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COURSE IN

MATHEMATICS

for the IIT-JEE & Other Engineering Entrance Examinations

2D Coordinate Geometry

K. R. Choubey
Ravikant Choubey
Chandrakant Choubey

ALWAYS LEARNING

PEARSON

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(FOR IIT JEE AND OTHER ENGINEERING ENTRANCE EXAMINATIONS)

COORDINATE GEOMETRY 2D

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PREFACE

When a new book is written on a well known subject like *Coordinate Geometry* for class XI/XII Academics/ AIEEE/IIT/State engineering entrance exams and NDA, several questions arise like—why, what, how and for whom? What is new in it? How is it different from other books? For whom is it meant? The answers to these questions are often not mutually exclusive. Neither are they entirely satisfactory except perhaps to the authors. We are certainly not under the illusion that there are no good books. There are many good books available in the market.

However, none of them caters specifically to the needs of students. Students find it difficult to solve most of the problems of any of the books in the absence of proper planning. This inspired us to write this book *Coordinate Geometry*, to address the requirements of students of class XI/XII CBSE and State Board Academics. In this book, we have tried to give a connected and simple account of the subject. It gives a detailed, lecture wise description of basic concepts with many numerical problems and innovative tricks and tips. Theory and problems have been designed in such a way that the students can themselves pursue the subject. We have also tried to keep this book self contained. In each lecture all relevant concepts, prerequisites and definitions have been discussed in a lucid manner and also explained with suitable illustrated examples including tests.

Due care has been taken regarding the Board (CBSE/ State) examination need of students and nearly 100 per cent articles and problems set in various examinations including the IIT-JEE have been included.

The presentation of the subject matter is lecturewise, intelligent and systematic, the style is lucid and rational, and the approach is comprehensible with emphasis on improving speed and accuracy. The basic motive is to attract students towards the study of mathematics by making it simple, easy and interesting and on a day-to-day basis. The instructions and method for grasping the lectures are clearly outlined topic wise. The presentation of each lecture is planned for better experiential learning of mathematics which is as follows:

1. Basic Concepts: Lecture Wise
2. Solved Subjective Problems (XII Board (C.B.S.E./State): For Better Understanding and Concept Building of the Topic.
3. Unsolved Subjective Problems (XII Board (C.B.S.E./State): To Grasp the Lecture Solve These Problems.
4. Solved Objective Problems: Helping Hand.
5. Objective Problem: Important Questions with Solutions.
6. Unsolved Objective Problems (Identical Problems for Practice)
   For Improving Speed with Accuracy.

7. Worksheet: To Check Preparation Level
8. Assertion-Reason Problems: Topic Wise Important Questions and Solutions with Reasoning
9. Mental Preparation Test: 01
10. Mental Preparation Test: 02
11. Topic Wise Warm Up Test: 01: Objective Test
12. Topic Wise Warm Up Test: 02: Objective Test
13. Objective Question Bank Topic Wise: Solve These to Master.

This book will serve the need of the students of class XI/XII board, NDA, AIEEE and SLEEE (state level engineering entrance exam) and IIT-JEE. We suggest each student to attempt as many exercises as possible without looking up the solutions. However, one should not feel discouraged if one needs frequent help of the solutions as there are many questions that are either tough or lengthy. Students should not get frustrated if they fail to understand some of the solutions in the first attempt. Instead they should go back to the beginning of the solution and try to figure out what is being done At the end of every topic, some harder problems with 100 per cent solutions and Question Bank are also given for better understanding of the subject.

There is no end and limit to the improvement of the book. So, suggestions for improving the book are always welcome.

We thank our publisher, Pearson Education for their support and guidance in completing the project in record time.

K.R. CHOUBEY
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Part A

Cartesian Coordinates

LECTURE

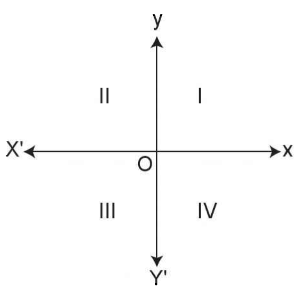
1

Cartesian Coordinates 1

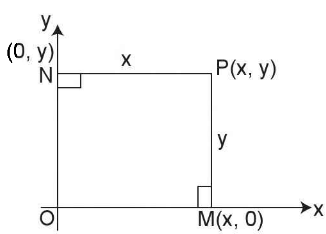
(Introduction, distance formula and its application, locus of a point)

BASIC CONCEPTS

1. **Quadrant** Two mutually perpendicular lines meeting at 'O' is called origin and these two lines are called axes. Horizontal line $X'\,OX$ is known as x-axis and vertical line $Y'OY$ is called y-axis. These two perpendicular lines divide the plane into four quadrants, viz., as follows their names are given in anti-clockwise sense.
XOY = First quadrant, $X'OY$ = Second quadrant
$X'OY'$ = Third quadrant, $Y'OX$ = fourth quadrant



1.1 Coordinate of a point



x = abscissa = Perpendicular distance of a point $P(x, y)$ from y-axis and is also known as x-coordinate of point P.

y = ordinate = Perpendicular distance of a point $P(x, y)$ from x-axis and is also known as y-coordinate of point P.

$M(x, 0)$ = Projection or foot of perpendicular of point $P(x, y)$ on x-axis.

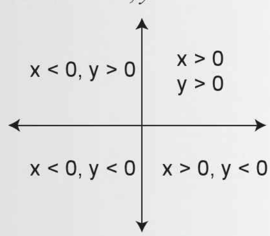
$N(0, y)$ = Projection or foot of perpendicular of point $P(x, y)$ on y-axis.

NOTE

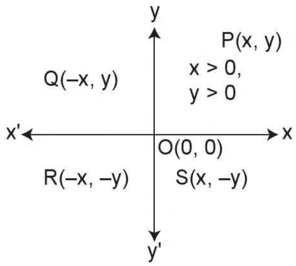
(i) y-coordinate of any point lying on the x-axis is always zero.

(ii) x-coordinate of any point lying on the y-axis is always zero.

In 1st quadrant $x > 0, y > 0$
IInd quadrant $x < 0, y > 0$
IIIrd quadrant $x < 0, y < 0$ and
IVth quadrant $x > 0, y < 0$



1.2 Quadrant wise sign of abscissa (x) and ordinate (y)



1.3 Equation of the coordinate axes

(a) Equation of x-axis is $y = 0$ (y-coordinate of every point lying on the x-axis is always zero).

(b) Equation of y-axis is $x = 0$ (x-coordinate of every point lying on the y-axis is alwas zero).

2. Distance Formula Distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(\textit{diff}.\text{ of abscissae})^2 + (\textit{diff}.\text{ of ordinates})^2}$$

Distance of (x_1, y_1) from origin $= \sqrt{x_1^2 + y_1^2}$

2.1 Use of Distance Formula

Nature of geometrical figures can be detected by Distance formula.

(i) **Equilateral triangle** All sides equal

(ii) **Isosceles triangle** Two sides equal

(iii) **Rhombus** All sides equal and no angle a right angle, but diagonals are at right angles and are unequal.

(iv) **Square** All sides equal and each angle is right angle. The diagonals are also equal.

(v) **Parallelogram** Opposite sides parallel and unequal diagonals bisect each other.

(vi) **Rectangle** Opposite sides equal and each angle a right angle. Diagonals are equal.

(vii) The line joining the mid-points of the two sides of a triangle is parallel to the third side and half its length.

(viii) Collinearity of three given points: $(AB + BC = AC, AC - AB = BC)$

3. Locus of a Point

The locus of a moving point is the path traced out by it under some given geometrical condition or conditions

Method of finding the locus of a point

To find the locus of a point proceed as follows:

Let (h, k) be the coordinates of the moving point say p. Now using given geometrical conditions find the relations between h, k and other known and unknown quantities to get a relation between h, k and the known quantities.

Now the locus of $p(h, k)$ is obtained by generalizing $(h. k)$ i.e., by putting x for h and y for k.

3.1 A point $P(x, y)$ moves such that the sum or difference of its distances from two fixed points $A(a, 0)$ and $A'(-a, 0)$ is always $2k$. i.e., $PA' \pm PA = 2k$ then its locus is:

$$x^2\left(1 - \frac{a^2}{k^2}\right) + y^2 = k^2 - a^2$$

or $$\frac{x^2}{k^2} + \frac{y^2}{k^2 - a^2} = 1$$

3.2 A point moves such that the sum or difference of its distances from the two fixed points $B(0, b)$ and $B'(0, -b)$ is always $2k$, then its locus is

$$x^2 + y^2\left(1 - \frac{a^2}{k^2}\right) = k^2 - a^2$$

or $$\frac{x^2}{k^2 - a^2} + \frac{y^2}{k^2} = 1$$

NOTE

The equation $\sqrt{(x-2)^2 + y^2} + \sqrt{(x+2)^2 + y^2} = 4$ represents a pair of coincident lines.

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find the equation of the locus of a point which moves so that its distance from the point $(a\,k, 0)$ is $k(k > 0, \neq 1)$ times its distance from the point $\left(\frac{a}{k}, 0\right)$.

Solution

Let $A\ (a\ k,\ 0)$ and $B\left(\frac{a}{k}, 0\right)$, then the point $P\ (x, y)$ will lie on the locus if $|AP| = k\ |\ BP|$,

i.e., if $|AP|^2 = k^2\ |BP|^2$,

i.e., if $(x - ak)^2 + (y - 0)^2$

$$= k^2\left[\left(x - \frac{a}{k}\right)^2 + (y-0)^2\right],$$

i.e., if $(x - ak)^2 + y^2 = k^2\left[\frac{(kx-a)^2}{k^2} + y^2\right]$,

i.e., $(x - ak)^2 + y^2 = (kx - a)^2 + k^2y^2$

i.e., if $x^2 - 2akx + y^2 + a^2k^2 = k^2x^2 - 2akx + a^2 + k^2y^2$,

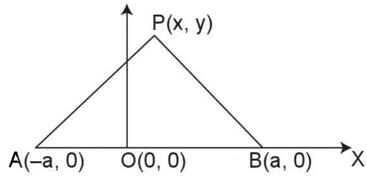
i.e., if $(1 - k^2)\,x^2 + (1 - k^2)y^2 = (1 - k^2)a^2$ dividing both sides by $(1 - k^2)$,

i.e., if $x^2 + y^2 = a^2$, which is the required equation.

2. Find the equation of the locus of a point such that the sum of its distances from two fixed points is constant and equal to $2c$.

Solution

Let two fixed points be A and B. Take O, AB itself as the x-axis and a line through O and perpendicular to AB as y-axis.



Let $|AB| = 2a$, then coordinates of A and B are $(-a, 0)$ and $(a, 0)$. Let $P(x, y)$ be any point on the locus.

According to the given $|PA| + |PB| = 2c$

$$\Rightarrow \quad \sqrt{(x+a)^2 + y^2} + \sqrt{(x-a)^2 + y^2} = 2c$$

$$\Rightarrow \quad \sqrt{(x+a)^2 + y^2} = 2c - \sqrt{(x-a)^2 + y^2}$$

Squaring both sides, we get

$$(x + a)^2 + y^2 = 4c^2 + (x - a)^2 + y^2 - 4c\sqrt{(x-a)^2 + y^2}$$

$$\Rightarrow \quad 4c\sqrt{(x-a)^2 + y^2} = 4c^2 - 4ax.$$

Dividing both sides by $4c$, we get

$$\sqrt{(x-a)^2 + y^2} = c - \frac{ax}{c},$$

Squaring again, we get $x^2 + a^2 - 2ax + y^2$

$$= c^2 + \frac{a^2x^2}{c^2} - 2ax$$

$$\Rightarrow \quad \left(1 - \frac{a^2}{c^2}\right)x^2 + y^2 = c^2 - a^2$$

$$\Rightarrow \quad \frac{c^2 - a^2}{c^2}x^2 + y^2 = c^2 - a^2.$$

Dividing throughout by $(c^2 - a^2)$, we get

$\frac{x^2}{c^2} + \frac{y^2}{c^2 - a^2} = 1$, which is the required equation of the locus.

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

EXERCISE 1

1. Find the distance between the pair of points (12, 10) and (6, 4).
2. Prove that the points (7, 10), (4, 5) and (10, 15) are the vertices of an isosceles triangle.
3. Prove that the points $O(4, 1)$ is the centre of a circle passing through the points $A(-2, 9)$, B (10, –7) and C (12, –5).
4. Find the locus of a point which is equidistant from both the axes.
5. Find the locus of a point whose abscissa is three times its ordinate.
6. Find the locus of a point which always remains at a distance of k units from
   (i) x-axis, (ii) y-axis
7. Find the locus of a point which remains at a distance of 5 from the point (2, 3).
8. Find the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ when ***[NCERT]***
   (i) PQ is parallel to the y-axis,
   (ii) PQ is parallel to the x-axis.
9. Without using the Pythagoras theorem, show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right angled triangle. ***[NCERT]***
10. Without using distance formula, show that points (–2, –1), (4, 0), (3, 3) and (–3, 2) are the vertices of a parallelogram. ***[NCERT]***
11. What point on y-axis is equidistant from (–5, –2) and (3, 2)? ***[KVS-2003]***
12. Show that (4, 4), (3, 5) and (–1, –1) are the vertices of a right triangle. ***[KVS-2003, NCERT]***
13. What point on x-axis is equidistant from the points $A(1, 3)$ and $B(2, -5)$?
14. The opposite angular points of a square are (2, 0) and (5, 1). Find the remaining points.
15. Without using Pythagoras theorem, show that the points (1, 2), (4, 5) and (6, 3) represent the vertices of a right triangle.

EXERCISE 2

1. What point on the x-axis is equidistant from (7,6) and (–3, 4)?
2. If (0, 0), (3, 0) and (x, y) are the vertices of an equilateral triangle, find x and y.
3. Find the distance between $(x - y, y - x)$, $(x + y, x + y)$.
4. Find the value of x such $|PQ| = |QR|$ when P, Q, and R are (6, –1), (1, 3) and $(x, 8)$ respectively.
5. Show that the points:
   (i) (a, a), $(-a, -a)$, $(\sqrt{3}a, -\sqrt{3}a)$ are the vertices of an equilateral triangle.
   (ii) (0, –1) (–2, 3), (6, 7), (8, 3) are the vertices of a rectangle.
   (iii) (1, 2), (5, 4), (3, 8), (–1, 6) are the angular points of a square.
6. Find the centre of the circle passing through the points $A(a, 0)$, $B(0, b)$ and $O(0, 0)$, a and b are non-zero real numbers.
7. A point moves so that its distance from the x-axis is half its distance from the origin. Find the equation of its locus.
8. A $(a, 0)$ and $B(-a, 0)$ are two fixed points, find the locus of a point P which moves such that $3.\ |PA| = 2.\ |PB|$.
9. Let A and A' be the points (5, 0) and (–5, 0) respectively. Find the equation of the locus of all points $P(x, y)$ such that $|AP| - |A'P| = 8$.
10. Find a point on the x-axis, which is equidistant from the points (7, 6) and (3, 4). ***[NCERT]***
11. Find the locus of a point whose distance from the origin is to its distance from (–2, –3) is as 5 : 7.

ANSWERS

EXERCISE 1

1. $6\sqrt{2}$
4. $x = \pm y$
5. $x = 3y$
6. (i) $y = \pm k$ (ii) $x = \pm k$
7. $(x - 2)^2 + (y - 3)^2 = 25$
8. (i) $|y_2 - y_1|$ (ii) $|x_2 - x_1|$
11. $(0, -2)$
13. $\left(\frac{19}{2}, 0\right)$
14. $(4, -1), (3, 2)$

EXERCISE 2

1. The required point is (3, 0)
2. $y = \pm\frac{3\sqrt{3}}{2}$ and $x = \frac{3}{2}$
3. $2\sqrt{x^2 + y^2}$
4. $(5, -3)$
6. $\left(\frac{a}{2}, \frac{b}{2}\right)$
7. $x^2 - 3y^2 = 0$
8. $5(x^2 + y^2) - 26ax + 5a^2 = 0$
9. $\frac{x^2}{16} - \frac{y^2}{9} = 1$
10. $(15/2, 0)$
11. $24(x^2 + y^2) - 100x - 150y - 325 = 0$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If a point P is at a equal distance from the points $A \equiv (a + b, b - a)$ and $B \equiv (a - b, a + b)$, then the locus of P is

(a) $ax = by$ (b) $bx = ay$
(c) $ax + by = 0$ (d) $bx + ay = 0$

[PET (Raj.)-93; MP-83, 94; IIT (Hyderabad)-2000; CET (Karnataka)-2003]

Solution

(b) Let $P \equiv (x, y)$. Then as given $PA = PB$

$\Rightarrow$ $[x - (a + b)]^2 + [y - (b - a)]^2 = [x - (a - b)]^2 + [y - (a + b)]^2$

$\Rightarrow$ $-2(a + b)x - 2(b - a)y = -2(a - b)x - 2(a + b)y$

$\Rightarrow$ $bx = ay$

2. ABC is an isosceles triangle. If the coordinates of the base are $B(1, 3)$ and $C(-2, 7)$, the coordinates of vertex A can be

(a) $(1, 6)$ (b) $\left(-\frac{1}{2}, 5\right)$

(c) $\left(\frac{5}{6}, 6\right)$ (d) None of these

[Orrissa JEE-2002; Pb. CET-2002]

Solution

(c) The vertex $A(x, y)$ is equidistant from B and C because ABC is an isosceles triangle. Therefore, $(x - 1)^2 + (y - 3)^2 = (x + 2)^2 + (y - 7)^2$

$\Rightarrow$ $6x - 8y + 43 = 0$

Thus, any point lying on this line can be the vertex A except the mid-point $\left(-\frac{1}{2}, 5\right)$ of BC.

3. If a and b are real numbers between 0 and 1 such that the points $(a, 1)$, $(1, b)$ and $(0, 0)$ form an equilateral triangle, then the value of a is

(a) $2 - \sqrt{3}$ (b) $-2 + \sqrt{3}$

(c) $1 + \sqrt{3}$ (d) $-1 + \sqrt{3}$

Solution

(a) $(a - 1)^2 + (1 - b)^2 = a^2 + 1 = b^2 + 1 = (\text{side})^2$

The last two imply $a = b$ or $a = -b$

The first two imply $b^2 - 2b - 2a + 1 = 0$

If $a = -b$, then $b^2 + 1 = 0$ not possible

If $a = b$, then $b^2 - 4b + 1 = 0$

$\therefore$ $b = 2 + \sqrt{3},\ 2 - \sqrt{3}$

Since $b < 1$

$\therefore$ $b = 2 - \sqrt{3} = a$.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The quadrilateral joining the points (1, –2); (3, 0); (1, 2) and (–1, 0) is
(a) parallelogram (b) rectangle
(c) square (d) rhombus

2. The perimeter of the triangle with vertices (0, 0); (4, 3) and (0, 3) is
(a) 4 (b) 5
(c) 10 (d) 12

3. The mid-points of sides *AB* and *AC* of ΔABC are (3,5) and (–3, –3) respectively. The length of its side *BC* is
(a) 10 (b) 20
(c) 15 (d) 5

4. The triangle with vertices $A(\sqrt{3},1); B(\sqrt{2},\sqrt{2})$ and $C(1,\sqrt{3})$ is
(a) Right angled (b) Equilateral
(c) Isosceles (d) None of these

5. If *P*, *Q*, *R* are collinear points such that *P*(7,7); *Q*(3,4) and *PR* = 10; then *R* is
(a) (1, –1) (b) (1,1)
(c) (–1,1) (d) (–1, –1)

6. The number of points on *x*-axis which are at a distance *c* units ($c < 3$) from (2, 3) is
(a) 1 (b) 2
(c) infinite points (d) no point

7. The locus of a points which moves in such a way that its distance from (0, 0) is three times its distance from *x*-axis is ***[MPPET-93]***
(a) $x^2 - 4y^2 = 0$ (b) $4x^2 - y^2 = 0$
(c) $x^2 - 8y^2 = 0$ (d) $x^2 + 8y^2 = 0$

8. Points *P*(2, 7); *Q*(4, –1) and *R*(–2, 6) are vertices of a ***[MPPET-97]***
(a) right angled triangle
(b) equilateral triangle
(c) isosceles triangle
(d) None of these

9. A point moves in such a way that its distance from the point (*a*, 0) is always equal to its distance from *y*-axis. Its locus is ***[MPPET-86]***
(a) $y^2 + 2ax + a^2 = 0$
(b) $y^2 + 2ay + a^2x^2 = 0$
(c) $y^2 - 2ax + a^2 = 0$
(d) None of these

10. If the difference between distances of a variable point from (3, 0) and (–3, 0) always remains 4, then its locus is
(a) $5x^2 + 4y^2 = 20$ (b) $4x^2 - 5y^2 = 20$
(c) $5x^2 - 4y^2 = 20$ (d) $4x^2 + 5y^2 = 20$

11. If a point *P* is at equal distance from three points *A*(1,3); *B*(–3, 5) and *C*(5, –1), then *PA* is equal to
(a) $5\sqrt{5}$ (b) $5\sqrt{10}$
(c) 5 (d) 25

12. (0, –1) and (0, 3) are two opposite vertices of a square. The other two vertices are
(a) (2, 1), (–2, 1) (b) (2, 2), (1, 1)
(c) (0, 1), (0, –3) (d) (3, –1), (0, 0)

13. If the distance between the point (*a*,2) and (3,4) be 8, then *a* is equal to ***[MNR-78]***
(a) $2+3\sqrt{15}$ (b) $2-3\sqrt{15}$
(c) $2\pm 3\sqrt{15}$ (d) $3\pm 2\sqrt{15}$

14. If *P* = (1, 0), *Q* = (–1, 0) and *R* = (2, 0) are three given points, then the locus of a points S satisfying the relation $SQ^2 + SR^2 = 2SP^2$ is ***[IIT-88]***
(a) A straight line parallel to *x*-axis
(b) A circle through origin
(c) A circle with centre at the origin
(d) A straight line parallel to *y*-axis

15. A point moves such that the sum of its distances from two fixed points (*ae*, 0) and (–*ae*, 0) is always 2*a*. Then equation of its locus is ***[MNR-81]***
(a) $\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$
(b) $\frac{x^2}{a^2}-\frac{y^2}{a^2(1-e^2)}=1$
(c) $\frac{x^2}{a^2(1-e^2)}+\frac{y^2}{a^2}=1$
(d) None of these

16. If a vertex of a equilateral triangle is on origin and second vertex is (4, 0), then its third vertex is

(a) $(2, \pm\sqrt{3})$ (b) $(3, \pm\sqrt{2})$

(c) $(2, \pm 2\sqrt{3})$ (d) $(3, \pm 2\sqrt{2})$

17. The equation of the locus of all points equidistant from the point (4, 2) and the *x*-axis, is **[CEE-1993]**

(a) $x^2 + 8x + 4y - 20 = 0$
(b) $x^2 - 8x - 4y + 20 = 0$
(c) $y^2 - 4y - 8x + 20 = 0$
(d) None of these

18. The distance between the points (a cos α, a sin α) and (a cos β, a sin β) is

(a) $a \cos\dfrac{\alpha-\beta}{2}$

(b) $2a \cos\dfrac{\alpha-\beta}{2}$

(c) $a \sin\dfrac{\alpha-\beta}{2}$

(d) $2a \sin\dfrac{\alpha-\beta}{2}$

19. The points (3a, 0), (0, 3b) and (a, 2b) are **[MPPET-82]**

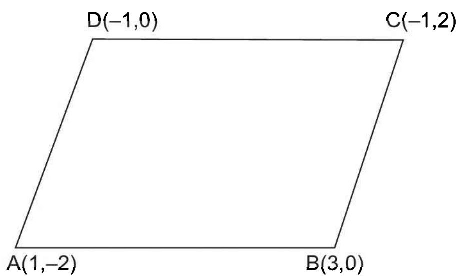
(a) Vertices of an equilateral triangle
(b) Vertices of an isosceles triangle
(c) Vertices of a right angled isosceles triangle
(d) Collinear

20. The points $(a/\sqrt{3}, a)$, $(2a/\sqrt{3}, 2a)$ and $(a/\sqrt{3}, 3a)$ are the vertices of

(a) An equilateral triangle
(b) An isosceles triangle
(c) A right angled triangle
(d) None of these

SOLUTIONS

1. (c)



$\therefore \quad AB = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$BC = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$CD = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$AD = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\therefore \quad AB = BC = CD = AD = 2\sqrt{2}$

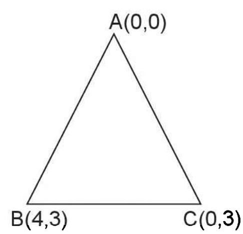
and diagonal $AC = \sqrt{0+16} = 4$

and $BD = \sqrt{16+0} = 4$

$\therefore \quad AC = BD$

Hence, the given points represents a square.

2. (d)



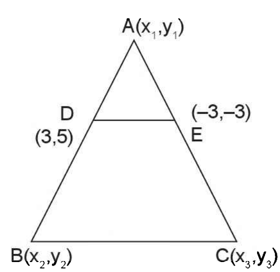
$\therefore$ side $AB = \sqrt{16+9} = \sqrt{25} = 5$

side $BC = \sqrt{16+0} = \sqrt{16} = 4$

side $AC = \sqrt{0+9} = \sqrt{9} = 3$

$\therefore$ Perimeter of triangle = 5 + 4 + 3 = 12

3. (b)



$DE = \sqrt{(3+3)^2 + (5+3)^2}$

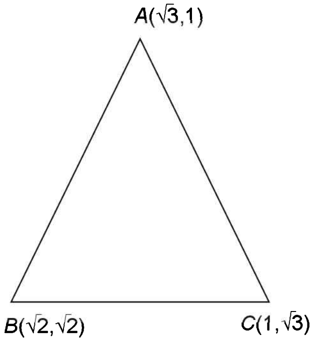
$= \sqrt{36+64} = \sqrt{100} = 10$

$\therefore$ the length of side BC

$= 2 \times DE$

$= 2 \times 10 = 20$

4. (c)



$\therefore$ side $AB = \sqrt{(\sqrt{3}-\sqrt{2})^2 + (\sqrt{2}-1)^2}$

$= \sqrt{3+2-\sqrt{6}+2+1-2\sqrt{2}}$

$= \sqrt{8-2\sqrt{6}-2\sqrt{2}}$

$BC = \sqrt{(\sqrt{2}-1)^2 + (\sqrt{2}-\sqrt{3})^2}$

$= \sqrt{8-2\sqrt{6}-2\sqrt{2}}$

$CA = \sqrt{(\sqrt{3}-1)^2 + (1-\sqrt{3})^2}$

$= \sqrt{3+1-2\sqrt{3}+3+1-2\sqrt{3}}$

$= \sqrt{8-4\sqrt{3}}$

Hence $AB = BC$

Hence the triangle is isosceles.

5. (a) $(a-3)^2 + (b-4)^2 = 100$

and $\dfrac{b-7}{3} = \dfrac{a-7}{4}$

Hence $(a, b) = (11, 10)$

Trick: Check with options. We find that the point (11, 10) satisfies both the conditions i.e.,

$AC = \sqrt{(11-3)^2 + (10-4)^2} = 10.$

Also this is collinear with A, B.

6. (d) Let $(h, 0)$ be such a point. Then as given $(h-2)^2 + 9 = c^2$

$\Rightarrow$ $h^2 - 4h + (13 - c^2) = 0$

Now h will be real

if $B^2 - 4AC \geq 0$

if $16 - 4(13 - c^2) \geq 0$

if $c^2 - 9 \geq 0$

if $|c| \geq 3$

so there is no point.

7. (c) Let (x, y) be the given point, then as given

$\sqrt{x^2 + y^2} = 3|y|$

$\Rightarrow$ $x^2 - 8y^2 = 0$

8. (a) Let given points be A, B, C respectively. Then

$AB^2 = 4 + 64 = 68, BC^2 = 36 + 49 = 85,$

$CA^2 = 16 + 1 = 17$

$\because$ $AB^2 + CA^2 = BC^2$,

so triangle is right angled.

9. (c) If the point be (x, y), then $(x-a)^2 + y^2 = |x|^2$

$\Rightarrow$ $y^2 - 2ax + a^2 = 0$

10. (c) Let the point be (x, y), then

$\sqrt{(x-3)^2 + y^2} - \sqrt{(x+3)^2 + y^2} = 4$

$\Rightarrow$ $x^2 + y^2 - 6x + 9 = x^2 + y^2 + 6x + 9 + 16 + 8\sqrt{x^2 + y^2 + 6x + 9}$

$\Rightarrow$ $(3x + 4)^2 = 4(x^2 + y^2 + 6x + 9)$

$\Rightarrow$ $5x^2 - 4y^2 = 20$

11. (b) Let $P \equiv (x, y)$, then

$(x-1)^2 + (y-3)^2 = (x+3)^2 + (y-5)^2 = (x-5)^2 + (y+1)^2$

$\Rightarrow$ $-2x - 6y + 10 = 6x - 10y + 34 = -10x + 2y + 26$

$\Rightarrow$ $8x - 4y + 24 = 0, 16x - 12y + 8 = 0$

$\Rightarrow$ $2x - y + 6 = 0, 4x - 3y + 2 = 0$

$\Rightarrow$ $x = -8, y = -10$

$\therefore$ $PA = \sqrt{(1+8)^2 + (3+10)^2} = \sqrt{250} = 5\sqrt{10}$

12. (a) The length of the diagonals of the square = 4 and their mid-point (i.e., the centre of the square) = (0, 1).

Only point of (a) satisfy these two properties. so they are the other two vertices (i.e., the end points of the other diagonal).

13. (d) Let the points be $A(a, 2)$ and $B(3, 4)$

$\therefore$ $AB = 8$

$\sqrt{(a-3)^2 + (2-4)^2} = 8$

$(a-3)^2 + (-2)^2 = 64$

$a^2 + 9 - 6a + 4 = 64$

$a^2 - 6a - 51 = 0$

$\therefore \quad a = \dfrac{6 \pm \sqrt{36+204}}{2}$

$\Rightarrow \quad a = \dfrac{6 \pm \sqrt{16\times5\times3}}{2}$

$= \dfrac{6 \pm 4\sqrt{15}}{2} = 3 \pm 2\sqrt{15}$

14. (a) If S is (x, y) then $SQ^2 + SR^2 = 2SP^2$

$\Rightarrow \quad (x+1)^2 + y^2 + (x-2)^2 + y^2 = 2((x-1)^2 + y^2)$

$\Rightarrow \quad 2x = -3$, a line parallel to Y-axis.

15. (a) $P(x, y)$, $A(ae, 0)$, $B(-ae, 0)$.

By assumption $PA + PB = 2a$

This

$\Rightarrow \quad \{(x-ae)^2 + y^2\}^{1/2} + \{(x+ae)^2 + y^2\}^{1/2} = 2a$ .....(1)

Evidently $\{(x-ae)^2 + y^2\} - \{(x+ae)^2 + y^2\} = -4aex$ ......(2)

Dividing (2) by (1),

$\sqrt{(x-ae)^2 + y^2} - \sqrt{(x+ae)^2 + y^2} = -2ex$ ...(3)

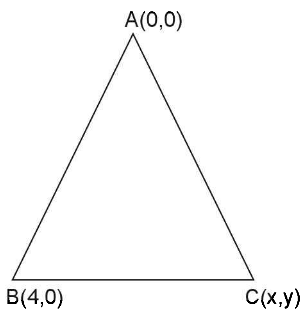
$(1) + (3) \Rightarrow \quad 2\sqrt{(x-ae)^2 + y^2} = 2(a - ex)$

Squaring, we get

$\Rightarrow \quad (x-ae)^2 + y^2 = (a-ex)^2$

$\Rightarrow \quad \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2(1-e^2)} = 1.$

16. (c) Let the third vertex be $C(x, y)$



According to the problem $BC = AC$

$\sqrt{(x-4)^2 + y^2} = \sqrt{x^2 + y^2}$

Squaring on both sides

$x^2 + 16 - 8x + y^2 = x^2 + y^2$

$8x = 16 \Rightarrow x = 2$

and also

$BC = AB$

$\sqrt{(x-4)^2 + y^2} = \sqrt{(4-0)^2}$

$x^2 + 16 - 8x + y^2 = 16$ ....(ii)

Put $x = 2$ in equation (ii)

$\therefore \quad 4 + 16 - 16 + y^2 = 16$

$y^2 = 12$

$y = \pm\sqrt{12} = \pm 2\sqrt{3}$

$\therefore$ third vertex $= (x, y) = (2, \pm 2\sqrt{3})$

17. (b) Let A be the point (4, 2) and $P(x, y)$ be any point on the locus. Let N be the foot of perpendicular from P on Y-axis so that N is $(x, 0)$

Now $PA = PN$

$\therefore \quad \sqrt{(x-4)^2 + (y-2)^2}$

$= \sqrt{(x-x)^2 + (y-0)^2}$

$\therefore \quad (x-4)^2 + (y-2)^2 = y^2$

$\therefore \quad x^2 + 16 - 8x + y^2 + 4 - 4y = y^2$

$x^2 - 8x - 4y + 20 = 0$

18. (d) Distance

$= \sqrt{a^2(\cos\alpha - \cos\beta)^2 + a^2(\sin\alpha + \sin\beta)^2}$

$= a\sqrt{\sin^2\alpha + \cos^2\alpha + \cos^2\beta + \sin^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$

$= \alpha\sqrt{2\{1-\cos(\alpha-\beta)\}} = 2a\sin\left(\dfrac{\alpha-\beta}{2}\right)$

Trick: Put $a = 1$, $\alpha = \dfrac{\pi}{2}, \beta = \dfrac{\pi}{6}$, then the points will be (0, 1) and $\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$.

Obviously, the distance between these two points is 1 which is given by (d).

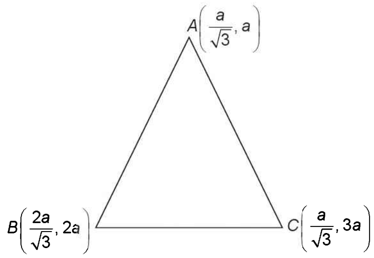
$$\left\{\begin{aligned} &\because 2a\sin\frac{\alpha-\beta}{2} = 2\times1\times \\ &\sin\frac{(\pi/2)-(\pi/6)}{2} = 2\times\frac{1}{2} = 1 \end{aligned}\right\}$$

19. (d) Area $= \dfrac{1}{2}\begin{vmatrix} 3a & 0 & 1 \\ 0 & 3b & 1 \\ a & 2b & 1 \end{vmatrix}$

$= \dfrac{1}{2}[3a(b) + 1(-3ab)] = 0$

$\therefore$ Points are collinear.

20. (b)



$$AB = \sqrt{\left(\frac{2a}{\sqrt{3}} - \frac{a}{\sqrt{3}}\right)^2 + (2a-a)^2}$$

$$= \sqrt{\frac{a^2}{3} + a^2} = \sqrt{\frac{4a^2}{3}} = \frac{2a}{\sqrt{3}}$$

$$BC = \sqrt{\left(\frac{2a}{\sqrt{3}} - \frac{a}{\sqrt{3}}\right)^2 + (2a-3a)^2}$$

$$= \sqrt{\left(\frac{a}{\sqrt{3}}\right)^2 + (-a)^2} = \sqrt{\frac{a^2}{3} + a^2}$$

$$= \sqrt{\frac{4a^2}{3}} = \frac{2a}{\sqrt{3}}$$

$$AC = \sqrt{\left(\frac{a}{\sqrt{3}} - \frac{a}{\sqrt{3}}\right)^2 + (3a-a)^2}$$

$$= \sqrt{0 + 4a^2} = 2a$$

∴ the given points are the vertices of isosceles triangle.

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If the distance of a variable point from *y*-axis is half of its distance from the origin, then its locus is

(a) $3y^2 + 4x^2 = 0$ (b) $3x^2 + 4y^2 = 0$

(c) $3y^2 - x^2 = 0$ (d) $3x^2 - y^2 = 0$

2. If $A(0, 0)$; $B(3, 4)$; $C(7, 7)$ and $D(4, 3)$ are vertices of a quadrilateral then $ABCD$ is a

(a) parallelogram (b) rectangle

(c) square (d) rhombus

3. $A(1, 0)$ and $B(-1, 0)$ are two points and Q is a point which satisfies the relation $AQ - BQ = \pm 1$. The locus of Q is

[MPPET-86]

(a) $12x^2 - 4y^2 = 3$

(b) $12x^2 + 4y^2 = 3$

(c) $12x^2 - 4y^2 + 3 = 0$

(d) $12x^2 + 4y^2 + 3 = 0$

4. Points $(2a, 4a)$; $(2a, 6a)$ and $((2+\sqrt{3})a, 5a)$ $(a > 0)$ are vertices of a

(a) equilateral triangle

(b) isosceles triangle

(c) right angled triangle

(d) None of these

5. The points $(0, 0)$, $(a, 0)$ and $\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right)$ are vertices of

(a) Isosceles triangle

(b) Equilateral triangle

(c) Scalene triangle

(d) None of these

6. The locus of a point, whose abscissa and ordinates are always equal is

(a) $x + y = 0$ (b) $x - y = 0$

(c) $x + y = 1$ (d) $x^2 + y^2 = k^2$

7. If the coordinates of vertices of ΔOAB are $(0, 0)$, $(\cos\alpha, \sin\alpha)$ and $(-\sin\alpha, \cos\alpha)$ respectively, then $OA^2 + OB^2$ is equal to

(a) 0 (b) 1

(c) 2 (d) 3

8. The points $A(-4, -1)$ $B(-2, -4)$, $C(4, 0)$ and $D(2, 3)$ are the vertices of

[Roorkee-73]

(a) Parallelogram

(b) Rectangle

(c) Rhombus

(d) None of these

9. If the points (1, 1), (–1, –1) and $(-\sqrt{3}, k)$ are vertices of a equilateral triangle then the value of k will be

(a) 1 (b) – 1

(c) $\sqrt{3}$ (d) $-\sqrt{3}$

10. The distance between the points $(am_1^2, 2am_1)$ and $(am_2^2, 2am_2)$ is

(a) $a(m_1 - m_2)\sqrt{(m_1 + m_2)^2 + 4}$

(b) $(m_1 - m_2)\sqrt{(m_1 + m_2)^2 + 4}$

(c) $a(m_1 - m_2)\sqrt{(m_1 + m_2)^2 - 4}$

(d) $(m_1 - m_2)\sqrt{(m_1 + m_2)^2 - 4}$

11. If the points A(6, –1), B(1, 3) and C (x, 8) be such that $AB = BC$, then $x =$

(a) –3, 5 (b) 3, –5

(c) –3, –5 (d) 3, 5

12. The distance between the points ($b + c$, $c + a$) and ($c + a$, $a + b$) is

(a) $\sqrt{a^2 + 2b^2 + c^2 - 2ab - 2bc}$

(b) $\sqrt{2a^2 + b^2 + c^2 - 2ab - 2bc}$

(c) $\sqrt{a^2 + 2b^2 + c^2 + 2ab + 2bc}$

(d) $\sqrt{a^2 + 2b^2 + c^2 - ab - bc}$

13. A point moves in a plane such that 4 times its distance from x-axis is equal to the square of its distance from the origin. Its locus is ***[Karnataka CET-04]***

(a) $x^2 + y^2 - 4x = 0$

(b) $x^2 + y^2 - 4\,|x| = 0$

(c) $x^2 + y^2 - 4y = 0$

(d) $x^2 + y^2 - 4\,|y| = 0$

14. The point on y-axis equidistant from the points (3, 2) and (–1, 3) is

(a) (0, –3) (b) (0, –3/2)

(c) (0, 3/2) (d) (0, 3)

15. The distance between the orthocentre and circumcentre of the triangle with vertices (1, 0), $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{-1}{2}, -\frac{\sqrt{3}}{2}\right)$ is ***[MPPET-2010]***

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{1}{3}$ (d) 0

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 14 minutes.
3. The worksheet consists of 14 questions . The maximum marks are 42.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If $A \equiv (0, 4)$; $B \equiv (0, -4)$ and $|AP - BP| = 6$ then the locus of P is
(a) $9x^2 - 7y^2 = 63$
(b) $9x^2 - 7y^2 + 63 = 0$
(c) $7x^2 - 9y^2 = 63$
(d) None of these

2. The quadrilateral formed by the vertices (–1, 1), (0, –3), (5, 2) and (4, 6) will be **[RPET-86]**
(a) Square (b) Parallelogram
(c) Rectangle (d) Rhombus

3. A point equidistant from the point (2, 0) and (0, 2) is
(a) (1, 4) (b) (2, 1)
(c) (1, 2) (d) (2, 2)

4. The equtaion to the locus of a point which moves so that its distance from x-axis is always one half its distance from the origin, is
(a) $x^2 + 3y^2 = 0$
(b) $x^2 - 3y^2 = 0$
(c) $3x^2 + y^2 = 0$
(d) $3x^2 - y^2 = 0$

5. The three points (–2, 2), (8, –2) and (–4, –3) are the vertices of
(a) An isosceles triangle
(b) an equilateral triangle
(c) A right angled triangle
(d) None of these

6. The point whose abscissa is equal to its ordinate and which is equidistant from the points (1, 0) and (0, 3) is
(a) (1, 1) (b) (2, 2)
(c) (3, 3) (d) (4, 4)

7. The locus of a point which moves so that its distance from the point (a, 0) is always four-times its distance from the axis of Y.
(a) $15x^2 - y^2 + 2ax = a^2$
(b) $15x^2 + y^2 + 2ax = a^2$
(c) $15x^2 - y^2 - 2ax = a^2$
(d) $15x^2 + y^2 + 2ax = a^2$

8. A triangle with vertices (4, 0), (–1, –1), (3, 5) is **[AIEEE-02]**
(a) Right angled
(b) Obtuse angled
(c) Equilateral
(d) None of these

9. If vertices of any quadrilateral are (0, –1), (2, 1), (0, 3) and (–2, 1), then it is a **[RPET-99]**
(a) Parallelogram (b) Square
(c) Rectangle (d) Collinear

10. Vertices of figure are (–2, 2), (–2, –1), (3, –1), (3, 2). It is a **[Karnataka CET-98]**
(a) Square (b) Rhombus
(c) Rectangle (d) Parallelogram

11. The common property of points lying on x-axis, is
(a) $x = 0$ (b) $y = 0$
(c) $a = 0, y = 0$ (d) $y = 0, b = 0$

12. The distance of the point ($b \cos \theta$, $b \sin \theta$) from origin is **[MPPET-84]**
(a) $b \cot \theta$ (b) b
(c) $b \tan \theta$ (d) $b\sqrt{2}$

13. A point moves in such a way that its distance from origin is always 4. Then the locus of the point is
(a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = 16$
(c) $x^2 + y^2 = 2$ (d) None of these

14. The points $A(5, -3)$, $B(-3, -2)$, $C(9, 12)$, $D(17, 11)$ taken in order are the vertices of a
(a) Rhombus (b) Square
(c) Rectangle (d) Parllelogram

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ
11. ⓐ ⓑ ⓒ ⓓ
12. ⓐ ⓑ ⓒ ⓓ
13. ⓐ ⓑ ⓒ ⓓ
14. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. (c) $|AP - BP| = 6$; Let P be (x, y) then

$$\left|\sqrt{x^2+(y-4)^2}-\sqrt{x^2+(y+4)^2}\right|=6$$

Solving we get $\frac{x^2}{7}-\frac{y^2}{9}=1 \Rightarrow 9x^2-7y^2=63$

4. (b) If given point is (x, y) then

Distance of point from x-axis $= |y|$

Distance of point from x origin $= \sqrt{x^2+y^2}$

$\therefore \quad |y| = \frac{1}{2}\sqrt{x^2+y^2}$

$\Rightarrow \quad y^2 = \frac{1}{4}(x^2+y^2) \; x^2 = 3y^2$

LECTURE 2

Cartesian Coordinates 2

(Section formula, area of triangle, area of quadrilateral)

BASIC CONCEPTS

1. Some important points connected with three points in a Plane

1.1 Section Formula Point (x, y) which divides the join of two given points (x_1, y_1) and (x_2, y_2) in a given ratio $m : n$ (internally and externally)

$$x = \frac{mx_2 + nx_1}{m+n}, y = \frac{my_2 + ny_1}{m+n} \quad \text{(internally)}$$

$$x = \frac{mx_2 - nx_1}{m-n}, y = \frac{my_2 - ny_1}{m-n}, m \neq n$$

(externally)

1.2 Coordinate of any point on the join of (x_1, y_1) and (x_2, y_2) dividing in the ratio $\lambda : 1$ can be taken as $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$

This point divides the given line in the ratio $\lambda : 1$. λ is + ve or –ve according as division is internal or external.

1.3 The coordinates of the mid-point of the line joining (x_1, y_1) and (x_2, y_2) is unique mid-point (for $\lambda = 1$) : $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

NOTE

1 : 1 external division is not defined.

1.4 Points of trisection The point which divide a line segment in the ratio 2 : 1 or 1 : 2 are called points of trisection. The coordinates of the points of trisection of the line joining A (x_1, y_1) and $B(x_2, y_2)$ are

$$P\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3}\right)$$

and $Q\left(\frac{x_1 + 2x_2}{3}, \frac{y_1 + 2y_2}{3}\right)$.

NOTE

If $AP = PQ = QB$, then mid-point of AB is same as that of PQ

1.5 x-axis divides the join of the points (x_1, y_1) and (x_2, y_2) in the ratio $-\left(\frac{y_1}{y_2}\right) = \lambda$.

1.6 y-axis divides the join of the points (x_1, y_1) and (x_2, y_2) in the ratio $-\left(\frac{x_1}{x_2}\right) = \lambda$.

1.7 Different conditions for three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ to be collinear are as follows

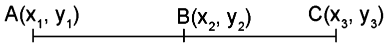
(i) $AB + BC = AC, AC - AB = BC$

(ii) If C (x_2, y_2) is a point on the line joing $A(x_1, y_1)$ and $B(x_2, y_2)$ then C divides AB in the ratio $\lambda = \frac{x_1 - x_2}{x_2 - x_3}$ or $\frac{y_1 - y_2}{y_2 - y_3}$

$\therefore$ **Condition for three points to be collinear is**

$$\frac{x_1 - x_2}{x_2 - x_3} = \frac{y_1 - y_2}{y_2 - y_3} \text{ (C.T.M.)}$$

(iii) Slope



Slope of AB = Slope of

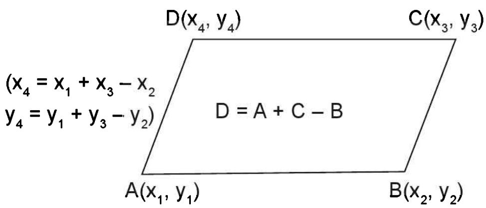
$$BC \Rightarrow \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

(iv) If the area of the triangle ABC be zero then the three points will be collinear.

1.8 The coordinates of the fourth vertex D of a square, rhombus, rectangle or parallelogram with three consecutive vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are

$D(x_4 = x_1 - x_2 + x_3, y_4 = y_1 - y_2 + y_3)$.

Since mid-point of any one diagonal is same as the mid-point of other.



2. Area of a Triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3)

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

or $$\frac{1}{2}\begin{vmatrix} x_1 - x_2 & x_1 - x_3 \\ y_1 - y_2 & y_1 - y_3 \end{vmatrix}$$

or $$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

2.1 Area of a quadratic lateral $ABCD$ whose vertices are (x_r, y_r)

$r = 1, 2, 3, 4 = \Delta_1 + \Delta_2 = \Delta ABC + \Delta BCD$ or

$\frac{1}{2}[x_1(y_2 - y_4) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + x_4(y_1 - y_3)]$

or $$\frac{1}{2}\begin{vmatrix} x_1 - x_3 & x_2 - x_4 \\ y_1 - y_3 & y_2 - y_4 \end{vmatrix}$$

2.2 Area of Pentagon = Area of quadrilateral + Area of Δ

2.3 If calculating the area of a polygon by dividing it in a number of Δ^S (triangles) we take the numerical value of area of each of the Δ.

2.4 Area is always positive but its sign may be positive or negative both. Therefore, in the formula of complete area we consider both signs + ve and –ve e.g., Complete area:

Area of $\Delta\, OAB$ = 10 units

$\Rightarrow\ |\Delta| = 10,\ \therefore\ \Delta = \pm 10$

3. If (x_1, y_1), (x_2, y_2) are the ends of the hypotenuse of a right angled isosceles triangle, then the third vertex is given by

$$\left(\frac{x_1 + x_2 \pm (y_1 - y_2)}{2}, \frac{y_1 + y_2 \mp (x_1 - x_2)}{2}\right)$$

e.g., let A (2, 0), B(0, 2) be the ends of the hypotenuse of a right angled isosceles triangle.

$\therefore$ third vertex is

$$\frac{(2+0) \pm (0-2)}{2}, \frac{(0+2) \mp (2-0)}{2}$$

i.e., (0, 0) or (2, 2)

4. Given two vertices (x_1, y_1) and (x_2, y_2) of an equilateral triangle, then its third vertex is given by

$$\left(\frac{x_1 + x_2 \pm \sqrt{3}(y_1 - y_2)}{2}, \frac{y_1 + y_2 \mp \sqrt{3}(x_1 - x_2)}{2}\right)$$

e.g., Let A(2, 0), B = (–2, 0) be the ends of an equilateral triangle, then its third vertex is

$$\left(\frac{2 - 2 \pm \sqrt{3}(0,-0)}{2}, \frac{0 \mp \sqrt{3}(2+2)}{2}\right) \text{ is}$$

$(0, \mp 2\sqrt{3})$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. The points (–1, 6) and (5, 4) are opposite vertices of a square. Find the coordinates of the remaining two vertices.

Solution

Let $A(-1, 6)$ and $C(5, 4)$ be the given vertices of a square $ABCD$.
Let the diagonal meet at M, then

$$M \equiv \left(\frac{-1+5}{2}, \frac{6+4}{2}\right) = (2, 5)$$

Let (x, y) be the coordinates of any of the remaining two vertices, then distance of (x, y) from M = distance of A from M (centre of a square is equidistant from the four vertices)

$$\Rightarrow \quad \sqrt{(x-2)^2+(y-5)^2} = \sqrt{(-1-2)^2+(6-5)^2}$$

$\Rightarrow \quad (x-2)^2 + (y-5)^2 = 10$

Also, (x, y) is equidistant from A and C.

$$\Rightarrow \quad \sqrt{(x+1)^2+(y-6)^2} = \sqrt{(x-5)^2+(y-4)^2}$$

$\Rightarrow \quad x^2 + 2x + 1 + y^2 - 12y + 36 = x^2 - 10x + 25 + y^2 - 8y + 16$

$\Rightarrow \quad 12x = 4y + 4 \Rightarrow y = 3x - 1$

Substituting the value of y from (2) in (1), we get $(x-2)^2 + (3x - 1 - 5)^2 = 10$

$\Rightarrow \quad (x-2)^2 + (3(x-2))^2 = 10$

$\Rightarrow \quad (1 + 9)(x-2)^2 = 10$

$\Rightarrow \quad (x-2)^2 = 1 \Rightarrow x - 2 = \pm 1$

$\Rightarrow \quad x = 2 \pm 1 \Rightarrow x = 1$ or 3

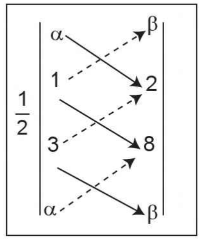
When $x = 1$, then from (2), $y = 3 \times 1 - 1 = 2$
and when $x = 3$, then for (2), $y = 3 \times 3 - 1 = 8$
Hence, the other two vertices are (1, 2) and (3, 8).

2. If $A(1, 2)$ and $B(3, 8)$ be two given points, find a point P such that $|PA| = |PB|$ and $\Delta PAB = 10$.

Solution

Let $P = (\alpha, \beta)$, then $|PA| = |PB| \Rightarrow PA^2 = PB^2$

$\Rightarrow \quad (\alpha - 1)^2 + (\beta - 2)^2 = (\alpha - 3)^2 + (\beta - 8)^2$



$\Rightarrow \quad \alpha^2 - 2\alpha + 1 + \beta^2 + 4 - 4\beta = \alpha^2 + 9 - 6\alpha + \beta^2 + 64 - 16\beta$

$\Rightarrow \quad 4\alpha + 12\beta = 68 \Rightarrow \alpha + 3\beta = 17 \quad \text{...... (1)}$

Also, area of $\Delta PAB = 10$

$\Rightarrow \quad \frac{1}{2}|2\alpha - \beta + 8 - 6 + 3\beta - 8\alpha| = 10$

$\Rightarrow \quad -6\alpha + 2\beta + 2 = \pm 20$

$\Rightarrow \quad -3\alpha + \beta + 1 = \pm 10$

$\Rightarrow \quad -3\alpha + \beta = -1 \pm 10$

$\Rightarrow \quad -3\alpha + \beta = -11 \quad \text{......... (2)}$

or $\quad -3\alpha + \beta = 9 \quad \text{.......... (3)}$

Solving (1) and (2), Solving (1) and (3), we obtain $\alpha = 5, \beta = 4$ we obtain $\alpha = -1, \beta = 6$
Hence, the point P is either (5, 4) or (–1, 6)

3. Prove that the area of the triangle with vertices $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$ is $a^2 |(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$ square units.

Solution

Required area

$$= \frac{1}{2}\begin{vmatrix} at_1^2 - 2at_1^2 & at_1^2 - at_3^2 \\ 2at_1 - 2at_2 & 2at_1 - 2at_3 \end{vmatrix}$$

$$= \frac{1}{2}(2a^2)\left|t_1^2t_2 - t_2^2t_1 + t_2^2t_3 + t_3^2t_2 + t_3^2t_1 - t_1^2t_3\right|$$

$$= a^2\left|t_1^2t_2 - t_1^2t_3 - t_2^2t_1 + t_3^2t_1 + t_2^2t_3 - t_3^2t_2\right|$$

$$= a^2\left|t_1^2(t_2 - t_3) - t_1(t_2^2 - t_3^2) + t_2t_3(t_2 - t_3)\right|$$

$$= a^2\left|(t_2 - t_3)\{t_1^2 - t_1(t_2 + t_3) + t_2t_3\}\right|$$

$$= a^2\left|(t_2 - t_3)(t_1 - t_2)(t_1 - t_3)\right|$$

$$= a^2\left|-(t_2 - t_3)(t_1 - t_2)(t_3 - t_1)\right|$$

$$= a^2\left|(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)\right|$$

($\because |(-x)| = |x|$ for all $x \in R$)

4. A rod of length $2l$ slides between two perpendicular lines. Find the locus of its middle-point.

Solution

Take the given perpendicular lines as the axes. Let AB be the line of length $2l$ meeting x-axis at A and y-axis at B. Let $OA = a$ and $OB = b$

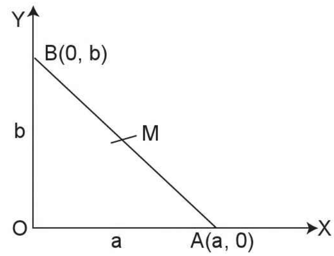
Since OAB is right angled triangle,

$\therefore \quad OA^2 + OB^2 = AB^2,$

i.e., $a^2 + b^2 = (2l)^2 = 4l^2$ .... (i)

Coordinates of A and B are $(a, 0)$, $(0, b)$ respectively.

Coordinates (x, y) of M, the mid-point of $[AB]$ are given by



$$x = \frac{a+0}{2} = \frac{a}{2},\ y = \frac{0+b}{2} = \frac{b}{2}$$

$\Rightarrow \quad a = 2x, b = 2y$

Putting these values in (i), we get

$4x^2 + 4y^2 = 4l^2 \Rightarrow x^2 + y^2 = l^2,$

which is the required equation of locus of M, the mid-point of $[AB]$.

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

EXERCISE 1

1. Find the coordinates of the point which divides the line segment joining the points (–1, 1) and (6, 8) in the ratio 3 : 4 internally.

2. Find the coordinates of the points which may divide internally and externally the line segments joining the points (–3, –4) and (–8, 7) in the ratio 7 : 5.

3. Find the coordinates of the points trisecting the line segments joining the point (1, –2) and (–3, 4).

4. In what ratio does the x-axis and the y-axis divide the line segment joining the points (5, 3) and (10, –8)?

5. Find the ratio in which (11, 15) divides the join of (15, 5) and (9, 20).

6. The coordinates of the end point of the diameter of a circle are (3, 5). The coordinates of the centre is (6, 6). Find the coordinates of the other end point of the diameter.

7. The coordinates of the centre of a circle are (–2, –3). If AB is a diameter of the circle and coordinates of A are (–1, 4). Find the coordinates of B.

8. Find out a point on x-axis, such that it is at the same distance from the points (–6, 4) and (2, –14).

9. Find the value of x for which the points $(x, –1)$, (2, 1) and (4, 5) are collinear.

[NCERT]

10. For what value(s) of x, the area of the triangle formed by the points (5, –1), $(x, 4)$ and (6, 3) is 5.5 square units.

[NCT-2003]

EXERCISE 2

1. The line segement joining the points (3, –1) and (–6, 5) is trisected. Find the coordinates of points of trisection.

2. Show that the points (0, 3), (6, 0) and (4, 1) lie on a straight line by (by using section formula).

3. In what ratio is the line segment joining the points (4, 5) and (1, 2) divided by the y-axis?

Find also the coordinates of the point of division.

4. In what ratio does the point (2, –5) divide the line segment joining the points (–3, 5) and (4, –9)?

5. By using area of triangle, show that the points $(a, b + c)$, $(b, c + a)$ and $(c, a + b)$ are collinear.

6. Find the area of the quadrilateral whose vertices, taken in order, are (1, 2), (5, 4), (3, 8) and (–1, 6).

7. Find the area of the triangle whose vertices are (2, 7), (3, –1) and (–5, 6).

8. Test whether the points (–3, 4), (2, –5) and (11, 18) are collinear or not.

9. If $P(x, y)$ is any point on the line joining the points $A(a, 0)$ and $B(0, b)$, then show that $\frac{x}{a}+\frac{y}{b}=1$

10. If vertices of a triangle are $(1, k)$, $(4, -3)$, $(-9, 7)$ and its area is 15 square units, find the value (s) of k.

ANSWERS

EXERCISE 1

1. (2, 4)
2. $\left(-\frac{71}{12},\frac{29}{12}\right),\left(-\frac{41}{2},\frac{69}{2}\right)$
3. $\left(-\frac{1}{3},0\right),\left(-\frac{5}{3},2\right)$
4. x-axis 3 : 8 internallly, y-axis 1 : 2 externally.
5. (2 : 1)
6. (9, 7)
7. (–3, –10)
8. $\left(\frac{37}{4},0\right)$
9. $x = 1$
10. $9,\frac{2}{7}$

EXERCISE 2

1. (–3, 3)
3. 4 : 1 externally; (0, 1)
4. 5 : 2
5. 0
6. 20 square units.
7. $\frac{57}{2}$ square units
8. Not collinear
10. $-3,\frac{21}{13}$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If (2, –1); (4, 3); (–1, 2) and (–3, –2) are vertices of a quadrilateral then its area is *[DCE-20003]*

(a) 36 (b) 18
(c) 54 (d) 27

Solution

(b) $\text{Area} = \frac{1}{2}\begin{bmatrix} 2 & 4 & -1 & -3 & 2 \\ -1 & 3 & 2 & -2 & -1 \end{bmatrix}$

$$=\frac{1}{2}\begin{bmatrix} 2-(-1) & 4-(-3) \\ -1-2 & 3-(-2) \end{bmatrix}=18\text{ sq. units.}$$

2. If the points $(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$ divides the line joining (x_1, y_1) and (x_2, y_2) internally, then *[EAMCET-2000]*

(a) $t < 0$ (b) $0 < t < 1$
(c) $t > 1$ (d) $t = 1$

Solution

(b) Coordinates of the dividing point can be written as $\left(\frac{tx_2+(1-t)x_1}{t+(1-t)},\frac{ty_2+(1-t)y_1}{t+(1-t)}\right)$

This shows that the division 'ratio' $=\frac{t}{1-t}$.

But this division is internal, so $\frac{t}{1-t} > 0$

$\Rightarrow \quad \frac{t}{t-1} < 0$

$\Rightarrow$ t and $(t-1)$ are in opposite signs.

$\Rightarrow \quad 0 < t < 1.$

3. If the points $(k, 2-2k)$; $(1-k, 2k)$ and $(-k-4, 6-2k)$ are collinear, then the possible values of k are ***[AMU-1978]***

(a) 1, –2 (b) 2, –1

(c) 1, –1/2 (d) –1, 1/2

Solution

(d) $\begin{vmatrix} k & 2-2k & 1 \\ 1-k & 2k & 1 \\ -k-4 & 6-2k & 1 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} k & 2-2k & 1 \\ 1-2k & 4k-2 & 0 \\ -2k-4 & 4 & 0 \end{vmatrix} = 0$

$\Rightarrow \quad 4 - 8k + (2k+4)(4k-2) = 0$

$\Rightarrow \quad 2k^2 + k - 1 = 0$

$\Rightarrow \quad k = -1, 1/2$

4. If area of a triangle with vertices $(2a, a)$; $(a, 2a)$ and (a, a); $(a > 0)$ is 18 units, then its centroid is ***[UPSEAT-99]***

(a) (8, 8) (b) (4, 4)

(c) (–4, –4) (d) None of these

Solution

(a) We have $|\, 2a\,(2a-a) + a\,(a-a) + a\,(a-2a)\,| = 36$

$\Rightarrow \quad 2a^2 - a^2 = 36 \Rightarrow a^2 = 36 \Rightarrow a = 6$

$\therefore \quad \text{Centroid} = \left(\frac{2a+a+a}{3}, \frac{a+2a+a}{3}\right)$

$= \left(\frac{4a}{3}, \frac{4a}{3}\right) = (8, 8)$

5. If $A(2, -1)$ and $B(6, 5)$ are two points the ratio in which the foot of the perpendicular from (4, 1) to AB divides it, is ***[EAMCET-2007]***

(a) 8 : 15 (b) 5 : 8

(c) – 5 : 8 (d) – 8 : 5

Solution

(b) Let $P(4, 1)$ and PD be a $\perp r$ on AB $A(2, -1)$, $B(6, 5)$

Equation of PD is $2x + 3y - 11 = 0$

$\therefore \quad D$ divides AB in the ratio $-L_{11} : L_{22} = 10 : 16$ $= 5 : 8$

6. If (1, 1); (3, 4); (5, –2) and (4, –7) are vertices of a quadrilateral then its area ***[IIT-1961]***

(a) $\frac{41}{2}$ units (b) $\frac{2}{41}$ units

(c) $\frac{43}{3}$ units (d) $\frac{42}{4}$ units

Solution

(a) $= \frac{1}{2}$ [(1) (4) – (3) (1) + 3 (–2) – 5 (4) + 5 (–7) – 4(–2) + 4(1) –1 (–7)]

$= \frac{1}{2}\,[4 - 3 - 6 - 20 - 35 + 8 + 4 + 7] = \frac{41}{2}$ units.

7. The point A divides the join of the points (–5, 1) and (3, 5) in the ratio $k : 1$ and coordinates of point B and C are (1, 5) and (7, –2) respectively. If the area of $\Delta\, ABC$ be 2 units, then k equals ***[IIT-1967]***

(a) 6, 7 (b) 7, 9

(c) 7, 31/9 (d) 9, 31/9

Solution

(c) $A \equiv \left(\frac{3k-5}{k+1}; \frac{5k+1}{k+1}\right)$

Area of $\Delta\, ABC$ = 2 units

$\Rightarrow \frac{1}{2}\left[\frac{3k-5}{k+1}(5+2) + 1\left(-2 - \frac{5k+1}{k+1}\right) + 7\left(\frac{5k+1}{k+1} - 5\right)\right] = \pm 2$

$\Rightarrow \quad 14\,k - 66 = \pm\, 4(k+1)$

$\Rightarrow \quad k = 7$ or $31/9$

8. The points (1, 1), $(0, \sec^2\theta)$, $(\text{cosec}^2\theta, 0)$ are collinear for ***[Roorkee-1963]***

(a) $\theta = \frac{n\pi}{2}$ (b) $\theta \neq \frac{n\pi}{2}$

(c) $\theta = n\pi$ (d) None of these

Solution

(b) The given points are collinear, if

$\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 0 & \sec^2\theta & 1 \\ \text{cosec}^2\theta & 0 & 1 \end{vmatrix} = 0$

$\Rightarrow$ $1(\sec^2\theta) + 1(\text{cosec}^2\theta) - 1(\text{cosec}^2\theta.\ \sec^2\theta) = 0$

$$\Rightarrow \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} - \frac{1}{\sin^2\theta\cos^2\theta} = 0$$

$$\Rightarrow \frac{1}{\cos^2\theta\sin^2\theta} - \frac{1}{\sin^2\cos^2\theta} = 0 \Rightarrow 0 = 0$$

Therefore, the points are collinear for all values of θ, except only $\theta = \frac{n\pi}{2}$ because at $\theta = \frac{n\pi}{2}, \sec^2\theta = \infty$

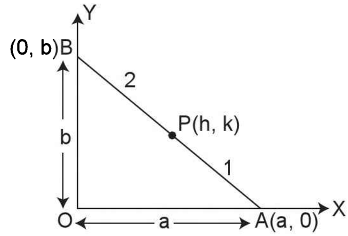
9. The ends of a rod of length l move on two mutually perpendicular lines. The locus of the point on the rod which divides it in the ratio 1 : 2 is

[IIT-1987; RPET-1997]

(a) $36x^2 + 9y^2 = 4l^2$
(b) $36x^2 + 9y^2 = l^2$
(c) $9x^2 + 36y^2 = 4l^2$
(d) None of these

Solution

(c) According to the figure



$AP : PB = 1 : 2$, then $h = \frac{1\times 0 + 2\times a}{1+2} = \frac{2a}{3}$

or $a = \frac{3h}{2}$,

Similarly, $b = 3k$

Now we have $OA^2 + OB^2 = AB^2$

$$\Rightarrow \left(\frac{3h}{2}\right)^2 + (3k)^2 = l^2$$

Hence locus of $P(h, k)$ is given by $9x^2 + 36y^2 = 4l^2$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If three vertices of a parallelogram in order are (1, 3); (2, 0) and (5, 1), then its fourth vertex is

(a) (3, 3) (b) (4, 4)
(c) (4, 0) (d) (0, –4)

2. The points which trisect the line segment joining (0,0) and (9,12) are

[RPET-86]

(a) (3, 4); (6, 8) (b) (4, 3); (6, 8)
(c) (4, 3); (8, 6) (d) (3, 4); (8, 6)

3. The points $(-a, -b)$; (0, 0); (a, b) and (a^2, ab) are

(a) collinear
(b) vertices of a parallelogram
(c) vertices of a rectangle
(d) None of these

4. If the middle points of the sides of a triangle are (0, 0); (1, 2) and (–3, 4), then area of the triangle is

(a) 10 (b) 20
(c) 40 (d) 60

5. Area of a triangle with vertices $(a, b + c)$; $(b, c + a)$ and $(c, a + b)$ is

[IIT-63; EAMCET-82; RPET-03]

(a) abc (b) $a^2 + b^2 + c^2$
(c) $ab + bc + ca$ (d) 0

6. If (3, 2); (6, 1) and (7, 3) are three vertices of a parallelogram, then its fourth vertex will be

(a) (4, 4) or (2, 0) (b) (–4, 4) or (10, 2)
(c) (10, 2) or (–2, 0) (d) (2, –2) or (4, 0)

7. The area of the quadrilateral with vertices (1, 1); (3, 4); (5, –2) and (4, –7) is

(a) 43/2 (b) 45/2
(c) 47/2 (d) 41/2

8. The point (5,8) divides AB internally in the ratio 2 : 1. If $A \equiv (3, 4)$, then B will be

(a) (6, 10) (b) (10, 6)
(c) (–7, 10) (d) (6, –10)

9. Each side of an equilateral triangle is equal to a. If its vertices are (x_1, y_1); (x_2, y_2) and (x_3, y_3); then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$ is equal to

(a) $3a^4$ (b) $3a^4/4$
(c) $4a^4$ (d) None of these

10. If points $(p + 1, 1)$; $(2p + 1, 3)$ and $(2p + 2, 2p)$ are vertices of a triangle, then *[MPPET-86]*

(a) $p \neq 1$ (b) $p \neq 2$
(c) $p \neq 0$ (d) $p \neq 3$

11. The line segment joining the points $(-3, -4)$ and $(1, -2)$ is divided by y-axis in the ratio

(a) 1: 3 (b) 2 : 3
(c) 3 : 1 (d) 3 : 2

12. If $A(2, -2)$; $B(8, 4)$; $C(5, 7)$ and $D(x, y)$ are vertices of a rectangle then D is

(a) (1, 1) (b) (1, –1)
(c) (–1, 1) (d) (–1, –1)

13. If points $(a - 2, a - 4)$; $(a, a + 1)$ and $(a + 4, 16)$ are collinear, then a is eaual to

(a) 5 (b) –5
(c) 7 (d) –7

14. The distance of the middle point of the line joining the points $(a \sin\theta, 0)$ and $(0, a \cos\theta)$ from the origin is *[MPPET-99]*

(a) a (b) $a/2$
(c) $a(\sin\theta + \cos\theta)2$ (d) $a/4$

15. If three different points $(x + 1, 2)$; $(1, x + 2)$ and $\left(\frac{1}{x+1}, \frac{2}{x+1}\right)$ are collinear, then x is *[RPET-02]*

(a) 4 (b) –4
(c) 0 (d) None of these

16. If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$, then the two triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and (a_1, b_1), (a_2, b_2), (a_3, b_3) must be

(a) Similar (b) Congruent
(c) Never congruent (d) None of these

17. Three points are $A(6, 3)$, $B(-3, 5)$, $C(4, -2)$ and $P(x, y)$ is a point, then the ratio of area of ΔPBC and ΔABC is *[IIT-83]*

(a) $\left|\frac{x+y-2}{7}\right|$ (b) $\left|\frac{x+y+2}{2}\right|$

(c) $\left|\frac{x-y-2}{7}\right|$ (d) None of these

18. The locus of P such that area of ΔPAB = 12 sq. units, where $A(2, 3)$ and $B(-4, 5)$ is *[EAMCET-89]*

(a) $(x + 3y - 1)(x + 3y - 23) = 0$
(b) $(x + 3y + 1)(x + 3y - 23) = 0$
(c) $(3x + y - 1)(3x + y - 23) = 0$
(d) $(3x + y + 1)(3x + y + 23) = 0$

19. Area of a triangle with vertices (a, b), (x_1, y_1) and (x_2, y_2), where a, x_1 and x_2 are in G.P. with common ratio r and b, y_1 and y_2 are in G.P. with common ratio s, is given by

(a) $ab(r - 1)(s - 1)(s - r)$
(b) $\frac{1}{2} ab (r + 1)(s + 1)(s - r)$
(c) $\frac{1}{2} ab (r - 1)(s - 1)(s - r)$
(d) $ab (r + 1)(s + 1)(r - s)$

20. If P and Q are points on the line joining $A(-2, 5)$ and $B(3, 1)$ such that $AP = PQ = QB$, then mid-point of PQ is

(a) (1/2, 3) (b) (–1/2, 4)
(c) (2, 3) (d) (–1, 4)

21. If $ABCD$ is a quadrilateral if the mid-point of consecutive sides AB, BC, CD and DA are combined by straight lines, then the quadrilateral $PQRS$ is always *[Orissa JEE-02]*

(a) Square (b) Parallelogram
(c) Rectangle (d) Rhombus

22. The length of altitude through A of the triangle ABC, where $A = (-3, 0)$; $B = (4, -1)$; $C = (5, 2)$, is *[Karnataka CET-01]*

(a) $2/\sqrt{10}$ (b) $4/\sqrt{10}$
(c) $11/\sqrt{10}$ (d) $22/\sqrt{10}$

23. If the points $A(1, -3)$, $B(2, 2)$, $C(5, 1)$ are collinear, then the coordinates of the point dividing AC in the ratio $AB : BC$ externally is

(a) (1, 5) (b) (–1, –5)
(c) (–1, 5) (d) (1, –5)

24. If the point $(x, -1)$, $(3, y)$, $(-2, 3)$ and $(-3, -2)$ be the vertices of a parallelogram, then
(a) $x = 2, y = 4$ (b) $x = 1, y = 2$
(c) $x = 4, y = 2$ (d) None of these

25. The A and B are the points $(-3, 4)$ and $(2, 1)$, then the coordinates of point C on AB produced such that $AC = 2BC$ are
(a) (2, 4) (b) (3, 7)
(c) (–1/2, 5/2) (d) (7, –2)

26. If the four points (1, 2), (–5, 6), (7, –4) and $(a, -2)$ are collinear. Find a is equal to
(a) 3 (b) 2
(c) 1 (d) 4

27. Line segment joining (5, 0) and $(10 \cos \theta, 10 \sin \theta)$ is divided by a point P in ratio 2:3. If θ varies then locus of P is a
(a) straight line
(b) circle
(c) pair of straight lines
(d) parabola

28. The mid-point of the line joining the points (–10, 8) and (–6, 12) divides the line joining the points (4, –2) and (–2, 4) in the ratio **[Kerala PET-2007]**
(a) 1 : 2 internally (b) 1 : 2 externally
(c) 2 : 3 externally (d) 2 : 1externally

29. The point dividing the line joining two points (–1, –2) and (3, 4) in the ratio 3 : 5 internally lies on the line **[MPPET-2007]**
(a) $4(x + y) = 3$ (b) $x + y = 3$
(c) $4x + 4y = 1$ (d) None of these

30. If (1, –2) and (1, 2) are end points of a diagonal of a square and $(3, h)$; $(-1, k)$ are end points of its other diagonal, then
(a) $h = -1, k = 1$ (b) $h = 1, k = -1$
(c) $h = k = 0$ (d) $h = k = 1$

31. The opposite angular points of a square are (3, 4) and (1, –1). Then the coordinates of other two vertices are
(a) $D(1/2, 9/2), B(-1/2, 5/2)$
(b) $D(-1/2, 9/2), B(1/2, 5/2)$
(c) $D(9/2, 1/2), B(-1/2, 5/2)$
(d) None of these

32. The area of the triangle whose vertices are (1, 0), (7, 0) and (4, 4) is **[MPPET-2009]**
(a) 8 (b) 10
(c) 12 (d) 14

SOLUTIONS

1. (b) Let the fourth vertex be (x, y). Since the mid-points of the two diagonals coincide, so $1 + 5 = 2 + x$ and $3 + 1 = 0 + y$
$\Rightarrow \quad x = 4, y = 4.$

2. (b) Let the given points be A, B and the required points be P, Q. Then P divides AB in the ratio 1 : 2.

So $$P \equiv \left(\frac{1.9 + 2.0}{1+2}, \frac{1.12 + 2.0}{1+2}\right) \equiv (3, 4)$$

Also Q is the mid-point of PB.
so $Q \equiv (6, 8)$

3. (a) If the given points are A, B, C, D taken in order, then the mid-point of $AC = (0, 0)$ the mid-point of
$BD = (a^2/a, ab/2)$.

Since these mid-points do not coincide so $ABCD$ cannot be a parallelogram or rectangle.
Also area of the quadrilateral $ABCD$

$$= \frac{1}{2}[0 - 0 + 0 - 0 + a^2b - a^2b - a^2b + a^2b] = 0$$

$\therefore \quad A, B, C, D$ are collinear.

Aliter : If given points be A, B, C, D; then slopes of AB, AC, AD are equal.
So A, B, C, D are collinear.

4. (b) If the given mid-points be D, E, F; then area of ΔDEF is given by $\frac{1}{2}\begin{vmatrix} 0 & -1 & -3 & -0 \\ 0 & -2 & 4 & -0 \end{vmatrix}$

$$= \frac{1}{2}[0 - 0 + 4 + 6 + 0] = 5$$

$\therefore \quad$ Area of the given triangle $= 4 \times 5 = 20$

5. (d) Area $= \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} \xrightarrow{c_2 + c_1}$

$$\frac{1}{2}\begin{vmatrix} a & a+b+c & 1 \\ b & a+b+c & 1 \\ c & a+b+c & 1 \end{vmatrix} = \left(\frac{a+b+c}{2}\right)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} = 0$$

6. (a) Let fourth vertex be (x, y). There are two possibilities that end points of two diagonals are (3, 2), (7, 3) and (6, 1), (x, y) or (3, 2), (6, 1) and (7, 3), (x, y).

In first case $3 + 7 = 6 + x$ and $2 + 3 = 1 + y$

$\Rightarrow \quad (x, y) = (4, 4)$

In second case $3 + 6 = 7 + x$ and $2 + 1 = 3 + y$

$\Rightarrow \quad (x, y) = (2, 0)$

7. (d) Area $= \frac{1}{2}\begin{vmatrix} 1-5 & 3-4 \\ 1-(-2) & 4-(-7) \end{vmatrix}$

$= \frac{1}{2}(-44 - (-3))$

$= 41/2$ units.

8. (a) Let $B \equiv (h, k)$, then $2h + 3 = 5 \times 3$, $2k + 4 = 8 \times 3$

$\therefore \quad B \equiv (6, 10)$

9. (b) Area of equilateral triangle $= \frac{\sqrt{3}}{4}$ (side)^2

$$\Rightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{\sqrt{3}}{4}a^2$$

$$\Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}$$

10. (b) $\frac{(p+1)-(2p+1)}{1-3} \neq \frac{(2p+1)-(2p+2)}{3-2p}$

$\Rightarrow \quad \frac{p}{2} \neq \frac{1}{2p-3}$

$\Rightarrow \quad 2p^2 - 3p - 2 \neq 0$

$\Rightarrow \quad p \neq 2, -1/2$

11. (c) The required ratio $= -x_1/x_2$

$= -(-3)/1$

$= 3 : 1$

12. (c) Step 1: The coordinates of the fourth vertex D rectangle with three consecutive vertices A (x_1, y_1), $B(x_2, y_2)$ and $C(x_3, y_3)$ are $D(x_4 = x_1 - x_2 + x_3, y_4 = y_1 - y_2 + y_3)$.

Step 2:

$x = x_1 - x_2 + x_3 = 2 - 8 + 5 = -1$

$y = y_1 - y_2 + y_3 = -2 - 4 + 7 = 1$

$\therefore$ Coordinate of $D = (x, y) = (-1, 1)$.

13. (a) $\begin{vmatrix} a-2 & a-4 & 1 \\ a & a+1 & 1 \\ a+4 & 16 & 1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} -6 & a-20 & 0 \\ -4 & a-15 & 0 \\ a+4 & 16 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad -6(a - 15) + 4(a - 20) = 0 \quad \Rightarrow a = 5.$

14. (a) Middle point M is

$$M\left(\frac{a\sin\theta}{2}, \frac{0 + a\cos\theta}{2}\right)$$

$$(OM)^2 = \left(\frac{a\sin\theta}{2}\right)^2 + \left(\frac{a\cos\theta}{2}\right)^2$$

$$= \frac{a^2}{4} = \left(\frac{a}{2}\right)^2 \quad \Rightarrow OM = \frac{a}{2}$$

15. (b) Collinearity

$$\Rightarrow \frac{x+1-1}{2-(x+2)} = \frac{(x+1) - \frac{1}{x+1}}{2 - \frac{2}{x+1}}$$

$$\Rightarrow \frac{x}{-x} = \frac{(x+1)^2 - 1}{2(x)} (x \neq -1)$$

$\Rightarrow \quad 2x^2 + x(x + 1)^2 - x = 0$

$\Rightarrow \quad x^3 + 4x^2 = 0$

$\Rightarrow \quad x^2(x + 4) = 0 \Rightarrow x = 0, -4$

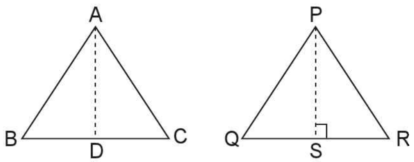
But when $x = 0$, two points will coincide.

Hence $x = -4$.

16. (a) We have $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

$\Rightarrow$ Area of triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is same as the area of triangle with vertices (a_1, b_1), (a_2, b_2), (a_3, b_3).



$$\frac{1}{2}\times AD\times BC=\frac{1}{2}\times PS\times OR$$

$$AD\times BC=PS\times QR$$

Hence, the two triangles are equal in area.

17. (a) A_1 = Area of $\Delta\, PBC=\frac{1}{2}\begin{vmatrix} x & y & 1\\ -3 & 5 & 1\\ 4 & -2 & 1\end{vmatrix}$

$=\frac{1}{2}[x(7)-y(-7)-14]=\frac{7}{2}[x+y-2]$ ... (1)

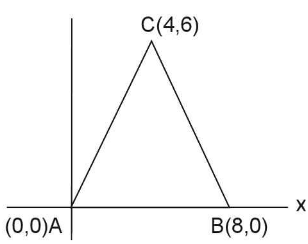
$=\frac{7}{2}[x+y-2]$ as $A_1 > 0$.

A_2 = Area of $\Delta\, ABC=\frac{1}{2}\begin{vmatrix} 6 & 3 & 1\\ -3 & 5 & 1\\ 4 & -2 & 1\end{vmatrix}$

$=\frac{1}{2}[6(7)-3(-7)+1(6-20)]=\frac{7}{2}[7]$ ... (2)

Dividing (1) by (2), $\frac{A_1}{A_2}=\left|\frac{x+y-2}{7}\right|$

18. (b) $\Delta\, PAB = 12 \Rightarrow \begin{vmatrix} x & y & 1\\ 2 & 3 & 1\\ -4 & 5 & 1\end{vmatrix}=\pm 24$



$\Rightarrow$ $-2x-6y+22=\pm 24$

$\Rightarrow$ $-2x-6y=2, -46$

$\Rightarrow$ $x+3y=-1, 23$

$\Rightarrow$ $x+3y+1=0, x+3y-23=0$

Locus of $p\,(x, y)$ is $(x+3y+1)(x+3y-23)=0$

19. (c) Since a, x_1, a_2 are in G.P. with common ratio r

$\therefore$ $x_1 = ar, x_2 = ar^2$

Similarly, $y_1 = bs, y_2 = bs^2$

$\therefore$ required area $=\frac{1}{2}\begin{vmatrix} a & b & 1\\ ar & bs & 1\\ ar^2 & bs^2 & 1\end{vmatrix}$

$=\frac{1}{2}\begin{vmatrix} a & b & 1\\ ar-a & bs-b & 0\\ ar^2-a & bs^2-b & 0\end{vmatrix}$

(Operate R_3-R_1, R_2-R_1)

$=\frac{1}{2}\,[a(r-1)\,b\,(s^2-1)-b\,(s-1)\,a\,(r^2-1)]$

$=\frac{1}{2}ab(r-1)(s-1)\left[\overline{s+1}-\overline{r+1}\right]$

$=\frac{1}{2}\,ab\,(r-1)\,(s-1)\,(s-t)$

20. (a) Mid-point of PQ = mid-point of

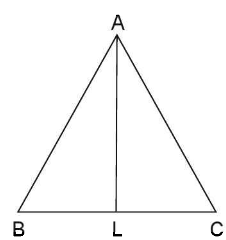
$AB=\left(\frac{-2+3}{2},\frac{5+1}{2}\right)=\left(\frac{1}{2},3\right)$

21. (b) It is obvious.

22. (d) In $\Delta\, ABC$, $A\equiv(-3, 0)$; $B\equiv(4, -1)$ and $C\equiv(5, 2)$

We know that $BC=\sqrt{(5-4)^2+(2+1)^2}$

$=\sqrt{1+9}=\sqrt{10}$



and area of $\Delta\, ABC$

$=\frac{1}{2}\,[-3\,(-1-2)+4\,(2-0)+5(0+1)]=11$

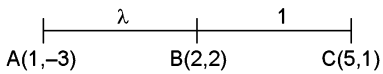
Therefore, altitude $AL = \dfrac{2\Delta ABC}{BC}$

$$= \frac{2\times 11}{\sqrt{10}} = \frac{22}{\sqrt{10}}$$

23. (b) Step 1 : If $B\ (x_2, y_2)$ is a point on the line joing $A\ (x_1, y_1)$ and $C\ (x_3, y_3)$ then C divides AB in the ratio

$$\lambda = \frac{x_1 - x_2}{x_2 - x_3} \text{ or } \frac{y_1 - y_2}{y_2 - y_3}$$

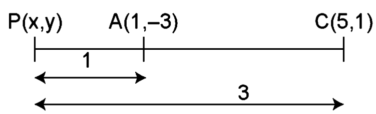
Step 2:



$$\frac{AB}{BC} = \frac{\lambda}{1} = \frac{1-2}{2-5} \text{ or } \frac{-3-2}{2-1} \text{ i.e., } \lambda = \frac{1}{3}$$

$\therefore$ $AB : BC$ externally = –1 : 3

Step 3:



Now by section formula we have

$$x = \frac{1\times 5 - 3\times 1}{1-3} = -1$$

$$y = \frac{1\times 1 - 3\times(-3)}{1-3} = -5$$

$P\ (x, y) = P\ (-1, -5)$

24. (a) Mid-points of the diagonals must be same.

$\Rightarrow$ $\dfrac{x-2}{2} = \dfrac{-3+3}{2} \Rightarrow x = 2$

and $\dfrac{-1+3}{2} = \dfrac{-2+y}{2} \Rightarrow y = 4$

25. (d) R divides PQ externally in the ratio 2 : 1

$\therefore$ R is $\left(\dfrac{4+3}{2-1}, \dfrac{2-4}{2-1}\right)$ i.e., (7, –2)

26. (a) Given points are
(1, 2), (–5, 6), (7, –4) and $(a, -2)$ are collinear
Area of quadrilateral = 0

$\therefore$ $(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4) = 0$

$\Rightarrow$ $(1 \times 6 + 5 \times 2) + (-5 \times (-4) - 7 \times 6) + (-14 + 4a) + (2a + 2) = 0$

$16 - 22 - 14 + 4a + 2a + 2 = 0$
$6a = 18 \Rightarrow a = 3$

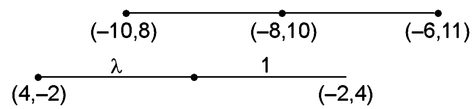
27. (b) Let $P \equiv (x, y)$, then

$$x = \frac{20\cos\theta + 15}{5},\ y = \frac{20\sin\theta + 0}{5}$$

$\Rightarrow$ $4\cos\theta = x - 3,\ 4\sin\theta = y$

$\Rightarrow$ $(x-3)^2 + y^2 = 16$, which is a circle.

28. (d)



$$\left(\frac{-2\lambda + 4}{\lambda + 1}, \frac{4\lambda - 2}{\lambda + 1}\right) \equiv (-8, 10)$$

$\Rightarrow$ $\dfrac{-2\lambda + 4}{\lambda + 1} = -8$

$\Rightarrow$ $-2\lambda + 4 = -8\lambda - 8$

$\Rightarrow$ $6\lambda = -12$

$\lambda = \dfrac{-2}{1}$ externally $\Rightarrow$ 2 : 1 externally.

29. (a) Point which divides the line joining the points (–1, –2) and (3, 4) in the ratio 3 : 5 is

$$x = \frac{3\times 3 + 5(-1)}{3+5} = \frac{4}{8} = \frac{1}{2}$$

$$y = \frac{3\times 4 + 5(-2)}{3+5} = \frac{2}{8} = \frac{1}{4}$$

Required point $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$

Slope of the line joining the points (–1, –2) and (3, 4) is $m = \dfrac{4+2}{3+1} = \dfrac{6}{4} = \dfrac{3}{2}$

Slope of the perpendicular line $m = -\dfrac{2}{3}$ perpendicular line passing through $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$

$$y - \frac{1}{4} = \frac{-2}{3}\left(x - \frac{1}{2}\right)$$

$\Rightarrow$ $\dfrac{4y-1}{4} = \dfrac{-1}{3}(2x-1)$ $12y - 3 = -8x + 4$

$\Rightarrow$ $8x + 12y = 7.$

30. (c) If the given diagonal is AC and the other diagonal is BD, then mid-points of AC and BD are same.

Hence, $(1, h + k) = (1, 0) \Rightarrow h + k = 0$ ... (1)

Also $AB \perp BC \Rightarrow \frac{h+2}{3-1}.\frac{2-h}{1-3} = -1$

$\Rightarrow \quad h^2 = 0$

$\Rightarrow \quad h = 0$ .... (2)

(1), (2) $\Rightarrow \quad h = k = 0$

31. (c) Middle point M of diagonal AC is

$$M\left(\frac{3+1}{2}, \frac{4-1}{2}\right) = M\left(2, \frac{3}{2}\right).$$

If D is $D(h, k)$ .... (1)

and $B(x_1, y_1)$, then $2 = \frac{x_1 + h}{2}, \frac{3}{2} = \frac{y_1 + k}{2}$

$\Rightarrow \quad x_1 = 4 - h, y_1 = 3 - k$

Now $B(x_1, y_1)$ is $(4 - h, 3 - k)$ .... (2)

Suppose slope of AB is m and slope of AC is

$\frac{4+1}{3-1} = \frac{5}{2}$

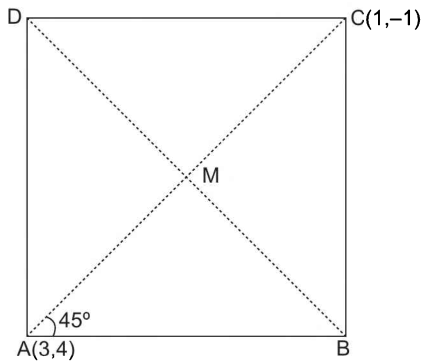
Then, $\tan(45°) = \left|\frac{m - \frac{5}{2}}{1 + \frac{5m}{2}}\right|$

$\Rightarrow \quad (2m - 5) = \pm (2 + 5m)$

$\Rightarrow \quad m = -\frac{7}{3}, \frac{3}{7}$

$\Rightarrow$ Equation of AB is

$$y - 4 = -\frac{7}{3}(x - 3) \text{ or, } 7x + 3y - 33 = 0$$



and equation of BC is $y + 1 = \frac{3}{7}(x - 1)$

or, $3x - 7y - 10 = 0$

Solving these two equations, we get

$$B\left(\frac{9}{2}, \frac{1}{2}\right)$$

This $\Rightarrow \frac{9}{2} = 4 - h, \frac{1}{2} = 3 - k,$ by (2)

$\Rightarrow \quad h = -\frac{1}{2}, k = \frac{5}{2}$

$\Rightarrow \quad D(h, k) = \left(-\frac{1}{2}, \frac{5}{2}\right)$

32. (c) Given vertices of a triangle are (1, 0), (7, 0) and (4, 4).

