Quick Study® ACADEMIC

GEOMETRY - Part 2

segments.lines.planes.geometric formulas

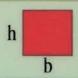
GEOMETRY HISTORY

Geometry means Earth measurement; early peoples used their knowledge of geometry to build roads, temples, pyramids and irrigation systems; Euclid (300 B.C.) organized Greek geometry into a 13-volume set of books named **The Elements**; thus, the formal geometry studied today is often called Euclidean geometry (also called plane geometry because the relationships deal with flat surfaces); geometry has undefined terms, defined terms, postulates (assumptions that have not been proven, but have "worked" for thousands of years), and theorems (relationships that have been mathematically and logically proven).

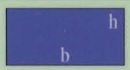
GEOMETRIC FORMULAS

Perimeter: The perimeter, P, of a two-dimensional shape is the sum of all side lengths. **Area:** The area, A, of a two-dimensional shape is the number of square units that can be put in the region enclosed by the sides. *Note: Area is obtained through some combination of multiplying heights and bases, which always form 90° angles with each other, except in circles.* **Volume:** The volume, V, of a three-dimensional shape is the number of cubic units that can be put in the region enclosed by all the sides.

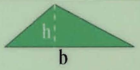
Square Area:
$A=hb$; if $h=8$ and $b=8$ also, as all sides are equal in a square, then: $A=64$ square units



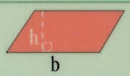
Rectangle Area:
$A=hb$, or $A=lw$; if $h=4$ and $b=12$, then: $A=(4)(12)$, $A=48$ square units



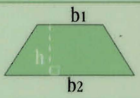
Triangle Area:
$A=\frac{1}{2}bh$; if $h=8$ and $b=12$, then: $A=\frac{1}{2}(8)(12)$, $A=48$ square units



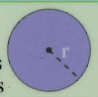
Parallelogram Area:
$A=hb$; if $h=6$ and $b=9$, then: $A=(6)(9)$, $A=54$ square units



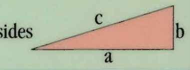
Trapezoid Area:
$A=\frac{1}{2}h(b_1+b_2)$; if $h=9$, $b_1=8$ and $b_2=12$, then: $A=\frac{1}{2}(9)(8+12)$, $A=\frac{1}{2}(9)(20)$, $A=90$ square units



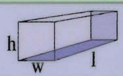
Circle Area:
$A=\pi r^2$; if $\pi=3.14$ and $r=5$, then: $A=(3.14)(5)^2$, $A=(3.14)(25)$, $A=78.5$ square units
Circumference: $C=2\pi r$, $C=(2)(3.14)(5)=31.4$ units



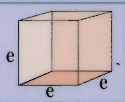
Pythagorean Theorem:
If a right triangle has hypotenuse *c* and sides *a* and *b*, then $c^2=a^2+b^2$



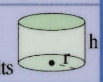
Rectangular Prism Volume:
$V=lwh$; if $l=12$, $w=3$ and $h=4$, then: $V=(12)(3)(4)$, $V=144$ cubic units



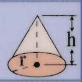
Cube Volume:
$V=e^3$; each edge length, *e*, is equal to the other edge in a cube; if $e=8$, then: $V=(8)(8)(8)$, $V=512$ cubic units

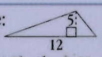


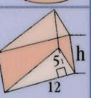
Cylinder Volume:
$V=\pi r^2h$; if radius $r=9$ and $h=8$, then: $V=\pi(9)^2(8)$, $V=3.14(81)(8)$, $V=2034.72$ cubic units



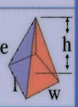
Cone Volume:
$V=\frac{1}{3}\pi r^2h$; if $r=6$ and $h=8$, then: $V=\frac{1}{3}\pi(6)^2(8)$, $V=\frac{1}{3}(3.14)(36)(8)$, $V=301.44$ cubic units



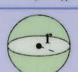
Triangular Prism Volume:
$V=$(area of triangle)h; if  has an area equal to $\frac{1}{2}(5)(12)$, then: $V=30h$ and if $h=8$, then: $V=(30)(8)$, $V=240$ cubic units



Rectangular Pyramid Volume:
$V=\frac{1}{3}$(area of rectangle)h; if $l=5$ and $w=4$, the rectangle has an area of 20, then: $V=\frac{1}{3}(20)h$ and if $h=9$, then: $V=\frac{1}{3}(20)(9)$, $V=60$ cubic units

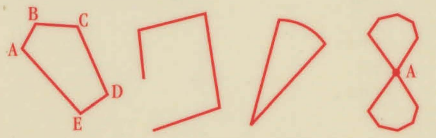


Sphere Volume:
$V=\frac{4\pi r^3}{3}$; if radius $r=5$, then: $V=\frac{4(3.14)(5)^3}{3}$, $V=\frac{1570}{3}$, 523.3 cubic units



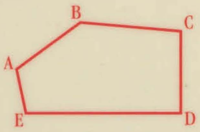
POLYGONS

A. **Polygons** are plane shapes that are formed by line segments that intersect only at the endpoints. These intersecting line segments create one and only one enclosed interior region.



Polygon ABCDE | Not a polygon; this shape is not closed. | Not a polygon; one side is not a line segment. | 2 polygons intersecting at point A.

B. Polygons are named by listing the endpoints of the line segments in order going either clockwise or counterclockwise, starting at any one of the endpoints.



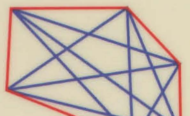
This may be called polygon ABCDE or polygon CBAED, or ...

C. The **sides** of polygons are line segments; polygons are all of the points on the sides (line segments) and vertices.

The sides of the polygon shown above are $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$ and $\overline{EA}$. The vertices are points A, B, C, D and E.

D. The **vertices** (or vertexes) of polygons are the endpoints of the line segments.

E. **Diagonals** of a polygon are line segments whose endpoints are vertices of the polygon, but diagonals are not line segments that are the sides of the polygons.

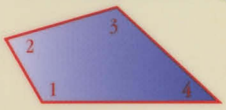


The red line segments are sides of the polygon. The blue line segments are the diagonals.

F. The **interior** of a polygon is all of the points in the region enclosed by the sides. The exterior of a polygon is all of the points on the plane of the polygon, but not on the sides nor in the interior of the polygon.

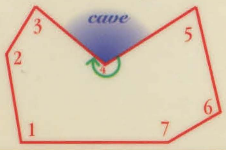
The points indicated in red are the points of the polygon.

G. The **interior angles** of a polygon are the angles that have the same vertex as one of the vertices of the polygon and have sides and interiors that are also sides and interiors of the polygon. Every polygon has as many interior angles as it has vertices.



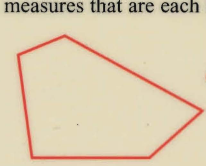
Angles 1, 2, 3 and 4 are interior angles of the polygon.

H. **Concave** polygons have at least one interior angle whose measure is more than 180 degrees.



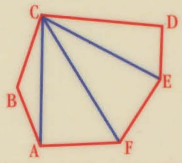
Interior angle 4 is more than 180°, so this polygon is concave. Notice the "cave" in concave and in the concave polygon.

