.0022	-.0044	-.0068	-.0115	-.0208	-.0368
170	-.0004	-.0004	-.0006	-.0011	-.0022	-.0034	-.0058	-.0106	-.0189
180	0	0	0	0	0	0	0	0	0

TABLE 12.35

C_{sm}			L_T/L_c = .400				X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ°									
0	-2.6379	-2.3318	-1.9876	-1.5919	-1.2712	-1.1151	-.9488	-.7736	-.6013
5	-2.2001	-2.0020	-1.7604	-1.4581	-1.1942	-1.0598	-.9126	-.7534	-.5933
10	-1.3126	-1.3136	-1.2703	-1.1581	-1.0157	-.9295	-.8259	-.7040	-.5728
15	-.6457	-.7503	-.8313	-.8616	-.8254	-.7854	-.7260	-.6444	-.5465
20	-.3231	-.4248	-.5332	-.6270	-.6578	-.6523	-.6289	-.5832	-.5173
25	-.1184	-.2000	-.3072	-.4303	-.5060	-.5271	-.5340	-.5206	-.4856
30	.0193	-.0455	-.1418	-.2725	-.3744	-.4145	-.4450	-.4592	-.4524
35	.0383	.0060	-.0581	-.1679	-.2731	-.3223	-.3677	-.4024	-.4192
40	.0192	.0142	-.0179	-.0985	-.1949	-.2468	-.3006	-.3499	-.3861
45	.0354	.0345	.0158	-.0447	-.1308	-.1826	-.2410	-.3009	-.3530
50	.0361	.0374	.0290	-.0120	-.0838	-.1320	-.1908	-.2568	-.3207
55	.0052	.0153	.0205	.0010	-.0527	-.0946	-.1500	-.2178	-.2895
60	.0005	.0097	.0180	.0104	-.0290	-.0644	-.1153	-.1825	-.2591
65	.0158	.0173	.0215	.0181	-.0108	-.0403	-.0860	-.1508	-.2296
70	.0045	.0075	.0135	.0168	-.0010	-.0239	-.0630	-.1232	-.2014
75	-.0096	-.0037	.0046	.0129	.0045	-.0123	-.0446	-.0989	-.1744
80	.0034	.0039	.0074	.0139	.0099	-.0023	-.0288	-.0771	-.1484
85	.0091	.0072	.0080	.0131	.0127	.0045	-.0163	-.0579	-.1237
90	-.0057	-.0032	.0007	.0082	.0122	.0080	-.0071	-.0416	-.1003
100	.0085	.0062	.0053	.0084	.0138	.0142	.0077	-.0137	-.0569
110	-.0073	-.0044	-.0014	.0038	.0116	.0154	.0161	.0072	-.0186
120	.0077	.0059	.0049	.0064	.0126	.0176	.0227	.0239	.0150
130	-.0034	-.0015	.0006	.0042	.0113	.0175	.0262	.0361	.0435
140	.0025	.0026	.0033	.0058	.0120	.0185	.0292	.0457	.0670
150	.0028	.0029	.0036	.0062	.0124	.0189	.0311	.0526	.0853
160	-.0030	-.0010	.0014	.0053	.0121	.0189	.0321	.0572	.0984
170	.0073	.0060	.0057	.0076	.0133	.0199	.0332	.0601	.1064
180	-.0054	-.0026	.0005	.0050	.0121	.0191	.0330	.0608	.1089

TABLE 12.36

C_{fm}				L_r/L_c = .400				X = 0	
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ø°									
0	0	0	0	0	0	0	0	0	0
5	-9.0612	-6.8751	-4.7739	-2.8376	-1.6469	-1.1871	-.7799	-.4370	-.1762
10	-9.8022	-7.8404	-5.7738	-3.6703	-2.2499	-1.6663	-1.1281	-.6548	-.2783
15	-5.3140	-4.8966	-4.1404	-3.0243	-2.0515	-1.5926	-1.1332	-.6955	-.3195
20	-2.6485	-2.9189	-2.8804	-2.4261	-1.8161	-1.4739	-1.0985	-.7097	-.3495
25	-2.1029	-2.2682	-2.3070	-2.0737	-1.6518	-1.3838	-1.0684	-.7191	-.3752
30	-.8884	-1.1654	-1.4174	-1.5025	-1.3381	-1.1780	-.9571	-.6809	-.3822
35	.2531	-.1602	-.5935	-.9386	-1.0014	-.9441	-.8189	-.6230	-.3796
40	.0117	-.1559	-.4110	-.6975	-.8119	-.7992	-.7263	-.5822	-.3799
45	-.2367	-.2195	-.3144	-.5146	-.6462	-.6645	-.6337	-.5365	-.3758
50	.2599	.1735	.0174	-.2354	-.4334	-.4952	-.5161	-.4743	-.3636
55	.2962	.2245	.1075	-.1022	-.2996	-.3777	-.4269	-.4233	-.3527
60	-.1686	-.0826	-.0458	-.1148	-.2474	-.3154	-.3696	-.3857	-.3441
65	-.0594	-.0118	.0125	-.0405	-.1609	-.2321	-.2993	-.3400	-.3307
70	.2523	.1928	.1413	.0539	-.0736	-.1507	-.2308	-.2940	-.3154
75	-.0068	.0112	.0298	.0154	-.0617	-.1226	-.1953	-.2639	-.3037
80	-.1984	-.1243	-.0566	-.0189	-.0552	-.1011	-.1652	-.2362	-.2913
85	.1005	.0734	.0585	.0429	-.0069	-.0545	-.1216	-.2019	-.2755
90	.1416	.0985	.0694	.0488	.0082	-.0327	-.0947	-.1763	-.2613
100	-.0681	-.0462	-.0248	-.0026	-.0044	-.0243	-.0673	-.1396	-.2348
110	-.0101	-.0075	-.0044	.0028	.0027	-.0097	-.0420	-.1067	-.2062
120	.0742	.0494	.0290	.0159	.0097	.0004	-.0244	-.0803	-.1771
130	-.1218	-.0829	-.0514	-.0266	-.0126	-.0118	-.0238	-.0648	-.1497
140	.1417	.0954	.0569	.0273	.0134	.0069	-.0070	-.0434	-.1187
150	-.1370	-.0929	-.0571	-.0302	-.0160	-.0120	-.0146	-.0358	-.0911
160	.1064	.0717	.0431	.0206	.0095	.0055	-.0012	-.0190	-.0596
170	-.0584	-.0396	-.0243	-.0128	-.0070	-.0051	-.0052	-.0116	-.0306
180	0	0	0	0	0	0	0	0	0

TABLE 12.37

C_{qm}				L_r/L_c = .400				X = 0	
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ø°									
0	-62.1528	-46.3842	-31.5480	-18.2282	-10.2682	-7.2619	-4.6437	-2.4849	-.8858
5	-30.0038	-23.7304	-17.2080	-10.6842	-6.3692	-4.6271	-3.0428	-1.6754	-.6134
10	19.1109	11.5482	5.5868	1.5718	.0654	-.2490	-.3649	-.3128	-.1526
15	25.0324	17.4571	10.5683	4.9268	2.0886	1.2067	.5730	.1873	.0233
20	6.1706	5.6958	4.3340	2.4420	1.1474	.6816	.3233	.0958	.0031
25	3.6888	4.2100	3.8175	2.5915	1.4597	.9794	.5626	.2483	.0647
30	8.9015	7.5251	5.8716	3.8263	2.2428	1.5726	.9705	.4823	.1530
35	2.2920	2.6892	2.7776	2.3028	1.5709	1.1743	.7718	.4086	.1379
40	-3.4110	-1.5650	-.0631	.7956	.8480	.7225	.5297	.3081	.1128
45	1.2677	1.2710	1.3908	1.3967	1.1408	.9323	.6746	.3963	.1492
50	2.6000	1.9523	1.5795	1.3423	1.0762	.8906	.6592	.3995	.1559
55	-2.3734	-1.5494	-.7206	.0362	.3662	.4083	.3715	.2621	.1143
60	-1.7017	-1.1773	-.6211	-.0429	.2650	.3257	.3180	.2380	.1092
65	2.2585	1.4671	.9048	.6225	.5377	.4889	.4077	.2824	.1256
70	.1360	.0268	-.0130	.0807	.2086	.2485	.2520	.2015	.0992
75	-2.3966	-1.6838	-1.0745	-.5123	-.1398	-.0050	.0867	.1136	.0694
80	.5871	.3399	.1448	.0634	.1067	.1396	.1596	.1435	.0781
85	1.6735	1.0854	.5987	.2682	.1751	.1670	.1620	.1372	.0739
90	-1.4478	-1.0169	-.6725	-.3869	-.1809	-.0869	-.0041	.0460	.0409
100	1.6180	1.0633	.5998	.2455	.1013	.0756	.0693	.0658	.0411
110	-1.7763	-1.2274	-.7822	-.4489	-.2648	-.1861	-.1076	-.0388	-.0010
120	1.3239	.8695	.4944	.2001	.0544	.0162	-.0007	.0004	.0046
130	-.9743	-.6848	-.4476	-.2709	-.1835	-.1486	-.1101	-.0663	-.0246
140	.1891	.1023	.0308	-.0268	-.0600	-.0691	-.0693	-.0553	-.0271
150	.2857	.1670	.0697	-.0059	-.0481	-.0624	-.0697	-.0625	-.0346
160	-.9856	-.6922	-.4515	-.2681	-.1784	-.1498	-.1248	-.0950	-.0491
170	1.1873	.7766	.4396	.1815	.0481	.0014	-.0353	-.0530	-.0374
180	-1.4738	-1.0224	-.6521	-.3690	-.2280	-.1829	-.1465	-.1096	-.0571

TABLE 12.38

	C_{mm}			L_r/L_c = 1.000			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
5	.3314	.3502	.3713	.3953	.4145	.4238	.4335	.4434	.4525
10	.1968	.2258	.2598	.3007	.3349	.3518	.3698	.3864	.4057
15	.1049	.1345	.1721	.2207	.2640	.2863	.3105	.3360	.3601
20	.0454	.0705	.1054	.1549	.2024	.2277	.2561	.2867	.3160
25	.0089	.0276	.0566	.1022	.1497	.1764	.2069	.2407	.2737
30	-.0099	.0017	.0231	.0616	.1059	.1321	.1631	.1983	.2334
35	-.0175	-.0122	.0014	.0312	.0701	.0945	.1244	.1595	.1953
40	-.0199	-.0191	-.0121	.0090	.0412	.0629	.0907	.1243	.1595
45	-.0191	-.0213	-.0196	-.0066	.0184	.0369	.0616	.0926	.1260
50	-.0160	-.0202	-.0225	-.0167	.0010	.0159	.0369	.0644	.0950
55	-.0129	-.0180	-.0229	-.0229	-.0120	-.0009	.0160	.0395	.0664
60	-.0105	-.0156	-.0219	-.0263	-.0214	-.0140	-.0013	.0177	.0404
65	-.0081	-.0129	-.0199	-.0274	-.0278	-.0239	-.0153	-.0012	.0168
70	-.0059	-.0102	-.0173	-.0271	-.0319	-.0310	-.0266	-.0173	-.0042
75	-.0048	-.0083	-.0150	-.0259	-.0341	-.0360	-.0353	-.0308	-.0229
80	-.0040	-.0068	-.0128	-.0241	-.0349	-.0392	-.0419	-.0420	-.0391
85	-.0029	-.0053	-.0107	-.0219	-.0345	-.0408	-.0466	-.0509	-.0530
90	-.0023	-.0043	-.0089	-.0196	-.0334	-.0413	-.0497	-.0579	-.0645
100	-.0019	-.0031	-.0064	-.0154	-.0298	-.0397	-.0520	-.0665	-.0812
110	-.0015	-.0024	-.0046	-.0118	-.0253	-.0359	-.0504	-.0692	-.0898
120	-.0012	-.0018	-.0034	-.0088	-.0207	-.0309	-.0461	-.0670	-.0910
130	-.0012	-.0016	-.0027	-.0066	-.0164	-.0256	-.0400	-.0609	-.0859
140	-.0007	-.0011	-.0020	-.0048	-.0124	-.0202	-.0329	-.0519	-.0753
150	-.0008	-.0010	-.0016	-.0034	-.0089	-.0149	-.0251	-.0407	-.0604
160	-.0003	-.0006	-.0010	-.0021	-.0057	-.0098	-.0168	-.0280	-.0421
170	-.0003	-.0004	-.0005	-.0011	-.0028	-.0048	-.0085	-.0142	-.0216
180	0	0	0	0	0	0	0	0	0

TABLE 12.39

	C_{sm}			L_r/L_c = 1.000			X = 0		
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ^o									
0	-2.0112	-1.7745	-1.5144	-1.2227	-.9924	-.8826	-.7680	-.6519	-.5460
5	-1.7824	-1.6057	-1.4006	-1.1571	-.9549	-.8557	-.7504	-.6420	-.5417
10	-1.2879	-1.2320	-1.1416	-1.0027	-.8645	-.7900	-.7065	-.6165	-.5302
15	-.8430	-.8744	-.8768	-.8328	-.7589	-.7109	-.6520	-.5835	-.5143
20	-.5385	-.6047	-.6569	-.6771	-.6547	-.6300	-.5940	-.5468	-.4953
25	-.3068	-.3860	-.4662	-.5318	-.5518	-.5479	-.5332	-.5069	-.4735
30	-.1368	-.2164	-.3083	-.4023	-.4545	-.4680	-.4722	-.4653	-.4495
35	-.0499	-.1127	-.1966	-.2981	-.3691	-.3951	-.4142	-.4238	-.4240
40	-.0079	-.0500	-.1176	-.2143	-.2945	-.3289	-.3594	-.3829	-.3973
45	.0264	-.0023	-.0559	-.1444	-.2283	-.2684	-.3077	-.3427	-.3696
50	.0394	.0228	-.0158	-.0911	-.1726	-.2153	-.2602	-.3041	-.3413
55	.0301	.0265	.0044	-.0535	-.1275	-.1701	-.2177	-.2675	-.3128
60	.0264	.0291	.0183	-.0247	-.0897	-.1306	-.1790	-.2327	-.2842
65	.0286	.0325	.0280	-.0028	-.0585	-.0966	-.1443	-.1998	-.2556
70	.0193	.0262	.0284	.0100	-.0348	-.0689	-.1140	-.1694	-.2273
75	.0092	.0181	.0254	.0175	-.0166	-.0461	-.0876	-.1412	-.1996
80	.0109	.0175	.0251	.0234	-.0019	-.0268	-.0642	-.1149	-.1723
85	.0105	.0153	.0229	.0260	.0088	-.0114	-.0441	-.0909	-.1457
90	.0024	.0082	.0172	.0250	.0157	.0002	-.0273	-.0691	-.1201
100	.0058	.0077	.0136	.0236	.0245	.0170	-.0004	-.0310	-.0715
110	-.0017	.0011	.0067	.0181	.0263	.0256	.0179	-.0002	-.0273
120	.0040	.0041	.0065	.0152	.0262	.0301	.0305	.0246	.0120
130	-.0006	.0006	.0032	.0111	.0236	.0310	.0383	.0438	.0458
140	.0018	.0021	.0035	.0092	.0214	.0308	.0433	.0585	.0740
150	.0019	.0022	.0032	.0077	.0192	.0298	.0461	.0692	.0962
160	-.0006	.0006	.0021	.0063	.0172	.0286	.0476	.0764	.1121
170	.0037	.0034	.0038	.0066	.0166	.0281	.0484	.0807	.1218
180	-.0015	.0000	.0017	.0054	.0157	.0275	.0484	.0820	.1250

TABLE 12.40

C_{fm}				L_r/L_c = 1.000				X = 0	
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ°									
0	0	0	0	0	0	0	0	0	0
5	-4.8099	-3.5690	-2.4230	-1.4096	-.8087	-.5818	-.3834	-.2186	-.0957
10	-5.8355	-4.5150	-3.2111	-1.9708	-1.1825	-.8704	-.5890	-.3474	-.1611
15	-4.2130	-3.5595	-2.7701	-1.8701	-1.2076	-.9213	-.6487	-.4016	-.2009
20	-2.9511	-2.7305	-2.3228	-1.7169	-1.1863	-.9349	-.6819	-.4402	-.2345
25	-2.3889	-2.2761	-2.0355	-1.5994	-1.1629	-.9400	-.7056	-.4715	-.2645
30	-1.4619	-1.5620	-1.5465	-1.3444	-1.0521	-.8801	-.6854	-.4781	-.2849
35	-.6217	-.8746	-1.0471	-1.0573	-.9092	-.7925	-.6441	-.4713	-.2992
40	-.4247	-.6219	-.7992	-.8787	-.8078	-.7270	-.6116	-.4660	-.3123
45	-.3165	-.4441	-.5978	-.7140	-.7035	-.6547	-.5712	-.4539	-.3215
50	.0238	-.1310	-.3234	-.5094	-.5725	-.5607	-.5143	-.4310	-.3253
55	.1192	-.0029	-.1729	-.3687	-.4692	-.4822	-.4637	-.4087	-.3275
60	-.0314	-.0587	-.1473	-.2955	-.3979	-.4227	-.4221	-.3888	-.3286
65	.0276	.0103	-.0601	-.1983	-.3143	-.3536	-.3728	-.3630	-.3261
70	.1560	.1139	.0358	-.1045	-.2335	-.2856	-.3226	-.3350	-.3210
75	.0432	.0473	.0188	-.0752	-.1873	-.2403	-.2851	-.3119	-.3158
80	-.0448	-.0086	.0001	-.0545	-.1482	-.1999	-.2499	-.2886	-.3089
85	.0686	.0670	.0544	-.0039	-.0973	-.1523	-.2100	-.2620	-.2993
90	.0779	.0709	.0610	.0168	-.0647	-.1176	-.1778	-.2385	-.8891
100	-.0191	-.0005	.0182	.0160	-.0303	-.0719	-.1285	-.1971	-.2665
110	-.0005	.0064	.0180	.0233	-.0033	-.0358	-.0871	-.1584	-.2395
120	.0317	.0243	.0237	.0275	.0125	-.0117	-.0558	-.1245	-.2097
130	-.0495	-.0321	-.0142	.0060	.0088	-.0047	-.0378	-.0976	-.1784
140	.0583	.0393	.0262	.0228	.0198	.0086	-.0189	-.0709	-.1440
150	-.0561	-.0377	-.0217	-.0045	.0058	.0026	-.0141	-.0522	-.1098
160	.0437	.0292	.0179	.0127	.0125	.0085	-.0042	-.0317	-.0732
170	-.0239	-.0161	-.0097	-.0031	.0016	.0015	-.0034	-.0163	-.0371
180	0	0	0	0	0	0	0	0	0

TABLE 12.41

C_{qm}				L_r/L_c = 1.000				X = 0	
γ	.02	.03	.05	.10	.20	.30	.50	1.00	3.00
ϕ°									
0	-31.7536	-23.1673	-15.3880	-8.6690	-4.7950	-3.3572	-2.1254	-1.1238	-.3945
5	-17.3813	-13.2664	-9.2573	-5.5108	-3.1804	-2.2751	-1.4713	-.7945	-.2841
10	5.4898	2.7088	.7776	-.2647	-.4778	-.4507	-.3635	-.2345	-.0958
15	10.5519	6.8034	3.7189	1.4750	.4932	.2270	.0611	-.0137	-.0197
20	4.3776	3.1981	1.9501	.8392	.2789	.1171	.0155	-.0264	-.0210
25	3.8961	3.1934	2.2354	1.1908	.5517	.3318	.1654	.0593	.0109
30	5.9630	4.7197	3.3307	1.8966	.9885	.6535	.3788	.1769	.0534
35	2.8636	2.6501	2.1608	1.4126	.8125	.5626	.3428	.1692	.0543
40	.0075	.6560	.9600	.8644	.5864	.4335	.2815	.1481	.0504
45	1.4429	1.4980	1.4191	1.1099	.7410	.5534	.3670	.1992	.0704
50	1.6132	1.4836	1.3352	1.0542	.7298	.5579	.3804	.2130	.0777
55	-.7033	-.1903	.2395	.4707	.4416	.3731	.2772	.1671	.0646
60	-.6161	-.2349	.1258	.3677	.3841	.3380	.2612	.1635	.0652
65	.9018	.7029	.6127	.5607	.4674	.3934	.2971	.1843	.0736
70	-.0213	.0260	.1382	.2707	.3036	.2812	.2301	.1524	.0693
75	-1.0856	-.7279	-.3703	-.0338	.1293	.1598	.1556	.1156	.0521
80	.1333	.0683	.0656	.1403	.1932	.1955	.1738	.1240	.0550
85	.5879	.3548	.2110	.1747	.1869	.1833	.1620	.1164	.0522
90	-.6825	-.5034	-.3270	-.1275	.0127	.0584	.0815	.0736	.0373
100	.5932	.3512	.1670	.0736	.0727	.0804	.0818	.0667	.0329
110	-.7865	-.5678	-.3917	-.2333	-.1134	-.0609	-.0170	.0086	.0104
120	.4923	.2955	.1312	.0163	-.0118	-.0081	.0011	.0082	.0067
130	-.4488	-.3315	-.2423	-.1753	-.1236	-.0948	-.0633	-.0333	-.0110
140	.0303	-.0078	-.0431	-.0750	-.0831	-.0772	-.0632	-.0417	-.0174
150	.0696	.0195	-.0238	-.0631	-.0828	-.0837	-.0754	-.0549	-.0249
160	-.4516	-.3294	-.2323	-.1654	-.1369	-.1243	-.1057	-.0753	-.0341
170	.4398	.2684	.1270	.0175	-.0465	-.0666	-.0746	-.0631	-.0316
180	-.6521	-.4636	-.3120	-.2032	-.1574	-.1414	-.1206	-.0872	-.0401

TABLE 12.42

12.4.4 Frame Analysis by the Dummy Load Method

The previous sections have dealt with "cookbook" methods for determining loads and moments in rings and frames. This section presents a method for analyzing any type of frame with any type of external loading and reaction system.

Generally, a frame will be redundant by more than one degree. This is especially true when a symmetrical frame is cut along its axis of symmetry and one-half is analyzed. This situation has three degrees of redundancy. The degree of redundancy can be said to be the number of external reactions which must be removed to make the structure statically determinant.

A redundant structure cannot be analyzed by the simple equations of statics, i.e., $\Sigma F = 0$ and $\Sigma M = 0$. Additional equations are necessary and these generally involve the deformation of the structure. The expression for the deflection at any point of a structure may be expressed as:

$$\delta = \int \frac{Mm\,dx}{EI} \qquad 12.22$$

where M is the bending moment in the structure caused by the external loads. The EI is the elastic modulus and the moment of inertia of the structure and m is the bending moment at any section of the structure caused by a "dummy" load of unity acting at the point of desired deflection and in the direction of desired deflection.

Another deformation equation, for rotation, may be expressed as:

$$\theta = \int \frac{Mm'\,dx}{EI} \qquad 12.23$$

where M and EI are the same as for equation 12.22 and m' is the bending moment at any section of the beam caused by a dummy moment of unity (1 in-lb) applied at any section where the change in slope is desired.

When using equations 12.22 and 12.23, it should be noted, that though m may be considered to be a bending moment, it is in fact a length in equation 12.22 and dimensionless in equation 12.23.

The dummy load method is described in Section 9.3.3 so only the procedure for solving a redundant frame will be presented here.

Figure 12.43 shows a frame with symmetrical external loads reacted by shear in the side skins. The frame is symmetrical about the vertical center line. The procedure is as follows:

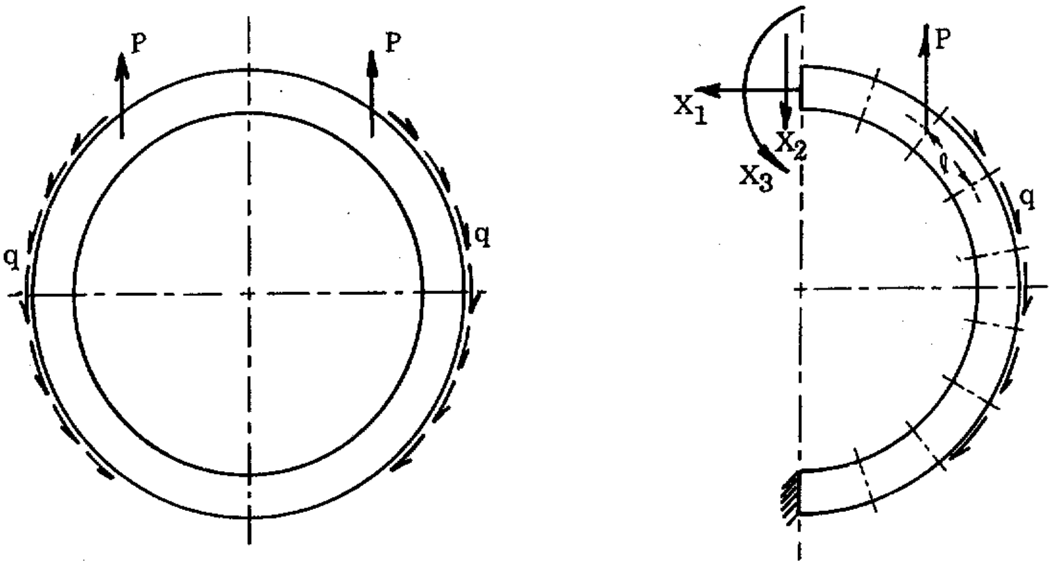


Figure 12.43 - Symmetrically Loaded Frame

1). Cut the frame about the centerline of symmetry. Apply the external loads and the balancing redundants as shown in Figure 12.43.

2). Cut the structure at approximately each 20^o making a cut at each point of reaction or load application or change in cross section. Number the cuts.

3). Calculate the cross sectional area, moment of inertia and neutral axis location at each cut.

4). Calculate the shear, axial load and bending moments at each cut for the applied load acting separately and reacted at the bottom centerline.

5). Calculate the shear, axial load and bending moments at each cut for each of the redundants applied separately and reacted at the bottom centerline.

The total strain energy, U, is determined from

$$U = \int M^2 dx/2EI + \int P^2 dx/2AE + \int V^2 dx/2AG \qquad 12.24$$

This expression contains the terms for bending, axial and shear strain energy. Generally bending is considerably larger than axial and shear combined so the last two terms in the previous equation are ignored. If, however, the combined effects of axial and shear are in excess of 10% of the total strain energy, consideration should be given to including them in the solution. In this procedure axial and shear will be ignored. It should be remembered that they can be included in a manner similar to bending.

The term for the bending moment at any point is

$$M = M_0 + x_1M_1 + x_2M_2 + x_3M_3 \qquad 12.25$$

Where M_0 = moment due to applied loads
M_1 = moment due to x_1
M_2 = moment due to x_2
M_3 = moment due to x_3

By expanding the equation for strain energy and applying Castigliano's Theorem, $\partial U/\partial x = 0$, three equations with three unknowns are obtained:

$$\frac{\partial U}{\partial x_1} = x_1\sum\frac{M_1^2dx}{EI} + x_2\sum\frac{M_1M_2dx}{EI} + x_3\sum\frac{M_1M_3dx}{EI} + \sum\frac{M_1M_0dx}{EI} \qquad 12.26$$

$$\frac{\partial U}{\partial x_2} = x_1\sum\frac{M_1M_2dx}{EI} + x_2\sum\frac{M_2^2dx}{EI} + x_3\sum\frac{M_2M_3dx}{EI} + \sum\frac{M_2M_0dx}{EI} \qquad 12.27$$

$$\frac{\partial U}{\partial x_3} = x_1\sum\frac{M_1M_3dx}{EI} + x_2\sum\frac{M_2M_3dx}{EI} + x_3\sum\frac{M_3^2dx}{EI} + \sum\frac{M_3M_0dx}{EI} \qquad 12.28$$

6). Calculate the values for the previous three equations using the moments and section properties previously determined.

7). Solve equations 12.26, 12.27 and 12.28 for x_1, x_2 and x_3.

8). The moments at each cut can be determined using equation 12.25.

A tabular form for the solution of a three-redundant frame is shown in Table 12.43.

Revision A

Beam	ℓ	EI	M_0	M_1	M_2	M_3	k	$\frac{kM_1^2\ell}{EI}$	k	$\frac{kM_1M_2\ell}{EI}$	k	$\frac{kM_1M_3\ell}{EI}$	k	$\frac{kM_1M_0\ell}{EI}$	k	$\frac{kM_2^2\ell}{EI}$	k	$\frac{kM_2M_3\ell}{EI}$	k	$\frac{kM_2M_0\ell}{EI}$
								Σ		Σ		Σ		Σ		Σ		Σ		Σ

k	$\frac{kM_3^2\ell}{EI}$	k	$\frac{kM_3M_0\ell}{EI}$	x_1M_1	x_2M_2	x_3M_3		$M_0 + x_1M_1 + x_2M_2 + x_3M_3$
	Σ		Σ					

NOTE: (1) Use equations 12.26, 12.27 and 12.28 to solve for x_1, x_2 and x_3.

(2) Reference section 9.3.3 for values of k.

Table 12.43 - Tabular Solution for Three-Redundant Frame.

SECTION 13

SANDWICH ANALYSIS

13.0 GENERAL

Structural sandwich is a layered composite, formed by bonding two thin facings to a thick core. It is a type of stressed-skin construction in which the facings resist nearly all of the applied edgewise (in-plane) loads and flatwise bending moments. The thin spaced facings provide nearly all of the bending rigidity to the construction. The core spaces the facings and transmits shear between them so that they are effective about a common neutral axis. The core also provides most of the shear rigidity of the sandwich construction. By proper choice of materials for facings and core, constructions with high ratios of stiffness to weight can be achieved.

A basic design concept is to space strong, thin facings far enough apart to achieve a high ratio of stiffness to weight; the lightweight core that does this also provides the required resistance to shear and is strong enough to stabilize the facings to their desired configuration through a bonding media such as an adhesive layer, braze or weld. The sandwich is analogous to an I-beam in which the flanges carry direct compression and tension loads, as do the sandwich facings, and the web carries shear loads, as does the sandwich core.

In order that sandwich cores be lightweight, they are usually made of low density material, some type of cellular construction (honeycomb-like core formed of thin sheet material) or of corrugated sheet material. As a consequence of employing a lightweight core, design methods account for core shear deformation because of the low effective shear modulus of the core. The main difference in design procedures for sandwich structural elements as compared to design procedures for homogeneous material is the inclusion of the effects of core shear properties on deflection, buckling and stress for the sandwich.

Because thin facings can be used to carry loads in a sandwich, prevention of local failure under edgewise direct or flatwise bending loads is necessary just as prevention of local crippling of stringers is necessary in the design of sheet stringer construction. Modes of failure that may occur in sandwich under edge load are shown in Figure 13.1.

Shear crimping failure, as shown in Figure 13.1, appears to be a local mode of failure, but is actually a form of general overall buckling in which the wavelength of the buckles is very small because of low core shear modulus. The crimping of the sandwich occurs suddenly and usually causes the core to fail in shear at the crimp; it may also cause shear failure in the bond between the facing and core.

Crimping may also occur in cases where the overall buckle begins to appear and then the crimp occurs suddenly because of severe local shear stresses at the ends of the overall buckle. As soon as the crimp appears, the overall buckle may disappear. Therefore, although examination of the failed sandwich indicates crimping or shear instability, failure may have begun by overall buckling that finally caused crimping.

If the core is of cellular (honeycomb) or corrugated material, it is possible for the facings to buckle or dimple into the spaces between core walls or corrugations

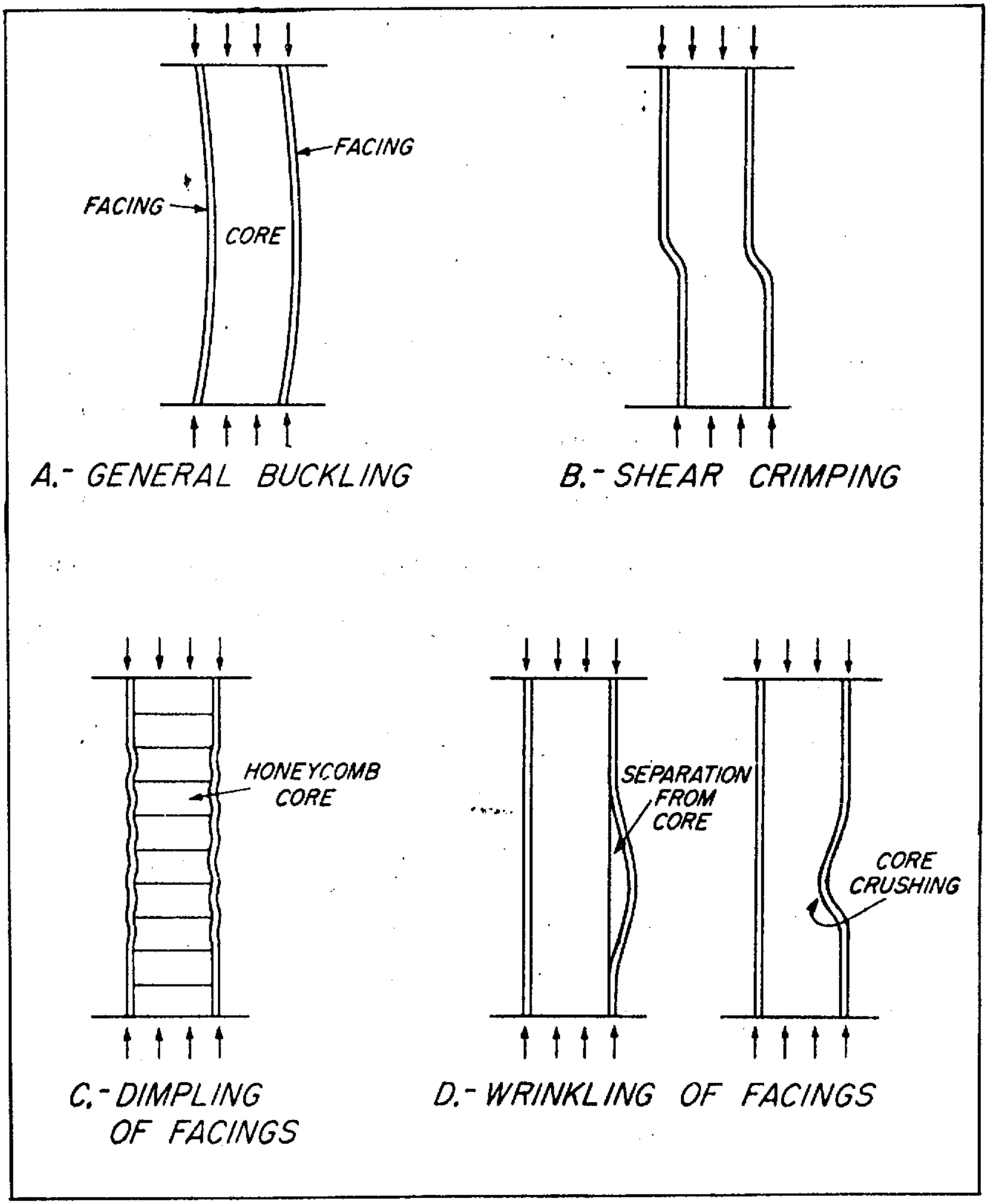


FIGURE 13.1 - POSSIBLE MODES OF FAILURE OF SANDWICH UNDER EDGEWISE COMPRESSION

as shown in Figure 13.1. Dimpling may be severe enough so that permanent dimples remain after removal of load and the amplitude of the dimples may be large enough to cause the dimples to grow across the cell walls and result in a wrinkling of the facings.

Wrinkling, as shown in Figure 13.1, may occur if a sandwich facing subjected to edgewise compression buckles as a plate on an elastic foundation. The facing may buckle inward or outward depending on the flatwise compressive strength of the core relative to the flatwise tensile strength of the bond between the facing and core. If the bond between facing and core is strong, facings can wrinkle and cause tension failure in the core. Thus, the wrinkling load depends upon the elasticity and strength of the foundation system; namely, the core and the bond between facing and core. Since the facing is never perfectly flat, the wrinkling load will also depend upon the initial eccentricity of the facing or original waviness.

The local modes of failure may occur in sandwich panels under edgewise loads or normal loads. In addition to overall buckling and local modes of failure, sandwich is designed so that facings do not fail in tension, compression, shear or combined stresses due to edgewise loads or normal loads and cores and bonds do not fail in shear, flatwise tension or flatwise compression due to normal loads.

The basic design principles can be summarized into four conditions as follows:

(1) Sandwich facings shall be at least thick enough to withstand chosen design stresses under design loads.

(2) The core shall be thick enough and have sufficient shear rigidity and strength so that overall sandwich buckling, excessive deflection and shear failure will not occur under design loads.

(3) The core shall have high enough modulus of elasticity and the sandwich shall have great enough flatwise tensile and compressive strength so that wrinkling of either facing will not occur under design loads.

(4) If the core is cellular (honeycomb) or of corrugated material and dimpling of the faces is not permissible, the cell size or corrugation spacing shall be small enough so that dimpling of either facing into the core spaces will not occur under design loads.

The facing to core bond shall have sufficient flatwise tensile and shear strength to develop the core strength in the expected environment. The environment includes effects of temperature, water or moisture, corrosive atmosphere and fluids, fatigue, creep and any other condition that affects material properties.

13.1 Materials

13.1.1 Facing Materials

The facings of a sandwich part serve many purposes, depending upon the application, but in all cases they carry the major applied loads. The stiffness, stability, configuration and, to a large extent, the strength of the part are determined by the characteristics of the facings as stabilized by the core. To perform these functions the facings must be adequately bonded to a core of acceptable quality. Facings sometimes have additional functions, such as providing a profile of proper aerodynamic smoothness, a rough nonskid surface, or a tough wear-resistant floor covering. To better fulfill these special functions, one facing of a sandwich is sometimes made thicker or of slightly different construction than the other.

Any thin sheet material can serve as a sandwich facing. A few of the materials usually used are shown in Table 13.1.

13.1.2 Core Materials

To permit an airframe sandwich construction to perform satisfactorily, the core of the sandwich must have certain mechanical properties, thermal characteristics and dielectric properties under conditions of use and still conform to weight limitations. Cores of densities ranging from 1.6 to 23 pounds per cubic foot have found use in airframe sandwich, but the usual density ranges from 3 to 10 pounds per cubic foot.

A wide variety of core materials can be constructed of thin sheet materials or ribbons formed to honeycomb-like configurations. By varying the sheet material, sheet thickness, cell size and cell shape cores of a wide range in density and properties can be produced. Most honeycomb cores can be formed to moderate amounts of single curvature, but some can easily be formed to fairly severe single curvature and to moderate compound curvature.

Honeycomb core cell size is determined by the diameter of a circle which can be inscribed in a cell. Two types of honeycomb core showing the cell size "s" are shown in Figure 13.2. Honeycomb core cell sizes used in airframes vary from about 1/16 to 7/16 inch, usually in multiples of 1/16 inch. Not all sheet materials are formed to all of these cell sizes because some sheet materials are so thick and stiff that they cannot be formed to core of cells less than 3/16 inch in size. For special use, such as an insert, honeycomb cores can be densified locally by under-expanding, crushing the core locally or by inserting a higher density core locally. Cores for airframe sandwich construction are presently being made of thin sheets of aluminum alloys, resin-treated glass fabric, resin-treated asbestos, resin-treated paper, stainless steel alloys, titanium alloys and refractory metals.

Honeycomb cores fabricated from nonmetallic materials have better thermal insulating characteristics than metallic honeycomb cores, even though both allow transmission of heat by radiation in the open cells. In considering thermal effects on sandwich structure, it should be understood that the sandwich can act as a reflective thermal insulator. The effective thermal conductivity of a honeycomb core depends upon conduction of the material of which the core is made, radiation between facings and connection within the core cell and can be computed approximately as

$$K_e = K_0 A_c + \frac{4\sigma t_c (1 - A_c) T_m^3}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 2 + \frac{2}{1 + F_{12}}} + t_c (1 - A_c)\eta \qquad 13.1$$

where K_e = effective conductivity
K_o = conductivity of core ribbon material
A_c = core solidity = W_c/W_o
W_c = core density
W_o = core ribbon material density
σ = Stefan-Boltzmann constant

FACING	YIELD STRENGTH $F_f \sim$psi	MODULUS OF ELASTICITY $E_f \sim$psi	$\lambda_f = 1 - \mu^2$	WEIGHT PER MIL THICKNESS LBS/FT^2
ALUMINUM:				
1100-H14	17,000	$10x10^6$	.89	.014
2024-T4	47,000	$10x10^6$	.89	.014
3003-H16	25,000	$10x10^6$	.89	.014
5052-H38	37,000	$10x10^6$	.89	.014
6061-T4	21,000	$10x10^6$	.89	.014
7075-T6	73,000	$10x10^6$	.89	.014
MILD CARBON STEEL	50,000	$30x10^6$	.91	.040
STAINLESS STEEL:				
316	60,000	$30x10^6$	.94	.040
17-7	200,000	$30x10^6$	.94	.040
TITANIUM:				
Ann.Ti-75A	80,000	$15x10^6$	.94	.0235
H.T. 6Al-4V	143,000	$16.8x10^6$	.94	.0230
FIBERGLASS PRE-PREG:				
EPOXY F155	63,000	$3.5x10^6$	.98	.0095
EPOXY F161	62,000	$3.5x10^6$	.98	.0088
PHENOLIC F120	48,000	$3.5x10^6$	.98	.0094
POLYESTER F141	48,000	$3.5x10^6$	.98	.010
PLYWOOD:				
DOUGLAS FIR	2,650	$1.8x10^6$	.99	.003
HARDBOARD	300	$0.4x10^6$	.99	.004

TABLE 13.1 - TYPICAL SANDWICH FACING MATERIALS

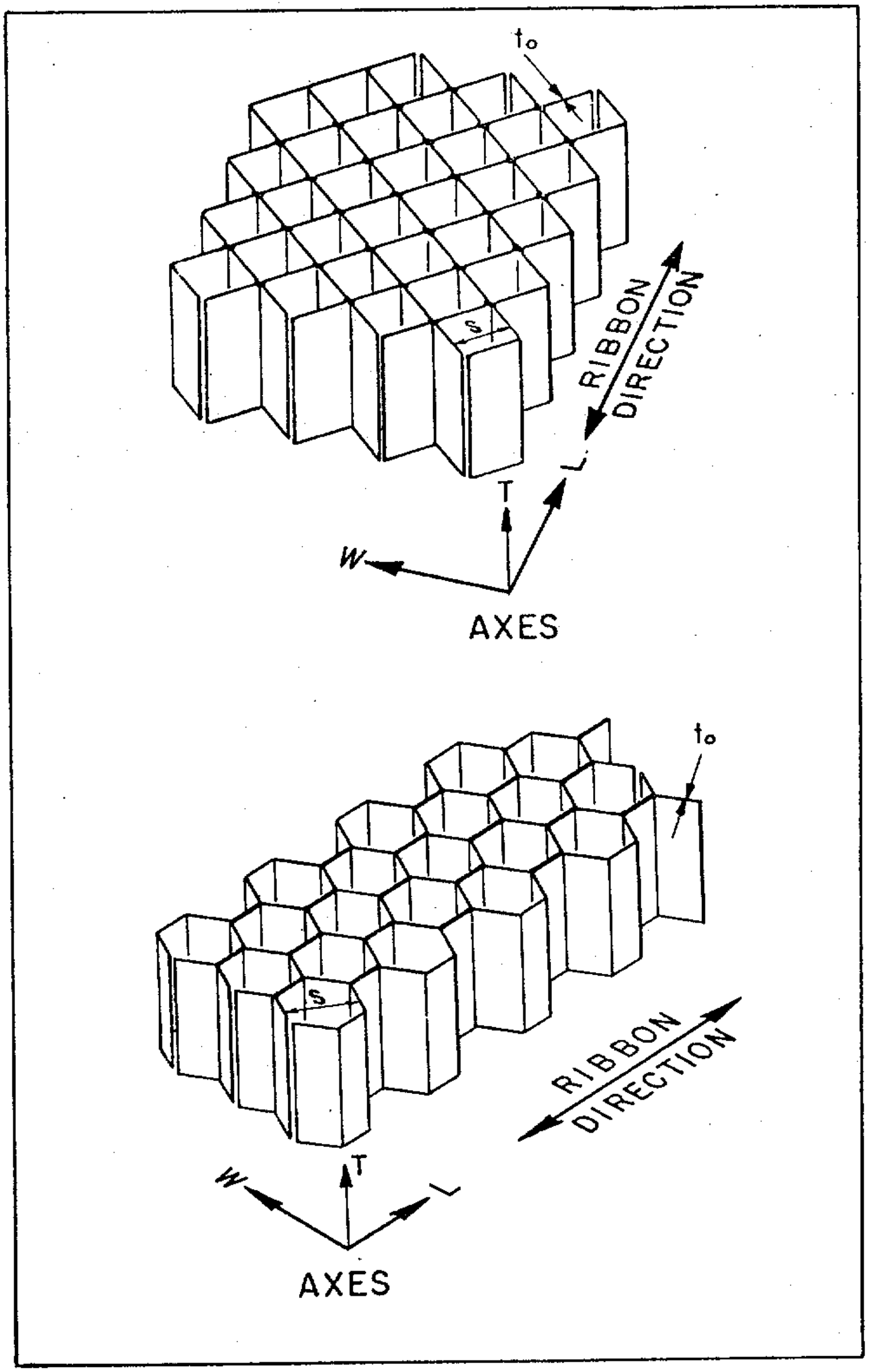


FIGURE 13.2 - HONEYCOMB CORE NOTATION

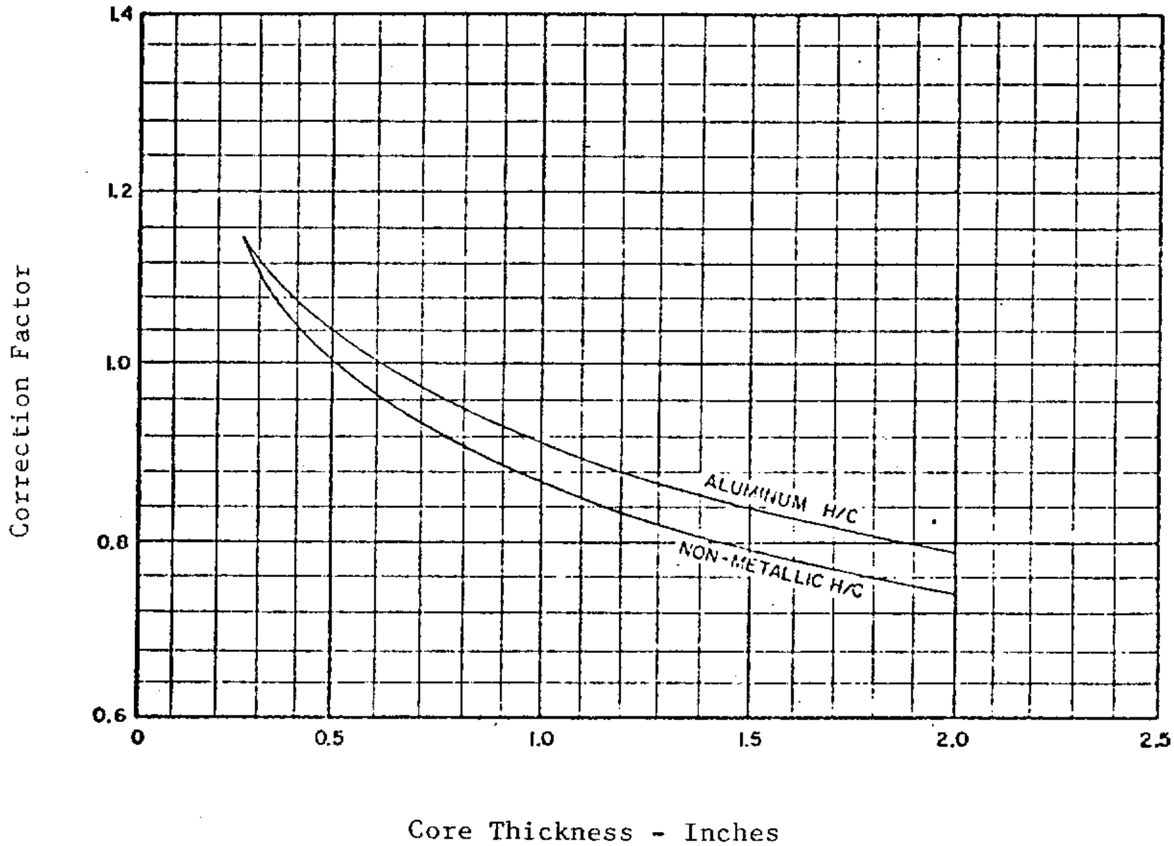


FIGURE 13.2a - SHEAR STRENGTH CORRECTION FACTORS

t_c = core thickness
T_m = mean absolute temperature of the two facings
ϵ_1 = emissivity of inside of facing 1
ϵ_2 = emissivity of inside of facing 2
F_{12} = geometric view factor between facings
η = connective heat transfer coefficient inside core cell

Sheets of corrugated metal foil are usually assembled with the corrugations parallel to form honeycomb cores. The foil may be perforated for use in core where solvents or gasses must be vented. Perforated foil in sandwich panels that are not sealed or are poorly sealed will allow penetration of moisture, etc., which may cause severe deterioration of the core. If the sheets are assembled with the corrugations in adjacent sheets perpendicular to each other, a well vented crossbanded core is produced. Crossbanded cores may be cut so that the corrugation flutes are at an angle of 45° to the sandwich facings, giving the core a trussed appearance.

Crossbanded cores are not as strong in compressions in the T direction or in shear in the TL or TW planes as honeycomb cores of the same density. Honeycomb cores, however, have negligible compressive strength in the W and L directions and shear strength in the WL plane, whereas crossbanded cores have considerable strength in these directions. Crossbanded core is not readily formed to curved surfaces because of its relatively high stiffness in all directions.

Many core materials and core configurations are available, but the aluminum honeycomb with a hexagonal cell is the most commonly used at Bell Helicopter. Table 13.2 shows the mechanical and physical properties for many of the available core materials. The metallics used at Bell are 5052-H39 and 5056-H39 aluminum. They should be procured in accordance with Bell Specification 299-947-059. The nonmetallic honeycomb materials are as specified in Bell Specifications 299-947-103 and 299-947-337. Regardless of the core material, the final bonded panel must meet Bell Specification 299-947-091. Figure 13.2a shows correction factors for core shear strength at various core thicknesses.

13.1.3 Adhesives

In the fabrication of sandwich, adhesives are used for bonding faces to core and bonding facings and fittings, reinforcing plates, edge strips and other inserts. The adhesives used are resin formulations especially developed to give high strength bonds over a wide range of exposure and stressing conditions. Adhesives can be used to bond many types of metal in highly stressed applications. They can also be formulated to have resistance to moderately elevated temperatures.

The intrinsic elastic properties and strength of adhesives have not been evaluated to any large extent. Instead, adhesive bonded lap joints are used to evaluate the strength of an adhesive. The standard test is .063 aluminum sheets bonded with an overlap of 1/2 inch. The lap joint strength is the load required to shear the bond divided by the bonded area. Table 13.3 shows typical lap joint strengths of several adhesives.

Lap joint strength is not considered of prime use for determining adequacy of adhesives for bonding sandwich facings to honeycomb cores. The need of an adhesive to form strong fillets at the ends of the core cells to produce satisfactory sandwich has prompted the evaluation of peel strength and sandwich flatwise tensile strength.

-059 Ty II & IV

$F_s = F_{sL} + F_{sW}$

5052 ALLOY HEXAGONAL ALUMINUM HONEYCOMB
AEROSPACE GRADE

HONEYCOMB DESIGNATION Cell-Material-Gage	Nominal Density pcf	COMPRESSIVE					Crush Strength psi	PLATE SHEAR					
		Bare		Stabilized				"L" Direction			"W" Direction		
		Strength psi		Strength psi		Modulus ksi		F_s Strength psi		Modulus ksi	Strength psi		Modulus ksi
		typ	min	typ	min	typical	typ	typ	min	typical	typ	min	typ
1/8-5052-.0007	3.1	270	200	290	215	75	130	210	155	45.0	130	90	22.0
1/8-5052-.001	4.5	520	375	545	405	150	260	340	285	70.0	220	168	31.0
1/8-5052-.0015	6.1	870	650	910	680	240	450	505	455	98.0	320	272	41.0
1/8-5052-.002	8.1	1400	1000	1470	1100	350	750	725	670	135.	455	400	54.0
5/32-5052-.0007	2.6	200	150	215	160	55	90	165	120	37.0	100	70	19.0
5/32-5052-.001	3.8	395	285	410	300	110	185	270	215	56.0	175	125	26.4
5/32-5052-.0015	5.3	690	490	720	535	195	340	420	370	84.0	270	215	36.0
5/32-5052-.002	6.9	1080	770	1130	800	285	575	590	540	114.	375	328	46.4
5/32-5052-.0025	8.4	1530	1070	1600	1180	370	800	760	690	140.	475	420	56.0
3/16-5052-.0007	2.0	130	90	135	100	34	60	120	80	27.0	70	46	14.3
3/16-5052-.001	3.1	270	200	290	215	75	130	210	155	45.0	130	90	22.0
3/16-5052-.0015	4.4	500	360	525	385	145	250	330	280	68.0	215	160	30.0
3/16-5052-.002	5.7	770	560	810	600	220	390	460	410	90.0	300	244	38.5
3/16-5052-.0025	6.9	1060	770	1130	800	285	575	590	540	114.	375	328	46.4
3/16-5052-.003	8.1	1400	1000	1470	1100	350	750	725	670	135	455	400	54.0
1/4-5052-.0007	1.6	85	60	95	70	20	40	85	60	21.0	50	32	11.0
1/4-5052-.001	2.3	165	120	175	130	45	75	140	100	32.0	85	57	16.2
1/4-5052-.0015	3.4	320	240	340	250	90	150	235	180	50.0	150	105	24.0
1/4-5052-.002	4.3	480	350	505	370	140	230	320	265	66.0	210	155	29.8
1/4-5052-.0025	5.2	670	500	690	510	190	335	410	360	82.0	265	200	35.4
1/4-5052-.003	6.0	850	630	880	660	235	430	495	445	96.0	315	265	40.5
1/4-5052-.004	7.9	1360	970	1420	1050	340	725	700	650	130.	440	390	52.8
3/8-5052-.0007	1.0	30	20	45	20	10	25	45	32	12.0	30	20	7.0
3/8-5052-.001	1.6	85	60	95	70	20	40	85	60	21.0	50	32	11.0
3/8-5052-.0015	2.3	165	120	175	130	45	75	140	100	32.0	85	57	16.2
3/8-5052-.002	3.0	260	190	270	200	70	120	200	145	43.0	125	85	21.2
3/8-5052-.0025	3.7	370	270	390	285	105	180	260	200	55.0	170	115	26.0
3/8-5052-.003	4.2	460	335	485	355	135	220	310	255	65.0	200	150	29.0
3/8-5052-.004	5.4	720	500	745	535	200	360	430	380	86.0	280	228	36.8
3/8-5052-.005	6.5	970	700	1020	750	265	505	545	500	105.	350	300	43.5

TABLE 13.2 - PROPERTIES OF TYPICAL SANDWICH CORES

5056 HEXAGONAL ALUMINUM HONEYCOMB
AEROSPACE GRADE

HONEYCOMB DESIGNATION Cell-Material-Gage	Nominal Density pcf	COMPRESSIVE Bare Strength psi		COMPRESSIVE Stabilized Strength psi		Stabilized Modulus ksi	Crush Strength psi	PLATE SHEAR "L" Direction Strength psi		"L" Direction Modulus	"W" Direction Strength psi		"W" Direction Modulus ksi
		typ	min	typ	min	typ	typ	typ	min	typ	typ	min	typ
1/8-5056-.0007	3.1	340	250	360	260	97	170	250	200	45.0	155	110	20.0
1/8-5056-.001	4.5	630	475	670	500	185	320	425	350	70.0	255	205	38.0
1/8-5056-.0015	6.1	1000	760	1100	825	295	535	640	525	102.	370	305	38.0
1/8-5056-.002	8.1	1520	1200	1700	1300	435	810	900	740	143.	520	440	51.0
5/32-5056-.0007	2.6	255	180	260	185	70	120	200	152	36.0	120	80	17.0
5/32-5056-.001	3.8	475	360	500	375	140	235	335	272	57.0	205	155	24.0
5/32-5056-.0015	5.3	820	615	865	650	240	420	530	435	85.0	310	250	33.0
5/32-5056-.002	6.9	1220	920	1340	1000	350	650	760	610	118	430	360	43.0
3/16-5056-.0007	2.0	155	110	160	120	45	75	140	105	27.0	85	50	13.0
3/16-5056-.001	3.1	340	250	360	260	97	170	255	200	45.0	155	110	20.0
3/16-5056-.0015	4.4	600	460	650	490	180	310	410	340	68.0	245	198	27.5
3/16-5056-.002	5.7	910	685	980	735	270	480	585	480	94.0	340	280	36.0
1/4-5056 -.0007	1.6	100	75	110	80	30	50	90	78	20.0	60	38	12.0
1/4-5056-.001	2.3	205	145	210	155	58	100	170	130	32.0	105	62	15.0
1/4-5056-.0015	3.4	395	300	420	315	115	200	290	230	50.0	175	130	22.0
1/4-5056-.002	4.3	580	440	620	465	172	300	400	325	67.0	240	190	27.0
1/4-5056-.0025	5.2	790	600	820	645	230	410	500	425	84.0	300	245	32.0
3/8-5056-.0007	1.0	35	25	50	35	15	35	60	45	15.0	35	25	9.0
3/8-5056-.001	1.6	100	75	110	80	30	50	90	78	20.0	60	38	12.0
3/8-5056-.0015	2.3	205	155	210	155	58	100	170	130	32.0	105	62	15.0
3/8-5056-.002	3.0	320	240	340	260	92	160	245	190	43.0	145	100	19.0
2024 HEXAGONAL ALUMINUM HONEYCOMB													
		typ	min	typ	min	typ	typ	typ	min	typ	typ	min	typ
1/8-2024-.0015	5.2	700	525	780	620	200	425	500	400	82.0	315	250	33.0
1/8-2024-.002	6.7	1100	825	1225	980	300	640	760	600	118	470	375	45.0
1/8-2024-.0025	8.0	1480	1100	1650	1320	380	840	960	770	148	590	470	54.0
1/8-2024-.003	9.5	1970	1475	2300	1725	480	1120	1150	950	170	650	585	64.0
3/16-2024-.0015	3.5	330	250	370	290	86	200	290	230	55.0	180	143	23.0
1/4-2024-.0015	2.8	220	165	250	165	40	110	200	140	42.0	120	88	19.0

TABLE 13.2 (CONTINUED) - PROPERTIES OF TYPICAL SANDWICH CORES

5052 ALUMINUM FLEX-CORE - AEROSPACE GRADE

HONEYCOMB DESIGNATION Material-Cell-Gage Count	Nominal Density pcf	COMPRESSIVE					Crush Strength pcf	PLATE SHEAR					
		Bare		Stabilized				"L" Direction			"W" Direction		
		Strength psi		Strength psi		Modulus psi		Strength psi			Strength psi		Modulus psi
		typ	min	typ	min	typ	typ	typ	min	typ	typ	min	typ
5052/F40-.0013	2.1	180	126	225	157	65	80	90	63	18	50	37	10
5052/F40-.0019	3.1	340	238	400	280	125	165	180	126	32	100	75	13
5052/F40-.0025	4.1	540	378	600	420	185	250	260	182	45	150	115	17
5052/F40-.0037	5.7	900	630	1000	700	290	380	400	280	68	230	170	23
5052/F80-.0013	4.3	575	-	650	-	195	-	280	-	45	160	-	18
5052/F80-.0019	6.0	1000	-	1050	-	310	-	440	-	72	240	-	24
5052/F80-.0025	8.0	1570	-	1600	-	400	-	620	-	98	345	-	31
		typ	min	typ	min	typ	typ	typ	min	typ	typ	min	typ
5056/F40-.0014	2.1	215	-	260	-	65	-	105	-	18	55	-	10
5056/F40-.0020	3.1	405	-	470	-	125	-	215	-	32	120	-	13
5056/F40-.0026	4.1	645	-	690	-	185	-	310	-	45	175	-	17
5656/F80-.0014	4.3	680	-	740	-	195	-	335	-	47	185	-	18
5656/F80-.0020	6.0	1150	-	1300	-	310	-	520	-	73	285	-	24
5056/F80-.0026	8.0	1730	-	1800	-	410	-	740	-	100	410	-	32
		typical		typical		typ	typ	typical		typ	typical		typ
ACG 1/4-.003	5.2	595		610		148	245	345		63	215		31
ACG 3/8-.003	3.6	325		340		92	175	210		40	130		20
ACG 3/4-.003	1.8	95		110		24	50	95		16	55		8

TABLE 13.2 (CONTINUED) - PROPERTIES OF TYPICAL SANDWICH CORES

HRP GLASS REINFORCED PHENOLIC HONEYCOMB

HONEYCOMB DESIGNATION Mat'l-Cell-Density	COMPRESSIVE					PLATE SHEAR					
	Bare		Stabilized			"L" Direction			"W" Direction		
	Strength		Strength		Modulus ksi	Strength psi		Modulus ksi	Strength psi		Modulus ksi
Hexagonal	typ	min	typ	min	typ	typ	min	typ	typ	min	typ
HRP-3/16-4.0	500	350	600	480	57	260	210	11.5	140	110	5.0
HRP-3/16-5.5	800	600	940	750	95	425	370	19.5	220	190	8.5
HRP-3/16-7.0	1150	900	1230	-	136	500	-	28.0	290	-	12.5
HRP-3/16-8.0	1400	1100	1600	1280	164	660	600	34.0	400	370	15.0
HRP-3/16-12.0	2280	1600	2300	-	260	940	-	55.0	570	-	25.0
HRP-1/4-3.5	350	260	500	400	46	230	170	9.0	120	100	3.5
HRP-1/4-4.5	630	450	700	560	70	300	250	14.0	170	140	6.0
HRP-1/4-5.0	700	510	820	660	84	340	-	17.0	200	-	7.5
HRP-1/4-6.5	1025	850	1180	900	120	450	-	25.0	260	-	11.0
HRP-3/8-2.2	150	105	200	145	13	105	75	5.0	60	45	2.0
HRP-3/8-3.2	320	245	440	350	38	200	160	8.0	105	85	3.0
HRP-3/8-4.5	610	450	690	550	65	300	260	14.0	170	150	6.0
HRP-3/8-6.0	900	750	1000	750	100	400	340	22.5	260	210	10.0
HRP-3/8-8.0	1060	920	1200	-	150	520	-	31.0	320	-	13.0
OX-CORE											
HRP/OX-1/4-4.5	520	350	625	-	43	210	-	8.0	250	-	15.2
HRP/OX-1/4-5.5	810	600	950	-	65	270	-	10.5	330	-	18.0
HRP/OX-1/4-7.0	1150	-	1230	-	84	395	-	14.0	450	-	20.0
HRP/OX-3/8-3.2	340	260	425	-	32	140	-	4.5	150	-	9.0
HRP/OX-3/8-5.5	700	580	820	-	60	240	-	10.0	300	-	17.0
FLEX-CORE											
HRP/F35-2.5	180	-	240	-	25	125	-	12.5	70	-	7.0
HRP/F35-3.5	320	-	400	300	37	200	140	15.0	105	75	10.0
HRP/F35-4.5	440	-	600	-	49	280	-	22.0	140	-	12.0
HRP/F50-3.5	300	-	425	300	37	195	140	20.0	100	75	10.0
HRP/F50-4.5	400	-	600	500	49	265	200	25.0	140	100	13.0
HRP/F50-5.5	600	-	880	-	61	390	-	31.5	205	-	16.0

TABLE 13.2 (CONTINUED) - PROPERTIES OF TYPICAL SANDWICH CORES

NP GLASS REINFORCED POLYESTER HONEYCOMB

HONEYCOMB DESIGNATION Mat'l-Cell-Fabric-Density	COMPRESSIVE					PLATE SHEAR					
	Bare		Stabilized			"L" Direction			"W" Direction		
	Strength psi		Strength psi		Modulus ksi	Strength psi		Modulus ksi	Strength psi		Modulus ksi
Hexagonal	typ	min	typ	min	typ	typ	min	typ	typ	min	typ
NP 3/16-4.5	520	470	670	470	80	280	195	13.5	130	90	5.2
NP 3/16-6.0	880	615	1050	735	116	330	230	15.0	155	110	5.8
NP 3/16-9.0	1700	1200	1800	1260	180	460	320	20.0	230	160	7.5
NP 1/4-4.0	420	295	560	390	68	260	180	13.0	120	85	5.0
NP 1/4-6.0	880	615	1050	736	116	330	230	15.0	155	110	5.8
NP 1/4-8.0	1400	980	1540	1080	160	410	290	18.0	205	145	7.0
NP 3/8-2.5	200	140	280	195	34	170	120	10.0	100	70	4.0
NP 3/8-4.5	520	365	670	470	80	280	195	13.5	13.5	90	5.2
OX-CORE											
NP/OX 1/4-4.0	350	-	-	-	-	160	-	5.0	190	-	12.0
NP/OX 1/4-6.0	700	-	-	-	-	275	-	7.5	375	-	19.5
NP/OX 3/8-4.5	420	-	-	-	-	190	-	5.5	285	-	15.0

HRH-327 GLASS REINFORCED POLYIMIDE HONEYCOMB

HONEYCOMB DESIGNATION Mat'l-Cell-Density	COMPRESSIVE			PLATE SHEAR					
	Stabilized			"L" Direction			"W" Direction		
	Strength psi		Modulus ksi	Strength psi		Modulus ksi	Strength psi		Modulus ksi
	typ	min	typ	typ	min	typ	typ	min	typ
HRH 327-3/16-4.0	440	-	50	280	-	29	130	-	10
HRH 327-3/16-4.5	520	400	58	320	220	33	150	110	11
HRH 327-3/16-5.0	600	-	68	370	-	37	180	-	25.5
HRH 327-3/16-6.0	780	625	87	460	345	45	230	170	15
HRH 327-3/16-8.0	1300	1000	126	650	500	62	410	330	22
HRH 327-1/4-4.0	440	-	50	280	-	29	130	-	10
HRH 327-1/4-5.0	600	-	68	370	-	37	180	-	12.5
HRH 327-3/8-4.0	440	325	50	280	195	29	150	100	12
HRH 327-3/8-5.5	680	540	78	420	300	41	210	160	13.5
HRH 327-3/8-7.0	1000	-	106	550	-	53	310	-	18.5

TABLE 13.2 (CONTINUED) - PROPERTIES OF TYPICAL SANDWICH CORES

ADHESIVE TYPE	AVERAGE ALUMINUM TO ALUMINUM LAP SHEAR BOND STRENGTH AT ROOM TEMPERATURE - (PSI)
(A) NITRILE PHENOLIC	3500
(B) VINYL PHENOLIC	4200
(C) EPOXY PHENOLIC	3400
(D) UNMODIFIED EPOXY	3100
(E) MODIFIED EPOXY - 250 CURE	4500
(F) MODIFIED EPOXY - 350 CURE	3300
(G) EPOXY POLYAMIDE	5500
(H) POLYAMIDE	3300

STRUCTURAL ADHESIVES
USEFUL TEMPERATURE RANGE, STRENGTH PROPERTIES
192 HR. EXPOSURE

ADHESIVE TYPE	USEFUL TEMP. RANGE (°F)	TYPICAL VALUES LAP SHEAR (PSI)	PEEL STRENGTH
(A) NITRILE PHENOLIC	-67 350	3500-4500 300-1700	GOOD TO EXCELLENT
(B) VINYL PHENOLIC	-67 225	2000-3000 100-1800	FAIR TO GOOD
(C) EPOXY PHENOLIC	-70 500	1300-5000 200-1900	POOR TO MEDIUM
(D) UNMODIFIED EPOXY	-67 300	1300-3000 800-3000	POOR TO MEDIUM
(E) MODIFIED EPOXY- 250 CURE	-250 180	1500-2500 1000-1900	GOOD
(F) MODIFIED EPOXY- 350 CURE	-67 250	3000-3500 1500-2500	GOOD
(G) EPOXY POLYAMIDE	-300 250	4000-5000 2800-3300	GOOD
(H) POLYAMIDE	UP TO 600	3300	POOR

TABLE 13.3 - ADHESIVE PROPERTIES

Peel strength is determined as the torque necessary to peel a facing from a sandwich core. Table 13.4 shows values of peel strength and fillet strength for several adhesives.

Adhesives are available in the form of liquid, paste, powders and supported or unsupported films and can be applied by spray, roller, spatula or hand lay-up. The form of the adhesive (liquid, paste or film) is chosen to suit the lay-up operation and glue line thickness requirement.

Structural adhesives have good shear and tensile strength, but resistance to peel stresses is relatively poor. Bonded joints should be designed to take advantage of the high shear and tensile strengths of the adhesive and avoid peel stresses in the bond where possible.

ADHESIVE	FILLET STRENGTH			PEEL STRENGTH		
	73°F lb/in	180°F lb/in	-67°F lb/in	73°F in-lb/in	180°F in-lb/in	-67°F in-lb/in
NITRILE ELASTOMER-PHENOLIC PLUS MODIFIED EPOXY FILM	73	51	84	66	32	21
POLYVINYL-PHENOLIC	42	35	49	25	22	14
EPOXY-POLYAMIDE	71	39	66	18	24	15
MODIFIED EPOXY-350°F CURE	62-86	27-65	61-87	16-97	13-82	12-49
MODIFIED EPOXY-250°F CURE	46-76	34-38	47-91	15-40	18-23	19-26
NITRILE EPOXIDE	61	33	89	27	18	28

TABLE 13.4 - STRENGTH OF ADHESIVES IN SANDWICH WITH HONEYCOMB CORE

13.2 Methods of Analysis

The analysis procedures described in this section apply to sandwich structures having isotropic facings and either orthotropic or isotropic cores. The isotropic materials are those having essentially constant properties in all directions. The orthotropic materials are those whose strength properties are not constant in all directions, such as honeycomb cores.

The assumption is made that adhesive failure does not occur, a reasonable assumption if proper care is taken in the selection of the adhesive system. This requires that the adhesive shear and flatwise tensile strength be greater than the respective core strength. This can be assured by specifying that the finished panel meet the requirements of Bell Specification 299-947-091.

If the sandwich has thin facings on a core of negligible bending stiffness, as is usually the case, and after assuming $\lambda_1 = \lambda_2 = \lambda$, the bending stiffness is given by the formula:

$$D = \frac{E_1' t_1 E_2' t_2 h^2}{(E_1' t_1 + E_2' t_2)\lambda} \qquad \text{(for unequal facings)}$$

$$D = \frac{E' t h^2}{2\lambda} \qquad \text{(for equal facings)}$$

Figure 13.3 shows the notation used for the analysis of sandwich panels in this section.

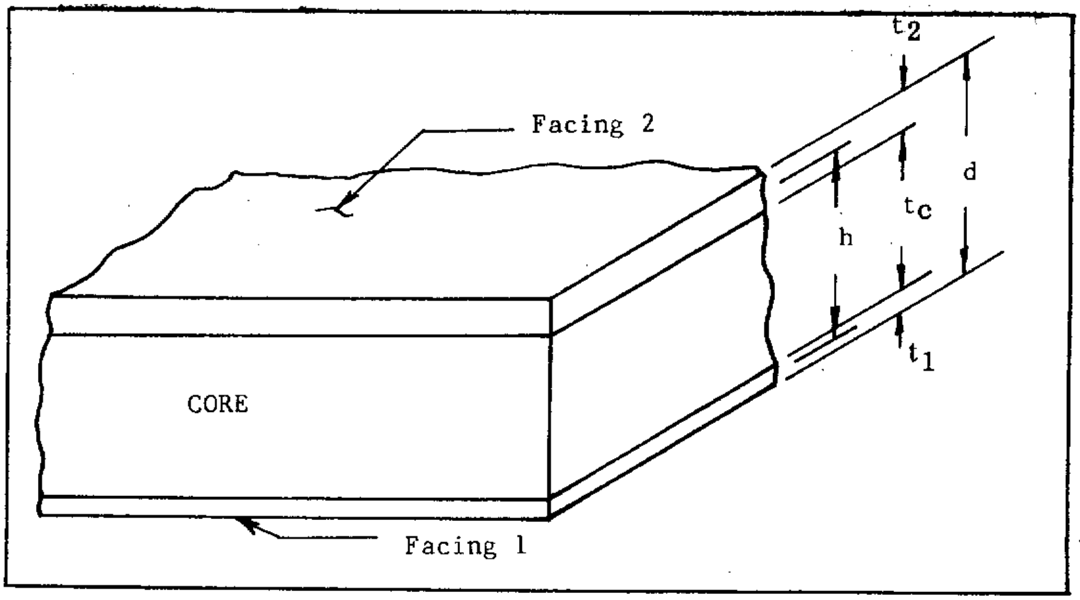


FIGURE 13.3 NOTATION FOR SANDWICH COMPOSITE

The notation used throughout this section is shown below:

1 - subscript denoting facing 1
2 - subscript denoting facing 2

a, b - length of panel edge; subscripts denoting parallel to a or b
c - subscript denoting core or compression
cr - subscript denoting critical
D - bending stiffness
d - total sandwich depth or thickness
E - modulus of elasticity
E' - effective modulus
F - allowable stress
f - applied stress
G - modulus of rigidity
h - distance between facings centroids
J - polar moment of inertia
K - a constant
L - length
M - bending moment
N - load per unit length of edge
P - load
p - distributed load
r - radius; subscript denoting reduced
R - ratio
S - shear load normal to surface of panel
s - core cell size; subscript denoting shear
T - torque
t - thickness; facing without subscript
U - transverse shear stiffness - - - - - - - $U = G_c h^2/t_c$
V - parameter relating shear and bending stiffness
W - weight
w - density
x - axis
y - axis perpendicular to x axis
z - axis normal to surface of sandwich
α - $\sqrt{E'_a/E'_b}$
β - $\alpha\mu_{ab} + 2\gamma$
γ - shear strain; elastic property parameter $= \lambda G'_{ba}/\sqrt{E'_a E'_b}$
δ - deflection
ϵ - strain
λ - $(1 - \mu^2)$
μ - Poisson's ratio

13.2.1 Wrinkling of Facings Under Edgewise Load

Wrinkling of sandwich facings, as shown in Figure 13.4, may occur if a sandwich facing buckles as a plate on an elastic foundation. It may buckle into the core or

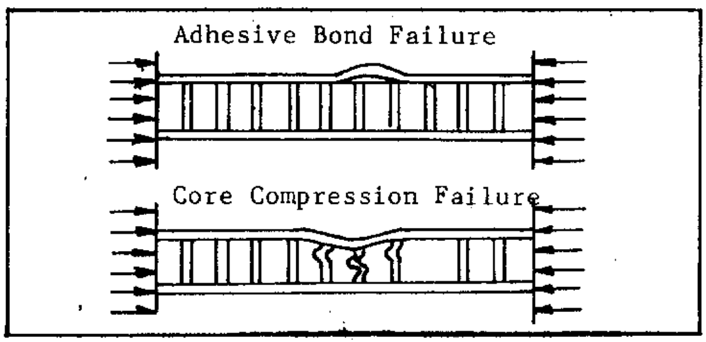


FIGURE 13.4 FACE WRINKLING

away from the core depending on the relative strengths of core in compression and adhesive in flatwise tension.

The facings of a sandwich shall not wrinkle under design load. The wrinkling stress formulas are given for two types of sandwich; sandwich with continuous cores and sandwich with honeycomb cores for which elastic-moduli in the plane of the core are very small compared with the elastic modulus in a direction normal to the core plane.

13.2.1.1 Continuous Core

(1) Determine the parameter q:

$$q = \frac{t_c}{t} G_c \left(\frac{\lambda}{E_f E_c G_c} \right)^{1/3} \qquad 13.2$$

where E_c = core flatwise modulus of elasticity
G_c = core modulus of rigidity

(2) Determine parameter K:

$$K = \frac{E_c \eta}{F_c t_c} \qquad 13.3$$

where η = total amplitude of initial facing waviness usually between .0005 - .005 inch
Fc = flatwise strength of core or bond, whichever is less

(3) Enter Figure 13.5 with q and K and determine value of parameter Q.

(4) Find the stress F_{cr} at which face wrinkling occurs:

$$F_{cr} = Q \left(\frac{E_f E_c G_c}{\lambda} \right)^{1/3} \qquad 13.4$$

(5) Computed compressive stress (f_f) must not exceed F_{cr}.

13.2.1.2 Honeycomb Core

(1) Enter Figure 13.6 with values of η/t_c and F_c/E_c to obtain K.

(2) Enter Figure 13.7 with value for K and $(E_c t/E_f t_c)^{1/2}$ to obtain F_{cr}/E_f.

(3) Solve for F_{cr}.

(4) Computed compressive stress (f_f) must not exceed F_{cr}.

13.2.2 Dimpling of Facings Under Edgewise Load

If the core of a sandwich construction is of cellular (honeycomb) material, it is possible for the facings to buckle or dimple into the spaces between core walls as shown in Figure 13.8. Dimpling of the facings may not lead to failure unless the amplitude of the dimples becomes large and causes the dimples or buckles to grow across core cell walls and result in wrinkling of the faces. Dimpling that does not

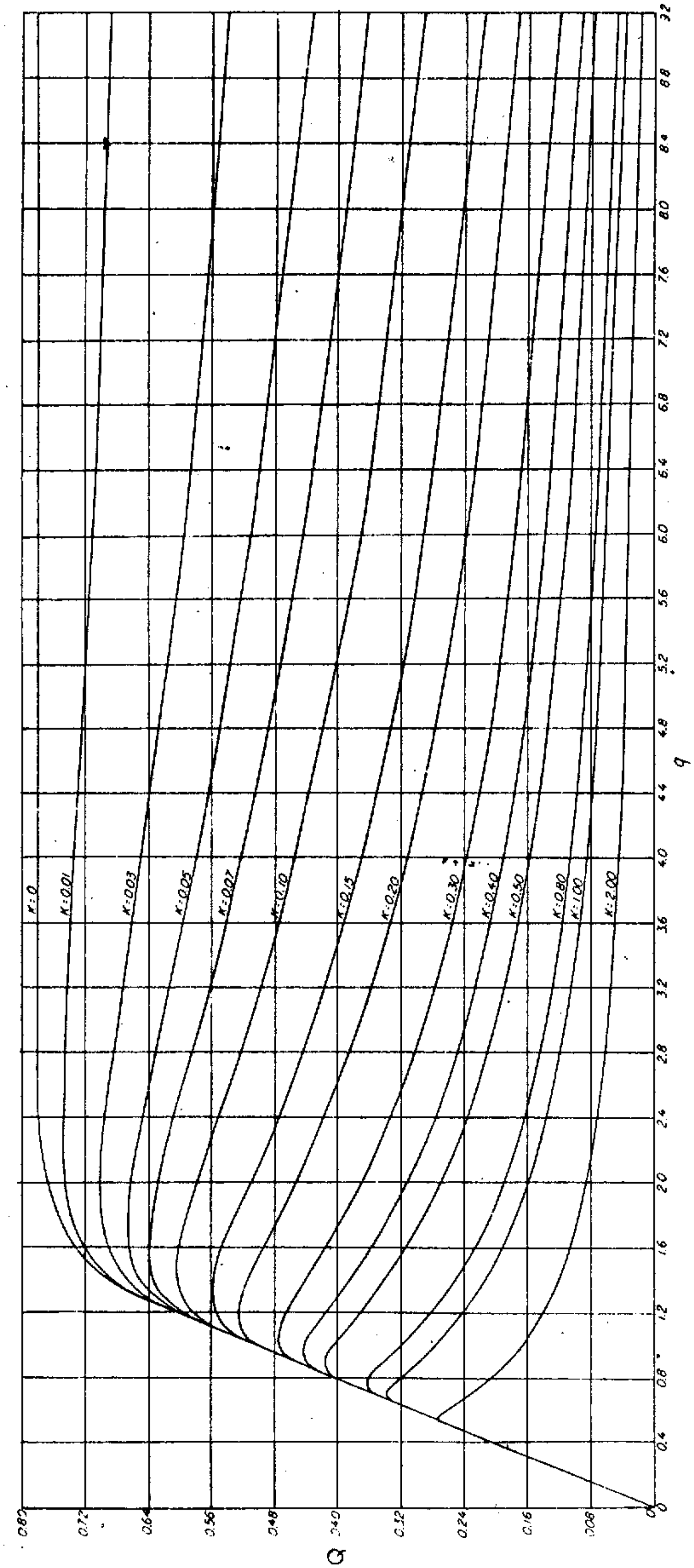


FIGURE 13.5 - PARAMETERS FOR DETERMINING WRINKLING OF FACINGS OF SANDWICH WITH CONTINUOUS CORES

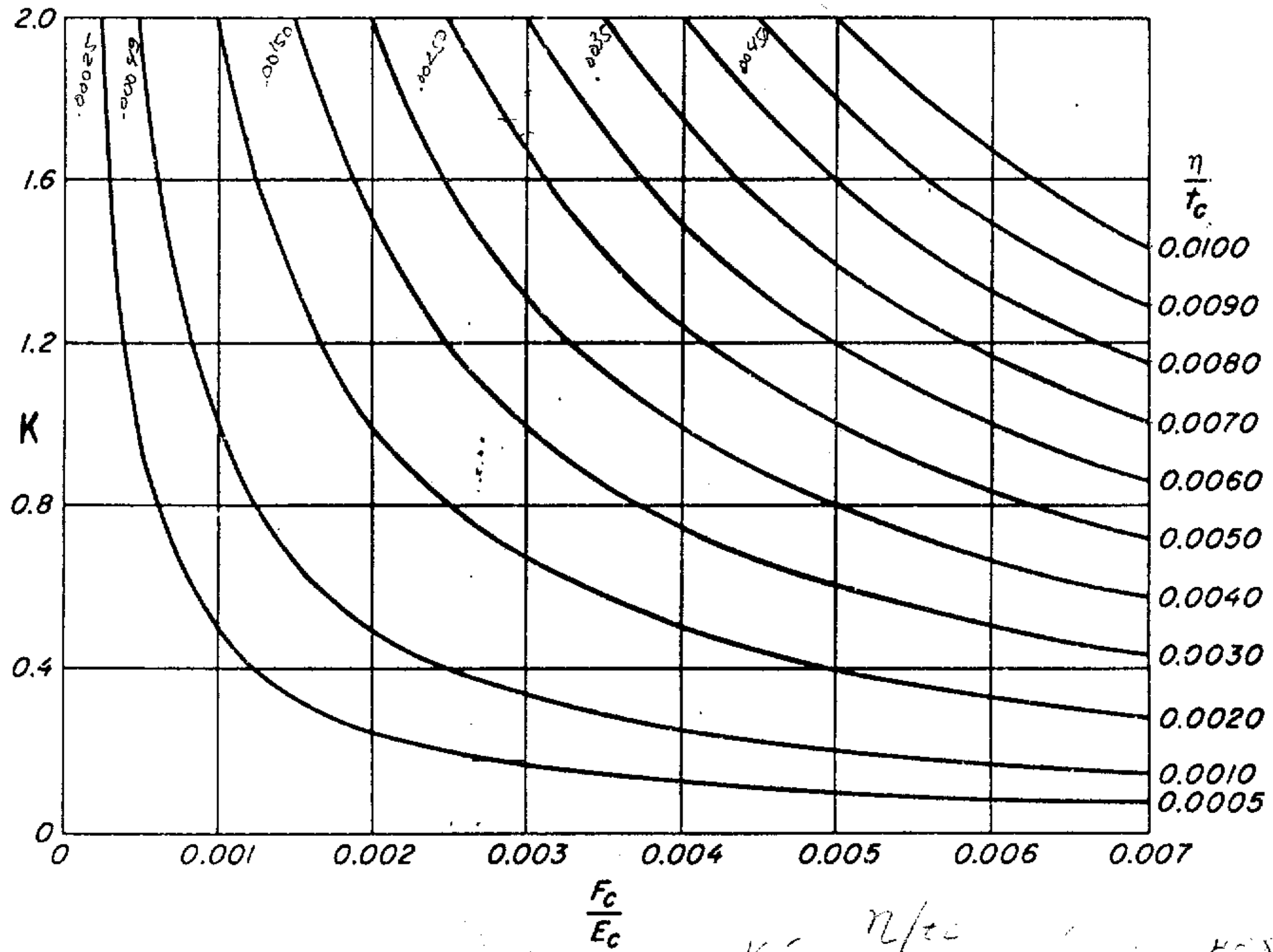


FIGURE 13.6 - RELATIONSHIP OF K TO CORE PROPERTIES
(F_c/E_c) AND FACING WAVINESS (δ/tc)

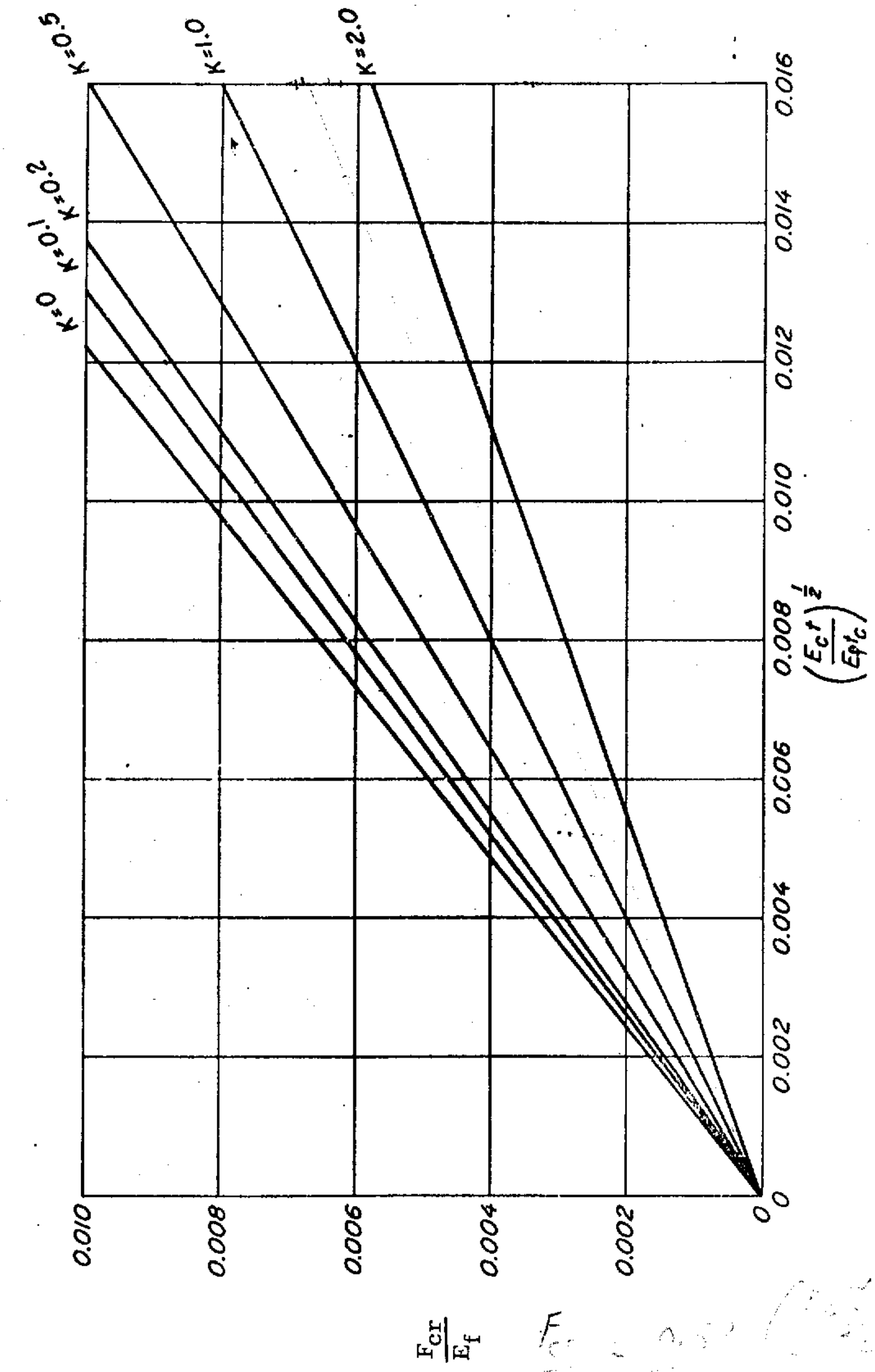


FIGURE 13.7 - GRAPH FOR THE WRINKLING STRESS OF FACINGS OF SANDWICH WITH HONEYCOMB CORES

cause total structural failure may, of course, be severe enough so that permanent dimples remain after removal of load.

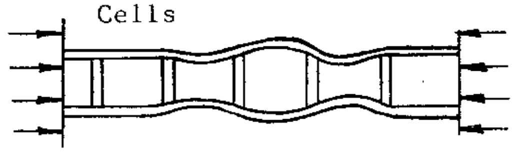


FIGURE 13.8 INTRACELL BUCKLING (FACE DIMPLING)

If dimpling of the facings is not permissible, the core cell size shall be small enough so that dimpling will not occur under design loads. It is assumed that failure in the facing-to-core bond cannot occur prior to dimpling.

Figure 13.9 can be used to determine the critical facing stress (stress at which dimpling will occur). The curves in Figure 13.9 represent a plot of equation 13.5 which can be used instead of Figure 13.9.

$$\frac{F_{cr}}{E_f} = \frac{2}{\lambda}\left(\frac{t}{s}\right)^2 \qquad 13.5$$

13.2.3 Flat Rectangular Panels with Edgewise Compression

The method presented here is used in design of a flat, rectangular sandwich panel subjected to edge compression. The panel is simply supported at the four edges and the load is applied equally and uniformly to the facings at two opposite edges as shown in Figure 13.10.