$\therefore$ Area of triangle $= \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 7 & 0 & 1 \\ 4 & 4 & 1 \end{vmatrix}$

$= \frac{1}{2}[1(0-4) + 1(28-0)]$ = 12 sq. units

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If (–6, –4); (1, –3/2); (–2, –1/2) are the mid-points of the sides of a triangle, then the coordinates of its vertices are
(a) (5, 2); (–9, –3); (–3, 5)
(b) (5, 2); (–9, 3); (–3, –5)
(c) (5, 2); (–9, –3); (–3, –5)
(d) (5, –2); (–9, –3); (–3, –5)

2. Three vertices of a rhombus taken in order are (2, –1); (3, 4) and (–2, 3), then its fourth vertex is
(a) (–3, 2)
(b) (3, –2)
(c) (3, 2)
(d) (–3, –2)

3. The coordinates of the points dividing internally the line joining points (4, –2) and (8, 6) in the ratio 7 : 5 will be

(a) (16, 18) (b) (18, 16)
(c) (19/3, 8/3) (d) (8/3, 19/3)

4. If (4, 3) and (–2, –1) are two opposite vertices of a parallelogram and its third vertex is (1, 0), then the product of the coordinates of its fourth vertex is

(a) 1 (b) 2
(c) 3 (d) 4

5. The points (a, b); (c, d) and $(a - c, b - d)$ are collinear if

(a) $ac = bd$ (b) $ad = bc$
(c) $ab = cd$ (d) None of these

6. The line joining points (2, –3) and (–5, 6) is divided by y-axis in the ratio

(a) 2 : 5 (b) 2 : 3
(c) 3 : 5 (d) 1 : 2

7. If (8, 5); (–7, –5) and (–5, 5) are three vertices of a parallelogram, then its fourth vertex is

(a) (10, 15) (b) (15, 10)
(c) (10, 5) (d) (5, 10)

8. The condition that three points $(a, 0)$, $(0, b)$ and (1, 1) are collinear is

(a) $\frac{1}{a}+\frac{1}{b}=2$ (b) $\frac{1}{a}+\frac{1}{b}=1$
(c) $\frac{1}{a}=\frac{1}{b}$ (d) $\frac{1}{a}+\frac{1}{b}=0$

9. If points (0, 4); $(2, k)$ and (4, 0) are collinear, then the value of k is

(a) 0 (b) 2
(c) 4 (d) 8

10. The points $(at_1^2, 2at_1), (at_2^2, 2at_2)$ and $(a, 0)$ will be collinear if

(a) $t_1 t_2 = 1$ (b) $t_1 t_2 = -1$
(c) $t_1 + t_2 = 1$ (d) $t_1 + t_2 = -1$

11. The vertices of the triangle ABC are (2, 1), (4, 3) and (2, 5). D, E, F are the mid-points of the sides. The area of the triangle DEF is

(a) 1 (b) 1 : 5
(c) 3 (d) 4

12. In what ratio does the x-axis divide the join of (–3, –4) and (1, –2)? ***[RPET-95]***

(a) 1 : 3 (b) 2 : 3
(c) – 2 : 1 (d) None of these

13. The points of trisection of the line segment joining the points (3, –2) and (–3, –4) are

(a) $\left(\frac{3}{2},-\frac{5}{2}\right),\left(-\frac{3}{2},-\frac{13}{4}\right)$

(b) $\left(-\frac{3}{2},\frac{5}{2}\right),\left(\frac{3}{2},\frac{13}{4}\right)$

(c) $\left(1,-\frac{8}{3}\right),\left(-1,-\frac{10}{3}\right)$

(d) None of these

14. If coordinates of the points A and B are (2, 4) and (4, 2) respectively and point M is such that $A - M - B$ also $AB = 3AM$, then the coordinates of M are

(a) $\left(\frac{8}{3},\frac{10}{3}\right)$ (b) $\left(\frac{10}{3},\frac{14}{4}\right)$
(c) $\left(\frac{10}{3},\frac{6}{3}\right)$ (d) $\left(\frac{13}{4},\frac{10}{4}\right)$

15. If $A(6, 3)$, $B(-3, 5)$, $C(4, -2)$ and $D(x, 3x)$ are four points. If the ratio of area of $\Delta\, DBC$ and $\Delta\, ABC$ is 1 : 2, then the value of x, will be

(a) 11/8 (b) 8/11
(c) 3 (d) None of these

16. The points which divides externally the line joining the points $(a + b, a - b)$ and $(a - b, a + b)$ in the ratio $a : b$, is

(a) $\left(\frac{a^2-2ab-b^2}{a-b},\frac{a^2+b^2}{a-b}\right)$

(b) $\left(\frac{a^2-2ab-b^2}{a-b},\frac{a^2-b^2}{a-b}\right)$

(c) $\left(\frac{a^2-2ab+b^2}{a-b},\frac{a^2+b^2}{a-b}\right)$

(d) None of these

17. If the points dividing internally the line segment joining the points (a, b) and (5, 7) in the rtaio 2 : 1 be (4, 6), then

(a) $a = 1, b = 2$
(b) $a = 2, b = -4$
(c) $a = 2, b = 4$
(d) $a = -2, b = 4$

18. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ is the conditions that the points (x_i, y_i), $i = 1, 2, 3$

(a) From an equilateral triangle
(b) are collinear
(c) From a right angled triangle
(d) (x_2, y_2) is the mid-point of the line joining (x_1, y_1), (x_3, y_3)

19. If the point $C(-1, 2)$ divide externally the line segment joining $A(2, 5)$ and B in the ratio 3 : 4 then coordinates of B are

(a) (3, –6) (b) (3, 6)
(c) (–3, –6) (d) (–3, 6)

20. Area of quadrilateral with consecutive vertices $(0, 0)$, (a, b), $(-a, -b)$, (a^2, b^2) is

(a) 0 (b) $\frac{1}{2}ab(a-b)$
(c) $\frac{1}{2}(a^2+b^2)$ (d) $a^2 b^2/2$

21. The area of the Δ with vertices $(x, 2x)$, $(-2, 6)$, $(3, 1)$ is 5 sq. units then the value of x is

(a) 2 or 2/3 (b) 1 or 2/3
(c) 2 or 1/2 (d) None of these

22. The locus of a point P which divides the line joining $(1, 0)$ and $(2 \cos \theta, 2 \sin \theta)$ internally in the ratio 2 : 3 for all θ is a ***[IIT-1986]***

(a) Straight line
(b) Circle
(c) Pair of straight lines
(d) Parabola

23. If $A(0, 0)$, $B(12, 0)$, $C(12, 2)$, $D(6, 7)$ and $E(0, 5)$ are vertices of the pentagon $ABCDE$, then its area in square units, is ***[Kerala PET-2008]***

(a) 58 (b) 60
(c) 61 (d) 63

24. If $A(2, 3)$, $B(1, 4)$, $C(0, -2)$ and $D(x, y)$ are the vertices of a parallelogram, then what is the value of (x, y)? ***[NDA-2008]***

(a) (1, –3) (b) (2, 4)
(c) (1, 1) (d) (0, 0)

25. If points (5, 5); (10, k) and (–5, 1) are collinear, then k equals ***[MPPET-94, 99; RPET-03]***

(a) 9 (b) 7
(c) 5 (d) 3

26. Point (1/2, –13/4) divides the line joining the points (3, –5) and (–7, 2) in the ratio of

(a) 1 : 3 internally (b) 3 : 1 internally
(c) 1 : 3 externally (d) 3 : 1 externally

27. The area of the triangle whose vertices are (1, 0), (7, 0) and (4, 4) is ***[MPPET-2009]***

(a) 8 (b) 10
(c) 12 (d) 14

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 21 minutes.
3. The worksheet consists of 21 questions. The maximum marks are 63.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If (1,2); (–2,3) and (–3, –4) are vertices of a triangle then its area is

(a) 11 (b) 12
(c) 22 (d) 24

2. The points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if

(a) $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_2 - y_1) = 0$
(b) $x_1(y_2 - y_3) + x_2(y_1 - y_3) + x_3(y_1 - y_2) = 0$
(b) $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$
(d) None of these

3. Point (11, 18) divides the line segment joining points (3, 4) and (7, 11) in the ratio

(a) 1 : 2 (b) 2 : 1
(c) – 2 : 1 (d) 1 : – 2

4. *D*, *E*, *F* are mid-points of the sides *AB*, *BC* and *CA* of the triangle formed by the points *A*(5, –1); *B*(–7, 6); *C*(1, 3); then the area of Δ *DEF* is

(a) 5/2 (b) 5
(c) 10 (d) 20

5. If the points (1, 2); (–1, *x*) and (3, 4) are collinear, then *x* is equal to

(a) 1 (b) –1
(c) 0 (d) 2

6. Point (–1/3, 0) divides the line segment joining points (1, –2) and (–3, 4) in the ratio

(a) 1 : 1
(b) 1 : 2
(c) 2 : 1
(d) 2 : 3

7. If the area of the triangle with vertices (*a*, 0); (1, 1) and (0, 2) is 4, then *a* is equal to

(a) 8 (b) –6
(c) –2 (d) –4

8. If the three vertices of a rectangle taken in order are the points (2, –2), (8, 4) and (5, 7). The coordinates of fourth vertex are **[CEE-93]**

(a) (1, 1) (b) (1, –1)
(c) (–1, 1) (d) None of these

9. If the middle point of the line segment joining the points (5, *a*) and (*b*, 7) be (3, 5), then (*a*, *b*) is equal to

(a) (3, 1) (b) (1, 3)
(c) (–2, –2) (d) (–3, –1)

10. If (3, –1), (2, 6), (–5, 7) are the mid-points of the sides of Δ *ABC*, then the area of Δ *ABC* is

(a) 24 (b) 48
(c) 72 (d) 96

11. Coordinates of the points which divide the line segment joining the points *P*(1, 7) and *Q* (6, –3) internally and externally in the ratio 2 : 3 in order

(a) (3, 3), (–9, 27) (b) (–3, 3), (9, 27)
(c) (3, –3), (–9, 27) (d) (3, 3), (9, 27)

12. If *P*(1, 2); *Q*(4, 6); *R*(5, 7) and *S*(*a*, *b*) are vertices of a parallelogram *PQRS*, then **[IIT-1998]**

(a) *a* = 2, *b* = 4 (b) *a* = 3, *b* = 4
(c) *a* = 2, *b* = 3 (d) *a* = 3, *b* = 5

13. If the points *A*(3, 4), *B*(7, 7), *C*(*a*, *b*) be collinear and *AC* = 10, then (*a*, *b*) is equal to

(a) (11, 10) (b) (10, 11)
(c) (11/2, 5) (d) (5, 11/2)

14. Three vertices of a parallelogram taken in order are (–1, –6), (2, –5) and (7, 2). The fourth vertex is **[Kerala Engg.-02]**

(a) (1, 4) (b) (4,1)
(c) (1, 1) (d) (4, 4)

15. If the points (–2, –5), (2, –2), (8, *a*) are collinear, then the value of *a* is

(a) –5/2 (b) 5/2
(c) 3/2 (d) 1/2

16. If the vertices of a triangle are (5,2) (2/3, 2) and (–4, 3), then the area of the triangle is **[Kurukshetra CEE-02]**

(a) 28/6 (b) 5/2
(c) 43 (d) 13/6

17. The line joining points (2, –3) and (–5, 6) is divided by y-axis in the ratio *[MPPET-99]*

(a) 2 : 5 (b) 2 : 3
(c) 3 : 5 (d) 1 : 2

18. The points of trisection of the line joining the points (0, 3) and (6, –3) are

(a) (2, 0) and (4, –1)
(b) (2, –1) and (4, 1)
(c) (3, 1) and (4, –1)
(d) (2, 1) and (4, –1)

19. The extremities of a diagonal of a parallelogram are the points (3, –4) and (–6, 5). If third vertex is (–2, 1), then fourth vertex is

(a) (1, 0) (b) (–1, 0)
(c) (1, 1) (d) None of these

20. The mid-points of sides of a triangle are (2, 1), (–1, –3) and (4, 5). Then the coordinates of its vertices are

(a) (7, 9), (–3, –7), (1, 1)
(b) (–3, –7), (1, 1), (2, 3)
(c) (1, 1), (2, 3), (–5, 8)
(d) None of these

21. The point (5, –1) divides the line segment joining points A and B in the ratio 2 : 3. If A is (11, –3), then B will be

(a) (4, 2) (b) (4, –2)
(c) (–4, 2) (d) (–4, –2)

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
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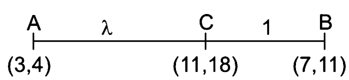
21. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

3. (c)



Let the ratio be λ : 1

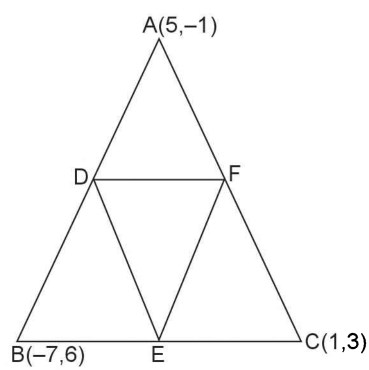
$$\therefore \quad \left(\frac{7\lambda+3}{\lambda+1}, \frac{11\lambda+4}{\lambda+1}\right) \equiv (11, 18)$$

$$\therefore \quad \frac{7\lambda+3}{\lambda+1} = 11$$

$\Rightarrow \quad 7\lambda + 3 = 11\lambda + 11$

$\Rightarrow \quad \lambda = -2$

4. (b)

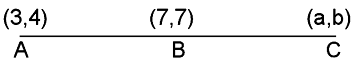


Area of $\Delta\, DEF = \frac{1}{4}$ (area of $\Delta\, ABC$)

$$= \frac{1}{4}\begin{vmatrix} 5 & -1 & 1 \\ -7 & 6 & 1 \\ 1 & 3 & 1 \end{vmatrix}$$

$$= \left|\frac{1}{4}(15 - 8 - 27)\right| = 5$$

13. (a)



$AC = 10,$

$AB = \sqrt{(7-3)^2 + (7-4)^2} = 5$

$\therefore \quad BC = AC - AB = 10 - 5 = 5$

$\therefore \quad \frac{AB}{AC} = \frac{5}{5} = 1$

$\therefore \quad B$ is mid-point of AC

$$B\left(\frac{a+3}{2}, \frac{b+4}{2}\right) \equiv (7, 7)$$

$\Rightarrow \quad a = 11,\ b = 10$

LECTURE

3

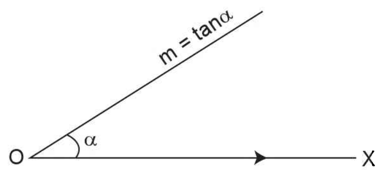
Cartesian Coordinates 3

(Slope of a line, special points in triangle (centroid, circumcentre centroid), orthocenter, incentre and excentre)

BASIC CONCEPTS

1. Slope of a Line

If a line is making an angle α from the positive direction of x-axis In anti-clock wise direction, then slope of the line is defined as tan α which is denoted by 'm'. Thus $m = \tan\alpha$.

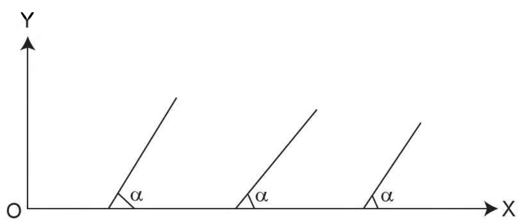


1.1. The slope of a line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is denoted by m and is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \tan\alpha$

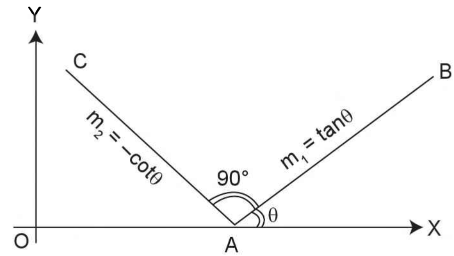
1.2 (a) The slope of x-axis is zero.

(b) The slope of y-axis is not defined (∞)

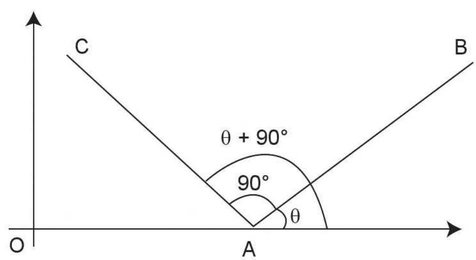
(c) Two lines (not parallel to y-axis) are parallel if their slopes are equal i.e., $m_1 = m_2$



(d) Two lines (not parallel to y-axis) with slopes m_1 and m_2 are perpendicular if $m_1 \times m_2 = -1$



(e) If inclination of a line is θ then inclination of a straight line perpendicular to it is $(\theta \pm 90°)$



(f) If angle between two lines having slope m_1 and m_2 is α then $\tan\alpha = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

2. Coordinates of Standard Point Connected with Triangle

2.1 Centroid of a triangle

(i) The centroid is the point of intersection of the medians (i.e., a line joining a vertex to the mid-point of the opposite side)

(ii) Centroid divides each median in the ratio 2 : 1 internally

(iii) The centroid G of the ΔABC formed by $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by

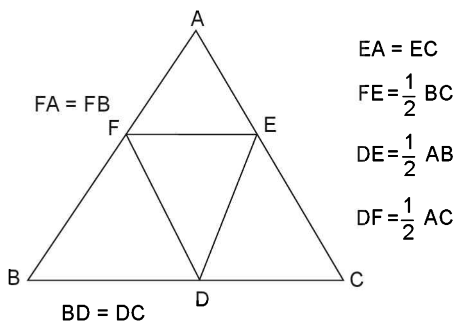
$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right).$$

2.2. If D, E, F are the mid-points of the sides BC, CA, AB of $\Delta\ ABC$, then coordinates of the vertices in terms of mid-points of the sides of Δ are obtained as follows

$$A = E + F - D$$

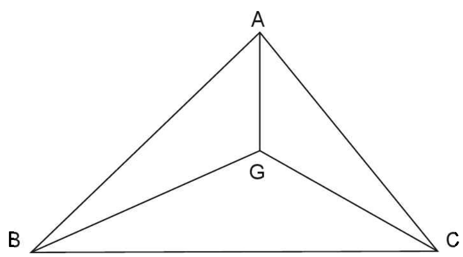
$$B = F + D - E$$

$C = D + E - F$ and also Area of $\Delta\ DEF = \frac{1}{4}$ area of ΔABC.



2.3 If D, E, F are the mid-points of the sides BC, CA, AB of ΔABC, then the centroid of ΔABC = centroid of ΔDEF.

2.4. Straight line joining the centroid of a Δ to its vertices divide it into three triangles of equal area.



i.e., Area of $\Delta\ ABG = \frac{1}{3}$ area of ΔABC.

2.5 If $A(x_1, y_1)$, $B(x_2, y_2)$ are any two vertices of $\Delta\ ABC$ and $G(h, k)$ is the centroid then coordiantes of the third vertex C be: $C\ (3h - x_1 - x_2, 3k - y_1 - y_2)$

2.6 If G be the centroid of a triangle ABC and O be any other point, then

(i) $AB^2 + BC^2 + CA^2 = 3\ (GA^2 + GB^2 + GC^2)$

(ii) $OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2$

2.7 In any Δ, four times the sum of square of the medians is equal to three times the sum of squares of the sides

i.e., $(AD)^2 + (BE)^2 + (CF)^2 = 3/4\ (AB^2 + BC^2 + CA^2)$

(i) In a right angled triangle the mid-point of the hypotenuse is equidistant from the three angular points.

(ii) The line segment joining the mid-points of two sides of a triangle is half of the third side.

2.8 Applonius Theorem

In any triangle prove that

$$AB^2 + AC^2 = 2(AD^2 + DC^2) = 2(AD^2 + BD^2)$$

where D is mid-point of BC.

2.9 Circumcentre of triangle:

This is a point which is equidistant from the three vertices of the triangle. It is also the point of intersection of right bisectors of the sides of the triangle (i.e., the lines through the mid-point of a side and perpendicular to it) It is the centre of O of the circle that passes through the vertices of the triangle.

(a) If the triangle be a right angled one, the circumcentre is mid-point of the hypotenuse.

2.10 Incentre of a triangle

(i) Is is equidistant from the sides of Δ. This is the centre of the circle which touches the sides of a given triangle.

(ii) It is the point of intersection of the internal bisectors of the angles of the triangle.

(iii) Internal bisectors divide the opposite sides in the ratio of the sides containing the angle.

(iv) Coordinates of the point D in which internal bisectors of the angle A meets the opposite side BC are

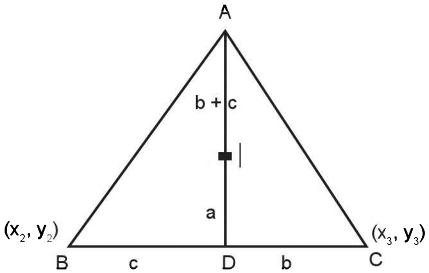
$$\left(\frac{bx_2+cx_3}{b+c}, \frac{by_2+cy_3}{b+c}\right)$$

(v) The incentre of $\Delta\ ABC$ with vertices $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ and side lengths a, b, c is

$$I\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

(vi) In centre of a Δ having vertices at $O(0, 0)$, $A(0, b)$ and $B(0, b)$ is

$$\left(\frac{ab}{a+b+\sqrt{a^2+b^2}}, \frac{ab}{a+b+\sqrt{a^2+b^2}}\right)$$



2.11 Orthocentre of a triangle

(i) Orthocentre H is the point of intersection of the altitudes

(ii) Altitudes are the straight lines through the vertices and perpendiculars to opposite sides.

(iii) In a right angled ΔABC, $\angle C = 90°$, the orthocentre H is the vertex C of the triangle where Δ is right angled.

2.12 Excentre of a triangle The point of the internal bisector of an angle A and external bisectors of the other two angles B and C is called excentre of Ex-circle opposite to vertex A. The excentre opposite to the vertex A is $I_1\left(\frac{-ax_1+bx_2+cx_3}{a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$

Similarly, taking $-b$ and $-c$, we will get centres of escribed circles drawn opposite to vertex B and vertex. C respectively Ex-centre opposite to the vertex B is

$$I_2\left(\frac{-ax_1-bx_2+cx_3}{a-b+c}, \frac{ay_1-by_2+cy_3}{a-b+c}\right).$$

The excentre opposite to the vertex C is

$$I_3\left(\frac{ax_1+bx_2-cx_3}{a+b-c}, \frac{ay_1+by_2-cy_3}{a+b-c}\right)$$

NOTE

I is the orthocentre of $\Delta I_1I_2I_3$

2.13 If (α, β), $(\bar{x}, \bar{y})$ and (p, q) are the coordinates of the circumcentre, centroid and orthocentre of a triangle, then $3\bar{x} = 2\alpha + p$ and $3\bar{y} = 2\beta + q$.

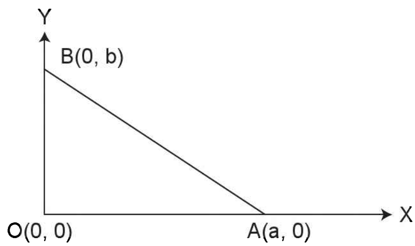
2.14 P, Q, R divide the sides of the a triangle ABC in the same ratio, then centroides of the triangles ABC and PQR coincide.

3. **Regular Polygon** A polygon is called a regular polygon if all its sides are equal and its angle are equal.

4. **Choice of Axes** Some times geometrical problems can be made simple with proper choice of axes. For the proper choice of axes, we follow the rules as given below

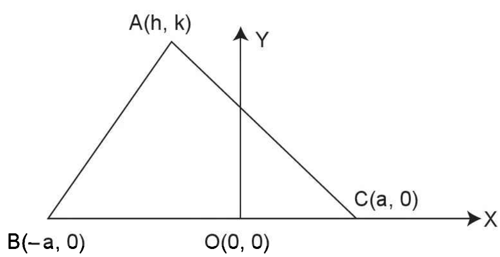
Rule I: Whenever the perpendicular lines are given in a problem, take these lines as coordinate axes.

For example Whenever a right-angled triangle is given in a problem, take two lines containing the right angle as coordinate axes.

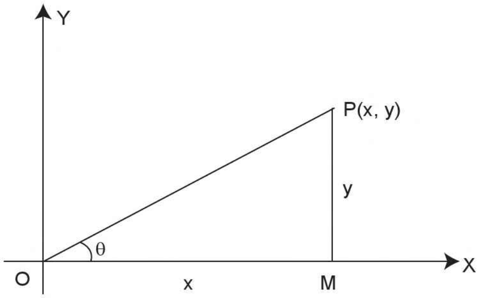


Rule II: Whenever the fixed points A and B are given in a problem, we take mid-point O of $[AB]$ as origin and $\overline{AB}$ as x-axis and a line through O and perpendicular to $\overline{AB}$ as y-axis. Thus if $|\overline{AB}| = 2a$, then B is $(a, 0)$ and A is $(-a, 0)$. [Points may also be taken along y-axis as $(0, b)$ and $(0, -b)$]

For example Whenever a ΔABC is given, we take mid-point O of the base $[BC]$ as origin and base $\overline{BC}$ as x-axis and a line through O and perpendicular to $\overline{BC}$ as y-axis. Thus, if $|BC| = 2a$, then C is $(a, 0)$ and B is $(-a, 0)$.



5. Polar coordinates System



If $\angle XOP = \theta$ and $OP = r$, then polar coordinate of P are given as (r, θ), where r is radius vector and its magnitude is $\sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$

5.1 Conversion formula

(i) From cartesian to polar coordinates

$$(r, \theta) = \left(\sqrt{x^2 + y^2}, \tan^{-1}\frac{y}{x}\right)$$

(ii) From polar to cartesian coordinates (x, y)
$(r\cos\theta, r\sin\theta)$

5.2 Distance between two polar coordinates viz $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ is given by:

$$AB = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

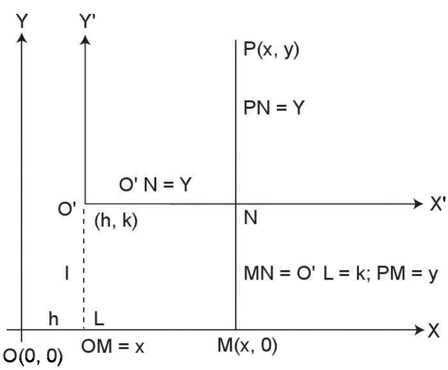
$$R = \frac{a}{2\sin\frac{\pi}{n}} = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n},$$

where a is the length of each

6. Transformation Axes

Is used to reduce the equation of any curve from its general form to its standard forms

Translation Shifting the origin to another point without changing the directions of the axes (i.e., parallel to the original axes) is called translation of axes



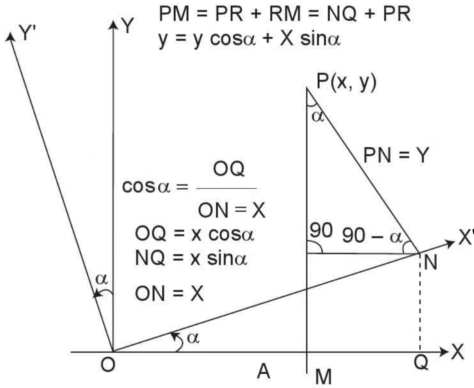
$x = OM = OM' + M'M = h + X$

$O'(h, k)$ – new origin, $(x, y) \to$ old

$x = X + h, y = Y + k$ $(X, Y) \to$ New axes.

Translation is used to remove the first degree terms and slope of the straight line is independent of the Translation.

Rotation without shifting the origin, if the axes are rotated through an angle α in anti-clockwise direction, then the transformations is called rotation of axes.



$OM = x, PM = y$

$$\cos(90° - \alpha) = \frac{RN}{Y} \Rightarrow RN = y\sin\alpha = MQ$$

$$\sin(90° - \alpha) = \frac{PR}{PN = y} \Rightarrow PR = y\cos\alpha$$

Thus
$$x = X\cos\alpha - Y\sin\alpha$$
$$y = X\sin\alpha + Y\cos\alpha$$

and
$$X = x\cos\alpha + y\sin\alpha$$
$$Y = -x\sin\alpha + y\cos\alpha$$

Let P is a point in the plane whose coordinates are (x, y) initially and (X, Y) in the new system. Rotation is used to remove the term containing xy (mixed term) from the equation of curve. For clockwise rotation, replace α by – α in above equation.

> **NOTE**
>
> **General Transformation** If the origin is shifted to (x_1, y_1) and axis are rotated through 'θ', the coordinates are related as follows:
>
> $$x = x_1 + X\cos\theta - Y\sin\alpha$$
> $$y = y_1 + X\sin\theta + Y\cos\theta$$

Examples

1. The origin is shifted to (–2, 3) then what are the coordinates of the point (3, –5) in the new position?

   Ans: $(x, y) \to (X + h, Y + k) = (X - 2, Y + 3)$
   $(3, -5) \to (X - 2, Y + 3)$
   $\therefore \quad (X, Y) = (5, -8)$.

2. If the origin is shifted to (1, –2), the coordinates of A become (2, 3). What are the original coordinates of A?

   Ans: Old New Pt. $x = X + h = 2 + 1 = 3$
   $y = Y + k = 3 - 2 = 1$
   $\therefore$ (3, 1)

3. What will be the coordinates of the point $(4, 2\sqrt{3})$ when the axes are rotated through an angle of 30° in clockwise sense?

   Ans: Old New Pt. $\theta = -30°$ (anti-cloclwise)
   $x = X\cos\theta - Y\sin\theta$ $y = X\sin\theta + Y\cos\theta$

   $$4 = X\frac{\sqrt{3}}{2} + \frac{Y}{2} \quad \text{or} \quad X\sqrt{3} + Y = 8$$

   $$2\sqrt{3} = -\frac{X}{2} + Y\frac{\sqrt{3}}{2} \quad \text{or} \quad -X + Y\sqrt{3} = 4\sqrt{3}$$

   Solving we get $X = \sqrt{3}, Y = 5$

   Hence, coordinates in new position are $(\sqrt{3}, 5)$.

7. **Some Important Problems**

7.1 If origin be shifted at (–2, 3) and new axes are parallel to old them: (i) new coordinates of point: (9, –11)

Ans. (11, –14)

(ii) new form of the equation $x^2 + y^2 + 4x - 6y - 15 = 0$

Ans: $x^2 + y^2 - 28 = 0$

7.2 Transform the equation $2x^2 + 4xy + 5y^2 - 4x - 6y - 15 = 0$ when the axes are translated to the point (–2, 3).

Ans: $2x^2 + 4xy + 5y^2 + 4 = 0$

7.3 If the axes be turned through an angle $\tan^{-1} 2$ what does the equation $4xy - 3x^2 = a^2$ becomes.

Ans: $x^2 - 4y^2 = a^2$

7.4 The new coordinates of a point (4, 5) when the origin is shifted to the point (1, –2) are

Ans: (3, 7)

7.5 If the origin is shifted to (1, –2), the coordinates of A become (2, 3). What are the original coordinates of A?

Ans: (3, 1)

7.6 If one vertex of a square is (–4, 5) and equation of its one diagonal is $7x - y + 8 = 0$, then the equation of the other diagonal is

Ans: $x + 7y = 31$

7.7 Let $0 < \alpha < \frac{\pi}{2}$ be a fixed angle. If $P = (\cos\theta, \sin\theta)$ and $Q(\cos(\alpha - \theta), \sin(\alpha - \theta))$ then $(\alpha - \theta)$ is obtained from P by

(a) clockwise rotation around origin through an angle α.
(b) anti-clockwise rotation around origin through an angle α.
(c) reflection in the line through origin with slope $\tan\alpha$.
(d) reflection in the line through origin with slope $\tan\frac{\alpha}{2}$

Hint $\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos\alpha\cos\theta + \sin\alpha\sin\theta \\ \sin\alpha\cos\theta - \cos\alpha\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha - \theta) \\ \sin(\alpha - \theta) \end{bmatrix}$$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. If G be the centroid of a triangle ABC and O be any other point, then prove that
   (i) $AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$
   (ii) $OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2$.

Solution

For the sake of convenience let us choose G as origin and the points A, B, C as (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively and the

other point O be taken as (h, k). Coordinates of G are $(\Sigma x_1/3, (\Sigma y_1/3)$
But we have chosen G as origin

$\therefore$ $(\Sigma x_1)/3 = 0, \Sigma y_1)/3 = 0$

or $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$ ...(1)

(i) $AB^2 + BC^2 + CA^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + ...+ ...... = 2(\Sigma x_1^2 + \Sigma y_1^2) - 2(\Sigma x_1 x_2 + \Sigma y_1 y_2) = 3(\Sigma x_1^2 + \Sigma y_1^2) - \Sigma x_1^2 - \Sigma y_1^2 - 2\Sigma x_1 x_2 - 2\Sigma y_1 y_2$

[By adding and subtracting $\Sigma x_1^2 + \Sigma y_1^2$]

$= 3(GA^2 + GB^2 + GC^2) - (x_1 + x_2 + x_3)^2 - (y_1 + y_2 + y_3)^2$

$= 3(GA^2 + GB^2 + GC^2)$

[$\because \Sigma x_1 = 0, \Sigma y_1 = 0$, by (1)]

and $GA^2 = (x_1 - 0)^2 + (y_1 - 0)^2 = x_1^2 + y_1^2$ etc.

(ii) $OA^2 + OB^2 + OC^2 = (h - x_1)^2 + (k - y_1)^2 + ...+.... (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + (x_3^2 + y_3^2) + (h^2 + k^2) + (h^2 + k^2) + (h^2 + k^2) - 2h(x_1 + x_2 + x_3) - 2k(y_1 + y_2 + y_3) = GA^2 + GB^2 + GC^2 + GO^2 + GO^2 + GO^2 - 0 - 0,$

By (1) $= GA^2 + GB^2 + GC^2 + 3GO^2$

[$\because GO^2 = (h - 0)^2 + (k - 0) = h^2 + k^2$].

2. If $\left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 0\right)$ be the middle points of the sides of a triangle, find the coordinates of its incentre.

Solution

Let the vertices of the triangle be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ and let $\left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 0\right)$ be the mid-points of sides $[BC]$, $[CA]$ and $[AB]$ respectively, then

$$\left(0, \frac{1}{2}\right) = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$= \left(\frac{x_3 + x_4}{2}, \frac{y_3 + y_1}{2}\right)$$

and $\left(\frac{1}{2}, 0\right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$\Rightarrow$ $x_2 + x_3 = 0$ .... (1)

$y_2 + y_3 = 1$ .... (2)

$x_3 + x_1 = 1$ .... (3)

$y_3 + y_1 = 1$ .... (4)

$x_1 + x_2 = 1$ .... (5)

and $y_1 + y_2 = 0$ .... (6)

Adding (1), (3) and (5), we get $2(x_1 + x_2 + x_3) = 2$

$\Rightarrow$ $x_1 + x_2 + x_3 = 1$ .... (7)

Subtracting (1), (3) and (5) from (7) in succession, we get $x_1 = 1, x_2 = 0, x_3 = 0$

Similarly, from (2), (4) and (6),
we shall find $y_1 = 0, y_2 = 0, y_3 = 1$

Hence, the vertices of the triangle are $A(1, 0)$, $B(0, 0)$ and $C(0, 1)$

Now $a = |BC| = 1$, $b = |CA| \sqrt{2}$

and $c = |AB| = 1$

If the incentre is (x, y), then

$$x = \frac{ax_1 + bx_2 + cx_3}{a + b + c}$$

$$= \frac{1 \times 1 + \sqrt{2} \times 0 + 1 \times 0}{1 + \sqrt{2} + 1} = \frac{1}{2 + \sqrt{2}}$$

$$= \frac{1(2 - \sqrt{2})}{(2 + \sqrt{2})(2 - \sqrt{2})}$$

$$= \frac{(2 - \sqrt{2})}{4 - 2} = \frac{2 - \sqrt{2}}{2}$$

and $$y = \frac{ay_1 + by_2 + cy_3}{a + b + c} = \frac{1 \times 0 + \sqrt{2} \times 0 + 1 \times 1}{1 + \sqrt{2} + 1}$$

$$= \frac{1}{2 + \sqrt{2}} = \frac{1(2 - \sqrt{2})}{(2 + \sqrt{2})(2 - \sqrt{2})}$$

$$= \frac{1(2 - \sqrt{2})}{4 - 2} = \frac{2 - \sqrt{2}}{2}$$

3. Prove analytically that in a right angled triangle, the mid-point of the hypotenuse is equidistant from the three angular points.

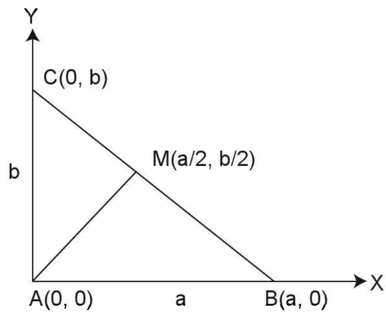
Solution

Let ABC be a right angled triangle with right angle at A.

Take AB and AC as the axes.

If $AB = a$ and $AC = b$, then coordinates of A, B, C will be $(0, 0)$, $(a, 0)$,$(0, b)$ respectively. Coordinates of M, the mid-point of $[BC]$ are $\left(\frac{a + 0}{2}, \frac{0 + b}{2}\right)$,

i.e., $\left(\frac{a}{2}, \frac{b}{2}\right)$



$$\therefore \quad |AM| = \sqrt{\left(\frac{a}{2}-0\right)^2 + \left(\frac{b}{2}-0\right)^2}$$

$$= \sqrt{\frac{a^2}{4}+\frac{b^2}{4}} = \frac{\sqrt{a^2+b^2}}{2}$$

$$|BM| = \sqrt{\left(\frac{a}{2}-a\right)^2 + \left(\frac{b}{2}-0\right)^2}$$

$$= \sqrt{\frac{a^2}{4}+\frac{b^2}{4}} = \frac{\sqrt{a^2+b^2}}{2}$$

$$\text{and} \quad |CM| = \sqrt{\left(\frac{a}{2}-0\right)^2 + \left(\frac{b}{2}-b\right)^2}$$

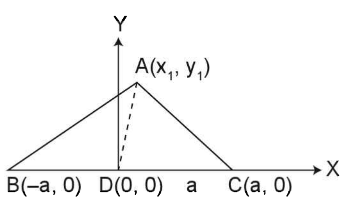
$$= \sqrt{\frac{a^2}{4}+\frac{b^2}{4}} = \frac{\sqrt{a^2+b^2}}{2}$$

$\therefore$ $|AM| = |BM| = |CM|$. Hence, M is equidistant from A, B, C.

4. In any triangle ABC, prove analytically that $AB^2 + AC^2 = 2(AD^2 + DC^2)$ where D is mid-point of $[BC]$.

Solution

Take D, the mid-point of $[BC]$ as origin; DC as the x-axis and $DY \perp DC$ as y-axis. Let $|BC| = 2a$, $(a > 0)$ then the coordinates of B and C are $(-a, 0)$ and $(a, 0)$ respectively.



Let the coordinates of A be (x_1, y_1), then

$AB^2 + AC^2 = (x_1 + a)^2 + y_1^2 + (x_1 - a)^2 + y_1^2$

$= 2\,(x_1^2 + y_1^2 + a^2)$ .... (i)

$2(AD^2 + DC^2) = 2(x_1^2 + y_1^2 + a^2)$ .... (ii)

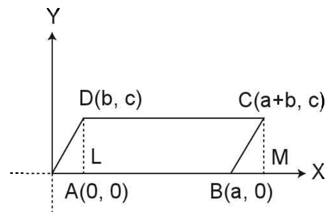
From (i) and (ii), we have $AB^2 + AC^2 = 2\,(AD^2 + DC^2)$

This is called median theorem.

5. Prove analytically that the diagonals of a parallelogram bisect each other.

Solution

Let $ABCD$ be the given parallelogram. Take AB as x-axis and A as origin, then the line through A and perpendicular to AB (in the plane of parallelogram) becomes y-axis and hence, A is (0,0). From the vertices C and D, draw CM and DL perpendiculars to y-axis.



Let $AB = a$, $AL = b$ and $LD = c$, then the coordinates of B and D are $(a, 0)$ and (b, c) respectively.

Since, $ABCD$ is a parallelogram, we have $MC = LD$

and $AL = BM$ ($\because$ Δs ALD and BMC are congruent)

$\Rightarrow$ $MC = c$ and $AM = AB + BM = AB + AL = a + b$

$\therefore$ coordinates of C are $(a + b, c)$.

Mid-point of diagonal $[AC]$ is

$$\left(\frac{0+a+b}{2}, \frac{0+c}{2}\right).$$

i.e, $$\left(\frac{a+b}{2}, \frac{c}{2}\right)$$

and mid-point of diagonal $[BD]$ is

$$\left(\frac{a+b}{2}, \frac{0+c}{2}\right)$$

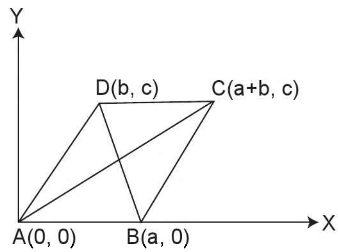
i.e., $$\left(\frac{a+c}{2}, \frac{c}{2}\right)$$

Therefore, the mid-points of the diagonals are the same, and hence, the diagonals of a parallelogram bisect each other.

6. Prove analytically that the diagonals of a rhombus bisector each other at right angles.

Solution

Let *ABCD* be a rhombus with each side of length *a*. Take *A* as origin and axes as shown in the figure, then $A \equiv (0, 0)$, $B \equiv (a, 0)$.



Let $D \equiv (b, c)$ then $C \equiv (a + b, c)$

As $|AB| = |AD|$, therefore

$$a = \sqrt{(b-0)^2 + (c-0)^2}$$

$\Rightarrow \quad a^2 = b^2 + c^2$ .... (1)

Now, the mid-point of [*AC*] is

$$\left(\frac{0+a+b}{2}, \frac{0+c}{2}\right) = \left(\frac{a+b}{2}, \frac{c}{2}\right)$$

and the mid-point of [*BD*] is

$$\left(\frac{a+b}{2}, \frac{0+c}{2}\right) = \left(\frac{a+b}{2}, \frac{c}{2}\right)$$

So, [*AC*] and [*BD*] have the same mid-point

$\Rightarrow$ Diagonals [*AC*] and [*BD*] bisect each other

Also, slope of $AC = \frac{c-0}{a+b-0} = \frac{c}{a+b}$

and slope of $BD = \frac{c-0}{b-a} = \frac{c}{b-a}$

$\Rightarrow$ (Slope of *AC*) × (Slope of *BD*)

$$= \frac{c}{a+b} \times \frac{c}{b-a}$$

$$= \frac{c^2}{b^2 - a^2} = \frac{c^2}{b^2 - (b^2 + c^2)} \text{ (using (1))}$$

$$= \frac{c^2}{c^2} = -1$$

$\Rightarrow$ *AC* and *BD* are at right angles.

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

EXERCISE 1

1. Find the area of triangles whose vertices are as (4, 4), (3, 2) and (–3, 16).

2. Find the area of the quadrilaterals (3, 4), (0, 5), (2, –1), (3, –2).

3. The points (1, 4), (*k*, –2) and (–3, 16) are collinear. Find the value of *k*.

4. Find the coordinates of the centroid of the triangle whose vertices are (–2, 2), (–1, –3) and (5, 7).

5. Two vertices of a triangle are (1, –6) and (–5, 2) and its centroid is (–2, 1), find the third vertex.

6. The vertices of ΔABC are $A(-36, 7)$, $B(20, 7)$ and $C(0, -8)$. Find the coordinates of its incentre.

7. Prove that points (*a*, *b*), (*a* + 3, *b* + 4), (*a* – 1, *b* + 7) and (*a* – 4, *b* + 3) are vertices of a square.

8. Prove that perimeter of triangle formed by points (1, 1), (–1, –1) and $(-\sqrt{3}, \sqrt{3})$ is $6\sqrt{2}$ unit.

9. For what value of *k* following points will be collinear (*k*, 2 – 2*k*), (1 – *k*, 2*k*) and (–4 – *k*, 6 – 2*k*).

10. Find a relation between *x* and *y* when the point (*x*, *y*) is equidistant from the points (6, –1) and (2, 3).

11. Draw a quadrilateral in the Cartesian plane, whose vertices are (–4, 5), (0, 7), (5, –5) and (–4, –2). Also, find its area.

[NCERT]

12. Plot the line $3x - 5 = 0$.

EXERCISE 2

1. Prove that the coordinates of the orthocentre of the triangle whose vertices are (0, 0), (2, –1) and (1, 3) is (–4/7, –1/7).

2. Two vertices of a triangle are (3, –5) and (–7, 4). If the centroid is (2, –1), find the third vertex.

3. If $A(-1, 3)$, $B(1, -1)$ and $C(5, 1)$ are the vertices of a triangle ABC, find the length of the median through A.

4. Find the point of intersection of the medians of the triangle with vertices at (–1, 0), (5, –2) and (8, 2).

5. Find the third vertex of a triangle if two of its vertices are (–1, 4) and (5, 2), and the medians meet at (0, 3).

6. Find the incentre of the triangle whose vertices are (7, –36), (7, 20) and (–8, 0).

7. Prove that the length of the line segment joining the middle points of two sides of a triangle is half the third side.

8. If $ABCD$ is a rectangle and P is any point (in the plane of rectangle), then $PA^2 + PC^2 = PB^2 + PD^2$. Prove

9. Prove that diagonals of a square are equal.

10. Find the equation of the locus of a point which moves so that its distance from the x-axis is five times its distance from the y-axis.

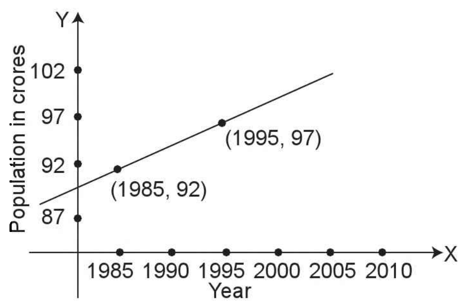
11. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle. ***[NCERT]***

12. Find y if the line containing the points $(3, y)$ and (2, 7) is parallel to the line containing the points (–1, 4) and (0, 6).

13. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points $P(0, -4)$ and $Q(8, 0)$. ***[NCERT]***

14. The slope of a line is double of the slope of another line. If tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines. ***[NCERT]***

15. Consider the following population and year graph.

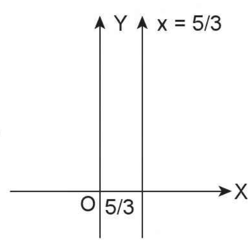


Find the slope of line AB and using it, find what will be the population in the year 2010? ***[NCERT]***

ANSWERS

EXERCISE 1

1. 13
2. 11
3. $K = 3$
4. $\left(\frac{2}{3}, 2\right)$
5. (–2, 7)
6. (–1, 0)
9. $k = -1, \frac{1}{2}$
10. $x - y - 3 = 0$
11. $\frac{121}{2}$ square unit
12.



EXERCISE 2

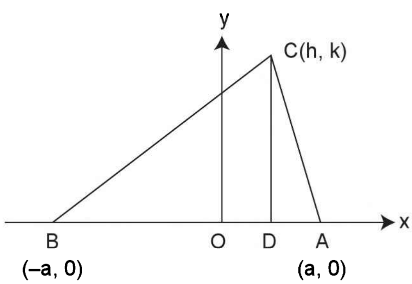
2. (10, –2)
3. 5 units
4. (4, 0)
5. (–4, 3)
6. (0, –1)
10. $y^2 = 25x^2$
11. $(0, a)$, $(0, -a)$ and $(-\sqrt{3}a, 0)$ or $(0, a)$, $(0, -a)$ and $(-\sqrt{3}a, 0)$
12. 9
13. – 1/2
14. 1 and 2 or 1/2 and 1, or – 1 and – 2 or – 1/2 and –1
15. 1/2, 104.5 crores.

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. A and B are two fixed points of ΔABC with side $AB = 2a$. If its vertex C moves in such way that $\cot A + \cot B = \lambda$, where λ is a constant, then the locus of the point C is ***[MP-81]***
   (a) $y\lambda = 2a$ (b) $ya = 2\lambda$
   (c) $y = \lambda a$ (d) None of these

Solution

(a) We may suppose that coordinates of two fixed points A, B are $(a, 0)$ and $(-a, 0)$ and variable point C is (h, k).



From the adjoining figure

$$\cot A = \frac{DA}{CD} = \frac{a-h}{k};\ \cot B = \frac{BD}{CD} = \frac{a+h}{k}$$

But $\cot A + \cot B = \lambda$, so we have

$$\frac{a-h}{k} + \frac{a+h}{k} = \lambda \Rightarrow \frac{2a}{k} = \lambda$$

Hence, locus of C is $y\lambda = 2a$.

2. The coordinates of the feet of perpendiculars from the vertices of a triangle on the opposite sides are (20, 25); (8, 16) and (8, 9). The orthocentre of the triangle lies at the point
   (a) (5, 10) (b) (10,15)
   (c) (15,30) (d) None of these

Solution

(b) Let given points be D, E, F. From geometry we know that the orthocentre of the given triangle is the incentre of its pedal triangle DEF.

Now $DE = \sqrt{12^2 + 9^2} = 15,\ EF = \sqrt{0 + 7^2} = 7,$

$FD = \sqrt{12^2 + 16^2} = 20$

$\therefore$ required point

$$= \left(\frac{20(7) + 8(20) + 8(15)}{7 + 20 + 15}, \frac{25(7) + 16(20) + 9(15)}{7 + 20 + 15}\right)$$

$$= \left(\frac{140 + 160 + 120}{42}, \frac{175 + 320 + 135}{42}\right) = (10, 15)$$

3. If the orthocentre and centroid of a triangle are (–3, 5) and (3, 3) respectively, then its circumcentre is ***[Haryana (CEE)-1999]***
   (a) (6, 2) (b) (6, –2)
   (c) (0, 4) (d) (0, 8)

Solution

(a) Let circumcentre be (α, β). Since centroid of a triangle divides the line segment joining its orthocentre and circumcentre in the ratio 2 : 1, so we have

$$\frac{2(\alpha) + 1(-3)}{2+1} = 3$$

and $$\frac{2(\beta) + 1(5)}{2+1} = 1$$

$\Rightarrow$ $2\alpha = 12$ and $2\beta = 4$

$\therefore$ Circumcentre is (6, 2)

4. If a vertex of a triangle is (1, 1) and the mid-points of two sides through this vertex are (–1, 2) and (3, 2); then the centroid of the triangle is ***[AIEEE-2005]***

(a) (1/3, 7/3) (b) (1, 7/3)
(c) (–1/3, 7/3) (d) (–1, 7/3)

Solution

(b) Let ABC be the given triangle and A be the given vertex (1, 1). Let $B \equiv (a, b)$ and $C \equiv (c, d)$. Then $a + 1 = -2$, $b + 1 = 4$

$\Rightarrow\ B \equiv (-3, 3)$

and $c + 1 = 6$, $d + 1 = 5$

$\Rightarrow\ C \equiv (5, 3)$

$\therefore$ Centroid

$$\equiv \left(\frac{1-3+5}{3}, \frac{1+3+3}{3}\right) = (1, 7/3)$$

5. The incentre of a triangle with vertices (7, 1) (–1, 5) and $(3+2\sqrt{3}, 3+4\sqrt{3})$ is

[J & K-2005]

(a) $\left(3+\frac{2}{\sqrt{3}}, 3+\frac{4}{\sqrt{3}}\right)$

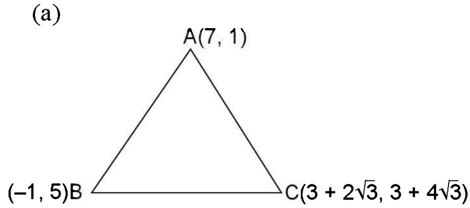
(b) $\left(1+\frac{2}{3\sqrt{3}}, 1+\frac{4}{3\sqrt{3}}\right)$

(c) (7, 1)

(d) None of these

Solution

(a)



$\because$ $AB = BC = CA = 4\sqrt{5}$ i.e., given triangle is equilateral.

(In centre of a triangle are same as the centroid when triangle is equilateral)

Hence, incentre

$$= \left(\frac{7-1+3+2\sqrt{3}}{3}, \frac{1+5+3+4\sqrt{3}}{3}\right)$$

$$= \left(3+\frac{2}{\sqrt{3}}, 3+\frac{4}{\sqrt{3}}\right)$$

6. The area (in square units) of the triangle formed by the points with polar coordinates (1, 0), (2, π/3) and (3, 2π/3) is

[EAMCET-2007]

(a) $\frac{11\sqrt{3}}{5}$ (b) $\frac{5\sqrt{3}}{4}$

(c) $\frac{5}{4}$ (d) $\frac{11}{4}$

Solution

(b) Area of the triangle formed by (1, 0), (2, π/3) and (3, 2π/3) is $\frac{1}{2}|\Sigma r_1 r_2 \sin(\theta_1 - \theta_2)|$

$$= \frac{1}{2}\left|2\sin\left(-\frac{\pi}{3}\right) + 6\sin\left(-\frac{\pi}{3}\right) + 3\sin\frac{2\pi}{3}\right|$$

$$= \frac{5\sqrt{3}}{4} \text{ sq. units.}$$

7. Let $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ be the vertices of a right angled triangle with AC as its hypotenuse. If the area of the triangle is 1 square unit, then the set of values which 'k' can take is given by

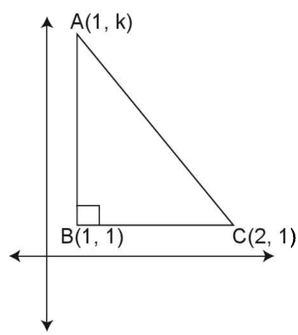
[AIEEE-2007]

(a) {–1, 3} (b) {–3, –2}
(c) {1, 3} (d) {0, 2}

Solution

(a) $\frac{1}{2} \times |k-1| \times 1 = 1$

$k = -1, 3.$



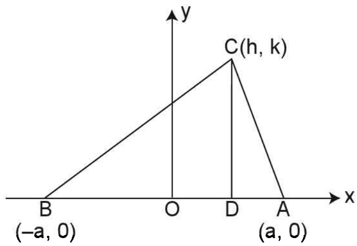
8. Two fixed points are $A(a, 0)$ and $B(-a, 0)$. Then the locus of the point C of $\Delta\, ABC$. If $\angle A - \angle B = \theta$, will be

[Roorkee-1982]

(a) $x^2 + y^2 + 2xy \tan\theta = a^2$
(b) $x^2 + y^2 + 2xy \cot\theta = a^2$
(c) $x^2 - y^2 + 2xy \tan\theta = a^2$
(d) $x^2 - y^2 + 2xy \cot\theta = a^2$

Solution

(d) Let $C \equiv (h, k)$. From adjoining figure



$$\tan A = \frac{h}{a-h}, \quad \tan B = \frac{h}{a+h}$$

Also $\angle A - \angle B = \theta$

$\Rightarrow \quad \tan(A - B) = \tan\theta$

$$\Rightarrow \quad \frac{\tan A - \tan B}{1 + \tan A \tan B} = \tan\theta$$

$$\Rightarrow \quad \frac{\left(\frac{h}{a-h}\right) - \left(\frac{h}{a+h}\right)}{1 + \left(\frac{h}{a-h}\right)\left(\frac{h}{a+h}\right)} = \tan\theta$$

$\Rightarrow \quad h^2 - k^2 + 2kh \cot\theta = a^2$

$\therefore \quad$ locus of C is $x^2 - y^2 + 2xy \cot\theta = a^2$

9. The vertices of a triangle are $[at_1t_2, a(t_1 + t_2)]$, $[at_2t_3, a(t_2 + t_3)]$, $[at_3t_1, a(t_3 + t_1)]$, then the coordinates of its orthocentre are **[IIT-1983]**

(a) $[a, a(t_1 + t_2 + t_3 + t_1t_2t_3)]$
(b) $[-a, a(t_1 + t_2 + t_3 + t_1t_2t_3)]$
(c) $[-a(t_1 + t_2 + t_3 + t_1t_2t_3), a]$
(d) None of these

Solution

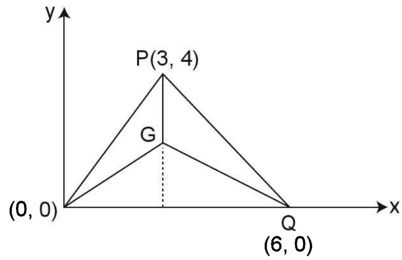
(b) $m_1 = \dfrac{-a(t_1t_2 - t_2t_3)}{a(t_1 - t_3)} = -t_2$, $m_2 = -t_3$ Therefore, perpendicular from point third $t_2x + y = a[t_1t_2t_3 + t_1 + t_3]$ and perpendicular from point first $t_3x + y = a[t_1t_2t_3 + t_1 + t_2]$ $x = -a, y = a[t_1t_2t_3 + t_1 + t_2 + t_3]$.

10. Let $O(0,0)$, $P(3,4)$, $Q(6,0)$ be the vertices of the triangle OPQ.The point R inside the triangle OPQ is such that triangles OPR, PQR, OQR are of equal area.The coordinates of R are **[IIT-2007]**

(a) (4/3,3) (b) (3,2/3)
(c) (3,4/3) (d) (4/3,2/3)

Solution

(c) $\Delta\, OPQ$ is an isosceles. $OP = OQ$.
Let G be the required point.Since, $\Delta\, OPQ$ is isosceles G will lie on $x = 3$. Now consider $\Delta\, OGQ$



$$\Delta\, OGQ = \frac{1}{3} \times \text{area } \Delta\, OPQ \;= \frac{1}{3} \times 6 \times \frac{4}{2}$$

or $\quad h = 4/3$ units

$\therefore \quad$ ordinate of $G = 4/3$ units

Required point is G (3,4/3) i.e., the centroid of the triangle

Also,area ($\Delta\, OGP$) = area ($\Delta\, GPQ$)

$= \frac{1}{3} \times \text{area } \Delta\, OPQ.$

11. For an equilateral ΔABC, $A(1, 2)$, $B(2, 3)$ and its incentre is $\left(\dfrac{9+\sqrt{3}}{6}, \dfrac{15-\sqrt{3}}{6}\right)$ then find the coordinates of vertex C. **[Gujrat CET-2007]**

(a) $\left(\dfrac{3+\sqrt{3}}{2}, \dfrac{5-\sqrt{3}}{2}\right)$

(b) $\left(\dfrac{3-\sqrt{3}}{2}, \dfrac{5+\sqrt{3}}{2}\right)$

(c) $\left(\dfrac{3+\sqrt{3}}{6}, \dfrac{5-\sqrt{3}}{6}\right)$

(d) None of these

Solution

(a) Let vertex c be (α, β)

$\because \quad$ In equiliateral triangle

Incentre = centroid

$\therefore \quad \left(\frac{\alpha+1+2}{3}, \frac{\beta+2+3}{3}\right)$

$\equiv \left(\frac{9+\sqrt{3}}{6}, \frac{15-\sqrt{3}}{6}\right)$

$\Rightarrow \quad (\alpha, \beta) \equiv \left(\frac{3+\sqrt{3}}{2}, \frac{5-\sqrt{3}}{2}\right)$

12. Consider three points $P = (-\sin(\beta-\alpha), -\cos\beta)$, $Q = (\cos(\beta-\alpha), \sin\beta)$ and $R = (\cos(\beta-\alpha+\theta), \sin(\beta-\theta))$, where $0 < \alpha, \beta, \theta < \pi/4$. Then,

[IIT-JEE-2008]

(a) P lies on the line segment RQ
(b) Q lies on the line segment PR
(c) R lies on the line segment QP
(d) P, Q, R are non-collinear

Solution

(d) We have $P \equiv (-\sin(\beta-\alpha), -\cos\beta) \equiv (x_1, y_1)$ say

$Q \equiv (\cos(\beta-\alpha), \sin\beta) \equiv (x_2, y_2)$ say

$R \equiv (\cos(\beta-\alpha+\theta), \sin(\beta-\theta))$

The x-coordinate and y-coordinate of R are

$X_R = x_2\cos\theta + x_1\sin\theta$

$Y_R = y_2\cos\theta + y_1\sin\theta$

The points

$\left(\frac{x_2\cos\theta + x_1\sin\theta}{\cos\theta+\sin\theta}, \frac{y_2\cos\theta + y_1\sin\theta}{\cos\theta+\sin\theta}\right)$

lies on PQ. But R doesn't lie on PQ. For it to lie on PQ, $\cos\theta$ and $\sin\theta$ must be related by $\cos\theta + \sin\theta = 1$, which is not given in the problem. Thus P, Q, R are non-collinear.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If (6,4) and (2,6) are two vertices of a triangle and its centroid is (4,6) then its third vertex will be

[RPET-96]

(a) (4, 8) (b) (8,4)
(c) (3, 3) (d) None of these

2. The incentre of the triangle with vertices (0,0); (3,0) and (0,4) is
(a) (1, 1) (b) (3, 3)
(c) (1, 0) (d) (2, 2)

3. The circumcentre of the triangle with vertices (0, 0); (3, 0) and (0, 4) is
(a) (2, 3/2) (b) (1, 1)
(c) (3/2, 2) (d) None of these

4. If (–2,3); (4, –3) and (4,5) are mid-points of the sides of a triangle, then coordinates of its centroid are
(a) (2, 5/3) (b) (6, 5)
(c) (–2, 5/3) (d) (5/3, 2)

5. If $(1, a)$; $(2, b)$ and $(c, -3)$ are vertices of a triangle, then its centroid will lie on x-axis if:
(a) $a+b+3=0$ (b) $a+b=0$
(c) $a+b=3$ (d) $c+3=0$

6. The distance between the circumcentre and orthocentre of the triangle with vertices (0, 0); (6, 0) and (0, 8) is
(a) 0 (b) 5
(c) 6 (d) 8

7. The centroid of a triangle with vertices (2, 1); (5, 2) and (3, 4) is

[IIT-1964]

(a) (8/3, 7/3) (b) (10/3, 7/3)
(c) (10/3, –7/3) (d) (–10/3, 7/3)

8. If the vertices of a triangle be (2, 1); (5, 2) and (3, 4), then its circumcentre is

[IIT-1964]

(a) (13/2, 9/2)
(b) (9/4, 13/4)
(c) (13/4, 9/4)
(d) None of these

9. The orthocentre of the triangle with vertices (0, 0); (3,0) and (0, 4) is

[MNR-82; RPET-97]

(a) (1, 4/3) (b) (0, 0)
(c) (3/2, 2) (d) None of these

10. If (–5, 30°); (7, 150°) and (11, 210°) are vertices of a triangle, then its area is

(a) $21\sqrt{3}/2$ (b) $3\sqrt{11}/2$
(c) $11\sqrt{2}/2$ (d) $2\sqrt{11}/2$

11. The incentre of the triangle formed by (0, 0); (5, 12); (16, 12) is

[EAMCET-84]

(a) (9, 7) (b) (7, 9)
(c) (–9, 7) (d) (–7, 9)

12. If *O* be the origin and if the coordinates of any two points Q_1 and Q_2 be (x_1, y_1) and (x_2, y_2) respectively, then $OQ_1 \cdot OQ_2 \cos Q_1OQ_2$ is equal to

[IIT-61]

(a) $x_1x_2 - y_1y_2$ (b) $x_1y_1 - x_2y_2$
(c) $x_1x_2 + y_1y_2$ (d) $x_1y_1 + x_2y_2$

13. The coordinates of the points *O*, *A* and *B* are (0, 0), (0, 4) and (6, 0) respectively. If a point *P* moves such that the area of ΔPOB, is twice the area of a *POA* then the equation to both parts of the locus of *P* is

[IIT-64, 85]

(a) $(x - 3y)(x + 3y) = 0$
(b) $(x - 3y)(x + y) = 0$
(c) $(3x - y)(3x + y) = 0$
(d) None of these

14. If the vertices of a triangle have integral coordintes, then the triangle is

[IIT-75; MPPET-83]

(a) Equilateral (b) Never equilateral
(c) Isosceles (d) None of these

15. The Coordinates of the centroid and circumcentre of a triangle are *G*(2, 3), *O*(–27/2, 39/2) then coordinates of its orthocentre are

(a) (33, –30) (b) (–38, 24)
(c) (0, 0) (d) (–38, –24)

16. The median *BE* and *AD* of a triangle with vertices *A*(0, *b*), *B*(0,0) and *C*(*a*, 0) are perpendicular to each other, if

(a) $a = b/2$ (b) $b = a/2$
(c) $ab = 1$ (d) $a = \pm\sqrt{2}\,b$

17. The vertices of a Δ are *A*(–1, –7): *B*(5, 1) and *C*(1, 4). The bisector of the angle Δ *ABC* meets the opposite side in point *D*; the coordinates of *D* are

(a) (1/3, 1/3) (b) (1/2, 1/3)
(c) (1/3, 1/4) (d) None of these

18. If the coordinates of a point be given by the equations $x = a(1 - \cos\theta)$, $y = a\sin\theta$, then the locus of the point will be

(a) A straight line (b) A circle
(c) A parabola (d) an ellipse

19. The coordinates of the points *A*, *B*, *C*, *D* are (2,a), (3,5), (3, 4) and (0, 6) respectively. If the lines *AC* and *BD* be perpendicular, then *a* is equal to

(a) 7 (b) 1
(c) –1 (d) –7

20. Locus of centroid of the triangle whose vertices are ($a\cos t$, $a\sin t$), ($b\sin t$, $-b\cos t$) and (1, 0), where *t* is a parameter, is

[AIEEE-2003]

(a) $(3x + 1)^2 + (3y)^2 = a^2 - b^2$
(b) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$
(c) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$
(d) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$

21. Which of the following polar coordinates are associated to the same point?

I : (2, 30°) II : (3, 150°)
III : (–2, 45°) IV : (–3, 330°)
V : (3, –210°) VI : (–3, –30°)

(a) I, III and IV (b) II, IV and VI
(c) II, IV, V AND VI (d) IV and VI

SOLUTIONS

1. (a) Let the third vertex be (*x*, *y*), the $x + 6 + 2 = 3(4)$ and $y + 4 + 6 = 3(6)$

$\Rightarrow \quad x = 4, y = 8$

2. (a) $\left(\dfrac{0.5 + 3.4 + 0.3}{5 + 4 + 3}, \dfrac{0.5 + 0.4 + 4.3}{5 + 4 + 3}\right)$

$\equiv (1, 1)$

3. (c) Triangle is right angled at (0, 0), so the mid-point of its hypotenuse i.e., (3/2, 2) is its circumcentre.

4. (a) Centroid will be the same as that of the triangle with given vertices.

5. (c) Let the centroid of triangle be $P(x, y)$

Given $(x_1, y_1) = (1, a)$

$(x_2, y_2) = (2, b)$

$(x_3, y_3) = (c, -3)$

and also given centroid lie on X-axis.

$\therefore \quad y = \frac{y_1 + y_2 + y_3}{3} = 0$

$\frac{a+b-3}{3} = \frac{0}{1} \Rightarrow a+b=3$

6. (b) Given triangle is right angled, hence its orthocentre is (0, 0).

Also its circumcentre = mid-point of hypotenuse = (3, 4)

$\therefore$ required distance

$= \sqrt{(3-0)^2+(4-0)^2} = 5$

7. (b) Let the centroid of triangle be $P(x, y)$

Given $(x_1, y_1) = (2, 1)$

$(x_2, y_2) = (5, 2)$

$(x_3, y_3) = (3, 4)$

$\therefore \quad x = \frac{x_1 + x_2 + x_3}{3} = \frac{5+2+3}{3} = \frac{10}{3}$

$y = \frac{y_1 + y_2 + y_3}{3} = \frac{1+2+4}{3} = \frac{7}{3}$

$\therefore$ Centroid of triangle $= \left(\frac{10}{3}, \frac{7}{3}\right)$

8. (c) Let circumcentre be (x, y), then $(x-2)^2 + (y-1)^2 = (x-5)^2 + (y-2)^2 = (x-3)^2 + (y-4)^2$

$\Rightarrow \quad -4x - 2y + 5 = -10x - 4y + 29 = -6x - 8y + 25$

$\Rightarrow \quad 6x + 2y = 24$ and $4x - 4y = 4$

$\Rightarrow \quad x = 13/4, y = 9/4$

9. (b) Triangle is right angled at (0, 0), so (0, 0) is the ortho-centre.

10. (a) Area $= \frac{1}{2}$ [(–5)(7) sin 120° + (7) (11) sin 60° + (11) (–5) sin (–180°)]

$= \frac{1}{2}\left[\frac{-35\sqrt{3}}{2} + \frac{77\sqrt{3}}{2}\right] = \frac{21\sqrt{3}}{2}$

11. (b) If given points be A, B, C; then a = 11, b = 20, c = 13

$\therefore$ Incentre

$= \left(\frac{11(0)+20(5)+13(16)}{11+20+13}, \frac{11(0)+20(12)+13(12)}{11+20+13}\right)$

$= (7, 9)$

12. (c) $\vec{OQ_1}.\vec{OQ_2} = (OQ_1)(OQ_2)\cos\theta$,

$\vec{OQ_1} = x_1\hat{i} + \hat{j}y_1, \vec{OQ_2} = \hat{i}x_2 + \hat{j}y_2$

$\Rightarrow \quad x_1x_2 + y_1y_2 = (OQ_1)(OQ_2)\cos\theta$

13. (a) $O = (0, 0), A = (0, 4), B = (6, 0), P(x, y)$

$A_1 = \text{Area of } \Delta POA = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 0 & 4 & 1 \end{vmatrix}$

$= \frac{1}{2}[-4x] = -2x$

$A_2 = \text{Area of } \Delta POB = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 6 & 0 & 1 \end{vmatrix}$

$= \frac{1}{2}[6y] = 3y$

By assumption, $A_1 = 2A_2$

$\Rightarrow \quad |-2x| = 3\,|3y|$

$\Rightarrow \quad \left|\frac{x}{3y}\right| = 1 \Rightarrow \frac{x}{3y} = 1, -1$

$\Rightarrow \quad x = 3y, x = -3y$

$\Rightarrow \quad (x - 3y)(x + 3y) = 0.$

14. (b) Let the vertex be (x_r, y_r), $r = 1, 2, 3$ where both x_r and y_r are integers.

Hence, its area $= \frac{1}{2}\Sigma x_1(y_2 - y_3) =$ rational number .... (1)

Also if a be its side then $a^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = a$ positive integer.

But the area of an equilateral triangle

$= \frac{\sqrt{3}}{4}a^2$

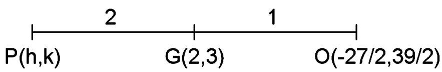
$\therefore$ Area $= \frac{\sqrt{3}}{4}a^2$, which is irrational, since a^2 is a positive integer. .... (2)

Thus the two statements (1) and (2) for area are contradictory. Therefore, if the vertices are integers, then that triangle cannot be an equilateral triangle.

NOTE

Students should remember this question as fact.

15. (a) Let the orthocentre be (h, k)



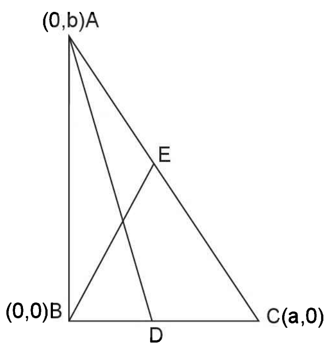
Then, $2 = \dfrac{h-27}{3} \Rightarrow h = 33$

and $\quad 3 = \dfrac{k+39}{3} \Rightarrow k = -30$

$\therefore$ Orthocentre $= (h, k) = (33, -30)$

16. (b, d) $D\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right) = D\left(\dfrac{a}{2}, 0\right)$

$$E\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = E\left(\frac{a}{2}, \frac{b}{2}\right)$$



Slope of line AD is

$$m_1 = \frac{y_2-y_1}{x_2-x_1} = \frac{b-0}{0-\dfrac{a}{2}} = -\frac{2b}{a}$$

Slope of line BE is $m_2 = \dfrac{\dfrac{b}{2}-0}{\dfrac{a}{2}-0} = \dfrac{b}{a}$

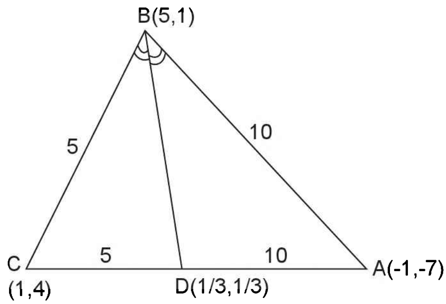
Now AD and BE are perpendicular if $m_1 m_2 = -1$

$\Rightarrow \quad \left(-\dfrac{2b}{a}\right)\left(\dfrac{b}{a}\right) = -1$

$\Rightarrow \quad 2b^2 = a^2$

$\Rightarrow \quad a = \pm b\sqrt{2}$

17. (a) $AB = 10$, $BC = 5$. The bisector of $\angle ABC$ will divide the opposite side AC in the ratio of arms of the angle.
i.e., 5 : 10. Point D by ratio formula is (1/3, 1/3).



18. (b) On eliminating θ, we get the required locus.

Since $x = a(1 - \cos\theta)$

$\Rightarrow \quad x - a = -a\cos\theta \qquad$ ....(i)

and $\quad y = a\sin\theta \qquad$ .... (ii)

Now adding the squares of (i) and (ii), we get $x^2 + y^2 - 2ax = 0$, which is equation of a circle.

19. (b) Slope of line $AC = \dfrac{4-a}{3-2} = (4-a)$

slope of line $BD = \dfrac{6-5}{0-3} = \dfrac{1}{-3} = \dfrac{-1}{3}$

given line AC and BD are perpendicular

then $(4-a)\left(\dfrac{-1}{3}\right) = -1$

$$(4-a) = 3$$

$$a = 1$$

20. (c) Let centroid be (x, y), then $3x = a\cos t + b\sin t + 1$, $3y = a\sin t - b\cos t$

$\Rightarrow \quad 3x - 1 = a\cos t + b\sin t \qquad$ .... (1)

$3y = a\sin t - b\cos t \qquad$ .... (2)

$(1)^2 + (2)^2$

$\Rightarrow \quad (3x-1)^2 + 9y^2 = a^2 + b^2.$

which is required locus.

21. (c) $\because (3, 150°) \equiv (3, 150° - 360°) = (3, -210°)$
$\equiv (-3, 180° + 150°) = (-3, 330°) \equiv (-3, -30°)$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If (6,0), (0,6) and (6,6) are vertices of a triangle, then distance between its circumcentre and centroid is

(a) $2\sqrt{2}$ (b) 2
(c) $\sqrt{2}$ (d) 1

2. The circumcentre of the triangle with vertices (–3, –1); (–1,3) and (6, 2) is

(a) (0, 0) (b) (1, –2)
(c) (2, –1) (d) (–2, 1)

3. Two vertices of a triangle are (5, 4) and (–2, 4); if its centroid is (5, 6), then its third vertex is

[MPPET-93]

(a) (10, 12) (b) (12,10)
(c) (–10, 12) (d) (12, –10)

4. If the vertices *P*, *Q*, *R*, of a triangle *PQR* are rational points, which of the following points of the triangle *PQR* is (are) always rational points (s)

(a) Centroid (b) Incentre
(c) Circumcentre (d) Orthocentre

5. The orthocentre of the triangle formed by (0, 0),(8, 0),(4, 6) is

[EAMCET-91]

(a) (4, 8/3) (b) (3, 4)
(c) (4, 3) (d) (–3, 4)

6. If (0, 1), (1, 1) and (1, 0) be the middle points of the sides of a triangle then its incentre is

(a) $(2+\sqrt{2}, 2+\sqrt{2})$
(b) $(2+\sqrt{2}, -2-\sqrt{2})$
(c) $(2-\sqrt{2}, 2+\sqrt{2})$
(d) $(2-\sqrt{2}, 2-\sqrt{2})$

7. If *A* (cos α, sin α), *B*(sin α, –cos α), *C*(1, 2) are the vertices of a Δ*ABC*, then as α varies, the locus of its centroid is

(a) $x^2 + y^2 - 2x - 4y + 1 = 0$
(b) $3(x^2 + y^2) - 2x - 4y + 1 = 0$
(c) $x^2 + y^2 - 2x - 4y + 3 = 0$
(d) None of these

8. If the line segement joining the points *A*(*a*, *b*) and *B*(*c*, *d*) subtends an angle θ at the origin, then cos θ is equal to

(a) $(ab + cd)/\sqrt{(a^2+b^2)(c^2+d^2)}$
(b) $(ac + bd)/\sqrt{(a^2+b^2)(c^2+d^2)}$
(c) $(ac - bd)/\sqrt{(a^2+b^2)(c^2+d^2)}$
(d) None of these

9. Slope of any line perpendicular to the line joining the points (3, 5) and (–4, 2) is

(a) 7/3 (b) –7/3
(c) 3/7 (d) –3/7

10. The point of intersection of the medians of the Δ with vertices at (–1, 0), (5, –2) and (8, 2)

(a) (–4, 0) (b) (–4, –4/3)
(c) (4, 0) (d) (12, 0)

11. The third vertex of a Δ if two of vertices are at (–1, 4) and (5, 2) and medians meet at (0, –3)

(a) (4, 15) (b) (–4, –15)
(c) (4, 6) (d) (4, 3)

12. Orthocentre of the triangle whose vertices are (0, 0), (2, –1) and (1, 3) is

[IIT-67,74]

(a) (4/7, 1/7) (b) (–4/7, –1/7)
(c) (–4, –1) (d) (4, 1)

13. The incentre of the triangle with vertices $(1, \sqrt{3})$, (0, 0) and (2, 0) is

(a) $\left(1, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$
(c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(1, \frac{1}{\sqrt{3}}\right)$

14. The circumcentre of the triangle with vertices (0, 30), (4, 0) and (30, 0) is

[Karnataka PET-2008]

(a) (10, 10) (b) (10,12)
(c) (12, 12) (d) (17, 17)

15. The centroid of the triangle *ABC* where A ≡ (2, 3), *B* ≡ (8, 10) and *C* ≡ (5, 5) is

[Karnataka CET-2008]

(a) (6, 6) (b) (15, 18)
(c) (5, 6) (d) (6, 5)

16. The line joining the points (x_1, y_1) and (x_2, y_2) subtends a right angle at the point (1, 1) then $x_1 + x_2 + y_1 + y_2 - 2$ is equal to
(a) $x_1 x_2 + y_1 y_2$
(b) $x_1 x_2 - y_1 y_2$
(c) $x_2 x_2 + y_1 y_2$
(d) None of these

17. Points (0, 8/3), (1, 3) and (82, 30) are vertices of a *[IIT-83; RPET-88]*
(a) right angled triangle
(b) acute angled triangle
(c) Obtuse angled triangle
(d) None of these

18. If $A(a, 0)$ and $B(-a, 0)$ are two fixed points and a point P moves such that $\angle APB = 90°$, then locus of P is
(a) $x^2 + y^2 = 2a^2$ (b) $x^2 + y^2 = a^2$
(c) $x^2 + y^2 + 2a^2 = 0$ (d) None of these

19. If two vertices of a triangle are (4, –3); (–9, 7) and its centroid is (1, 4); then its area is
(a) 181/2 (b) 183/2
(c) 185/2 (d) None of these

20. If $A(2, 2)$; $B(-4, 4)$; $C(5, -8)$ are vertices of a triangle then the length of the median through C is equal to *[RPET-88]*
(a) $\sqrt{65}$ (b) $\sqrt{157}$
(c) $\sqrt{85}$ (d) $\sqrt{113}$

21. Two vertices of a triangle are (5, –1) and (–2,3). If orthocentre is the origin then coordinates of the third vertex are *[IIT-79-83]*
(a) (7, 4) (b) (–4, 7)
(c) (4, –7) (d) (–4, –7)

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The incentre of the triangle with vertices $(1, \sqrt{3})$, (0,0) and (2,0) is **[IIT-Sc.-2000]**

(a) $(1, \sqrt{3}/2)$ (b) $(2/3, 1/\sqrt{3})$
(c) $(2/3, \sqrt{3}/2)$ (d) $(1, 1/\sqrt{3})$

2. If the vertices of a triangle be $(a, 1)$, $(b, 3)$ and $(4, c)$, then the centroid of the triangle will lie on x-axis if

(a) $a + c = -4$
(b) $a + b = -4$
(c) $c = -4$
(d) $b + c = -4$

3. If the coordinates of the points A and B be (6, 2) and $(3, a)$, respectively and the slope of the line AB be 3, then a is equal to

(a) 7 (b) 11
(c) –7 (d) None of these

4. If the vertices of a triangle be $(a, b - c)$, $(b, c - a)$ and $(c, a - b)$, then the centroid of the triangle lies

(a) At origin (b) On x-axis
(c) On y-axis (d) None of these

5. If the middle points of the sides of a triangle be (–2, 3), (4, –3) and (4, 5), then the centroid of the triangle is

(a) (5/3, 2) (b) (5/6, 1)
(c) (2, 5/3) (d) (1, 5/6)

6. $P(2, 1)$, $Q(4, -1)$, $R(3, 2)$ are the vertices of a triangle and if through P and R lines parallel to opposite sides are drawn to intersect in S, then the area of $PQRS$ is

(a) 6 (b) 4
(c) 8 (d) 12

7. The orthocentre of the triangle with vertices $\left(2, \frac{\sqrt{3}-1}{2}\right)\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(2, -\frac{1}{2}\right)$ is **[IIT-1993]**

(a) $\left(\frac{3}{2}, \frac{\sqrt{3}-3}{6}\right)$ (b) $\left(2, -\frac{1}{2}\right)$
(c) $\left(\frac{5}{4}, \frac{\sqrt{3}-2}{4}\right)$ (d) $\left(\frac{1}{2}, -\frac{1}{2}\right)$

8. The centroid of the triangle is (2, 7) and two of its vertices are (4, 8) and (–2, 6). The third vertex is **[Kerala Engg.-02]**

(a) (0, 0) (b) (4, 7)
(c) (7, 4) (d) 7, 7)

[RPET-87]

9. If $A(-a, 0)$ and $B(a, 0)$ are two fixed points, then the locus of the point on which the line AB subtends the right angle, is

(a) $x^2 + y^2 = 2a^2$ (b) $x^2 - y^2 = a^2$
(c) $x^2 + y^2 + a^2 = 0$ (d) $x^2 + y^2 = a^2$

10. The points (1, 2), (2, 3) and (3, 4) form the vertices of a triangle. The centroid of the triangle has coordinates **[MPPET-2007]**

(a) (4, 5) (b) (3, 2)
(c) (2, 3) (d) (6, 9)

11. The orthocentre of the triangle with vertices $(2/\sqrt{3} - 1/2)$, $(1/2, -1/2)$ and $(2, -1/2)$ is

(a) $\left(\frac{3}{2}, \frac{\sqrt{3}-3}{6}\right)$ (b) $\left(2, -\frac{1}{2}\right)$
(c) $\left(\frac{5}{4}, \frac{(\sqrt{3}-2)}{4}\right)$ (d) $\left(\frac{1}{2}, -\frac{1}{2}\right)$

12. If $A(1, 4)$; $B(3, 0)$ and $C(2, 1)$ are vertices of a triangle then the length of the median through C is

(a) 1 (b) 2
(c) $\sqrt{2}$ (d) $\sqrt{3}$

13. If O be the origin and $A(x_1, y_1)$, $B(x_2, y_2)$ are two points, then what is (OA) (OB) cos $\angle AOB$? **[NDA-2008]**

(a) $x_1^2 + x_2^2$ (b) $y_1^2 + y_2^2$
(c) $x_1x_2 + y_1y_2$ (d) $x_1y_1 + x_2y_2$

14. The locus of a point P which moves in such a way that the segement OP, where O is the origin, has slope $\sqrt{3}$, is

(a) $x-\sqrt{3}y=0$ (b) $x+\sqrt{3}y=0$

(c) $\sqrt{3}x+y=0$ (d) $\sqrt{3}x-y=0$

15. If the coordinates of the vertices of a triangle be $(1, a)$, $(2, b)$ and $(c^2, 3)$, then the centroid of the triangle

(a) Lies at the origin
(b) Cannot lie on x-axis
(c) Cannot lie on y-axis
(d) None of these

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ
11. ⓐ ⓑ ⓒ ⓓ
12. ⓐ ⓑ ⓒ ⓓ
13. ⓐ ⓑ ⓒ ⓓ
14. ⓐ ⓑ ⓒ ⓓ
15. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

2. (c) Centroid $\equiv\left(\dfrac{a+b+4}{3}, \dfrac{1+3+c}{3}\right)$

$$\equiv\left(\frac{a+b+4}{3}, \frac{c+4}{3}\right)$$

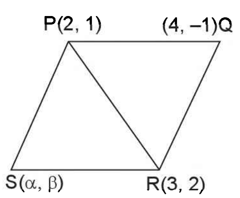
$\because$ Centroid is on x axis, $\dfrac{c+4}{3}=0 \Rightarrow c=-4$

6. (b) Here $PQRS$ is a parallelogram.

$\therefore$ Area = 2 area (ΔPQR)

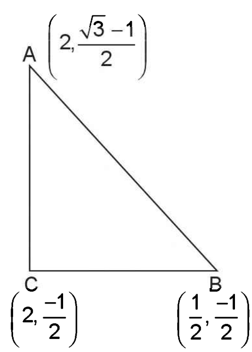
$$=2\times\frac{1}{2}\begin{vmatrix}2 & 1 & 1\\ 3 & 2 & 1\\ 4 & -1 & 1\end{vmatrix}$$

$=|[2 \times 3 - 1(-1) + (-3 - 8)]|$
$= |[6 + 1 - 11]| = 4$



7. (b) Given a ΔABC

$$AB=\sqrt{\left(2-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}-1}{2}+\frac{1}{2}\right)^2}=\sqrt{3}$$



Similarly, $AC=\dfrac{\sqrt{3}}{2}, BC=\dfrac{3}{2}$

We can see that $AC^2 + BC^2 = (AB)^2$

$\therefore$ Δ is right angled at C.

$\therefore$ Orthocentre = Vertex $C=\left(2, -\dfrac{1}{2}\right)$

PART B

Straight Line

LECTURE

1

Straight Line 1

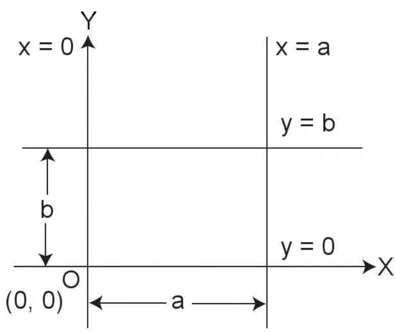
(Some important results connected with one straight line, point-slope form, symmetric form or distance form, two points form, intercept form equation of the straight lines)

BASIC CONCEPTS

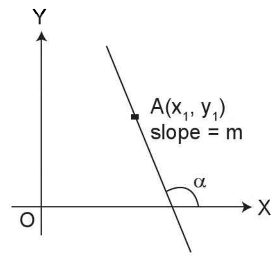
1. **Some Important Results Connected with One Straight Line**
An equation (First degree in x and y) of the form $ax + by + c = 0$ always represents a straight line

Various forms of equation of straight line

1. The equation of x-axis is $y = 0$.
2. The equation of y-axis is $x = 0$.
3. The equation of a straight line parallel to x-axis at a distance b from origin is $y = b$.
4. The equation of a straight line parallel to y-axis at a distance a from origin is $x = a$.



2. **Point Slope Form** The equation of straight line, which has slope m and passes through a fixed point $A(x_1, y_1)$, is:
$y - y_1 = m(x - x_1)$.



$$m = \tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

2.1 The equation of a line through the point (x_1, y_1) and parallel to x-axis is $y = y_1$.

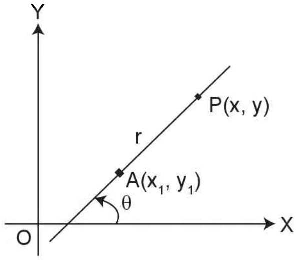
2.2 The equation of a line through the point (x_1, y_1) and parallel to y-axis is $x = x_1$.

3. Any non–vertical line through origin having slope m is $y = mx$

NOTES

Constant term is zero in equation of a line passing through origin.

4. **Parametric form (or symmetric Form or Distance form)**
A straight line passing through a fixed point $A(x_1, y_1)$ and inclined at an angle θ to x-axis may also be given by
$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$ or $x = x_1 + r\cos\theta$,
$y = y_1 + r\sin\theta$



NOTES

(i) 'r' is called the parameter and represents the distance of the variable point $P(x, y)$ from the fixed point $A(x_1, y_1)$.

(ii) 'r' is positive for all points lying on one side of the given point A and negative for all points lying on the other side of the point A.

(iii) The coordinates of any point on such a line may be written as $(x_1 + r\cos\theta, y_1 + r\sin\theta)$, where r is the distance of this point from the given point $A(x_1, y_1)$.

Illustration

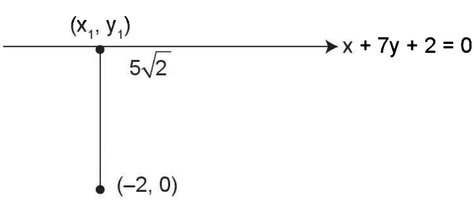
Points on the line $x + 7y + 2 = 0$ at a distance $5\sqrt{2}$ from the point $(-2, 0)$ are

[Gujrat CET-2007]

(a) $(-9, 1)$ and $(5, -1)$
(b) $(12, -2)$ and $(19, -3)$
(c) $(-1, 9)$ and $(1, -5)$
(d) None of these

Solution

(a) The line is $x + 7y + 2 = 0$
Distance from $(-2, 0)$ is $5\sqrt{2}$
Distance between (x_1, y_1) & $(-2, 0)$



$(5\sqrt{2})^2 = (x_1 + 2)^2 + (y_1 - 0)^2$

$25 \times 2 = x_1^2 + 2^2 + 4x_1 + y_1^2$

$\Rightarrow \quad x_1^2 + 4x_1 + 4 + y_1^2 = 50$

$\Rightarrow \quad x_1^2 + 4x_1 + y_1^2 = 46 \quad$ ....(1)

(x_1, y_1) is on $x_1 + 7y_1 + 2 = 0 \quad$ ....(2)

$\Rightarrow \quad x_1 = -2 - 7y_1$

substitute in (1) $(-2 - 7y_1)^2 + 4(-2 - 7y_1) + y_1^2 = 46$

$\Rightarrow \quad 4 + 49y_1^2 + 28y - 8 - 28y_1 + y_1^2 = 46$

$\Rightarrow \quad 50y_1^2 - 4 = 46 \Rightarrow 50y_1^2 = 50$

$\Rightarrow \quad y_1^2 = 1 \Rightarrow y_1 = \pm 1$

$\therefore$ substitute in (2) $x_1 = -2 - 7 = -9$
and $x_1 = -2 + 7 = 5$

$\therefore$ The points are $(5, -1)$ $(-9, +1)$.

5. Slope of any line through (x_1, y_1) and (x_2, y_2) is $\dfrac{y_2 - y_1}{x_2 - x_1}$

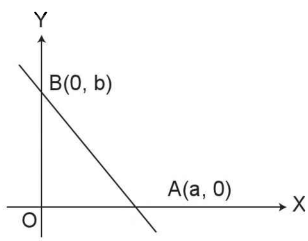
5.1 Two point form The equation of the line through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

NOTES

Through two points one and only straight line can be made to pass.

6. Intercept form The equation of a straight line, which makes a and b, intercepts on x-axis and y-axis respectively is $\dfrac{x}{a} + \dfrac{y}{b} = 1$



6.1 Important points connected with Triangle Δ *OAB* enclosed by this Straight line and co-ordinate axes

(i) This straight line always passes through the fixed points $A(a, 0)$ and $B(0, b)$.

(ii) The length of the interapted part (AB) of the straight line between the coordinate axes $= \sqrt{a^2 + b^2}$.