I. **Convex** polygons have no interior angles that are more than 180 degrees. All interior angles have measures that are each less than 180 degrees.



Convex polygons have no "caves."

1. **Theorem:** The sum of the measures of the interior angles of a convex polygon with n sides is (n - 2) 180 degrees.



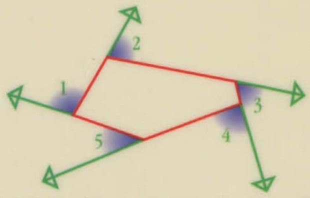
Pick any one vertex and draw all diagonals from that one vertex. There will always be 2 less triangles than number of sides of the polygon. Since each triangle has angles that total 180°, multiply the number of triangles by 180°; thus, the formula $(n - 2)\ 180°$. In polygon ABCDEF, $n = 6$ because there are 6 sides. Using C and drawing all diagonals from point C creates 4 triangles, so $(n - 2)\ 180° = (6 - 2)\ 180° = (4)180° = 720°$.

Note: To find the measures of the interior angles of a regular polygon, find the sum of all of the interior angles and divide by the number of interior angles. Thus, the formula:

$$\frac{(n - 2)\ 180°}{n}$$

If hexagon ABCDEF, above, were a regular hexagon, all angles would be equal, so $\frac{(n - 2)\ 180°}{n} = \frac{720}{6} = 120°$ each.

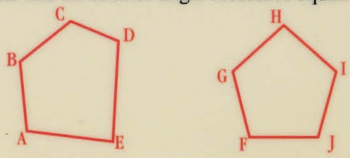
J. **Exterior angles** of polygons are formed when the sides of the polygon are extended. Each exterior angle has a vertex and one side which are also a vertex and one side of the polygon. The second side of the exterior angle is the extension of the polygon sides.



Angles 1, 2, 3, 4 and 5 are exterior angles of the polygon, and their sum equals 360°.

1. **Theorem:** The sum of the measures of the exterior angles of any convex polygon, using one exterior angle at each vertex, is 360 degrees.

K. **Regular polygons** are polygons with all side lengths equal and all interior angle measures equal.



Polygon ABCDE | Regular polygon FGHIJ because $FG = GH = HI = IJ = JF$ and $m\angle F = m\angle G = m\angle H = m\angle I = m\angle J$.

L. Classifications of Polygons
1. Polygons are classified by the number of sides, which is equal to the number of vertices.
2. The side lengths are not necessarily equal unless the word "regular" is also used to name the polygon. A regular polygon has equal side lengths and equal interior angle measurements.

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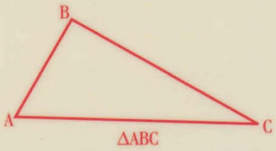
3. Categories

Polygon Name	Number of Sides
Triangles	3
Quadrilaterals	4
Pentagons	5
Hexagons	6
Heptagons (Septagons)	7
Octagons	8
Nonagons	9
Decagons	10
n-gons	n

M. Special Polygons

1. Triangles

a. **Triangles** are polygons with 3 sides; each triangle also has 3 vertices; the symbol for triangle is Δ.

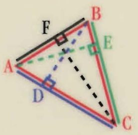


ΔABC

b. Altitudes and bases of triangles are used to find the areas.

i. The **altitude** (**height**) of a triangle is the line segment whose endpoints are a vertex of the triangle and the point on the line containing the opposite side of the triangle where a 90-degree angle is formed; the altitude is perpendicular to the line containing the side opposite the vertex of the triangle. Since a triangle has 3 vertices, every triangle has 3 altitudes.

ii. The **base** of a triangle is the side of the triangle that is on the line that is perpendicular to the altitude. Since every triangle has 3 altitudes, every triangle has 3 bases. Each altitude has a different side that is the base.



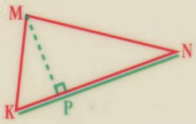
In ΔABC, $\overline{BD}$ is the altitude for base $\overline{AC}$, $\overline{AE}$ is the altitude for base $\overline{BC}$, and $\overline{CF}$ is the altitude for $\overline{AB}$. N c c m 90° ng w

iii. The **area of a triangle** may be found by applying this formula: $A = \frac{1}{2}ab$, where

A means area

a means altitude

b means base. Any base and its altitude may be used.

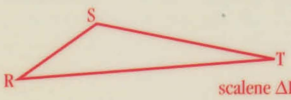


In ΔKMN, if base KN=18 and altitude MP=12, then $A=\frac{1}{2}\cdot 12\cdot 18=108$ units².

c. Triangles are **classified** in 2 ways, by side lengths and by angle measurements.

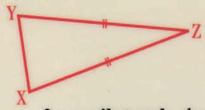
i. When classified by **side lengths**, a triangle is either:

1. **scalene**; that is, no side lengths are equal,



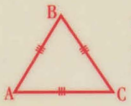
scalene ΔRST

2. **isosceles**; that is, at least 2 side lengths are equal,



Isosceles ΔXYZ because YZ=ZX. Notice: The marks on the sides indicate lengths are equal.

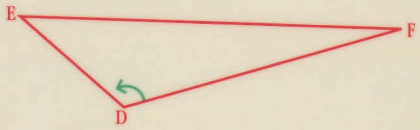
3. **equilateral**; that is, all 3 side lengths are equal. *Note: An equilateral triangle is also an isosceles triangle.*



ΔABC is equilateral because AB=BC=CA. ΔABC is also isosceles because at least 2 sides are equal.

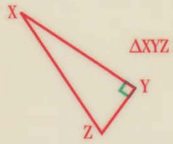
i. When classified by **angle measurements**, a triangle is either:

1. **obtuse**; that is, one angle measurement is more than 90 degrees,



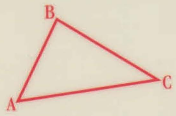
ΔDEF is obtuse because $m\angle D>90°$. Notice that both $m\angle E<90°$ and $m\angle F<90°$.

2. **right**; that is, one angle measurement is equal to 90 degrees,



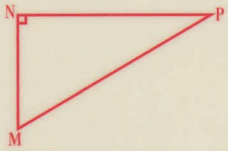
ΔXYZ is a right triangle because $m\angle Y=90°$.

3. **acute**; that is, all 3 angle measurements are less than 90 degrees. *Note: If all 3 angles are equal, then the triangle is called equiangular.*

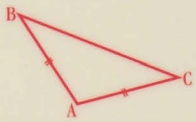


ΔABC is an acute triangle.

Note: Triangles are classified using one side classification name and one angle classification name; therefore, a triangle classification uses two words.



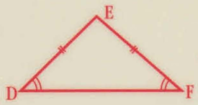
ΔMNP is a right scalene Δ.



ΔABC is an obtuse isosceles Δ.

d. **Isosceles triangles**

i. The **vertex angle** of an isosceles triangle is the angle whose sides are the two congruent sides of the triangle.



∠E is the vertex angle. ∠D and ∠F are the base angles. $\overline{DF}$ is the base of ΔDEF. $m\angle D=m\angle F$.

Note: The equal number of curves in angles indicates that they are equal in measurements.

ii. The base of an isosceles triangle is the side that does not have the same length as the other two sides, unless the triangle is equilateral. The base is not necessarily the side on the bottom of the triangle.