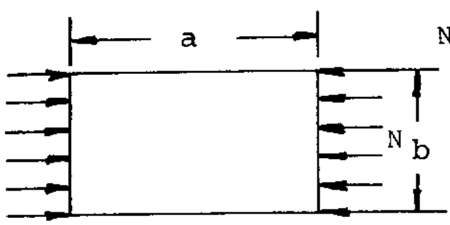


FIGURE 13.10 COMPRESSION PANELS

Overall buckling of the sandwich or dimpling or wrinkling of the facings cannot occur without possible total collapse of the panel. Detailed procedures follow giving theoretical formulas and graphs for determining dimensions of the facings and core, as well as necessary core properties. Facing modulus of elasticity, E, and stress values, F, shall be compression values at the conditions of use; that is, if

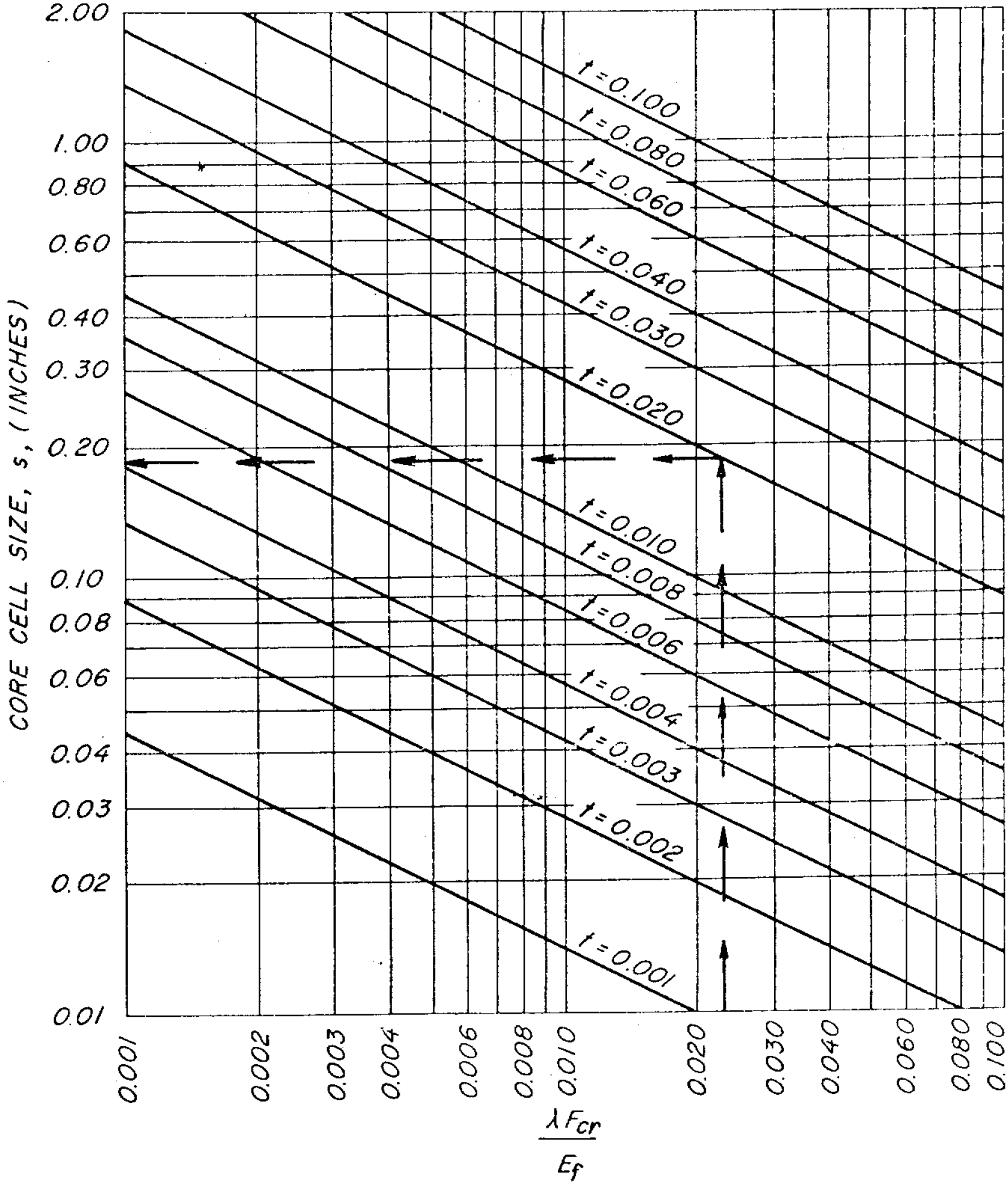


FIGURE 13.9 - CHART FOR DETERMINING CELLULAR CORE CELL SIZE SUCH THAT DIMPLING (INTRACELL BUCKLING) OF SANDWICH FACING WILL NOT OCCUR

application is at elevated temperature, then facing properties at elevated temperature shall be used in design. The facing modulus of elasticity is the effective value at the facing stress. If this stress is beyond the proportional limit value, an appropriate tangent, reduced or modified compression modulus shall be used.

(1) Choose an allowable design compressive stress (F_f) and determine the required facing thickness from

$$t_1 F_{f1} + t_2 F_{f2} = N; \text{ unequal faces} \qquad 13.6$$

$$t = N/2F_f; \text{ equal faces} \qquad 13.7$$

When the elastic modulus of one face is different from the elastic modulus of the other face, equation 13.6 must be satisfied, but also the stresses F_{f1} and F_{f2} must be chosen so that

$$\frac{F_{f1}}{E_1} = \frac{F_{f2}}{E_2} \qquad 13.8$$

The lower of the ratios in equation 13.8 must be used for design, otherwise the face with the lower ratio will be overstressed.

(2) The critical facing stress (F_{cr}) at which buckling of the panel will occur is

$$F_{cr} = \pi^2 K \frac{E_{f1} t_1 E_{f2} t_2}{(E_{f1} t_1 + E_{f2} t_2)^2} \left(\frac{h}{b}\right)^2 \left(\frac{E_f}{\lambda}\right) \qquad 13.9$$

where E_f and λ are values for the facing with least F_f/E_f ratio as determined from equation 13.8.

If the facings are of equal thickness and of the same material, equation 13.9 becomes

$$F_{cr} = \frac{\pi^2 K}{4} \left(\frac{h}{b}\right)^2 \left(\frac{E_f}{\lambda}\right) \qquad 13.10$$

In equations 13.9 and 13.10

$$K = K_M + K_F \qquad 13.11$$

K is determined first by going through the following steps 3 to 8.

(3) Determine the value of parameters:

$$a/b \text{ or } b/a, \text{ whichever is } < 1 \qquad 13.12$$

$$\frac{E_{f2}t_2}{E_{f1}t_1} \qquad 13.13$$

$$\frac{F_f\lambda}{E_f} \qquad 13.14$$

(4) Enter the appropriate figure (13.11, 13.12, 13.13 or 13.14) with equation 13.12 at V = .01 (Choosing a low finite value of V to start with since V = 0 gives h as a minimum and G_c as infinite). Project laterally to parameter 13.13 then vertically down to parameter 13.14 and horizontally to h/a. Evaluate h.

(5) Determine core thickness from

$$t_c = h - \frac{t_1 + t_2}{2}\text{; for unequal faces} \qquad 13.15$$

$$t_c = h - t\text{; for equal faces} \qquad 13.16$$

(6) Determine constant K' in the equation $V = K'/G_c$ from

$$K' = \frac{\pi^2 t_c E_{f1} t_1 E_{f2} t_2}{\lambda a^2 (E_{f1}t_1 + E_{f2}t_2)}\text{; for unequal faces} \qquad 13.17$$

$$K' = \frac{\pi^2 t_c E_f t}{2\lambda a^2}\text{; for equal faces} \qquad 13.18$$

(7) Determine tentative core modulus of rigidity (G_c) from $G_c = K'/V$ for V = .01. If this value of G_c is not within the range available in the desired core material and type, enter Figure 13.15. Project diagonally along the line $V = K'/G_c$ until a practical value is reached. For the new value of V, repeat steps (4), (5) and (6).

(8) From the appropriate Figure (13.16 through 13.27) find the value of K_M. From Figure 13.28 obtain the value of K_{MO}. Find K_F

$$K_F = \frac{(E_{f1}t_1^3 + E_{f2}t_2^3)(E_{f1}t_1 + E_{f2}t_2)}{12E_{f1}t_1E_{f2}t_2h^2} K_{MO} \qquad 13.19$$

$$K_F = \frac{t^2 K_{MO}}{3h^2}\text{; for equal faces} \qquad 13.20$$

For values of b/a greater than those in Figure 13.28 assume $K_F = 0$.

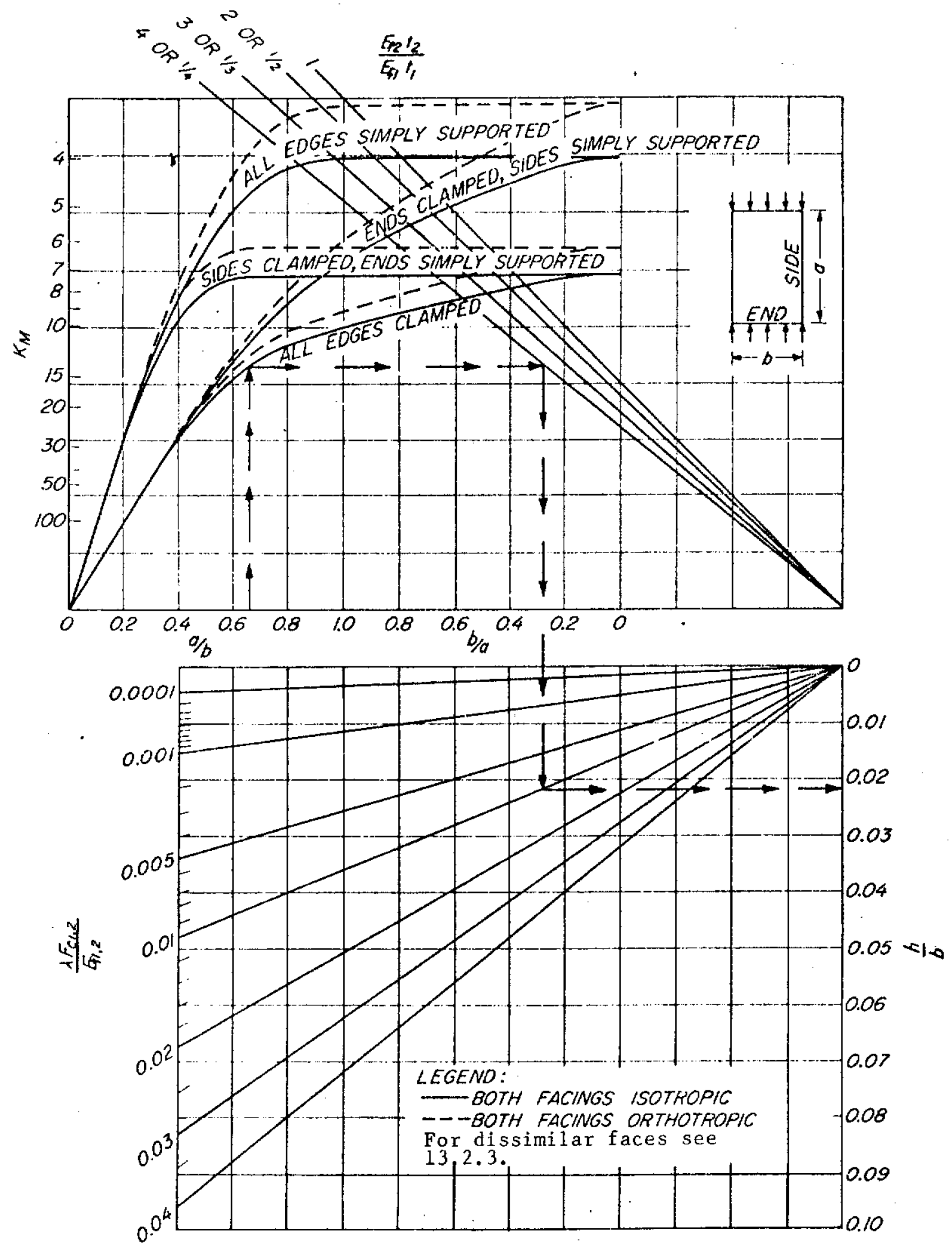


FIGURE 13.11 - CHART FOR DETERMINING h/b RATIO (V=D) SUCH THAT A SANDWICH PANEL WILL NOT BUCKLE UNDER EDGEWISE COMPRESSION LOAD

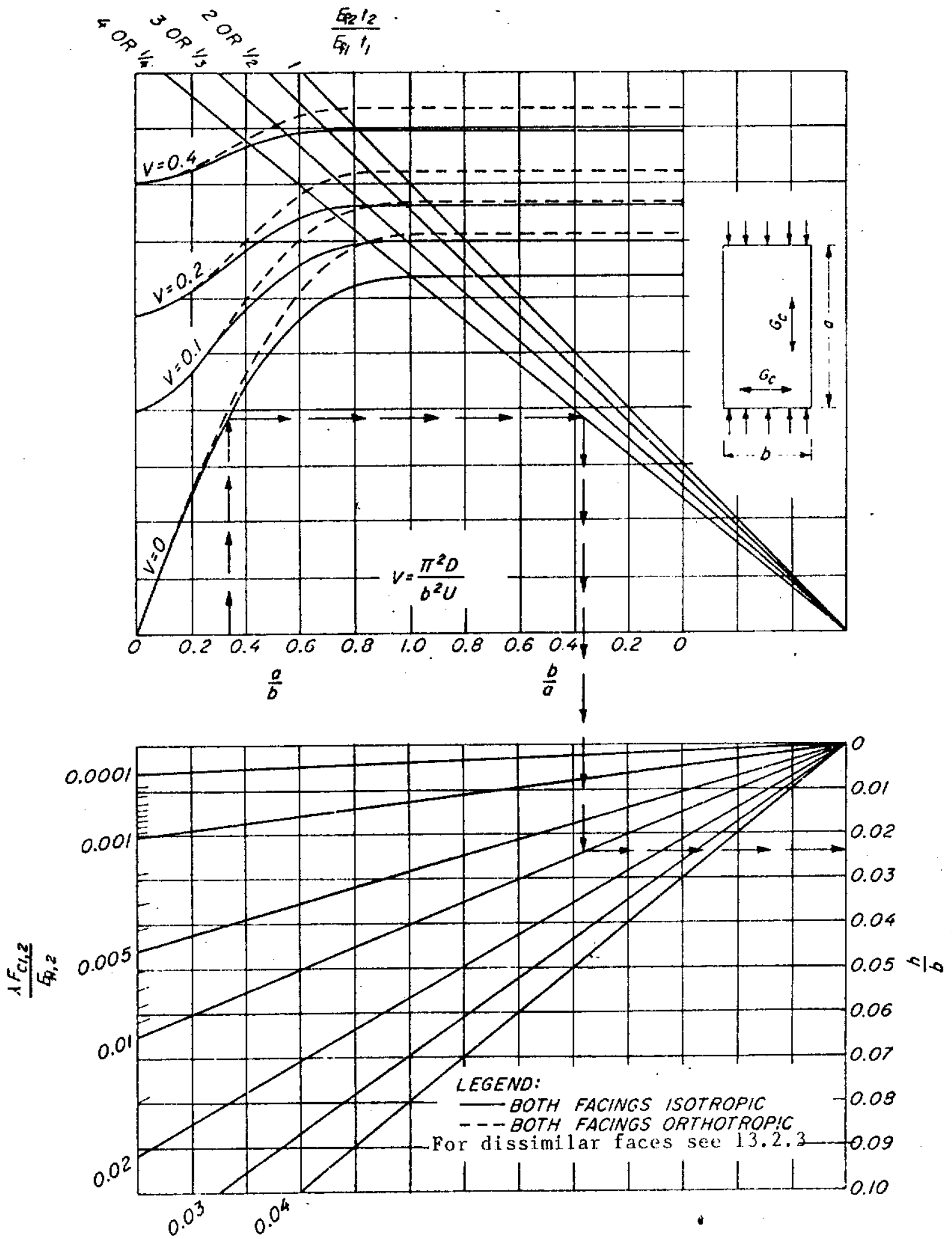


FIGURE 13.12 - CHART FOR DETERMINING h/b RATIO SUCH THAT A SIMPLY SUPPORTED SANDWICH PANEL WITH ISOTROPIC ($G_{cb} = G_{ca}$) WILL NOT BUCKLE UNDER EDGEWISE COMPRESSION LOAD

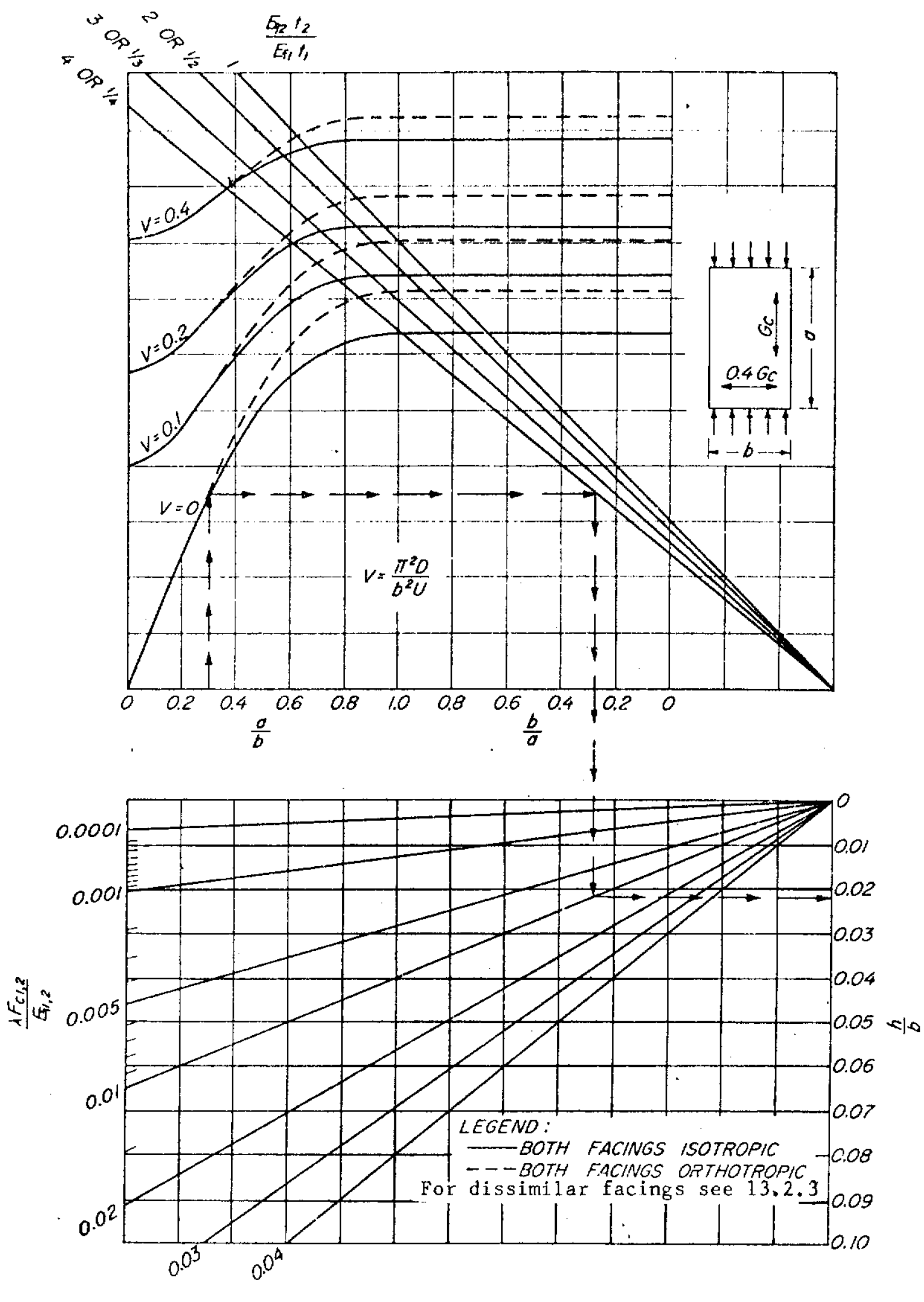


FIGURE 13.13 - CHART FOR DETERMINING h/b RATIO SUCH THAT A SIMPLY SUPPORTED SANDWICH PANEL WITH ORTHOTROPIC CORE ($G_{cb} = 0.4\ G_{ca}$) WILL NOT BUCKLE UNDER EDGEWISE COMPRESSION LOAD.

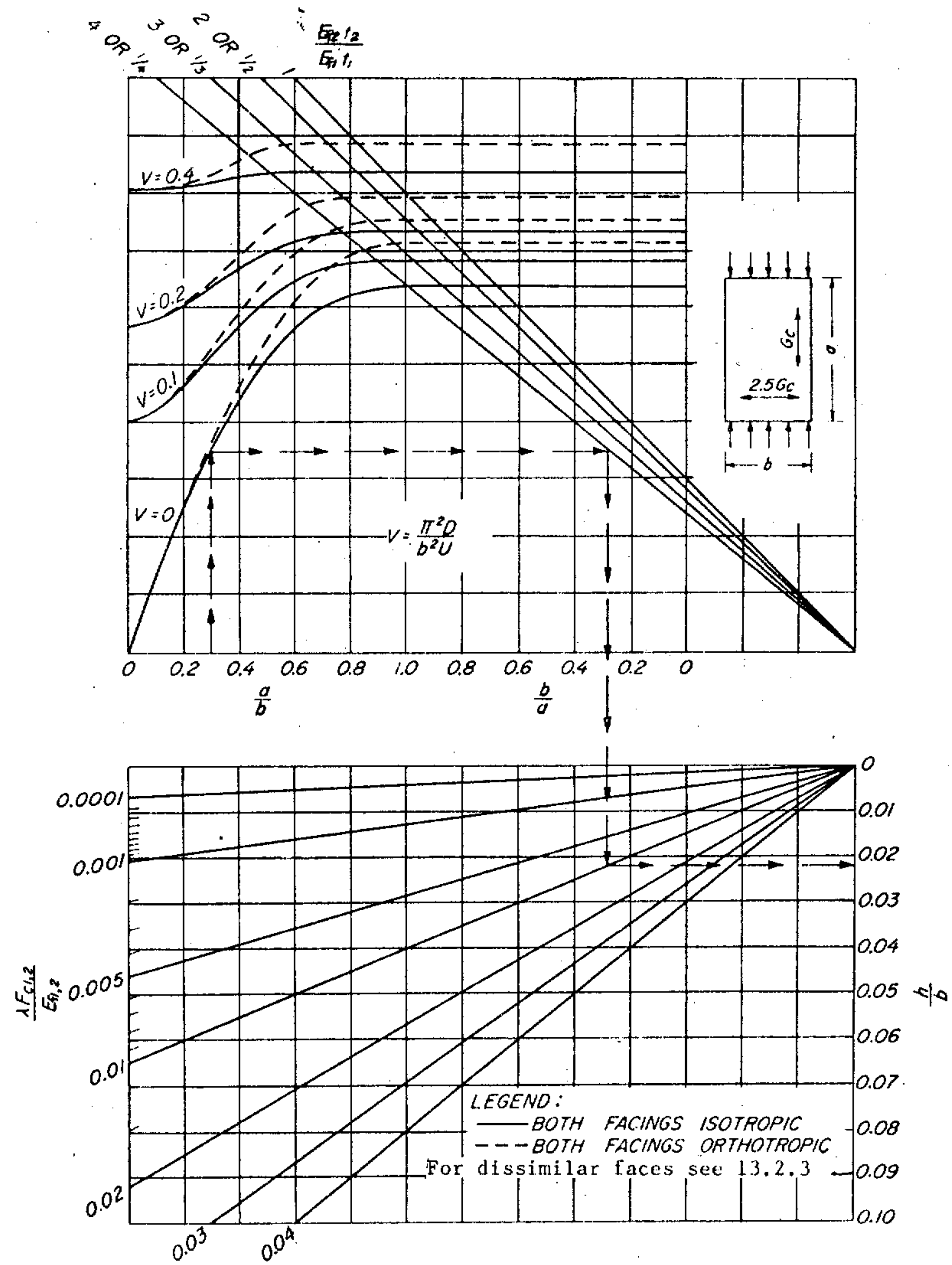


FIGURE 13.14 - CHART FOR DETERMINING h/b RATIO SUCH THAT A SIMPLY SUPPORTED SANDWICH PANEL WITH ORTHOTROPIC CORE ($G_{cb} = 2.5\ G_{ca}$) WILL NOT BUCKLE UNDER EDGEWISE COMPRESSION LOAD.

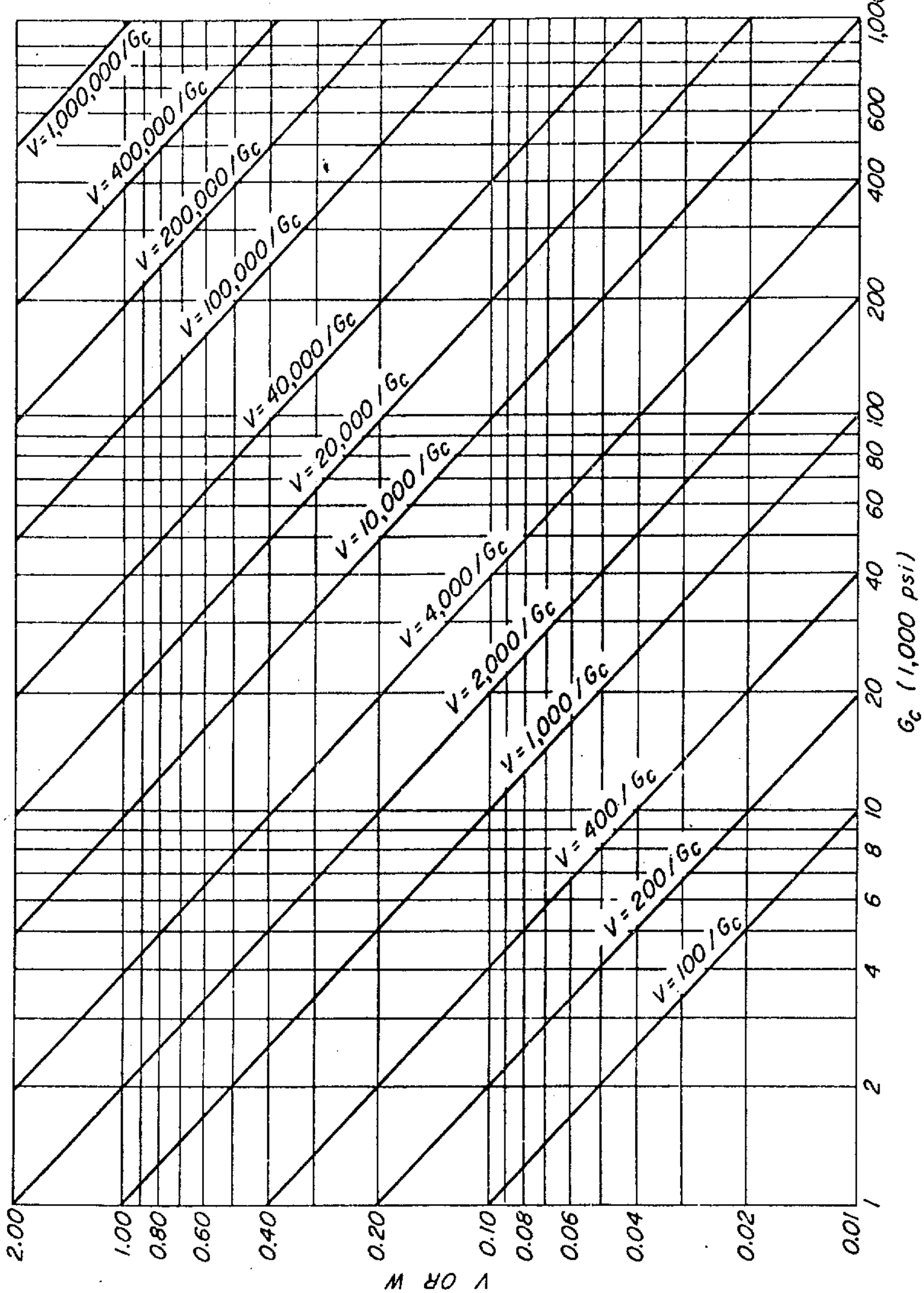


FIGURE 13.15 - CHART FOR DETERMINING V OR W AND G_c FOR SANDWICH IN EDGEWISE COMPRESSION

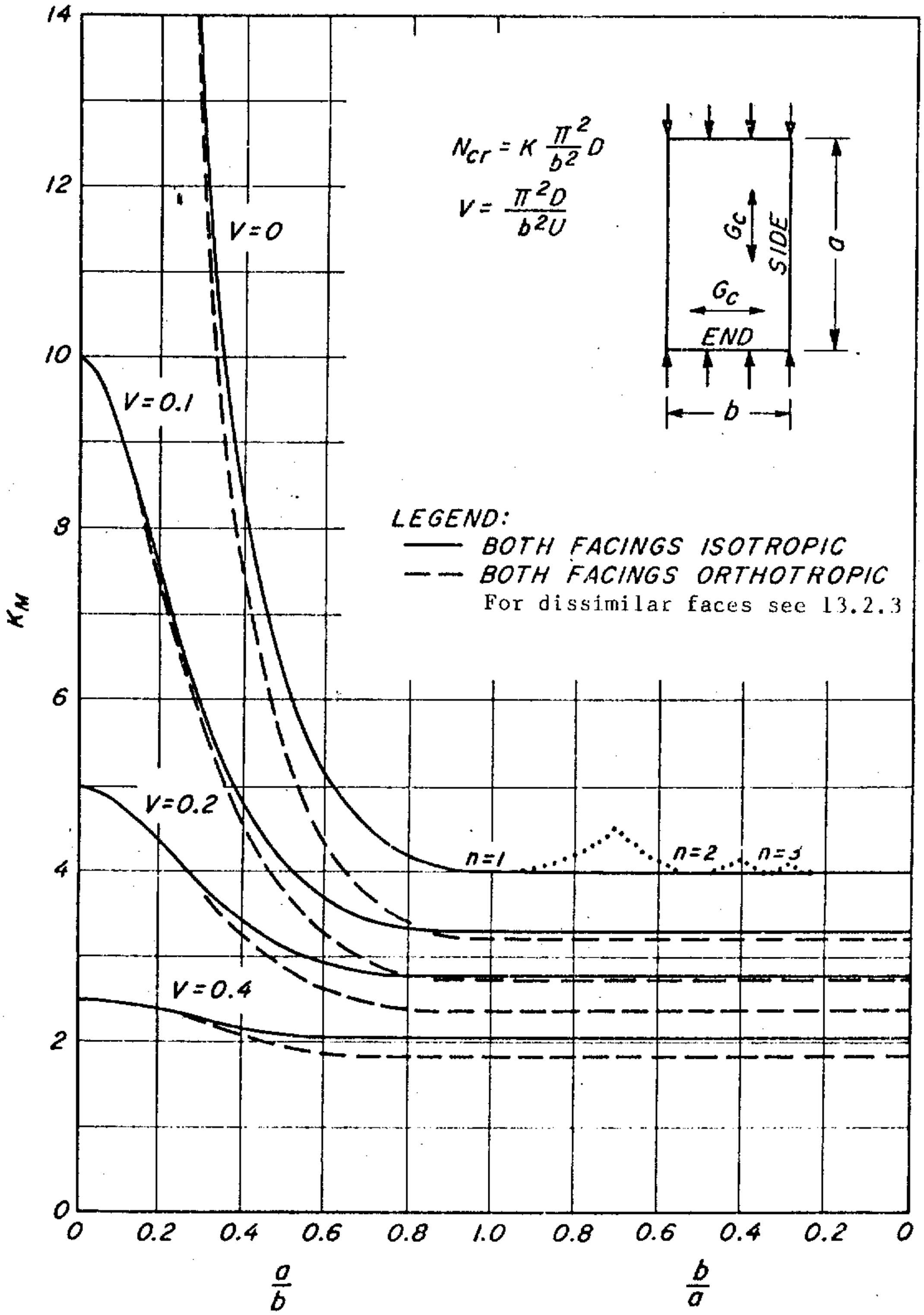


FIGURE 13.16 - K_M FOR SANDWICH PANEL WITH ENDS AND SIDES SIMPLY SUPPORTED AND ISOTROPIC CORE. ($G_{cb} = G_{ca}$).

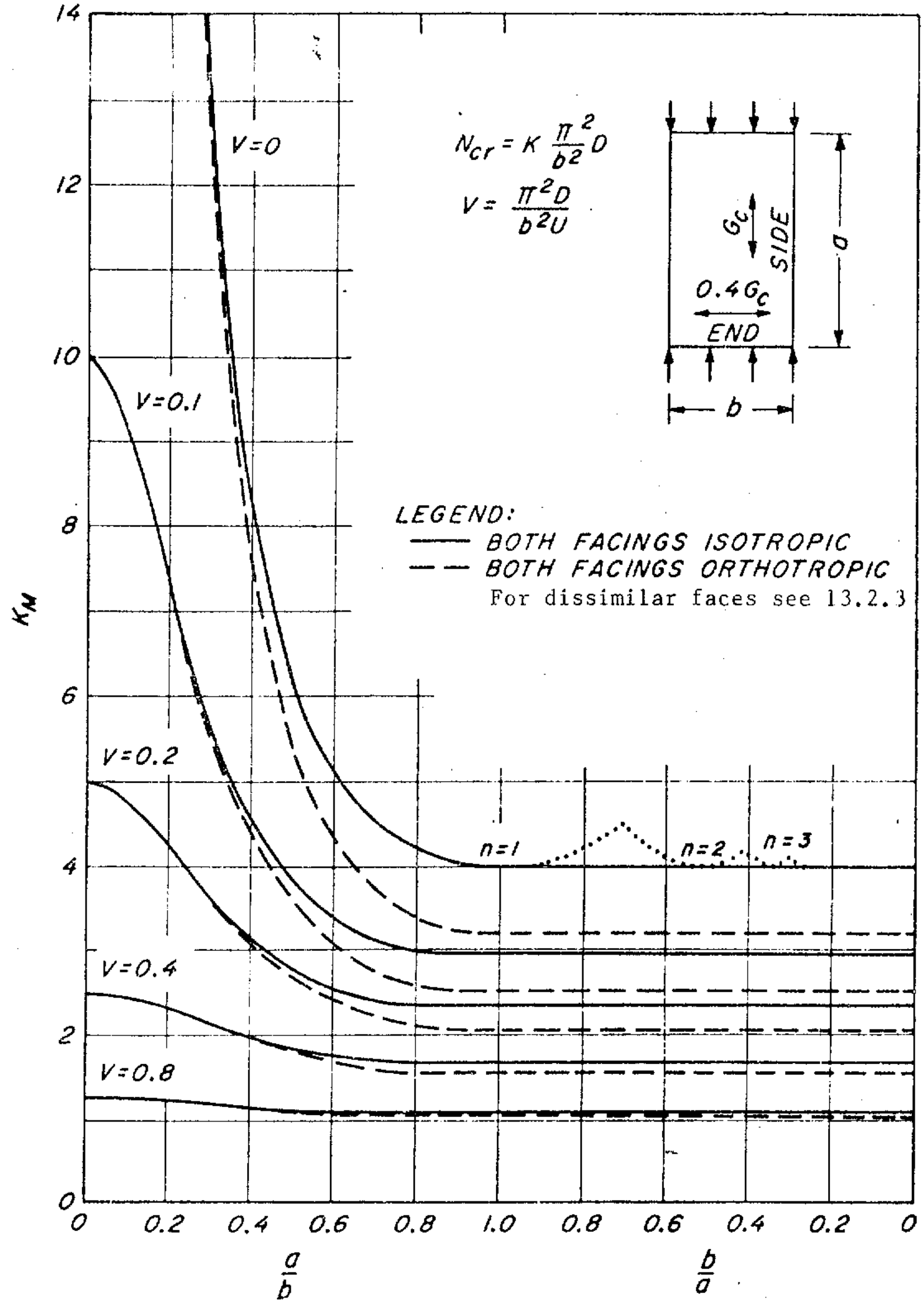


FIGURE 13.17 - K_M FOR SANDWICH PANEL WITH ENDS AND SIDES SIMPLY SUPPORTED AND ORTHOTROPIC CORE. ($G_{cb} = 0.4\ G_{ca}$).

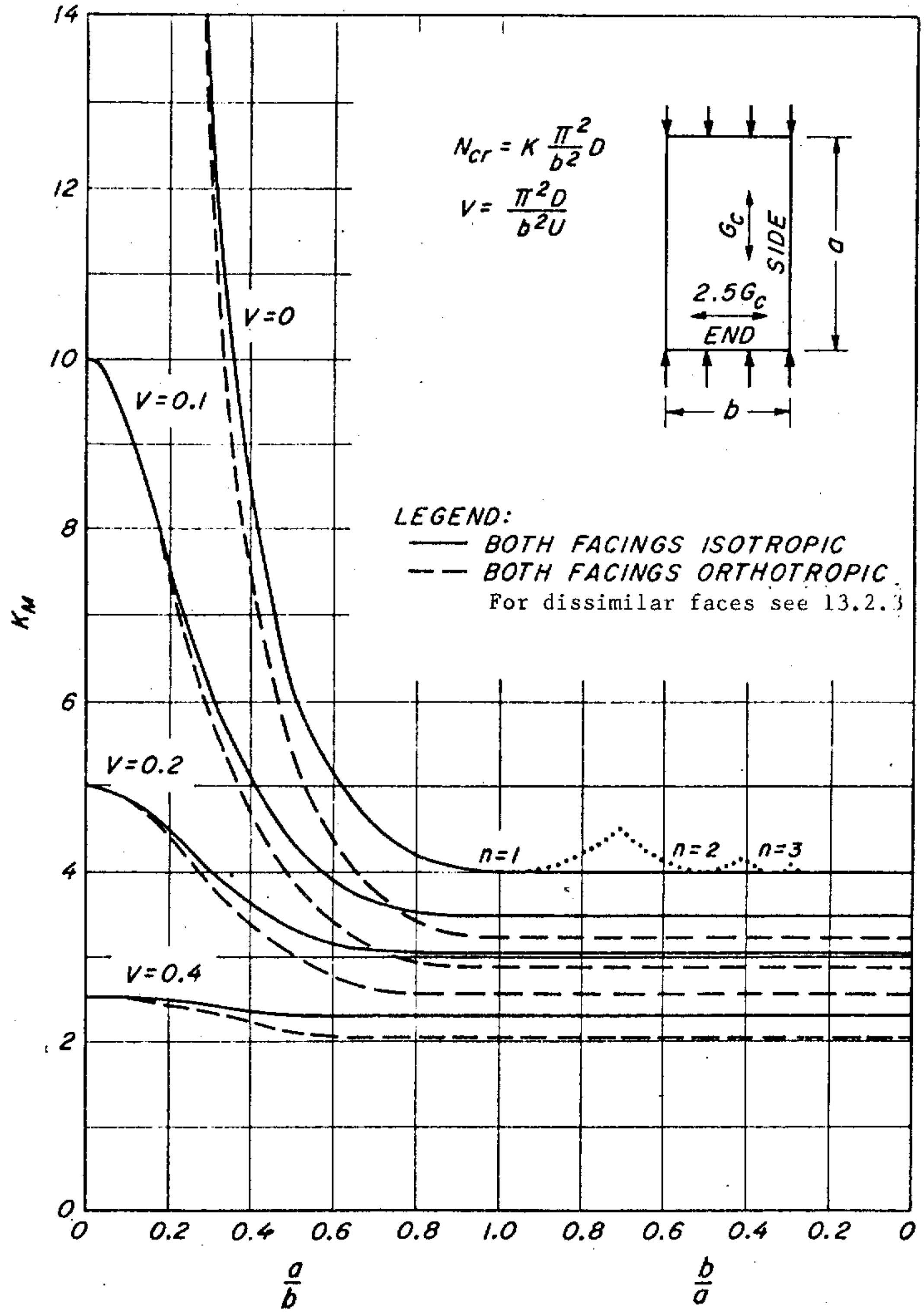


FIGURE 13.18 - K_M FOR SANDWICH PANEL WITH ENDS AND SIDES SIMPLY SUPPORTED AND ORTHOTROPIC CORE ($G_{cb} = 2.5\ G_{ca}$).

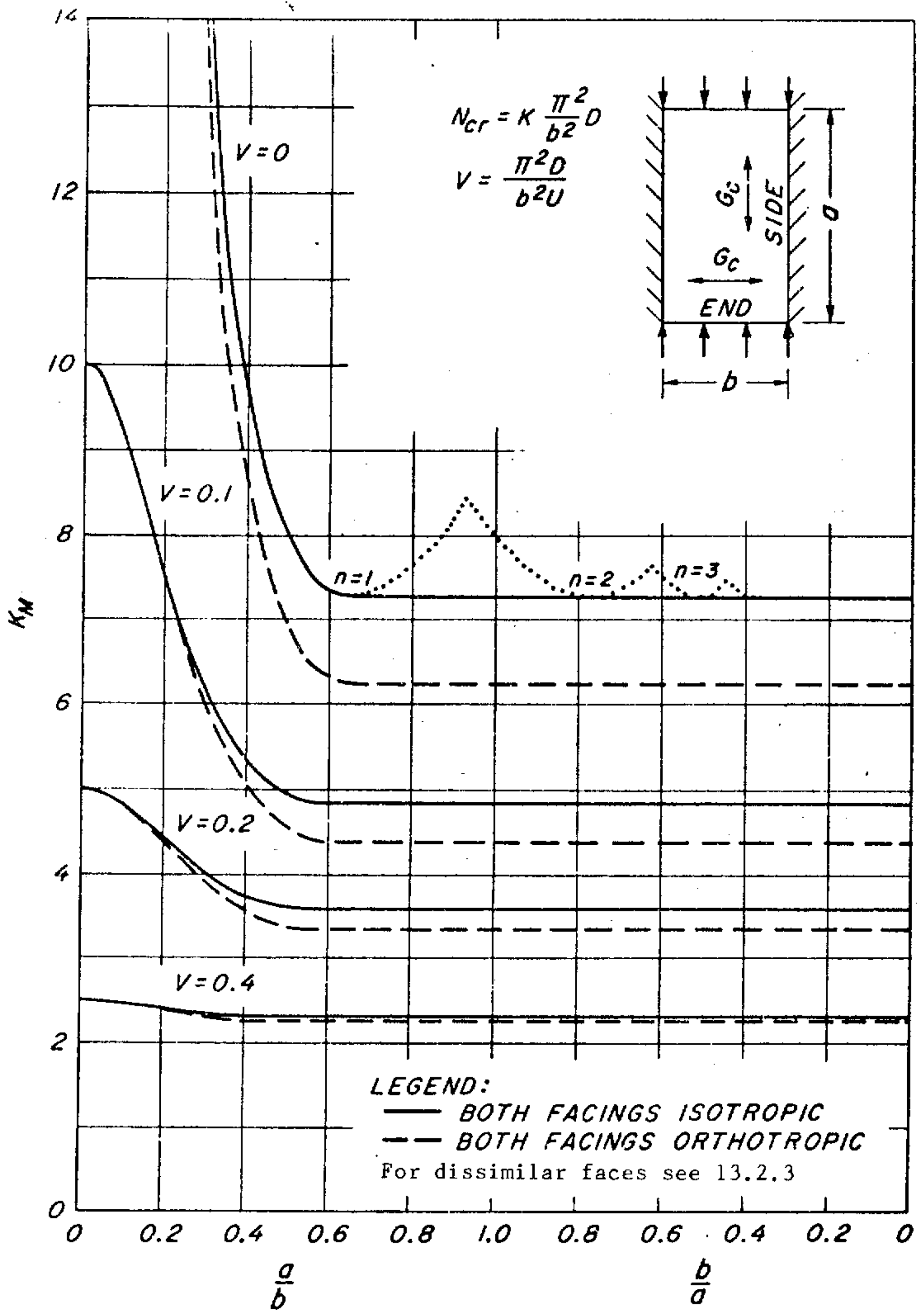


FIGURE 13.19 - K_M FOR SANDWICH PANEL WITH ENDS SIMPLY SUPPORTED AND SIDES CLAMPED AND ISOTROPIC CORE, ($G_{cb}=G_{ca}$).

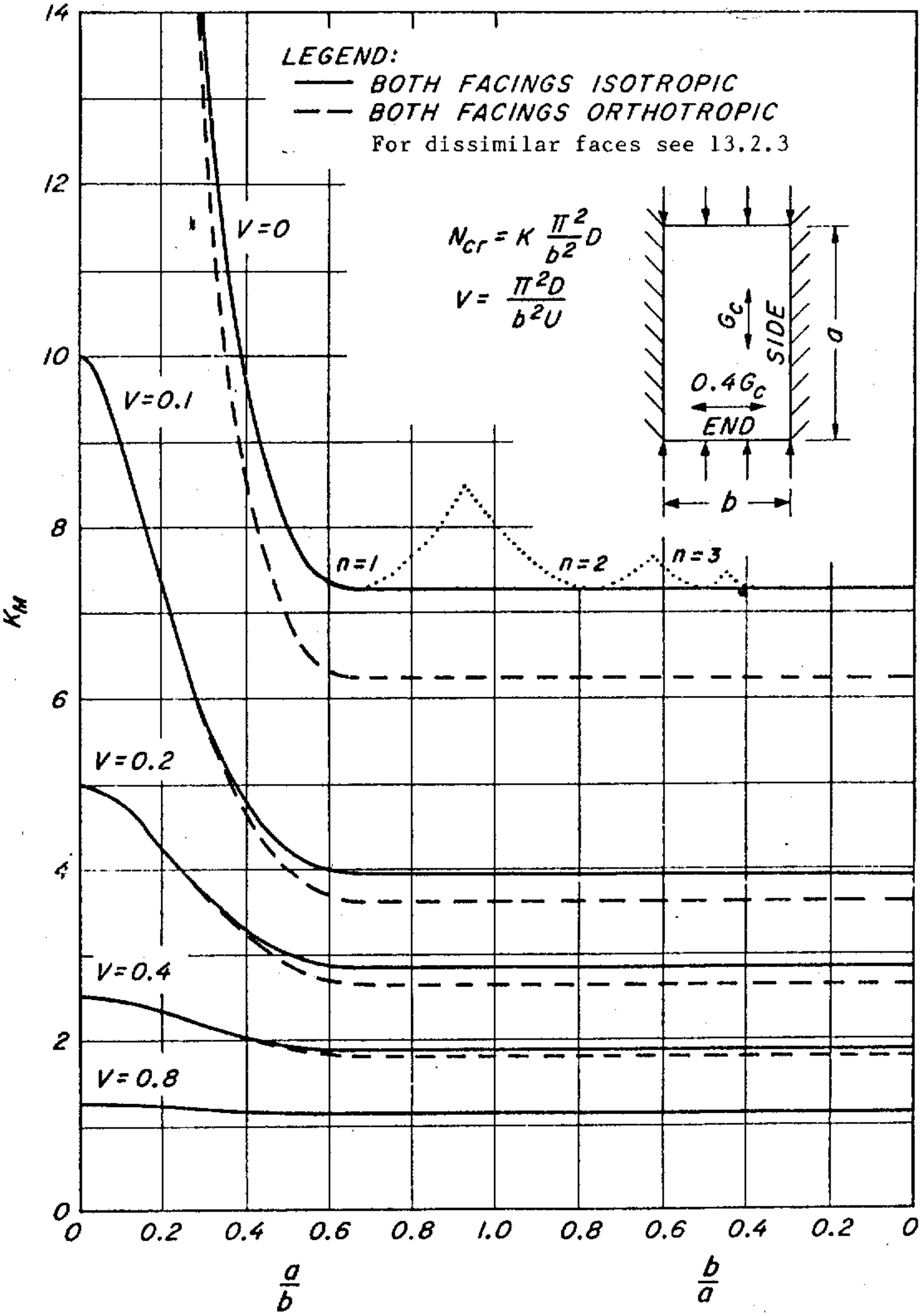


FIGURE 13.20 - K_M FOR SANDWICH PANEL WITH ENDS SIMPLY SUPPORTED AND SIDES CLAMPED AND ORTHOTROPIC CORE (G_{cb}=0.4 G_{ca}).

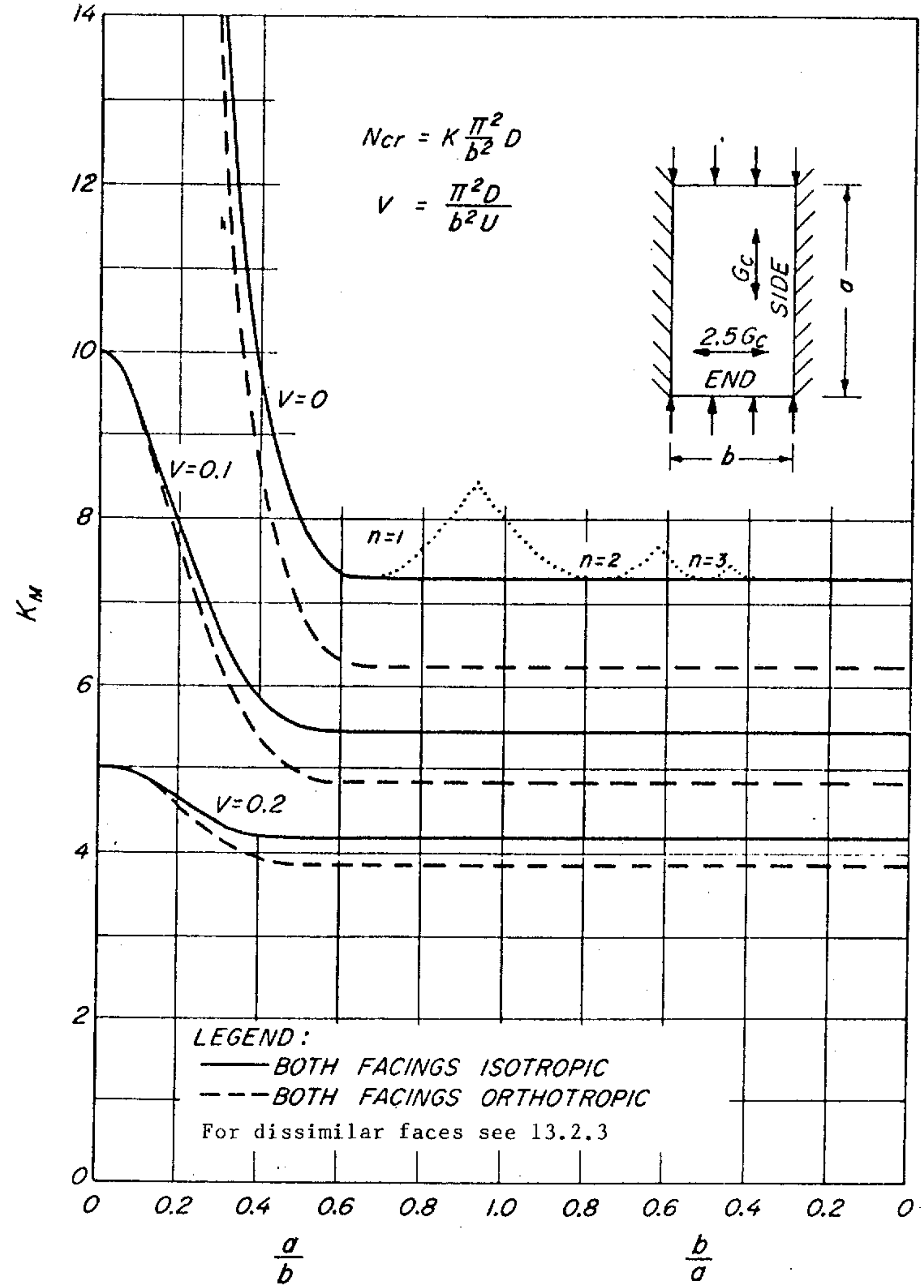


FIGURE 13.21 - K_M FOR SANDWICH PANEL WITH ENDS SIMPLY SUPPORTED AND SIDES CLAMPED AND ORTHOTROPIC CORE ($G_{cb}=2.5\ G_{ca}$).

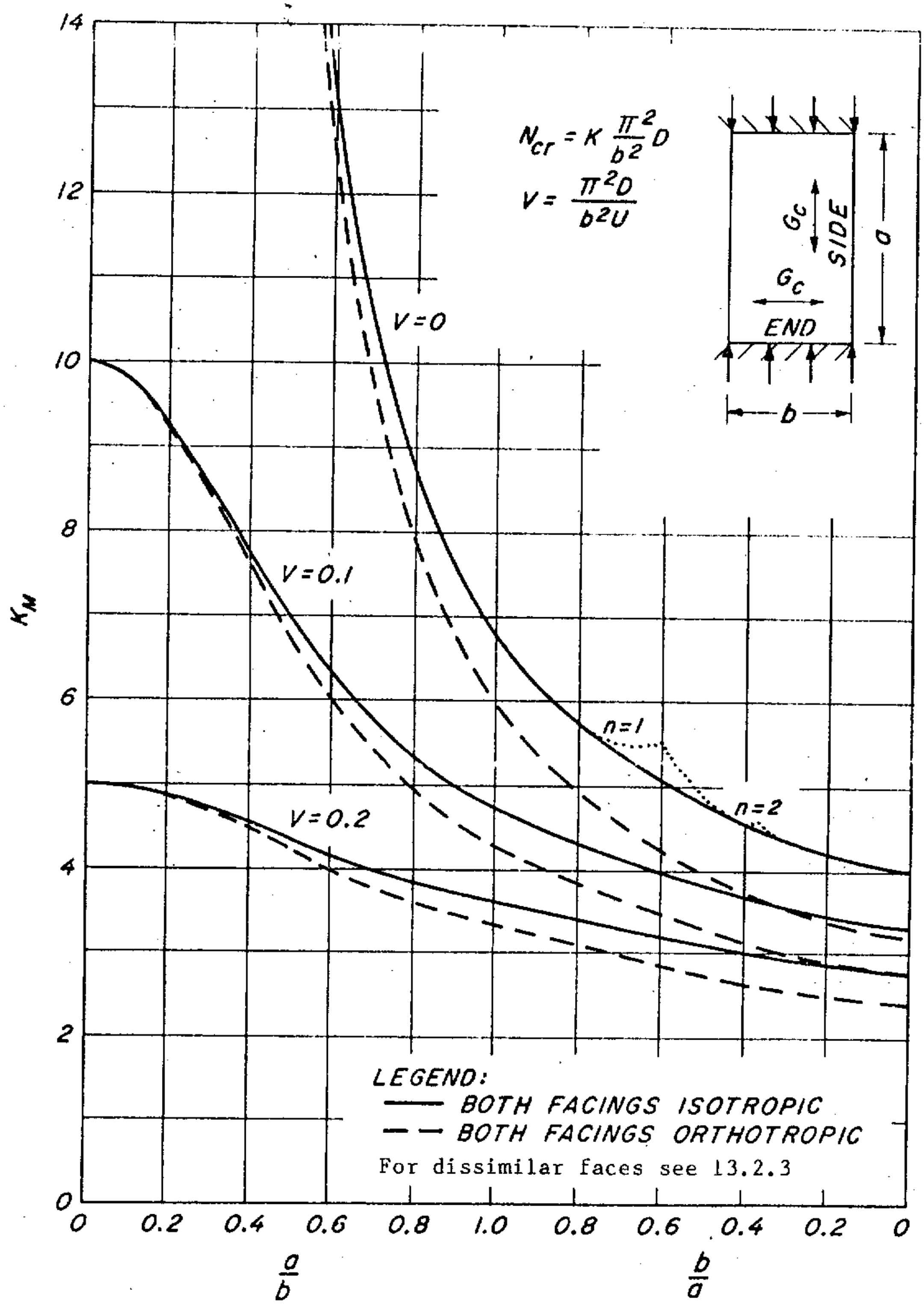


FIGURE 13.22 - K_M FOR SANDWICH PANEL WITH ENDS CLAMPED AND SIDES SIMPLY SUPPORTED AND ISOTROPIC CORE (G_{cb}=G_{ca}).

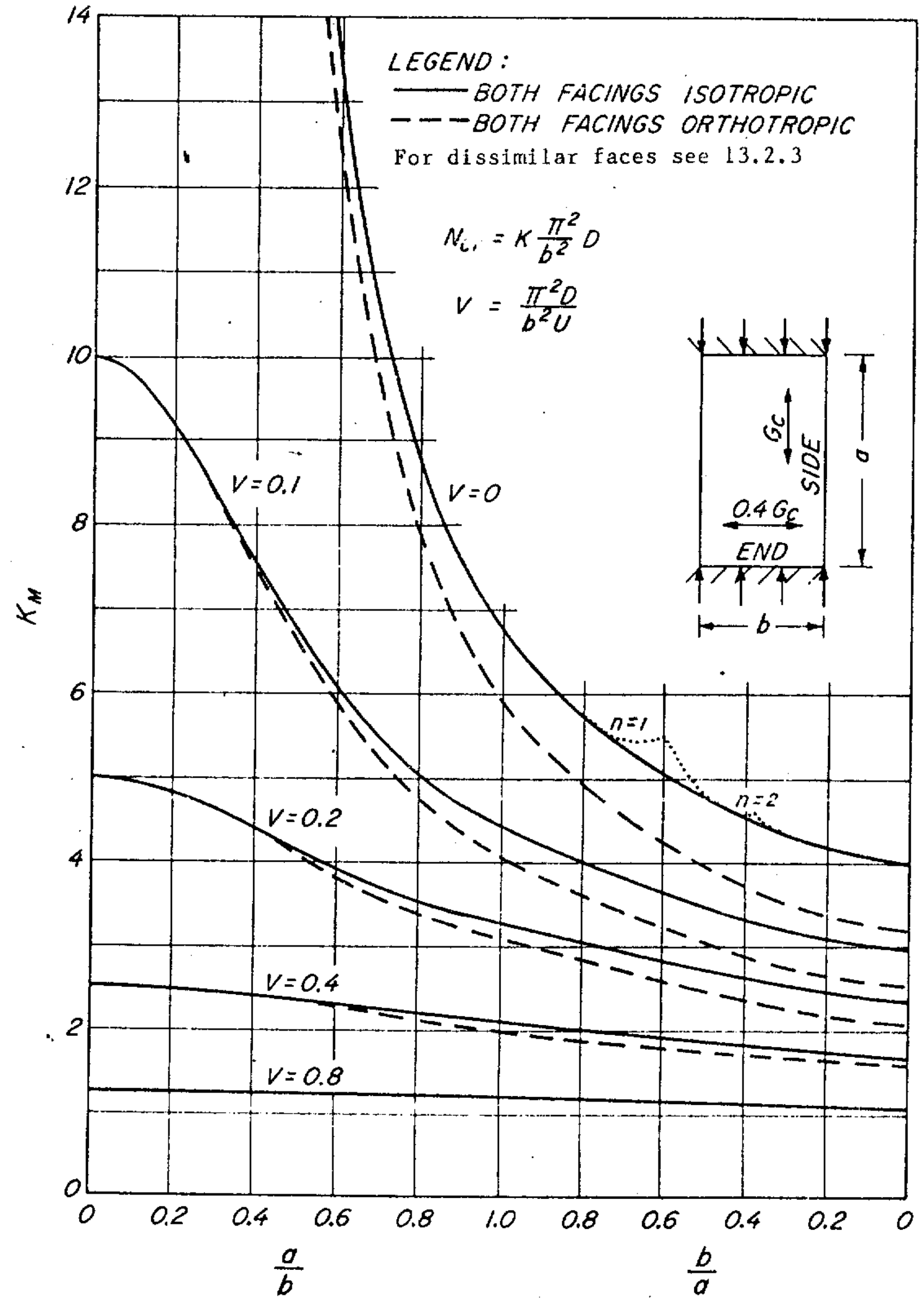


FIGURE 13.23 - K_M FOR SANDWICH PANEL WITH ENDS CLAMPED AND SIDES SIMPLY SUPPORTED AND ORTHOTROPIC CORE ($G_{cb} = 0.4\ G_{ca}$)

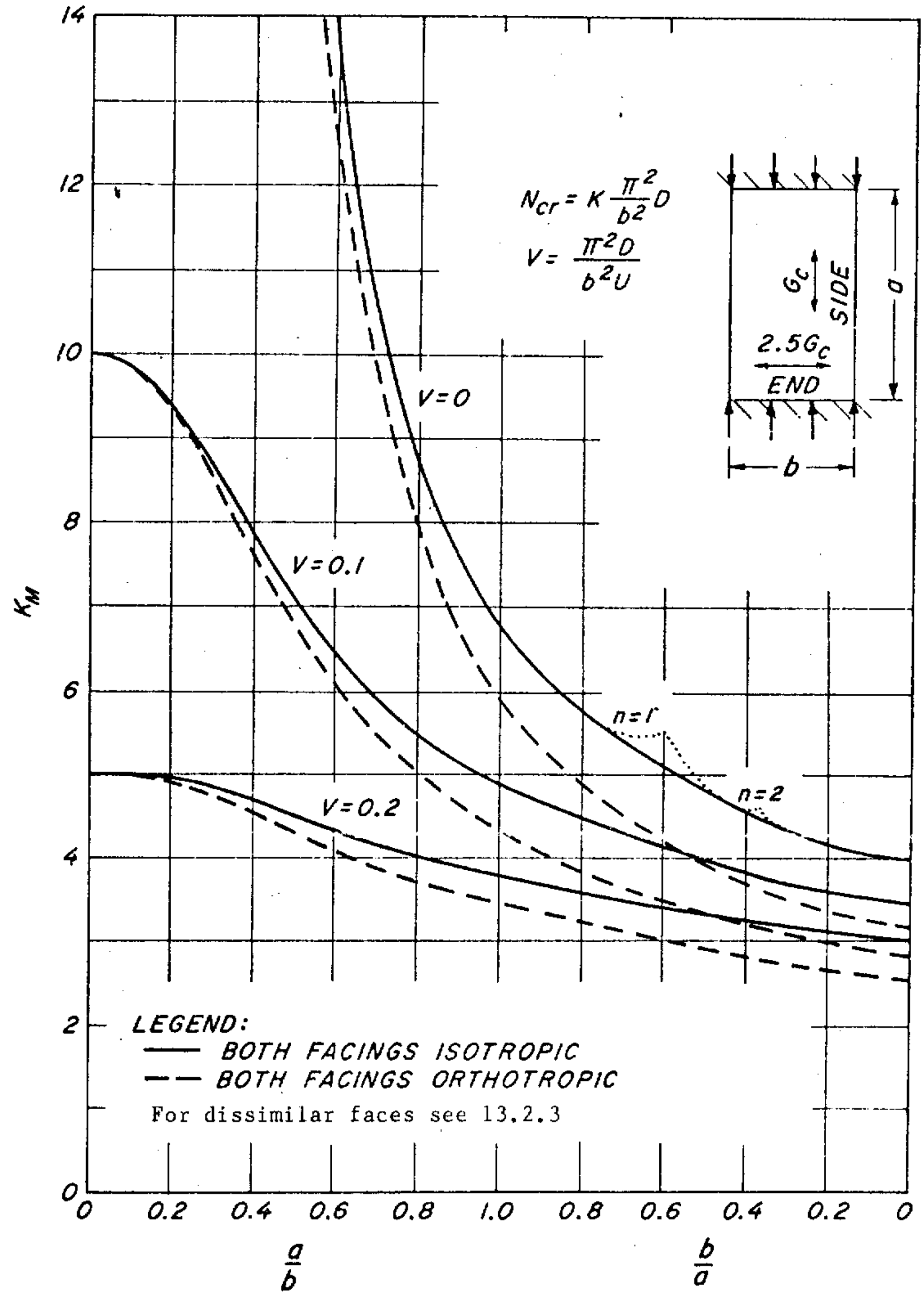


FIGURE 13.24 - K_M FOR SANDWICH PANEL WITH ENDS CLAMPED AND SIDES SIMPLY SUPPORTED AND ORTHOTROPIC CORE (G_{cb}=2.5 G_{ca})

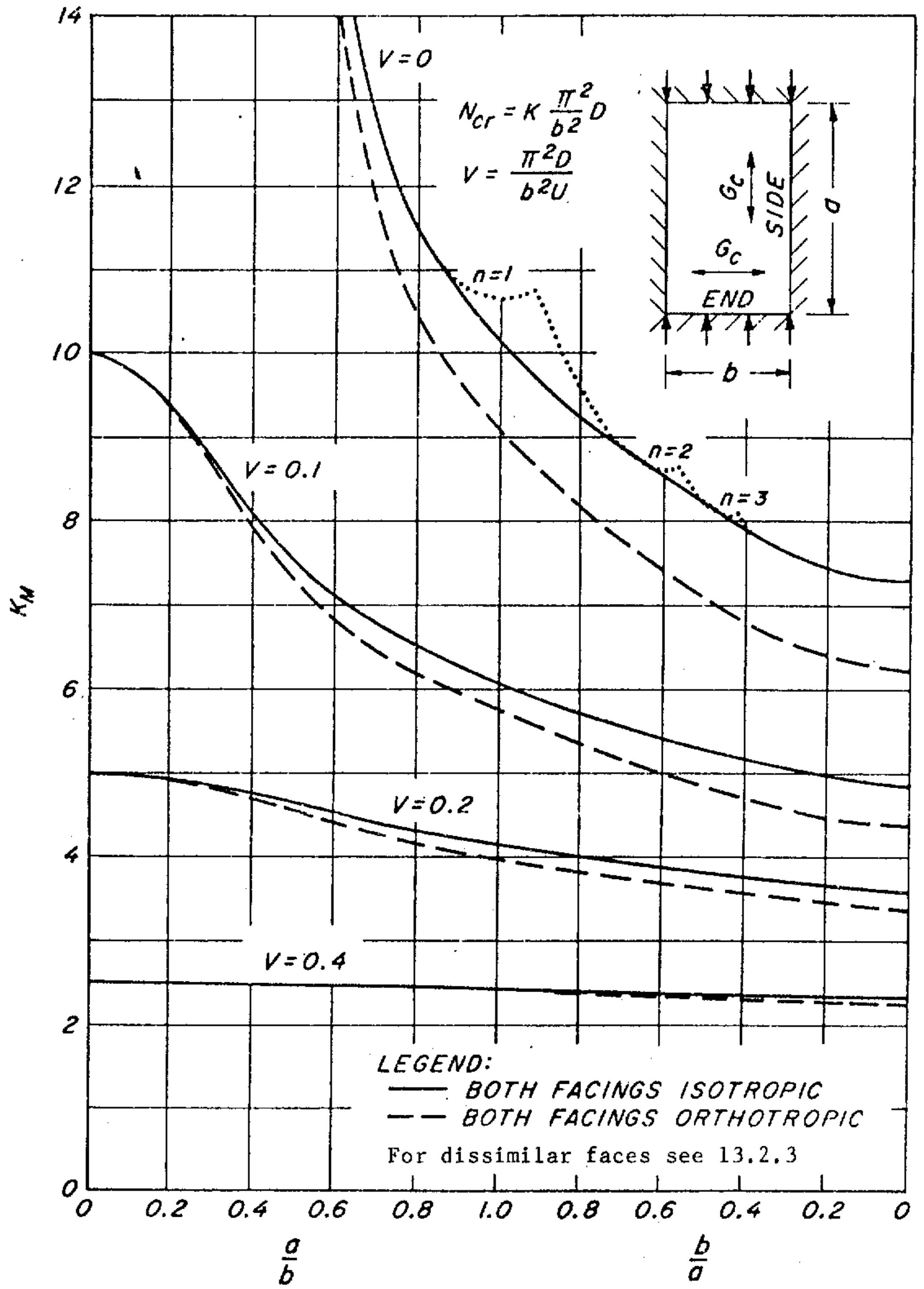


FIGURE 13.25 - K_M FOR SANDWICH PANEL WITH ENDS AND SIDES CLAMPED AND ISOTROPIC CORE ($G_{cb}=G_{ca}$).

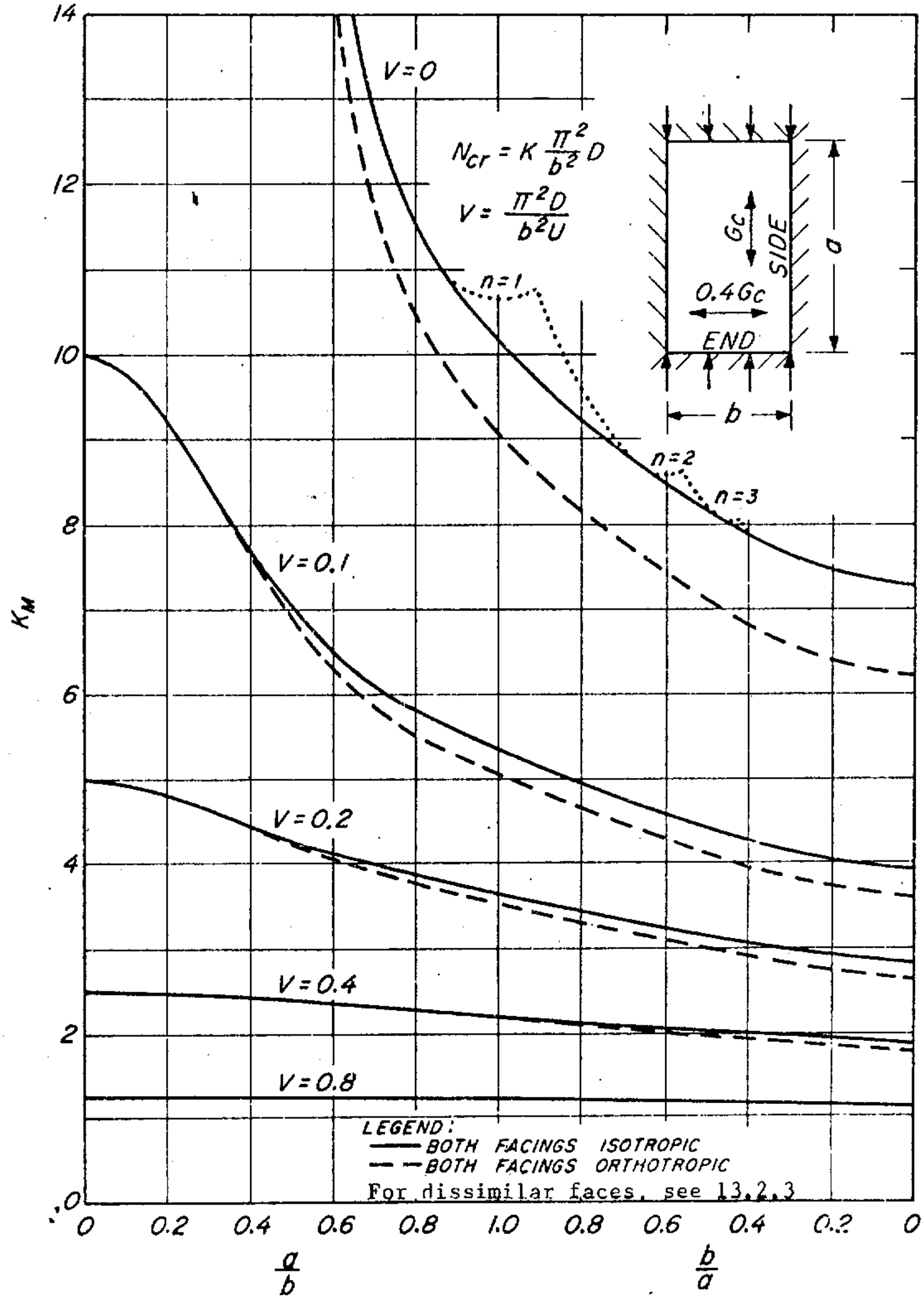


FIGURE 13.26 - K_M FOR SANDWICH PANEL WITH ENDS AND SIDES CLAMPED AND ORTHOTROPIC CORE (G_{cb}= 0.4 G_{ca})

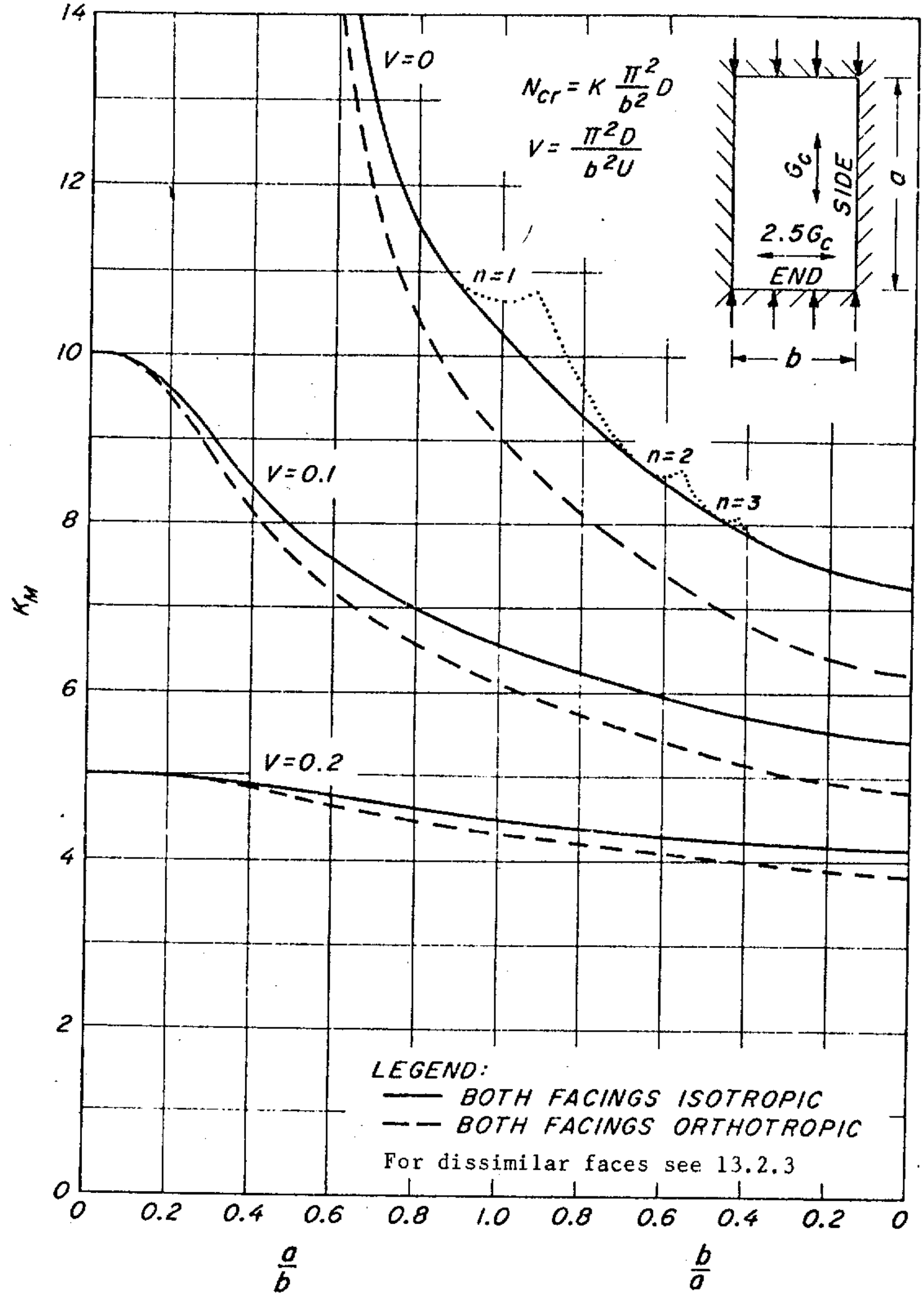


FIGURE 13.27 - K_M FOR SANDWICH PANEL WITH ENDS AND SIDES CLAMPED AND ORTHOTROPIC CORE ($G_{cb} = 2.5\ G_{ca}$)

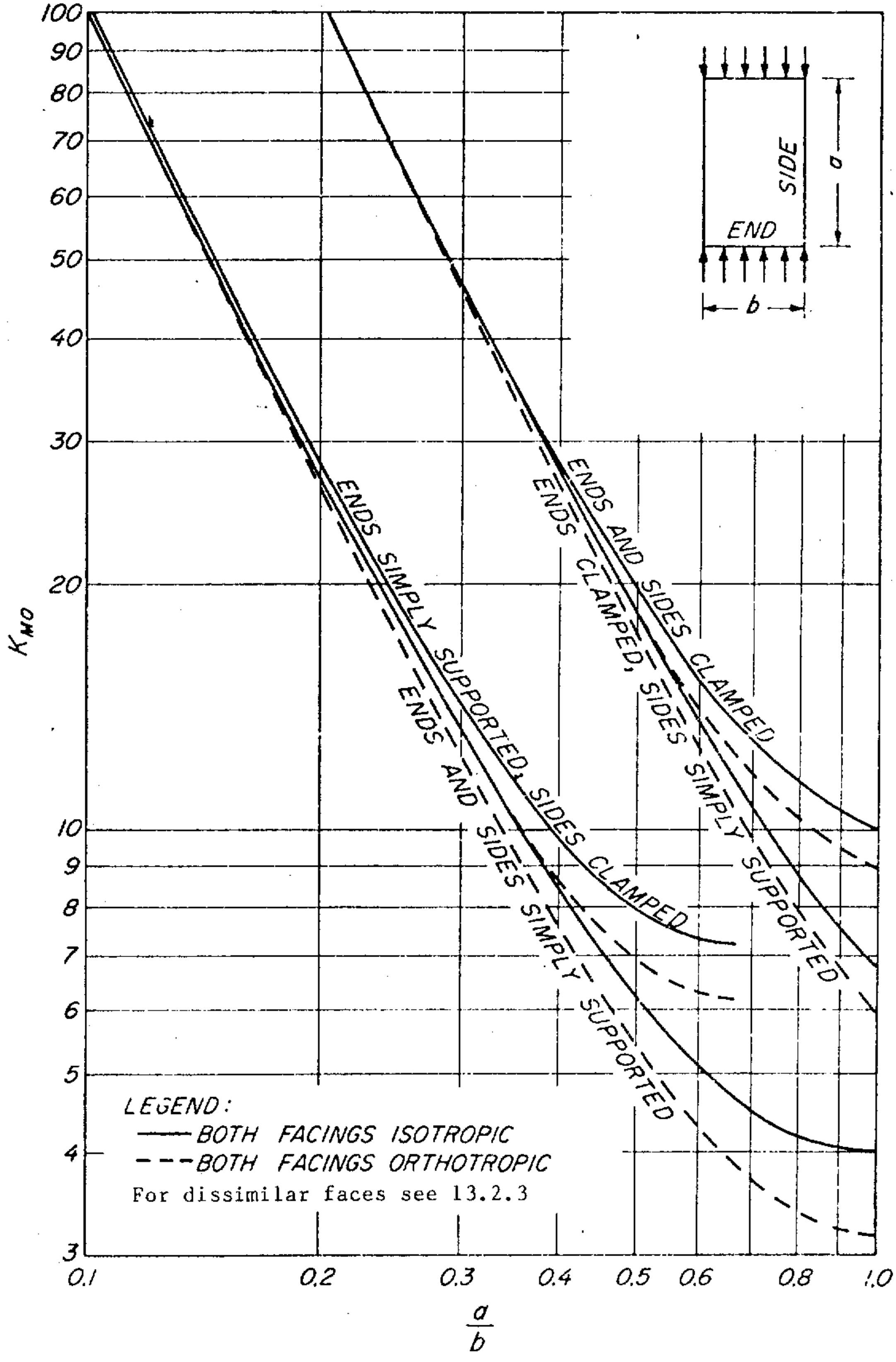


FIGURE 13.28 - VALUES OF K_{MO} FOR SANDWICH PANELS IN EDGEWISE COMPRESSION

(9) Evaluate F_{cr} by using the relationships in step (2). If the applied stress exceeds F_{cr}, repeat steps (3) through (9) for a stronger panel.

(10) Analyze for face wrinkling, Section 13.2.1.

(11) Analyze for intracell buckling, Section 13.2.2.

13.2.4 Flat Rectangular Panels Under Edgewise Shear

The following method is used in the design of flat sandwich shear panels. It is assumed that the shear load is equally and uniformly distributed over the edges of the panel as shown in Figure 13.29.

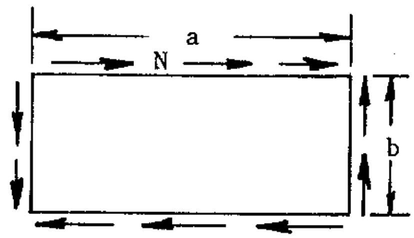


FIGURE 13.29 SHEAR PANELS

Overall buckling of the sandwich or dimpling or wrinkling of the facings cannot occur without possible total collapse of the panel. Detailed procedures follow, giving theoretical formulas and graphs for determining dimensions of the facings and core, as well as necessary core properties. Facing modulus of elasticity, E, shear modulus, G, and stress values, F, shall be values at the conditions of use; for example, if application is at elevated temperature, then facing properties at elevated temperature shall be used in design. The facing shear modulus or modulus of elasticity is the effective value at the facing stress. If this stress is beyond the proportional limit value, an appropriate tangent, reduced or modified value shall be used.

(1) Choose an allowable shear stress (F_{sf}). Determine the facing thickness (t) using

$$t_1F_{sf1} + t_2F_{sf2} = N;\ \text{unequal faces} \qquad 13.21$$

$$t = N/2F_{sf};\ \text{equal faces} \qquad 13.22$$

When the shear modulus of one face is different from the shear modulus of the other face, the face stresses are balanced by the ratio

$$\frac{F_{sf1}}{G_{s1}} = \frac{F_{sf2}}{G_{s2}} \qquad 13.23$$

The lower of the ratios in equation 13.23 must be used for design, otherwise the face with the lower ratio will be overstressed.

(2) The critical facing stress ($F_{s_{cr}}$) at which panel buckling will occur is given by

$$F_{s_{cr}} = \pi^2 K \frac{E_{f1}t_1E_{f2}t_2}{(E_{f1}t_1 + E_{f2}t_2)^2} \left(\frac{h}{b}\right)^2 \left(\frac{E_f}{\lambda}\right) \quad 13.24$$

where E_f and λ are values for the facing with least F_{sf}/G_s ratio as determined from equation 13.23.

If the facings are of equal thickness and of the same material, equation 13.24 becomes

$$F_{s_{cr}} = \frac{\pi^2 K}{4} \left(\frac{h}{b}\right)^2 \left(\frac{E_f}{\lambda}\right) \quad 13.25$$

In equations 13.24 and 13.25

$$K = K_M + K_F \quad 13.26$$

(3) Evaluate the following parameters

$$b/a \quad 13.27$$

$$(E_{f2}t_2)/(E_{f1}t_1) \quad 13.28$$

$$(\lambda F_{sf})/E_f \quad 13.29$$

where equation 13.29 uses the values of the facing with the minimum ratio from equation 13.23.

(4) Enter the appropriate chart (Figure 13.30, 13.31 or 13.32) with parameter b/a (13.27) to V = .01. (Choose a low finite value to start since V = 0 gives h as a minimum and G_c as infinite). Move laterally to parameter, equation 13.28, and then downward to equation 13.29. Project laterally and read value of h/b. Determine h.

(5) Evaluate core thickness from

$$t_c = h - \frac{t_1 + t_2}{2} \text{ ; unequal facings} \quad 13.30$$

$$t_c = h - t\text{; equal facings} \quad 13.31$$

(6) Determine the value of K' from

$$K' = \frac{\pi^2 t_c E_{f1}t_1E_{f2}t_2}{\lambda b^2(E_{f1}t_1 + E_{f2}t_2)} \text{ ; unequal facings} \quad 13.32$$

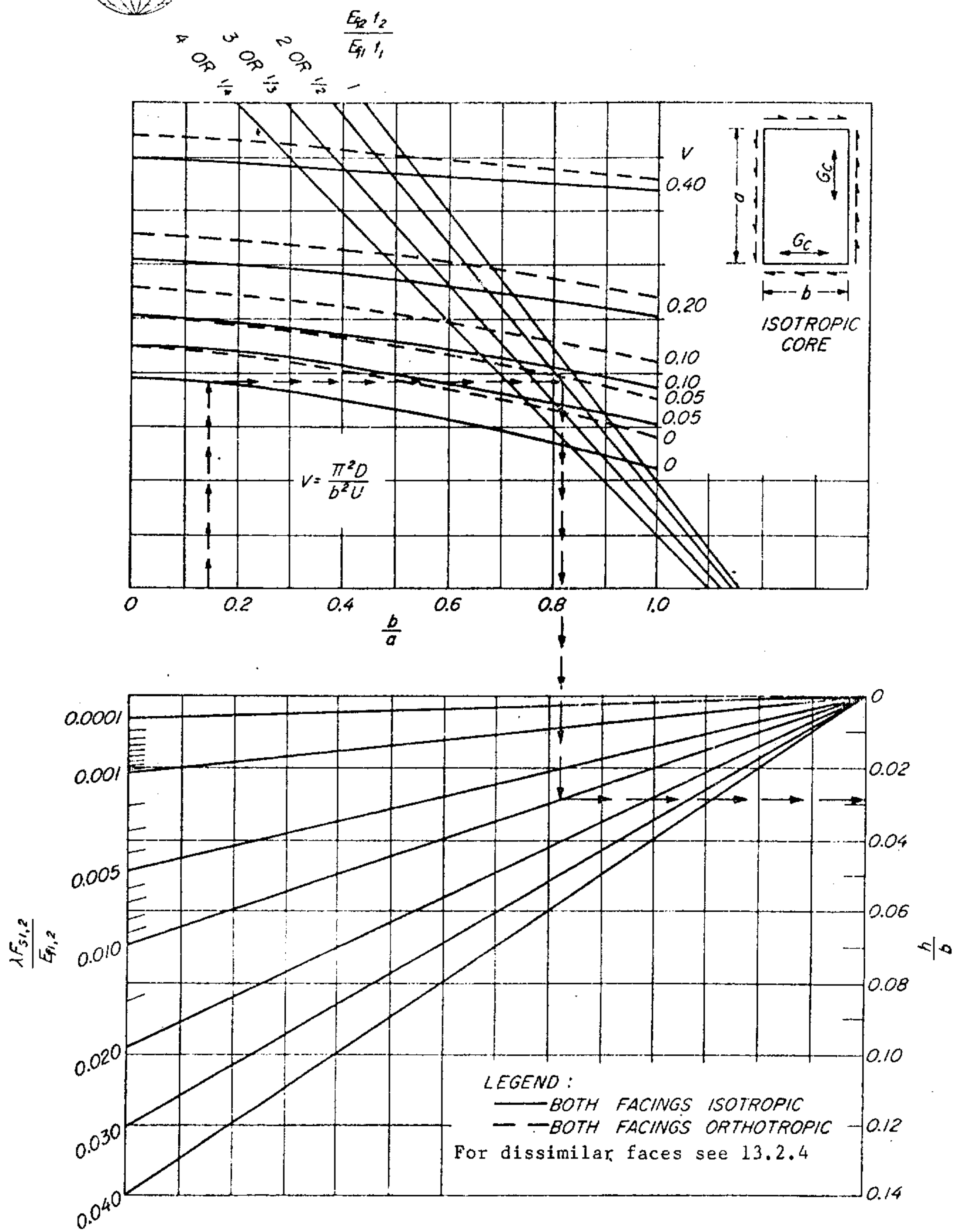


FIGURE 13.30 - CHART FOR DETERMINING h/b RATIO SUCH THAT A SIMPLY SUPPORTED SANDWICH PANEL WITH ISOTROPIC CORE WILL NOT BUCKLE UNDER EDGE-WISE SHEAR LOAD

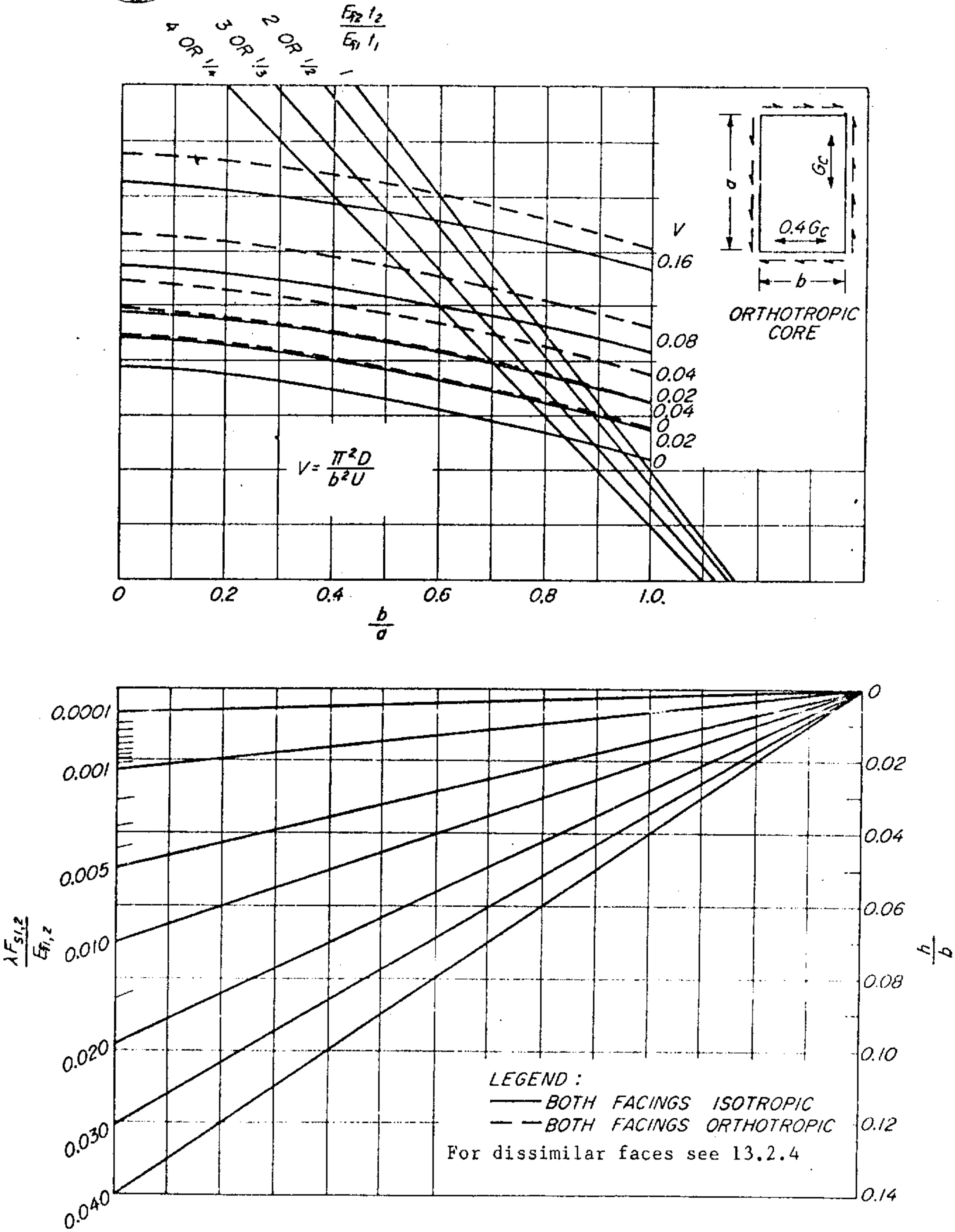


FIGURE 13.31 - CHART FOR DETERMINING h/b RATIO SUCH THAT A SIMPLY SUPPORTED SANDWICH PANEL WITH ORTHOTROPIC CORE WILL NOT BUCKLE UNDER EDGEWISE SHEAR LOAD ($G_{cb} = 0.4\ G_{ca}$).