(iii) Co-ordinates of the mid-point of the intercepted part $AB = \left(\dfrac{a}{2}, \dfrac{b}{2}\right)$

(iv) Area of $\Delta OAB = \dfrac{1}{2}|ab|$

(v) Centroid of $\Delta\, OAB = (a/3, b/3)$

(vi) Circumcentre of $\Delta\, OAB = (a/2, b/2)$

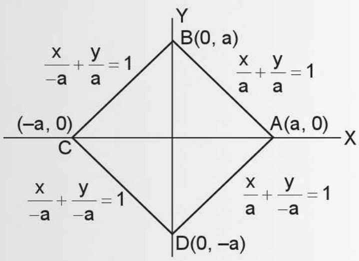
(vii) Orthocentre of $\Delta\, OAB = (0, 0)$

(viii) Incentre of

$$\Delta OAB = \left(\frac{ab}{a+b\sqrt{a^2+b^2}};\frac{ab}{a+b\sqrt{a^2+b^2}}\right)$$

NOTES

1. Algebraic sum of the slopes of the straight lines which are equally inclined with co-ordinate axes is zero i.e., $m_1 + m_2 = 0$.
2. Number of straight line cutting equal intercepts of constant length from co-ordinate axes is 04 Combined equations of such four straight lines is: $\pm x \pm y = a$ or $|x| + |y| = a$. Gradients of such straight line = ± 1.



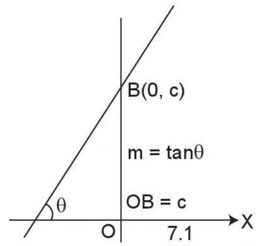
3. Incident ray and reflected ray are equally inclined to the normal to the surface, i.e., algebraic sum of the slopes of these rays is zero i.e., $m_1 + m_2 = 0$ then its area is:

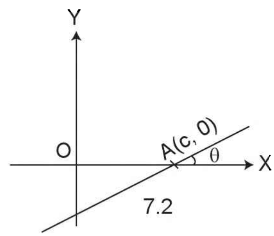
(ix) The intercept of a line between the axes is divided by the point (x_1, y_1) in the ratio $m : n$. Its equation is

$$\frac{x}{\frac{(m+n)x_1}{n}} + \frac{y}{\frac{(m+n)y_1}{m}} = 1$$

7. Slope Intercept Form:

7.1 If m be the slope of a line, which cuts an intercept c on axis of y, then the equation of the line is $y = mx + c$.





7.2 Equation of a line having slope m and cutting intercept c on x-axis is : $y = m(x - c)$.

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. A straight line moves so that the sum of the reciprocals of its intercepts on axes is constant, show that it passes through a fixed point.

Solution

Let the intercepts of the line be a and b such that $\frac{1}{a} + \frac{1}{b} = a$ constant $= \frac{1}{k}$ (say), where k is a fixed real.

$$\Rightarrow \quad \frac{k}{a} + \frac{k}{b} = 1$$

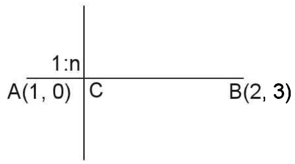
$\Rightarrow$ (k, k) lies on the line $\frac{x}{a} + \frac{y}{b} = 1$

Note that (k, k) is a fixed point.

2. A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio $1 : n$. Find the equation of the line, $(n \neq -1)$ ***[NCERT]***

Solution

The given points are $A(1, 0)$ and $B(2, 3)$. The required line divides the segment $[AB]$ in the ratio $1 : n$ $(n \neq -1)$.



$\therefore$ Dividing point C is

$$\left(\frac{1\times2+n\times1}{1+n},\frac{1\times3+n\times0}{n+1}\right)$$

$$=\left(\frac{2+n}{1+n},\frac{3}{n+1}\right)$$

$\because$ Slope of line $AB=\frac{3-0}{2-1}=3$

$\therefore$ Slope of the required line ($\perp AB$) = –1/3

As this line passes through C, therefore, its equation is $y-\frac{3}{n+1}=-\frac{1}{3}\left(x-\frac{n+2}{n+1}\right)$

Point slope form

or $\frac{1}{3}x+y-\frac{3}{n+1}-\frac{n-2}{3(n+1)}=0$

or $\frac{1}{3}x+y-\left(\frac{9+n+2}{3(n+1)}\right)=0$

or $(n+1)x+3(n+1)y-(n+11)=0$

3. Find equation of the passing through the point (2, 2) and cutting off intercepts on the axes whose sum is 9.

[NCERT]

Solution

Let the intercepts of the line on the axes be a and $9-a$, then its equation is $\frac{x}{a}+\frac{y}{9-a}=1$

Intercept form .... (1)

This line passes through (2, 2) if

$$\frac{2}{a}+\frac{2}{9-a}=1$$

or $18-2a+2a=9a-a^2$

$\Rightarrow$ $a^2-9a+18=0$

$\Rightarrow$ $a=\frac{9\pm\sqrt{81-72}}{2\times1}=\frac{9\pm3}{2}=6,3$

Taking $a=6$, we find that the equation of the line is (from (i)) $\frac{x}{6}+\frac{y}{3}=1$ or $3x+6y-18=0$

Taking $a=3$, we find that the equation of the line is (from (i)) $\frac{x}{3}+\frac{y}{6}=1$ or $6x+3y-18=0$

4. Point $R(h, k)$ divides a line segment between the axes in the ratio 1 : 2. Find equation of the line.

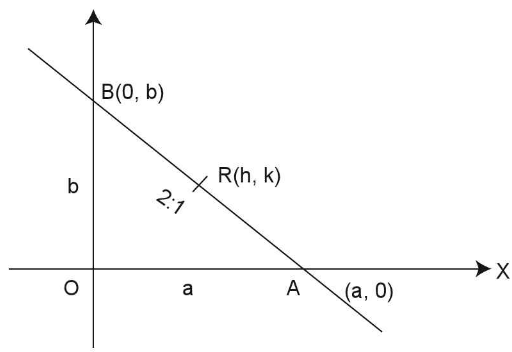
[NCERT]

Solution

Let the intercepts of the line be a and b, then the equation of the line is $\frac{x}{a}+\frac{y}{b}=1$ .... (i)

The line meets x-axis in the points $A(a, 0)$ and $B(0, b)$

It is given that $R(h, k)$ divides $[AB]$ in the ratio 1 : 2



$\therefore$ $(h, k)=\left(\frac{2\times a+1\times0}{1+2},\frac{2\times0+1\times b}{1+2}\right)$

$\Rightarrow$ $h=\frac{2a}{3}$ and $k=\frac{b}{3}$

$\Rightarrow$ $a=\frac{3h}{2}$ and $b=3k$

Substituting these values in (1), we get

$$\frac{x}{\frac{3h}{2}}+\frac{y}{3k}=1$$

or $\frac{2x}{h}+\frac{y}{k}=3$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

EXERCISE 1

1. Find the value of k, if the straight line $2x + 3y + 4 + k(6x - y + 12) = 0$ is perpendicular to the line $7x + 5y - 4 = 0$. *[KVS-2006]*
2. Find the equations of lines which cut off intercepts on the axes whose sum and product are 1 and –6 respectively. *[NCERT]*
3. The extremities of the base of an isosceles triangle are the points $(2a, 0)$ and $(0, a)$ and the equation of one of the equal sides is $x = 2a$. Find the equation of the other equal side.
4. Prove that the line through the point (x_1, y_1) and parallel to the line $Ax + By + C = 0$ is $A(x - x_1) + B(y - y_1) = 0$. *[NCERT]*
5. Find the slope of the line, which makes an angle of 30° with the positive direction of y-axis measured anti-clockwise. *[NCERT]*
6. Write the equations for the x-axes and y- axes. *[NCERT]*
7. Find the equation of line passing through the points (–1, 1) and (2, –4) *[NCERT]*
8. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point (2, 3). *[NCERT]*

EXERCISE 2

1. Given the vertices $A(10, 4)$, $B(-4, 9)$ and $C(-2, -1)$ of $\Delta\, ABC$, find (i) the equation of the side AB (ii) the equation of the median through A.
2. Reduce the equation $3x + 5y - 7 = 0$ to the slope intercept form and find the intercept it makes on y-axis.
3. Reduce the equation $3x - 4y + 12 = 0$ to intercept form. Hence find the length of the portion of the line intercepted between the axes.
4. Find the equation of a line whose intercepts on the axes are thrice as long as those of the line $2x + 3y + 11 = 0$
5. Find the equation of line passing through $(2, 2\sqrt{3})$ and inclined with the x-axis at an angle of 75°. *[NCERT]*
6. Find the equation of line intersecting the x-axis at a distance of 3 units to the left of origin with slope – 2. *[NCERT]*
7. Find the equation of line intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the x-axis. *[NCERT]*
8. $P(a, b)$ is the mid-point of a line segment between axes. Show that equation of the line is $\frac{x}{a} + \frac{y}{b} = 2$. *[NCERT]*
9. Reduce the following equations into slope-intercept form and find their slopes and the y-intercepts, *[NCERT]*
   (i) $x + 7y = 0$
   (ii) $6x + 3y - 5 = 0$
   (iii) $y = 0$
10. Find the equations of the lines which cut off intercepts on the axes whose sum and product are 1 and –6, respectively. *[NCERT]*

ANSWERS

EXERCISE 1

1. $-\frac{29}{37}$
2. $2x - 3y - 6 = 0$, $3x - 2y + 6 = 0$
3. $3x - 4y + 4a = 0$
5. $-\sqrt{3}$
6. $y = 0$ and $x = 0$
7. $5x + 3y + 2 = 0$
8. $x + y = 5$

EXERCISE 2

1. (i) $5x + 14y - 106 = 0$
   (ii) $y - 4 = 0$

2. $y = -\frac{-3}{5}x + \frac{7}{5}$ and $\frac{7}{5}$

3. 5

4. $2x + 3y + 33 = 0$

5. $(2+\sqrt{3})\ x - y - 4 = 0$

6. $2x + y + 6 = 0$

7. $x - \sqrt{3}y + 2\sqrt{3} = 0$

9. (i) The slope of the line $= -1/7$ and its y-intercept $= 0$
   (ii) The slope of the line $= -2$ and its y-intercept $= 5/3$
   (iii) The slope of the line $= 0$ and its y-intercep t $= 0$

10. $2x - 3y = 6, -3x + 2y = 6$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

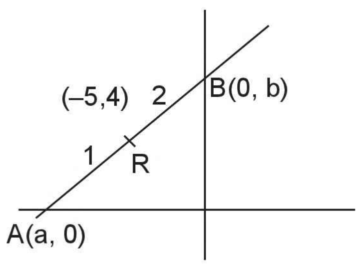
1. The intercept of a line between the coordinate axes is divided by point $(-5, 4)$ in the ratio $1 : 2$. The equation of the line will be

[IIT-1986]

(a) $5x - 8y + 60 = 0$
(b) $8x - 5y + 60 = 0$
(c) $2x - 5y + 30 = 0$
(d) None of these

Solution

(b) Suppose equation of line AB is $\frac{x}{a} + \frac{y}{b} = 1$, then $A(a, 0)$, $B(b, 0)$. $R\,(-5, 4)$ is point on line AB such that it divides AB in the ratio $1 : 2$.



This $\Rightarrow -5 = \frac{2a+0}{1+2}, 4 = \frac{2(0)+1(b)}{1+2}$

$\Rightarrow \quad a = -\frac{15}{2}, b = 12$

Put this in $\frac{x}{a} + \frac{y}{b} = 1$, we get $\frac{x}{-15/2} + \frac{y}{12} = 1$

$\Rightarrow \quad -8x + 5y = 60 \Rightarrow 8x - 5y + 60 = 0$

2. If the coordinates of the points A and B be $(3, 3)$ and $(7, 6)$, then the length of the portion of the line AB intercepted between the axes is

(a) $5/4$
(b) $\sqrt{10}/4$
(c) $\sqrt{13}/3$
(d) None of these

Solution

(a) Equation of line AB is $y - 3 = \frac{6-3}{7-3}(x-3)$

$\Rightarrow \quad 3x - 4y + 3 = 0 \Rightarrow \frac{x}{-1} + \frac{y}{3/4} = 1$

Hence, required length is $\sqrt{(-1)^2 + \left(\frac{3}{4}\right)^2} = \frac{5}{4}$

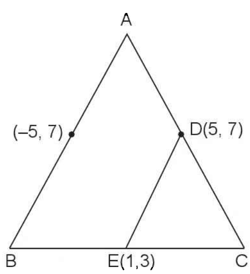
3. If the middle points of the sides BC, CA and AB of the triangle ABC be $(1, 3)$, $(5, 7)$ and $(-5, 7)$, then the equation of the side AB is

(a) $x - y - 2 = 0$
(b) $x - y + 12 = 0$
(c) $x + y - 12 = 0$
(d) None of these

Solution

(b) Slope of $DE = \frac{7-3}{5-1} = 1$

$\Rightarrow$ Slope of $AB = 1$



Hence, equation of AB is

$y - 7 = (x + 5)$

$\Rightarrow \quad x - y + 12 = 0$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The equation of the straight line which passes through the point (1, –2) and cuts off equal intercepts from axes, is ***[MNR-1978]***
(a) $x + y = 1$ (b) $x - y = 1$
(c) $x + y + 1 = 0$ (d) $x - y - 2 = 0$

2. A line passes through the point (3, 4) and cuts off intercepts from the coordinates axes such that their sum is 14. The equation of the line is
(a) $4x - 3y = 24$ (b) $4x + 3y = 24$
(c) $3x - 4y = 24$ (d) $3x + 4y = 24$

3. The equation of the line joining the origin to the point (–4, 5), is ***[MP-PET-1984]***
(a) $5x + 4y = 0$ (b) $3x + 4y = 2$
(c) $5x - 4y = 0$ (d) $4x - 5y = 0$

4. The equation of the line which cuts off an intercept 3 units on *OX* and an intercept –2 unit on *OY*, is ***[MP PET-1984]***
(a) $\frac{x}{3} - \frac{y}{2} = 1$ (b) $\frac{x}{3} + \frac{y}{2} = 1$
(c) $\frac{x}{2} + \frac{y}{3} = 1$ (d) $\frac{x}{2} - \frac{y}{3} = 1$

5. The equation of the line passing through (4, –6) and makes an angle 45° with positive *x*-axis, is ***[RPET-1988]***
(a) $x - y - 10 = 0$ (b) $x - 2y - 16 = 0$
(c) $x - 3y - 22 = 0$ (d) None of these

6. Line passing through (1, 2) and (2, 5) is ***[RPET-1995]***
(a) $3x - y + 1 = 0$ (b) $3x + y + 1 = 0$
(c) $y - 3x + 1 = 0$ (d) $3x + y - 1 = 0$

7. A straight line makes an angle of 135° with *x*-axis and cuts *y*-axis at a distance of –5 from the origin. The equation of the line is ***[MP PET-98; Pb. CET-2001]***
(a) $2x + y + 5 = 0$ (b) $x + 2y + 3 = 0$
(c) $x + y + 5 = 0$ (d) $x + y + 3 = 0$

8. If the line $\frac{x}{a} + \frac{y}{b} = 1$ passes through the points (2, –3) and (4, –5), then (*a, b*) is equal to
(a) (1, 1) (b) (–1, 1)
(c) (1, –1) (d) (–1, –1)

9. Area of a ΔABC = 20 units and its verticess *A* and *B* are (–5, 0) and (3, 0) respectively. If its vertex *C* lines on the line $x - y = 2$, then *C* is
(a) (3, 5) (b) (–3, –5)
(c) (–5, 7) (d) None of these

10. A straight line moves so that the sum of the reciprocals of its intercepts on two perpendicular lines is constant, then the line passes through ***[IIT-1977]***
(a) a fixed point
(b) a variable point
(c) origin
(d) None of these

11. For what values of *a* and *b* the intercepts cut off on the coordinate axes by the line $ax + by + 8 = 0$ are equal in length but opposite in signs to those cut off by the line $2x - 3y + 6 = 0$ on the axes ***[MP PET-1983]***
(a) $a = \frac{8}{3}, b = -4$ (b) $a = -\frac{8}{3}, b = -4$
(c) $a = \frac{8}{3}, b = 4$ (d) $a = -\frac{8}{3}, b = 4$

12. Equation to the straight line cutting off an intercept 2 from the negative direction of the axis of *y* and inclined at 30° to the positive direction of axis of *x*, is ***[MP PET-2003]***
(a) $y + x - \sqrt{3} = 0$
(b) $y - x + 2 = 0$
(c) $y - \sqrt{3}x - 2 = 0$
(d) $\sqrt{3}y - x + 2\sqrt{3} = 0$

13. If the coordinates of the middle point of the portion of a line intercepted between coordinate axes (3, 2), then the equation of the line will be ***[RPET-1985; MP PET-1984]***
(a) $2x + 3y = 12$
(b) $3x + 2y = 12$
(c) $4x - 3y = 6$
(d) $5x - 2y = 10$

14. A straight line through the point $A(3, 4)$ is such that its intercept between the axes is bisected at A. Its equation is

[AIEEE-2006]

(a) $3x - 4y + 7 = 0$ (b) $4x + 3y = 24$
(c) $3x + 4y = 25$ (d) $x + y = 7$

15. A straight line through the point (1, 1) meets the x-axis at 'A' and the y-axis at 'B'. The locus of the mid-point of AB is

[UPSEAT-2004;MPPET-2006]

(a) $2xy + x + y = 0$ (b) $x + y - 2xy = 0$
(c) $x + y + 2 = 0$ (d) $x + y - 2 = 0$

SOLUTIONS

1. (c) Let equation of the line be $x + y = a$. since it passes through (1, –2), so $1 - 2 = a$
$\Rightarrow$ $a = -1$.
Hence, line is $x + y + 1 = 0$.

2. (b) Given $a + b = 14 \Rightarrow a = 14 - b$
Hence the equation of straight line is $\frac{x}{14-b}+\frac{y}{b}=1$
Also, it passes through (3, 4)
$\therefore$ $\frac{3}{14-b}+\frac{4}{b}=1 \Rightarrow b = 8$ or 7

Therefore, equations are $4x + 3y = 24$ and $x + y = 7$
Trick: This question can be checked with the options as the line $4x + 3y = 24$ passes through (3, 4) and also cuts the intercepts from the axes whose sum is 14.

3. (a) $y-0=\left(\frac{5-0}{-4-0}\right)(x-0) \Rightarrow 5x+4y=0$

4. (a) Here, $a = 3$, $b = -2$ (given)

Now $\frac{x}{a}+\frac{y}{b}=1$ gives $\frac{x}{3}-\frac{y}{2}=1$

5. (a) $y - y_1 = m(x - x_1)$
$\Rightarrow$ $y + 6 = (\tan 45°)(x - 4)$
$\Rightarrow$ $y + 6 = x - 4$
$\Rightarrow$ $x - y - 10 = 0$

6. (c) $y-2=\left(\frac{5-2}{2-1}\right)(x-1) \Rightarrow y-2=3(x-1)$
$\Rightarrow$ $y - 3x + 1 = 0$

7. (c) Given Angle made with x axis $\theta = 135°$
Intercept on y-axis $c = -5$ formula used $y = (\tan\theta)x + c$ $y = (\tan 135°)x - 5$ or $y = -x - 5$
Required line $x + y + 5 = 0$

Other Similar concept
If intercept on x-axis is given then $y = \tan\theta\,(x + c)$

8. (d) $\frac{2}{a}-\frac{3}{b}=1$ and $\frac{4}{a}-\frac{5}{b}=1$
$\Rightarrow$ $b = -1, a = -1$.

9. (b) Let the third vertex be (p, q). Since, it lies on the line $x - y = 2$, then $p - q = 2$ ...... (1)
Also area of the triangle $= \pm 20$

$\frac{1}{2}[-5(0-q)+3(q-0)+p(0-0)]=\pm 20$
$5q + 3q = \pm 40$
$\Rightarrow$ $8q = \pm 40$
$\Rightarrow$ $q = \pm 5$
If $q = 5$ then $p = 7$
if $q = -5$ then $p = -3$
$\Rightarrow$ the coordinates of third vertex is either (7, 5) or (–3, –5)

10. (a) Take two perpendicular lines as x and y-axis.

Equation of line AB is $\frac{x}{a}+\frac{y}{b}=1$ .... (1).

Its intercepts are a, b.

Given that $\frac{1}{a}+\frac{1}{b}=\frac{1}{k}=$ const.

or, $\frac{k}{a}+\frac{k}{b}=1$ .... (2)

By comparing (1) and (2), we find that the line (1) always passes through a fixed point (k, k).

11. (d) $ax + by + 8 = 0$

$\Rightarrow$ $\frac{x}{\left(\frac{-8}{a}\right)}+\frac{y}{\left(\frac{-8}{b}\right)}=1$ .... (1)

$$2x - 3y + 6 = 0$$

$$\Rightarrow \quad \frac{2x}{(-6)} - \frac{3y}{(-6)} = 1$$

$$\Rightarrow \quad \frac{x}{(-3)} + \frac{y}{2} = 1 \qquad \text{......... (2)}$$

According to the given condition, by (1) and (2),

$$-\frac{8}{a} = -(-3), -\frac{8}{b} = -2$$

$$\Rightarrow \quad a = -\frac{8}{3}, b = 4$$

12. (d) Given $\theta = 30°$ and $c = -2$ from, $y = \tan\theta\, x + c$, $y = (\tan 30°)\, x - 2$

$$y = \frac{1}{\sqrt{3}}x - 2$$

or $\quad \sqrt{3}y - x + 2\sqrt{3} = 0$

13. (a) Line is $\frac{x}{a} + \frac{y}{b} = 1$.

It meets axes at $A(a, 0)$, $B(0, b)$ and middle point of line AB is $\left(\frac{a}{2}, \frac{b}{2}\right) = (3, 2)$, given

$\Rightarrow \quad a = 6, b = 4.$

Put in $\frac{x}{a} + \frac{y}{b} = 1$, we get $\frac{x}{6} + \frac{y}{4} = 1$

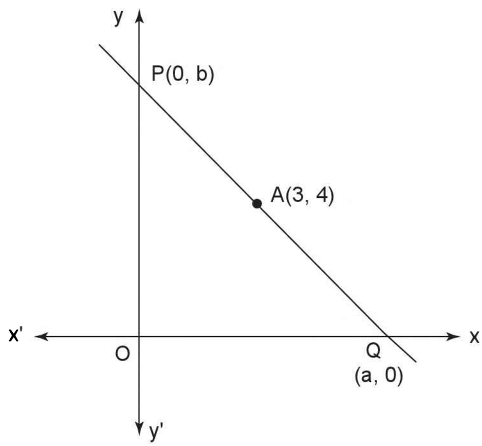
or $\quad 2x + 3y = 12$

14. (b) Since, A is the mid-point of line PQ.

$$\therefore \quad 3 = \frac{a+0}{2} \Rightarrow a = 6$$

and $\quad 4 = \frac{0+b}{2} \Rightarrow b = 8$

Hence, the equation of line is



$$\frac{x}{6} + \frac{y}{8} = 1$$

or $\quad 4x + 3y = 24$

15. (b) A line AB meeting x-axis at $A(a, 0)$ and y-axis at $B(0, b)$ is given by $\frac{x}{a} + \frac{y}{b} = 1$ .... (1)

It passes through (1, 1).

$$\therefore \quad \frac{1}{a} + \frac{1}{b} = 1 \qquad \text{......... (2)}$$

Mid-point $M(x_1, y_1)$ of AB is given by

$$x_1 = \frac{a+0}{2}, y_1 = \frac{0+b}{2}$$

This $\Rightarrow a = 2x_1$, $b = 2y_1$.

Putting this in (2),

$$\frac{1}{2x_1} + \frac{1}{2y_1} = 1$$

$\Rightarrow \quad$ Locus of (x_1, y_1) is $\frac{1}{2x} + \frac{1}{2y} = 1$

$\Rightarrow \quad y + x - 2xy = 0.$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If the intercept made by the line between the axis is bisected at the point (5, 2), then its equation is

[RPET96]

(a) $5x + 2y = 20$ (b) $2x + 5y = 20$
(c) $5x - 2y = 20$ (d) $2x - 5y = 20$

2. The equation of a straight line passing through (–3, 2) and cutting an intercept equal in magnitude but opposite in sign from the axes is given by *[RPET-1984; MP PET-1993]*

(a) $x - y + 5 = 0$ (b) $x + y - 5 = 0$
(c) $x - y - 5 = 0$ (d) $x + y + 5 = 0$

3. The intercept cut off from y-axis is twice that from x-axis by the line and line is passes through (1, 2) then its equation is *[AMU-1972; RPET-1985]*

(a) $2x + y = 4$ (b) $2x + y + 4 = 0$
(c) $2x - y = 4$ (d) $2x - y + 4 = 0$

4. The equation of a straight line passing through the points (–5, –6) and (3, 10), is *[MNR-1974]*

(a) $x - 2y = 4$ (b) $2x - y + 4 = 0$
(c) $2x + y = 4$ (d) None of these

5. A straight line through P (1, 2) is such that its intercept between the axes is bisected at P. Its equation is *[EAMCET-1994]*

(a) $x + 2y = 5$ (b) $x - y + 1 = 0$
(c) $x + y - 3 = 0$ (d) $2x + y - 4 = 0$

6. The line joining the points (–1, 3) and (4, –2) will pass through the point (p, q) if

(a) $p - q = 1$ (b) $p + q = 1$
(c) $p - q = 2$ (d) $p + q = 2$

7. Slope of a line which cuts intercepts of equal lengths on the axes is *[MP PET-1986]*

(a) – 1 (b) 0
(c) 2 (d) $\sqrt{3}$

8. The equation of the line which makes right angled triangle with axes whose area is 6 sq. units and whose hypotenuse is of 5 units, is

(a) $\frac{x}{4} + \frac{y}{3} = \pm 1$ (b) $\frac{x}{4} - \frac{y}{3} = \pm 3$
(c) $\frac{x}{6} + \frac{y}{1} = \pm 1$ (d) $\frac{x}{1} - \frac{y}{6} = \pm 1$

9. The number of straight lines which is equally inclined to both the axes is *[RPET-2002]*

(a) 4 (b) 2
(c) 3 (d) 1

10. The area of the triangle formed by the line $x \sin \alpha + y \cos \alpha = \sin 2\alpha$ and the coordinates axes is

(a) $\sin 2\alpha$ (b) $\cos 2\alpha$
(c) $2 \sin 2\alpha$ (d) $2 \cos 2\alpha$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 10 minutes.
3. The worksheet consists of 10 questions. The maximum marks are 30.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If the portion of a straight line intercepted between the co-ordinate axis is bisected at (2, 2), then the equation of the line is

[SCRA-2007]

(a) $x + y = 4$
(b) $x + 2y = 6$
(c) $2x + y = 6$
(d) $3x - y = 4$

2. The area of triangle formed by the lines $x = 0$, $y = 0$ and $\frac{x}{a}+\frac{y}{b}=1$, is ***[RPET-1984]***

(a) ab
(b) $ab/2$
(c) $2ab$
(d) $ab/3$

3. The equation of the line whose slope is 3 and which cuts off an intercept 3 from the positive x-axis is

(a) $y = 3x - 9$
(b) $y = 3x + 3$
(c) $y = 3x + 9$
(d) None of these

4. Equation of the line which passes through the point (–4, 3) and the portion of the line intercepted between the axes is divided internally in the ratio 5 : 3 by this point, is

[AMU-1973; Dhanbad Engg.-1971]

(a) $9x + 20y + 96 = 0$
(b) $20x + 9y + 96 = 0$
(c) $9x - 20y + 96 = 0$
(d) None of these

5. The equation of the line joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

(a) $y(t_1 + t_2) + 2x = 2at_1t_2$
(b) $y(t_1 + t_2) - 2x = at_1t_2$
(c) $y(t_1 + t_2) - 2x = 2at_1t_2$
(d) None of these

6. The line passes through (1, 0) and $(-2, \sqrt{3})$ makes an angle of ..... with x-axis

(a) 60° (b) 120°
(c) 150° (d) 135°

7. Two points (a, 0) and (0, b) are joined by a straight line, Another point on this line is

[Orissa JEE-2005]

(a) $(3a, -2b)$
(b) (a^2, ab)
(c) $(-3a, 2b)$
(d) (a, b)

8. If the co-ordinates of the point A and B be $(1, 0)$ and $(2, \sqrt{3})$, then the angle made by the line AB with x-axis is

(a) 30° (b) 45°
(c) 60° (d) 75°

9. What does an equation of the first degree containing one arbitrary parameter passing through a fixed point represent?

[NDA–2009]

(a) Circle (b) Straight line
(c) Parabola (d) Ellipse

10. Consider the following statements:

1. The equation to a straight line parallel to the axis of x is $y = d$, where d is a constant.
2. The equation to the axis of x is $x = 0$.

Which of the statements given above is/are correct? ***[NDA–2009]***

(a) 1 only
(b) 2 only
(c) Both 1 and 2
(d) Neither 1 nor 2

ANSWER SHEET

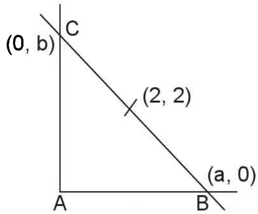
1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. (a) Let equation of straight line be $\frac{x}{a}+\frac{y}{b}=1$

Also $\left(\frac{a+0}{2}, \frac{b+0}{2}\right) \equiv (2, 2)$

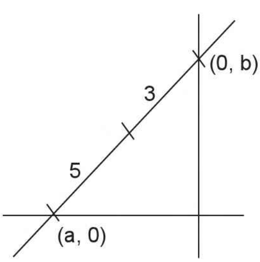
$(a, b) \equiv (4, 4)$



$\therefore$ Equation of straight line

$$\frac{x}{4}+\frac{y}{4}=1 \Rightarrow x+y=4$$

4. (c) Let equation of line be $\frac{x}{a}+\frac{y}{b}=1$



$\therefore \left(\frac{3a}{8}, \frac{5b}{8}\right) \equiv (-4, 3)$

$\Rightarrow (a, b) \equiv \left(\frac{-32}{3}, \frac{24}{5}\right)$

$\therefore$ Equation of straight line $\frac{3x}{-32}+\frac{y}{\frac{24}{5}}=1$

$\Rightarrow -9x + 20y = 96$

7. (a) Clearly, equation of straight line is $\frac{x}{a}+\frac{y}{b}=1$ satisfy each option to get

8. (c) $y-y_1=\left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)$

$\Rightarrow y-0=\left(\frac{\sqrt{3}-0}{2-1}\right)(x-1)$

$\Rightarrow y=\left(\frac{\sqrt{3}}{1}\right)(x-1)$

$\Rightarrow$ slope $=\tan\theta=\sqrt{3}=\tan 60° \Rightarrow \theta=60°$

9. (b) The required equation represents a straight line.

10. (a) We know that, the equation of x-axis is $y = 0$.
Thus, only statement 1 is correct.

LECTURE

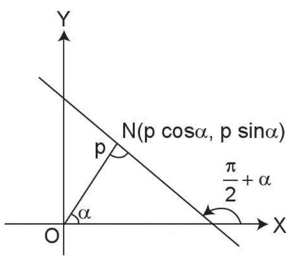
2

Straight Line 2

(Normal form equation of the straight line, the general form equation of the straight line, reduction of the general form into different cases, position of points with respect to the straight line $ax + by + c$ and the perpendicular distance of point from the line $ax + by + c = 0$)

BASIC CONCEPTS

1. **Normal Form of a Straight line** The equation of the straight line on which the length of perpendicular from the origin is p and the angle which this perpendicular makes with the x-axis is α is $x \cos \alpha + y \sin \alpha = p$.



This coordinates of foot of perpendicular N are $(p \cos \alpha, p \sin \alpha)$ and the slope of the line is

$$\tan\left(\frac{\pi}{2}+\alpha\right) = -\cot\alpha,\ 0 \le \alpha \le 360°,\ p > 0$$

NOTE

$\cos \alpha$ and $\sin \alpha$ are direction cosines of the perpendicular drawn from origin on the line.

2. **The General Form of the Equation of a Straight Line**
Thus the general form of the equation of straight line is $ax + by + c = 0$.
Reduction or transformation of general equation of $ax + by + c = 0$ into different cases i.e., if

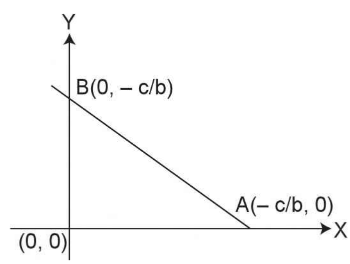
(i) **Slope intercept form:**
If $a \neq 0, b \neq 0$, the equation may be written as $\left(y = -\frac{a}{b}x - \frac{c}{b}\right)$. Thus the slope of the line is $-a/b = -\left(\frac{\text{coefficient of } x}{\text{coefficient of } y}\right)$

(ii) Line is parallel to x-axis then $a = 0$ $\left(-\frac{a}{b} = 0\right)$.

(iii) Line is paralel to y-axis then $b = 0$ $\left(-\frac{a}{b} \to \infty\right)$.

(iv) If the straight line $ax + by + c = 0$ passes through origin the $c = 0$ i.e., constant term = 0.

(v) **Intercept form**
If $a \neq 0$, $b \neq 0$, the equation (i) may be written as $\frac{x}{-\frac{c}{a}} + \frac{y}{-\frac{c}{b}} = 1$



Thus, the x-intercept of the line is $-c/a$ and the y-intercept of the line is $-c/b$.

(vi) **Normal form**

Divide the equation (i) by $\sqrt{a^2+b^2}$, we get,

$$\frac{a}{\sqrt{a^2+b^2}}x+\frac{b}{\sqrt{a^2+b^2}}y=\pm\left(\frac{-c}{\sqrt{a^2+b^2}}\right)$$

out of ± sign we consider that sign which will make right hand side + ve (positive).

3. Equation of straight line passing through point $A(x_1, y_1)$ and (i) Parallel to $ax + by + c = 0$ is

$$y-y_1=-\frac{a}{b}(x-x_1)$$

or $a(x-x_1)+b(y-y_1)=0$

or $ax + by = ax_1 + by_1$

or $ax + by + k = 0$

NOTE

In equation of parallel straight lines only constant terms are different.

(ii) Perpendicular to $ax + by + c = 0$ is

$$y-y_1=\frac{b}{a}(x-x_1)$$

or $b(x-x_1)-a(y-y_1)=0$

or $bx - ay = bx_1 - ay_1$

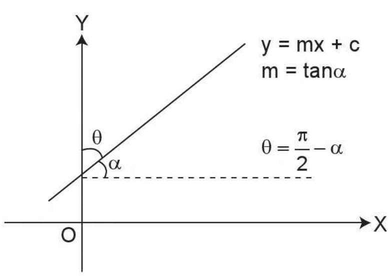
or $bx - ay + k = 0$

Here, k is constant whose value is obtained by given conditions.

(iii) Equation of the perpendicular bisector of a line segment joining points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$y-\frac{y_1+y_2}{2}=-\left(\frac{x_2-x_1}{y_2-y_1}\right)\left(x-\frac{x_1+x_2}{2}\right).$$

4. Angle of a straight line $y = mx + c$ with y-axis or a line parallel to y-axis may be given by $\left|\tan^{-1}\frac{1}{m}\right|$.



5. Any line parallel to a given straight line $ax + by + c = 0$ is given by $ax + by + k = 0$, where k is to be determined from given conditions.

Example

$3x + 4y + 5 = 0$, $3x + 4y + 1 = 0$,
$6x + 8y + 3 = 0$

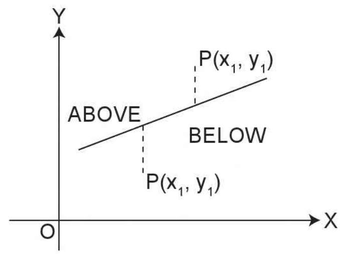
6. Any line perpendicular to a given straight line $ax + by + c = 0$ is given by $bx - ay + k = 0$, where k is to be determined from given conditions.

Example

$3x + 4y + 5 = 0$ and $4x - 3y + 5 = 0$
or $4x - 3y + 1 = 0$

7. Position of a Given Point (x_1, y_1) Relative to a Given Line $ax + by + c = 0$

A point $P(x_1, y_1)$ will lie above or below this line according as $\frac{ax_1+by_1+c}{b}$ is positive or negative.



8. Position of a Point with Respect to a Line

A point (x_1, y_1) lies on the **origin side** of the line $ax + by + c = 0$ if $ax_1 + by_1 + c$ and c have the **same sign** and (x_1, y_1) lies on the **opposite side of the origin** if $ax_1 + by_1 + c$ and c have the opposite signs.

9. The ratio λ, in which a line $ax + by + c = 0$ divides the line joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$\lambda=-\left(\frac{ax_1+by_1+c}{ax_2+by_2+c}\right).$$ λ is positive or negative according as division is internal or external.

10. Relative positions of the points (x_1, y_1) and (x_2, y_2) w.r.t. the line $ax + by + c = 0$ The points (x_1, y_1) and (x_2, y_2) lies on the same or opposite side of the line $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same sign or opposite signs.

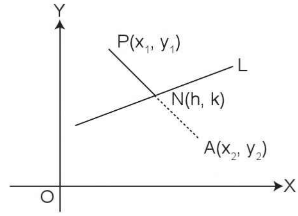
10.1 (x_1, y_1) lies in the acute angle between the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$:

$$\frac{a_1a_2 + b_1b_2}{(a_1x_1 + b_1y_1 + c_1)\,(a_2x_1 + b_2y_1 + c_2)} < 0$$

(x_1, y_1) lies in the obtuse angle:

$$\frac{a_1a_2 + b_1b_2}{(a_1x_1 + b_1y_1 + c_1)(a_2x_1 + b_2y_1 + c_2)} > 0$$

11. The perpendicular distance of the point $P(x_1, y_1)$ from the given line $ax + by + c = 0$ is



$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find equation of the line through the point (0, 2) making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also, find the equation of line parallel to it and crossing the y-axis at a distance of 2 units below the origin.

[NCERT]

Solution

Slope of the line $= \tan\frac{2\pi}{3} = \tan 120°$

$= \tan(180° - 60°) = -\tan 60° = -\sqrt{3}$

$\therefore$ The equation of the line through (0, 2) and with slope $-\sqrt{3}$ is $y - 2 = -\sqrt{3}(x - 0)$

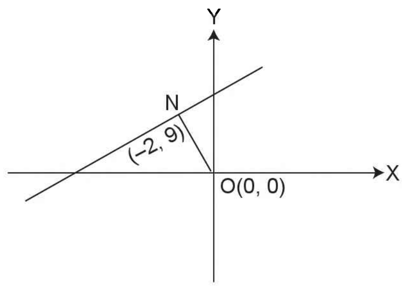
Point slope from or $\sqrt{3}\ x + y - 2 = 0$

Also the line parallel to this line has the slope $-\sqrt{3}$ and if it passes through (0, –2), then its equation is $y - (-2) = -\sqrt{3}\ (x - 0)$ Point slope from or $\sqrt{3}\ x + y + 2 = 0$

2. The perpendicular from the origin from the origin to a line meets it at the point (–2, 9), find the equation of the line.

Solution

Let N be the foot of perpendicular from $O(0, 0)$ upon the line in reference.



Now slope of $ON = \frac{9 - 0}{-2 - 0} = -\frac{9}{2}$

Slope of the required line = 2/9

Hence, its equation is $y - 9 = \frac{2}{9}\ (x - (-2))$

Point slope from

or $\quad 9y - 81 = 2x + 4$ or $2x - 9y + 85 = 0$

3. The length L (in centimeters) of a copper rod is a linear function of its Celsius temperature C. In an experiment if $L = 124.942$ when $C = 20$ and $L = 125.134$ when $C = 110$, express L in terms of C.

Solution

Since L, is a linear function of C, therefore, we can take $L = \text{m}\ C + b$

Compare it with $y = mx + b$ .... (i) when $C = 20, L = 124.942$,

$\therefore \quad 124.942 = 20\,m + b$ .... (ii)

and when $C = 110, L = 125.134$

$\therefore \quad 125.134 = 110\,m + b$ .... (iii)

Subtracting (ii) from (iii), we get

$125.134 - 124.942 = 90m$

$\Rightarrow \quad 0.192 = 90m$

$\Rightarrow \quad m = \dfrac{0.192}{90} = \dfrac{19.2}{90000} = 0.00213$ ... (iv)

substituting this value of m in (ii), we get

$124.942 = 20 \times 0.00713 + b$

$\Rightarrow \quad b = 124.942 - 0.0426 = 124.8994$

Substituting $m = 0.00213$ and $b = 124.8994$ in (i) we find that the required relation between L and C is

$L = 0.00213\,C + 124.8994$

4. The owner of a milk store finds that, he can sell 980 liters of milk each week at Rs 14/ litre and 1220 liters of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many liters could he sell weekly at Rs 17/litre?

[NCERT]

Solution

Let the milk store owner sell L litre of milk at Rs p per litre and let the linear relation ship between L and p be

$L = mp + b$ .... (i)

when $p = 14, L = 980$

$\therefore \quad 980 = 14m + b$ .... (ii)

when $p = 16, L = 1220$

$\therefore \quad 1220 = 16m + b$ .... (iii)

Subtracting (ii) from (iii), we get $240 = 2m$

$\Rightarrow \quad m = \dfrac{240}{2} = 120.$

Substituting this value of m in (ii), we get

$980 = 14 \times 120 + b \Rightarrow 980 - 1680 = -700$

Substituting $m = 120$ and $b = -700$ in (i), we obtain $L = 120\,p - 700$.

when $p = 17$, then $L = 120 \times 17 - 700 = 2040 - 700 = 1340$

Hence, the man can sell 1340 litres of milk at Rs 17 per litre.

5. By using the concept of equation of a line, prove that the three points (3, 0), (–2, –2) and (8, 2) are collinear.

[NCERT]

Solution

The given points are $A(3, 0)$, $B(-2, -2)$ and $C(8, 2)$.

Using two point form, we find that the equation of the line AB as $y - 0 = \dfrac{-2-0}{-2-3}(x-3)$

or $\quad y = \dfrac{2}{5}(x-3)$

or $\quad 5y = 2x - 6$

or $\quad 2x - 5y - 6 = 0$

The point C (8, 2) lies on this line if $2 \times 8 - 5 \times 2 - 6 = 0$

i.e., if $16 - 16 = 0$ which is true

Hence, the three given points are collinear and lie on the line whose equation is $2x - 5y - 6 = 0$.

UNSOLVED SUBJECTIVE PROBLEMS (XI BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

EXERCISE 1

1. Find the equation of a line that cuts off equal intercepts on the co-ordinate axes and passes through the point (2, 3).

[KVS-2004, NCERT]

2. Find the equation of the line passing through (–3, 5) and perpendicular to the line through the points (2, 5) and (–3, 6). ***[KVS-2005]***

3. A straight line passes through the point (2, 3) and its segment intercepted between the axes is bisected at that point. Find its equation.

[MSE-2006, NCERT]

4. Reduce the general equation of a line $ax + by + c = 0$ in tangent form and, hence, find the slope of the line and the y-intercept.

5. Find the equation of a line which is perpendicular to the line $4x - y + 8 = 0$ and passes through the mid-point of the line segment joining (1, 5) and (3, 11).

6. Find equation of the line passing through the point (2, 1) and perpendicular to the line $2x - 4y + 7 = 0$.

7. Find the distance of the line $4x + 7y + 5 = 0$ from the point (1, 2) along the line $2x - y = 0$. ***[NCERT]***

8. Find the equation of the straight line joining the point (a, b) to the point of intersection of the lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$
Also prove that sum of intercepts made by this line on co-ordinate axes is $\frac{(b-a)(a^2-b^2)}{ab}$.

9. Find the angle between the x-axis and the line joining the points (3, –1) and (4, –2) ***[NCERT]***

10. If three points (h, 0), (a, b) and $(0, k)$ lie on a line, show that $\frac{a}{h}+\frac{b}{k}=1$. ***[NCERT]***

11. Reduce the following equations into intercept form and find their intercepts on the axes. ***[NCERT]***
    (i) $3x + 2y - 12 = 0$
    (ii) $4x - 3y = 6$
    (iii) $3y + 2 = 0$

12. Find the points on the x-axis, whose distances from the line $\frac{x}{3}+\frac{y}{4}=1$ are 4 units. ***[NCERT]***

13. Prove that the line through the point (x_1, y_1) and parallel to the line $Ax + By + C = 0$ is $A(x - x_1) + B(y - y_1) = 0$. ***[NCERT]***

EXERCISE 2

1. Find the equation of the line for which $p = 2$ and $\sin \alpha = 4/5$, where p and α have usual meaning.

2. In what ratio is the line segment joining (1, 3) and (2, 7) divided by $3x + y - 9 = 0$?

3. Prove that the straight line $ax + by + c = 0$ divides the line segment joining (x_1, y_1) and (x_2, y_2) in the ratio $-\frac{ax_1+by_1+c}{ax_2+by_2+c}$
Hence, show that the points (x_1, y_1) and (x_2, y_2) lie on the same side or on opposite sides of the line $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of same sign or of opposite signs.

4. Find it equation of line whose perpendicular distance from the origin in 5 units and the angle made by the perpendicular with the positive x-axis is 30°. ***[NCERT]***

5. Reduce the following equations into normal form. Find their perpendicular distances from the origin and angle between perpendicular and the positive x-axis. ***[NCERT]***
    (i) $x - \sqrt{3}\,y + 8 = 0$ (ii) $y - 2 = 0$

6. Find equation of the line parallel to the line $3x - 4y + 2 = 0$ and passing through the point (–2, 3). ***[NCERT]***

7. Find equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x-intercept 3. ***[NCERT]***

8. Find the values of k for which the line $(k - 3)\,x - (4 - k^2)\,y + k^2 - 7k + 6 = 0$ is
    (a) parallel to the x-axis
    (b) parallel to the y-axis
    (c) passing through the origin ***[NCERT]***

9. Find the values of θ and p, if the equation $x \cos\theta + y \sin\theta = p$ is the normal form of the line $\sqrt{3}\,x + y + 2 = 0$. ***[NCERT]***

10. Find the equation of line passing through the point (–4, 3) with slope 1/2. ***[NCERT]***

11. (i) Show that (1, 2) and (–3, –2) lie on the same side of $2x - 3y + 5 = 0$.
    (ii) Show that the points (3, 7) and (–3, –1) are on the same side of the straight line $3x - 8y = 7$.

12. Find the distance of the point (–1, 1) from the line $12(x + 6) = 5(y - 2)$. ***[NCERT]***

13. Find the equation of the right bisector of the line segment joining the points (3, 4) and (–1, 2). ***[NCERT]***

14. In what ratio, the line joining (–1, 1) and (5, 7) is divided by the line $x + y = 4$? ***[NCERT]***

15. What are the points on y-axis whose distance from the line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units. ***[NCERT]***

ANSWERS

EXERCISE 1

1. $x + y - 5 = 0$
2. $5x - y + 20 = 0$
3. $3x + 2y - 12 = 0$
4. $y=-\frac{a}{b}x-\frac{c}{b}$; slope $= -\frac{a}{b}$; y-intercept $= -\frac{c}{b}$
5. $x + 4y - 34 = 0$
6. $2x + y = 5$
7. $\frac{23\sqrt{5}}{18}$
8. $a^2y - b^2x = ab(a - b)$
9. 135°
10. (i) $x \cos 120° + y \sin 120° = 4$, 4, 120°
    (ii) $x \cos 90° + y \sin 90° = 2$, 2, 90°
11. (i) $\frac{x}{4}+\frac{y}{6}=1, 4, 6$
    (ii) $\frac{x}{3/2}+\frac{y}{-2}=1, \frac{3}{2}, -2$
    (iii) $y=-\frac{2}{3}$, intercept with y-axis $= -\frac{2}{3}$ and no intercept with x-axis.
12. (–2, 0) and (8, 0)

EXERCISE 2

1. $3x + 4y = 10$ or $3x - 4y + 10 = 0$
2. 3 : 4 internally
4. $\sqrt{3}x + y - 10 = 0$
5. (i) $x \cos 120° + y \sin 120° = 4$, 4, 120°
   (ii) $x \cos 90° + y \sin 90° = 2$, 2, 90°
6. $3x - 4y + 18 = 0$
7. $y + 7x = 21$
8. (a) 3
   (b) ± 2
   (c) 6 or 1
9. $7\pi/6$, 1
10. $x - 2y + 10 = 0$
12. 5 units
13. $2x + y = 5$
14. 1 : 2
15. (0, –8/3) and (0, 32/3)

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The equation of the line passing through $(-a, 0)$ which makes with the axes a triangle of area T is ***[PET (Raj.)-1987]***

(a) $2Tx + a^2y + 2aT = 0$
(b) $2Tx - a^2y + 2aT = 0$
(c) $2Tx - a^2y - 2aT = 0$
(d) None of these

Solution

(a) Any line passing through $(-a, 0)$ is $y = m(x + a)$ .... (1)

$\Rightarrow \quad -mx + y = ma$

$\Rightarrow \quad -\frac{x}{a}+\frac{y}{ma}=1$

Since it makes with axes a triangle of area T,

so $\frac{1}{2}(-a)(ma) = T$

$\Rightarrow \quad m = -2T/a^2$

Putting it in (1) we get the required equation as $2T(x + a) + a^2y = 0$

$\Rightarrow \quad 2Tx + a^2y + 2aT = 0$

2. $A(2, -3)$ and $B(-2, 1)$ are vertices of ΔABC. If its centroid lies on the line $2x + 3y = 1$, then locus of the vertex C is

(a) $3x + 2y = 5$
(b) $2x - 3y = 7$
(c) $2x + 3y = 9$
(d) $3x - 2y = 3$

[AIEEE-2004]

Solution

(c) Let $C \equiv (x, y)$ and centroid $G = (h, k)$. Then $h = \frac{2-2+x}{3} = \frac{x}{3}$ and $k = \frac{-3+1+y}{3} = \frac{y-2}{3}$

Since, G lies on $2x + 3y = 1$, so $2h + 3k = 1$

$\Rightarrow \quad 2\left(\frac{x}{3}\right)+3\left(\frac{y-2}{3}\right)=1$

$\Rightarrow \quad 2x + 3y = 9$

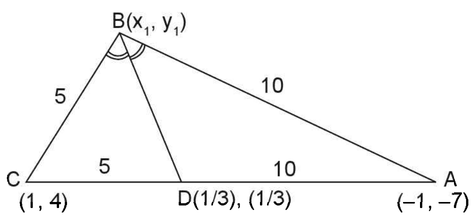
which is the required locus of C.

3. The vertices of a triangle are $A(-1, -7)$, $B(5, 1)$ and $C(1, 4)$.
The equation of the bisector of the angle $\angle ABC$ is ...... ***[IIT-1993]***

(a) $x - 7y + 3 = 0$
(b) $x - 7y + 2 = 0$
(c) $x + 2y + 3 = 0$
(d) $x - 2y + 3 = 0$

Solution

(b) $x - 7y + 2 = 0$. $AB = 10$, $BC = 5$. The bisector of $\angle ABC$ will divide the opposite side AC in the ratio of arms of the angle.
i.e., 5 : 10. Point D by ratio formula is (1/3, 1/3).



$\therefore$ Equation of BD by two point formula is $x - 7y + 2 = 0$.

4. The equation to the straight line passing through the points $(a \cos^3\theta, a \sin^3\theta)$ and perpendicular to the line $x \sec\theta + y \operatorname{cosec}\theta = a$ is

(a) $x \cos\theta - y \sin\theta = a \cos 2\theta$
(b) $x \cos\theta + y \sin\theta = a \cos 2\theta$
(c) $x \sin\theta + y \cos\theta = a \cos 2\theta$
(d) $x \sin\theta + y \cos\theta = -a \cos 2\theta$

Solution

(b) Slope of given line $= -\frac{\sec\theta}{\operatorname{cosec}\theta} = -\frac{\sin\theta}{\cos\theta}$

$\therefore$ Slope of a line perpendicular to it will be $\frac{\sin\theta}{\cos\theta}$

The line passes through the point $(a \cos^3\theta, a \sin^3\theta)$ and hence its equation is $(y - a \sin^3\theta) = (\cos\theta/\sin\theta)(x - a\cos^3\theta)$ or $x \cos\theta - y\sin\theta = a(\cos^4\theta - \sin^4\theta) = a \cos 2\theta$.

$\therefore$ $\cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta) = (\cos 2\theta).\ 1 = \cos 2\theta$.

5. The point $P(a, b)$ lies on the straight line $3x + 2y = 13$ and the point $Q(b, a)$ lies on the straight line $4x - y = 5$, then the equation of line PQ is ***[MP PET-1999]***

(a) $x - y = 5$
(b) $x + y = 5$
(c) $x + y = -5$
(d) $x - y = -5$

Solution

(b) Point $P(a, b)$ is on $3x + 2y = 13$

So, $3a + 2b = 13$ ..... (i)

Point $Q(b, a)$ is on $4x - y = 5$

So, $4b - a = 5$ ..... (ii)

By solving (i) and (ii), $a = 3$, $b = 2$

$P(a, b) \rightarrow (3, 2)$ and $Q(b, a) \rightarrow (2, 3)$

Now equation of PQ

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$\Rightarrow \quad y - 2 = \frac{3-2}{2-3}(x-3)$

$\Rightarrow \quad y - 2 = -(x - 3)$

$\Rightarrow \quad x + y = 5$

6. The line passing through $(1, \pi/2)$ and perpendicular to $\sqrt{3}\sin\theta + 2\cos\theta = 4/r$ is ***[EAMCET-2003]***

(a) $2 = \sqrt{3}r\cos\theta - 2r\sin\theta$
(b) $5 = -2\sqrt{3}r\sin\theta + 4r\cos\theta$
(c) $2 = \sqrt{3}r\cos\theta + 2r\cos\theta$
(d) $5 = 2\sqrt{3}r\sin\theta + 4r\cos\theta$

Solution

(a) Perpendicular to $\sqrt{3}\sin\theta + 2\cos\theta = \frac{4}{r}$ is

$$\sqrt{3}\sin\left(\frac{\pi}{2}+\theta\right)+2\cos\left(\frac{\pi}{2}+\theta\right)=\frac{k}{r}$$

It is passing through $(-1, \pi/2)$;

$$\sqrt{3}\sin\pi + 2\cos\pi = \frac{k}{-1} \Rightarrow k = 2$$

$$\therefore \quad \sqrt{3}\cos\theta - 2\sin\theta = \frac{2}{r}$$

$$\Rightarrow \quad 2 = \sqrt{3}r\cos\theta - 2r\sin\theta.$$

7. If the equation of base of an equilateral triangle is $2x - y = 1$ and the vertex is $(-1, 2)$, then the length of the side of the triangle is

[Kerala (Engg.)-2005]

(a) $\sqrt{\frac{20}{3}}$ (b) $\frac{2}{\sqrt{15}}$

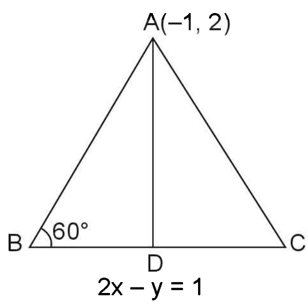
(c) $\sqrt{\frac{8}{15}}$ (d) $\sqrt{\frac{15}{2}}$

Solution

$$\text{(a) } AD = \left|\frac{-2-2-1}{\sqrt{(2)^2+(-1)^2}}\right| = \left|\frac{-5}{\sqrt{5}}\right| = \sqrt{5}$$

$$\because \quad \tan 60° = \frac{AD}{BD} \Rightarrow \sqrt{3} = \frac{\sqrt{5}}{BD}$$

$$\Rightarrow BD = \sqrt{\frac{5}{3}}$$



$$\therefore \quad BC = 2BD = 2\sqrt{\frac{5}{3}} = \sqrt{\frac{50}{3}}$$

8. If (a, a^2) falls inside the angle made by the lines $y = x/2$, $x > 0$ and $y = 3x$, $x > 0$; then a belongs to

[AIEEE-2006]

(a) $(3, \infty)$ (b) $(1/2, 3)$

(c) $(-3, -1/2)$ (d) $(0, 1/2)$

Solution

(b) Obviously (1, 1) lies inside the angle made by given lines. So (1, 1) and (a, a^2) must lie on the same side of two lines. Hence for $y = x/2$

$1 - 1/2 > 0 \Rightarrow a^2 - a/2 > 0$

$\Rightarrow \quad a(2a - 1) > 0$

$\Rightarrow \quad a < 0 \text{ or } a > \frac{1}{2}$ .... (1)

For $y = 3x$

$1 - 3 < 0 \Rightarrow a^2 - 3a < 0$

$\Rightarrow \quad a(a - 3) < 0$

$\Rightarrow \quad 0 < a < 3$ .... (2)

(1) and (2) hold together when $a \in \left(\frac{1}{2}, 3\right)$

9. The perpendicular bisector of the line segment joining $P(1, 4)$ and $Q(k, 3)$ has y-intercept-4. Then a possible value of k is

[AIEEE-2008]

(a) –4 (b) 1

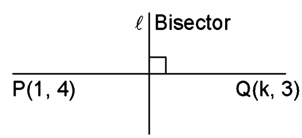
(c) 2 (d) –2

Solution

(a) The slope of

$$l = -\frac{1}{\text{the slope of the original line PQ}}$$

$$= -\frac{1}{\frac{3-4}{k-1}} = (k-1)$$



$$\text{The mid-point} = \left(\frac{k+1}{2}, \frac{7}{2}\right)$$

The equation to the bisector l is

$$\left(y - \frac{7}{2}\right) = (k-1)\left(x - \frac{k+1}{2}\right)$$

As $x = 0$, $y = -4$ satisfies it, we have

$$\left(-4 - \frac{7}{2}\right) = (k-1)\left(0 - \frac{k+1}{2}\right)$$

$$\Rightarrow \quad -\frac{15}{2} = -\frac{k^2-1}{2}$$

$\Rightarrow \quad k^2 - 1 = 15 \Rightarrow k^2 = 16$

$\therefore \quad k = \pm 4$

10. The equation of straight line equally inclined to the axis and equidistant from the points (1, –2) and (3, 4) is $ax + by + c = 0$ where ***[Orissa JEE-2008]***

(a) $a = 1, b = 1, c = 1$
(b) $a = 1, b = -1, c = -1$
(c) $a = 1, b = 1, c = 2$
(d) None of these

Solution

(b) Slope of given line $ax + by + c = 0$ is $-\dfrac{a}{b}$

$\therefore\ -\dfrac{a}{b} = \pm 1 \Rightarrow a = \pm b$ .... (i)

distance of line $ax + by + c = 0$ from (1, –2) $= \dfrac{|a-2b+c|}{\sqrt{a^2+b^2}}$ distance of line $ax + by + c = 0$ from (3, 4) $= \dfrac{|3a+4b+c|}{\sqrt{a^2+b^2}}$ according to given problem

$$= \frac{|a-2b+c|}{\sqrt{a^2+b^2}} = \frac{|3a+4b+c|}{\sqrt{a^2+b^2}}$$

$\Rightarrow\ 3a + 4b + c = \pm (a - 2b + c)$

$\Rightarrow\ a + 3b = 0$ .... (ii) if taking positive

$2a + b + c = 0$ .... (iii) if taking negative

From (i) and (ii) we get $a = b = 0$ which is not possible so taking (i) and (iii), we get (taking $a = -b$ from (i))

$a + c = 0 \Rightarrow c = -a$

$a : b : c = a : -a : -a = 1 : -1 : -1$

or $a = 1, b = -1, c = -1$

If $2a + b + c = 0$ .... (iii)

$a = b$ .... (i) taking positive

on adding $3a + c = 0 \Rightarrow c = -3a$

$a : b : c = a : a : -3a = 1 : 1 : -3$

from option $a = 1, b = -1, c = -1$ is correct.

11. The locus of the orthocentre of the triangle formed by the lines $(1 + p)x - py + p(1 + p) = 0$, $(1 + q)x - qy + q(1 + q) = 0$ and $y = 0$, where $p \neq q$ is ***[IIT-2009]***

(a) a hyperbola (b) a parabola
(c) an ellipse (d) a straight line

Solution

(d) Intersection point of $y = 0$ with first line is $B(-p, 0)$

Intersection point of $y = 0$ with second line is $A(-q, 0)$

Intersection point of the two lines is $C(pq, (p+1)(q + 1))$

Altitude from C to AB is $x = pq$

Altitude from B to AC is $y = -\dfrac{q}{1+q}(x+p)$

Solving these two we get $x = pq$ and $y = -pq$

$\therefore$ locus of orthocentre is $x + y = 0$, which is a straight line

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If the coordinates of A and B be (1,1) and (5, 7), then the equation of the perpendicular bisector of the line segment AB is

(a) $2x + 3y = 18$ (b) $2x - 3y + 18 = 0$
(c) $2x + 3y - 1 = 0$
(d) $3x - 2y + 1 = 0$

2. The equation of a line through (3, –4) and perpendicular to the line $3x + 4y = 5$ is ***[RPET-1981, 84, 86; MP PET-1984]***

(a) $4x + 3y = 24$
(b) $y - 4 = (x + 3)$
(c) $3y - 4x = 24$
(d) $y + 4 = 4/3\ (x - 3)$

3. Equation of the line passing through (1, 2) and parallel to the line $y = 3x - 1$ is ***[MP PET-1984]***

(a) $y + 2 = x + 1$
(b) $y + 2 = 3(x + 1)$
(c) $y - 2 = 3(x - 1)$
(d) $y - 2 = x - 1$

4. For specifying a straight line how many geometrical parameters should be known ***[MP PET-1982]***

(a) 1 (b) 2
(c) 4 (d) 3

5. Equation of a straight line on which length of perpendicular from the origin is four units and the line makes an angle of 120° with the x-axis, is *[MNR-1986]*
   (a) $x\sqrt{3}+y+8=0$
   (b) $x\sqrt{3}-y=8$
   (c) $x\sqrt{3}+y=8$
   (d) $x-\sqrt{3}y+8=0$

6. If we reduce $3x + 3y + 7 = 0$ to the form $x \cos \alpha + y \sin \alpha = p$, then the value of p is *[MP PET-2001]*
   (a) $\frac{7}{2\sqrt{3}}$ (b) $\frac{7}{3}$
   (c) $\frac{3\sqrt{7}}{2}$ (d) $\frac{7}{3\sqrt{2}}$

7. The equation of the lines on which the perpendiculars from the origin make 30° angle with x-axis and which form a triangle of area $\frac{50}{\sqrt{3}}$ with axes, are
   (a) $x+\sqrt{3}y\pm10=0$ (b) $\sqrt{3}x+y\pm10=0$
   (c) $x\pm\sqrt{3}y-10=0$ (d) None of these

8. A line L is perpendicular to the line $5x - y = 1$ and the area of the triangle formed by the line L and coordinate axes is 5. The equation of the line L is *[IIT-1980; RPET-1997]*
   (a) $x + 5y = 5$
   (b) $x+5y=\pm5\sqrt{2}$
   (c) $x - 5y = 5$
   (d) $x-5y=5\sqrt{2}$

9. The equation of the line perpendicular to the line $\frac{x}{a}-\frac{y}{b}=1$ and passing through the point at which it cuts x-axis, is *[RPET-1996; Kerala (Engg.)-2002]*
   (a) $\frac{x}{a}+\frac{y}{b}+\frac{a}{b}=0$ (b) $\frac{x}{b}+\frac{y}{d}=\frac{b}{a}$
   (c) $\frac{x}{b}+\frac{y}{a}=0$ (d) $\frac{x}{b}+\frac{y}{a}=\frac{a}{b}$

10. The equation of line perpendicular to $x = c$ is *[RPET-2001]*
    (a) $y = d$ (b) $x = d$
    (c) $x = 0$ (d) None of these

11. A line AB makes zero intercepts on x-axis and y-axis and it is perpendicular to another line CD, $3x + 4y + 6 = 0$. The equation of line AB is *[Karnataka CET-2001]*
    (a) $y = 4$ (b) $4x - 3y + 8 = 0$
    (c) $4x - 3y = 0$ (d) $4x - 3y + 6 = 0$

12. Equation of a line through the origin and perpendicular to the line joining $(a, 0)$ and $(-a, 0)$, is *[MPPET-84]*
    (a) $y = 0$ (b) $x = 0$
    (c) $x = -a$ (d) $y = -a$

13. If a, b, c are in $A.P.$, then the fixed point through which the straight line $ax + 2by + c = 0$ will always pass, is
    (a) $(1, -2)$ (b) $(-1, 1)$
    (c) $(1, -1)$ (d) $(-1, 2)$

14. For which value of α from the following given values slope of the line $x \cos \alpha + y \sin \alpha = P$ is $\sqrt{3}$. $(P \neq 0)$ *[Gujrat CET-2007]*
    (a) $5\pi/6$ (b) $-5\pi/6$
    (c) $\pi/3$ (d) $-\pi/3$

15. The points on the x-axis whose perpendicular distance from the line $\frac{x}{a}+\frac{y}{b}=1$ is a, are *[RPET-2001; MP PET-2003]*
    (a) $\left[\frac{a}{b}(b\pm\sqrt{a^2+b^2}),0\right]$
    (b) $\left[\frac{b}{a}(b\pm\sqrt{a^2+b^2}),0\right]$
    (c) $\left[\frac{a}{b}(a\pm\sqrt{a^2+b^2}),0\right]$
    (d) None of these

16. The point on the line $x + y = 4$ which lie at a unit distance from the line $4x + 3y = 10$, are *[IIT-1976]*
    (a) (3, 1), (–7, 11) (b) (3,1), (7, 11)
    (c) (–3,1), (–7,11) (d) (1,3), (–7, 11)

17. The length of perpendicular from (3, 1) on line $4x + 3y + 20 = 0$, is *[RPET-1989; MP PET-1984]*
    (a) 6 (b) 7
    (c) 5 (d) 8

18. The length of perpendicular from the point $(a \cos \alpha, a \sin \alpha)$ upon the straight line $y = x \tan \alpha + c$, $c > 0$ is *[MP PET-2004]*

(a) $c \cos \alpha$ (b) $c \sin^2 \alpha$
(c) $c \sec^2 \alpha$ (d) $c \cos^2 \alpha$

19. A point equidistant from the lines $4x + 3y + 10 = 0$, $5x + 12y + 26 = 0$ and $7x + 24y - 50 = 0$ is ***[EAMCET-1994]***
(a) $(1, -1)$ (b) $(1, 1)$
(c) $(0, 0)$ (d) $(0, 1)$

20. Which pair of points lie on the same side of $3x - 8y - 7 = 0$ ***[Roorkee-1990]***
(a) $(0, -1)$ and $(0, 0)$
(b) $(4, -3)$ and $(0,1)$
(c) $(-3, -4)$ and $(1, 2)$
(d) $(-1, -1)$ and $(3, 7)$

21. A point moves such that its distance from the point $(4, 0)$ is half that of its distance from the line $x = 16$. The locus of this point is ***[AMU-1980]***
(a) $3x^2 + 4y^2 = 192$ (b) $4x^2 + 3y^2 = 192$
(c) $x^2 + y^2 = 192$ (d) None of these

22. If p and p' be the distance of origin from the lines $x \sec \alpha + y \operatorname{cosec} \alpha = k$ and $x \cos \alpha - y \sin \alpha = k \cos 2\alpha$, then $4p^2 + p'^2$ is equal to
(a) k (b) $2k$
(c) k^2 (d) $2k^2$

23. $(\sin \theta, \cos \theta)$ and $(3, 2)$ lies on the same side of the line $x + y = 1$, then θ lies between ***[DCE-2005]***
(a) $(0, \pi/2)$ (b) $(0, \pi)$
(c) $(\pi/4, \pi/2)$ (d) $(0, \pi/4)$

24. The equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is $(2, -1)$. The length of the side of the triangle is ***[IIT-1973, 83; MP PET-1995; RPET-1999, 2000]***
(a) $\sqrt{3/2}$ (b) $\sqrt{2}$
(c) $\sqrt{2/3}$ (d) None of these

25. If the slope of a line passing through the point $A(3, 2)$ be $3/4$, then the points on the line which are 5 units away from A, are ***[IIT-1965]***
(a) $(5, 5), (-1, -1)$
(b) $(7, 5), (-1, -1)$
(c) $(5, 7), (-1, -1)$
(d) $(7, 5), (1, 1)$

26. The equation of the line bisecting perpendicularly the segment joining the points $(-4, 6)$ and $(8, 8)$ is ***[Karnataka CET-03]***
(a) $6x + y - 19 = 0$
(b) $y = 7$
(c) $6x + 2y - 19 = 0$
(d) $x + 2y - 7 = 0$

27. Equation of the line passing through the point $(1, 2)$ and perpendicular to $3x + 4y + 5 = 0$ is ***[MPPET-2009]***
(a) $3y = 4x - 2$ (b) $3y = 4x + 2$
(c) $3y = 4x + 3$ (d) $3y = 4x - 3$

28. The angle between the lines $x + 2y = 11$ and $2x - y = 9$ is ***[MPPET-2009]***
(a) 30° (b) 45°
(c) 60° (d) 90°

29. If $(0, \beta)$ lies on or inside the triangle with sides $y + 3x + 2 = 0$, $3y - 2x - 5 = 0$ and $4y + x - 14 = 0$, then ***[MPPET-2010]***
(a) $0 \le \beta \le \frac{7}{2}$ (b) $0 \le \beta \le \frac{5}{2}$
(c) $\frac{5}{3} \le \beta \le \frac{7}{2}$ (d) None of these

SOLUTIONS

1. (a) Mid-point is $(3, 4)$

Slope of perpendicular $= \frac{-1}{6/4} = \frac{-2}{3}$

Hence the line is $2x + 3y = 18$.

2. (d) $3x + 4y = 5 \Rightarrow$ slope $= -\frac{3}{4}$

$\Rightarrow$ slope of a line perpendicular to it is $\frac{4}{3}$

Equation of such line through $(3, -4)$ is

$y + 4 = \frac{4}{3}(x - 3)$.

3. (c) Required line is $y - 2 = 3(x - 1)$

4. (b) $y = mx + c$, two parameters m and c.

5. (a) Slope of given line is

$m = \tan(120°) = \tan(\pi - 60°) = -\sqrt{3}$

Equation of this line is $y = mx + c$

$\Rightarrow \quad -x\sqrt{3} - y + c = 0$

$\Rightarrow \quad x\sqrt{3} + y - c = 0$ .... (1)

Length of perpendicular from (0, 0) on (1) is 4.

$\therefore \quad \left|\frac{0-c}{\sqrt{3+1}}\right| = 4$

$\Rightarrow \quad |c| = 8 \Rightarrow c = \pm 8$

Put this in (1), we get $x\sqrt{3} + y \pm 8 = 0$, one value of it is $x\sqrt{3} + y + 8 = 0$.

6. (d) On changing $3x + 3y = 7$ into $x \cos\alpha + y \sin\beta = P$ from we get

$$\frac{3x}{\sqrt{3^2+3^2}} + \frac{3y}{\sqrt{3^2+3^2}} = \frac{7}{\sqrt{3^2+3^2}}$$

$$\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \frac{7}{3\sqrt{2}}$$

7. (b) Let the length of the perpendicular from origin on the given line be p. Then its equation in normal form is

$x \cos 30° + y \sin 30° = p$

$\Rightarrow \quad \sqrt{3}x + y = 2p$

It meets x-axis at $A\left(\frac{2p}{\sqrt{3}}, 0\right)$ and y-axis at $B(0, 2p)$

Given $\Delta\, OAB = \frac{50}{\sqrt{3}}$

$\Rightarrow \quad \frac{1}{2}\left(\frac{2p}{\sqrt{3}}\right)(2p) = \frac{50}{\sqrt{3}}$

$\Rightarrow \quad p^2 = 25 \Rightarrow p = \pm 5$

Hence, required lines are $\sqrt{3}x + y \pm 10 = 0$

8. (b) Any line $\perp$ to $5x - y = 1$ is $x + 5y = c$ .... (1)

This is line L.

$1 \Rightarrow \frac{x}{c} + \frac{y}{c/5} = 1$

$\Rightarrow \quad A(c, 0), B\left(0, \frac{c}{5}\right)$

$\Rightarrow \quad$ Area of $\Delta\, OAB = \frac{1}{2}c\left(\frac{c}{5}\right) = 5$ (given)

$\Rightarrow \quad c^2 = 50$

$\Rightarrow c = \pm 5\sqrt{2}$.

Put this in (1), $x + 5y = \pm 5\sqrt{2}$.

9. (d) $\frac{x}{a} - \frac{y}{b} = 1$ .... (1)

its slope $= \frac{b}{a}$, it cuts x-axis at $A(a, 0)$.

Equation of line perpendicular to (1) and through $A(a, 0)$ is

$$y - 0 = -\frac{a}{b}(x - a)$$

$\Rightarrow \quad by + ax = a^2$, dividing by ab, we get

$$\frac{x}{b} + \frac{y}{a} = \frac{a}{b}$$

10. (a) It is obvious.

11. (c) Obviously line passes through (0, 0) and perpendicular to $3x + 4y + 6 = 0$.
so its equation will be $4x - 3y = 0$.

12. (b) Slope of line joining $(a, 0)$ to $(-a, 0)$ is

$$m_1 = \frac{0-0}{a+a} = 0, m_1 = 0 = \tan\theta \Rightarrow \theta = 0.$$

$\therefore$ Any line perpendicular to it is parallel to y-axis but it passes through (0, 0) and so its equation is $x = 0$ (y-axis).

13. (c) $\because 2b = a + c$, so equation of the line is

$ax + (a + c)y + c = 0$

$\Rightarrow \quad a(x + y) + c(y + 1) = 0$

This always passes through the point of intersection of $x + y = 0$ and $y + 1 = 0$ which is $(1, -1)$.

14. (a) Slope $= -\frac{\cos\alpha}{\sin\alpha}$

$\Rightarrow \quad \sqrt{3} = -\cot\alpha$

$\Rightarrow \quad \cot\alpha = -\sqrt{3} = -\cot\frac{\pi}{6}$

$= \cot\left(\pi - \frac{\pi}{6}\right)$

$\Rightarrow \quad \alpha = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

15. (c) Point situated at X-axis → $(x_1, 0)$
According to question distance from

$\frac{x}{a}+\frac{y}{b}-1=0$ is 'a'

$$\therefore \quad \frac{\frac{x_1}{a}+0-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=\pm a$$

$$\frac{x_1}{a}-1=\frac{\pm a\sqrt{a^2+b^2}}{ab}$$

$$x_1=a\pm\frac{a}{b}\sqrt{a^2+b^2}$$

∴ required points $\left\{a\pm\frac{a}{b}\sqrt{a^2+b^2},0\right\}$

16. (a) Any point P on line $x + y = 4$ is $P\,(a, 4 - a)$ ...... (1)
Length of ⊥ from on line $4x + 3y - 10 = 0$ is one.

This $\Rightarrow \frac{4a+3(4-a)-10}{\sqrt{4^2+3^2}}=1\Rightarrow a+2=\pm 5$

⇒ $a = 3, a = -7$.

Put this in (1), required points are (3, 1), (–7, 11).

17. (b) Length of perpendicular

$$=\frac{4(3)+3(1)+20}{\sqrt{(4^2+3^2)}}=\frac{35}{3}=7$$

18. (c) Point is $(a\cos\alpha, a\sin\alpha)$
line is $x\tan\alpha - y + c = 0$
∴ $x\sin\alpha - y\cos\alpha + c\cos\alpha = 0$
Length of perpendicular

$$=\frac{a\cos\alpha\sin\alpha-a\sin\alpha\cos\alpha+c\cos\alpha}{\sqrt{\tan^2\alpha+1}}$$

$$=\frac{c\cos\alpha}{\sec\alpha}=c\cos^2\alpha$$

19. (c) Evidently the length of perpendicular from (0, 0) on the given lines are each equal to 2.

20. (d) Line is $3x - 8y - 7 = 0$
$L(-1, -1) = -3 + 8 - 7 < 0$
$L(3, 7) = 9 - 56 - 7 < 0$
Since both have same signs w.r.t. the line. Hence, they lie on the same side.

21. (a) $\sqrt{(h-4)^2+k^2}=\frac{1}{2}\left(\frac{h-16}{\sqrt{1^2+0}}\right)$

Replace (h, k) by (x, y),
we get $3x^2 + 4y^2 = 192$.

22. (c) Here $p=\left|\frac{-k}{\sqrt{\sec^2\alpha+\text{cosec}^2\alpha}}\right|$,

$$p'=\left|\frac{-k\cos 2\alpha}{\sqrt{\cos^2\alpha+\sin^2\alpha}}\right|$$

Hence $4p^2+p'^2=\frac{4k^2}{\sec^2\alpha+\text{cosec}^2\alpha}$

$$+\frac{k^2(\cos^2\alpha-\sin^2\alpha)^2}{1}$$

$= 4k^2\sin^2\alpha\cos^2\alpha + k^2(\cos^4\alpha + \sin^4\alpha) - 2k^2\cos^2\alpha\sin^2\alpha$
$= k^2(\sin^2\alpha + \cos^2\alpha)^2 = k^2$

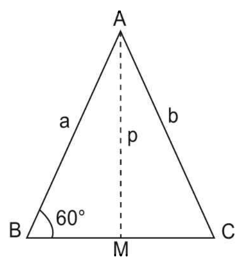
23. (d) $\sin\theta + \cos\theta - 1$ and $3 + 2 - 1$ must be of the same sign
So $\sin\theta + \cos\theta - 1 > 0$
⇒ $\sin\theta + \cos\theta > 1$

⇒ $\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)>1$

⇒ $\sin\left(\theta+\frac{\pi}{4}\right)>\frac{1}{\sqrt{2}}$

⇒ $\frac{\pi}{4}<\theta+\frac{\pi}{4}<\frac{\pi}{2}\Rightarrow 0<\theta<\frac{\pi}{4}$

24. (c) Length of perpendicular from vertex (2, –1) to base $x + y - 2 = 0$ of triangle is



$$p=AM=\left|\frac{2-1-2}{\sqrt{1^2+1^2}}\right|=\frac{1}{\sqrt{2}}$$

from ΔABM $\frac{AB}{AM}$ = cosec 60°

or $AB = AM$ cosec 60°

length of side $a=\frac{1}{\sqrt{2}}\times\frac{2}{\sqrt{3}}=\sqrt{\frac{2}{3}}$

25. (b) $m = \tan\theta = \frac{3}{4}$, Equation of line through A with slope m is

$y - 2 = m(x - 3)$

$\Rightarrow \quad \frac{x-3}{\cos\theta} = \frac{y-2}{\sin\theta} = r = \pm 5$

as $AP = r = 5 \Rightarrow x = \pm 5 \cos$

$\Rightarrow \quad + 3 = \pm 5\left(\frac{4}{5}\right) + 3 = 7, -1$

and $y = r\sin\theta + 2 = \pm 5\left(\frac{3}{5}\right) + 2 = 5, -1$

$\therefore \quad P(x, y) = P(7, 5), P(-1, -1)$.

26. (a) Equation of the line passing through (–4, 6) and (8, 8) is

$$y - 6 = \left(\frac{8-6}{8+4}\right)(x+4)$$

$\Rightarrow \quad y - 6 = \frac{2}{12}(x+4)$

$\Rightarrow 6y - 36 = x + 4$

$\Rightarrow 6y - x - 40 = 0$ .... (i)

Now equation of any line perpendicular to it is

$6x + y + \lambda = 0$ .... (ii)

The line passes through the mid-point of (–4, 6) and (8, 8)

i.e., $(2, 7) \Rightarrow 6 \times 2 + 7 + \lambda = 0$

$\Rightarrow \quad 19 + \lambda = 0 \Rightarrow \lambda = -19$

From (ii) the equation of required line is $6x + y - 19 = 0$.

27. (b) Let equation of line perpendicular to $3x + 4y + 5 = 0$ is $4x - 3y + \lambda = 0$.

It passes through the point (1, 2).

$\therefore \quad 4 - 6 + \lambda = 0 \Rightarrow \lambda = 2$

$\therefore$ Required equation of line is $4x - 3y + 2 = 0$.

30. (d) Given lines are $x + 2y = 11$ and $2x - y = 9$. Whose slopes are

$m_1 = -\frac{1}{2}$ and $m_2 = 2$

$\therefore \quad \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{2 + \frac{1}{2}}{1 + \left(-\frac{1}{2}\right) \times 2}$

$\Rightarrow \quad \tan\theta = \frac{5/2}{0} = \infty$

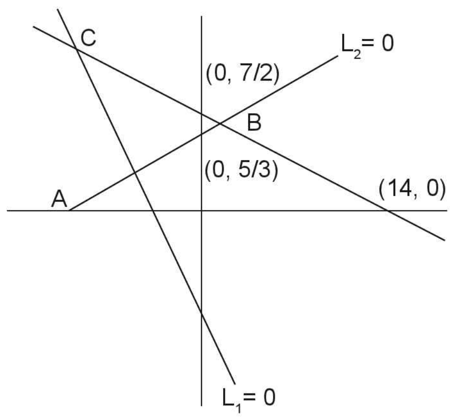
$\Rightarrow \quad \theta = 90°$

29. (c) $L_1 \equiv y + 3x + 2 = 0$

or $\quad \frac{x}{\frac{-2}{3}} + \frac{y}{-2} = 1$

$L_2 \equiv 3y - 2x - 5 = 0$

or $\quad \frac{x}{\frac{-5}{2}} + \frac{y}{\frac{5}{3}} = 1$



$L_3 \equiv 4y + x - 14 = 0$

or $\quad \frac{x}{14} + \frac{y}{\frac{7}{2}} = 1$

$\because \quad (0, \beta)$ lies inside ΔABC

$\therefore \quad \beta \in \left(\frac{5}{3}, \frac{7}{2}\right)$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The equation of line passing through (c, d) and parallel to $ax + by + c = 0$, is ***[RPET-1987]***

(a) $a(x + c) + b(y + d) = 0$
(b) $a(x + c) - b(y + d) = 0$
(c) $a(x - c) + b(y - d) = 0$
(d) None of these

2. A line passing through (2, 2) and is perpendicular to the line $3x + y = 3$. Its y-intercept is

(a) 1/3 (b) 2/3
(c) 1 (d) 4/3

3. The number of lines that are parallel to $2x + 6y + 7 = 0$ and have an intercept of length 10 between the coordinate axes is

(a) 1 (b) 2
(c) 4 (d) Infinitely many

4. The equation of the line passing through the point (1, 2) and perpendicular to the line $x + y + 1 = 0$ is ***[MNR-1981]***

(a) $y - x + 1 = 0$ (b) $y - x - 1 = 0$
(c) $y - x + 2 = 0$ (d) $y - x - 2 = 0$

5. Equation of the perpendicular bisector of the line segment joining the points (7, 4) and (–1, –2), is ***[AMU-1979]***

(a) $4x - 3y = 15$ (b) $3x + 4y = 15$
(c) $4x + 3y = 15$ (d) None of these

6. Equation of a line passing through (1, –2) and perpendicular to the line $3x - 5y + 7 = 0$ is ***[RPET-2003]***

(a) $5x + 3y + 1 = 0$ (b) $3x + 5y + 1 = 0$
(c) $5x - 3y - 1 = 0$ (d) $3x - 5y + 1 = 0$

7. A line perpendicular to the line $ax + by + c = 0$ and passes through (a, b) the equation of the line is ***[RPET-1984; RPET-90; MP PET-1991]***

(a) $bx - ay + (a^2 - b^2) = 0$
(b) $bx - ay - (a^2 - b^2) = 0$
(c) $bx - ay = 0$
(d) None of these

8. The equation of the line passing through (1, 1) and parallel to the line $2x + 3y - 7 = 0$ is ***[RPET-1996]***

(a) $2x + 3y - 5 = 0$
(b) $3x + 2y - 5 = 0$
(c) $3x - 2y - 7 = 0$
(d) $2x + 3y + 5 = 0$

9. If the length of the perpendicular drawn from the origin to the line whose intercepts on the axes are a and b be p, then ***[Karnataka CET-2003]***

(a) $a^2 + b^2 = p^2$ (b) $a^2 + b^2 = \frac{1}{p^2}$
(c) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{2}{p^2}$ (d) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$

10. The perpendicular distance of the straight line $12x + 5y = 7$ from the origin is given by ***[Pb. CET-2002; MP PET-1993]***

(a) 7/13 (b) 12/13
(c) 5/13 (d) 1/13

11. The length of the perpendicular drawn from origin upon the straight line $\frac{x}{3} - \frac{y}{4} = 1$ is ***[MP PET-1997]***

(a) $2\frac{2}{5}$ (b) $3\frac{1}{5}$
(c) $4\frac{2}{5}$ (d) $3\frac{2}{5}$

12. The position of the points (3, 4) and (2, –6) with respect to the line $3x - 4y = 8$ are ***[Roorkee-1972; MP PET-1984]***

(a) On the same side of the line
(b) On different side of the line
(c) One point on the line and the other outside the line
(d) Both points on the Line

13. A point moves so that square of its distance from the point (3, –2) is numerically equal to its distance from the line $5x - 12y = 13$. The equation of the locus of the point is ***[Roorkee-1974]***

(a) $13x^2 + 13y^2 - 83x + 64y + 182 = 0$
(b) $x^2 + y^2 - 11x + 16y + 26 = 0$
(c) $x^2 + y^2 - 11x + 16y = 0$
(d) None of these

14. If $2p$ is the length of the perpendicular from the origin to the lines $\frac{x}{a}+\frac{y}{b}=1$, then a^2, $8p^2$, b^2 are in

(a) A.P. (b) G.P.
(c) H.P. (d) None of these

15. If points (1, 2) and (3, 4) are on opposite sides of the line $3x - 5y + 9 = 0$, then

(a) $7 < a < 12$ (b) $7 < a < 11$
(c) $7 < a > 11$ (d) $7 > a > 11$

16. Which pair of points lie on the same side of $3x - 8y - 7 = 0$

(a) (0, –1) and (0, 0)
(b) (4, –3) and (0, 1)
(c) (–3, –4) and (1, 2)
(d) (–1, –1) and (3, 7)

17. If p be the length of the perpendicular from the origin on the straight line $x + 2by = 2p$, then what is the value of b? ***[NDA-2007]***

(a) 1/2 (b) $\sqrt{3}/2$
(c) $1/p$ (d) p

18. In what ratio the line $y - x + 2 = 0$ divides the line joining the points (3, –1) and (8, 9) ***[Karnataka CET-2002]***

(a) 1 : 2 (b) 2 : 1
(c) 2 : 3 (d) 3 : 4

19. A variable line $\frac{x}{a}+\frac{y}{b}=1$ is such that $a + b = 4$. The locus of the mid-point of the portion of the line intercepted between the axes is ***[Karnataka CET-2008]***

(a) $x + y = 1$ (b) $x + y = 2$
(c) $x + y = 4$ (d) $x + y = 8$

20. A straight line is passing through the points represented by the complex numbers $a + ib$ and $\frac{1}{-a+ib}$, where $(a, b) \neq (0,0)$. Which one of the following is correct? ***[NDA-2008]***

(a) It passes through the origin
(b) It is parallel to the x-axis
(c) It is parallel to the y-axis
(d) It passes through $(0, b)$

21. What is the area of the triangle formed by the lines $y - x = 0$, $y + x = 0$, $x = c$? ***[NDA-2009]***

(a) $c/2$ (b) c^2
(c) $2c^2$ (d) $c^2/2$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 17 minutes.
3. The worksheet consists of 17 questions. The maximum marks are 51.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The equation of the line passes through (a, b) and parallel to the line $\frac{x}{a}+\frac{y}{b}=1$ is **[RPET-1986, 95]**

(a) $\frac{x}{a}+\frac{y}{b}=3$ (b) $\frac{x}{a}+\frac{y}{b}=2$

(c) $\frac{x}{a}+\frac{y}{b}=0$ (d) $\frac{x}{a}+\frac{y}{b}+2=0$

2. The area of the triangle bounded by the straight line $ax + by + c = 0$, $(a, b, c \neq 0)$ and the coordinate axes is **[AMU-2000]**

(a) $\frac{1}{2}\frac{a^2}{|bc|}$ (b) $\frac{1}{2}\frac{c^2}{|ab|}$

(c) $\frac{1}{2}\frac{b^2}{|ac|}$ (d) 0

3. The equation of the straight line passing through (4, 5) and parallel to the line $2x - 3y = 5$, is

(a) $2x - 3y = 7$ (b) $2x - 3y + 7 = 0$

(c) $3x - 2y = 2$ (d) $3x + 2y - 22 = 0$

4. The equation of the line which is such that the portion of line segment intercepted between the coordinate axes is bisected at (4, –3) is **[Keral PET-2007]**

(a) $3x + 4y = 24$ (b) $3x - 4y = 12$

(c) $3x - 4y = 24$ (d) $4x - 3y = 24$

5. If the straight line $ax + by + c = 0$ always passes through (1, –2), then a, b, c are **[AMU-2000]**

(a) In A.P. (b) In H.P.

(c) In G.P. (d) None of these

6. The equation of the straight line passing through the point (3, 2) and perpendicular to the line $y = x$ is **[MNR-1979; MPPET-2002]**

(a) $x - y = 5$ (b) $x + y = 5$

(c) $x + y = 1$ (d) $x - y = 1$

7. The length of perpendicular from a point (1, 2) to the straight line $3x + 4y + 4 = 0$ is **[MPPET-07]**

(a) 5 (b) 3

(c) 7/5 (d) None of these

8. In what ratio does the line $y - x + 2 = 0$ cut the line joining (3, –1) and (8, 9)? **[NDA-2007]**

(a) 3 : – 2 (b) 1 : 2

(c) 2 : 3 (d) 3 : 2

9. The vertices of a triangle ABC are (–1, –2), (0, 1) and (2, 0) respectively. The equation of the median through (–1, –2) is

(a) $5x - 4y - 3 = 0$ (b) $x + 2y - 2 = 0$

(c) $4x + y - 1 = 0$ (d) None of these

10. The equation of the straight line whose intercepts on the axes are twice the intercepts of the straight line $3x + 4y = 6$ on the axes is

(a) $3x + 44y = 3$ (b) $3x + 4y = 12$

(c) $6x + 8y = 9$ (d) None of these

11. The equations of the straight lines which pass through the origin and trisect the intercepts of line $3x + 4y = 12$ between the axes are

(a) $x - 2y = 0$ and $x - 6y = 0$

(b) $2x - 3y = 0$ and $2x - 7y = 0$

(c) $3x + 8y = 0$ and $3x + 2y = 0$

(d) $3x - 8y = 0$ and $3x - 2y = 0$

12. The equation of the straight line cutting off an intercept 8 from x-axis and making an angle of 60° with the positive direction of y-axis is

(a) $x + \sqrt{3}y = 8$ (b) $x - \sqrt{3}y = 8$

(c) $y - \sqrt{3}x = 8$ (d) None of these

13. The equation of the straight line which pass through the point (3, 4) and whose intercept on x-axis is twice that on y-axis, is

(a) $2x - y = 10$ (b) $x + 2y = 10$

(c) $2x + y = 10$ (d) None of these

14. The equation of the straight line whose intercept on x-axis and y-axis are respectively twice and thrice of those by the line $3x + 4y = 12$, is

(a) $9x + 8y = 72$ (b) $9x - 8y = 7$
(c) $8x + 9y = 72$ (d) None of these

15. The equation of the straight line upon which the length of perpendicular from origin is $3\sqrt{2}$ units and this perpendicular makes an angle of 75° with the positive direction of x-axis, is

(a) $(\sqrt{3} - 1)x + (\sqrt{3} + 1)y - 12 = 0$
(b) $(\sqrt{3} - 1)x + (\sqrt{3} + 1)y + 12 = 0$
(c) $(\sqrt{3} + 1)x + (\sqrt{3} - 1)y - 12 = 0$
(d) None of these

16. The coordinates of the points at a distance $4\sqrt{2}$ units from the point (–2, 3) in the direction making an angle of 45° with the positive direction of x-axis are

(a) (2, 7), (–6, –1) (b) (2, 7), (6, –1)
(c) (2, –7), (–6, –1) (d) None of these

17. If $x \cos\theta + y \sin\theta = 2$ is perpendicular to the line $x - y = 3$, then what is one of the values of θ? ***[NDA-2009]***

(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/2$ (d) $\pi/3$

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**16.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ	**17.** ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

3. (b) Straight line passes through (4, 5) and parallel to the line $2x - 3y = 5$

$\therefore$ equation is $y - 5 = \frac{2}{3}(x - 4)$

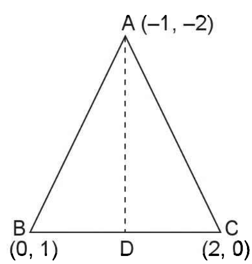
$2x - 3y + 7 = 0$

8. (c) $\lambda = -\left(\dfrac{ax_1 + by_1 + c}{ax_2 + bx_2 + c}\right)$

$$\lambda = -\left(\frac{-1-3+2}{9-8+2}\right) = \frac{2}{3}$$

9. (a) Median through A passes through mid-point of BC.