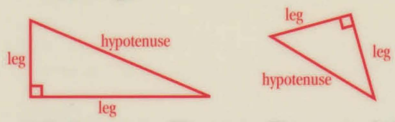
iii. The **base angles** of an isosceles triangle are the angles that have the base of the triangle as one of their sides.

iv. The base angles of an isosceles triangle are always equal in measure.

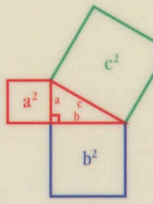
e. **Right triangles**

i. The **hypotenuse** of a right triangle is opposite the right angle and is the longest side of the right triangle.

ii. The other two sides of a right triangle are called **legs**.

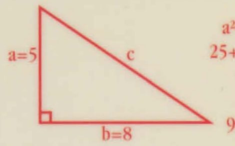


iii. **Pythagorean Theorem**: The sum of the areas of the squares on the legs of a right triangle is equal to the area of the square on the hypotenuse; that is, the sum of the squares of the legs is equal to the square of the hypotenuse, or $a^2 + b^2 = c^2$ where a and b are the lengths of the legs and c is the length of the hypotenuse.



Notice the area of the small square is a², the medium square, b², and the largest square, c². The areas of the 2 smaller squares equals the area of the biggest square. leg²+leg²=hypotenuse² or $a^2+b^2=c^2$

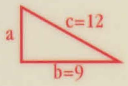
For example: Find the length of the hypotenuse of a right triangle when the lengths of the legs are 5 and 8.



$a^2+b^2= c^2$
$25+64 = c^2$
$89 = c^2$
$\sqrt{89} = c$
$9.43 \doteq c$

Note: When looking for the hypotenuse, knowing both legs, ***add*** *the squares and square root. When knowing the hypotenuse and looking for one leg,* ***subtract*** *the squares and square root the value.*

For example: Find the length of a leg of a right triangle when the length of the hypotenuse is 12 and one of the legs is 9.

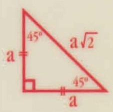


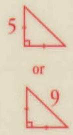
$a^2 + 81 = 144$
$a^2 = 144 - 81$
$a^2 = 63$
$a = \sqrt{63}$
$a \doteq 7.94$

iv. **Theorem:** If the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

v. **Theorem:** If the square of the longest side of the triangle is greater than the sum of the squares of the other two sides, then the triangle is obtuse; if it is less than the sum of the squares of the other two sides, then the triangle is acute.

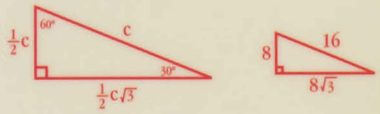
vi. **45-45-90 Theorem**: In a 45-45-90 triangle, the legs have equal lengths and the length of the hypotenuse is $\sqrt{2}$ times the length of one of the legs.





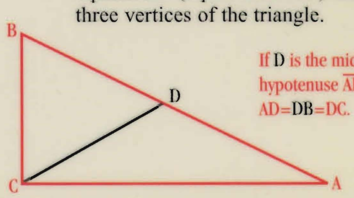
For example: If the legs are 5, then the hypotenuse is 5 times $\sqrt{2}$ or $5\sqrt{2}$. or If the hypotenuse is 9, each leg is $9/\sqrt{2}$ or $\frac{9}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{9\sqrt{2}}{\sqrt{4}}=\frac{9\sqrt{2}}{2}$.

vii. **30-60-90 Theorem**: In a 30-60-90 triangle, the length of the shortest leg is ½ of the length of the hypotenuse, and the length of the longer leg is $\sqrt{3}$ times the length of the shortest leg.



For example: If the hypotenuse is 16, the shortest leg is 8 and the longest leg is $8\sqrt{3}$.

viii. **Theorem:** The midpoint of the hypotenuse of a right triangle is equidistant (equal distances) from the three vertices of the triangle.

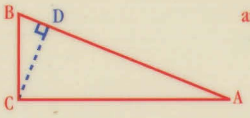


If D is the midpoint of hypotenuse $\overline{AB}$, then AD=DB=DC.

ix. **Theorem:** When an altitude is drawn to the hypotenuse of a right triangle,

1. the two triangles that are formed are similar to each other and to the original right triangle,

$\overline{CD}$ is the altitude to hypotenuse $\overline{AB}$, so $\Delta BDC \sim \Delta CDA \sim \Delta BCA$



and $\frac{BD}{CD}=\frac{CD}{DA}$ and $\frac{BA}{CB}=\frac{CB}{BD}$ and $\frac{BA}{CA}=\frac{CA}{AD}$.

2. the altitude is the geometric mean between the lengths of the two segments of the hypotenuse,

For example: In the right Δabove, if BD=4 and DA=15, then

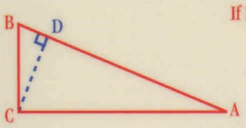
$\frac{4}{CD} = \frac{CD}{15}$

$CD^2 = 60$

$CD = \sqrt{60}$

$CD \doteq 7.75.$

3. each leg is the geometric mean between the hypotenuse and the length of the segment of the hypotenuse that is adjacent (touches) to the leg.



If BD=4 and BA=19, then

$\frac{4}{BC}=\frac{BC}{19}$

$BC^2= 4 \bullet 19=76$

$BC = \sqrt{76}$

$BC = 8.72.$

If BD=4, BA=19 and DA=15, then

$\frac{AD}{CA}=\frac{CA}{BA}$

$\frac{15}{CA}=\frac{CA}{19}$

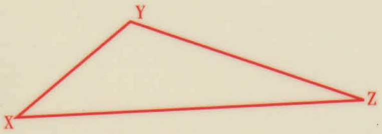
$CA^2= 15 \bullet 19=285$

$CA = \sqrt{285}$

$CA = 16.88.$

f. **Postulates and theorems**

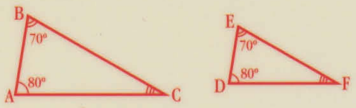
i. The sum of the 3-angle measurements of a triangle is 180 degrees.



$m\angle X+m\angle Y+m\angle Z=180°$

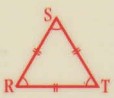
If $m\angle X=50°$ and $m\angle Y=35°$, then $m\angle Z=95°$.

ii. If two angle measurements of one triangle are equal to two angle measurements of another triangle, then the measurements of the third angles are also equal.



If $m\angle A=m\angle D=80°$ and $m\angle B=m\angle E=70°$, then $m\angle C=m\angle F=30°$.

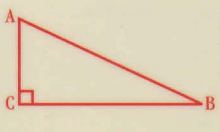
iii. Each angle of an equilateral triangle is 60º.



In ΔRST, RS=ST=TR, then $m\angle R=m\angle S=m\angle T$ and since the angles are equal and total 180°, each angle must equal 60°, so $m\angle R=m\angle S=m\angle T=60°$.

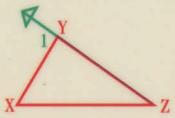
iv. There can be no more than one right or obtuse angle in any one triangle.

v. The acute angles of a right triangle are **complementary**.