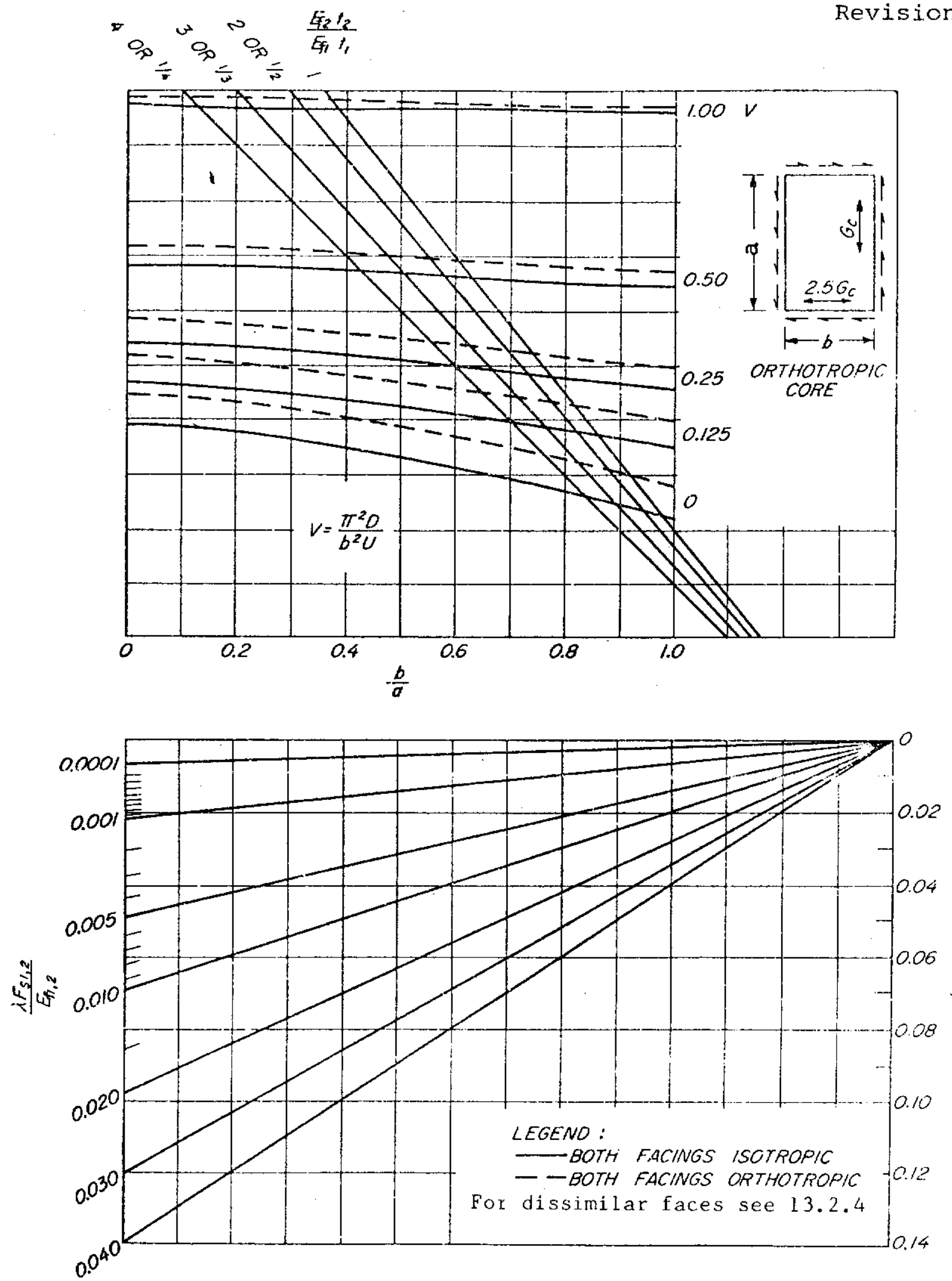


FIGURE 13.32 - CHART FOR DETERMINING h/b RATIO SUCH THAT A SIMPLY SUPPORTED SANDWICH PANEL WITH ORTHOTROPIC CORE WILL NOT BUCKLE UNDER EDGEWISE SHEAR LOAD ($G_{cb} = 2.5\ G_{ca}$).

$$K' = \frac{\pi^2 t_c E_f t}{2\lambda b^2} \text{ ; equal facings} \qquad 13.33$$

(7) Determine tentative core modulus of rigidity (G_c) from 13.34 for V = .01. If this value

$$G_c = K'/V \qquad 13.34$$

of G_c is not within the range available in the desired core material and type, enter chart in Figure 13.33 along line $V = K'/G_c$ until a practical value is reached. For the new value of V repeat steps (4), (5) and (6).

(8) From appropriate charts in Figures 13.34 through 13.39, read directly the values of K_M and K_{Mo}. For determining K_{Mo}, assume V = 0. Evaluate K_F by using

$$K_F = \frac{(E_{f1}t_1^3 + E_{f2}t_2^3)(E_{f1}t_1 + E_{f2}t_2)}{12E_{f1}t_1E_{f2}t_2h^2} K_{Mo} \qquad 13.35$$

$$K_F = \frac{t^2}{3h^2} K_{Mo}\text{; equal facings} \qquad 13.36$$

Determine the value for K from

$$K = K_M + K_F \qquad 13.37$$

(9) Substitute the value of K into (2) and solve for F_{scr}. This stress must be greater than the allowable stresses F_{sf1} and F_{sf2} determined by step (1).

(10) Check for face wrinkling as outlined in Section 13.2.1.

(11) Check the panel for intercell buckling, Section 13.2.2.

13.2.5 Flat Panels Under Uniformly Distributed Normal Load

This section gives procedures for determining sandwich facing and core thickness and core shear modulus so that design facing stresses and allowable panel deflections will not be exceeded. This procedure is used in the design of a flat sandwich panel with equal facings, simply supported at the four edges and subjected to uniform normal loading. Facings are isotropic; core may be isotropic or orthotropic. In the case of an orthotropic core, G_{ca} is the modulus of rigidity associated with the shear distortion observed in a cross section parallel to side a. Correspondingly, G_{cb} is associated with side b.

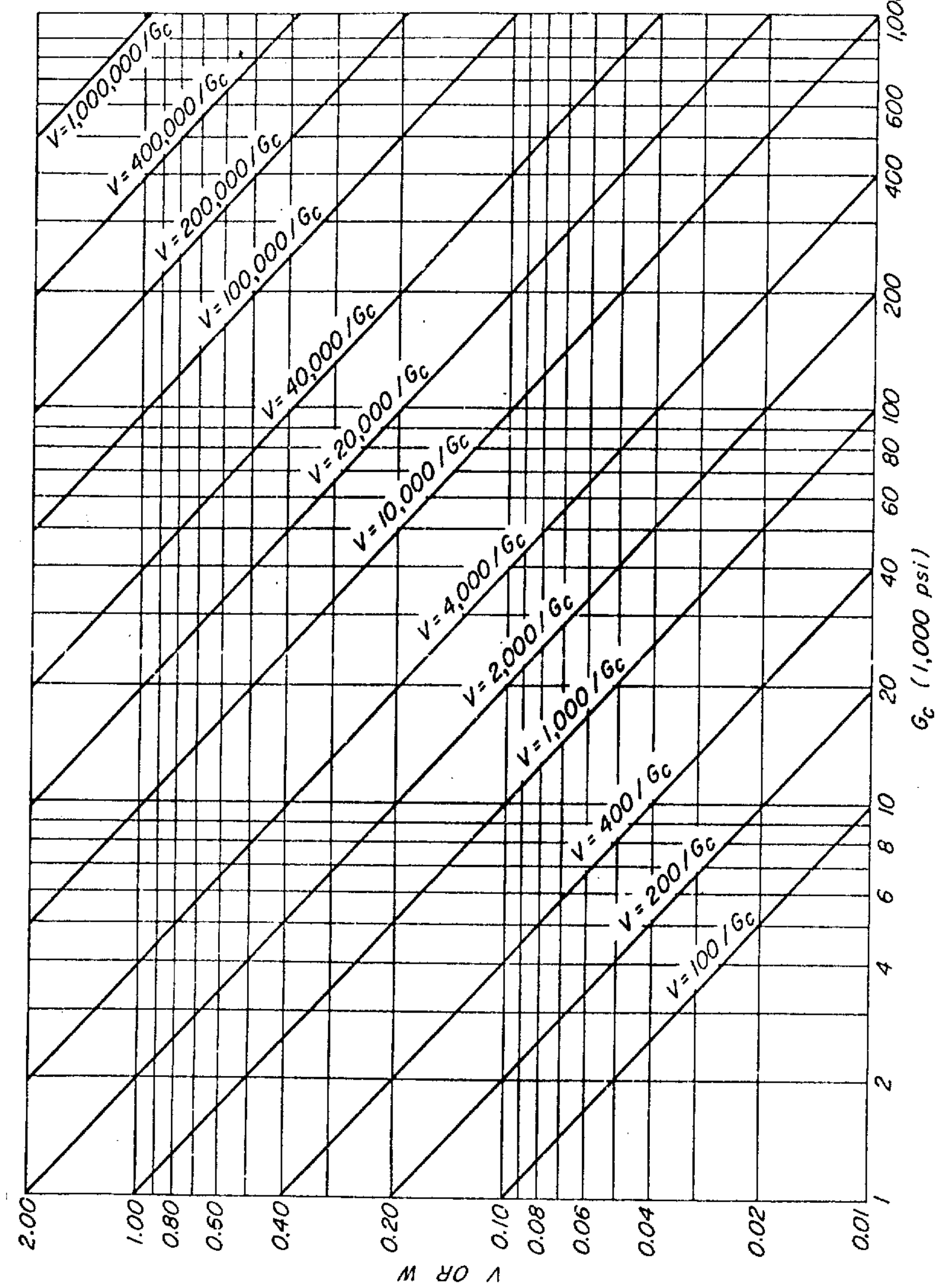


FIGURE 13.33 - CHART FOR DETERMINING V OR W AND G_c FOR SANDWICH IN EDGEWISE SHEAR

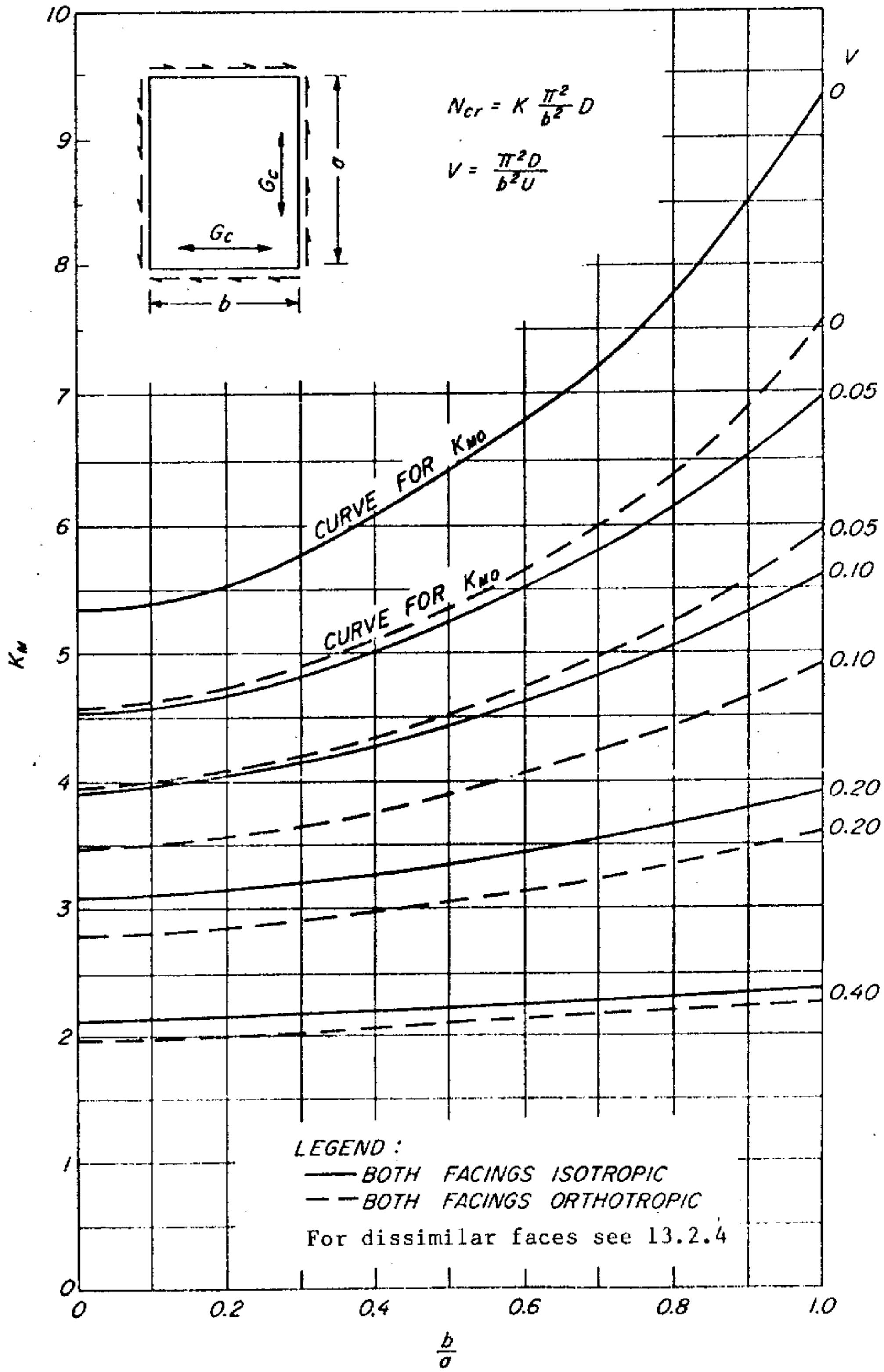


FIGURE 13.34 - K_M FOR SANDWICH PANEL WITH ALL EDGES SIMPLY SUPPORTED, AND ISOTROPIC CORE.

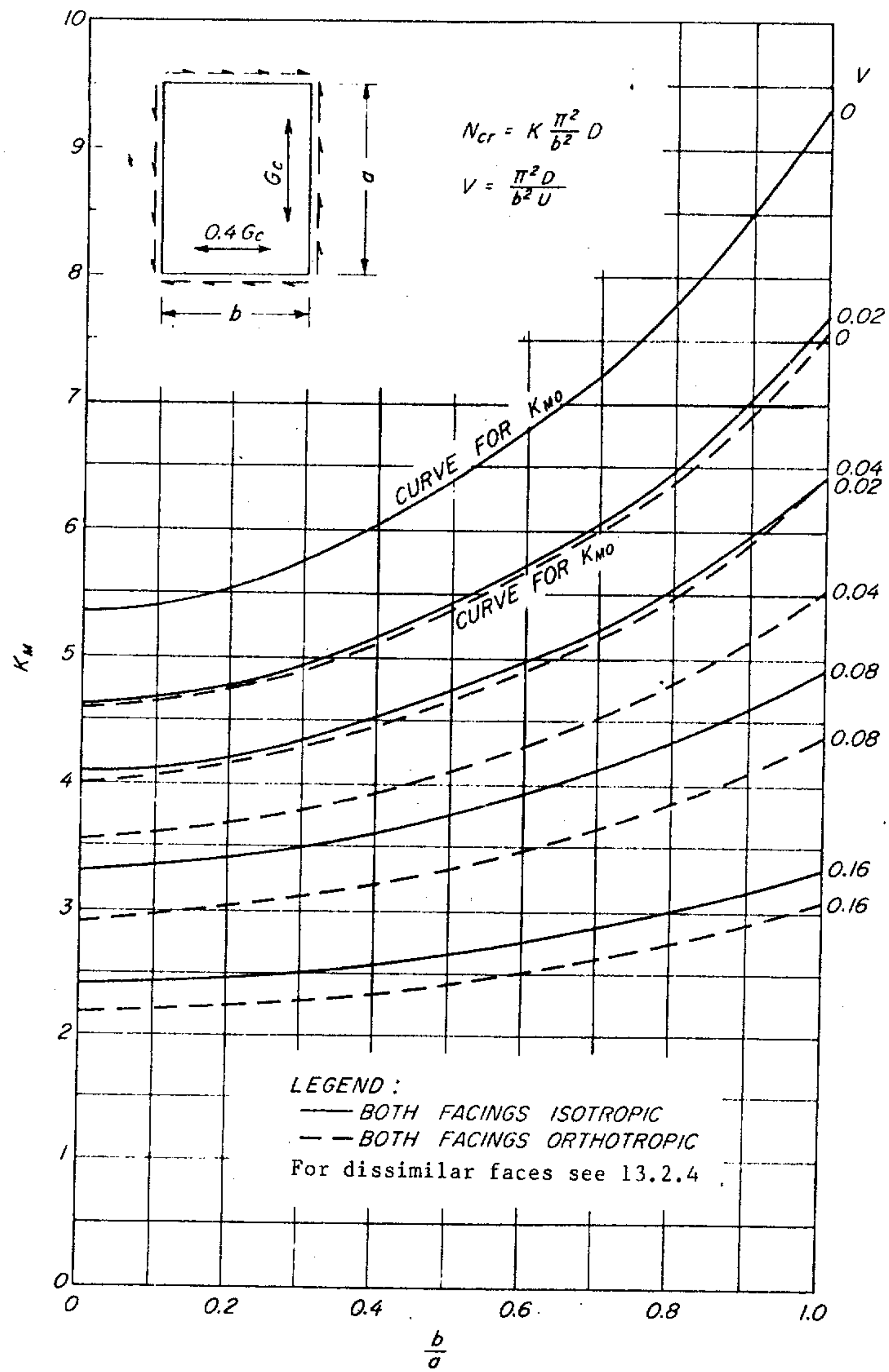


FIGURE 13.35 - K_M FOR SANDWICH PANEL WITH ALL EDGES SIMPLY SUPPORTED, AND ORTHOTROPIC CORE. (G_{cb}= 0.4 G_{ca}).

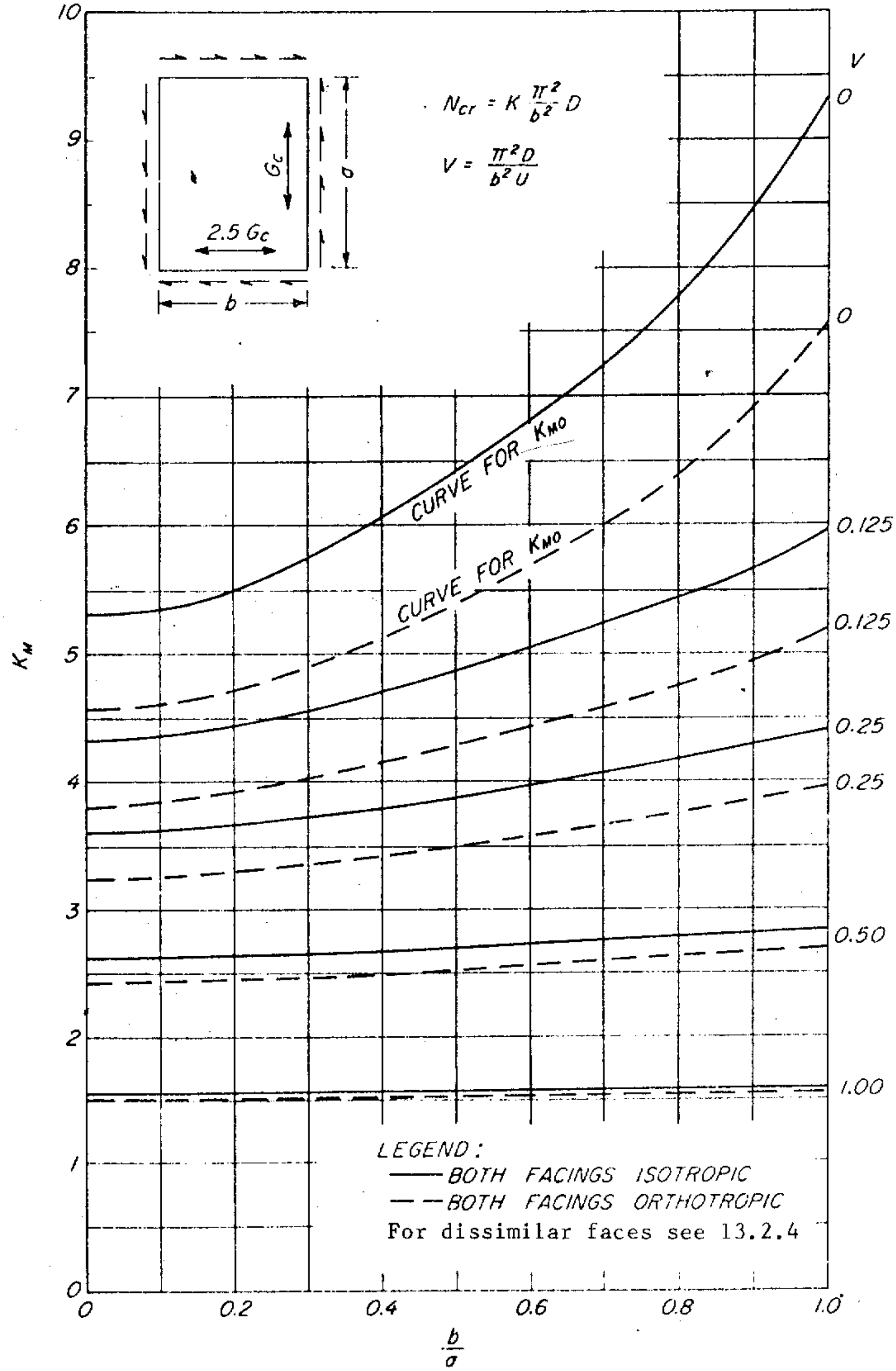


FIGURE 13.36 - K_M FOR SANDWICH PANEL WITH ALL EDGES SIMPLY SUPPORTED, AND ORTHOTROPIC CORE. (G_{cb} = 2.5 G_{ca}).

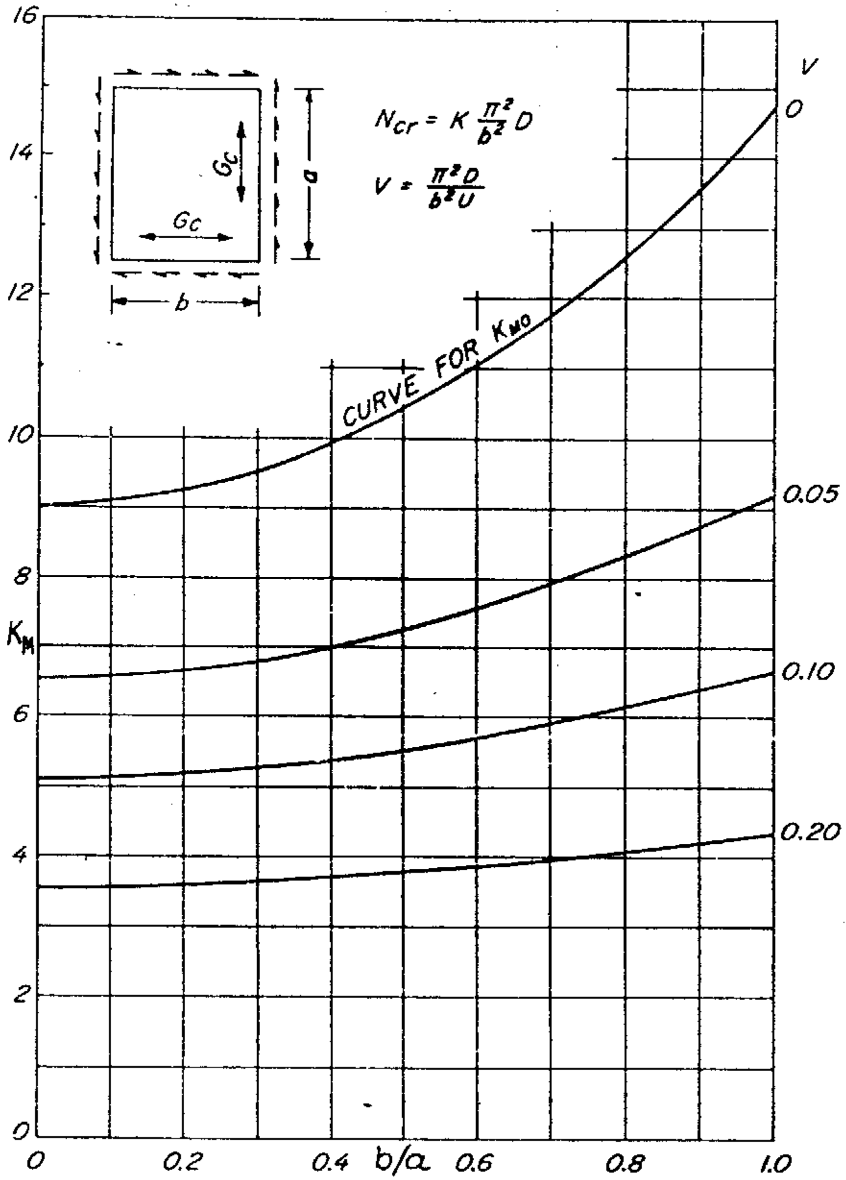


FIGURE 13.37 - K_M FOR SANDWICH PANEL WITH ALL EDGES CLAMPED, ISOTROPIC FACES AND ISOTROPIC CORE

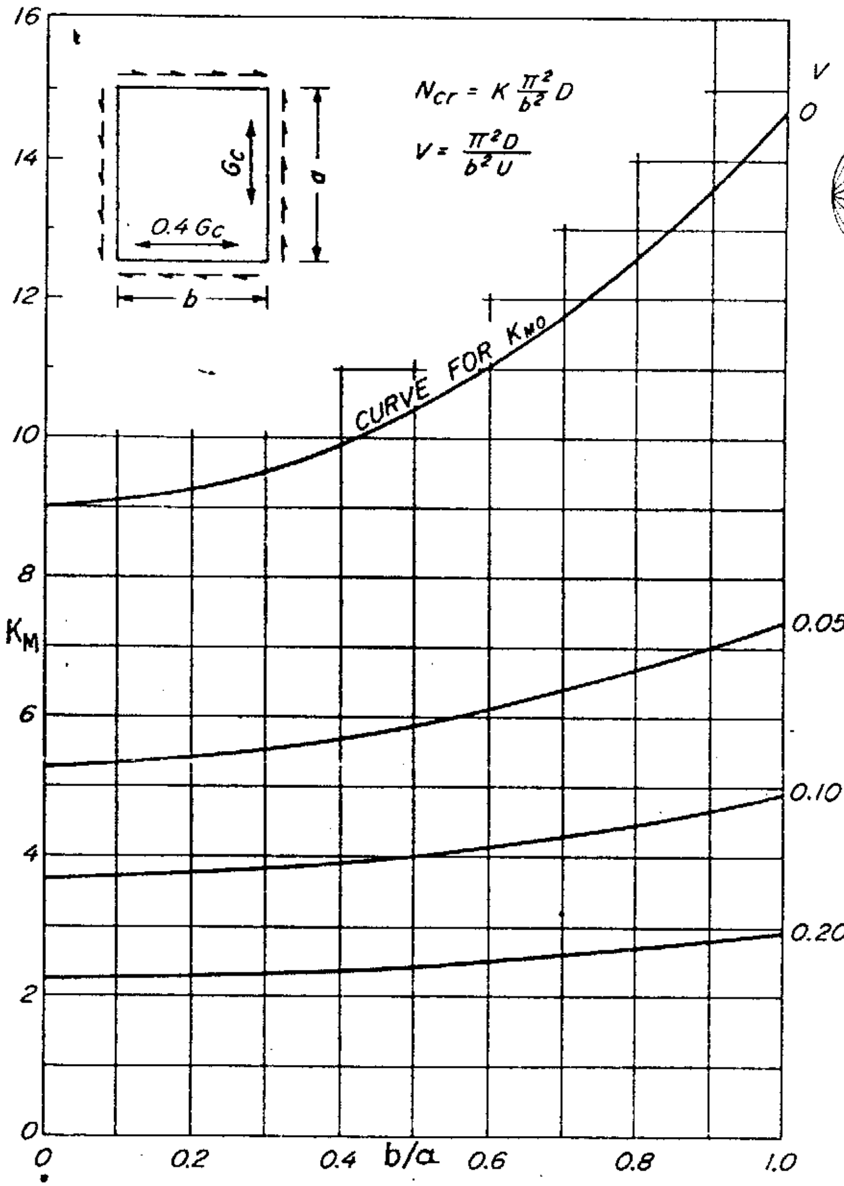


FIGURE 13.38 - K_M FOR SANDWICH PANEL WITH ALL EDGES CLAMPED, ISOTROPIC FACES AND ORTHOTROPIC CORE ($G_{cb} = 0.4\ G_{ca}$)

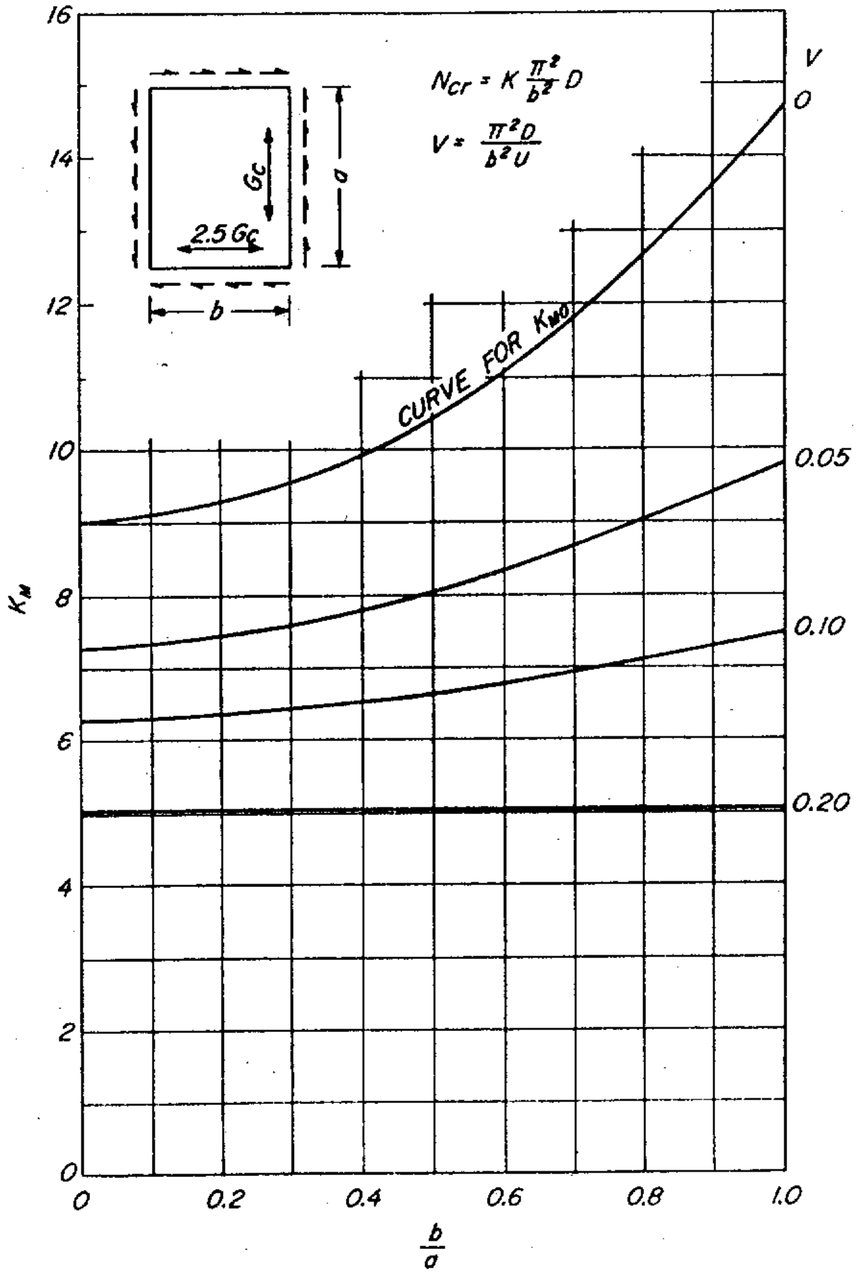


FIGURE 13.39 - K_M FOR SANDWICH PANEL WITH ALL EDGES CLAMPED, ISOTROPIC FACES AND ORTHOTROPIC CORE (G_{cb}= 2.5 G_{ca})

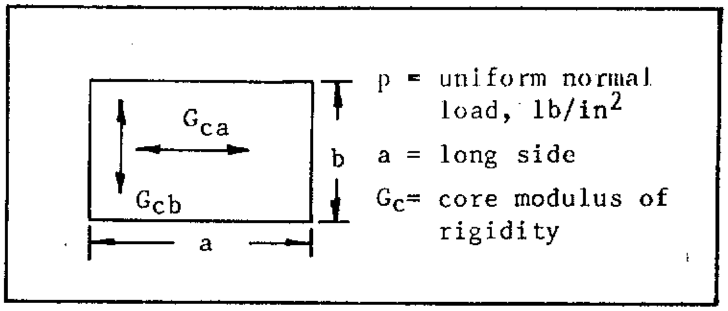


FIGURE 13.40 SIMPLY SUPPORTED FLAT PANEL WITH UNIFORMLY DISTRIBUTED NORMAL LOAD

(1) Evaluate the maximum bending moment per inch using the equation given in Figure 13.41.

(2) Tentatively select panel materials and establish allowable stresses.

(3) Determine facing thickness in the following weight minimizing expression.

$$t = \sqrt{\frac{W_c M}{2W_f F_f}} \qquad 13.38$$

where F_f = allowable facing stress, psi
W_c = density of core, #/in^3
W_f = facing density, #/in^3

Increase t to nearest standard gage.

(4) Determine core thickness (t_c) from

$$t_c = \sqrt{\frac{2W_f M}{W_c F_f}} \qquad 13.39$$

If practical considerations require unequal facings or different t_c, make the necessary changes at this point.

(5) For a panel configuration thus determined, evaluate the parameter V

$$V = \frac{\pi^2 E_{f1} t_1 E_{f2} t_2 t_c}{\lambda(E_{f1} t_1 + E_{f2} t_2) b^2 G_{ca}} \text{ ; unequal faces} \qquad 13.40$$

$$V = \frac{\pi^2 E_f t t_c}{2\lambda b^2 G_{ca}} \text{ ; equal faces} \qquad 13.41$$

Enter the appropriate charts in Figures 13.42 through 13.46 with b/a and V to determine the value for constants C_2 and C_3.

(6) The maximum bending moments occur at the panel center and are determined by the following expression

$$M_a = \frac{16pb^2}{\pi^4}(C_3 + \mu C_2); \text{ across width} \quad 13.42$$

$$M_b = \frac{16pb^2}{\pi^4}(C_2 + \mu C_3); \text{ across length} \quad 13.43$$

Moments obtained are per unit width and length of panel respectively.

(7) Calculate the resulting facing stress from

$$f_f = \frac{2M_a}{t(d + t_c)} \quad 13.44$$

$$f_f = \frac{2M_b}{t(d + t_c)} \quad 13.45$$

The equations 13.44 and 13.45 are based on faces made of the same material and of equal thickness. If materials or thicknesses are different, the stresses must be calculated using Mc/I. If the facing stress is greater than the chosen allowable design stress or if considerably below, iterate the previous procedure to obtain a more nearly optimum design.

(8) The maximum shear loads occur at the mid-length of the panel edges and are determined from

$$S_a = \frac{16pb}{\pi^3} C_4; \text{ shear on side a} \quad 13.46$$

$$S_b = \frac{16pb}{\pi^3} C_5; \text{ shear on side b} \quad 13.47$$

Enter chart in Figures 13.47 through 13.50 to determine C_4 and C_5.

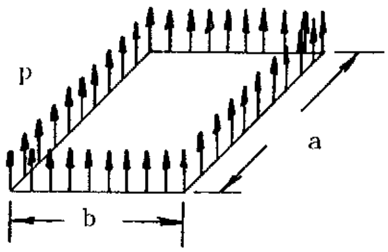
(9) Evaluate shear stresses

$$f_{sa} = \frac{2S_a}{d + t_c} \quad 13.48$$

$$f_{sb} = \frac{2S_b}{d + t_c} \quad 13.49$$

Choose an available core to meet the stress requirement of 13.48 and 13.49.

(10) If panel deflection is limited by the design criteria, it may be determined by



$M_{max} = \beta p b^2$ = moment @ middle of panel

a/b	1	1.2	1.4	1.6	1.8	2	3	4	5	∞
β	.048	.063	.075	.086	.095	.102	.119	.124	.125	.125

FIGURE 13.41 - MOMENT AND DEFLECTION IN THE CENTER OF A RECTANGULAR PANEL WITH UNIFORMLY DISTRIBUTED NORMAL LOAD

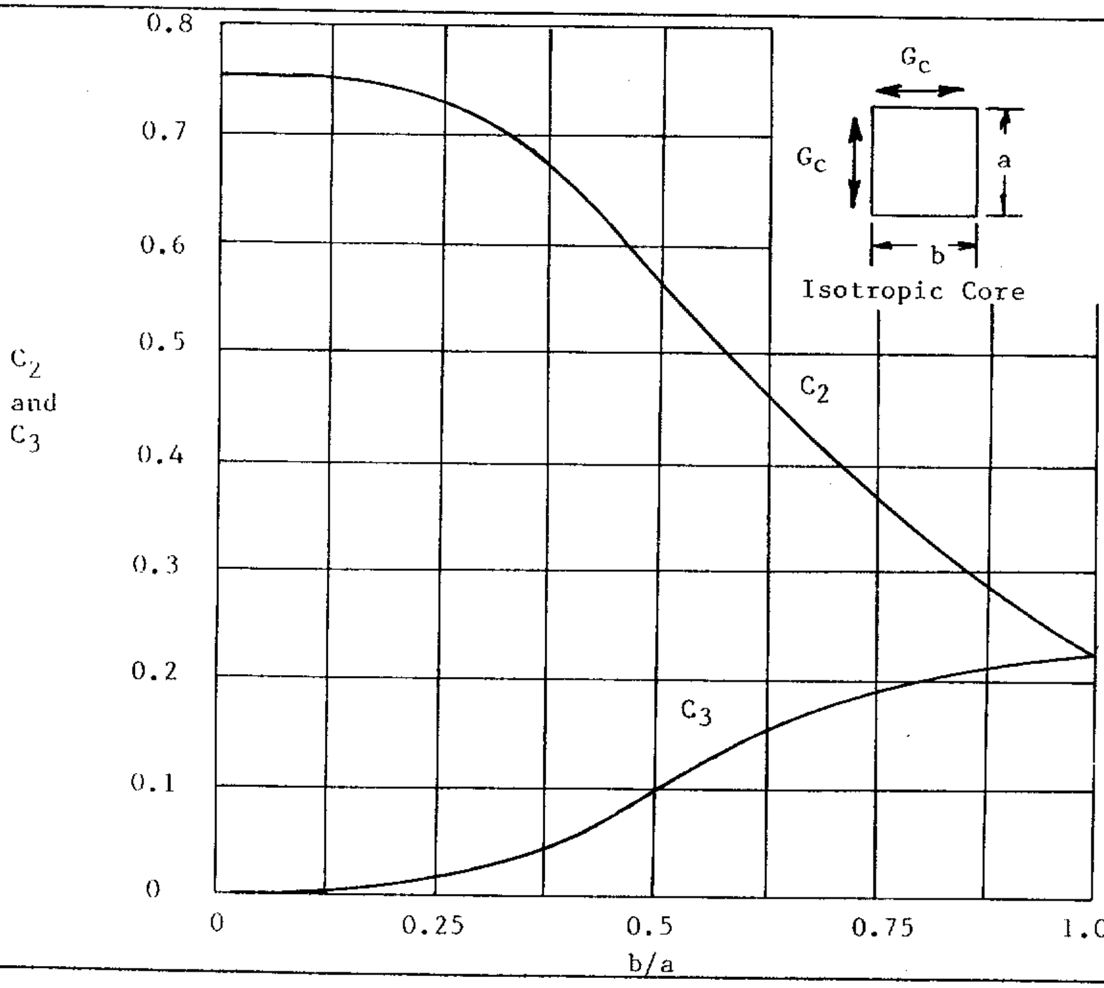


FIGURE 13.42 - MOMENT CONSTANTS FOR FLAT PANEL WITH ISOTROPIC CORE UNDER NORMAL LOAD

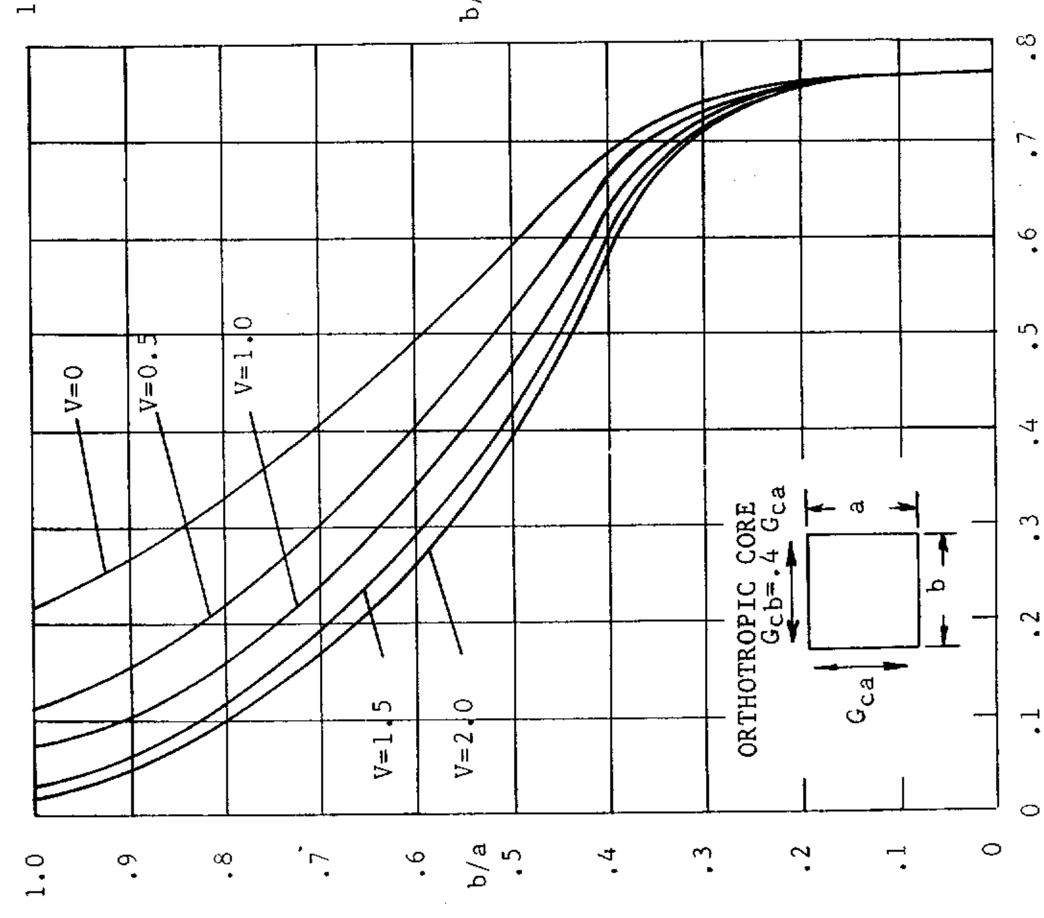


FIGURE 13.43 - MOMENT CONSTANT, C_2, FOR FLAT PANEL WITH ORTHOTROPIC CORE UNDER NORMAL LOAD

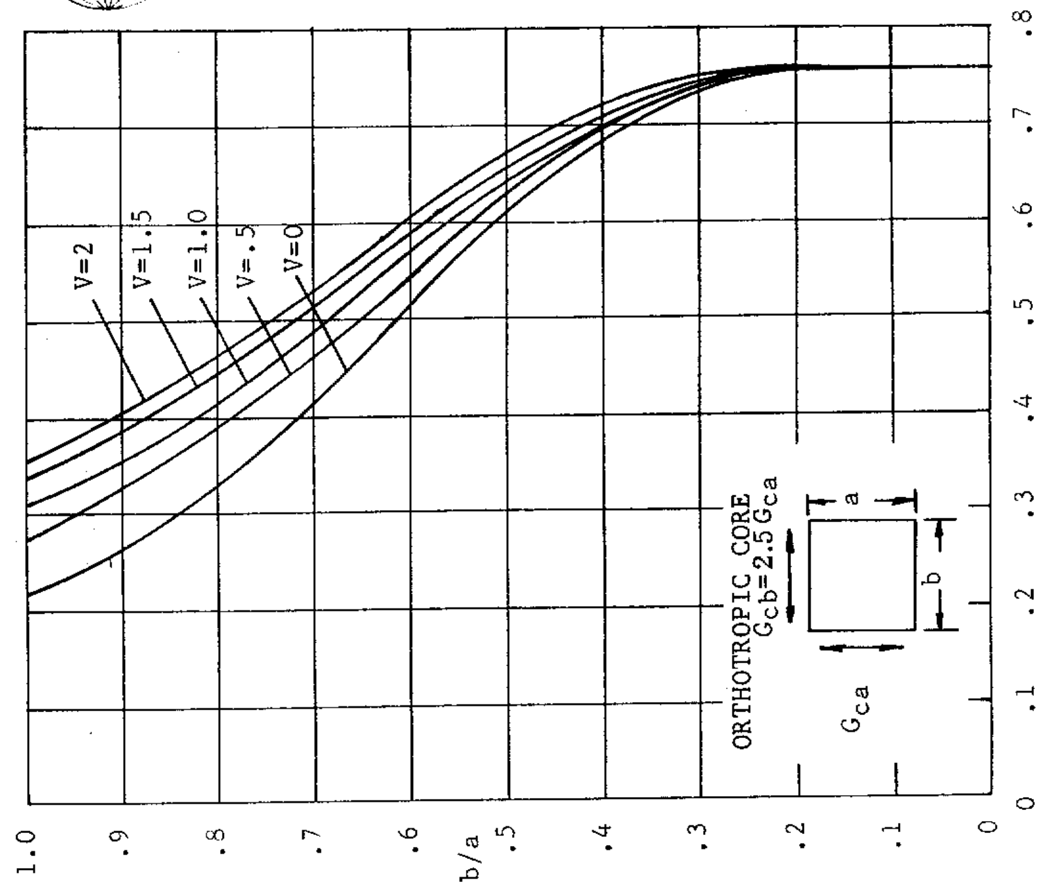


FIGURE 13.44 - MOMENT CONSTANT, C_2, FOR FLAT PANEL WITH ORTHOTROPIC CORE UNDER NORMAL LOAD

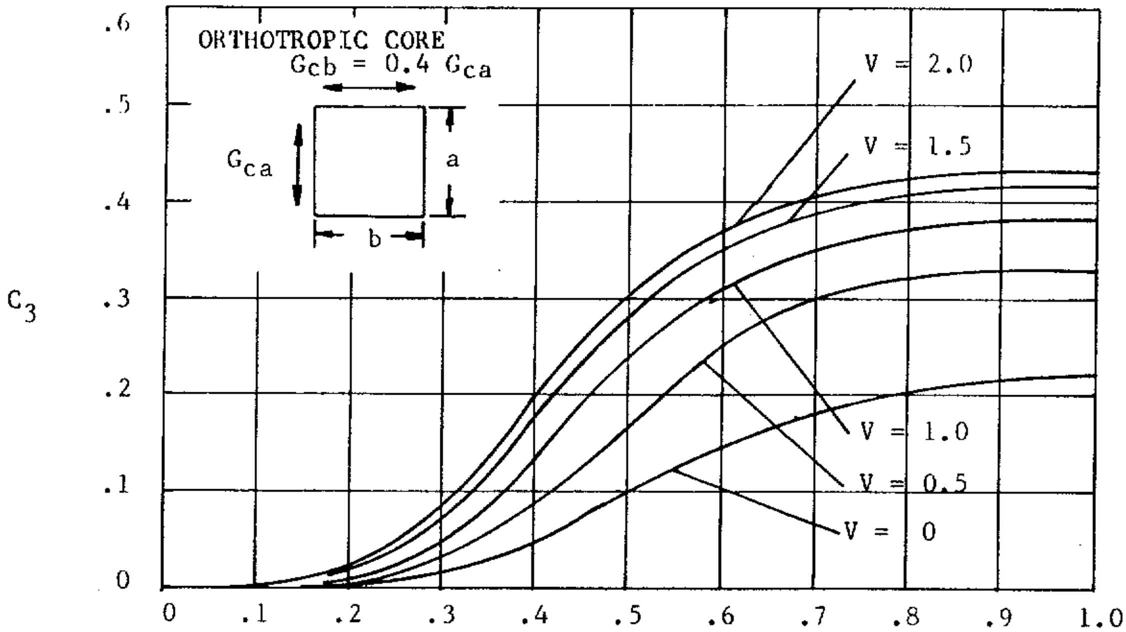


FIGURE 13.45 - MOMENT CONSTANT, C_3, FOR FLAT PANEL WITH ORTHOTROPIC CORE UNDER NORMAL LOAD

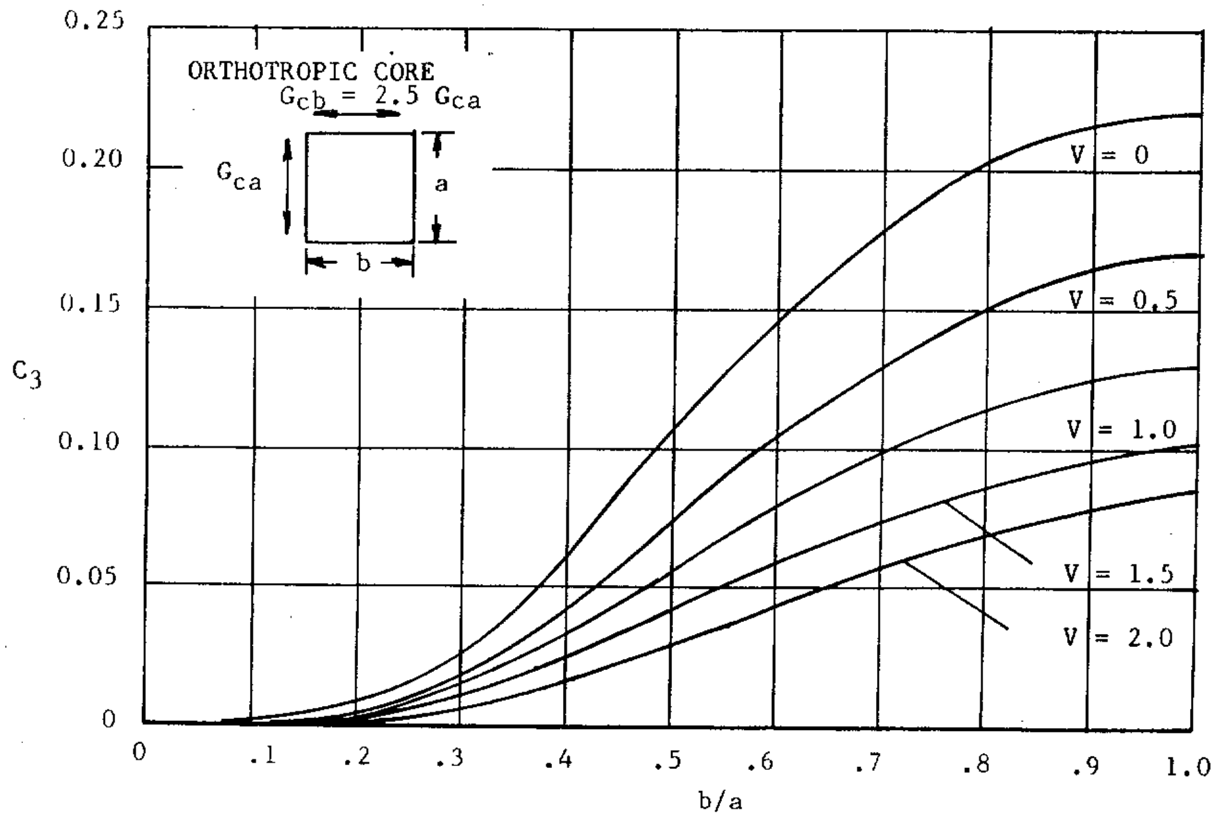


FIGURE 13.46 - MOMENT CONSTANT, C_3, FOR FLAT PANEL WITH ORTHOTROPIC CORE UNDER NORMAL LOAD

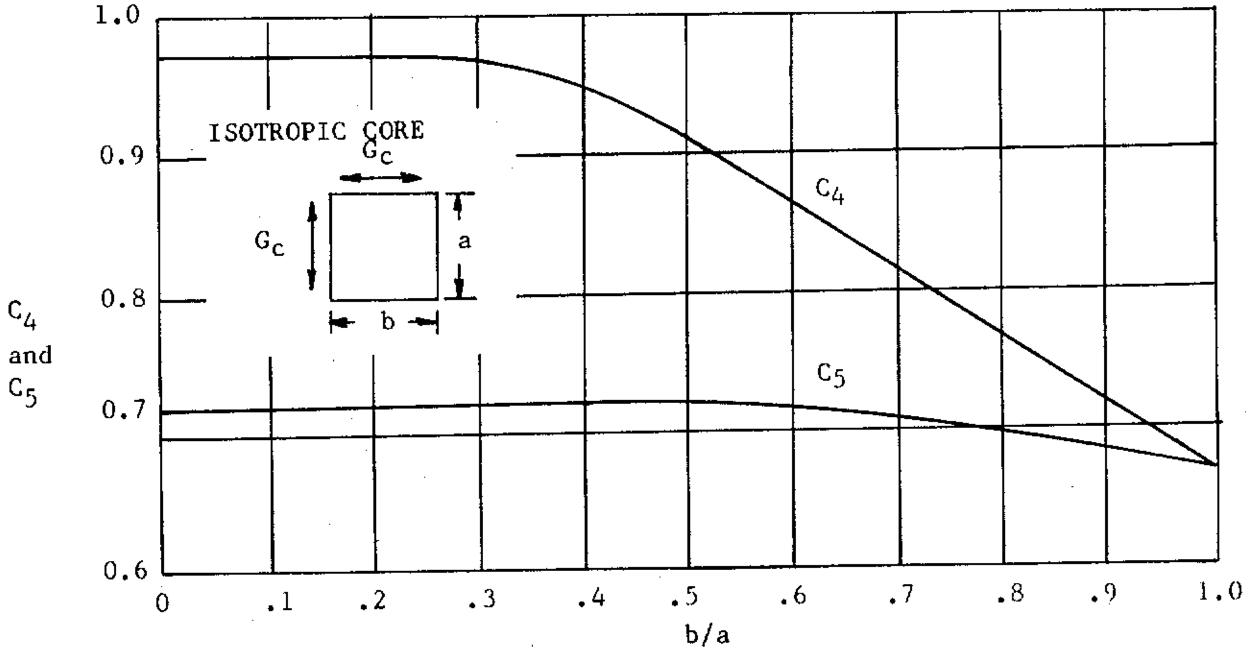


FIGURE 13.47 - SHEAR CONSTANTS, C_4 AND C_5, FOR FLAT PANELS WITH ISOTROPIC CORE AND NORMAL LOAD

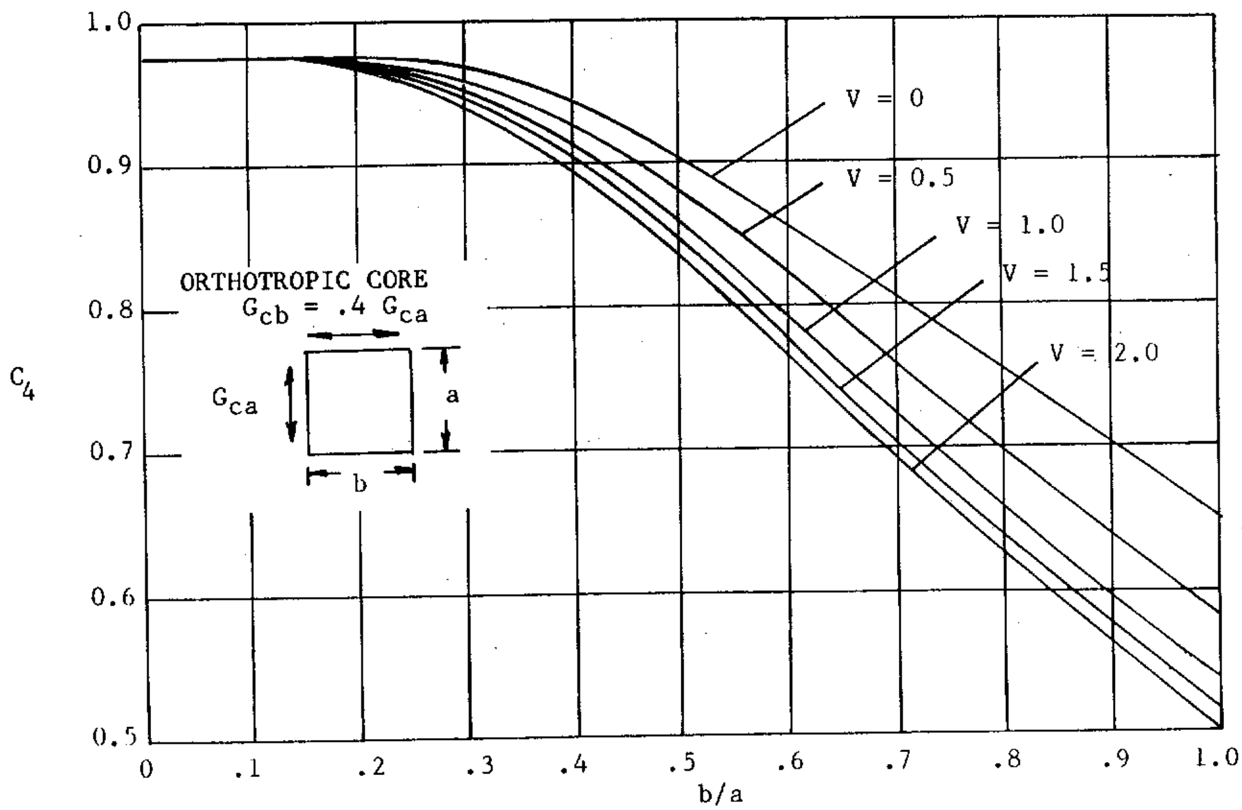


FIGURE 13.48 - SHEAR CONSTANT, C_4, FOR FLAT PANELS WITH ORTHOTROPIC CORE UNDER NORMAL LOAD

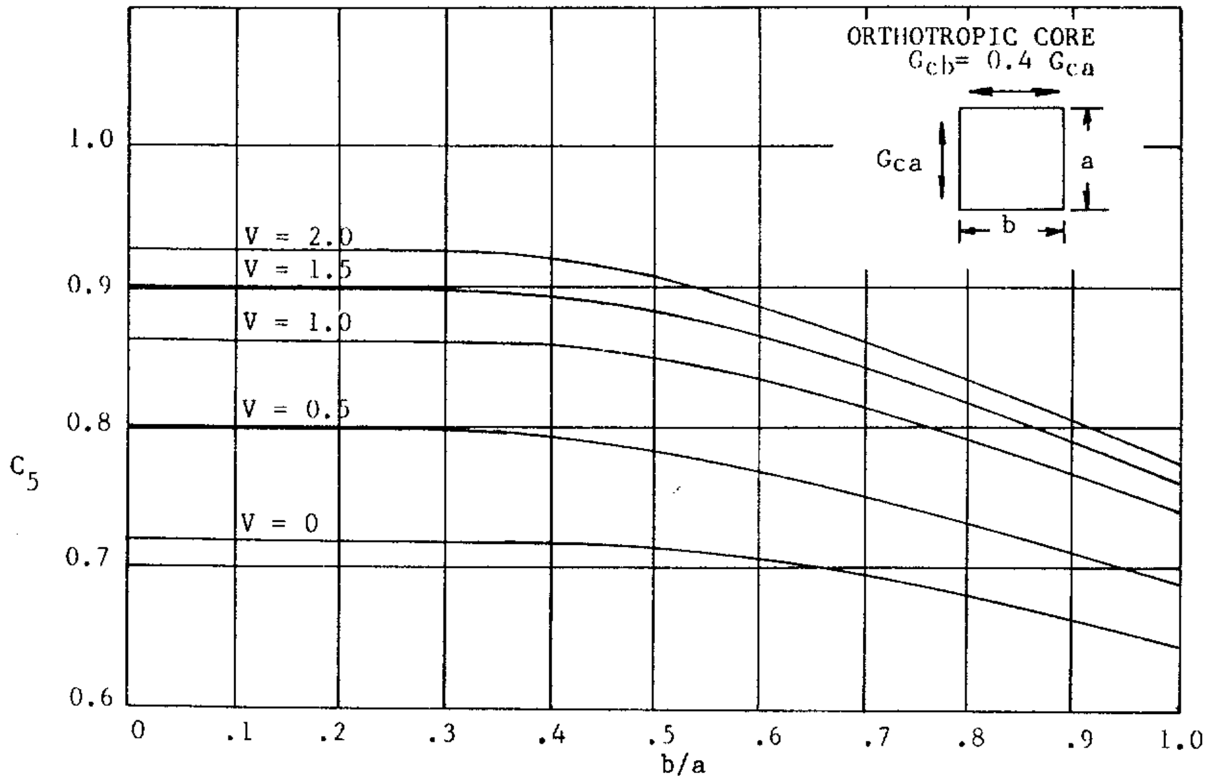


FIGURE 13.49 - SHEAR CONSTANT, C_5, FOR FLAT PANELS WITH ORTHOTROPIC CORE UNDER NORMAL LOAD

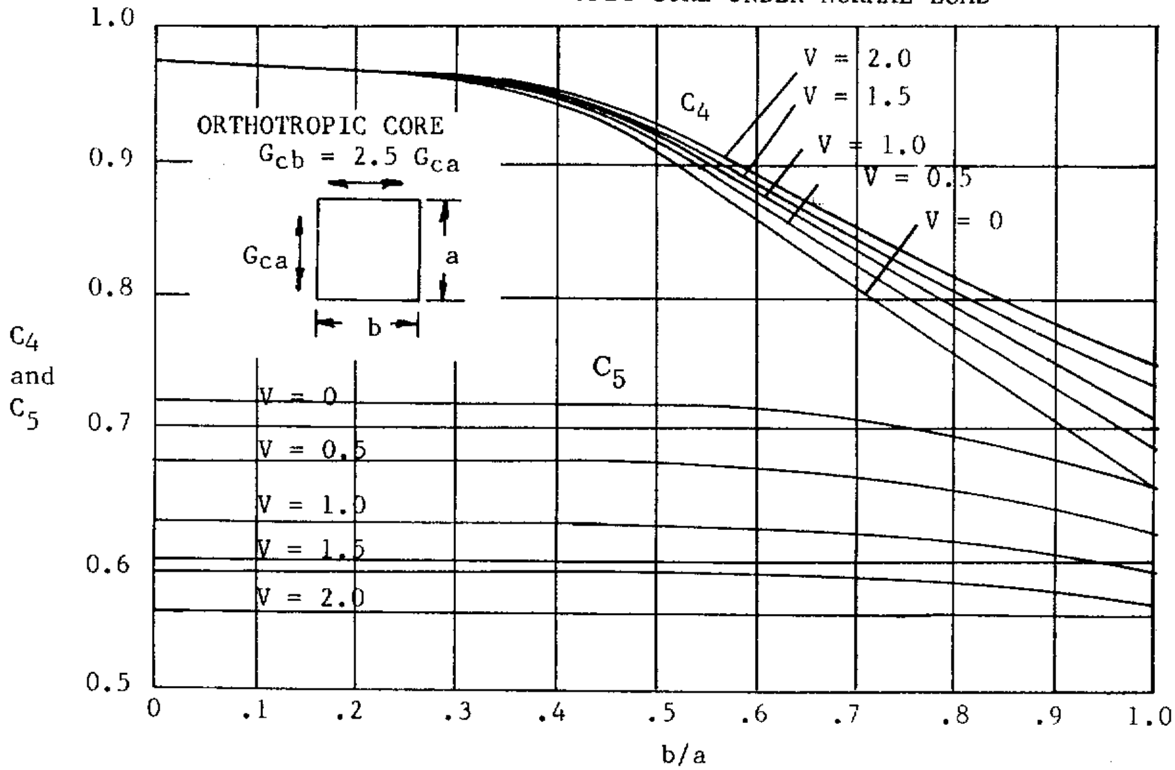


FIGURE 13.50 - SHEAR CONSTANTS, C_4 AND C_5, FOR FLAT PANELS WITH ORTHOTROPIC CORE UNDER NORMAL LOADS

$$\delta = \frac{16pb^4C_1}{\pi^6 D} \qquad 13.50$$

where

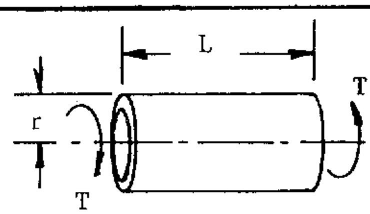
$$D = \frac{E_{f1}t_1E_{f2}t_2h^2}{E_{f1}t_1\lambda_2 + E_{f2}t_2\lambda_1} \text{ ; unequal faces} \qquad 13.51$$

$$D = \frac{E_f}{12\lambda}\left[d^3 - tc^3\left(1 - \frac{E_c'}{E_f}\right)\right] \text{ ; equal faces} \qquad 13.52$$

and C_1 is determined from charts in Figures 13.51 through 13.53, interpolating between values when necessary. If δ exceeds the design limit, increase the core thickness, and if necessary, the facing thickness until the deflection is acceptable. Repeat steps (5) through (9) to determine new, lower stresses.

13.2.6 Sandwich Cylinders Under Torsion

This section gives the procedure for determining core thickness and core shear modulus so that overall buckling of the sandwich walls of the cylinder will not occur. Buckling of the sandwich walls, dimpling or wrinkling of the fairings or sidewise buckling of the cylinder cannot occur without possible total collapse of the cylinder. Detailed procedures giving theoretical formulas and graphs for determining dimensions of the facings and core, as well as necessary core properties follow.



r = radius of outer surface in inches

FIGURE 13.54 CYLINDERS IN TORSION

(1) As a first approximation in determining the required facing thickness assume each face carries half of the shear load. Then

$$t = \frac{1.25T}{4\pi r^2 F_s} \qquad 13.53$$

where T = torque
t = thickness of either face
r = radius of outside surface

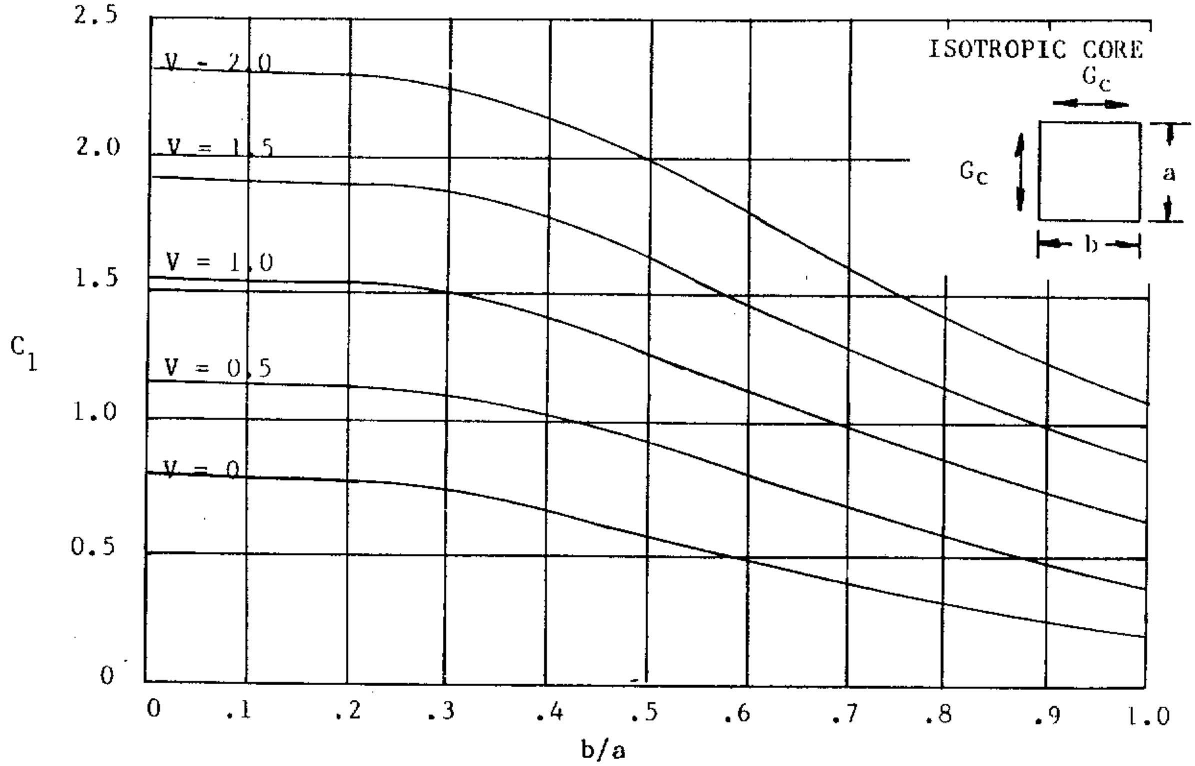


FIGURE 13.51 - DEFLECTION CONSTANT, C_1, FOR FLAT PANELS

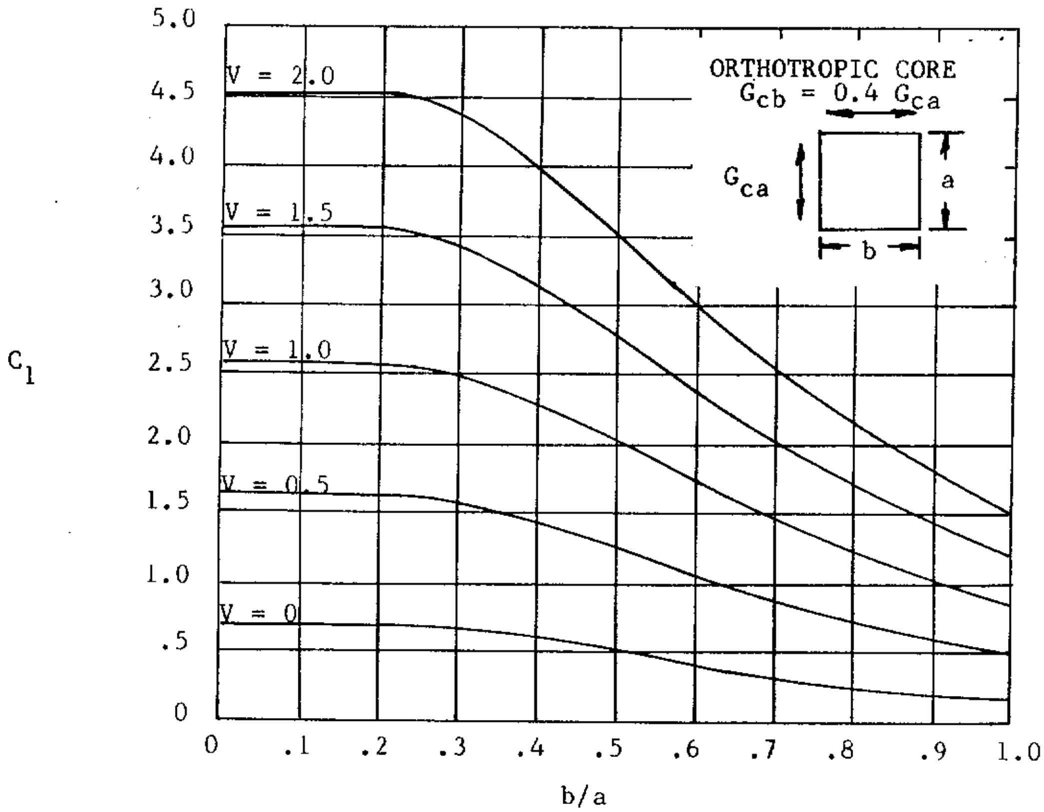


FIGURE 13.52 - DEFLECTION CONSTANT, C_1, FOR FLAT PANELS
WITH ISOTROPIC CORE UNDER NORMAL LOAD

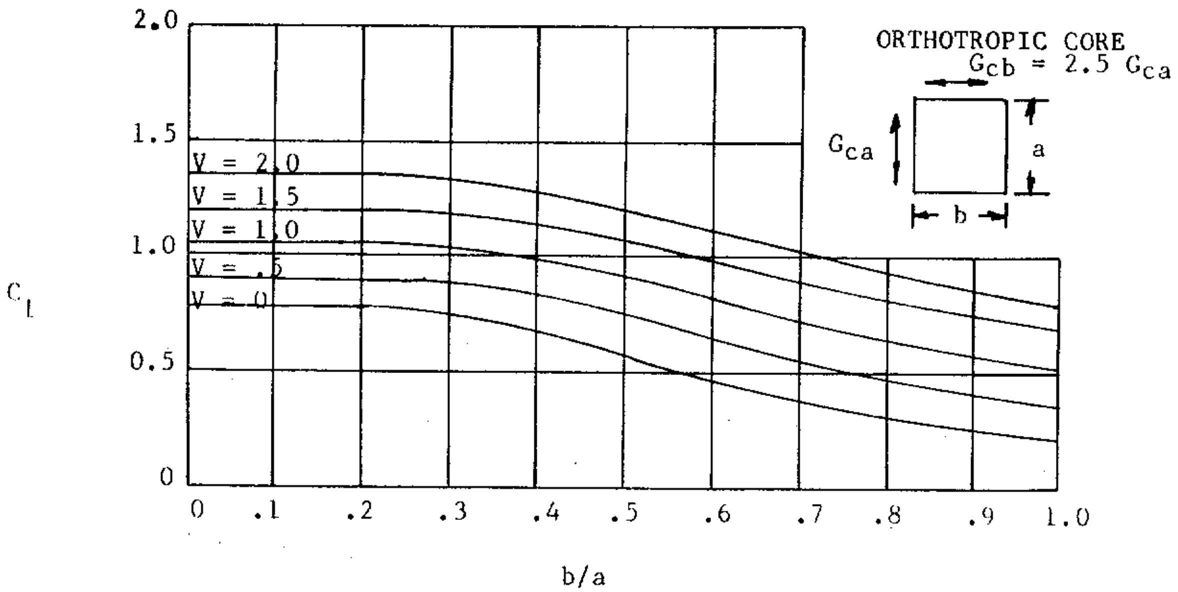


FIGURE 13.53 - DEFLECTION CONSTANT, C_1, FOR FLAT PANELS WITH ORTHOTROPIC CORE UNDER NORMAL LOAD

(2) Choose a practical core depth and density.

(3) For the previous configuration determine the facing stresses by

$$f_{so} = \frac{Tr_o}{J} \text{ ; outer facing} \qquad 13.54$$

$$f_{si} = \frac{Tr_i}{J} \text{ ; inner facing} \qquad 13.55$$

where r_o = radius to midline of outer facing
r_i = radius to midline of inner facing
J = polar moment of inertia of cylinder
$= 2\pi t(r_i^3 + r_o^3)$

(4) Calculate the shear load on the cylinder by

$$N_s = \frac{T}{2\pi r_c^2} \qquad 13.56$$

where

$$r_c = \frac{r_o + r_i}{2} \qquad 13.57$$

(5) Determine the critical buckling load from

$$N_{cr} = \frac{2KE_f t t_c}{r_c} \qquad 13.58$$

where K is determined by entering the appropriate chart in Figures 13.55, 13.56 or 13.57 with parameters

$$J' = L^2/dr_c \qquad 13.59$$

$$V = \frac{t t_c E_f}{2\lambda r_c d G_c} \qquad 13.60$$

where G_c is the circumferential core shear modulus.

(6) If N_{cr} is smaller than the shear load N_s calculated in step (4), increase the sandwich strength and repeat steps (1) through (6).

(7) Analyze the design for intercell buckling per Section 13.2.2.

13.2.7 Sandwich Cylinders Under Axial Compression

This section gives the procedures for determining core thickness and core shear modulus so that overall buckling of the sandwich walls of the cylinder will not occur.

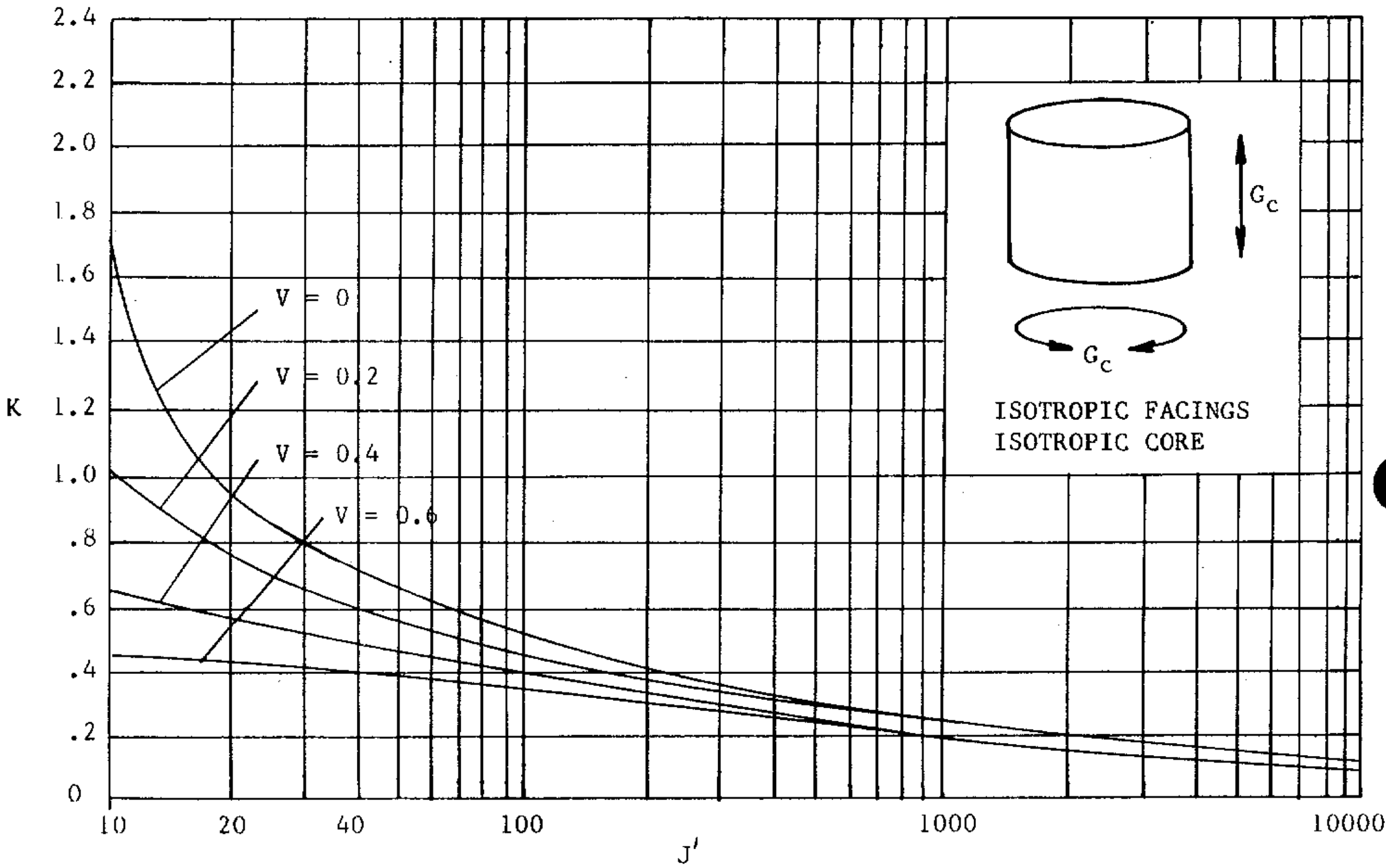


FIGURE 13.55 - BUCKLING CONSTANT, K, FOR CYLINDERS WITH ISOTROPIC FACINGS AND ISOTROPIC CORE AND TORSIONAL LOADING

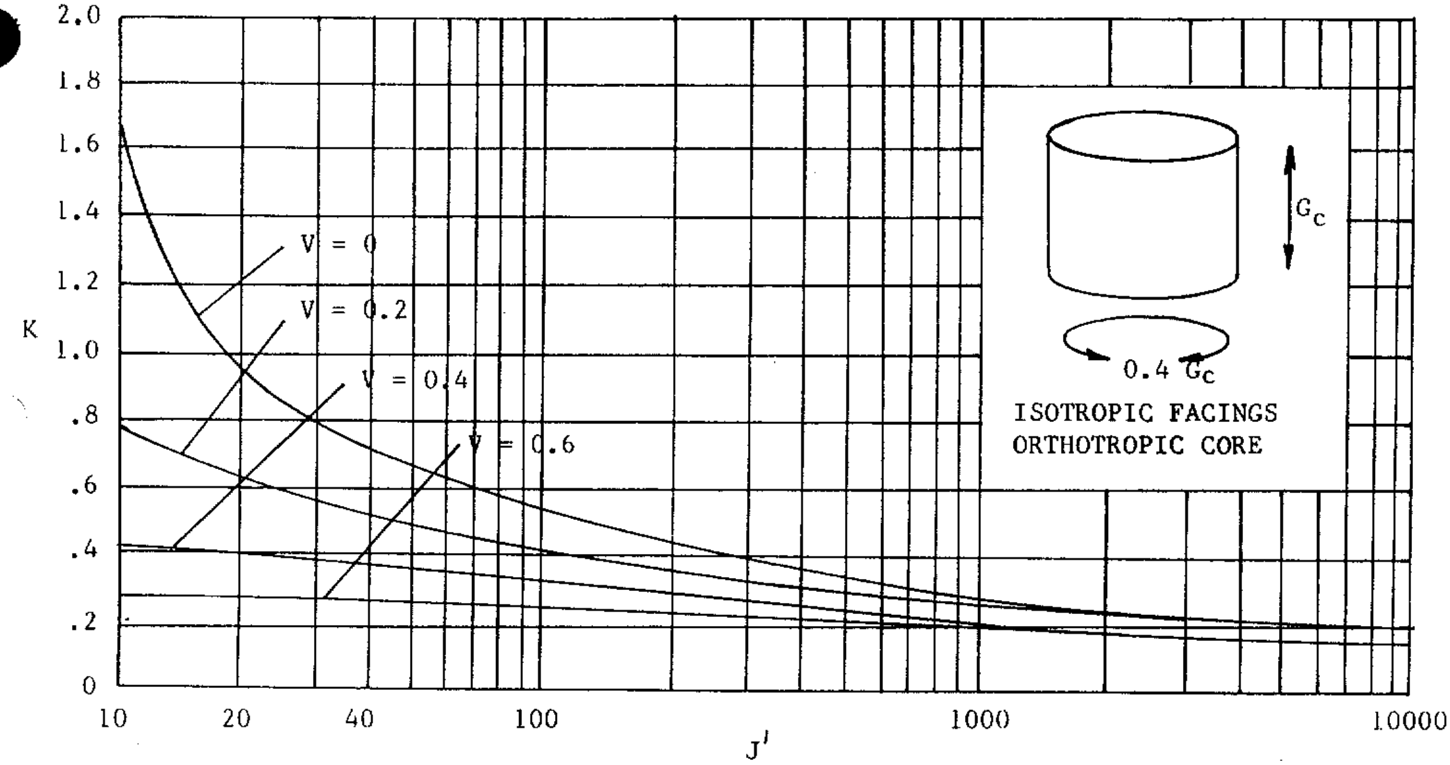


FIGURE 13.56 - BUCKLING CONSTANT, K, FOR CYLINDERS WITH ISOTROPIC FACINGS AND ORTHOTROPIC CORE AND TORSIONAL LOADING

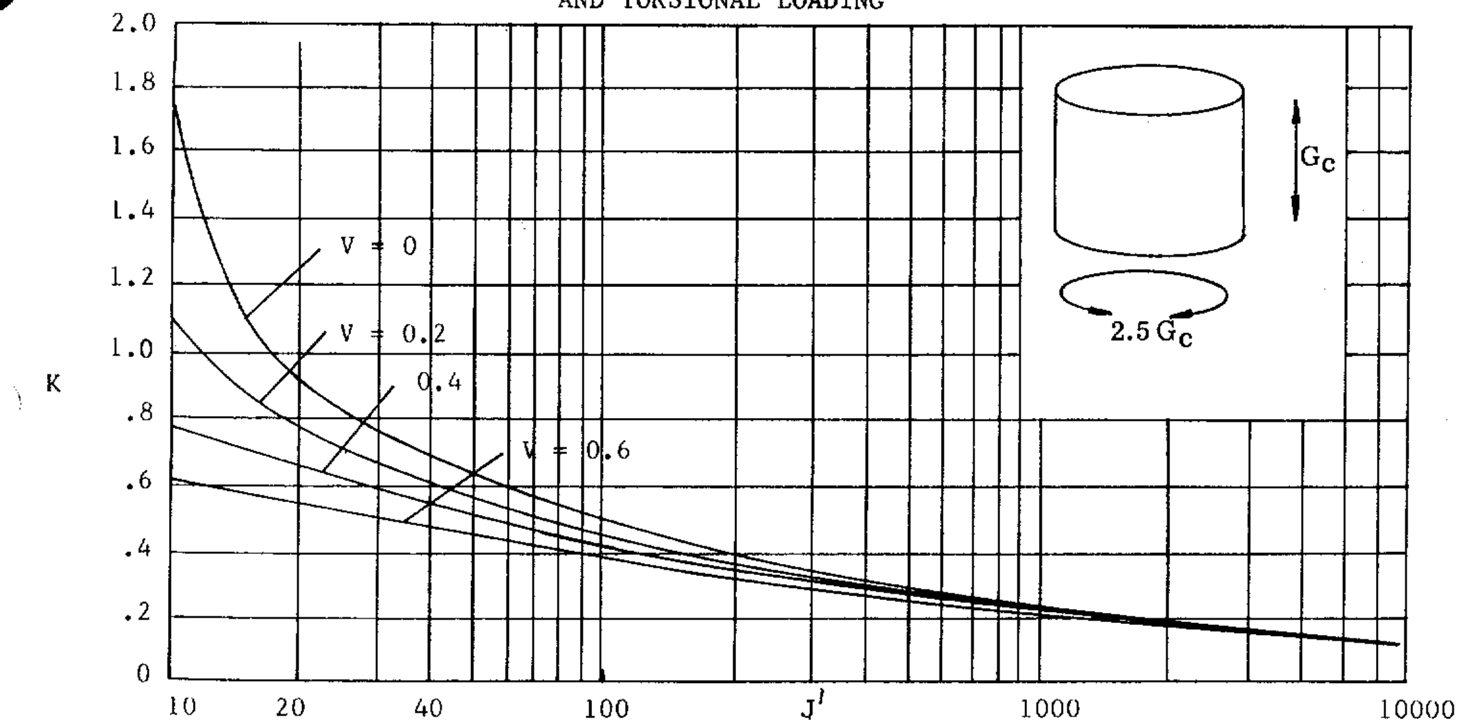


FIGURE 13.57 - BUCKLING CONSTANT, K, FOR CYLINDERS WITH ISOTROPIC FACINGS AND ORTHOGRAPHIC CORE AND TORSIONAL LOADING

Theoretical formulas are based on buckling loads for classical sine-wave buckling. The theory defines the parameters involved rather than determining exact coefficients for computing buckling loads. Large discrepancies exist between theory and tests and, unfortunately, the test values for buckling of thin walled cylinders in axial compression are much lower than expected by theory. Design information based on large deflection theory and diamond shaped buckles give results less than one-half the buckling loads given by classical theory.

Until sufficient test data are available, the two methods of analysis, large deflection, theory, and small deflection or classical theory must be used. The two methods are presented in this section. The designer may take his choice, but this choice should be dictated somewhat by the application of the structure.

A. Large Deflection Theory

The following method is used in the design of a sandwich cylinder subjected to axial compression loading. Assume the load is applied uniformly to both facings. Either the outside or the inside diameter is given. The sandwich has isotropic faces and isotropic or orthotropic core.

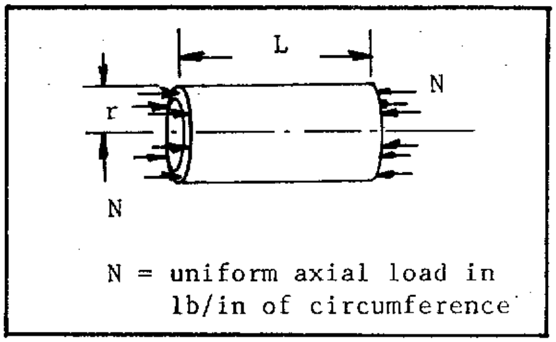


FIGURE 13.58 - CYLINDERS UNDER AXIAL COMPRESSION

(1) Choose an allowable compressive stress (F_f) for the facings and determine the approximate required thicknesses by

$$t_1F_{f1} + t_2F_{f2} = N; \text{ unequal faces} \qquad 13.61$$

$$t = N/2F_f; \text{ equal faces} \qquad 13.62$$

For facings of different materials, maintain the ratio

$$F_{f1}/E_{f1} = F_{f2}/E_{f2} \qquad 13.63$$

(2) Determine the following parameters, assigning subscripts in such a manner that equation 13.64 is $\geq$ 1.

$$E_{f2}t_2/E_{f1}t_1 \qquad 13.64$$

$$\frac{F_f\sqrt{\lambda}}{E_f} \qquad 13.65$$

(3) Enter chart in Figure 13.59 with $V' = 0.1$. Project vertically upward to parameter determined by equation 13.64. Proceed horizontally to appropriate cone shear modulus (G_c) curve, then downward to parameter of equation 13.64 and across to equation 13.65. Project upward to h/r and read the value; use it to determine a tentative sandwich configuration ($r \approx r_o$).

$$h = r\,(h/r) \qquad 13.66$$

Determine core thickness (t_c) from

$$t_c = h - (t_1 + t_2)/2; \text{ unequal faces} \qquad 13.67$$

$$t_c = h - t; \text{ equal faces} \qquad 13.68$$

(4) Estimate the value of r, the radius to the centroid of the cylinder wall, from

$$r = (r_o + r_i)/2 \qquad 13.69$$

This is true for equal facings only but is sufficiently accurate for most practical cases involving unequal faces.