Let $D = \left(\dfrac{2+0}{2}, \dfrac{1+0}{2}\right) \equiv (1, 1/2)$



$\therefore$ Equation of median AD

$$y-\frac{1}{2}=\frac{\frac{1}{2}+2}{1+1}(x-1)$$

$\Rightarrow \quad y-\frac{1}{2}=\frac{5}{4}(x-1)$

$4y-2=5x-5$

$\Rightarrow \quad 5x-4y-3=0$

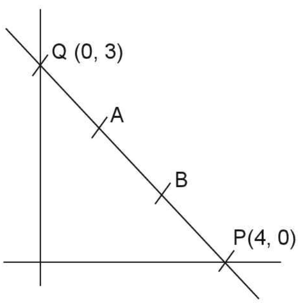
11. (d) Given line $3x+4y=12$

or $\quad \frac{x}{4}+\frac{y}{3}=1$

Point of trisection are A and B

$$\frac{BP}{BQ}=\frac{1}{2}$$

$\therefore \quad B\equiv\left(\frac{8}{3},1\right)$



Similarly, $A=\left(\frac{4}{3},2\right)$

Line passing through O, A is $y=\frac{3}{2}x$

Line passing through O, B is

$$y=\frac{3}{8}x\Rightarrow 3x-8y=0$$

12. (b) Line passes through (8, 0) and slope

$$=\tan(90°-60°)=\frac{1}{\sqrt{3}}$$

($\because$ angle made with the x axis = 90°– 60°)

Equation of line $y-0=\frac{1}{\sqrt{3}}(x-8)$

$\Rightarrow \quad x-\sqrt{3}y-8=0$

17. (b) Since, slope of line $x\cos\theta+y\sin\theta=2$ is $-\cot\theta$ and slope of line $x-y=3$ is 1. Also, these lines are perpendicular to each other

$\therefore \quad (-\cot\theta)(1)=-1$

$\Rightarrow \quad \cot\theta=1=\cot\frac{\pi}{4}$

$\Rightarrow \quad \theta=\frac{\pi}{4}$

LECTURE

3

Straight Line 3

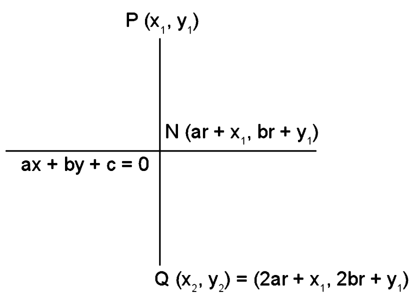
(Foot of perpendicular, reflection point or image, some important results connected with two straight lines, angle between two straight lines)

BASIC CONCEPTS

1. Co-ordinates of the foot of perpendicular (N) and image or reflection of the point (x_1, y):

1.1 Coordinates of **foot of perpendicular** $N(h, k)$ from the point $P(x_1, y_1)$ on the line $ax + by + c = 0$ are given by $(ar + x_1, br + y_1)$ where $r = -\left(\dfrac{ax_1 + by_1 + c}{a^2 + b^2}\right)$

1.2 Coordinates of the **image** $Q(x_2, y_2)$ of the point $P(x_1, y_1)$ in the line mirror $ax + by + c = 0$ are given by $(2ar + x_1, 2br + y_1)$ where $r = -\left(\dfrac{ax_1 + by_1 + c}{a^2 + b^2}\right)$



NOTE

Straight line $ax + by + c = 0$ is perpendicular bisector of a line segment joining P and Q i.e., point and its image with respect to the line.

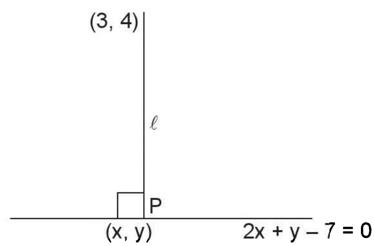
***Illustration*:**

The co-ordinates of the foot of the perpendicular drawn from the point (3, 4) on the line $2x + y - 7 = 0$ is **[Karnataka CET-2007]**

(a) (9/5, 17/5) (b) (1, 5)
(c) (–5, 1) (d) (1, –5)

Solution

(a) Let P be (x, y) and $2x + y - 7 = 0$ .....(1)
As l is perpendicular on $2x + y - 7 = 0$ and slope of $2x + y - 7 = 0$ is –2
($\therefore y = -2x + 7$) therefore, slope of line l is $\dfrac{-1}{-2} = \dfrac{1}{2}$ $[\because m_1 m_2 = -1]$



Equation of line l is $(y - 4) = \dfrac{1}{2}(x - 3)$

$2y - 8 = x - 3 \Rightarrow x - 2y = -5$ ....(ii)

Solving (i) and (ii)

$2x + y = 7$ and $2x - 4y = -10$

$\therefore \quad 5y = 17 \Rightarrow y = 17/5$

Substitute the value of y in x

we get, $2x + \left(\frac{17}{5}\right) = 7$

$\Rightarrow \quad 2x = 7 - \frac{17}{5} \Rightarrow 2x = \frac{35-17}{5}$

$2x = \frac{18}{5}$

$\therefore \quad x = \frac{18}{5\times 2} = \frac{9}{5}, P = \left(\frac{9}{5}, \frac{17}{5}\right)$.

2. Some Important Results Connected with Two Straight Lines

$a_1x + b_1y + c_1 = 0$ ... (i)

$a_2x + b_2y + c_2 = 0$ ... (ii)

2.1 The point of intersection Point of intersection of the two given straight lines can be found by solving equations (i) and (ii). Thus,

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}, \text{provided}$$

$a_1b_2 - a_2b_1 \neq 0 \Rightarrow \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

That is the point of intersection of the lines is

$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$, provided $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

OR $\left(\frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}\right)$

2.2 Any line through the point of intersection of lines $ax_1 + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $a_1x + b_1y + c_1 + \lambda\,(a_2x + b_2y + c_2) = 0$ ...(i) where, λ is constant and its value is obtained by given condition

NOTES

1. For all values of λ straight line $(a_1 + \lambda\, a_2)\, x + (b_1 + \lambda\, b_2)\, y + c_1 + \lambda\, c_2 = 0$ always passes through point of intersection of straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.
2. Slope of the straight line $a_1x + b_1y + c_1 + \lambda\,(a_2x + b_2y + c_2) = 0$ is equal to $= -\left(\frac{a_1 + \lambda a_2}{b_1 + \lambda b_2}\right)$

3. Angle Between Two Straight Lines

Case 1: $y = m_1x + c_1$ .....(iii)

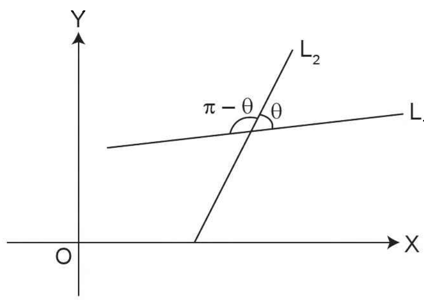
$y = m_2x + c_2$ ...(iv)

i.e., slope of two straight lines be m_1 and m_2 then the acute angle (θ) between the lines is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|,$$

$$\sin\theta = \frac{m_1 - m_2}{\sqrt{(1+m_1^2)(1+m_2^2)}}$$

The lines are parallel if $m_1 = m_2$.
The lines are perpendicular if $1 + m_1m_2 = 0 \Rightarrow m_1m_2 = -1$



Case 2: $a_1x + b_1y + c_1 = 0$ .... (i)

$a_2x + b_2y + c_2 = 0$ ...(ii)

(a) angle (θ) between them

$$\tan\theta = \left|\frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}\right|,$$

$$\cos\theta = \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}}$$

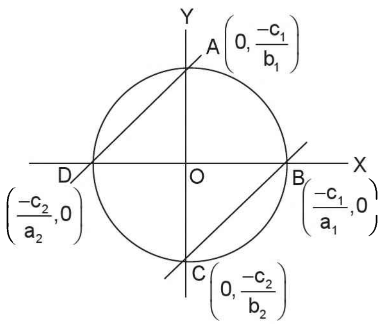
(b) the lines are paralle if $a_1b_2 - a_2b_1 = 0 \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(c) the lines are coincident if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

(d) the line are perpendicular if $a_1a_2 + b_1b_2 = 0$

(e) the lines are intersecting if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(f) the lines intersect co-ordinate axes in concycle points if $a_1a_2 = b_1b_2$.



BC and AD are two given straight lines A, B, C, D are concyclic points $A\left(\frac{-c_1}{a_1}, 0\right)$, $B\left(\frac{-c_2}{a_2}, 0\right)$, $C\left(0, \frac{-c_1}{b_1}\right)$, $D\left(0, \frac{-c_2}{b_2}\right)$

$\therefore \quad OA \times OC = OC \times OD$

$a_1a_2 = b_1b_2$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find perpendicular distance of the line joining the points (cos θ, sin θ) and (cos ϕ, sin ϕ) from the origin. ***[NCERT]***

Solution

Let the points be A and B i.e., A = (cos θ, sin θ), B = (cos ϕ, sin ϕ)

Equation of line AB is

$$y - \sin\theta = \frac{\sin\phi - \sin\phi}{\cos\phi - \cos\theta}(x - \cos\theta)$$

$$\text{or } y - \sin\theta = \frac{2\cos\frac{\theta+\phi}{2}\sin\frac{\phi-\theta}{2}}{-2\sin\frac{\phi+\theta}{2}\sin\frac{\phi-\theta}{2}}(x - \cos\theta)$$

$$\text{or} \quad y\sin\frac{\theta+\phi}{2} - \sin\theta\sin\frac{\theta+\phi}{2}$$

$$= -x\cos\frac{\theta+\phi}{2} + \cos\theta\cos\frac{\theta+\phi}{2}$$

$$\text{or} \quad x\cos\frac{\theta+\phi}{2} + y\sin\frac{\theta+\phi}{2}$$

$$-\left\{\cos\theta\cos\frac{\theta+\phi}{2} + \sin\theta\sin\frac{\theta+\phi}{2}\right\} = 0$$

or

$$x\cos\left(\frac{\theta+\phi}{2}\right) + y\sin\frac{\theta+\phi}{2} - \cos\left(\theta - \frac{\theta+\phi}{2}\right) = 0$$

$$\text{or} \quad x\cos\frac{\theta+\phi}{2} + y\sin\frac{\theta+\phi}{2} - \cos\frac{\theta-\phi}{2} = 0$$

Distance of this line from the origin

$$= \frac{\left|0 + 0 - \cos\left(\frac{\theta-\phi}{2}\right)\right|}{\sqrt{\cos^2\left(\frac{\theta+\phi}{2}\right) + \sin^2\left(\frac{\theta+\phi}{2}\right)}}$$

$$= \left|\cos\left(\frac{\theta-\phi}{2}\right)\right| + y$$

2. If p and q are the lengths of perpendiculars from the origin to the lines x cos θ – y sin θ = k cos 2θ and x sec θ + y cosec θ = k, respectively, prove that $p^2 + 4q^2 = k^2$. ***[NCERT]***

Solution

$$\text{Here, } p = \frac{|0\cos\theta - 0\sin\theta - k\cos 2\theta|}{\sqrt{(\cos\theta)^2 + (-\sin\theta)^2}}$$

$$\text{or} \quad p = \frac{|k\cos 2\theta|}{1} \quad \text{or} \quad p = |k\cos 2\theta| \quad .....(i)$$

$$\text{and} \quad q = \frac{|0\sec\theta + 0\,\text{cosec}\,\theta - k|}{\sqrt{\sec^2\theta + \text{cosec}^2\theta}}$$

$$= \frac{|k|}{\sqrt{\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}}}$$

$$= |k\cos\theta\sin\theta| = \frac{|k\sin 2\theta|}{2}$$

(∵ sin 2θ = 2 sin θ cos θ)

$$\Rightarrow \quad 2q = |k\sin 2\theta| \quad ....(ii)$$

Squaring (i) and (ii) and adding

or $p^2 + 4q^2 = (k \cos 2\theta)^2 + (k \sin 2\theta)^2$

or $p^2 + 4q^2 = k^2 (\cos^2 2\theta + \sin^2 2\theta)$

or $p^2 + 4q^2 = k^2$.

3. If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$. ***[NCERT]***

Solution

Equation of the line whose intercepts are a and b is $\frac{x}{a} + \frac{y}{b} = 1$ or $bx + ay - ab = 0$

Also, p = length of perpendicular from origin upon the line (i)

$$\Rightarrow \quad p = \frac{|b \times 0 + a \times 0 - ab|}{\sqrt{b^2 + a^2}}$$

$$\Rightarrow \quad p^2 = \frac{a^2b^2}{a^2 + b^2}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{a^2 + b^2}{a^2b^2}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{a^2}{a^2b^2} + \frac{b^2}{a^2b^2}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{a^2}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

EXERCISE 1

1. The foot of the perpendicular drawn from the point (2, 3) to a line is (3, –1). Find the equation of the line. Also find the equation of a line parallel to the line and passing through the origin. ***[NCT-2004]***

2. If the angle between two lines is $\frac{\pi}{4}$ and the slope of one of the lines is $\frac{1}{2}$, find the slope of the other line. ***[KVS-2005, NCERT]***

3. Find the angle between the lines whose equations are $ax + by + c = 0$ and $(a + b)x = (a - b)y$.

4. Transform the equation of the line $\sqrt{3}x + y - 8 = 0$ to

(i) slope intercept form and find its slope and y–intercept

(ii) intercept form and find intercepts on the coordinate axes,

(iii) normal form and find the inclination of the perpendicular segment from the origin on the line with x-axis and also find its length. ***[NCERT]***

5. Find the equation of the line passing through the intersection of the lines $3x - 4y + 1 = 0$ and $5x + y - 1 = 0$ and cutting off equal intercepts on the coordinate axes.

6. Find the equation of the line passing through the intersection of the lines $x + y + 1 = 0$ and $x - y + 1 = 0$ and whose distance from the origin is 1.

7. The perpendicular from the origin to the line $y = mx + c$ meets it at the point (–1, 2). Find the values of m and c. ***[NCERT]***

8. If three lines whose equations are $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent, then show that $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$. ***[NCERT]***

EXERCISE 2

1. A straight line is drawn through the point $P(\sqrt{3}, 2)$ making an angle of 30° with x-axis. Determine the length of the line measured from this point where it meets the line $\sqrt{3}x - 4y + 8 = 0$.

2. Find the point of intersection of the lines $x - 4y = 3$ and $6x - y = 11$.

3. If $ax - 2y - 1 = 0$ and $6x - 4y + b = 0$ represent the same line, find the values of a and b.

4. Find the equation of the perpendicular from the point (1, –2) on the line $3y = 4x - 5$. Also find the co–ordinates of the foot of the perpendicular.

5. Find the co-ordinates of the foot of perpendicular from the point (2, 3) on the line $x = 3y + 4$.

6. Find angles between the lines $\sqrt{3}x + y = 1$ and $x + \sqrt{3}y = 1$. *[NCERT]*

7. The line through the points (h, 3) and (4, 1) intersects the line $7x - 9y - 19 = 0$ at right angle. Find the value of h. *[NCERT]*

8. Two lines passing through the point (2, 3) intersect each other at an angle of 60°. If slope of one line is 2, find equation of the other line. *[NCERT]*

9. If the segment joining the points (a, b) and (c, d) subtends a right angle at the origin, show that $ac + bd = 0$.

10. Find the coordinates of the foot of perpendicular from the point (–1, 3) to the line $3x - 4y - 16 = 0$. *[NCERT]*

ANSWERS

EXERCISE 1

1. $x - 4y = 0$
2. $3, -\frac{1}{3}$
3. 45°
4. (i) Slope $= -\sqrt{3}$, Intercept = 8,
   (ii) $\frac{x}{\frac{8}{\sqrt{3}}} + \frac{y}{8} = 1$
   (iii) $\frac{\sqrt{3}}{2}x + \frac{1}{2}y\ 4, \alpha = \frac{\pi}{6}$, $p = 4$
5. $23x + 23y = 11$
6. $x + 1 = 0$
7. $m = \frac{1}{2}, c = \frac{5}{2}$

EXERCISE 2

1. 6 units
2. $\left(\frac{41}{23}, -\frac{7}{23}\right)$
3. $a = 3, b = -2$
4. $\left(\frac{1}{5}, -\frac{7}{5}\right)$
5. $\left(\frac{31}{10}, -\frac{3}{10}\right)$
6. 30° and 150°
7. 22/9
8. $(\sqrt{3} + 2)x + (2\sqrt{3} - 1)y = 8\sqrt{3} + 1$ and $(\sqrt{3} - 2)x + (2\sqrt{3} + 1)y = 8\sqrt{3} - 1$
10. (68/25, –49/25)

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The coordinates of the foot of the perpendicular from (x_1, y_1) to the line $ax + by + c = 0$ are *[Dhanbad engg.-1973]*

(a) $\left(\frac{b^2x_1 - aby_1 - ac}{a^2 + b^2}, \frac{a^2y_1 - abx_1 - bc}{a^2 + b^2}\right)$

(b) $\left(\frac{b^2x_1 + aby_1 + ac}{a^2 + b^2}, \frac{a^2y_1 + abx_1 + bc}{a^2 + b^2}\right)$

(c) $\left(\frac{ax_1 + by_1 + ab}{a + b}, \frac{ax_1 - by_1 - ab}{a + b}\right)$

(d) None of these

Solution

(a) It is a fundamental concept.

NOTE

Students should remember this question as a formula.

2. If for a variable line $\frac{x}{a}+\frac{y}{b}=1$, the condition $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$ (c is a constant) is satisfied, then locus of foot of perpendicular drawn from origin to the line is ***[RPET-1999]***
(a) $x^2 + y^2 = c^2/2$ (b) $x^2 + y^2 = 2c^2$
(c) $x^2 + y^2 = c^2$ (d) $x^2 - y^2 = c^2$

Solution

(c) Equation of perpendicular drawn from origin to the line $\frac{x}{a}+\frac{y}{b}=1$ is $y-0=\frac{a}{b}(x-0)$

[$\because$ m of given line = $\frac{-b}{a}$, $\therefore$ m of perpendicular = $\frac{a}{b}$]

$\Rightarrow$ $by - ax = 0$

$\Rightarrow$ $\frac{x}{b}-\frac{y}{a}=0$

Now, the locus of foot of perpendicular is the intersection point of line $\frac{x}{a}+\frac{y}{b}=1$ .... (i)

and $\frac{x}{b}-\frac{y}{a}=0$ .... (ii)

To find locus, squaring and adding (i) and (ii)

$$\left(\frac{x}{a}+\frac{y}{b}\right)^2+\left(\frac{x}{b}-\frac{y}{a}\right)^2=1$$

$$\Rightarrow x^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+y^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=1$$

$$\Rightarrow x^2\left(\frac{1}{c^2}\right)+y^2\left(\frac{1}{c^2}\right)=1, \left[\because \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}\right]$$

$\Rightarrow$ $x^2 + y^2 = c^2$

3. A ray of light coming from the point (1, 2) is reflected at a point A on the x-axis and then passes through the point (5, 3). The coordinates of the point A are ***[Orissa JEE-2003]***
(a) (13/5, 0) (b) (5/13, 0)
(c) (–7, 0) (d) None of these

Solution

(a) Let the coordinates of A be (a, 0). Then the slope of the reflected ray is $\frac{3-0}{5-a}=\tan\theta$, (say)
The slope of the incident ray

$=\frac{2-0}{1-a}=\tan(\pi-\theta)$

Since, $\tan\theta + \tan(\pi - \theta) = 0$

$\Rightarrow$ $\frac{3}{5-a}+\frac{2}{1-a}=0$

$\Rightarrow$ $13 - 5a = 0 \Rightarrow a = 13/5$

Thus the coordinate of A are $\left(\frac{13}{5}, 0\right)$.

4. A line L passes through the points (1, 1) and (2, 0) and another line L' passes through $\left(\frac{1}{2}, 0\right)$ and perpendicular to L. Then the area of the triangle formed by the lines L. L' and y-axis, is ***[RPET-1991]***
(a) 15/8 (b) 25/4
(c) 25/8 (d) 25/16

Solution

(d) Here, $L \equiv x + y = 2$ and $L' \equiv 2x - 2y = 1$.
Equation of y-axis is $x = 0$
Hence the vertices of the triangle are $A(0, 2)$, $B\left(0, -\frac{1}{2}\right)$ and $C\left(\frac{5}{4}, \frac{3}{4}\right)$. Therefore, the area of the triangle is

$$\frac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 0 & -\frac{1}{2} & 1 \\ \frac{5}{4} & \frac{3}{4} & 1 \end{vmatrix}=\frac{25}{16}$$

5. The foot of the perpendicular drawn from the origin on the line $x + y = 2$ and the reflection of the origin in this line is:
(a) (1, 1) (2, 2) (b) (1, 2) (1, 2)
(c) (2, 1) (1, 1) (d) (2, 1) (2, 1)

Solution

(a) Let foot of the perpendicular be (h, k), and reflection point be (h', k'). Then

$\frac{h-0}{1}=\frac{k-0}{1}=-\frac{0+0-2}{1+1} \Rightarrow h = 1, k = 1$

$\therefore$ Foot of the perpendicular $= (1, 1)$

Also $\frac{h'-0}{1} = \frac{k'-0}{1} = -\frac{2(0+0-2)}{1+1}$

$\Rightarrow$ $h' = 2, k' = 2$

$\therefore$ Reflection point $= (2, 2)$.

6. The equations of two lines through $(0, a)$ which are at distance 'a' from the point $(2a, 2a)$ are **[Dhanbad Engg.-1972]**
   (a) $y - a = 0$ and $4x - 3y - 3a = 0$
   (b) $y - a = 0$ and $3x - 4y + 3a = 0$
   (c) $y - a = 0$ and $4x - 3y + 3a = 0$
   (d) None of these

Solution

(c) Equation of any line through $(0, a)$ is $y - a = m(x - 0)$ or $mx - y + a = 0$ ...... (i)
If the length of perpendicular from $(2a, 2a)$ to the line (i) is 'a', then

$$a = \pm\frac{m(2a) - 2a + a}{\sqrt{m^2+1}} \Rightarrow m = 0, \frac{4}{3}.$$

Hence, the required equations of lines are $y - a = 0$, $4x - 3y + 3a = 0$.

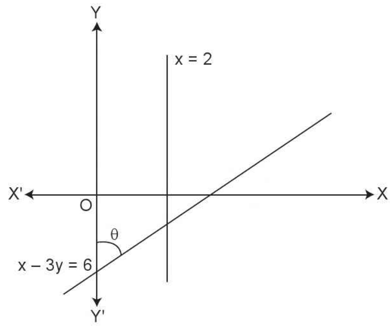
7. Angle between $x = 2$ and $x - 3y = 6$ is **[MNR-1988]**
   (a) ∞ (b) $\tan^{-1}(3)$
   (c) $\tan^{-1}\left(\frac{1}{3}\right)$ (d) None of these

Solution

(b) $\theta = 90° - \tan^{-1}\left(\frac{1}{3}\right)$

$$\tan\theta = \cot\left[\tan^{-1}\left(\frac{1}{3}\right)\right] = 3$$

$\Rightarrow$ $\theta = \tan^{-1}(3)$.



8. The straight lines $x + y = 0$, $3x + y - 4 = 0$ and $x + 3y - 4 = 0$ form a triangle which is **[MPPET-2006]**
   (a) right angled
   (b) equilateral
   (c) isosceles
   (d) None of these

Solution

(c) The given equations of straight lines are

$x + y = 0$ .... (i),
$3x + y = 4$ ...... (ii)
and $x + 3y = 4$ .... (iii)

On solving these equations we get the vertices of triangle ABC which are A $(2, -2)$, B $(1, 1)$ and C $(-2, 2)$.

$\therefore$ $AB = \sqrt{(1-2)^2 + (1+2)^2} = \sqrt{1+9} = \sqrt{10}$

$BC = \sqrt{(-2-1)^2 + (2-1)^2} = \sqrt{9+1} = \sqrt{10}$

and $CA = \sqrt{(2+2)^2 + (-2-2)^2}$
$= \sqrt{16+16} = \sqrt{32}$

This shows that triangle is isosceles triangle.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The equation of a line through the intersection of lines $x = 0$ and $y = 0$ and through the point $(2, 2)$, is **[MP PET-1984]**
   (a) $y = x - 1$ (b) $y = -x$
   (c) $y = x$ (d) $y = -x + 2$

2. The straight line passes through the point of inter section of the straight lines $x + 2y - 10 = 0$ and $2x + y + 5 = 0$, is
   (a) $5x - 4y = 0$ (b) $5x + 4y = 0$
   (c) $4x - 5y = 0$ (d) $4x + 5y = 0$

3. The equation to the straight line passing through the point of intersection of the lines $5x - 6y - 1 = 0$ and $3x + 2y + 5 = 0$ and perpendicular to the line $3x - 5y + 11 = 0$ is **[MP PET-1994]**

(a) $5x + 3y + 8 = 0$
(b) $3x - 5y + 8 = 0$
(c) $5x + 3y + 11 = 0$
(d) $3x - 5y + 11 = 0$

4. The equation of straight line passing through the intersection of the lines $x - 2y = 1$ and $x + 3y = 2$ and parallel to $3x + 4y = 0$ is **[MP PET-2000]**

(a) $3x + 4y + 5 = 0$
(b) $3x + 4y - 10 = 0$
(c) $3x + 4y - 5 = 0$
(d) $3x + 4y + 6 = 0$

5. If a and b are two arbitrary constants, then the straight line $(a - 2b)x + (a + 3b)y + 3a + 4b = 0$ will pass through **[RPET-1990]**

(a) $(-1, -2)$ (b) $(1, 2)$
(c) $(-2, -3)$ (d) $(2, 3)$

6. The angle between the lines $y = (2 - \sqrt{3})x + 5$ and $y = (2 + \sqrt{3})x - 7$ is **[MP PET-1997]**

(a) 30° (b) 60°
(c) 45° (d) 90°

7. Angle between the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{a} - \frac{y}{b} = 1$ is

(a) $2\tan^{-1}\frac{b}{a}$ (b) $\tan^{-1}\frac{2ab}{a^2 + b^2}$
(c) $\tan^{-1}\frac{a^2 - b^2}{a^2 + b^2}$ (d) None of these

8. The angle between the lines $x \cos \alpha_1 + y \sin \alpha_1 = p_1$ and $x \cos \alpha_2 + y \sin \alpha_2 = p_2$ is

(a) $(\alpha_1 + \alpha_2)$ (b) $(\alpha_1 - \alpha_2)$
(c) $2\alpha_1$ (d) $2\alpha_2$

9. To which of the following types the straight lines represented by $2x + 3y - 7 = 0$ and $2x + 3y - 5 = 0$ belong

(a) Parallel to each other
(b) Perpendicular to each other
(c) Inclined at 45° to each other
(d) Coincident pair of straight lines

10. The obtuse angle between the lines $y = -2$ and $y = x + 2$ is

(a) 120° (b) 135°
(c) 150° (d) 160°

11. The angle between the straight lines $x - y\sqrt{3} = 5$ and $\sqrt{3}x + y = 7$ is **[MP PET-2003]**

(a) 90° (b) 60°
(c) 75° (d) 30°

12. The line passing through the points $(3, -4)$ and $(-2, 6)$ and a line passing through $(-3, 6)$ and $(9, -18)$ are **[AMU-1974]**

(a) Perpendicular
(b) Parallel
(c) Makes an angle 60° with each other
(d) None of these

13. The distance of the point of intersection of the lines $2x - 3y + 5 = 0$ and $3x + 4y = 0$ from the line $5x - 2y = 0$ is

(a) $\frac{130}{17\sqrt{29}}$ (b) $\frac{13}{7\sqrt{29}}$
(c) $\frac{130}{17}$ (d) None of these

14. The coordinates of the foot of perpendicular from the point $(2, 3)$ on the line $x + y - 11 = 0$ are **[MP PET-1986]**

(a) $(-6, 5)$ (b) $(5, 6)$
(c) $(-5, 6)$ (d) $(6, 5)$

15. The pedal points of a perpendicular drawn from origin on the line $3x + 4y - 5 = 0$, is **[RPET-1990]**

(a) $\left(\frac{3}{5}, 2\right)$ (b) $\left(\frac{3}{5}, \frac{4}{5}\right)$
(c) $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ (d) $\left(\frac{30}{17}, \frac{19}{17}\right)$

16. The image of a point $A(3, 8)$ in the line $x + 3y - 7 = 0$, is **[RPET-1991]**

(a) $(-1, -4)$ (b) $(-3, -8)$
(c) $(1, -4)$ (d) $(3, 8)$

17. The reflection of the point $(4, -13)$ in the line $5x + y + 6 = 0$ is **[EAMCET-94]**

(a) $(-1, -14)$ (b) $(3, 4)$
(c) $(1, 2)$ (d) $(-4, 13)$

18. If (–2, 6) is the image of the point (4, 2) with respect to line $L = 0$, then L is equal to ***[EAMCET-2002]***

(a) $3x - 2y + 5$
(b) $3x - 2y + 10$
(c) $2x + 3y - 5$
(d) $6x - 4y - 7$

19. For the straight lines given by the equation $(2 + k)x + (1 + k)y = 5 + 7k$, for different values of k which of the following statements is true ***[IIT-1971]***

(a) Lines are parallel
(b) Lines pass through the point (–2, 9)
(c) Lines pass through the point (2, –9)
(d) None of these

20. Coordinates of the vertices of a quadrilateral are (2, –1), (0, 2), (2, 3) and (4, 0). The angle between its diagonals will be ***[IIT-1986]***

(a) 90° (b) 0°
(c) $\tan^{-1}(2)$ (d) $\tan^{-1}\left(\frac{1}{2}\right)$

21. The ratio in which the line segment joining (1,2) and (3,4) is divided by the line joining points (2, 3) and (4, 1) is

(a) 1 : 2 (b) 1 : 1
(c) 2 : 1 (d) 2 : 3

22. If a line with y-intercept 2, is perpendicular to the line $3x - 2y = 6$, then its x-intercept is ***[Kerala PET-2008]***

(a) 1 (b) 2
(c) – 4 (d) 3

23. If the line passing through (4, 3) and (2, k) is perpendicular to $y = 2x + 3$, then k is equal to ***[RPET-1985; MP PET-1999]***

(a) –1 (b) 1
(c) –4 (d) 4

24. The lines $p(p^2 + 1)x - y + q = 0$ and $(p^2 + 1)^2 x + (p^2 + 1)y + 2q = 0$ are perpendicular to a common line for ***[AIEEE-2009]***

(a) no value of p
(b) exactly one value of p
(c) exactly two values of p
(d) more than two values of p

SOLUTIONS

1. (c) Intersection of $x = 0$ and $y = 0$ is $P(0, 0)$. Equation of line joining (0, 0) to (2, 2) is

$$y - 0 = \left(\frac{2-0}{2-0}\right)(x - 0) \Rightarrow y = x \Rightarrow x - y = 0$$

2. (b) Intersection of given lines $x + 2y - 10 = 0$ .... (1)
and $2x + y + 5 = 0$ .... (2)
is $\left(-\frac{20}{3}, \frac{25}{3}\right)$.
This point only satisfies the line $5x + 4y = 0$.

3. (a) Intersection of lines $5x - 6y - 1 = 0$ ... (1)
and $3x + 2y + 5 = 0$ .... (2)
is $P(-1, -1)$. Equation of a line through $P(-1, -1)$ and perpendicular to $3x - 5y + 11 = 0$ is

$$y + 1 = -\frac{5}{3}(x + 1) \Rightarrow 5x + 3y + 8 = 0$$

4. (c) Intersection point of given lines is (7/5, 1/5) gradient of $3x + 4y = 0$ is $m = \frac{-3}{4}$ gradient of praralel line $m = \frac{-3}{4}$ by $y - y_1 = m(x - x_1)$ $y - \frac{1}{5} = \frac{-3}{4}\left(x - \frac{7}{5}\right)$ or $3x + 4y - 5 = 0$

5. (a) $(a - 2b)x + (a + 3b)y + 3a + 4b = 0$
$\Rightarrow$ $a(x + y + 3) + b(-2x + 3y + 4) = 0$ is true $\forall\, a$ and $\forall\, b$,
$\therefore$ $x + y + 3 = 0, -2x + 3y + 4 = 0$
Solving them we get $(x, y) = (-1, -2)$.

6. (c) Compare given lines with $y = mx + c$

$m_1 = (2 - \sqrt{3})$ for $y = (2 - \sqrt{3})x + 5$ &
$m_2 = 2 + \sqrt{3}$ for $y = (2 + \sqrt{3})x - 7$

Angle between two lines

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\therefore \quad \tan\theta = \frac{2+\sqrt{3}-(2-\sqrt{3})}{1+(2+\sqrt{3})(2-\sqrt{3})}$$

$$\tan\theta = \frac{2\sqrt{3}}{1+4-3} = \sqrt{3} \quad \therefore \theta = 60°$$

7. (a) Lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{a}-\frac{y}{b}=1$ have slopes

$$m_1 = -\frac{b}{a}, m_2 = \frac{b}{a}$$

$$\Rightarrow \quad m_1 = -m_2 = \frac{b}{a}$$

Angle θ between them is given by

$$\tan\ \theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\theta = \left|\frac{m_1 + m_1}{1 - m_1^2}\right| = \frac{2m_1}{1 - m_1^2}$$

if $\quad m_1 = \frac{b}{a} = \tan\alpha$

$$= \left|\frac{2\tan\alpha}{1-\tan^2\alpha}\right| = \tan(2\alpha)$$

$$\Rightarrow \quad \theta = 2\alpha \Rightarrow \theta = 2\tan^{-1}\left(\frac{b}{a}\right)$$

8. (b) $\theta = \tan^{-1}\left[\frac{-\cot\alpha_1 + \cot\alpha_2}{1+\cot\alpha_1 \cot\alpha_2}\right]$

$$= \tan^{-1}\left[\frac{\tan\alpha_2 - \tan\alpha_1}{1+\tan\alpha_2 \tan\alpha_1}\right] = (\alpha_2 \sim \alpha_1)$$

Aliter: Obviously, first line makes angle $\frac{\pi}{2}+\alpha_1$ with the x-axis and second line makes the angle $\frac{\pi}{2}+\alpha_2$.

Therefore, angle between these two lines is $\alpha_1 \sim \alpha_2$.

9. (a) Here $m_1 = -\frac{2}{3} = m_2$.

10. (b) $y = -2$ is a line parallel to x-axis and $y = x + 2$.

slope $= 1 = \tan\left(\frac{\pi}{4}\right)$ makes $\frac{\pi}{4}$ angle with x-axis.

Acute angle between them is $\frac{\pi}{4}$ and obtuse angle is $\pi - \frac{\pi}{4} = 135°$.

11. (b) Gradient of $x - \sqrt{3}\,y = 5$ is $m_1 = \frac{1}{\sqrt{3}}$ gradient of $\sqrt{3}\ x + y = 7$ is $m_2 = -\sqrt{3}$ by formula

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan\theta = \frac{\frac{1}{\sqrt{3}} - (-\sqrt{3})}{1 + \frac{1}{\sqrt{3}}(-\sqrt{3})}$$

$$\tan\ \theta = \infty \text{ or } \theta = 90°$$

Trick: $m_1 . m_2 = \frac{1}{\sqrt{3}}(-\sqrt{3}) = -1$

$\therefore$ Lines will be perpendicular

12. (b) $m_1 = \frac{6+4}{-2-3} = \frac{10}{-5} = -2$

and $\quad m_2 = \frac{-18-6}{9-(-3)} = -2$

Hence, the lines are parallel.

13. (a) Points of intersection are

$$y = \frac{15}{17}, x = \frac{-4\times 15}{3\times 17} = \frac{-20}{17}$$

Therefore, $D = \left|\frac{5\left(\frac{-20}{17}\right) - 2\left(\frac{15}{17}\right)}{\sqrt{29}}\right| = \frac{130}{17.\sqrt{29}}$

14. (b) Given line $x + y - 11 = 0$ ...... (i)

Any line perpendicular to $x + y - 11 = 0$ are $x - y + k = 0$ and passes through (2, 3) then $2 - 3 + k = 0 \Rightarrow k = 1$.

$\therefore \quad x - y + 1 = 0$ ......(ii)

Solving equation (i) and (ii), we get

$2x = 10 \Rightarrow x = 5$ and $y = 6$

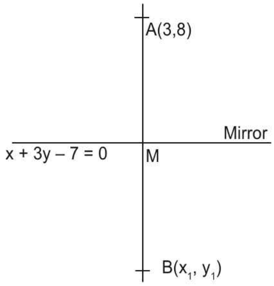
$\therefore$ Coordinate are (5, 6).

15. (b) $3x + 4y - 5 = 0 \Rightarrow 3x + 4y = 5$

$\Rightarrow \frac{3}{5}x + \frac{4}{5}y = 1$, slope $= \tan\theta = -\frac{3}{4}$

Comparing this with $x\cos\alpha + y\sin\alpha = p$, we find that $\left(\frac{3}{5}, \frac{4}{5}\right)$ is pedal point lying on the line.

16. (a)



$B(x_1, y_1)$ is image of point $A(3, 8)$

w.r.t. line $x + 3y - 7 = 0$ ....(1)

if (i) $M\left(\frac{x_1+3}{2}, \frac{y_1+8}{2}\right)$, the middle point of AB lies on (1)

(ii) line AMB is $\perp$ to line (1)

(i) $\Rightarrow \left(\frac{x_1+3}{2}\right) + 3\left(\frac{y_1+8}{2}\right) - 7 = 0$

$\Rightarrow x_1 + 3y_1 + 13 = 0$ ....(2)

(ii) $\Rightarrow m_1m_2 = -1$

$\Rightarrow \left(\frac{y_1-8}{x_1-3}\right)\left(-\frac{1}{3}\right) = -1$

$\Rightarrow y_1 - 8 = 3(x_1 - 3)$

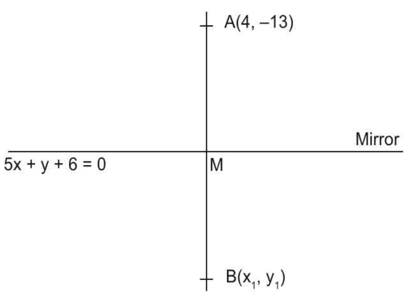
$\Rightarrow y_1 - 8 = 3x_1 - 9$

$\Rightarrow 3x_1 - y_1 = 1$ .... (3)

Solving (2) and (3) we get

$\therefore \; x_1 = -1, y_1 = -4$

17. (c) $B(x_1, y_1)$ is image of point $A(4, -13)$



w.r.t. line $5x + y + 6 = 0$ .... (1)

if (i) $M\left(\frac{x_1+4}{2}, \frac{y_1-13}{2}\right)$, the middle point of AB lies on (1).

(ii) line AMB is perpendicular to line (1)

(i) $\Rightarrow 5\left(\frac{x_1+4}{2}\right) + \left(\frac{y_1-13}{2}\right) + 6 = 0$

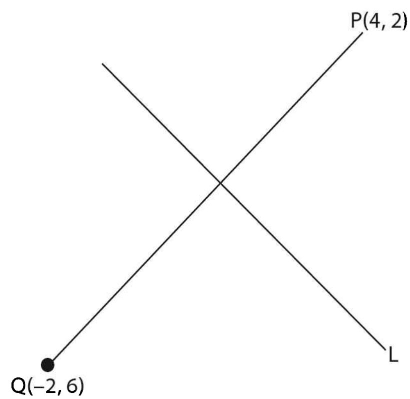
$\Rightarrow 5x_1 + y_1 + 19 = 0$ .... (2)

(ii) $\Rightarrow m_1m_2 = -1 \Rightarrow \left(\frac{y_1+13}{x_1-4}\right)(-5) = -1$

$\Rightarrow x_1 - 5y_1 = 69$ .... (3)

Solving (2) and (3), we get $(x_1, y_1) = (-1, -14)$.

18. (a) The mid-point of $P(-2, 6)$ and $Q(4, 2)$ is $\left(\frac{-2+4}{2}, \frac{6+2}{2}\right)$ i.e., (1, 4) and the gradient of line $PQ = \frac{2-6}{4+2} = \frac{-2}{3}$



$\therefore$ The slope of $L = \frac{3}{2}$

Hence, the equation of line which passes through point (1, 4) is

$y - 4 = \frac{3}{2}(x - 1) \Rightarrow 3x - 2y + 5 = 0$

19. (b) $(2 + k)x + (1 + k)y = 5 + 7k$ .... (1)

is true for every value of k

Putting, $k = 1, k = 2$ in (1), we get $3x + 2y = 12$ .... (2)

$4x + 3y = 19$ .... (3)

Obviously they are not parallel. Intersection point of (2) and (3) is (–2, 9).

20. (c) $A(2,-1), B(0,2), C(2,3), D(4,0), m_1 =$ slope of line diagonal AC $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3+1}{2-2} = \infty$,

$m_2 =$ slope of diagonal $BD = \dfrac{2-0}{0-4} = -\dfrac{1}{2}$

If θ is angle between them, then $\theta = \left|90° - \tan^{-1}\left(-\dfrac{1}{2}\right)\right| = \left|\cot^{-1}\left(-\dfrac{1}{2}\right)\right| = |\tan^{-1}$

$(2)| = |-\tan^{-1}(2)| = \tan^{-1}(2)$.

21. (d) Equation of line passing through (2, 3) and (4, 1) then $y - 3 = \dfrac{1-3}{4-2}(x-2)$

$$y - 3 = \frac{-2}{2}(x-2)$$

$y - 3 = -x + 2$

$x + y = 5$ ....(1)

and $A(1, 2)$ and $B(3, 4)$

Suppose line AB is divided by

line $x + y = 5$ .... (1)

into the ratio $= \lambda : 1$.

Then $x = \dfrac{3\lambda+1}{\lambda+1}, y = \dfrac{4\lambda+2}{\lambda+1}$

Put this in (1), we get

$$\frac{3\lambda+1}{\lambda+1} + \frac{4\lambda+2}{\lambda+1} = 5$$

$7\lambda + 3 = 5\lambda + 5$

$2\lambda = 2$

$$\lambda = \frac{2}{2} = 1$$

22. (d) Any line at right angles to $3x - 2y = 6$ is $2x + 3y + k = 0$ .... (1)

$\Leftrightarrow \quad 2x + 3y = -k$

$\Leftrightarrow \quad \dfrac{2x}{-k} + \dfrac{3y}{-k} = 1$

$\Leftrightarrow \quad \dfrac{x}{\frac{-k}{2}} + \dfrac{y}{\frac{-k}{3}} = 1$

But y-intercept = 2

$\Rightarrow \quad \dfrac{-k}{3} = 2 \Rightarrow k = -6$

and then x-intercept $= \dfrac{-k}{2} = \dfrac{-(-6)}{2} = 3$

23. (d) Gradient of line joining the points (4, 3), (2, k) is

$m_1 = \dfrac{k-3}{2-4} \left(\text{by } m = \dfrac{y_2 - y_1}{x_2 - x_1}\right)$

$m_1 = \dfrac{k-3}{-2}$ .... (1)

Gradient of $y = 2x + 3$ is $m_2 = 2$ .... (2)

If lines are perpendicular then $m_1 . m_2 = -1$ or $\left(\dfrac{k-3}{-2}\right)2 = -1$ or $k = 4$.

24. (b) Lines must be parallel, therefore slope are equal

$\Rightarrow \quad p(p^2 + 1) = -(p^2 + 1)$

$\Rightarrow \quad p = -1$

∴ There is exactly one value of p.

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The equation of line passing through the point of intersection of the lines $4x - 3y - 1 = 0$ and $5x - 2y - 3 = 0$ and parallel to the line $2y - 3x + 2 = 0$, is ***[RPET-1985, 86, 88]***

(a) $x - 3y = 1$ (b) $3x - 2y = 1$

(c) $2x - 3y = 1$ (d) $2x - y = 1$

2. The equation of straight line passing through point of intersection of the straight lines $3x - y + 2 = 0$ and $5x - 2y + 7 = 0$ and having infinite slope is ***[UPSEAT-2001]***

(a) $x = 2$

(b) $x + y = 3$

(c) $x = 3$

(d) $x = 4$

3. The angle between the two lines $y - 2x = 9$ and $x + 2y = -7$, is ***[RPET-1981, 85, 86; MP PET-1984]***

(a) 60° (b) 30°

(c) 90° (d) 45°

4. If $\frac{1}{ab'}+\frac{1}{ba'}=0$, then lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b'}+\frac{y}{a'}=1$ are

(a) Parallel
(b) Inclined at 60° to each other
(c) Perpendicular to each other
(d) Inclined at 30° to each other

5. If the lines $y=(2+\sqrt{3})x+4$ and $y = kx + 6$ are inclined at an angle 60° to each other, then the value of k will be

(a) 1 (b) 2
(c) –1 (d) –2

6. The lines $y = 2x$ and $x = -2y$ are ***[MP PET-1993]***

(a) Parallel
(b) Perpendicular
(c) Equally inclined to axes
(d) Coincident

7. The coordinates of the foot of the perpendicular from the point (2, 3) on the line $y = 3x + 4$ are given by ***[MP PET-1984]***

(a) $\left(\frac{37}{10},-\frac{1}{10}\right)$ (b) $\left(-\frac{1}{10},\frac{37}{10}\right)$

(c) $\left(\frac{10}{37},-10\right)$ (d) $\left(\frac{2}{3},-\frac{1}{3}\right)$

8. The projection of the point (2, 1) with respect to line $x - 2y + 3 = 0$ is

(a) (3/5, 17/5) (b) (2/5, 17/5)
(c) (4/5, 17/5) (d) None of these

9. The equation of line passing through point of intersection of lines $3x - 2y - 1 = 0$ and $x - 4y + 3 = 0$ and the point (π, 0), is

(a) $x-y=\pi$
(b) $x-y=\pi(y+1)$
(c) $x-y=\pi(1-y)$
(d) $x+y=\pi(1-y)$

10. If the line $2x + 3ay - 1 = 0$ and $3x + 4y + 1 = 0$ are mutually perpendicular, then the value of a will be ***[MNR-1975]***

(a) 1/2 (b) 2
(c) – 1/2 (d) None of these

11. The angle between lines $x \cos\alpha + y \sin\alpha = a$ and $x \sin\beta - y\cos\beta = a$ is ***[MPPET-2010]***

(a) $\frac{x}{2}+\beta+\alpha$ (b) $\frac{\pi}{2}-\beta+\alpha$

(c) $\frac{\pi}{2}-\beta-\alpha$ (d) None of these

12. If vertices of a parallelogram are respectively (0, 0), (1, 0), (2, 2) and (1, 2), then angle between diagonals is ***[RPET-1996]***

(a) π/3 (b) π/2
(c) 3π/2 (d) π/4

13. Equations of lines which passes through the points of intersection of the lines $4x - 3y - 1 = 0$ and $2x - 5y + 3 = 0$ and are equally inclined to the axes are

(a) $y \pm x = 0$
(b) $y - 1 = \pm 1(x - 1)$
(c) $x - 1 = \pm 2(y - 1)$
(d) None of these

14. If the line segment joining the points $P(a, b)$ and $Q(c, d)$ subtends an angle θ at the origin, then the value of cos θ is ***[Kerala PET-2008]***

(a) $\frac{ab+cd}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$

(b) $\frac{ac}{\sqrt{a^2+b^2}}+\frac{bd}{\sqrt{c^2+d^2}}$

(c) $\frac{ac+bd}{\sqrt{(a^2+b^2)}\sqrt{(c^2+d^2)}}$

(d) $\frac{ac-bd}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$

15. The lines $(a + 2b)x + (a - 3b)y = a - b$ for different values of a and b pass through the fixed point whose coordinates are ***[Kerala PET-08]***

(a) $\left(\frac{2}{5},\frac{2}{5}\right)$ (b) $\left(\frac{3}{5},\frac{3}{5}\right)$

(c) $\left(\frac{1}{5},\frac{1}{5}\right)$ (d) $\left(\frac{2}{5},\frac{3}{5}\right)$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 14 minutes.
3. The worksheet consists of 14 questions. The maximum marks are 42.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The equation of a line passing through the point of intersection of the lines $x + 5y + 7 = 0$, $3x + 2y - 5 = 0$, and perpendicular to the line $7x + 2y - 5 = 0$, is given by ***[RPET-87; MP PET-1993; Pb. CET-2000]***

(a) $2x - 7y - 20 = 0$
(b) $2x + 7y - 20 = 0$
(c) $-2x + 7y - 20 = 0$
(d) $2x + 7y + 20 = 0$

2. Equation of a line passing through the point of intersection of lines $2x - 3y + 4 = 0$, $3x + 4y - 5 = 0$ and perpendicular to $6x - 7y + 3 = 0$, then its equation is ***[RPET-2000]***

(a) $119x + 102y + 125 = 0$
(b) $119x + 102y = 125$
(c) $119x - 102y = 125$
(d) None of these

3. The equation of the straight line joining the origin to the point of intersection of $y - x + 7 = 0$ and $y + 2x - 2 = 0$ is

(a) $3x + 4y = 0$ (b) $3x - 4y = 0$
(c) $4x - 3y = 0$ (d) $4x + 3y = 0$

4. The angle between the lines $2x - y + 3 = 0$ and $x + 2y + 3 = 0$ is ***[Kerala (Engg.)-2002]***

(a) 90° (b) 60°
(c) 45° (d) 30°

5. The lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular to each other, if ***[MP PET-1996]***

(a) $a_1b_2 - b_1a_2 = 0$ (b) $a_1a_2 + b_1b_2 = 0$
(c) $a_1^2b_2 + b_1^2a_2 = 0$ (d) $a_1b_1 + a_2b_2 = 0$

6. The angle between the lines $xy = 0$ is equal to ***[Pb. CET-2003]***

(a) 45° (b) 60°
(c) 90° (d) 180°

7. The foot of the coordinates drawn from (2, 4) to the line $x + y = 1$ is ***[Roorkee-1995]***

(a) $\left(\frac{1}{3}, \frac{3}{2}\right)$ (b) $\left(-\frac{1}{2}, \frac{3}{2}\right)$
(c) $\left(\frac{4}{3}, \frac{1}{2}\right)$ (d) $\left(\frac{3}{4}, -\frac{1}{2}\right)$

8. The angle between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, is ***[MPPET-94]***

(a) $\tan^{-1}\frac{a_1b_2 + a_2b_1}{a_1a_2 - b_1b_2}$ (b) $\cot^{-1}\frac{a_1a_2 + b_1b_2}{a_1b_2 - a_2b_1}$
(c) $\cot^{-1}\frac{a_1b_1 - a_1b_2}{a_1a_2 + b_1b_2}$ (d) $\tan^{-1}\frac{a_1b_1 - a_2a_2}{a_1a_2 + b_1b_2}$

9. Two lines represented by equations $x + y = 1$ and $x + ky = 0$ are mutually orthogonal if k is ***[MPPET-2007]***

(a) 1 (b) –1
(c) 0 (d) None of these

10. A line passes through the point of intersection of $2x + y = 5$ and $x + 3y + 8 = 0$ and parallel to the line $3x + 4y = 7$ is ***[RPET-1984; MP PET-1991]***

(a) $3x + 4y + 3 = 0$ (b) $3x + 4y = 0$
(c) $4x - 3y + 3 = 0$ (d) $4x - 3y = 3$

11. If O is at the origin, OA is along the x-axis and (–40, 9) is point on OB, then the value of sin $\angle AOB$ is ***[Kerala PET-2008]***

(a) $\frac{13}{40}$ (b) $\frac{7}{41}$
(c) $\frac{9}{41}$ (d) $\frac{9}{40}$

12. Angle between the lines $2x - y - 15 = 0$ and $3x + y + 4 = 0$ is ***[RPET-2003]***

(a) 90° (b) 45°
(c) 180° (d) 60°

13. The angle between the line joining the points (1, –2), (3, 2) and the line $x + 2y - 7 = 0$ is *[EAMCET-2007]*

(a) π (b) π/2
(c) π/3 (d) π/6

14. What is the foot of the perpendicular from the point (2, 3) on the line $x + y - 11 = 0$? *[NDA-2009]*

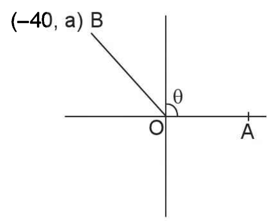
(a) (1, 10) (b) (5, 6)
(c) (6, 5) (d) (7, 4)

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ
11. ⓐ ⓑ ⓒ ⓓ
12. ⓐ ⓑ ⓒ ⓓ
13. ⓐ ⓑ ⓒ ⓓ
14. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

11. (c) Here $\tan\theta = m_{OB} = \dfrac{-9}{40} \therefore \sin\theta = \dfrac{9}{41}$ (θ is in II quadrants)



14. (b) The equation of line perpendicular to given line $x + y - 11 = 0$ .... (i)

is $-x + y + \lambda = 0$ .... (ii)

This equation passes through (2, 3).

$\therefore\ -2 + 3 + \lambda = 0 \Rightarrow \lambda = -1$

$\therefore$ From equation (ii),

$-x + y - 1 = 0 \Rightarrow y = x + 1$

$\therefore$ From equation (i),

$x + x + 1 - 11 = 0 \Rightarrow 2x = 10$

$\Rightarrow x = 5$

Hence, coordinates of foot of perpendicular from (2, 3) to given line is (5, 6).

LECTURE

4

Straight Line 4

(Distance between two parallel lines; position of origin (0, 0) with respect to angle between two lines, angular bisectors of two given lines, some important points connected with three straight lines)

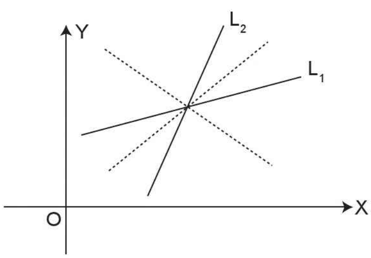
BASIC CONCEPTS

1. The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 =$ is given by $\left|\dfrac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$

NOTE

Coefficients of x and y are made identical before finding distance between two parallel lines.

2. **Position of origin (0, 0) with respect to angle between two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ be determined in following two steps if c_1 and c_2 are of the same sign**
   i.e., $c_1c_2 > 0$.
   step 1: If $a_1a_2 + b_1b_2 > 0$ then origin lies in obtuse angle.
   step 2: If $a_1a_2 + b_1b_2 < 0$ then origin lies in actue angle.
3. **Angular Bisectors of Two Given Lines**
   Let the equations of two given lines be $a_1x + b_1y + c_1 = 0$
   and $a_2x + b_2y + c_2 = 0$.



The equations of the bisectors of the angle between two lines is given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

(preferably $c_1c_2 > 0$).
The above equations represent two equations, one of which is acute angle bisector and the other is obtuse angle bisector. Bisector of the angle containing the origin is taken with position sign and bisector of the angle not containing the origin is taken with negative sign.

NOTE

To find the bisector of acute or obtuse angle
If θ be the angle between one of the given lines and any one bisector, then find $\tan\theta$. If $|\tan\theta| < 1$, it is the bisector of the acute angle and if $|\tan\theta| > 1$, then it is the bisector of the obtuse angle.

4. **Bisector of the angle containing the origin and that of not containing the origin**
   We rewrite the given equations
   $a_1x + b_1y + c_1 = 0$
   $a_2x + b_2y + c_2 = 0$ so that $c_1, c_2 > 0$.
   Then

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = + \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

is the **bisector of angle containing the origin** and $\dfrac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\dfrac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$

is the bisector of the angle not containing the origin.

5. **Method to find acute angle bisector and obtuse angle bisector**

If $a_1a_2 + b_1b_2 > 0$, then origin is situated in an obtuse angle region and the bisector of this obtuse angle is $\dfrac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\dfrac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$

$(c_1, c_2 > 0)$ and $\dfrac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\dfrac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$

is the other bisector.

If $a_1a_2 + b_1b_2 < 0$, then origin is situated in **acute angle** region and the bisector of this angle is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=+\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

and $\dfrac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=-\dfrac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$

is the other bisector.

6. **Family of Straight Lines**

(i) The family of parallel lines is given by $y = mx + c$, where m is constant and c is variable.

(ii) The family of concurrent lines is given by $y - y_1 = m(x - x_1)$, where m is variable and x_1, y_1 are constant.

(iii) The equation $ax + by + c$ represents a family of concurrent lines if a relation between the coefficients of the form $\lambda a + \mu b + \nu c = 0$ exists, where $\lambda, \mu, \nu \in R$.

The point of concurrency is

$$\left(\frac{\lambda}{\nu}, \frac{\mu}{\nu}\right), \nu \neq 0.$$

7. **Some important points connected with three straight lines**

7.1 **Condition of concurrency of three lines**

Let the equations of three given lines be $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$

These three lines are concurrent, i.e., they pass through a common point if

(i) Point of intersection of any two lines lies on the third

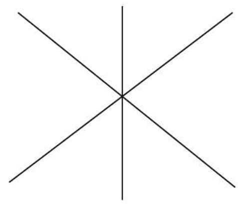
OR

(ii) The coefficients of the equations of three lines satisfy the determinant,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \text{ OR}$$

(Condition is necessary only)

(iii) Three non-zero constants λ, μ, ν can be found such that $\lambda L_1 + \mu L_2 + \nu L_3 = 0$ (OR $L_1 + L_2 + L_3 = 0$ for $\lambda = \mu = \nu = 1$)



a

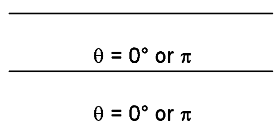
(Three concurrent lines)

where, $L_1 = a_1x + b_1y + c_1 = 0$

$L_2 = a_2x + b_2y + c_2 = 0$

$L_3 = a_3x + b_3y + c_3 = 0$

(iv)

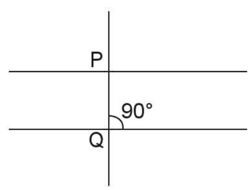


b

(Three parallel lines)

PQ = minimum intercept

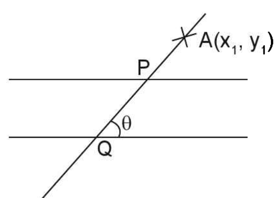
(v)



c

(Two lines parallel and third line is perpendicular to them)

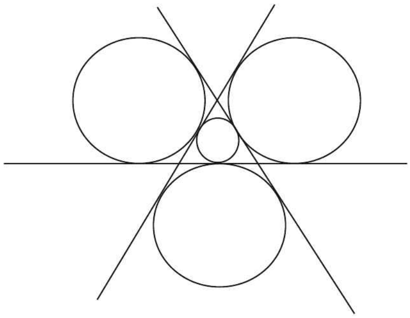
(vi)



d

(Two lines parallel and third line intercepting length PQ from given two lines)

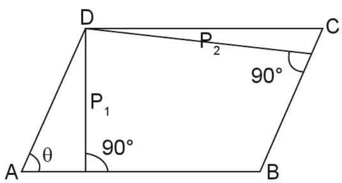
(vii)



e

(Number of circles touching three straight lines)

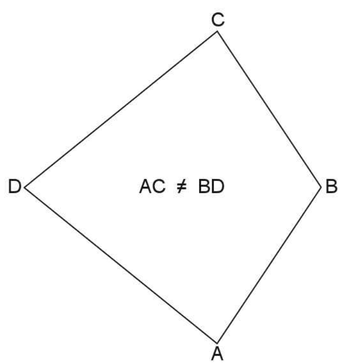
8. **Area of Parallelogram** If P_1 and P_2 be the distance, between the parallel sides, then are $a = \dfrac{P_1P_2}{\sin\theta}$. where θ is the angle between two adjacent sides.



8.1 Area of rhombus

Area $= \dfrac{1}{2} \times d_1 \times d_2$ where d_1 and d_2 are the lengths of two perpendicular diagonals of a rhombus.

$AC = d_1, BD = d_2, AC \perp^r BD$



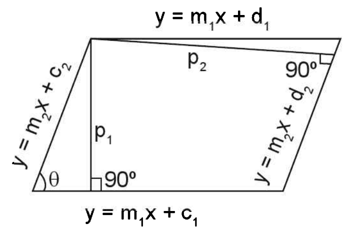
9. The equations of the sides of a triangle are $y = m_1x + c_1$, $y = m_2x + c_2$ and $x = 0$; then its area is $\dfrac{1}{2}\dfrac{(c_1-c_2)^2}{m_1-m_2}$.

10. Area of the triangle formed by the straight lines $y = m_1x + c_1$, $y = m_2x + c_2$, and $y = m_3x + c_3$ is

$$\frac{1}{2}\left[\frac{(c_1-c_2)^2}{m_1-m_2}+\frac{(c_2-c_3)^2}{m_2-m_3}+\frac{(c_3-c_1)^2}{m_3-m_1}\right].$$

11. If the sides of a parallelogram be $y = m_1x + c_1$, $y = m_1x + d_1$, $y = m_2x + c_2$ and $y = m_2x + d_2$ then its area is:

$$\left|\frac{(d_1-c_1)(d_2-c_2)}{m_1-m_2}\right|$$



Area $= \dfrac{p_1p_2}{\sin\theta}$

12. Area of the parallelogram formed by the lines $a_1x + b_1y + c_1 = 0$, $a_1x + b_1y + d_1 = 0$ and $a_2x + b_2y + c_2 = 0$, $a_2x + b_2y + d_2 = 0$ is

(a) $\dfrac{(d_1-c_1)(d_2-c_2)}{[(a_1^2+b_1^2)(a_2^2+b_2^2)]^{1/2}}$

(b) $\dfrac{(d_1-c_1)(d_2-c_2)}{a_1a_2-b_1b_2}$

(c) $\dfrac{(d_1+c_1)(d_2+c_2)}{a_1a_2+b_1b_2}$

(d) $\dfrac{(d_1-c_1)(d_2-c_2)}{a_1b_2-a_2b_1}$

Solution

(d) Area $= \dfrac{p_1p_2}{\sin\theta} = p_1p_2 \operatorname{cosec}\theta$, where

$$p_1 = \frac{d_1-c_1}{\sqrt{(a_1^2+b_1^2)}};$$

$$p_2 = \frac{d_2-c_2}{\sqrt{(a_2^2+b_2^2)}}$$

Also $\tan\theta = \dfrac{a_1b_2-a_2b_1}{a_1a_2+b_1b_2}$.

Since $\text{cosec}^2\theta = 1 + \cot^2\theta$

or $$\text{cosec}^2\,\theta = \frac{(a_1a_2 + b_1b_2)^2 + (a_1b_2 - a_2b_1)^2}{(a_1b_2 - a_2b_1)^2}$$

$$= \frac{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}{(a_1b_2 - a_2b_1)^2}$$

Putting for p_1, p_2 and cosec θ,

we get Area $= \dfrac{(d_1 - c_1)(d_2 - c_2)}{a_1b_2 - a_2b_1}$

Illustration

Show that the area of the parallelogram formed by the lines $3y - 2x = a$, $2y - 3x + a = 0$, $2x - 3y + 3a = 0$ and $3x - 2y = 2a$ is $2a^2/5$.

Solution

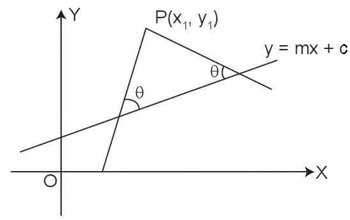
The set of paralle sides are

$3x - 2y - a = 0$, $3x - 2y - 2a = 0$

and $2x - 3y + a = 0$, $2x - 3y + 3a = 0$.

$$\therefore \quad \text{Area} = \left|\frac{(2a-a)(3a-a)}{\begin{vmatrix} 3 & -2 \\ 2 & -3 \end{vmatrix}}\right| = \frac{2a^2}{5}$$

13. Equation of straight lines through a given point $P(x_1, y_1)$ and inclined at an angle θ to a given line $y = mx + c$ are



$$y - y_1 = \frac{m + \tan\theta}{1 - m\tan\theta}(x - x_1)$$

and $$y - y_1 = \frac{m - \tan\theta}{1 + m\tan\theta}(x - x_1)$$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find the equation of the family of lines with x-intercept-3.

Solution

The x-intercept-3 implies that the members of the family of lines pass through the point $(-3, 0)$.

Hence, by using point slope form, the equation of such a family of lines is $y - 0 = m(x - (-3))$

$\Rightarrow \quad y = m(x + 3)$ where m is the parameter.

Remark: The above equation does not give the equation of the vertical line through the point $(-3, 0)$.

However, the equation of this line is $x = -3$, i.e., $x + 3 = 0$.

2. The lines $ax + 2y + 1 = 0$, $bx + 3y + 1 = 0$ and $cx + 4y + 1 = 0$ are concurrent, show that a, b, c are in A.P.

Solution

The equation of any line through the intersection of

$ax + 2y + 1 = 0$ and $bx + 3y + 1 = 0$ is

$ax + 2y + 1 + k(bx + 3y + 1) = 0$

We find k so that (1) and $cx + 4y + 1 = 0$ represent the same straight line, so we get

$$\frac{a + bk}{c} = \frac{2 + 3k}{4} = \frac{1 + k}{1}$$

From the last two members of (3), we get $4 + 4k = 2 + 3k$

$\Rightarrow \quad k = -2$

$\therefore \quad$ From (3), we get

$$\frac{a + (-2)b}{c} = \frac{1 - 2}{1}$$

$\Rightarrow \quad a - 2b = -c$

$\Rightarrow \quad a + c = 2b$

$\Rightarrow \quad a, b, c$ are in A.P.

3. Find the direction in which a straight line must be drawn through the point (–1, 2) so that its point of intersection with the line $x + y = 4$ may be at a distance of 3 units from this point.

[NCERT]

Solution

Let the required direction make an angle θ with the positive direction of x-axis, then the equation of the line through (–1, 2) and in the above said direction is $\frac{x-(-1)}{\cos\theta} = \frac{y-2}{\sin\theta} = r$ | distance form

Here | r | is the distance of (x, y) from (–1, 2)

Any point on this line is $(x, y) = (r\cos\theta - 1, r\sin\theta + 2)$

$$(\because \frac{x+1}{\cos\theta} = r \Rightarrow x = r\cos\theta - 1 \text{ and } \frac{y-2}{\sin\theta} = r \Rightarrow y = r\sin\theta + 2\ )$$

We are given that when this point lies on $x + y = 4$, then $|r| = 3$,

$\therefore$ $r\cos\theta - 1 + r\sin\theta + 2 = 4, |r| = 3$

$\Rightarrow$ $r(\cos\theta + \sin\theta) = 3, r^2 = 9$

$\Rightarrow$ $(\cos\theta+\sin\theta)^2 = \frac{9}{r^2}, r^2 = 9$

$\Rightarrow$ $\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta = \frac{9}{9}$

$\Rightarrow$ $1 + \sin 2\theta = 1$

$\Rightarrow$ $\sin 2\theta = 0$

$\Rightarrow$ $2\theta = 0°$ or $180°$

$\Rightarrow$ $\theta = 0°$ or $90°$.

$\therefore$ The required line is either parallel to x-axis or parallel to y-axis.

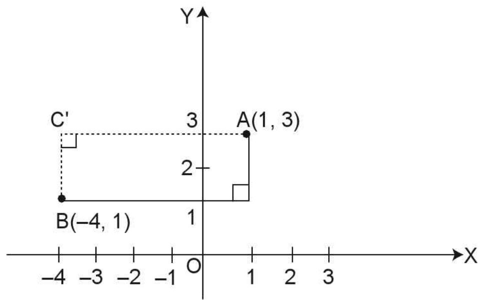
4. The hypotenuse of a right triangle has its ends at the points (1, 3) and (–4, 1). Find the equation of the legs (perpendicular sides) of the triangle if they are parallel to the axes.

[NCERT]

Solution

According to the conditions of the problem, we can have two possible right triangles as shown in the figure.

Either the sides are AC and BC with respective equations $x = 1$ and $y = 1$ or they are AC' and BC' with respective equations



$y = 3$

and $x = -4$.

5. If sum of the perpendicular distance of a variable point $P(x, y)$ from the lines $x + y - 5 = 0$ and $3x - 2y + 7 = 0$, is always 10. Show that P must move on a line. ***[NCERT]***

Solution

Given lines are $x + y - 5 = 0$ ......(i)

and $3x - 2y + 7 = 0$ ......(ii)

We are given that distance of (i) from $P(x, y)$ + distance of (ii) from $P(x, y) = 10$

$$\Rightarrow \frac{|x+y-5|}{\sqrt{1^2+1^2}} + \frac{|3x-2y+7|}{\sqrt{3^2+2^2}} = 10$$

$$\text{or} \quad \pm\frac{x+y-5}{\sqrt{2}} \pm \frac{3x-2y+7}{\sqrt{13}} = 10,$$

which is a set of four lines.

($\because$ A first degree equation in x and y always represents a line)

6. Find equation of the line mid way between the parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$. ***[NCERT]***

Solution

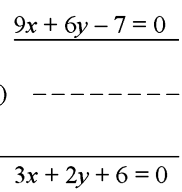
Given lines are

$3x + 2y + 6 = 0$ ....(i)

and $9x + 6y - 7 = 0$

or $3x + 2y - \frac{7}{3} = 0$ .....(ii)

(Dividing by 3)



Note that the lines (i) and (ii) are parallel

Any line parallel to these can be taken as

$3x + 2y + k = 0$ ....(iii)

This is mid-parallel to (i) and (ii) if it is equidistant from them.

$\therefore$ Distances of any point on (iii) from (i) and (ii) are equal.

We choose a point on (iii), by taking $x = 0$ so that $2y + k = 0$

$\Rightarrow \quad y = -\dfrac{k}{2}$

$\therefore$ The point $\left(0, -\dfrac{k}{2}\right)$ lies on (iii).

(iii) is mid–parallel to (i) and (ii) if distance of $\left(0, -\dfrac{k}{2}\right)$ from (i)

= distance of $\left(0, -\dfrac{k}{2}\right)$ from (ii)

i.e., if $\dfrac{\left|3\times 0+2\left(-\dfrac{k}{2}\right)+6\right|}{\sqrt{3^2+2^2}}$

$= \dfrac{\left|3\times 0+2\left(-\dfrac{k}{2}\right)-\dfrac{7}{3}\right|}{\sqrt{3^2+2^2}}$

i.e., if $|-k+6| = \left|-k-\dfrac{7}{3}\right|$

i.e., if $-k+6 = \pm\left(-k-\dfrac{7}{3}\right)$

i.e., if $-k+6 = -k-\dfrac{7}{3}$

or $-k+6 = k+\dfrac{7}{3}$

i.e., if $2k = 6-\dfrac{7}{3} = \dfrac{11}{3}$

i.e., if $k = \dfrac{11}{6}$

$\left(\because \quad -k+6 = -k-\dfrac{7}{3} \quad \textit{i.e.,} \quad 6 = -\dfrac{7}{3} \text{ is not possible}\right)$

$\therefore$ Required line is $3x+2y+\dfrac{11}{6}=0$

[from (iii)]

or $18x + 12y + 11 = 0$.

7. Show that the equation of the line through the origin and making an angle θ with the line $y = mx + c$ is $\dfrac{y}{x} = \dfrac{m+\tan\theta}{1-m\tan\theta}$ or $\dfrac{y}{x} = \dfrac{m-\tan\theta}{1+m\tan\theta}$.

[NCERT]

Solution

Slope of the given line $y = mx + c$ is m. Let M be the slope of the required line, then

$$\tan\theta = \left|\frac{M-m}{1+Mm}\right|$$

$$\Rightarrow \quad \tan\theta = \pm\left|\frac{M-m}{1+Mm}\right|$$

Case I When $\tan\theta = \dfrac{M-m}{1+Mm}$ then $\tan\theta + Mm$

$\tan\theta = M - m$

or $Mm\tan\theta - M = -m - \tan\theta$

$$\Rightarrow \quad M \equiv \frac{m+\tan\theta}{1-m\tan\theta}$$

$\therefore$ Equation of the required line through the origin is $y - 0 = M(x - 0)$

i.e., $\dfrac{y}{x} = \dfrac{m+\tan\theta}{1-m\tan\theta}$

Case II When $\tan\theta = -\left(\dfrac{M-m}{1+Mm}\right)$, then $\tan\theta$

$+ Mm\tan\theta = -M + m$

$\Rightarrow \quad M(1 + m\tan\theta) = m - \tan\theta$

$$\Rightarrow \quad M = \frac{m-\tan\theta}{1+m\tan\theta}.$$

$\therefore$ Equation of the required line through the origin is $y - 0 = M(x - 0)$

i.e., $\dfrac{y}{x} = \dfrac{m-\tan\theta}{1+m\tan\theta}$.

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

EXERCISE 1

1. A line cuts x-axis at point A and y-axis at point B. The point (2, 2) divides AB in the ratio 2 : 1. Find the equation of the line. *[NCT-2003]*

2. Find the value of p, so that three lines $3x + y - 2 = 0$, $px + 2y - 3 = 0$ and $2x - y = 3$ are concurrent. *[KVS-2004, NCERT]*

3. Find the equation of perpendicular bisector of the line segment joining the points (0, 3) and (–4, 1).

4. Find the equation of the line such that segment intercepted by the axes is divided by the point (–5, 4) in the ratio 1 : 2.

5. Find the equation of the line which has length of perpendicular segment from the origin to the line as 5 and the inclination of perpendicular segment with the positive direction of x-axis is 30°. *[NCERT]*

6. The base of an equilateral triangle is $x + 2y = 3$ and the vertex is (1, 1). Find the equations of the remaining sides.

7. If the lines whose equations are $y = m_1x + a_1$, $y = m_2x + a_2$ and $y = m_3x + a_3$ meet in a point then prove that $m_1(a_2 - a_3) + m_2(a_3 - a_1) + m_3(a_1 - a_2) = 0$. *[NCERT]*

8. The sides AB, BC, CD, DA of a quadrilateral have the equations $x + 2y = 3$, $x = 1$, $x - 3y = 4$, $5x + y = -12$ respectively. Find the angle between the diagonals AC and BD.

9. Find the equation of a line passing through the intersection of the lines $x - 3y + 1 = 0$, $2x + 5y - 9 = 0$ and whose distance from the origin is 2 units. *[NCT-2003]*

10. Find the image of the point (3, 8) with respect to the line $x + 3y = 7$ assuming the line to be a plane mirror. *[NCERT]*

11. If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined to the line $y = mx + 4$, find the value of m. *[NCERT]*

12. Find the equations of the lines passing through the point (3, –2) and inclined at an angle of 60° to line $\sqrt{3}x + y = 1$ *[MSE-2005]*

13. A ray of light passing through the point (1, 2) reflects on the x-axis at point A and the reflected ray passes through the point (5, 3). Find the coordinates of A. *[NCERT]*

EXERCISE 2

1. Show that the straight lines $7x - 2y + 10 = 0$, $7x + 2y - 10 = 0$ and $y + 2 = 0$ from an isosceles triangle and find its area.

2. Prove that the three lines $2x - 3y = 7$, $3x - 4y = 13$ and $8x - 11y = 33$ are concurrent.

3. For what value of k will the three lines $x - 2y + 1 = 0$, $2x - 5y + 3 = 0$ and $5x - 9y + k = 0$ be concurrent

4. Find the incentre of the triangle formed by the lines $y - 15 = 0$, $3x + 4y = 0$ and $12y - 5x = 0$

5. Find the equation of the line that is perpendicular to $3x + 2y - 8 = 0$ and passes through the mid-point of the line segment joining the points (5, –2) and (2, 2)

6. Find the equations of the two straight lines passing through the point (1, –1) and inclined at an angle of 45° to the line $2x - 5y + 7 = 0$

7. Find the equation of the line through the intersection of the lines $2x - y + 3 = 0$ and $x - 5y + 3 = 0$ and passing through the point (8, 5).

8. Find the equation of a line through the intersection of the lines $2x + 3y - 2 = 0$ and $x - 2y + 1 = 0$ and having x-intercept equal to 3.

9. Find the equation of the line through the point of intersection of the lines $5x - 3y = 1$ and $2x + 3y = 23$, and perpendicular to the line $5x - 3y - 1 = 0$

10. Find the distance between parallel lines
 (i) $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$
 (ii) $l(x + y) + p = 0$, and $l(x + y) - r = 0$ *[NCERT]*

11. Find the equation of a line drawn perpendicular to the line $\frac{x}{4}+\frac{y}{6}=1$ through the point, where it meets the y-axis. *[NCERT]*

12. Find the area of the triangle formed by the lines $y - x = 0$, $x + y = 0$ and $x - k = 0$. *[NCERT]*

13. Find the equations of the lines which pass through (4, 5) and make an angle of 45° with the line $2x + y + 1 = 0$.

14. Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines $x - 7y + 5 = 0$ and $3x + y = 0$. *[NCERT]*

15. Find the equation of the lines through the point (3, 2), which make an angle of 45° with the line $x - 2y = 3$. *[NCERT]*

ANSWERS

EXERCISE 1

1. $x + 2y - 6 = 0$
2. $p = 5$
3. $2x + y + 2 = 0$
4. $8x - 5y + 60 = 0$
5. $\sqrt{3}x+y-10=0$
6. $(2\sqrt{3}-1)x-(2+\sqrt{3})y+(3-\sqrt{3})=0,$
 $(2\sqrt{3}+1)x-(\sqrt{3}-2)y-(\sqrt{3}+3)=0$
7. $(-2, 3)$
8. 90°
9. $3x + 4y - 11 = 0$
10. $(-1, -4)$
11. $\frac{1\pm5\sqrt{2}}{7}$
12. $y + 2 = 0$; $\sqrt{3x}-y-3\sqrt{3}-2=0$
13. $\left(\frac{13}{5},0\right)$.

EXERCISE 2

1. 14 sq. units
3. $k = 4$
4. (1, 8)
5. $2x - 3y - 7 = 0$
6. $7x - 3y - 10 = 0$ and $3x + 7y + 4 = 0$
7. $x - 2y + 2 = 0$
8. $x + 5y - 3 = 0$
9. $63x + 105y - 781 = 0$
10. (i) 65/17 units
 (ii) $\frac{1}{\sqrt{2}}\left|\frac{p+r}{l}\right|$ units
11. $2x - 3y + 18 = 0$
12. k^2 square units
13. $3x - y - 7 = 0$ and $x + 3y - 19 = 0$
14. $x = -5/22$
15. $3x - y = 7$ & $x + 3y = 9$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. In the equation $y - y_1 = m(x - x_1)$ if m and x_1 are fixed and different lines are drawn for different values of y_1, then *[MP PET-1986]*
 (a) The lines will pass through a single point
 (b) There will be a set of parallel lines
 (c) There will be one line only
 (d) None of these

Solution

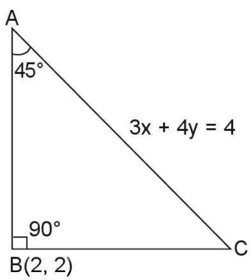
(b) Since m (gradient) and x_1 are fixed and y_1 is variate, then they formed a set of parallel lines because gradient of every line remains 'm'.

2. Equation of one of the sides of an isosceles right angled triangle whose hypotenuse is $3x + 4y = 4$ and the opposite vertex of the hypotenuse is (2, 2), will be ***[MNR-1986]***

(a) $x - 7y + 12 = 0$ (b) $7x + y - 12 = 0$
(c) $x - 7y + 16 = 0$ (d) $7x + y + 16 = 0$

Solution

(a) Since $\angle A = \angle C = 45°$. We have to find equation of AB.
Here let gradient of AB be m, then equation of AB is $y - 2 = m(x - 2)$ .... (i)
But angle between $3x + 4y = 4$ and (i) is $45°$.



So, $\tan 45° = \dfrac{m + \frac{3}{4}}{1 - \frac{3m}{4}} \Rightarrow m = \dfrac{1}{7}$

Hence, the required equation is $x - 7y + 12 = 0$ {By putting the value of m in (i)}

3. The equation of bisectors of the angles between the lines $|x| = |y|$ are ***[Orissa JEE-2002]***

(a) $y = \pm x$ and $x = 0$
(b) $x = 1/2$ and $y = 1/2$
(c) $y = 0$ and $x = 0$
(d) None of these

Solution

(c) The equation of lines are $x + y = 0$ and $x - y = 0$

$\therefore$ The equation of bisectors of the angles between these lines are $\dfrac{x+y}{\sqrt{1+1}} = \pm \dfrac{x-y}{\sqrt{1+1}}$

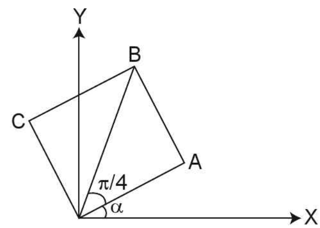
$\Rightarrow \quad x + y = \pm (x - y)$
Taking + ve sign, we get $y = 0$
Taking – ve sign, we get $x = 0$
Hence the equation of bisectors are
$x = 0, y = 0$

4. A square of side a lies above the x-axis and has one vertex at the origin. The side passing through the origin makes an angle α, $\left(0 < \alpha < \dfrac{\pi}{4}\right)$ with the positive direction of x-axis. The equation of its diagonal not passing through the origin is ***[AIEEE-2003]***

(a) $y(\cos\alpha - \sin\alpha) - x(\sin\alpha - \cos\alpha) = a$
(b) $y(\cos\alpha + \sin\alpha) - x(\sin\alpha - \cos\alpha) = a$
(c) $y(\cos\alpha + \sin\alpha) + x(\sin\alpha + \cos\alpha) = a$
(d) $y(\cos\alpha + \sin\alpha) + x(\sin\alpha - \cos\alpha) = a$

Solution

(b) Co-ordinates of $A = (a\cos\alpha, a\sin\alpha)$



Equation of OB, $y = \tan\left(\dfrac{\pi}{4} + \alpha\right) x \because CA \perp^r$ to OB

$\therefore$ Slope of $CA = -\cot\left(\dfrac{\pi}{4} + \alpha\right)$

Equation of CA, $y - a\sin\alpha = -\cot\left(\dfrac{\pi}{4} + \alpha\right)(x - a\cos\alpha)$

$\Rightarrow \quad y(\sin\alpha + \cos\alpha) + x(\cos\alpha - \sin\alpha) = a$
$\Rightarrow \quad y(\cos\alpha + \sin\alpha) - x(\sin\alpha - \cos\alpha) = a$

5. The equations of the sides of a triangle are $y = m_1x + c_1$, $y = m_2x + c_2$ and $x = 0$; its area is

(a) $\dfrac{1}{2}\dfrac{(c_1 - c_2)^2}{m_1 - m_2}$ (b) $\dfrac{1(c_1 + c_2)^2}{2m_1 - m_2}$

(c) $\dfrac{1(c_1 + c_2)^2}{2m_1 + m_2}$ (d) $\dfrac{1(c_1 - c_2)^2}{2m_1 + m_2}$

Solution

(a) Solving in pairs the coordinates of the vertices are $\left(\dfrac{c_1 - c_2}{m_1 - m_2}, \dfrac{m_2c_1 - m_1c_2}{m_2 - m_1}\right)$, $(0, c_1)$ and $(0, c_2)$

$$\text{Area} = \frac{1}{2}(c - d)\,h = \frac{1}{2}\frac{c_1 - c_2}{m_2 - m_1}\cdot(c_1 - c_2)$$
$$= \frac{1}{2}\frac{(c_1 - c_2)^2}{m_2 - m_1}.$$

6. The diagonals of the parallelogram whose sides are $lx + my + n = 0$, $lx + my + n' = 0$, $mx + ly + n = 0$
$mx + ly + n' = 0$ include an angle.
[EAMCET-1994]
(a) π (b) $\pi/2$
(c) $2\pi/3$ (d) $2\pi/5$

Solution

(b) The distances between two set of parallel sides is each $\frac{n - n'}{\sqrt{(l^2 + m^2)}}$ i.e., $p_1 = p_2$ hence the parallelogram is rhombus whose diagonals we know intersect at right angles.

7. Equations of the two straight lines passing through the point (3, 2) and making an angle of 45° with the line $x - 2y = 3$, are ***[AMU-1978]***
(a) $3x + y + 7 = 0$ and $x + 3y + 9 = 0$
(b) $3x - y - 7 = 0$ and $x + 3y - 9 = 0$
(c) $x + 3y - 7 = 0$ and $x + 3y - 9 = 0$
(d) None of these

Solution

(b) Slope of given line is 1/2.

Thus $\tan 45° = \pm\frac{m - \frac{1}{2}}{1 + m.\frac{1}{2}} \Rightarrow m = \pm 3$ Hence option (b) is correct

8. If the given lines $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ be concurrent, then
(a) $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$
(b) $m_1(c_2 - c_1) + m_2(c_3 - c_2) + m_3(c_1 - c_3) = 0$
(c) $c_1(m_2 - m_3) + c_2(m_3 - m_1) + c_3(m_1 - m_2) = 0$
(d) None of these

Solution

(a) $\begin{vmatrix} m_1 & -1 & c_1 \\ m_2 & -1 & c_2 \\ m_3 & -1 & c_3 \end{vmatrix} = 0$

$\Rightarrow m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$

NOTE

Students should remember this question as a formula.

9. If one of the diagonals of a square is along the line $x = 2y$ and one of its vertices is (3, 0), then its sides through this vertex are given by the equations
(a) $y - 3x + 9 = 0$, $3y + x - 3 = 0$
(b) $y + 3x + 9 = 0$, $3y + x - 3 = 0$
(c) $y - 3x + 9 = 0$, $3y - x + 3 = 0$
(d) $y - 3x + 3 = 0$, $3y + x + 9 = 0$

Solution

(a) $\tan(\pm 45°) = \frac{m - 1/2}{1 + m.1/2}$

$\therefore\ m = 3, -1/3.$

10. Circumcentre of a triangle represented by lines $x + y = 0$, $x - y = 0$ and $x - 7 = 0$ is
[Gujrat CET-2007]
(a) (7, 0) (b) (7/2, 0)
(c) (0, 7) (d) (7/2, 7/2)

Solution

(a) Let ABC be the triangle whose sides BC, CA, AB have the equations $x + y = 0$, $x - y = 0$ and $x - 7 = 0$

Solving these equations pairwise we can obtain the coordinates of the vertices A, B, C as $A(0, 0)$, $B(7, -7)$ and $C(7, 7)$ respectively. Let (x, y) be the coordinates to the circumcentre.

Centre O (say) then $OA = OB = OC$

Now $OA = OB \Rightarrow OA^2 = OB^2$

$x^2 + y^2 = x^2 + 49 - 14x + y^2 + 49 + 14y$

$14x - 14y - 98 = 0$ ....(i)

and $OB = OC \Rightarrow OB^2 = OC^2$

$(x - 7)^2 + (y + 7)^2 = (x - 7)^2 + (y - 7)^2$

$x^2 + 49 - 14x + y^2 + 49 + 14y$

$= x^2 + 49 - 14x + y^2 + 49 - 14y$

$14y = -14y \Rightarrow 28y = 0$

$\therefore\ y = 0$ ....(ii)

Substitute (ii) in (i), $14x - 98 = 0$

$\therefore\ x = 7$

Therefore coordinate of the circumcentre is (7, 0).

11. A straight line through the point (2, 2) intersects the lines $\sqrt{3}\,x + y = 0$ and $\sqrt{3}\,x - y = 0$

at the points A and B. The equation to the line AB so that the triangle OAB is equilateral is ***[DCE-2007]***

(a) $x - 2 = 0$ (b) $y - 2 = 0$
(c) $x + y - 4 = 0$ (d) None of these

Solution

(b) $\sqrt{3}\,x + y = 0$ makes an angle of 120° with OX and $\sqrt{3}\,x - y = 0$ makes an angle of 60° with OX.
So, the required line is $y - 2 = 0$.

12. Line L has intercepts a and b on the coordinate axes. When the axes are rotated through a given angle keeping the origin fixed, the same line L has intercepts p and q, then ***[IIT-90; Kurukshetra CEE-98; AIEEE-2002]***

(a) $a^2 + b^2 = p^2 + q^2$

(b) $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$

(c) $a^2 + p^2 = b^2 + q^2$

(d) $\frac{1}{a^2}+\frac{1}{p^2}=\frac{1}{b^2}+\frac{1}{q^2}$

Solution

(b) Suppose we ratate the coordinate axes in the anti-clockwise direction through an angle α. The eqaution of the line L with respect to old axes is $\frac{x}{a}+\frac{y}{b}=1$. In this question replacing x by $x\cos\alpha - y\sin\alpha$ and y by $x\sin\alpha + y\cos\alpha$, the equation of the line with respect to new axes is

$$\frac{x\cos\alpha - y\sin\alpha}{a}+\frac{x\sin\alpha + y\cos\alpha}{b}=1$$

$$\Rightarrow\ x\left(\frac{\cos\alpha}{a}+\frac{\sin\alpha}{b}\right)+y\left(\frac{\cos\alpha}{b}-\frac{\sin\alpha}{a}\right)=1 \quad \text{..............(i)}$$

The intercepts made by (i) on the co-ordinate axes are given as p and q.

Therefore, $\frac{1}{p}=\frac{\cos\alpha}{a}+\frac{\sin\alpha}{b}$

and $\frac{1}{q}=\frac{\cos\alpha}{b}-\frac{\sin\alpha}{a}$

Squaring and adding, we get

$$\frac{1}{p^2}+\frac{1}{q^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

NOTE

Students should remember this question as a formula.