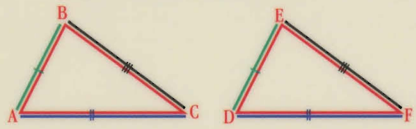
$m\angle C+m\angle A+m\angle B=180$ and $m\angle C=90$, so, $90+m\angle A+m\angle B=180$, so, $m\angle A+m\angle B=90$ and $\angle A$ and $\angle B$ are complementary.

vi. The measurement of an exterior angle of a triangle is equal to the sum of the measurements of the two remote (not having the same vertex as the exterior angle) interior angles of the triangle.



The exterior angle, $\angle 1$, is equal in measure to $m\angle X$ and $m\angle Z$ because they all have different vertices, so $m\angle 1=m\angle X+m\angle Z$ and if $m\angle 1=110°$ and $m\angle X=60°$, then $110°=60°+m\angle Z$ and $50°=m\angle Z$, so $m\angle Y=70°$.

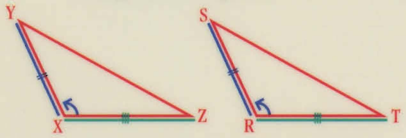
vii. **SSS** Postulate: If three sides of one triangle are equal in length to three sides of another triangle, then the triangles are congruent (same shape and same size).



If AB=DE, BC=EF, and AC=DF, then $\Delta ABC \cong \Delta DEF$, therefore, $m\angle A=m\angle D$, $m\angle B=m\angle E$, and $m\angle C=m\angle F$.

Notice: In $\Delta ABC \cong \Delta DEF$, matching vertices are put in the same order; that is, $\Delta ABC \cong \Delta DEF$.

viii. **SAS** Postulate: If two sides and the included angle of one triangle are equal in measure to two sides and the included angle of another triangle, then the triangles are congruent.

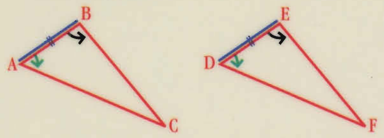


If XY=RS, XZ=RT, and $m\angle X=m\angle R$, then $\Delta XYZ \cong \Delta RST$.

Notice: $\angle X$ is between XY and XZ and $\angle R$ is between SR and RT.

For example: If XY=SR=8, XZ=RT=15 and $m\angle X=m\angle R=110°$, then $m\angle Y=m\angle S$, $m\angle Z=m\angle T$ and YZ=ST.

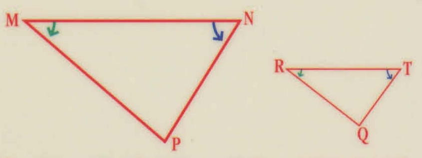
ix. **ASA** Postulate: If two angles and the included side of one triangle are equal in measure to two angles and the included side of another triangle, then the triangles are congruent.



If $m\angle A=m\angle D$, $m\angle B=m\angle E$ and AB=DE, then $\Delta ABC \cong \Delta DEF$.

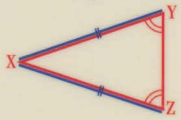
Notice: The side $\overline{AB}$ has the vertices of the 2$\angle$s, A and B as endpoints, and $\overline{DE}$ has the vertices of the 2$\angle$s, D and E as endpoints.

x. **AA** Similarity Postulate: If two angles of one triangle are equal in measure to two angles of another triangle, then the triangles are similar (same shape but not necessarily the same size).



If $m\angle M=m\angle R$ and $m\angle N=m\angle T$, then $m\angle P$ must equal $m\angle Q$ because the sum of the angles of a Δ=180°. If the angles of one Δ equal the angles of another Δ, the shapes have to be the same, but the side lengths don't have to be equal. However, the sides must be proportional, so $\frac{MN}{RT}=\frac{NP}{TQ}=\frac{PM}{QR}$. For example: If MN=12, RT=7and NP=10, then $\frac{MN}{RT}=\frac{NP}{TQ}$, so $\frac{12}{7}=\frac{10}{TQ}$, so $TQ=7 \bullet 10 \div 12=5.83$ and if RQ=9, then $\frac{MN}{RT}=\frac{MP}{RQ}$, so $\frac{12}{7}=\frac{MP}{9}$, so $MP=12 \bullet 9 \div 7=15.43$.

xi. **Theorem:** If two sides of a triangle are equal in measure, then the angles opposite those sides are also equal in measure; and, if two angles of a triangle are equal in measure, then the sides opposite those angles are also equal in measure.

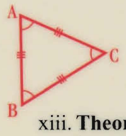


If XY=XZ, then $m\angle Y=m\angle Z$.

or

If $m\angle Y=m\angle Z$, then XY=XZ.

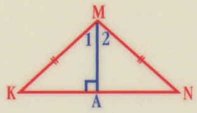
xii. **Theorem:** An equilateral triangle is also equiangular and an equiangular triangle is also equilateral.



If AB=BC=CA, then $m\angle A=m\angle B=m\angle C$ and since $m\angle A+m\angle B+m\angle C=180°$, each angle must equal 60°.

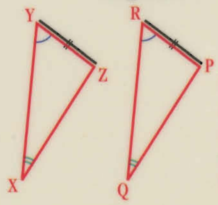
xiii. **Theorem:** An equilateral triangle has three 60-degree angles.

xiv. **Theorem:** The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base of the triangle.



In isosceles ΔKMN, if KM=MN, then $\angle M$ is the vertex angle. If $m\angle 1=m\angle 2$, KA=AN and $\overline{MA} \perp \overline{KN}$.

xv. **AAS** Theorem: If two angles and a non-included side of one triangle are equal in measure to the two corresponding (matching if placed on top of the other shape) angles and non-included side of another triangle, then the triangles are congruent.

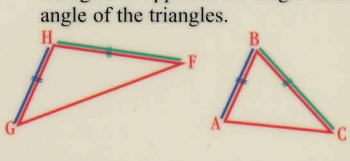


If $m\angle X=m\angle Q$, $m\angle Y=m\angle R$, and YZ=RP, then $\Delta XYZ \cong \Delta QRP$. Notice $\overline{YZ}$ is not located between $\angle X$ and $\angle Y$. $\overline{RP}$ is not located between $\angle R$ and $\angle Q$.

xvi. **HL** Theorem: If the hypotenuse and one leg of a right triangle are equal in measure to the hypotenuse and the corresponding leg of another right triangle, then the two right triangles are congruent.

Remember, the hypotenuse is the side opposite the 90° angle and is the longest side of a right triangle.

xvii. **SAS** Inequality Theorem: If two sides of one triangle are equal in length to two sides of another triangle, but the included angle of one triangle is larger than the included angle of the other triangle, then the longer third side of the triangles is opposite the larger included angle of the triangles.



If GH=AB and HF=BC, but $m\angle H>m\angle B$, then GF>AC because GF is opposite the larger of the two angles H and B.

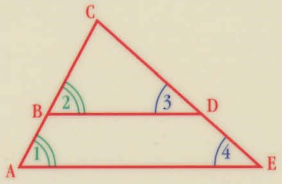
xviii. **SSS** Inequality Theorem: If two sides of one triangle are equal in length to two sides of another triangle, but the third side of one triangle is longer than the third side of the other triangle, then the larger included angle (included between the two equal sides) is opposite the longer third side of the triangles.