(5) Determine constant K', relating V' to G_c by

$$K' = \frac{2\,E_{f1} t_1 t_c}{3\lambda r d} \qquad 13.70$$

For unequal facings evaluate K' for each facing; use lower value.

(6) Enter chart in Figure 13.63 with $V' = 0.1$. Project horizontally to approximate $V' = K'/G_c$ diagonal and read the value for G_c. If this value of G_c is impractical, move diagonally to a desired value. Read the new V'. For the new value of V' repeat Steps (3) through (6), iterating until a satisfactory solution is reached.

NOTE: For values of $V' \geq 1.0$ use charts in Figures 13.60, 13.61, and 13.62.

(7) For sandwich buckling analysis, evaluate the parameters

$$V = \frac{2}{3\lambda} \frac{E_{f1} t_1 E_{f2} t_2 t_c}{(E_{f1} t_1 + E_{f2} t_2)\, d r\, G_c}; \text{ unequal faces} \qquad 13.71$$

$$V = \frac{E_f t t_c}{3\lambda d r G_c}; \text{ equal faces} \qquad 13.72$$

$$t_c/d \qquad 13.73$$

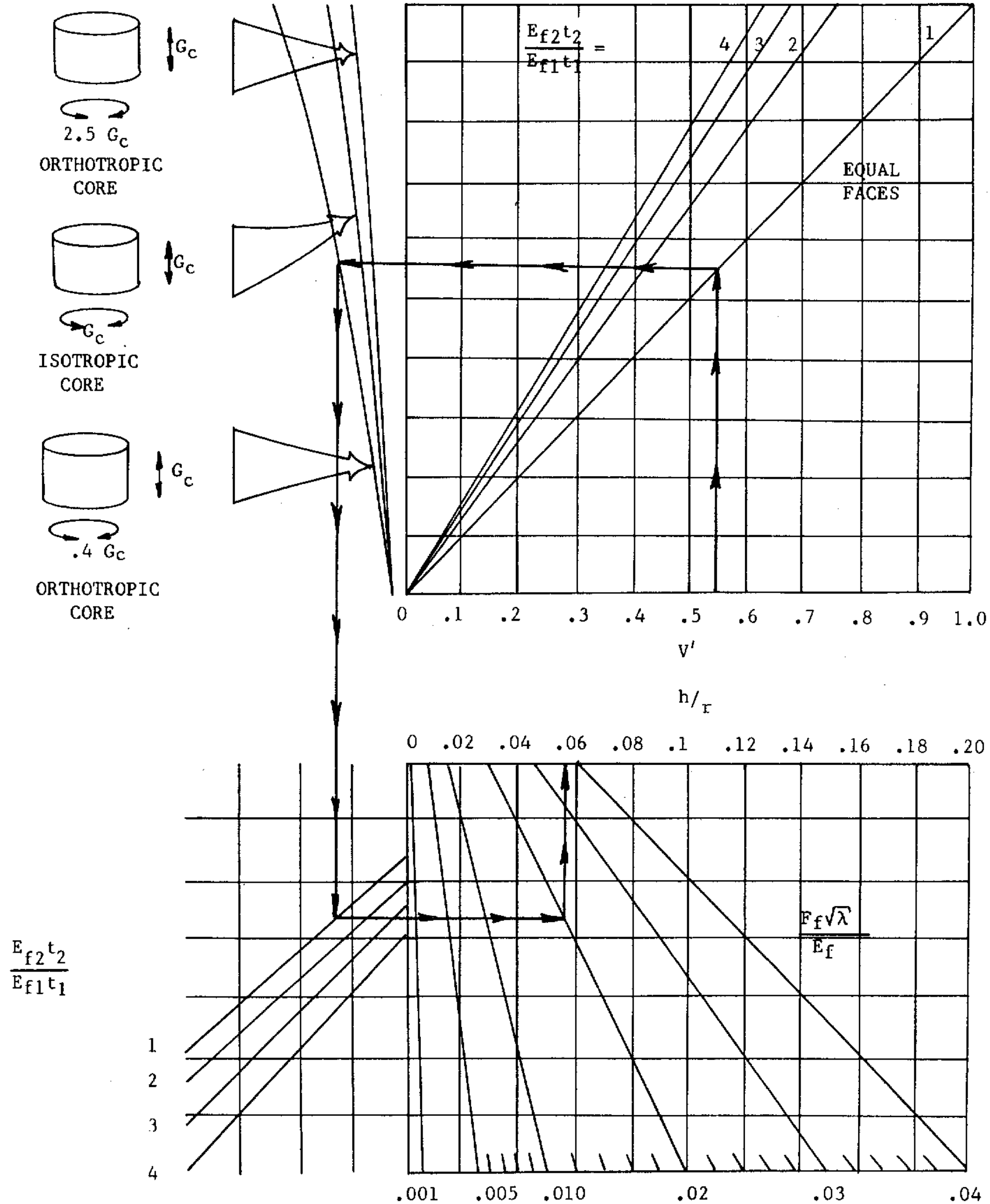


FIGURE 13.59 - APPROXIMATE DESIGN CURVE FOR CYLINDERS UNDER AXIAL COMPRESSION

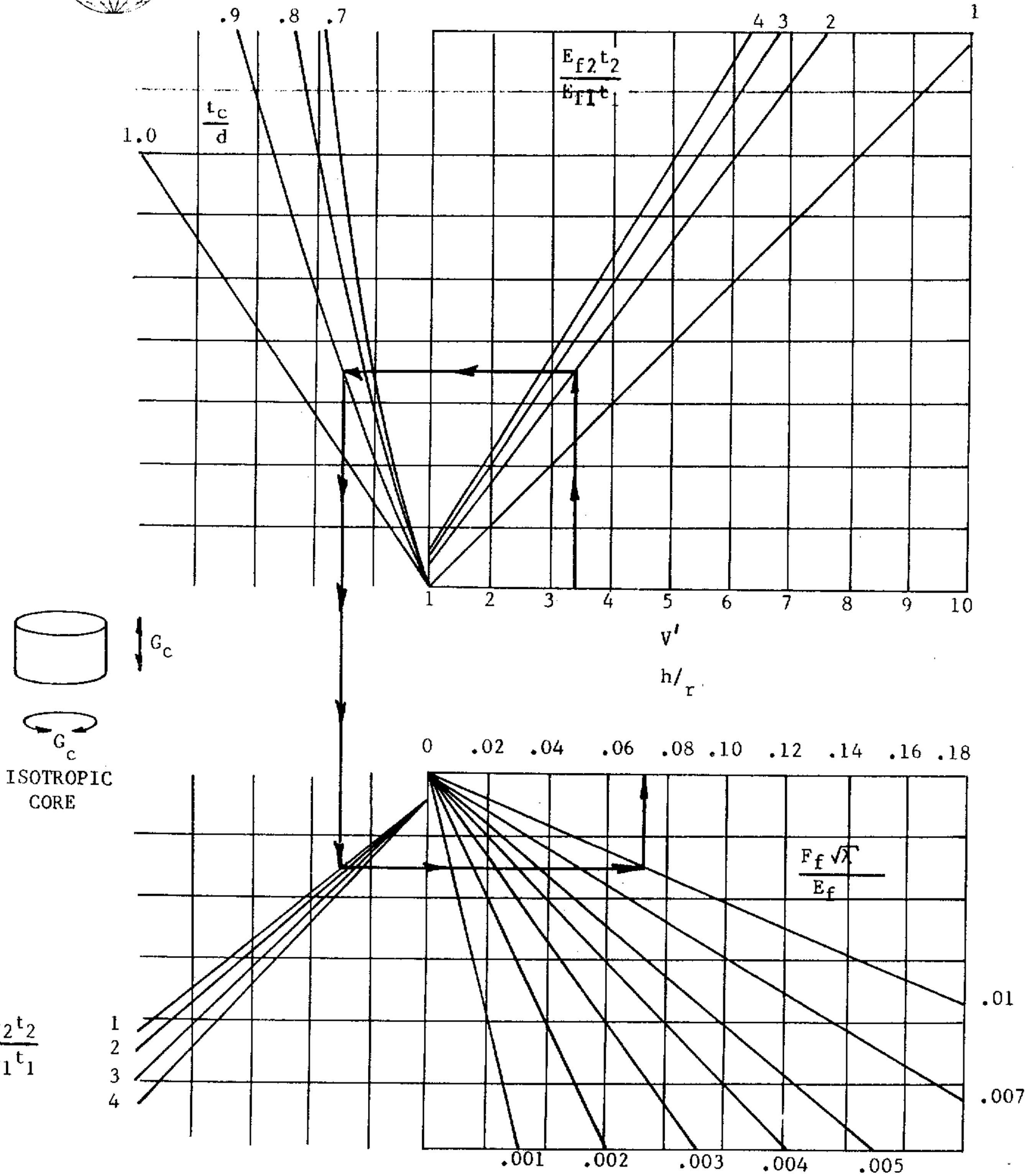


FIGURE 13.60 - DESIGN CURVE FOR CYLINDERS WITH ISOTROPIC CORE UNDER AXIAL COMPRESSION

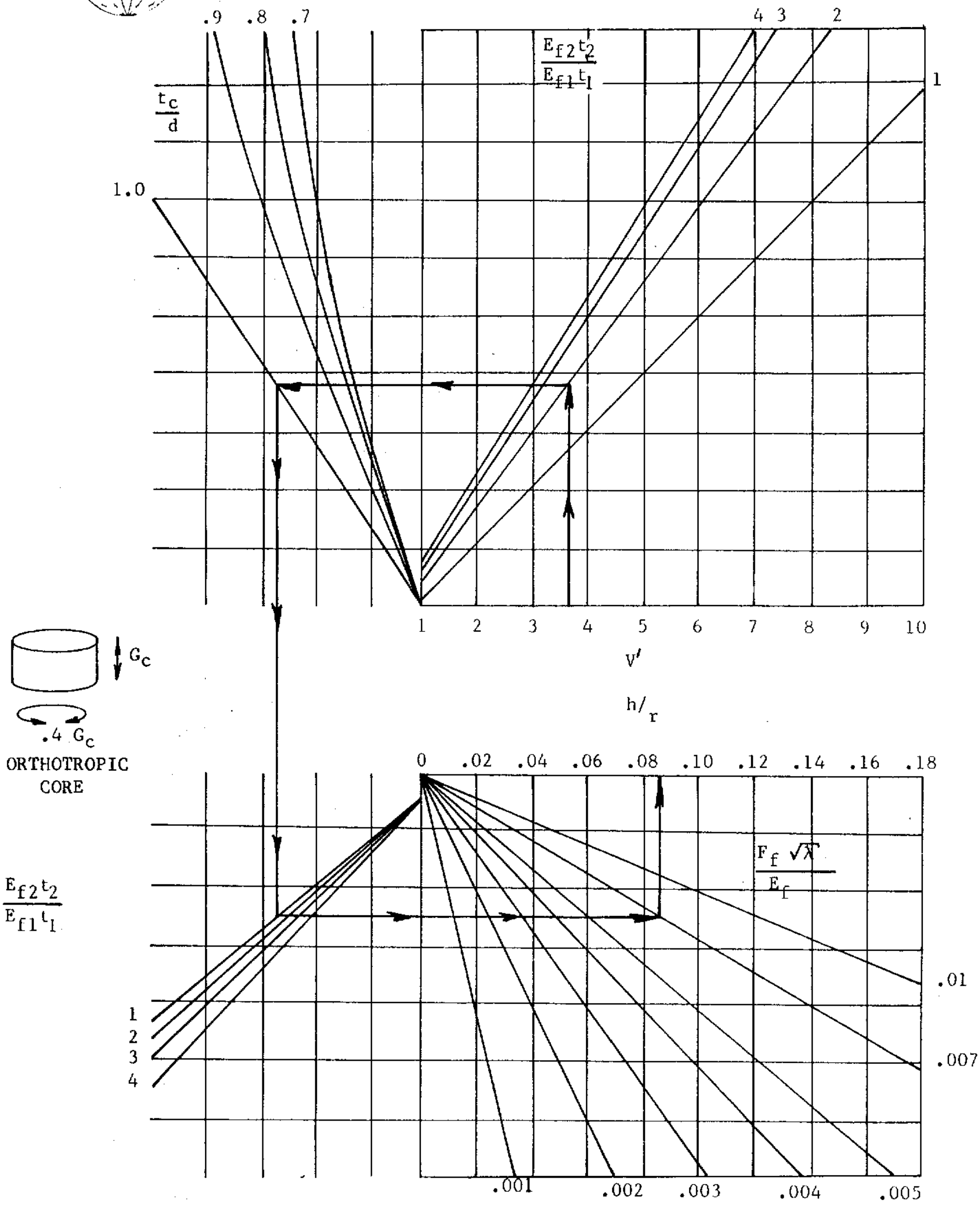


FIGURE 13.61 - DESIGN CURVE FOR CYLINDERS WITH ORTHOTROPIC CORE UNDER AXIAL COMPRESSION

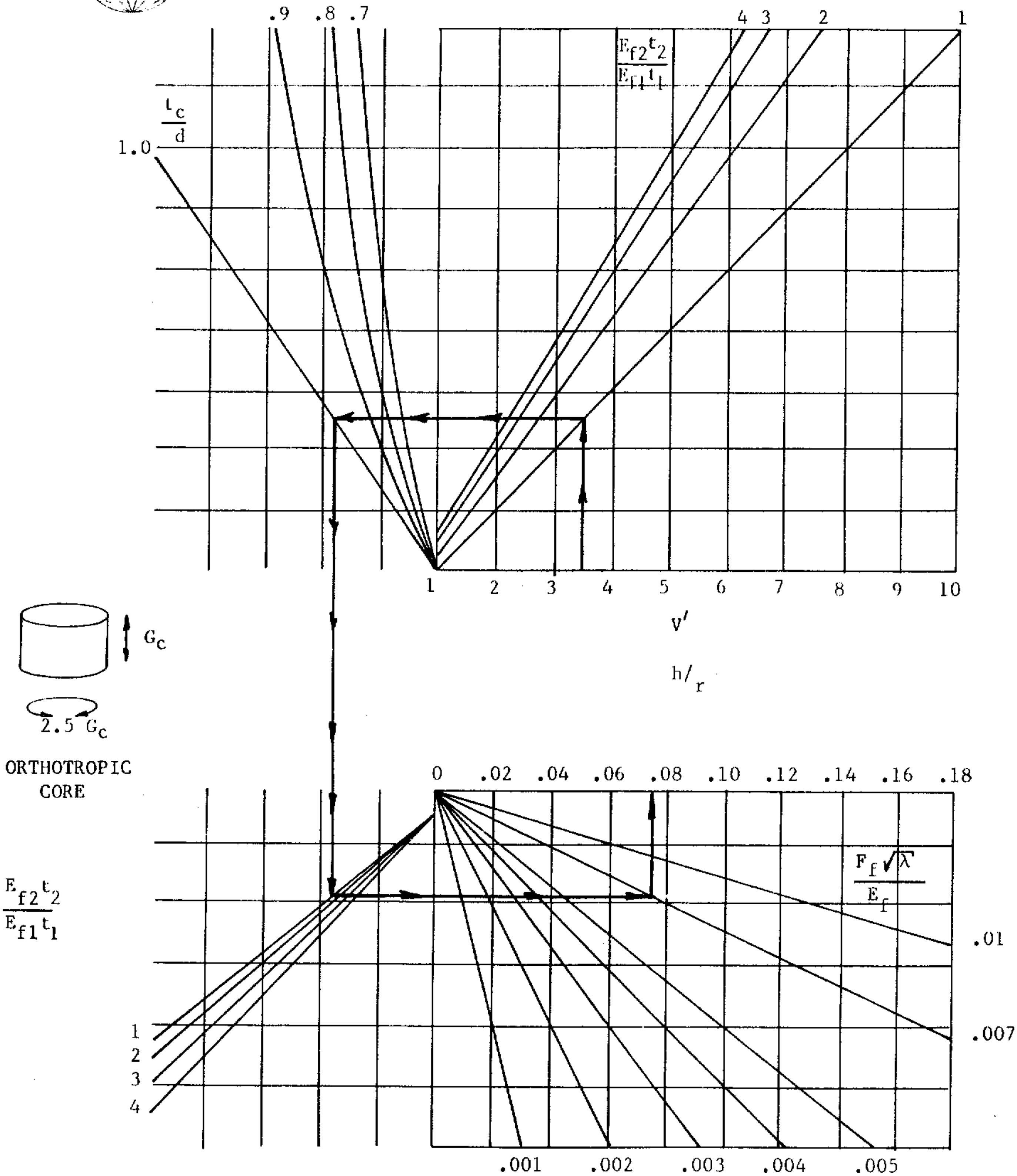


FIGURE 13.62 - DESIGN CURVE FOR CYLINDERS WITH ORTHOTROPIC CORE UNDER AXIAL COMPRESSION

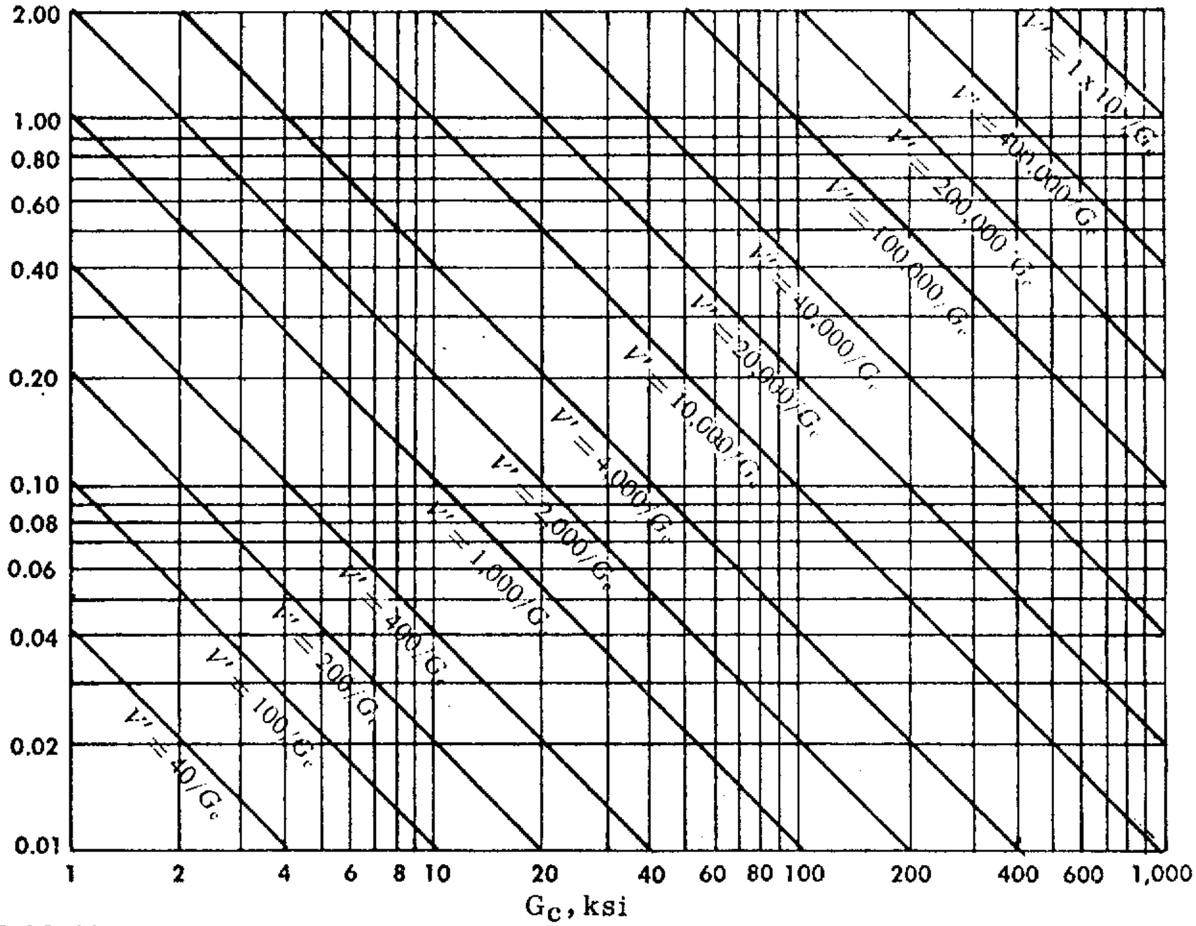
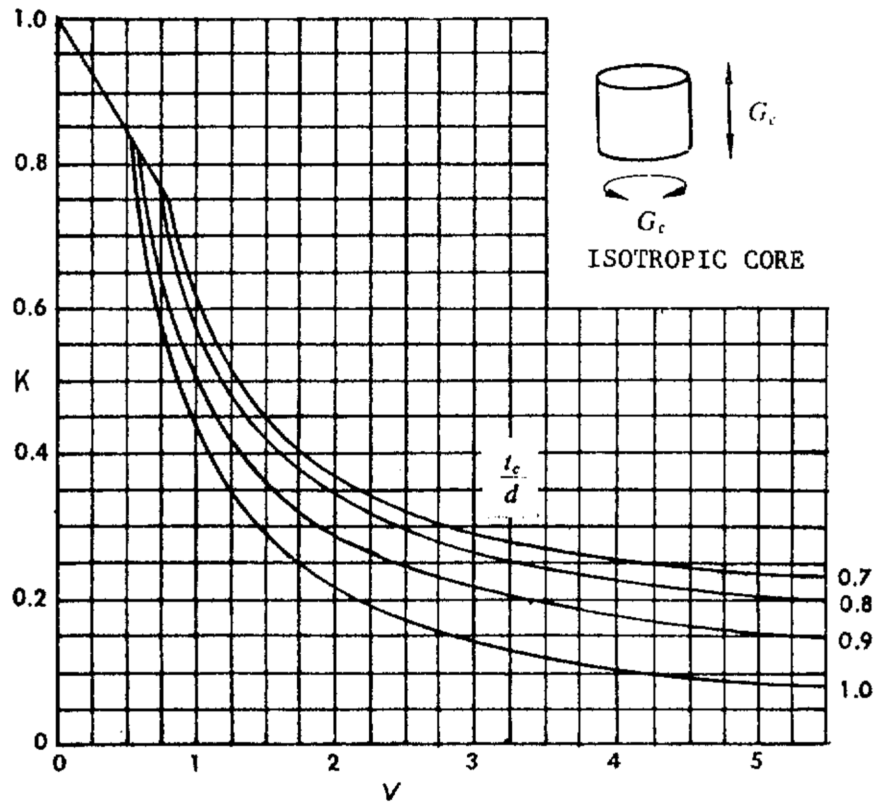


FIGURE 13.63 - CHART FOR DETERMINING G_C FOR CYLINDERS UNDER AXIAL COMPRESSION

FIGURE 13.64 - CHART FOR DETERMINING K FOR CYLINDERS UNDER AXIAL COMPRESSION WITH ISOTROPIC CORE



Enter the appropriate chart in Figures 13.64, 13.65, or 13.66. Obtain the value of K.

(8) Determine the ratio of the facing stiffness to the sandwich stiffness

$$R_F = \frac{(E_1 t_1^3 + E_2 t_2^3)(E_1 t_1 + E_2 t_2)}{12\, E_1 t_1 E_2 t_2 h^2} \quad ; \text{ unequal faces} \qquad 13.74$$

$$R_F = t^2/3h^2 \quad ; \text{ equal faces} \qquad 13.75$$

(9) The value of facing stress (F_{cr}) at which buckling of sandwich wall will occur is

$$F_{ccr} = \frac{4}{5} K \frac{Eh\sqrt{E_1 t_1 E_2 t_2}}{r\,(E_1 t_1 + E_2 t_2)} \sqrt{\frac{1+R_F}{\lambda}} \quad ; \text{ unequal faces} \qquad 13.76$$

$$F_{ccr} = \frac{2}{5} K \frac{Eh}{r} \sqrt{\frac{1+R_F}{\lambda}} \quad ; \text{ equal faces} \qquad 13.77$$

Unequal faces must both be checked to insure that $F_{ccr} > f_f$.

(10) Check for overall column buckling using

$$N_{ccr} = \frac{\pi^2 r^2 (E_1 t_1 + E_2 t_2)}{2\, L^2} \quad ; \text{ unequal faces} \qquad 13.78$$

$$N_{ccr} = \frac{\pi^2 r^2 E t}{L^2} \quad ; \text{ equal faces} \qquad 13.79$$

(11) Check the cylinder faces for intracell buckling per Section 13.2.2.

B. Small Deflection or Classical Theory

Proceed through step (6) in the previously described large deflection method. This will give a satisfactory first approximation.

(7) For sandwich buckling analysis evaluate the parameters

$$V = \frac{E_{f1} E_{f2} t_c \sqrt{t_1 t_2}}{(E_{f1} + E_{f2})\sqrt{\lambda}\, hr\, G_{xz}} \qquad 13.80$$

FIGURE 13.65

CHART FOR DETERMINING K FOR CYLINDERS UNDER AXIAL COMPRESSION WITH ORTHOTROPIC CORE

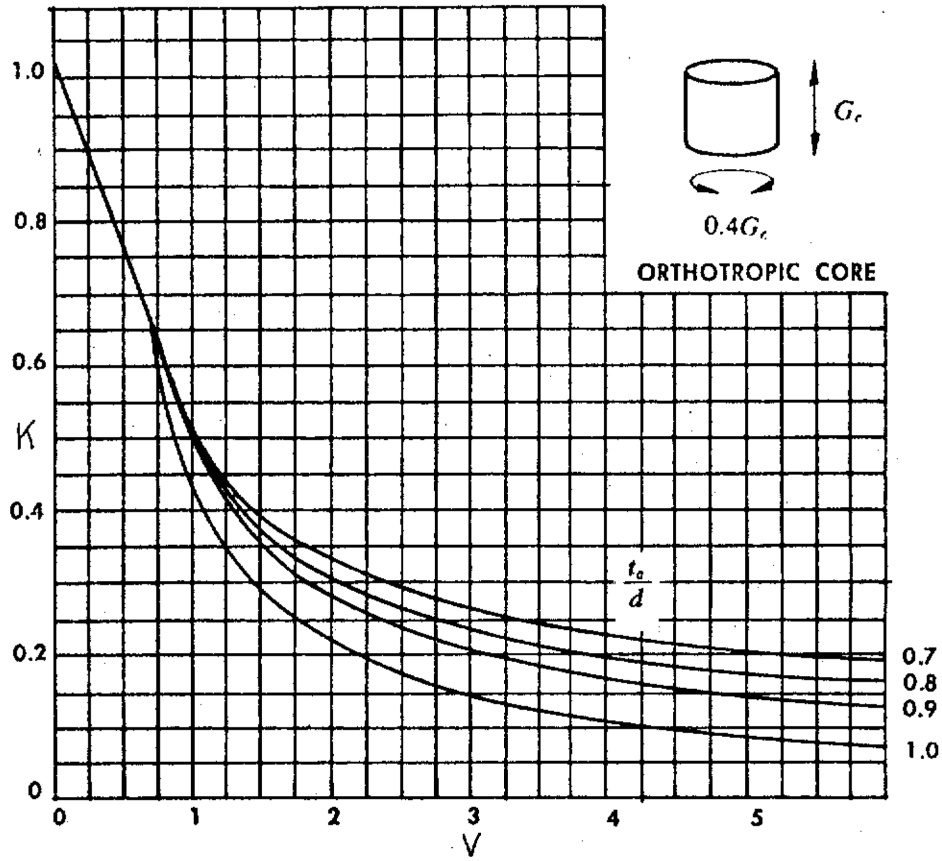
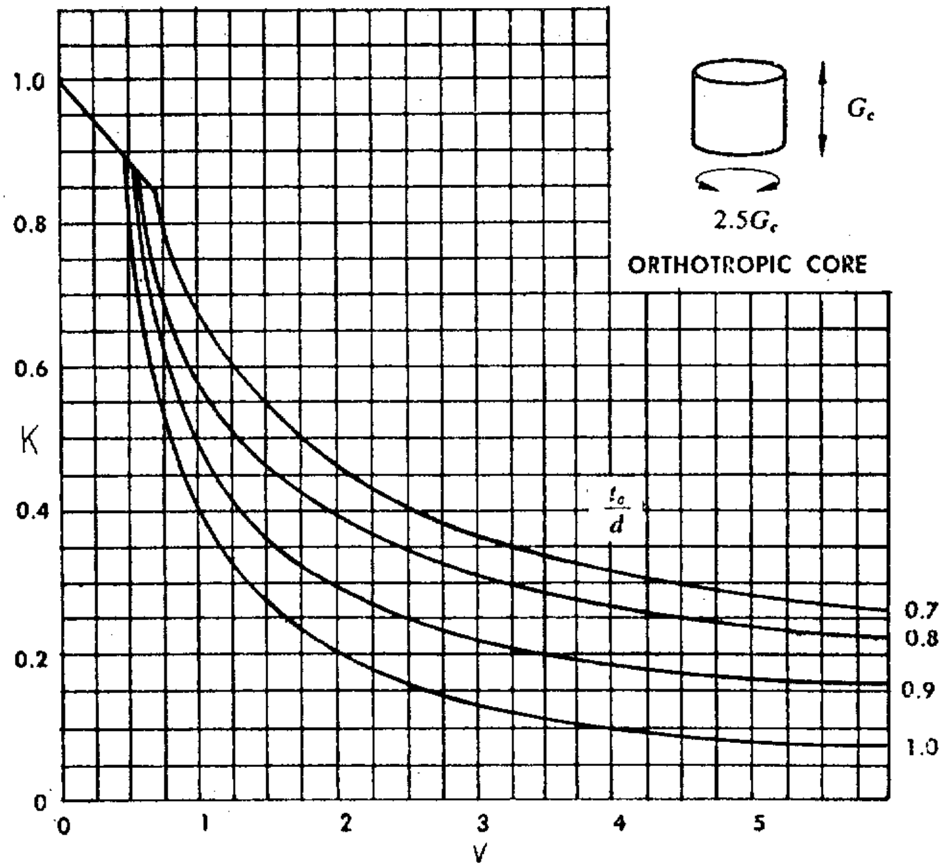


FIGURE 13.66

CHART FOR DETERMINING K FOR CYLINDERS UNDER AXIAL COMPRESSION WITH ORTHOTROPIC CORE



$$V = \frac{E_f t_c t}{2\sqrt{\lambda}\, h r G_{xz}} \text{ ; equal faces} \qquad 13.81$$

(8) Enter chart in Figure 13.67 and obtain a value for K.

(9) The value of F_{cr} at which buckling of the sandwich will occur is

$$F_{cr} = \frac{4KE_{f1}E_{f2}h\sqrt{t_1 t_2}}{r\sqrt{\lambda}(t_1 t_2)} \text{ ; unequal faces} \qquad 13.82$$

$$F_{cr} = \frac{KE_f h}{r\sqrt{\lambda}} \text{ ; equal faces} \qquad 13.83$$

(10) and (11) Same as steps (10) and (11) for large deflection.

(12) Check for face wrinkling per section 13.2.1.

13.2.8 Cylinders Under Uniform External Pressure

This section presents formulas, theoretical equations, and a design procedure for determining the sandwich facing thickness, core thickness, and core shear modulus such that overall buckling of a sandwich cylinder will not occur at the facing design stresses. The following method is used in the analysis and design of sandwich cylinders subjected to uniform external pressure. The facings are isotropic, but may be of different materials and different thicknesses. The core may be either isotropic or orthotropic. The outside diameter and cylinder length are given as part of the design criteria.

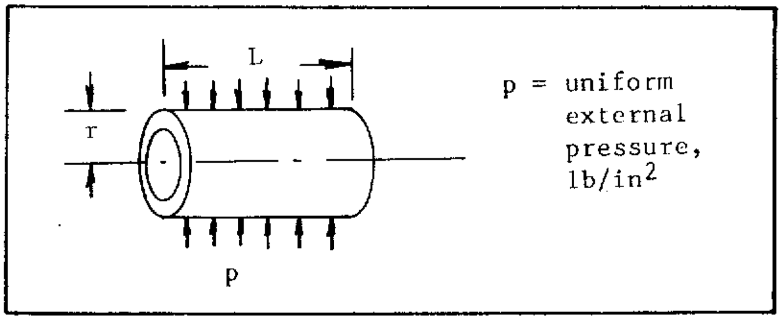


FIGURE 13.68 - CYLINDER UNDER UNIFORM EXTERNAL PRESSURE

(1) Select a tentative sandwich configuration given cylinder length (L), the outside diameter (D_o) and external pressure (p).

Choose:

a. Facing materials and thicknesses.

b. Core material: The flatwise compressive strength of the core, (F_c), must satisfy $F_c \geq 1.5$ 13.84

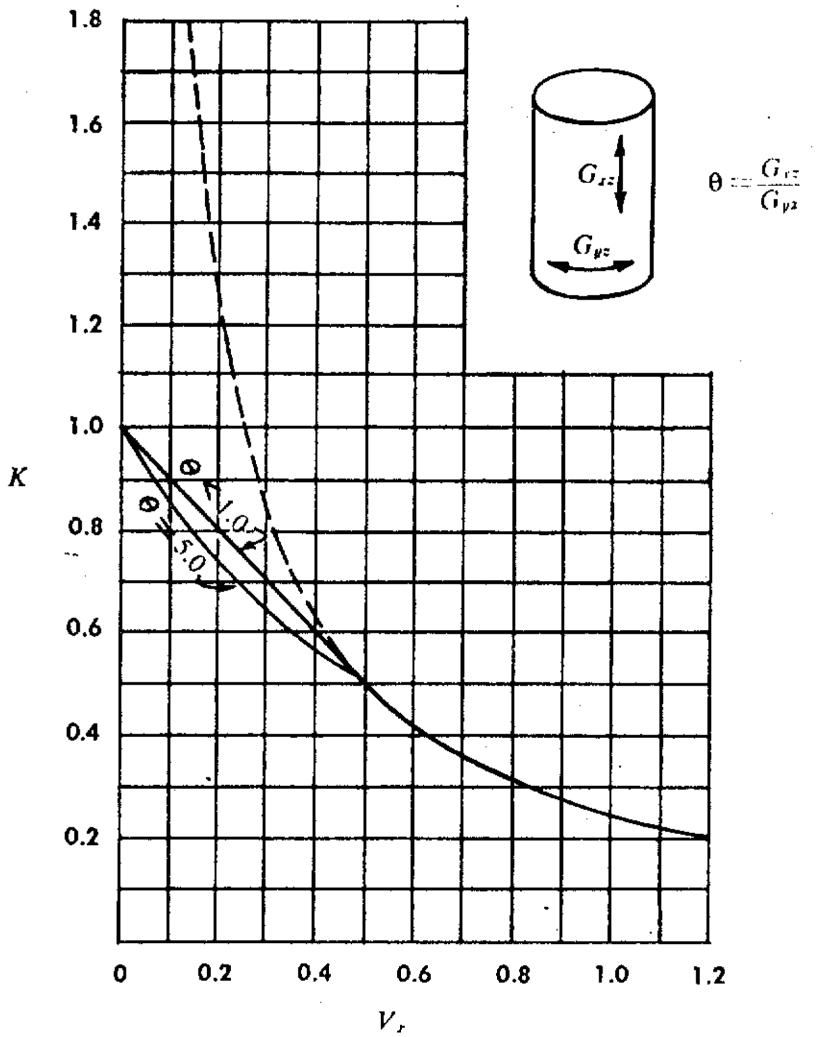


FIGURE 13.67 - CLASSICAL BUCKLING COEFFICIENT - ISOTROPIC FACES AND ORTHOTROPIC CORE

c. A tentative value for the centroidal distance between facings (h).

(2) Calculate the total depth of the sandwich (d) and the mean radius (r_c).

$$d = h + \frac{(t_1 + t_2)}{2} \text{ ; unequal faces} \qquad 13.85$$

$$r_c = \frac{D_o - d}{2} \qquad 13.86$$

$$d = h + t\text{; equal faces} \qquad 13.87$$

(3) Calculate the parameter (R).

$$R = \frac{E_1 t_1}{E_2 t_2} \text{ ; unequal faces} \qquad 13.88$$

$$R = 1\text{; equal faces} \qquad 13.89$$

(4) Calculate the parameter (V).

$$V = \frac{2E_1 t_1 E_2 t_2}{3r_c \lambda (E_1 t_1 + E_2 t_2) G_{r\theta}} \text{; unequal faces} \qquad 13.90$$

$$V = \frac{E t}{3r_c \lambda G_{r\theta}} \text{ ; equal faces} \qquad 13.91$$

where $G_{r\theta}$ = Core modulus of rigidity in the radial and tangential direction.

(5) Calculate the parameters (α^2) and (L/r_c).

L/r_c = the ratio of cylinder length to the mean radius 13.92

$$\alpha^2 = \tfrac{1}{2} (h/r_c)^2 \qquad 13.93$$

h/r_c = the ratio of the centroidial distance between faces to the mean radius, where the mean radius is the distance to the center of the core.

(6) Enter charts in Figures 13.69 through 13.84 with the values of α^2, L/r_c and R. Determine k. If there is no exact chart for the given values, interpolate between adjacent charts.

A simpler but more conservative method based on the assumption of a very long cylinder ($L/r_c > 100$) may also be used to determine the value for k directly.

$$k = \frac{12R\,\alpha^2}{(R+1)^2\,(1 - 12\,V\alpha)} \text{ ; unequal faces} \qquad 13.94$$

$$k = \frac{3\,\alpha^2}{1 - 12\,V\alpha} \text{ ; equal faces} \qquad 13.95$$

(7) Calculate the critical buckling pressure (q_{cr}).

$$q_{cr} = \left[\frac{E_1 t_1 + E_2 t_2}{r_c\,\lambda}\right] k\text{; unequal faces} \qquad 13.96$$

$$q_{cr} = \frac{2Etk}{r_c\lambda} \text{ ; equal faces} \qquad 13.97$$

(8) If q_{cr} is less than the external pressure, p, select a new sandwich configuration and repeat steps (1) through (7).

13.2.9 Beams

This section contains the procedure for the design of sandwich members used as beams. The load is applied normal to the face of the sandwich and the member has reaction points at the ends. The edge of the beam is assumed to have no support.

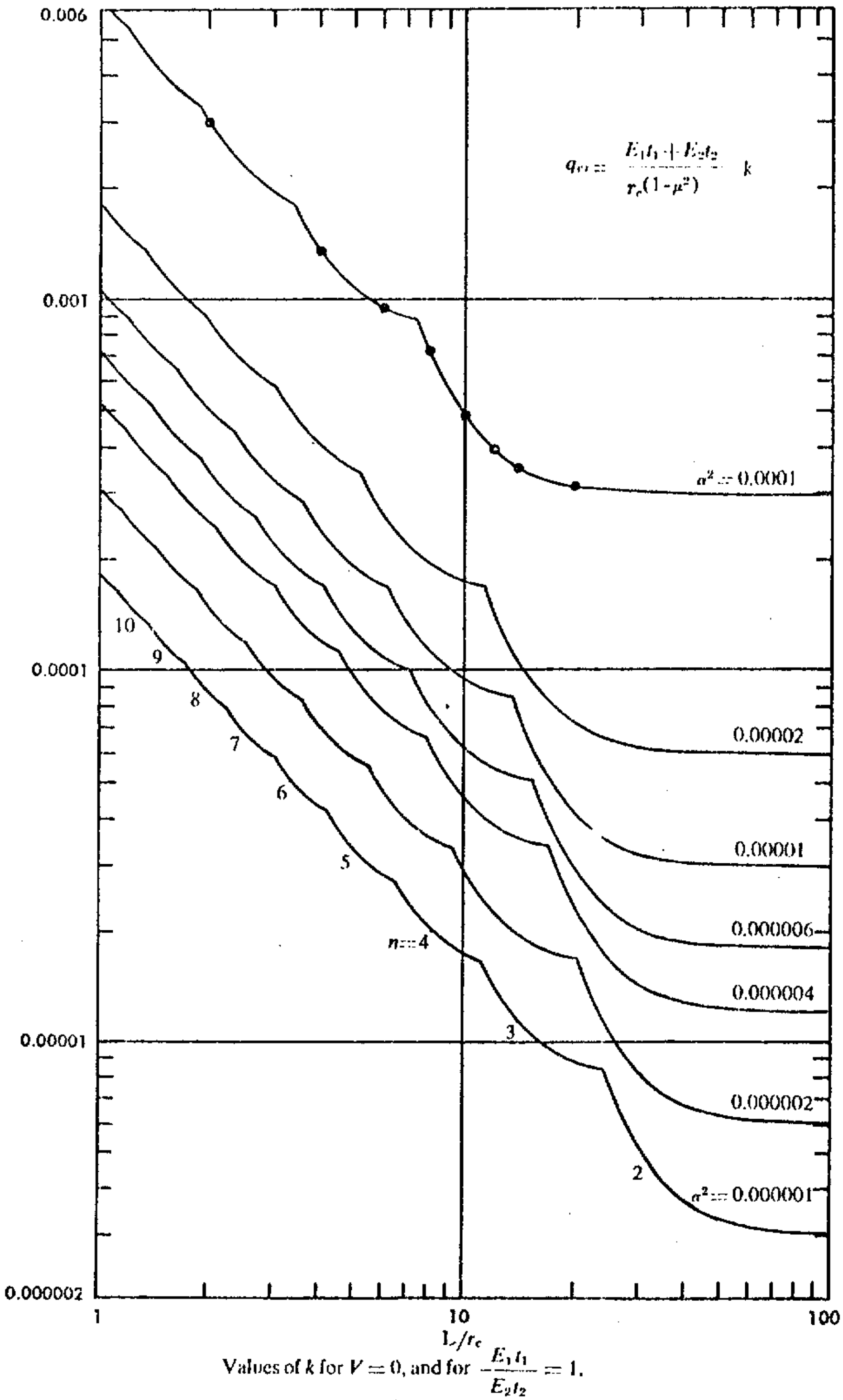
(1) Determine the maximum bending moment (M) and maximum beam shear (S) for the design loading and end support conditions. Figure 13.85 shows some commonly used beams with maximum moments and shears.

(2) Choose an allowable design facing stress (F_f) which does not exceed either the tensile or compressive yield stress of the face material.

(3) Calculate the required section modulus per unit width (Z) from

$$Z = \frac{M}{F_f b} \qquad 13.98$$

where b = beam width.



Values of k for $V = 0$, and for $\frac{E_1t_1}{E_2t_2} = 1$.

FIGURE 13.69 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE

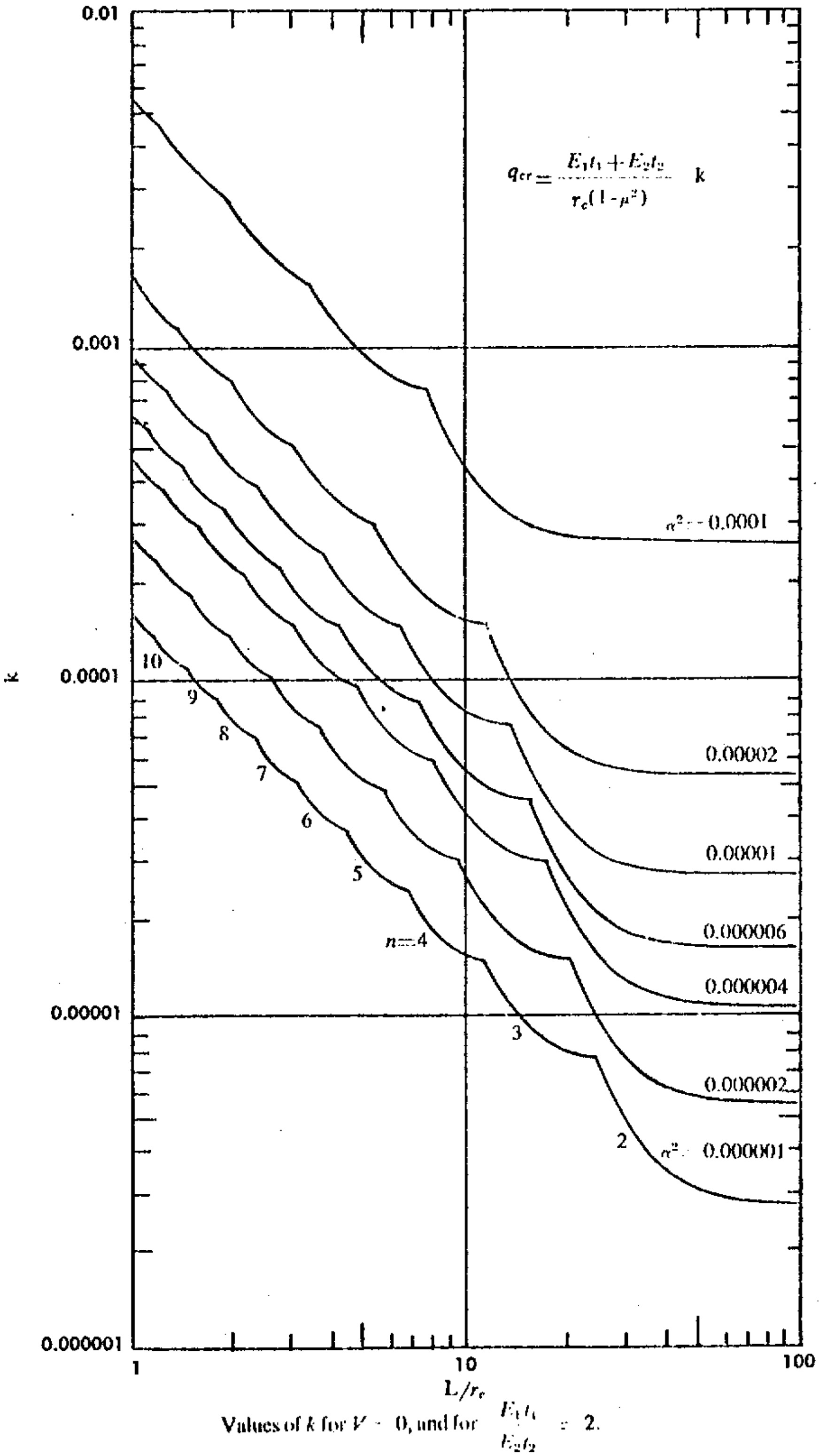
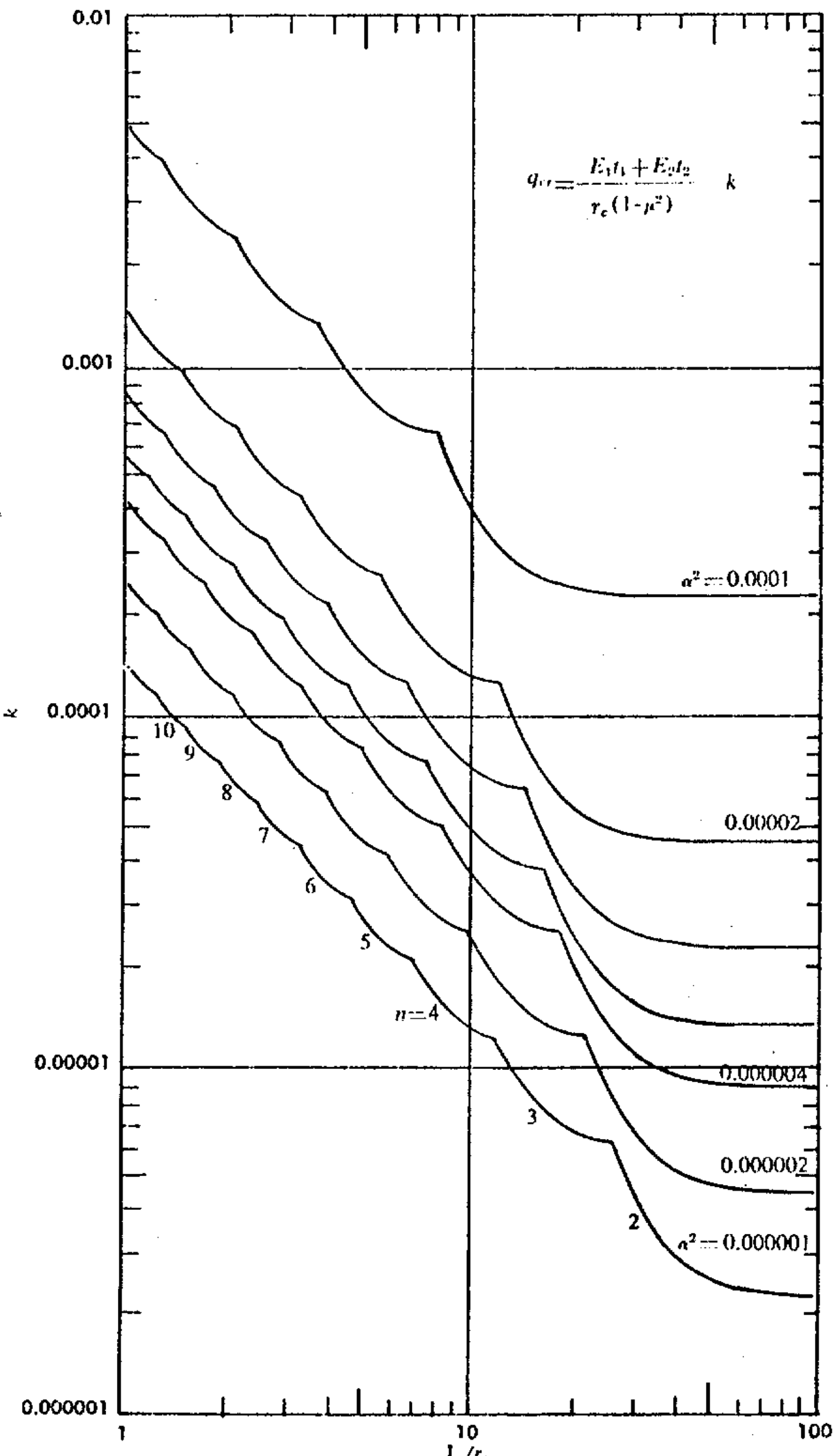
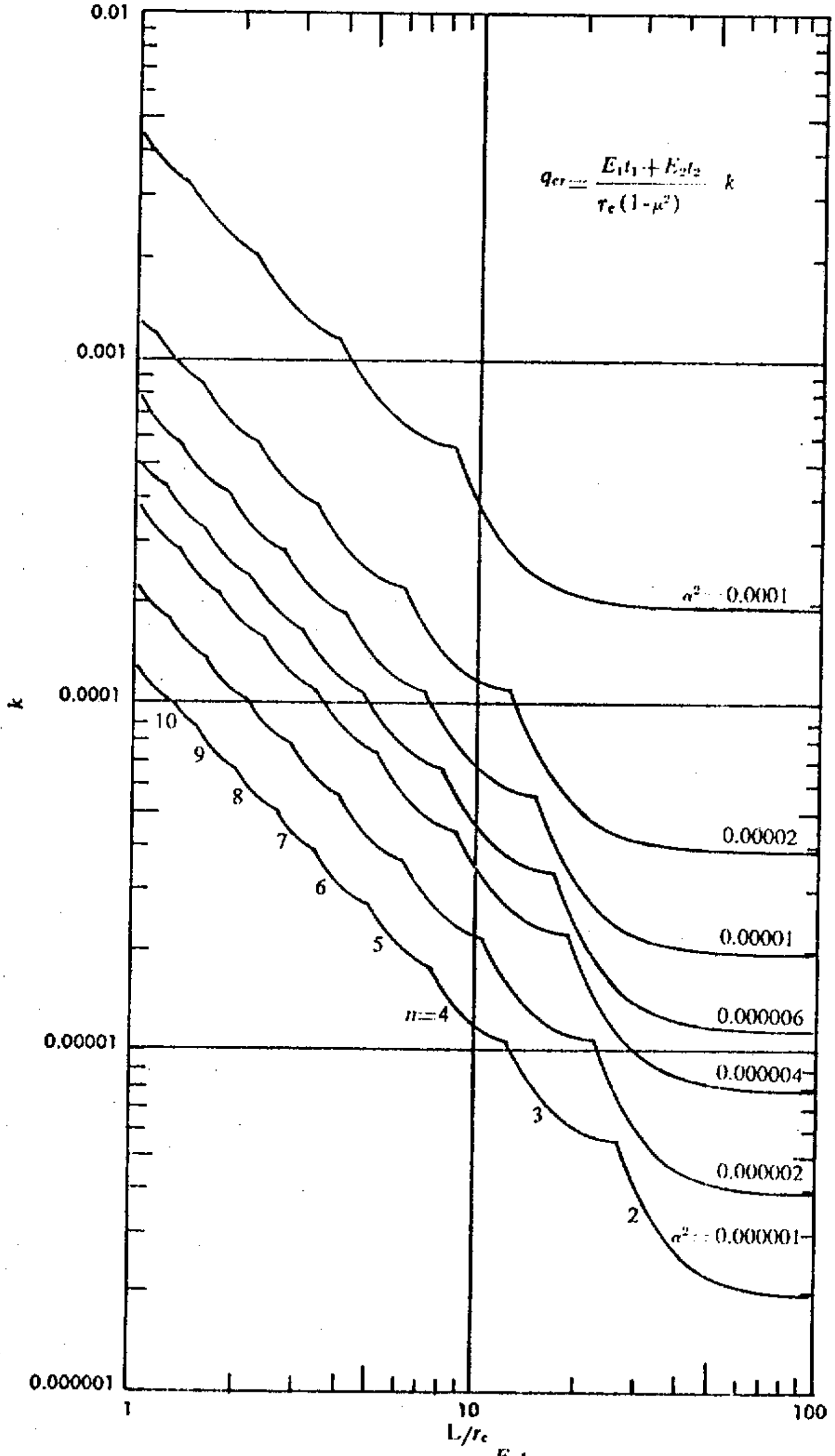


FIGURE 13.70 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE



Values of k for $V = 0$, and for $\frac{E_1 t_1}{E_2 t_2} = 3$.

FIGURE 13.71 - BUCKLING COEFFICIENT FOR CYLINDERS
UNDER UNIFORM EXTERNAL PRESSURE



Values of k for $V = 0$, and for $\frac{E_1 t_1}{E_2 t_2} = 4$.

FIGURE 13.72 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE

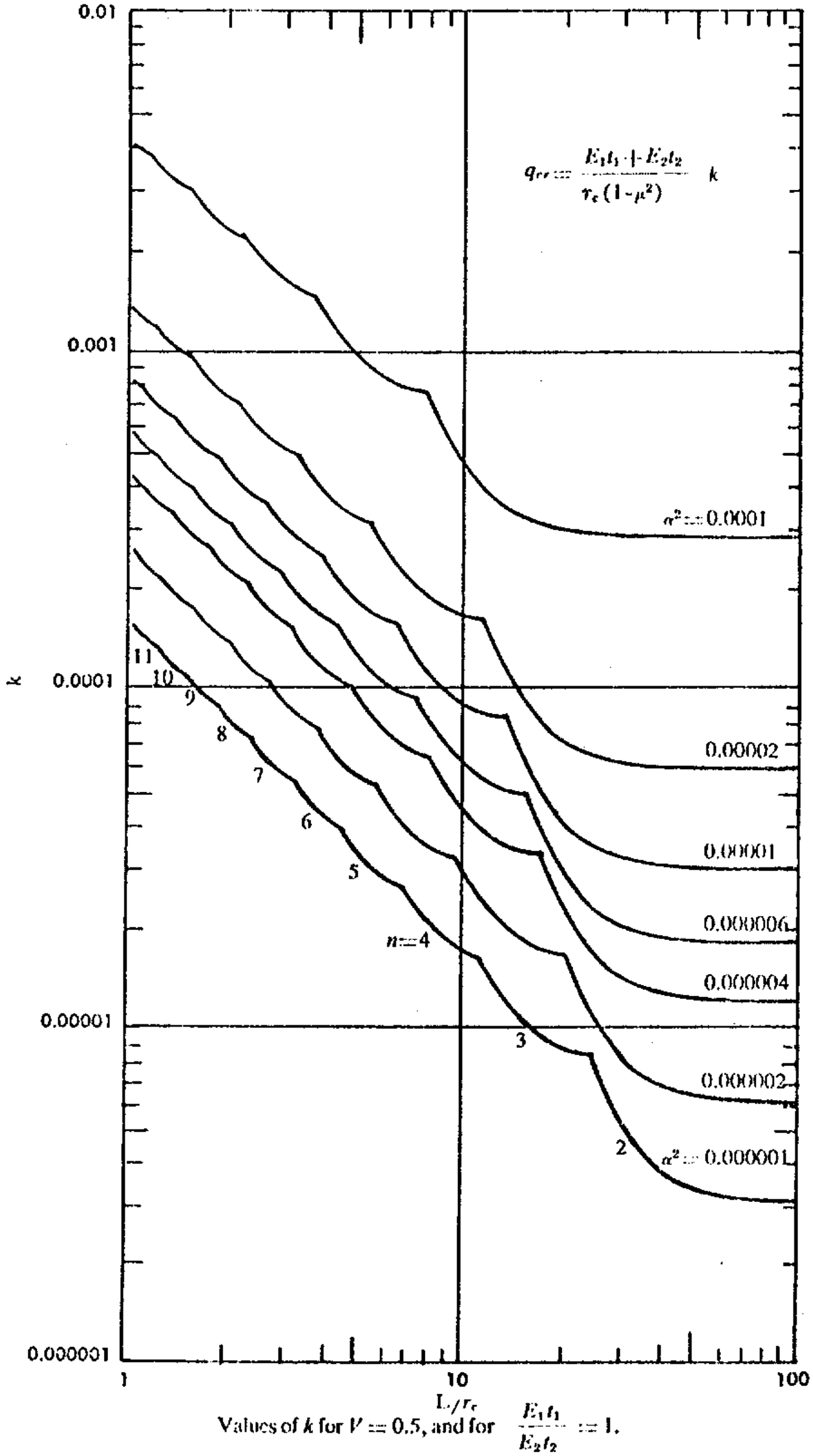
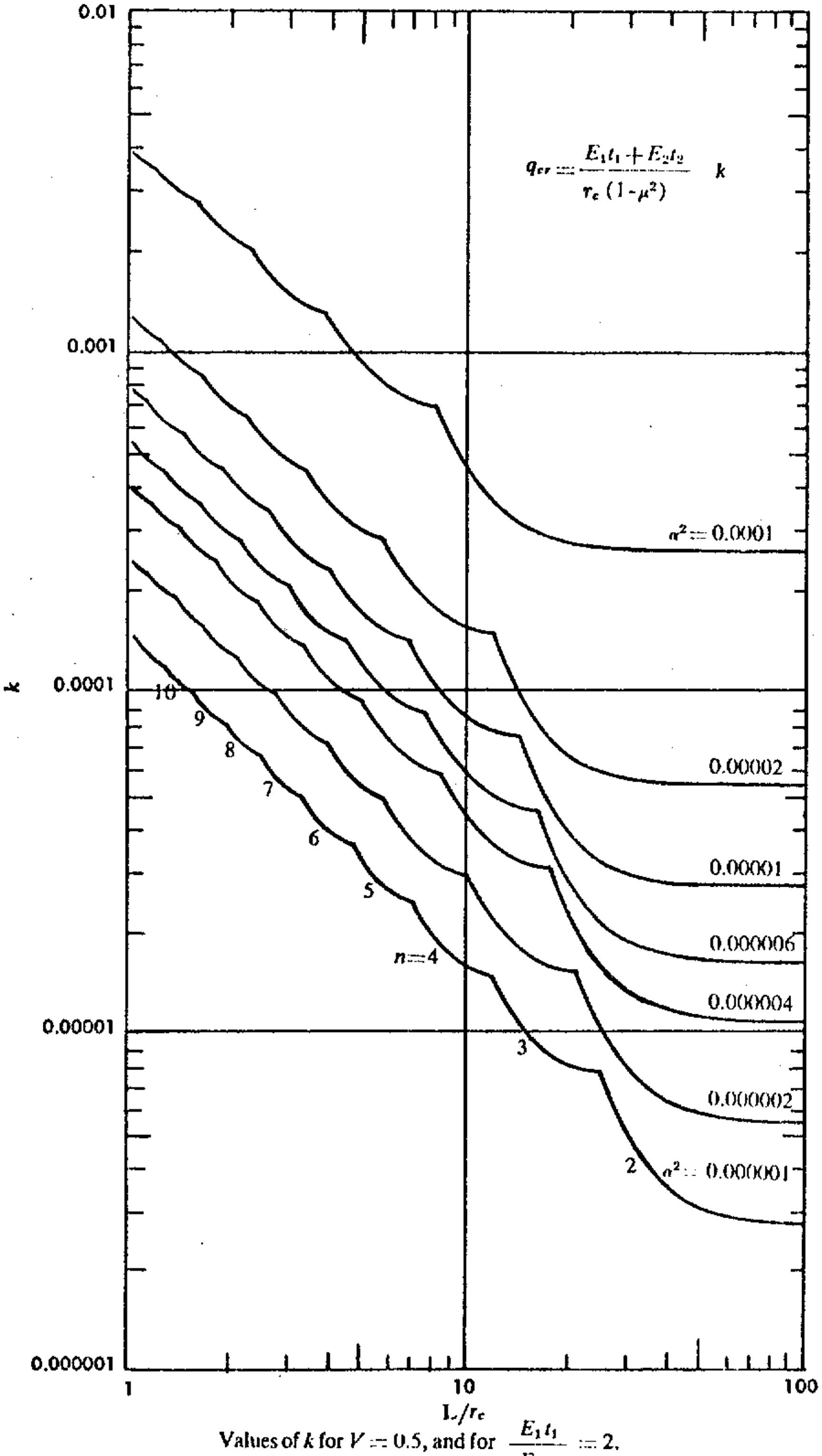
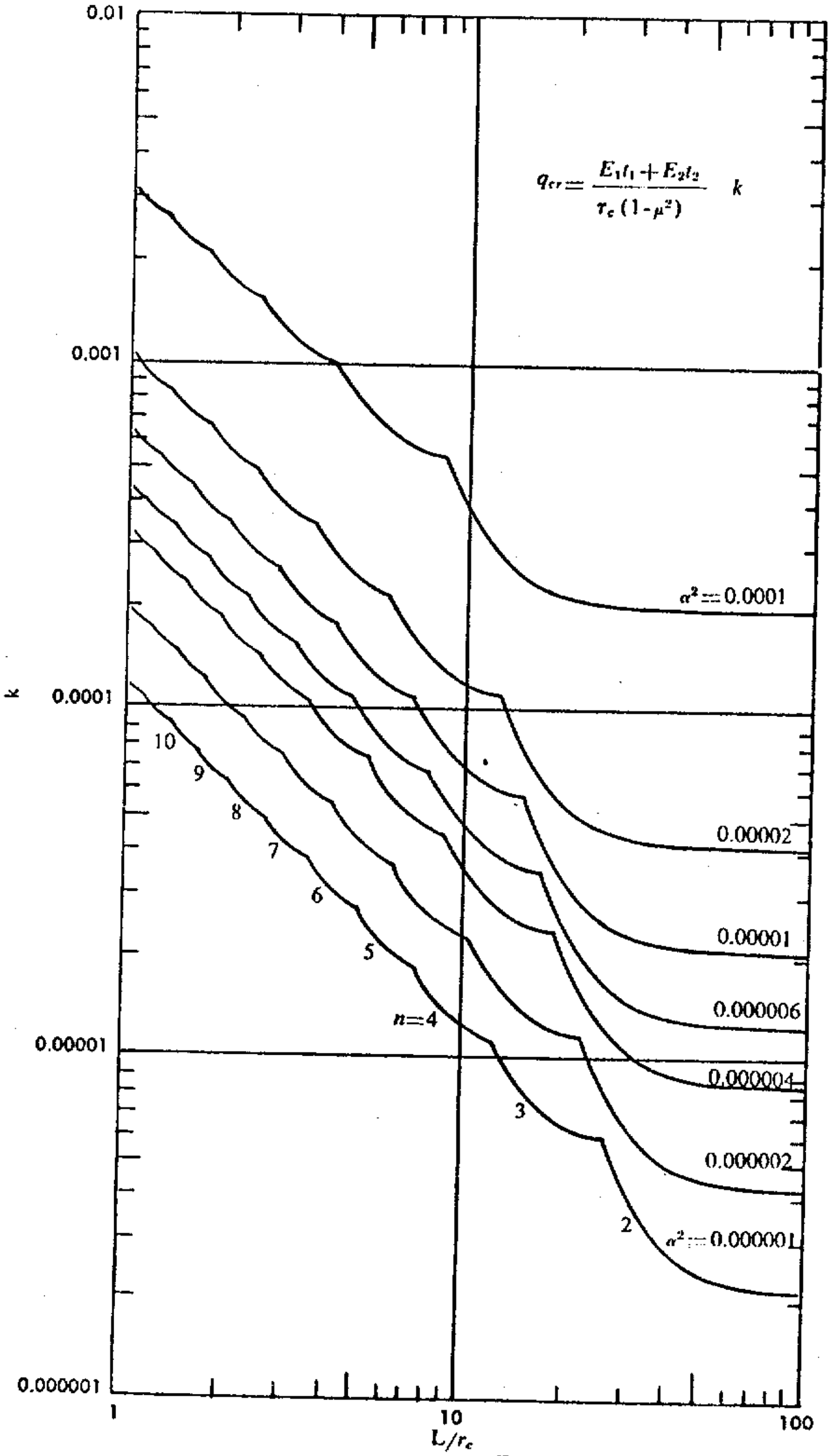


FIGURE 13.73 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE



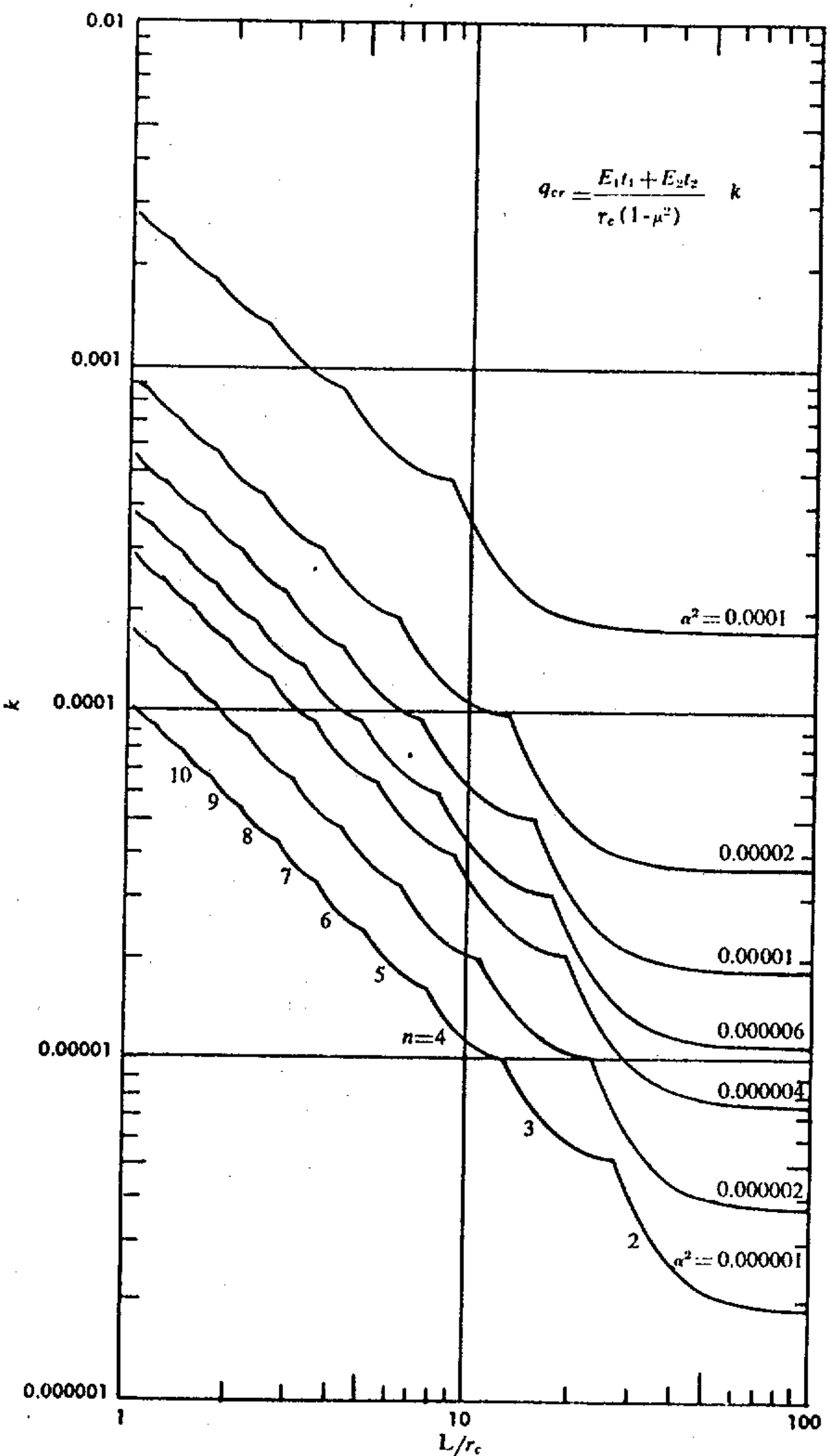
Values of k for $V = 0.5$, and for $\frac{E_1 t_1}{E_2 t_2} = 2$.

FIGURE 13.74 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE



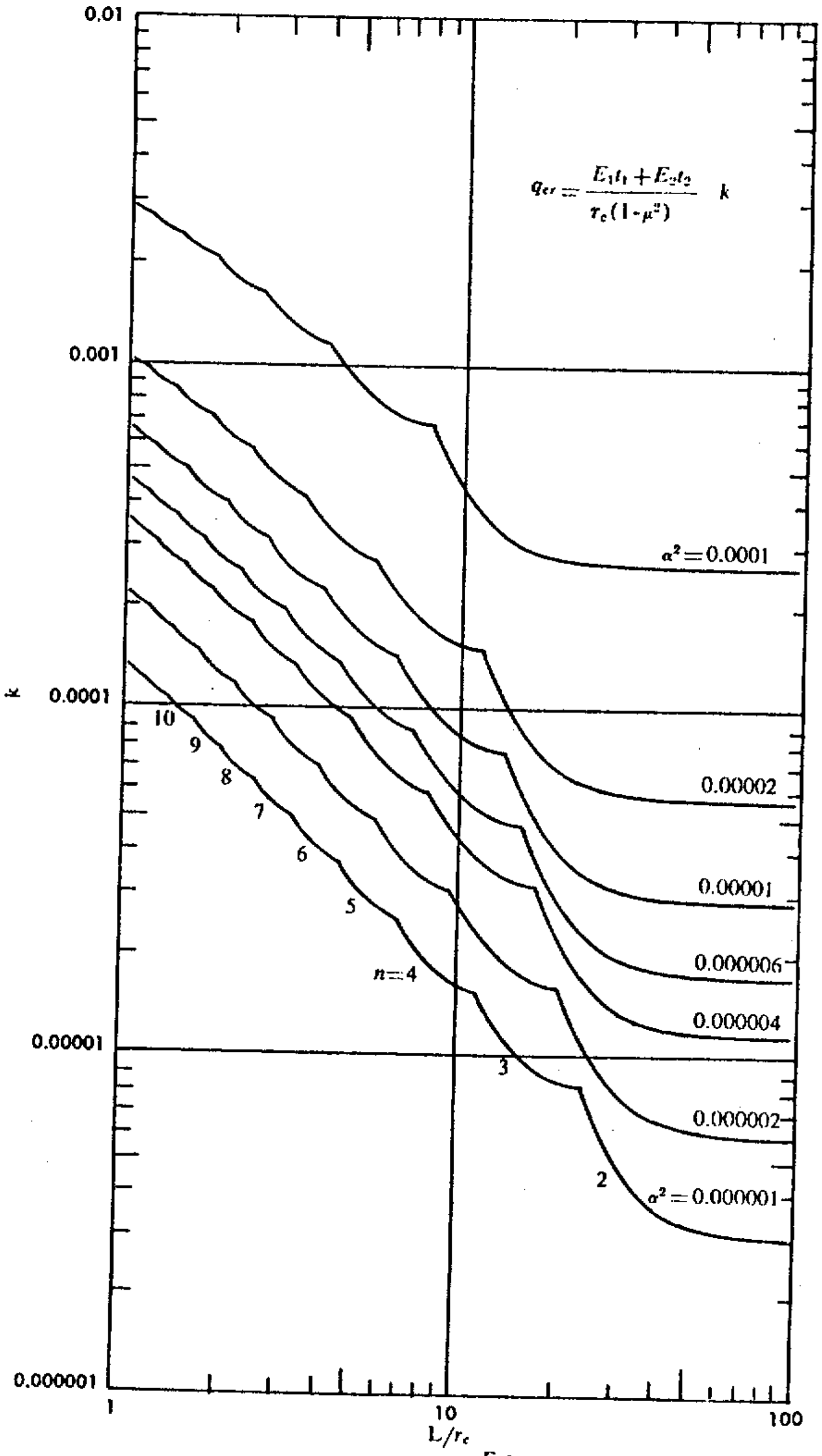
Values of k for $V = 0.5$, and for $\frac{E_1 t_1}{E_2 t_2} = 3$.

FIGURE 13.75 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE



Values of k for $V = 0.5$, and for $\frac{E_1 t_1}{E_2 t_2} = 4$.

FIGURE 13.76 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE



Values of k for $V = 1$, and for $\frac{E_1 t_1}{E_2 t_2} = 1$.

FIGURE 13.77 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE

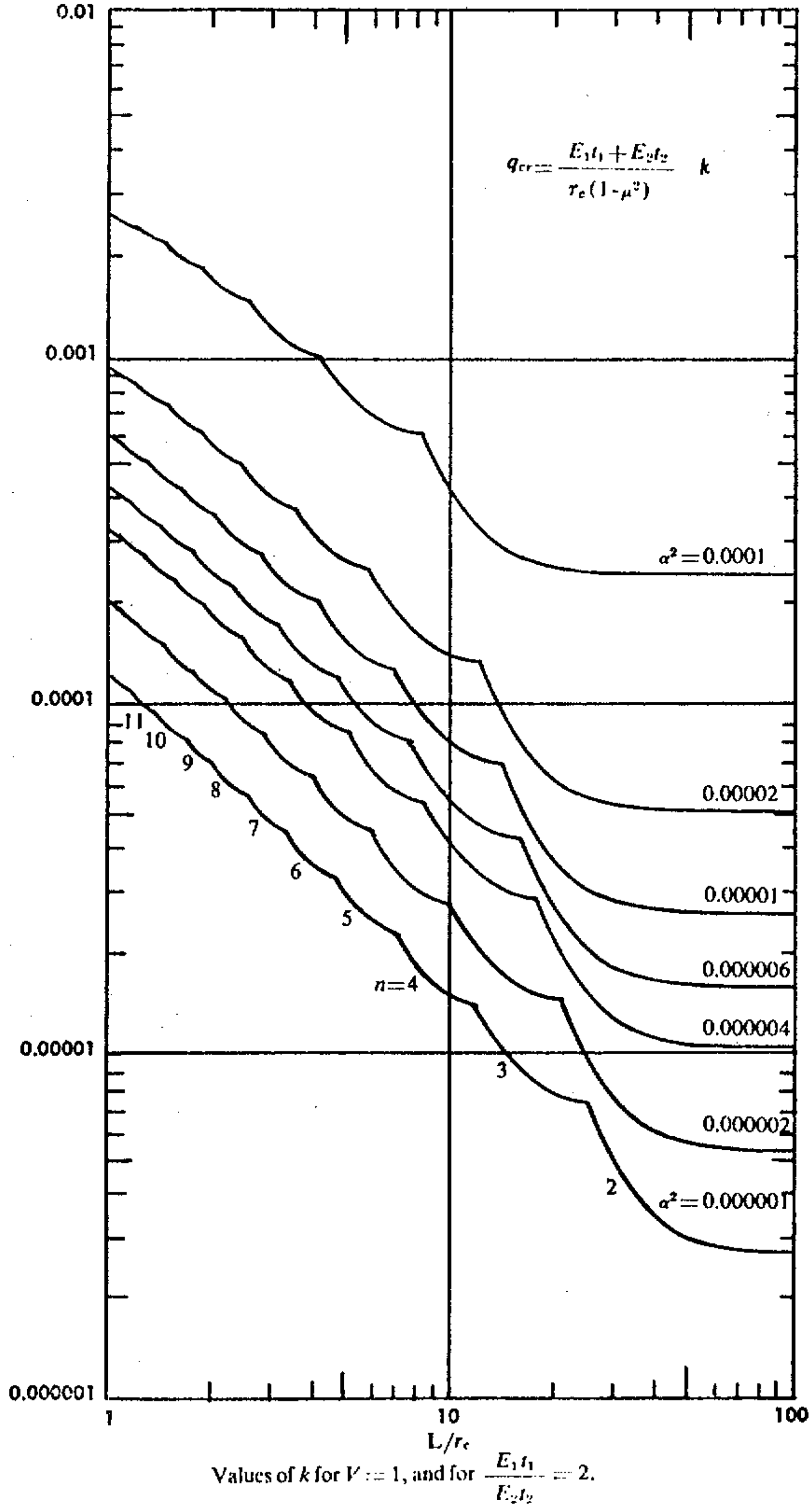


FIGURE 13.78 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE

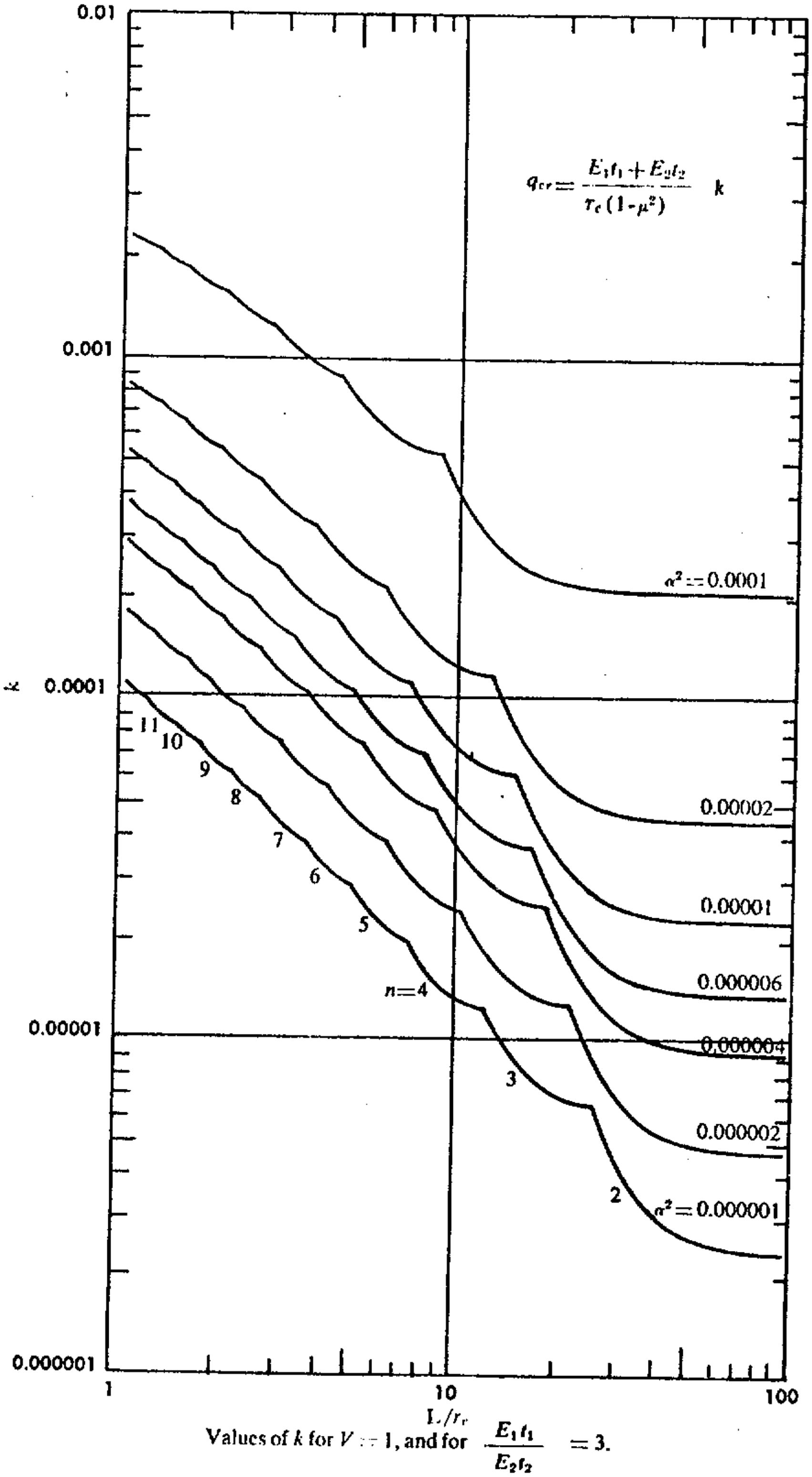
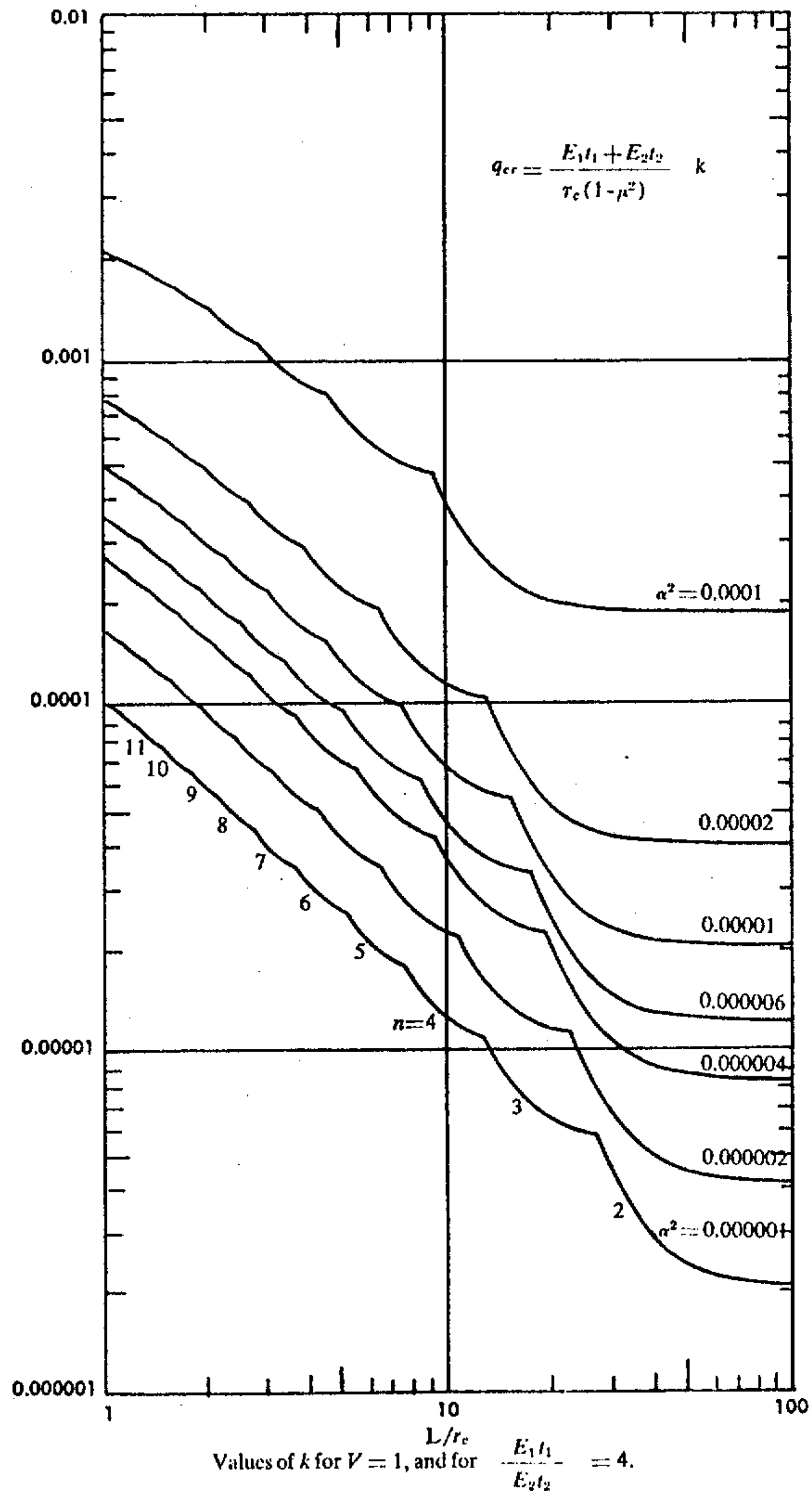
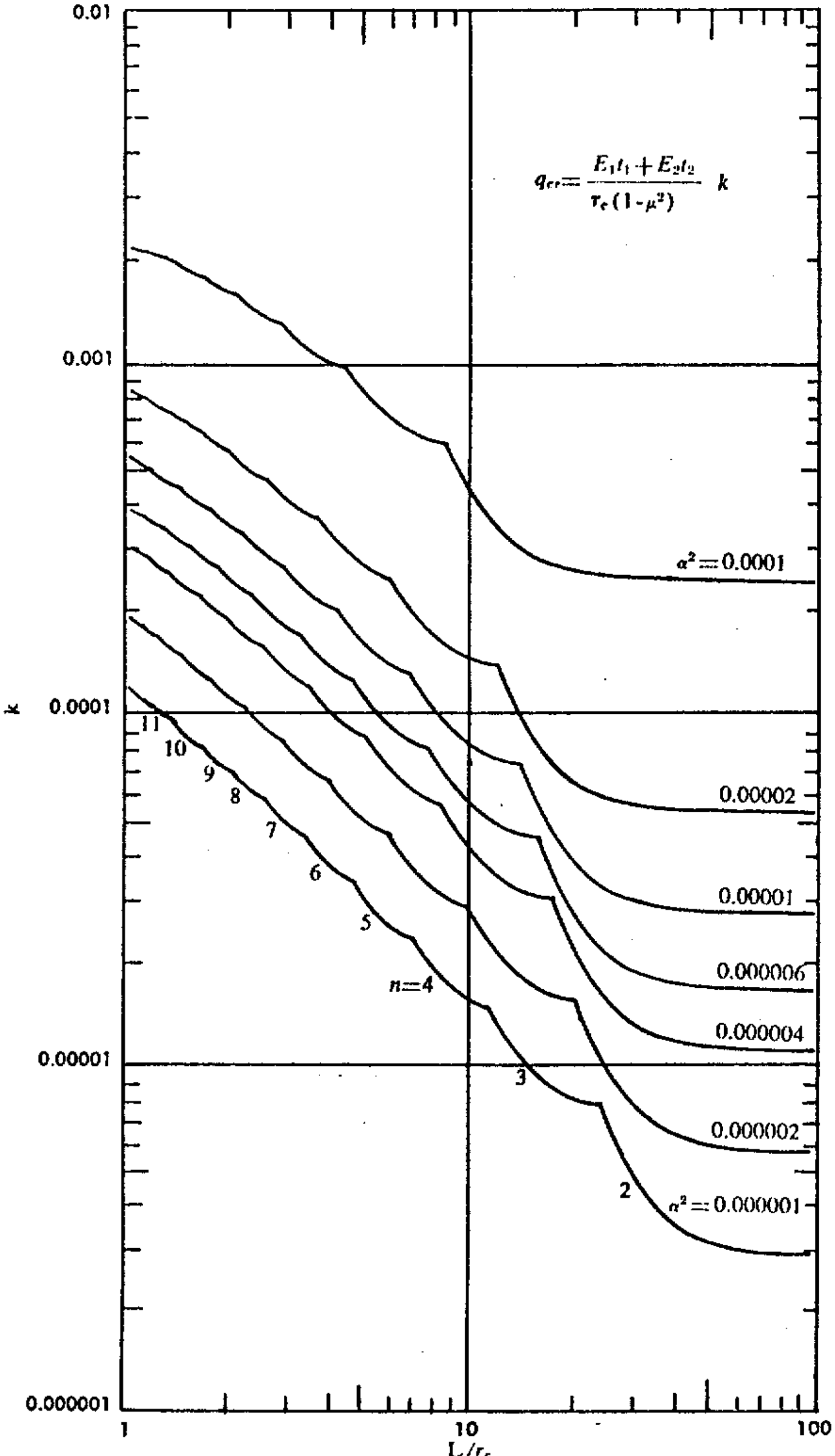


FIGURE 13.79 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE



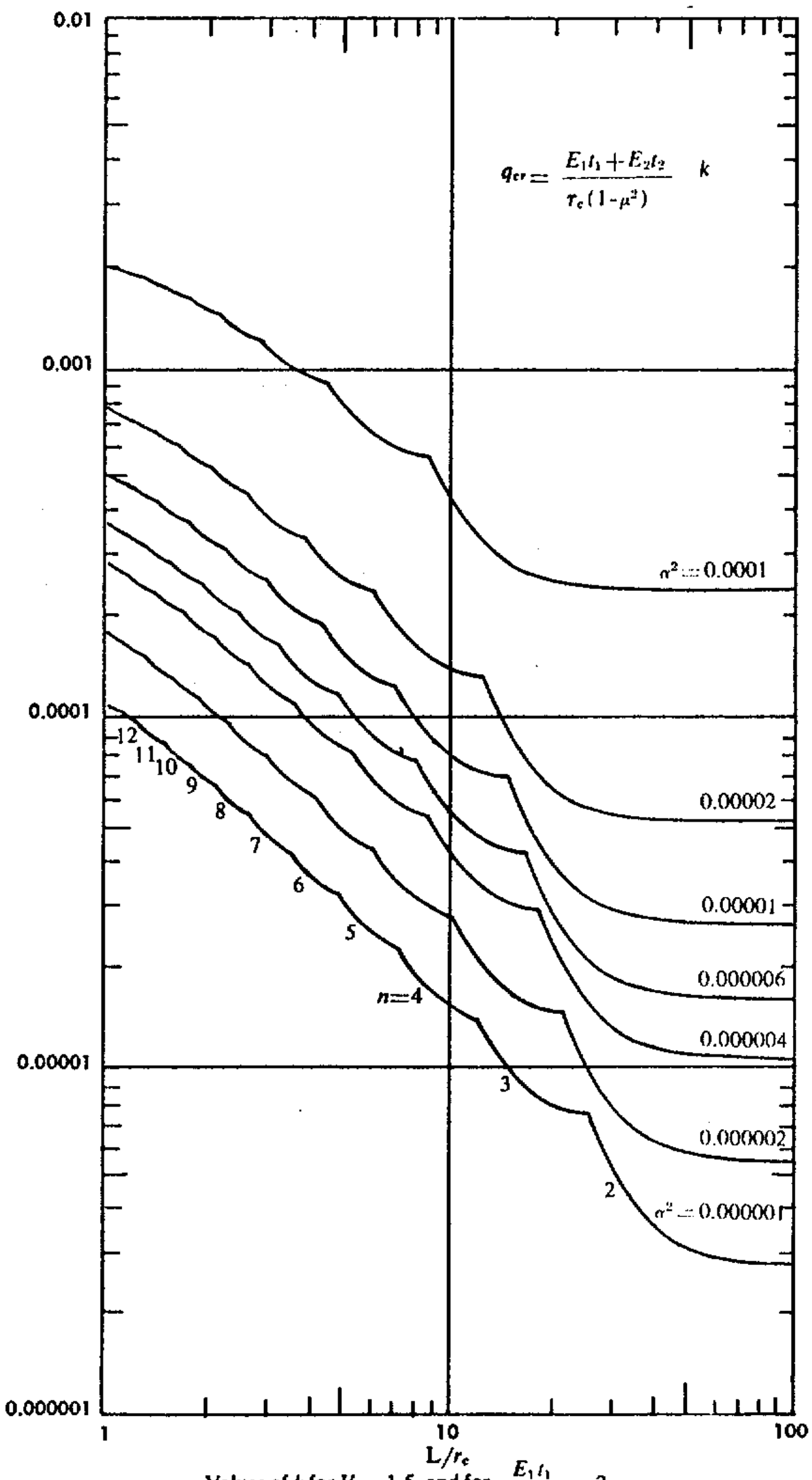
Values of k for $V = 1$, and for $\frac{E_1 t_1}{E_2 t_2} = 4$.