13. Let L be the line $2x + y = 2$. If the axes are rotated by 45°, then the intercepts made by the line L on the new axes are respectively ***[Roorkee Qualifying-1998]***

(a) $\sqrt{2}$ and 1

(b) 1 and $\sqrt{2}$

(c) $2\sqrt{2}$ and $2\sqrt{2}/3$

(d) $2\sqrt{2}/3$ and $2\sqrt{2}$

Solution

(c, d) Suppose the axes are rotated in the anticlockwise direction through an angle 45°. To find the equation of L w.r.t. the new axes, we replace x by $x\cos\alpha - y\sin\alpha$ and by $x\sin\alpha + y\cos\alpha$, so that equation of line w.r.t. new axes is

$\Rightarrow$ 1/1($x\cos 45° - y\sin 45°$) + $\frac{1}{2}$($x\sin 45° + y\cos 45°$) = 1

Since p, q are the intercept made by the line on the coordinate axes. we have on putting (p, 0) and the (0, q)

$$\Rightarrow\ \frac{1}{p}=\frac{1}{a}\cos\alpha+\frac{1}{b}\sin\alpha$$

$$\Rightarrow\ \frac{1}{q}=-\frac{1}{a}\sin\alpha+\frac{1}{b}\cos\alpha$$

$$\Rightarrow\ \frac{1}{p}=\frac{1}{1}\cos 45°+\frac{1}{2}\sin 45°$$

$$\Rightarrow\ \frac{1}{p}=\frac{1}{\sqrt{2}}+\frac{1}{2}\cdot\frac{1}{\sqrt{2}}=\frac{3}{2\sqrt{2}}$$

$$\therefore\ p=\frac{2\sqrt{2}}{3};\ \therefore \frac{1}{q}=-\frac{1}{1}\sin 45°+\frac{1}{2}\cos 45°$$

$$\frac{1}{q}=\frac{-1}{\sqrt{2}}+\frac{1}{2\sqrt{2}}=-\frac{1}{2\sqrt{2}},\ \therefore q=2\sqrt{2}$$

So intercept made by is assume on the new axis ($2\sqrt{2}/3$, $2\sqrt{2}$). If the rotation is assume in clockwise direction, so intercept made by the line on the new axes would be ($2\sqrt{2}$, $2\sqrt{2}/3$).

14. The triangle formed by the lines $x + y - 4 = 0$, $3x + y = 4$, $x + 3y = 4$ is

[RPET-02; IIT-1983; MNR-1992; UPSEAT-01]

(a) Isosceles (b) Equilateral
(c) Right-angled (d) None of these

Solution

(a) The vertices of triangle are the intersection points of these given lines. The vertices of Δ are $A(0, 4)$, $B(1, 2)$, $C(4, 0)$

Now, $AB = \sqrt{(0-1)^2 + (4-1)^2} = \sqrt{10}$

$BC = \sqrt{(1-4)^2 + (0-1)^2} = \sqrt{10}$

$AC = \sqrt{(0-4)^2 + (0-4)} = 4\sqrt{2}$

$\because$ $AB = BC$; $\therefore$ Δ is isosceles.

15. If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined to the line $y = mx + 4$, then $m =$ ***[ISM Dhanbad-1976]***

(a) $\frac{1+3\sqrt{2}}{7}$ (b) $\frac{1-3\sqrt{2}}{7}$
(c) $\frac{1\pm 3\sqrt{2}}{7}$ (d) $\frac{1\pm 5\sqrt{2}}{7}$

Solution

(d) $m_1 = 3$, $m_2 = 1/2$ and $m_3 = m$
Let the angle between first and third line is θ_1 and between second and third is θ_2, then

$$\tan\theta_1 = \frac{3-m}{1+3m} \text{ and } \tan\theta_2 = \frac{m-\frac{1}{2}}{1+\frac{m}{2}}$$

$$\text{But } \theta_1 = \theta_2 \Rightarrow \frac{3-m}{1+3m} = \frac{m-\frac{1}{2}}{1+\frac{m}{2}}$$

$$\Rightarrow \quad 7m^2 - 2m - 7 = 0 \Rightarrow m = \frac{1\pm 5\sqrt{2}}{7}$$

16. The area of the parallelogram formed by the lines $y = mx$, $y = mx + 1$, $y = nx$ and $y = nx + 1$ equals

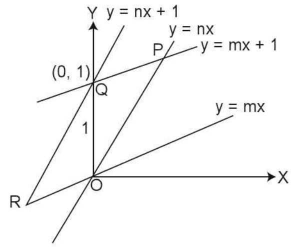
[IIT Screening-01]

(a) $\frac{|m+n|}{(m-n)^2}$ (b) $\frac{2}{|m+n|}$
(c) $\frac{1}{|m+n|}$ (d) $\frac{1}{|m-n|}$

Solution

(d) Solving $y = nx$ and $y = mx + 1$, we get

$$P = \left(\frac{1}{n-m}, \frac{n}{n-m}\right)$$



$\therefore$ Area of parallelogram $= 2 \times$ (area of ΔPOQ)

$$= 2\times\left|\frac{1}{2}\times OQ\times\frac{1}{n-m}\right| = \frac{1}{|n-m|} = \frac{1}{|m-n|}$$

17. Find the coordinates of the orthocentre of the trangle formed by the lines $3x - 2y - 6 = 0$, $3x + 4y + 12 = 0$ and $3x - 8y + 12 = 0$.

(a) $\left(-\frac{1}{6}, -\frac{23}{9}\right)$ (b) $\left(\frac{1}{6}, -\frac{23}{9}\right)$
(c) $\left(\frac{1}{6}, \frac{23}{9}\right)$ (d) $\left(\frac{-1}{6}, \frac{23}{9}\right)$

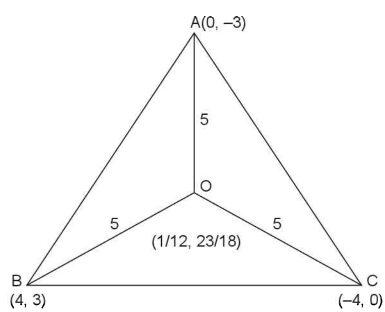
Solution

(a) Solving in pairs the coordinates of the vertices of the triangle are $A(0, -3)$, $B(4, 3)$, $C(-4, 0)$. If G be the centroid, then

$$G = \left(\frac{\Sigma x}{3}, \frac{\Sigma y}{3}\right) = (0, 0) \qquad ....(1)$$

If O be the circumcentre, then $OA = OB = OC$ gives $2x + 3y - 4 = 0$ and $8x - 6y + 7 = 0$.

Solving the above point O is $\left(\frac{1}{12}, \frac{23}{18}\right)$. If $H(\alpha, \beta)$ be the orthocentre, then G divides OH in the ratio $1 : 2$.



$$0 = \frac{\alpha + 2.(1/12)}{1+2} \text{ and } \frac{\beta + 2.(23/18)}{1+2}$$

$\therefore$ (α, β) is $\left(-\frac{1}{6}, -\frac{23}{9}\right)$.

18. Let α be the distance between the lines $-x + y = 2$ and $x - y = 2$, and β be the distance between the lines $4x - 3y = 5$ and $6y - 8x = 1$, then ***[J & K-2005]***

(a) $20\sqrt{2}\beta = 11\alpha$ (b) $20\sqrt{2}\alpha = 11\beta$

(c) $11\sqrt{2}\beta = 20\alpha$ (d) None of these

Solution

(a) Distance between lines $-x + y = 2$ and $x - y = 2$ is, $\alpha = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = 2\sqrt{2}$

Distance between lines $4x - 3y = 5$ and $6y - 8x = 1$ is, $\beta = \frac{1}{10} + \frac{5}{5} = \frac{11}{10}$

Therefore $\frac{\alpha}{\beta} = \frac{2\sqrt{2}}{11/10} = 20\sqrt{2}\beta = 11\alpha$.

19. The equation of the bisector of the actue angles between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$ is ***[MPPET-06]***

(a) $99x - 27y - 81 = 0$
(b) $11x - 3y + 9 = 0$
(c) $21x + 77y - 101 = 0$
(d) $21x + 77y + 101 = 0$

Solution

(c) The equations of the straight lines are $3x - 4y + 7 = 0$ .... (i) and $12x + 5y - 2 = 0$ ...... (ii) The equations of the bisectors of the angles between these lines are

$$\frac{3x - 4y + 7}{\sqrt{3^2 + 4^2}} = \pm \frac{12x + 5y - 2}{\sqrt{12^2 + 5^2}}$$

$\therefore$ Required equation of the bisector of the acute angle between these lines is

$$\frac{3x - 4y + 7}{5} = \frac{12x + 5y - 2}{13}$$

$\Rightarrow$ $39x - 52y + 91 = 60x + 25y - 10$

$\therefore$ $21x + 77y - 101 = 0$

20. The points (4, 1) undergoes the following three transformations successively

(i) Reflection about the line $y = x$
(ii) Translation through a distance 2 units along the positive direction of x-axis.
(iii) Rotation through an angle $\pi/4$ about the origin in the anti-clockwise direction. The final position of the point is given by the coordinates

[IIT-1980]

(a) $\left(\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (b) $(-\sqrt{2}, 7\sqrt{2})$

(c) $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (d) $(\sqrt{2}, 7\sqrt{2})$

Solution

(c) $(x_1, y_1) \to \left(\frac{y_1 - 1}{x_1 - 4}\right) = -1$

and $\frac{x_1 + 4}{2} = \frac{y_1 + 1}{2}$

$\Rightarrow$ $x_1 + y_1 = 5$ and $x_1 - y_1 = -3$

$\Rightarrow$ $x_1 = 1, y_1 = 4$

2nd operation $\Rightarrow$ (3, 4)

3rd operation

$$\Rightarrow \left(\frac{3}{\sqrt{2}} - \frac{4}{\sqrt{2}}, \frac{3}{\sqrt{2}} + \frac{4}{\sqrt{2}}\right) = \left(\frac{-1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$$

21. The equation to the sides of a triangle are $x - 3y = 0$, $4x + 3y = 5$ and $3x + y = 0$. The line $3x - 4y = 0$ passes through ***[EAMCET-1994]***

(a) The incentre
(b) The centroid
(c) The circumcentre
(d) The orthocentre of the triangle

Solution

(d) Two sides $x - 3y = 0$ and $3x + y = 0$ of the given triangle are perpendicular to each other. Therefore its orthocentre is the point of intersection of $x - 3y = 0$ and $3x + y = 0$ i.e., (0, 0). Clearly (0, 0) satisfies $3x - 4y = 0$.

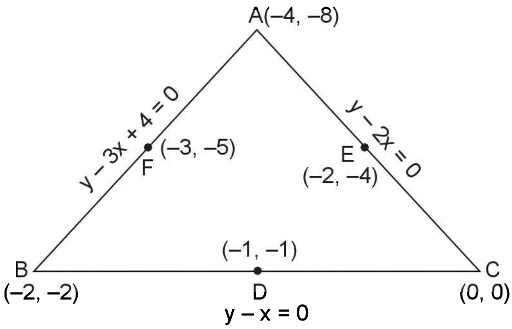
22. The sides of a triangle are $y = x$, $y = 2x$ and $y = 3x + 4$. Find the coordinates of its centroid.

(a) $\left(-2, \frac{-10}{3}\right)$ (b) $\left(2, \frac{10}{3}\right)$

(c) $\left(2, \frac{-10}{3}\right)$ (d) $\left(-2, \frac{10}{3}\right)$

Solution

(a) Solving the given equations in pairs the coordinates of the vertices of the triangle are $A(-4, -8), B(-2, -2), C(0, 0)$.



The mid-points of the sides of the triangle are $D(-1, -1), E(-2, -4), F(-3, -5)$.

Equation to median AD joining $(-4, -8)$ and $(-1, -1)$ is

$$y + 8 = \frac{-1-(-8)}{-1-(-4)}(x+4)$$

or $\quad 3(y + 8) = 7(x + 4)$

or $\quad 7x - 3y - 4 = 0 \qquad$ ....(1)

Similarly, the equations to medians BE and CF are respectively $x + 2 = 0 \qquad$ ....(2)

and $\quad 5x - 3y = 0 \qquad$ ....(3)

NOTE

If $y - y_1 = m(x - x_1)$

or $\dfrac{y-y_1}{x-x_1} = m = \infty$, then $x - x_1 = 0$

Solving any two of (1), (2) and (3), we get the point of intersection as $(-2, -10/3)$ which clearly satisfies the third. Hence the three medians are concurrent and the point of concurrency is called centroid of the triangle.

We can, however, find the coordinates of the centroid by using the formula

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

i.e., $\left(-2, \dfrac{-10}{3}\right)$.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The equation of the bisector of the acute angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$ is

[IIT-1975, 1983; RPET-2003; UPSEAT-2004]

(a) $21x + 77y - 101 = 0$
(b) $11x - 3y + 9 = 0$
(c) $31x + 77y + 101 = 0$
(d) $11x - 3y - 9 = 0$

2. Equation of angle bisectors between x and y-axes are ***[MP PET-1984]***

(a) $y = \pm x$ (b) $y = \pm 2x$
(c) $y = \pm\dfrac{1}{\sqrt{2}}x$ (d) $y = \pm 3x$

3. The distance between the lines $3x + 4y = 9$ and $6x + 8y = 15$ is

[MNR-1982; RPET-1995; MP PET-2002]

(a) 3/2 (b) 3/10
(c) 6 (d) None of these

4. The distance between the lines $3x - 2y = 1$ and $6x + 9 = 4y$ is ***[MP PET-1998]***

(a) $1/\sqrt{52}$ (b) $11/\sqrt{52}$
(c) $4/\sqrt{13}$ (d) $6/\sqrt{13}$

5. The position of the point $(8, -9)$ with respect to the lines $2x + 3y - 4 = 0$ and $6x + 9y + 8 = 0$ is

(a) Point lies on the same side of the lines
(b) Point lies on the different sides of the lines
(c) Point lies on one of the line
(d) None of these

6. If the lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ be concurrent, then

[IIT-1985; DCE-2000, 02]

(a) $a^3 + b^3 + c^3 + 3abc = 0$
(b) $a^3 + b^3 + c^3 - abc = 0$
(c) $a^3 + b^3 + c^3 - 3abc = 0$
(d) None of these

7. The lines $15x - 18y + 1 = 0$, $12x + 10y - 3 = 0$ and $6x + 66y - 11 = 0$ are ***[AMU-1978]***
(a) Parallel (b) Perpendicular
(c) Concurrent (d) None of these

8. If the lines $ax + 2y + 1 = 0$, $bx + 3y + 1 = 0$ and $cx + 4y + 1 = 0$ are concurrent, then a, b, c are in
(a) A.P. (b) G.P.
(c) H.P. (d) None of these

9. The straight lines $4ax + 3by + c = 0$ where $a + b + c = 0$, will be concurrent, if point is ***[RPET-2002]***
(a) (4, 3) (b) (1/4, 1/3)
(c) (1/2, 1/3) (d) None of these

10. The area of a parallelogram formed by the lines $ax \pm by \pm c = 0$, is ***[IIT-1973]***
(a) $\frac{c^2}{ab}$ (b) $\frac{2c^2}{ab}$
(c) $\frac{c^2}{2ab}$ (d) None of these

11. Locus of the points which are at equal distance from $3x + 4y - 11 = 0$ and $12x + 5y + 2 = 0$ and which is near the origin is ***[MNR-1987]***
(a) $21x - 77y + 153 = 0$
(b) $99x + 77y - 133 = 0$
(c) $7x - 11y = 19$
(d) None of these

12. The equations of the lines through the origin making an angle of 60° with the line $x + y\sqrt{3} + 3\sqrt{3} = 0$ are
(a) $y = 0, x - y\sqrt{3} = 0$
(b) $x = 0, x - y\sqrt{3} = 0$
(c) $x = 0, x + \sqrt{3} = 0$
(d) $y = 0, x + y\sqrt{3} = 0$

13. The equation of the bisectors of the angles between the lines $3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$ is
(a) $99x - 77y + 51 = 0$, $21x + 27y - 131 = 0$
(b) $99x - 77y + 131 = 0$, $21x + 27y - 51 = 0$
(c) $99x - 77y + 51 = 0$, $21x + 27y + 131 = 0$
(d) None of these

14. The angle between two lines is $\pi/4$. If the slope of one of them be 1/2, then the slope of the other line is
(a) 1, –1/3 (b) – 1, 1/2
(c) – 1/3, 3 (d) None of these

15. Which of the following lines is concurrent with the lines $3x + 4y + 6 = 0$ and $6x + 5y + 9 = 0$
(a) $2x + 3y + 5 = 0$ (b) $3x + 3y + 5 = 0$
(c) $7x + 9y + 3 = 0$ (d) None of these

16. Distance between the two parallel lines $y = 2x + 7$ and $y = 2x + 5$ is ***[Orissa JEE-2004]***
(a) $\frac{\sqrt{5}}{2}$ (b) $\frac{2}{5}$
(c) $\frac{2}{\sqrt{5}}$ (d) $\frac{1}{\sqrt{5}}$

17. Coordinates of the orthocentre of the triangle whose sides are $x = 3$, $y = 4$ and $3x + 4y = 6$, will be
(a) (0, 0) (b) (3, 0)
(c) (0, 4) (d) (3, 4)

18. Two vertices of a triangle are (4, –3) and (–2, 5). If the orthocentre of the triangle is at (1, 2), then the third vertex is
(a) (–33, –26) (b) (33, 26)
(c) (26, 33) (d) None of these

19. The line L given by $\frac{x}{5} + \frac{y}{b} = 1$ passes through the point (13, 32). The line K is parallel to L and has the equation $\frac{x}{c} + \frac{y}{3} = 1$. Then the distance between L and K is ***[AIEEE-2010]***
(a) $\sqrt{17}$ (b) $\frac{17}{\sqrt{15}}$
(c) $\frac{23}{\sqrt{17}}$ (d) $\frac{23}{\sqrt{15}}$

SOLUTIONS

1. (b) Equation of bisector is

$$\frac{3x-4y+7}{5}=\frac{(-12x-5y+2)}{13} \quad \text{.......... (1)}$$

Here $a_1a_2 + b_1b_2 = 3\,(-12) - 4\,(-5) = -36 + 20 < 0$

Hence bisector given by (1) is acute bisector.

Now (1) $\Rightarrow 39x - 52y + 91 = -60x - 25y + 10$

$\Rightarrow\ 99x - 27y + 81 = 0$

$\Rightarrow\ 11x - 3y + 9 = 0$

2. (a) Bisector of the angle between x and y-axes will make either 45° or (180° – 45°) with x-axis.

Hence their joint equation is $y = \pm x$.

3. (b) Distance between two parallel lines $d = |P_1 - P_2|$

$$d=\left|\frac{-9}{\sqrt{3^2+4^2}}-\frac{(-15)}{\sqrt{6^2+8^2}}\right|$$

$$d=\left|\frac{-9}{5}+\frac{15}{10}\right| \text{ or } d=\frac{3}{10}$$

4. (b) Distance between parallel lines $d = |p_1 - p_2|$ where p_1, p_2 are lengths of perpendiculars from origin to lines length of perpendiculars from (0, 0) to $3x - 2y - 1 = 0$

$$p_1=\frac{-1}{\sqrt{9+4}}=\frac{-1}{\sqrt{13}}$$

similarly $$p_2=\frac{9}{\sqrt{36+16}}=\frac{9}{\sqrt{52}}=\frac{9}{2\sqrt{13}}$$

$$d=\left|\frac{-1}{\sqrt{13}}-\frac{9}{2\sqrt{13}}\right|=\frac{11}{2\sqrt{13}}$$

5. (a) $L_1(8, -9) = 2(8) + 3(-9) - 4 = -15$

$L_2(8, -9) = 6(8) + 9(-9) + 8 = -25$

Hence point lies on same side of the lines.

6. (c) Lines are concurrent if $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$

$\Rightarrow\ a^3 + b^3 + c^3 - 3abc = 0.$

7. (c) $3\,(12x + 10y - 3) - 2(15x - 18y + 1) = 6x + 66y - 11 = 0$

Hence the lines are concurrent.

8. (a) It is given that the lines $ax + 2y + 1 = 0$, $bx + 3y + 1 = 0$ and $cx + 4y + 1 = 0$ are concurrent, therefore $\begin{vmatrix} a & 2 & 1 \\ b & 3 & 1 \\ c & 4 & 1 \end{vmatrix}=0$

$\Rightarrow\ -a + 2b - c = 0$

$\Rightarrow\ 2b = a + c$

$\Rightarrow\ a, b, c$ are in A.P.

9. (b) Given equation can be written as

$3ax + 4\,(-a - c)y + c = 0$

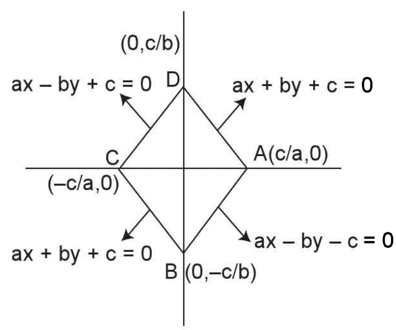
$\Rightarrow\ a\,(3x - 4y) + c\,(1 - 4y) = 0$

Hence fixed point is given by $3x - 4y = 0$ and $1 - 4y = 0$

i.e., (1/3, 1/4).

10. (b) Clearly given parallelogram is a rhombus whose area $=\frac{1}{2}$ (Product of diagonals)

Area $=\frac{1}{2}\,(AC.BD)$



$$=\frac{1}{2}\left(\frac{2c}{a}\right)\left(\frac{2c}{b}\right)=\frac{2c^2}{|ab|}$$

11. (b) Given $3x + 4y - 11 = 0$ .... (1),

$12x + 5y + 2 = 0$ .... (2)

According to the given conditions for locus, internal bisector of (1) and (2) is the required locus.

$$\therefore\quad \frac{-3x-4y+11}{\sqrt{3^2+4^2}}=\frac{(12x+5y+2)}{\sqrt{(12^2+5^2)}},$$

here $a_1a_2 + b_1b_2 = -3\,(12) - 4\,(5) < 0$

$\therefore$ origin lies in acute angle.

This $\Rightarrow 13(-3x - 4y + 11) = 5(12x + 5y + 2)$

$\Rightarrow\ 99x + 77y - 133 = 0.$

12. (b) Since the line $x+y\sqrt{3}+3\sqrt{3}=0$ makes an angle of 150° with x-axis. Therefore, the required lines will make angles of 90° and 210° i.e., 30° with the positive direction of x-axis.

Hence the lines are $x = 0$ and $y=\frac{1}{\sqrt{3}}x$

13. (a) Equations of the lines are

$3x - 4y + 7 = 0$ ....(i)

and $12x - 5y - 8 = 0$ ....(ii)

∴ Equations of the bisectors are

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

$$\Rightarrow \frac{3x-4y+7}{\sqrt{3^2+4^2}}=\pm\frac{-12x+5y+8}{\sqrt{12^2+5^2}}$$

$$\Rightarrow \frac{3x-4y+7}{5}=\pm\frac{-12x+5y+8}{13}$$

$\Rightarrow$ $39x - 52y + 91 = \pm(-60x + 25y + 40)$

Now, taking positive sign,

$39x - 52y + 91 = -60x + 25y + 40$

$\Rightarrow$ $99x - 77y + 51 = 0$ .... (3)

It is the equation of one bisector.

Again, taking negative sign,

$39x - 52y + 91 = 60x + 25y + 40$

$\Rightarrow$ $21x + 27y - 131 = 0$ .... (4)

It is the equation of the other bisector.

14. (c) Given $\theta=\frac{\pi}{4}, m_1=\frac{1}{2}$

$$\therefore \tan\theta=\frac{m_1-m_2}{1+m_1m_2}$$

$$\tan\frac{\pi}{4}=\frac{\frac{1}{2}-m_2}{1+\frac{m_2}{2}}$$

$$\Rightarrow 1+\frac{m_2}{2}=\frac{1}{2}-m_2$$

$$m_2+\frac{m_2}{2}=\frac{1}{2}-1$$

$$\frac{3m_2}{2}=\frac{-1}{2}$$

$$m_2=\frac{-1}{3}$$

OR

$$1+\frac{m_2}{2}=m_2-\frac{1}{2}$$

$$1+\frac{1}{2}=\frac{m_2}{2}$$

$$\frac{3}{2}=\frac{m_2}{2} \Rightarrow m_2=3$$

15. (b) Check by options

From option (a), $\begin{vmatrix}3&4&6\\6&5&9\\2&3&5\end{vmatrix}=3(25-27)-4(12)$

$+ 6(8) \neq 0$

From option (b), $\begin{vmatrix}3&4&6\\6&5&9\\2&3&5\end{vmatrix}=3(25-27)-4(3)$

$+ 6(3) = 0$.

16. (c) Given lines are $2x - y + 7 = 0$

and $2x - y + 5 = 0$

Both the lines are one same side of origin.

Distance between two parallel lines

$$=\frac{7-5}{\sqrt{2^2+1^2}}=\frac{2}{\sqrt{5}}$$

17. (d) Obviously it is a right angled at (3, 4). Hence the orthocentre is (3, 4).

18. (b) Slope of $AD=\frac{k-2}{h-1}$, slope of

$$BC=\frac{5+3}{-2-4}=-\frac{4}{3}.$$

$$AD\perp BC \Rightarrow \left(\frac{k-2}{h-1}\right)\left(-\frac{4}{3}\right)=-1$$

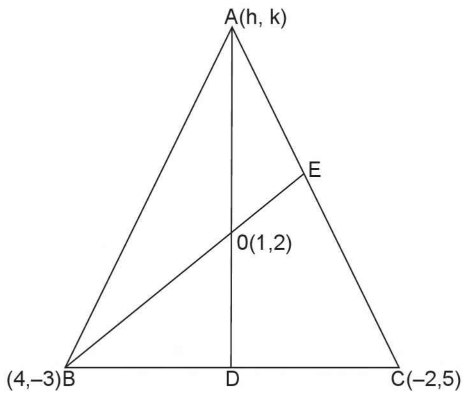
$\Rightarrow$ $3h - 4k + 5 = 0$ .... (1)

Slope of BO or BE is $\frac{-3-2}{4-1}=-\frac{5}{3}$

Slope of AC is $\frac{k-5}{h+2}$.

Again $AC \perp BE$

$$\Rightarrow \left(\frac{k-5}{h+2}\right)\left(-\frac{5}{3}\right)=-1$$



$\Rightarrow \quad 3h - 5k + 31 = 0$ .... (2)

Solving (1) and (2), we get $C(33, 26)$.

19. (c) $\frac{x}{5}+\frac{y}{b}=1$

$$\frac{13}{5}+\frac{32}{b}=1 \Rightarrow \frac{32}{b}=-\frac{8}{5}\Rightarrow b=-20$$

$$\frac{x}{5}-\frac{y}{20}=1\Rightarrow 4x-y=20$$

Line K has same slope

$$\Rightarrow \quad -\frac{3}{c}=4$$

$$c=-\frac{3}{4}\Rightarrow 4x-y=-3$$

$$\text{Distance } =\frac{23}{\sqrt{17}}$$

Hence correct option is (c).

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The three straight lines $ax + by = c$, $bx + cy = a$ and $cx + ay = b$ are concurrent, if ***[MP PET-2004]***

(a) $a + b + c = 0$ (b) $b + c = a$
(c) $c + a = b$ (d) $a + b = c$

2. Equation of angle bisector between the lines $3x + 4y - 7 = 0$ and $12x + 5y + 17 = 0$ are ***[RPET-1995]***

(a) $\frac{3x+4y-7}{\sqrt{25}}=\pm\frac{12x+5y+17}{\sqrt{169}}$

(b) $\frac{3x+4y+7}{\sqrt{25}}=\frac{12x+5y+17}{\sqrt{169}}$

(c) $\frac{3x+4y+7}{\sqrt{25}}=\pm\frac{12x+5y+17}{\sqrt{169}}$

(d) None of these

3. The bisector of the acute angle formed between the lines $4x - 3y + 7 = 0$ and $3x - 4y + 14 = 0$ has the equation ***[Pb. CET-2004]***

(a) $x + y + 3 = 0$
(b) $x - y - 3 = 0$
(c) $x - y + 3 = 0$
(d) $3x + y - 7 = 0$

4. The distance between two parallel lines $3x + 4y - 8 = 0$ and $3x + 4y - 3 = 0$, is given by ***[MP PET-1984]***

(a) 4 (b) 5
(c) 3 (d) 1

5. Distance between the lines $5x + 3y - 7 = 0$ and $15x + 9y + 14 = 0$ is ***[Kerala (Engg.)-2002]***

(a) $\frac{35}{\sqrt{34}}$ (b) $\frac{1}{3\sqrt{34}}$
(c) $\frac{35}{3\sqrt{34}}$ (d) $\frac{35}{2\sqrt{34}}$

6. Choose the correct statement which describe the position of the point $(-6, 2)$ relative to straight lines $2x + 3y - 4 = 0$ and $6x + 9y + 8 = 0$ ***[MP PET-1983]***

(a) Below both the lines
(b) Above both the lines
(c) In between the lines
(d) None of these

7. For what value of 'a' the lines $x = 3, y = 4$ and $4x - 3y + a = 0$ are concurrent ***[RPET-1984]***

(a) 0 (b) –1
(c) 2 (d) 3

8. Three lines $3x - y = 2$, $5x + ay = 3$ and $2x + y = 3$ are concurrent, then a equals

[MP PET-1996]

(a) 2 (b) 3
(c) – 1 (d) –2

9. The area of the parallelogram formed by lines $4y - 3x - a = 0$, $3y - 4x + a = 0$, and $4y - 3x - 3a = 0$, $3y - 4x + 2a = 0$ is

(a) $\frac{2}{7}a^2$ (b) $\frac{2}{3}a^2$
(c) $\frac{2}{5}a^2$ (d) None of these

10. The orthocentre of the triangle formed by the lines $4x - 7y + 10 = 0$, $x + y = 5$ and $7x + 4y = 15$, is

(a) (1, 2) (b) (1, –2)
(c) (–1, –2) (d) (–1, 2)

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 10 minutes.
3. The worksheet consists of 10 questions. The maximum marks are 30.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The incentre of triangle formed by the lines $x = 0, y = 0$ and $3x + 4y = 12$ is
(a) (1/2, 1/2)
(b) (1, 1)
(c) (1, 1/2)
(d) (11/2, 1)

2. If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ (a, b, c being distinct and different from 1) are concurrent, then $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=$
(a) 0 (b) 1
(c) $\frac{1}{a+b+c}$ (d) None of these

3. A straight line$(\sqrt{3}-1)x = (\sqrt{3}+1)y$ makes an angle 75° with another straight line which passes through origin. Then the equation of the line is
(a) $x = 0$ (b) $y = 0$
(c) $x + y = 0$ (d) $x - y = 0$

4. The ratio in which the line $3x + 4y + 2 = 0$ divides the distance between $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$, is
(a) 7 : 3 (b) 3 : 7
(c) 2 : 3 (d) None of these

5. A straight line through the origin O meets the parallel lines $4x + 2y = 9$ and $2x + y + 6 = 0$ at points P and Q respectively. Then the point O divides the segment PQ in the ratio
(a) 1 : 2 (b) 3 : 4
(c) 2 : 1 (d) 4 : 3

6. If a line joining two points $A(2, 0)$, $B(3, 1)$ is rotated about A in anti-clockwise direction through an angle 15° such that the point B goes to C in the new position, then the coordinates of C are
(a) $\left(2+\frac{1}{\sqrt{2}}, \sqrt{3}\right)$ (b) $\left(2+\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$
(c) $\left(2+\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(2-\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{2}\right)$

7. Coordinate axes are rotated in anticlockwise direction at 60° angle. If the intercepts made by line $x + y = 1$ on new coordinates axes be p and q then $\frac{1}{p^2}+\frac{1}{q^2}$ is equal to
(a) 1 (b) 2
(c) 3 (d) 4

8. The straight line $3x + 4y - 5 = 0$ and $4x = 3y + 15$ intersect at the point P. On these lines the points Q and R are chosen so that $PQ = PR$. The slopes of the lines QR passing through (1, 2) are

[Keral PET-2007]

(a) – 7, 1/7 (b) 7, 1/7
(c) 7, –1/7 (d) 3, –1/3

9. The point (4, 1) undergoes the following two successive transformation. (i) Reflection about the line $y = x$ (ii) Translation through a distance 2 units along the positive x-axis Then the final coordinates of the point are

[MNR-1987; UPSEAT-2000]

(a) (4, 3)
(b) (3, 4)
(c) (1, 4)
(d) (7/2, 7/2)

10. The value of k for which the lines $7x - 8y + 5 = 0$, $3x - 4y + 5 = 0$ and $4x + 5y + k = 0$ are concurrent is given by

[MP PET-1993]

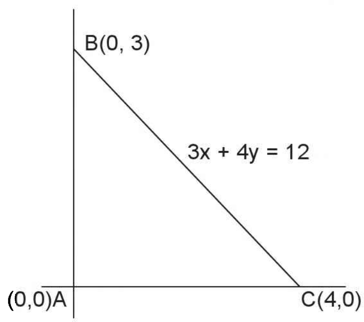
(a) – 45 (b) 44
(c) 54 (d) – 54

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. (b) Here $a = BC = 5$, $b = AC = 4$, $c = AB = 3$



Hence incentre is

$$\left(\frac{0+0+3\times4}{5+4+3}, \frac{0+4\times3+0}{5+4+3}\right) = (1,1)$$

2. (b) If the given lines are concurrent, then

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$$

{Apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$}

$\Rightarrow$ $a(b-1)(c-1) - (b-1)(1-a) - (c-1)(1-a) = 0$

$$\Rightarrow \frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

{Divide by $(1-a)(1-b)(1-c)$}

$$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

3. (a) We know that $\frac{\sqrt{3}+1}{\sqrt{3}-1} = \tan 75°$

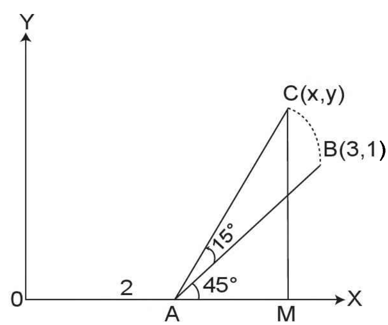
Hence, the line makes an angle of 75° with y-axis, so the equation of y-axis is $x = 0$.

5. (d) Equation of the line passing through origin perpendicular to given lines is $x - 2y = 0$
Required ratio = ratio between distance of the two lines from the origin

$$= \frac{\frac{6}{\sqrt{5}}}{\frac{9}{\sqrt{120}}} = \frac{12}{9} = \frac{4}{3}$$

6. (d) $\angle BAX = \tan^{-1}\left(\frac{1-0}{3-2}\right) = 45°$

$\Rightarrow$ $\angle CAX = 60°$.



$$AC = AB = \sqrt{(1+1)} = \sqrt{2}$$

$\therefore$ $x = OA + AM = 2 + \sqrt{2}\cos 60° = 2 + \sqrt{2}/2 = (4 + \sqrt{2})/2$, and $y = CM = \sqrt{2}\sin 60° = \sqrt{2}\,.\,\sqrt{3}/2 = \sqrt{6}/2$

$\therefore$ Coordinates of C are

$$\left[\frac{1}{2}(4+\sqrt{2}), \frac{1}{2}\sqrt{6}\right]$$

7. (b) Since the origin remains the same.
So, lengths of the perpendicular from the origin on the line in its positions $x + y = 1$ and $\frac{x}{p}+\frac{y}{q}=1$ are equal.

Therefore, $\frac{1}{\sqrt{1^2+1^2}} = \frac{1}{\sqrt{\frac{1}{p^2}+\frac{1}{q^2}}}$

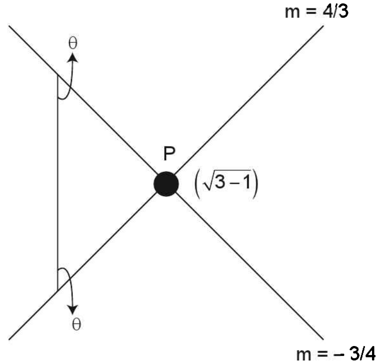
$\Rightarrow \quad \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{\frac{1}{p^2}+\frac{1}{q^2}}}$

$\Rightarrow \quad \frac{1}{p^2}+\frac{1}{q^2} = 2$

8. (a) $3x + 4y = 5$
$4x - 3y = 15$
their point of intersection
$\Rightarrow \quad x = 3$ and $y = -1$

$$\frac{\frac{4}{3}-m}{1+\frac{4}{3}\times m} = \frac{m+\frac{3}{4}}{1+\left(-\frac{3}{4}\right)m}$$

$\Rightarrow \quad \frac{4-3m}{3+4m} = \frac{4m+3}{4-3m}$



$16 + 9m^2 - 24m = 16m^2 + 9 + 24m$
$7m^2 + 48m - 7 = 0$
$7m^2 + 49m - m - 7 = 0$
$7m(m + 7) - 1(m + 7) = 0$

$\Rightarrow \quad m = -7$ and $m = \frac{1}{7}$

$\Rightarrow \quad -7, \frac{1}{7}$

9. (d) (i) On taking reflection of point (4, 1) w.r.t. line $y = x$, it becomes (1, 4).
(ii) Now we move only 2 unit along x-axis.
$\therefore \quad (1 + 2, 4 + 0) = (3, 4)$

LECTURE 5

Straight Line 5

(Miscellaneous questions, revision of straight lines, some harder problems)

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. In the triangle ABC with vertices $A(2, 3)$, $B(4, -1)$ and $C(1, 2)$, find the equation and length of altitude from the vertex A.

[NCERT]

Solution

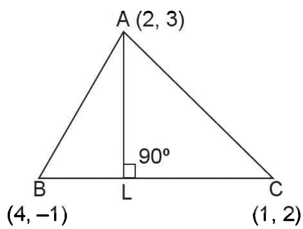
Slope of $BC = \dfrac{2-(-1)}{1-4} = -\dfrac{3}{3} = -1$

$\therefore$ Equation of BC is

$y - (-1) = -1(x - 4)$ [Point-slope form]

or $y + 1 = -x + 4$

or $x + y - 3 = 0$



Also, slope of $AL = 1$

$\therefore$ Equation of AL is $y - 3 = 1(x - 2)$ or $x - y + 1 = 0$

Length of perpendicular from $A(2, 3)$ on BC

$\dfrac{|2+3-3|}{\sqrt{1^2+1^2}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}.$

2. Prove that the product of the lengths of the perpendiculars drawn from the points $(\sqrt{a^2-b^2}, 0)$ and $(-\sqrt{a^2-b^2}, 0)$ to the line $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta - 1$ is b^2.

[NCERT]

Solution

Distance of the given line from

$$(\sqrt{a^2-b^2}, 0) = \frac{\left|\dfrac{\sqrt{a^2-b^2}}{a}\cos\theta - 1\right|}{\sqrt{\left(\dfrac{\cos\theta}{a}\right)^2 + \left(\dfrac{\sin\theta}{b}\right)^2}} = p_1 \text{(say)}$$

and distance of the given line from

$$(-\sqrt{a^2-b^2}, 0) = \frac{\left|-\dfrac{\sqrt{a^2-b^2}}{a}\cos\theta - 1\right|}{\sqrt{\left(\dfrac{\cos\theta}{a}\right)^2 + \left(\dfrac{\sin\theta}{b}\right)}} = p_2 \text{(say)}$$

Product of these two perpendiculars $= p_1p_2$

$$= \frac{\left|\left(\dfrac{\sqrt{a^2-b^2}}{a}\cos\theta - 1\right)\left(-\dfrac{\sqrt{a^2-b^2}}{a}\cos\theta - 1\right)\right|}{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}$$

$$= \frac{\left|\left(\dfrac{a^2-b^2}{a^2}\right)\cos^2\theta - 1\right|}{\dfrac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2}}$$

$$= \frac{|a^2\cos^2\theta - b^2\cos^2\theta - a^2|\,a^2b^2}{a^2(a^2\sin^2\theta + b^2\cos^2\theta)}$$

$$= \frac{|-(a^2\sin^2\theta + b^2\cos^2\theta)|\,b^2}{a^2\sin^2\theta + b^2\cos^2\theta}$$

($\because$ $a^2\cos^2\theta - a^2 = a^2(\cos^2\theta - 1) = -a^2\sin^2\theta$)

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)b^2}{a^2\sin^2\theta + b^2\cos^2\theta} = b^2$$

3. The straight line $2x + 3y + 1 = 0$ bisects the angle between two straight lines, one of which is $3x + 2y + 4 = 0$. Find the equation of the other straight line.

Solution

The given line is $3x + 2y + 4 = 0$ ....(1)
and bisector is $2x + 3y + 1 = 0$ .... (2)

Since the other line passes through the intersection of (1) and (2), its equation is $2x + 3y + 1 + k(3x + 2y + 4) = 0$,
i.e., $(2 + 3k)x + (3 + 2k)y + (1 + 4k) = 0$

Consider the point $P(1, -1)$ which lies on the bisector (2) and it does not lie on (1)

Since, P lies on one bisector length of perpendicular from P on (1) = length of perpendicular from P on (3)

$$\Rightarrow \quad \frac{|3.1 + 2.(-1) + 4|}{\sqrt{9+4}} = \frac{|(2+3k).1 + (3+2k)(-1) + (1+4k)|}{\sqrt{(2+3k)^2 + (3+2k)^2}}$$

$$\Rightarrow \quad \frac{5}{\sqrt{13}} = \frac{5|k|}{\sqrt{13k^2 + 24k + 13}}$$

$$\Rightarrow \quad 13k^2 + 24k + 13 = 13k^2$$

$$\Rightarrow \quad 24k + 13 = 0 \Rightarrow k = -\frac{13}{24}$$

Substituting this value of k in (3), we get

$$\left(2 - \frac{13}{8}\right)x + \left(3 - \frac{13}{12}\right)y + \left(1 - \frac{13}{6}\right) = 0$$

$$\Rightarrow \quad \frac{3}{8}x + \frac{23}{12}y - \frac{7}{6} = 0$$

4. Find the distance of the line $4x + 7y + 5 = 0$ from the point (1, 2) along the line $2x - y = 0$. ***[NCERT]***

Solution

Note that the point $P(1, 2)$ lies on the line $2x - y = 0$. To find the required distance, we need the point of intersection of lines $2x - y = 0$ ......(i)

and $4x + 7y + 5 = 0$ ....(ii)

Multiplying (i) by 7 and adding it to (ii), we get

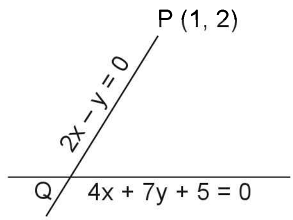
$$14x + 4x + 5 = 0$$

$$\Rightarrow \quad x = -\frac{5}{18}.$$

Substituting $x = -\frac{5}{18}$ in (i), we get

$$y = 2x = 2 \times -\frac{5}{18} = -\frac{5}{9}.$$

$\therefore$ The lines (i) and (ii) meet in the point $Q\left(-\frac{5}{18}, -\frac{5}{9}\right)$



$\therefore$ Required distance

$$= |PQ| = \sqrt{\left(-\frac{5}{18} - 1\right)^2 + \left(-\frac{5}{9} - 2\right)^2}$$

$$= \sqrt{\frac{529}{324} + \frac{529}{81}} = \sqrt{\frac{529}{81}}\sqrt{\frac{1}{4} + 1}$$

$$= \frac{23}{9}\sqrt{\frac{5}{4}} = \frac{23\sqrt{5}}{18}.$$

5. A person standing at the junction (crossing) of two straight paths represented by the equations $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$, wants to reach the path whose equation is $6x - 7y + 8 = 0$ in the least time. Find equation of the path that he should follow. ***[NCERT]***

Solution

Given lines are

$2x - 3y - 4 = 0$ .... (i)

$3x + 4y - 5 = 0$ .... (ii)

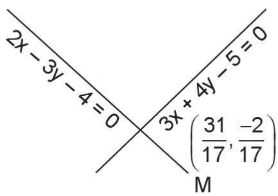
and $6x - 7y + 8 = 0$ .... (iii)

First, we find the junction of (i) and (ii) i.e. the point common to (i) and (ii).

Multiplying (i) by 4 and (ii) by 3 and adding

$8x - 12y - 16 + 9x + 12y - 15 = 0$

$\Rightarrow \quad 17x = 31$



$\Rightarrow x = \frac{31}{17}$ Substituting $x = \frac{31}{17}$ in (i), we get

$$2\left(\frac{31}{17}\right) - 3y - 4 = 0$$

$$\Rightarrow \quad 3y = \frac{62-68}{17} = -\frac{6}{17}$$

$$\Rightarrow \quad y = \frac{-2}{17} \quad \therefore M \text{ is } \left(\frac{31}{17}, \frac{-2}{17}\right)$$

To reach the line (iii) in least possible time, the man must move along the perpendicular from M to (iii). Now slope of (iii) is $\frac{-6}{-7} = \frac{6}{7}$.

Slope of the required path $= \frac{-7}{6}$

Its equation is $y - \left(-\frac{2}{17}\right) = \frac{-7}{6}\left(x - \frac{31}{17}\right)$

Point slope form

or $\quad 6\left(y + \frac{2}{17}\right) = -7\left(x - \frac{31}{17}\right)$

or $\quad 6y = \frac{12}{17} = -7x + \frac{217}{17}$

$\Rightarrow \quad 7x + 6y = \frac{205}{17}$

or $\quad 119x + 102y = 205.$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. Which of the lines $x - y + 3 = 0$ and $x - 4y - 7 = 0$ is farther from origin.

2. The vertices of triangle PQR are $P(2, 1)$, $Q(-2, 3)$ and $R(4, 5)$. Find equation of the median through the vertex R. ***[NCERT]***

3. Given the triangle A (10, 4), $B(-4, 9)$ and $C(-2, -1)$, find the equation of the altitude through A.

4. Find the bisector of the obtuse angle between the lines $3x - 4y + 5 = 0$ and $5x + 12y - 8 = 0$

5. Find the equations of straight lines parallel to $3x + 4y + 1 = 0$ and at a distance of 2 units from the point (–1, 2).

6. Find the equation of the line through the intersection of the lines $x + 2y = 3$ and $4x - y + 7 = 0$, and which is parallel to $5x + 4y - 20 = 0$

7. Obtain the equation of the line passing through the intersection of the lines $2x - 3y + 4 = 0$ and $3x + 4y = 5$, and drawn parallel to y-axis.

8. Find the equation of the line through the intersection of $3x - 4y + 1 = 0$ and $5x + y - 1 = 0$ which cuts off equal intercepts on the axes.

9. Find the value of p so that the three lines $3x + y - 2 = 0$, $px + 2y - 3 = 0$ and $2x - y - 3 = 0$, may intersect at one point. ***[NCERT]***

10. Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$, that has equal intercepts on the axes. ***[NCERT]***

ANSWERS

1. $x - 4y - 7 = 0$
2. $3x - 4y + 8 = 0$
3. $x - 5y + 10 = 0$
4. $64x + 8y + 25 = 0$
5. $3x + 4y + 5 = 0$,
 $3x + 4y - 15 = 0$
6. $15x + 12y - 7 = 0$
7. $17x + 1 = 0$
8. $23x + 23y - 11 = 0$
9. 5
10. $13x + 13y = 6$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. A straight line passes through a fixed point (h, k) the locus of the foot of perpendicular on it drawn from the origin is **[EAMCET-2002]**

(a) $x^2 + y^2 - hk - ky = 0$
(b) $x^2 + y^2 + hk + ky = 0$
(c) $3x^2 + 3y^2 + hk - ky = 0$
(d) None of these

Solution

(a) $y - k = m(x - h)$ and $y - 0 = -\frac{1}{m}(x - 0)$.

Eliminate m and replace (h, k) by (x, y), we get $x^2 + y^2 - hx - ky = 0$, which is the required locus of the point.

2. If A is (2, 5), B is (4, –11) and C lies on $9x + 7y + 4 = 0$, then the locus of the centroid of the ΔABC is a straight line parallel to the straight line is **[MPPET-1986]**

(a) $7x - 9y + 4 = 0$
(b) $9x - 7y - 4 = 0$
(c) $9x + 7y + 4 = 0$
(d) $7 + 9y + 4 = 0$

Solution

(c) According to question, $x_1 = \frac{2+4+x}{3}$

$\Rightarrow \quad x = 3x_1 - 6$

$y_1 = \frac{5-11+y}{3}$

$\Rightarrow \quad y = 3y_1 + 6$

$\therefore \quad 9(3x_1 - 6) + 7(3y_1 + 6) + 4 = 0$

Hence locus is $27x + 21y - 8 = 0$, which is parallel to $9x + 7y + 4 = 0$.

3. Equation of a diameter of the circumcircle of a rectangle $ABCD$ is $2x - y + 4 = 0$. If A, B are (4, 6) and (1, 9) respectively, then area of this rectangle is **[JEE (WB)-2005]**

(a) 9 (b) 18
(c) 12 (d) 16

Solution

(b) $AB = \sqrt{9+9} = 3\sqrt{2}$. If E b the mid point of AB, then $E = (5/2, 15/2)$

Equation of the diameter passing through E (it is perpendicular to AB) is:

$$y - \frac{15}{2} = \frac{3}{3}\left(x - \frac{5}{2}\right)$$

$\Rightarrow \quad x - y + 5 = 0$

Centre of the circle is the point of intersection of $2x - y + 4 = 0$ and $x - y + 5 = 0$ which is $P(1, 6)$.

Now $BC = 2PE = 2 \cdot \frac{3}{2}\sqrt{2} = 3\sqrt{2}$

$\therefore \quad$ Area of rectangle $= (AB)(BC) = (3\sqrt{2})(3\sqrt{2}) = 18$.

4. The line $2x + 3y = 12$ meets the x-axis at A and y-axis at B. the line through (5, 5) perpendicular to AB meets the x-axis, y–axis and the AB at C, D and E respectively. If O is the origin of coordinates, then the area of $OCEB$ is **[IIT-1976]**

(a) 23 sq. units (b) 23/2 sq. units
(c) 23/3 sq. units (d) None of these

Solution

(c) Here O is the point (0, 0). The line $2x + 3y = 12$ meets the y-axis at B and so B is the point (0, 4). The equation of any line perpen-

dicular to the line $2x + 3y = 12$ and passes through (5, 5) is $3x - 2y = 5$ The line (i) meets the x-axis at C and so coordinates of C are (5/3,0). Similarly the coordiantes of E are (3, 2) by solving the line AB and (i). Thus $O(0, 0)$, C (5/3,0), $E(3, 2)$ and $B(0, 4)$.
Now the area of $\Delta\, OEB$ = 23/3 sq. units.

5. The number of integral values of m, for which the x-coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer is ***[IIT Screening-2001]***
(a) 2 (b) 0
(c) 4 (d) 1

Solution

(a) Solving $3x + 4y = 9, y = mx + 1$ we get $x = \frac{5}{3+4m}$ x is an integer if $3 + 4m = 1, -1, 5, -5$

$\therefore \quad m = \frac{-2}{4}, \frac{-4}{4}, \frac{2}{4}, \frac{-8}{4}$. So, m has two integral values.

6. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then ***[IIT JEE-2008]***

(a) $\frac{1}{PS}+\frac{1}{ST}<\frac{2}{\sqrt{QS\times SR}}$

(b) $\frac{1}{PS}+\frac{1}{ST}>\frac{2}{\sqrt{QS\times SR}}$

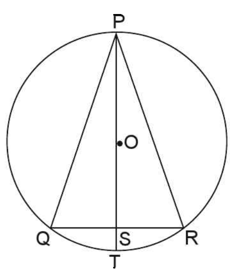
(c) $\frac{1}{PS}+\frac{1}{ST}<\frac{4}{QR}$

(d) $\frac{1}{PS}+\frac{1}{ST}>\frac{4}{QR}$

Solution

(b, d) We have, from property of intersecting chords of a circle

$PS\,.\,ST = QS\,.\,SR$



Applying $AM - GM$ inequality to $\frac{1}{PS}$ and $\frac{1}{ST}$

We have $\frac{\frac{1}{PS}+\frac{1}{ST}}{2} \ge \sqrt{\frac{1}{PS}.\frac{1}{ST}}$

The equality occuring when $PS = ST$ .... (1)

Using $PS\,.\,ST = QS\,.\,SR$, we have

$$\frac{1}{PS}+\frac{1}{ST} \ge \frac{2}{\sqrt{QS.SR}} \quad \text{....... (2)}$$

Again from A.M.–G.M. inequality applied to QS and SR, we have $\frac{QS+SR}{2} \ge \sqrt{QS.SR}$

$$\Rightarrow \quad \frac{QR}{2} \ge \sqrt{QS.SR} \quad \text{......... (3)}$$

equality occuring when $QS = SR$ .... (4)

Using (2) and (3) $\frac{1}{PS}+\frac{1}{ST} \ge \frac{2}{QR/2}$

$$\Rightarrow \quad \frac{1}{PS}+\frac{1}{ST} \ge \frac{4}{QR} \quad \text{........ (5)}$$

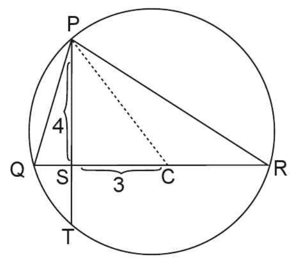
Note that the equality holds in (5) when both (1) and (4) are satisfied i.e. $PS = ST$ and $QS = SR$. But then S become the centre of the circle, while the problem says that S is not the centre of the circumcircle. Hence in (5) only strict inequality holds, i.e. $\frac{1}{PS}+\frac{1}{ST} > \frac{4}{QR}$

But in (2) equality can hold, i.e.

$$\frac{1}{PS}+\frac{1}{ST} \ge \frac{2}{\sqrt{QS.SR}}$$

As an example take a circle of radius 5 and the chord PT.

C = centre of circle



Length 8, S being the mid-point of the chord (S is not the centre of the circumcircle) We have $PS = 4 = ST$

$$SC = \sqrt{PC^2 - PS^2} = 3,$$

$$SR = SC + CR = 3 + 5 = 8$$

$$QS = QC - SC = 5 - 3 = 2$$

$$\frac{1}{PS} + \frac{1}{ST} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\frac{2}{\sqrt{QS.QR}} = \frac{2}{\sqrt{2.8}} = \frac{2}{4} = \frac{1}{2}$$

Thus $\frac{1}{PS} + \frac{1}{ST} = \frac{2}{\sqrt{QS.SR}}$ can hold.

So choice be in the paper must be

Thus $\frac{1}{PS} + \frac{1}{ST} \geq \sqrt{\frac{2}{QS.SR}}$ that is, one cannot rule out the case of equality.

7. The area enclosed by the pair of lines $xy = 0$, the line $x - 4 = 0$ and $y + 5 = 0$ is

[Karnataka CET-2007]

(a) 20 sq. units (b) 10 sq. units
(c) 5/4 sq. units (d) 0 sq. units

Solution

(a) $xy = 0$ ....(i)

$y = 0$

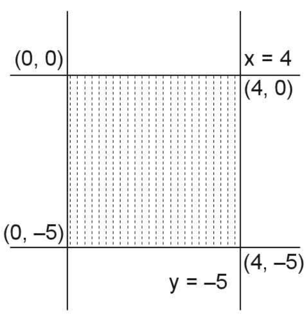


Figure formed is a rectangle of length 5 units, breadth = 4 units.

$\therefore$ Area = 4 × 5 = 20 sq. units.

8. The equation of the bisector of that angle between the lines $x + 2y - 11 = 0$, $3x - 6y - 5 = 0$ which contains the point (1, –3) is

(a) $3x = 19$ (b) $3y = 7$
(c) $3x = 19$ and $3y = 7$ (d) None of these

Solution

(a) Since, the origin and the point (1, –3) lie on the same side of $x + 2y - 11 = 0$ and on the opposite side of $3x - 6y - 5 = 0$.

Therefore, the bisector of the angle containing (1, –3) is the bisector of that angle which does not contain the origin and is given by

$$\frac{-x-2y+11}{\sqrt{5}} = -\left(\frac{-3x+6y+5}{\sqrt{45}}\right) \text{i.e., } 3x = 19.$$

9. The lines $(p - q)x + (q - r)y + (r - p) = 0$
$(q - r)x + (r - p)y + (p - q) = 0$
$(r - p)x + (p - q)y + (q - r) = 0$ are

(a) Parallel
(b) Perpendicular
(c) Concurrent
(d) None of these

Solution

(c) $\begin{vmatrix} p-q & q-r & r-p \\ q-r & r-p & p-q \\ r-p & p-q & q-r \end{vmatrix}$

$$= \begin{vmatrix} 0 & q-r & r-p \\ 0 & r-p & p-q \\ 0 & p-q & q-r \end{vmatrix} = 0$$

Hence the lines are concurrent.

Alternate: Since sum of the coefficient of x, y and the constant term is zero, hence the lines are concurrent.

10. Consider the lines given by $L_1 : x + 3y - 3 = 0$, $L_2 : 3x - ky - 1 = 0$, $L_3 : 5x + 2y - 12 = 0$

Match the following columns:

[IIT-JEE-2008]

Column I	Column II
(a) L_1, L_2, L_3 are concurrent, if	(p) $k = -9$
(b) One of L_1, L_2, L_3 is parallel to at least one of the other two, if	(q) $k = -6/5$
(c) L_1, L_2, L_3 form a triangle, if	(r) $k = 5/6$
(d) L_1, L_2, L_3 do not form a triangle, if	(s) $k = 5$

Solution

(a) – s; (b) – p, q; (c) – r; (d) = p, q, s

In line $x + 3y - 5 = 0$ and $5x + 3y - 12 = 0$ intersect at (2, 1).

For L_1 and L_2 to be parallel, the corresponding coefficient must be proportional, giving $\frac{1}{3} = \frac{3}{-k} \Rightarrow k = -9$

Similarly, for L_2 and L_3 to be parallel

$$\frac{3}{5} = -\frac{k}{2} \Rightarrow k = -\frac{6}{5}$$

When $k = 5, -9, -6/5$, the lines will not form a triangle

When $k \neq 5, -9, -6/5$, the lines will form a triangle.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. Two points A and B have coordinates (1, 1) and (3, –2) respectively. The co–ordinates of a point distant $\sqrt{85}$ from B on the line through B perpendicular to AB are ***[AMU-2000]***
   (a) (4, 7) (b) (7, 4)
   (c) (5, 7) (d) (–5, –3)

2. The distance of the lines $2x - 3y = 4$ from the point (1, 1) measured parallel to the line $x + y = 1$ is ***[Orissa JEE-2002]***
   (a) $\sqrt{2}$ (b) $5/\sqrt{2}$
   (c) $1/\sqrt{2}$ (d) 6

3. Two lines are drawn through (3, 4), each of which makes angle of 45° with the line $x - y = 2$, then area of the triangle formed by these lines is ***[RPET-2000]***
   (a) 9 (b) 9/2
   (c) 2 (d) 2/9

4. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is ***[IIT-1992; Karnataka CET-1999; DCE-2000, 01]***
   (a) Square
   (b) Circle
   (c) Straight line
   (d) Two intersecting lines

5. The area enclosed within the curve $|x|+|y|=1$ is ***[RPET-1990, 1997; IIT-1981; UPSEAT-2003]***
   (a) $\sqrt{2}$ (b) 1
   (c) $\sqrt{3}$ (d) 2

6. The straight line passing through the point of intersection of the straight lines $x - 3y + 1 = 0$ and $2x + 5y - 9 = 0$ and having infinite slope and at a distance of 2 units from the origin, has the equation ***[AMU-1980]***
   (a) $x = 2$ (b) $3x + y - 1 = 0$
   (c) $y = 1$ (d) None of these

7. If the equation of the locus of a point equidistant from the points (a_1, b_1) and (a_2, b_2) is $(a_1 - a_2)x + (b_1 - b_2)y + c = 0$, then the value of c is ***[IIT Screening-2003]***
   (a) $\frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$
   (b) $a_1^2 - a_2^2 + b_1^2 - b_2^2$
   (c) $\frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2)$
   (d) $\sqrt{a_1^2 + b_1^2 - a_2^2 - b_2^2}$

8. The equation of line, which bisect the line joining two points (2, –19) and (6, 1) and perpendicular to the line joining two points (–1, 3) and (5, –1), is ***[RPET-87]***
   (a) $3x - 2y = 30$ (b) $2x - y - 3 = 0$
   (c) $2x + 3y = 20$ (d) None of these

9. If a, b, c are in harmonic progression, then straight line $\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0$ always passes through a fixed point, that point is ***[MPPET-99; AIEEE-2005]***
   (a) (–1, –2) (b) (–1, 2)
   (c) (1, –2) (d) (1, –1/2)

10. Equations of diagonals of square formed by lines $x = 0, y = 0, x = 1$ and $y = 1$ are ***[MPPET-84]***

(a) $y = x, y + x = 1$ (b) $y = x, x + y = 2$
(c) $2y = x, y + x = 1/3$ (d) $y = 2x, y + 2x = 1$

11. The diagonal passing through origin of a quadrilateral formed by $x = 0, y = 0, x + y = 1$ and $6x + y = 3$, is

(a) $3x - 2y = 0$ (b) $2x - 3y = 0$
(c) $3x + 2y = 0$ (d) None of these

12. If the straight line drawn through the point $P(\sqrt{3}, 2)$ and making an angle $\pi/6$ with x-axis meets the line $\sqrt{3}x - 4y + 8 = 0$ at Q then the length PQ is

(a) 7 (b) 6
(c) 5 (d) 4

13. The locus of the mid-point of the portion intercepted between the axes by the line $x\cos\alpha + y\sin\alpha = p$ where p is constant is ***[AIEEE-02]***

(a) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{4}{p^2}$ (b) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{2}{p^2}$

(c) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{p^2}$ (d) $\frac{1}{x^2}-\frac{1}{y^2}=\frac{3}{p^2}$

14. The locus of the point of intersection of the lines $x\cos\alpha + y\sin\alpha = a$ and $x\sin\alpha - y\cos\alpha = b$ is

(a) $x^2 + y^2 = a^2 - b^2$
(b) $x^2 + y^2 = a^2 + b^2$
(c) $x^2 - y^2 = a^2 - b^2$
(d) $x^2 - y^2 = a^2 + b^2$

15. The slope of the line which is drawn through the point (1, 2). So that its point of intersection with the line $x + y + 3 = 0$ is at a distance $3\sqrt{2}$ is

(a) $1/\sqrt{3}$ (b) $\sqrt{3}$
(c) 1 (d) $(\sqrt{3} - 1)/3$

16. The equation of a straight line passing through the point $(-5, 4)$ and which cuts off an intercept of $\sqrt{2}$ units between the line $x + y + 1 = 0$ and $x + y - 1 = 0$ is

(a) $x - 2y + 13 = 0$ (b) $2x - y + 14 = 0$
(c) $x - y + 9 = 0$ (d) $x - y + 10 = 0$

17. The lines $y = mx$, $y + 2x = 0$, $y = 2x + k$ and $y + mx = k$ from a rhombus if m equals

(a) -1 (b) $1/2$
(c) 1 (d) 2

18. The graph of $y = |x|$ consists of a pair of straight lines lying ***[MPPET-2007]***

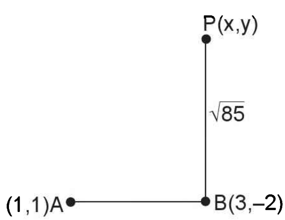
(a) To the left of y-axis
(b) Above x-axis
(c) Below x-axis
(d) To the right of y-axis

19. The points (1, 3) and (5, 1) are the opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$, then the value of c will be ***[Pb. CET-2003; IIT-1981]***

(a) 4 (b) -4
(c) 2 (d) -2

SOLUTIONS

1. (c) Trick: From option (c)



$$BP = \sqrt{(5-3)^2 + (7+2)^2}$$
$$= \sqrt{4+81} = \sqrt{85}$$

2. (a) The slope of line $x + y = 1$ is -1

$\therefore$ It makes an angle of 135° with x-axis.

The equation of line passing through (1, 1) and making an angle of 135° is,

$$\frac{x-1}{\cos 135°} = \frac{y-1}{\sin 135°} = r$$

$$\Rightarrow \frac{x-1}{-1/\sqrt{2}} = \frac{y-1}{1/\sqrt{2}} = r$$

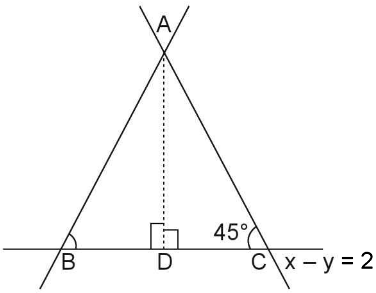
Co–ordinates of any point on this line are

$$\left(1-\frac{r}{\sqrt{2}}, 1+\frac{r}{\sqrt{2}}\right)$$

If this point lies on $2x - 3y = 4$, then

$$2\left(1-\frac{r}{\sqrt{2}}\right)-3\left(1+\frac{r}{\sqrt{2}}\right)=4 \Rightarrow r=\sqrt{2}$$

3. (b) $\tan 45° = \dfrac{AD}{\frac{BC}{2}} \Rightarrow BC = 2AD$



$$AD = \left|\frac{3-4-2}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$$

Area

$$= \frac{1}{2}(BC)\times AD = \frac{1}{2}\times(2AD)(AD) = AD^2 = \frac{9}{2}$$

4. (a) Take the two perpendicular lines as x and y axes and the point as $P(x, y)$. Then sum of distances of $P(x, y)$ from x and y axes is 1.

This $\Rightarrow |x| + |y| = 1$

$\Rightarrow \quad x + y = 1, -x + y = 1, \ x - y = 1, -x - y = 1$

$\Rightarrow$ which form a square.

5. (a) $|x| + |y| = 1 \Rightarrow x + y = 1$ .... (1)

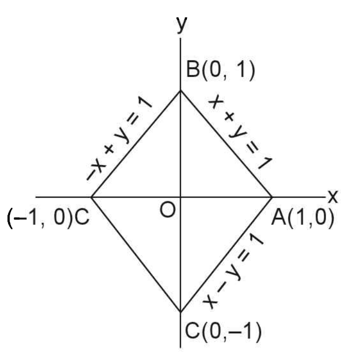
$x - y = 1$ .... (2)

$-x + y = 1$ .... (3)

and $-x - y = 1$ .... (4)

Evidently $ABCD$ is square.

Its area $= (AB)^2 = (\sqrt{2})^2 = 2$.



6. (a) The intersection point of $x - 3y + 1$ and $2x + 5y - 9 = 0$ is (2, 1) and $m = \dfrac{1}{0}$. So the required line is $y - 1 = \dfrac{1}{0}(x-2) \Rightarrow x = 2$

Trick: Check by option.

7. (a) Let $P(\alpha, \beta)$ be the point which is equidistant to $A(a_1, b_1)$ and $B(a_2, b_2)$.

$\therefore \quad PA = PB$

$\Rightarrow \quad (\alpha - a_1)^2 + (\beta - b_1)^2 = (\alpha - a_2)^2 + (\beta - b_2)^2$

$\Rightarrow \quad \alpha^2 + a_1^2 - 2\alpha a_1 + \beta^2 + b_1^2 - 2\beta b_1$

$= \alpha^2 + a_2^2 - 2\alpha a_2 + \beta^2 + b_2^2 - 2\beta b_2$

$\Rightarrow \quad 2(a_2 - a_1)\alpha + 2(b_2 - b_1)\beta + (a_1^2 + b_1^2 - a_2^2 - b_2^2) = 0$

Thus the equation of locus (α, β) is

$(a_2 - a_1)x + (b_2 - b_1)y + \dfrac{1}{2}(a_1^2 + b_1^2 - a_2^2 - b_2^2) = 0$

But the given equation is $(a_2 - a_1)x + (b_2 - b_1)y - c = 0$

$$\therefore \quad c = -\frac{1}{2}(a_1^2 + b_1^2 - a_2^2 - b_2^2)$$

$$= \frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$$

8. (a) Bisecting point = (4, –9) and slope of required line $= -\dfrac{6}{-4} = \dfrac{3}{2}$

$\therefore$ Its equation is $y + 9 = \dfrac{3}{2}(x-4)$

$\Rightarrow \quad 3x - 2y = 30$

9. (c) As a, b, c are in H.P.

$$\therefore \quad \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

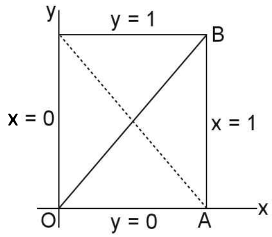
or $\dfrac{1}{a} - \dfrac{2}{b} + \dfrac{1}{c} = 0$ comparing it with

$\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{1}{c} = 0$ then $x = 1, y = -2$.

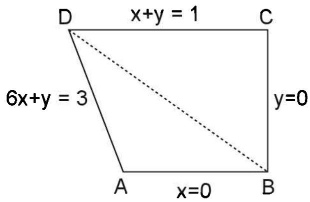
Thus line will pass through (1, –2).

10. (a) Diagonal OB is $y = x$. A (1, 0), C (0, 1).

Then diagonal AC is $\frac{x}{1}+\frac{y}{1}=1$ or $x + y = 1$



11. (a) Intersection of AB and BC is $B(0, 0)$.



Thus B is origin. Intersection AD and CD is $D\left(\frac{2}{5},\frac{3}{5}\right)$

Diagonal BD is passing through B (origin) is

$$y-0=\frac{\frac{3}{5}-0}{\frac{2}{5}-0}(x-0)\Rightarrow 3x-2y=0$$

12. (b) The equation of any line passing through $(\sqrt{3}, 2)$ and making an angle $\frac{\pi}{6}$ with x-axis is

$$\frac{x-\sqrt{3}}{\cos 30°}=\frac{y-2}{\sin 30°}=r \text{ (say)} \quad ....(1)$$

where 'r' represents the distance of any point Q on this line from the given points $P(\sqrt{3}, 2)$. The coordinates (x, y) of any point Q on line (1) are $(\sqrt{3} + r\cos 30°, 2 + r\sin 30°)$

i.e., $\left(\sqrt{3}+\frac{r\sqrt{3}}{2}, 2+\frac{r\times 1}{2}\right)$

If the point lies on the line $\sqrt{3}x - 4y + 8 = 0$ then

$$\sqrt{3}\left(\sqrt{3}+\frac{r\sqrt{3}}{2}\right)-4\left(2+\frac{r}{2}\right)+8=0$$

$$\Rightarrow \quad 3+\frac{3r}{2}-8-2r+8=0$$

$$\Rightarrow \quad \frac{3r}{2}-2r=-3$$

$$\Rightarrow \quad \frac{-r}{2}=-3\Rightarrow r=6$$

$\therefore$ length of $PQ = 6$

13. (a) $x\cos\alpha + y\sin\alpha = p$

$$\Rightarrow \quad \frac{x}{p\sec\alpha}+\frac{y}{p\operatorname{cosec}\alpha}=1,$$

meets axes at A ($p\sec\alpha$, 0) and B (0, $p\operatorname{cosec}\alpha$). Let (h, k) be mid-point of AB.

Then $h=\frac{p\sec\alpha}{2}, k=\frac{p\operatorname{cosec}\alpha}{2}$

$$\Rightarrow \quad \frac{2h}{p}=\sec\alpha, \frac{2k}{p}=\operatorname{cosec}\alpha$$

$$\Rightarrow \quad \left(\frac{p}{2h}\right)^2+\left(\frac{p}{2k}\right)^2=\cos^2\alpha+\sin^2\alpha=1$$

Locus of (h, k) is $\frac{p^2}{4x^2}+\frac{p^2}{4y^2}=1$

$$\Rightarrow \quad \frac{1}{x^2}+\frac{1}{y^2}=\frac{4}{p^2}$$

14. Eliminate α by squaring and adding.

15. (c) Let θ be the angle made by the line with positive direction of x-axis, then its equation is $\frac{x-1}{\cos\theta}=\frac{y-2}{\sin\theta}=3\sqrt{2}$

Any point on this line is $(1 + 3\sqrt{2}\cos\theta, 2 + 3\sqrt{2}\sin\theta)$

Since it lies on the line $x + y + 3 = 0$

$\therefore \quad (1+3\sqrt{2}\cos\theta)+(2+3\sqrt{2}\sin\theta)+3=0$

$\Rightarrow \quad 3\sqrt{2}(\cos\theta+\sin\theta)=-6$

$\Rightarrow \quad \cos\theta+\sin\theta=-\sqrt{2}$

Squaring both sides, we get $\cos^2\theta + \sin^2\theta + \sin 2\theta = 2$

$\Rightarrow \quad \sin 2\theta = 1$

$$\Rightarrow \quad 2\theta=\frac{\pi}{2}\Rightarrow\theta=\frac{\pi}{4}$$

Hence, slope of the line is $\tan\frac{\pi}{4}=1$

16. (c) Since, the ⊥ distance between the given lines is $\sqrt{2}$

$\therefore$ the required line is a straight line ⊥ to the given parallel lines and passing through (–5, 4).

Any line ⊥ to the given lines is $x - y + K = 0$

This passes through (–5, 4)

$\therefore \quad -5 - 4 + K = 0$

$\therefore \quad K = 9$

Hence, required line is $x - y + 9 = 0$

17. (d) Clearly $m = 2$ ($y + 2x = 0$ and $y + 2x = K$ are parallel and $y = 2x$ and $y = 2x + K$ are parallel)

18. (b) $y = |x|$ i.e., y is +ve for all x

$\therefore$ above x-axis.

19. (b) The mid-point $M\left(\frac{1+5}{2}, \frac{3+1}{2}\right) = M(3, 2)$ of the diagonal lies on the line $y = 2x + c$

$\therefore \quad 2 = 6 + c$ or $c = -4$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The locus of a point so that sum of its distance from two given perpendicular lines is equal to 2 unit in first quadrant, is ***[Bihar CEE-1994]***

(a) $x + y + 2 = 0$ (b) $x + y = 2$
(c) $x - y = 2$ (d) None of these

2. Two point moves such that the area of the triangle formed by it with the points (1, 5) and (3, –7) is 21 sq. unit. The locus of the point is ***[Kerala (Engg.)-2002]***

(a) $6x + y - 32 = 0$ (b) $6x - y + 32 = 0$
(c) $x + 6y - 32 = 0$ (d) $6x - y - 32 = 0$

3. If the coordinates of the vertices of the triangle ABC be (–1, 6), (–3, –9) and (5, –8) respectively, then the equation of the median through C is

(a) $13x - 14y - 47 = 0$
(b) $13x - 14y + 47 = 0$
(c) $13x + 14y + 47 = 0$
(d) $13x + 14y - 47 = 0$

4. If $u = a_1x + b_1y + c_1 = 0$, $v = a_2x + b_2y + c_2 = 0$ and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the curve $u + kv = 0$ is

(a) The same straight line u
(b) Different straight line
(c) It is not a straight line
(d) None of these

5. The distance of the point (3, 5) from the line $2x + 3y - 14 = 0$ measured parallel to the line $x - 2y = 1$ is

(a) $7/\sqrt{5}$ (b) $7/\sqrt{13}$
(c) $\sqrt{5}$ (d) $\sqrt{13}$

6. The lines $x + (a - 1)y + 1 = 0$ and $2x + a^2y - 1 = 0$ are perpendicular if

(a) $|a| = 2$ (b) $0 < a < 1$
(c) $-1 < a < 0$ (d) $a = -1$

7. Distance between the parallel lines $3x + 4y + 7 = 0$ and $3x + 4y - 5 = 0$ is ***[RPET-2003]***

(a) 2/5 (b) 12/5
(c) 5/12 (d) 3/5

8. The three lines $lx + my + n = 0$, $mx + ny + l = 0$, $nx + ly + m = 0$ are concurrent if ***[Pb. CET-2002]***

(a) $l = m + n$ (b) $m = l + n$
(c) $n = l + m$ (d) $l + m + n = 0$

9. The distance between the two parallel straight lines $y = 2x + 3$ and $2y = 4x + 9$ is equal to

(a) 6 (b) 12
(c) $3\sqrt{5}/10$ (d) $3\sqrt{5}/2$

10. If the lines $x + q = 0$, $y - 2 = 0$ and $3x + 2y + 5 = 0$ are concurrent, then value of q will be ***[DCE-2002]***

(a) 1 (b) 2
(c) 3 (d) 5

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 13 minutes.
3. The worksheet consists of 13 questions. The maximum marks are 39.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The line $x + y = 4$ divides the line joining the points $(-1, 1)$ and $(5, 7)$ in the ratio
(a) 2 : 1 (b) 1 : 2
(c) 1 : 2 externally (d) None of these

2. The distance between $4x + 3y = 11$ and $8x + 6y = 15$, is *[AMU-1979; MNR-1987; UPSEAT-2000]*
(a) 7/2 (b) 4
(c) 7/10 (d) None of these

3. The lines $2x + y - 1 = 0$, $ax + 3y - 3 = 0$ and $3x + 2y - 2 = 0$ are concurrent for *[EAMCET-1994]*
(a) All a (b) $a = 4$ only
(c) $-1 \le a \le 3$ (d) $a > 0$ only

4. The value of λ for which the lines $3x + 4y = 5$, $5x + 4y = 4$ and $\lambda x + 4y = 6$ meet at a point is *[Kerala (Engg.)-2002]*
(a) 2 (b) 1
(c) 4 (d) 3

5. The lines $ax + by + c = 0$, where $3a + 2b + 4c = 0$ are concurrent at the point *[IIT-1982]*
(a) (1/2, 3/4) (b) (1, 3)
(c) (3, 1) (d) (3/4, 1/2)

6. If lines $4x + 3y = 1$, $y = x + 5$ and $5y + bx = 3$ are concurrent, then b equals
(a) 1 (b) 3
(c) 6 (d) 0

7. The equation of the lines through the point (3, 2) which makes an angle of 45° with the line $x - 2y = 3$ are *[Kerala PET-2007]*
(a) $3x - y = 7$ and $x + 3y = 9$
(b) $x - 3y = 7$ and $3x + y = 9$
(c) $x - y = 3$ and $x + y = 2$
(d) $2x + y = 7$ and $x - 2y = 9$

8. The distance between the lines $y = mx + c_1$ and $y = mx + c_2$ is
(a) $\dfrac{c_1 - c_2}{\sqrt{m^2+1}}$ (b) $\dfrac{c_2 - c_1}{\sqrt{m^2+1}}$
(c) $\dfrac{c_1 \sim c_2}{\sqrt{m^2+1}}$ (d) 0

9. The area of the triangle formed by the lines $7x - 2y + 10 = 0$, $7x + 2y - 10 = 0$ and $y + 2 = 0$ is
(a) 8 sq. units (b) 12 sq. units
(c) 14 sq. units (d) None of these

10. The vertex of an equilateral triangle is $(2, -1)$ and the equation of its base in $x + 2y = 1$. The length of its sides is *[UPSEAT-2003]*
(a) $4/\sqrt{15}$ (b) $2/\sqrt{15}$
(c) $4/3\sqrt{3}$ (d) $1/\sqrt{5}$

11. The image of the point $(4, -3)$ with respect to the line $y = x$ is *[RPET-2002]*
(a) $(-4, -3)$ (b) $(3, 4)$
(c) $(-4, 3)$ (d) $(-3, 4)$

12. The x-axis, y-axis and a line passing through the point $A(6, 0)$ form a triangle ABC. If $\angle A = 30°$, then the area of the triangle, in sq. units, is *[Kerala PET-2007]*
(a) $6\sqrt{3}$ (b) $12\sqrt{3}$
(c) $4\sqrt{3}$ (d) $8\sqrt{3}$

13. If there is a triangle whose vertices are $A(3, -5)$, $B(7, 3)$ and $C(-5, 5)$ then the equations $6x - y = 23$ represents *[SCRA-2007]*
(a) the line AB
(b) the line joining the mid-points of BC and AC
(c) the median through A
(d) altitudes from A to BC

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	6. ⓐ ⓑ ⓒ ⓓ	11. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	7. ⓐ ⓑ ⓒ ⓓ	12. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	8. ⓐ ⓑ ⓒ ⓓ	13. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	9. ⓐ ⓑ ⓒ ⓓ	
5. ⓐ ⓑ ⓒ ⓓ	10. ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

5. (d) Here $3a + 2b + 4c = 0$

or $c = -\left(\frac{3a+2b}{4}\right)$

Put in line to get $ax + by - \left(\frac{3a+2b}{4}\right) = 0$

$$a\left(x - \frac{3}{4}\right) + b\left(y - \frac{2}{4}\right) = 0$$

or $\left(x - \frac{3}{4}\right) + \frac{b}{a}\left(y - \frac{1}{2}\right) = 0$

Equation is in form of $L_1 + \lambda L_2 = 0$

∴ It will pass through point of intersection of $x - \frac{3}{4} = 0$ and $y - \frac{1}{2} = 0$

∴ Point is $\left(\frac{3}{4}, \frac{1}{2}\right)$

7. (a) Slope of line can be = tan (θ ±45°)

Where, $\tan\theta = \frac{1}{2}$

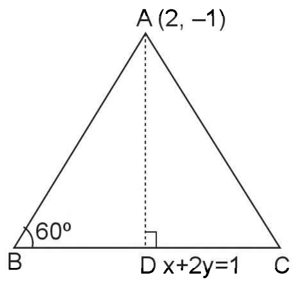
∴ Slope $= \frac{\tan\theta \pm 1}{1 \mp \tan\theta} = 3, -\frac{1}{3}$

∴ Equation of line
$y - 2 = 3(x - 3)$

or $3x - y = 7$

or $y - 2 = \frac{-1}{3}(x - 3)$ or $x + 3y = 9$

10. (b) Let length of side = a

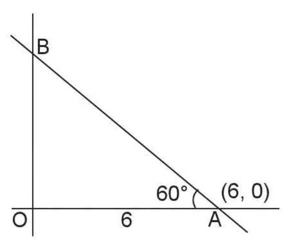


∴ $AD = a \sin 60° = \left|\frac{2 + 2(-1) - 1}{\sqrt{1^2 + 2^2}}\right|$

$$a.\frac{\sqrt{3}}{2} = \frac{1}{\sqrt{5}} \Rightarrow a = \frac{2}{\sqrt{15}}$$

12. (a) $\tan 30° = \frac{OB}{OA}$

$$OB = 6.\frac{1}{\sqrt{3}} = 2\sqrt{3}$$



Area $= \frac{1}{2} \times OA \times OB = 6\sqrt{3}$

Part C

Pair of Straight Lines

LECTURE

1

Pair of Straight Lines 1

(Homogeneous equations of second degree and their various forms)

BASIC CONCEPTS

1. Homogeneous Equations of Second Degree

1.1 An equation of the type $ax^2 + 2hxy + by^2 = 0$ is a homogeneous equation of second degree.

NOTE

The sum of the powers of x and y in every term is the same and is equal to 2.

1.2 The homogeneous equation of the second degree always represents a pair of straight lines passing through the origin.

1.3 Both straight lines represented by $ax^2 + 2hxy + by^2 = 0$ are given by:

$$ax + [h + \sqrt{h^2 - ab}]\, y = 0 \qquad \text{........ (i)}$$

$$ax + [h - \sqrt{h^2 - ab}]\, y = 0 \qquad \text{......... (ii)}$$

(a) Slopes of both straight lines are respectively

$$m_1 = -\left(\frac{a}{h + \sqrt{h^2 - ab}}\right),$$

$$m_2 = -\left(\frac{a}{h - \sqrt{h^2 - ab}}\right)$$

(b) Difference of slopes i.e.,

$$|m_1 - m_2| = \frac{2\sqrt{h^2 - ab}}{b}$$

(c) Ratio of the slopes $= \dfrac{h + \sqrt{h^2 - ab}}{h - \sqrt{h^2 - ab}}$ or $\dfrac{h - \sqrt{h^2 - ab}}{h + \sqrt{h^2 - ab}}$

(d) If $h^2 > ab$, the two lines are **Real and different.**

(e) If $h^2 = ab$, the two lines are **COINCIDENT**.

(f) If $h^2 < ab$, the two lines are **IMAGINARY** having origin as their real point of intersection.

2. Homogeneous equation of second degree in terms of gradients $(m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$

If $ax^2 + 2hxy + by^2 = 0$ represents the pair of lines $y = m_1x$ and $y = m_2x$. Then $ax^2 + 2hxy + by^2 = (y - m_1x)(y - m_2x) = m_1m_2 x^2 - (m_1 + m_2)xy + y^2 = 0$ comparing the coefficients we get

$$m_1 + m_2 = -\frac{2h}{b} = \tan\alpha + \tan\beta \quad \textit{[MPPET-1988]}$$

$$\text{and } m_1m_2 = \frac{a}{b} = \tan\alpha.\tan\beta \quad \textit{[CEE-1993]}$$

$|(m_1 - m_2)| = \dfrac{2\sqrt{h^2 - ab}}{b}$ = difference of the slopes = $|\tan\alpha - \tan\beta|$

$(1 + m_1^2)(1 + m_2^2) = \dfrac{(a-b)^2 + 4h^2}{b^2}$ α and β are angle made by straight lines ($y = m_1x$, $y = m_2x$) with positive direction of x-axis.

3. The equation of the straight lines passing through origin and whose slopes are

(i) negative (equal in magnitude but opposite in sign) of the gradients of the lines $ax^2 + 2hxy + by^2 = 0$ is $ax^2 - 2hxy + by^2 = 0$ $[y = -m_1x; y = -m_2x]$

(ii) reciprocals of the gradients of the lines $ax^2 + 2hxy + by^2 = 0$ is $bx^2 + 2hxy + ay^2 = 0$.

4. The equation to the pair of lines through the origin and perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$. ***[MPPET-1989]***

NOTE

If straight lines represented by $ax^2 + 2hxy + by^2 = 0$ by $y = m_1x$ and $y = m_2x$ then straight lines represented by

(i) $ax^2 - 2hxy + by^2 = 0$ are $y = -m_1x$, $y = -m_2x$
(ii) $bx^2 + 2hxy + ay^2 = 0$ are $m_1y = x$ and $m_2y = x$
(iii) $bx^2 - 2hxy + ay^2 = 0$ are $m_1y + x = 0$ and $m_2y + x = 0$

5. If $y = mx$ be one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ then $a + 2hm + bm^2 = 0$ i.e., (slopes of the both lines represented by $ax^2 + 2hxy + by^2 = 0$ are the roots of the equation $a + 2hm + bm^2 = 0$).

6. The equation to the pair of lines through (x_1, y_1) and

(i) Parallel to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $a(x - x_1)^2 + 2h(x - x_1)(y - y_1) + b(y - y_1)^2 = 0$

(ii) Perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $b(x - x_1)^2 - 2h(x - x_1)(y - y_1) + a(y - y_1)^2 = 0$

7. The condition that the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$

(i) be λ times the slope of the other is $ab(1+\lambda)^2 = 4h^2\lambda$.

(ii) be square of the slope of the other is $ab(a + b) - 6abh + 8h^3 = 0$

NOTE

If ratio of the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $m : n$ then required condition is: $(m + n)^2 ab = 4mnh^2$.

8. If the separate equations of the pair of lines $ax^2 + 2hxy + by^2 = 0$ are $l_1x + m_1y = 0$, $l_2x + m_2y = 0$ then $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, and $m_1m_2 = b$

9. Equation of the pair of straight lines through the origin each of two straight lines through the origin

(i) at a distance a from the point (x_1, y_1)
$(x_1y - y_1x)^2 = a^2(x^2 + y^2)$.

(ii) making angle α with the straight line $y = x$.
$x^2 + y^2 - 2xy \sec 2\alpha = 0$

10. An angle θ between the pair of lines $ax^2 + 2hxy + by^2 = 0$ is given by $\cos\theta = \dfrac{a+b}{\sqrt{(a-b)^2 + 4h^2}}$

or

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b};\ \sin\theta = \frac{2\sqrt{h^2 - ab}}{\sqrt{(a-b)^2 + 4h^2}}$$

[RPET-1995]

11. If $h^2 = ab$ then the two lines given by $ax^2 + 2hxy + by^2 = 0$ are coincident and $ax^2 + 2hxy + by^2 + 0$ is a perfect square.

11.1 If $a + b = 0$, the two lines will be perpendicular.

12. The equation of the pair of bisectors of angles between the pair lines $ax^2 + 2hxy + by^2 = 0$ is
$$\frac{x^2 - y^2}{a-b} = \frac{xy}{h}$$
or $h(x^2 - y^2) = (a - b)xy$

13. The $y = mx$ bisects the angle between the lines $ax^2 + 2hxy + by^2 = 0$ if $h(m^2 - 1) + m(a - b) = 0$.

14. The lines bisecting the angle between the bisectors of the angles between the lines $ax^2 + 2hxy + by^2 = 0$ are given by $(a - b)(x^2 - y^2) + 4hxy = 0$ (Equation of bisector of bisector)

14.1 If pair of lines $(a - b)(x^2 - y^2) + 4hxy = 0$ and $ax^2 + 2hxy + by^2 = 0$ be identical or same then $a + b = 0$.

Some miscellaneous results

15. The area of the triangle formed by $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is given by
$$\left|\frac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}\right|$$

16. The orthocentre of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is $(\lambda l, \lambda m)$

where $\lambda = \dfrac{-n(a+b)}{am^2 - 2hlm + bl^2}$

17. Triangle enclosed by the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ be isosceles if $\dfrac{l^2 - m^2}{a-b} = \dfrac{lm}{h}$

NOTE

If $a + b = 0$ then area of the right angled isosceles triangle is $\dfrac{n^2}{l^2 + m^2}$

18. The equation of the pair of lines through $O(0, 0)$ and forming an equilateral triangle with the line $lx + my + n = 0$ is $(l^2 - 3m^2)x^2 + lmxy + (m^2 - 3l^2)y^2 = 0$ (i) and area of the equilateral triangle then enclosed is $\dfrac{n^2}{\sqrt{3}(l^2+m^2)}$

$(lx + my)^2 - 3(ly - mx)^2 = 0$

NOTE

Combined equation of pair of straight line (i) can also be written as follows:

19. The product of the perpendiculars from a point $P(x_1, y_1)$ to the pair of lines $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$

SOLVED SUBJECTIVE PROBLEMS (XI-XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find the area of the triangle, equation of whose sides are given by $y^2 - 9xy + 18x^2 = 0$ and $y = 9$.

Solution

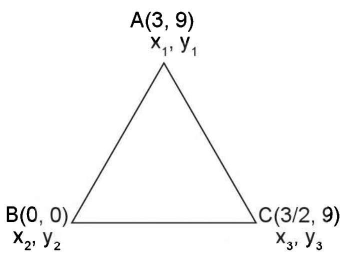
$y^2 - 9xy + 18x^2 = 0$

$\Rightarrow$ $y^2 - 3xy - 6xy + 18x^2 = 0$

$\Rightarrow$ $y(y - 3x) - 6x(y - 3x) = 0$

$\Rightarrow$ $(y - 3x)(y - 6x) = 0$

$\therefore$ $y - 3x = 0$ and $y - 6x = 0$



Now, equation of sides of triangle are

$y - 3x = 0$ .... (1) $\rightarrow AB$

$y - 6x = 0$ .... (2) $\rightarrow BC$

and $y = 9$ .... (3) $\rightarrow CA$

Solving equations (1) and (2) the vertex B is (0, 0)

Solving equations (2) and (3) the vertex C is (3/2, 9)

Solving equations (1) and (3) the vertex A is (3, 9)

$A[\Delta ABC] = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$

$$= \frac{1}{2}\left[\begin{matrix}(3\times 0) - (0\times 9) + (0\times 9) - \\ \left(\frac{3}{2}\times 0\right) + \left(\frac{3}{2}\times 9\right) - (3\times 9)\end{matrix}\right]$$

$$= \frac{1}{2}\left[0 - 0 + 0 - 0 + \frac{27}{2} - 27\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} - 1\right]27 = \frac{27}{2}\left[\frac{1-2}{2}\right]$$

$\Rightarrow$ $A[\Delta ABC] = \dfrac{-27}{2}$

$\because$ Area cannot be –ve,

$\therefore$ $A[\Delta ABC] = \dfrac{27}{2}$ square units.