This is the converse of the SAS Inequality Theorem above. It indicates that if GF>AC, then $m\angle H>m\angle B$.

xix. **SSS** Similarity Theorem: If the sides of one triangle are proportional to the corresponding sides of another triangle, then the triangles are similar.

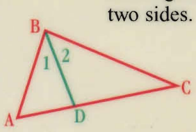
xx. **SAS** Similarity Theorem: If two sides of one triangle are proportional to two sides of another triangle and the included angles of each triangle are congruent, then the triangles are similar. *(See AA Postulate)*

xxi. **Triangle Proportionality** Theorem: If a line is parallel to one side of a triangle and intersects the other two sides, then it divides those two sides proportionally, and creates 2 similar triangles. *(See AA Postulate)*



If $BD \parallel AE$, then $m\angle 1 = m\angle 2$, $m\angle 4 = m\angle 3$, $\frac{BC}{AC} = \frac{DC}{EC} = \frac{BC}{BA} = \frac{DC}{DE}$, and $\triangle BCD \sim \triangle ACE$.
For example: If BC=20, AC=28, and CD=22, then $\frac{BC}{AC} = \frac{DC}{CE}$, so $\frac{20}{28} = \frac{22}{CE}$, so $CE = 28 \cdot 22 \div 20 = 30.8$

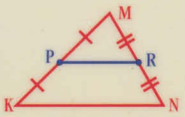
xxii. **Theorem:** If a ray bisects an angle of a triangle, it divides the opposite side into segments proportional to the other two sides.



If $\overrightarrow{BD}$ bisects $\angle B$, then $m\angle 1 = m\angle 2$ and $\frac{AD}{DC} = \frac{AB}{BC}$.
For example: If AB=20, BC=24, and AD=13, then $\frac{13}{DC} = \frac{20}{24}$, so $DC = 24 \cdot 13 \div 20 = 15.6$ and AC=28.6.

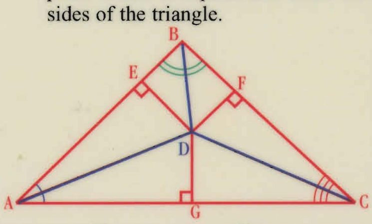
xxiii. **Theorem:** The line segment that joins the midpoints of two sides of a triangle has two properties:

1. it is parallel to the third side, and
2. it is half the length of the third side.



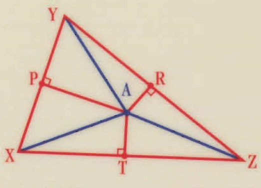
If P is the midpoint of $\overline{KM}$, and R is the midpoint of $\overline{MN}$, then $\overline{PR} \parallel \overline{KN}$ and $PR = \frac{1}{2}KN$, so, if KN=30, then PR=15.

xxiv. **Theorem:** The 3 bisectors of the angles of a triangle intersect in one point which is equidistant from the 3 sides of the triangle.



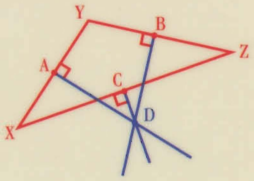
If $\overrightarrow{AD}$ bisects $\angle A$, $\overrightarrow{BD}$ bisects $\angle B$, and $\overrightarrow{CD}$ bisects $\angle C$, then they intersect in one point, D, and point D is equal distances from $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$, so DE=DF=DG. Remember, the distances from D to the sides must be $\perp$.

xxv. **Theorem:** The perpendicular bisectors of the sides of a triangle intersect at one point which is equidistant from the 3 vertices of the triangle.

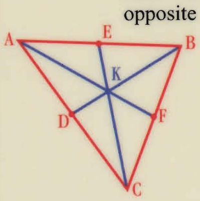


If $\overline{PA} \perp$ bisector of $\overline{XY}$, $\overline{RA} \perp$ bisector of $\overline{ZY}$, and $\overline{TA} \perp$ bisector of $\overline{XZ}$, then they all intersect at one point, A, which is equidistant from points X, Y, and Z, so XA=YA=ZA.

For obtuse triangles, the point of intersection is in the exterior of the triangle. Point D is equidistant from points X, Y, and Z, so XD=YD=ZD.

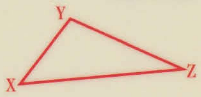


xxvi. **Theorem:** The **medians** (line segements whose endpoints are one vertex of the triangle and the midpoint of the side that is opposite that vertex) of a triangle intersect in one point that is two thirds of the distance from each vertex to the midpoint of the opposite side.



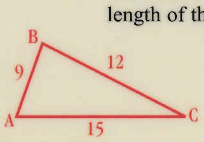
If points D, E and F are midpoints of the sides of $\triangle ABC$, then all medians intersect at one point, K. So, $KB = \frac{2}{3}DB$, $KC = \frac{2}{3}EC$, and $KA = \frac{2}{3}AF$. It is also true that KB=2DK, KC=2EK, and KA=2FK.

xxvii. **Theorem:** If two sides of a triangle are unequal in length, then the angles opposite those sides are unequal and the larger angle is opposite the longer side; and, conversely, if two angles of a triangle are unequal, then the sides opposite those angles are unequal and the longer side is opposite the larger angle.



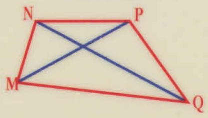
In scalene $\triangle XYZ$, the largest angle is $\angle Y$ and the longest side is opposite $\angle Y$, $\overline{XZ}$. The smallest angle is $\angle Z$, so the shortest side is opposite $\angle Z$, $\overline{XY}$.

xxviii. **Theorem:** The sum of the lengths of any two sides of a triangle is greater than the length of the third side and the difference of the lengths of any two sides of a triangle is less than the length of the third side.



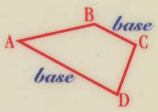
Notice that 9+12>15, 15+12>9 and 9+15>12. Also, 12–9<15, 15–12<9, and 15–9<12.

2. Quadrilaterals
   a. **Quadrilaterals** are 4-sided polygons.
   b. Quadrilaterals have 2 diagonals and 4 vertices.

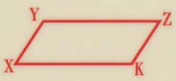


Quadrilateral MNPQ has sides $\overline{MN}$, $\overline{NP}$, $\overline{PQ}$ and $\overline{QM}$, with vertices M, N, P, and Q, and diagonals $\overline{NQ}$ and $\overline{MP}$.

   c. Special quadrilaterals
      i. **Trapezoids** are quadrilaterals with exactly one pair of parallel sides (called the bases), never more than one pair of parallel sides.

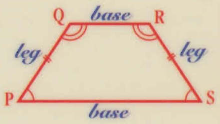


If $\overline{AD} \parallel \overline{BC}$, then quadrilateral ABCD is a trapezoid.



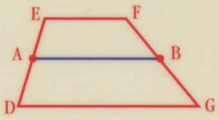
If $\overline{XY} \parallel \overline{KZ}$ and $\overline{YZ} \parallel \overline{XK}$, then quadrilateral XYZK is not a trapezoid.