FIGURE 13.80 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE



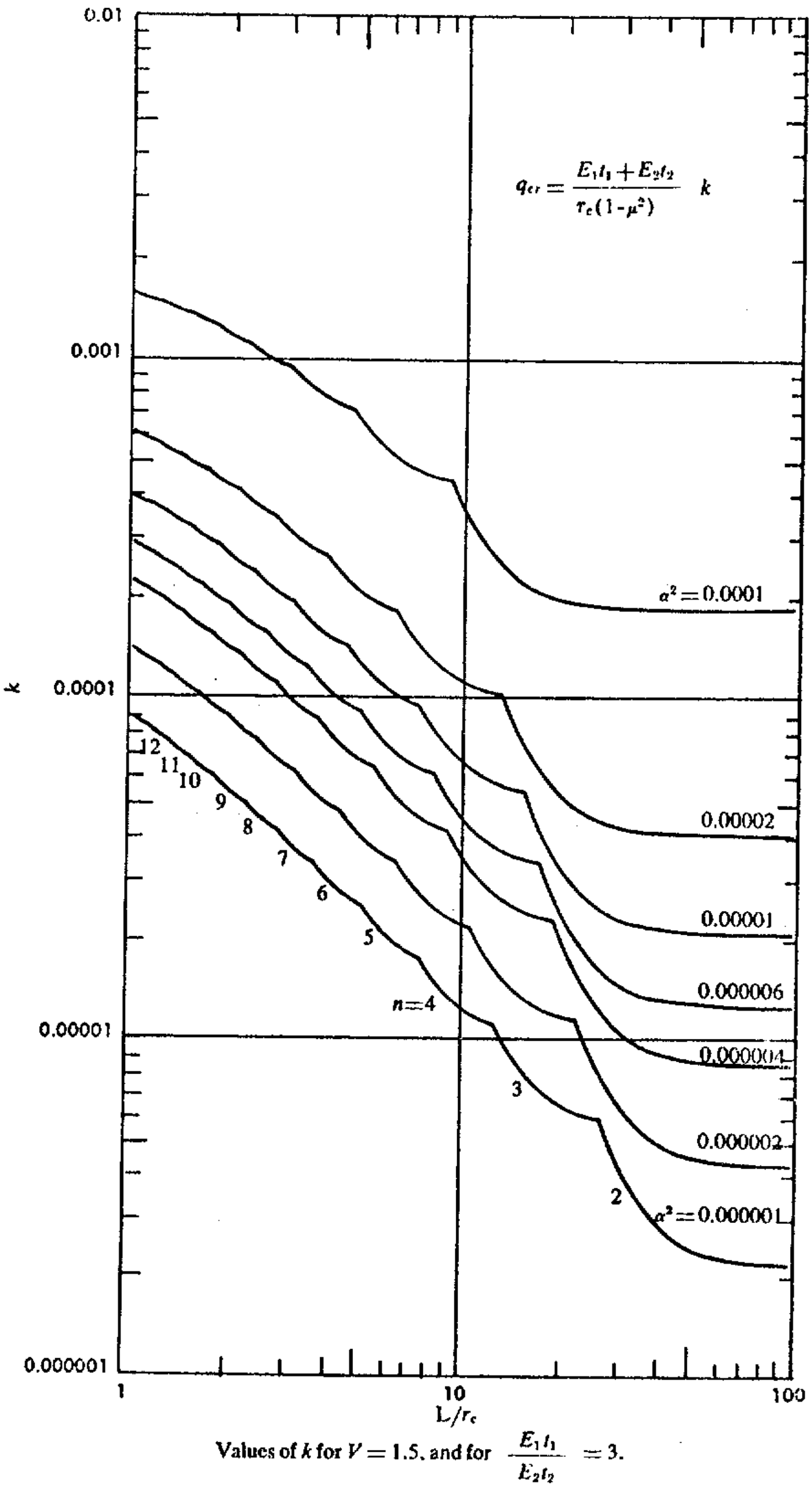
Values of k for $V = 1.5$, and for $\frac{E_1 t_1}{E_2 t_2} = 1$.

FIGURE 13.81 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE



Values of k for $V = 1.5$, and for $\frac{E_1 t_1}{E_2 t_2} = 2$.

FIGURE 13.82 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE



Values of k for $V = 1.5$, and for $\frac{E_1 t_1}{E_2 t_2} = 3$.

FIGURE 13.83 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE

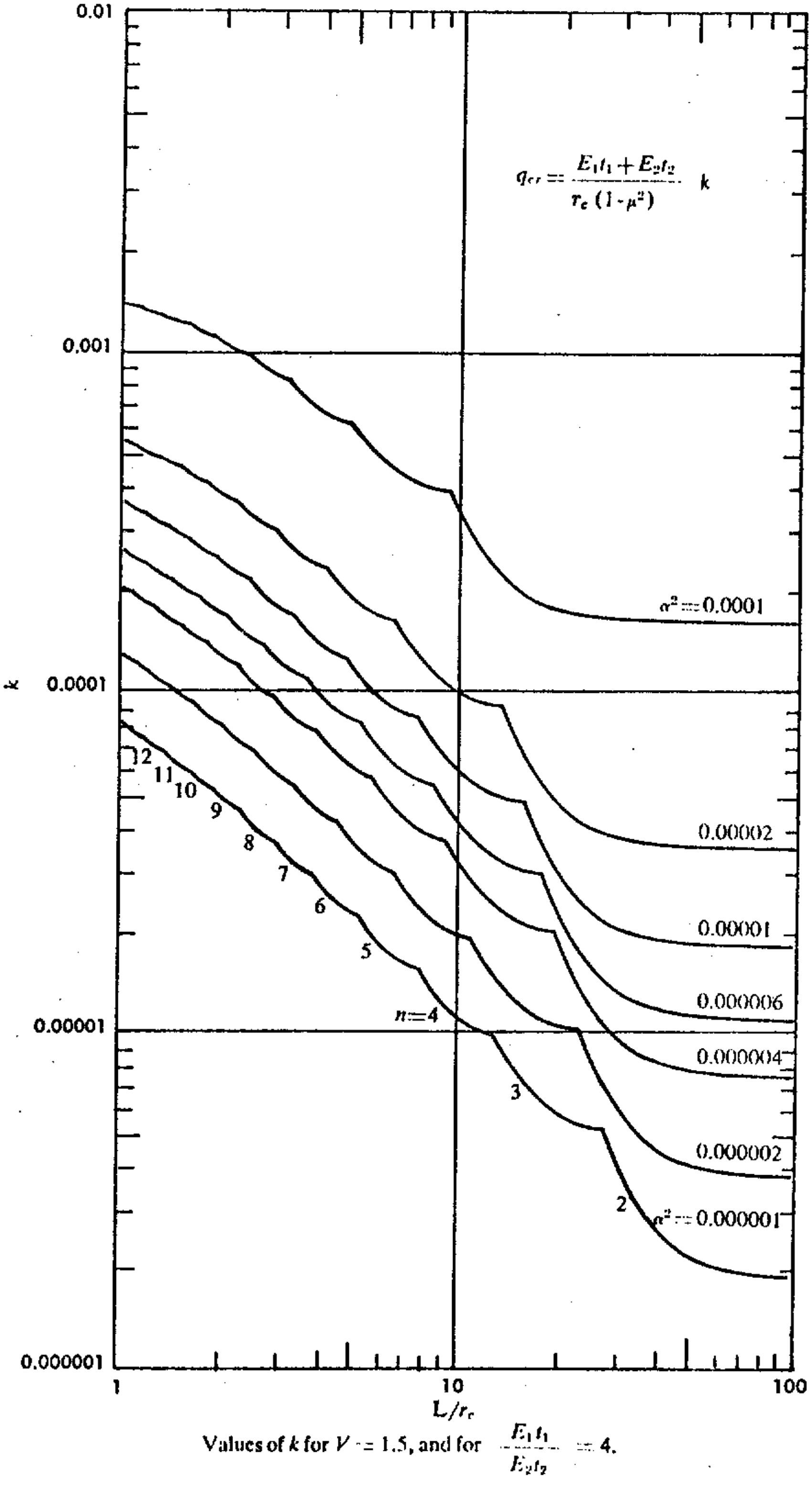


FIGURE 13.84 - BUCKLING COEFFICIENT FOR CYLINDERS UNDER UNIFORM EXTERNAL PRESSURE

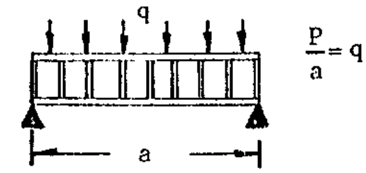
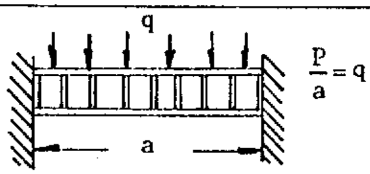
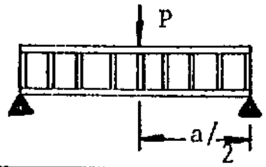
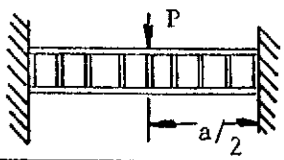
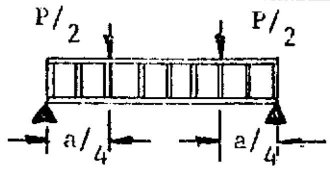
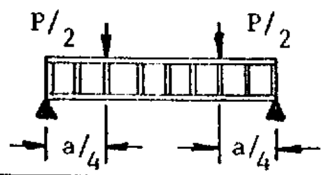
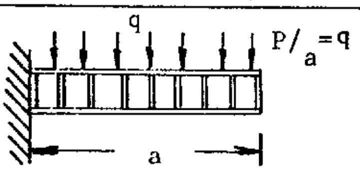
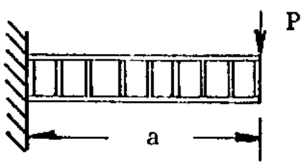
TYPE OF LOADING AND END SUPPORTS	POINT OF DEFLECTION	M	V
 UNIFORM LOAD SIMPLY SUPPORTED	MIDSPAN	$\frac{Pa}{8}$	$\frac{P}{2}$
 UNIFORM LOAD FIXED ENDS	MIDSPAN	$\frac{Pa}{12}$	$\frac{P}{2}$
 POINT LOAD AT MIDSPAN SIMPLY SUPPORTED	MIDSPAN	$\frac{Pa}{4}$	$\frac{P}{2}$
 POINT LOAD AT MIDSPAN FIXED ENDS	MIDSPAN	$\frac{Pa}{8}$	$\frac{P}{2}$
 POINT LOADS AT QUARTER SPAN SIMPLY SUPPORTED	MIDSPAN	$\frac{Pa}{8}$	$\frac{P}{2}$
 POINT LOADS AT QUARTER SPAN SIMPLY SUPPORTED	LOAD	$\frac{Pa}{8}$	$\frac{P}{2}$
 UNIFORM AND CANTILEVER	FREE END	$\frac{Pa}{2}$	P
 POINT LOAD AT FREE END CANTILEVER	FREE END	Pa	P

FIGURE 13.85(a) - BEAM MOMENTS

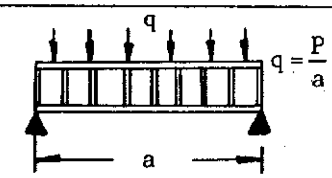
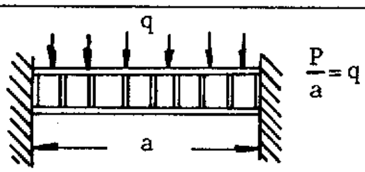
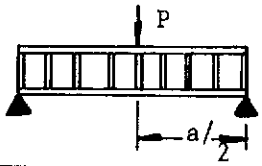
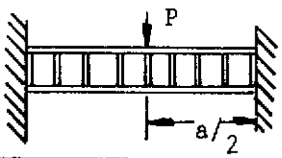
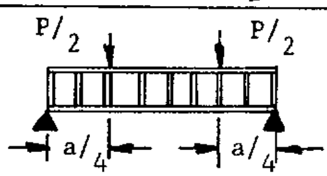
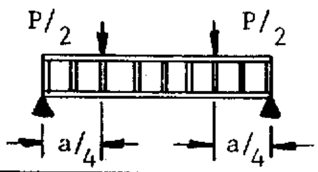
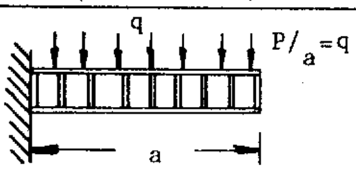
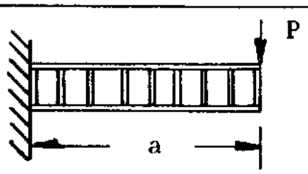
TYPE OF LOADING AND END SUPPORTS		POINT OF DEFLECTION	K_B	K_S
	UNIFORM LOAD SIMPLY SUPPORTED	MIDSPAN	$\frac{5}{384}$	$\frac{1}{8}$
	UNIFORM LOAD FIXED ENDS	MIDSPAN	$\frac{1}{384}$	$\frac{1}{8}$
	POINT LOAD AT MIDSPAN SIMPLY SUPPORTED	MIDSPAN	$\frac{1}{48}$	$\frac{1}{4}$
	POINT LOAD AT MIDSPAN FIXED ENDS	MIDSPAN	$\frac{1}{192}$	$\frac{1}{4}$
	POINT LOADS AT QUARTER SPAN SIMPLY SUPPORTED	MIDSPAN	$\frac{11}{768}$	$\frac{1}{8}$
	POINT LOADS AT QUARTER SPAN SIMPLY SUPPORTED	LOAD	$\frac{1}{96}$	$\frac{1}{8}$
	UNIFORM AND CANTILEVER	FREE END	$\frac{1}{8}$	$\frac{1}{2}$
	POINT LOAD AT FREE END CANTILEVER	FREE END	$\frac{1}{3}$	1

FIGURE 13.85(b) - DEFLECTION CONSTANTS

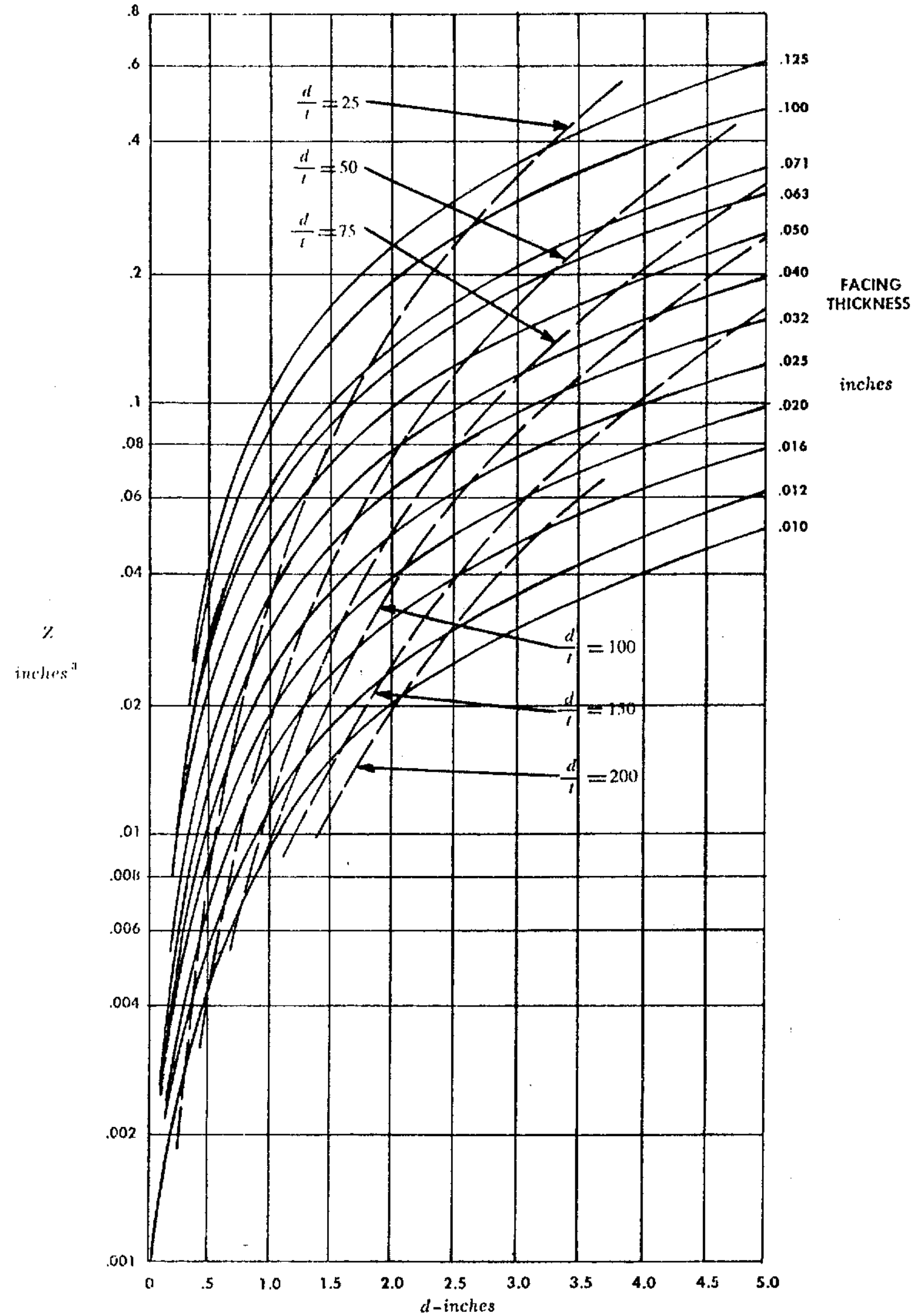


FIGURE 13.86 - DESIGN CHART FOR BEAMS WITH EQUAL THICKNESS FACES

(4) Calculate a starting value of d/t from the weight minimizing relation

$$d/t = 1 + 2\, w_f/w_c \qquad 13.99$$

where w_f and w_c are the densities of chosen facing and core materials respectively.

(5) From Figure 13.86 find the value of d associated with the value of Z determined in step (3), and move horizontally (Z = constant) to the nearest standard sheet gage. Read the corresponding panel thickness d.

(6) With the values of d and t thus determined, check the facing stress,

$$f_f = \frac{M}{bt(d-t)} \qquad 13.100$$

This value of f_f should equal F_f (step (2)).

(7) Determine core thickness

$$t_c = d - 2t \qquad 13.101$$

(8) Solve for core shear stress from

$$f_s = \frac{2S}{b(d + t_c)} \qquad 13.102$$

This value of f_s should not exceed the allowable shear strength of the chosen core.

(9) When stiffness is an important consideration, determine the two stiffness parameters, D and U

$$D = \frac{b\,E_f}{12\lambda}\left[d^3 - t_c^{\,3}\left(1 - \frac{E'_c}{E_f}\right)\right] \qquad 13.103$$

where E'_c = Elastic modulus of the core in the spanwise direction

E_f = Elastic modulus of face material

$\lambda = 1 - \mu^2$

μ = Poisson's ratio for face material

For beams with cellular cores, E'_c is often very low in comparison to E_f, and the ratio E'_c/E_f is then assumed to be equal zero.

Calculate shear stiffness

$$U = \frac{h^2 G_c b}{t_c} \qquad 13.102$$

where $h = d-t = t_c + t$

G_c = core modulus of ridigity

(10) Compute the deflection (δ) from

$$\delta = \frac{K_B PL^3}{D} + \frac{K_S PL}{U} \qquad 13.103$$

where P = applied load

L = length of beam

The coefficients K_B and K_S are given in Figure 13.85 for various beam loadings and end support conditions. If the computed value of δ is greater than that compatible with the design criteria or good design practices, the beam's stiffness may be increased by increasing core thickness, or by using a core with a higher modulus of rigidity, or both. Any of the above calculations affected by the change should be repeated.

(11) Determine the flexure induced core compressive stress

$$f_c = \frac{2f_f^2}{E_f(d/t - 1)} \qquad 13.104$$

(12) The core should also be analyzed for local crushing due to concentrated loadings, either applied or at reaction points.

13.3 Attachment Details

All sandwich parts must be attached to the framework of the airframe and often to other similar parts; therefore, means for transferring the concentrated loads imposed at these attachments must be provided. Occasionally, on very lightly loaded parts, unreinforced bolt holes or subsequently inserted reinforcements will suffice, but in most structural applications, local reinforcements must be incorporated during fabrication.

13.3.1 Edge Design

Sandwich parts are normally joined over a framing member. The edge configuration is often dictated by the loads to be transferred, core, smoothness requirement, fasteners, facings, panel usage, etc. Figure 13.87 shows some commonly used edge configurations. Care should be used in selecting the edge design. If the methods of Section 13.2 are used in the design of the panel, both faces are capable of reacting load. Then, in order to fully utilize the sandwich concept, the edges must be designed to be compatible.

Some of the edge configurations have beveled edges, such as a 45° chamfer with fiberglass closure. This is a commonly used configuration at Bell. The load that is introduced into the inner face at the edges is only what can be transferred through the fiberglass edging or shear lagged through the core.

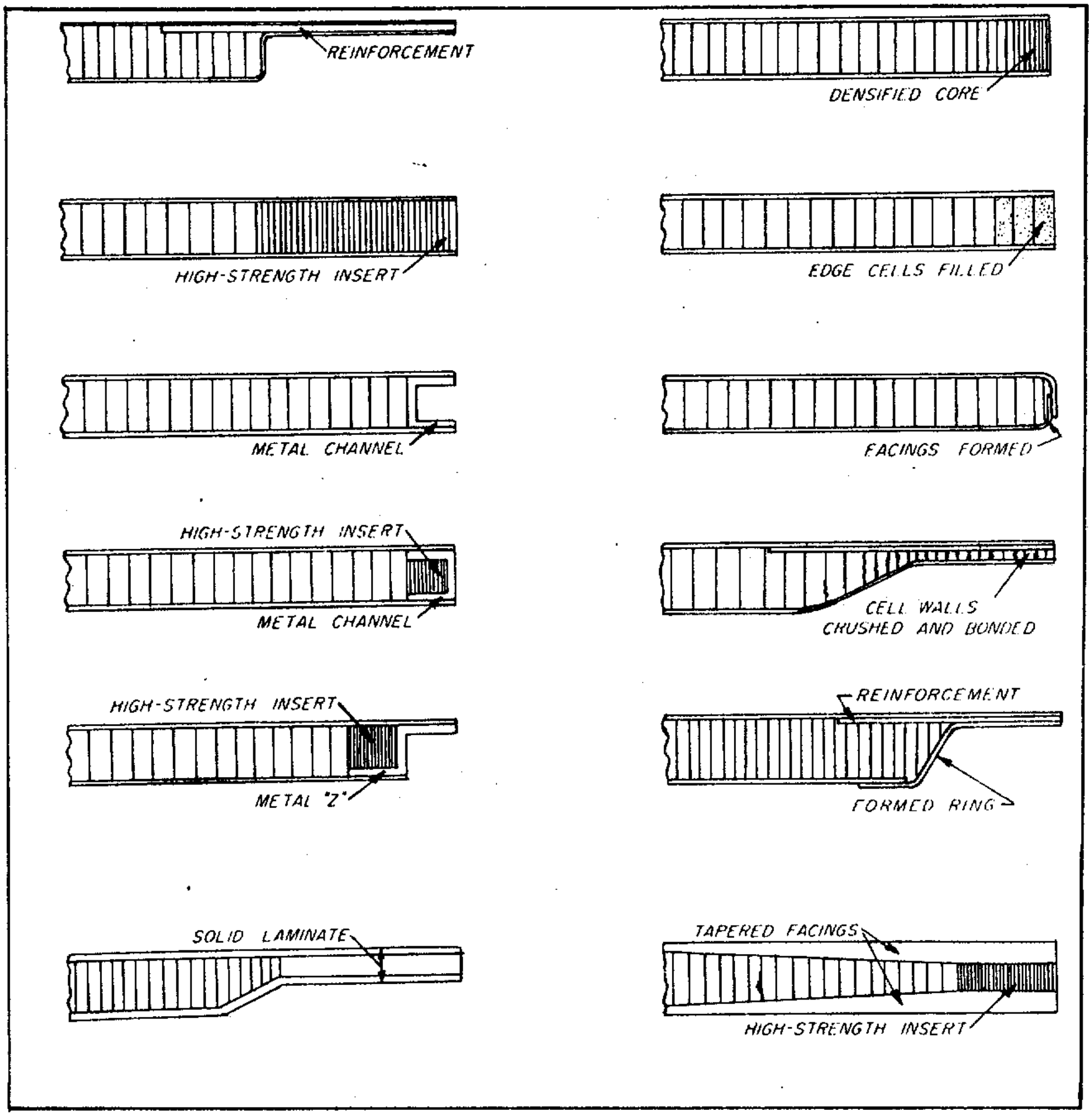


FIGURE 13.87 - SOME TYPICAL EDGE DESIGNS

Regardless of the type of configuration, the edge design should be such as to keep moisture out of the core. This can be accomplished by the use of potting compounds or fiberglass closures.

The facings which have been sized for the type of failure modes discussed in Section 13.2 will not necessarily be thick enough to develop the fasteners. An edge reinforcement must be installed with a thickness sufficient to develop the fasteners. The reinforcement is a doubler, either internally or externally. In some cases, it can be chem-etched integral with one of the faces.

When the loads on a sandwich panel are normal to the panel, the edging doubler may have to react these loads. In this case, it must be thick enough and wide enough to develop the bending moment in the edge.

13.3.2 Doublers and Inserts

The design of a sandwich structure may be such that loads must be transferred to or from individual parts at points other than at their edges. Inserts in the part are required at these attachment points if the loads are of appreciable nagnitude. Typical methods of introducing high loads into a sandwich panel are shown in Figure 13.88. These may be in the form of strips (metal, wood, foam), inserted continuously across the panel or as local reinforcements under individual bolt patterns. Shear loads on attachment bolts may require additional reinforcement to provide adequate attachment bearing area. This can be in the form of a doubler which can be installed internally or externally.

One method of densification (increasing the density of the core so that concentrated loads can be introduced) is to cut out an area of the core and insert a piece of denser core. Another method of densification is to compress the core in a local area so that the cell size is smaller than the main body of the core.

13.3.3 Attachment Fittings

Accessories, such as shelves, fittings, mounting brackets, are often fastened to the sandwich panels. Figure 13.89 shows some examples of how fittings can be attached to sandwich panels.

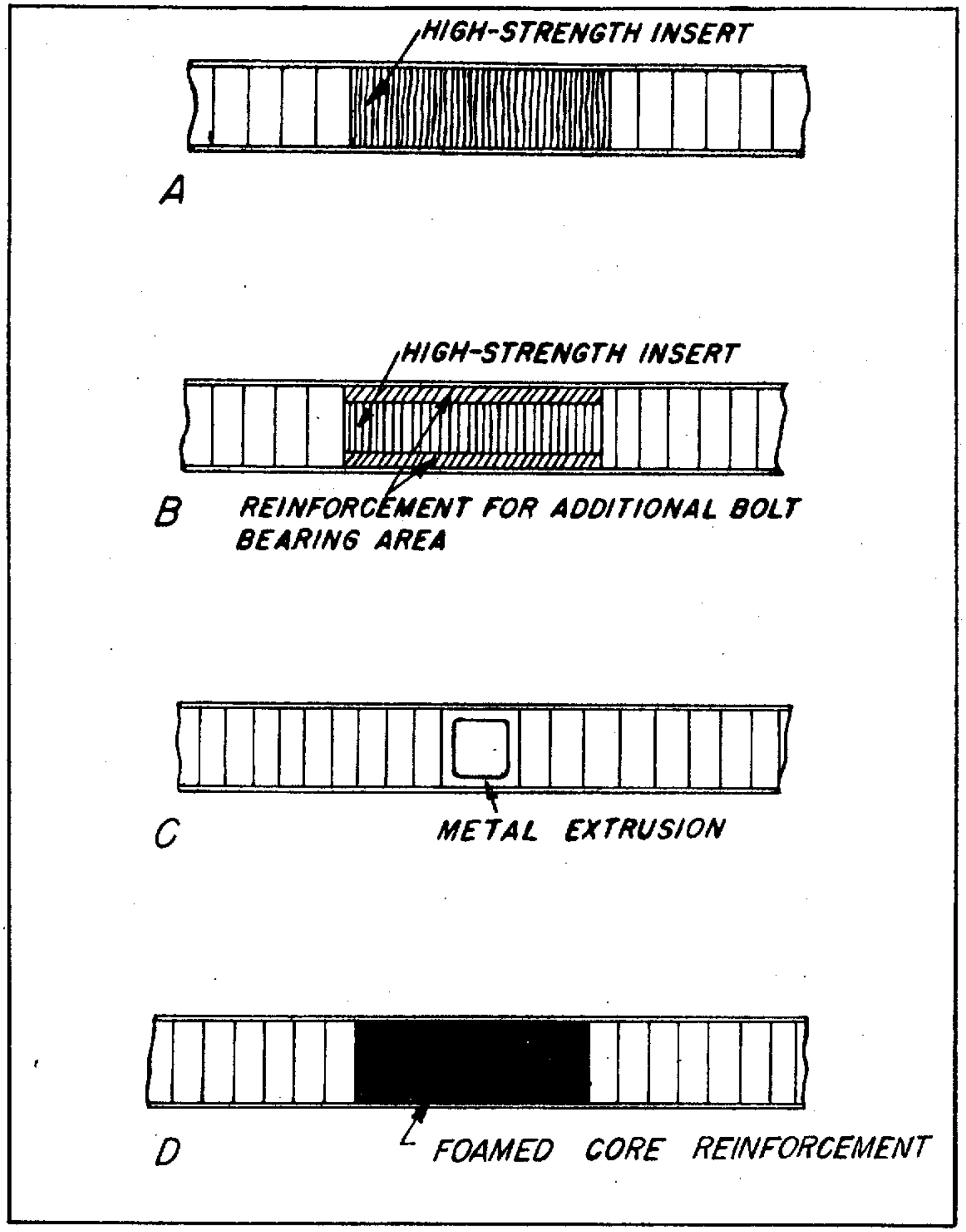


FIGURE 13.88 - SOME TYPICAL HIGH-STRENGTH INSERT DESIGNS

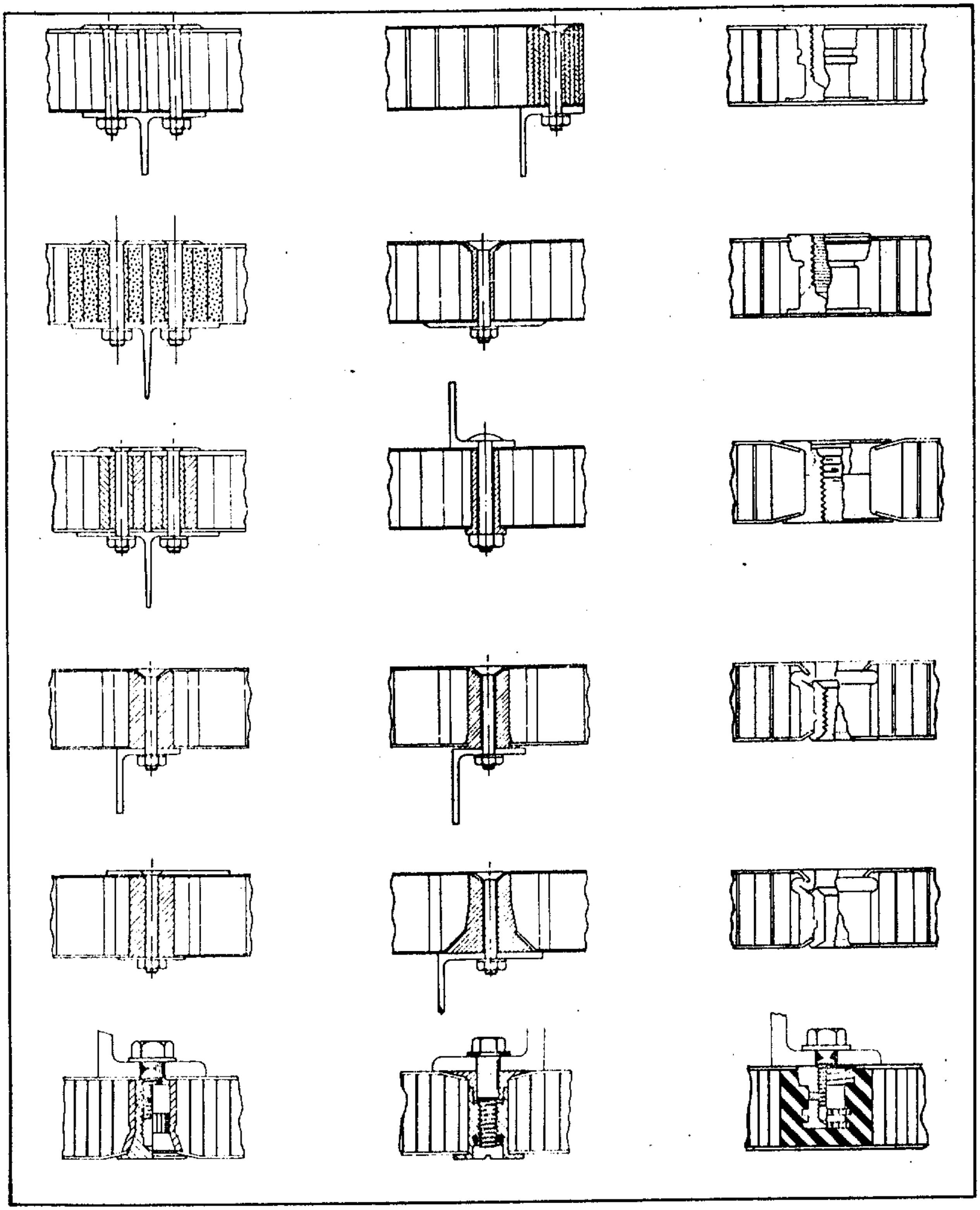


FIGURE 13.89 - SOME TYPICAL ATTACHMENT FITTINGS

SECTION 14

SPRINGS

14.0 GENERAL

The proper design of springs requires an understanding of spring materials, design formulas, stress analysis and manufacturing procedures. Various aids are available to the analyst including special spring slide rules, tables of constants, curves, charts and nomographs. All are helpful, but an understanding of the basic fundamental formulas and experience in their use is essential to good design. The purpose of this section is to describe the design methods used for each type of spring commonly used.

14.1 Abbreviations and Symbols

The following abbreviations and symbols are used throughout this section unless otherwise specified.

A = constant for rectangular wire
B = constant for rectangular wire
b = width, in
C = Spring Index = D/d
CL = compressed length, in
D = mean coil diameter, in
d = diameter of wire or side of square, in
E = modulus of elasticity in tension, psi
F = deflection, for N coils with load P, in
F^o= deflection, for N coils, rotary, degrees
FL = free length, unloaded spring, in
f = deflection, for one active coil, at load P, in
G = modulus of elasticity in torsion, psi
ID = inside diameter, in
in. = inch
K = stress correction factor for curvature
L = active length subject to deflection, in
l = length, in
lb = pound
M = bending moment, in-lb
N = total active coils
n'= vibration per minute
OD = outside diameter
P = load, lb
P_1= applied load, lb, (also P_2, etc.)
p = pitch, in
psi = pounds per sq. in.
R = distance from load to central axis, in
r = spring rate load per inch, lb/in
r_t= torsional spring rate, in lb/deg
S_b= bending stress, psi
S_t= torsional stress, psi

S_{it}= torsional stress due to initial tension, psi
SG= squared and ground
SH= solid height (or SL = solid length), in
s = height load is dropped, in
T = torque = P x R, in lb
TC = total coils
t = thickness, in
U = number of revolutions = $F^o \div 360^o$
W = weight, (also applied dynamic load), lb
x = multiplied by
Y = constant for Belleville springs
Z_1= constant for Belleville springs
Z_2= constant for Belleville springs
α = angle of movement, deg.
π = 3.1416
μ = Poisson's ratio

14.2 Compression Springs

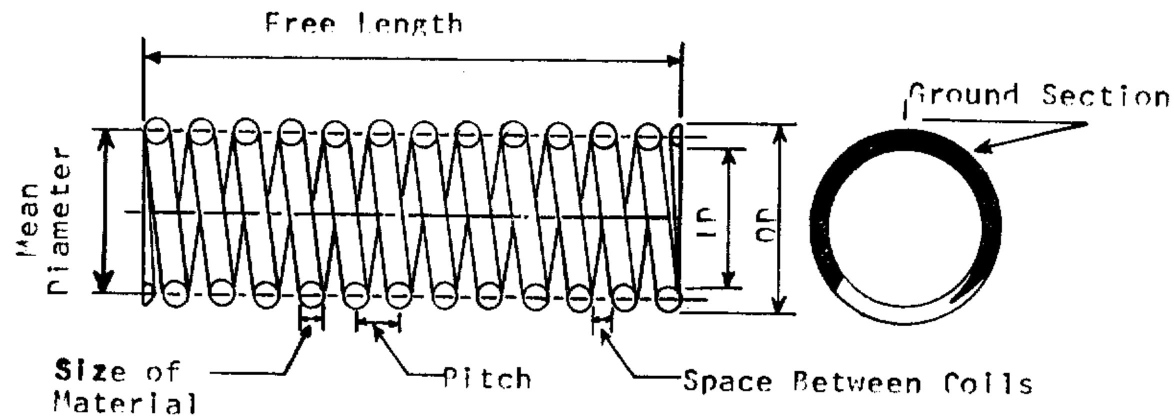
Most compression springs are open coil, helical springs which offer resistance to loads acting to reduce the length of the spring. The longitudinal deflection of the springs produce shearing stresses in the spring wire. Where particular load deflection characteristics are desired, springs with varying pitch diameters may be used. These springs may have any number of configurations, including cone, barrel and hourglass. They may be made from wire of round, square or rectangular cross section.

Figure 14.1 shows a typical compression spring with the nomenclature and a description of the four types of ends which can be made. The ends can be finished into 1) open not ground; 2) closed not ground; 3) open ground and 4) closed ground.

Open ends not ground; sometimes called plain ends, has the largest eccentricity of loading. These are used only when accuracy of loads is not important. This type is seldom used because such springs tangle severely during shipping.

Closed ends not ground; also called squared ends, cost approximately the same as open end types and have less eccentricity. This type is often used on light wire springs under 1/32 in. dia. wire and for heavier wire where the index exceeds 13.

Open ends ground; also called plain ends ground, are seldom used because they cost about the same as the closed ends ground, but have high eccentricities of loading and tangle during shipping. They are sometimes used where the solid height is very limited and it is necessary to have as many active coils as possible in the least space.



TYPES OF END FINISHES

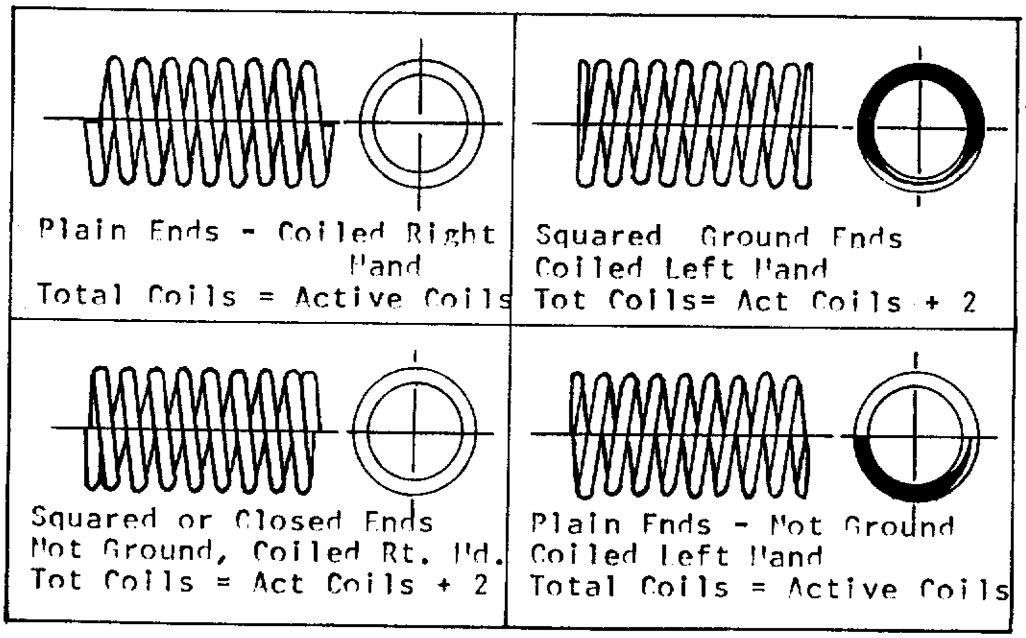


FIGURE 14.1 COMPRESSION SPRINGS

Closed ends ground; sometimes called squared and ground, is the most popular type as it provides a level seat and reduces the tendency to buckle. This is the most expensive type and should be avoided for springs made of very light wire. Each end coil is ground for 270° plus or minus 30°.

14.2.1 Design Formulas

Formulas for design of helical compression and extension springs without initial tension are given in Table 14.1. Dimensional characteristics for the four end finishes of compression springs are given in Table 14.2.

14.2.1.1 Diameter Changes in Compression Springs

When a helical compression spring is compressed, an increase in the outside diameter occurs because the angularity of the coils changes so that it is nearly at a right angle to the axis. The outside diameter when the spring is compressed solid, can be obtained from the following formula:

$$OD_c = \sqrt{D^2 + (p^2 - d^2)/\pi^2} + d \qquad 14.1$$

14.2.2 Buckling

Compression springs having a free length greater than four (4) times their mean diameter become critical in lateral stability. When deflected beyond a certain percentage of the free length, a spring will buckle. Figure 14.2 shows the maximum deflection which may be expected without buckling if the ends of the spring are closed and ground. Buckling can be reduced, space permitting, by redesign using a heavier size wire and increasing the diameter of the coil. Buckling causes an undesirable reduction of the load and may cause early spring failure. If properly guided in a cylinder or over a rod, buckling can be reduced, although friction against the guiding member will affect the load and shorten the spring life.

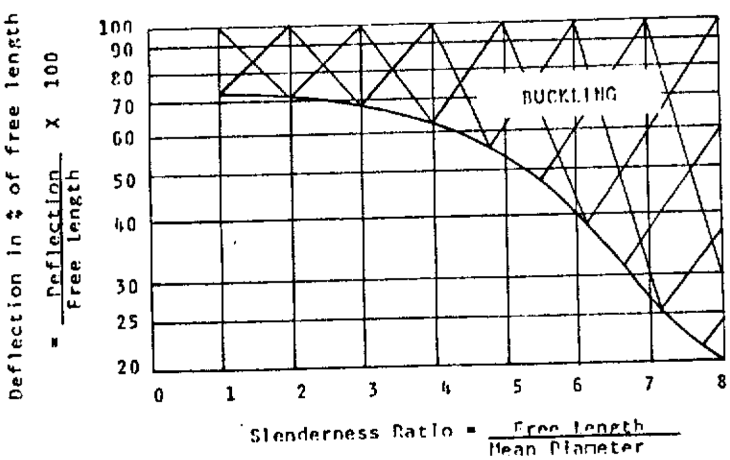


FIGURE 14.2 MAXIMUM DEFLECTION WITHOUT BUCKLING OF SPRINGS WITH CLOSED AND GROUND ENDS

TABLE 14.1 FORMULAS FOR COMPRESSION AND EXTENSION SPRINGS WITHOUT INITIAL TENSION

Property	Round wire	Square wire	Rectangular wire *
Torsional stress, psi	$\frac{P D}{0.393\ d^3}$	$\frac{P D}{0.416\ d^3}$	$\frac{P D}{B\ b\ t^2}$
S_t	$\frac{G d F}{\pi N\ D^2}$	$\frac{G d F}{2.32\ N\ D^2}$	$\frac{A\ G\ t\ F}{N\ D^2}$
Deflection, in.	$\frac{8\ P\ N\ D^3}{G\ d^4}$	$\frac{5.58\ P\ N\ D^3}{G\ d^4}$	$\frac{S_t\ N\ D^2}{A\ G\ t}$
F	$\frac{\pi S_t\ N\ D^2}{G\ d}$	$\frac{2.32\ S_t\ N\ D^2}{G\ d}$	....
Change in load lb $P_2 - P_1$ Compression springs only	$\frac{L_1 - L_2}{\frac{F}{P}}$	$\frac{L_1 - L_2}{\frac{F}{P}}$	$\frac{L_1 - L_2}{\frac{F}{P}}$
Change in load lb $P_2 - P_1$ Extension springs only	$\frac{L_2 - L_1}{\frac{F}{P}}$	$\frac{L_2 - L_1}{\frac{F}{P}}$	$\frac{L_2 - L_1}{\frac{F}{P}}$
Stress due to initial tension, psi S_{it}	$\frac{S_t}{P} \times IT$	$\frac{S_t}{P} \times IT$	$\frac{S_t}{P} \times IT$
Rate lb/in. r	$\frac{P}{F}$	$\frac{P}{F}$	$\frac{P}{F}$

* See Figure 14.3

TABLE 14.2 COMPRESSION SPRING FORMULAS FOR DIMENSIONAL CHARACTERISTICS

Dimensional characteristics	Type of ends			
	Open or plain (not ground)	Open or plain with ends ground	Square or closed (not ground)	Closed and ground
Pitch (p)	$\frac{FL - d}{N}$	$\frac{FL}{TC}$	$\frac{FL - 3d}{N}$	$\frac{FL - 2d}{N}$
Solid Height (SH)	$(TC + 1)\ d$	$TC \times d$	$(TC + 1)\ d$	$TC \times d$
Active Coils (N)	$N = TC$ or $\frac{FL - d}{p}$	$N = TC - 1$ or $\frac{FL}{p} - 1$	$N = TC - 2$ or $\frac{FL - 3d}{p}$	$N = TC - 2$ or $\frac{FL - 2d}{p}$
Total Coils (TC)	$\frac{FL - d}{p}$	$\frac{FL}{p}$	$\frac{FL - 3d}{p} + 2$	$\frac{FL - 2d}{p} + 2$
Free Length (FL)	$(p \times TC) + d$	$p \times TC$	$(p \times N) + 3d$	$(p \times N) + 2d$

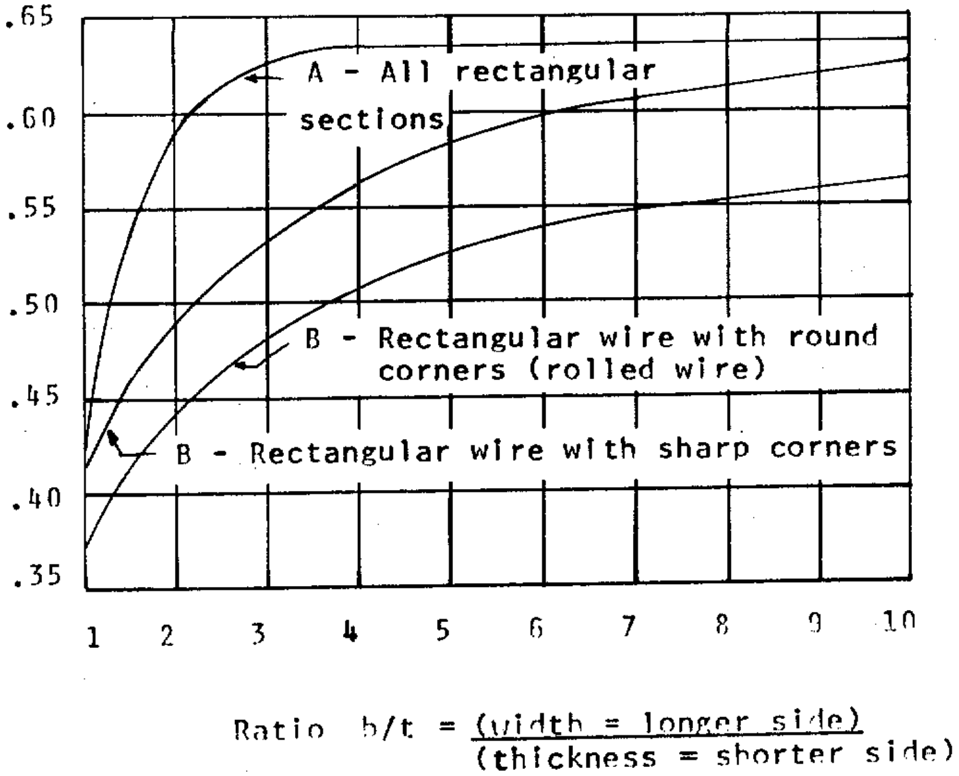


FIGURE 14.3 CONSTANTS A AND B FOR RECTANGULAR OR ROLLED WIRE

Slowly applied load	Suddenly applied load
$F = \frac{W}{r}$	$F = \frac{2W}{r}$
Applied load dropped vertically (spring initially compressed)*	Applied load with striking velocity of V in/sec. (spring in horizontal plane)
$F = \frac{W - P_1 + \sqrt{(W - P_1)^2 + 2Wrs}}{r}$	$F = \frac{-P_1 + \sqrt{P_1^2 + \frac{WrV^2}{386}}}{r}$

* If spring is not initially compressed, disregard P_1

TABLE 14.3 LOAD-DEFLECTION FORMULAS FOR COMPRESSION AND EXTENSION SPRINGS

14.2.3 Helix Direction

Unless functional requirements dictate a definite direction, the helix of compression and extension springs should be specified optional. To prevent intermeshing of coils when springs operate one inside the other, the helixes should be specified as opposite hand. For the same reason, springs which operate to slide freely overscrew threads should have the helix specified opposite to that of the screw threads.

14.2.4 Natural Frequency, Vibration and Surge

The use of springs for loads which are applied dynamically; i.e., with impact or rapidly repeated, will be in error if the spring is designed on the basis of static or slow loading. The load, stress, deflection, etc., will have been calculated for applications where the load is applied and held, or the rate of load application is below the natural frequency of the spring. Because of the inertia effect of the coils in instances where the load is suddenly applied, the load on the spring does not have time to distribute itself uniformly throughout the mass of the spring. This non-uniform loading causes deflection or a surge wave in a few coils of the spring which results in a high stress in this area and a lower stress in the remainder of the spring. In applications of high rate of repeated loading, non-uniform load distribution occurs in the same manner as suddenly applied loads and the natural frequency of vibration of the spring may be excited. The excitation of the natural frequency of vibration, in some instances, may be of such magnitude as to cause the spring coils to clash causing the spring to destroy its constraint on the mechanism. This is known as spring surge. The following methods may be used to prevent spring surge:

1. Stiffen the spring
   a. Increase wire diameter
   b. Decrease mean diameter
   c. Decrease number of coils
   d. Use square or rectangular wire

2. Use nested springs

3. Use conical spring

4. Reduce or vary the pitch of the coils near the end of the spring.

5. Use stranded wire

The formulas for calculating the natural frequency of steel springs are:

$$n' = 761{,}500d/ND^2\text{; UNLOADED SPRING} \qquad 14.2$$

$$n' = 187.6 \sqrt{1/F}\text{ ; LOADED SPRING} \qquad 14.3$$

If the frequency of the spring and its harmonics are too low, the spring will surge causing the coils to clash. In general, if the natural frequency of the spring is at least thirteen (13) times that of the maximum frequency of the applied load, the design should be satisfactory.

14.2.5 Impact

If the load is suddenly applied, dropped vertically from a known height or strike the spring with a known velocity, the deflection can be calculated using the equations in Table 14.3. The applied stress can then be determined from Table 14.1.

14.2.6 Spring Nests

The nesting (one inside the other) of helical compression springs is a method of obtaining maximum energy storage in a limited space. An example is shown in Figure 14.4. It is desirable to design the springs for equal life with 60 to 70 percent of the load on the outer spring. Maximum energy storage is obtained in a spring nest when the value of the spring indexes are between 5 and 7, when solid lengths of all the springs are approximately the same and when the working stroke (L_2-L_1) is of constant magnitude.

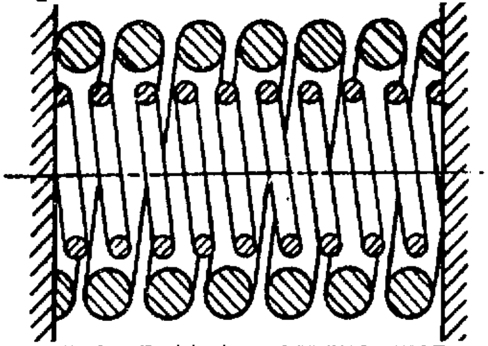


FIGURE 14.4 - SPRING NEST

14.2.7 Spring Index (D/d)

The spring index is the ratio of the mean coil diameter of the spring to the wire diameter (D/d). This ratio is one of the most important considerations in spring design inasmuch as the deflection, stress, number of coils and selection of material depends upon this ratio. The best proportioned springs have an index of 7 to 9. Ratios of 4 to 7 and 9 to 16 require more than standard tolerances for manufacturing; those with values less than 5 are difficult to coil on automatic coiling machines.

14.2.8 Stress Correction Factors for Curvature

The equations for stress in Table 14.1 are based on the assumption that the magnitude of the stress varies directly with the distance from the center of the wire. Actually, the stress is greater on the inside of the cross section due to the curvature of the spring coil. A correction factor has been determined to account for the increase in stress level due to curvature. This correction factor gives the effect of both torsion and direct shear. For helical compression and extension springs, the curvature stress correction factor (K) is determined from the following equation:

$$K = \frac{4C-1}{4C-4} + \frac{.615}{C} \quad 14.4$$

The total stress becomes

$$S_{max} = S_t \times K \quad 14.5$$

where S_t is determined from Table 14.1. This is the stress which should be compared to the allowable stress to determine whether or not the spring is safely designed and is the sole use of K. The stress determined in equation 14.5 should not be used in calculating the deflection or number of coils. The stress determined from Table 14.1 is used without correction for these purposes.

14.2.9 Keystone Effect

When square and rectangular wire are coiled into springs, a change in shape occurs. This change takes place because some of the material on the outside diameter is drawn into the spring and the material on the inside diameter upsets, thereby changing the wire into a trapezoidal shape. This is known as Keystone effect. The original thickness of the curve is maintained at or near the mean diameter of the coil. It is necessary to take into account this upsetting of the material in determining the solid height of the spring. The dimensional change depends upon the spring index and the thickness of the material and may be determined from the following

$$t' = 0.48t \left(\frac{OD}{D} + 1\right) \quad 14.6$$

where t' is the new thickness of the inner edge after coiling and t is the thickness before coiling. Equation 14.6 can be used for both rectangular and square wire.

14.2.10 Design Guidelines for Compression Springs

(a) Compression springs ordinarily should not be permitted to go solid.
(b) Whenever practicable, springs should be designed so that if they were compressed to solid height, the corrected stress still would not exceed the minimum elastic limit.
(c) The length of a compression spring at maximum working deflection must not be too close to the solid length. As a minimum, a clearance of 10% of the wire diameter should exist between coils.
(d) The selection of springs for continuous cycling should be made on the basis of the fatigue allowables given in Section 14.7.

(e) The outside diameter of a compression spring when compressed must be less than the minimum hole diameter, if the spring operates in a hole. When operating over a guide, the minimum inside diameter of the spring must be larger than the maximum diameter of the guide.
(f) The possibility of buckling must be investigated and guides used, if necessary.
(g) Use compression springs in preference to other types since they are easier to produce, less expensive and have a deflection limiting feature in the solid height.
(h) The best proportioned springs from the standpoint of manufacture and design have a spring index between 7 and 9, although indexes of 5 to 16 are commonly used.
(i) For indexes less than 5 in the larger diameter wires, it may be necessary to use annealed material and harden after forming.
(j) Specify baking immediately after plating to relieve hydrogen embrittlement.
(k) Springs operating in parallel have a total spring rate equal the summation of individual spring rates; i.e., $r_t = \Sigma r_i$
(l) Springs operating in series have a total spring rate equal the reciprocal of the summation of the reciprocals of the individual spring rates; i.e., $1/r_t = \Sigma 1/r_i$

14.3 Extension Springs

Helical extension (sometimes called tension) springs differ from helical compression springs only in that they are usually closely coiled helixes with ends formed to permit their use in applications requiring resistance to tensile forces. It is also possible for the spring to be wound so that it is preloaded; that is, the spring is capable of resisting an initial tensile load before the coils separate. This load does not affect the spring rate.

14.3.1 Design Formulas

The same procedure as described in Section 14.2 is used for extension springs. The difference is in the end design and reload.

14.3.2 End Design

Various types of ends which can be obtained on a tension spring are shown in Figure 14.5. Loading an extension spring having hook ends causes the hooks to deflect. The amount of this deflection depends on the type of hook used. For a half hook the deflection per hook is equivalent to .1 of a full coil and the total number of active coils for design purposes will be N + .2. When a full hook is turned up from a full coil, the deflection per hook is equivalent to .5 of a full coil and the total number of active coils for design purposes will be N + 1.

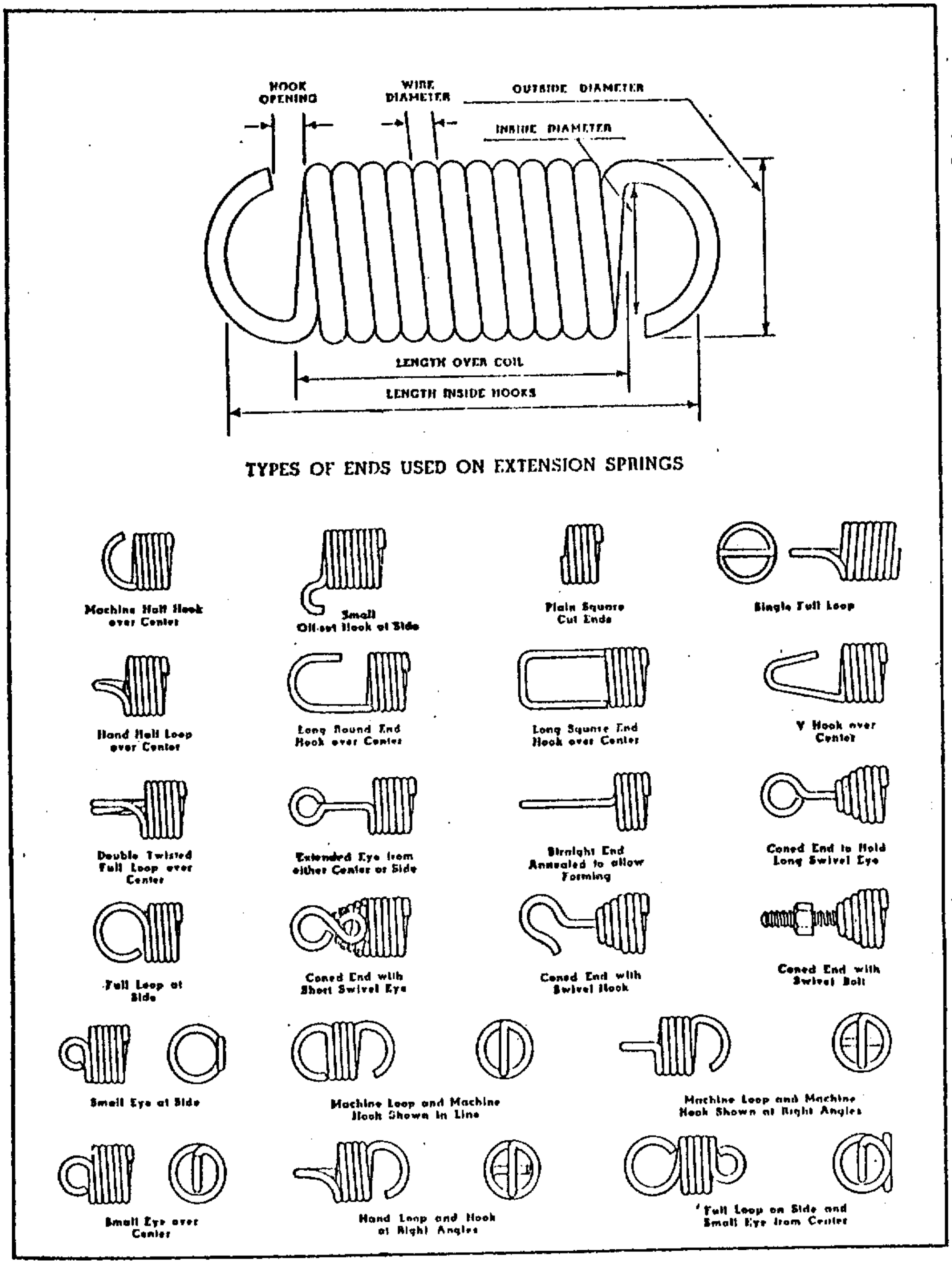


FIGURE 14.5 EXTENSION SPRINGS

The hooks at the ends of extension springs are subjected to both tension (bending) and torsional stresses. These combined stresses are frequently the limiting factor which determines the characteristics of the spring. These stresses occur at the base of the hooks and their magnitude is higher than the stress in the body. This, then, is the weakest point in an extension spring and the stresses should be calculated. The allowable working stresses should not exceed those shown in Section 14.8.

Figure 14.6 shows a typical hook end. The bending stress at Section A is

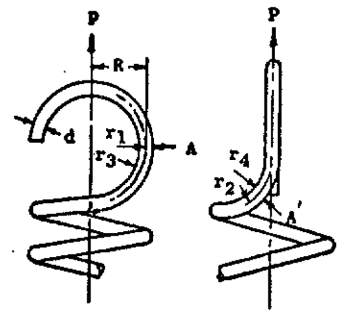


FIGURE 14.6. TYPICAL HOOK END

calculated using :

$$S_b = \frac{32PR}{\pi d^3} \times \frac{r_1}{r_3} \qquad 14.7$$

The torsional stress at section A' is calculated using :

$$S_t = \frac{16PR}{\pi d^3} \times \frac{r_2}{r_4} \qquad 14.8$$

Where 1 = mean radius of hook, in.
2 = mean radius of bend, in.
3 = inside radius of hook, in.
4 = inside radius of bend, in.

For best results, the inside radius should be at least twice the wire diameter. Special ends can be used when high stresses occur in the hooks. By using a smaller diameter for the last few coils before the hook, the magnitude of PR is reduced. Thus, the stress is reduced in direct proportion to the decrease in PR. By using as large radii for r_3 and r_4 as the design will permit, the stress is further reduced.

14.3.3 Initial Tension (Preload)

Initial tension is a load in pounds which opposes the opening of the coils by an external force. It is wound into the springs during the coiling operation. Extension springs will have a uniform rate after the applied load overcomes the load due to initial tension. The number of coils do not affect the amount of initial tension except when the weight of the coils is heavier than the initial tension. The amount of initial tension is dependent on the spring index (D/d);

the smaller the index the larger the initial tension. Initial tension does not increase the ultimate load or capacity of the spring, but causes a larger portion thereof to be exerted during the initial deflection. For example; if the initial tension is 4 lbs. and the spring rate is 9 lbs/in. then, at 1 inch deflection the load is

$$(1 \times 9) + 4 = 13 \text{ lbs.}$$

3 inches of deflection gives a load of

$$(3 \times 9) + 4 = 31 \text{ lbs.}$$

In computing the total torsional stress, add the torsional stress caused by initial tension to the torsional stress caused by deflection. Figure 14.7 shows the amount of initial tension in terms of torsional stress (without application of curvature stress correction factor) which can be coiled into extension springs made of music wire, oil tempered, corrosion resisting steel and hard drawn spring steel. Reduce these values 20 percent for springs made from nickel-base alloys such as Monel and Inconel. Hot rolled springs and those made of annealed materials cannot be wound with initial tension. Springs which require stress relieving will lose 25 to 50 percent of their initial tension. This loss can be compensated for during the coiling operation by winding more initial tension into the spring and, thus, obtain the required initial tension after stress relieving.

14.3.4 Design Guidelines for Extension Springs

(a) Avoid using enlarged, extended or specially shaped hooks or loops; they may double the cost of the spring and have high stress concentrations.
(b) If a plug must screw into the end of a spring, the spring should be coiled right hand.
(c) Nearly all extension springs are wound with enough initial tension to keep the spring together. Always figure at least 5 to 10 percent of the final load as initial tension, unless otherwise specified.
(d) Electroplating does not deposit a good coating on the inside of, or between, the coils of extension springs.
(e) Hooks on extension springs deflect under load. Each half hook, made by bending one half of a coil, deflects an amount equivalent to 0.1 of an active coil. Each full hook is equivalent to 0.5 of an active coil. Allowance for this deflection must be considered.
(f) If the relative position of the ends is not important, note this fact on the drawing.
(g) For standard hooks keep the OD of the hook the same as the OD of the spring and the distance from the end of the body, or from the last coil, to the inside of the hook about 75 to 85 percent of the ID of the spring.
(h) The body length or closed portion of an extension spring equals the number of coils in the body plus one, multiplied by the wire diameter.

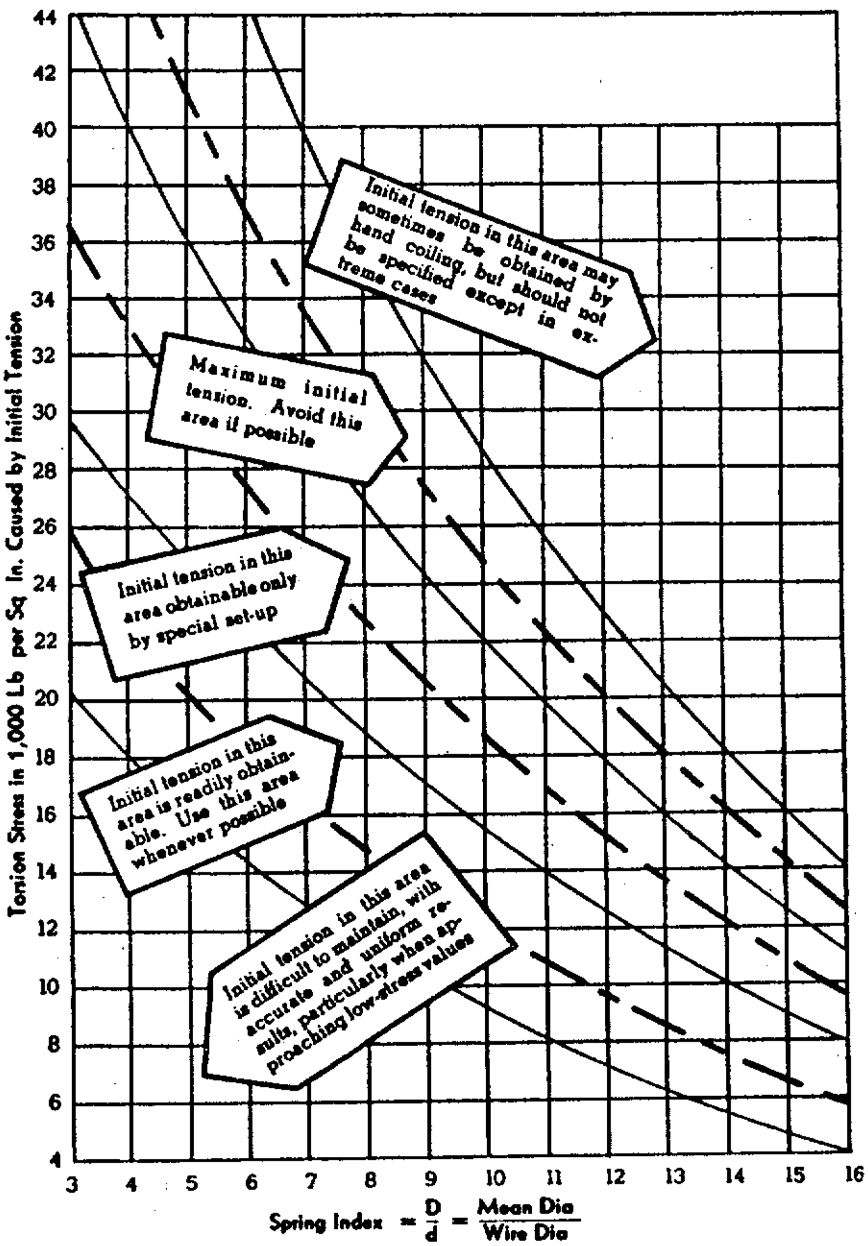


FIGURE 14.7 PERMISSIBLE TORSIONAL STRESS RESULTING FROM INITIAL TENSION IN COILED EXTENSION SPRINGS FOR DIFFERENT D/d RATIOS

(i) When deflected $1\frac{1}{4}$ times the maximum deflection as assembled, the total stress should be less than the minimum elastic limit shown by the curves in Section 14.7 as modified by their multiplying constants.

14.4 Torsion Springs

A helical spring can be loaded by a torque about the axis of the helix. Such loading is similar to the torsional loading of a shaft. The torque about the axis of the helix acts as a bending moment on each section of the wire. Ordinarily, round wire is used, but where added elastic resistance is needed in a limited amount of space, square or rectangular wire is frequently used.

The design theory for helical torsion springs is the same as beam theory. The wire in the torsion spring, as in the beam, is essentially in a state of bending. The analysis is simplified by assuming a constant moment throughout the wire cross section and a moment equal to the product of the load and the distance from its point of application to the central axis of the spring coil.

14.4.1 Design Formulas

The stress in helical torsion springs is a bending or tensile stress. The stress caused by a load should be compared with the elastic limit in tension of the material to determine the allowable stress. Comparison should also be made with the curves of allowable stresses, corrected for torsion springs, as shown in Section 14.7. Table 14.4 shows the various formulas for designing helical torsion springs.

14.4.2 End Design

Frequently, the limiting stress value in helical torsion springs is the stress value in the ends. When a helical torsion spring has an eye or bent off end as shown in Figure 14.8, the stress at the inside of the bend is a tensile stress. The sharp curvature causes the neutral axis to move inward toward the center of the curve and the tensile stress becomes that of a cantilever multiplied by a constant (K). The formula for determining the stress in the bend of the eye in Figure 14.8 is

$$S_b = \frac{32PRK}{\pi d^3} \qquad 14.9$$

where R = mean radius of eye, in

$$= \frac{\text{ID of eye} + d}{2}$$

$$= \frac{\text{OD of eye} - d}{2} \qquad 14.10$$

$$K = \frac{4C^2 - C - 1}{4C(C-1)}$$

Property	Round wire	Square wire	Rectangular wire *
Torque, lb in. T (also, PR)	$\frac{E d^4 F^\circ}{4{,}000\, N D}$	$\frac{E d^4 F^\circ}{2{,}375\, N D}$	$\frac{E b t^3 F^\circ}{2{,}375\, N D}$
	$\frac{S_b d^3}{10.2}$	$\frac{S_b d^3}{6}$	$\frac{S_b b t^2}{6}$
Bending stress, psi S_b	$\frac{10.2\, P R}{d^3}$	$\frac{6\, P R}{d^3}$	$\frac{6\, P R}{b t^2}$
	$\frac{E d F^\circ}{392\, N D}$	$\frac{E d F^\circ}{392\, N D}$	$\frac{E t F^\circ}{392\, N D}$
Deflection, F°	$\frac{4{,}000\, P R N D}{E d^4}$	$\frac{2{,}375\, P R N D}{E d^4}$	$\frac{2{,}375\, P R N D}{E b t^3}$
	$\frac{392\, S_b N D}{E d}$	$\frac{392\, S_b N D}{E d}$	$\frac{392\, S_b N D}{E t}$
Change in moment $T_2 - T_1$	$\frac{F^\circ_2 - F^\circ_1}{\frac{F^\circ}{T}}$	$\frac{F^\circ_2 - F^\circ_1}{\frac{F^\circ}{T}}$	$\frac{F^\circ_2 - F^\circ_1}{\frac{F^\circ}{T}}$
ID after deflection in. ID_1	$\frac{N\,(ID\ free)}{N + \frac{F^\circ}{360}}$	$\frac{N\,(ID\ free)}{N + \frac{F^\circ}{360}}$	$\frac{N\,(ID\ free)}{N + \frac{F^\circ}{360}}$
Rate r_t lb. in./Deg	$\frac{T}{F^\circ}$	$\frac{T}{F^\circ}$	$\frac{T}{F^\circ}$

TABLE 14.4 FORMULAS FOR HELICAL TORSION SPRINGS

When a spring has (makes) several complete revolutions, $F^\circ = 360^\circ$ multiplied by the number of revolutions.

* Rectangular wire may be coiled on edge or on flat, but b is always parallel to the axis of the spring and t is always perpendicular to the axis.

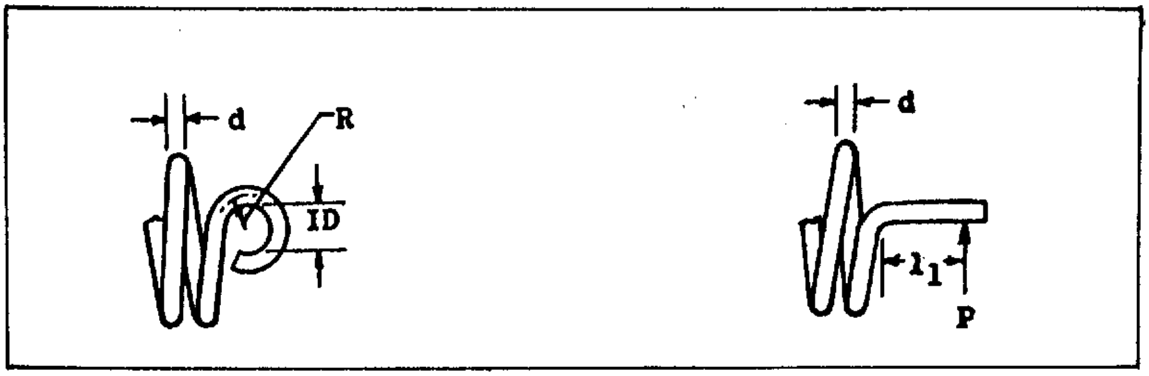


FIGURE 14.8 TORSION SPRING END DESIGN

For bends of the coil as shown in Figure 14.8 the stress value in the bend is

$$S_b = \frac{32 P l_1 K}{\pi d^3} \qquad 14.11$$

Where l_1 = distance from center of bend to load

K = curvature correction factor as defined by Equation 14.10

When the length of the material in the arms of a helical torsion spring approaches the length of material in one coil, the deflection of the arms will cause the deflection under applied loads to be in error. As shown in Figure 14.9, such ends deflect as a cantilever and may be calculated as such or the

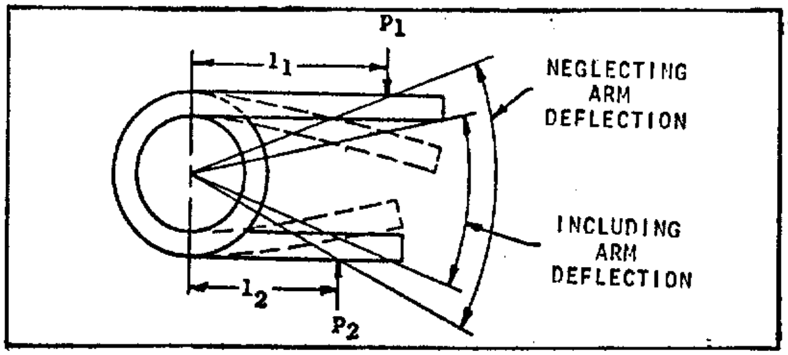


FIGURE 14.9 - CANTILEVER ENDS

formula for spring rate including arms may be used. The formula for spring rate when the deflection of the arms should be included is :

$$r_t = \frac{Ed^4}{1170(L + l_1/3 + l_2/3)} \qquad 14.12$$

where l_1 = length of arms from center of coil to point of load (P_1), in
l_2 = length of arm from center of coil to the point of load (P_2), in
L = active length of material = πDN

In springs with a large number of coils and short arms, the deflection of the arms is neglected. However, short arms should be avoided since this causes difficulty in coiling and forming.

14.4.3 Change in Diameter and Length

When a helical torsion spring is deflected, a reduction in diameter and an increase in length occurs. In order to prevent binding or scuffing, which reduces spring life, sufficient space must be provided when operating over a rod or in a cylinder. The new inside diameter ID, is obtained from Table 14.4. The change in length is due to the increase in the number of coils at the deflected position. If the spring makes one complete revolution, the increase in length is equal to one thickness of wire, plus an allowance for the space between coils, if any.

14.4.4 Helix of Torsion Springs

The direction of coiling (helix) should always be specified for torsion springs. A torsion spring should be so designed that the applied load tends to wind up the spring and increase its length. In springs operating under high stress it is desirable to design the springs with open coils. A slight space of about 1/64 inch or 20 to 25 percent of the wire diameter will eliminate friction between coils and reduce stress concentration which will lengthen the spring life. When long helical torsion springs are used there exists the possibility of buckling. Since buckling will cause abrasion between coils, erratic loads and early spring failure, it should be avoided. Buckling may be reduced in varying amounts by providing some means of lateral support such as:

1. Mounting the spring over a rod or guide.
2. Mounting the spring in a tube.
3. Clamping the ends.
4. Winding the spring with a small amount of initial tension.

14.4.5 Torsional Moment Estimation

Table 14.5 is an aid to quickly determine the torque (T or PR) that can be applied to a wire diameter at the suggested basic stress listed. For example, what wire diameter is required to support a torque of 10.5 in lbs? From the table it will be observed at 0.090 diameter music wire or corrosion resisting steel; 0.0915 diameter carbon or alloy steel and 0.125 diameter copper and nickel alloys could be considered. The final determination must be made by use of formulas,but Table 14.5 gives a good starting point. The basic stress indicated is a bending stress S_b caused by a torque T or PR, corrected for curvature.

14.4.6 Design Guidelines for Torsion Springs

a. Always try to support a torsion spring by a rod running through the center of the spring. Torsion springs unsupported or held by clamps or lugs alone are unsteady, will buckle and cause additional stresses in the wire.
b. Torsion springs should be designed and installed so that the deflection increases the number of coils. This increase should be allowed for in the design of space requirements.
c. The inside diameter reduces during deflection and should be computed to determine the clearance over the supporting rod.
d. Use as few bends in the ends as possible. They are often formed in separate operations, are expensive and cause concentrations of stress and frequent breakage.
e. Consider tolerances on diameters when determining clearances over rods.
f. Always specify the direction of coiling as either right-hand or left-hand on drawings.
g. Springs may be closely or loosely wound, but they should be wound tightly except when frictional resistance between the coils is desired.

TABLE 14.5 MOMENT VS. WIRE SIZE CHART

MUSIC WIRE			CARBON & ALLOY STEELS			COPPER & NICKEL ALLOYS		
Corrected Moment lb.-in.	Wire Diam. in.	Basic Stress psi	Corrected Moment lb.-in.	Wire Diam. in.	Basic Stress psi	Corrected Moment lb.-in.	Wire Diam. in.	Basic Stress psi
.0101	.008	201,000	1.013	.041	149,500	.00358	.008	71,200
.0143	.009	200,000	1.555	.0475	148,000	.00509	.009	71,000
.0196	.010	199,500	2.37	.054	146,700	.00694	.010	70,800
.0259	.011	198,500	3.47	.0625	145,000	.00925	.011	70,600
.0337	.012	198,000	5.24	.072	143,000	.0152	.013	70,500
.0425	.013	196,800	7.12	.080	141,500	.0189	.014	70,200
.0527	.014	195,500	10.49	.0915	139,800	.0281	.016	69,800
.0780	.016	194,000	11.21	.0937	139,000	.0398	.018	69,500
.1103	.018	193,000	15.72	.1055	136,600	.0546	.020	69,300
.1505	.020	192,000	23.05	.1205	134,000	.0827	.023	69,100
.199	.022	190,000	25.5	.125	133,000	.1056	.025	69,000
.256	.024	189,000	31.8	.135	131,700	.1635	.029	68,200
.323	.026	187,500	41.4	.1483	129,200	.219	.032	67,900
.443	.029	185,000	48.0	.1562	128,100	.310	.036	67,700
.533	.031	182,000	53.1	.162	127,100	.422	.040	67,200
.638	.033	181,000	68.0	.177	124,800	.598	.045	66,900
.755	.035	179,500	79.8	.1875	123,100	.866	.051	66,500
.886	.037	178,000	85.0	.192	122,400	1.194	.057	65,600
1.030	.039	177,000	104.6	.207	120,200	1.664	.064	64,600
1.185	.041	175,500	116.5	.2187	118,400	2.35	.072	64,100
1.350	.043	173,000	131.0	.2253	117,000	3.17	.080	63,200
1.535	.045	171,500	163	.2437	115,000	4.61	.091	62,300
1.73	.047	170,000	175	.250	114,000	6.42	.102	61,600
1.95	.049	169,000	199	.2625	112,200	8.82	.114	60,600
2.18	.051	167,500	239	.2812	109,600	11.5	.125	59,800
2.70	.055	165,000	301	.3065	106,400	12.2	.128	59,500
3.26	.059	162,000	315	.3125	105,300	17.2	.144	58,500
3.95	.063	161,000	367	.331	103,000	23.9	.162	57,200
4.70	.067	159,500	405	.3437	101,700	33.4	.182	56,200
5.50	.071	157,000	464	.3625	99,000	45.9	.204	55,100
6.39	.075	154,000	506	.375	97,800	62.5	.229	53,800
7.68	.080	153,000	575	.3938	95,800	88.7	.258	52,600
9.04	.085	150,000	622	.4062	94,600	122	.289	51,300
10.56	.090	147,500	752	.4375	91,800	169	.325	50,100
12.30	.095	146,500	898	.4687	88,800	234	.365	48,800
14.35	.100	146,000	1060	.500	86,500	320	.410	47,300
16.98	.106	145,000	1445	.5625	82,700	441	.460	46,200
19.80	.112	143,600	1910	.625	79,800			

NOTE: The values for Music Wire may also be used for Corrosion Resisting Steels.

h. Avoid using double torsion springs. Two single torsion springs, one coiled left-hand and one coiled right-hand, usually can perform the same action as a double torsion spring, at less than half the cost.
i. When deflected 1-1/4 times the maximum deflection as assembled, the total stress should be less than the Minimum Elastic Limit shown in Section 14.7 modified by their multiplying constants.

14.5 Coned Disc (Belleville) Springs

The coned disc (Belleville) spring or washer is a plain dished washer of a particular diameter, sectional profile, and height suited for an intended purpose. It is used in a variety of applications, all having the common characteristic of necessity for short range of motion and attendant high loads. In order to calculate the free spring height and required thickness of stock in a relatively simple manner, it is necessary to know the outside diameter (OD), inside diameter (ID) and the load (P) for a specific deflection.

14.5.1 Design Formulas

By obtaining the value for constant (Y) from the proper curve, Figure 14.10, the following formula can be used to calculate the load-deflection characteristics:

$$P = \frac{Ef}{(1-\mu^2)\ Y\ a^2}\left[\left(h-\frac{f}{2}\right)(h-f)t + t^3\right] \qquad 14.13$$

where a = one half the outside diameter, in
h = free height minus thickness, in

By obtaining the values for constants Z_1 and Z_2 from the proper curves, Figure 14.10, the following formula can be used to calculate stress

$$S_b = \frac{Ef}{(1-\mu^2)Ya^2}\left(Z_1\left(h-\frac{f}{2}\right) + Z_2 t\right) \qquad 14.14$$

It is possible for the term (h-f/2) to become negative if f is large. When this occurs, the terms inside the bracket should be changed to read $Z_1(h-f/2)-Z_2t$. This means, in this instance, the maximum stress is a tensile stress. For a spring life of less than one-half million (500,000) cycles, a stress of 200,000 psi can be substituted for S_b, even though this limit might be slightly beyond the elastic limit of the steel. This is because the stress is calculated at the point of greatest intensity, which is on an extremely small part of the disc. Immediately surrounding this area is a much lower stressed portion which so supports the higher stressed point that very little yielding results at atmospheric temperatures. For higher than atmospheric temperatures and long spring life, lower stresses must be employed.

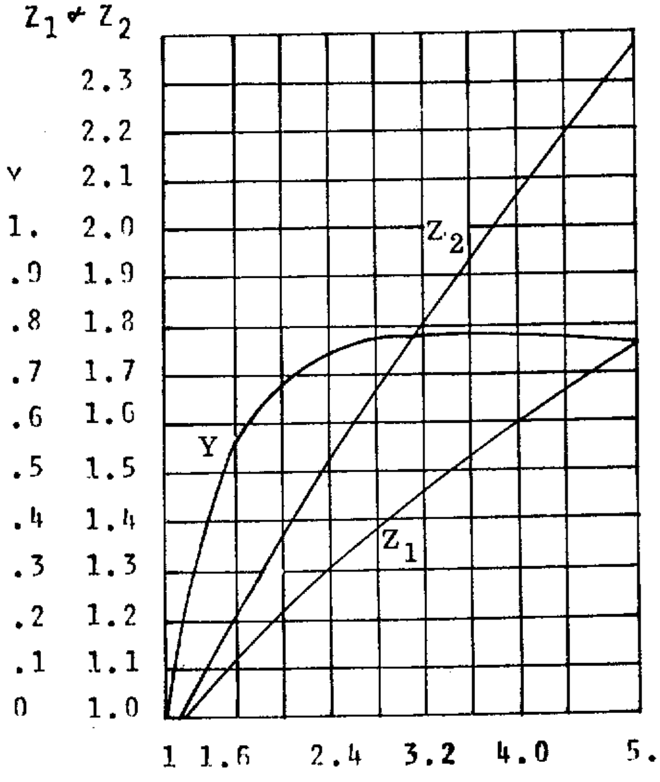


FIGURE 14.10 STRESS AND DEFLECTION CONSTANTS FOR BELLEVILLE WASHERS OF UNIFORM THICKNESS

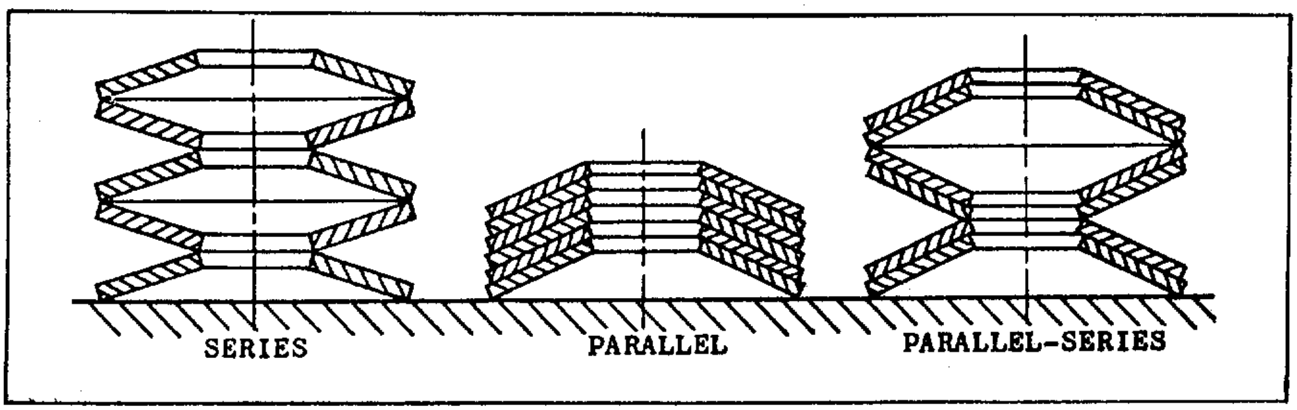


FIGURE 14.11 METHODS OF STACKING CONED DISC (BELLEVILLE) SPRINGS

Belleville springs can be stacked to obtain various load/deflection relationships. Figure 14.11 shows typical arrangements of these stacks. When fine-coned disc (Belleville) springs are stacked in series as in Figure 14.11, they have a spring rate only one-fifth that of one disc and the solid load will be the same as for one disc.