2. Find the combined equation of straight lines which pass through (1, 2) and perpendicular to the line $3x^2 - 8xy + 5y^2 = 0$.

Solution

Given equation is $3x^2 - 8xy + 5y^2 = 0$

$\Rightarrow$ $3x^2 - 5xy - 3xy + 5y^2 = 0$

$\Rightarrow$ $x(3x - 5y) - y(3x - 5y) = 0$

$\Rightarrow$ $(3x - 5y)(x - y) = 0$

$\therefore$ Equation of lines are: $3x - 5y = 0$ .... (1)

and $x - y = 0$ ... (2)

Let perpendicular lines on equations (1) and (2) be respectively $5x + 3y = \lambda_1$ and $x + y = \lambda_2$

If these lines pass through (1, 2), then

$5 + 3 \times 2 = \lambda_1$ and $1 + 2 = \lambda_2$

$\Rightarrow$ $\lambda_1 = 11$ and $\lambda_2 = 3$

$\therefore$ Equation of required lines are:

$5x + 3y = 11$ and $x + y = 3$

$\therefore$ Their combined equation is

$(5x + 3y - 11)(x + y - 3) = 0$

$\Rightarrow$ $5x^2 + 5xy - 15x + 3xy + 3y^2 - 9y - 11x - 11y + 33 = 0$

$\Rightarrow$ $5x^2 + 8xy + 3y^2 - 26x - 20y + 33 = 0$

3. Show that the lines $x^2 + 4xy + y^2 = 0$ and $x - y = 0$ form an equilateral triangle.

Solution

Angle between the two lines $x^2 + 4xy + y^2 = 0$ is

$$\theta = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a+b}\right)$$

$$= \tan^{-1}\left(\frac{2\sqrt{4-1}}{1+1}\right)$$

$$= \tan^{-1}\sqrt{3} = 60°$$

Again, $x^2 + 4xy + y^2 = 0$

$\Rightarrow$ $x^2 + 4xy + 4y^2 - 3y^2 = 0$

$\Rightarrow$ $(x + 2y)^2 - 3y^2 = 0$

$\Rightarrow$ $(x+2y)^2 - (\sqrt{3}y)^2 = 0$

$\Rightarrow$ $(x+2y+\sqrt{3}y)(x+2y-\sqrt{3}y) = 0$

$\Rightarrow$ $[x+(2+\sqrt{3})y][x+(2-\sqrt{3})y] = 0$

$\therefore$ Equation of the lines represented by $x^2 + 4xy + y^2 = 0$ are

$x+(2+\sqrt{3})y = 0$ .... (1)

and $x+(2-\sqrt{3})y = 0$ .... (2)

Now, angle between the line $x - y = 4$ and line (2) is

$$\theta = \tan^{-1}\left(\frac{m_1 - m_2}{1+m_1m_2}\right)$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{1+\frac{1}{2+\sqrt{3}}}{1-\frac{1}{2+\sqrt{3}}}\right)$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{2+\sqrt{3}+1}{2+\sqrt{3}-1}\right)$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{3+\sqrt{3}}{\sqrt{3}+1}\right)$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{3}(\sqrt{3}+1)}{(\sqrt{3}+1)}\right)$$

$$\Rightarrow \theta = \tan^{-1}\sqrt{3}$$

$\Rightarrow$ $\theta = 60°$

Similarly, angle between the line $x - y = 4$ and line (2) is 60°.

i.e., angle between the lines taking any two of the given three lines be 60°.

Thus, the triangle formed by these lines is an equilateral triangle. **Proved.**

4. Prove that the equation of the straight line passing through the origin and making an angle θ with the line $y + x = 0$ is $x^2 + 2xy \sec\theta + y^2 = 0$.

Solution

Equation of a line passing through origin is

$y = mx$ .... (1)

If this line makes an angle θ with the line $y + x = 0$, then

$$\tan\theta = \pm\frac{m_1 - m_2}{1+m_1m_2}$$

$$\Rightarrow \tan\theta = \pm\frac{(m+1)}{(1-m)}$$

$\Rightarrow$ $(1 - m)^2 \tan^2\theta = (m + 1)^2$

$\Rightarrow$ $\tan^2\theta + m^2\tan^2\theta - 2m\tan^2\theta - m^2 - 1 - 2m = 0$

$\Rightarrow$ $(\tan^2\theta - 1) - 2m(1 + \tan^2\theta) + m^2(\tan^2\theta - 1) = 0$

$$\Rightarrow 1 - 2m\frac{1+\tan^2\theta}{\tan^2\theta - 1} + m^2 = 0$$

$$\Rightarrow 1 + 2m\frac{1+\tan^2\theta}{1-\tan^2\theta} + m^2 = 0$$

$\Rightarrow$ $1 + 2m\sec 2\theta + m^2 = 0$

$$\left[\because \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$$

$$\Rightarrow 1 + 2\frac{y}{x}\sec 2\theta + \left(\frac{y}{x}\right)^2 = 0,$$

{from equation (1)}

$\Rightarrow$ $x^2 + 2xy\sec 2\theta + y^2 = 0$ **Proved.**

5. Prove that the lines $12x^2 + 7xy - 12y^2 = 0$ and $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$ are sides of a square.

Solution

Given lines are $12x^2 + 7xy - 12y^2 = 0$ ... (1)

and $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$ ... (2)

From equation (1),

$12x^2 + 7xy - 12y^2 = 0$

$\Rightarrow$ $12x^2 - 9xy + 16xy - 12y^2 = 0$

$\Rightarrow$ $3x(4x - 3y) + 4y(4x - 3y) = 0$

$\Rightarrow$ $(4x - 3y)(3x + 4y) = 0$

$\therefore$ Lines denoted by equation (1) are

$4x - 3y = 0$ and $3x + 4y = 0$

Let the lines denoted by equation (2) be

$3x + 4y + l = 0$ and $4x - 3y + m = 0$

$\therefore$ $12x^2 + 7xy - 12y^2 - x + 7y - 1 = (3x + 4y + l)(4x - 3y + m) = 12x^2 + 7xy - 12y^2 + (4l + 3m)x + (-3l + 4m)y + lm$

Comparing the coefficient of x and y and constant terms,

$4l + 3m = -1$

$-3l + 4m = 7$

$lm = -1$

Solving them, we have

$l = -1, m = 1$

Therefore, lines denoted by equation (2) are

$3x + 4y - 1 = 0$ and $4x - 3y + 1 = 0$

Thus, sides of the square are

$4x - 3y = 0$ .... (3)

$3x + 4y - 1 = 0$ .... (4)

$4x - 3y + 1 = 0$ .... (5)

and $3x + 4y = 0$ .... (6)

Obviously, opposite sides are equations (3), (5) and equations (4), (6) are parallel and any pair of adjacent sides are perpendicular. This, the lines are side of a square.

6. Show that the pair of lines $2x^2 + 6xy + y^2 = 0$ is equally inclined to the pair of lines $4x^2 + 18xy + y^2 = 0$.

Solution

Equation of bisectors of the angle between the lines represented by $2x^2 + 6xy + y^2 = 0$ be

$$\frac{x^2 - y^2}{2-1} = \frac{xy}{3}$$

$\Rightarrow$ $3(x^2 - y^2) = xy$

$\Rightarrow$ $3x^2 - xy - 3y^2 = 0$

Again, equation of bisectors of the angle between the lines represented by $4x^2 + 18xy + y^2 = 0$ be $\dfrac{x^2 - y^2}{4-1} = \dfrac{xy}{9}$

$$\Rightarrow \quad \frac{x^2 - y^2}{3} = \frac{xy}{9}$$

$\Rightarrow$ $3x^2 - xy - 3y^2 = 0$

Since the lines (1) and (2) are same, therefore angle between the lines is equal to the other pair of lines.

UNSOLVED SUBJECTIVE PROBLEMS (XI-XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

EXERCISE 1

1. Find the straight lines given by the equations
(i) $x^2 - 3x + 2 = 0$
(ii) $x^2 + 2xy \operatorname{cosec}\theta + y^2 = 0$

2. Find the angle between the pair of straight lines represented by
(i) $\lambda x^2 + (1 - \lambda^2)xy - \lambda y^2 = 0$
(ii) $4x^2 - 24xy + 11y^2 = 0$

3. Find the equation of the bisectors of the angle between each of the following pairs of lines
(i) $x^2 - 10xy - y^2 = 0$
(ii) $x^2 - 2xy \sec\theta + y^2 = 0$

4. If $4ab = 3h^2$ then find the ratio of the gradients of the lines denoted by $ax^2 + 2hxy + by^2 = 0$.

5. If angle between the lines denoted by $y^2 + \lambda xy - x^2 \tan\theta = 0$ is 2θ, then find the value of λ.

6. Show that the lines represented by $3x^2 - 4xy + 5y^2 = 0$ are perpendicular to the lines given by $5x^2 + 4xy + 3y^2 = 0$.

7. One of the bisector of the angle between the lines denoted by $x^2 + 2hxy - 2y^2 = 0$ is $x - 2y = 0$, then find the value of h.

8. Find the difference of gradient of lines denoted by $x^2(\sec^2\theta - \sin^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$

EXERCISE 2

1. If $(a + 3b)(3a + b) = 4h^2$, then find the angle between the lines $ax^2 + 2hxy + by^2 = 0$.

2. Find the combined equation represented by the following equations
(i) $x = 0$ and $y = 0$
(ii) $x - 4 = 0$ and $y - 3 = 0$

3. If the ratio of the gradient of the lines $ax^2 + 2hxy + by^2 = 0$ is 1 : 3, then prove that $\frac{h^2}{ab} = \frac{4}{3}$.

4. Show that the angle between the lines $3x^2 - 4xy + 5y^2 = 0$ is equal to the angle between the line $5x^2 + 4xy + 3y^2 = 0$.

5. Find the area of the triangle formed by the lines $x^2 - 4y^2 = 0$ and $x = k$.

6. If sum of the gradients of the lines denoted by $x^2 - 2xy \tan A - y^2$ is 4, then find $\angle A$.

ANSWERS

EXERCISE 1

1. (i) $x = 1$ and $x = 2$
(ii) $x \sin\theta + y(1 \pm \cos\theta) = 0$

2. (i) 90°
(ii) $\tan^{-1}(4/3)$

3. (i) $5x^2 + 2xy - 5y^2 = 0$
(ii) $x^2 - y^2 = 0$

4. 1 : 3

5. 0

7. 2

8. 2

EXERCISE 2

1. $\theta = 60°$.

2. (i) $xy = 0$
(ii) $xy - 3x - 4y + 12 = 0$

5. $\frac{k^2}{2}$

6. $\tan^{-1}(-2)$

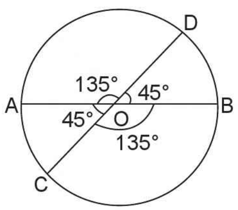
SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If the pair of lines $ax^2 + 2(a + b)xy + by^2 = 0$ lie along diameters of a circle and divide the circle into four sectors such that the rea of one of the sectors is thrice the area of another sector, then ***[AIEEE-2005]***
(a) $3a^2 + 2ab + 3b^2 = 0$
(b) $3a^2 + 10ab + 3b^2 = 0$
(c) $3a^2 - 2ab + 3b^2 = 0$
(d) $3a^2 - 10ab + 3b^2 = 0$

Solution

(a) Let *AOB* and *COD* be two given diameter lines which divide the circle into four sectors such that

Area *AOD* = 3 Area *BOD* (given)



$\Rightarrow \quad \angle BOD = \pi/4$

$\Rightarrow$ Angle between the pair of given lines = $\pi/4$

$$\therefore \quad \tan\frac{\pi}{4} = \frac{2\sqrt{(a+b)^2 - ab}}{a+b}$$

$$\Rightarrow \quad 1 = \frac{2\sqrt{a^2 + b^2 + ab}}{a+b}$$

$\Rightarrow \quad 3a^2 + 3b^2 + 2ab = 0.$

2. The locus of the point $P(x, y)$ satisfying the relation

$$\sqrt{(x-3)^2 + (y-1)^2} + \sqrt{(x+3)^2 + (y-1)^2} = 6$$

is

[Orissa JEE-2002]

(a) Straight line
(b) Pair of straight lines
(c) Circle
(d) Ellipse

Solution

(b) $\sqrt{(x-3)^2+(y-1)^2}+\sqrt{(x+3)^2+(y-1)^2}=6$

$\sqrt{(x-3)^2+(y-1)^2}=6-\sqrt{(x+3)^2+(y-1)^2}$

Squaring both sides. We get,

$12x+36=12\sqrt{(x+3)^2+(y-1)^2}$

Again squaring we get the given equation is pair of straight lines.

3. The image of the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ by the line mirror $y = 0$ is

(a) $ax^2 – 2hxy – by^2 = 0$
(b) $bx^2 – 2hxy + ay^2 = 0$
(c) $bx^2 + 2hxy + ay^2 = 0$
(d) $ax^2 – 2hxy + by^2 = 0$

Solution

(d) Let $y = m_1x$ and $y = m_2x$ be the lines represented by $ax^2 + 2hxy + by^2 = 0$. Then their images in $y = 0$ are $y = – m_1x$ and $y = – m_2x$ and so their combined equation is $y^2 + m_1m_2x^2 + xy(m_1+ m_2) = 0$ or $y^2+\frac{a}{b}x^2+xy\left(-\frac{2h}{b}\right)=0$

$\left(\because m_1+m_2=-\frac{2h}{b},\ m_1m_2=\frac{a}{b}\right)$

or $ax^2 – 2hxy + by^2 = 0$

4. If $x^2 – 2pxy – y^2 = 0$ and $x^2 – 2qxy – y^2 = 0$ bisect angles between each other, then
[CEE (Delhi)-1999; AIEEE-2003]

(a) $p + q = 1$ (b) $pq = 1$
(c) $pq + 1 = 0$ (d) $p^2 + pq + q^2 = 0$

Solution

(c) The equation of the bisectors of the anlges between the lines $x^2 – 2pxy – y^2 = 0$ is

$$\frac{x^2-y^2}{1-(-1)}=\frac{xy}{-p}$$

$$\frac{x^2-y^2}{2}=\frac{-xy}{p}$$

$\Rightarrow px^2 + 2xy – py^2 = 0$

This must be same as $x^2 – 2qxy – y^2 = 0$

5. The pair of lines $\sqrt{3}x^2-4xy+\sqrt{3}y^2=0$ are rotated about the origin by $\pi/6$ in anti-clockwise sense. The equation of the pair in the new position is:

(a) $\sqrt{3}x^2-xy=0$ (b) $x^2-\sqrt{3}xy=0$
(c) $xy-\sqrt{3}y^2=0$ (d) None of these

Solution

(a) We have, $\sqrt{3}x^2-4xy+\sqrt{3}y^2=0$

$\Rightarrow (\sqrt{3}x-y)(x-\sqrt{3}y)=0$

$\Rightarrow \sqrt{3}x-y=0,\ x-\sqrt{3}y=0$

$\Rightarrow y=\sqrt{3}x,\ y=\frac{1}{\sqrt{3}}x$

These lines makes 60° and 90° angles respectively with x-axis. If they are rotated about the origin by $\pi/6$ i.e., 30° in anti-clockwise direction, then they make 90° and 60° angles respectively with x-axis. So, their equations in new position are $x = 0$ and $y=\sqrt{3}x$. The combined equation of these two lines is

$x(\sqrt{3}x-y)=0$ or $\sqrt{3}x^2-xy=0$

6. The pair of lines represented by $3ax^2 + 5xy + (a^2 – 2)y^2 = 0$ are perpendicular to each other for

(a) Two values of a (b) all values of a
(c) one value of a (d) no value of a

Solution

(a) We have, $3ax^2 + 5xy + (a^2 – 2)y^2 = 0$

This will represent a pair of perpendicular lines, if Coeff. of x^2 + Coeff. of $y^2 = 0$

$\Rightarrow 3a + a^2 – 2 = 0$

$\Rightarrow a=\frac{-3\pm\sqrt{17}}{2}$

Hence, there are two values of a.

7. If the slope of one line of the pair of lines represented by $ax^2 + 4xy + y^2 = 0$ is three times the slope of the other, then a is equal to

(a) 1 (b) 2
(c) 3 (d) 4

Solution

(c) Let m at $3m$ be the slope of the lines represented by the equation $ax^2 + 4xy + y^2 = 0$. Then, $m + 3m = –4$ and $m \times 3m = a$

$\Rightarrow$ m $=-1$ and $m^2=\frac{a}{3}\Rightarrow\frac{a}{3}=1\Rightarrow a=3$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. A second degree homogenous equation in x and y always represents
(a) A pair of straight lines
(b) A circle
(c) A conic section
(d) None of these

2. The nature of straight lines represented by the equation $4x^2 + 12xy + 9y^2 = 0$ is *[MP PET-1988]*
(a) Real and coincident
(b) Real and different
(c) Imaginary and different
(d) None of these

3. If one of the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ be $y = mx$, then *[UPSEAT-1999]*
(a) $bm^2 + 2hm + a = 0$
(b) $bm^2 + 2hm - a = 0$
(c) $am^2 + 2hm + b = 0$
(d) $bm^2 - 2hm + a = 0$

4. The equation of the lines passing through the origin and having slopes 3 and –1/3 is
(a) $3y^2 + 8xy - 3x^2 = 0$
(b) $3x^2 + 8xy - 3y^2 = 0$
(c) $3y^2 - 8xy + 3x^2 = 0$
(d) $3x^2 + 8xy + 3y^2 = 0$

5. The equation of the lines represented by the equation $x^2 - 5xy + 6y^2 = 0$ are
(a) $y + 2x = 0, y - 3x = 0$
(b) $y - 2x = 0, y - 3x = 0$
(c) $y + 2x = 0, y + 3x = 0$
(d) None of these

6. The equation of one of the line represented by the equation $x^2 + 2xy \cot\theta - y^2 = 0$, is
(a) $x - y \cot\theta = 0$
(b) $x + y \tan\theta = 0$
(c) $x \sin\theta + y(\cos\theta + 1) = 0$
(d) $x \cos\theta + y(\sin\theta + 1) = 0$

7. The pair of straight lines passes through the point (1, 2) and perpendicular to the pair of straight lines $3x^2 - 8xy + 5y^2 = 0$, is
(a) $(5x + 3y + 11)(x + y + 3) = 0$
(b) $(5x + 3y - 11)(x + y - 3) = 0$
(c) $(3x + 5y - 11)(x + y + 3) = 0$
(d) $(3x - 5y + 11)(x + y - 3) = 0$

8. If the ratio of gradients of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is 1 : 3, then the value of the ratio h^2 : ab is *[MP PET-1998]*
(a) 1 / 3 (b) 3 / 4
(c) 4 / 3 (d) 1

9. Difference of slopes of the lines represented by equation $x^2(\sec^2\theta - \sin^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$ is *[Kurukshetra CEE-2002]*
(a) 4 (b) 3
(c) 2 (d) None of these

10. If one of the lines given by $6x^2 - xy + 4cy^2 = 0$ is $3x + 4y = 0$, then c equals *[AIEEE-2004]*
(a) –3 (b) –1
(c) 3 (d) 1

11. If the sum of the slopes of the lines given by $x^2 - 2cxy - 7y^2 = 0$ is four times their product, then c has the value *[AIEEE-2004]*
(a) – 2 (b) – 1
(c) 2 (d) 1

12. Angles made by the lines represented by the equation $xy + y = 0$ with y-axis are
(a) 0° and 90° (b) 0° and 30°
(c) 30° and 60° (d) 30° and 90°

13. If the lines represented by the equation $2x^2 - 3xy + y^2 = 0$ make angles α and β with x-axis, then $\cot^2\alpha + \cot^2\beta$ is equal to
(a) 0 (b) 3/2
(c) 7/4 (d) 5/4

14. Which of the equation represents the pair of perpendicular straight lines
(a) $y^2 + xy - x^2 = 0$ (b) $y^2 - xy + x^2 = 0$
(c) $x^2 + xy + y^2 = 0$ (d) $x^2 + xy - 2y^2 = 0$

15. If the angle 2θ is acute, then the acute angle between $x^2(\cos\theta - \sin\theta) + 2xy\cos\theta + y^2(\cos\theta + \sin\theta) = 0$ is *[EAMCET-2002]*
(a) 2θ (b) $\theta/3$
(c) θ (d) $\theta/2$

16. The lines $x - 2y = 0$ will be a bisector of the angle between the lines represented by the equation $x^2 - 2hxy - 2y^2 = 0$, if h is equal to
(a) 1/2 (b) 2
(c) –2 (d) –1/2

17. If the bisectors of the angles between the pairs of lines given by the equation $ax^2 + 2hxy + by^2 = 0$ and $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ be coincident then λ is equal to

(a) a (b) b
(c) h (d) any real number

18. The combined equation of the bisectors of the angle between the lines represented by $(x^2 + y^2)\sqrt{3} = 4xy$ is ***[MP PET-1992]***

(a) $y^2 - x^2 = 0$ (b) $xy = 0$
(c) $x^2 + y^2 = 2xy$ (d) $\frac{x^2 - y^2}{\sqrt{3}} = \frac{xy}{2}$

19. If the equation $ax^2 + 2hxy + by^2 = 0$ has the one line as the bisector of angle between the coordinate axes, then ***[Bihar CEE-1990]***

(a) $(a - b)^2 = h^2$ (b) $(a + b)^2 = h^2$
(c) $(a - b)^2 = 4h^2$ (d) $(a + b)^2 = 4h^2$

20. The equation of the pair of straight lines, each of which makes an angle α with the line $y = x$ is ***[MP PET-1990]***

(a) $x^2 + 2xy \sec 2\alpha + y^2 = 0$
(b) $x^2 + 2xy \operatorname{cosec} 2\alpha + y^2 = 0$
(c) $x^2 - 2xy \operatorname{cosec} 2\alpha + y^2 = 0$
(d) $x^2 - 2xy \sec 2\alpha + y^2 = 0$

21. The figure formed by the lines $x^2 + 4xy + y^2 = 0$ and $x - y = 4$, is ***[Roorkee 1980]***

(a) A right angled triangle
(b) An isosceles triangle
(c) An equilateral triangle
(d) None of these

22. Locus of the points equidistant from, the lines represented by $x^2 \cos^2\theta - xy \sin 2\theta - y^2 \sin^2\theta = 0$ is

(a) $x^2 + y^2 + 2xy \sec 2\theta = 0$
(b) $x^2 + y^2 + 2xy \operatorname{cosec} 2\theta = 0$
(c) $x^2 - y^2 + 2xy \sec 2\theta = 0$
(d) $x^2 - y^2 + 2xy \operatorname{cosec} 2\theta = 0$

23. The equation $x^2 - 7xy + 12y^2 = 0$ represents ***[BIT Ranchi-1991]***

(a) Circle
(b) Pair of parallel straight lines
(c) Pair of perpendicular straight lines
(d) Pair of non-perpendicular intersecting straight lines

24. The equation $x^2 + ky^2 + 4xy = 0$ represents two coincident lines, if k is equla to ***[MP PET-1995]***

(a) 0 (b) 1
(c) 4 (d) 16

25. If the bisectors of the angles of the lines represented by $3x^2 - 4xy + 5y^2 = 0$ and $5x^2 + 4xy + 3y^2 = 0$ are same, then the angle made by the lines respresented by first with the second, is

(a) 30° (b) 60°
(c) 45° (d) 90°

26. If the bisectors of angles represented by $ax^2 + 2hxy + by^2 = 0$ and $a'x^2 + 2h'xy + b'y^2 = 0$ are same, then

(a) $(a - b) h' = (a' - b') h$
(b) $(a - b) h = (a' - b') h'$
(c) $(a + b) h' = (a' - b') h$
(d) $(a - b) h' = (a' + b') h$

27. Area of the triangle formed by the lines $y^2 - 9xy + 18x^2 = 0$ and $y = 9$ is

(a) 27/4 sq. units
(b) 27 sq. units
(c) 27/2 sq. units
(d) None of these

28. The angle between the pair of straight lines $y^2 \sin^2\theta - xy \sin^2\theta + x^2(\cos^2\theta - 1) = 1$, is ***[MNR-1985; UPSEAT-2000; Kerala (Engg.)-2005]***

(a) $\pi/3$ (b) $\pi/4$
(c) $2\pi/3$ (d) None of these

29. A pair of $\perp^r$ straight lines is drawn through the origin and forming with the line $2x + 3y = 6$ an isosceles Δ right angled at the origin. Find the equation of the pair of straight lines and area of the Δ

(a) $5x^2 - 24xy - 5y^2 = 0$, 36/13 sq. unit
(b) $5x^2 + 24xy - 5y^2 = 0$, 36/13 sq. unit
(c) $5x^2 - 24xy - 6y^2 = 0$, 36/12 sq. unit
(d) $5x^2 - 24xy - 5y^2 = 0$, 35/13 sq. unit

SOLUTIONS

1. (a) A second degree homogeneous equation in x and y always represents a pair of straight lines passing through origin.

But general equation of pair of straight lines, circle and conics are not homogeneous.

2. (a) $4x^2 + 12xy + 9y^2 = 0$

$\Rightarrow \quad (2x + 3y)^2 = 0$

$\Rightarrow \quad 2x + 3y = 0, 2x + 3y = 0$

$\Rightarrow$ Real and coincident.

3. (a) $ax^2 + 2hxy + by^2 = 0$

i.e., $b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$

Since, $y = mx \therefore m = \frac{y}{x}$

$\therefore \quad bm^2 + 2hm + a = 0$

4. (b) $m_1 = 3, m_2 = -\frac{1}{3}$.

Hence, the lines are $y = 3x, \ y = -\frac{1}{3}x$.

Multiplying both the lines, we get

$(y - 3x)(3y + x) = 0$

$\Rightarrow \quad 3x^2 + 8xy - 3y^2 = 0$

5. (d) $x^2 - 5xy + 6y^2 = 0$

$\Rightarrow \quad x^2 - 2xy - 3xy + 6y^2 = 0$

$\Rightarrow \quad x(x - 2y) - 3y(x - 2y) = 0$

$\Rightarrow \quad (x - 2y) = 0$ and $x - 3y = 0$

6. (c) The lines represented by the equation $x^2 + 2xy \cot\theta - y^2 = 0$ are

$$ax + hy \pm y\sqrt{h^2 - ab} = 0$$

$$\Rightarrow \quad x + y\cot\theta \pm y\sqrt{\cot^2\theta + 1} = 0$$

$$\Rightarrow \quad x + y\left(\frac{\cos\theta}{\sin\theta} \pm \frac{1}{\sin\theta}\right) = 0$$

$\Rightarrow \quad x\sin\theta + y(\cos\theta \pm 1) = 0$

Hence, one line is $x\sin\theta + y(\cos\theta + 1) = 0$

7. (b) The equation of lines represented by the equation $3x^2 - 8xy + 5y^2 = 0$ are $3x - 5y = 0$ and $x - y = 0$.

Therefore, equation of lines passing through (1, 2) and perpendicular to given lines are $x + y - 3 = 0$ and $5x + 3y - 11 = 0$.

8. (c) $ax^2 + 2hxy + by^2 = 0$ represents one of its lines as $y = mx$

Then $ax^2 + 2hx \cdot mx + bm^2x^2 = 0$

$\Rightarrow \quad bm^2 + 2mh + a = 0$ .... (1)

According to the given condition, roots of (1) are $\alpha, 3\alpha$

$$\alpha + 3\alpha = -\frac{2h}{b} \quad \text{.......... (2),}$$

$$\alpha.(3\alpha) = \frac{a}{b} \quad \text{......... (3)}$$

$$(2) \Rightarrow \quad \alpha = -\frac{h}{2b}.$$

Put this in (3), $3.\left(-\frac{h}{2b}\right)^2 = \frac{a}{b}$

$$\Rightarrow \quad \frac{h^2}{ab} = \frac{4}{3}$$

9. (c) We know that

$$m_1 - m_2 = \sqrt{(m_1 + m_2)^2 - 4m_1m_2}$$

$$= \sqrt{\left(\frac{2\tan\theta}{\sin^2\theta}\right)^2 - 4\left(\frac{\sec^2\theta - \sin^2\theta}{\sin^2\theta}\right)}$$

$$= \sqrt{\frac{4\tan^2\theta}{\sin^4\theta} - 4(\sec^2\theta\operatorname{cosec}^2\theta - 1)} = 2$$

10. (a) $3x + 4y = 0$

$$\Rightarrow \quad y = -\frac{3}{4}x$$

Substituting $y = -\frac{3}{4}x$ in the given equation,

we get $6x^2 - x\left(-\frac{3}{4}x\right) + 4c\left(-\frac{3}{4}x\right)^2 = 0$ for all x

or $\quad 6 + \frac{3}{4} + \frac{9c}{4} = 0 \Rightarrow c = -3$

11. (c) If the slopes are m_1, m_2; then they are the roots of $-7m^2 - 2cm + 1 = 0$ (Replacing y by m and x by 1)

$\Rightarrow \quad m_1 + m_2 = \frac{-2c}{7}$ and $m_1m_2 = -\frac{1}{7}$

As $m_1 + m_2 = 4\, m_1m_2$, therefore,

$$\frac{-2c}{7} = 4\left(-\frac{1}{7}\right) \Rightarrow c = 2$$

12. (a) $xy + y = 0$

$\Rightarrow \quad (x + 1)y = 0$

$\Rightarrow \quad x + 1 = 0, y = 0$

$\Rightarrow \quad x = -1, y = 0$

$\Rightarrow$ line parallel to y-axis, x-axis

Angles made by lines with y-axis = 0, 90°.

13. (d) $2x^2 - 3xy + y^2 = 0$, let $y = mx$ be one of its line.

Then $2x^2 - 3x \cdot mx + m^2x^2 = 0$

$\Rightarrow \quad m^2 - 3m + 2 = 0$

$\Rightarrow \quad (m - 2)(m - 1) = 0$

$\Rightarrow \quad m = 1, m = 2.$

According to given condition $\tan\alpha = 1$, $\tan\beta = 2$

Then $\cot^2\alpha + \cot^2\beta = 1 + \frac{1}{4} = \frac{5}{4}$.

14. (a) Coefficient of x^2 + Cofficient of $y^2 \geq 0$

15. (d) Here,

$$\theta = \tan^{-1}\frac{2\sqrt{(\cos\theta)^2 - (\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}}{(\cos\theta - \sin\theta) + (\cos\theta + \sin\theta)}$$

$$= \frac{\pi}{4}$$

16. (c) Bisector of $x^2 - 2hxy - 2y^2 = 0$ is one of the lines is $x = 2y$ put in equation to get

$$\frac{x^2 - y^2}{1 - (-2)} = \frac{xy}{-h}$$

$\Rightarrow \quad hx^2 - hy^2 + 3xy = 0$

$\Rightarrow \quad 4h - h + 6 = 0 \Rightarrow h = -2$

17. (d) Bisectors of $ax^2 + 2hxy + by^2 = 0$ are

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \qquad \text{......... (i)}$$

and of $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$

i.e., $(a + \lambda)x^2 + 2hxy + (b + \lambda)y^2 = 0$ are

$$\frac{x^2 - y^2}{(a + \lambda) - (b + \lambda)} = \frac{xy}{h} \qquad \text{......... (ii)}$$

Which is the same equation as equation (i).

Hence, for any λ belonging to real numbers, the lines will have same bisectors.

18. (a) Bisectors of $(x^2 + y^2)\sqrt{3} - 4xy = 0$ are

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

$$\Rightarrow \quad \frac{x^2 - y^2}{\sqrt{3} - \sqrt{3}} = \frac{xy}{-2} \Rightarrow x^2 - y^2 = 0$$

19. (d) Bisector of angle between x and y axes is $y = x$.(1)

It is one of the lines of $ax^2 + 2hxy + by^2 = 0$

$\Rightarrow \quad ax^2 + 2hx^2 + bx^2 = 0$

$\Rightarrow \quad a + b = -2h$

$\Rightarrow \quad (a + b)^2 = 4h^2$

20. (d) Any line through origin is $y = mx$. If this line makes an angle α with the line $y = x$, then

$$\pm\tan\alpha = \frac{m_1 - m_2}{1 + m_1m_2}$$

$$\Rightarrow \quad \frac{m - 1}{1 + m} = \pm\tan\alpha$$

$\Rightarrow \quad (m - 1)^2 = (1 + m)^2 \tan^2\alpha$

$\Rightarrow \quad m^2(1 - \tan^2\alpha) - 2m(1 + \tan^2\alpha) + (1 - \tan^2\alpha) = 0,$

Dividing by $(1 - \tan^2\alpha)$,

$$\Rightarrow \quad m^2 - 2m\left(\frac{1 + \tan^2\alpha}{1 - \tan^2\alpha}\right) + 1 = 0$$

$$\Rightarrow \quad m^2 - \frac{2m}{\cos(2\alpha)} + 1 = 0.$$

But $\frac{y}{x} = m$, $\therefore \left(\frac{y}{x}\right)^2 - 2(\sec 2\alpha)\left(\frac{y}{x}\right) + 1 = 0$

$\Rightarrow \quad x^2 + y^2 - 2xy\sec(2\alpha) = 0.$

21. (c) $x^2 + 4xy + y^2 = 0$ .... (1)

represents one of its line as $y = mx$

Then (1) $\Rightarrow \quad (1 + 4m + m^2)x^2 = 0$

$\Rightarrow \quad m^2 + 4m + 1 = 0$

$$\Rightarrow \quad m = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

Take $m_1 = -2+\sqrt{3}, m_2 = -2-\sqrt{3}$

If θ is angle between lines (1), then

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b} = \frac{2\sqrt{(2)^2-1}}{1+1} = \sqrt{3} = \tan(60°)$$

∴ one angle between sides = 60°.

Third line is $x - y = 4$, its slope $m_3 = 1$. If α is angle between lines (1) and third line, then

$$\tan\alpha = \left|\frac{m_1 - m_3}{1+m_1m_3}\right| = \left|\frac{(-2+\sqrt{3})-1}{1+(-2+\sqrt{3})}\right|$$

$$= \left|\frac{-3+\sqrt{3}}{-1+\sqrt{3}}\right| = \sqrt{3} = \tan 60°$$

⇒ α = 60°. Remaining third angle is 180° – (60 + 60°) = 60°

⇒ Δ is equilateral.

22. (d) We know that point lie on the bisector of angle between the lines represented by any curve is always equidistant from the lines. Therefore, equation of bisectors will be the required locus.

$$\frac{x^2-y^2}{xy} - \frac{1}{\frac{-\sin^2\theta}{2}}$$

⇒ $x^2 - y^2 + 2xy \operatorname{cosec}^2\theta = 0$

23. (d) Here $h^2 = \left(-\frac{7}{2}\right) = \frac{49}{4}$, $ab = 1.(12) = 12$

⇒ $h^2 > ab$

⇒ alternative (d)

24. (c) Lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident if $h^2 = ab$ in equation $x^2 + ky^2 + 4xy = 0$

$a = 1, b = k, h = 2$

∴ $(2)^2 = 1 \times k$ or $k = 4$.

25. (d) $3x^2 - 4xy + 5y^2 = 0$ .... (1)

$5x^2 + 4xy + 3y^2 = 0$ .... (2)

Bisectors of (1) are $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$

⇒ $\frac{x^2-y^2}{3-5} = \frac{xy}{-2}$

⇒ $x^2 - y^2 - xy = 0$ .... (3)

Bisectors of (2) are $\frac{x^2-y^2}{5-3} = \frac{xy}{2}$

⇒ $x^2 - y^2 - xy = 0$ .... (4)

Evidently (3) and (4) are the same.

Angle between lines given by (3) are perpendicular.

∴ θ = 90°

26. (a) Since bisectors are same, therefore

$$\frac{a-b}{h} = \frac{a'-b'}{h'}$$

⇒ $(a - b)h' = (a' - b')h$.

27. (a) $y^2 - 9xy + 18x^2 = 0$

⇒ $(y - 6x)(y - 3x) = 0$

⇒ $y = 6x, y = 3x$

Sides forming triangles are $y = 9, y = 6x, y = 3x$

Their intersection points are

$$\left(\frac{3}{2}, 9\right), (0,0), (3,9)$$

$$\text{Are} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{3}{2} & 9 & 1 \\ 3 & 9 & 1 \end{vmatrix} = \frac{1}{2}\left(\frac{27}{2} - 27\right)$$

$$= \frac{1}{2}\left(-\frac{27}{2}\right) = -\frac{27}{4}$$

Since area is positive

∴ Area $= \frac{27}{4}$

28. (d) $y^2 \sin^2\theta - xy \sin^2\theta + x^2(\cos^2\theta - 1) = 1$

⇒ $y^2 \sin^2\theta - xy \sin^2\theta - x^2 \sin^2\theta = 1$

Here $a = -\sin^2\theta$, $b = \sin^2\theta$

Then, $a + b = 0$

⇒ lines are perpendicular

⇒ $\theta = \frac{\pi}{2}$

29. (c) $2x + 3y = 6$, its slope $= m_1 = -\frac{2}{3}$.

Let $y = mx$ be the equation of a line through origin.

Then $m_2 = m$. These lines form a right angled isosceles.

Hence its angles are $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$.

$$\tan\frac{\pi}{4} = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\Rightarrow \quad 1 = \left|\frac{-\frac{2}{3} - m}{1 - \frac{2m}{3}}\right|$$

$$\Rightarrow \quad (2 + 3m)^2 = (3 - 2m)^2$$

$$\Rightarrow \quad 5m^2 + 24m - 5 = 0$$

But $m = \frac{y}{x}$.

$$\therefore \quad 5\left(\frac{y}{x}\right)^2 + 24\left(\frac{y}{x}\right) - 5 = 0$$

$$\Rightarrow \quad 5y^2 + 24xy - 5x^2 = 0$$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If the equation $2x^2 - 2hxy + 2y^2 = 0$ represents two coincident straight lines passing through the origin, then h is equal to

(a) ± 6 (b) $\sqrt{6}$
(c) $-\sqrt{6}$ (d) ± 2

2. If the sum of slopes of the pair of lines represented by $4x^2 + 2hxy - 7y^2 = 0$ is equal to the product of the slopes, then the value of h is ***[Karnataka CET-1999]***

(a) -6 (b) -2
(c) -4 (d) 4

3. The gradient of one of the lines of $ax^2 + 2hxy + by^2 = 0$ is twice that of the other, then ***[MP PET-2000; Pb. CET-2002]***

(a) $h^2 = ab$ (b) $h = a + b$
(c) $8h^2 = 9ab$ (d) $9h^2 = 8ab$

4. Separate equations of lines, for a pair of lines, whose equation is $x^2 + xy - 12y^2 = 0$, are ***[Karnataka CET-2001; Pb. CET-2000]***

(a) $x + 4y = 0$ and $x + 3y = 0$
(b) $2x - 3y = 0$ and $x - 4y = 0$
(c) $x - 6y = 0$ and $x - 3y = 0$
(d) $x + 4y = 0$ and $x - 3y = 0$

5. The equation to the pair of straight lines through the origin which are perpendicular to the lines $2x^2 - 5xy + y^2 = 0$, is ***[MP PET-1990]***

(a) $2x^2 + 5xy + y^2 = 0$
(b) $x^2 + 2y^2 + 5xy = 0$
(c) $x^2 - 5xy + 2y^2 = 0$
(d) $2x^2 + y^2 - 5xy = 0$

6. If $\frac{x^2}{a} + \frac{y^2}{b} + \frac{2xy}{h} = 0$ represent pair of straight lines and slope of one line is twice the other. Then $ab : h^2$ is ***[DCE-2005]***

(a) 9 : 8 (b) 8 : 9
(c) 1 : 2 (d) 2 : 1

7. Angle between the lines represented by the equation $x^2 + 2xy \sec\theta + y^2 = 0$ is

(a) θ (b) 2θ
(c) $\theta/2$ (d) None of these

8. Pair of straight lines perpendicular to each other represented by ***[Roorkee-1990]***

(a) $2x^2 = 2y(2x + y)$ (b) $x^2 + y^2 + 3 = 0$
(c) $2x^2 = y(2x + y)$ (d) $x^2 = 2(x - y)$

9. Then angle between the pair of lines represented by $2x^2 - 7xy + 3y^2 = 0$, is ***[Kurukshetra CEE-2002]***

(a) 60° (b) 45°
(c) $\tan^{-1}(7/6)$ (d) 30°

10. The equation of the bisectors of the angle between lines represented by equation $4x^2 - 16xy - 7y^2 = 0$ is

(a) $8x^2 + 11xy - 8y^2 = 0$
(b) $8x^2 - 11xy - 8y^2 = 0$
(c) $16x^2 + 11xy - 16y^2 = 0$
(d) $16x^2 + 11xy + 16y^2 = 0$

11. The equation of the bisectors of the angle between the lines represented by the equation $x^2 - y^2 = 0$, is

(a) $x = 0$ (b) $y = 0$
(c) $xy = 0$ (d) None of these

12. If $y = mx$ be one of the bisectors of the angle between the lines $ax^2 - 2hxy + by^2 = 0$, then
(a) $h(1 + m^2) + m(a - b) = 0$
(b) $h(1 - m^2) + m(a + b) = 0$
(c) $h(1 - m^2) + m(a - b) = 0$
(d) $h(1 + m^2) + m(a + b) = 0$

13. The area of the triangle formed by the lines $x^2 - 4y^2 = 0$ and $x = a$, is
(a) $2a^2$ (b) $a^2/2$
(c) $\sqrt{3}a^2/2$ (d) $2a^2/\sqrt{3}$

14. If $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$ represents two straight lines represented by $ax^2 + 2hxy + by^2 = 0$, then $(l_1^2 + m_1^2)(l_2^2 + m_2^2)$ is equal to
(a) $\dfrac{(a-b)^2+4h^2}{4b^2}$ (b) $(a - b)^2 - 4h^2$
(c) $(a - b)^2 + 4h^2$ (d) $\dfrac{(a-b)^2-4h^2}{4b^2}$

15. The equation of one of the line represented by the equation $pq(x^2 - y^2) + (p^2 - q^2)xy = 0$, is
(a) $px + qy = 0$ (b) $px - qy = 0$
(c) $p^2x + q^2y = 0$ (d) $q^2x + p^2y = 0$

16. If $(a + 3b)(3a + b) = 4h^2$, then the angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$ is
(a) 30° (b) 45°
(c) 60° (d) $\tan^{-1}(1/2)$

17. If the sum of the slopes of the lines represented by the equation $x^2 - 2xy\tan A - y^2 = 0$ be 4, then $\angle A$ is equal to
(a) 0° (b) 45°
(c) 60° (d) $\tan^{-1}(-2)$

18. If the lines represented by the equation $ax^2 - bxy - y^2 = 0$ make angles α and β with the x-axis, then $\tan(\alpha + \beta)$ is equal to
(a) $\dfrac{b}{1+a}$
(b) $\dfrac{-b}{1+a}$
(c) $\dfrac{a}{1+b}$
(d) None of the above

19. If the acute angles between the pairs of lines $3x^2 - 7xy + 4y^2 = 0$ and $6x^2 - 5xy + y^2 = 0$ be θ_1 and θ_2 respectively, then
(a) $\theta_1 = \theta_2$ (b) $\theta_1 = 2\theta_2$
(c) $2\theta_1 = \theta_2$ (d) None of these

20. If the pair of lines $ax^2 + acxy + cy^2 = 0$ and $\left(3+\dfrac{1}{c}\right)x^2+\lambda xy+\left(3+\dfrac{1}{a}\right)y^2=0$ are equally inclined for all real $a \neq 0, c \neq 0$, then λ is equal to
(a) 1 (b) 2
(c) 3 (d) 4

21. The combined equation to a pair of the straight lines passing through the origin and inclined at 30° and 60° respectively with
(a) $\sqrt{3}(x^2 + y) = 4xy$
(b) $4(x^2 + y^2) = \sqrt{3}xy$
(c) $x^2 + \sqrt{3}y^2 - 2xy = 0$
(d) $x^2 + 3y^2 - 2xy = 0$

22. The equation $4x^2 - 24xy + 11y^2 = 0$ represents ***[Orissa JEE-2003]***
(a) Two parallel lines
(b) Two perpendicular lines
(c) Two lines through the origin
(d) A circle

23. If the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is 5 times the other, then ***[Karnataka CET-2003]***
(a) $5h^2 = ab$ (b) $5h^2 = 9ab$
(c) $9h^2 = 5ab$ (d) $h^2 = ab$

24. The angle between the lines represented by the equation $\lambda x^2 + (1 + \lambda)^2 xy - \lambda y^2 = 0$ is
(a) 30° (b) 45°
(c) 60° (d) 90°

25. The angle between the pair of straight lines $x^2 + 4y^2 - 7xy = 0$, is
(a) $\tan^{-1}\left(\dfrac{1}{3}\right)$ (b) $\tan^{-1} 3$
(c) $\tan^{-1}\dfrac{\sqrt{33}}{5}$ (d) $\tan^{-1}\dfrac{5}{\sqrt{33}}$

26. Condition that the two lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ to be perpendicular is ***[Kurukshetra CEE-1998; MP PET-2001]***
(a) $ab = -1$ (b) $a = -b$
(c) $a = b$ (d) $ab = 1$

27. The angle between the lines $x^2 + 4xy + y^2 = 0$ is ***[Karnataka CET-2001; Pb. CET-2001]***
(a) 60° (b) 15°
(c) 30° (d) 45°

28. If the pair of straight lines $x^2 - 4xy - y^2 = 3$ and $x^2 - 2pxy - y^2 = 0$ be such that each pair bisects the angle between the other, then p is equal to

(a) 1 (b) –2
(c) –1/2 (d) 2

29. Two lines represented by equation $x^2 + xy + y^2 = 0$ are

(a) Collinear
(b) Parallel
(c) Mutually perpendicular
(d) Imaginary

30. The triangle formed by $x^2 - 9y^2 = 0$ and $x = 4$ is ***[Orissa JEE-2004]***

(a) Isosceles (b) Equilateral
(c) Right angled (d) None of these

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 16 minutes.
3. The worksheet consists of 16 questions. The maximum marks are 48.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The angle between the lines represented by the equation $4x^2 - 24xy + 11y^2 = 0$ are *[MP PET-1990]*

(a) $\tan^{-1}\frac{3}{4}, \tan^{-1}\left(-\frac{3}{4}\right)$

(b) $\tan^{-1}\frac{1}{3}, \tan^{-1}\left(-\frac{1}{3}\right)$

(c) $\tan^{-1}\frac{4}{3}, \tan^{-1}\left(-\frac{4}{3}\right)$

(d) $\tan^{-1}\frac{1}{2}, \tan^{-1}\left(-\frac{1}{2}\right)$

2. The angle between the pair of straight lines given by equation $x^2 + 2xy - y^2 = 0$, is *[MNR-1983]*

(a) $\pi/3$ (b) $\pi/6$
(c) $\pi/2$ (d) 0

3. The angle between the lines given by $x^2 - y^2 = 0$ is *[MP PET-1999]*

(a) 15° (b) 45°
(c) 75° (d) 90°

4. The equation $x^2 + k_1y^2 + k_2 xy = 0$ represents a pair of perpendicular lines if

(a) $k_1 = -1$ (b) $k_1 = 2k_2$
(c) $2k_1 = k_2$ (d) None of these

5. If the equation $3x^2 + 2hxy + 3y^2 = 0$ represents two coincident lines, then h is equal to

(a) 2 (b) 3
(c) 4 (d) 1

6. The combined equation of bisectors of angles between coordinate axes is

(a) $x^2 + y^2 = 0$ (b) $x^2 - y^2 = 0$
(c) $xy = 0$ (d) $x + y = 0$

7. If the slope of one line represented by $ax^2 + 10xy + y^2 = 0$ is four times that of the other, then a is equal to

(a) 16 (b) 6
(c) 4 (d) 1

8. If $4ab = 3h^2$, then the ratio of slopes of the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ will be

(a) $\sqrt{2}:1$ (b) $\sqrt{3}:1$
(c) 2 : 1 (d) 1 : 3

9. The angle between the lines $xy = 0$ is *[MP PET-1990, 92]*

(a) 45° (b) 60°
(c) 90° (d) 180°

10. If the lines represented by the equation $6x^2 + 41xy - 7y^2 = 0$ make angles α and β with x-axis, then $\tan\alpha$, $\tan\beta$ is equal to

(a) – 6/7 (b) 6/7
(c) 7/6 (d) – 7/6

11. If the lines $(p - q)x^2 + 2(p + q)xy + (q - p)y^2 = 0$ are mutually perpendicular, then

(a) $p = q$
(b) $q = 0$
(c) $p = 0$
(d) p and q may have any value

12. The lines $x^2 - 3y^2 = 0$ and $x = a$ form a triangle which is

(a) right angled
(b) isosceles
(c) equilateral
(d) None of these

13. If straight lines bisecting the angle between the bisectors of the angles between the lines $ax^2 + 2hxy + by^2 = 0$ are given by $\lambda(x^2 - y^2) + 4hxy = 0$, then λ is equal to

(a) $a + b$ (b) $2ab$
(c) $a - b$ (d) $a^2 - b^2$

14. The equation $x^2 + ky^2 + 4xy = 0$ represents two coincident lines, if k is equal to

(a) 0 (b) 1
(c) 4 (d) 16

15. The pair of straight lines $a^2 x^2 + 2h(a + b)y + b^2 y^2 = 0$ and $ax^2 + 2hxy + by^2 = 0$ are such that

(a) one of the lines coincide
(b) one of the lines of first is perpendicular to one of lines of second pair
(c) the lines represented by two pairs are equally inclined
(d) None of these

16. The coordinates of the orthocentre of the triangle formed by lines $xy = 0$ and $x + y = 1$ is.... ***[Orissa JEE-2008]***

(a) (1, 1) (b) (0, 0)
(c) (1, 0) (d) (1, 2)

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**16.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ	
6. ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

9. (c) Line represented by $xy = 0$ are $x = 0$ and $y = 0$

$\therefore$ Angle between them = 90°

11. (d) $\because$ Sum of coefficient $= p - q + q - p = 0$

$\therefore$ Line will be mutually perpendicular for all values of p and q.

12. (c) Vertices of Δ formed by $x^2 - 3y^2 = 0$ and $x = a$ will be $A(0, 0)$, $B\left(a, \frac{a}{\sqrt{3}}\right)$ and $C\left(a, \frac{-a}{\sqrt{3}}\right)$

Here we can see that

$AB = BC = CA$

$\therefore$ ΔABC is a equilateral.

LECTURE

2

Pair of Straight Lines 2

(Some important results connected with two homogenous pair of straight lines, general equation of second degree)

BASIC CONCEPTS

1. **Some important results connected with two pairs of straight lines:**

$ax^2 + 2hxy + by^2 = 0$ ....(i)

$a'x^2 + 2h'xy + b'y^2 = 0$ ....(ii)

1.1 If both the pair of lines are identical then $\frac{a}{a'} = \frac{h}{h'} = \frac{b}{b'}$

1.2 If both the pair of lines have one line in common then

$$\begin{matrix} a & b & 2h \\ a' & b' & 2h' \end{matrix}$$

or $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}^2 = \begin{vmatrix} b & 2h \\ b' & 2h' \end{vmatrix}\begin{vmatrix} 2h & a \\ ah' & a' \end{vmatrix}$ or

$(ab' - ba')^2 = 4(ha' - h'a)(h'b - hb')$

1.3 If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is perpendicular to one of the lines represented by $a'x^2 + 2h'xy + b'y^2 = 0$ then

$$\begin{matrix} a & b & 2h \\ a' & b' & 2h' \end{matrix}$$

or $\begin{vmatrix} a & b \\ b' & a' \end{vmatrix}^2 = \begin{vmatrix} b & 2h \\ a' & -2h' \end{vmatrix}\begin{vmatrix} 2h & a \\ -2h' & b' \end{vmatrix}$ or

$(aa' - bb')^2 + 4(ha' + h'b)(h'b + ha') = 0$

1.4 Two pairs of lines are said to be equally inclined to each other if they possess the same pair of angle bisectors.

2. If the pairs of lines $ax^2 + 2hxy - ay^2 = 0$ and $bx^2 + 2hxy - by^2 = 0$ be such that each pair bisects the angle between the other then hg + ab = 0.

3. **General Equation of Second Degree:**

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$ **[MPPET-2007]**

4. If the separate equations of the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are $y = m_1 x + c_1$ and $y = m_2 x + c_2$ then

$m_1 + m_2 = \frac{-2h}{b}, m_1 m_2 = \frac{a}{b}, \quad m_1 c_2 + m_2 c_1 = \frac{2g}{b},$

$c_1 + c_2 = \frac{-2f}{b}, c_1 c_2 = \frac{c}{b},$

$|m_1 - m_2| = \frac{2\sqrt{h^2 - ab}}{b},$

$|m_1 c_2 - m_2 c_1| = \frac{2\sqrt{g^2 - ac}}{b},$

$|c_1 - c_2| = \frac{2\sqrt{f^2 - bc}}{b}$

$(1 + m_1^2)(1 + m_2^2) = \frac{(a-b)^2 + 4h^2}{b^2}$

5. Both lines represented by $ax^2 + 2hxy + by^2 + 2hx + 2fy + c = 0$ are $ax + hy + g = \pm \sqrt{(h^2 - ab)y^2 + (gh - af)y + (g^2 - ac)}$

5.1 If both lines are real and distinct then $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

5.2 If both lines are real and coincident then h^2 = ab, gh = af and g^2 = ac.

5.3 If both lines are parallel then h^2 = ab, gh = af

or $\frac{a}{h}=\frac{h}{b}=\frac{g}{f}$ or $af^2 = bg^2$

6. The distance between the parallel lines represented by $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ is

$$2\sqrt{\frac{g^2-ac}{a(a+b)}} \text{ or } 2\sqrt{\frac{f^2-bc}{b(a+b)}}.$$

7. Angle between the lines represented by $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ is θ, then

$$\theta = \tan^{-1}\left|\frac{2\sqrt{h^2-ab}}{a+b}\right| = \sin^{-1}\frac{2\sqrt{h^2-ab}}{\sqrt{(a-b)^2+4h^2}}$$

$$= \cos^{-1}\frac{a+b}{\sqrt{(a-b)^2+4h^2}}$$

7.1 Note that the angle between the pair of lines $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ is same as the angle between the lines $ax^2 + 2hxy$ + by$^2 = 0$.

7.2 Condition of perpendicularty : $a + b = 0$

8. The equation to the pair of straight lines through the origin and (i) parallel to the pair of lines represented by $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ is: $ax^2 + 2hxy$ + by$^2 = 0$

(ii) perpendicular to the pair of lines represented by $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ is : $bx^2 - 2hxy + ay^2 = 0$

9. The equation to the pair of lines through (α, β) and parallel to the pair of lines $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ is

a $(x - \alpha)^2 + 2h(x - \alpha)(y - \beta)$ + b $(y - \beta)^2 = 0$

10. The equation to the pair of lines through (α, β) and perpendicular to the pair of lines

$ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ is

$b(x - \alpha)^2 - 2h(x - \alpha)(y - \beta)$ + a $(y - \beta)^2 = 0$

11. **Point of intersection of the lines**:
The point of intersection of the two lines represented by $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ is

$$\left(\frac{bg-hf}{h^2-ab}, \frac{af-gh}{h^2-ab}\right)$$

or $\left(\sqrt{\frac{f^2-bc}{h^2-ab}}, \sqrt{\frac{g^2-ca}{h^2-ab}}\right)$

11.1 If the pair of lines $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ intersect at (α, β) then (α, β) satisfy the equation aα + $h\beta + g = 0$, $h\alpha + b\beta + f = 0$ and $g\alpha + b\beta + c = 0$

11.2 The pair of lines $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ intersect on x-axis, then g^2 = ac and $2fgh = af^2 + ch^2$

11.3 The pair of lines $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy$ +c = 0 intersect on y-axis, then f^2 = bc and $2fgh = bg^2 + ch^2$

12. If p is the distance of the point of intersection of $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ from the origin then, $p^2 = \frac{c(a+b)-f^2-g^2}{ab-h^2}$.

13. **Bisectors:** The equations of the bisectors of the angles between the lines represented by $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ are given by

$$\frac{(x-x')^2-(y-y')^2}{a-b} = \frac{(x-x')(y-y')}{h}$$

where (x', y') is the point of intersection of the lines represented by $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$.

14. The product of the perpendiculars from (α, β) to the pair of lines $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ is: $\frac{a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c}{\sqrt{(a-b)^2+4h^2}}$

[Bihar CEE-1994]

15. If the lines represented by the equation $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ will be equidistant from the origin if $f^4 - g^4$ = c $(bf^2 - ag^2)$.

16. Equation of the pair of straight lines passing through point of intersection of pair of straight lines $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c = 0$ ..(i) and $a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$ ..(ii)

is $ax^2 + 2hxy$ + by$^2 + 2gx + 2fy + c + \lambda$ $(a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c') = 0$
where λ is constant whose value is obtained by given condition:

16.1 If both pairs of straight lines (i) and (ii) are parallel then $\frac{a}{a'}=\frac{h}{h'}=\frac{b}{b'}$

16.2 If both pairs of straight lines are coincident then

$$\frac{a}{a'}=\frac{h}{h'}=\frac{b}{b'}=\frac{g}{g'}=\frac{f}{f'}=\frac{c}{c'}$$

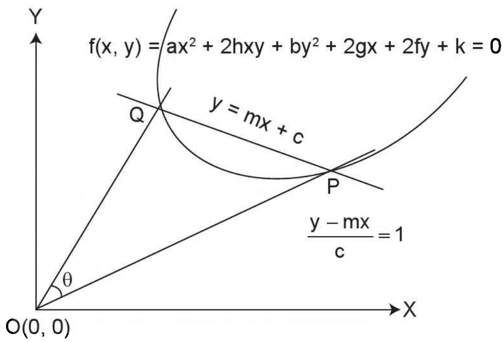
17. Equation of the lines joining the origin to the points of intersection of a given line and a given curve:

Let the straight line be,

$$y = mx + c \text{ or } \frac{y - mx}{c} = 1 \qquad \text{.... (1)}$$

and the curve be,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + k = 0 \qquad \text{.... (2)}$$



Let the line cuts the curve at the points P and Q, then the joint equation of OP and OQ is

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)$$

$$\left(\frac{y - mx}{c}\right) + k\left(\frac{y - mx}{c}\right)^2 = 0 \qquad \text{..(3)}$$

NOTE

If OP and OQ are equally inclined with coordinate axes then $m_1 + m_2 = -\frac{2h}{b} = 0$

This is done by making the equation of the curve homogenous with the help of the equation of the line.

18. The pair of lines which join the origin to the points of intersection of the line $y = mx + c$ with the curve $x^2 + y^2 = a^2$ are at right angles, if: $2c^2 = a^2(1 + m^2)$

19. The straight lines joining the origin to the points of intersection of the two curves $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$ are at right angles, if:

$$\frac{a + b}{g} = \frac{a' + b'}{g'}$$

20. The equation of the diagonal not passing through origin of the quadrilateral formed by the pairs of lines

$ax^2 + 2hxy + by^2 = 0$ ...... (1) and

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...... (2)

is obtained by subtracting (1) from the (2) i.e. $2gx + 2fy + c = 0$.

21. If centroid of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$ is

$$\left(\frac{2(bl - hm)}{3(bl^2 - 2hlm + am^2)}, \frac{2(am - hl)}{3(bl^2 - 2hlm + am^2)}\right)$$

22. The two pair of straight lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and

$ax^2 + 2hxy + by^2 = 0$ form a:

(a) square if $(a - b)fg + h(f^2 - g^2) = 0, a + b = 0$

(b) rectangle if $(a - b)fg + h(f^2 - g^2) \neq 0, a + b = 0$

(c) rhombus if $(a - b)fg + h(f^2 - g^2) = 0, a + b \neq 0$

(d) parallelogram if $(a - b)fg + h(f^2 - g^2) \neq 0, a + b \neq 0$

23. If (x_1, y_1) is the midpoint of the intercept made by a line L between:

(a) the pair of lines $S \equiv ax^2 + 2hxy + by^2 = 0$, then the equation to the line L is $S_1 = S_{11}$ i.e. $axx_1 + h(xy_1 + x_1y) + byy_1$ [$T = S_1$] $= ax_1^2 + 2h_1y_1 + by_1^2$

(b) the pair of lines $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then the equation to the line L is $S_1 = S_{11}$ i.e.,

$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c$

$= ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Show that the equation $6x^2 + 5xy - 4y^2 + 7x + 13y - 3 = 0$ represents a pair of straight lines.

Solution

Comparing the given equation with the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ .... (1)

We have,

$$a = 6, b = -4,\ h = \frac{5}{2},$$

$$g = \frac{7}{2},\ f = \frac{13}{2}, c = -3$$

If the general equation (1) represents a pair of straight lines, then

$$\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Hence, if the given equation represents a pair of straight lines, then

$$\Delta = 6(-4)(-3) + 2.\frac{13}{2}.\frac{7}{2}.\frac{5}{2}$$

$$-6.\frac{169}{4} - (-4).\frac{49}{4} - (-3).\frac{25}{4}$$

$$= 72 + \frac{455}{4} - \frac{1014}{4} + 49 + \frac{75}{4}$$

$$= \frac{288 + 455 - 1014 + 196 + 75}{4}$$

$$= \frac{0}{4} = 0$$

Hence, the given equation represents a pair of straight lines. **Proved.**

2. Prove that the equation $6x^2 + 8xy - 8y^2 + 23x - 10y + 7 = 0$ represents a pair of straight lines. Also, find the point of intersection of the lines and the angle between them.

Solution

Comparing the given equation with the general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$,

We have,

$$a = 6, b = -8, h = 4,\ g = \frac{23}{2}, f = -5, c = 7$$

Now $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 6(-8)7$

$$+ 2(-5).\frac{23}{2}..4 - 6(-5)^2 - (-8)\left(\frac{23}{2}\right)^2 - 7(4)^2$$

$= -336 - 460 - 150 + 1058 - 112$

i.e., $\Delta = 0$

This shows that the given equation represents a pair of straight lines.

Given equation is $6x^2 + 8xy - 8y^2 + 23x - 10y + 7 = 0$

or $\quad 6x^2 + 8xy + 23x = 8y^2 + 10y - 7$

Multiplying by 6, we get

$$36x^2 + 6(8y + 23)x = 48y^2 + 60y - 42$$

On completing the square L.H.S., add $\left(\frac{8y+23}{2}\right)^2$ to both the sides, Hence,

$$36x^2 + 6(8y + 23)x + \left(\frac{8y+23}{2}\right)^2$$

$$= 48y^2 + 60y - 42 + \left(\frac{8y+23}{2}\right)^2$$

$$\Rightarrow \left(6x + \frac{8y+23}{2}\right)^2$$

$$= \frac{192y^2 + 240y - 168 + 64y^2 + 368y + 529}{4}$$

$$\Rightarrow \left(\frac{12x+8y+23}{2}\right)^2 = \frac{256y^2 + 608y + 361}{4}$$

$$\Rightarrow \left(\frac{12x+8y+23}{2}\right)^2 = \left(\frac{16y+19}{2}\right)^2$$

$\therefore \quad 12x + 8y + 23$

$\quad = \pm(16y + 19)$

Taking positive sign, we have

$$12x + 8y + 23 = \pm(16y + 19)$$

$\Rightarrow \quad 12x - 8y + 4 = 0$

$\Rightarrow \quad 3x - 2y + 1 = 0$ .... (1)

Taking negative sign, we have

$$12x + 8y + 23 = -16y - 19$$

$\Rightarrow \quad 12x + 24y + 42 = 0$

$\Rightarrow \quad 2x + 4y + 7 = 0$ .... (2)

These are the equations of two different straight lines Proved.

To find the point of intersection, multiplying by 2 to equation (1) and add with equation (2), we get

$$8x + 9 = 0 \Rightarrow x = \frac{-9}{8}$$

Now, put the value of x in equation (1) or (2),

$$y = \frac{-19}{16}$$

So, the point of intersection of lines is $\left(\frac{-9}{8}, \frac{-19}{16}\right)$.

Angle between two lines can also be known by two methods.

First Method: Gradient (m_1) of line $3x - 2y + 1 = 0$ is $\frac{3}{2}$ and gradient (m_2) of line $2x + 4y + 7 = 0$ is $\frac{-1}{2}$

If the angle between the two lines is θ, then

$$\tan\theta = \frac{m_1 \sim m_2}{1 + m_1 m_2}$$

$$\Rightarrow \quad \tan\theta = \frac{\frac{3}{2} + \frac{1}{2}}{1 - \frac{3}{2} \cdot \frac{1}{2}}$$

$\Rightarrow \quad \tan\theta = 8$ or $\theta = \tan^{-1}(8)$.

3. Show that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel lines, if $bg^2 = af^2$ and $h^2 = ab$ and distance between these parallel lines is $2\sqrt{\frac{g^2 - ac}{a(a+b)}}$

Solution

Given equation is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\Rightarrow \quad a^2x^2 + 2ahxy + aby^2 + 2agx + 2afy + ac = 0$

$\Rightarrow \quad a^2x^2 + 2ax(hy + g) = -aby^2 - 2afy - ac$

On completing square of L.H.S., adding both the sides by $(hy + g)^2$

hence,

$$a^2x^2 + 2ax(hy + g) + (hy + g)^2 = -aby^2 - 2afy - ac + (hy + g)^2$$

$\Rightarrow \quad (ax + hy + g)^2 = y^2(h^2 - ab) + 2y(hg - af) + g^2 - ac$

$\Rightarrow \quad ax + hy + g = \pm\sqrt{y^2(h^2 - ab) + 2y(hg - af) + g^2 - ac}$

Above equation shows the two parallel lines if coefficient of y^2 and y under square root be zero, then

	$h^2 - ab = 0$	and	$hg - af = 0$
$\Rightarrow$	$h^2 = ab$	and	$hg = af$
$\Rightarrow$	$h^2 = ab$	and	$h^2g^2 = a^2f^2$
$\Rightarrow$	$h^2 = ab$	and	$abg^2 = a^2f^2$
$\Rightarrow$	$h^2 = ab$	and	$bg^2 = af^2$.

Proved

Therefore, parallel lines are

$$ax + hy + g = \pm\sqrt{g^2 - ac}$$

$\Rightarrow \quad ax + hy + g + \sqrt{g^2 - ac} = 0$ .... (1)

$\Rightarrow \quad ax + hy + g - \sqrt{g^2 - ac} = 0$ .... (2)

Now, perpendicular on line (1) and (2) drawn from origin (0,0) are:

$$P_1 = \frac{g + \sqrt{g^2 - ac}}{\sqrt{a^2 + h^2}}$$

and $P_2 = \frac{g - \sqrt{g^2 - ac}}{\sqrt{a^2 + h^2}}$

$\therefore$ Distance between these lines $= P_1 - P_2$

$$= \frac{g + \sqrt{g^2 - ac}}{\sqrt{a^2 + h^2}} - \frac{g - \sqrt{g^2 - ac}}{\sqrt{a^2 + h^2}}$$

$$= \frac{2\sqrt{g^2 - ac}}{\sqrt{a^2 + h^2}} = 2\sqrt{\frac{g^2 - ac}{a^2 + ab}}, \{\because h^2 = ab\}$$

$$= 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$$

4. Prove that the lines joining the point of intersection of $y - 3x = 2$ and the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ to the origin are inclined at an angle $\tan^{-1}\frac{2\sqrt{2}}{3}$.

Solution

The equation of the given line may be written as $\frac{y-3x}{2}=1$ .... (1)

and equation of curve is

$x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ .... (2)

The equation of the lines joining the origin to the points of intersection of the curve and the line is obtained by making equation (2) homogenous with the help of equation (1). Thus the required equation is

$x^2 + 2xy + 3y^2 + 4(x + 2y).$

$$\frac{y-3x}{2} - 11\left(\frac{y-3x}{2}\right)^2 = 0$$

$\Rightarrow$ $x^2 + 2xy + 3y^2 + 2(xy - 3x^2 + 2y^2 - 6xy)$

$-\frac{11}{4}(y^2 - 6xy + 9x^2) = 0$

or $4x^2 + 8xy + 12y^2 + 8xy - 24x^2 + 16y^2$

$-48xy - 11y^2 + 66xy - 99x^2 = 0$

or $-119x^2 + 34xy + 17y^2 = 0$

or $7x^2 - 2xy - y^2 = 0$

Comparing it with $ax^2 + 2hxy + by^2 = 0$, we get $a = 7, h = -1, b = -1$

If θ is the angle between the lines, then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{2\sqrt{1+7}}{7-1}$$

$$= \frac{2\sqrt{8}}{6} = \frac{4\sqrt{2}}{6}$$

or $\tan\theta = \frac{2\sqrt{2}}{3}$

$\therefore$ $\theta = \tan^{-1}\frac{2\sqrt{2}}{3}$

5. Show that the pair of lines joining the origin to the points of intersection of the curves $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$ will be at right angles to one another, if $g'(a + b) = g(a' + b')$. ***[MP–93,97]***

Solution

The equations of the given curves are

$ax^2 + 2hxy + by^2 + 2gx = 0$ .... (1)

and $a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$ .... (2)

The equation representing a pair of lines joining the origin to the points of intersection of the given curves is obtained by making equation (1) homogenous with the help of equation (2) by eliminating the first degree terms.

So, multiplying equation (1) by g′, equation (2) by g and subtracting, we get the required equation as $(ag' - a'g)x^2 + 2(hg' - h'g)xy + (bg' - b'g)y^2 = 0$ ... (3)

Now, the two lines given by equation (3) will be perpendicular, if (coefficient of x^2) + (coefficient of y^2) = 0

i.e., $(ag' - a'g) + (bg' - b'g) = 0$

or $g(a' + b') = g'(a + b)$ Proved.

6. Prove that the lines joining the origin to the points of intersection of $y = mx + c$ with $x^2 + y^2 = a^2$ will be perpendicular to each other, if $2c^2 = a^2(1 + m^2)$. ***[MP-93, 94, 97]***

Solution

The equation of the given line may be written as $\frac{y-mx}{c}=1$ .... (1)

The equation of given curve is

$x^2 + y^2 = a^2$ .... (2)

The required equation is obtained by making the equation of curve (2) homogenous with the help of equation (1)

$\therefore$ The required equation is

$$x^2 + y^2 = a^2\left(\frac{y-mx}{c}\right)^2$$

or $(x^2 + y^2)c^2 = a^2(y^2 + m^2x^2 - 2xym)$

or $x^2(c^2 - a^2m^2) + 2a^2 mxy + y^2(c^2 - a^2) = 0$

The line represented by this equation are perpendicular to each other if the sum of coefficients of x^2 and y^2 is zero, i.e.,

$c^2 - a^2m^2 + c^2 - a^2 = 0$

or $2c^2 = a^2(1 + m^2)$ Proved.

7. Find the value of λ so that the lines joining the origin to the point of intersection of the line $x + y = 1$ with the curve $x^2 + y^2 + x - 2y + \lambda = 0$ may be at right angles.

Solution

Equation of the line is

$x + y = 1$ .... (1)

and equation of the curve is

$x^2 + y^2 + x - 2y + \lambda = 0$ .... (2)

Making equation (2) homogenous by equation (1), we get the equation of the lines joining the origin to the points of intersection of equations (1) and (2),

$x^2 + y^2 + (x - 2y)(x + y) + \lambda(x + y)^2 = 0$

$\Rightarrow$ $x^2 + y^2 + x^2 + xy - 2xy - 2y^2 + \lambda(x^2 + 2xy + y^2) = 0$

$\Rightarrow$ $x^2(2 + \lambda) + y^2(\lambda - 1) - xy + 2\lambda xy = 0$ .... (3)

Lines denoted by equation (3) will be perpendicular to each other if

Coefficient of x^2 + Coefficient of $y^2 = 0$.

$\Rightarrow$ $(2 + \lambda) + (\lambda - 1) = 0$

$\Rightarrow$ $2\lambda + 1 = 0$

$\Rightarrow$ $\lambda = -\dfrac{1}{2}$

8. Prove that the area of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is $\dfrac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}$.

Solution

Let the equation $ax^2 + 2hxy + by^2 = 0$ shows two lines

$y = m_1x$ .... (1)

and $y = m_2x$ .... (2)

$\therefore$ $m_1 + m_2 = -\dfrac{2h}{b}$ and $m_1m_2 = \dfrac{a}{b}$

Third line is

$lx + my + n = 0$ .... (3)

Intersection of lines (1) and (2) is A(0, 0).

From equations (1) and (3), we have

$lx + m(m_1x) + n = 0$

$\Rightarrow$ $(l + mm_1)x = -n$

$\Rightarrow$ $x = -\dfrac{n}{1 + mm_1}$

$\therefore$ From equation (1), $y = -\dfrac{m_1n}{1 + mm_1}$

$\therefore$ Intersection of lines (1) and (3) is:

$B\left(-\dfrac{n}{1 + mm_1}, -\dfrac{m_1n}{1 + mm_1}\right)$

Similarly, intersection of lines (2) and (3) is:

$C\left(-\dfrac{n}{1 + mm_2}, -\dfrac{nm_2}{1 + mm_2}\right)$

$\therefore$ Area of required triangle ABC

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\left[0 + \left(-\frac{n}{l + mm_1}\right)\left(-\frac{nm_2}{l + mm_2} - 0\right) - \frac{n}{l + mm_2}\left(0 + \frac{nm_1}{l + mm_1}\right)\right]$$

$$= \frac{n^2}{2}\left[\frac{m_2 - m_1}{(l + mm_1)(l + mm_2)}\right]$$

$$= \frac{n^2}{2}\left[\frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{l^2 + lm(m_1 + m_2) + m_1m_2m^2}\right]$$

$$= \frac{n^2}{2}\left(\frac{\sqrt{\frac{4h^2}{b^2} - 4.\frac{a}{b}}}{l^2 - \frac{2h}{b}lm + \frac{a}{b}m^2}\right)$$

$$= \frac{2n^2\sqrt{h^2 - ab}}{2(bl^2 - 2hlm + am^2)}$$

$$= \frac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}$$ Proved.

9. If orthocentre of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$ is (x_1, y_1) then show that:

$$\frac{x_1}{l} = \frac{y_1}{m} = \frac{a + b}{am^2 - 2hlm + bl^2}$$

Solution

Let $ax^2 + 2hxy + by^2 = 0$ shows two lines

$y = m_1x$ .... (1)

and $y = m_2x$ .... (2)

$\therefore$ $m_1 + m_2 = -\dfrac{2h}{b}$ and $m_1m_2 = \dfrac{a}{b}$ equation of third line is

$lx + my = 1$

Obviously, intersection of equations (1) and (2) is point (0, 0), intersection of equations (1) and (3) is $A\left(\dfrac{1}{l + mm_1}, \dfrac{m_1}{l + mm_1}\right)$

and intersection of equations (2) and (3) is

$B\left(\dfrac{1}{l + mm_2}, \dfrac{m_2}{l + mm_2}\right)$.

Equation of the line passing through the point (0, 0) and perpendicular to the line (3) is $mx - ly = 0$

$\Rightarrow \quad \dfrac{x}{l} = \dfrac{y}{m}$

$\because$ Orthocentre of the triangle is (x_1, y_1)

$\therefore \quad \dfrac{x_1}{l} = \dfrac{y_1}{m} = r$ (say) .... (4)

Again, equation of the line perpendicular on line (2) is

$x + m_2y = \lambda$

If it passes through point

$$A\left(\frac{1}{l+mm_1}, \frac{m_1}{l+mm_1}\right)$$

then $\dfrac{1}{l+mm_1} + \dfrac{m_1m_2}{l+mm_1} = \lambda \Rightarrow \lambda = \dfrac{1+m_1m_2}{l+mm_1}$

$\therefore$ Equation of the line perpendicular on line (2) is

$$x + m_2y = \frac{1+m_1m_2}{l+mm_1}$$

$\because$ Point (x_1, y_1) also lies on this line

$$\therefore \quad x + m_2y = \frac{1+m_1m_2}{l+mm_1}$$

$$\Rightarrow \quad lr + mm_2r = \frac{1+m_1m_2}{l+mm_1}$$

{from equation (4)}

$$\Rightarrow \quad r = \frac{1+m_1m_2}{(l+mm_1)(l+mm_2)}$$

$$= \frac{1+m_1m_2}{l^2+(m_1+m_2)lm+m^2m_1m_2}$$

$$\frac{1+\dfrac{a}{b}}{l^2 - \dfrac{2h}{b}lm + \dfrac{a}{b}m^2} = \frac{a+b}{bl^2-2hlm+am^2}$$

Thus, from equation (4) and by this result, we have

$$\frac{x}{l} = \frac{y_1}{m} = \frac{a+b}{bl^2-2hlm+am^2}$$ **Proved.**

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

EXERCISE 1

1. For what value of k the following represent a pair of straight lines:
   (i) $kx^2 + 12xy + 9y^2 + 12x + 18y + 5 = 0$
   (ii) $kx^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$
2. Prove that the equation $4x^2 - 12xy + 9y^2 + 8x - 12y - 5 = 0$ represents two parallel lines and find the distance between them.
3. Find the equation of the different lines represented by $6x^2 + 5xy - 4y^2 + 7x + 13y - 3 = 0$ and find the angle between them.
4. For what value of m, the lines joining the origin to the points of intersection of $y = mx + c$ and $y^2 = 4ax$ are perpendicular.
5. Prove that the lines joining the points of intersection of $7x^2 - 4xy + 8y^2 + 2x - 4y - 8 = 0$ and line $3x - y = 2$ to the origin are perpendicular.
6. If the distance between the lines denoted by $(x - 2y)^2 + k(x - 2y) = 0$ is 3, then find the value of k
7. Prove that the equation $3x^2 + 17xy + 10y^2 + 2x - 16y - 8 = 0$ represents two lines. Find the angle between these lines.
8. If $x^2 - 3xy \pm \lambda y^2 + 3x - 5y + 2 = 0$ shows two lines, where λ is any real number and angle between these lines is θ, then find $\text{cosec}^2\,\theta$.

EXERCISE 2

1. Prove that the equation $x^2 + 2\sqrt{3}xy + 3y^2 - 3 - 3x - 3\sqrt{3}y - 4 = 0$ represents two parallel straight lines, also find distance between them. ***[MP – 1997]***
2. Prove that each of the following equations represent a pair of straight lines:
   (i) $x^2 - 2xy + y^2 - x + y = 0$
   (ii) $6x^2 - xy - 12y^2 - 8x + 29y - 14 = 0$

3. Find the equation of the different lines represented by $6x^2 - 5xy - 6y^2 - 22x + 7y + 20 = 0$ and find their point of intersection and the angle between them.

4. Find the equation of the pair of lines joining the origin of coordinates to the point of intersection of the line $y = x + 2$ with the circle $x^2 + y^2 = 4$

5. Show that the lines joining the points of intersection of the line $4x - 3y = 10$ with the circle $x^2 + y^2 + 3x - 6y - 20 = 0$ to the origin are perpendicular.

6. If the lines joining the origin to the points of intersection of $y = mx + 2$ and $x^2 + y^2 - 4 = 0$ are perpendicular then find the value of m.

ANSWERS

EXERCISE 1

1. (i) 4 (ii) 2
2. $\dfrac{6}{\sqrt{13}}$
3. $2x - y + 3 = 0$ and $3x + 4y - 1 = 0$, $\tan^{-1}\left(\dfrac{11}{2}\right)$
4. $-\dfrac{c}{4a}$
6. $\pm 3\sqrt{5}$
7. 45°.
8. 10

EXERCISE 2

1. $\dfrac{5}{2}$ units.
3. $2x - 3y -$ [illegible] 0 and $3x + 2y - 5 = 0$, [illegible], $\left(\dfrac{23}{13}, -\dfrac{2}{13}\right)$
4. $x = 0, y = 0$
6. ± 1.

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The area of the triangle formed by the lines $x^2 - 4xy + y^2 = 0$ and $x + y + 4\sqrt{6} = 0$ is: ***[Roorkee, 1983]***

(a) $8\sqrt{3}$ units (b) $16\sqrt{3}$ units

(c) $32\sqrt{3}$ units (d) None of these

Solution

(b) $x^2 - 4xy + y^2 = 0 \equiv [x - (2+\sqrt{3})y][x - (2-\sqrt{3})y] = 0 = 0$

Hence sides of given triangle are

$$x - (2+\sqrt{3})y = 0,$$
$$x - (2-\sqrt{3})y = 0, \quad x + y + 4\sqrt{6} = 0$$

∴ Its area is given by

$$\Delta = \frac{1}{2(3-\sqrt{3})(-3-\sqrt{3})2\sqrt{3}} \begin{vmatrix} 1 & -2-\sqrt{3} & 0 \\ 1 & -2+\sqrt{3} & 0 \\ 1 & 1 & 4\sqrt{6} \end{vmatrix}^2$$

$$= \frac{1}{24\sqrt{3}}[4\sqrt{6}(2\sqrt{3})]^2$$

$$= \frac{64 \times 18}{24\sqrt{3}} = 16\sqrt{3} \text{ units}$$

2. The equation of the bisectors of angle between the lines represented by equation $(y - mx)^2 = (x + my)^2$ is

(a) $mx^2 + (m^2 - 1)\,xy - my^2 = 0$
(b) $mx^2 - (m^2 - 1)\,xy - my^2 = 0$
(c) $mx^2 + (m^2 - 1)\,xy + my^2 = 0$
(d) None of these

Solution

(a) The equation is $y^2 + m^2x^2 - 2mxy - x^2 - m^2y^2 - 2mxy = 0$

$\Rightarrow \quad x^2(m^2 - 1) + y^2(1 - m^2) - 4mxy = 0$

Therefore, the equation of bisectors is $\dfrac{x^2 - y^2}{xy}$

$$= \frac{(m^2 - 1) - (1 - m^2)}{-2m}$$

$\Rightarrow \quad mx^2 + (m^2 - 1)\,xy - my^2 = 0$

3. If the bisectors of the angles of the lines represented by $3x^2 - 4xy + 5x^2 = 0$ and $5x^2 + 4xy + 3y^2 = 0$ are same, then the angle made by the lines represented by first with the second, is:

(a) 30° (b) 60°
(c) 45° (d) 90°

Solution

(d) $\because$ Both lines have same bisector.
So, under the given condition it is clear that angle is 90°

4. If the pair of straight lines given by $Ax^2 + 2Hxy + By^2 = 0$ $(H^2 > AB)$ forms an equilateral triangle with line $ax + by + c = 0$, then $(A + 3B)(3A + B)$ is: ***[EAMCET 2003]***

(a) H^2 (b) $-H^2$
(c) $2H^2$ (d) $4H^2$

Solution

(d) We know that the pair of lines $(a^2 - 3b^2) x^2 + 8abxy + (b^2 - 3a^2) y^2 = 0$ with the line $ax + by + c = 0$ form an equilateral triangle. Hence comparing with $Ax^2 + 2Hxy + By^2 = 0$, then $A = a^2 - 3b^2$, $B = b^2 - 3a^2$, $2H = 8ab$

Now $(A + 3B)(3A + B) = (-8a^2)(-8b^2) = (8ab)^2 = (2H)^2 = 4H^2$

5. The area (*in* square units) of the quadrilateral formed by the two pairs of lines $l^2x^2 - m^2y^2 - n(lx + my) = 0$ and $l^2x^2 - m^2y^2 + n(lx - my) = 0$ is ***[EAMCET 2003]***

(a) $\dfrac{n^2}{2|lm|}$ (b) $\dfrac{n^2}{|lm|}$

(c) $\dfrac{n}{2|lm|}$ (d) $\dfrac{n^2}{4|lm|}$

Solution

(a) Given lines are (*on* factorising)

$lx + my = 0,\ lx + my + n = 0$

$lx - my = 0,\ lx + my - n = 0$

$$\text{Area} = \left|\frac{(c_1 - d_1)(c_2 - d_2)}{(a_1b_2 - a_2b_1)}\right|$$

$$= \left|\frac{(0 - n)(0 + n)}{(-lm - lm)}\right| = \frac{n^2}{2|lm|}$$

6. The lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be equidistant from the origin, if

(a) $f^2 + g^2 = c(b - a)$
(b) $f^4 + g^4 = c(bf^2 + ag^2)$
(c) $f^4 - g^4 = c(bf^2 - ag^2)$
(d) $f^2 + g^2 = af^2 + bg^2$

Solution

(c) Let the equations represented by

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ be

$lx + my + n = 0$ and $l'x + m'y + n' = 0$

Then the combined equation reprsented by these lines is given by $(lx + my + n)(l'x + m'y + n') = 0$

So, it must be similar with the given equation. On comparing, we get $ll' = a$, $mm' = b$ $nn' = c$, $lm' + ml' = 2h$, $ln' + l'n = 2g$, $mn' + nm' = 2f$

According to the condition, the length of perpendiculars drawn from origin to the lines are same.

$$\text{So,}\quad \frac{n}{\sqrt{l^2 + m^2}} = \frac{n'}{\sqrt{l'^2 + m'^2}}$$

$$= \frac{(nn')^2}{(l^2 + m^2)(l'^2 + m'^2)}$$

Now on eliminating l, m, l', m' and n, n', we get the required condition $f^4 - g^4 = c(bf^2 - ag^2)$

7. The circumcentre of the triangle formed by the lines $xy + 2x + 2y + 4 = 0$ and $x + y + 2 = 0$: ***[EAMCET 1994]***

(a) (0, 0) (b) (–2, –2)
(c) (–1, –1) (d) (–1, –2)

Solution

(c) The separate equations of the lines given by $xy + 2x + 2y + 4 = 0$ are $(x + 2)(y + 2) = 0$ or $x + 2 = 0, y + 2 = 0$. Solving the equations of the sides of the triangle, we obtain the coordinates of the vertices as $A(-2,0)$, $B(0, -2)$ and $C(-2, -2)$. Clearly, ΔABC is a right angled triangle with right angle at C. Therefore the centre of the circum–circle is the mid-point of AB whose coordinates are (–1, –1).

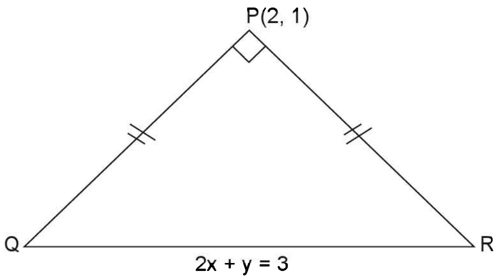
8. Let PQR be a right angled isosceles triangle, right angled at $P(2, 1)$. If the equation of the line QR is $2x + y = 3$, then the equation representing the pair of lines PQ and PR is ***[IIT 1999]***

(a) $3x^2 - 3y^2 + 8xy + 20x + 10y + 25 = 0$
(b) $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$

(c) $3x^2 - 3y^2 + 8xy + 10x + 15y + 20 = 0$
(d) $3x^2 - 3y^2 - 8xy - 10x - 15y - 20 = 0$

Solution

(b) Parametric equation of line passing through point P, $\frac{x-2}{\cos\theta} = \frac{y-1}{\sin\theta} = r$



hence, the point is $r\cos\theta + 2$, $r\sin\theta + 1$ If this point lie on QR, then $2(r\cos\theta + 2) + r\sin\theta + 1 = 3$

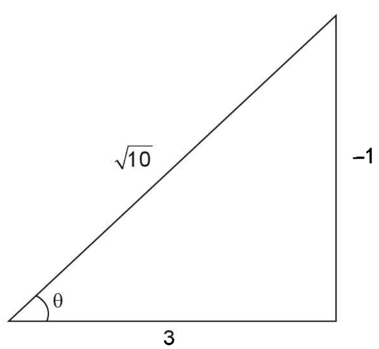
$r(2\cos\theta + \sin\theta) = -2$

$\Rightarrow \quad 2\cos\theta + \sin\theta = \frac{-2}{r}$ ....(i)

If θ be the inclination of PQ, then $\left(\frac{\pi}{2} + \theta\right)$ will be the inclination of PR.

Hence $2\cos\left(\frac{\pi}{2} + \theta\right) + \sin\left(\frac{\pi}{2} + \theta\right) = \frac{-2}{r}$

$\Rightarrow \quad -2\sin\theta + \cos\theta = \frac{-2}{r}$ ....(ii)



From (i) and (ii), we get $2\cos\theta + \sin\theta = -2\sin\theta + \cos\theta$

$\Rightarrow \quad \cos\theta = -3\sin\theta$

$\Rightarrow \quad \tan\theta = -\frac{1}{3}$

Hence the gradients of PQ and PR are –1/3 and 3

Therefore the equations of PQ and PR are

$$y - 1 = -\frac{1}{3}(x-2);\ y - 1 = 3(x-2)$$

and their combine equation is

$[3(y-1) + (x-2)]\,[(y-1) - 3(x-2)] = 0$

or $3(y-1)^2 - 8(y-1)(x-2) - 3(x-2)^2 = 0$

or $3(x-2)^2 + 8(x-2)(y-1) - 3(y-1)^2 = 0$

or $3(x^2 - 4x + 4) + 8(xy - x - 2y + 2) - 3(y^2 - 2y + 1) = 0$

or $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$

9. Mixed term xy is to be removed from the general equation $ax^2 + by^2 + 2hxy + 2fy + 2gx + c = 0$. One should rotate the axes through an angle θ given by $\tan 2\theta$ equal to

(a) $\frac{a-b}{2h}$ (b) $\frac{2h}{a+b}$

(c) $\frac{a+b}{2h}$ (d) $\frac{2h}{a-b}$

Solution

(d) Let (x′, y′) be the coordinates on new axes, then put $x = x'\cos\theta - y'\sin\theta$, $y = x'\sin\theta + y'\cos\theta$ in the equation.

Then the coefficient of xy in the transformed equation is 0.

So $2(b-a)\sin\theta\cos\theta + 2h\cos 2\theta = 0$

$\Rightarrow \quad \tan 2\theta = \frac{2h}{a-b}$

10. If the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on the y-axis, then:

(a) $2fgh = bg^2 + ch^2$ (b) $bg^2 \neq ch^2$
(c) $abc = 2fgh$ (d) None of these

Solution

(a) $f(x, y) = ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$

Points of intersection of lines $\frac{\partial f(x,y)}{\partial x} = 0$

$2ax + 2hy + 2g = 0$

Since, $x = 0$, intersects on y-axis $y = -g/h$.

Thus putting this value in $f(x, y)$.

We get, $\frac{bg^2}{h^2} + 2f(-g/h) + c = 0$

or $bg^2 + ch^2 = 2fgh$.