1. **Isosceles trapezoids** have nonparallel sides that are the same length and are called legs. The angles whose vertices are the endpoints of the same base of an isosceles trapezoid are called base angles, and they are equal in measure.



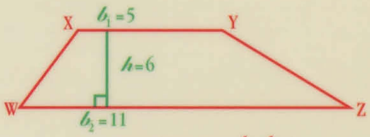
In trapezoid PQRS, if QP=RS, then the trapezoid is isosceles. Base angles of isosceles trapezoids are equal in measure, so $m\angle P = m\angle S$ and $m\angle Q = m\angle R$.

2. **Theorem:** The **median** of a trapezoid (the line segment whose endpoints are the midpoints of the 2 nonparallel sides of the trapezoid) is parallel to the bases, and has a length equal to half the sum of the lengths of the 2 bases (that is, equal to the average of the lengths of the 2 parallel sides, bases).



If A and B are midpoints of $\overline{DE}$ and $\overline{FG}$, then $\overline{AB} \parallel \overline{EF} \parallel \overline{DG}$ and $AB = \frac{1}{2}(EF + DG)$, so, if EF=10 and DG=18, then $AB = \frac{1}{2}(10+18) = 14$.

3. The area of a trapezoid may be calculated by averaging the length of the bases and multiplying by the height (altitude—the line segment that forms 90 degree angles with the bases); thus, the formula

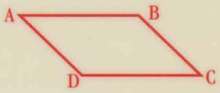


$A = \frac{1}{2}(b_1 + b_2)h$ or $A = \frac{b_1 + b_2}{2}h$

For example, in this trapezoid, the parallel sides, that is, the bases, are 5 and 11, so the average of the bases, (5 + 11)/2, is 8. Multiply 8 by the height of 6. Thus, the area is 48 square units.

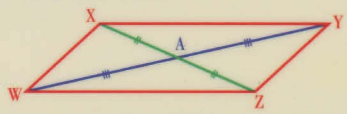
ii. **Parallelograms** are quadrilaterals with 2 pairs of parallel sides.

1. **Theorem:** Opposite sides are parallel and equal in length.
2. **Theorem:** Opposite angles are equal in measure.
3. All 4 interior angle measures total 360º.
4. Consecutive interior angles (their vertices are endpoints for the same side of the parallelogram) are supplementary (measures total 180 degrees).



$\overline{AB} \parallel \overline{DC}$, $\overline{AD} \parallel \overline{BC}$, AB=DC, AD=BC, $m\angle A = m\angle C$ and $m\angle D = m\angle B$
$m\angle A + m\angle D = 180$
$m\angle A + m\angle B = 180$
$m\angle B + m\angle C = 180$
$m\angle C + m\angle D = 180$

5. **Theorem:** Diagonals of a parallelogram bisect each other.

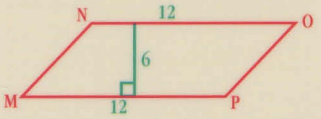


In ▱WXYZ, diagonals $\overline{XZ}$ and $\overline{WY}$ bisect each other at A, so WA=AY and XA=AZ.

6. **Theorem:** If one pair of opposite sides of a quadrilateral are equal in length and parallel, then the quadrilateral is a parallelogram.
7. **Theorem:** If both pairs of opposite sides of a quadrilateral are equal in length, then the quadrilateral is a parallelogram.
8. **Theorem:** If both pairs of opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.

9. **Theorem:** If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

10. The **area of a parallelogram** can be calculated by multiplying the base and the height; that is, A=bh or A=hb. Note: Since opposite sides of a parallelogram are both parallel and equal in length, any side can be the base. The height (altitude) is any line segment that forms 90-degree angles with the base and whose endpoints are on the base and the opposite side of the parallelogram.

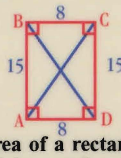


The base=12 and the height=6, so A=6•12=72 units².

11. Special Parallelograms

a. **Rectangles** are parallelograms with 4 right angles (90 degrees each).

1) **Theorem:** The diagonals of a rectangle are equal.



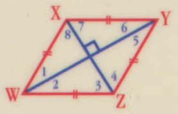
$\overline{AC}=\overline{BD}$

• 20 units²

2) The **area of a rectangle** is calculated by multiplying any 2 consecutive sides (sides that share a common endpoint). Since all angles are 90 degrees, any 2 consecutive sides are the base (length) and the height (width or altitude) of the rectangle; thus, A = bh or A = hb or A = lw.

b. **Rhombuses (or Rhombi)** are parallelograms with 4 sides equal in length. The 4 interior angles can have any measures, but opposite angles have equal measures and all 4 angle measures total 360º.

1) **Theorem:** The diagonals of a rhombus are perpendicular.



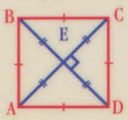
$\overline{XZ} \perp \overline{WY}$

$m\angle 1=m\angle 2=m\angle 5=m\angle 6$

$m\angle 3=m\angle 4=m\angle 7=m\angle 8$

2) **Theorem:** Each diagonal of a rhombus bisects the pair of opposite angles whose vertices are the endpoints of the diagonal.

c. **Squares** have 4 equal sides and 4 equal angles (each 90 degrees); therefore, every square is both a rectangle and a rhombus. Each square has 4 right angles, just as all rectangles do, and each square has 4 equal sides, just as all rhombi do. The diagonals of a square are equal in length, bisect each other, are perpendicular to each other, and bisect the interior angles of the square.



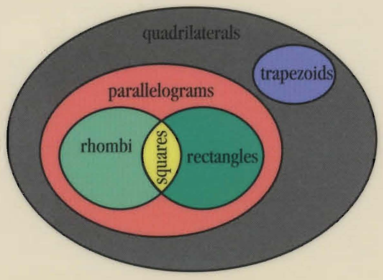
AB=BC=CD=DA

BD=AC

BD ⊥ AC

AE=ED=EC=EB

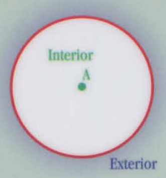
Note: This Venn diagram indicates the relationships of squares to other quadrilaterals.



CIRCLES

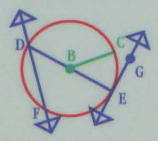
A. Defined Terms

1. A **circle** is the set of points in a plane that are the same distance from one point in the plane, which is called the **center** of the circle. The center of the circle lies in the interior of the circle and is not a point on the circle. ⊙ means circle.
2. The **radius** is the distance that each point on the circle is from the center of the circle; or, a radius is a line segment whose endpoints are the center of the circle and a point on the circle.
3. A **chord** is a line segment whose endpoints are 2 points on the circle.
4. A **diameter** is a chord that contains the center of the circle; or, a diameter is the length of the chord that contains the center of the circle (the distance from one point on the circle to another point on the circle, going through the center of the circle).



Point A is the center of ⊙A and is in the interior of the circle, not on the circle.

5. A **secant** is a line that intersects a circle in two points.
6. A **tangent** is a line that is coplanar with a circle and intersects the circle in one point only. That point is called the **point of tangency**.