When six discs are stacked in parallel as in Figure 14.11, they will have a spring rate and a solid load six times that of one disc, disregarding friction.

When six discs are stacked in parallel-series as in Figure 14.11, they will have a spring rate only two-thirds that of one disc and the solid load will be twice that of one disc, disregarding friction.

14.6 Flat Springs

Load requirements are intimately connected with spring dimensioning and the space available for the spring. The point of load application, deflection, length, width and thickness should be clearly specified. Formulas for designing various flat spring characteristics are given in Table 14.6.

The stress in flat springs is in bending and should be compared with the elastic limit in tension which is shown in Section 14.7.

14.7 Material Properties

Various materials are available for springs. The selection of material is based on wire size, temperature of operation, load application (i.e. impact or slowly applied), cycles of load, corrosion, environment, etc. Table 14.7 shows a list of commonly used spring materials and the recommended usage.

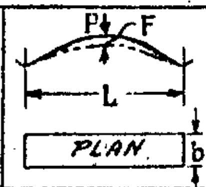
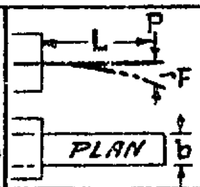
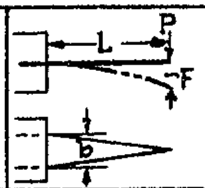
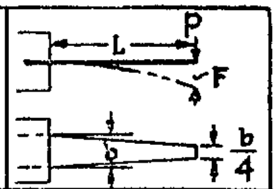
14.7.1 Fatigue Strength

The fatigue strength curves shown in Figures 14.12 through 14.18 are for the most popular spring materials. These are for compression springs, based on the minimum torsional elastic limit of each material. The values may be increased 25 percent for springs that have been properly stress relieved, cold set and shot peened. Table 14.8 shows the proper usage of the allowable curves. Figures 14.12 through 14.18 show the torsional minimum elastic limit and maximum solid stress at which a spring can be stressed. In addition, fatigue curves are shown for three different service conditions; light service, average service and severe service. These service conditions are defined as:

14.7.1.1 Light Service - This includes springs subjected to static loads or small deflections and seldom used springs such as those in bomb fuzes, projectiles and safety devices. This service is for 1,000 to 10,000 cycles.

14.7.1.2 Average Service - This includes springs in general use in machine tools, mechanical products and electrical components. Normal frequency of deflections not exceeding 3600 per hour permit such springs to withstand 100,000 to 1,000,000 cycles.

TABLE 14.6 FORMULAS FOR FLAT SPRINGS*

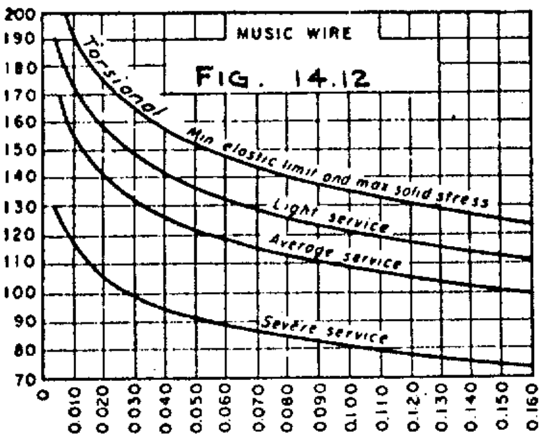
PROPERTY				
Deflection F Inches	$\frac{P L^3}{4 E b t^3}$	$\frac{4 P L^3}{E b t^3}$	$\frac{6 P L^3}{E b t^3}$	$\frac{5.22 P L^3}{E b t^3}$
	$\frac{S_b L^2}{6 E t}$	$\frac{2 S_b L^2}{3 E t}$	$\frac{S_b L^2}{E t}$	$\frac{.87 S_b L^2}{E t}$
Load P Pounds	$\frac{2 S_b b t^2}{3 L}$	$\frac{S_b b t^2}{6 L}$	$\frac{S_b b t^2}{6 L}$	$\frac{S_b b t^2}{6 L}$
	$\frac{4 E b t^3 F}{L^3}$	$\frac{E b t^3 F}{4 L^3}$	$\frac{E b t^3 F}{6 L^3}$	$\frac{E b t^3 F}{5.22 L^3}$
Stress S_b Bending psi	$\frac{3 P L}{2 b t^2}$	$\frac{6 P L}{b t^2}$	$\frac{6 P L}{b t^2}$	$\frac{6 P L}{b t^2}$
	$\frac{6 E t F}{L^2}$	$\frac{3 E t F}{2 L^2}$	$\frac{E t F}{L^2}$	$\frac{E t F}{.87 L^2}$
Thickness t Inches	$\frac{S_b L^2}{6 E F}$	$\frac{2 S_b L^2}{3 E F}$	$\frac{S_b L^2}{E F}$	$\frac{.87 S_b L^2}{E F}$
	$\sqrt[3]{\frac{P L^3}{4 E b F}}$	$\sqrt[3]{\frac{4 P L^3}{E b F}}$	$\sqrt[3]{\frac{6 P L^3}{E b F}}$	$\sqrt[3]{\frac{5.22 P L^3}{E b F}}$

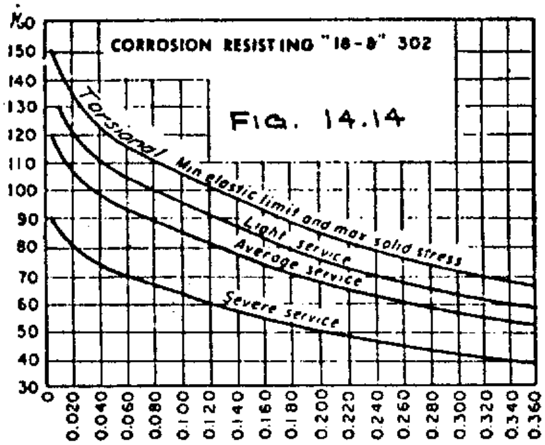
* Based on standard beam formulas where the deflection is small.

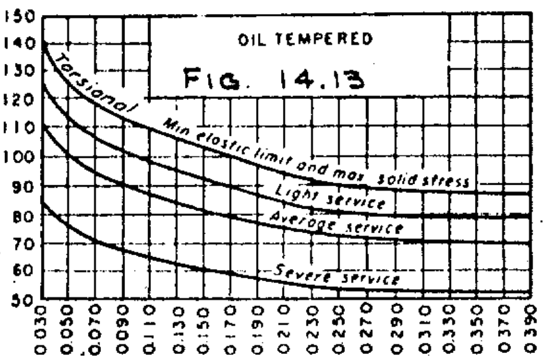
Material	Specification	Use
Inconel Wire	QQ-W-390 Cond. C	This material is excellent for applications requiring good corrosion resistance and an ability to withstand operation at temperatures from sub-zero to 650°F.
Inconel Sheet & Strip Spring Temper	MIL-N-6840 Condition S	Inconel possesses high electrical resistance and should not be used as a conductor; it is non-magnetic.
Inconel X Wire Spring Temper	Jan-W-562 Class 2	The applicable divisional staff unit must be consulted before release of spring designs calling for this material.
Inconel X Wire No. 1 Temper	Jan-W-562 Class 1	This material is like Inconel except that it is heat treatable. The spring temper wire is to be used whenever maximum mechanical properties are desired and the maximum temperature of operation will not exceed 800°F. This material is also excellent for sub-zero applications. The Class 1 temper, when properly heat treated, may be used up to 950°F. It's resistance to relaxation is superior to the Class 2 temper and should be given preference for applications when this characteristic is desired.

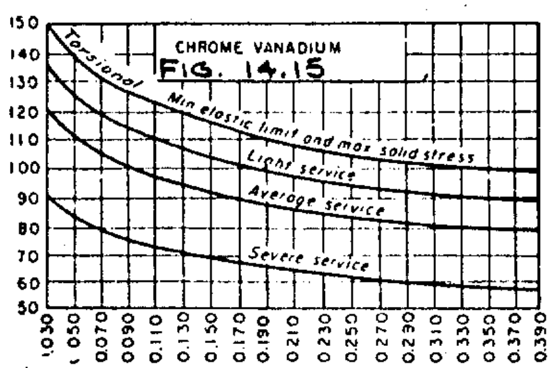
TABLE 14.7 APPLICATION OF COMMONLY USED SPRING MATERIALS

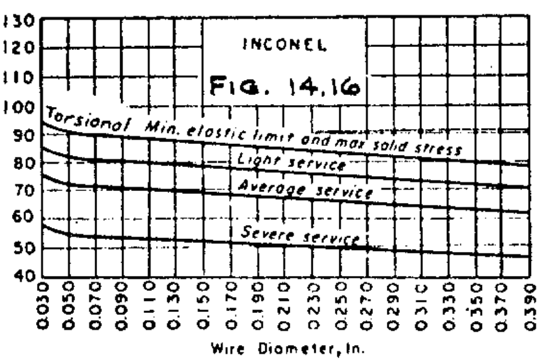
Torsional Stress (Corrected) 1000 Psi

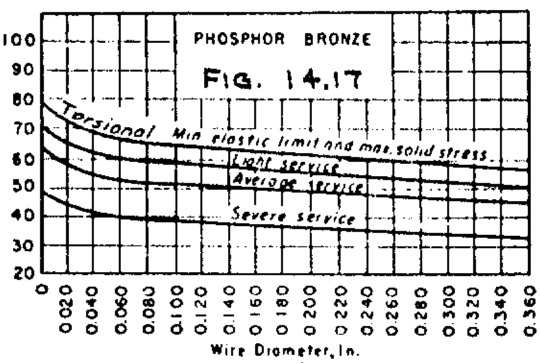




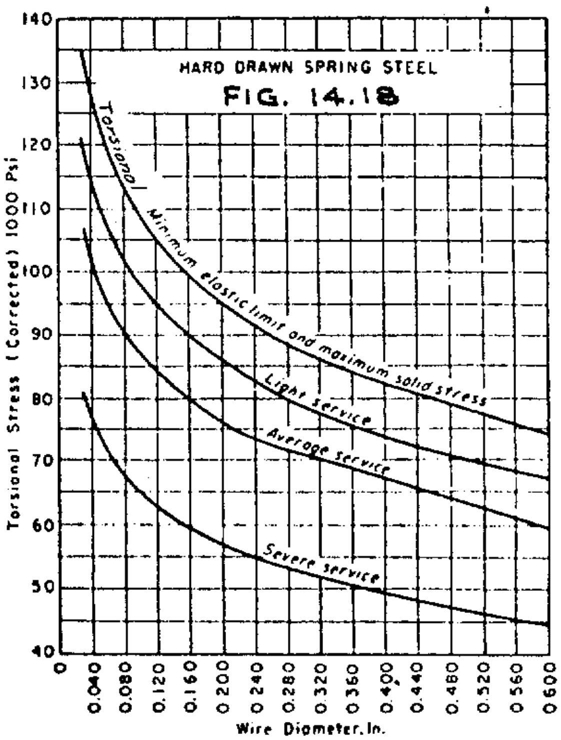








Torsional Stress (Corrected) 1000 Psi



FIGURES 14.12 - 14.18 FATIGUE STRENGTH CURVES - RECOMMENDED MAXIMUM WORKING STRESSES FOR COMPRESSION SPRINGS

TABLE 14.8 CRITICAL STRESS DATA*

COMPRESSION SPRING	EXTENSION SPRING	TORSION SPRING
1. Torsion Stress - Compare calculated stress in coils with service curve of Figures 14.12-14.18. 2. Solid Stress - Compare torsion stress in coils when compressed solid with minimum elastic limit curve.	1. Torsion Stress(Coils) Compare calculated design stress in coils with service curve of Figures 14.12-14.18 multiplied by .85. 2. Torsion Stress (Hooks) Compare calculated design stress in hooks with service curve multiplied by .85. 3. Bending Stress (Hooks) Compare calculated design stress in hooks with service curve multiplied by 1.5. 4. Torsion Stress (Coils) at Max Extended Length-Compare calculated stress at this length with min elastic limit curve multiplied by 8.5. 5. Torsion Stress (Hooks) at Max Extended Length-Compare calculated stress at this length with min elastic limit curve multiplied by .85. 6. Bending Stress (Hooks) at Max Extended Length-Compare calculated stress at this length with min elastic limit curve multiplied by 1.5.	1. Bending Stress (Coils) Compare calculated design stress in coils with service curve of Figures 14.12-14.18 multiplied by 1.5. 2. Bending Stress (Ends) Compare calculated design stress in ends with service curve multiplied by 1.5. 3. Bending Stress in Coils at Maximum Deflection - Compare calculated stress in coils at this deflection with min elastic limit of Figures 14.12-14.18 multiplied by 1.5. 4. Bending Stress in Ends at Maximum Deflection Compare calculated stress in ends at this deflection with min elastic limit of Figures 14.12-14.18 multiplied by 1.5.

*Note 1: After tentative spring configuration has been determined, use data in above table in association with Figures 14.12-14.18, to ascertain that allowable stresses are not exceeded.

Note 2: The above referenced "calculated design stresses" are TOTAL STRESSES. They include curvature stress-correction factors, except for extension spring hook stresses which include the correction factor in the basic formulas.

14.7.1.3 Severe Service - This includes springs subjected to rapid deflections over long periods of time and to shock loading such as in pneumatic tools, hydraulic controls and valves. This service is for 1,000,000 cycles and above. Lowering the values 10 percent permits 10,000,000 cycles.

14.7.2 Other Materials

For materials not shown on the curves of Figures 14.12 through 14.18, the following multiplying factors may be applied:

a. Beryllium Copper - multiply the values of the phosphor bronze curves by 1.20.
b. Spring Brass - multiply the values of the phosphor bronze curves by 0.75.
c. Monel - multiply the values of the inconel curves by 0.82.
d. K-Monel - multiply the values of the inconel curves by 0.90.
e. Duranickel - use the same values as for inconel.
f. Inconel-X, (Drawn to spring temper and precipitation hardened) - multiply the values of inconel curves by 1.25.
g. Silico-Manganese - multiply the values of the Chrome-Vanadium curves by 0.90.
h. Chrome-Silicon - multiply the values of the Chrome-Vanadium curves by 1.20.
i. Value Spring Quality Wire - Use the same values as for Chrome Vanadium.
j. Corrosion Resisting Steels Type FS304 and FS420 - multiply the values of the Corrosion Resisting Steel curves by 0.95.
k. Corrosion Resisting Steel Type FS316 - multiply the values of the Corrosion Resisting Steel curves by 0.90.
l. Corrosion Resisting Steel Type AISI431 and 17-7PH - multiply the values of the Music Wire curves by 0.90.

14.7.3 Elevated Temperature Operation

Springs used at elevated temperatures exert less load and have larger deflections under load than at room temperature. Compression and extension springs subjected to the temperatures and stresses shown in Table 14.9 will have a loss of 5 percent or less (or if the load remains constant they will deflect an additional 5 percent) in 48 hours. Elastic limits and modulus values are also reduced, thus necessitating these lower allowable working stresses.

Spring Material	Permissible Elevated temperature F deg	Maximum Recommended working stress S_t PSI
Brass Spring Wire	150	30,000
Phosphor Bronze	225	35,000
Music Wire	250	75,000
Beryllium-Copper	300	40,000
Hard Drawn Steel Wire	325	50,000
Carbon Spring Steels	375	55,000
Alloy Spring Steels	400	65,000
Monel	425	40,000
K-Monel	450	45,000
Duranickel	500	50,000
Corrosion Resisting FS-302	550	55,000
Corrosion Resisting AISI 431	600	50,000
Inconel	700	50,000
High Speed Steel	775	70,000
Cobenium, Elgiloy	800	75,000
Inconel X	850	55,000
Chrome-Moly-Vanadium	900	55,000

TABLE 14.9 - Permissible Elevated Temperatures for Compression and Extension Springs (Loss of load at these temperatures is less than 5% in 48 hours.)

14.7.4 Exact Fatigue Calculation

The curves shown in Figures 14.12 through 14.18 show allowable stresses for three types of service conditions. The exact life of a spring cannot be determined from these curves. If for some reason the exact life is necessary, a Goodman diagram can be combined with an S-N curve for the spring material and an exact life predicted.

First, an S-N curve is drawn from the data in Tables 14.10 and 14.11 and the strength properties of the materials. There will be a separate S-N curve for

Structures in Bending	Structures in Torsion
10^4 cycles - 80% of F_{tu}	10^4 cycles - 45%* of F_{tu}
10^5 cycles - 53% of F_{tu}	10^5 cycles - 35% of F_{tu}
10^6 cycles - 50% of F_{tu}	10^6 cycles - 33% of F_{tu}
10^7 cycles - 48% of F_{tu}	10^7 cycles - 30% of F_{tu}
	*35% for Phosphor Bronze or AISI302

TABLE 14.10 - Design Stresses for Cyclic Service - Springs Not Shot Peened

Structures in Bending	Structures in Torsion
10^4 cycles - 80% of F_{tu}	10^4 cycles - 45% of F_{tu}
10^5 cycles - 62% of F_{tu}	10^5 cycles - 42% of F_{tu}
10^6 cycles - 60% of F_{tu}	10^6 cycles - 40% of F_{tu}
10^7 cycles - 58% of F_{tu}	10^7 cycles - 36% of F_{tu}

TABLE 14.11 - Design Stresses for Cyclic Service - Springs Shot Peened

each material by size and also by stress type (torsion or bending). The abscissa is also used as a double scale - a log scale for the number of cycles, N, and a linear scale labeled "minor stress" of the same rate as the ordinate or "major stress" scale. Figure 14.19 shows the diagram. A 45 degree line is drawn from the origin of the plot. On this line, a point "A" is marked corresponding to the ultimate strength of the material. This is the tensile strength for structures in bending and the torsional strength for structures in torsion. Torsional strength can be taken as two-thirds of the tensile strength. These two lines constitute the combined S-N and Goodman diagrams for the given material and the pertinent stress type.

If a minimum service life must be met, draw a vertical line from the appropriate value on the "N" scale to its intersection with the S-N curve "B". At the intersection, draw a horizontal line to the intersection with the ordinate or "major stress" scale "C". From that point, draw a straight line to the tensile strength point "A". Along this line "AC" lie the combinations of major and minor stress that will meet the desired life.

14.8 Spring Manufacture

Certain processes may be employed during manufacture of the spring to greatly enhance the performance of the spring. Information on a few essential operations is given here.

14.8.1 Stress Relieving

The usual types of hardening and tempering ovens are used for stress relieving. Springs made from prehardened wire such as Music Wire, Oil Tempered, Hard Drawn, Corrosion Resisting 18-8 and similar materials are stress relieved by heating at low temperatures from 400 to 650°F to reduce the residual stresses trapped in the wire during the coiling operation. Springs made from annealed wire are hardened and tempered in a manner somewhat similar to tool steel. Precipitation hardening materials such as Beryllium-Copper, K-Monel, Inionel-X, 17-7PH and others are heated at varying temperatures, depending upon composition, for extended times from 1 to 16 hours.

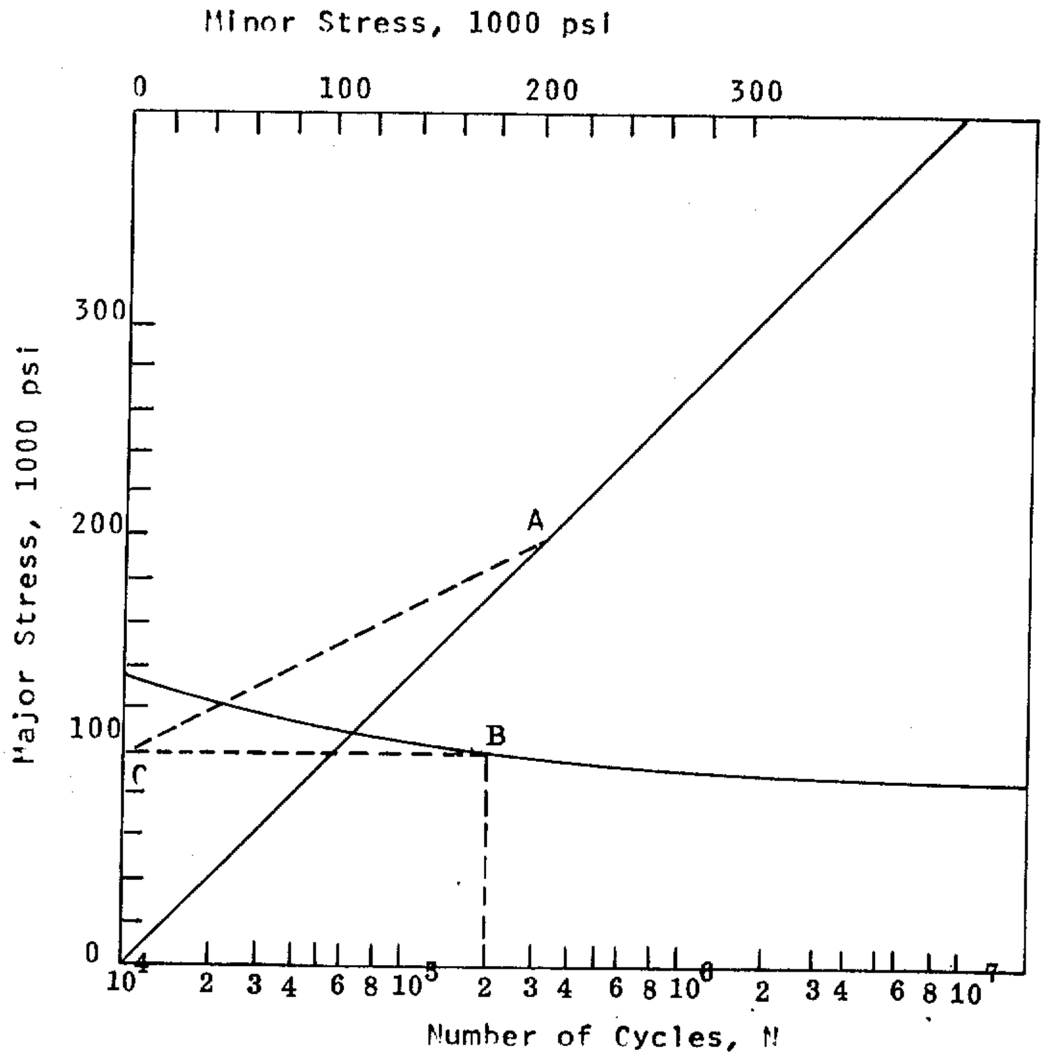


FIGURE 14.19 GOODMAN DIAGRAM COMBINED WITH SN CURVE

14.8.2 Cold Set to Solid

This process is used to stabilize the free length of a compression spring, so that subsequent inadvertant or intentional compression to solid height will not change the loads at working deflections.

a. If a compression spring is designed so that the elastic limit is not exceeded when the spring is compressed to solid height, no appreciable permanent set will occur, other than removal of small kinks in the wire. The note "Cold Set to Solid" should be specified on the drawing of such springs.
b. If a spring is designed so that the elastic limit is exceeded when the spring is closed solid, permanent set will occur and the free length will be decreased. Residual stresses of opposite sign will be set up in the wire when the load is released so that if the spring is again closed solid, it will withstand a higher calculated stress than the stress corresponding with the elastic limit. If the initial free length of the spring is made greater than the calculated free length by the proper amount, overstressing the spring beyond the elastic limit by compressing it to solid height will stabilize the characteristics and produce the desired loads at working deflections. Additional cycles of compressing the spring to solid height and releasing the load will not further change the free length. However, there is a limit to this process. After a certain initial free length has been reached for a particular spring, the final free length after compression to a solid height will remain constant no matter what increases are made in initial free length.
c. When a spring is designed so that the stress at solid height is so far above the elastic limit that the spring will not have the desired loads at working deflections if cold set to solid; the note "Shall compress to.......in. without permanent set" should be placed on the drawing. The computed stress at the specified length (equal to or less than the final assembled length) must be less than the stress at the elastic limit. Whenever practicable, this design of spring should be discarded in favor of a spring having a solid stress within limits that will permit closing solid without permanent set.

14.8.3 Grinding

End coils of compression springs are ground whenever it is necessary for the springs, (1) to stand upright, (2) to obtain a good seat against a contacting part, (3) to reduce buckling, and (4) to cause the springs to exert more uniform pressures under a diaphram or against a mating part. This operation is expensive and should be avoided whenever it is practicable to do so - especially on light springs with wire diameters under 1/32 inch and where a large spring index ratio prevails, such as 13 or larger.

14.8.4 Shot Peening

Spring life can be increased at least 30 percent and has often been increased from two to ten times by shot peening. This process may be applied to all highly stressed springs made from steel and non-ferrous materials usually over

1/16 inch wire diameter. Extension springs and closely wound torsion springs are difficult to shot peen because the tiny steel shot is frequently trapped between the coils and is difficult to remove. The large increase in fatigue life of helical springs due to shot peening is accomplished by a combination of effects:

a. Small surface irregularities seen only by the microscope are hammered smooth.
b. The surface of the wire is thoroughly cleaned and sharp burrs are made dull.
c. This additional cold work hardens the surface of the wire and raises the physical properties where the stress is highest.
d. Cold forging traps beneficial compression stresses near the wire surface which must be overcome by the destructive tensile stresses that cause fatigue failure before breakage can occur. All heat-treating of springs and all stress-relieving processes should be done prior to shot peening, except in those instances where electroplating is used; it is then necessary to reheat after plating. Heating the springs above 500°F after shot peening counteracts much of the beneficial effects of the trapped compression stresses produced by shot peening.

Springs made from annealed, oil tempered or alloy steels that must be electroplated can be shot peened principally to clean the surface, thus avoiding the necessity of soaking them in acid solutions to remove scale. The slightly roughened surface of shot peened springs does not produce a bright glossy electroplated coating.

14.8.5 Protective Coatings

Uncoated or oil dipped springs are satisfactory where corrosive conditions are not a factor. Black japanning is often used as it is a flexible, inexpensive finish suitable for many applications. Enamels, lacquers and paint are occasionally used. Cadmium with supplementary chromate treatment provides one of the best electro-deposited coatings because it is both flexible and corrosion resistant.

14.8.6 Hydrogen Embrittlement

Steel, particularly hardened steel, is susceptible to embrittlement resulting from hydrogen introduced by acid pickling, electroplating or cathodic electrocleaning operations. Absorbed hydrogen results in brittle behavior, particularly under sustained loading in the presence of stress concentrations. Baking to relieve hydrogen embrittled springs should be accomplished.

SECTION 15

THERMAL STRESS ANALYSIS

15.0 GENERAL

When a structural element is subjected to a change in temperature, it will either expand or contract depending on whether the temperature change is an increase or decrease. If the element is restrained, such as is common in airframes, the attempt at expansion or contraction will induce stresses into the structure. Not only will the individual element be affected but the surrounding structure will have induced loads from the temperature change. In the absence of constraints at boundaries, thermal stresses in a body are self equilibrating.

Except for a few simple cases, the solution of the thermoelasticity problem becomes intractable. Therefore, for thermal stress analysis, approximations leading to the strength of materials and finite element methods are used extensively. Depending on its geometry, a structural element is classified as; rod, beam, curved beam, plate or shell. If a structure consists of one of these elements or some simple combination of them, the method of strength of materials will yield good results. If a structure has a complex geometry, the finite element method is easier to use and the results are satisfactory. The finite element method used at Bell Helicopter is NASTRAN. It should be used on an idealized structure which consists of a large number of smaller simpler elements to provide approximately the configuration of the actual structure.

In a constrained structure, compressive stresses resulting from thermal, or thermal and mechanical, loading may produce instability of the structure. The linear thermoelastic solution of the problem excludes the question of large deflections. Thus, for buckling, or for structures where loads depend on deformations, nonlinearity that is due to large deformations must be incorporated in the problem formulation. The extreme difficulty involved in solving the nonlinear thermoelasticity problem has led to the approximate methods of strength of materials and finite elements.

The strength of materials solutions for simple structural shapes are presented in this section. The finite element solutions must be obtained by the use of NASTRAN and is not within the scope of this manual.

15.1 Strength of Materials Solutions

The assumption that a plane section normal to the reference axis before thermal loading remains normal to the deformed reference axis and plane after thermal loading, along with neglecting the effect on stress distribution of lateral contraction, lays the foundation of the approximate methods of strength of materials. Materials solution improves with the reduction of depth-to-span ratio, if the variation of temperature along the length of the beam is smooth. As in the case of mechanical loads, a considerable error results in the vicinity of abrupt changes in the cross sections. If the temperature is either uniform or linear along the length of the beam, the assumption of a plane section is valid and the strength of materials method gives the same results as those by the plane stress thermoelastic method.

Thermal stresses are induced in structures as a result of

a. Heating or cooling of an element which has some restraints (elements with no restraints have self equilibrating stresses).
b. Heating or cooling of a structure composed of elements with different coefficients of thermal expansion.
c. Unequal heating or cooling causing a non-linear or non-uniform temperature distribution within a beam or plate.
d. Unsymmetrical heating or cooling through the thickness of a plate or beam producing bending moments, with or without external restraints.

Thermal stresses can be added linearly to mechanical stresses if the total is below the proportional limit of the material. Above the proportional limit, the sum of the thermal and mechanical stresses can be obtained using a strain analysis.

15.1.1 General Stresses and Strains

The following equations will give the strains in the x, y and z directions.

$$\epsilon_x = 1/E\,[\,\sigma_x - \mu(\sigma_y + \sigma_z)] + \alpha(T-T_0) \qquad 15.1$$

$$\epsilon_y = 1/E\,[\,\sigma_y - \mu(\sigma_x + \sigma_z)] + \alpha(T-T_0) \qquad 15.2$$

$$\epsilon_z = 1/E\,[\,\sigma_z - \mu(\sigma_x + \sigma_y)] + \alpha(T-T_0) \qquad 15.3$$

where: α = coefficient of thermal expansion
μ = Poisson's ratio
E = modulus of elasticity
T_0 = reference temperature (zero thermal stress)
T = temperature at point in question

The equations for stress in the x, y and z directions are

$$\sigma_x = \frac{E\mu}{(1+\mu)(1-2\mu)}(\epsilon_x + \epsilon_y + \epsilon_z) + \frac{E}{(1+\mu)}\epsilon_x - \frac{\alpha E(T-T_0)}{1-2\mu} \qquad 15.4$$

$$\sigma_y = \frac{E\mu}{(1+\mu)(1-2\mu)}(\epsilon_x + \epsilon_y + \epsilon_z) + \frac{E}{(1+\mu)}\epsilon_y - \frac{\alpha E(T-T_0)}{1-2\mu} \qquad 15.5$$

$$\sigma_z = \frac{E\mu}{(1+\mu)(1-2\mu)}(\epsilon_x + \epsilon_y + \epsilon_z) + \frac{E}{(1+\mu)}\epsilon_z - \frac{\alpha E(T-T_0)}{1-2\mu} \qquad 15.6$$

In the plane stress case ($\sigma_z = 0$), these equations become

$$\sigma_x = \frac{E}{1-\mu^2}(\epsilon_x + \mu\epsilon_y) - \frac{E\alpha\,(T-T_0)}{1-\mu} \qquad 15.7$$

$$\sigma_y = \frac{E}{1-\mu^2}(\epsilon_y + \mu\epsilon_x) - \frac{E\alpha\,(T-T_0)}{1-\mu} \qquad 15.8$$

For uniaxial conditions ($\sigma_y = \sigma_z = 0$)

$$\sigma_x = E\,\epsilon_x - E\alpha(T-T_0) = E\,[\,\epsilon_x - \alpha(T-T_0)] \qquad 15.9$$

15.2 Uniform Heating

Following are some typical beam elements using the previous equations to determine the effects of uniform heating.

15.2.1 Bar Restrained Against Lengthwise Expansion

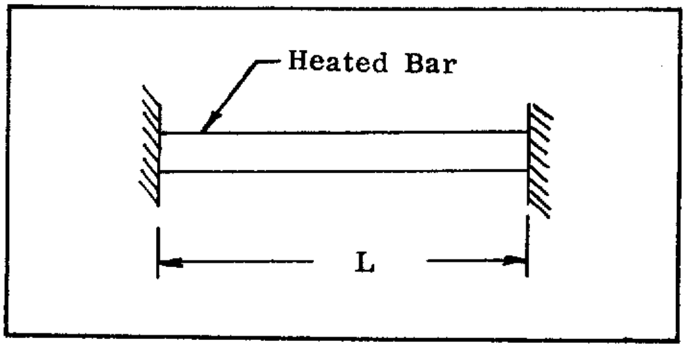


FIGURE 15.1 - FULL RESTRAINT, UNIFORM HEAT

$$P = -AE\alpha(T-T_0) \quad (15.10)$$
$$\sigma = -E\alpha(T-T_0) \quad (15.11)$$

15.2.2 Restrained Bar With a Gap at One End

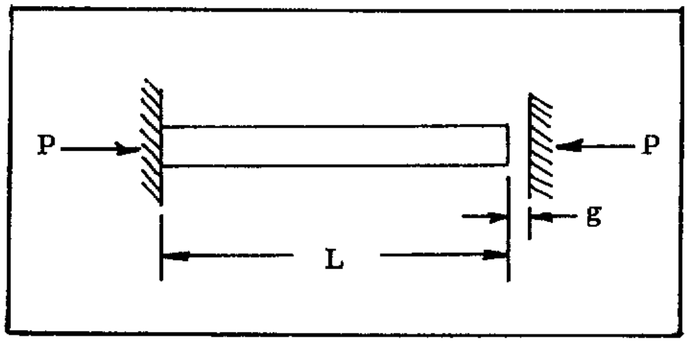


FIGURE 15.2 - FULL RESTRAINT WITH GAP, UNIFORM HEAT

If $g/L \geq \alpha(T-T_0)$, $P = 0$

If $g/L < \alpha(T-T_0)$, $P = -EA\alpha(T-T_0) + gAE/L$ 15.12

and $\sigma = -E\alpha(T-T_0) + gE/L$ 15.13

15.2.3 Partial Restraint

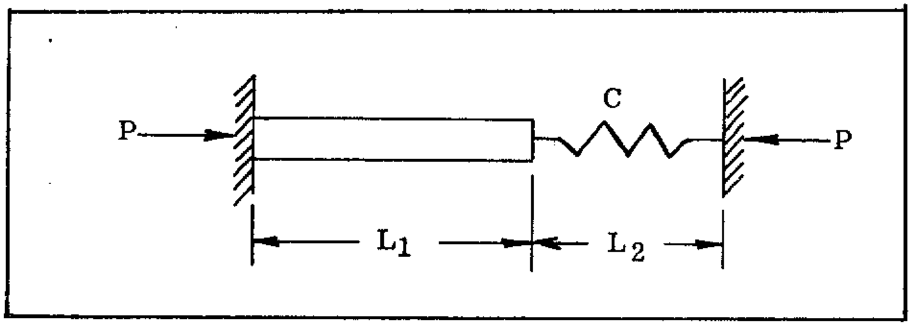


FIGURE 15.3 - RESTRAINT WITH SPRING, UNIFORM HEAT

$$P = \frac{-\alpha_1 L_1 (T-T_0)_1 + \alpha_2 L_2 (T-T_0)_2}{1/C + L_1/A_1 E_1}$$

where: C = spring rate for L_2 at its final temperature T. This spring may be real or represent another structure.

15.2.4 Two Bars at Different Temperatures

The bars are attached such that the cold bar restrains the expansion of the hot bar. The bars remain straight with no bending.

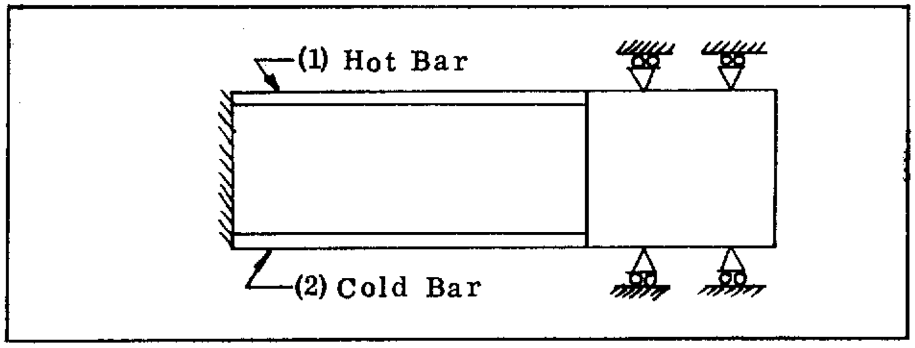


FIGURE 15.4 - TWO BARS AT DIFFERENT TEMPERATURES

$$\sigma_1 = -E_1\,\alpha_1(T_1-T_0)C_1 \qquad 15.14$$

$$\sigma_2 = -A_1/A_2\,\sigma_1 \qquad 15.15$$

$$C_1 = \left[\frac{E_2A_2}{E_1A_1 + E_2A_2}\right]\left[\frac{1 - \alpha_2(T_2-T_0)}{\alpha_1(T_1-T_0)}\right] \qquad 15.16$$

15.2.5 Three Bars at Different Temperatures

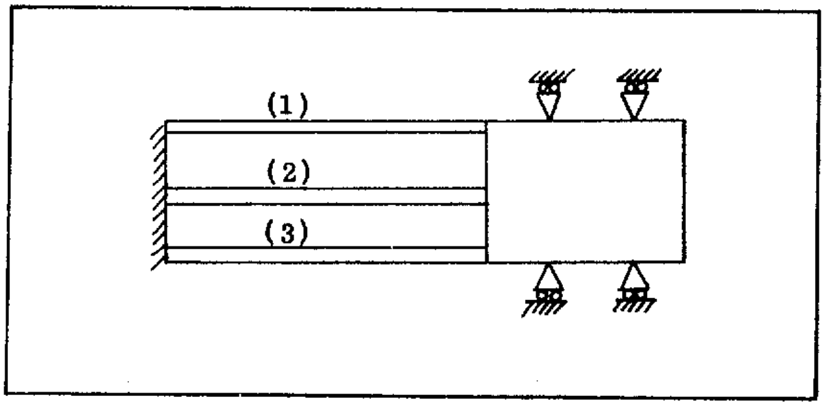


FIGURE 15.5 - THREE BARS AT DIFFERENT TEMPERATURES

$$\sigma_1 = -E_1\ \alpha_1(T_1 - T_0)C_1 \qquad 15.17$$

$$\sigma_2 = -E_2\ \alpha_2(T_2 - T_0)C_2 \qquad 15.18$$

$$\sigma_3 = 1/A_3(\sigma_1 A_1 + \sigma_2 A_2) \qquad 15.19$$

$$C_1 = \left|\frac{A_2E_2 + A_3E_3}{A_1E_1 + A_2E_2 + A_3E_3}\right|\left|\frac{1 - A_2\alpha_2E_2(T_2 - T_0) + A_3\alpha_3E_3(T_3 - T_0)}{\alpha_1(T_1 - T_0)(A_2E_2 + A_3E_3)}\right| \qquad 15.20$$

$$C_2 = \left|\frac{A_1E_1 + A_3E_3}{A_1E_1 + A_2E_2 + A_3E_3}\right|\left|\frac{1 - A_1\alpha_1E_1(T_1 - T_0) + A_3\alpha_3E_3(T_3 - T_0)}{\alpha_2(T_2 - T_0)(A_1E_1 + A_3E_3)}\right| \qquad 15.21$$

15.2.6 General Equations for Bars at Different Temperatures

$$\sigma_i = E_i \left[\frac{\sum_{i=1}^{n} E_i A_i\ {}_i(T_i - T_0)}{\sum_{i=1}^{n} E_i A_i} + \frac{P}{\sum_{i=1}^{n} E_i A_i} - \alpha_i(T_i - T_0) \right] \qquad 15.22$$

where i refers to the bar in question and P is the externally applied axial load.

15.3 Non-Uniform Temperatures

The following are equations which can be used to determine the stress in beams with temperatures varying through the depth. Figure 15.6 shows beams with uniform and non-uniform thickness.

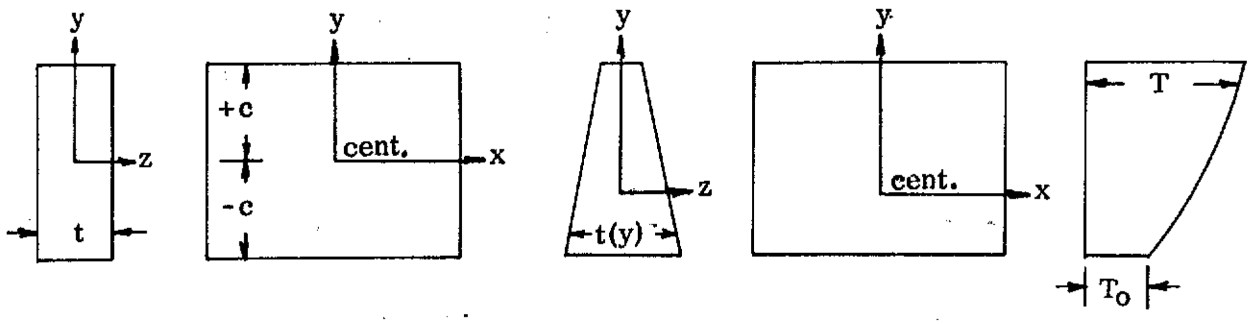


FIGURE 15.6 - TYPICAL BEAMS WITH VARYING TEMPERATURES

15.3.1 Uniform Thickness

$$\sigma_x = -\alpha E(T-T_0) + 1/2C \int_{-C}^{+C} \alpha E(T-T_0)dy + 3y/2C^3 \int_{-C}^{+C} \alpha E(T-T_0)ydy \qquad 15.23$$

Notes:

(1) If the beam is restrained from expanding and bending, drop the last two terms.

(2) If the beam is restrained from expansion but is free to bend, drop the middle term.

(3) If the beam is restrained from bending, drop the last term.

15.3.2 Varying Thickness

If the beam has a varying width through its depth and is symmetrical about its vertical centerline, the previous equation becomes

$$\sigma_x = -\alpha E(T-T_0) + 1/A \int_A \alpha E(T-T_0)tdy + y/I_z \int_A \alpha E(T-T_0)ytdy \qquad 15.24$$

where I_z is moment of inertia about the centroidal axis and t is a function of y as shown in Figure 15.6.

15.4 Linear Temperature Variations

The following are equations which can be used to determine the stress in beams with linear temperature variations between the two faces.

15.4.1 Restrained Rectangular Beam, Uniform Face Temperatures

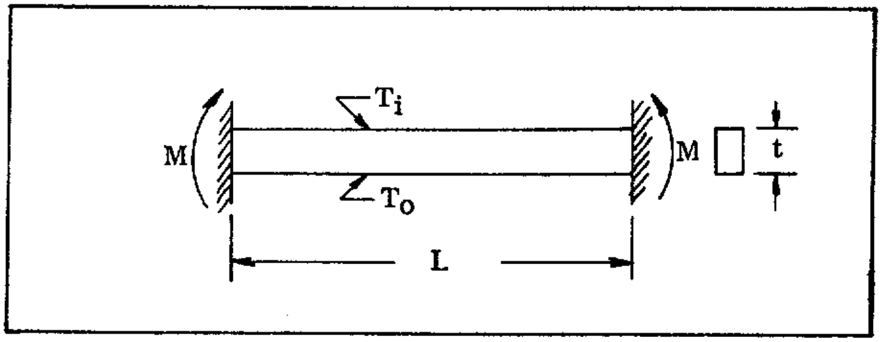


FIGURE 15.7 - RESTRAINED, DIFFERENT FACE TEMPERATURES

$$M = EI\alpha(T_1 - T_0)/t \qquad 15.25$$

$$\sigma_{b_{max}} = \pm E\alpha(T_1 - T_0)/2 \qquad 15.26$$

15.4.2 Pin-Ended Beam

The following equations are for a pin-ended beam with rectangular cross section and different uniform face temperatures.

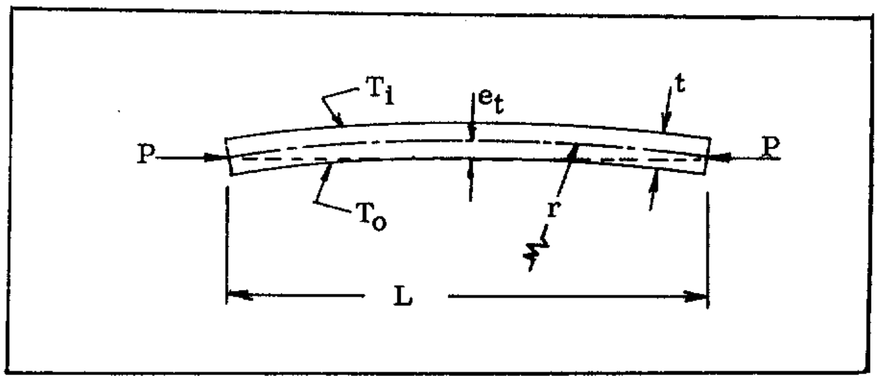


FIGURE 15.8 - PIN-ENDED COLUMN, DIFFERENT FACE TEMPERATURES

$$e_T \approx \alpha(T_1 - T_0)L^2/8t \qquad 15.27$$

$$P' = \pi^2 EI/L^2 \qquad 15.28$$

$$e_{final} = (e_T \pm e_i)/(1 - P/P') \qquad 15.29$$

$$M_{max} = P(e_{final}) \qquad 15.30$$

where: e_T = eccentricity due to temperature

e_i = any initial eccentricity

15.5 Combined Mechanical and Thermal Stresses

If a member is subjected to external loads and moments with temperature variations in two planes as shown in Figure 15.9, the stresses can be calculated as follows.

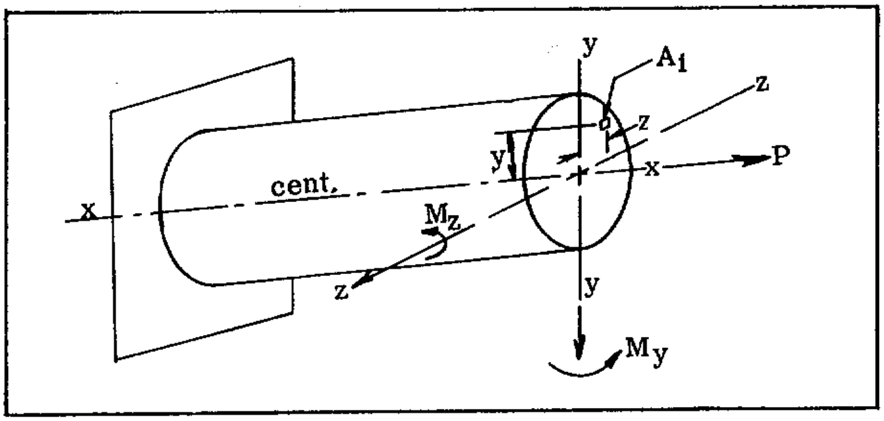


FIGURE 15.9 - COMBINING MECHANICAL AND THERMAL STRESSES

$$ {}_i = E_i \left[Y_i \left(\frac{M_z + \sum_{i=1}^{n} Y_i E_i A_i \, \alpha_i (T_i - T_0)}{\sum_{i=1}^{n} Y_i^2 E_i A_i} \right) + Z_i \left(\frac{M_y + \sum_{i=1}^{n} Z_i E_i A_i \, \alpha_i (T_i - T_0)}{\sum_{i=1}^{n} Z_i^2 E_i A_i} \right) + \left(\frac{P + \sum_{i=1}^{n} E_i A_i \, \alpha_i (T_i - T_0)}{\sum_{i=1}^{n} E_i A_i} \right) \right] - E_i \, \alpha_i (T_i - T_0) \qquad 15.31 $$

where: M_z = moment about z-axis

M_y = moment about y-axis

If no bending in the x-z plane is assumed the quantity in the second parenthesis is eliminated.

15.6 Flat Plates

The following equation is the general expression for flat plates with the temperature varying through the thickness and independent of the length or width. Figure 15.10 shows the nomenclature for this equation.

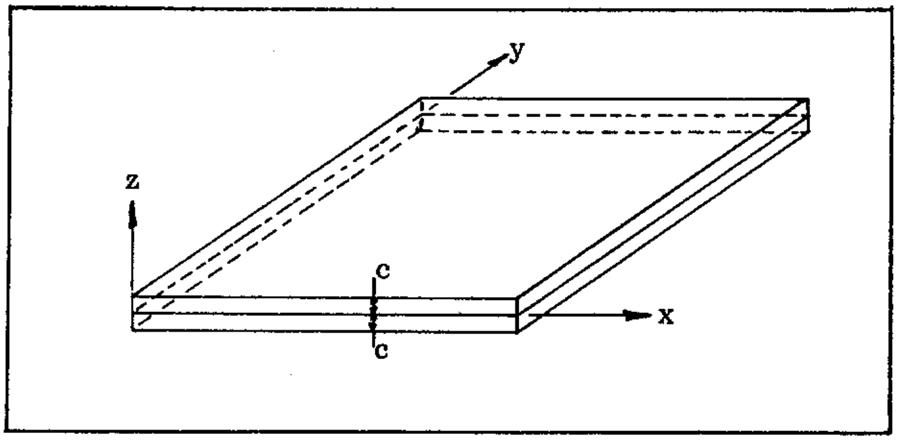


FIGURE 15.10 - FLAT PLATE, NON-UNIFORM TEMPERATURE VARIATION

$$\sigma_x = \sigma_y = \frac{-\alpha E(T-T_0)}{(1-\mu)} + \frac{1}{2C(1-\mu)} \int_{-C}^{C} \alpha E(T-T_o)dz$$

$$+ \frac{3z}{2C^3(1-\mu)} \int_{-C}^{C} \alpha E(T-T_0)zdz \qquad 15.32$$

15.6.1 Plate of General Shape

The following equation is for a flat plate of any shape, rotationally restrained at the edges, with linear temperature gradient between the two faces, both at different uniform temperatures.

$$\sigma_{max} = \pm \frac{E\alpha(T_1-T_0)}{2(1-\mu)} \qquad 15.33$$

15.6.2 Square Plates

The following equations are for a square plate, rotationally restrained at the edges, with linear temperature variation between the two faces, both at uniform temperature.

Near the edges: $\sigma_{bmax} = E\,\alpha(T_1-T_0)/2$ 15.34

Away from the edges: $\sigma_{bmax} = \dfrac{E\alpha(T_1-T_0)}{2(1-\mu)}$ 15.35

15.6.3 Flat Plates with Uniform Heating

The following equations assume that there is uniform heating, no bending and that the edges remain straight and parallel.

1) Uniformly heated rectangular plate restrained in the x and y directions only.

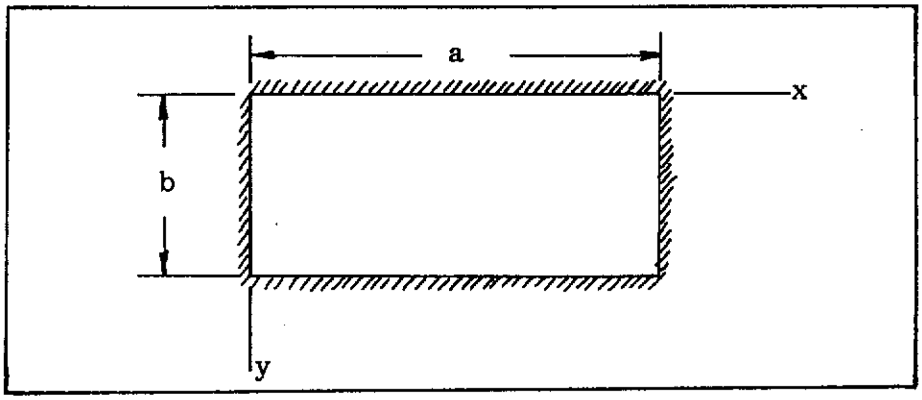


FIGURE 15.11 - RESTRAINED PLATE, UNIFORMLY HEATED

$$\sigma_x = \sigma_y = \frac{-E\alpha(T-T_0)}{(1-\mu)} \qquad 15.36$$

2) Partial restraint of a uniformly heated square plate.

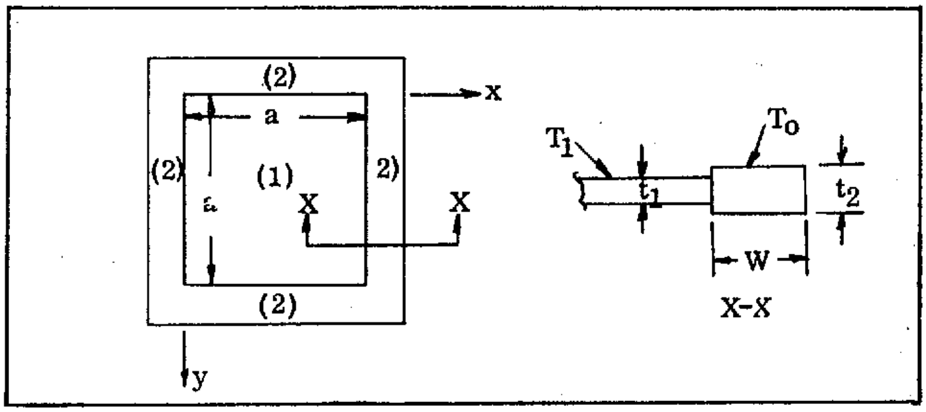


FIGURE 15.12 - PARTIALLY RESTRAINED SQUARE PLATE, UNIFORMLY HEATED

$$\sigma_{x_1} = \sigma_{y_1} = \frac{-E_1\,\alpha_1(T_1-T_0)K}{(1-\mu)} \qquad 15.37$$

$$\sigma_2 = -A_1/A_2\ \sigma_{x_1} = -A_1/A_2\ \sigma_{y_1} \qquad 15.38$$

where: $A_1 = at_1$ and $A_2 = 2wt_2$

$$K = \frac{1 - \dfrac{\alpha_2(T-T_0)}{\alpha_1(T_1-T_0)}}{1 + \dfrac{E_1A_1}{(1-\mu)(E_2A_2)}} \qquad 15.39$$

3) Partial restraint of an uniformly heated rectangular plate.

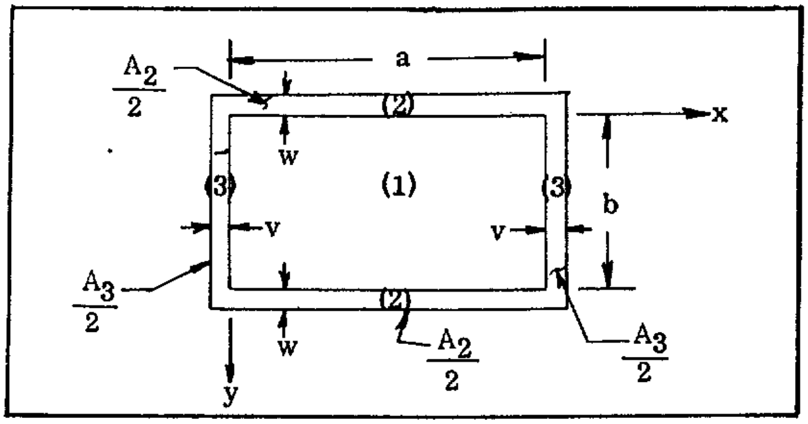


FIGURE 15.13 - PARTIALLY RESTRAINED RECTANGULAR PLATE, UNIFORMLY HEATED

$$\sigma_{x_1} = E_1/(1-\mu^2)\ [\ \epsilon_x + \mu\epsilon_y - (1+\mu)\alpha_1(T_1-T_0)] \qquad 15.40$$

$$\sigma_{y_1} = E_1/(1-\mu^2)\ [\ \epsilon_y + \mu\epsilon_x - (1+\mu)\alpha_1(T_1-T_0)] \qquad 15.41$$

$$\sigma_2 = E_2[\epsilon_x - \alpha_2(T_2-T_0)] \qquad 15.42$$

$$\sigma_3 = E_3\ [\epsilon_y - \alpha_3(T_3-T_0)] \qquad 15.43$$

$$\Sigma F_x = \sigma_{x_1}A_{x_1} + \sigma_2 A_2 = 0 \qquad 15.44$$

$$\Sigma F_y = \sigma_{y_1}A_{y_1} + \sigma_3 A_3 = 0 \qquad 15.45$$

where: $A_{x_1} = bt$

$A_2 = 2wt_2$

$A_{y_1} = at$

$A_3 = 2vt_3$

Equations 15.44 and 15.45 can be solved simultaneously for E_x and E_y. Substituting back into equations 15.40, 15.41, 15.42 and 15.43, the stress at the edge of the members can be obtained.

15.7 Temperature Effects on Joints

Temperature affects the preload in a fastener and also induces loads into the clamped sheets. These effects are obtained using the following equations.

15.7.1 Preload Effects Due to Temperature

Joints with bolts, or other threaded fasteners, which are exposed to a temperature change after installation will have a change in preload. The change can either be an increase or decrease. Figure 15.14 shows a typical joint.

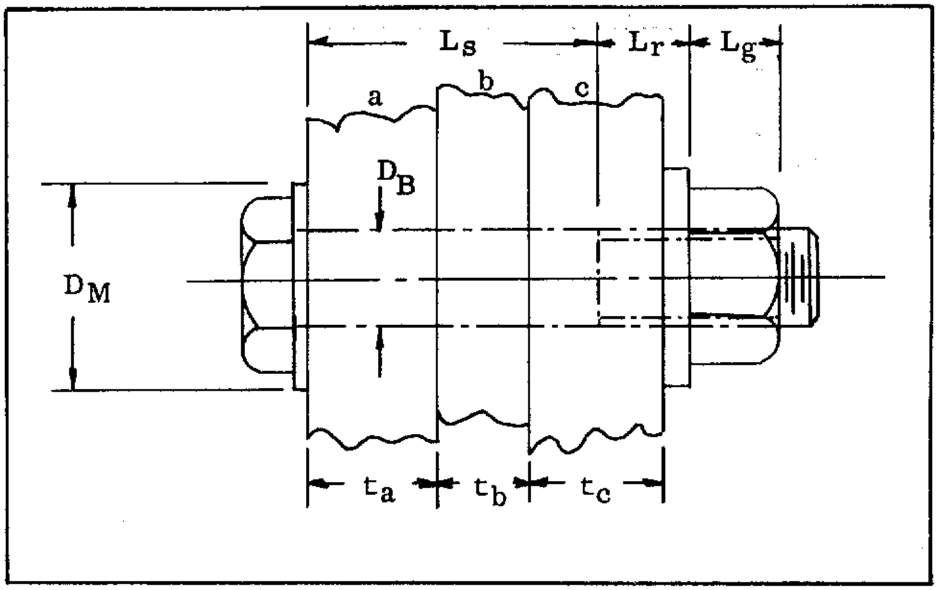


FIGURE 15.14 - JOINT NOMENCLATURE

The following equations assume no washer deformation and no gap exists.

$$e_B = \alpha_B L_B (T-T_0) + \frac{P}{E_B}\left(\frac{L_s}{A_s} + \frac{L_r}{A_r} + \frac{L_g}{2A_r}\right) \quad 15.46$$

$$e_M = (T-T_0)(\alpha_a t_a + \alpha_b t_b + \alpha_c t_c) - P\left(\frac{t_a}{A_a E_a} + \frac{t_b}{A_b E_b} + \frac{t_c}{A_c E_c}\right) \quad 15.47$$

$$e_B = e_M \quad 15.48$$

$$P_T = \frac{(T-T_0)(\alpha_a t_a + \alpha_b t_b + \alpha_c t_c - \alpha_B L_B)}{\frac{1}{E_B}\left(\frac{t_s}{A_s} + \frac{t_r}{A_r} + \frac{t_g}{2A_r}\right) + \left(\frac{t_a}{A_a E_a} + \frac{t_b}{A_b E_b} + \frac{t_c}{A_c E_c}\right)} \quad 15.49$$

P_t will be tension if $T > T_0$ and $\alpha_M > \alpha_b$.

(+) is tension in bolt, (-) would unload a preloaded bolt.

If all three materials are the same, equation 15.49 becomes

$$P_T = \frac{(T-T_0)(\alpha_M t_M - \alpha_B L_B)}{\frac{1}{E_b}\left(\frac{t_s}{A_s} + \frac{t_r}{A_r} + \frac{t_g}{2A_r}\right) + \frac{t_M}{A_M E_M}} \qquad 15.50$$

If a gap exists, no preload in bolt, equations 15.49 and 15.50 become

$$P_T = \frac{-g + (T-T_0)(\alpha_a t_a + \alpha_b t_b + \alpha_c t_c - \alpha_B L_B)}{\frac{1}{E_B}\left(\frac{t_s}{A_s} + \frac{t_r}{A_r} + \frac{t_g}{2A_r}\right) + \left(\frac{t_a}{A_a E_a} + \frac{t_b}{A_b E_b} + \frac{t_c}{A_c E_c}\right)} \qquad 15.51$$

(+) is tension in bolt, (-) would unload a preloaded bolt

$$P_T = \frac{-g(T-T_0)(\alpha_M t_M - \alpha_B L_B)}{\frac{1}{E_B}\left(\frac{t_s}{A_s} + \frac{t_r}{A_r} + \frac{t_g}{2A_r}\right) + \frac{t_M}{A_M E_M}} \qquad 15.52$$

(+) is tension in bolt, (-) would unload a preloaded bolt

where: $A_M = 3\pi D_B^2/4$
- A_r = Area of root of bolt
- A_s = Area of shank of bolt
- D_B = Bolt diameter
- D_M = Effective diameter of material exerting thermal load on bolt; assumed to be 2 D_B
- e_B = Deformation of bolt over length, L_B
- e_M = Deformation of material over thickness, L_M
- g = Gap
- L_B = Effective length of bolt = $L_s + L_r + L_g/2$
- T = Final temperature
- T_o = Initial temperature
- t_M = Total material thickness = $t_a + t_b + t_c$
- E = Modulus of elasticity
- α = Coefficient of thermal expansion

15.7.2 Thermally Induced Loads in Material

When dissimilar materials are subjected to a uniform temperature change from the same initial temperature, T_0, to a final temperature T_1 and T_2, the loads induced into the clamped materials are as follows. Figure 15.15 shows the general arrangement of the joint. The equations assume that the bolts are concentric in the

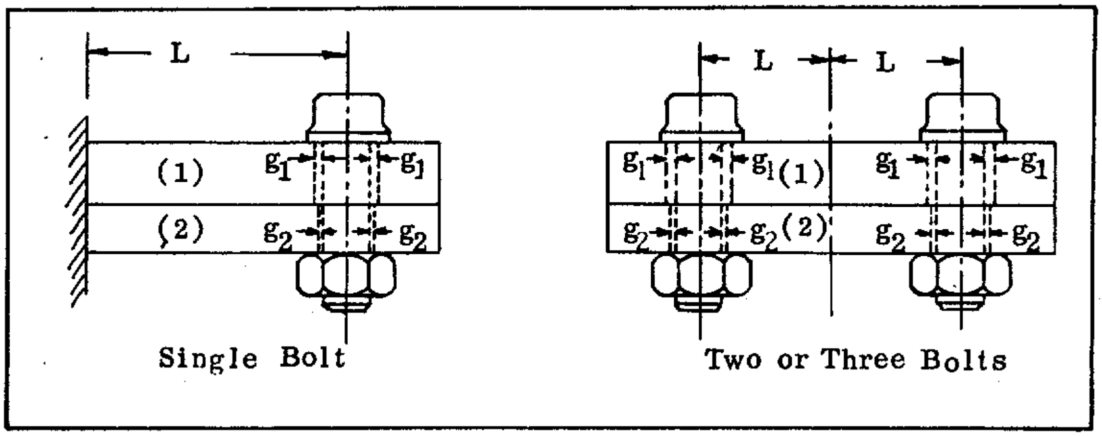


FIGURE 15.15 - JOINTS WITH DISSIMILAR MATERIALS

holes, i.e., the gap is equal all the way around the bolt.

For heating of sheet 1 or cooling of sheet 2;

$$P = \frac{\alpha_1(T-T_0) - \alpha_2(T_2-T_0) - (g_1 + g_2)/L}{1/A_1E_1 + 1/A_2E_2} \qquad 15.53$$

The force P will be compressive in 1 and tensile in 2.

For heating of sheet 2 or cooling of sheet 1;

$$P = \frac{\alpha_2(T_2-T_0) - \alpha_1(T_1-T_0) - (g_1 + g_2)/L}{1/A_1E_1 + 1/A_2E_2} \qquad 15.54$$

The force P will be compressive in 2 and tensile in 1.

If the joint is a single lap joint as shown in Figure 15.16, the equations for

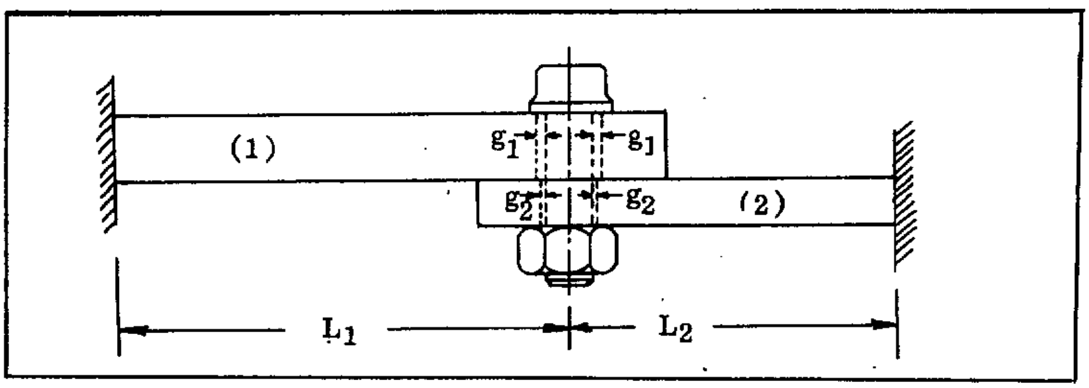


FIGURE 15.16 - SINGLE LAP JOINT

loads in the materials are as follows.

$$P = \frac{\alpha_1 L_1(T_1-T_0) - \alpha_2 L_2(T_2-T_0) + g_1 + g_2}{L_1/A_1E_1 + L_2/A_2E_2} \qquad 15.55$$

For riveted joints, g_1 and g_2 are set equal to zero in equations 15.53, 15.54 and 15.55.

15.8 Thermal Buckling

Thermally induced strains can induce buckling in beams and plates. It is assumed that this buckling occurs in the elastic range. The following general equation can be used to determine the temperature differential which would initiate buckling of a column.

$$[\alpha(T-T_0)]_{CR} = \frac{C\,\pi^2\,\rho^2}{L^2 C_1} \qquad 15.56$$

where: $T-T_0$ = Temperature change
C = Column fixity coefficient
$C_1 = C_2/(C_2 + AE/L)$
C_2 = Stiffness of restraining structure

If the structure being heated is a flat plate with uniform heating the temperature differential which would initiate buckling is

Fully restrained in one direction:

$$(T_1-T_0)_{CR} = \frac{K\,\pi^2}{12(1+\mu)\alpha}\left(\frac{t}{b}\right)^2 \qquad 15.57$$

where K is the non-dimensional buckling constant shown in Section 10.

$$(T_1-T_0)_{CR} = \frac{\pi^2}{12(1-\mu^2)}\left(\frac{t}{b}\right)^2 \left|\frac{(1-\mu)}{\alpha}(1 + b^2/a^2)\right| \qquad 15.58$$

Fully restrained in two directions, clamped edges:

$$(T_1-T_0)_{CR} = \frac{\pi^2}{12(1-\mu^2)}\left(\frac{t}{b}\right)^2 \frac{4(1-\mu)}{3\,\alpha\left(1 + \frac{a^2}{b^2}\right)}\left(\frac{3b^2}{a^2} + \frac{3a^2}{b^2} + 2\right) \qquad 15.59$$

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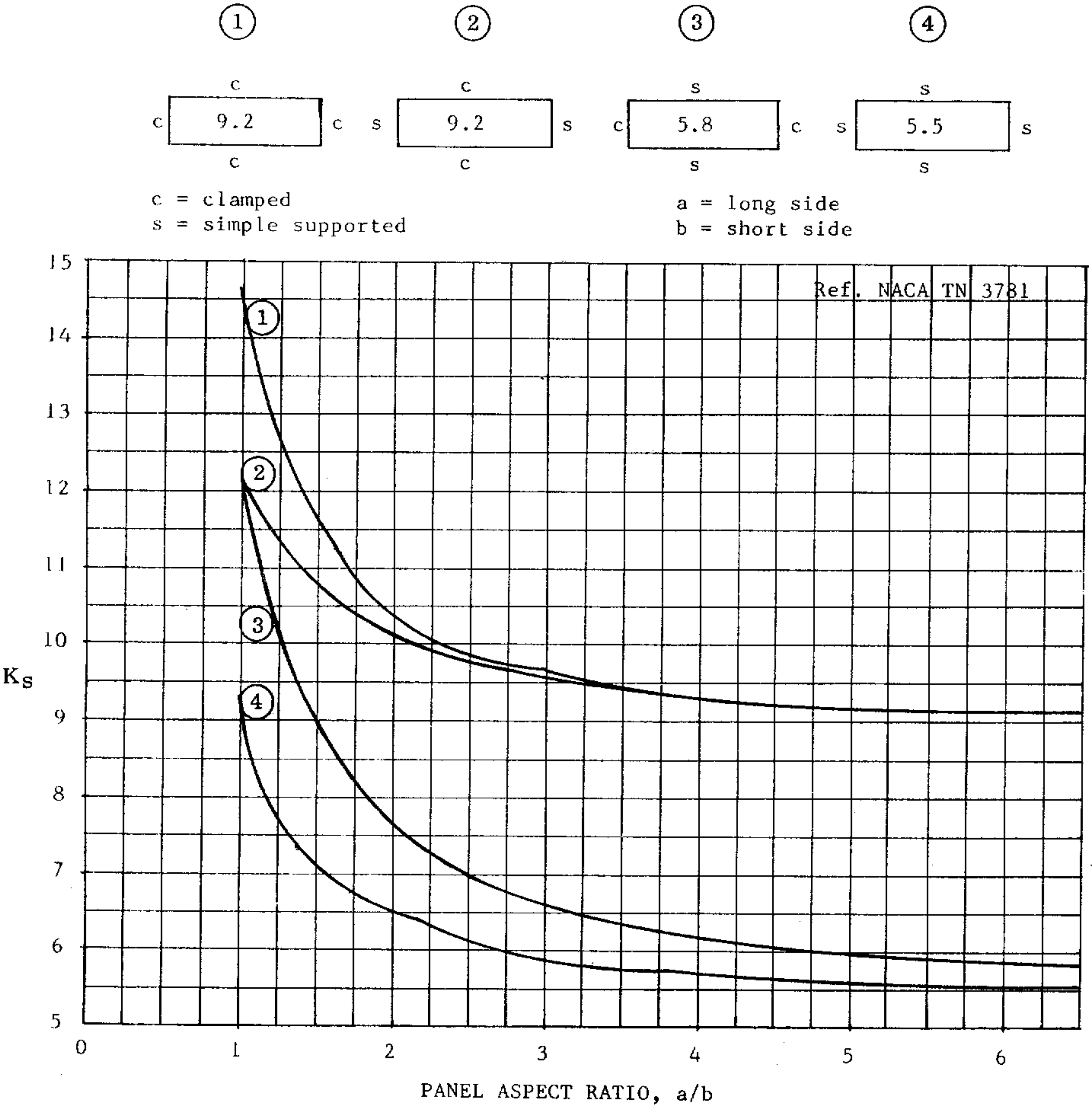

FIGURE 10.18 - SHEAR BUCKLING COEFFICIENT, K_s

Revision A

10.4 SHEAR BUCKLING

The critical shear stress at which a plate first buckles is given by the equation:

$$\tau_{cr} = \frac{K_s \pi^2 \eta E}{12(1-\mu^2)} \left(\frac{t}{b}\right)^2$$

where K_s (Fig. 10.18) is the non-dimensional shear buckling coefficient and is a function of the plate geometry and edge restraints. The values of K_s and μ are always the elastic values since the plasticity correction factor, η, contains all changes in those terms resulting from inelastic behavior. The term b is the smaller dimension of the panel.

A great deal of work has been done relative to the value of the plasticity correction factor. The expression for η must involve a measure of the stiffness of the material in the elastic and inelastic ranges. A simple means of obtaining a value of η is to take the ratio of the shear secant modulus to the shear modulus.

$$\eta = \frac{G_s}{G} = \frac{\text{shear secant modulus}}{\text{shear modulus}}$$

10.4.1 CRITICAL BUCKLING STRESS WITH AXIAL LOADS

When axial loads are present the actual shear buckling stress defined in paragraph 10.4 will be different. The presence of compressive stresses together with shear stresses causes the panel to buckle at a lower value of shear than if no compression were present. Tension causes the panel to buckle at a higher shear stress.

When shear and compression are present the panel buckles according to the interaction

$$f_c/F_{c_{cr}} + (f_s/F_{s_{cr}})^2 = 1.0$$

where $F_{c_{cr}}$ and $F_{s_{cr}}$ are the critical panel buckling stresses for pure compression and pure shear. From chapter 7, section 7.3 the buckling stress for a panel under compression is

$$F_{c_{cr}} = \frac{\pi^2 \eta k_c E}{12(1 - \mu^2)} \left(\frac{t}{b}\right)^2$$

For any particular panel

$$F_{c_{cr}}/F_{s_{cr}} = A, \text{ (a constant)}$$

From conventional means the applied compressive stress, f_c, and the applied shear stress, f_s, can be calculated. These stresses will have a constant relationship with each other until the panel buckles, after which the compressive stress no longer increases. Thus

The shear stresses are calculated from the shear flow equation:

$$f_s = \frac{V}{(h_c)(t)} = q/t = f_{s_{DT}} + f_{s_s}$$

$$f_{s_{DT}} = (k)(f_s); \; f_{s_s} = (1-k)(f_s)$$

As the load increases beyond the initial buckling load, a higher percentage of the total shear is carried by tension field. This causes the ratio $f_s/f_{s_{cr}}$ to become an important parameter.

Methods of analysis for three specific types of tension field beams are given:

1. Flat tension field beams with single uprights.
2. Flat tension field beams with single uprights and access holes.
3. Curved tension field beams.

The curves given for use in these analyses yield results with a reasonable assurance of conservative strength predictions, provided that normal design practices and proportions are used.

10.5.1 Effective Area of the Uprights

In order to make the design curves apply to both single and double uprights, it is necessary to define an effective upright area A_{ue}.

For double uprights, which are symmetric with respect to the web:

$A_{ue} = A_u$ = total cross-sectional area of the uprights.

For single uprights:

$$A_{ue} = \frac{A_u}{1 + \left(\frac{e}{\rho}\right)^2}$$

where ρ = radius of gyration of the stiffener and
e = distance from the centroid of the stiffener to the center of the web.

If the upright has a very deep web, A_{ue} should be taken to be the sum of the cross-sectional area of the attached leg and an area 12 t_u, where t_u is the upright thickness.

10.5.2 Moment of Inertia of the Uprights

The uprights must have a sufficient moment of inertia to prevent buckling of the web system as a whole before formation of the tension field, in addition to preventing column failure due to the loads imposed upon the upright by the tension field. Forced crippling failure, caused by the waves of the buckled web and possibly most critical, must also be prevented by the upright. The required moment of inertia of the upright may be determined by iterating through the appropriate Table 10.1, 10.2, 10.3, or 10.4.

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10.5.3 Effective Column Length

The effective column (upright) length is calculated by the equations:

If $d_c < 1.5\ h_u$, $$L_e = \frac{h}{\sqrt{1 + k^2\ (3 - \frac{2d}{h})}}$$

If $d_c > 1.5\ h_u$, $$L_e = h$$

10.5.4 Discussion of the End Panel of a Beam

The following analyses are concerned with the "interior" bays of a beam. The uprights in these areas are subjected, primarily, only to axial compressive loads. The end panel, however, is a special case. Since the diagonal tension effect results in an inward pull on the end upright, bending, in addition to the usual compressive axial load, is also produced. There are three general ways of dealing with the edge member subjected to bending.

1. Sufficiently strengthen the edge member so it can carry all of the loads (which is inefficient, weight-wise, for long unsupported lengths).

2. Increase the thickness of the end panel either to make it non-buckling or to reduce k, which would reduce the running load producing bending in the edge member. (This is usually inefficient for large panels.)

3. Additional uprights may be provided to support the edge member and thus reduce its bending moment.

10.5.5 Analysis of a Flat Tension Field Beam with Single Uprights

Table 10.1 is a step-by-step procedure which yields the stresses in the flanges, webs, rivets, and uprights of a flat tension field beam with single uprights (Figure 10.19).

Table 10.1 is based on a single web with parallel flanges and parallel uprights. Most beams consist of more than one web. At various locations in the following table adjacent panels must be considered. Such a situation occurs for rivet load, stringer axial stress, upright stress and moment in stringer.

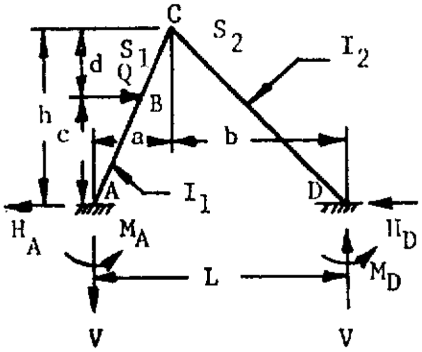
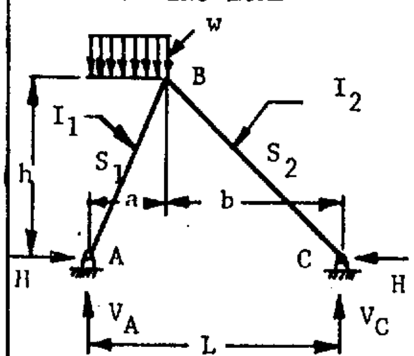
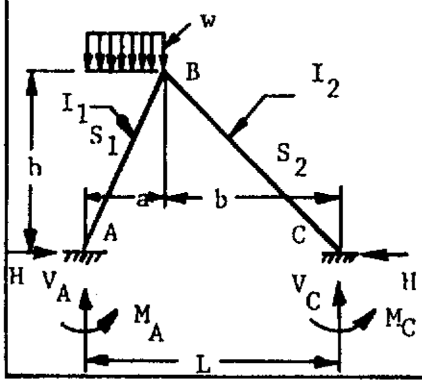
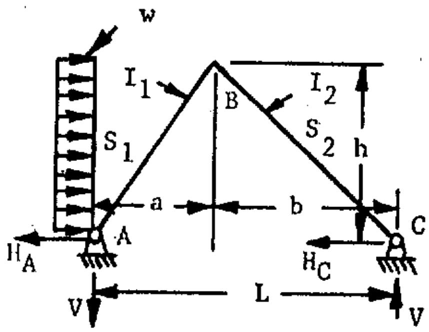
4. HORIZ. CONCENTRATED LOAD 	$V = \frac{Qc}{L}\left[1 - \frac{d(h+d)}{2h^2}\right]$; $H_A = Q - H_D$ $H_D = \frac{Qc}{Lh}\left\{b + \frac{d}{6h^2(K+1)}\left[(h+d)(-b[3K+4] - 2L) + 2(2L+b)(h+e) + 3ac\right]\right\}$ $M_A = \frac{Qcd}{6h^2(K+1)}\left[(h+d)(3K+4) - 2(h+c)\right]$ $M_D = \frac{Qcd}{6h^2(K+1)} \quad (h + 2c + d)$
5. VERTICAL UNIFORM RUNNING LOAD 	$V_A = wa\left[1 - \frac{a}{2L}\right]$ $V_C = \frac{wa^2}{2L}$ $H = \frac{wa^2}{8h}\left[\frac{4b}{L} + \frac{1}{1 + K}\right]$
6. VERTICAL UNIFORM RUNNING LOAD 	$V_A = wa\left[1 - \frac{3a}{8L}\right]$: $V_C = \frac{3wa^2}{8L}$ $H = \frac{wa^2}{24Lh(K + 1)}\left[b(10 + 9K) + 2L + a\right]$ $M_A = \frac{wa^2(3K + 2)}{24(K + 1)}$ $M_c = \frac{wa^2}{24(K + 1)}$

TABLE 12.4 (CONT'D) - REACTIONS AND CONSTRAINING MOMENTS IN TRIANGULAR FRAMES

7. HORIZ. UNIFORM RUNNING LOAD

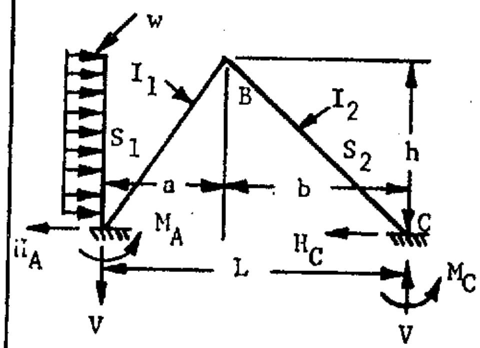


$$V = \frac{wh^2}{2L}$$

$$H_A = wh - H_c$$

$$H_C = \frac{wh}{8}\left[\frac{4b}{L} + \frac{1}{K+1}\right]$$

8. HORIZ. UNIFORM RUNNING LOAD



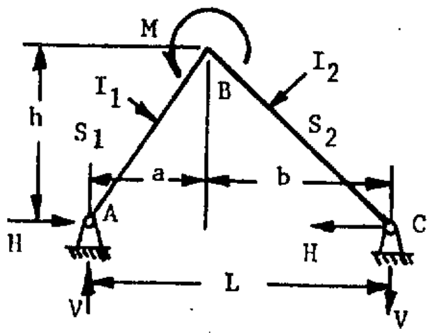
$$V = \frac{3wh^2}{8L}$$

$$M_A = wh - M_C$$

$$M_C = \frac{wh}{8L(K+1)}\ [\,b(3K+4) + a\,]$$

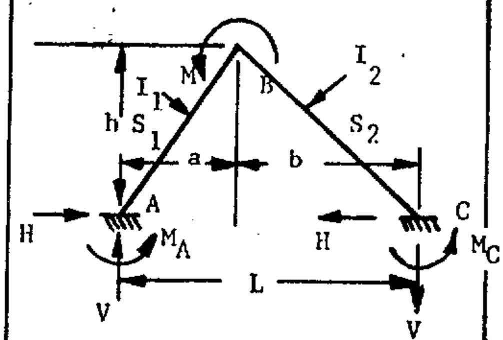
$$M_A = \frac{wh^2(3K+2)}{24(K+1)} \qquad M_C = \frac{wh^2}{24(K+1)}$$

9. APPLIED MOMENT AT APEX



$$V = \frac{M}{L}$$

$$H = \frac{M}{hL}\left[\frac{a - bK}{K+1}\right]$$



$$V = \frac{3M}{2L}$$

$$H = \frac{3M(a - bK)}{2hL(K+1)}$$

$$M_A = \frac{KM}{2(K+1)}$$

$$M_C = \frac{M}{2(K+1)}$$

TABLE 12.4 (CONT'D) - REACTIONS AND CONSTRAINING MOMENTS IN TRIANGULAR FRAMES

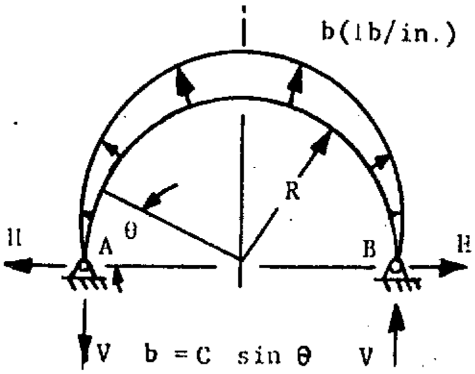
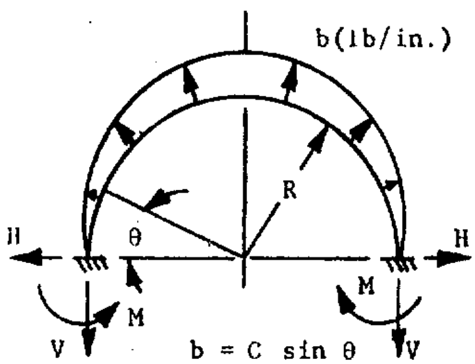
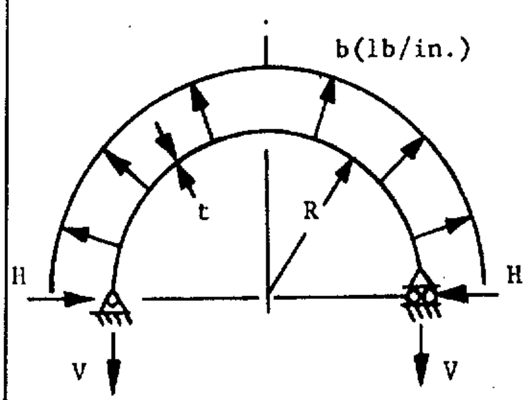
1. Sinusoidal Normal Pressure  	$V = \frac{C\pi R}{4}$ $H = \frac{CR}{4}$ $M_\theta = \frac{CR^2}{4}\left[(\pi - 2\theta)\cos\theta - \pi + 3\sin\theta\right]$ (Positive moment acts clockwise on section ahead.)
2. Sinusoidal Normal Pressure  	$V = \frac{C\pi R}{4}$ $H = \frac{CR}{4}\left[\frac{3\pi^2 - 32}{8 - \pi^2}\right] = .31974CR$ $M = \frac{CR^2}{4}\left[\frac{\pi^3 - 10\pi}{8 - \pi^2}\right] = .05478CR^2$ $M_\theta = CR^2\left[.81974\sin\theta - .84018 + \frac{\cos\theta}{2}\left(\frac{\pi}{2} - \theta\right)\right]$ (Positive moment acts clockwise on section ahead.)
3. Uniform Normal Pressure  	M = 0 at all points since pin points permit a uniform hoop tension. T, where: $T = V = bR$ $H = 0$

TABLE 12.5 - REACTIONS AND CONSTRAINING MOMENTS IN SEMICIRCULAR FRAMES OR ARCHES

12.4 Analysis of Rings

Tables and figures are presented for the analysis of rings and ring-supported shells. Sections 12.4.1 and 12.4.2 show analysis methods for rings which are rigid with respect to the resisting structure for out-of-plane loads. The plane of the ring remains plane and the supporting structure deforms.

Only bending is considered in the deflection curves for the in-plane load cases given in Figures 12.4 through 12.29. Refer to Figures 12.30 through 12.33 to include the effects of shear and normal forces.

Section 12.4.3 shows methods of analysis for circular cylindrical shells supported by "flexible" rings.

12.4.1 Analysis of Rigid Rings with In-Plane Loading

Coefficients to obtain slope, deflection, bending moment, shear, and axial force along with equations for these values are given for some of the frequently-used load cases. Figure 12.4 shows an index for the various load cases presented in Figures 12.5 through 12.29.

The sign convention used throughout the rigid frame analysis in-plane load cases is shown in Figure 12.3. It basically consists of: moments which produce tension

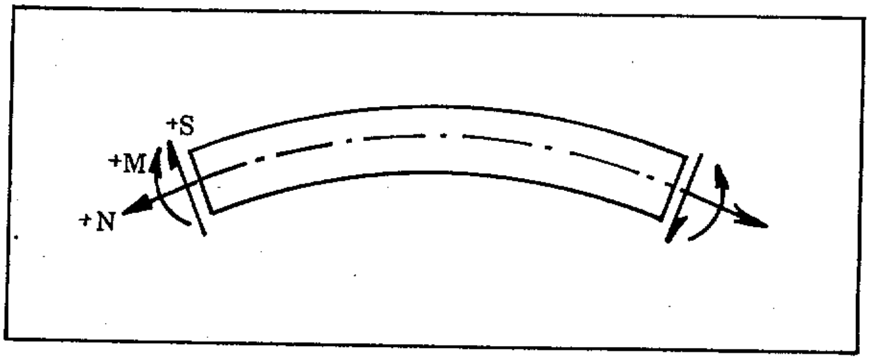


FIGURE 12.3 - SIGN CONVENTION FOR RIGID RINGS WITH IN-PLANE LOADS

on the inner fibers are positive, transverse forces which act upward to the left of the cut are positive and axial forces which produce tension in the frame are positive.

Deflections in Figures 12.5 through 12.29 are based on bending only. Deflection curves for the three basic load cases due to shear and concentrated loads are shown in Figures 12.31 through 12.33. A shape factor (β) that is to be used with the curves for shear deflection of various cross sections is shown in Figure 12.30.