Step 1: Co-ordinates of the point of intersection of given pair of straight lines are given by

$$\left(\sqrt{\frac{f^2-bc}{h^2-ab}}, \sqrt{\frac{g^2-ac}{h^2-ab}}\right) \ldots\ldots \quad \text{(i)}$$

and condition for representing a pair of straight lines' is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ .... (ii)

Step 2: Given that x co-ordinate of point of intersection is zero i.e., $f^2 - bc = 0$

$f^2 = bc$ or ...(iii)

Substituting $f^2 = bc$ in ....(ii)

we find:

$abc + 2fgh - abc - bg^2 - ch^2 = 0$

or $2fgh = bh^2 + ch^2$

11. The lines joining the origin to the point of intersection of the circle $x^2 + y^2 = 3$ and the line $x + y = 2$ are ***[Roorkee 1995]***

(a) $y-(3+2\sqrt{2})x=0$

(b) $x-(3+2\sqrt{2})y=0$

(c) $x-(3-2\sqrt{2})y=0$

(d) $y-(3-2\sqrt{2})x=0$

Solution

(a,b,c,d) Homogenising the equation of circle, we get

$$x^2 - 6xy + y^2 = 0$$

$$x = \frac{6y \pm \sqrt{(36-4)y^2}}{2}$$

$$= \frac{6y \pm 4y\sqrt{2}}{2} = 3y \pm 2\sqrt{2}y$$

Hence the equations are $x=(3+2\sqrt{2})y$ and $x=(3-2\sqrt{2})y$

Also after rationalizing these equations becomes $y-(3+2\sqrt{2})x=0$ and $y-(3-2\sqrt{2})x$ $x = 0$

12. The value of 'p' for which the equation $x^2 + pxy + y^2 - 5x - 7y + 6 = 0$ represents a pair of straight lines is: ***[MPPET – 2006]***

(a) 5/2 (b) 5

(c) 2 (d) 2/5

Solution

(a) The equation $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ will represesnt a pair of straight lines,

if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

So, the equation $x^2 + pxy + y^2 - 5x - 7y + 6 = 0$ will represent a pair of straight lines, if

$$1.1.6 + 2\times\left(\frac{-7}{2}\right)\times\left(\frac{-5}{2}\right)\left(\frac{p}{2}\right) - 1.$$

$$\left(\frac{-7}{2}\right)^2 - 1.\left(-\frac{5}{2}\right)^2 - 6\left(\frac{p}{2}\right)^2 = 0$$

$$\Rightarrow \quad 6 + \frac{35p}{4} - \frac{49}{4} - \frac{25}{4} - \frac{6p^2}{4} = 0$$

$\Rightarrow \quad 24 + 35p - 74 - 6p^2 = 0$

$\Rightarrow \quad 6p^2 - 35p + 50 = 0$

$\Rightarrow \quad 6p^2 - 20p - 15p + 50 = 0$

$\Rightarrow \quad 2p(3p - 10) - 5(3p - 10) = 0$

$\Rightarrow \quad (2p - 5)(3p - 10) = 0$

$\Rightarrow \quad p = \frac{5}{2}$ or $\frac{10}{3}$

13. The lines joining the points of intersection of the curve $(x - y)^2 + (y - k)^2 - c^2 = 0$ and the line $kx + hy = 2hk$ to the origin are perpendicular, then:

(a) $c = h \pm k$

(b) $c^2 = h^2 + k^2$

(c) $c^2 = (h + k)^2$

(d) $4c^2 = h^2 + k^2$

Solution

(b) The line is $\frac{x}{2h} + \frac{y}{2k} = 1$ and circle is, $x^2 + y^2 - 2(hx + ky) + (h^2 + k^2 - c^2) = 0$ making it homogenous, we get

$$\Rightarrow \quad (x^2 + y^2) - 2(hx + ky)\left(\frac{x}{2h} + \frac{y}{2k}\right) + (h^2 + k^2 - c^2)\left(\frac{x}{2h} + \frac{y}{2k}\right)^2 = 0$$

If these lines be perpendicular, then $A + B = 0$

$$\left[1 - 1 + \frac{(h^2 + k^2 - c^2)}{4h^2}\right] +$$

$$\left[1 - 1 + \frac{(h^2 + k^2 - c^2)}{4k^2}\right] = 0$$

or $\quad (h^2 + k^2 - c^2)\left(\dfrac{h^2 + k^2 - c^2}{4h^2k^2}\right) = 0$

$\therefore \quad h^2 + k^2 = c^2.$

14. Two lines are given by $(x-2y)^2 + k(k-2y) = 0$. The value of k so that the distance between them is 3, is

(a) $1/\sqrt{5}$

(b) $\pm 2/\sqrt{5}$

(c) $\pm 3\sqrt{5}$

(d) None of thses

Solution

(c) Applying the formula, the distance between them is

$$\left|2\sqrt{\frac{(k^2/4)-0}{1.(1+4)}}\right| = \left|\frac{k}{\sqrt{5}}\right|, \quad \therefore \left|\frac{k}{\sqrt{5}}\right| = 3$$

$\Rightarrow \quad k \pm 3\sqrt{5}.$

15. If the portion of the line $lx + my = 1$ falling inside the circle $x^2 + y^2 = a^2$ subtends an angle of 45° at the origin, then:

(a) $4[a^2(l^2+m^2)-1] = a^2(l^2+m^2)$

(b) $4[a^2(l^2+m^2)-1] = a^2(l^2+m^2)-2$

(c) $4[a^2(l^2+m^2)-1] = [a^2(l^2+m^2)-2]^2$

(d) None of these

Solution

(c) Making the equation of circle homogenous with the help of the line $lx + my = 1$, we get $x^2 + y^2 - a^2(lx + my)^2 = 0$

$\Rightarrow \quad (a^2l^2-1)x^2 + (a^2m^2-1)y^2 + 2a^2lmxy = 0$

Now

$$\tan 45° = \frac{2\sqrt{(a^2lm)^2-(a^2l^2-1)(a^2m^2-1)}}{a^2l^2+a^2m^2-2}$$

$\Rightarrow \quad \sqrt{a^2l^2+a^2m^2-1} = a^2l^2 + a^2m^2 - 2$

On squaring both sides, we get

$$4[a^2(l^2+m^2)-1] = [a^2(l^2+m^2)-2]^2$$

16. The angle between the pair of straight lines formed by joining the points of intersection of $x^2 + y^2 = 4$ and $y = 3x + c$ to the origin is a right angle. Then $c^2 =$ ***[EAMCET – 2007]***

(a) 20 (b) 13

(c) 1/5 (d) 5

Solution

(a) Homogenising, $x^2 + y^2 = 4\left(\dfrac{y-3x}{c}\right)^2$

or $\quad c^2(x^2+y^2) - 4(9x^2 + y^2 - 6xy) = 0.$

These lines are $\perp$.

$\therefore$ Coefficient of x^2 + coefficient of $y^2 = 0$ i.e., $c^2 = 20$.

17. If the lines $x^2 + 2xy - 35y^2 - 4x + 44y - 12 = 0$ and $5x + \lambda y - 8 = 0$ are concurrent, then the value of λ is ***[EAMCET – 2007]***

(a) 0 (b) 1

(c) –1 (d) 2

Solution

(d) of intersection $\left(\dfrac{4}{3}, \dfrac{2}{3}\right)$ and the lines are concurrent

$$\Rightarrow \quad 5\left(\frac{4}{3}\right) + \lambda\left(\frac{2}{3}\right) - 8 = 0 \Rightarrow \lambda = 2.$$

18. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents parallel straight lines, then: ***[DCE – 2007]***

(a) $hf = bg$

(b) $h^2 = bc$

(c) $af^2 = bg^2$

(d) None of these

Solution

(a) The given equation represents a pair of parallel straight lines, if $h^2 = ab$ and $bg^2 = af^2$

Now, $bg^2 = af^2$

$\Rightarrow \quad abf^2 = b^2g^2$ [Multiplying both sides by b]

$\Rightarrow \quad h^2f^2 = b^2g^2 \qquad [\because h^2 = ab]$

$\Rightarrow \quad hf = bg.$

19. If the lines $ax^2 + 2hxy + by^2 = 0$ represents the adjacent sides of a parallelogram, then the equation of second diagonal if one is $lx + my = 1$, will be:

(a) $(am + hl)x = (bl + hm)y$

(b) $(am - hl)x = (bl - hm)y$

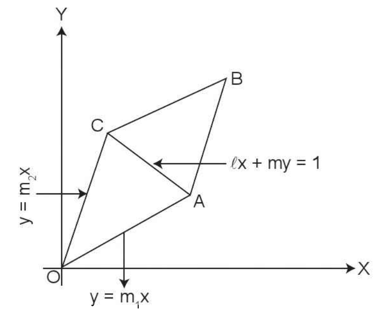
(c) $(am - hl)x = (bl + hm)y$

(d) None of these

Solution

(b) Let the equation of lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1x = 0$ and $y - m_2x = 0$ and one diagonal AC be $lx + my = 1$.

Therefore, $m_1 + m_2 = \dfrac{-2h}{b}$ and $m_1m_2 = \dfrac{a}{b}$.



Now on solving the equation of OA and OC with the line AC, we get the coordinates of

$$A\left(\frac{1}{l+mm_1}, \frac{m_1}{l+mm_1}\right)$$

and $C\left(\frac{1}{l+mm_2}, \frac{m_2}{l+mm_1}\right)$

Now find the coordinates of mid–point of AC and the equation of diagonal through this mid–point and origin.

The required equation is $x(am - hl) = (lb - mh)y$.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The condition of representing the coincident lines by the general quadratic equation $f(x, y) = 0$, is:
(a) $\Delta = 0$ and $h^2 = ab$
(b) $\Delta = 0$ and $a + b = 0$
(c) $\Delta = 0$ and $h^2 = ab$, $g^2 = ac$, $f^2 = bc$
(d) $h^2 = ab$, $g^2 = ac$ and $f^2 = bc$

2. The value of λ, for which the equation $x^2 - y^2 - x - \lambda y - 2 = 0$ represent a pair of straight line, are: ***[MP PET 2004]***
(a) 3, –3 (b) –3, 1
(c) 3, 1 (d) –1, 1

3. The straight line joining the origin to the points of intersection of the line $2x + y = 1$ and curve $3x^2 + 4xy - 4x + 1 = 0$ include an angle: ***[MP PET 1993]***
(a) $\pi/2$ (b) $\pi/3$
(c) $\pi/4$ (d) $\pi/6$

4. The acute angle formed between the lines joining the origin to the points of intersection of the curves $x^2 + y^2 - 2x - 1 = 0$ and $x + y = 1$, is: ***[MP PET 1998]***
(a) $\tan-^{1}(-1/2)$ (b) $\tan^{-1} 2$
(c) $\tan^{-1}(1/2)$ (d)60°

5. The point of intersection of the lines represented by the equation $2x^2 + 3y^2 + 7xy + 8x + 14y + 8 = 0$ is:
(a) (0, 2) (b) (1, 2)
(c) (–2, 0) (d) (–2, 1)

6. Distance between the lines represented by the equation $x^2 + 2\sqrt{3}xy + 3y^2 - 3x - 3\sqrt{3}y - 4 = 0$ is:

[Roorkee 1989]
(a) 5 / 2 (b) 5 / 4
(c) 5 (d) 0

7. The equation of pair of straight lines joining the point of intersection of the curve $x^2 + y^2 = 4$ and $y - x = 2$ to the origin, is:
(a) $x^2 + y^2 = (y - x)^2$
(b) $x^2 + y^2 + (y - x)^2 = 0$
(c) $x^2 + y^2 = 4(y - x)^2$
(d) $x^2 + y^2 + 4(y - x)^2 = 0$

8. The lines joining the points of intersection of line $x + y = 1$ and curve $x^2 + y^2 - 2y + \lambda = 0$ to the origin are perpendicular, then the value or λ will be:
(a) 1 / 2 (b) – 1 / 2
(c) $1/\sqrt{2}$ (d) 0

9. If the distance of two lines passing through origin from the point (x_1, y_1) is 'd', then the equation of lines is:
(a) $(xy_1 - yx_1)^2 = d^2(x^2 + y^2)$
(b) $(x_1y_1 - xy)^2 = (x^2 + y^2)$
(c) $(xy_1 + yx_1)^2 = (x^2 - y^2)$
(d) $(x^2 - y^2) = 2(x_1 + y_1)$

10. The equation of lines passing through the origin and parallel to the lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is:

(a) $m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$
(b) $m_1m_2x^2 + (m_1 + m_2)xy + y^2 = 0$
(c) $m_1m_2y^2 - (m_1 + m_2)xy + x^2 = 0$
(d) $m_1m_2y^2 + (m_1 + m_2)xy + x^2 = 0$

11. The equation of the perpendiculars drawn from the origin to the lines represented by the equation $2x^2 - 10xy + 12y^2 + 5x - 16 - 3 = 0$ is:
(a) $6x^2 + 5xy + y^2 = 0$ (b) $6y^2 + 5xy + x^2 = 0$
(c) $6x^2 - 5xy + y^2 = 0$ (d) None of these

12. Equation of bisectors of the angles between the lines represented by $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$ is:
(a) $x^2 - 6xy - 5y^2 = 0$
(b) $5(x-2)^2 - 6(x-2)(y-1) - 5(y-1)^2 = 0$
(c) $5(x-2)^2 + 6(x-2)(y-1) - 5(y-1)^2 = 0$
(d) $4(x-2)^2 - 5(x-2)(y-1) + (y-1)^2 = 0$

13. The pair of lines $(a - b)(x^2 - y^2) + 4hxy = 0$ and $h(x^2 - y^2) - (a - b)xy = 0$ are:
(a) Parallel
(b) Perpendicular
(c) Equally inclined
(d) Each pair bisects the angle between other pairs

14. If the pair of straight lines $xy - x - y + 1 = 0$ and the line $ax + 2y - 3 = 0$ are concurrent, then a:
(a) -1 (b) 0
(c) 3 (d) 1

15. If $x^2 + y^2 + 2gx + 2fy + 1 = 0$ is the equation of a pair of straight lines, then L:
(a) $f^2 + g^2 = 1$ (b) $g^2 - f^2 = 1$
(c) $f^2 + g^2 = 1/2$ (d) $f^2 - g^2 = 1$

16. The lines joining the origin to the points of intersection of the line $3x - 2y = 1$ and the curve $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$, are:
(a) Parallel to each other
(b) Perpendicular to each other
(c) Inclined at 45° to each other
(d) None of these

17. The square of distance between the point of intersection of the lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and origin, is:
(a) $\dfrac{c(a+b) - f^2 - g^2}{ab - h^2}$
(b) $\dfrac{c(a-b) + f^2 + g^2}{\sqrt{ab - h^2}}$
(c) $\dfrac{c(a+b) - f^2 - g^2}{ab + h^2}$
(d) None of these

18. One of the lines represented by the equation $x^2 + 6xy = 0$ is:
(a) Parallel to x-axis (b) Parallel to y-axis
(c) x-axis (d) y-axis

19. Angle between the lines joining the origin to the points of intersection of the curves $2x^2 + 3y^2 + 10x = 0$ and $3x^2 + 5y^2 + 16x = 0$ is:
(a) $\tan^{-1}\dfrac{3}{2}$ (b) $\tan^{-1}\dfrac{4}{3}$
(c) 90° (d) None of these

20. The equation $4x^2 + 12xy + 9y^2 + 2gx + 2fy + c = 0$ will represents two real parallel straight lines, if:
(a) $g = 4, f = 9, c = 0$
(b) $g = 2, f = 3, c = 1$
(c) $g = 2, f = 3$, c is any number
(d) $g = 4, f = 9, c > 1$

21. The equation of the line joining the origin to the points of intersection of the curve $x^2 + y^2 = a^2$ and $x^2 + y^2 - ax - ay = 0$ is:
(a) $x^2 - y^2 = 0$ (b) $xy = 0$
(c) $xy - x^2 = 0$ (d) $y^2 + xy = 0$

22. The equation of the pair of straight lines passing through the point (2, 3) and parallel to x and y-axis is:
(a) $x^2 + y^2 + xy + 6 = 0$
(b) $xy - 3x - 2y + 6 = 0$
(c) $xy + 3x + 2y + 6 = 0$
(d) None of these

23. If $ax^2 + by^2 + hx + hy = 0$, $h \neq 0$ represents a pair of straight lines, then: ***[MPPET – 2008]***
(a) $a + b = 0$ (b) $a + h = 0$
(c) $b + h = 0$ (d) $ab - h^2 = 0$

24. The line $ax^2 + 2hxy + by^2 = 0$ are equally inclined to the lines $ax^2 + 2hxy + by^2 + k(x^2 + y^2) = 0$ for: ***[Orissa JEE – 2008]***
(a) $k = 2$ (b) $k = 1$
(c) any value of k (d) None of these

25. If $3x^2 + xy - y^2 - 3x + 6y + K = 0$ represents a pair of lines, then $K =$ ***[Karnataka CET – 2008]***
(a) 1 (b) -9
(c) 0 (d) 9

26. If the pairs of lines $3x^2 - 5xy + py^2 = 0$ and $6x^2 - xy - 5y^2 = 0$ have one line in common then $p =$

(a) 2, 25/4 (b) –2, 25/4
(c) –2, –25/4 (d) None of these

27. If the equation $12x^2 + 7xy - py^2 - 18x + qy + 6 = 0$ represents a pair of perpendicular straight lines, then: ***[Kurukshetra CEE 2002]***

(a) $p = 12, q = 1$ (b) $p = 1, q = 12$
(c) $p = -1, q = 12$ (d) $p = 1, q = -12$

28. The lines joining origin to the points of intersection of the line $x + y = 1$ with the curve $x^2 + 3y^2 - kx + 1 = 0$, will be perpendicular if k is equal to:

(a) 7 (b) 6
(c) 5 (d) 4

29. The area bounded by the angle bisectors of the lines $x^2 - y^2 + 2y = 1$ and the line $x + y = 3$, is ***[IIT – SC. 2004]***

(a) 2 (b) 3
(c) 4 (d) 6

30. If one of the lines of $my^2 + (1 - m^2)xy - mx^2 = 0$ is a bisector of the angle between the lines $xy = 0$, then m is: ***[AIEEE – 2007]***

(a) 1 (b) 2
(c) –1/2 (d) –2

SOLUTIONS

1. (a) Equation can be found by homogenising the curve w.r.t. line i.e., $x^2 + y^2 = 4\left(\dfrac{y-x}{2}\right)^2$
or $x^2 + y^2 = (y - x)^2$.

2. (c) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will represent a pair of straight line if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{...... (i)}$$

for given equation $x^2 - y^2 - x + \lambda y - 2 = 0$

$a = 1, b = -1, c = -2, h = 0,$

$$g = -\frac{1}{2}, f = \frac{\lambda}{2}$$

∴ from equation (i)

$$2 + 0 - \frac{\lambda}{4} + \frac{1}{4} = 0 \Rightarrow \frac{\lambda^2}{4} = \frac{9}{4} \Rightarrow \lambda = \pm 3$$

3. (a) Make equation of curve homogenous with the help of

$2x + y = 1$

$3x^2 + 4xy - 4x(2x + y) + (2x + y)^2 = 0$

or $-x^2 + 4xy + y^2 = 0$

∵ $a + b = (-1) + (1) = 0$

Hence lines are perpendicular so $\theta = 90°$.

4. (b) Homogenous equation of 2*nd* degree through intersection of curve and line is

$x^2 + y^2 - 2x(x + y) - 1(x + y)^2 = 0$

i.e., $x^2 + 2xy = 0$

for $ax^2 + 2hxy + by^2 = 0$

angle between two lines is

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} \quad \because a = 1, h = 1, b = 0$$

$$\therefore \quad \tan\theta = \frac{2\sqrt{1-0}}{1+0} \text{ or } \theta = \tan^{-1} 2$$

5. (c) If the equation is general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Then points of intersection are given by

$$\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)$$

Hence point is (–2, 0).

6. (a) First check for parallel lines

i.e., $\dfrac{a}{h} = \dfrac{h}{b} = \dfrac{g}{f} \Rightarrow \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3} = \dfrac{-3/2}{-3\sqrt{3}/2}$

which is true, hence line are parallel.
Applying formula for distance is

$$2\sqrt{\frac{g^2 - ac}{a(a+b)}} = 2\sqrt{\frac{\frac{9}{4} + 4}{4}} = \frac{5}{2}$$

7. (a) Equation can be found by homogenising the curve w.r.t. line i.e., $x^2 + y^2 = 4\left(\dfrac{y-x}{2}\right)^2$
or $x^2 + y^2 = (y - x)^2$

8. (d) Making the equation of curve homogenous with the help of line $x + y = 1$, we get

$$x^2 + y^2 - 2y(x + y) + \lambda(x + y)^2 = 0$$

$$\Rightarrow \quad x^2(1 + \lambda) + y^2(-1 + \lambda) - 2yx + 2\lambda xy = 0$$

Therefore the lines be perpendicular, if $A + B = 0$

$$\Rightarrow \quad 1 + \lambda - 1 + \lambda = 0 \Rightarrow \lambda = 0.$$

9. (a) If the equation of line is y = mx and the length of perpendicular drawn on it from the point (x_1, y_1) is d, then $\frac{y_1 - mx_1}{\sqrt{1+m^2}} = \pm d$

$$\Rightarrow \quad (y_1 - mx_1)^2 = d^2(1 + m^2).$$

But $m = \frac{y}{x}$, therefore on eliminating 'm', the required equation is $(xy_1 - yx_1)^2 = d^2(x^2 + y^2)$.

10. (a) Passing through origin and is parallel to given lines are $y = m_1x$ and $y = m_2x$. If represented as pair of straight lines, we get $(y - m_1x)(y - m_2x) = 0$

$$\Rightarrow \quad m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$$

11. (a) We know that the equation of perpendicular drawn from origin on $ax^2 + 2hxy +$ by$^2 + 2gx + 2fy + c = 0$ is

$$bx^2 - 2hxy + ay^2 = 0$$

Therefore, the required equation is given by $12x^2 + 10xy + 2y^2 = 0$ or $6x^2 + 5xy + y^2 = 0$.

12. (b) Point of intersection of $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$ is obtained as follows:

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

$$2x - 5y + 1 = 0; -5x + 8y + 2 = 0$$

Solving we get $x = 2, y = 1$

point of intersection (2, 1)

∴ Angle bisector's equation i.e.,

$$\frac{(x-2)^2 - (y-1)^2}{1-4} = \frac{(x-2)(y-1)}{-5/2}$$

$$5(x - 2)^2 - 5(y - 1)^2\ 6(x - 2)(y - 1) = 0$$

13. (d) ∵ each pair of line is bisector of the other, these lines bisect the pair of other.

14. (d) $xy - x - y + 1 = 0$

$$\Rightarrow \quad x(y - 1) - (y - 1) = 0$$

$$\Rightarrow \quad (x - 1)(y - 1) = 0$$

$$\Rightarrow \quad x = 1, y = 1$$

These are two ⊥ lines meeting at (1, 1).

Given lines are concurrent if (1, 1) lies on $ax + 2y - 3 = 0$

$$\Rightarrow \quad a + 2 - 3 = 0$$

$$\Rightarrow \quad a = 1$$

15. (d) $x^2 + y^2 + 2gx + 2fy + 1 = 0$ represents pair of straight lines

$$\Rightarrow \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 0 & g \\ 0 & 1 & f \\ g & f & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (f^2 - 1) + g(-g) = 0$$

$$\Rightarrow \quad f^2 - g^2 = 1.$$

16. (b) Making the given equation homogenous with the help of equation of line, we get $9x^2 + 10xy - 9y^2 = 0$

Hence the lines are perpendicular.

17. (a) Let the lines represented by given equation be

$$y = m_1x + c_1 \text{ and } y = m_2x + c_2.$$

Then $ax^2 + 2hxy +$ by$^2 + 2gx + 2fy + c = 0$

$$= b(y - m_1x - c_1)(y - m_2x - c_2) = 0$$

Comparing the coefficients of x^2, xy, x, y and constant term, we get

$$m_1m_2 = \frac{a}{b}, m_1m_2 = \frac{-2h}{b},$$

$$m_1c_2 + m_2c_1 = \frac{2g}{b},$$

$$c_1 + c_2 = \frac{2f}{b} \text{ and } c_1c_2 = \frac{c}{b}$$

Also the point of intersection of $y = m_1x + c_1$ and $y = m_2x + c_2$ is $\left(\frac{c_2 - c_1}{m_1 - m_2}, \frac{m_1c_2 - m_2c_1}{(m_1 - m_2)}\right)$

Therefore, the square of distance of this point from origin is

$$\left(\frac{c_2 - c_1}{m_1 - m_2}\right)^2 + \frac{(m_1c_2 - m_2c_1)^2}{(m_1 - m_2)^2}$$

$$= \frac{[(c_1 + c_2)^2 - 4c_1c_2] + [(m_1c_2 + m_2c_1)^2 - 4m_1m_2c_1c_2]}{(m_1 + m_2)^2 - 4m_1m_2}$$

Now putting the value defined above, we get the required distance i.e., $\frac{-c(a+b) + f^2 + g^2}{h^2 - ab}$

18. (d) $x^2 + 6xy = 0$

$\Rightarrow \quad x(x + 6y) = 0$

$\Rightarrow \quad x = 0, x + 6y = 0$

But $x = 0 \Rightarrow y$-axis.

19. (c) The equation of any curve through the points of intersection of the given curves is $2x^2 + 3y^2 + 10x + \lambda(3x^2 + 5y^2 + 16x) = 0$ .... (i)

If this equation represents two straight lines through the origin, then this must be homogenous equation of second degree i.e., coefficient of x in (i) must vanish

$\therefore \quad 10 + 16\lambda = 0$

$\Rightarrow \quad \lambda = \dfrac{-10}{16} = \dfrac{-5}{8}$

Substituting this value of λ in (i), we get the equation of pair of straight lines $x^2 - y^2 = 0$. Hence the lines represented by the equation (ii) are mutually perpendicular.

20. (c) The lines are parallel, if $h^2 -$ ab, $af^2 = bg^2$

or $\quad \dfrac{a}{h} = \dfrac{h}{b} = \dfrac{g}{f} \Rightarrow 4f^2 = 9g^2$

$\Rightarrow \quad f = \dfrac{3}{2}g$

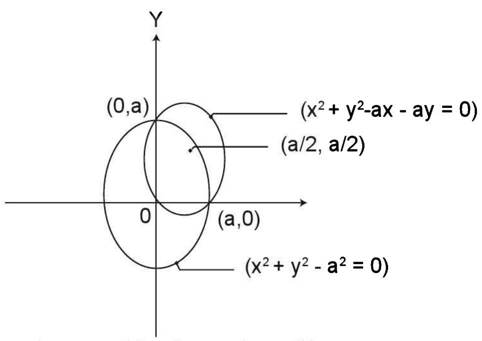
$\Rightarrow \quad g = 2, f = 3$ (let)

Now $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\Rightarrow \quad 4 \times 9 \times c + 2 \times 3 \times 2 \times 6 - 4(3)^2 - 9(2)^2 - c(6)^2 = 0$

$\Rightarrow \quad c$ is any number.

21. (b) Clearly, from figure $xy = 0$ represents the pair of straight lines



i.e., combined equation of both axes.

22. (b) Required lines is $(x - 2)(y - 3) = 0$

$xy - 3x - 2y + 6 = 0$

23. (a) For general equation condition to represent straight line is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

Here $C = 0, h = 0, f = g = \dfrac{h}{2}$

$\therefore \quad 0 + 0 - \dfrac{ah^2}{2} - \dfrac{bh^2}{2} = 0$

$(a+b)\dfrac{h^2}{2} = 0 \Rightarrow a + b = 0$

24. (c) $\because$ Bisector of both pair of lines are same

$\therefore \quad$ Both pair are equiinclined each other $\forall\, k \in R$

25. (b) Compare the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Here, $a = 3,\ h = \dfrac{1}{2},\ b = -1,$

$g = -\dfrac{3}{2},\ f = 3$ and $c = k$

Given equation represents a pair of lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

i.e., if $3 \times (-1)k + 2 \times 3 \times \left(-\dfrac{3}{2}\right) \times \dfrac{1}{2}$

$- 3 \times 3^2 - (-1)\left(-\dfrac{3}{2}\right)^2 - k\left(\dfrac{1}{2}\right)^2 = 0$

i.e., if $-\dfrac{13}{4}k = \dfrac{117}{4} \Rightarrow k = -9$

26. (d) $6x^2 - xy - 5y^2 = 0$ or $6x^2 - 6xy + 5xy - 5y^2 = 0$

$= (6x + 5y)(x - y) = 0$

$\therefore \quad$ Lines are $6x + 5y = 0$ or $y = x$

$\therefore \quad$ If $y = \dfrac{-6}{5}x$ is common with $3x^2 - 5xy + py^2 = 0$

$\Rightarrow \quad 3x^2 - 5x\left(\dfrac{-6}{5}x\right) + \left(\dfrac{36}{25}x^2\right) = 0$

or $\quad 3 + 6 + \dfrac{36p}{25} = 0$

$\Rightarrow \quad p = \dfrac{-25}{4}$

Or If $y = x$ is common

$3x^2 - 5x^2 + px^2 = 0$

$\Rightarrow \quad p = 2$

$\therefore \quad p = 2, \dfrac{-25}{4}$

27. (a) ∵ The given equation represents a pair of straight lines therefore, $-72p-\frac{63}{2}q-3q^2+8\ p$
$81p-\frac{147}{2}=0$ .... (i)

Also, since given equation represents perpendicular lines

∴ $12-p=0 \Rightarrow p=12$

∴ from (i), $2q^2+21q-23=0 \Rightarrow q=1$

28. (b) Equation of line joining origin with point of intersection line and curve.

$x^2+3y^2-kx(x+y)+(x+y)^2=0$

$\Rightarrow$ $x^2+3y^2-k(x^2+ky)+x^2+y^2+2xy=0$

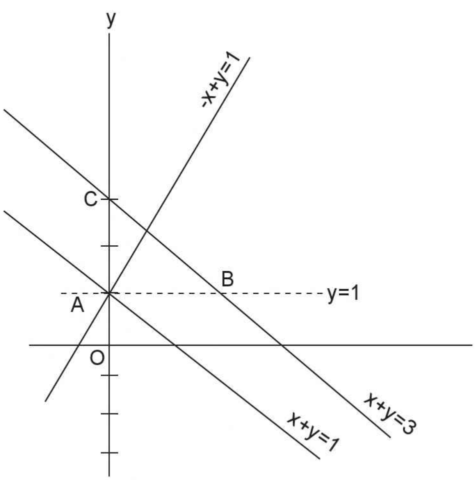
∴ Angle between them = 90°

Coff. of x^2 + Coff. of $y^2=0$

$\Rightarrow$ $1-k+1+3+1=0 \Rightarrow 1=0$

29. (a) $x^2-y^2+2y+1=0$

$\Rightarrow$ $x^2=(y-1)^2$



$\Rightarrow$ $x=\pm(y-1)$

$\Rightarrow$ $-x+y=1, x+y=1$

as shown in the figure. Its bisector are $y=1$, y-axis ($x=0$)

line $x+y=3$ is shown figure. Triangle ABC is bounded by bisectors $y=1, x=0$ and $x+y=3$.

Here AC = 2, BC = 2 as A(0, 1), C(0, 3), B (2, 1)

Area $=\frac{AB.AC}{2}=\frac{2\times 2}{2}=2$

30. (a) Given pair of lines is $my^2+(1-m^2)xy-mx^2=0$

or $m\left(\frac{y}{x}\right)^2+(1-m)^2\frac{y}{x}m=0$ .... (1)

Lines $xy=0$ are $x=0$ and $y=0$, i.e., the axes. Bisectors of angles between the *axes* are $y=x$ and $y=-x$, i.e., $\frac{y}{x}=1$

or $\frac{y}{x}=-1.$

If $\frac{y}{x}=1$ is represented by (1), then

$m(1)^2+(1-m^2)(-1)-m=0$

$\Rightarrow$ $1-m^2=0$

$\Rightarrow$ $m=\pm 1$

If $\frac{y}{x}=-1$ is represented by (1), then

$m(-1)^2+(1-m^2)(-1)-m=0$

$\Rightarrow$ $1-m^2=0$

$\Rightarrow$ $m=\pm 1$

In both cases either $m=1$ or $m=-1$.

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The angle between the pair of lines $2x^2+5xy+2y^2+3x+3y+1=0$ is: ***[EAMCET 1994]***

(a) $\cos^{-1}(4/5)$ (b) $\tan^{-1}(4/5)$
(c) 0 (d) $\pi/2$

2. The lines joining the points of intersection of curve $5x^2+12xy-8y^2+8x-4y+12=0$ and the line $x-y=2$ to the origin, makes the angles with the *axes*:

(a) 30° and 45° (b) 45° and 60°
(c) Equal (d) Parallel to *axes*

3. The distance between the parallel lines $9x^2-6xy+y^2+18x-6y+8=0$ is: ***[Roorkee 1989]***

(a) $1/\sqrt{10}$ (b) $2/\sqrt{10}$
(c) $4/\sqrt{10}$ (d) $\sqrt{10}$

4. The area of the triangle formed by the line $4x^2 - 9xy - 9y^2 = 0$ and $x = 2$ is:
(a) 2 (b) 3
(c) 10/3 (d) 20/3

5. Equation $3x^2 + 7xy + 2y^2 + 5x + 3y + 2 = 0$ represents:
(a) Pair of st. lines (b) Ellipse
(c) Hyperbola (d) None of these

6. If the point (2, – 3) lies on $Kx^2 - 3y^2 + 2x + y - 2 = 0$,then K is equal to:
(a) 1/7 (b) 16
(c) 7 (d) 12

7. The angle between the two straight lines $2x^2 - 5xy + 2y^2 - 3x + 3y + 1 = 0$ is:
(a) 45° (b) 60°
(c) $\tan^{-1}\frac{4}{3}$ (d) $\tan^{-1}\frac{3}{4}$

8. A line passing through (1, 0) intersects the curve $2x^2 + 5y^2 - 7x = 0$ in A and B. Then AB subtends at the origin an angle:
(a) 30° (b) 45°
(c) 60° (d) 90°

9. If the equation $Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$ represents a pair of straight lines, then $B^2 - AC$:
(a) < 0 (b) $= 0$
(c) > 0 (d) None of these

10. The equation $8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$ represents a pair of straight lines. The distance between them is:
(a) $7/\sqrt{5}$ (b) $7/2\sqrt{5}$
(c) $\sqrt{7}/5$ (d) None of these

11. If the angle between the two lines represented by $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ is $\tan^{-1} m$, then m = ***[MNR 1993]***
(a) 1/5 (b) 1
(c) 7/5 (d) 7

12. The point of intersection of the lines represented by the equation $6x^2 + 5xy - 21y^2 + 13x + 38y - 5 = 0$ is:
(a) (1, 1) (b) (5/23, – 7/23)
(c) (17/23, 32/23) (d) (–32/23, 17/23)

13. If $ax^2 - y^2 + 4x - y = 0$ represents a pair of lines then a = ***[Karnataka CET 2004]***
(a) –16 (b) 16
(c) 4 (d) –4

14. If the angle between the pair of straight lines represented by the equation $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$ is $\tan^{-1}(1/3)$ where 'λ' is a non negative real number. Then λ is:
[Orissa JEE 2002]
(a) 2 (b) 0
(c) 3 (d) 1

15. The lines joining the origin to the points of intersection of the line $y = mx + c$ and the circle $x^2 + y^2 = a^2$ will be mutually perpendicular, if: ***[Roorkee – 1977]***
(a) $a^2(m^2 + 1) = c^2$
(b) $a^2(m^2 - 1) = c^2$
(c) $a^2(m^2 + 1) = 2c^2$
(d) $a^2(m^2 - 1) = 2c^2$

16. Distance between the pair of lines represented by the equation $x^2 - 6xy + 9y^2 + 3x - 9y - 4 = 0$ is:
(a) $\frac{15}{\sqrt{10}}$ (b) $\frac{1}{2}$
(c) $\sqrt{\frac{5}{2}}$ (d) $\frac{1}{\sqrt{10}}$

17. The joint equation of the straight lines $x + y = 1$ and $x - y = 4$ is:
(a) $x^2 - y^2 = -4$
(b) $x^2 - y^2 = 4$
(c) $(x + y - 1)(x - y - 4) = 0$
(d) $(x + y + 1)(x - y + 4) = 0$

18. The point of intersection of the lines represented by equation $2(x + 2)^2 + 3(x + 2)(y - 2) - 2(y - 2)^2 = 0$ is:
(a) (2, 2) (b) (–2, – 2)
(c) (–2, 2) (d) (2, – 2)

19. If $\lambda x^2 - 5xy + 6y^2 + x - 3y = 0$ represents a pair of straight lines, then their point of intersection is: ***[MPPET-2010]***
(a) (1, 3) (b) (–1, – 3)
(c) (3, 1) (d) (–3, –1)

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The angle between the lines $x^2 - xy - 6y^2 - 7x + 31y - 18 = 0$ is: ***[Karnataka CET 2003]***
(a) 45° (b) 60°
(c) 90° (d) 30°

2. The equation $12x^2 + 7xy + ay^2 + 13x - y + 3 = 0$ represents a pair of perpendicular lines. Then the value of 'a' is ***[Karnataka CET 2001]***
(a) 7 / 2 (b) – 19
(c) – 12 (d) 12

3. If the slope of one of the lines represented by $ax^2 + 2hxy + b^2 = 0$ be the square of the other, then:
(a) $a^2b + ab^2 - 6abh + 8h^3 = 0$
(b) $a^2b + ab^2 + 6abh + 8h^3 = 0$
(c) $a^2b + ab^2 - 3abh + 8h^3 = 0$
(d) $a^2b + ab^2 - 6abh - 8h^3 = 0$

4. The angle between the lines joining the points of intersection of line $y = 3x + 2$ and the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ to the origin, is:
(a) $\tan^{-1}(3/2\sqrt{2})$ (b) $\tan^{-1}(2\sqrt{2}/3)$
(c) $\tan^{-1}(\sqrt{3})$ (d) $\tan^{-1}(2\sqrt{2})$

5. The lines represented by the equation $x^2 + 2\sqrt{3}xy + 3y^2 - 3x - 3\sqrt{3}y - 4 = 0,$ are:
(a) Perpendicular to each other
(b) Parallel
(c) Inclined at 45° to each other
(d) None of these

6. The equations of the lines represented by the equation $ax^2 + (a + b)xy + by^2 + x + y = 0$ are:
(a) $ax + by + 1 = 0, x + y = 0$
(b) $ax + by - 1, x + y = 0$
(c) $ax + by + 1 = 0, x - y = 0$
(d) None of these

7. Point of intersection of the straight lines $6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0$ is:
(a) (–2,0) (b) (2, 0)
(c) (1, 1) (d) (–2, 1)

8. If equation $3x^2 + xy - y^2 - 3x + 6y + k = 0$ represents a pair of lines, then k is equal to:
(a) 9 (b) 1
(c) 0 (d) – 9

9. The pair of straight lines joining the origin to the points of intersection of the line $y = 2\sqrt{2}\,x + c$ and the circle $x^2 + y^2 = 2$ are at right angles, if:
(a) $c^2 - 4 = 0$ (b) $c^2 - 8 = 0$
(c) $c^2 - 9 = 0$ (d) $c^2 - 10 = 0$

10. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel straight lines, then:
(a) $a/h = h/b = g/f$
(b) distance between them $= 2\sqrt{\dfrac{g^2 - ac}{a(a+b)}}$
(c) distance between them $= 2\sqrt{\dfrac{f^2 - bc}{b(a+b)}}$
(d) All are correct

11. Pair of lines $ax^2 + 2hxy + by^2 = 0$ is equally inclined to the pair of lines which is:
(a) $a^2x^2 + 2hxy + b^2y^2 = 0$
(b) $ax^2 + 2h(a + b)x + by^2 = 0$
(c) $ax^2 + 2h(a + b)xy + by^2 + \lambda(x^2 + y^2) = 0$
(d) $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$

12. A second degree hemogeneous equation in x and y always represents:
(a) A pair of st. lines (b) A circle
(c) A conic section (d) None of these

13. The equation of the pair of straight lines through the (2, 1) and perpendicular to the lines $5x^2 - 7xy - 3y^2 = 0$ is: ***[SCRA – 2007]***
(a) $3x^2 - 7xy - 5y^2 = 0$
(b) $3x^2 + 7xy - 5y^2 = 0$
(c) $3x^2 - 7xy + 5y^2 = 0$
(d) None of these

14. Equation of pair of straight lines drawn through (1, 1) and $\perp^r$ to the pair of lines $3x^2 - 7xy - 2y^2 = 0$:

(a) $2x^2 + 7x - 11x + 6 = 0$

(b) $2(x-1)^2 + 7(x-1)(y-1) - 3y^2 = 0$

(c) $2(x-1)^2 - 7(x-1)(y-1) - 3(y-1)^2 = 0$

(d) None of these

15. Distance between two lines represented by the pair of straight lines $x^2 - 6xy + 9y^2 + 3x - 9y - 4 = 0$ is: ***[MPPET–2009]***

(a) $\frac{1}{2}$ (b) $\frac{5}{\sqrt{2}}$

(c) $\sqrt{10}$ (d) $\sqrt{\frac{5}{2}}$

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**6.** ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

3. (a) Let slopes of pair of lines $ax^2 + 2hxy + by^2 = 0$ be m and m^2

$$\therefore \quad m + m^2 = \frac{-2h}{b} \qquad \text{............(i);}$$

$$m.m^2 = \frac{a}{b}$$

Cubing (i) we get

$$m^3 + (m^2)^3 + 3m.m^2(m + m^2) = \frac{-8h^3}{b^3}$$

$$\frac{a}{b} + \left(\frac{a}{b}\right)^2 + \frac{3a}{b}\left(\frac{-2h}{b}\right) = \frac{-8h^3}{b^3}$$

$$ab^2 + a^2b - 6abh + 8h^3 = 0$$

4. (b) Homegenising we get

$$x^2 + 2xy + 3y^2 + 4x\left(\frac{y-3x}{2}\right) + 8y\left(\frac{y-3x}{2}\right) - 11\left(\frac{y-3x}{2}\right)^2 = 0$$

$$\Rightarrow \quad x^2 + 2xy + 3y^2 + 2xy - 6x^2 + 4y^2 - 12xy - \frac{11}{4}(y^2 + 9x^2 - 6xy) = 0$$

$$\Rightarrow \quad x^2\left[1 - 6 - \frac{99}{4}\right] + y^2\left[3 + 4 - \frac{11}{4}\right] + xy\left(-8 + \frac{66}{4}\right) = 0$$

$$\Rightarrow \quad x^2\left(\frac{-119}{4}\right) + y^2\left(\frac{17}{4}\right) + xy\left(\frac{34}{4}\right) = 0$$

$$\Rightarrow \quad -7x^2 + xy + y^2 = 0$$

$$\Rightarrow \quad \tan\theta = \left|\frac{2\sqrt{1^2 - (-7)}}{-7+1}\right| \Rightarrow \left|2\frac{\sqrt{8}}{-6}\right| = \frac{2\sqrt{2}}{3}$$

6. (a) Given equation $ax^2 + (a+b)xy + by^2 + x + y = 0$

$$ax^2 + bxy + x + axy + by^2 + y = 0$$

$$x(ax + by + 1) + y(ax + by + 1) = 0$$

$$(x + y)(ax + by + 1) = 0$$

$\therefore$ Lines are $x + y = 0$ and $ax + by + 1 = 0$

Part D

Circle

LECTURE 1

Circle 1

(Equation of circle in various forms)

BASIC CONCEPTS

EQUATION OF A CIRCLE IN DIFFERENT CASES

1. **Simplest or standard form** The equation of a circle whose centre is (0, 0) and radius a is $x^2 + y^2 = a^2$
2. **Central form** The equation of a circle whose centre is (h, k) and radius a is $(x-h)^2 + (y-k)^2 = a^2$
3. **Zero or Point circle** The equation of a circle whose radius is zero is $x^2 + y^2 = 0$ or $(x-h)^2 + (y-k)^2 = 0$
4. The equation of a circle whose centre is (h, k) and the circle passes through the point (p, q) is $(x-h)^2 + (y-k)^2 = (p-h)^2 + (q-k)^2$
5. The equation of a circle whose centre is (h, k) and the circle touches x-axis ($k = a$ = radius and $h \neq k$) is $(x-h)^2 + (y-k)^2 = k^2$
6. The equation of a circle whose centre is (h, k) and the circle touches y-axis ($h = a$ = radius and $h \neq k$) is $(x-h)^2 + (y-k)^2 = h^2$
7. The equation of circle whose centre is (h, k) and the circle touches both the axis ($h = k = a$) is $(x-a)^2 + (y-a)^2 = a^2$

NOTE

If a is constant then the number of circles in each touching both axes one in each quadrant will be 4.

8. The equation of circle which touches both the axis and line $x = 2c$ is $(x-c)^2 + (y \mp c)^2 = c^2$
9. The equation of circle which touches both the axis and line $x = 2c$ and $y = 2c$ is $(x-c)^2 + (y-c)^2 = c^2$
10. The equation of circle which passes through origin (0, 0) and cuts the intercepts $2a$ and $2b$ on both the axes is $(x-a)^2 + (y-b)^2 = a^2 + b^2$ or $x^2 + y^2 - 2ax - 2by = 0$ centre (a, b), radius $= \sqrt{a^2+b^2}$

NOTE

This circle also passes through three points (0, 0), $(2a, 0)$ and $(0, 2b)$.

11. The equation of circle which touches x-axis at a distance of a unit from the origin and cut the intercept b on y-axis is $(x-a)^2 + \left(y - \frac{\sqrt{4a^2+b^2}}{2}\right)^2 = \left(\frac{\sqrt{4a^2+b^2}}{2}\right)^2$

    centre $\left(a, \frac{\sqrt{4a^2+b^2}}{2}\right)$, radius $= \frac{\sqrt{4a^2+b^2}}{2}$

NOTE

This circle passes through point $(a, 0)$

12. The centre and radius of circle which touches y-axis at point $(0, a)$ and cuts the intercept b on x-axis is

centre $\left(\dfrac{\sqrt{4a^2+b^2}}{2}, a\right)$, radius $= \dfrac{\sqrt{4a^2+b^2}}{2}$

13. **Diameter form of a circle**

The equation of circle whose diameter ends coordinates are (x_1, y_1) and $B(x_2, y_2)$ is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

centre $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

radius $= \dfrac{1}{2}\,[(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}$

NOTES

1. Radius of circle which passes through points $A(x_1, y_1)$ and $B(x_2, y_2)$ is minimum if A and B are ends of diameter.
2. $y = mx$ be a chord of the circle $x^2 + y^2 = 2ax$ then, equation of the circle taking this chord as diameter is $x^2 + y^2 - \dfrac{2a}{1+m^2}x - \dfrac{2am}{1+m^2}y = 0$

14. **General equation of a circle**

$x^2 + y^2 + 2gx + 2fy + c = 0$

centre $(-g, -f)$, radius

$= \sqrt{g^2 + f^2 - c}$

NOTES

1. If circle is real then $g^2 + f^2 = c$
2. If circle is point circle then $g^2 + f^2 = c$
3. If circle is imaginary then $g^2 + f^2 < c$
4. The circle is real, point circle, imaginary circle if radius of circle will be real, zero, and imaginary respectively.
5. Three geometrical conditions (parameter) are required for defining a circle.
6. Centre and radius of the circle $ax^2 + ay^2 + 2gx + 2fy + c = 0$ are as follows:

   centre $\left(-\dfrac{g}{a}, -\dfrac{f}{a}\right)$,

   radius $= \dfrac{\sqrt{g^2 + f^2 - ac}}{a}$
7. General equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle if $a = b$; $h = 0$ i.e., coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.
8. Before finding the centre and radius of general equation of circle, the coefficient of x^2 and y^2 is are made unity if they are not already.
9. General equation of circle whose centre is on x-axis is $x^2 + y^2 + 2gx + c = 0$

   centre $(-g, 0)$; radius $= \sqrt{g^2 - c}$
10. General equation of circle whose centre is on X-axis and passes through origin is $x^2 + y^2 + 2gx = 0$ centre $(-g, 0)$, radius $= g$
11. General equation of circle touching X-axis at the origin and of radius f is $x^2 + y^2 + 2fy = 0$ centre $(0, -f)$, radius $= f$.
12. Maximum distance of point $P(x_1, y_1)$ from circle having centre c and radius $r = Cp + r$ and minimum distance $= Cp - r$.

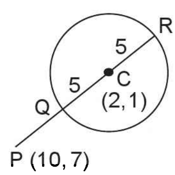
SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. The least and the greatest distances of the point (10, 7) from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ are 5 and 15. Is this statement true or false?

Solution

True. C is (2, 1), P (10, 7) and $r = 5$

$\therefore \quad PC = 10$



If the diameter through P cuts the circle in Q and R, then PQ is least and PR is greatest.

$\therefore \quad PQ = PC - r = 10 - 5 = 5$ and
$PR = PC + r = 10 + 5 = 15$

2. Find the equations of a circles which pass through the points (7, 10) and (–7, –4) and have radius equal to 10 units.

Solution

Let the centre be (h, k), then distances of this point from the given points are equal, each equal to the radius 10, i.e.,

$(h - 7)^2 + (k - 10)^2 = 10^2$ .... (1)

and $(h + 7)^2 + (k + 4)^2 = 10^2$ .... (2)

Subtracting (ii) from (i), we get

$-28h - 20k - 8k + 100 - 16 = 0$

or $h + k = 3$ .... (3)

Substituting the value of $k = 3 - h$ from (3) in (1), we obtain $(h - 7)^2 + (3 - h - 10)^2 = 10^2$

$\Rightarrow (h - 7)^2 + (-7 - h)^2 = 10^2$

$\Rightarrow 2h^2 + 98 = 100$

$\Rightarrow 2h^2 = 2$

$\Rightarrow h = \pm 1$

When $h = 1$, then from (3), $k = 3 - h = 3 - 1 = 2$

Hence, the centre of the circle is (1, 2) and its equation is $(x - 1)^2 + (y - 2)^2 = 10^2$.

When $h = -1$, then from (3), $k = 3 - h = 3 - (-1) \Rightarrow k = 4$.

Hence, the centre of the circle is (–1, 4) and its equation is $(x - (-1))^2 + (y - 4)^2 = 10^2$.

3. If the distance of a point P from (6, 0) is twice its distance from the point (1, 3) prove that the locus of P is a circle. Find its centre and radius.

Solution

Let the given points be A (6, 0) and B(1, 3).

Let, $P(x, y)$ be any point such that $|PA| = 2\,|PB|$

$\Rightarrow PA^2 = 4PB^2$

$\Rightarrow (x - 6)^2 + (y - 0)^2 = 4\{(x - 1)^2 + (y - 3)^2\}$

$\Rightarrow 3x^2 + 3y^2 + 4x - 24y + 4 = 0$

$\Rightarrow x^2 + y^2 + \frac{4}{3}x - 8y + \frac{4}{3} = 0$, which is of the form

$x^2 + y^2 + 2gx + 2fy + c = 0$, with

$2g = \frac{4}{3}, 2f = -8, c = \frac{4}{3}$

i.e., $g = \frac{2}{3}, f = -4, c = \frac{4}{3}$

Since, $g^2 + f^2 - c = \frac{4}{9} + 16 - \frac{4}{3} = \frac{136}{9} > 0$

therefore, locus of P is a circle with centre $(-g, -f) = \left(-\frac{2}{3}, 4\right)$ and radius $= \sqrt{\frac{136}{9}} = \frac{2}{3}\sqrt{34}$

$= \sqrt{\frac{136}{9}} = \frac{2}{3}\sqrt{34}$

4. Find the equation of the circle passing through the points (4, 1) and (6, 5) and whose centre is on the line $4x + y = 16$.

[NCERT]

Solution

Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ .... (i)

As the points (4, 1) and (6, 5) lie on (i), therefore, $4^2 + 1^2 + 8g + 2f + c = 0$

$\Rightarrow 8g + 2f + c = -17$ .... (ii)

and $6^2 + 5^2 + 12g + 10f + c = 0$

$\Rightarrow 12g + 10f + c = -61$ .... (iii)

Subtracting (iii) from (ii), we get

$-4g - 8f = -17 + 61$

$\Rightarrow 4g + 8f = -44$

$\Rightarrow g + 2f = -11$ .... (iv)

Also, the centre $(-g, -f)$ of (i) lies on the line $4x + y = 16$

$\Rightarrow 4(-g) - f = 16$

$\Rightarrow 4g + f = -16$ .... (v)

Multiplying (v) by 2 and subtracting it from (iv), we obtain $g + 2f - 2(4g + f) = -11 - 2(-16)$

$\Rightarrow -7g = 21$

$\Rightarrow g = -3$

Substituting this value of g in (iv), we get $-3 + 2f = -11$

$\Rightarrow 2f = -11 + 3$

$\Rightarrow f = -\frac{8}{2} = -4$

Substituting $g = -3, f = -4$ in (ii), we obtain

$c = -17 - 8g - 2f = -17 - 8(-3) - 2(-4) = 15$

Hence, the required equation of the circle in reference is $x^2 + y^2 + 2(-3)x + 2(-4)y + 15 = 0$ (from (i))

or $x^2 + y^2 - 6x - 8y + 15 = 0$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

EXERCISE 1

1. Find the equation of the circle whose centre is (2, –3) and radius is 8.
2. Find the centre and radius of each of the following circles
   (i) $(x + 1)^2 + (y - 1)^2 = 4$
   (ii) $x^2 + y^2 + 6x - 4y + 4 = 0$.
3. Find the equation of the circle which touches
   (i) the x-axis at the origin and whose radius is 5
   (ii) both the axes and whose radius is 5
4. Find the centre and radius of the circle given by the equation $2x^2 + 2y^2 + 3x + 4y + = 0$.
5. Centre of a circle is in first quadrant. If circle touches the Y-axis at the point (0, 2) and passing through the point (1, 0), then find its equation.
6. Find the equation of the circle which passes through the origin and cuts off intercepts 3 and 4 from the positive parts of the axes respectively.
7. Prove that a circle touches the X-axis and its centre is (1, 2) be $x^2 + y^2 - 2x - 4y + 1 = 0$.

EXERCISE 2

1. Find the equation of a circle of radius 5 whose centre lies on x-axis and passes through the point (2, 3).
2. Find the centre and radius of each of the following circles
   (i) $x^2 + (y + 2)^2 = 9$
   (ii) $x^2 + y^2 - 4x + 6y = 12$
3. Find the equation of the circle which touches:
   (i) the x-axis and whose centre is (3, 4)
   (ii) the lines $x = 0, y = 0$ and $x = a$.
4. Find the equaion of the circle, the coordinates of the end points of whose diameter are (–1,2) and (4, –3).
5. Find the equation of circle whose centre is $(-a, -b)$ and radius is $\sqrt{a^2 - b^2}$.
6. Find the equation to the circles which touch axis of X, axis of Y and the line $x = a$.
7. Find the equation of the circle which passes through two points on the x-axis which are at distances 4 from the origin and whose radius is 5.
8. Find the equation of a circle which touches y-axis at a distance of 4 units from the origin and cuts an intercept of 6 units along the positive direction of x-axis.
9. Find equation of a circle whose centre is (2, 3) and passes through the point of intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$.

ANSWERS

EXERCISE 1

1. $x^2 + y^2 - 4x + 6y - 51 = 0$
2. (i) (–1, 1) and radius 2
   (ii) Centre (–3, 2) and radius 3
3. (i) $x^2 + y^2 - 10y = 0$
   (ii) $x^2 + y^2 \pm 10x \pm 10y + 25 = 0$
4. Centre (– 3/4, –1) and radius 1.
5. $x^2 + y^2 - 5x - 4y + 4 = 0$
6. $\left(x - \frac{3}{2}\right)^2 + (y - 2)^2 = \left(\frac{5}{2}\right)^2$

EXERCISE 2

1. $x^2 + y^2 + 4x - 21 = 0$,
   $x^2 + y^2 - 12x + 11 = 0$,

2. (i) Centre (0, –2) and radius 3
(ii) Centre (2, –3) and radius 5

3. (i) $x^2 + y^2 - 6x - 8y + 9 = 0$

(ii) $\left(x-\frac{a}{2}\right)^2+\left(y\mp\frac{a}{2}\right)^2=\left(\frac{a}{2}\right)^2$

4. $x^2 + y^2 - 3x + y - 10 = 0$

5. $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$

6. $x^2+y^2-ax\pm ay+\frac{1}{4}a^2=0$

7. $x^2 + y^2 \mp 6y - 16 = 0$

8. $x^2 + y^2 - 10x \mp 8y + 16 = 0$

9. $x^2 + y^2 - 4x - 6y - 12 = 0$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

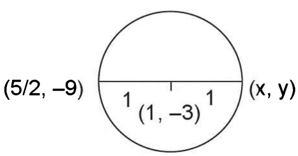
1. If one of the end point of a diameter of the circle $x^2 + y^2 - 2x + 6y - 15 = 0$ is (5/2, –9) then find its other end point

[Gujarat CET-2007]

(a) (–1/2, –3) (b) (1/2, 0)
(c) (1/2, –3) (d) (–1/2, 3)

Solution

(c) Given equation of the circle is $x^2 + y^2 - 2x + 6y - 15 = 0$
So centre is $(-g, -f) = (1, -3)$
One end of the diameter is (5/2, –9)
centre divides the diameter in 1 : 1



$$1=\frac{1\times\frac{5}{2}+1\times x}{1+1}$$

$$\Rightarrow\quad 1=\frac{\frac{5}{2}+x}{2}$$

$$\Rightarrow\quad 2-\frac{5}{2}=x$$

$$\Rightarrow\quad \frac{-1}{2}=x$$

$$-3=\frac{1\times-9+1\times y}{1+1}\Rightarrow -3\times 2=-9+y$$

$$\Rightarrow\quad -6+9=y\Rightarrow y=3$$

$$\therefore\quad (x, y)\text{ is }\left(\frac{-1}{2},3\right)$$

2. The equation of the circle of radius 3 that lies in the fourth quadrant and touching the lines $x = 0$ and $y = 0$ is ***[EAMCET-2007]***

(a) $x^2 + y^2 - 6x + 6y + 9 = 0$
(b) $x^2 + y^2 - 6x - 6y + 9 = 0$
(c) $x^2 + y^2 + 6x - 6y + 9 = 0$
(d) $x^2 + y^2 + 6x + 6y + 9 = 0$

Solution

(a) As the circle touching both the coordinate axes in 4th quadrant of radius 3 units, $(x-3)^2 + (y+3)^2 = 3^2$ or $x^2 + y^2 - 6x + 6y + 9 = 0$

3. The equation of the circle which passes through (1, 0) and (0, 1) and has its radius as small as possible, is

(a) $x^2 + y^2 - 2x - 2y + 1 = 0$
(b) $x^2 + y^2 - x - y = 0$
(c) $2x^2 + 2y^2 - 3x - 3y + 1 = 0$
(d) $x^2 + y^2 - 3x - 3y + 2 = 0$

Solution

(b) The radius will be minimum, if the given points are the end points of a diameter.

4. The abscissae of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. The radius of the circle with AB as a diameter will be

[IIT-84; Roorkee-89]

(a) $\sqrt{a^2+b^2+p^2+q^2}$

(b) $\sqrt{a^2+b^2-p^2-q^2}$

(c) $\sqrt{b^2+q^2}$

(d) $\sqrt{a^2+p^2}$

Solution

Let $A \equiv (\alpha, \beta); B \equiv (\gamma, \delta)$.

Then, $\alpha + \gamma = -2a, \alpha\gamma = -b^2$

and $\beta + \delta = -2p, \beta\delta = -q^2$

Now equation of the required circle is

$(x - \alpha)(x - \gamma) + (y - \beta)(y - \delta) = 0$

$\Rightarrow x^2 + y^2 - (\alpha + \gamma)x - (\beta + \delta)y + \alpha\gamma + \beta\delta = 0$

$\Rightarrow x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$

Its radius $= \sqrt{a^2 + p^2 + b^2 + q^2}$

5. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3} + 1$ apart. If the chords subtend at the centre, angles of $\frac{\pi}{k}$ and $\frac{2\pi}{k}$, where $k > 0$, then the value of $[k]$ is *[IIT (2)-2010]*

NOTE

$[k]$ denotes the largest integer less than or equal to k

Solution

Since distance between parallel chords is greater than radius, therefore both chords lie on opposite side of centre.

$$2\cos\frac{\pi}{2k} + 2\cos\frac{\pi}{k} = \sqrt{3} + 1.$$

Let $\frac{\pi}{2k} = \theta$

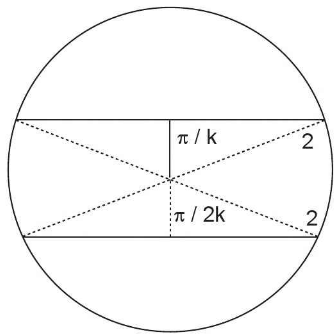
$\therefore \quad 2\cos\theta + 2\cos 2\theta = \sqrt{3} + 1$

$\Rightarrow \quad 2\cos\theta + 2(2\cos^2\theta - 1) = \sqrt{3} + 1$

$\Rightarrow \quad 4\cos^2\theta + 2\cos\theta - (3 + \sqrt{3}) = 0$

$$\therefore \quad \cos\theta = \frac{-2 \pm \sqrt{4 + 16(3 + \sqrt{3})}}{2(4)}$$

$$= \frac{-2 \pm 2\sqrt{1 + 12 + 4\sqrt{3}}}{2(4)}$$



$$= \frac{-1 \pm \sqrt{(\sqrt{12} + 1)^2}}{4} = \frac{-1 \pm (2\sqrt{3} + 1)}{4}$$

$\Rightarrow \quad \cos\frac{\pi}{2k} = \frac{\sqrt{3}}{2}, \frac{-(\sqrt{3} + 1)}{2}$ Rejected

$\Rightarrow \quad \frac{\pi}{2k} = \frac{\pi}{6} \Rightarrow k = 3 \Rightarrow [k] = 3$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If the line $x + 2by + 7 = 0$ is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$ then b is equal to *[MPPET-1991]*

   (a) 3 (b) –5
   (c) –1 (d) 5

2. The area of the circle whose centre is at (1, 2) and which passes through the point (4, 6) is *[DCE-2000]*

   (a) 5π (b) 10π
   (c) 25π (d) None of these

3. The centres of the circles $x^2 + y^2 = 1$, $x^2 + y^2 + 6x - 2y = 1$ and $x^2 + y^2 - 12x + 4y = 1$ are *[PET-86]*

   (a) Same (b) Collinear
   (c) Non-collinear (d) None of these

4. The circles $x^2 + y^2 + 4x - 4y + 4 = 0$ touches *[PET-88]*

   (a) x-axis (b) y-axis
   (c) x-axis and y-axis (d) None of these

5. The equation of the circle which touches both axes and whose centre is (x_1, y_1) is **[PET-88]**

(a) $x^2 + y^2 + 2x_1(x + y) + x_1^2 = 0$
(b) $x^2 + y^2 - 2x_1(x + y) + x_1^2 = 0$
(c) $x^2 + y^2 = x_1^2 + y_1^2$
(d) $x^2 + y^2 + 2xx_1 + 2yy_1 = 0$

6. The equation of the cirlce which touches x-axis and whose centre is (1, 2) is **[PET-84]**

(a) $x^2 + y^2 - 2x + 4y + 1 = 0$
(b) $x^2 + y^2 - 2x - 4y + 1 = 0$
(c) $x^2 + y^2 + 2x + 4y + 1 = 0$
(d) $x^2 + y^2 + 4x + 2y + 4 = 0$

7. If the radius of the circle $x^2 + y^2 - 18x + 12y + k = 0$ be 11, then k is equal to **[PET-87]**

(a) 347 (b) 4
(c) –4 (d) 49

8. The equation of a circle which touches both axes and the line $3x - 4y + 8 = 0$ and lies in the third quadrant is **[PET-86]**

(a) $x^2 + y^2 - 4x + 4y - 4 = 0$
(b) $x^2 + y^2 - 4x + 4y + 4 = 0$
(c) $x^2 + y^2 + 4x + 4y + 4 = 0$
(d) $x^2 + y^2 - 4x - 4y - 4 = 0$

9. If one end of a diameter of the cirlce $x^2 + y^2 - 4x - 6y + 11 = 0$ be (3, 4) then the other end is **[PET-86; RANCHI BIT-91]**

(a) (0, 0) (b) (1, 1)
(c) (1, 2) (d) (2, 1)

10. The equation of the circle having centre (1, –2) and passing through the point of intersection of lines $3x + y = 14$, $2x + 5y = 18$ is **[PET-90]**

(a) $x^2 + y^2 - 2x + 4y - 20 = 0$
(b) $x^2 + y^2 - 2x - 4y - 20 = 0$
(c) $x^2 + y^2 + 2x - 4y - 20 = 0$
(d) $x^2 + y^2 + 2x + 4y - 20 = 0$

11. Equation of the circle which touches the lines $x = 0$, $y = 0$ and $3x + 4y = 4$ is **[PET-91]**

(a) $x^2 - 4x + y^2 + 4y + 4 = 0$
(b) $x^2 - 4x + y^2 - 4y + 4 = 0$
(c) $x^2 + 4x + y^2 + 4y + 4 = 0$
(d) $x^2 + 4x + y^2 - 4y + 4 = 0$

12. The centre and radius of the circle $2x^2 + 2y^2 - x = 0$ are **[PET-84, 87]**

(a) (1/4, 0) and 1/4
(b) (–1/2, 0) and 1/2
(c) (1/2, 0) and 1/2
(d) (0, –1/4) and 1/4

13. The equation of the circle which touches x-axis at (3, 0) and passes through (1, 4) is given by **[PET-93]**

(a) $x^2 + y^2 - 6x - 5y + 9 = 0$
(b) $x^2 + y^2 + 6x + 5y - 9 = 0$
(c) $x^2 + y^2 - 6x + 5y - 9 = 0$
(d) $x^2 + y^2 + 6x - 5y + 9 = 0$

14. The equation of circle whose diameter is the line joining the points (–4, 3) and (12, –1) is **[IIT-71; AMU-79; MPPET-84; Roorkee-69]**

(a) $x^2 + y^2 + 8x + 2y + 51 = 0$
(b) $x^2 + y^2 + 8x - 2y - 51 = 0$
(c) $x^2 + y^2 + 8x + 2y - 51 = 0$
(d) $x^2 + y^2 - 8x - 2y - 51 = 0$

15. The area of a circle whose centre is (h, k) and radius a is **[PET-94]**

(a) $\pi(h^2 + k^2 - a^2)$ (b) πa^2hk
(c) πa^2 (d) None of these

16. If the equation $\frac{K(x+1)^2}{3} + \frac{(y+2)^2}{4} = 1$ represents a circle, then K is equal to **[PET-94]**

(a) 3/4 (b) 1
(c) 4/3 (d) 12

17. If (α, β) is the centre of a circle passing through the origin, then its equation is **[MPPET-1999]**

(a) $x^2 + y^2 - \alpha x - \beta y = 0$
(b) $x^2 + y^2 + 2\alpha x + \beta y = 0$
(c) $x^2 + y^2 - 2\alpha x - 2\beta y = 0$
(d) $x^2 + y^2 + \alpha x + \beta y = 0$

18. If the area of the circle $4x^2 + 4y^2 - 8x + 16y + k = 0$ is 9π sq. units, then the value of k is **[MPPET-2005]**

(a) 4 (b) 16
(c) –16 (d) ±16

19. Circle $x^2 + y^2 - 8x + 4y + 4 = 0$ touches **[CET(Karnataka)-99, 04]**

(a) y-axis (b) both the axes
(c) None of the axes (d) x-axis

20. The minimum distance of the point (2, –7) from the circle $x^2 + y^2 - 14x - 10y - 151 = 0$ is ***[NDA-2004]***

(a) 2 (b) 3
(c) 5 (d) 7

21. For which value of a in equation $x^2 + y^2 + (a^2 - 4)xy + 2x + 2y + a = 0$ represents circle? ***[Gujarat CET-2007]***

(a) 3 (b) 2
(c) 4 (d) None of these

22. A circle touches the y-axis at the point (0, 4) and cuts the x-axis in a chord of length 6 units. The radius of the circle is ***[PET-92]***

(a) 3 (b) 4
(c) 5 (d) 6

23. The equation of a diameter passing through origin of circle $x^2 + y^2 - 6x + 2y = 0$

(a) $x + 3y = 0$ (b) $x - 3y = 0$
(c) $3x + y = 0$ (d) $3x - y = 0$

SOLUTIONS

1. (d) Here the center of circle (3, –1) must lies at the line $x + 2by + 7 = 0$. Therefore, $3 - 2b + 7 = 0$
$\Rightarrow \quad b = 5$.

2. (c) Obviously radius $= \sqrt{(1-4)^2 + (2-6)^2} = 5$
Hence, the area is given by $\pi r^2 = 25\pi$ sq.units.

3. (b) Centers are (0, 0), (–3, 1) and (6, –2) and a line passing through any two points say (0, 0) and (–3, 1) is $y = -\frac{1}{3}x$ and point (6, –2) lies on it. Hence points are collinear.

4. (c) Both axis, as center is (–2, 2) and radius is 2.

5. (b) The equation will be $x^2 + y^2 - 2x_1(x + y) + x_1^2 = 0$.

6. (b) Center (1, 2) and since circle touches x-axis, therefore, radius is equal to 2
Hence, the equation is $(x - 1)^2 + (y - 2)^2 = 2^2$
$\Rightarrow \quad x^2 + y^2 - 2x - 4y + 1 = 0$.

7. (c) $(\text{radius})^2 = g^2 + f^2 - c$ or $121 = 81 + 36 - k$
$\Rightarrow \quad k = -4$.

8. (c) The equation of a circle in third quadrant touching the coordinate axis with center $(-a, -a)$ and radius a is $x^2 + y^2 + 2ax + 2ay + a^2 = 0$ and we know $\left|\frac{3(-a) - 4(-a) + 8}{\sqrt{9+16}}\right| = a$
$\Rightarrow \quad a = 2$.
Hence the required equation is $x^2 + y^2 + 4x + 4y + 4 = 0$.

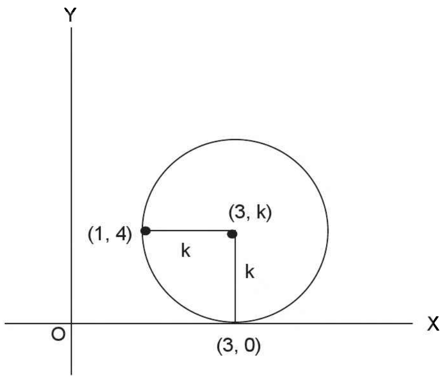
9. (c) Center is (2, 3), one end is (3, 4). P_2 divides the joins of P, and O in ratio of 2 : 1.
Hence P_2 is $\left(\frac{4-3}{2-1}, \frac{6-4}{2-1}\right) \equiv (1, 2)$

10. (a) The point of intersection of $3x + y - 14 = 0$ and $2x + 5y - 18 = 0$
are $x = \frac{-18+70}{15-2}$, $y = \frac{-28+54}{13}$
$\Rightarrow \quad x = 4, y = 2$
i.e., point is (4, 2)
Therefore radius is $\sqrt{(9)+(16)} = 5$ and equation is $x^2 + y^2 - 2x + 4y - 20 = 0$.

11. (b) Let center of circle be (h, k). Since it touches both axis therefore $h = k = a$. Hence, equation can be $(x - a)^2 + (y - a)^2 = a^2$. But it also touches the line $3x + 4y = 4$.
Therefore, $\frac{3a + 4a - 4}{5} = a \Rightarrow a = 2$. Hence the required equation of circle is $x^2 + y^2 - 4x - 4y + 4 = 0$.

12. (a) centre $(-g, -f) = \left(\frac{1}{4}, 0\right)$
and $R = \sqrt{\frac{1}{16} + 0 - 0} = \frac{1}{4}$

13. (a) $k^2 = (3 - 1)^2 + (k - 4)^2$
$\Rightarrow \quad k^2 = 4 + (k - 4)^2$
$\Rightarrow \quad k = \frac{5}{2}$.



Hence required equation of circle is

$$(x-3)^2+\left(y-\frac{5}{2}\right)^2=\left(\frac{5}{2}\right)^2$$

$\Rightarrow$ $x^2+y^2-6x-5y+9=0$

14. (d) Required equation is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$(x+4)(x-12)+(y-3)(y+1)=0$

$=x^2+y^2-8x-2y-51=0$

15. (c) Since area $=\pi r^2$, where $r=a$

$\Rightarrow$ area $=\pi a^2$.

16. (a) It represent a circle, if $a=b$

$\Rightarrow$ $\frac{3}{k}=4 \Rightarrow k=\frac{3}{4}$.

17. (c) Radius = Distance from origin $=\sqrt{\alpha^2+\beta^2}$.

$\therefore$ $(x-\alpha)^2+(y-\beta)^2=\alpha^2+\beta^2$

$\Rightarrow$ $x^2+y^2-2\alpha x-2\beta y=0$

18. (c) The equation of the circle is

$$x^2+y^2-2x+4y+\frac{k}{4}=0$$

$\therefore$ Radius of circle $=\sqrt{1+4-\frac{k}{4}}$

$\Rightarrow$ $\pi\left(5-\frac{k}{4}\right)=9\pi$

$\Rightarrow$ $5-9=\frac{k}{4}$

$\Rightarrow$ $k=-16$

19. $g=-4, f=2, c=4$

$\Rightarrow$ $f^2=c$

$\Rightarrow$ circle touches y-axis.

20. Centre of the circle = (7, 5), radius = 15

Distance of the given point from centre = 13

$\therefore$ required distance = 15 – 13 = 2

21. (b) General equation of a circle is $x^2+y^2+2gx+2fy+c=0$

Equation $x^2+y^2+(a^2-4)xy+2x+2y+a=0$ represents a circle if coefficient of $xy=0$

$\Rightarrow$ $a^2-4=0$

$\Rightarrow$ $a^2=4$ $\therefore a=\pm 2$

22. (c) $r=\frac{\sqrt{4a^2+b^2}}{2}=\frac{\sqrt{64+36}}{2}=5$.

Here $a=4$, $b=6$

23. (a) Center (3, –1). Line through it and origin is $x+3y=0$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. What is the equation of a circle, whose centre lies on the x-axis at a distance h from the origin and the circle passes through the origin? ***[NDA-2007]***

(a) $x^2+y^2-2hx=0$

(b) $x^2+y^2-2hx+h^2=0$

(c) $x^2+y^2+2hxy=0$

(d) $x^2+y^2-h^2=0$

2. A circle touches the axes at the point (3, 0) and (0, –3). The centre of the circle is ***[PET-92]***

(a) (3, –3)

(b) (0, 0)

(c) (–3, 0)

(d) (6, –6)

3. If $g^2 + f^2 = c$ then the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ will represent

[MPPET-03]

(a) A circle of radius g
(b) A circle of radius f
(c) A circle of diameter $\sqrt{c}$
(d) A circle of radius 0

4. The equation of the circle in the first quadrant touching each coordinate axis at a distance of one unit from the origin is

[RPET-91]

(a) $x^2 + y^2 - 2x - 2y + 1 = 0$
(b) $x^2 + y^2 - 2x - 2y - 1 = 0$
(c) $x^2 + y^2 - 2x - 2y = 0$
(d) None of these

5. Radius of circle is $(x - 5)(x - 1) + (y - 7)(y - 4) = 0$ is

(a) 3 (b) 4
(c) 5/2 (d) 7/2

6. The equation of the circle whose diameters have the end points $(a, 0)$, $(0, b)$ is given by

[PET-93]

(a) $x^2 + y^2 - ax - by = 0$
(b) $x^2 + y^2 + ax - by = 0$
(c) $x^2 + y^2 - ax + by = 0$
(d) $x^2 + y^2 + ax + by = 0$

7. $ax^2 + 2y^2 + 2bxy + 2x - y + c = 0$ represents a circle through the origin, if

[PET-84]

(a) $a = 0, b = 0, c = 0$
(b) $a = 2, b = 2, c = 0$
(c) $a = 2, b = 2, c = 0$
(d) $a = 2, b = 0, c = 0$

8. Equation of a circle whose centre is origin and radius is equal to the distance between the lines $x = 1$ and $x = -1$ is

[PET-84]

(a) $x^2 + y^2 = 1$
(b) $x^2 + y^2 = \sqrt{2}$
(c) $x^2 + y^2 = 4$
(d) $x^2 + y^2 = -4$

9. A circle has its equation in the form $x^2 + y^2 + 2x + 4y + 1 = 0$. Choose the correct coordinates of its centre and the right value of its radius from the following

[PET-82]

(a) Centre $(-1, -2)$, radius = 2
(b) Centre $(2, 1)$, radius = 1
(c) Centre $(1, 2)$, radius = 3
(d) Centre $(-1, 2)$, radius = 2

10. The area of the curve $x^2 + y^2 = 2ax$ is

(a) πa^2 (b) $2\pi a^2$
(c) $4\pi a^2$ (d) $\frac{1}{2}\pi a^2$

11. Which of the following line is a diameter of the circle $x^2 + y^2 - 6x - 8y - 9 = 0$

(a) $3x - 4y = 0$
(b) $4x - 3y = 9$
(c) $x + y = 7$
(d) $x - y = 1$

12. The equation of circle passing through the points of intersection of circles $x^2 + y^2 - 6x + 8 = 0$ and $x^2 + y^2 = 6$ and point $(1, 1)$ is

(a) $x^2 + y^2 - 6x + 4 = 0$
(b) $x^2 + y^2 - 3x + 1 = 0$
(c) $x^2 + y^2 - 4y + 2 = 0$
(d) None of these

13. The radius of the circle passing through the point $(6, 2)$ and two of whose diameters are $x + y = 6$ and $x + 2y = 4$ is

[Karnataka CET-04]

(a) 4 (b) 6
(c) 20 (d) $\sqrt{20}$

14. Let, $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points such that their abscissa x_1 and x_2 are the roots of the equation $x^2 + 2x - 3 = 0$ while the ordinates y_1 and y_2 are the roots of the equation $y^2 + 4y - 12 = 0$. The centre of the circle with PQ as diameter is

[Orissa JEE-2005]

(a) $(-1, -2)$ (b) $(1, 2)$
(c) $(1, -2)$ (d) $(-1, 2)$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The coordinates of the centre and the radius of the circle $x^2 + y^2 + 4x - 6y - 36 = 0$ are, respectively, given by **[NDA-2001]**

(a) $(-4, 6)$ and 6 (b) $(4, -6)$ and 7
(c) $(2, -3)$ and 6 (d) $(-2, 3)$ and 7

2. If r_1, r_2 and r_3 are the radii of the circles $x^2 + y^2 - 4x + 6y = 5$, $x^2 + y^2 + 6x - 4y = 3$ and $x^2 + y^2 - 2x + 4y = 8$, respectively, then **[NDA-2002]**

(a) $r_1 > r_2 > r_3$ (b) $r_2 > r_3 > r_1$
(c) $r_3 > r_1 > r_2$ (d) $r_1 > r_3 > r_2$

3. Centre of the circle $(x - 3)^2 + (y - 4)^2 = 5$ is **[MPPET-88]**

(a) (3, 4) (b) (–3, –4)
(c) (4, 3) (d) (–4, –3)

4. Radius of the circle $x^2 + y^2 + 2x \cos \theta + 2y \sin \theta - 8 = 0$, is **[MNR-74]**

(a) 1 (b) 3
(c) $2\sqrt{3}$ (d) $\sqrt{10}$

5. The radius of the circle $x^2 + y^2 + 4x + 6y + 13 = 0$ is **[Karnataka CET-05]**

(a) $\sqrt{26}$ (b) $\sqrt{13}$
(c) $\sqrt{23}$ (d) 0

6. From three non-collinear points we can draw

(a) Only one circle (b) Three circle
(c) infinite circles (d) No circle

7. The equation of the circle which touches both the axes and whose radius is a, is **[PET-1984]**

(a) $x^2 + y^2 - 2ax - 2ay + a^2 = 0$
(b) $x^2 + y^2 + ax + ay - a^2 = 0$
(c) $x^2 + y^2 + 2ax + 2ay - a^2 = 0$
(d) $x^2 + y^2 - ax - ay + a^2 = 0$

8. The equation $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ will represent a circle, if **[MNR-79; PET-88, 97]**

(a) $a = b = 0$ and $c = 0$
(b) $f = g$ and $h = 0$
(c) $a = b \neq 0$ and $h = 0$
(d) $f = g$ and $c = 0$

9. The equation $x^2 + y^2 = 0$ denotes **[PET-84]**

(a) A point (b) A circle
(c) x-axis (d) y-axis

10. The equation of the circle in the first quadrant which touches each axis at a distance 5 from the origin is

(a) $x^2 + y^2 + 5x + 5y + 25 = 0$
(b) $x^2 + y^2 - 10x - 10y + 25 = 0$
(c) $x^2 + y^2 - 5x - 5y + 25 = 0$
(d) $x^2 + y^2 - 12y + 27 = 0$

11. The equation of the circle of radius 3 that lies in the fourth quadrant and touching the lines $x = 0$ and $y = 0$ is **[EAMCET-2007]**

(a) $x^2 + y^2 - 6x + 6y + 9 = 0$
(b) $x^2 + y^2 - 6x - 6y + 9 = 0$
(c) $x^2 + y^2 + 6x - 6y + 9 = 0$
(d) $x^2 + y^2 + 6x + 6y + 9 = 0$

12. Equation of a circle passing through origin is $x^2 + y^2 - 6x + 2y = 0$. What is the equation of one of its diameters? **[NDA-2008]**

(a) $x + 3y = 0$ (b) $x + y = 0$
(c) $x = y$ (d) $3x + y = 0$

13. If the coordinates of one end of the diameter of the circle $x^2 + y^2 - 8x - 4y + c = 0$ are $(-3, 2)$, then the coordiantes of other end are **[Roorkee-95]**

(a) (5, 3) (b) (6, 2)
(c) (1, –8) (d) (11, 2)

14. If area of the circle $4x^2 + 4y^2 - 8x + 16y + k = 0$ is 9π, then k is equal to

(a) –16 (b) 16
(c) ± 16 (d) 4

15. If x-axis is tangent to the circle $x^2 + y^2 + 2gx + 2fy + k = 0$, then which one of the following is correct ? ***[NDA-2009]***

(a) $g^2 = k$ (b) $g^2 = f$
(c) $f^2 = k$ (d) $f^2 = g$

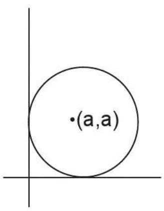
ANSWER SHEET

1. (a) (b) (c) (d)	**6.** (a) (b) (c) (d)	**11.** (a) (b) (c) (d)
2. (a) (b) (c) (d)	**7.** (a) (b) (c) (d)	**12.** (a) (b) (c) (d)
3. (a) (b) (c) (d)	**8.** (a) (b) (c) (d)	**13.** (a) (b) (c) (d)
4. (a) (b) (c) (d)	**9.** (a) (b) (c) (d)	**14.** (a) (b) (c) (d)
5. (a) (b) (c) (d)	**10.** (a) (b) (c) (d)	**15.** (a) (b) (c) (d)

HINTS AND EXPLANATIONS

4. (b) radius $= \sqrt{\cos^2\theta + \sin^2\theta - (-8)} = 3$

7. (a) If circle touches both axis (Let it be in 1st quadrant)



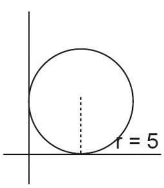
Centre (a, a) radius $= a$

$\therefore$ equation is $(x - a)^2 + (y - a)^2 = a^2$

$x^2 + y^2 - 2ax - 2ay + a^2 = 0$

$\therefore$ In all quadrant equation can be $x^2 + y^2 \pm 2ax \pm 2ay + a^2 = 0$

10. (b) Here centre is (r, r)
Where radius $= r$
Distance of point of contact with axis from origin = 5 = radius

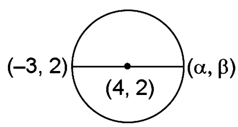


Equation is $(x - 5)^2 + (y - 5) = 5^2$

$x^2 + y^2 - 10x - 10y + 25 = 0$

12. (a) Centre of circle is
Diameter always passes through centre (3, –1)
Satisfy each option to get (a)

13. (d) Centre (4, 2)
Let other end be (α, β)



$\therefore \quad \left(\dfrac{\alpha - 3}{2}, \dfrac{\beta + 2}{2}\right) \equiv (4, 2) \quad (\alpha, \beta) \equiv (11, 2)$

14. (a) Area $= 9\pi$

$\pi r^2 = 9\pi \Rightarrow r = 3$

Equation of circle is $yx^2 + 4y^2 - 8x + 16y + k = 0$

or $\quad x^2 + y^2 - 2x + 4y + \dfrac{k}{4} = 0$

$\therefore \quad \sqrt{g^2 + f^2 - c} = 3$

$\Rightarrow \quad \sqrt{(-1)^2 + 2^2 - \dfrac{k}{4}} = 3$

15. (a) Since, x-axis is a tangent to the given circle. It means circle touches the x-axis.

$\therefore \quad \sqrt{g^2 - k} = 0$

$g^2 = k$

LECTURE

2

Circle 2

(Relative position of point with respect to circle, parametric form of equation of circle, relative position of line and circle)

BASIC CONCEPTS

1. Position of a Point with Respect to Circle

(i) **Position of Point $A(x_1, y_1)$ with respect to the circle $x^2 + y^2 = a^2$ is**

(i) out side if $x_1^2 + y_1^2 - a^2 > 0$

(ii) on the circle if $x_1^2 + y_1^2 - a^2 = 0$

(iii) inside the circle if $x_1^2 + y_1^2 - a^2 < 0$

(ii) **Position of Point $A(x_1, y_1)$ with respect to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is**

(i) outside the circle if $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$

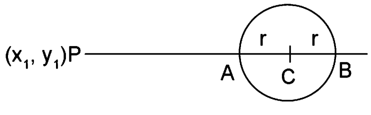
(ii) on the circle if $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

(iii) inside the circle if $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$

NOTE

In equation of circle which passes through origin, $c = 0$ i.e., constant term is zero.

2. The Least and Greatest Distance of a Point from a Circle



The diameter through point $P(x_1, y_1)$ cuts the circle at point A and B, then the point P from the circle is at the

(i) Least distance $= PA = PC - CA$

(ii) greatest distance $= PB = PC + CB$

3. Parametric Equation of a Circle

The number of variables is reduced by the parameteric equations and both co-ordiantes of a point are expressed in terms of one variable only.

(i) For the circle $x^2 + y^2 = a^2$, parametric equation are $x = a \cos\theta, y = a \sin\theta$, Hence for all values of θ, the point θ ($a \cos\theta$, $a \sin\theta$) lies on the circle $x^2 + y^2 = a^2$.

(ii) For the circle $(x - h)^2 + (y - k)^2 = a^2$ parametric equations are $x = h + a \cos\theta$; $y = k + a \sin\theta$

Hence for all values of θ, the point θ ($h + a \cos\theta$, $k + a \sin\theta$) is on the circle $(x - h)^2 + (y - k)^2 = a^2$

(iii) For the general equation of circle $x^2 + y^2 + 2gx + 2fy + c = 0$, Parametric equations are

$$x = -g + \sqrt{g^2 + f^2 - c}\cos\theta;$$

$$y = -f + \sqrt{g^2 + f^2 - c}\sin\theta.$$

4. Circle Passing Through Three Given Points

The equation of circle passing through three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

NOTES

1. The circle passing through three points can be obtained by Inverse test or negative test or verification method or substitution method.
2. Through three non-collinear points one and only one circle can be made to pass.
3. Condition that given four points be concylic
The given four points P, Q, R and S are satid to be concylic if the fourth point also lies on the circle passing through any three points of the given four points.

5. **Position of the Coordinate Axes with Respect to General Circle**
i.e., $x^2 + y^2 + 2gx + 2fy + c = 0$

(i) If Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ intersects x-axis in two real points if $g^2 > c$ and length of x-intercepts

$$= 2\sqrt{g^2 - c}\,.$$

(ii) Circle $x^2 + y^2 + 2gx + 2fy + c = 0$, touches x-axis if $g^2 = c$

(iii) The circle, intersects x-axis in two imaginary points i.e., circle and x-axis are separate if $g^2 < c$.

(iv) (a) Circle $x^2 + y^2 + 2gx + 2fy + c = 0$, intersects Y-axis in two real points $f^2 > c$ and the length of its y-intercept is

$$= 2\sqrt{f^2 - c}$$

(b) Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ touches y-axis if $f^2 = c$.

(c) Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and y-axis will be separate i.e., intersect in imaginary point if $f^2 < c$.

6. **Position of the straight line $y = mx + c$ with respect to the circle $x^2 + y^2 = a^2$**

6.1 If the line $y = mx + c$ is a chord of circle $x^2 + y^2 = a^2$ (i.e., the line meets the circle in two real and distinct points) then $c^2 < a^2(1 + m^2)$ and length of chord
= length of intercept on line by the circle

$$= 2\sqrt{a^2 - \frac{c^2}{1+m^2}} = 2\sqrt{(\text{radius})^2 - \begin{pmatrix}\text{Perpendicular distance} \\ \text{from centre to the chord}\end{pmatrix}^2}$$

NOTES

1. If the line $y = mx + c$ is a chord of circle $x^2 + y^2 = a^2$ then the coordinates of mid-point of chord $\left(\frac{-cm}{1+m^2}, \frac{c}{1+m^2}\right)$.
2. The co-ordinates of mid-point of the chord of the circle $x^2 + y^2 = a^2$ be (h, k) then the equation of chord is $hx + ky = h^2 + k^2$

6.2 The line $y = mx + c$ is a tangent to circle $x^2 + y^2 = a^2$ (i.e., the length of perpendicular from the centre to the line is equal to its radius) then $c = \pm a\sqrt{1+m^2}$ and the co-ordinate of point of contact is $\left(\frac{-cm}{1+m^2}, \frac{c}{1+m^2}\right)$

NOTE

For all values of m the straight line $y = mx \pm a\sqrt{1+m^2}$ is a tangent of the circle $x^2 + y^2 = a^2$.

6.3 The line $y = mx + c$ intersects the circle in two imaginary points i.e., both are separate if $c^2 > a^2(1 + m^2)$

NOTE

A line will intersect, touch or neither intersect nor touch the circle according as the length of perpendicular from centre to the line is less than, equal to, or more than the radius.

7. **Equation of Tangent**

(i) Equation of the tangent to the circle $x^2 + y^2 = a^2$ at (x_1, y_1) is $xx_1 + yy_1 = a^2$

(ii) The parametric equation of the tangent to the circle $x^2 + y^2 = a^2$ at a point $(a \cos\theta, a \sin\theta)$ is $x\cos\theta + y\sin\theta = a$.

(iii) Area of the triangle enclosed by the tangent to the circle $x^2 + y^2 = a^2$ at point θ $(a\cos\theta, a\sin\theta)$ and coordinate axes is $= \frac{a^2}{\sin 2\theta}$

(iv) If the line $lx + my + n = 0$ is a tangent to the circle $x^2 + y^2 = a^2$ then condition is $n^2 = a^2(l^2 + m^2)$ and co-ordinates of point of contact is $\left(-l\frac{a^2}{n}, -m\frac{a^2}{n}\right)$

(v) Equation of tangent at point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

NOTES

1. The equation of tangent at point (0, 0) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $gx + fy = 0$
2. The equation of tangent of slope m to the circle $x^2 + y^2 = a^2$ is $y = mx \pm a\sqrt{1+m^2}$ and the coordinate of the point of contact are
$\left(-\frac{am}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$ or $\left(\frac{am}{\sqrt{1+m^2}}, \frac{-a}{\sqrt{1+m^2}}\right)$
3. The length of the perpendicular from centre of the circle to the tangent is equal to the radius of circle.

7.1 The number of tangents drawn from any point to the circle is
(a) zero if the point is inside the circle
(b) one if the point is on the circle
(c) Two if the point is outside the circle

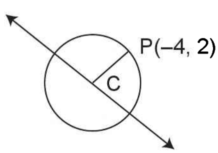
SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find the equation of circle of lowest size which passes through the point (–4, 2) and whose centre lies on the line $3x + 2y = 5$. ***[MP-1997]***

Solution

The length of perpendicular from the point (–4, 2) to the line $3x + 2y = 5$ must be equal to radius of circle of lowest size.