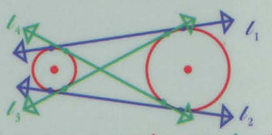


$\overline{BC}$ is the radius.
$\overline{DE}$ is a chord and a diameter.
$\overline{DF}$ is a chord.
$\overleftrightarrow{EG}$ is a tangent with E as the point of tangency.
$\overleftrightarrow{DF}$ is a secant.

a. A **common tangent** is a line that is tangent to 2 coplanar circles.

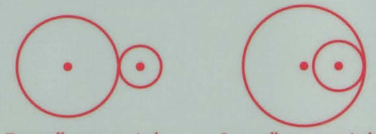
i. **Common internal tangents** intersect between the two circles.

ii. **Common external tangents** do not intersect between the circles.



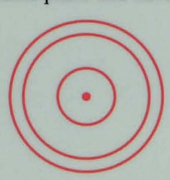
ℓ_1 and ℓ_2 are common external tangents. ℓ_3 and ℓ_4 are common internal tangents.

iii. Two circles are tangent to each other when they are coplanar and share the same tangent line at the same point of tangency. They may be externally tangent or internally tangent.



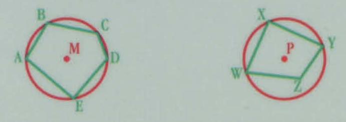
Externally tangent circles   Internally tangent circles

7. **Equal circles** are circles that have equal length radii (plural of radius).
8. **Concentric circles** are circles that lie in the same plane and have the same center.



3 concentric circles

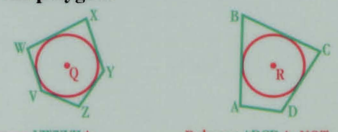
9. An **inscribed polygon** is a polygon whose vertices are points on the circle. In this same situation, the circle is said to be **circumscribed about the polygon.**



Polygon ABCDE is inscribed in ⊙M. ⊙M is circumscribed about polygon ABCDE.

Polygon WXYZ is not inscribed in ⊙P because vertex Z is not on ⊙P.

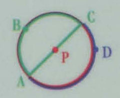
10. A **circumscribed polygon** is a polygon whose sides are segments of tangents to the circle; i.e., the sides of the polygon each contain exactly one point on the circle. In this same situation, the circle is said to be **inscribed in the polygon**.



Pentagon VWXYZ is circumscribed about ⊙Q because each side is tangent to ⊙Q. ⊙Q is inscribed in pentagon VWXYZ.

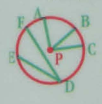
Polygon ABCD is NOT circumscribed about ⊙R because side $\overline{AD}$ is NOT tangent to ⊙R. ⊙R is NOT inscribed in polygon ABCD.

11. An **arc** is part of a circle.
    a. A **semicircle** is an arc whose endpoints are the endpoints of a diameter. Three points must be used to name a semicircle.
    b. A **minor** arc is an arc whose length is less than the length of the semicircle. Only two points may be used to name a minor arc.
    c. A **major** arc is an arc whose length is more than the length of the semicircle. Three points must be used to name a major arc.



Arc ABC or $\overparen{ABC}$ is a semicircle because chord $\overline{AC}$ is a diameter of ⊙P.
$\overparen{ADC}$ is also a semicircle.
$\overparen{CD}$ and $\overparen{AD}$ are minor arcs.
$\overparen{AD}=\overparen{DA}$ and $\overparen{CD}=\overparen{DC}$
$\overparen{DAB}$, $\overparen{BAD}$, $\overparen{BAC}$, $\overparen{BCA}$, $\overparen{DAC}$, $\overparen{DCA}$ are major arcs.
$\overparen{DAC}=\overparen{DBC}=\overparen{CBD}=\overparen{CAD}$; $\overparen{BAC}=\overparen{BDC}=\overparen{CAB}=\overparen{CDB}$

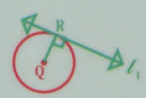
12. A **central angle** of a circle is an angle whose vertex is the center of the circle.
13. An **inscribed angle** is an angle whose vertex is on a circle and whose sides contain chords of the circle.



∠APB, ∠BPC, and ∠APC are central angles because the vertex, P, is the center of the circle. ∠FDE is an inscribed angle because the vertex, D, is on the circle.

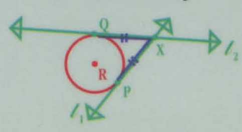
B. **Theorems**

1. If a line is tangent to a circle, then the line is perpendicular to the radius whose endpoint is the point of tangency.



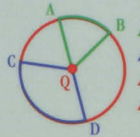
ℓ_1 is a tangent to ⊙Q at point R, so radius $\overline{QR} \perp \ell_1$.

2. If a line in the plane of a circle is perpendicular to a radius at its outer endpoint, then the line is tangent to the circle.
3. If two tangents intersect, then the line segments whose endpoints are the point of intersection and the two points of tangency are equal in length; or, line segments drawn from a coplanar exterior point of a circle to points of tangency on the circle are equal in length.



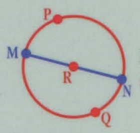
ℓ_1 and ℓ_2 are tangent to ⊙R at points Q and P, so QX=PX.

4. The measure of a minor arc is equal to the measure of its central angle.



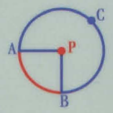
$m\overset{\frown}{AB}=m\angle AQB$
$m\overset{\frown}{CD}=m\angle CQD$
$m\overset{\frown}{AC}=m\angle CQA$
$m\overset{\frown}{BD}=m\angle BQ$

5. The measure of a semicircle is 180 degrees.



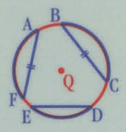
$\overline{MN}$ is a diameter, so $\overset{\frown}{MPN}$ and $\overset{\frown}{MQN}$ are semicircles. $m\overset{\frown}{MPN}=m\overset{\frown}{MQN}=180°$

6. The measure of a major arc is equal to 360 degrees minus the measure of its corresponding minor arc.



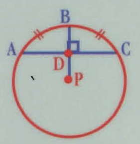
$m\overset{\frown}{ACB}=360°-m\overset{\frown}{AB}$
If $m\overset{\frown}{AB}=90°$, then $m\overset{\frown}{ACB}=270°$.

7. In the same circle or in equal circles, equal chords have equal arcs and equal arcs have equal chords.



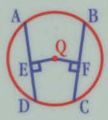
If chords $\overline{AF}=\overline{BC}$, then $m\overset{\frown}{FA}=m\overset{\frown}{BC}$, but $\overline{ED}\neq\overline{BC}$, so $m\overset{\frown}{ED}\neq m\overset{\frown}{BC}$ and $m\overset{\frown}{ED}\neq m\overset{\frown}{AF}$.

8. A diameter or radius that is perpendicular to a chord bisects the chord and its arc.



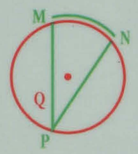
Radius $\overline{PB}\perp\overline{AC}$, so $\overline{AD}=\overline{DC}$ and $m\overset{\frown}{AB}=m\overset{\frown}{BC}$.

9. In the same circle or in equal circles, equal chords are the same distance from the center, and chords that are the same distance from the center are equal.



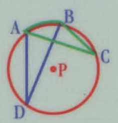
AD=BC if QE=QF because both chords are equal distances from the center, Q. Remember, distance is always measured at 90° angles.