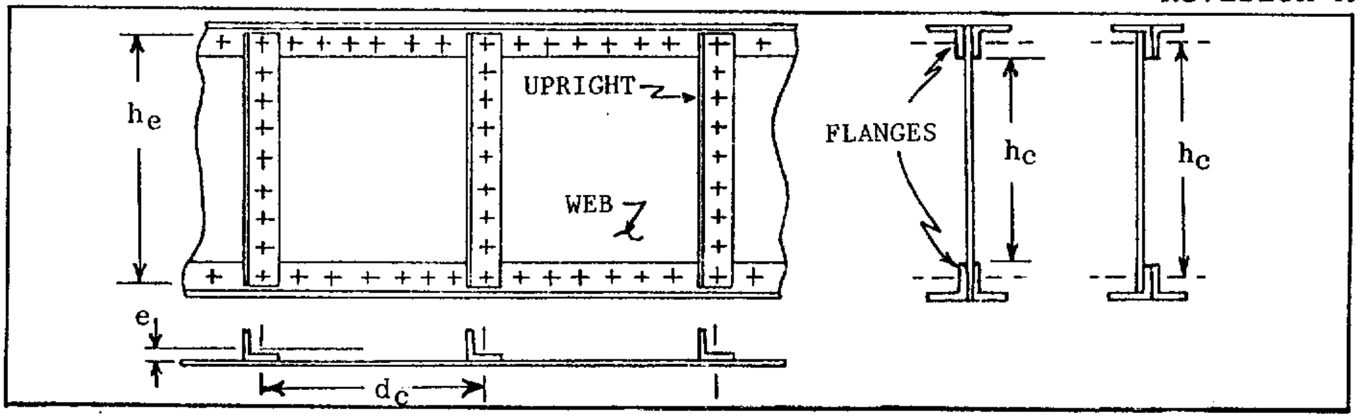


FIGURE 10.19 - FLAT TENSION FIELD BEAM WITH SINGLE UPRIGHTS

Description	Variable and Equation	Numerical Value
(1) Elastic modulus	E_c	
(2) Upright spacing, (NA to NA)	d	
(3) Clear web between uprights (rivet to rivet)	d_c	
(4) Distance from median plane of web to centroid of upright	e	
(5) Clear web between flanges (rivet to rivet)	h_c	
(6) Distance between flange centroids	h_e	
(7) Length of upright between upright to flange rivets	h_u	
(8) Web thickness	t	
(9) Upright thickness	t_u	
(10) Flange thickness	t_f	
(11) Upright area	A_u	
(12) Flange area	A_f	
(13) Radius of gyration of upright	ρ	
(14) Moment of inertia of upright	I_u	
(15) Moment of inertia of flange	I_F	
(16) Applied load - upright	P_u	
(17) Applied load - flange	P_f	
(18) Applied web shear flow	q	
(19) Web shear stress	$\tau = q/t =$ (18) / (8)	

TABLE 10.1 - ANALYSIS OF FLAT TENSION FIELD BEAM WITH SINGLE UPRIGHT

⑳ Effective area of upright	A_{ue} = ⑪ / 1+(④^2/ ⑬^2)	
㉑ Parameter	$A_{ue}/d_c t$ = ⑳ / ③⑧	
㉒ Parameter	$h_e t$ = ⑥⑧	
㉓ Parameter	d_c/h_u = ③ / ⑦	
㉔ Parameter	t_f/t = ⑩ / ⑧	
㉕ Parameter	t_u/t = ⑨ / ⑧	
㉖ Parameter	h_c/d_c = ⑤ / ③	
㉗ Parameter	d_c/h_c = 1 / ㉖	
㉘ Parameter	t/d_c = ⑧ / ③	
㉙ Parameter	t/h_c = ⑧ / ⑤	
㉚ Upright restraint coefficient	R_h, Figure 10.20(b)	
㉛ Flange restraint coefficient	R_d, Figure 10.20(b)	
㉜ Theoretical buckling coefficient	k_{ss}, Figure 10.20(a)	
㉝ Elastic buckling stress: $d_c < h_c$	τ_{cr_e} = ㉜①㉘^2 |㉚ + ½(㉛ - ㉚) ㉗^3|	
$d_c > h_c$	τ_{cr_e} = ㉜①㉙^2 |㉛ + ½(㉚ - ㉛) ㉖^3|	
㉞ Initial buckling stress	τ_{cr}, Figure 10.21 (See Note 2)	
㉟ Stress ratio	τ/τ_{cr} = ⑲ / ㉞	
㊱ Diagonal tension factor	k, Figure 10.22 @ $300td_c/12h_c$ = 0	
㊲ Parameter	$\frac{A_{ue}}{d_c t} + \frac{1}{2}(1-k)$ = ㉑ + (1- ㊱)/2	
㊳ Ratio of upright stresses	$\sigma_{u_{max}}/\sigma_u$, Figure 10.23	
㊴ Ratio of upright to shear stresses	σ_u/τ, Figure 10.24	
㊵ Diagonal tension angle	Tanα, Figure 10.25(a)	
㊶ Stress in median plane upright/web	σ_u = - ㊱ ⑲ ㊵ / ㊲	
㊷ Upright average stress	$\sigma_{u_{avg}}$ = ㊶ ⑳ / ⑪	
㊸ Upright maximum stress	$\sigma_{u_{max}}$ = ㊶ ㊲	
㊹ Effective column length: If ㉓ < 1.5	L_e = ⑦ / |1+ ㊱^2(3-2 ㉓)|$^{1/2}$	
If ㉓ > 1.5	$L_e = h_u$ = ⑦	
㊺ Slenderness ratio	$L_e/2\rho$ = ㊹ /2 ⑬	
㊻ Column allowable	$\sigma_{co} = \pi^2$ ① / ㊺^2 or Section 11	
㊼ Proportional limit	F_{pl}, Section 5	
㊽ Strain, if ㊶ > ㊼	σ_u/E = ㊶ / ①	

TABLE 10.1 (cont'd) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH SINGLE UPRIGHT

(49) From stress strain curve	F_c, use (48) to determine all.	
(50) Margin of Safety: column yield (41) > (47) (41) < (47)	 MS = (49) / (41) - 1 MS = (47) / (41) - 1	
(51) MS - Column	MS = (46) / (42) -1	
(52) Parameter	$k^{2/3}(t_u/t)^{1/3}$ = (36) $^{2/3}$ (25) $^{1/3}$	
(53) Upright allowable (forced crippling)	σ_o, Figure 10.26	
(54) MS - Forced crippling	MS = (53) / (43) -1	
(55) Parameter	wd_c = .7 (3) ((8) /2 (15) (6)) $^{1/4}$	
(56) Parameter	C_1, Figure 10.27	
(57) Parameter	C_2, Figure 10.28	
(58) Maximum web stress	τ'_{max} = (19) (1 + (36) ^{2} (56))(1 + (36) (57))	
(59) Web allowable	τ^*_{all}, Figure 10.29 @ α_{PDT} = 45°	
(60) MS - Web	MS = (59) / (58) -1	
(61) Parameter	C_3, Figure 10.28	
(62) Secondary bending in flange	M_{SB} = (1/12)((36) (61) (18) (3) ^{2} (40))	
(63) Distance from NA to extreme fiber of flange	C_f	
(64) Distance - NA to near fiber of flange	D_f	
(65) Flange applied stress	σ_a = (17) / (12)	
(66) Diagonal tension stress-flange (comp)	σ_{DT} = -((36) (19) / (40)) / [2 (12) / (22) + .5(1 - (36))]	
(67) Secondary bending stress-flange (comp)	σ_{SB} = - (62) (63) / (15)	
(68) Secondary bending stress-flange (tension)	σ_{SB} = (67) (64) / (63)	
(69) Flange stress-inside fiber	σ_{tot} = (65) + (66) + (67)	
(70) Flange stress-extreme fiber	σ_{tot} = (65) + (66) + (68)	
(71) Allowable crippling stress-flange	F_{cc}	
(72) Allowable tension stress-flange	F_{tu} or F_{ty}	
(73) MS - Flange (tension)	MS = (72) / (70) -1	
(74) MS - Flange (compression)	MS = (71) / (69) -1	
(75) Rivet factor	R = 1 + 0.414 (36)	
(76) Rivet load-web to flange	R" = qR = (18) (75)	

TABLE 10.1 (cont'd) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH SINGLE UPRIGHT

(77) Allowable rivet shear load	P_{af}		
(78) MS - Flange rivets	MS = (77) / (76) -1		
(79) Rivet load-upright to flange	P_u = (41) (20)		
(80) Allowable rivet load-upright	P_{au}		
(81) MS - Upright rivets	MS = (80) / (79) -1		
(82) Interrivet buckling allowable	F_{ir}, Section 10.6		
(83) MS - Interrivet buckling	MS = (82) / (43) -1		
(84) Ultimate tensile stress of web	F_{tu}, Section 5		
(85) Rivet tensile stress-upright/web	σ_R = .22(8)(84)		
(86) Rivet allowable tensile stress	F_{RT}, Section 6		
(87) MS - Rivet tension	MS = (86) / (85) -1		

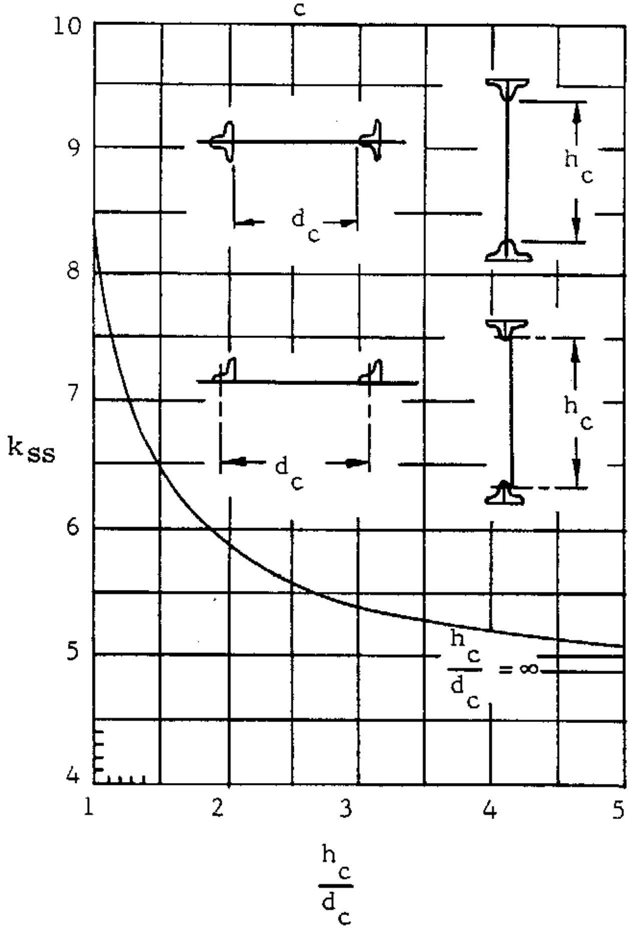
NOTES:

(1) If any of the margins of safety are less than zero, the design is inadequate. The deficient area must be corrected and this table repeated.

(2) If the web is subjected to tension or compression as well as shear, the initial buckling stress of the web must be modified according to the method described in Section 10.4.1.

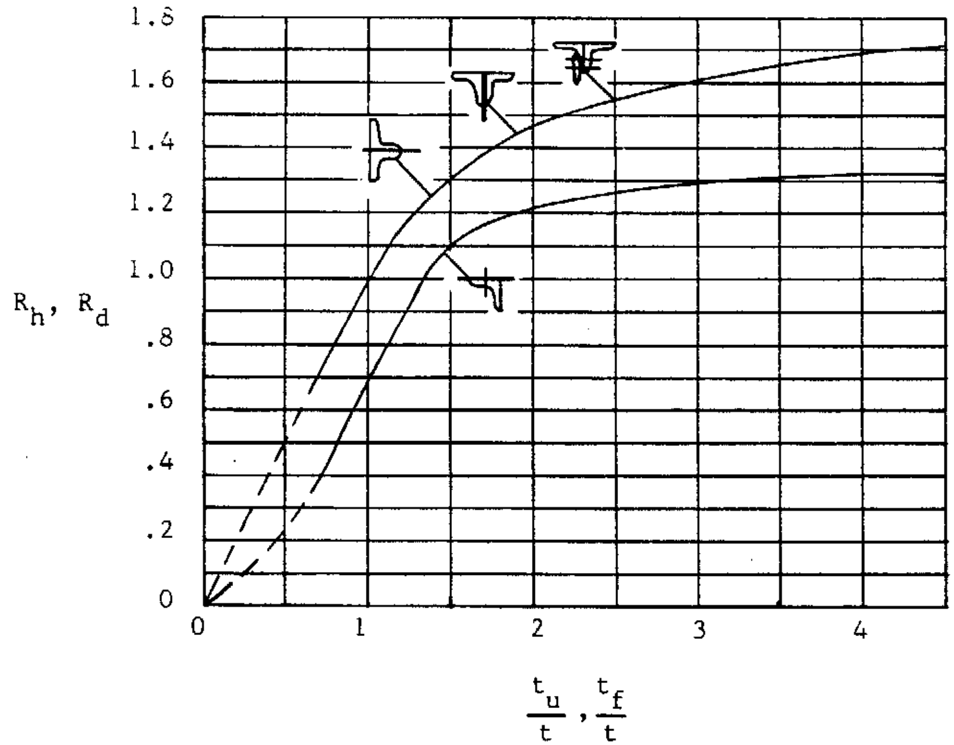
TABLE 10.1 (cont'd) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH SINGLE UPRIGHT

THIS CURVE IS SHOWN FOR $d_c < h_c$.
IF $d_c > h_c$ THE ABSCISSA SHOULD BE READ AS $\frac{d_c}{h_c}$



(A) THEORETICAL COEFFICIENTS FOR PLATES WITH SIMPLY SUPPORTED EDGES.

R_h = UPRIGHT COEFFICIENT
R_d = FLANGE COEFFICIENT



(B) EMPIRICAL RESTRAINT COEFFICIENTS

FIGURE 10.20 - GRAPHS FOR CALCULATING BUCKLING STRESS OF WEBS

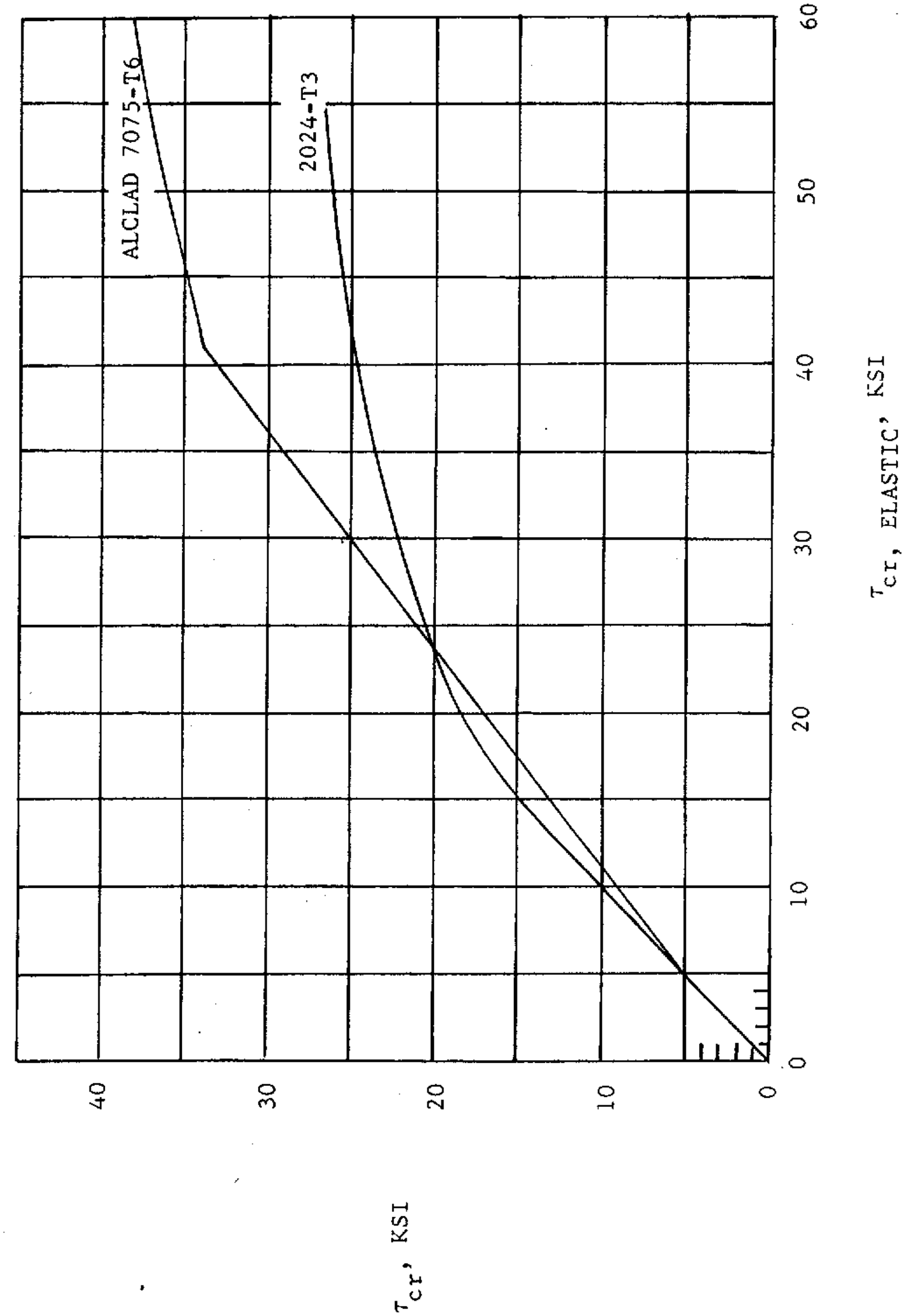


FIGURE 10.21 - GRAPHS FOR CALCULATING BUCKLING STRESS OF WEBS

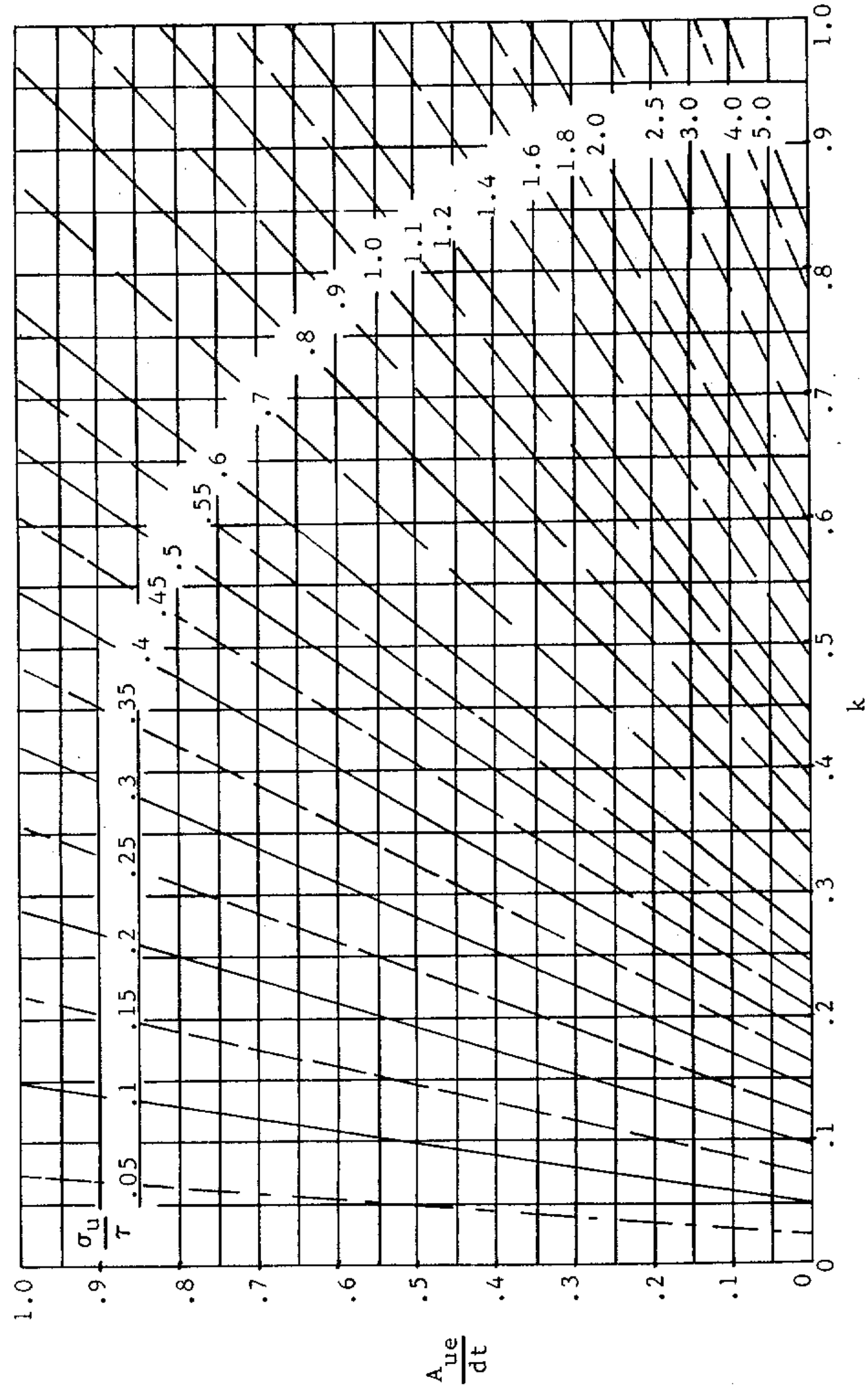
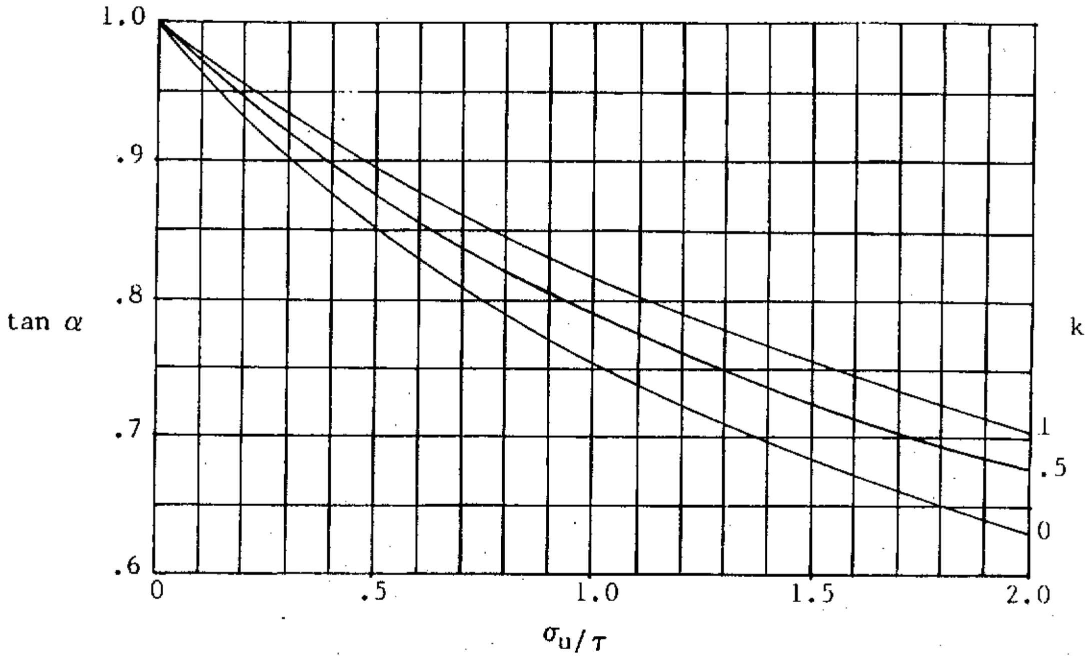
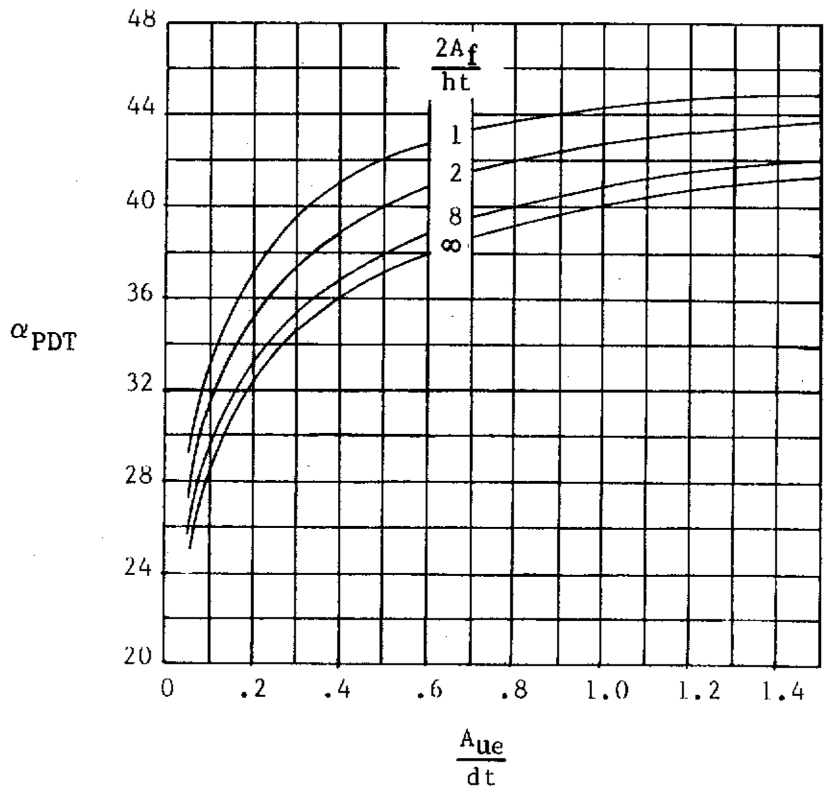


FIGURE 10.24 - DIAGONAL TENSION ANALYSIS



(a) INCOMPLETE DIAGONAL TENSION



(b) PURE DIAGONAL TENSION

FIGURE 10.25 - ANGLE OF DIAGONAL TENSION

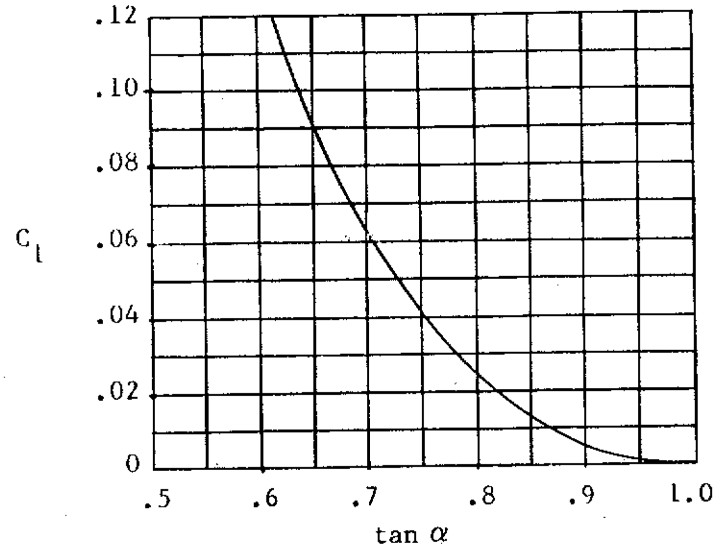


FIGURE 10.27 - ANGLE FACTOR C_1

$$wd = 0.7d \left(\frac{t}{(I_C + I_T) h e} \right)^{1/4}$$

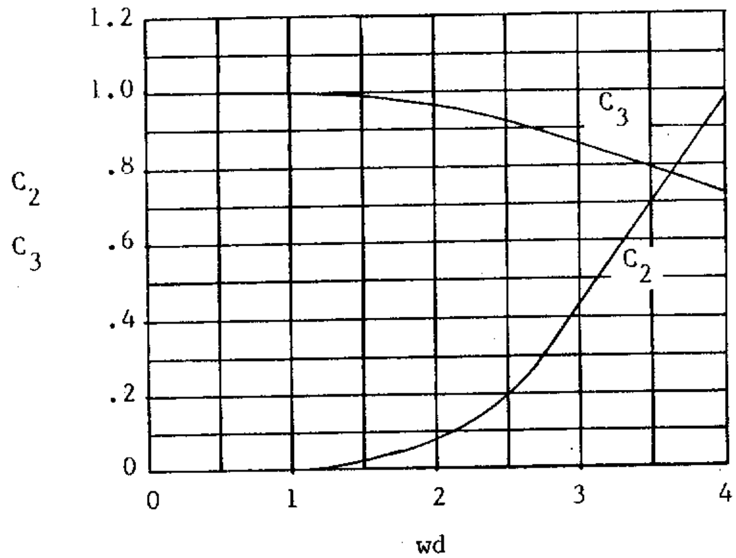
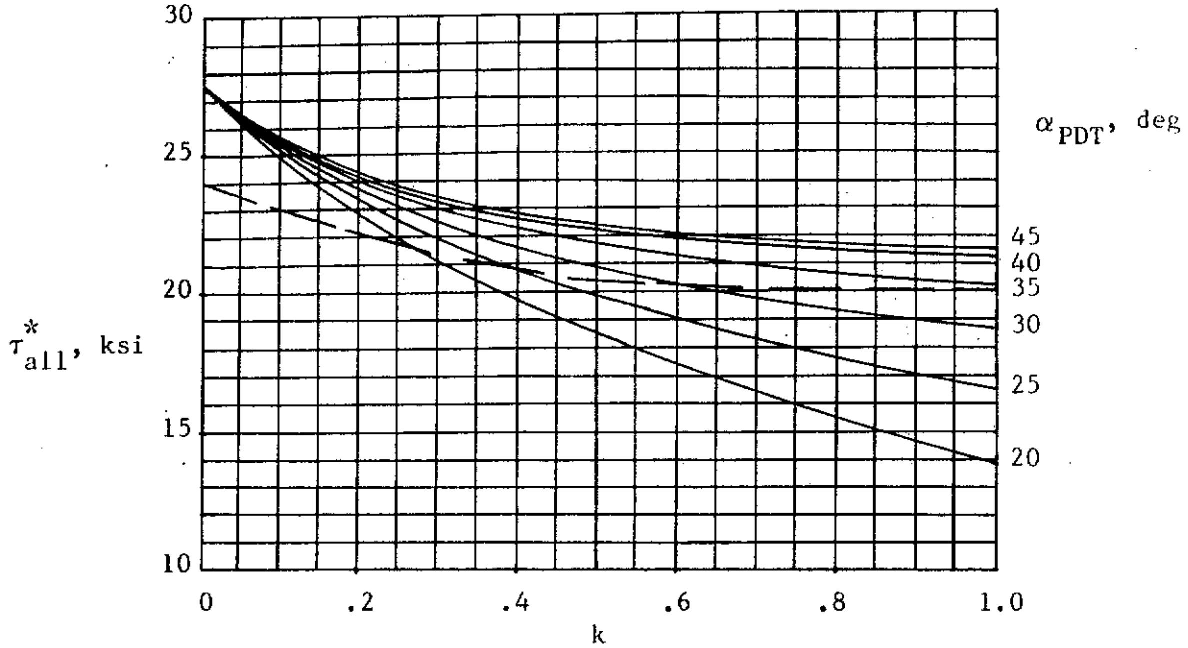
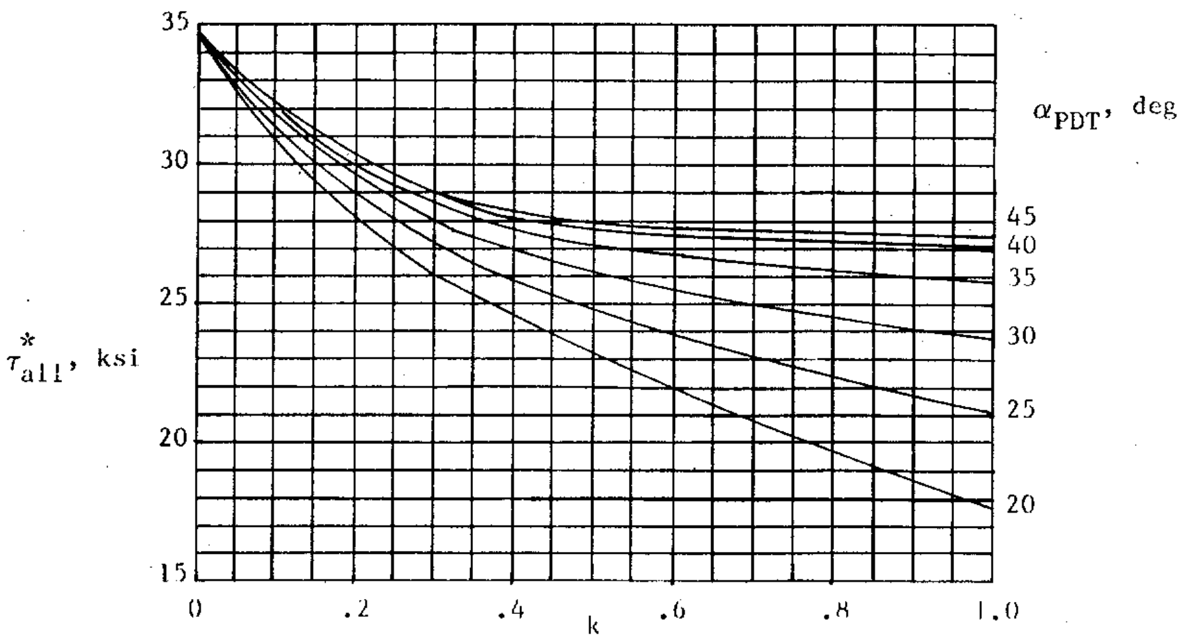


FIGURE 10.28 - STRESS CONCENTRATION FACTORS, C_2 AND C_3



2024 ALUMINUM ALLOY. F_{tu} = 62000 psi

DASHED LINE IS ALLOWABLE YIELD STRESS



7075 ALUMINUM ALLOY. F_{tu} = 72000 psi

FIGURE 10.29 - BASIC ALLOWABLE VALUES OF τ_{MAX}

(20) EFFECTIVE AREA OF UPRIGHT	$A_{ue} = A_u$	
(21) PARAMETER	$A_{ue}/d_c t =$ (20) / (3)(8)	
(22) PARAMETER	$h_e t =$ (6)(8)	
(23) PARAMETER	$d_c/h_u =$ (3) / (7)	
(24) PARAMETER	$t_f/t =$ (10) / (8)	
(25) PARAMETER	$t_u/t =$ (9) / (8)	
(26) PARAMETER	$h_c/d_c =$ (5) / (3)	
(27) PARAMETER	$d_c/h_c = 1$ / (26)	
(28) PARAMETER	$t/d_c =$ (8) / (3)	
(29) PARAMETER	$t/h_c =$ (8) / (5)	
(30) UPRIGHT RESTRAINT COEFFICIENT	R_h, FIGURE 10.20 (b)	
(31) FLANGE RESTRAINT COEFFICIENT	R_d, FIGURE 10.20 (b)	
(32) THEORETICAL BUCKLING COEFFICIENT	k_{ss}, FIGURE 10.20 (a)	
(33) ELASTIC BUCKLING STRESS: $d_c < h_c$	$\tau_{cr_e} = (32)(1)(28)^2 \left[(30) + \frac{1}{2}\left((31) - (30)\right)(27)^3\right]$	
$d_c > h_c$	$\tau_{cr_e} = (32)(1)(29)^2 \left[(30) + \frac{1}{2}\left((30) - (31)\right)(26)^3\right]$	
(34) INITIAL BUCKLING STRESS	τ_{cr}, FIGURE 10.21 (See Note 2)	
(35) STRESS RATIO	$\tau/\tau_{cr} =$ (19) / (34)	
(36) DIAGONAL TENSION FACTOR	k, FIGURE 10.22 @ $300td_c/12h_c = 0$	
(37) PARAMETER	$\frac{A_{ue}}{d_c t} + \frac{1}{2}(1-k) =$ (21) + $\frac{(1-(36))}{2}$	
(38) RATIO OF UPRIGHT STRESSES	σ_{uMAX}/σ_u, FIGURE 10.23	
(39) RATIO OF UPRIGHT TO SHEAR STRESSES	σ_u/τ, FIGURE 10.24	
(40) DIAGONAL TENSION ANGLE	$\tan\alpha$, FIGURE 10.25 (a)	
(41) STRESS IN MEDIAN PLANE UPRIGHT/WEB	$\sigma_u = -$ (36)(19)(40) / (37)	
(42) UPRIGHT AVERAGE STRESS	$\sigma_{uAVG} =$ (41)(20) / (11)	
(43) UPRIGHT MAXIMUM STRESS	$\sigma_{uMAX} =$ (41)(37)	
(44) EFFECTIVE COLUMN LENGTH: IF (23) < 1.5	$L_e = (7) / \left[1 + (36)^2(3 - 2(23))\right]^{\frac{1}{2}}$	
(23) > 1.5	$L_e = h_u =$ (7)	

TABLE 10.2,(CONT'D) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH DOUBLE UPRIGHTS

(45)	SLENDERNESS RATIO	L_e/ρ = (44) / (13)	
(46)	COLUMN ALLOWABLE	$\sigma_{co} = \pi^2$ (1) / (45)^2 or SECTION 11	
(47)	PROPORTIONAL LIMIT	F_{pl}, SECTION 5	
(48)	STRAIN, IF (41) > (47)	σ_u/E = (41) / (1)	
(49)	FROM STRESS STRAIN CURVE	F_c, USE (48) TO DETERMINE ALLOWABLE	
(50)	MARGIN OF SAFETY: (41) > (47) COLUMN YIELD (41) < (47)	MS = (49) / (41) - 1 MS = (47) / (41) - 1	
(51)	MS - COLUMN	MS = (46) / (42) - 1	
(52)	PARAMETER	$k^{2/3}(t_u/t)^{1/3}$ = (36)$^{2/3}$ (25)$^{1/3}$	
(53)	UPRIGHT ALLOWABLE (FORCED CRIPPLING)	σ_o, FIGURE 10.26	
(54)	PLASTICITY CORRECTION: IF (53) > (47)	σ_o, = (E_{SEC}/(1)) (53)	
(55)	MS - FORCED CRIPPLING	MS = (54) / (43) - 1	
(56)	PARAMETER	Wd_c = .7 (3) $\left\|(8)/2(15)(6)\right\|^{1/4}$	
(57)	PARAMETER	C_1, FIGURE 10.27	
(58)	PARAMETER	C_2, FIGURE 10.28	
(59)	MAXIMUM WEB STRESS	τ'_{MAX} = (19) (1 + (36)^2 (57))(1 + (36) (58))	
(60)	WEB ALLOWABLE	τ^*_{all}, FIGURE 10.29 @ $\alpha_{PDT} = 45^o$	
(61)	MS - WEB	MS = (60) / (59) -1	
(62)	PARAMETER	C_3, FIGURE 10.28	
(63)	SECONDARY BENDING IN FLANGE	M_{SB} = (1/12)((36) (62) (18) (3)^2 (40))	
(64)	DISTANCE FROM N.A. TO EXTREME FIBER OF FLANGE	C_f	
(65)	DISTANCE - N.A. TO NEAR FIBER OF FLANGE	D_f	
(66)	FLANGE APPLIED STRESS	σ_a = (17) / (12)	
(67)	DIAGONAL TENSION STRESS - FLANGE (COMP)	σ_{DT} = -(36) (19) / (40) / $\left\|2\right.$ (12)/(22) + .5(1-(36))$\left.\right\|$	
(68)	SECONDARY BENDING STRESS - FLANGE (COMP)	σ_{SB} = -(63) (64) / (15)	
(69)	SECONDARY BENDING STRESS - FLANGE (TENSION)	σ_{SB} = (68) (65) / (64)	

TABLE 10.2 (CONT'D) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH DOUBLE UPRIGHTS

(70) FLANGE STRESS - INSIDE FIBER	σ_{tot} = (66) + (67) + (68)	
(71) FLANGE STRESS - EXTREME FIBER	σ_{tot} = (66) + (67) + (69)	
(72) ALLOWABLE CRIPPLING STRESS - FLANGE	F_{cc}	
(73) ALLOWABLE TENSION STRESS - FLANGE	F_{tu} or F_{ty}	
(74) MS - FLANGE (TENSION)	MS = (73) / (71) - 1	
(75) MS - FLANGE (COMP)	MS = (72) / (70) - 1	
(76) RIVET FACTOR	R = 1 + 0.414 (36)	
(77) RIVET LOAD - WEB TO FLANGE	R'' = qR = (18) (76)	
(78) ALLOWABLE RIVET SHEAR LOAD	P_{af}	
(79) MS - FLANGE RIVETS	MS = (78) / (77) - 1	
(80) RIVET LOAD - UPRIGHT TO FLANGE	P_u = (41) (20)	
(81) ALLOWABLE RIVET LOAD	P_{au}	
(82) MS - UPRIGHT RIVETS	MS = (81) / (80) - 1	
(83) STATIC MOMENT OF CROSS SECT. OF ONE UPRIGHT ABOUT MEDIAN PLANE OF WEB	Q	
(84) WIDTH OF OUTSTANDING LEG OF UPRIGHT	b	
(85) UPRIGHT COLUMN YIELD STRESS	F_{coy}, SECTION 11	
(86) RIVET LOAD - UPRIGHT TO WEB	R_R = 2 (83) (85) (7) / (84) (44)	
(87) RIVET ALLOWABLE LOAD	P_{ar}	
(88) MS - RIVET, UPRIGHT TO WEB	MS = (87) / (86) - 1	
(89) ULTIMATE TENSILE STRESS OF WEB	F_{tu}, SECTION 5	
(90) RIVET TENSILE STRESS, UPRIGHT/WEB	σ_R = .15 (8) (89)	
(91) RIVET ALLOWABLE TENSILE STRESS	F_{RT}, SECTION 6	
(92) MS - RIVET TENSION	MS = (91) / (90) - 1	

NOTES:

(1) If any of the margins of safety are less than zero, the design is inadequate. The deficient area must be corrected and this table repeated.

(2) If the web is subjected to tension or compression as well as shear, the initial buckling stress of the web must be modified according to the method described in Section 10.4.1.

TABLE 10.2 (CONT'D) - ANALYSIS OF FLAT TENSION FIELD BEAMS WITH DOUBLE UPRIGHTS

Revision A

10.5.7 Analysis of a Flat Tension Field Beam with Single Uprights and Access Holes

The following step-by-step procedure (in Table 10.3) is an analysis of a flat tension field beam with single uprights and access holes (Figure 10.31).

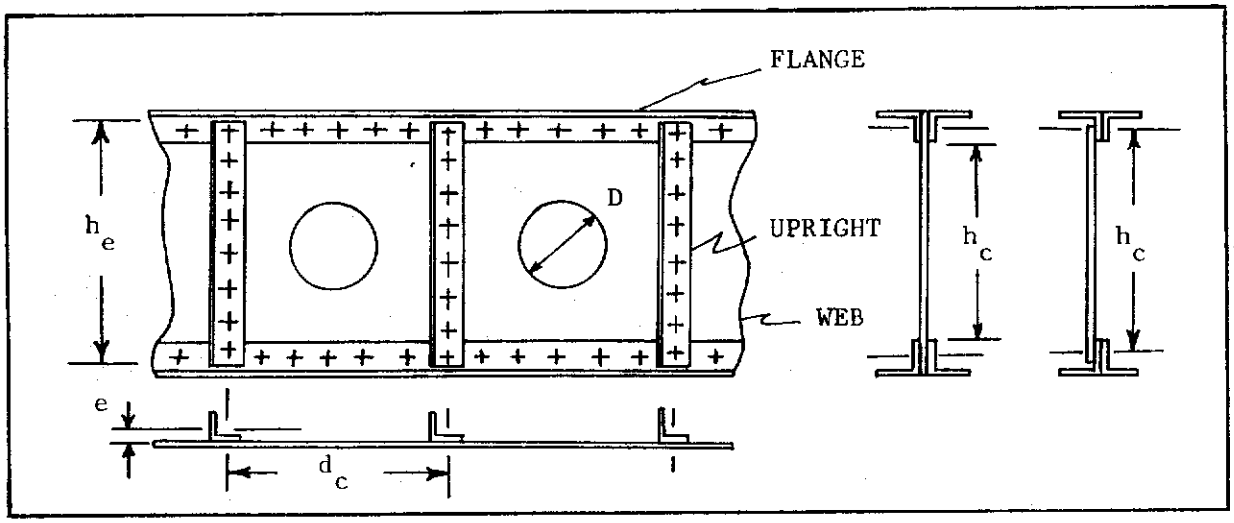


FIGURE 10.31 - FLAT TENSION FIELD BEAM WITH SINGLE UPRIGHTS AND ACCESS HOLES

Table 10.3 is based on a single web with parallel flanges and parallel uprights. Most beams consist of more than one web. At various locations in the following table adjacent panels must be considered. Such a situation occurs for rivet load, stringer axial stress, upright stress and moment in stringer.

(42)	Diagonal Tension Coefficient	k, Figure 10.22 @ $300\, td_c/12 h_c = 0$	
(43)	Parameter	(22) + ½(1 − (42))	
(44)	Ratio of Upright Stresses	σ_{umax}/σ_u, Figure 10.23	
(45)	Ratio of Upright to Shear Stress	σ_u/τ, Figure 10.24	
(46)	Diagonal Tension Angle	$\tan\alpha$, Figure 10.25 (a)	
(47)	Stress in Median Plane-Upright to Web	σ_u = − (42) (20) (46) / (43)	
(48)	Upright Average Stress	σ_{uavg} = (47) (21) / (12)	
(49)	Upright Maximum Stress	σ_{umax} = (47) (44)	
(50)	Effective Column Length: If (24) < 1.5	L_e = (7) / [1 + (42)^2(3 − 2 (24))]$^{1/2}$	
	If (24) > 1.5	$L_e = h_u$ = (7)	
(51)	Slenderness Ratio	$L_e/2\rho$ = (50)/2 (14)	
(52)	Column Allowable	$\sigma_{co} = \pi^2$ (1)/(51)^2 or Section 11	
(53)	Proportional Limit	F_{pl}, Section 5	
(54)	Strain, If (47) > (53)	σ_u/E = (47) / (1)	
(55)	From Stress-Strain Curve	F_c, use (54) to determine allowable	
(56)	Margin of Safety: Column Yield (47)>(53)	MS = (55) / (47) − 1	
	(47)<(53)	MS = (53) / (47) − 1	
(57)	MS - Column	MS = (52) / (48) − 1	
(58)	Parameter	$k^{2/3}(t_u/t)^{1/3}$ = (42)$^{2/3}$ (26)$^{1/3}$	
(59)	Parameter	C_4, Figure 10.32	
(60)	Parameter	C_5, Figure 10.32	
(61)	Parameter	(12) (32) (59)	
(62)	Parameter	(34) + (61)	
(63)	Parameter	(12) / (3)(9)	
(64)	Upright Allowable (Without Access Hole)	σ_o, Figure 10.26	

TABLE 10.3 (CONT'D) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH ACCESS HOLE AND SINGLE UPRIGHTS

(65) Upright Allowable (With Access Hole)	$\sigma_o' = \textcircled{64} / \textcircled{33}$	
(66) MS Forced Crippling	$MS = \textcircled{65} / \textcircled{49} - 1$	
(67) Parameter	$wd_c = .7 \textcircled{3} \left[\textcircled{9} / 2 \textcircled{16} \textcircled{6} \right]^{1/4}$	
(68) Parameter	C_1, Figure 10.27	
(69) Parameter	C_2, Figure 10.28	
(70) Maximum Web Stress	$\tau'_{max} = \textcircled{20} \left[\frac{(1 + \textcircled{42}^2 \textcircled{68})}{(1 + \textcircled{42} \textcircled{69})} \right]$	
(71) Web Allowable (Without Access Hole)	τ^*_{all}, Figure 10.29 @ $\alpha_{PDT} = 45^\circ$	
(72) Web Allowable (With Access Hole)	$\tau_s = \textcircled{71} \textcircled{63} / \textcircled{60}$	
(73) MS - Web	$MS = \textcircled{72} / \textcircled{70} - 1$	
(74) Parameter	C_3, Figure 10.28	
(75) Secondary Bending in Flange	$M_{SB} = (1/12) (\textcircled{42} \textcircled{74} \textcircled{19} \textcircled{3}^2 \textcircled{46})$	
(76) Distance From N.A. to Extreme Fiber of Flange	C_f	
(77) Distance - N.A. to Near Fiber of Flange	D_f	
(78) Flange Applied Stress	$\sigma_a = \textcircled{17} / \textcircled{13}$	
(79) Diagonal Tension Stress Flange (Comp)	$\sigma_{DT} = -\left[\frac{(\textcircled{42} \textcircled{20} / \textcircled{46})}{2 \textcircled{13} / \textcircled{23} + .5 (1 - \textcircled{42})} \right]$	
(80) Secondary Bending Stress - Flange (Comp)	$\sigma_{SB} = - \textcircled{75} \textcircled{76} / \textcircled{16}$	
(81) Secondary Bending Stress - Flange (Tension)	$\sigma_{SB} = \textcircled{80} \textcircled{77} / \textcircled{76}$	
(82) Flange Stress - Inside Fibers	$\sigma_{tot} = \textcircled{78} + \textcircled{79} + \textcircled{80}$	
(83) Flange Stress - Extreme Fibers	$\sigma_{tot} = \textcircled{78} + \textcircled{79} + \textcircled{81}$	
(84) Allowable Crippling Stress - Flange	F_{cc}	
(85) Allowable Tensile Stress - Flange	F_{tu} or F_{ty}	

TABLE 10.3 (CONT'D) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH ACCESS HOLE AND SINGLE UPRIGHTS

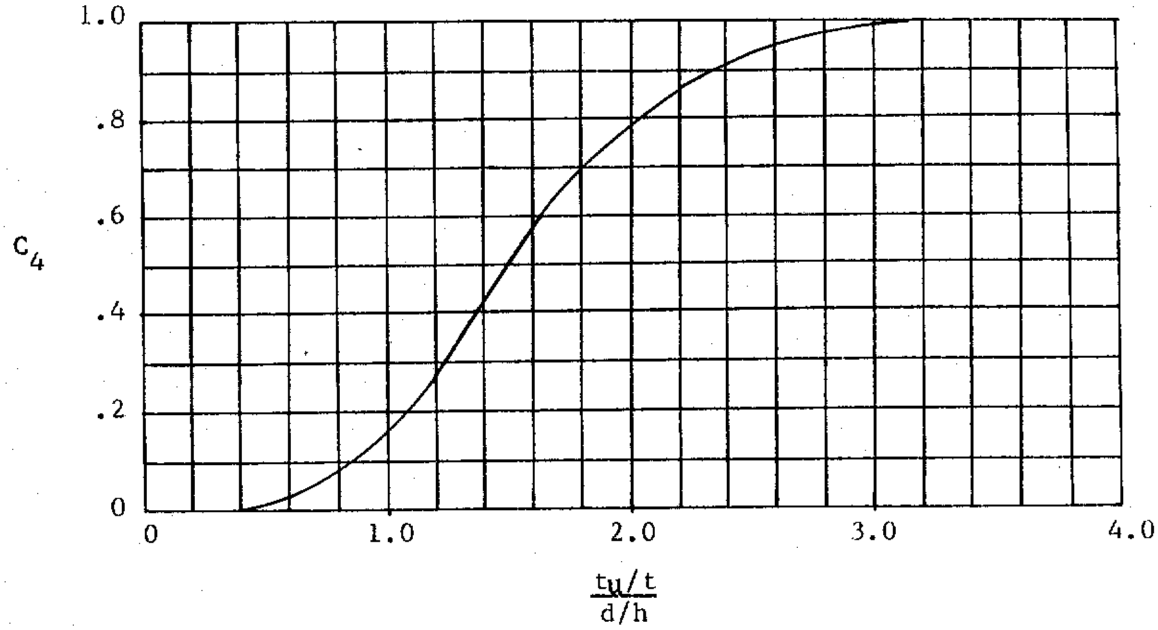
(86) MS - Flange (Tension)	MS = (85) / (83) - 1	
(87) MS - Flange (Comp)	MS = (84) / (82) - 1	
(88) Rivet Factor	R = 1 + 0.414 (42)	
(89) Rivet Load - Web to Flange	R" = qR = (19) (88)	
(90) Allowable Rivet Shear Load	P_{af}	
(91) MS - Flange Rivets	MS = (90) / (89) - 1	
(92) Rivet Load - Upright to Flange	P_u = (47) (21)	
(93) Allowable Rivet Upright Load	P_{au}	
(94) MS - Upright Rivets	MS = (93) / (92) - 1	
(95) Inter Rivet Buckling Allowable	F_{IR}, Section 10.6	
(96) MS - Inter Rivet Buckling	MS = (95) / (49) - 1	
(97) Ultimate Tensile Stress of Web	F_{tu}, Section 5	
(98) Rivet Tensile Stress - Upright to Web	σ_R = .22 (9) (97)	
(99) Rivet Allowable Tensile Stress	F_{RT}, Section 6	
(100) MS - Rivet Tension	MS = (99) / (98) - 1	

NOTES:

(1) If any of the margins of safety are less than zero, the design is inadequate. The deficient area must be corrected and this table repeated.

(2) If the web is subjected to tension or compression as well as shear, the initial buckling stress of the web must be modified according to the method described in Section 10.4.1.

TABLE 10.3 (CONT'D) - ANALYSIS OF FLAT TENSION FIELD BEAM WITH ACCESS HOLE AND SINGLE UPRIGHTS



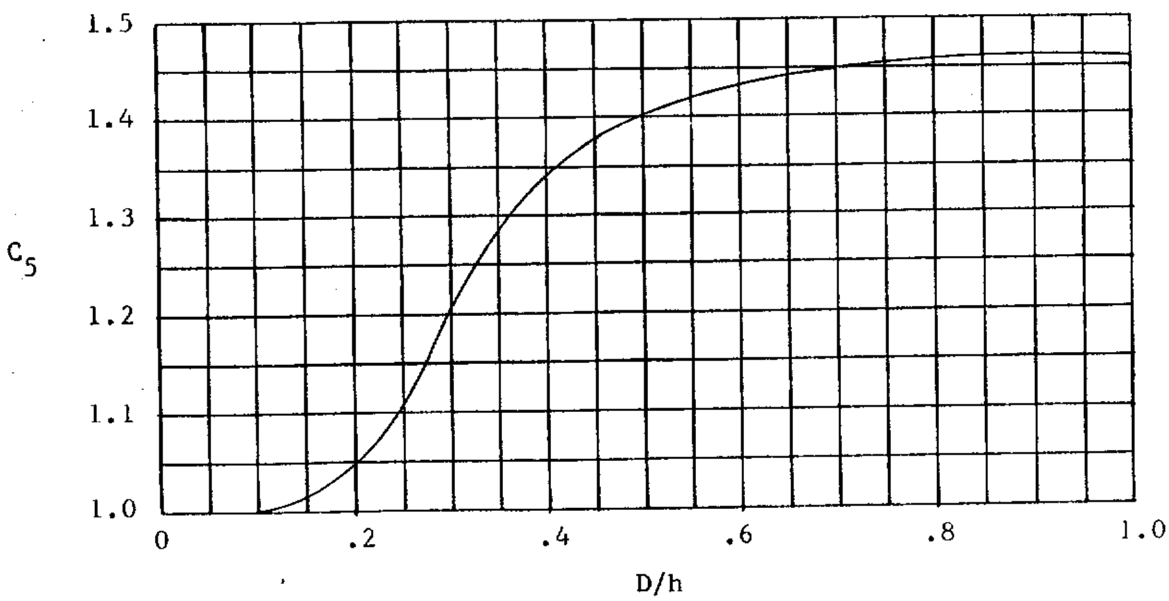


FIGURE 10.32 - ACCESS HOLE REDUCTION FACTORS

10.7 COMPRESSIVE CRIPPLING

Introduction

Compressive crippling or local buckling is defined as an inelastic distortion of the cross-section of a structural element in its own plane (rather than along the longitudinal axis, as in column buckling). The crippling stress, which is the maximum average stress developed by a structural shape, is a function of the cross-sectional area rather than the length. The crippling stress for a given cross-section is calculated by assuming that a uniform stress is acting over the entire section, $P_{cc} = F_{cc} \cdot A$. In reality, however, the stress is not uniform over the entire cross-section. Parts of the section will buckle at a stress below the gross area crippling stress, while the more stable areas, such as intersections and corners, reach a higher stress than the buckled elements.

Method of Analysis

The allowable crippling stress may be obtained from the procedure outlined below.

1. Divide the section into individual segments as shown in Figures 10.43 and 10.44. Define for each segment a width b and a thickness t. Each segment will have either zero or one edge free.

2. The allowable crippling stress, F_{cc}, for each segment is obtained from the compressive crippling curves of Figures 10.43 or 10.44.

3. The allowable crippling stress for the entire section is found by taking a weighted average of the allowable stresses for each segment:

$$F_{cc} = \frac{b_1 t_1 F_{cc1} + b_2 t_2 F_{cc2} + \ldots}{b_1 t_1 + b_2 t_2 + \ldots} = \frac{\Sigma b_n t_n F_{ccn}}{\Sigma b_n t_n}$$

The same procedure is used to analyze formed and extruded sections. Care must be taken in segmenting an unbalanced extruded section. When the thicknesses of the segments differ by a factor of 3.0 or more, the excess thickness should be discounted in calculating both the crippling stress of the segment and the effective load carrying area of the section. Also note that the bend radii of formed sections are ignored. For formed sections with lips, Figure 10.45 may be used to determine whether the lip provides sufficient stability to the adjacent segment.

FIGURE 10.43 - COMPRESSIVE CRIPPLING FORMED SECTIONS, GENERAL SOLUTION

$$F_{cc} = \frac{\Sigma\, b_N\, t_N\, F_{ccN}}{\Sigma\, b_N\, t_N}$$

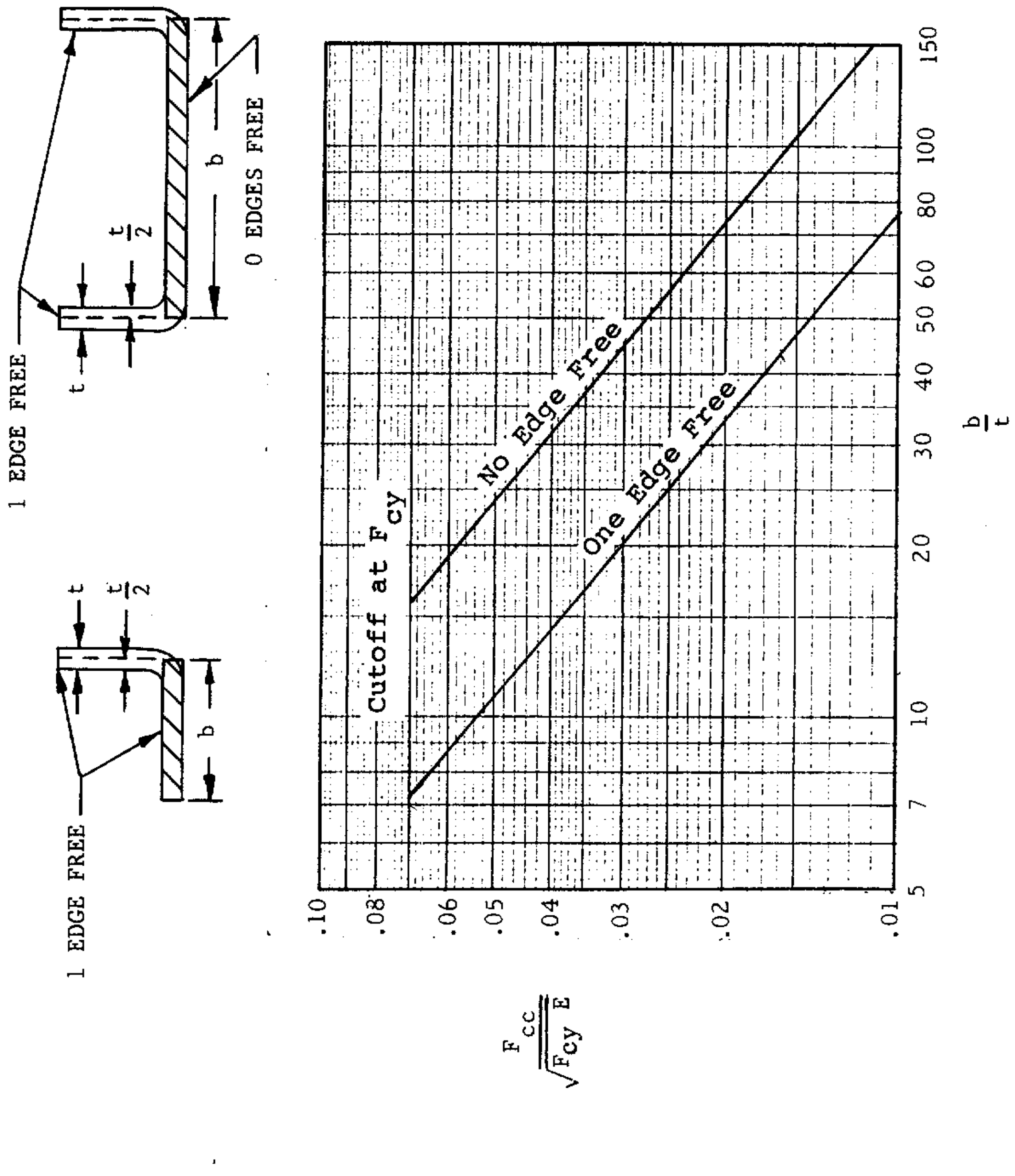
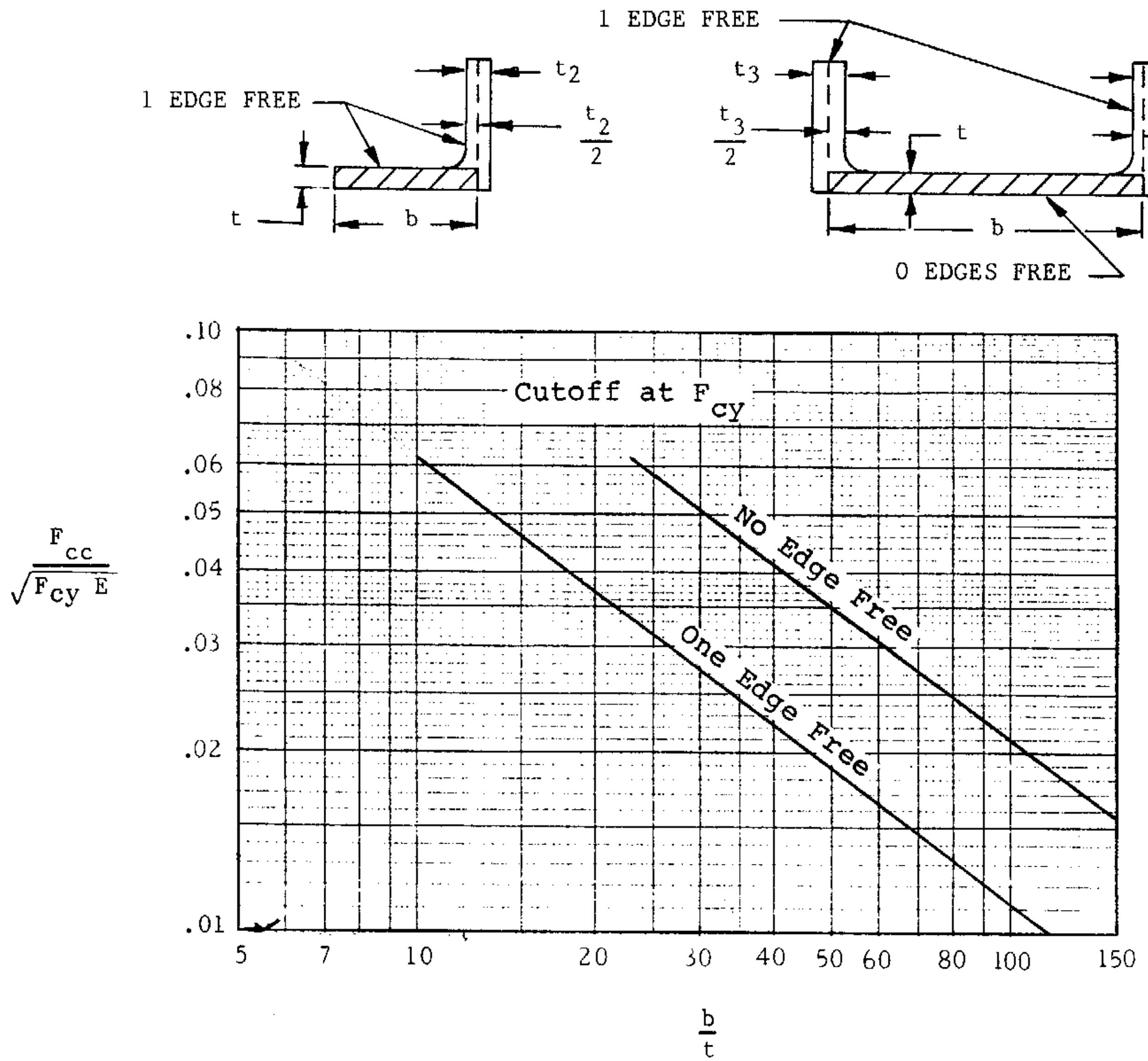
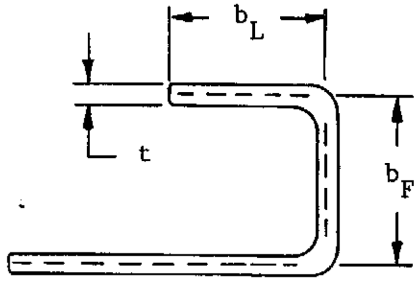


FIGURE 10.44 - COMPRESSIVE CRIPPLING OF EXTRUSIONS, GENERAL SOLUTION

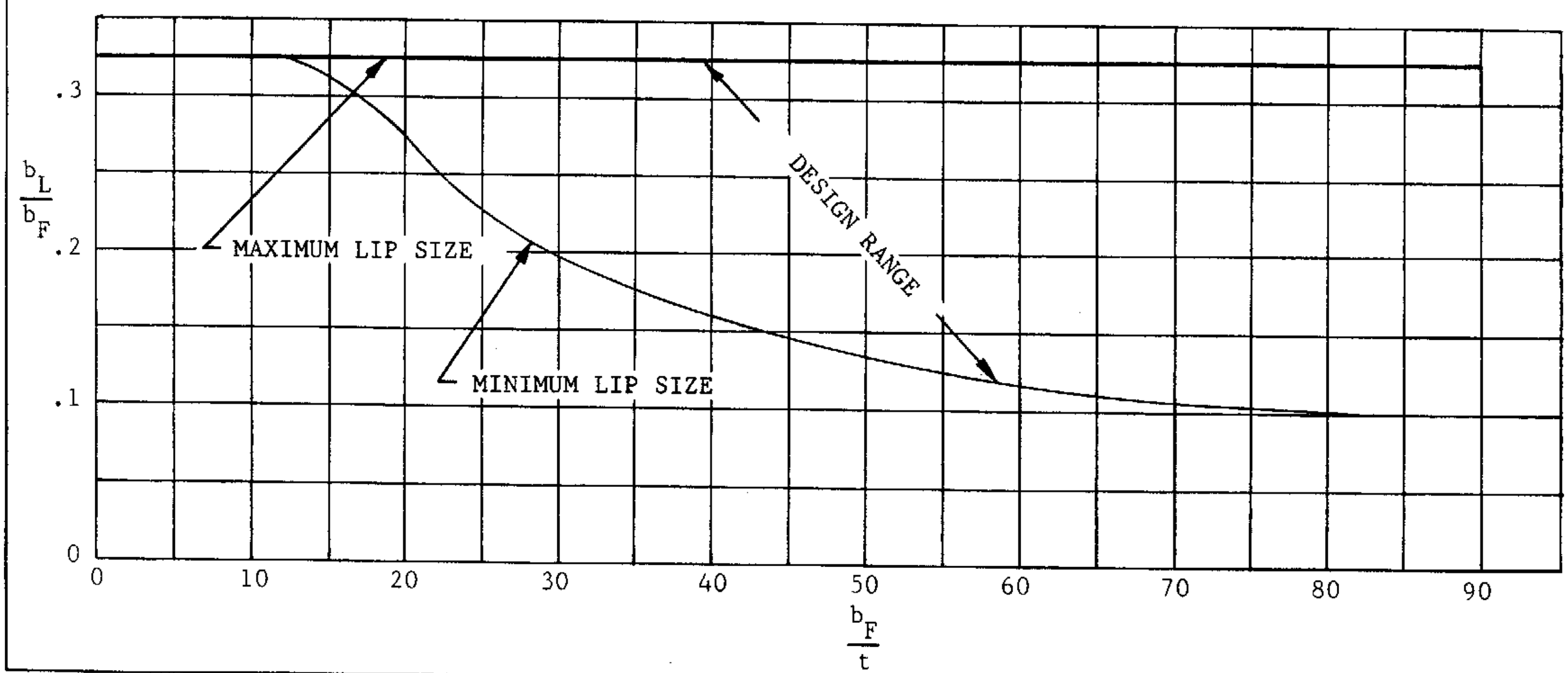


$$F_{cc} = \frac{\Sigma\, b_N\, t_N\, F_{cc_N}}{\Sigma\, b_N\, t_N}$$

FIGURE 10.45 - DESIGN RANGE FOR LIPS ON FORMED SECTIONS



ABOVE DESIGN RANGE:	LIP BUCKLES BEFORE FLANGE
BELOW DESIGN RANGE:	LIP TOO SMALL TO PROVIDE SIMPLE SUPPORT TO FLANGE
WITHIN DESIGN RANGE:	LIP PROVIDES SIMPLE SUPPORT TO ADJACENT ELEMENT AND IS TREATED AS A FLANGE IN CRIPPLING ANALYSIS



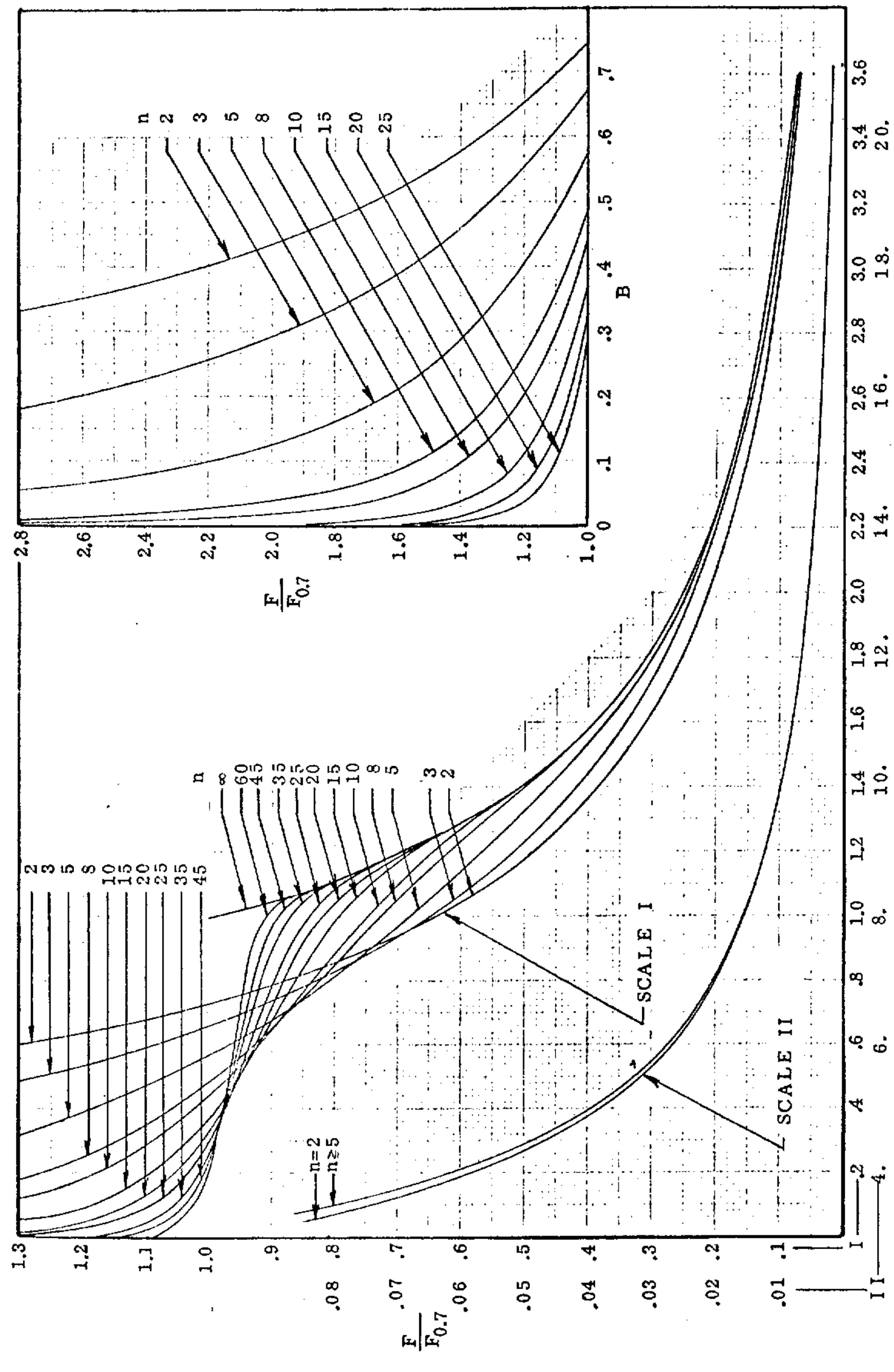


FIGURE 11.8 - COLUMN BUCKLING

11.1.3 Columns With Varying Cross Section

The conventional Euler critical column stress equation :

$$F_{crit} = \frac{\pi^2 E}{\left(L'/\rho\right)^2}$$

is only valid for a straight column under compression with constant bending rigidity (EI) and a constant area along its length. When the bending rigidity varies along the length of the column, determination of the Euler load becomes more difficult. In this section column buckling coefficient charts and the appropriate formulas for the Euler loads are given for numerous columns of varying cross section. CPS Program SC5001 is a computer analysis of stepped columns.

The critical buckling load for variable section columns in the elastic range is given by general equations of the form

$$P_{cr} = MEI/L^2 \qquad (11.13)$$

where M is the column buckling coefficient and is a function of the column geometry, bending rigidity and end restraint. Values of the column buckling coefficient, M, for various stepped columns shown in Figure 11.10 are given in Figures 11.11 through 11.29.

For tapered columns with the moment of inertia varying at the ends according to

$$I_x = I_2\,(x/b)^n \qquad (11.14)$$

where b, x, Ix and I_2 are defined in Figure 11.9, the values of the coefficient, M, to be used in Equation 11.13 are obtained from Figures 11.11 through 11.29 for the cases given in Figure 11.10.

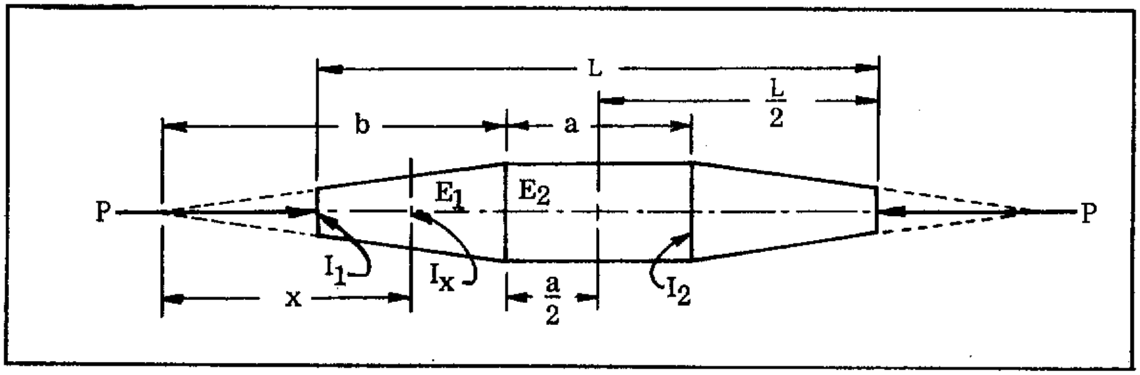


Figure 11.9 - Column with Varying Cross-Sections

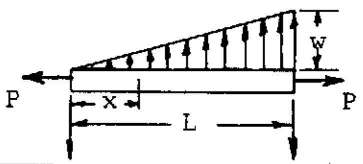
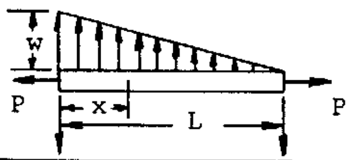
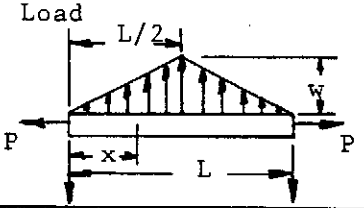
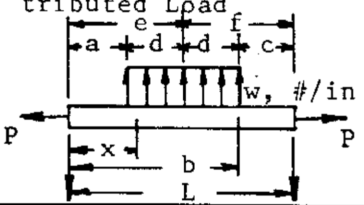
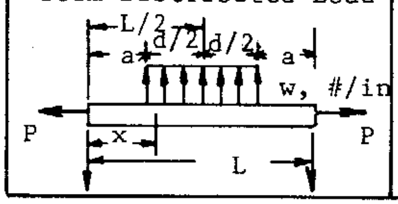
LOADING	C_1	C_2	f(w)	MAXIMUM MOMENT
Uniform Increasing Load 	$\frac{wj^2}{Sinh(L/j)}$	0	$\frac{-wj^2 x}{L}$	Occurs at: $Cosh(x/j)=(j/L)Sinh(L/j)$ Solve for x/j and x, Substitute into Equation 11.19
Uniform Decreasing Load 	$\frac{-wj^2}{Tanh(L/j)}$	wj^2	$-wj^2(1-x/L)$	Occurs at: $Cosh(\frac{L-x}{j})=(j/L)Sinh(L/j)$ Solve for x/j and x, Substitute into Equation 11.19
Symmetrical Triangular Load 	x < L/2: $\frac{2wj^3}{LCosh(L/2j)}$ x > L/2: $\frac{-2wj^3 Cosh(L/j)}{LCosh(L/2j)}$	0 $\frac{4wj^3}{L} Sinh(L/2j)$	$\frac{-2wj^2 x}{L}$ $-2wj^2(1-x/L)$	$\frac{2wj^3}{L}Tanh(\frac{L}{2j}) - wj^2$
Partial Uniformly Distributed Load 	x < a: $\frac{-2wj^2 Sinh(d/2j)Sinh(f/j)}{Sinh(L/j)}$ a < x < b: $\frac{2wj^2 Sinh(d/2j)Sinh(e/j)}{Tanh(L/j)} - wj^2 Sinh(b/j)$ b < x < L: $\frac{2wj^2 Sinh(d/2j)Sinh(e/f)}{Tanh(L/j)}$	0 $wj^2 Cosh(a/j)$ $-2wj^2 Sinh(d/j)Sinh(e/f)$	0 $-wj^2$ 0	See Note 6
Symmetrical Partial Uniform Distributed Load 	x < a: $\frac{-wj^2 Sinh(d/2j)}{Cosh(L/2j)}$ a < x < L-a: $-wj^2 Cosh(a/j)Tanh(L/2j)$ L-a < x < L: $\frac{wj^2 Sinh(d/2j)Cosh(L/j)}{Cosh(L/2j)}$	0 $-wj^2 Cosh(a/j)$ $-2wj^2 Sinh(d/2j)Sinh(L/2j)$	0 $-wj^2$ 0	$wj^2\left[\frac{Cosh(a/j)}{Cosh(L/2j)} - 1\right]$

TABLE 11.4 (CONT'D) - BEAM COLUMNS WITH AXIAL TENSION

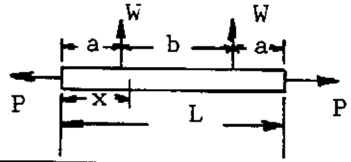
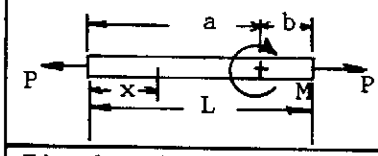
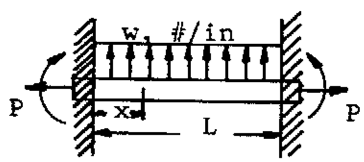
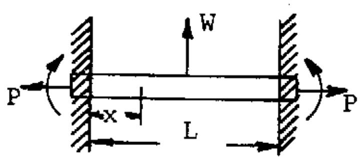
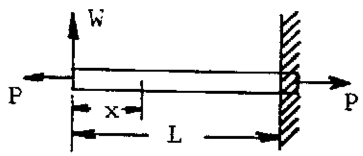
LOADING	C_1	C_2	f(w)	MAXIMUM MOMENT
Two Symmetrical Concentrated Loads 	x<a: $\frac{-WjCosh(b/2j)}{Cosh(L/2j)}$ a<x<L-a: WjSinh(a/j)Tanh(L/2j) L-a<x<L: $\frac{WjCosh(L/j)Cosh(b/2j)}{Cosh(L/2j)}$	0 -WjSinh(a/j) $\frac{-WjSinh(L/j)Cosh(b/2j)}{Cosh(L/2j)}$	0 0 0	$-Wj\frac{Sinh(a/j)}{Cosh(L/2j)}$
Concentrated Moment 	x<a: $\frac{-MCosh(b/j)}{SinhIL/j)}$ x>a: $\frac{-MCosh(a/j)}{Tanh(L/j)}$	0 MCosh(a/j)	0 0	See Note 6
Fixed End Beam - Uniform Load 	$\frac{-wLj}{2}$	$\frac{wLj}{2Tanh(L/2j)}$	$-wj^2$	At x = 0 $wj^2\left[\frac{L/2j}{Tanh(L/2j)} -1\right]$
Fixed End Beam-Concentrated Load at Center 	x < L/2: $\frac{-Wj}{2}$ x > L/2: $\frac{Wj}{2}[2Cosh(L/2j) -1]$	$\frac{Wj}{2}\left[\frac{Cosh(L/2j) -1}{Sinh(L/2j)}\right]$ $\frac{Wj}{2}\left[\frac{Cosh(L/2j)-Cosh(L/j)}{Sinh(L/2j)}\right]$	0 0	$\frac{Wj}{2}\left[\frac{1 - Cosh(L/2j)}{Sinh(L/2j)}\right]$
Cantilever - Concentrated End Load 				WjTanh(L/j)

TABLE 11.4 (CONT'D) - BEAM COLUMNS WITH AXIAL TENSION

11.3 Torsional Instability of Columns

The previous sections have assumed that the column was torsionally stable; i.e., the column would either fail by bending in a plane of symmetry of the cross section, by crippling or by a combination of crippling and bending. There are cases when a column will fail either by twisting or by a combination of bending and twisting. These torsional buckling failures occur when the torsional rigidity of the section is very low. Thin walled open sections, for instance, can buckle by twisting at loads well below the Euler load. Often in thin open sections the centroid and shear center do not coincide, therefore, torsion and flexure interact.

In this section, it will be assumed that the plane cross sections of the column warp, but their geometric shape does not change during buckling; that is, the theories consider primary failure of columns and not secondary failures, characterized by distortion of the cross sections. There is coupling of primary and secondary failures but no method has been developed to handle them so secondary failures will be ignored.

11.3.1 Centrally Loaded Columns

Centrally loaded columns can buckle in one of three possible modes: (1) they can bend in the plane of one of the principal axes; (2) they can twist about the shear center axis; or (3) they can bend and twist simultaneously. Bending in the plane of one of the principal axes has been discussed previously. The latter two modes will be discussed here.

11.3.1.1 Two Axes of Symmetry

When the cross section has two axes of symmetry or is point symmetric, the shear center and centroid coincide. In this case, the purely torsional buckling load about the shear center is given by

$$P_\phi = 1/r_o^2 \, \{ GJ + E\Gamma \pi^2/l^2 \} \tag{11.27}$$

where:

r_o = polar radius of gyration of the section about its shear center

G = shear modulus of elasticity

J = torsion constant (Section 8.0)

E = modulus of elasticity

Γ = warping constant of the section (Figure 11.85)

l = effective length of the member

For a cross section with two axes of symmetry there are three critical values of the axial load. They are the flexural buckling loads about the principal axes, P_x and P_y and the purely torsional buckling load, P_ϕ. One of these loads will be minimum and will determine the mode of failure. In this case there is no interaction and the column fails either in pure bending or in pure twisting. Shapes in this category include I-sections, Z-sections and cruciforms.

11.3.1.2 General Cross Section

In general a thin walled open section buckling occurs by a combination of torsion and bending. Purely flexural or purely torsional failure cannot occur. Consider a general section with the x and y axes the principal centroidal axes of the cross section and x_o and y_o the coordinates of the shear center. The cross section will undergo translation and rotation during buckling. The translation is defined by the deflections of the shear center u and v in the x and y directions. During translation of the cross section, point 0 moves to 0' and point C to C' where 0 is the shear center and C is the centroid. The cross section rotates an angle ϕ about the shear center. Equilibrium of a longitudinal element yields three simultaneous equations, the solution of which results in the following cubic equation for calculating the critical value of buckling load.

$$r_o^2(P_{cr}-P_y)(P_{cr}-P_x)(P_{cr}-P_\phi) - P_{cr}^2 y_o^2(P_{cr}-P_x) - P_{cr}^2 x_o^2(P_{cr}-P_y) = 0 \quad (11.28)$$

where

$$P_x = \pi^2 EI_x/L^2 \quad (11.29)$$

$$P_y = \pi^2 EI_y/L^2 \quad (11.30)$$

$$P_\phi = 1/r_o^2(GJ + E\Gamma\,\pi^2/L^2) \quad (11.31)$$

Solution of the cubic equation, 11.28, gives three values of the critical load, P_{cr}, of which the smallest will be used. The lowest value of P_{cr} will always be less than P_x, P_y, or P_ϕ. By use of the effective length, L, various end conditions can be considered.

11.3.1.3 Cross Sections With One Axis of Symmetry

A number of singly symmetric sections are shown in Figure 11.76. If the x-axis is considered to be the axis of symmetry, the $y_o = 0$ and the equation for a general section reduces to

$$(P_{cr} - P_y)\left\{ r_o^2(P_{cr} - P_x)(P_{cr} - P_\phi) - P_{cr}^2 x_o^2 \right\} = 0 \quad (11.32)$$

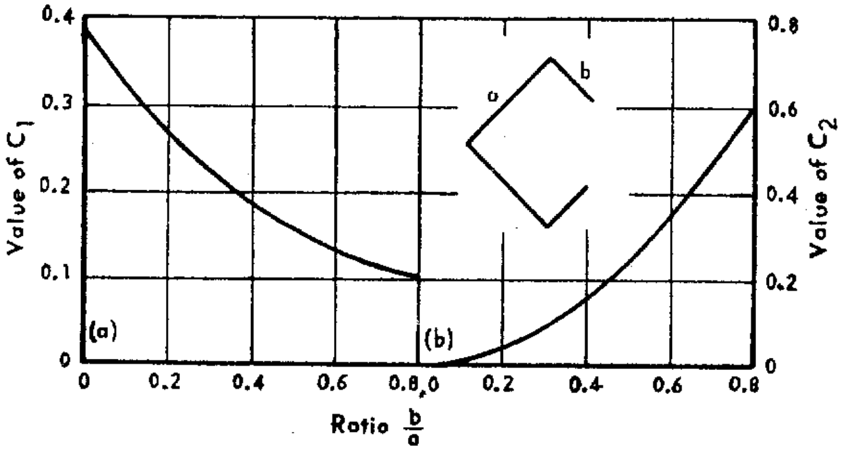


FIGURE 11.82 - TORSIONAL BUCKLING COEFFICIENTS C_1 AND C_2 FOR ANGLES

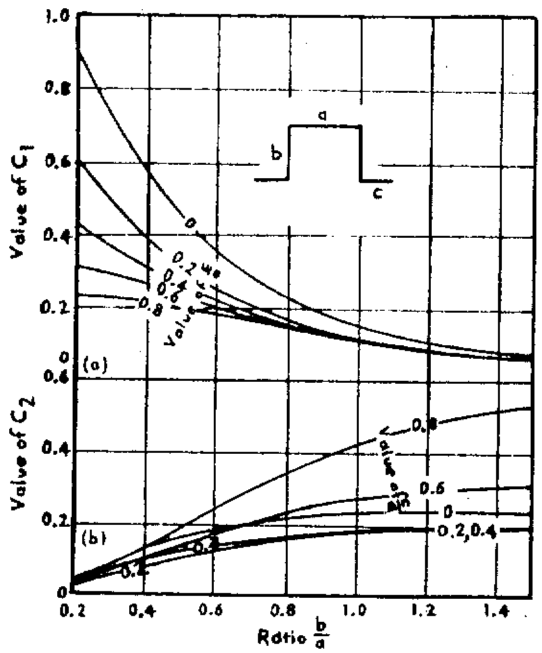


FIGURE 11.83 - TORSIONAL BUCKLING COEFFICIENTS C_1 AND C_2 FOR HAT SECTIONS.

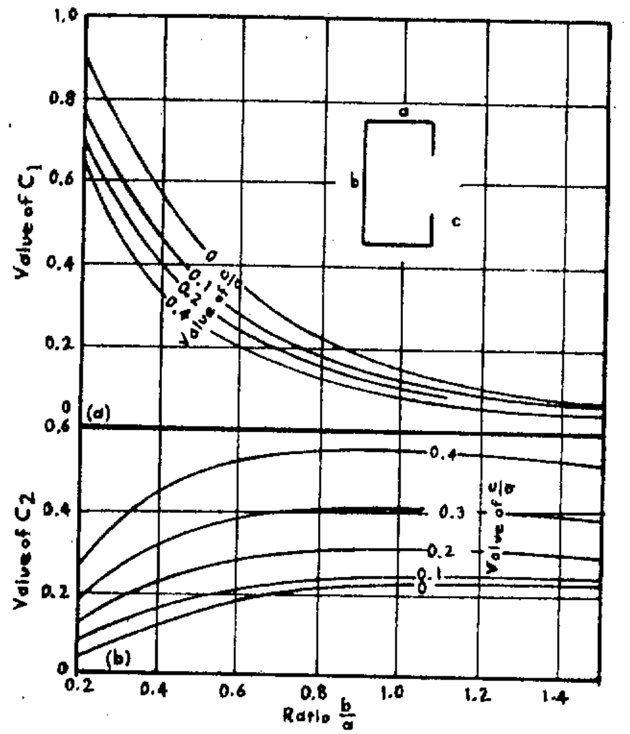


FIGURE 11.84 - TORSIONAL BUCKLING COEFFICIENTS C_1 AND C_2 FOR CHANNEL SECTIONS.