10. An inscribed angle is equal in measure to half of the measure of its intercepted arc (the arc which lies in the interior of the inscribed angle and whose endpoints are on the sides of the angle).



$m\angle MPN=1/_2 m\overset{\frown}{MN}$

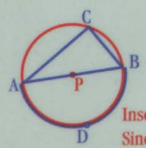
11. If two inscribed angles intercept the same arc, then the angles are equal in measure.



$\angle ADB$ and $\angle ACB$ are inscribed angles because the vertices, D and C, are on the circle. $m\angle ADB=1/_2 m\overset{\frown}{AB}$ and $m\angle ACB=1/_2 m\overset{\frown}{AB}$, so $m\angle ADB=m\angle ACB$.

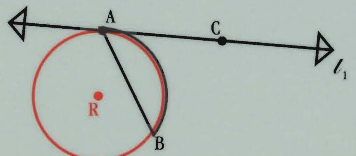
12. If a quadrilateral is inscribed in a circle, then opposite angles are supplementary.

13. An angle inscribed in a semicircle is always a right angle.



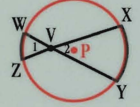
Inscribed $\angle ACB$ intercepts semicircle $\overset{\frown}{ADB}$. Since $m\overset{\frown}{ADB}=180°$, so $m\angle ACB=1/_2 m\overset{\frown}{ADB}=90°$.

14. The measure of an angle formed by a chord and a tangent is equal to half of the measure of its intercepted arc.



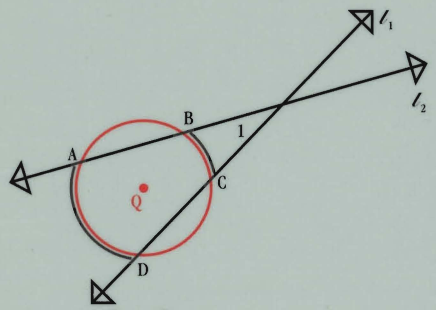
ℓ_1 is tangent to ⊙R at A, so $m\angle BAC=1/_2 m\overset{\frown}{AB}$.
If $m\overset{\frown}{AB}=110°$, then $m\angle BAC=55°$.

15. The measure of an angle formed by two chords intersecting inside a circle is equal to half the **sum** of the intercepted arcs.

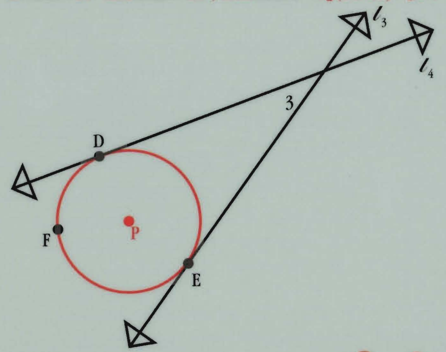


$\angle 1$ and $\angle 2$ are vertical angles, so
$m\angle 1=m\angle 2$
$m\angle 1=m\angle 2=1/_2\ (m\overset{\frown}{WZ}+m\overset{\frown}{XY})$.
If $m\overset{\frown}{ZW}=30°$ and $m\overset{\frown}{XY}=100°$, then
$m\angle 1=m\angle 2=1/_2\ (30+100)=65°$

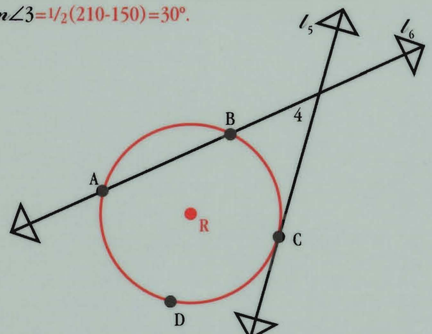
16. The measure of an angle formed by two secants, or two tangents, or a secant and a tangent, that intersect at a point outside of the circle, is equal to half the **difference** of the intercepted arcs.



$m\angle 1=1/_2(m\overset{\frown}{AD}-m\overset{\frown}{BC})$
If $m\overset{\frown}{AD}=80°$ and $m\overset{\frown}{BC}=20°$, then $m\angle 1=1/_2(80-20)=30°$.

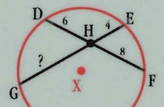


ℓ_3 and ℓ_4 are tangents to ⊙P, so $m\angle 3=1/_2(m\overset{\frown}{DFE}-m\overset{\frown}{DE})$.
If $m\overset{\frown}{DFE}=210°$, then $m\overset{\frown}{DE}=360°-210°=150°$ and $m\angle 3=1/_2(210-150)=30°$.



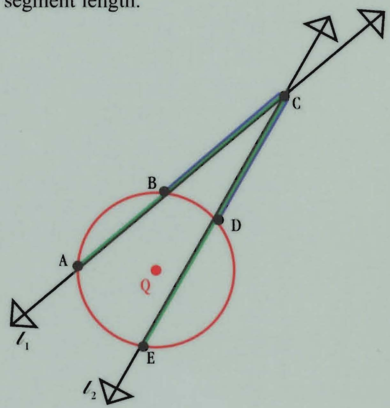
ℓ_5 is a tangent to ⊙R. ℓ_6 is a secant.
$m\angle 4=1/_2(m\overset{\frown}{ADC}-m\overset{\frown}{BC})$. If $m\overset{\frown}{ADC}=170°$ and $m\overset{\frown}{BC}=84°$, then $m\angle 4=1/_2(170-84)=43°$.

17. When two chords intersect inside a circle, the product of the segment lengths of one chord is equal to the product of the segment lengths of the other chord.



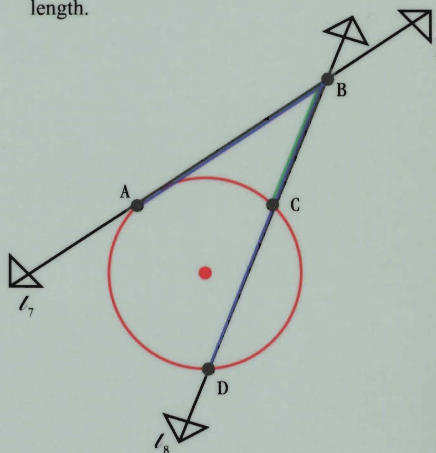
(DH)(HF)=(GH)(HE)
If DH=6, HF=8 and HE=4, then (6)(8)=(GH)(4), so GH=48÷4=12.

18. When two secants are drawn to a circle from the same exterior point, the product of one secant and its external segment length equals the product of the other secant and its external segment length.



(BC)(AC)=(DC)(EC)
If AC=26, BC=14, and DC=10, then (14)(26)=(10)(EC), so EC=14•26÷10=36.4 and since ED=EC-DC; ED=36.4-10=26.4.

19. When a tangent and a secant are drawn to a circle from the same exterior point, the square of the length of the tangent segment is equal to the product of the secant and its external segment length.



ℓ_7 is tangent at point A. ℓ_8 is a secant. (AB)(AB)=(DB)(CB) or $(AB)^2$=(DB)(CB).
If AB=30, CB=20, then $(30)^2$=(DB)(20), so DB=$(30)^2$÷20=900÷20=45. DC=DB-CB=45-20=25.

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