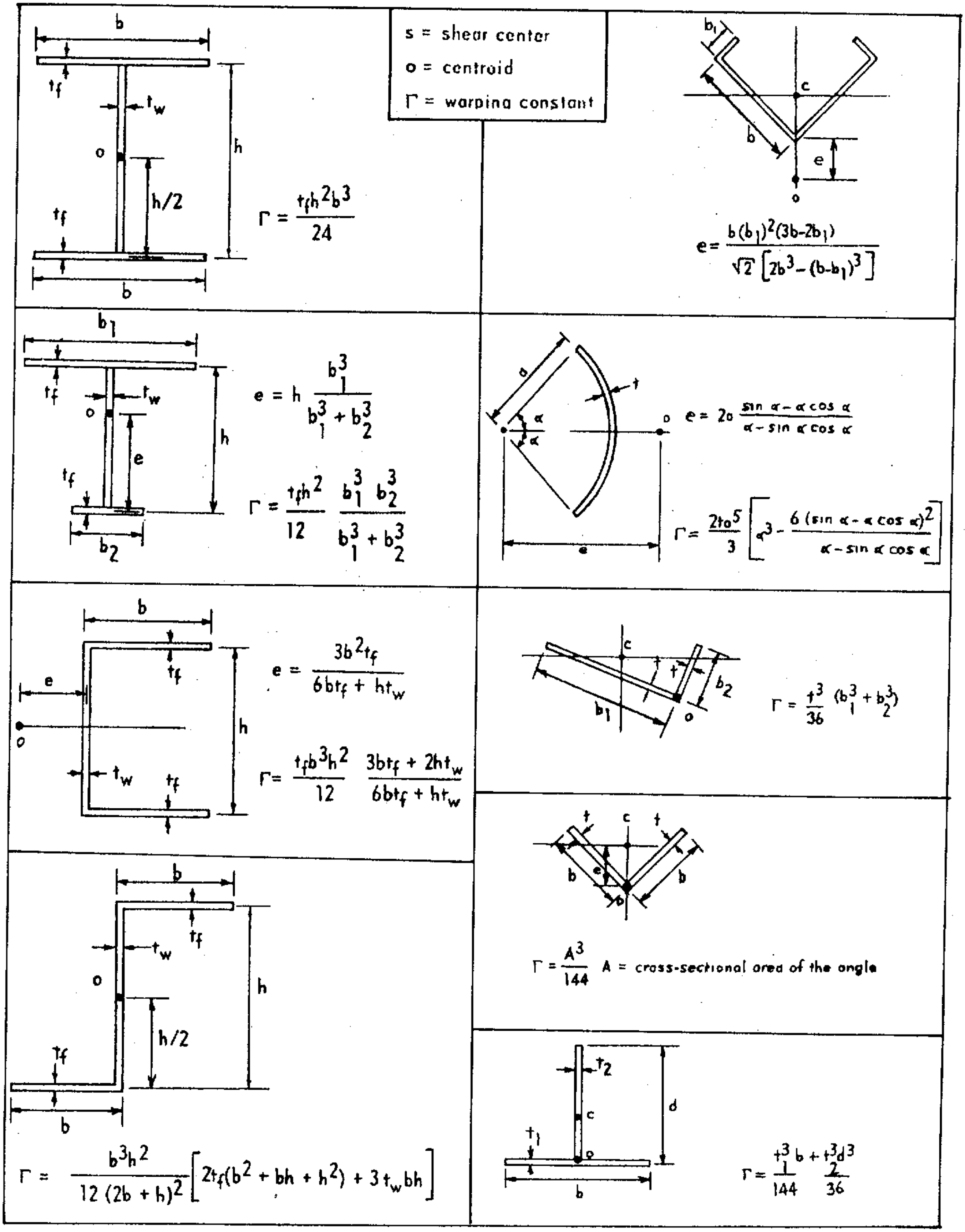
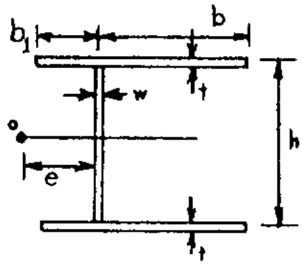
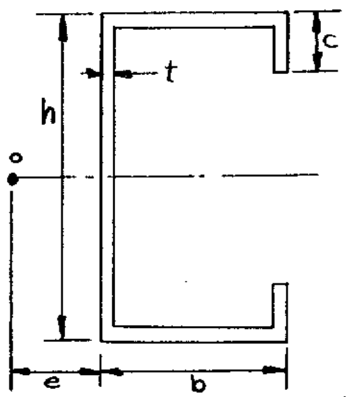


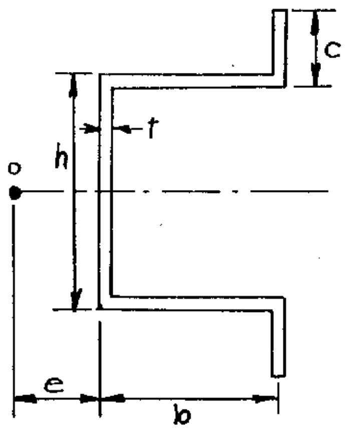
FIGURE 11.85 - SHEAR CENTER LOCATIONS AND WARPING CONSTANTS



$$e = \frac{3(b^2 - b_1^2)}{(w/t)h + 6(b + b_1)} \quad ; \; b_1 < b$$



Values of e/h							
c/h \ b/h	1.0	0.8	0.6	0.4	0.2		
0	.430	.330	.236	.141	.055		
0.1	.477	.280	.280	.183	.087		
0.2	.530	.425	.325	.222	.115		
0.3	.575	.470	.365	.258	.138		
0.4	.610	.503	.394	.280	.155		
0.5	.621	.517	.405	.290	.161		



Values of e/h							
c/h \ b/h	1.0	0.8	0.6	0.4	0.2		
0	.430	.330	.236	.141	.055		
0.1	.464	.367	.270	.173	.080		
0.2	.474	.377	.280	.182	.090		
0.3	.453	.358	.265	.172	.085		
0.4	.410	.320	.235	.150	.072		
0.5	.355	.275	.196	.123	.056		
0.6	.300	.225	.155	.095	.040		

Figure 11.85 (Cont'd) - Shear Center Locations and Warping Constants.

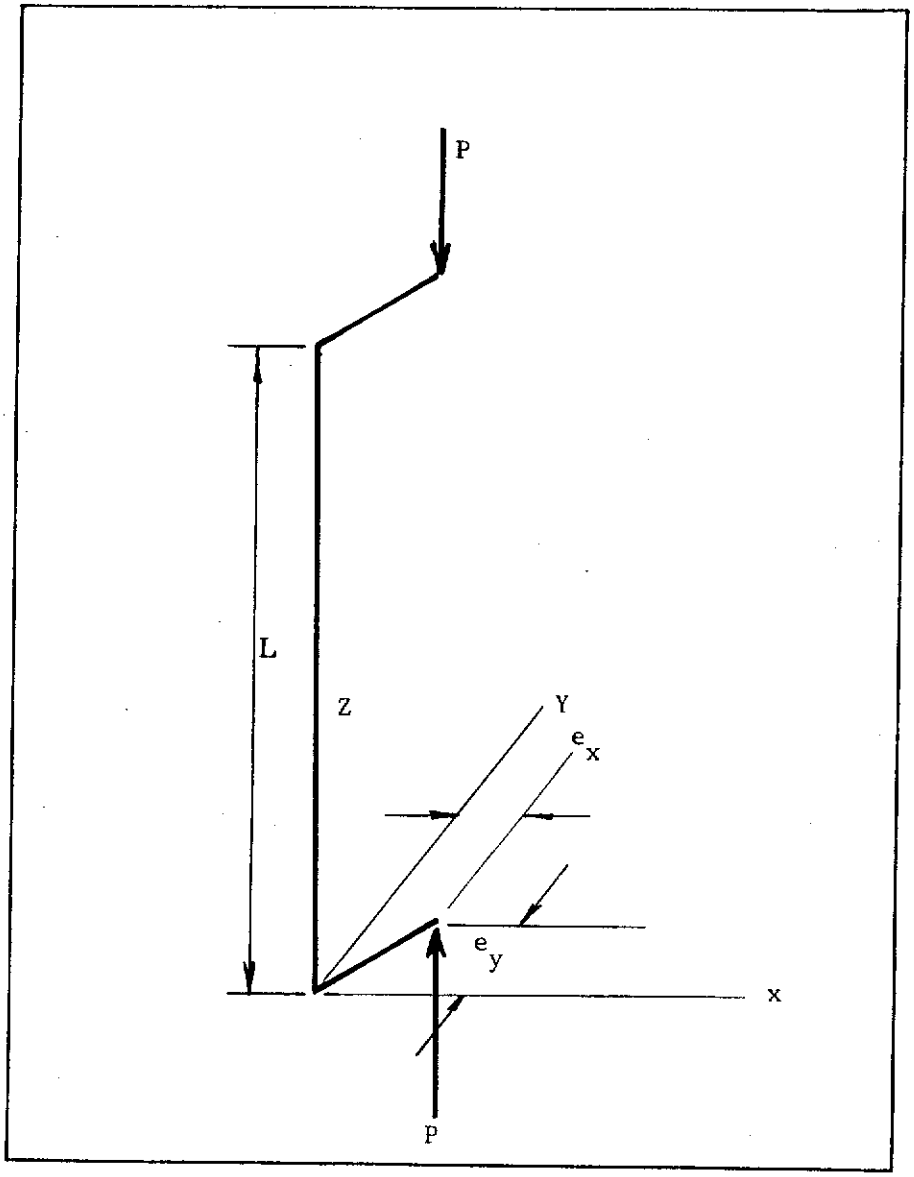


FIGURE 11.86 - ECCENTRICALLY APPLIED LOAD

FACSIMILE MESSAGE

Sender: Leth and Associates Phone: (206)622-4546 Fax No: (206)392-4482	Reference: 20614, RANGER Action: X Info:
Name: OLE LETH	Date: 5 OCT 93 — Page 1 of 11
Operator Instructions: Please forward IMMEDIATELY.	

To: GUY LAMBERT
BHTC
C. S. ARAFAT
B. COLLINS

Telefax No: (514) 437-6382

Message: SUBJECT: L-4 AFT FUSELAGE (SECTION 11)

PLEASE REVIEW THE ENCLOSED PAPER RELATING TO YOUR PROCEDURES FOR CALCULATING COMPRESSIVE STRESSES @ THE ROD ENDS FOR AXIAL LOADS COMBINED WITH DIAGONAL TENSION LOADS. WITH REGARD TO OUR DISCUSSION TODAY, EQUATION (4) IS THE BASIS FOR MY POSITION.

AT ANY RATE, I NEED TO UNDERSTAND THE INTERACTION EQUATION FOR CALCULATING THE M.S. VALUE FOR CRIPPLING.

YOUR EARLIEST RESPONSE IS APPRECIATED.

REGARDS,
OLE LETH

Analysis of Stiffened Curved Panels Under Shear and Compression

M. A. MELCON* AND A. F. ENSRUD†

Lockheed Aircraft Corporation

SUMMARY

This paper presents a method of analysis for curved panels reinforced by longitudinal stiffeners subjected to shear and compression. New formulas have been derived for the critical buckling stress of curved panels in shear and for the various effects of the diagonal tension field in the postbuckling state of the sheet. These latter effects are expressed by fictitious compressive stresses in the stiffeners which are then combined by an interaction equation with the compression stresses due to the external loading. In addition, a formula is given for the ultimate shear strength of the sheet. The method proposed in this paper has been compared with available experimental data, and satisfactory agreement is found. The method has been put in a form that requires the solution of certain mathematical relations and a minimum of chart reading and, hence, is readily adaptable to IBM or other high-speed computing techniques. A sample interaction diagram is included showing how the results of this method may be presented for practical application.

INTRODUCTION

IN SPITE OF GREAT ADVANCEMENTS in the design of aircraft structures, the analysis of stiffened shells still presents a challenging problem. The difficulty arises mainly from the fact that no practical mathematical treatment of the stress pattern in the postbuckling state has been developed as yet. Therefore, for the time being, the designer has to be satisfied with semi-empirical solutions that show sufficient agreement with test results.

Based on a study of the available literature giving theoretical and experimental data pertaining to this subject, a method has been developed for the prediction of the ultimate strength of sheet and longitudinal stiffeners in a curved panel.

NOTATION

A_{st} = stiffener area
a = ring spacing
b = stiffener spacing
C_r = rivet factor, ratio of net to gross area of web
F_c' = allowable compression stress for stiffener alone; F_c' is the lower of either the column allowable (use a fixity of 2 for stiffeners continuous across rings) or the crippling cutoff of the stiffener
F_c = allowable compression stress for stiffener plus effective skin
f_c' = effective compression stress in stiffener due to diagonal tension
f_c'' = a measure of stiffener bending stiffness required to break up shell into panels
f_c = applied compression stress based on stiffener area plus effective width of skin
f_{cg} = applied compression stress based on stiffener area plus total area of skin
F_s = Allowable gross area shear stress for web failure.
F_{s0} = basic allowable gross area shear stress (for homogeneous diagonal tension field at 45°)
F_s' = critical buckling shear stress for shell
F_s'' = critical buckling shear stress for flat panel
F_{scr} = $F_s' + F_s''$ = critical buckling shear stress for curved panel
F_{su} = ultimate shear stress of the material
f_s = applied gross area shear stress
f_{sDT} = part of applied shear stress carried in diagonal tension
F_{tu} = ultimate tensile stress of the material
I_{st} = moment of inertia of stiffener about centroidal axis parallel to the tangent of the skin contour
J_{st} = torsional stiffness factor: for open sections, $J = A_{st}t_{st}^2/3$; for closed sections $J = 4A^2t_{st}/p$, where A is the enclosed area and p is the perimeter
S = shell parameter
R = radius of curvature of panel
t = thickness of web
t_{st} = thickness of stiffener
r = a factor reflecting the various effects of the diagonal tension field on the stiffener
η = a factor indicating the intensity of diagonal tension in web
λ = a factor reducing the critical shear stress under combined loading

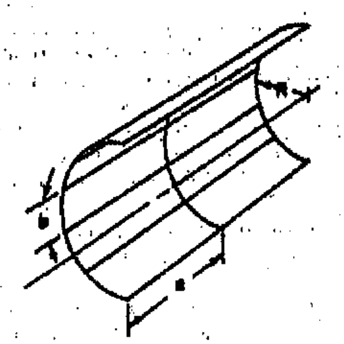


FIGURE 1.
TYPICAL SHELL STRUCTURE

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Method of Analysis

A semimonocoque shell reinforced by rings and longitudinal stiffeners (Fig. 1) may fail in three ways: (1) failure of stiffeners, (2) failure of sheet, and (3) failure of rings. If several components fail simultaneously, the type of failure is usually referred to as general instability. It is assumed that the attachments between skin and supporting members will not shear or pop and thus reduce the ultimate load carrying capacity of the structure.

The following deliberations are limited to an investigation of the first two types of failure. The results of this investigation are then checked against the test data of specimens that failed by buckling of stiffeners or tearing of the skin. Formulas derived in this paper apply specifically only to thin-walled shells of circular cross section with constant values for shear flow, stiffener area, stiffener spacing, and sheet thickness. Adaptations for variation in these items are easily made.

The proposed method may be outlined as follows:

The critical buckling stress of a curved panel is given as the sum of the strength of the shell and that of the panel. If the applied shear stress is below the critical shell-buckling stress, no longitudinal stiffeners are required. On the other hand, it is obvious that the stiffeners must have a certain amount of bending stiffness in order to subdivide the shell into panels. As soon as the applied shear exceeds the critical buckling stress for a curved panel, a rather complex interaction between sheet and supporting members is started. The mathematical treatment becomes extremely involved, if not impossible.

At the start, the shell between the rings bows outward like a barrel. With increasing shear flow, the radial components of the diagonal tension field will pull the stringers inward. In addition, a nonlinear relationship between the applied torque and the compressive stress in the stiffeners can be observed. Failure usually occurs from an individual buckle in the shell forcing the stiffener out of its original location, thus bringing about its collapse by an intricate beam-column action.

Critical Shear Buckling Stress

A simplified expression has been derived for the critical buckling stress of a curved panel. For a flat panel the critical buckling stress may be expressed by

$$F_s'' = KE(t/b)^2 \qquad (1)$$

where the magnitude of the coefficient K depends on the material, the degree of edge restraint, and the aspect ratio of the panel. For aluminum alloys and standard construction, this value is taken equal to 5.25 irrespective of aspect ratio. The variation in panel length is taken care of in the term that reflects the effect of curvature. The critical buckling stress of the curved panel may be expressed by

$$F_{s_{cr.}} = KE(t/b)^2$$

where in this case $K = 5.25 + (\pi/4)(b/a)^2S^{3/4}$ and $S = a^4/Rt$ denotes the shell parameter. This formula for the critical buckling stress of a curved panel may be transformed to the form

$$F_{s_{cr.}} = F_s' + F_s'' \qquad (2)$$

where

$$F_s' = (\pi/4\sqrt[4]{S})(Et/R)$$
$$F_s'' = 5.25E(t/b)^2$$

Postbuckling Behavior of a Shear Panel

With the introduction of thin sheet construction for aircraft structure, engineers began to accept the new idea that the buckling of structural components did not necessarily indicate failure. Since that time, a great deal has been written on the postbuckling behavior of structures. Of foremost importance in the study of this subject, is the theory of the incomplete diagonal tension field.

The stress pattern in a flat sheet subjected to shear forces beyond its buckling strength is known as diagonal tension. A rigorous formulation of the transition from the unbuckled state to Wagner's[7] ideal diagonal tension field has not yet been accomplished. Semiempirical formulas developed by Kuhn[8] are most widely used.

When the stress in a plane web starts to exceed its initial buckling strength, the applied shear forces are gradually taken by a combined truss action of the web and stiffeners. The sheet acts more and more like a diagonal, while the stiffeners take the place of uprights. There is a tendency for a buckle to form from corner to corner of the panel, provided this pattern is compatible with the deformation of the stiffeners supporting the edges of the panel.

The various methods proposed for the analysis of flat panels in the postbuckling state differ in the assumption of magnitude of compression stress the sheet is able to sustain in its buckled shape. Additional complications arise when the panel is curved. The diagonal tension in the sheet tends to reduce its curvature in the direction of the wrinkles. This action induces nonuniform radial loads on the longitudinal stiffeners.

Analysis of Longitudinal Stiffeners

The longitudinal stiffeners in a reinforced shell perform three functions: (1) They subdivide the shell into panels; (2) they sustain axial and radial loads induced by the tension field; and (3) they sustain directly applied axial loads. As long as the shell stiffened by rings only is able to sustain the applied shear flow without buckling, no stiffeners are required for the purpose of reducing the panel size. With increasing shear flow, it usually becomes more desirable to raise the initial buckling stress by the addition of longitudinal stiffeners than by thickening of the shell. The required bending stiffness of the stiffeners which will raise the initial

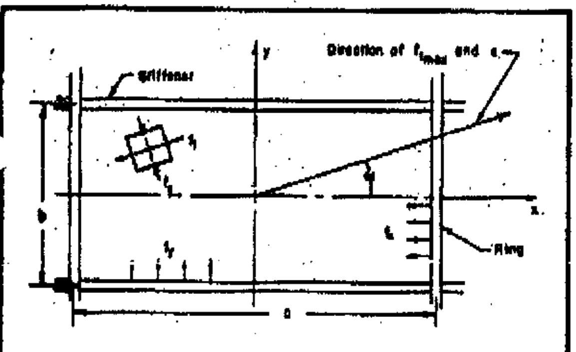

FIGURE 2.

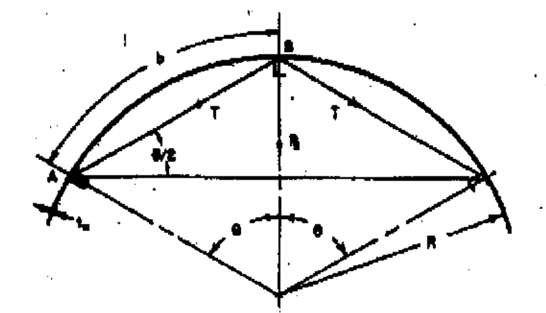

FIGURE 3.

uckling stress to the full value of shell plus panel may be determined according to Seydel's[10] formula for the critical shear stress of flat orthotropic plates,

$$F_{s_{cr}} = (32/ta^2)\sqrt[4]{D_1 D_2^3}$$

where

$$D_1 = D$$
$$D_2 = EI_{st}/b$$
$$D = Et^3/12(1 - \mu^2)$$

Setting this expression equal to F_s' and eliminating on both sides, there follows the expression for the portion of the moment of inertia, ΔI_{st}, of the stiffener required to divide the shell into panels,

$$\Delta I_{st} = (bt^3/5)(a/b)^{3/4}$$

As mentioned previously, this requirement will be transformed into a fictitious compression stress indicating the portion of the stiffener strength needed for effective subdivision into panels. It is assumed that the ratio of the stiffener moment of inertia required for this purpose to its total moment of inertia may be taken equal to the ratio of an additional area to the total area of the stiffener. The following relation for the fictitious compression stress f_c'' is established:

$$f_c'' A_{st} = F_c' \Delta A_{st}$$

then,

$$f_c'' = F_c' \frac{\Delta A_{st}}{A_{st}} = F_c' \frac{\Delta I_{st}}{I_{st}}$$

where $F_c' = \pi^2 EI_{st}/a^2 A_{st}$ reflects the column strength of the stiffener. Combining and solving these equations gives

$$f_c'' = F_c''(at/A_{st})\sqrt[8]{0.053b/a} \quad (3)$$

This equation indicates that heavy skin and wide ring spacing require strong stiffeners in order to avoid excessive values of f_c''. However, this portion of the total effective compression stress in the stiffener in most structures is usually small.

The effects of the diagonal tension field on the stiffeners in a buckled shell are rather complex. The axial load build-up in the longitudinal stiffeners caused by the diagonal tension (Fig. 2) is given by

$$P = (f_s - F_{s_{cr}})bt \cot \alpha$$

In addition, there are radial components of the diagonal tension field which produce bending moments in the stiffeners. The pull per running inch (Fig. 3) exerted by the tension field along this chord line is

$$T = (f_s - F_{s_{cr}})t \tan \alpha$$

and the radial load per running inch is

$$P_R = Tb/R = (f_s - F_{s_{cr}})bt(\tan \alpha/R)$$

The unknown angle α may be found by trial and error. However, this standard approach to the analysis of longitudinal stiffeners does not agree with tests for many reasons, and, hence, a different approach to the problem was chosen.

Fig. 4 pictures a stiffened shell subjected to torsion. Assuming a diagonal tension field of 45° and no bending in the stiffeners, the induced compression stress in the stiffeners is given by

$$f_c' = (f_s - F_{s_{cr}})(bt/A_{st}) \quad (4)$$

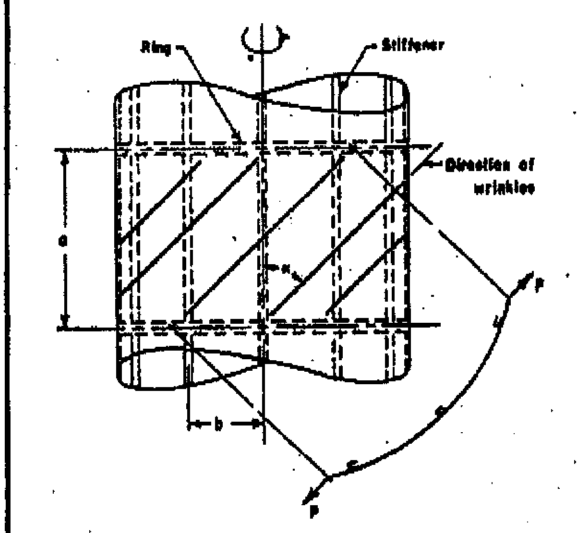

FIGURE 4.

The actual angle of the tension field is usually smaller than 45°; in addition, there is a pull P along the direction of the wrinkles which forces the stringers toward the center of the cylinder. It can be seen from Fig. 4 that the pull P has a greater effect on stringers of smaller flexibility, because the shell support in this case starts at a later state. Furthermore, it is obvious that this effect is increased with sharper curvature. These, in addition to other considerations, led to the determination of the empirical factor ν, which was introduced into Eq. (4) to give a fictitious compression stress that is a measure of required stiffener strength to sustain the effects of diagonal tension. Eq. (4) then transforms into

$$f_c' = \nu(f_s - F_{s_{cr.}})(bt/A_{st}) \qquad (5)$$

where

$$\nu = 1 + (a/R)\sqrt[4]{(I_{st}/J_{st})(t/b)}$$

This formula gives satisfactory results for shells with $R \leq 100$ in. The total effective or fictitious compression stress for which the stiffener must be designed becomes, then,

$f_c' + f_c''$ = total effective compression stress in stiffener

The allowable stress to be used for determining the strength of the stiffener is F_c', which is the lower of the column allowable (using a fixity of 2 for stiffeners that are continuous across rings) or the crippling cutoff for the stiffener alone. Allowables determined from tests on the stiffener by itself may often be unsuitable for this analysis, since the stiffener may fail in a mode not possible for a stiffener that is attached to the shell. Table A summarizes the effective compression stress to be used at different values of f_s when the panel is subjected to shear alone.

TABLE A

f_c' and f_c'' When Panel Is Subjected to Shear Only

f_c'	f_c''
$f_c' = 0$, when $f_s < F_{s_{cr.}}$	$f_c'' = 0$, when $f_s < F_s'$
$f_c' = \frac{\nu(f_s - F_{s_{cr.}})bt}{A_{st}}$, when $f_s \geq F_{s_{cr.}}$	$f_c'' = \left(\frac{f_s - F_s'}{F_s''}\right)\left(\frac{F_s''at}{A_{st}}\right) \times \sqrt[3]{\frac{0.053b}{a}}$, when $F_s' \leq f_s < F_{s_{cr.}}$
	$f_c'' = \frac{F_s''at}{A_{st}} \sqrt[3]{\frac{0.053b}{a}}$, when $f_s \geq F_{s_{cr.}}$

If, in addition to shear, the panel is also subjected to direct compression, the effects of shear on the stiffener are combined by an interaction equation with the effects of the direct compression. The only exception to the method as previously outlined for determining the effects of shear on the stiffener is that the critical shear buckling stresses, in this case, must be reduced because of the effects of compression. There are well-known interaction equations that give the initial buckling of panels under combined loadings. However, for the purpose at hand, these equations are not satisfactory, since they will not properly account for the amount of the shear being carried in diagonal tension nor will the angle of diagonal tension be the same as when the sheet buckled under shear alone. In other words, it would be conservative to assume that, after buckling occurs in combined shear and compression, any additional shear is carried in the same manner as if the buckling were due to shear alone. If linear interaction were assumed for initial buckling in compression and shear, the following equation may be written:

$$(f_s/F_{s_{cr.}}) + (f_{cg}/F_{c_{cr.}}) = 1$$

Then the shear stress at which the panel buckles becomes

$$f_s = \frac{1}{1 + (f_{cg}/f_s)(F_{s_{cr.}}/F_{c_{cr.}})} F_{s_{cr.}}$$

Based on this reasoning, a factor λ was arbitrarily selected to reflect the reduction in the critical shear buckling stress due to compression,

$$\lambda = \sqrt[4]{1/[1 + (f_{cg}/f_s)]} \qquad (6)$$

where f_{cg} is the applied compression stress based on stiffener area plus total area of skin.

The prediction of stiffener failure due to the combined action of shear and direct compression is based upon the following interaction equation:

$$\left(\frac{f_c' + f_c''}{F_c'}\right)^{1.125} + \left(\frac{f_c}{F_c}\right)^{1.125} = 1 \qquad (7)$$

where f_c is the direct compression stress based on stiffener area plus effective width of skin and F_c is the allowable compression stress based on the same area. In other words, the ratio f_c/F_c is the ratio of the applied compression load to the allowable compression load of the stiffener plus skin. Table B summarizes the effective compression stress to be used at different values of f_s when the panel is subjected to shear and compression. For $\lambda = 1$, Table B is identical with Table A.

TABLE B

f_c' and f_c'' When Panel Is Subjected to Shear and Compression

f_c'	f_c''
$f_c' = 0$, when $f_s < \lambda F_{s_{cr.}}$	$f_c'' = 0$, when $f_s < \lambda F_s'$
$f_c' = \frac{\nu(f_s - \lambda F_{s_{cr.}})bt}{A_{st}}$, when $f_s \geq \lambda F_{s_{cr.}}$	$= \left(\frac{f_s - \lambda F_s'}{F_s''}\right)\left(\frac{\lambda F_s''at}{A_{st}}\right) \times \sqrt[3]{\frac{0.053b}{a}}$, when $\lambda F_s' \leq f_s < \lambda F_{s_{cr.}}$
	$f_c'' = \frac{\lambda F_s''at}{A_{st}} \sqrt[3]{\frac{0.053b}{a}}$, when $f_s \geq \lambda F_{s_{cr.}}$

Little test data could be found for shells reinforced by stiffeners with closed section. Whether the factor ν still applies in this case could therefore not be substantiated by comparison with test data. The only

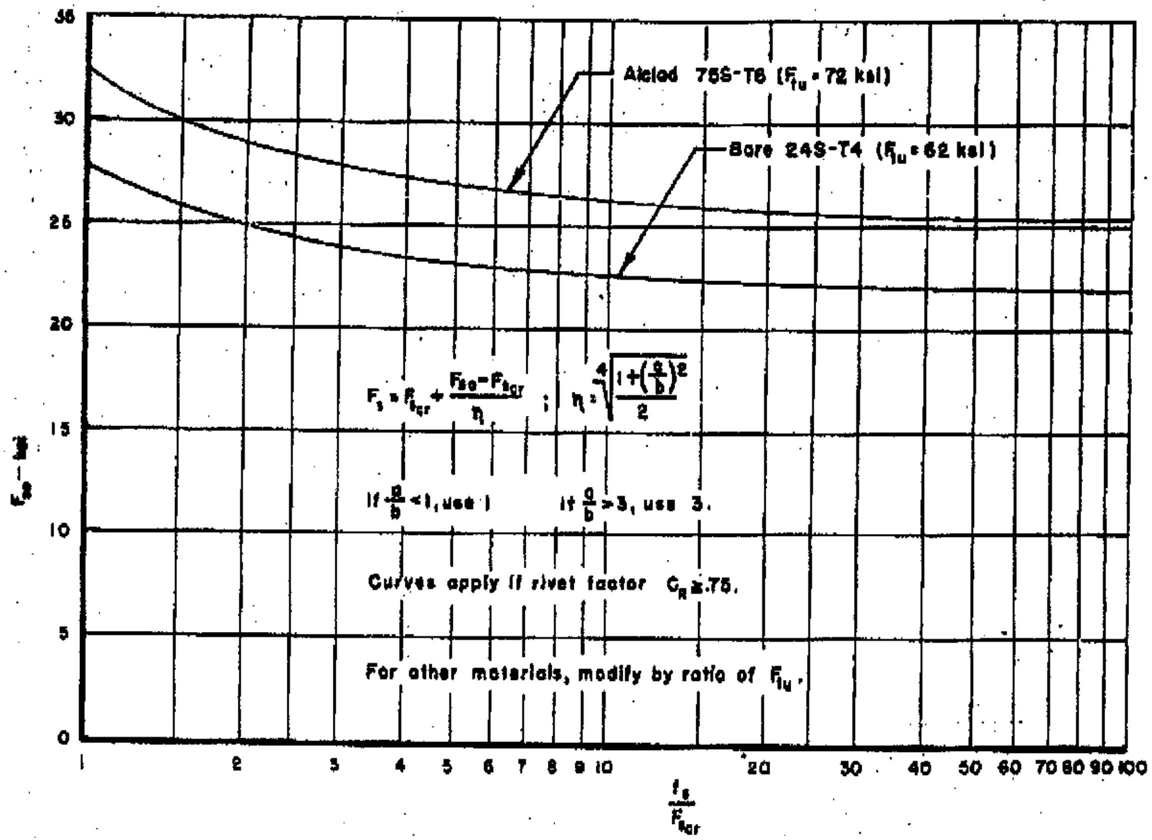

FIGURE 5 - ULTIMATE ALLOWABLE, GROSS AREA SHEAR STRESS.

two specimens that had closed stiffeners failed by tearing of the web.

Analysis of Shear Web

The formulas for the allowable gross area shear stress presented here are based on the assumption that the net area of the sheet is equal to, or greater than, 75 per cent of its gross area. In the unbuckled condition of the web, the maximum tension stress in the sheet is equal to the shear stress. For aluminum alloys, the ultimate tensile strength is approximately 1.67 times the ultimate shear strength of the material. This indicates a marginal tension capacity of 67 per cent. In the ideal tension field, where the compression strength of the sheet is assumed equal to zero, the maximum tension of the sheet is equal to twice the shear stress. This indicates that in the homogeneous tension field the sheet cannot develop the ultimate shear strength of the material. The curve of Fig. 5, giving the basic allowable shear stress F_{so}, takes account of this. However, a correction is also needed to provide for an angle α different from 45° and for wrinkles forming from corner to corner of the panel. This correction is reflected by the empirical factor

$$\eta = \sqrt[4]{[1 + (a/b)^2]/2}$$

if $a/b < 1$, use 1; if $a/b > 3$, use 3. The ultimate allowable gross area web stress is, therefore,

$$F_s = F_{s_{cr}} + [(F_{so} - F_{s_{cr}})/\eta] \qquad (9)$$

where F_{so} is obtained from Fig. 5.

The curves of Fig. 5 are similar to the lower curves of reference 11, Fig. 9. The curves of this reference were modified so that for $f_s/F_{s_{cr}} = 1$ the basic allowable shear stress becomes $F_{so} = 0.75F_{su}$. This correction is made so that net area shear stress will never exceed F_{su}.

In exceptional cases where the net area of the sheet is less than 75 per cent of the gross area, an additional check is recommended. The circular interaction for combined tension and shear is given by the formula

$$\left[\frac{f_t}{C_R F_{tu}}\right]^2 + \left[\frac{\eta(f_s - F_{s_{cr}})}{C_R F_{su}}\right]^2 = 1 \qquad (10)$$

where C_R denotes the rivet factor.

If compression is superimposed on the shear stress in the panel, the initial buckling stress is again multiplied by the factor λ in both Eqs. (9) and (10).

Analysis of Attachments

At either end of the sheet or at splices, the load in the sheet must be transferred by rivets or fasteners to other

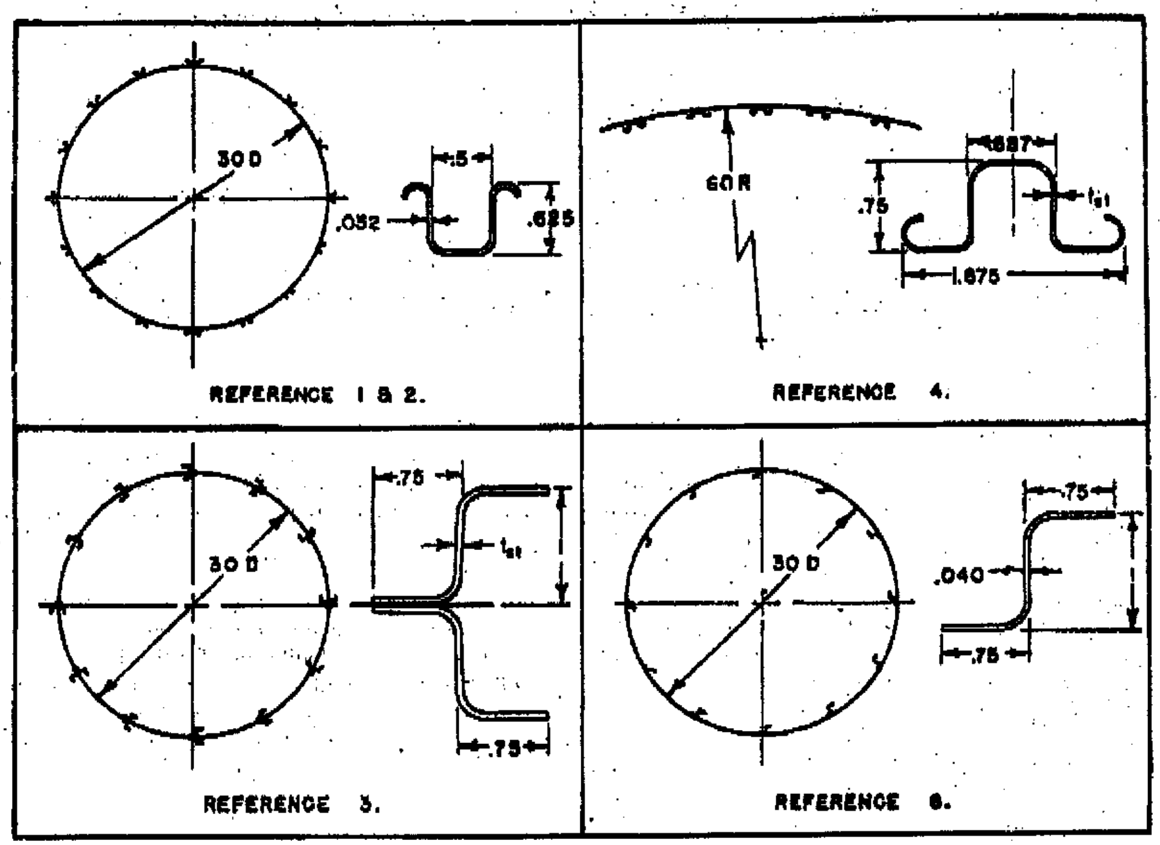

FIGURE 6 – TYPES OF SPECIMENS TESTED

parts of the structure. The shear load for which each fastener should be checked is given by

$$P = qp\sqrt{1 + \left[\frac{\eta(f_s - F_{s_{cr}})}{f_s}\right]^2} \qquad (11)$$

where p = pitch of fasteners and q = shear flow in sheet. Again, reduce $F_{s_{cr}}$ by the factor λ if compression is present.

Comparison with Test Data

The substantiation for the methods proposed herein are based upon the test work performed by the N.A.C.A.[3,6], the Aluminum Company of America,[1,2] Douglas Aircraft Company, Inc.,[4] and the National Bureau of Standards.[5] A sum total of 54 test specimens has been analyzed in this investigation. Fig. 6 shows the general configurations of the types of specimens tested. As can be observed, some of the specimens are complete cylinders stiffened with longitudinal stiffeners and rings, while others are portions of cylinders. The radius of specimens varied from 15 to 60 in.; skin gage, from 0.020 to 0.051; and the factor ν had a range on the specimens examined from approximately 1.10 to 3.

Critical Buckling Shear Stress

Table 1 gives a comparison of the critical buckling shear stress computed by the method of this paper, by the method of reference 9, and observed test values. Twenty-six panels were checked. Test values averaged 2.0 per cent higher than predicted by the proposed method and 5.8 per cent higher than by the method of reference 9. Figure 7 shows graphically the relationship between test and predicted values. There is no significant difference between the predictions of the two methods. The advantage of the proposed method lies in the fact that the buckling stress of the curved panel is expressed explicitly in terms of parameters of the panel and that reference to charts is not required.

Panels Subjected to Shear Only (Stiffener Failures)

A total of 24 specimens that were subjected to shear only failed in the stiffener. Fig. 8 shows the relationship between the shear strength predicted by the method of this paper and test shear strength. Including all specimens, the average conservatism of the prediction is 7.8 per cent. The results from specimen 2 of reference 2 and specimen 10 of reference 4 were overly conservative. In the first case the stiffener apparently

TABLE 1

Comparison of Initial Buckling Stress Computed by $F_{scr.} = F_s' + F_s''$ by Method of Reference 9 and Observed Test Values. Panels Subjected to Shear Only

Reference	Specimen	Initial Buckling Stress: Proposed method, $F_{scr.} = F_s' + F_s''$	Initial Buckling Stress: Method of ref. 9	Initial Buckling Stress: Observed by test	Ratio of Test to Predicted: Proposed method	Ratio of Test to Predicted: Method of ref. 9
1	14	2,869	2,840	2,950	1.028	1.039
	20	2,844	2,750	2,870	1.009	1.044
	21	3,455	3,230	4,120	1.192	1.276
2	2	3,018	3,160	3,400	1.146	1.005
	5	4,128	3,730	4,130	1.000	1.107
	8	4,854	4,480	4,660	0.960	1.040
	11	5,442	5,400	5,160	0.948	0.956
	3	4,245	4,790	4,720	1.112	0.985
	6	5,580	5,700	6,040	1.082	1.060
	9	5,755	5,480	5,840	1.015	1.066
	12	6,710	6,120	7,150	1.066	1.168
3	1	3,318	2,950	3,200	0.964	1.085
	2	4,897	4,580	5,000	1.021	1.092
	3	3,643	3,480	3,700	1.016	1.063
	4	4,908	4,600	4,800	0.978	1.043
	5	6,317	6,700	6,400	1.013	0.955
	6	8,383	8,360	8,500	1.014	1.017
	7	7,115	6,530	6,700	0.942	1.026
	8	8,508	8,640	8,800	1.034	1.019
	9	3,939	3,940	3,040	0.774	0.772
	10	2,326	2,350	2,380	1.023	1.014
	11	7,552	7,110	7,900	1.046	1.111
	12	6,472	6,170	7,270	1.123	1.179
4	8	1,480	1,470			
	28	1,212	1,230			
	27	1,480	1,460			
	30	2,200	2,200			
	16	1,212	1,230			
	3	1,480	1,400			
	10	2,200	2,200			
5	20a	14,420	12,780	12,200*	0.846	0.955
	30a	12,258	10,850	13,150	1.073	1.212
6	1	3,418	3,280	3,700	1.083	1.128

* This observed test value is probably low because of some initial condition, since identical panels with larger radii buckled at a higher stress.

developed a fixity of 4 instead of 2 as determined from the compression stresses read in the strain gages, and in the second case the test is questioned, since this specimen was one of a family of specimens and the test results were incompatible with the rest. If these two specimens are excluded, the average conservatism becomes 5.8 per cent.

Panels Subjected to Shear Only (No Stiffener Failures)

As a negative check of the method for predicting stiffener failures, 15 specimens that did not fail in the stiffener were analyzed for stiffener failure. The purpose of this was to determine to what extent stiffener failures would be predicted by the proposed method when they did not actually occur in the test panel. Fig. 9 is a graph of predicted shear stress vs. test shear stress and indicates that for most of the specimens the test stress is lower than the predicted stress. This is as it should be, since these panels did not fail in the stiffener.

Those panels that had a test stress close to the [...]cted stress based on stiffener failure failed by popping of the rivets, and it is questionable whether the panels would have carried much more stress had the rivets not failed.

Panels Subjected to Combined Shear and Compression (Stiffener Failures)

A total of 15 specimens subjected to shear and compression were analyzed. Fig. 10 shows the interaction between effective compression in stiffener due to shear and the direct compression for the test specimens analyzed.

The results indicate an average conservatism for the method of prediction of 5.0 per cent. The test result from one specimen from reference 4 with a high conservatism was considered to be questionable, since it was not compatible with other tests in the same series of specimens. If this specimen is neglected, the average conservatism drops to 3.8 per cent. The actual conservatism in an interaction diagram between shear flow in skin and compression load on stiffener would be less than the above value, since the conservatism enters into only that portion of the shear flow supported by the stiffeners.

Panels Subjected to Shear Only (Web Failures)

Nine specimens failed in the web. Fig. 11 shows a graph of test stress vs. predicted stress for web failure. The results indicate that the method is conservative by an average of 0.3 per cent. There is reason to question

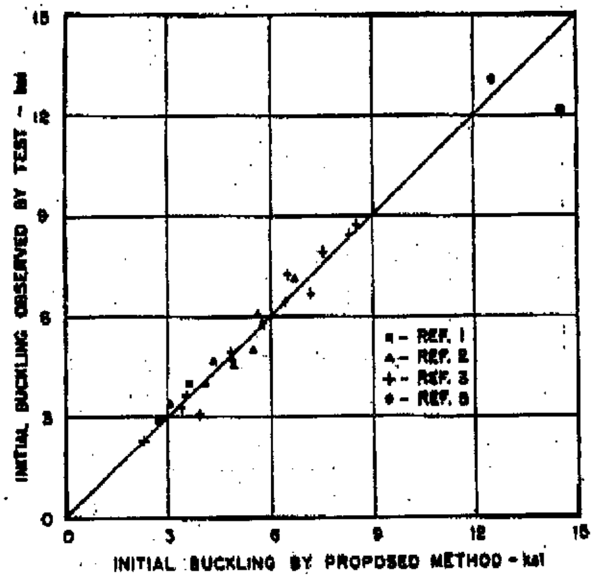

FIGURE 7.- INITIAL SHEAR BUCKLING STRESS IN CURVED PANELS.

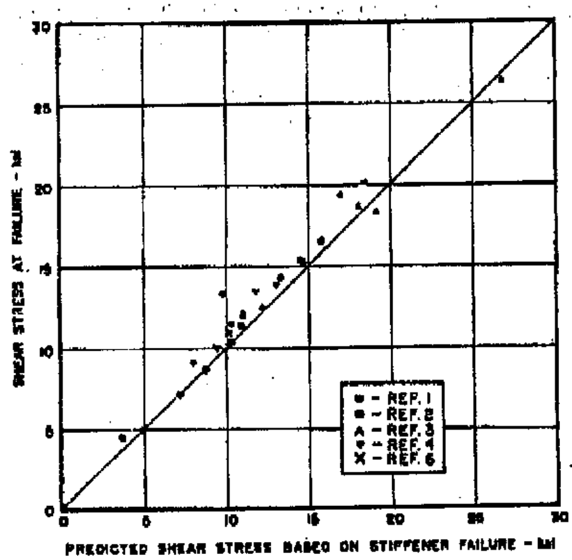

FIG 8 - SHEAR STRESS AT FAILURE vs PREDICTED SHEAR STRESS FOR PANELS WITH STIFFENER FAILURES. PANELS SUBJECTED TO SHEAR ONLY.

the results from three of the specimens. Specimens 20a and 30a of reference 5 failed by tearing the sheet in the end panels and was precipitated by shearing of rivets connecting stringers to jig. In testing specimen 1 of reference 3, the loading jig bound, and the applied torque was actually about 10 per cent greater than the test gage indicated. If these three specimens were omitted, the average conservatism becomes 4.0 per cent.

Typical Application

A typical application to which this method of analysis may be applied is the determination of allowable strengths for a fuselage shell structure. The following is an outline of the procedure for constructing an interaction curve of combined shear and compression on a curved panel. An example of such curves is shown in Fig. 12.

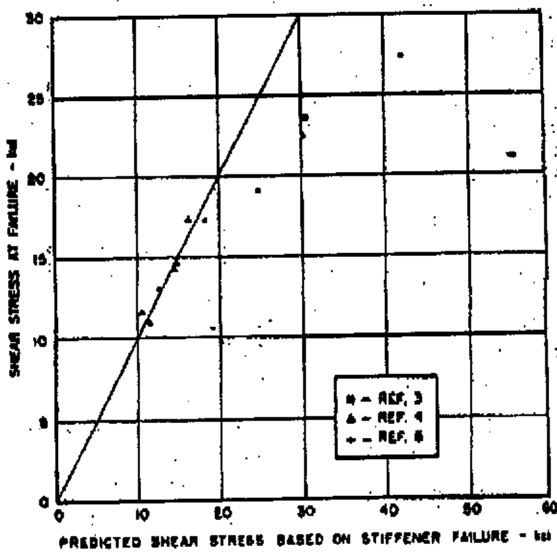

FIG 9 - SHEAR STRESS AT FAILURE vs PREDICTED SHEAR STRESS FOR PANELS WITHOUT STIFFENER FAILURES. PANELS SUBJECTED TO SHEAR ONLY.

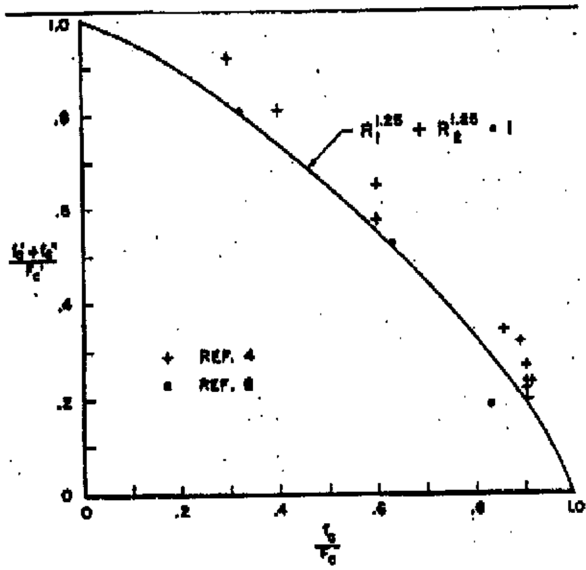

FIGURE 10 - INTERACTION OF COMPRESSION DUE TO SHEAR AND DIRECT COMPRESSION FOR PANELS WITH STIFFENER FAILURES. PANELS SUBJECTED TO SHEAR AND COMPRESSION.

Interaction for Stiffener Failure

(1) The ring spacing, stiffener spacing, stiffener section, skin gage, and the radius of the panel are known; hence, the factor ν may be calculated.

(2) Determine F_c' and F_s. These may be calculated by standard procedures. The allowable compression strength of a panel is often determined as a load per stiffener and skin rather than as a stress on stiffener plus effective skin. For the rest of this example, it will be assumed that P is the allowable compression load and p is the applied compression load.

(3) Assume a value for p.

(4) For the assumed value p, assume several values of f_s and obtain corresponding values of λ from Eq. (6). [λ involves f_{st}, which is determined by $p/(A_{st} + bt)$.]

(5) From Table B, determine f_s' and f_s'' for assumed values of f_s and corresponding values λ.

(6) Determine value of f_s which makes

$$\left(\frac{f_s' + f_s''}{F_c'}\right)^{1.25} = 1 - \left(\frac{p}{P}\right)^{1.25}$$

(7) This value of f_s times t is then the shear flow that may be applied with the compression load p. Repeat this procedure for other values of p to complete the interaction for stiffener failure.

Interaction of Web Failure

The effect of compression on the ultimate shear strength of the web is small, since its only influence is to reduce the critical buckling shear stress, $F_{s_{cr}}$, by the factor λ. The web failure curve with the effect of compression included may be obtained as follows:

(1) Compute η.

(2) Perform steps 3 and 4 under section above.

(3) Determine the value of f_s such that $f_s = F_s$, where F_s is given by Eq. (9). It should be noted that in these determinations $F_{s_{cr}}$ must be multiplied by the factor λ. If the rivet factor $C_R < 0.75$, then an additional check must be made in accordance with Eq. (10). The value f_s times t is then the shear flow that will produce web failure.

The envelope of the stiffener failure curve and web failure curve forms the complete interaction curve for the panel. In plotting the curves, it is recommended that $p = 0$ be the first point investigated, since in this case λ is always equal to one and the trial-and-error solution involved in steps 4, 5, and 6 of the section above is eliminated. Once the value of f_s at $p = 0$ has been determined, it is possible to make a close first approximation as to what f_s will be at other values of p, and again labor of trial and error will be greatly reduced.

Conclusions

(1) Simplified formulas for the calculation of critical t[hick] buckling stress of curved panels have been derived. Test values averaged 2.0 per cent higher than predicted values.

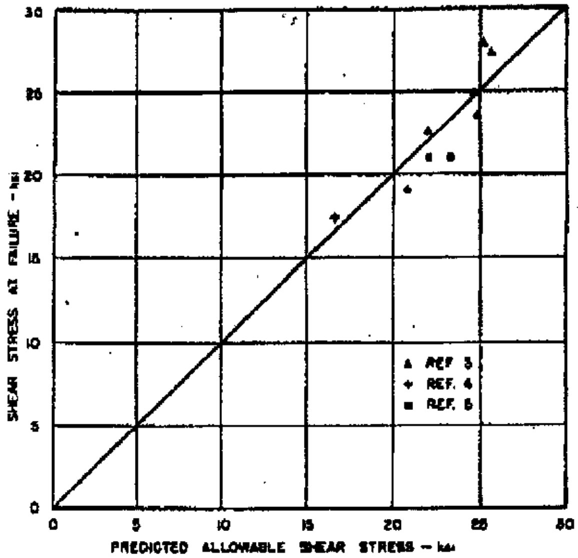

FIGURE 11. - SHEAR STRESS AT FAILURE vs PREDICTED ALLOWABLE SHEAR STRESS FOR PANELS WITH WEB- FAILURES. PANELS SUBJECTED TO SHEAR ONLY.

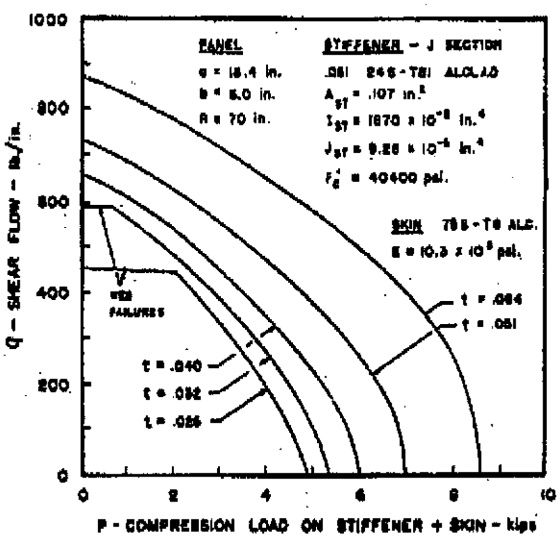

FIGURE 12. - EXAMPLE OF INTERACTION CURVES OF COMBINED SHEAR AND COMPRESSION ON CURVED PANELS.

(2) A method has been determined for predicting the ultimate strength of a stiffened curved panel subjected to shear. The average prediction is approximately 6 per cent conservative for longitudinal stiffener failure and 4 per cent conservative for web failure.

(3) A method has been determined for predicting the ultimate strength of a stiffened curved panel subjected to combined shear and compression. The average prediction is slightly less than 3.8 per cent conservative for longitudinal stiffener failure.

(*Concluded on page 130*)

or

$$\frac{1}{c}\int\int Py\,dS = \frac{U_0}{i\omega c}\int\int Pw\,dS \qquad (56)$$

The integral on the left side is the rolling moment that gives the required result.

Conclusions

A general relation between linearized solutions of lifting-surface problems in direct and reverse flow has been established for compressible nonsteady flows. This relation is a direct extension of that already known for steady-flow solutions. On the basis of the analysis leading to this result, an adjoint variational principle, also the counterpart of one already known for steady flows, has been established. This may be useful in the approximate solution of lifting-surface problems in nonsteady flow. Several applications of the general theorem to problems in nonstationary wing theory have been given. These included determination of relations between certain aerodynamic coefficients for plan forms in direct and reverse flow and establishment of influence functions for total lift, pitching moment, and rolling moment for wings oscillating with arbitrary motion and deformation of the plan form. The influence functions were found to be certain simple solutions for the plan form in reverse flow.

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Analysis of Stiffened Curved Panels Under Shear and Compression

(Concluded from page 110)

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INTER-OFFICE MEMO

G. LAMBERT

FYI

Tim

2 May 1991
81:KET:rlc/42-005

MEMO TO: **Airframe and Landing Gear Design**

COPY TO: **B. Alapic, R. Alsmiller, J.O. Clark, J. Fila, T. Fox (BHTC), L. Graff, G. Grimes, J. Lang, T. Meyers, G. Moore, D. Newland, T. Pekurney, R. Seago, W. Taylor, W. Thomas**

SUBJECT: **MIL-S-81733 vs. MIL-S-8802 Sealant**

In a meeting held on 5-1-91 between representatives of Airframe Design, Airframe Structures, Chem. Lab, Materials and Processes, and Methods and Materials Lab it was decided that, for all future airframe applications, corrosion inhibiting sealant per MIL-S-81733 will be used in preference to sealant per MIL-S-8802, unless fuel resistance is the primary consideration. MIL-S-8802 will continue to be used to seal fuel cell areas.

When new assembly or installation dash numbers containing sealant are created on existing drawings, the sealant should be changed to MIL-S-81733 at that time.

Kurt

Kurt E. Tessnow
Group Engineer
Airframe & Landing Gear Design

H-69-30

MIL-S-81733C
13 March 1980
SUPERSEDING
MIL-S-81733B(AS)
7 June 1976

MILITARY SPECIFICATION

SEALING AND COATING COMPOUND, CORROSION INHIBITIVE

This specification is approved for use by all Departments and Agencies of the Department of Defense.

1. SCOPE

1.1 Scope. This specification covers accelerated, room temperature curing synthetic rubber compounds used in the sealing and coating of metal components on weapons and aircraft systems for protection against corrosion. The sealant is effective over a continuous operating temperature range of -54° to +93°C (-65° to +200°F).

1.2 Classification. The sealing compound shall be of the following types as specified (see 6.2):

Type I - For brush or dip applications
Type II - For extrusion application, gun or spatula
Type III - For spray gun application
Type IV - For faying surface application, gun or spatula

1.2.1 Dash numbers. The following dash numbers shall be used to designate the minimum application time in hours.

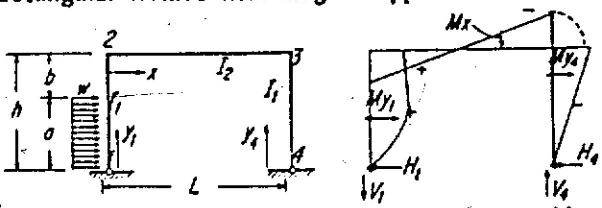
Type I - Dash numbers shall be -1/2 and -2
Type II - Dash numbers shall be -1/2, -2 and -4
Type III - Dash number shall be -1
Type IV - Dash numbers shall be -12, -24, -40 and -48

Example - Type I-½ shall designate a brushable material having an application time of ½ hour. Type I-2 shall designate an application time of 2 hours. All other types and dash numbers shall be designated in a similar manner.

FSC 8030

THIS DOCUMENT CONTAINS 31 PAGES.

Rectangular frames with hinges supports—vertical loading.



7.

$M_2, M_3 = \frac{wa^2}{4}\left[\pm 1 - CK\left(1-\frac{a^2}{2h^2}\right)\right] \qquad H_4 = -\frac{M_3}{h}$

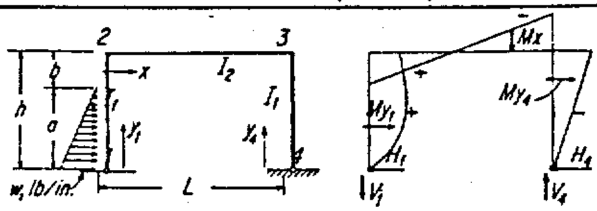
Use plus sign for M_2, minus sign for M_3

$$H_1 = H_4 - wa \qquad V_4 = \frac{wa^2}{2L} = -V_1$$

When $y_1 \leqq a$, $M_{y1} = \frac{way_1}{2}\left(\frac{b}{h}+\frac{a-y_1}{a}\right)+M_2\frac{y_1}{h}$

When $y_1 > a$, $M_{y1} = \frac{wa^2}{2}\left(1-\frac{y_1}{h}\right)+M_2\frac{y_1}{h}$

$$M_{y4} = M_3\frac{y_4}{h} \qquad M_x = M_2\left(1-\frac{x}{L}\right)+M_3\frac{x}{L}$$



8.

$M_2, M_3 = \frac{wa^2}{12}\left[\pm 1 - \frac{CK\left(10-3\frac{a^2}{h^2}\right)}{10}\right] \qquad H_4 = -\frac{M_3}{h}$

Use plus sign for M_2, minus sign for M_3

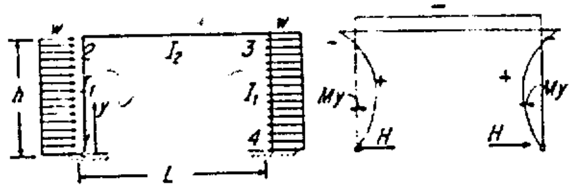
$$H_1 = H_4 - \frac{wa}{2} \qquad V_4 = \frac{wa^2}{6L} \qquad V_1 = -V_4$$

When $y_1 \leqq a$,

$$M_{y1} = \frac{way_1}{6}\left[\frac{b}{h}+\frac{(a-y_1)(2a-y_1)}{a^2}\right]+M_2\frac{y_1}{h}$$

When $y_1 > a$, $M_{y1} = \frac{wa^2}{6}\left(1-\frac{y_1}{h}\right)+M_2\frac{y_1}{h}$

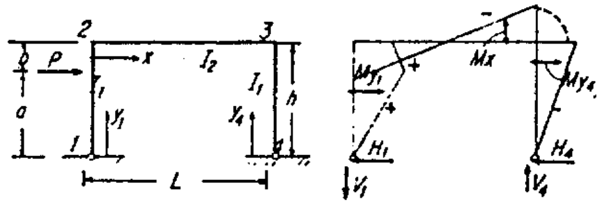
$$M_{y4} = M_3\frac{y_4}{h} \qquad M_x = M_2\left(1-\frac{x}{L}\right)+M_3\frac{x}{L}$$



9.

$$M_2 = M_3 = -\tfrac{1}{4}wh^2CK \qquad V_1 = V_4 = 0$$

$$M_y = \frac{why}{2}+M_2\frac{y}{h} \qquad H = -\frac{wh}{2}-\frac{M_2}{h}$$



10.

$M_2, M_3 = \frac{Pa}{2}\left[\pm 1 - CK\left(1-\frac{a^2}{h^2}\right)\right]$

Use plus sign for M_2, minus sign for M_3

$$H_4 = -\frac{M_3}{h} \qquad H_1 = H_4 - P$$

$$V_4 = \frac{Pa}{L} \qquad V_1 = -V_4$$

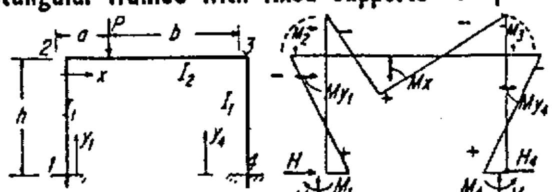
When $y_1 \leqq a$ $\quad M_{y1} = \frac{y_1}{h}(M_2+Pb)$

When $y_1 > a$ $\quad M_{y1} = Pa\left(1-\frac{y_1}{h}\right)+M_2\frac{y_1}{h}$

$$M_{y4} = M_3\frac{y_4}{h} \qquad M_x = M_2\left(1-\frac{x}{L}\right)+M_3\frac{x}{L}$$

Rectangular frames with fixed supports

Rectangular frames with fixed supports—simple loading.



11.

$M_1, M_4 = \frac{Pab}{2L}\left[\frac{1}{D} \pm \frac{L-2a}{LE}\right] \qquad H = \frac{3Pab}{2hD}$

For $\pm$ signs: $M_1 = -$; $M_4 = +$

$$M_2, M_3 = -\frac{Pab}{L}\left[\frac{1}{D} \pm \frac{L-2a}{2LE}\right]$$

Where $M_2 = +$; $M_3 = -$.

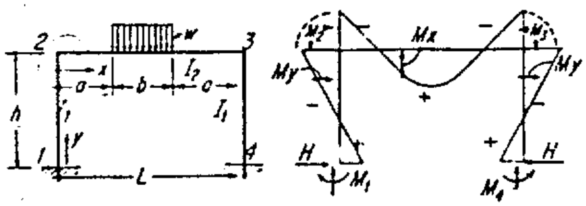
$$V_4 = \frac{Pa}{L}\left[1-\frac{b-(2ab/L)}{LE}\right] \qquad V_1 = P - V_4$$

When $x \leqq a$, $M_x = (Pb+M_3)\frac{x}{L}+M_2\left(1-\frac{x}{L}\right)$

When $x > a$, $M_x = (Pa+M_2)\left(1-\frac{x}{L}\right)+M_3\frac{x}{L}$

$$M_{y1} = M_2\frac{y_1}{h}+M_1\left(1-\frac{y_1}{h}\right)$$

$$M_{y4} = M_4\left(1-\frac{y_4}{h}\right)+M_3\frac{y_4}{h}$$



12.

$$M_1 = M_4 = \frac{wb(3L^2-b^2)}{24LD} \qquad H = \frac{3M_1}{h}$$

$$M_2 = M_3 = -2M_1 \qquad V = \frac{wb}{2}$$

When: $x \leqq a$, $M_x = \frac{wbx}{2}+M_2$

$$a < x \leqq a+b, \quad M_x = \frac{wb}{2}\left[x-\frac{(x-a)^2}{b}\right]+M_2$$

$$x > a+b, \quad M_x = \frac{wb}{2}(L-x)+M_2$$

$$M_y = M_2\frac{y}{h}+M_1\left(1-\frac{y}{h}\right)$$

BELL HELICOPTER COMPANY

Engineering Department

June 15, 1972

STRUCTURES INFORMATION MEMO NO. 1

TO: Mr. N. J. Mackenzie

COPIES TO: SIM Distribution

SUBJECT: PROCEDURE FOR STRUCTURES INFORMATION MEMO (SIM)

REFERENCES: (a) As required
(b) As required

ENCLOSURES: (a) SIM Index
(b) SIM Distribution

This memo is written to establish a procedure for making new and unique structural design information available to members of the Structures groups and appropriate design groups. Much useful information is either generated or collected by members of the Structures groups during the normal performance of their duties. This information is usually available to a limited number of persons and is often filed away and forgotten. In order to prevent valuable information from becoming useless and forgotten, the Structures Information Memo is hereby established as the vehicle for conveying this information.

The Methods and Materials Structures Group Engineer will be the coordinator for all SIM's and will assist in determining what information is valuable enough to publish. He will retain all originals, will assign SIM index number, and update the index and distribution list as required.

Each SIM shall include a cover memo, addressed to the Chief of Structural Design, giving a brief synopsis of the material. The memo shall be signed by the originator and approved by the SIM coordinator. The originator of each SIM shall be responsible for establishing the credibility and accuracy of his information and for preparing the SIM for distribution. Each SIM shall "stand on its own" and be thoroughly checked and referenced. Format of the material is left to the discretion of the originator, however, it should be remembered that all SIM's will be considered for incorporation in a Structures Manual to be issued at a later date. Similar significant structural information originating in any design group will be welcomed and handled in the same manner.

Any additions or deletions to the distribution list should be directed to the SIM coordinator.

M. J. McGuigan

M. J. McGuigan
Chief of Structural Design
Ext. 3147

Add dwg. number part being analyzed in this box.

TITLE HERE

Leave space for revision letter later.

DETAIL PART NOMENCLATURE HERE

State geometry, loads, detail and location or reference Sect. A, Pg._____ where this is shown. See #3.

Compute the actual ultimate stress level, generally from limit loads, referencing a report or page number for the loads.

It may be necessary to compute section properties, loads on the section being analyzed or to determine and show a static balance prior to computation of the stress level.

Compute an allowable or state a reference for the allowables being used.

State limit loads and yield allowables when these are used to prove structure is non-yielding at limit load.

State the margin of safety.

This is the purpose and conclusion of the analysis. Be sure to include fitting factors, casting factors in the M.S. and so state, i.e., "using 1.15 fitting factor". State which formula is used such as,

$$M.S. = \frac{1}{(R_1 + R_2)(Factor)} - 1 = +.xx$$

Confine the analysis within these limits and thereby preserve the neatness of the report.

BELL HELICOPTER COMPANY

Engineering Department

August 28, 1973

STRUCTURES INFORMATION MEMO NO. 3

TO: Mr. R. Lynn

COPIES TO: SIM Distribution

SUBJECT: LOAD SHEETS FOR STATIC TEST OF CASTINGS

This memo is written to standardize the data furnished on the Load Sheets prepared for the Mechanical Laboratory and used by them in the static tests of castings.

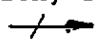
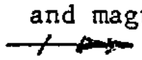
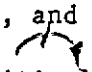
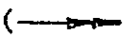
The following information, as a minimum, should be included on all Load Sheets.

1. Title, part no., and name, thus:

   CASTING LOAD SHEET

   Part No. 204-XXX-XXX-X, Bellcrank, Cyclic. Control

2. Indicate loads as "limit" or "ultimate". Ultimate preferred.

3. Draw sketches of part showing the external load application, direction and magnitude, and the reactions (usually designated by , , and ). Sufficient views shall be used to completely define the critical loading condition. Each view shall show the reactions necessary to place the part, with its applied external load(s), in a state of static equilibrium. The loads and reactions shall be the same as those used in the structural analysis to insure that the part will be tested in the same manner as it was analyzed. Where moment vectors () are used a note shall be included to indicate whether the right or left hand rule is applicable.

4. When available the report number from which loads and reactions were obtained shall be referenced, thus:

   Ref. Report 205-XXX-XXX

5. A note shall give a brief description of the loading condition, thus:

   Loading Condition: 8G Forward Crash

6. A note shall indicate the casting factor, thus:

   Loads include a 1.33 casting factor

   Where the casting factor is unity, so indicate, thus:

   Casting factor of 1.00 is applicable

7. Any other special information necessary to assure that the casting will be tested as it was analyzed.

8. All Load Sheets shall be prepared on stress pad paper.

An example Load Sheet is attached. Note that the rule for the moment vectors was omitted.

M. J. McGuigan

M. J. McGuigan
Chief of Structural Design
Ext. 3147

BELL HELICOPTER COMPANY

Engineering Department

27 February 1974

STRUCTURES INFORMATION MEMO NO. 4

TO: Mr. R. Lynn

COPIES TO: SIM Distribution, B. Alapic, J. Duppstadt, J. Garrison, J. Gilday, W. Rollings, C. Sloan, K. Wernicke

SUBJECT: BOLTS IN MOVABLE CONTROL SYSTEM JOINTS

In order to avoid the possibility of installing an understrength bolt and to provide increased resistance to repeated loads, the following policy shall be implemented on the Model 409, Model D306, Model 301; the production series of the Model 214 and Model 206L; and all future designs.

- NAS quality bolts shall be installed in the movable portion of all control system joints.

M. J. McGuigan
Chief of Structural Design
Ext. 3147

BELL HELICOPTER COMPANY

Engineering Department

12 March 1974

STRUCTURES INFORMATION MEMO NO. 5

TO: Mr. R. Lynn

COPIES TO: SIM Distribution

SUBJECT: JUMP TAKEOFF LOADS

Recently, it has come to my attention that we are not addressing the rotor tilt for the jump takeoff conditions in a consistent manner. In order to provide a uniform approach, the following procedure shall be followed:

- Assume the helicopter has landed on a slope of specified magnitude in any direction (normally 6°) and executes a vertical takeoff at maximum load factor for this condition. The rotor tilt will be that which is necessary to execute this maneuver.

Orvill Baker

O. Baker
Senior Group Engineer
Airframe Structures

BELL HELICOPTER COMPANY

Engineering Department

2 August 1974

STRUCTURES INFORMATION MEMO NO. 6

TO: Mr. R. Lynn

COPIES TO: SIM Distribution

SUBJECT: DETERMINATION OF FAILURE MODES

ENCLOSURE: Suggested Form for Recording Failure Modes

Beginning with the Model 222, and effective for all future design activity, the Airframe and Dynamic Structures Groups will establish and maintain a notebook which shows the first and second predicted failure modes for all structural elements. The maintenance of these notebooks will be the responsibility of the lead structures engineer for each project.

The determination of these failure modes will consider static and dynamic loads along with other contributing factors, such as temperature, corrosion, and fabrication effects. The primary control will be maintained at the subassembly level (i.e., engine mount, bulkhead, main beam, etc.). Primary and secondary failure modes for static and fatigue loadings will be determined for each subassembly. For those elements which are subjected to static or fatigue testing, the results of those tests will be entered in the notebook. In addition, any service problems encountered in the production cycle of the element will be entered. A suggested form for these records is enclosed.

To aid the designer in his determination of these failure modes, the structural design groups will supply the designer with the critical loads for the structural element under consideration. These will be supplied in the form of a sketch or free body of the element with the applied loads and reactions. These loads will be updated as the mathematical model is refined during the design process.

The establishment and maintenance of these records can mean much in establishing the rationale for a particular design, tracking its performance and guiding similar designs in the future. Your cooperation in implementing the procedure is essential to its success.

M. J. McGuigan

M. J./McGuigan
Chief of Structural Design
Ext. 3147

DWG. NO.	DESCRIPTION	PREDICTED FAILURE MODE		TEST RESULTS	SERVICE HISTORY
		PRIMARY	SECONDARY		

BELL HELICOPTER COMPANY
Engineering Department

8 August 1974

STRUCTURES INFORMATION MEMO NO. 7

TO: Mr. R. Lynn

COPIES TO: SIM Distribution

SUBJECT: STRUCTURES APPROVAL OF ENVELOPE, SOURCE CONTROL, AND SPECIFICATION CONTROL DRAWINGS

It has recently come to my attention that some of the subject type drawings do not always contain adequate information to allow us to properly validate the item to the government or the FAA. For example, castings may be purchased from a Source Control Drawing without proper inspection or test requirements being fully met within the company. The Source Control Drawing may make no reference to x-ray requirements, static test requirements or any other special inspections required on castings.

Therefore, all Structural Design personnel who have occasion to sign Envelope, Source Control, or Specification Control Drawings shall, as a minimum, establish that the drawing adequately defines the following:

- Configuration
- Mounting and mating dimensions
- Dimensional limitations (interferences)
- Performance (loads, environment, life, etc.)
- Weight limitations
- Reliability requirements
- Interchangeability requirements
- Test requirements
- Verification requirements (analysis or test)
- Material limitations (example, no castings allowed, etc.)
- Casting classification if allowed (also casting factor)
- Primary Part designation
- Reference to applicable specification

Also, if special inspections and tests such as x-rays and static tests are required, Project should be alerted so that plans can be made to procure parts for the required tests.

"Approved Sources of Supply" or "Suggested Sources of Supply" shall not be approved by Structures until we are completely satisfied that the proposed vendor item does meet all structural requirements. This may mean vendors must submit stress analyses of their design or test data as a part of their proposal.

On all Primary Parts or other items with significant structural requirements, the Structures Engineer shall retain a copy of the approved design, vendor stress analysis and test data and file this information in the proper Drawing Check Notebook.

It is hoped that other design groups will use this or some other check list for processing these type drawings.

M. J. McGuigan

M. J. McGuigan
Chief of Structural Design

Bell Helicopter TEXTRON
A Subsidiary of Textron Inc

INTER-OFFICE MEMO

13 August 1984
81: GRG:MLZ:db- 1650

MEMO TO: Airframe Designers, Stress Analysts, Checkers, E. Ryba

COPY TO B. Alapic, O. Baker, T. Eidson, W. Fontain, K. Tessnow

SUBJECT: BONDED PANEL INSERTS

REFERENCE: (a) IOM 81:JMS/DET:jhb-460

Reference (a) specified three types of potted inserts, NAS1832C, NAS1834C and NAS1835C, which are acceptable for use with GR/EP composite panels. It is the opinion of Airframe Structures that the use of these inserts be limited to non-structural and structural shear applications only. These inserts must be considered non-structural for tension applications (unless modified with an enclosing doubler), because when installed there is no bearing surface on the composite facings of the panels.

A vendor is investigating the manufacture of the 80-007 insert using 300 series CRES material. This insert should be available for callout on drawings presently in work. Also under consideration is a flat head 80-013 plug and sleeve type insert that is domed on one end for use in fuel cell panels. These inserts are considered to be structural for tension and shear applications. Information regarding these inserts will be forwarded to the cognizant personnel as soon as it becomes available.

G. R. Grimes
Group Engineer
Airframe Structures

M. L. Zaleski
Airframe Structures

BELL HELICOPTER TEXTRON INC.

Engineering Department

24 January 1984

STRUCTURES INFORMATION MEMO NO. 17

TO: SIM Distribution

SUBJECT: LATERAL LOAD CRITERIA FOR COLLECTIVE CONTROL

To preclude inadvertent damage from handling, the following additional criteria will be met on all future collective control systems:

> 170 pound limit load applied separately in a horizontal plane inboard or outboard at the center of the collective handgrip.

O. K. McCaskill
for O. Baker
Manager of Structural Analysis

F. Wagner
Director of Vehicle Design

STRUCTURES INFORMATION MEMO NO. 14

To: SIM DISTRIBUTION

Subject: STRUCTURAL APPROVAL POLICY

Reference: Structures Information Memo No. 7 - "Structures Approval of Envelope, Source Control and Specification Control Drawings"

Structures Group approval of any drawing is defined as structural approval of all parts called out on that drawing regardless of whether or not they are Bell designed parts.

It is therefore the responsibility of the Structures Engineer who signs a drawing to satisfy himself that all components of that drawing, including vendor part numbers, standard parts and specification controlled items, meet Bell's structural requirements for that particular installation. For components that are defined by Bell Procurement Specification, Specification Control Drawing or Source Control Drawing, the guidelines of SIM No. 7, as amplified here, are to be followed. The Structures Engineer must be assured that the controlling Bell specification or drawing contains adequate requirements for vendor stress analysis and/or structural test proposal and results report to assure that strength requirements are met. Provision should be made for FAA conformity and for Bell witness of testing, if required.

In the case of a product defined entirely by vendor's drawings and procured by their part number, the Structures Engineer must notify the Project Engineer in writing of the extent of structural substantiation by analysis or testing required from the vendor. Provision should be made for FAA conformity and for Bell witness of testing, if required. It must be made clear that drawing approval is contingent upon successful completion of analysis or testing and submittal of these data for structures approval. If Bell testing is indicated, EWAs and schedules must be written to establish these tests.

David Poster
Dave Poster
Director of Design Engineering

Orville Baker
Orville Baker
Manager of Structural Analysis

BELL HELICOPTER TEXTRON

Engineering Department

STRUCTURES INFORMATION MEMO NO. 13

To: SIM Distribution

Subject: FITTING FACTORS, THEIR DEFINITION AND APPLICABILITY

Reference: FAR 29.623, 29.619

A fitting factor is a 1.15 load factor, applied to limit loads, and is in addition to the 1.50 factor of safety. It accounts for uncertanties such as deterioration in service, manufacturing process variables and unaccountability in the inspection processes.

For design considerations, a fitting shall be defined as part(s) used in a primary structural load path whose principal function is to provide a load path through the joint of one member to another. The connecting means is generally a single fastener.

A fitting factor is applicable to the fitting, the fastener bearing on the joined members, as well as the attachments joining the fitting(s) to the structure. It is particularly considered when failure of such fitting would not allow load redistribution in a manner that would provide continued safe flight and that load redistribution cannot be verified by analysis or test. Obviously then, a fitting factor is applied to non-redundant connecting members in primary load path applications the failure of which may affect safety of the aircraft and its occupants. It is applied until the load is distributed into the surrounding back-up structure to which the fitting is attached.

A fitting factor is not applicable to:

a) Crash load factors that are the only design condition and/or crash load factors that exceed limit load factors x 1.5 x 1.15.

b) A continuous riveted joint(s) in basic structure when section properties remain consistent throughout the joint and the joint consists of approved practices and methods such as splices of main beam caps - riveted door post caps to bulkheads, riveted skin splice doublers, continuous riveted skins to longerons, continuous riveted structure such as bulkheads to beams or intercostals, or frames, etc.

c) An integral fitting beyond the point where section properties become typical of the part. Example, integrally fabricated lug on a forging, or machining.

d) Welded joints.

e) To a member when a larger load factor is used such as a larger special bearing factor, a 1.25 casting factor, a 1.33 fatigue factor, a 1.33 retention factor of seats and safety belts.

f) Systems or structure when they are verified by limit or ultimate load tests. The fixed control system is an example of this exception.

g) Bonded inserts and/or fittings in sandwich panels.

h) A fitting in redundant connecting members.

Orville Baker
Manager of Structural Analysis
Ext. 3147

D. Poster
Director of Design Engineering

BELL HELICOPTER TEXTRON
ENGINEERING DEPARTMENT

7 August 1981

STRUCTURES INFORMATION MEMO NO. 12

To: Mr. R. Lynn

Copies to: SIM Distribution, Engineering Design Groups, Check Group, D. May

Subject: 7050-T73 RIVETS IN LIEU OF 2024-T31 (ICE BOX) RIVETS

7050-T73 rivets will be utilized in lieu of 2024-T31 "DD" (ice box) rivets as of 20 July 1981. The 7050 rivets can be stored at room temperature, thereby eliminating numerous problems that exist with the 2024 rivets.

The following policy will be implemented.

1. Manufacturing will utilize the 7050-T73 rivets to supersede the MS 20426DD and MS 20470DD rivets (Reference, the SUPER-CESSION LIST, BHT Standard 170-001, Revision "G"), effective the target date of 20 July 1981.

2. The 100° flush and protruding head 7050 aluminum alloy rivets are delineated in BHT Standards 110-174 and 110-175, respectively.

3. All new drawings initiated after this date will call out the 7050 rivets for 3/16 and 1/4 inch diameters. Approval for other diameters MUST be obtained from applicable Structures and Design Group Engineers prior to utilization.

4. The 7050 rivets "will not" be utilized to replace "AD" rivets (generally used in 5/32 inch diameters and smaller) at this time (not cost effective).

5. The driven shear strengths for both the 7050 and 2024 rivets are established for an F_{su} = 41 ksi. Until MIL-HDBK-5 allowables are available, 2024-T31 MIL-HDBK-5 data in the 3/16 and 1/4 inch diameters, for both protruding and 100° flush heads, are acceptable for 7050-T73 installations and should be so identified for report referencing. It is anticipated that 7050-T73 MIL-HDBK-5 allowables will be available during the 1981-1982 time frame.

Orville Baker
O. Baker
Manager of Structural Analysis

D. Poster
D. Poster
Director of Design Engineering

BELL HELICOPTER TEXTRON
ENGINEERING DEPARTMENT

16 January 1980

STRUCTURES INFORMATION MEMO NO. 11

To: SIM Distribution

Subject: Emergency Float Kit Loads

In addition to the existing design conditions, emergency float kit loads must be developed for the following conditions:

1. Floats in the water at 0.8 bag buoyancy and combined with salt water drag for 20 knots forward speed. These loads will be treated as limit loads. These loads will be applied at angles corresponding to the righting moments, but not to exceed 20^{o}.

2. For skid mounted floats;

   a) A computer drop will be done in a tail down attitude for limit sink speed. Skids will be checked for a positive M.S. at yield.

   b) Crosstubes will not yield with the helicopter in the water, floats inflated and no rotor lift.

M. J. McGuigan, Jr.
M. J. McGuigan, Jr.
Manager of Structures Technology

BELL HELICOPTER TEXTRON
ENGINEERING DEPARTMENT

STRUCTURES INFORMATION MEMO NO. 10

To: SIM Distribution

Subject: Design Criteria for Doors and Hatches

Unless otherwise specified in a Detail Specification or Structural Design Criteria Report, the Structural Design Criteria presented herein should be used on new designs for the following:

1. Access doors
2. Hinged or sliding canopies
3. Sliding doors
4. Passenger doors
5. Crew doors
6. Cargo compartment doors
7. Emergency doors
8. Escape hatches

All loads associated with the use and operation of doors and hatches terminate in the latches and hinges and their attachment to the airframe. The sources of these loads are:

1. Open canopy during approach or taxi operation
2. Gusts
3. Outward push from personnel
4. Air loads
5. Rough handling

1. Open canopy during approach or taxi operation

If a sliding or hinged canopy is used, it should be designed to withstand an air load from taxi operations of up to 60 kt.

2. Gusts

All doors that are subject to damage by ground gusts and wind loads from other helicopters being run up or taxied nearby or flown close overhead, should be provided with a means to absorb the energy resulting from a 40 kt ground gust occurring during opening or closing. Doors and access doors or panels that have a positive hold-open feature should be capable of withstanding gust loads to 65 kt when the door or panel is in the open position and unattended.

3. Outward push from personnel

Due to possible inadvertent loading by personnel, passenger doors should be capable of withstanding an outward load of 200 lb. without opening. Also, doors between occupied compartments shall be capable of withstanding a load of 200 lb. in either direction without opening. These loads are assumed to be applied upon a 10 sq. in. area at any point on the surface of the door. Yielding and excessive deflections are permitted but the door must not open.

4. Air loads

The air loads on doors and hatches for helicopters probably are minimal when compared to the many personnel-oriented loads. The air loads, however, should be investigated, including the application of the appropriate gust criteria. All doors should be capable of withstanding air loads up to V_D in the closed position. All sliding cargo and passenger doors should be capable of withstanding air loads up to 120 knots in the full open position and up to 80 knots in any partially open position.

5. Rough handling

All doors and hatches that are likely to receive rough handling during their lifetime should be capable of withstanding loads they are expected to receive in operation. Passenger and crew doors should withstand a 150 pound load applied downward at the most critical location without permanent deformation. All other doors that are unlikely to be stepped on or used as a handhold or which are marked with a "NO STEP" or "NO HANDHOLD" decal should withstand a 50 pound load parallel to the hinge pin axes and a 50 pound load perpendicular to the surface without permanent deformation.

M. J. McGuigan, Jr.
M. J. McGuigan, Jr.
Chief of Structures Technology

Bell Helicopter TEXTRON
Division of Textron Inc.

INTER-OFFICE MEMO

16 February 1979
81:GRA:jo-831

Memo To: Production Airframe Stress Group

Copies To: Messrs. O. Baker, M. Glass, W. Kuipers, O.K. McCaskill, J. McGuigan, G. McLeod, R. Scoma

Subject: Analysis of Castings Where Foundry Weld Repair is Allowed

Reference: SIM No. 9

The referenced SIM specifies the amount of reduction to be applied to the allowables of five (5) commonly used casting alloys when the foundry is allowed to make weld repair of casting flaws per BPS 4470.

Our Final stress analyses should take these factors into account in the following manner.

1. List basic allowables at the beginning of the analysis of the part in question.
2. If weld repair is allowed in area being analyzed, state "Weld Repair Allowed," and reduce the allowable by the amount shown in SIM No. 9. Note that the reduction factor is for weld repair.

This memo should be attached to your copy of SIM No. 9.

G. R. Alsmiller, Jr.
Group Engineer
Production Airframe Stress

BELL HELICOPTER TEXTRON
Engineering Department

4 May 1976

STRUCTURES INFORMATION MEMO NO. 9

To: SIM Distribution

Subject: Mechanical Properties Reduction Factors for Castings with Foundry Weld Repair

References:
a) BHC Report 599-233-909, "The Effect of Weld Repair on the Static and Fatigue Strengths of Various Cast Alloys"

b) BPS FW 4470 - In Process Welding of Castings

c) ASM Technical Report W 6-6.3, "Static and Fatigue Properties of Repair Welded Aluminum and Magnesium Premium Quality Castings"

Future casting drawings should have a note that permits the in process welding of castings per BPS 4470. To allow for this weld repair, parts should be analyzed using the following reductions in allowables.

Material	Reduction Factors for Foundry Weld Repair			
	Ultimate Tensile	Yield Tensile	Elongation	Endurance Limit
356-T6	10%	5%	0	10%
A356-T6	10%	13%	0	10%
AZ91-T6	25%	22%	50%	10%
ZE41-T5	10%	0	50%	10%
17-4 PH	0	2%	30%	10%

In those circumstances where the part cannot be sized to allow for weld repair throughout the part, a weld map should be provided on the drawing to indicate those areas which may receive weld repair.

If the entire part is so critical that no weld repair can be permitted and the part cannot be redesigned, the drawing and all analysis should be clearly marked "No Weld Repair Allowed." All 201 Aluminum Alloy castings shall be marked "No Weld Repair Allowed".

M. J. McGuigan, Jr.
Chief of Structures Technology

SIM
DISTRIBUTION

BELL HELICOPTER COMPANY

INTER-OFFICE MEMO

Date 2 June 1976

81:WCF:bb-092

TO: Engineering Design Chiefs, & Group Engineers

COPIES TO: Messrs: A. Green, L. Hochreiter, J. McGuigan, D. Poster, M. P. Smith, Jr., J. Weathers

Reference: (a) S.I.M. No. 9 dtd 4 May 1976
(b) BPS 4470 - In-Process Welding of Casting

Subject: IN-PROCESS REPAIR WELDING OF ROUGH CASTING

In accordance with Reference (a), future casting drawings will be analyzed to allow for in-process repair welding of rough castings. New casting drawings shall have one of the following notes.

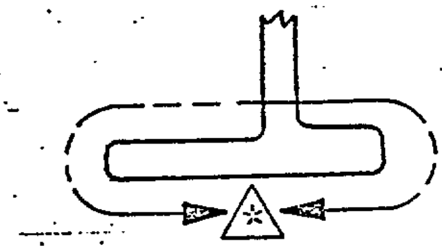
If the entire casting can be repair welded, the following general note shall be added:

In-process repair welding permissible per BPS 4470.

In those circumstances where the casting cannot be sized to allow for repair weld throughout the part, the drawing shall indicate those areas which may receive repair weld. This area will be flagged with the following note:

△ In-process repair welding permissible per BPS 4470 in this area only.

Example of callout on F/D.



If the entire casting is so critical that no repair weld can be permitted, the following note shall be added:

No repair welding allowed.

Please circulate to all Design Personnel. This procedure will be added to the DRM at the next revision.

W. C. Fountain

W. C. Fountain
Chief Draftsman

BELL HELICOPTER COMPANY
Engineering Department

26 February 1975

STRUCTURES INFORMATION MEMO NO. 8

TO: Mr. R. Lynn

COPIES TO: SIM Distribution, Engineering Design Groups, Check Group

SUBJECT: EDGE DISTANCE REQUIREMENTS FOR NAS 1738 AND NAS 1739 BLIND RIVET INSTALLATION

As stated in MIL-HDBK-5B, paragraph 8.1.4, Blind Fasteners, "The strength values were established from test data and are applicable to joints having values of e/D equal to or greater than 2.0. Where e/D values less than 2.0 are used, tests to substantiate yield and ultimate strengths must be made." On page 1-11 of MIL-HDBK-5B, e is defined as the distance from a hole centerline to the edge of the sheet and D is the hole diameter.

The ultimate and yield strength values for NAS 1738 locked spindle blind rivets are based on a hole diameter of 0.144 for a 1/8 rivet, 0.177 for a 5/32 rivet, and 0.2055 for a 3/16 rivet, reference MIL-HDBK-5B, Table 8.1.4.1.2(d). The shank diameter for the NAS1738 and NAS 1739 rivets are 0.140 for a 1/8 rivet, 0.173 for a 5/32 rivet, and 0.201 for a 3/16 rivet.

Loft and sometimes Engineering Design will dimension edge distances and parts for the NAS 1738 blind rivet based on two times the 5/32 value (.31) rather than two times the 0.177 MIL-HDBK-5B value (.36), for example. This practice results in a rivet edge distance of less than 2.0; therefore, the MIL-HDBK-5B strength values in Table 8.1.4.1.2(1) for NAS 1738B rivets are not applicable.

In conclusion, to ensure the correct edge distance is used when planned patterns of NAS 1738 and NAS 1739 rivets are installed, Structures Group recommends that the correct edge distance dimension be specified on the face of the drawing for rivet patterns rather than using the drawing note that states rivet e/D is equal to two times the rivet shank diameter. Also, special attention must be given to skin overlaps, and bulkhead and stiffener flange dimensioning. The edge distance for the countersunk NAS 1739 rivet of 2.5 times the rivet shank diameter is valid because MIL-HDBK-5B values for the NAS 1739 rivet are based on two times the hole diameter. The table below summarizes the recommended minimum nominal edge distance values for NAS 1738 and NAS 1739 blind spindle locked rivets.

Rivet Size	EDGE DISTANCE NAS 1738	NAS 1739
1/8	.29	.32
5/32	.36	.39
3/16	.41	.47

J. McGuigan
Chief of Structures Technology

D. Poster
Manager of Design Engineering