$\therefore$ Radius of circle $= \frac{-3(-4) - 2(2) + 5}{\sqrt{(-3)^2 + (-2)^2}}$



$= \frac{12 - 4 + 5}{\sqrt{9+4}} = \frac{13}{\sqrt{13}} = \sqrt{13}$

Given line is $3x + 2y = 5$ .... (1)

Equation of any line perpendicular to line (1) is

$2x - 3y + \lambda = 0$

It passes through (–4, 2)

$\therefore$ $2(-4) - 3(2) + \lambda = 0$

$\Rightarrow$ $\lambda = 14$

$\therefore$ Equation of perpendicular line is

$2x - 3y + 14 = 0$ .... (2)

Solving equations (1) and (2), we get $x = -1$ and $y = 4$.

Hence, centre of circle is (–1, 4).

Therefore, required equation is

$(x + 1)^2 + (y - 4)^2 = (\sqrt{13})^2$

$\Rightarrow$ $x^2 + 2x + 1 + y^2 + 16 - 8y = 13$

$\Rightarrow$ $x^2 + y^2 + 2x - 8y + 4 = 0$

2. Find the co-ordinates of the points of intersection of the line $5x - y + 2 = 0$ and the circle $x^2 + y^2 - 13x - 4y - 9 = 0$ and also find the length of the intercepted chord.

Solution

Equation of line is $5x - y + 2 = 0$ .... (1)

Equation of circle is $x^2 + y^2 - 13x - 4y - 9 = 0$ .... (2)

Putting the value of y in equation (2) from equation (1),

we get $x^2 + (5x + 2)^2 - 13x - 4(5x + 2) - 9 = 0$

$\Rightarrow$ $26x^2 - 13x - 13 = 0$

$\Rightarrow$ $2x^2 - x - 1 = 0$

$\Rightarrow$ $(x - 1)(2x + 1) = 0$

$\therefore$ $x = 1$ or $x = -\frac{1}{2}$

When $x = 1, y = 5(1) + 2 = 7$

When $x = -\frac{1}{2}$,

$y = 5\left(-\frac{1}{2}\right) + 2 = -\frac{1}{2}$

Coordinates of points of intersection are $(1, 7)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

Length of intercepted chord

$$= \sqrt{\left(1+\frac{1}{2}\right)^2 + \left(7+\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{9}{4}+\frac{225}{4}} = \sqrt{\frac{234}{4}} = \sqrt{\frac{117}{2}}$$

3. If the circle $x^2 + y^2 = a^2$ cuts off an intercept of length $2b$ from the line $y = mx + c$ then prove that $c^2 = (1 + m^2)(a^2 - b^2)$.

Solution

Here, centre and radius of the circle $x^2 + y^2 = a^2$ are (0, 0) and $OP = a$ respectively.

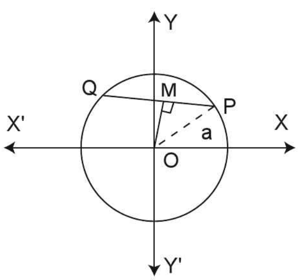
Draw perpendicular OM on chord PQ, then by geometry $PM = QM$

and $PQ = 2PM$

Now, length of the perpendicular OM on the line

$y = mx + c$ (i.e., PQ)

$$= \frac{c}{\sqrt{1+m^2}}$$



∴ Chord $PQ = 2PM = 2\sqrt{OP^2 - OM^2}$

$$= 2\sqrt{a^2 - \frac{c^2}{1+m^2}}$$

According to question, $PQ = 2b$

$$\Rightarrow \quad 2\sqrt{a^2 - \frac{c^2}{1+m^2}} = 2b$$

$$\Rightarrow \quad a^2 - \frac{c^2}{1+m^2} = b^2$$

$$\Rightarrow \quad -\frac{c^2}{1+m^2} = b^2 - a^2$$

$$\Rightarrow \quad \frac{c^2}{1+m^2} = a^2 - b^2$$

$\Rightarrow \quad c^2 = (1 + m^2)(a^2 - b^2)$ **Proved.**

4. If a line passes through $(-\sqrt{8}, \sqrt{8})$ and makes an angle 135° with X-axis. It meets the circle $x = 5\cos\theta$, $y = 5\sin\theta$ at the points A and B, then find the length of AB.

Solution

Parametric equation of the circle be $x = 5\cos\theta$, $y = 5\sin\theta$

∴ Its Cartesian equation be

$x^2 + y^2 = 25$ ....(1)

Equation of the line passing through $(-\sqrt{8}, \sqrt{8})$ and making an angle 135° with X-axis be

$(y - \sqrt{8}) = \tan 135° \,(x + \sqrt{8})$

$\Rightarrow \quad y - \sqrt{8} = -(x + \sqrt{8})$ ....(2)

$\Rightarrow \quad y = -x$

Putting $y = -x$ in equation (1), we get

$x^2 + x^2 = 25$

$\Rightarrow \quad 2x^2 = 25$

$$\Rightarrow \quad x^2 = \frac{25}{2}$$

$$\Rightarrow \quad x = \pm\frac{5}{\sqrt{2}}$$

$$\Rightarrow \quad y = \pm\frac{5}{\sqrt{2}}$$

∴ From equation (2).

Therefore, intersection of line and circle be

$$A\left(\frac{5}{\sqrt{2}}, -\frac{5}{\sqrt{2}}\right)$$

and $B\left(-\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right)$

∴ Length of the chord

$$= \sqrt{\left(\frac{5}{\sqrt{2}}+\frac{5}{\sqrt{2}}\right)^2 + \left(-\frac{5}{\sqrt{2}}-\frac{5}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\left(\frac{10}{\sqrt{2}}\right)^2 + \left(-\frac{10}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{100}{2}+\frac{100}{2}}$$

$$= \sqrt{100} = 10.$$

5. Find the parametric representation of the circle $3x^2 + 3y^2 + 4x - 6y - 4 = 0$

Solution

The given equation is

$3x^2 + 3y^2 + 4x - 6y - 4 = 0$

$\Rightarrow \quad x^2 + y^2 + \frac{4}{3}x - 2y - \frac{4}{3} = 0$

$\Rightarrow \quad x^2 + \frac{4}{3}x + y^2 - 2y = \frac{4}{3}$

$\Rightarrow \quad x^2 + \frac{4}{3}x + \left(\frac{2}{3}\right)^2 + y^2 - 2y + 1^2$

$= \frac{4}{3} + \left(\frac{2}{3}\right)^2 + 1^2$

$\Rightarrow \quad \left(x + \frac{2}{3}\right)^2 + (y-1)^2 = \frac{25}{9}$

$\Rightarrow \quad \left(x - \left(-\frac{2}{3}\right)\right)^2 + (y-1)^2 = \left(\frac{5}{3}\right)^2$

which is of the form $(x - h)^2 + (y - k)^2 = r^2$,

where, $h = -\frac{2}{3}, k = 1, r = \frac{5}{3}$

$\therefore$ Parametric equations of the given circle are $x = -\frac{2}{3} + \frac{5}{3}\cos\theta,\ y = 1 + \frac{5}{3}\sin\theta$, $0 \le \theta < 2\pi$.

$(x = h + r\cos\theta, y = k + r\sin\theta)$

6. Find the cartesian equation of the curve whose parametric equations are

$$x = \frac{2at}{1+t^2}, y = \frac{a(1-t^2)}{1+t^2}, t \in R, a > 0$$

Solution

Given equations are $x = \dfrac{2at}{1+t^2}$ .... (1)

$y = \dfrac{a(1-t^2)}{1+t^2}$ .... (2)

$\Rightarrow$ put $t = \tan\theta$ to get

$x = a\sin 2\theta, y = a\cos 2\theta$

Squaring and adding these equations, we get

$x^2 + y^2 = a^2(\sin^2 2\theta + \cos^2 2\theta)$

$\Rightarrow \quad x^2 + y^2 = a^2$

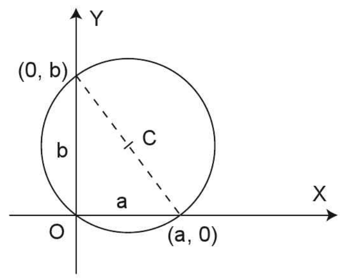
7. Find the equation of the circle passing through (0, 0) and making intercepts a and b on the coordinate axes.

[NCERT]

Solution

As the circle passes through the origin and makes intercepts a and b on x-axis and y-axis respectively, therefore, the points $A(a, 0)$ and $B(0, b)$ also lie on the circle in reference.

As $\angle AOB = 90°$, therefore, $[AB]$ is a diameter of the circle. Using diameter form, the equation of the circle is $(x - a)(x - 0) + (y - 0)(y - b) = 0$ | Diameter form



or $\quad x^2 + y^2 - ax - by = 0$

8. Find the co-ordinates of the points of intersection of the line $2x + y - 1 = 0$ and the circle $x^2 + y^2 + 6x - 4y + 8 = 0$ and show that given line is a tangent to the circle.

Solution

Equation of line is $2x + y - 1 = 0$

$\Rightarrow \quad y = 1 - 2x$ .... (1)

Equation of circle is $x^2 + y^2 + 6x - 4y + 8 = 0$ .... (2)

Putting the value of y in equation (2) from equation (1), we get

$x^2 + (1 - 2x)^2 + 6x - 4(1 - 2x) + 8 = 0$

$\Rightarrow \quad x^2 + 1 - 4x + 4x^2 + 6x - 4 + 8x + 8 = 0$

$\Rightarrow \quad 5x^2 + 10x + 5 = 0$

$\Rightarrow \quad x^2 + 2x + 1 = 0$

$\Rightarrow \quad (x + 1)^2 = 0$

$\Rightarrow \quad (x + 1) . (x + 1) = 0$

$\therefore \quad x = -1, -1$

$\therefore \quad y = 1 - 2(-1) = 1 + 2 = 3$

$\therefore$ Points of intersection are (–1, 3) and (–1, 3)

Hence points of intersection are coincident i.e., one point. Therefore by definition of tangency, given line $2x + y - 1 = 0$ is tangent to the circle $x^2 + y^2 + 6x - 4y + 8 = 0$.

Proved.

9. Find the value of p if the line $x \cos\alpha + y \sin\alpha = p$ touches the circle $x^2 + y^2 - 12x = 0$.

Solution

Given circle is, $x^2 + y^2 - 12x = 0$

Here, $2g = -12$, $2f = 0$ and $c = 0$

$\Rightarrow$ $g = -6, f = 0$ and $c = 0$

$\therefore$ Centre of the circle $= (-g, -f) = (6, 0)$ and

Radius of the circle $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{36 + 0 - 0} = 6$

Since the line $x \cos\alpha + y \sin\alpha = p$ touches the circle, therefore length of the perpendicular drawn from centre (6, 0) on this line is equal to radius 6.

$\therefore$ $\dfrac{6\cos\alpha + 0 - p}{\sqrt{\cos^2\alpha + \sin^2\alpha}} = \pm 6$

$\Rightarrow$ $6\cos\alpha - p = \pm 6$

$\Rightarrow$ $p = 6\cos\alpha \pm 6$

$\Rightarrow$ $= 6(\cos\alpha - 1), 6(\cos\alpha + 1)$

$\Rightarrow$ $p = 6(\cos\alpha + 1)$,

($\because$ p is always positive)

$= 6\left(2\cos^2\dfrac{\alpha}{2} - 1 + 1\right)$

$\Rightarrow$ $p = 12\cos^2\dfrac{\alpha}{2}$

10. Point (3, –1) is centre of a circle. This circle cuts off a chord of length 6 units on the line $2x + 5y + 18 = 0$. Find the equation of the circle.

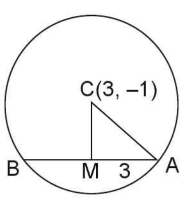
Solution

Given, centre of the circle is (3, –1) and equation of the chord AB is $2x + 5y + 18 = 0$ .... (1)

Draw a perpendicular CM on the line AB,

then $MA = \dfrac{1}{2}AB = \dfrac{1}{2} \times 6 = 3$

Now, CM = Length of the perpendicular drawn from $C(3, -1)$ on the line (1).



$= \dfrac{2\times 3 + 5\times(-1) + 18}{\sqrt{4+25}} = \dfrac{6 - 5 + 18}{\sqrt{29}} = \dfrac{19}{\sqrt{29}}$

$\therefore$ Radius of the circle. $CA = \sqrt{CM^2 + MA^2}$

$= \sqrt{\left(\dfrac{19}{\sqrt{29}}\right)^2 + 3^2} = \sqrt{\dfrac{361}{29} + 9}$

$= \sqrt{\dfrac{361 + 261}{29}}$

$= \sqrt{\dfrac{622}{29}}$

$\therefore$ Equation of the circle is $(x - 3)^2 + (y + 1)^2 = \dfrac{622}{29}$

$\Rightarrow$ $29(x^2 - 6x + 9 + y^2 + 2y + 1) = 622$

$\Rightarrow$ $29x^2 + 29y^2 - 174x + 58y + 290 - 622 = 0$

$\Rightarrow$ $29x^2 + 29y^2 - 174x + 58y - 332 = 0$

11. Find the tangents to the circle $x^2 + y^2 = a^2$ which make the triangle with axes of area a^2.

Solution

Any tangent to the circle $x^2 + y^2 = a^2$ be

$y = mx \pm a\sqrt{1+m^2}$ .... (1)

This line cuts the X-axis at A. when $y = 0$

$\therefore$ $0 = mx \pm a\sqrt{1+m^2}$

$\Rightarrow$ $x = \mp\dfrac{a\sqrt{1+m^2}}{m}$

Again, tangent (1) cuts the Y-axis at B, where $x = 0$

$\therefore$ $y = \pm a\sqrt{1+m^2}$

$\Rightarrow$ $OB = \pm a\sqrt{1+m^2}$

$\therefore$ Area of the triangle formed by axes and tangent

$= \dfrac{1}{2}OA.OB = \dfrac{1}{2}\left(\mp\dfrac{a\sqrt{1+m^2}}{m}\right).\left(\pm a\sqrt{1+m^2}\right)$

$= \mp\dfrac{1}{2}a^2.\dfrac{1+m^2}{m}$

According to question, $\pm\dfrac{a^2(1+m^2)}{2m} = a^2$

$\Rightarrow$ $1 + m^2 = \pm 2m$

$\Rightarrow$ $1 + m^2 \mp 2m = 0$

$\Rightarrow \quad (1 \mp m)^2 = 0$

$\Rightarrow \quad m = \pm 1$

Putting this value of m in equation (1),

$$y = \pm x \pm a\sqrt{1+1}$$

$\Rightarrow \quad y = \pm x \pm a\sqrt{2}$

Which are the equation of tangents.

12. If the line $3x + 4y - 1 = 0$ touches the circle $(x - 1)^2 + (y - 2)^2 = r^2$, then find the value of r.

Solution

Given line is $3x + 4y - 1 = 0$ .... (1)

Given circle is $(x - 1)^2 + (y - 2)^2 = r^2$ ..... (2)

Its centre is (1, 2) and radius is 2.

If line (1) touches the circle (2), then length of the perpendicular drawn from centre (1, 2) to the line (1) is equal to radius r.

i.e., $r = \dfrac{3\times1+4\times2-1}{\sqrt{9+16}} = \dfrac{3+8-1}{\sqrt{25}} = \dfrac{10}{5} = 2$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

EXERCISE 1

1. If the equations of the two diameters of a circle are $x - y = 5$ and $2x + y = 4$ and the radius of the circle is 5, find the equation of the circle.

2. If $y = 2x$ is a chord of the circle $x^2 + y^2 - 10x = 0$, find the equaion of a circle with this chord as diameter.

3. Find the lengths of the intercepts of the circle $x^2 + y^2 - 5x - 13y - 14 = 0$ on the axes of co-ordinates.

4. Find the co-ordinates of the points of intersection of the line $2x + y - 1 = 0$ and the circle $x^2 + y^2 + 6x - 4y + 8 = 0$ and show that given line is a tangent to the circle.

5. Find the value of k so that striaght line $4x + 3y + k = 0$ be a tangent to the circle $x^2 + y^2 = 4$.

6. Find the equation of the tangent to the circle $x^2 + y^2 = 2$ at (1, 1).

7. Find the equation of the tangents to the circle $x^2 + y^2 - 12x + 4y - 16 = 0$ which are perpendicular to the line $4x - 3y + 8 = 0$.

8. Find the equation of the circle which passes through the points (1, – 2) and (4, – 3) and has its centre on the line $3x + 4y = 7$.

EXERCISE 2

1. Find the equation of the circle that passes through the point (1, 0), (–1 , 0) and (0, 1).

2. Show that the points (9, 1), (7, 9) (–2, 12) and (6, 10) are concyclic.

3. To find the position of the points
   (i) (–4, 2),
   (ii) origin, with respect to the circle $2x^2 + 2y^2 + 16x - 8y - 41 = 0$.

4. Find the equation of the circle whose centre is on the line $y - x + 1 = 0$ and passing through the point (7, 3) and its radius is 3.

5. Find the equations of the tangent to the circle $x^2 + y^2 = 25$ at points whose abscissa is 4.

6. The equation of the tangents at point (α, β) to the circle $x^2 + y^2 = r^2$ cuts the axes at points A and B. Show that the area of triangle OAB is $\dfrac{1}{2}\dfrac{r^4}{\alpha\beta}$, where O is origin.

7. Show that the line $y = x + a\sqrt{2}$ touches the circles $x^2 + y^2 = a^2$ at the point $\left(-\dfrac{a}{\sqrt{2}}, \dfrac{a}{\sqrt{2}}\right)$.

8. Find the equation of tangents to the circle $x^2 + y^2 = 25$, which make an angle 30° with the axis of X.

9. Find out the equation of tangents to the circle $x^2 + y^2 = 4$ which are parallel to the straight line $x + 2y + 3 = 0$.

ANSWERS

EXERCISE 1

1. $x^2 + y^2 - 6x + 4y - 12 = 0$
2. $x^2 + y^2 - 2x - 4y = 0$
3. 15.
4. (–1, 3) & (-1, 3)
5. $k = \pm 10$
6. $x + y = 2$.
7. $3x+4y-5\sqrt{68}-10=0$
8. $15(x^2 + y^2) - 94x + 18y + 55 = 0$

EXERCISE 2

1. $x^2 + y^2 = 1$
3. (i) and (ii) point inside the circle.
4. $x^2 + y^2 - 14x - 12y + 76 = 0$.
5. $4x - 3y = 25$.
8. $\sqrt{3}y = x \pm 10$
9. $x+2y\pm 2\sqrt{5}=0$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the center. Then the locus of the centroid of the Δ PAB as P moves on the circle.

[IIT-2001]

(a) A parabola
(b) A circle
(c) An ellipse
(d) A pair of straight lines

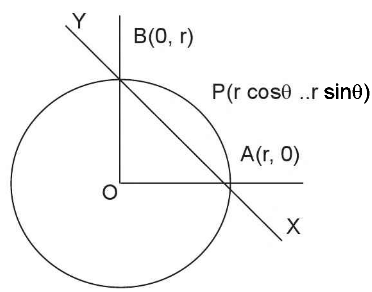
Solution

(b) Let the centroid = (α,β) Then

$$\alpha = \frac{r + r\cos\theta}{3}, \beta = \frac{r + r\sin\theta}{3}$$

or $$\left(\alpha - \frac{r}{3}\right)^2 + \left(\beta - \frac{r}{3}\right)^2 = \frac{r^2}{9}$$

∴ The locus is $\left(x - \frac{r}{3}\right)^2 + \left(y - \frac{r}{3}\right)^2 = \left(\frac{r}{3}\right)^2$, which is a circle.



2. If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the y-axis, then the length of PQ is

[IIT-2002]

(a) 4
(b) $2\sqrt{5}$
(c) 5
(d) $3\sqrt{5}$

Solution

(c) Let $P = (x_1, y_1)$. The tangent at P is

$xx_1 + yy_1 + 3(x + x_1) + 3(y + y_1) - 2 = 0$ ....(i)

Coordinates of Q satisfy (i), $5x - 2y + 6 = 0$, $x = 0$.

So, $3x_1 + 6y_1 + 7 = 0$ and $Q = (0, 3)$

$\therefore \quad PQ^2 = x_1^2 + (y_1 - 3)^2 = x_1^2 + y_1^2 - 6y_1 + 9$

$= 11 - 6x_1 - 12y_1$,

$(\because \; x_1^2 + y_1^2 + 6x_1 + 6y_1 - 2 = 0)$

$= 11 - 2(3x_1 + 6y_1) = 11 - 2(-7) = 25.$

So, $PQ = 5$.

3. Under which one of the following conditions does the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ meet the x-axis in two points on opposite sides of the origin? ***[NDA-2007]***

(a) $c > 0$ (b) $c < 0$
(c) $c = 0$ (d) $c \le 0$

Solution

(b) The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ meets x-axis ($y = 0$) in two points on opposite sides of origin.

If mean $x^2 + 2gx + c = 0$

$\Rightarrow \quad x = \dfrac{-2g \pm \sqrt{4g^2 - 4c}}{2}$

$\Rightarrow \quad x = -g \pm \sqrt{g^2 - c}$

Circle meets the x-axis in two points on opposite side of origin.

Hence, $-g + \sqrt{g^2 - c} > 0$

$-g - \sqrt{g^2 - c} < 0$

$\Rightarrow \quad \sqrt{g^2 - c} > g \Rightarrow g^2 - c > g^2$

$\Rightarrow \quad -c > 0 \; \therefore c < 0$

4. If points (0, 0); (1, 0); (0, 2) and (0, a) are concyclic, then a is equal to ***[PET (Raj.)-97]***

(a) 0, 1 (b) 0, 2
(c) − 1, 2 (d) None of these

Solution

(b) Obviously (0, 0); (1, 0) and (0, 2) are vertices of a right angled triangle, so equation of the circle passing through these points is

$(x - 1)x + y(y - 2) = 0$

$\Rightarrow \quad x^2 + y^2 - x - 2y = 0$

Since this also passes through (0, a),

so $\quad a^2 - 2a = 0 \Rightarrow a = 0, 2$

5. The side of an equilateral triangle drawn in the circle $x^2 + y^2 = a^2$ is ***[UPSEAT-99]***

(a) $\sqrt{2a}$
(b) $\sqrt{3a}$
(c) $\sqrt{3/2a}$
(d) None of these

Solution

(b) Using cosine formula in ΔAOB

$$\cos 120° = \frac{a^2 + a^2 - AB^2}{2a.a}$$

$\Rightarrow AB = \sqrt{3}a$

6. Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then which of the following equations can represent L_1 ***[IIT-1999]***

(a) $x + y = 0$
(b) $x - y = 0$
(c) $x + 7y = 0$
(d) $x - 7y = 0$

Solution

(b, c) Let the equation of line passing through origin be $y = mx$. Therefore

We know that chord of equal length are equidistant from centre.

$\therefore$ Distance of both chord are equal centre

$$\frac{\dfrac{1-3m}{1+m^2}}{\sqrt{\left(\dfrac{1-3m}{1+m^2}\right)^2 + m^2\left(\dfrac{1-3m}{1+m^2}\right)^2}}$$

Only solving we get

$\Rightarrow \quad 7m^2 - 6m - 1 = 0$

$\Rightarrow \quad (7m + 1)(m - 1) = 0$

$\Rightarrow \quad m = 1, -\dfrac{1}{7}$

$\therefore \quad$ Lines are $y = x, y = -\dfrac{1}{7}x$

or $\quad x - y = 0$,

$x + 7y = 0$.

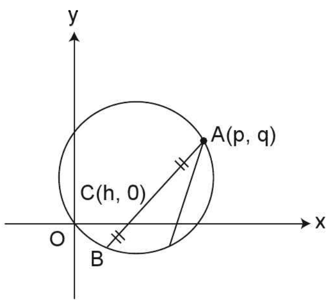
7. If two distinct chord, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where, $pq \neq 0$) are bisected by the x-axis, then

[IIT-99; 2M]

(a) $p^2 = q^2$
(b) $p^2 = 8q^2$
(c) $p^2 < 8q^2$
(d) $p^2 > 8q^2$

Solution

(d) From the equation of circle it is clear that circle passes through origin. Let AB is chord of the circle.



$A \equiv (p, q)$. C is mid-point and Coordinate of C is $(h, 0)$

Then co-ordinates of B are $(-p + 2h, -q)$.

and B lies on the circle $x^2 + y^2 = px + qy$, we have

$$(-p + 2h)^2 + (-q)^2 = p(-p + 2h) + q(-q)$$

$\Rightarrow \quad p^2 + 4h^2 - 4ph + q^2 = -p^2 + 2ph - q^2$

$\Rightarrow \quad 2p^2 + 2q^2 - 6ph + 4h^2 = 0$

$\Rightarrow \quad 2h^2 - 3ph + p^2 + q^2 = 0 \quad \ldots (1)$

There are given two distinct chords which are bisected at x-axis then, there will be two distinct values of h satisfying (1). So discriminant of this quadratic equation must be > 0.

$\Rightarrow \quad D > 0 \Rightarrow (-3p)^2 - 4.2\,(p^2 + q^2) > 0$

$\Rightarrow \quad 9p^2 - 8p^2 - 8q^2 > 0$

$\Rightarrow \quad p^2 - 8q^2 > 0$

$\Rightarrow \quad p^2 > 8q^2$ Therefore, (d) is the answer.

8. Consider, L_1: $2x + 3y + p - 3 = 0$,

[IIT-JEE-2008]

L_2: $2x + 3y + p + 3 = 0$

where p is a real number, and

C: $x^2 + y^2 + 6x - 10y + 30 = 0$

Statement 1: If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C.

Statement 2: If line L_1 is a diameter of circle C, then line L_2 is not a chord of circle C.

Solution

(c) The circle is $(x + 3)^2 + (y - 5)^2 = 4$

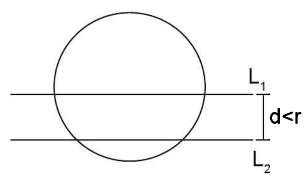
The distance between L_1: $2x + 3y + p - 3 = 0$ and

$L_2 = 2x + 3y + p + 3 = 0$ is

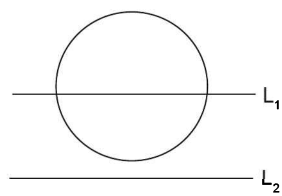
$$d = \frac{6}{\sqrt{2^2 + 3^2}} = \frac{6}{\sqrt{13}} <$$

the radius of the circle (= 2)

Statement 2 is false. If L_1 is a diameter then L_2 has to intersect the circle in two distinct points and then it will be a chord.



Statement 1 is true. If L_1 is a chord, L_2 may lie outside the circle and so need not always be a diameter.



OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The equation of the chord of the circle $x^2 + y^2 = a^2$ having (x_1, y_1) as its mid-points is ***[IIT-83; PET-86; CET-03]***
   (a) $xy_1 + yx_1 = a^2$
   (b) $x_1 + y_1 = a$
   (c) $xx_1 + yy_1 = x_1^2 + y_1^2$
   (d) $xx_1 + yy_1 = a^2$

2. *ABCD* is a square the length, of whose side is *a*. Taking *AB* and *AD* as the coordinate axes, the equation of the circle passing through the vertices of the square is
   (a) $x^2 + y^2 + ax + ay = 0$
   (b) $x^2 + y^2 - ax - ay = 0$
   (c) $x^2 + y^2 + 2ax + 2ay = 0$
   (d) $x^2 + y^2 - 2ax - 2ay = 0$

3. The locus of the middle points of those chords of the circle $x^2 + y^2 = 4$ which subtend a right angle at the origin is ***[IIT-84; PET-97]***
   (a) $x^2 + y^2 - 2x - 2y = 0$
   (b) $x^2 + y^2 = 4$
   (c) $x^2 + y^2 = 2$
   (d) $(x - 1)^2 + (y - 1)^2 = 5$

4. $y = mx$ is a chord of a circle of radius a and the diameter of the circle lies among *x*-axis and one end of this chord is origin. The equation of the circle described on this chord as diameter is ***[PET-90]***
   (a) $(1 + m^2)(x^2 + y^2) - 2ax = 0$
   (b) $(1 + m^2)(x^2 + y^2) - 2a(x + my) = 0$
   (c) $(1 + m^2)(x^2 + y^2) + 2a(x + my) = 0$
   (d) $(1 + m^2)(x^2 + y^2) - 2a(x - my) = 0$

5. A point inside the circle $x^2 + y^2 + 3x - 3y + 2 = 0$ is ***[PET-88]***
   (a) $(-1, 3)$ (b) $(-2, 1)$
   (c) $(2, 1)$ (d) $(-3, 2)$

6. Position of the point (1, 1) with respect to the circle $x^2 + y^2 - 2x + y - 1 = 0$ is ***[PET-86, 90]***
   (a) outside the circle (b) Upon the cirlce
   (c) Inside the circle (d) None of these

7. The equation to the tangents to the circle $x^2 + y^2 - 6x + 4y = 12$ which are parallel to the straight line $4x + 3y + 5 = 0$, are ***[ISM Dhanbad-73; PET-91]***
   (a) $3x - 4y - 19 = 0, 3x - 4y + 31 = 0$
   (b) $4x + 3y - 19 = 0, 4x + 3y + 31 = 0$
   (c) $4x + 3y + 19 = 0, 4x + 3y - 31 = 0$
   (d) $3x - 4y + 19 = 0, 3x - 4y + 31 = 0$

8. The straight line $x - y - 3 = 0$ touches the circle $x^2 + y^2 - 4x + 6y + 11 = 0$ at the point whose co-ordinates are ***[PET-93]***
   (a) $(1, -2)$ (b) $(1, 2)$
   (c) $(-1, 2)$ (d) $(-1, -2)$

9. At which point the line $y = x + \sqrt{2}$ a touches to the circle $x^2 + y^2 = a^2$ or Line $y = x + a\sqrt{2}$ is a tangent to the circle $x^2 + y^2 = a^2$ at
   (a) $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$ (b) $\left(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$
   (c) $\left(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$ (d) $\left(-\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$

10. The centre of the circle $x = -1 + 2\cos\theta, y = 3 + 2\sin\theta$ is ***[PET-1995]***
    (a) $(1, -3)$ (b) $(-1, 3)$
    (c) $(1, 3)$ (d) None of these

11. $x = 7$ touches the circle $x^2 + y^2 - 4x - 6y - 12 = 0$, then the coordinates of the point of contact are
    (a) (7, 3) (b) (7, 4)
    (c) (7, 8) (d) (7, 2)

12. The equation of circle with centre (1, 2) and tangent $x + y - 5 = 0$ is ***[PET-01]***
    (a) $x^2 + y^2 + 2x - 4y + 6 = 0$
    (b) $x^2 + y^2 - 2x - 4y + 3 = 0$
    (c) $x^2 + y^2 - 2x + 4y + 8 = 0$
    (d) $x^2 + y^2 - 2x - 4y + 86 = 0$

13. The equation to the tangents to the circle $x^2 + y^2 = 4$, which are parallel to $x + 2y + 3 = 0$ are ***[PET-2003]***
    (a) $x - 2y = 2$ (b) $x + 2y = \pm 2\sqrt{3}$
    (c) $x + 2y = \pm 2\sqrt{5}$ (d) $x - 2y = \pm 2\sqrt{5}$

14. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB, equation of the circle on AB as a diameter is

[AIEEE-04]

(a) $x^2 + y^2 + x - y = 0$
(b) $x^2 + y^2 - x + y = 0$
(c) $x^2 + y^2 + x + y = 0$
(d) $x^2 + y^2 - x - y = 0$

15. If the chord $y = mx + 1$ of the circle $x^2 + y^2 = 1$ subtends an angle of measure 45° at the measure segment of the circle then value of m is ***[AIEEE-2002]***

(a) 2 (b) –2
(c) –1 (d) None of these.

16. The radius of any circle touching the lines $3x - 4y + 5 = 0$ and $6x - 8y - 9 = 0$ is

[MPPET-2005]

(a) 1.9 (b) 0.95
(c) 2.9 (d) 1.45

17. The value of c, for which the line $y = 2x + c$ is a tangent to the circle $x^2 + y^2 = 16$ is

[MPPET-2004]

(a) $-16\sqrt{5}$ (b) $4\sqrt{5}$
(c) $16\sqrt{5}$ (d) 20

18. The locus of mid-point of the chords of the circle $x^2 + y^2 - 2x - 2y - 2 = 0$ which makes an angle of 120° at the centre is

[MNR-1994]

(a) $x^2 + y^2 - 2x - 2y - 1 = 0$
(b) $x^2 + y^2 + x + y - 1 = 0$
(c) $x^2 + y^2 - 2x - 2y - 1 = 0$
(d) None of these

19. The length of the chord joining the points in which the straight line $\frac{x}{3} + \frac{y}{4} = 1$ cuts the circle $x^2 + y^2 = \frac{169}{25}$ is

[Orissa JEE-2003]

(a) 1 (b) 2
(c) 4 (d) 8

20. If the line $3x - 2y = k$ meets the circle $x^2 + y^2 = 4r^2$ only at one point, then k is equal to

[CET (Karnataka)-2003]

(a) $52\ r^2$ (b) $20\ r^2$
(c) $(52/9)\ r^2$ (d) $(20/9)\ r^2$

21. Circle passing through the points $(t, 1)$, $(1, t)$ and (t, t) for all values of t passes through the point

[Orissa JEE-2007]

(a) (0, 0) (b) (1, 1)
(c) (1, –1) (d) (–1, 1)

22. If the equation of the tangent to the circle $x^2 + y^2 - 2x + 6y - 6 = 0$ parallel to $3x - 4y + 7 = 0$ is $3x - 4y + k = 0$, then the values of k are

[Kerala (Engg.)-05]

(a) 5, –35 (b) –5, 35
(c) 7, –32 (d) –7, 32

23. If the straight line $y = mx$ is outside the circle $x^2 + y^2 - 20y + 90 = 0$, then

[Roorkee-1999]

(a) $m > 3$ (b) $m < 3$
(c) $|m| > 3$ (d) $|m| < 3$

24. The number of distinct tangents that can be drawn from the origin to the circle $x^2 + y^2 = 2(x + y)$ is

[ICS-90]

(a) 0 (b) 1
(c) 2 (d) 3

25. Consider a family of circles which are passing through the point (–1, 1) and are tangent to x-axis. If (h, k) are the co-ordinates of the centre of the circles, then the set of values of k is given by the interval

[AIEEE-2007]

(a $0 < k < \frac{1}{2}$ (b) $k \geq \frac{1}{2}$

(c) $-\frac{1}{2} \leq k \leq \frac{1}{2}$ (d) $k \leq \frac{1}{2}$

26. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if

[AIEEE-2010]

(a) $-35 < m < 15$
(b) $15 < m < 65$
(c) $35 < m < 85$
(d) $-85 < m < -35$

SOLUTIONS

1. (c) $T = S_1$ is the equation of desired chord, hence $xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$
$\Rightarrow \quad xx_1 + yy_1 = x_1^2 + y_1^2$

2. (b) According to the figure $A(0, 0)$, $B(a, 0)$ and $D(0, a)$ and centre is $\left(\frac{a}{2}, \frac{a}{2}\right)$. Therefore, the equation of circle is $\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{2}$
$\Rightarrow \quad x^2 + y^2 - ax - ay = 0.$

3. (c) Let the mid-point of chord is (h, k). Also radius of circle is 2. Therefore, $\frac{oc}{ob} = \cos 45°$
$\Rightarrow \quad \frac{\sqrt{h^2 + k^2}}{2} = \frac{1}{\sqrt{2}}$
$\Rightarrow \quad h^2 + k^2 = 2$
Hence locus is $x^2 + y^2 = 2$

4. (b) Here the equation of circle is $(x - a)^2 + (y - 0)^2 = a^2$
$\Rightarrow \quad x^2 + y^2 - 2ax = 0$
Now the point of intersection of circle and chord i.e., O and B are $0(0, 0)$ and $B\left(\frac{2a}{1+m^2}, \frac{2am}{1+m^2}\right)$
Hence the equation of circle are chord OB's diameter is $(x^2 + y^2)(1 + m)^2 - 2a(x + my) = 0$.

NOTE

It may be used as formula.

5. (b) For point $(-2, 1)$, $S_1 < 0$

6. (a) $S_1 > 0$, Hence point lies outside the circle.

7. (c) Let equation of tangent be $4x + 3y + k = 0$,
then $\sqrt{9+4+12} = \left|\frac{4(3) + 3(-2) + k}{\sqrt{16+9}}\right|$
$\Rightarrow \quad 6 + k = \pm 25$
$\Rightarrow \quad k = 19$ and -31
Hence the tangents are $4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$.

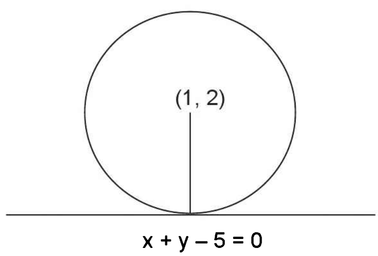
8. (a) The point must satisfy the line and the circle simultaneously. Obviously the point of contact is $(1, -2)$.

9. (d) Suppose that point be (h, k). Tangent at (h, k) is $hx + ky = a^2 \equiv x - y = -\sqrt{2}a$
or $\quad \frac{h}{1} = \frac{k}{-1} = \frac{a^2}{-\sqrt{2}a}$
or $\quad h = -\frac{a}{\sqrt{2}}, k = \frac{a}{\sqrt{2}}.$

10. (b) Given that $\frac{x+1}{2} = \cos\theta$.
Also $\frac{y-3}{2} = \sin\theta$
$\Rightarrow \quad \left(\frac{x+1}{2}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$
$\Rightarrow \quad (x+1)^2 + (y-3)^2 = 4.$

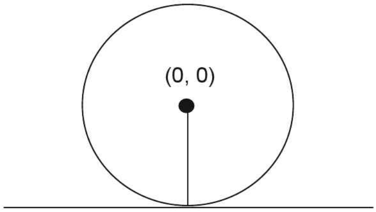
11. (a) Putting $x = 7$, we get $y^2 - 6y + 9 = 0$
$\Rightarrow \quad y = 3$. Hence the point of contact is $(7, 3)$.

12. (b) $\because$ Radius of circle = Perpendicular distance of tangent from the centre of circle.



$\Rightarrow \quad r = \frac{1+2-5}{\sqrt{1+1}} = \sqrt{2}$
Hence the equation of required circle is
$(x - 1)^2 + (y - 2)^2 = (\sqrt{2})^2$
$\Rightarrow \quad x^2 + y^2 - 2x - 4y + 3 = 0$
OR
Verification Method: Centre of circles in options are as follows
(a) $(-1, 2)$ (b) $(1, 2)$
(c) $(1, -2)$ (d) $(-1, -2)$
So (b) can be the only answer

13. (c)



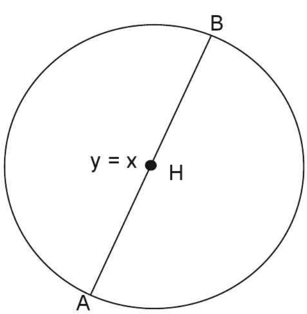
Centre of $x^2 + y^2 = 4$ is (0, 0).
Tangents which are parallel to $x + 2y + 3 = 0$ is
$x + 2y + \lambda = 0$ ....(i)
Perpendicular distance from (0, 0) to $x + 2y + \lambda = 0$ should be equal to radius of circle, (Clearly radius = 2).

$$\therefore \quad \frac{0+2\times 0+\lambda}{\sqrt{1^2+2^2}} = \pm 2$$

$\Rightarrow \quad \lambda = \pm 2\sqrt{5}$ Put the value of λ in (i), tangents of circle are $x + 2y = \pm 2\sqrt{5}$

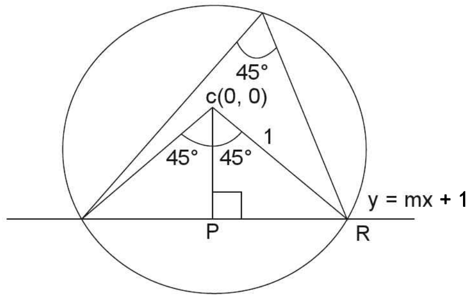
OR
Only (b) and (c) lines are parallel to $x + 2y + 3 = 0$.
Also the line is a tangent to $x^2 + y^2 = 4$.
Its distance from (0, 0) should be 2.
Therefore, (c) is the answer.

14. (d) Given, circle is $x^2 + y^2 - 2x = 0$ .... (i),
and line is $y = x$ ..... (ii)
Putting $y = x$ in (i), We get $2x^2 - 2x = 0$ $\Rightarrow x = 0, 1$
From (i), $y = 0, 1$
Let $A = (0, 0)$, $B = (1, 1)$
Equation of Required circle is $(x - 0)(x - 1) + (y - 0)(y - 1) = 0$
or $\quad x^2 + y^2 - x - y = 0$.



15. (c) Given circle is $x^2 + y^2 = 1$
$C(0, 0)$ and radius = 1 and chord is $y = mx + 1$

$$\cos 45° = \frac{CP}{CR}$$



CP = Perpendicular distance from (0, 0) to chord $y = mx + 1$

$$CP = \frac{1}{\sqrt{m^2+1}} \quad (CR = \text{radius} = 1)$$

$$\cos 45° = \frac{1/\sqrt{m^2+1}}{1} \Rightarrow \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{m^2+1}}$$

$m^2 + 1 = 2 \Rightarrow m = \pm 1$.

16. (b) Distance between the lines $3x - 4y + 5 = 0$ and $6x - 8y - 9 = 0$ is $\left|\frac{5+\frac{9}{2}}{\sqrt{3^2+4^2}}\right| = \left(\frac{19}{10}\right) = 1.9$

1.9 is the diameter of circle

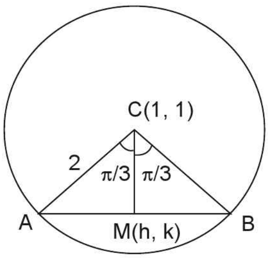
$\therefore \quad$ Radius of circle $= \frac{1.9}{2} = 0.95$

17. (b) Given that $y = 2x + c$ ....(i)
and $x^2 + y^2 = 16$ ....(ii)
We know that $y = mx + c$ is tangent to the circle $x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1+m^2}$
$m = 2, a = 4$

$\therefore \quad c = \pm 4\sqrt{1+2^2} = \pm 4\sqrt{5}$

18. (a) The centre of given circle is (1, 1) and its radius is $\sqrt{2}$ From the figure, if $M(h, k)$ be the middle point of chord AB subtending an angle $\frac{2\pi}{3}$ at C, then $\frac{CM}{AC} = \cos\frac{\pi}{3} = \frac{1}{2}$

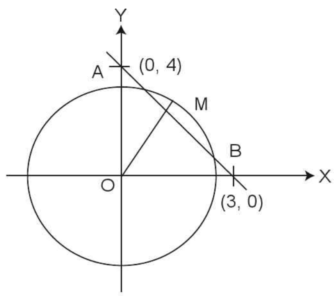
$\Rightarrow \quad 4CM^2 = AC^2$
or $\quad 4\,[(h-1)^2 + (k-1)^2\,] = 4$



$\Rightarrow \quad h^2 + k^2 - 2h - 2k + 2 = 1$

Hence, the locus is $x^2 + y^2 - 2x - 2y + 1 = 0$.

19. (b) OM = length of the perpendicular to the line $4x + 3y = 12$ from $(0, 0) = 12/5$



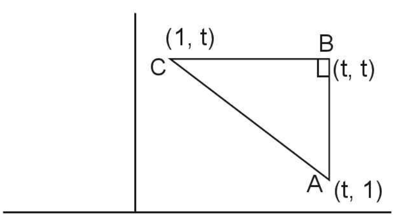
Radius of the circle is $\frac{13}{5}$.

Required length $= 2\sqrt{\frac{169}{25} - \frac{144}{25}} = 2$

20. Line is a tangent to the circle, so

$$p = a \Rightarrow \frac{k}{\sqrt{13}} = 2r$$

21.



3 points form right angled triangle
AC is diameter of circumcircle
equation of circle is
$(x - t)(x - 1) + (y - 1)(y - t) = 0$
which clearly pairs through $(1, 1)$

22. (a) Equation of circle is, $x^2 + y^2 - 2x + 6y - 6 = 0$

$$(x - 1)^2 + (y + 3)^2 = (4)^2$$

Radius of circle = 4

And centre of circle = $(1, -3)$

Equation of tangent $3x - 4y + k = 0$

$$\therefore \quad \frac{3 \times 1 - 4 \times (-3) + k = 0}{\sqrt{(3)^2 + (-4)^2}} = \pm 4 .$$

Hence $k = 5, -35$

23. (d) If the straight line $y = mx$ is outside the given circle then distance from centre of circle > radius of circle $\frac{10}{\sqrt{1+m^2}} > \sqrt{10}$

$\Rightarrow \quad (1 + m^2) < 10 \Rightarrow m^2 < 9$

$\Rightarrow \quad |m| < 3.$

24. **(b)** Origin lies on the circle, so number of distinct tangents is 1.

25. (b) $k^2 = (h + 1)^2 + (k - 1)^2$

$\Rightarrow \quad 2k = h^2 + 2h + 2$

$\Rightarrow \quad 2k = (h + 1)^2 + 1$

$\Rightarrow \quad 2k \geq 1$

$\Rightarrow \quad k \geq 1/2.$

26. (a) $r = \sqrt{4 + 16 + 5} = 5$

$$\left|\frac{6 - 16 - m}{5}\right| = 5$$

$\Rightarrow \quad -25 < m + 10 < 25$

$\Rightarrow \quad -35 < m < 15$

Hence correct option is (a)

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If $3x + y + k = 0$ is a tangent to the circle $x^2 + y^2 = 10$, the values of k are ***[Karnataka CET-2007]***

(a) ± 7
(b) ± 5
(c) ± 10
(d) ± 9

2. What is the radius of the circle passing through the points (0, 0), (a, 0) and (0, b)? ***[NDA-2006]***

(a) $\sqrt{a^2 - b^2}$ (b) $\sqrt{a^2 + b^2}$
(c) $\frac{1}{2}\sqrt{a^2 + b^2}$ (d) $2\sqrt{a^2 + b^2}$

3. If $2x - 4y = 9$ and $6x - 12y + 7 = 0$ are common tangents to the circle, then radius of the circle is ***[DCE-2005]***

(a) $\frac{\sqrt{3}}{5}$ (b) $\frac{17}{6\sqrt{5}}$
(c) $\frac{\sqrt{2}}{3}$ (d) $\frac{17}{3\sqrt{5}}$

4. A point inside the circle $x^2 + y^2 + 3x - 3y + 2 = 0$ is ***[PET-88]***

(a) $(-1, 3)$ (b) $(-2, 1)$
(c) $(2, 1)$ (d) $(-3, 2)$

5. The point of intersection of the circles $x^2 + y^2 = 25$ and $x^2 + y^2 - 8x + 7 = 0$ are ***[PET-88]***

(a) (4, 3) and (4, –3)
(b) (4, –3) and (–4, –3)
(c) (–4, 3) and (4, 3)
(d) (4, 3) and (3, 4)

6. The length of tangent from the point (5, 1) to the circle $x^2 + y^2 + 6x - 4y - 3 = 0$, is

(a) 81 (b) 29
(c) 7 (d) 21

7. The equation of the circle having centre (1, –2) and passing through the point of intersection of lines $3x + y = 14$, $2x + 5y = 18$ is ***[PET-90]***

(a) $x^2 + y^2 - 2x + 4y - 20 = 0$
(b) $x^2 + y^2 - 2x - 4y - 20 = 0$
(c) $x^2 + y^2 + 2x - 4y - 20 = 0$
(d) $x^2 + y^2 + 2x + 4y - 20 = 0$

8. The equation to the tangents to the circle $x^2 + y^2 = 4$, which are parallel to $x + 2y + 3 = 0$ are ***[PET-2003]***

(a) $x - 2y = 2$ (b) $x - 2y = \pm 2\sqrt{3}$
(c) $x + 2y = \pm 2\sqrt{5}$ (d) $x - 2y = \pm 2\sqrt{5}$

9. If the line $y = \sqrt{3}x + k$ touches the circle $x^2 + y^2 = 16$, then k is equal to

(a) 0 (b) 2
(c) 4 (d) 8

10. If the circles $x^2 + y^2 + 2ax + 8y + 16 = 0$ touches x-axis, then the value of a is

(a) ± 16 (b) ± 4
(c) ± 8 (d) ± 1

11. A circle touches the y-axis at the point (0, 4) and cuts the x-axis in a chord of length 6 units. The radius of the circle is ***[PET-92]***

(a) 3 (b) 4
(c) 5 (d) 6

12. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then the radius of the circle is ***[IIT-84, PET-94]***

(a) 3/2 (b) 3/4
(c) 1/10 (d) 1/20

13. Equation of the tangent to the circle $x^2 + y^2 = a^2$ which is perpendicular to the straight line $y = mx + c$ is

(a) $y = -\frac{x}{m} \pm a\sqrt{1 + m^2}$
(b) $x + my = \pm a\sqrt{1 + m^2}$
(c) $x + my = \pm a\sqrt{1 + (1/m)^2}$
(d) $x - my = \pm a\sqrt{1 + m^2}$

14. A circle touches the axes at the point (3, 0) and (0, – 3). The centre of the circle is ***[PET-92]***

(a) (3, – 3) (b) (0, 0)
(c) (–3, 0) (d) (6, – 6)

15. The line $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 square units. The equation of the circle is **[IIT-89; Kerala Engg.-2005; AIEEE-2003]**
(a) $x^2 + y^2 + 2x - 2y = 62$
(b) $x^2 + y^2 - 2x + 2y = 47$
(c) $x^2 + y^2 + 2x - 2y = 47$
(d) $x^2 + y^2 - 2x + 2y = 62$

16. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along diameters of a circle of circumference 10π, then the equation of the circle is **[AIEEE-2004]**
(a) $x^2 + y^2 + 2x - 2y - 23 = 0$
(b) $x^2 + y^2 - 2x - 2y - 23 = 0$
(c) $x^2 + y^2 + 2x + 2y - 23 = 0$
(d) $x^2 + y^2 - 2x + 2y - 23 = 0$

17. The number of tangents that can be drawn from (0, 0) to the circle $x^2 + y^2 + 2x + 6y - 15 = 0$ is **[MP-92]**
(a) 0 (b) 1
(c) 2 (d) infinite

18. The lines $2x + 3y + 6 = 0$ and $3x - 2y - 18 = 0$ are tangents resepectively at the points (0, – 2) and (6, 0) on a circle. Then the centre of the circle is **[Kerala PET-2008]**
(a) $\left(\frac{36}{13}, -28\right)$ (b) $\left(\frac{-72}{13}, 28\right)$
(c) $\left(\frac{74}{13}, \frac{-28}{13}\right)$ (d) $\left(\frac{36}{13}, \frac{28}{13}\right)$

19. The points of contact of the circle $x^2 + y^2 + 2x + 2y + 1 = 0$ and the co-ordinate axes are
(a) (1, 0), (0, 1) (b) (–1, 0), (0, 1)
(c) (–1, 0), (0, –1) (d) (1, 0), (0, –1)

20. If the straight line $y = mx + c$ touches the circle $x^2 + y^2 - 2x - 4y + 3 = 0$ at the point (2, 3), then c is equal to
(a) –3 (b) 4
(c) 5 (d) –2

21. Middle point of the chord of the circle $x^2 + y^2 = 25$ intercepted on the line $x - 2y = 2$ is
(a) $\left(\frac{3}{5}, \frac{4}{5}\right)$ (b) (–2, –2)
(c) $\left(\frac{2}{5}, -\frac{4}{5}\right)$ (d) $\left(\frac{8}{3}, \frac{1}{3}\right)$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 21 minutes.
3. The worksheet consists of 21 questions. The maximum marks are 63.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. Position of the point (1, 1) with respect to the circle $x^2 + y^2 - x + y - 1 = 0$ is **[MPPET-86, 90]**

(a) Outside the circle (b) Upon the circle
(c) Inside the circle (d) None of these

2. Length of the tangent from (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is **[EAMCET-80]**

(a) $\left(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c\right)^{1/2}$
(b) $\left(x_1^2 + y_1^2\right)^{1/2}$
(c) $[(x_1 + g)^2 + (y_1 + f)^2]^{1/2}$
(d) None of these

3. Circle $ax^2 + ay^2 + 2gx + 2fy + c = 0$ touches x-axis if

(a) $f^2 > ac$ (b) $g^2 > ac$
(c) $f^2 = bc$ (d) $g^2 = ac$

4. The geometric means of the abscissa of the points of intersection of the circle $x^2 + y^2 - 4x - 6y + 7 = 0$ and the line $y = 1$ is **[SCRA-2007]**

(a) $\sqrt{7}$ (b) $\sqrt{2}$
(c) $\sqrt{14}$ (d) 1

5. The locus of the mid-point of the portion of any tangent to the circle $x^2 + y^2 = a^2$ intercepted between the co-ordinate axes is **[SCRA-2007]**

(a) $\frac{1}{x^2} - \frac{1}{y^2} = \frac{1}{a^2}$ (b) $\frac{1}{x^2} - \frac{1}{y^2} = \frac{4}{a^2}$
(c) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{a^2}$ (d) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{a^2}$

6. Consider a circle of radius R. What is the length of a chord which subtends an angle θ at the centre? **[NDA-2007]**

(a) $2R \sin(\theta/2)$ (b) $2R \sin\theta$
(c) $2R \tan(\theta/2)$ (d) $2R \tan\theta$

7. The equation of two circles which touch the Y-axis at (0, 3) and make an intercept of 8 units on X-axis are **[Karnataka CET-2007]**

(a) $x^2 + y^2 \pm 10x - 6y + 9 = 0$
(b) $x^2 + y^2 \pm 6x - 10y + 9 = 0$
(c) $x^2 + y^2 - 8x \pm 10y + 9 = 0$
(d) $x^2 + y^2 + 10x \pm 6y + 9 = 0$

8. If $x^2 + y^2 - 2cx - 2cy + c^2 = 0$ represents a circle touching coordinate axes and $\frac{x}{3} + \frac{y}{4} = 1$ and its centre is in 1st quadrant then value of C is **[MP PET-2007]**

(a) 1 (b) 2
(c) 3 (d) 6

9. $2x - y - 2 = 0$ is a line and $x^2 + y^2 - 6x + 2y + 5 = 0$ and $5x^2 + 5y^2 = 4$ are two circles. The line touches **[UPSC (Rly)-98]**

(a) first circle but not second
(b) second circle but not first
(c) both circles
(d) None of these

10. The point diametrically opposite to the point $P(1, 0)$ on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ is **[AIEEE-2008]**

(a) (3, 4) (b) (3, – 4)
(c) (–3, 4) (d) (–3, – 4)

11. If (3, – 2) is the centre of a circle and $4x + 3y + 19 = 0$ is a tangent to the circle, then the equation of the circle is **[Kerala PET-2008]**

(a) $x^2 + y^2 - 6x + 4y + 25 = 0$
(b) $x^2 + y^2 - 6x + 4y + 12 = 0$
(c) $x^2 + y^2 - 6x + 4y - 12 = 0$
(d) $x^2 + y^2 - 6x + 4y + 13 = 0$

12. The equation of the smallest circle passing through the points (2, 2) and (3, 3) is ***[Karnataka CET-2008]***

(a) $x^2 + y^2 + 5x - 5y + 12 = 0$
(b) $x^2 + y^2 - 5x + 5y - 12 = 0$
(c) $x^2 + y^2 + 5x + 5y + 12 = 0$
(d) $x^2 + y^2 - 5x - 5y + 12 = 0$

13. The point (5, – 7) lies outside the circle ***[Karnataka CET-2008]***

(a) $x^2 + y^2 - 5x + 7y - 1 = 0$
(b) $x^2 + y^2 - 8x + 7y - 2 = 0$
(c) $x^2 + y^2 - 8x = 0$
(d) $x^2 + y^2 - 5x + 7y = 0$

14. Point (1, 2) relative to the circle $x^2 + y^2 + 4x - 2y - 4 = 0$ is which one of the following? ***[NDA-2008]***

(a) Exterior point
(b) Interior point, but not centre
(c) Boundary point
(d) Centre

15. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, $(c > 0)$ touches the y-axis, then which one of the following is correct?

(a) $g = -\sqrt{c}$ only (b) $g = \pm\sqrt{c}$
(c) $f = \sqrt{c}$ only (d) $f = \pm\sqrt{c}$

[NDA-2008]

16. The lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to the same circle then radius of this circle is......... ***[Orissa JEE-2008]***

(a) 1/2 (b) 3/4
(c) 4/3 (d) 5/4

17. If the centre of a circle is (–6, 8) and it passes through the origin, then equation to its tangent at the origin, is ***[MNR-76]***

(a) $2y = x$ (b) $4y = 3x$
(c) $3y = 4x$ (d) $3x + 4y = 0$

18. If the line $3x + 4y - 1 = 0$ touches the circle $(x - 1)^2 + (y - 2)^2 = r^2$, then the value of r will be ***[RPET-86]***

(a) 2 (b) 5
(c) 12/5 (d) 2/5

19. The angle between the tangents drawn at the points (5, 12) and (12, – 5) to the circle $x^2 + y^2 = 169$ is ***[MPPET-2009]***

(a) 45° (b) 60°
(c) 30° (d) 90°

20. The line $ax + by + c = 0$ is normal to the circle $x^2 + y^2 + 2gx + 2fy + d = 0$, if ***[MPPET-2009]***

(a) $ag + bf + c = 0$ (b) $ag + bf - c = 0$
(c) $ag - bf + c = 0$ (d) $ag - bf - c = 0$

21. If the lines $3x - 4y - 7 = 0$ and $2x - 3y - 5 = 0$ are two diameters of a circle whose area is 49π sq. units, then the equation of the circle is ***[MPPET-2009]***

(a) $x^2 + y^2 + 2x - 2y - 47 = 0$
(b) $x^2 + y^2 - 2x + 2y - 47 = 0$
(c) $x^2 + y^2 + 2x - 2y - 51 = 0$
(d) $x^2 + y^2 - 2x + 2y - 51 = 0$

ANSWER SHEET

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HINTS AND EXPLANATIONS

4. (b) For abscissa of points of intersection. Put $y = 1$ in circle

$x^2 + y^2 - 4x - 6y + 7 = 0$

$\Rightarrow \quad x^2 + 1^2 - 4x - 6 \times 1 + 7 = 0$

$\Rightarrow \quad x^2 - 4x + 2 = 0$; let roots are x_1, x_2

$\therefore \quad$ G.M. of x_1 and $x_2 = \sqrt{x_1 x_2} = \sqrt{2}$

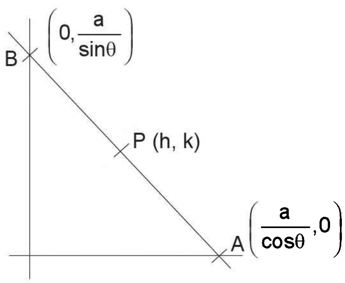
5. (d) Let equation of tangent be

$x \cos\theta + y \sin\theta = a$

or $\quad \dfrac{x}{\dfrac{a}{\cos\theta}} + \dfrac{y}{\dfrac{a}{\sin\theta}} = 1$

$P(h, k)$ = mid-point of AB

$$(h,k) = \left(\frac{a}{2\cos\theta}, \frac{a}{2\sin\theta}\right)$$

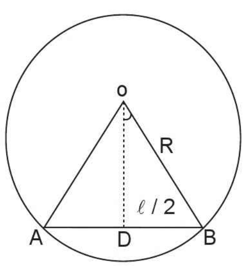


$$\cos\theta = \frac{a}{2h},\ \sin\theta = \frac{a}{2k}$$

Squaring and adding

$$\left(\frac{a}{2h}\right)^2 + \left(\frac{a}{2k}\right)^2 = 1 \quad \frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{a^2}$$

6. (a) $\sin\dfrac{\theta}{2} = \dfrac{BD}{OB}$



$\Rightarrow \quad \ell = 2k\sin\dfrac{\theta}{2}$ ℓ = length of chord

12. (d) For smallest circle (2, 2) and (3, 3) must be end points of diameter.

$\therefore \quad$ Equation of circle $(x - 2)(x - 3) + (y - 2)(y - 3) = 0$

$x^2 + y^2 - 5x - 5y + 12 = 0$

16. (b) $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are parallel tangents

$\therefore \quad$ distance between them = Diameter = $2r$

$$2r\left|\frac{8-(-7)}{\sqrt{6^2+(-8)^2}}\right| = \frac{3}{2}\ r = \frac{3}{4}$$

8. (a, d) Centre (C, C);

radius $= \sqrt{C^2 + C^2 - C^2} = C$

$\dfrac{x}{3} + \dfrac{y}{4} = 1$ or $4x + 3y - 12 = 0$ is tangent

$\therefore \quad$ Distance from centre of the tangent = radius

$$\left|\frac{4C + 3C - 12}{\sqrt{4^2+3^2}}\right| = C$$

$\Rightarrow \quad 7C - 12 = 5C$ or $7C - 12 = -5C$

$\Rightarrow \quad C = 6$ or $C = 1$

19. (d) Since, both the points lie on the circle. At (5, 12), equation of tangent is

$5x + 12y = 169$ .... (i)

At (12, –5), equation of tangent is

$12x - 5y = 169$ .... (ii)

It is clear that equations (i) and (ii) are perpendicular to each other.

Hence, angle between them is 90°.

20. (b) The centre of given circle is $(-g, -f)$.

If the given line $ax + by + c = 0$ is normal to the circle, then it passes through the centre of circle.

$a(-g) + b(-f) + c = 0$

$\Rightarrow \quad ag + bf - c = 0$

21. (b) The intersection of given two lines is (1, –1) which is the centre of circle.

Also, $\quad \pi r^2 = 49\pi$

$\Rightarrow \quad r = 7$

$\therefore$ Equation of circle is

$(x - 1)^2 + (y + 1)^2 = 7^2$

$\Rightarrow \quad x^2 + y^2 - 2x + 2y - 47 = 0$

LECTURE 3

Circle 3

(Relative position of circles, pair of tangents and chord of contact draw from an enternal point)

BASIC CONCEPTS

1. Position of One Circle with Respect to Other

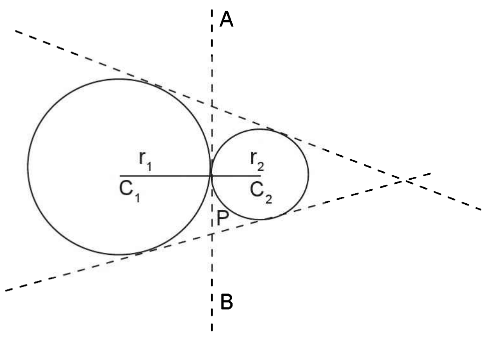
Let the equation of two circles be

$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ .... (1)

$S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ .... (2)

whose centre is c_1 and c_2 and radius is r_1 and r_2 respectively and $c_1c_2 = d =$ distance between their centres.

(i) **Two Circles Touch Externally** if the distance d between their centres is equal to the sum of their radius i.e., $c_1c_2 = d = r_1 + r_2$ and the number of common tangent lines is 3 and the point of contact P divides the line of centres c_1c_2 internally in the ratio $r_1 : r_2$



$C_1C_2 = r_1 + r_2$, $C_1C_2 = C_1P + C_2P$

$d = r_1 + r_2$

Equation of the common tangent AB is $S_1 - S_2 = 0$

Point of contact $P\left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$

Example

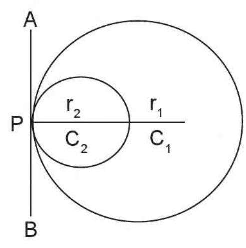
$S_1 = x^2 + y^2 - 2x - 4y - 4 = 0$

$S_2 = x^2 + y^2 + 4x + 4y + 4 = 0$

$c_1 = (1, 2); r_1 = 3, n = 3$ $c_1\,c_2 = d = 5$

$c_2 = (-2, -2), r_2 = 2$

(ii) **Two circles touch internally** if the distance d between their centres is equal to the difference of their radii, i.e., $c_1c_2 = d = r_1 - r_2$ and the number of common tangent line is one and point of contact P divides the line of centres c_1c_2 externally in the ratio $r_1 : r_2$



$P\left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$,

$r_1 \neq r_2$

$C_1C_2 = r_1 - r_2 = d$

NOTE

Equation of the common tangent AB is $S_1 - S_2 = 0$

Examples

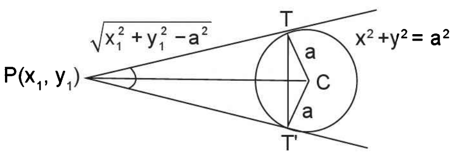
$S_1 = x^2 + y^2 - 4x - 6y - 87 = 0$
$S_2 = x^2 + y^2 + 4x - 21 = 0$
$c_1\ (2, 3), r_1 = 10, n = 1\ c_1c_2 = d = 5$
$c_2\ (-2, 0), r_2 = 5$

(iii) **Concentric circles** The circle having same centre but different radius are called concentric circles.

1. $x^2 + y^2 = r_1^2;\ x^2 + y^2 = r_2^2;\ r_1 \neq r_2$
2. $(x - a)^2 + (y - b)^2 = r_1^2;\ (x - a)^2 + (y - b)^2 = r_2^2$
3. $x^2 + y^2 + 2gx + 2fy + c_1 = 0$
   $x^2 + y^2 + 2gx + 2fy + c_2 = 0$

2. Pair of Tangents and Chord of Contact

2.1 (i) The length of tangent drawn to the circle $x^2 + y^2 = a^2$ from an external point (x_1, y_1) is $= \sqrt{x_1^2 + y_1^2 - a^2}$



(ii) The length of tangent drawn to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from external point (x_1, y_1) is $= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$ $= \sqrt{S_1}$, where S_1 is known as **power of point** (x_1, y_1)

(v) The equation of a pair of tangents drawn to the circle $x^2 + y^2 = a^2$ from an external point (x_1, y_1) is $SS_1 = T^2$ where $S = x^2 + y^2 - a^2$, $S_1 = x_1^2 + y_1^2 - a^2$ and $T = xx_1 + yy_1 - a^2$

(vi) The equation of a pair of tangents drawn to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from external point (x_1, y_1) is $SS_1 = T^2$ where, $S = x^2 + y^2 + 2gx + 2fy + c$, $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ and $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

2.2 If the angle between a pairs of tangents drawn from an external point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ be θ then $\tan\theta$

$$= \frac{2\times a\times\sqrt{x_1^2+y_1^2-a^2}}{(x_1^2+y_1^2-a^2)-a^2}$$

$$= \frac{2\times\text{radius}\times\text{length of tangent}}{(\text{length of tangent})^2-(\text{radius})^2}$$

If two tangents intersect at right angle, then condition is $x_1^2 + y_1^2 = 2a^2$ (Angle between two circles is = 90°).

3. Let $S_1 = 0$, $S_2 = 0$ be equation of circles and $L = 0$ is equation of a line, then

(a) $S_1 + \lambda S_2 = 0$ is equation of circle passing through point of intersection of S_1 and S_2 $(\lambda \neq -1)$

(b) $S_1 + \lambda L = 0$ is equation of circle passing through point of intersection of line L and circle S_1.

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Prove that the circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ will touch each other, if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$

[MP-1983, 89S; MNR-87]

Solution

Equation $x^2 + y^2 + 2ax + c = 0$ represents a circle whose centre is $C_1(-a, 0)$ and radius $r_1 = \sqrt{a^2 - c}$ .... (1)

Equation $x^2 + y^2 - 2by + c = 0$ represents a circle whose centre is $C_2\ (0, -b)$ and radius $r_2 = \sqrt{b^2 - c}$ .... (2)

If circle touch each other, then

Their distance between the centres = Sum of their radii

or $C_1C_2 = r_1 \pm r_2$

$$\sqrt{(-a-0)^2+(0+b)^2}$$
$$= \sqrt{a^2-c} \pm \sqrt{b^2-c}$$

Squaring both the sides, we get

$$a^2 + b^2 = a^2 - c + b^2 - c \pm 2\sqrt{(a^2 - c)(b^2 - c)}$$

$\Rightarrow \quad 2c = \pm 2\sqrt{(a^2 - c)(b^2 - c)}$

$\Rightarrow \quad 4c^2 = 4(a^2 - c)(b^2 - c)$

$\Rightarrow \quad c^2 = a^2b^2 - a^2c - b^2c + c^2$

$\Rightarrow \quad a^2b^2 = c(a^2 + b^2) \Rightarrow \frac{1}{c} = \frac{1}{a^2} + \frac{1}{b^2}$ **Proved**

2. Show that the circles $x^2 + y^2 - 14x - 10y + 58 = 0$ and $x^2 + y^2 - 2x + 6y - 26 = 0$ touch each other externally.

Solution

Given equation of the circles are

$x^2 + y^2 - 14x - 10y + 58 = 0$ .... (1)

and $x^2 + y^2 - 2x + 6y - 26 = 0$ .... (2)

Centre and radius of the circle (1) are respectively $(-g, -f)$ or $(7, 5)$ and $\sqrt{g^2 + f^2 - c}$

$= \sqrt{49 + 25 - 58} = \sqrt{16} = 4$

Centre of the circle (2) is $(-g, -f)$ or $(1, -3)$ and its radius $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{1 + 9 + 26} = \sqrt{36} = 6$

Now, distance between the centre of these two circles

$= \sqrt{(7-1)^2 + (5+3)^2} = \sqrt{36 + 64}$

$= \sqrt{100} = 10 = 4 + 6$

= Sum of their radius

Hence, circles touch each other externally.

3. Lines $x = 4$, $y = 2$, $x = -2$ and $y = -3$ represent sides of rectangle. Find the equation of a circle whose one diameter is diagonal of this rectangle.

Solution

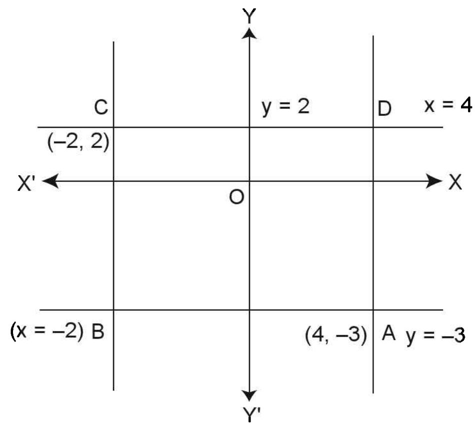
Let equation of the sides AB, BC, CD and DA of the rectangle are respectively

$y = -3$ ..... (1), $x = -2$ .... (2),

$y = 2$ ......(3) and $y = 4$ .... (4)

Intersection of line (1) and (4) is $A(4, -3)$

Intersection of line (2) and (3) is $C(-2, 2)$



$\therefore$ Ends of the diagonal AC are $A(4, -3)$ and $C(-2, 2)$

$\therefore$ Equation of the circle whose diameter is AC be

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$\Rightarrow \quad (x - 4)(x + 2) + (y + 3)(y - 2) = 0$

$\Rightarrow \quad x^2 - 2x + y^2 + y - 8 - 6 = 0$

$\Rightarrow \quad x^2 + y^2 - 2x + y - 14 = 0$

Again, co-ordinates of points B and D are respectively $(-2, -3,)$ and $(4, 2)$.

$\therefore$ Equation of the cirlce whose diameter is BD be

$(x + 2)(x - 4) + (y + 3)(y - 2) = 0$

$\Rightarrow \quad x^2 - 2x - 8 + y^2 + y - 6 = 0 \Rightarrow x^2 + y^2 - 2x + y - 14 = 0$

Therefore, equation of the circle whose two diameters are diagonal of rectangle be

$x^2 + y^2 - 2x + y - 14 = 0$

4. Find the equation of the circle in which the chord joining the points (a, b) and $(b, -a)$ subtends an angle of 45° at any point on the circumference of the circle.

Solution

Given points are $A(a, b)$ and $B(b, -a)$ and $P(h, k)$ is any point on the circumference of the circle.

Then, gradient of $PA = m_1 = \frac{k - b}{h - a}$

and gradient of $PB = m_2 = \frac{k + a}{h - b}$

$\therefore \quad \angle APB = 45° \quad \therefore \quad \tan 45° = \frac{m_1 - m_2}{1 + m_1 m_2}$

$$\Rightarrow \quad 1 = \frac{\frac{k-b}{h-a} - \frac{k+a}{h-b}}{1 + \frac{k-b}{h-a} \cdot \frac{k+a}{h-b}}$$

$$\Rightarrow \quad 1 = \frac{(k-b)(h-b)-(k+a)(h-a)}{(h-a)(h-b)+(k-b)(k+a)}$$

$\Rightarrow$ $(h - a)(h - b) + (k - b)(k + a)$
$= (k - b)(h - b) - (k + a)(h - a)$

$\Rightarrow$ $h^2 - h(a + b) + ab + k^2 + k(a - b) - ab$
$= (hk - bh - bk + b^2) - (hk + ah - ak - a^2)$

$\Rightarrow$ $h^2 + k^2 - h(a + b) + k(a - b)$
$= -h(a + b) + k(a - b) + a^2 + b^2$

$\Rightarrow$ $h^2 + k^2 = a^2 + b^2$

Thus, equation of the circle is locus of the point $P(h, k)$

$x^2 + y^2 = a^2 + b^2$.

5. Prove that the tangents to the circle $x^2 + y^2 = 5$ at point $(1, -2)$ also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$.

Solution

Given circle is $x^2 + y^2 = 5$

$\Rightarrow$ $x^2 + y^2 = (\sqrt{5})^2$ .... (1)

Its centre is $(0, 0)$ and radius is $\sqrt{5}$

$\therefore$ Equation of the tangent to the circle (1) at the point $(1, -2)$ is $x \times 1 + y \times (-2) = 5$

$\Rightarrow$ $x - 2y = 5$ .... (2)

Equation of the given second circle is

$x^2 + y^2 - 8x + 6y + 20 = 0$ .... (3)

Here, $2g = -8$, $2f = 6$ and $c = 20$

$\Rightarrow$ $g = -4, f = 3$ and $c = 20$

$\therefore$ Centre of this circle is $(-g, -f)$ or $(4, -3)$

and radius $= \sqrt{(4)^2 + (-3)^2 - 20}$

$= \sqrt{16 + 9 - 20} = \sqrt{5}$

If line (2) touches the circle (3), then length of the perpendicular drawn from its centre $(4, -3)$ to the line will equal to its radius $\sqrt{5}$.

Now, length of the perpendicular drawn from centre $(4, -3)$ to the line (2) is

$$= \frac{4 - 2 \times (-3) - 5}{\sqrt{1+4}} = \frac{4 + 6 - 5}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

= radius of the circle (3) **Proved.**

6. Abscissa of two points A and B are roots of the equation $x^2 + 2x - a^2 = 0$ and ordinate are roots of the equation $y^2 + 4y - b^2 = 0$. Find the equation of that circle whose diameter is AB. Find its centre and radius.

Solution

Let abscissa of the points A and B are x_1 and x_2 respectively.

$\because$ x_1 and x_2 are roots of the equation $x^2 + 2x - a^2 = 0$

$\therefore$ $x_1 + x_2 = -2$ and $x_1 x_2 = -a^2$ .... (1)

Again, let ordinate of the points A and B are y_1 and y_2 respectively.

$\because$ y_1 and y_2 are roots of the equation $y^2 + 4y - b^2 = 0$

$\therefore$ $y_1 + y_2 = -4$ and $y_1 y_2 = -b^2$ .... (2)

Thus, the points be $A(x_1, y_1)$ and $B(x_2, y_2)$

$\therefore$ Equation of the circle whose a diameter is AB be

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$\Rightarrow$ $x^2 - x(x_1 + x_2) + x_1 x_2 + y^2 - y(y_1 + y_2) + y_1 y_2 = 0$

$\Rightarrow$ $x^2 - (-2)x - a^2 + y^2 - (-4)y - b^2 = 0$

$\Rightarrow$ $x^2 + y^2 + 2x + 4y - (a^2 + b^2) = 0$

Here, $2g = 2$, $2f = 4$ and $c = -(a^2 + b^2)$

$\Rightarrow$ $g = 1, f = 2$ and $c = -(a^2 + b^2)$

$\therefore$ Centre $= (-g, -f)$ or $(-1, -2)$

and radius $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{1 + 4 + (a^2 + b^2)} = \sqrt{a^2 + b^2 + 5}$

7. Tangent to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ are drawn from the point $(4, 5)$ then find the area of the triangle formed by these tangents and two radii.

Solution

Given equation of the circle be $x^2 + y^2 - 4x - 2y - 11 = 0$

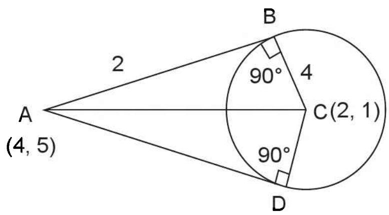
Here, $2g = -4$, $2f = -2$, $c = -11$

$\Rightarrow$ $g = -2, f = -1, c = -11$

$\therefore$ Centre of this circle $= (-g, -f) = (2, 1)$

and radius $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{4 + 1 + 11} = \sqrt{16} = 4$



Length of the tangent drawn from the point (4, 5) to the circle (1) be $AB = \sqrt{S_1}$

$= \sqrt{4^2 + 5^2 - 4\times4 - 2\times5 - 11}$

$= \sqrt{16+25-16-10-11} = \sqrt{4} = 2$

$\because \quad ar(\Delta ABC) = ar(\Delta ACD)$

$\therefore \quad ar(\Delta ABCD) = ar(\Delta ABC) + ar(\Delta ACD)$

$= 2 \times ar(\Delta ABC)$

$= 2 \times \frac{1}{2} \times AB \times BC = AB \times BC = 2 \times 4$

$= 8$ sq. units.

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

EXERCISE 1

1. Find the equation of the circle concentric with the circle $2x^2 + 2y^2 + 8x + 10y - 39 = 0$ and having its area equal to 16π square units.
2. Find the equation of a circle which passes through the points (2, 3) and (3, – 2) and the chord formed by these points makes the angle of 90° at the centre of circle.
3. Show that $x = a \cos \alpha + b \sin \alpha$, $y = a \sin \alpha - b \cos \alpha$ represents a circle for all values of α.
4. Tangents are drawn to the circle $x^2 + y^2 = 10$ from the point (4, – 2). Prove that they are perpendicular.
5. Find the length of tangents drawn from the point (2, 6) to the circle $x^2 + y^2 - 2x - 3y - 1 = 0$ and also find the power of the point.
6. Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ externally at the point (5, 5).
7. Find the equation of the circle whose centre is at the point (4, 5) and which passes through the centre of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$.

EXERCISE 2

1. Find the equation of the circle drawn on the intercept made by the line $2x + 3y = 6$ between the coordinate axes as diameter.
2. Find the equation of the circle which passes through the centre of the circle $x^2 + y^2 + 8x + 10y - 7 = 0$ and concentric with the circle $2x^2 + 2y^2 - 8x - 12y - 9 = 0$. 3
3. Find the equation of the circle circumscribing the triangle formed by the lines $x + y = 6$, $2x + y = 4$ and $x + 2y = 5$.
4. Find the length of tangent drawn from (2, 5) to the circle $x^2 + y^2 - 2x - 3y - 1 = 0$.
5. Find the equation of a circle of radius 5 which lies within the circle $x^2 + y^2 + 14x + 10y - 26 = 0$ and which touches the given circle at the point (–1, 3).
6. A circle of radius 2 lies in the first quadrant and touches both the axes. Find the equation of the circle with centre at (6, 5) and touching the above circle externally.

ANSWERS

EXERCISE 1

1. $4x^2 + 4y^2 + 16x + 20y - 23 = 0$
2. $x^2 + y^2 - 10x - 2y + 13 = 0$.
5. $\sqrt{17}; 17$.
6. $(x - 9)^2 + (y - 8)^2 = 5^2$
7. $x^2 + y^2 - 8x - 10y - 9 = 0$.

EXERCISE 2

1. $x^2 + y^2 - 3x - 2y = 0$
2. $x^2 + y^2 - 4x - 6y - 87 = 0$.
3. $x^2 + y^2 - 17x - 19y + 50 = 0$
4. $\sqrt{9} = 3$.
5. $(x + 4)^2 + (y + 1)^2 = 5^2$
6. $(x - 6)^2 + (y - 5)^2 = 3^2$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. Let PQ and RS be tangents at the extremeties of the diameter PR of a circle of radius r. If PS and RQ intersect at a point X on the circumference of the circle, then $2r$ equals **[IIT-01]**

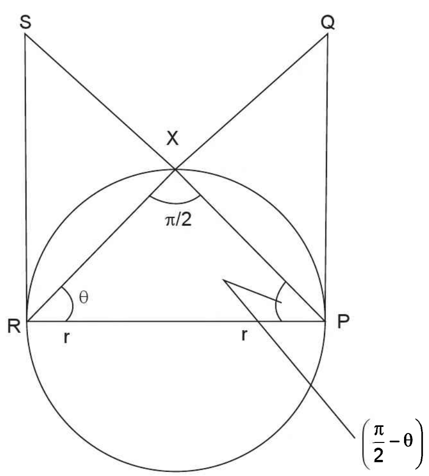
(a) $\sqrt{PQ.RS}$ (b) $\dfrac{PQ+RS}{2}$

(c) $\dfrac{2PQ.RS}{PQ+RS}$ (d) $\sqrt{\dfrac{PQ^2+RS^2}{2}}$

Solution

(a) $\tan\theta = \dfrac{PQ}{PR} = \dfrac{PQ}{2r}$ .....(i),

Also $\tan\theta\left(\dfrac{\pi}{2}-\theta\right) = \dfrac{RS}{2r}$ ......(ii)



i.e., $\cot\theta = \dfrac{RS}{2r}$. Multiplying (i) and (ii), side-wise we find

$\therefore \quad \therefore \tan\theta, \cot\theta = \dfrac{PQ.RS}{4r^2}$ i.e.,

$\Rightarrow \quad 4r^2 = PQ.RS \Rightarrow 2r = \sqrt{(PQ)(RS)}$.

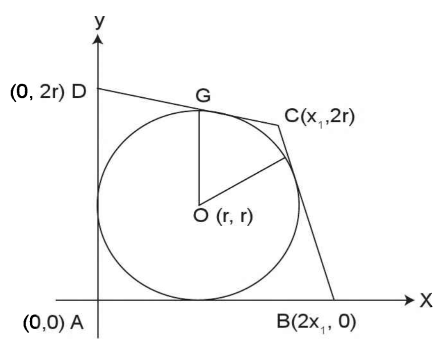
2. Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is

(a) 3 (b) 2
(c) 3/2 (d) 1

[IIT-2007]

Solution

(b)



According to figure.

$$\frac{1}{2}\times 3x_1 \times 2r = 18 \text{ or } x_1 \times r = 6 \quad ....(1)$$

Equation of BC is $y = -\frac{2r}{x_1}(x - 2x_1)$.

BC is tangent to circle $(x - r)^2 + (y - r)^2 = r^2$ Therefore, the perpendicular distance of BC from centre = radius.

$$\frac{\left|r + \frac{2r}{x_1}(r - 2x_1)\right|}{\sqrt{1 + \frac{4r^2}{x_1^2}}} = r$$

or $\quad \frac{2r^2}{x_1} - 3r = r\sqrt{1 + \frac{4r^2}{x_1^2}}$

or $\quad (2r - 3x_1)^2 = x_1^2 + 4r^2$

$$r.\,x_1 = \frac{2}{3}x_1^2 \text{ or } 3r = 2x_1 \quad ....(2)$$

From (1) and (2) $r = 2$ units.

3. Each question contains Statement-1 (Assertion) and Statement-2 (Reason). Of these statements, mark correct choice if:

(a) Statements-1 and 2 are true and Statement-2 is a correct explanation for statement-1

(b) Statements-1 and 2 are true and Statement-2 is not a correct explanation for statement-1

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true.

Tangents are drawn from the point (17, 7) to the circle $x^2 + y^2 = 169$.

[IIT-2007]

Statement 1: The tangents are mutually perpendicular because.

Statement 2: The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2 + y^2 = 338$.

Solution

(a) $\quad x^2 + y^2 = 169$ Equation of a pair of tangents from (h, k) to the given circle is $SS_1 = T^2$

$(hx + ky - 169)^2 = (h^2 + k^2 - 169)$

$(x^2 + y^2 - 169)$ or $h^2x^2 + k^2y^2 + (169)^2 + 2hxy - 338\,(hx + ky)$

$= x^2(h^2 + k^2 - 169) + y^2(h^2 + k^2 - 169) - 169\,(h^2 + k^2 - 169)$

$\therefore \quad x^2(k^2 - 169) + y^2(h^2 - 169) - 2hxy + 338\,(hx + ky) - 169\,(h^2 + k^2) = 0.$

For the pair of lines to be mutually perpendicular, coeff. of x^2 + coeff. $y^2 = 0$

or $\quad h^2 + k^2 - 338 = 0$

$\Rightarrow \quad h^2 + k^2 = 338,$

which is the equation of a director circle. Point (17, 7) satisfies the above equation.

Hence, statement 1 and 2 are true, statement 2 is a correct explanation of statement 1.

4. The locus of the centre of a circle of radius 2 which rolls on the outside of circle $x^2 + y^2 + 3x - 6y - 9 = 0$, is

(a) $x^2 + y^2 + 3x - 6y + 5 = 0$

(b) $x^2 + y^2 + 3x - 6y - 31 = 0$

(c) $x^2 + y^2 + 3x - 6y + \frac{29}{4} = 0$

(d) None of these

Solution

(b) Let (h, k) be the centre of the circle which rolls on the outside of the given circle. Centre of the given circle is $(-3/2, 3)$ and its radius $= \sqrt{\frac{9}{4} + 9 + 9} = \frac{9}{2}$.

Clearly, (h, k) is always at a distance equal to the sum $\left(\frac{9}{2} + 2\right) = \frac{13}{2}$ of the radii of two circle from $\left(-\frac{3}{2}, 3\right)$. Therefore,

$$\left(h + \frac{3}{2}\right)^2 + (k - 3)^2 = \left(\frac{13}{2}\right)^2$$

$\Rightarrow \quad h^2 + k^2 + 3h - 6k + \frac{9}{4} + 9 - \frac{169}{4} = 0$

$\Rightarrow \quad$ Hence locus of (h, k) is $x^2 + y^2 + 3x - 6y - 31 = 0$.

5 If OA and OB be the tangents to the circle $x^2 + y^2 - 6x - 8y + 21 = 0$ drawn from the origin O, then AB is equal to

(a) 11 (b) $\frac{4}{5}\sqrt{21}$

(c) $\sqrt{\frac{17}{3}}$ (d) None of these

Solution

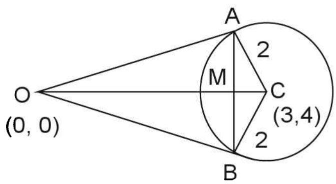
(b) Here the equation of AB (chord of contact) is

$0 + 0 - 3(x + 0) - 4(y + 0) + 21 = 0$

$\Rightarrow \quad 3x + 4y - 21 = 0$ .... (i)

CM = perpendicular distance from (3, 4) to line (i) is

$$\frac{3\times3+4\times4-21}{\sqrt{9+16}} = \frac{4}{5},$$



$$AM = \sqrt{AC^2 - CM^2}$$

$$= \sqrt{4-\frac{16}{25}} = \frac{2}{5}\sqrt{21}$$

$\therefore \quad AB = 2AM = \frac{4}{5}\sqrt{21}$.

6. The equation of the circle which passes through the intersection of $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ and whose centre lies on $13x + 30y = 0$ is ***[DCE-2001]***

(a) $x^2 + y^2 + 30x - 13y - 25 = 0$
(b) $4x^2 + 4y^2 + 30x - 13y - 25 = 0$
(c) $2x^2 + 2y^2 + 30x - 13y - 25 = 0$
(d) $x^2 + y^2 + 30x - 13y + 25 = 0$

Solution

(b) The equation of required circle is $S_1 + \lambda S_2 = 0$

$\Rightarrow \quad x^2(1 + \lambda) + y^2(1 + \lambda) + x(2 + 13\lambda) - y\left(\frac{7}{2}+3\lambda\right) - \frac{25}{2} = 0$

$$\text{Centre} = \left(\frac{-(2+13\lambda)}{2}, \frac{\frac{7}{2}+3\lambda}{2}\right)$$

$\because \quad$ Centre lies on $13x + 30y = 0$

$$\Rightarrow \quad -13\left(\frac{2+13\lambda}{2}\right) + 30\left(\frac{\frac{7}{2}+3\lambda}{2}\right) = 0$$

$\Rightarrow \quad \lambda = 1$

Hence the equation of required circle is

$4x^2 + 4y^2 + 30x - 13y - 25 = 0$

7. If P is a point such that the ratio of the squares of the lengths, of the tangents from P to the circles $x^2 + y^2 + 2x - 4y - 20 = 0$ and $x^2 + y^2 - 4x + 2y - 44 = 0$ is 2 : 3, then the locus of P is a circle with centre

[EAMCET-2003]

(a) (7, –8) (b) (–7, 8)
(c) (7, 8) (d) (–7, –8)

Solution

(b) $$\frac{x^2+y^2+2x-4y-20}{x^2+y^2-4x+2y-44} = \frac{2}{3}$$

$\Rightarrow \quad x^2 + y^2 + 14x - 16y + 28 = 0$

$\therefore \quad$ Centre = (–7, 8)

8. A circle C_1 of radius 2 touches both x-axis and y-axis. Another circle C_2 whose radius is greater than 2 touches circle C_1 and both the axes. Then the radius of circle C_2 is

[AMU-2005]

(a) $6 - 4\sqrt{2}$ (b) $6 - 4\sqrt{2}$
(c) $6 - 4\sqrt{3}$ (d) $6 + 4\sqrt{3}$

Solution

(b) First circle touches both axes and radius is 2 unit.

Hence centre of circle is (2, 2).

Let radius of other circle be a and this circle also touches both the axes.

Hence centre of circle is (a, a). This circle touches first circle

Hence, $\sqrt{(a-2)^2+(a-2)^2} = a + 2$ squaring both the sides,

$\Rightarrow \quad a^2 - 12a + 4 = 0,$

$$a = \frac{12 \pm \sqrt{(12)^2 - 4\times4\times1}}{2}$$

$$= \frac{12 \pm \sqrt{128}}{2} = 6 \pm 4\sqrt{2}$$

But $a > 2$, Hence $a = 6 - 4\sqrt{2}$ is neglected.

Hence, $a = 6 + 4\sqrt{2}$.

9. The locus of the centre of a circle which touches externally the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches the y-axis, is given by the equation ***[IIT-1993; DCE-2000]***

(a) $x^2 - 6x - 10y + 14 = 0$
(b) $x^2 - 10x - 6y + 14 = 0$
(c) $y^2 - 6x - 10y + 14 = 0$
(d) $y^2 - 10x - 6y + 14 = 0$

Solution

(d) Let the centre be (h, k), then radius $= h$
Also $CC_1 = R_1 + R_2$

or $\sqrt{(h-3)^2 + (k-3)^2} = h + \sqrt{9+9-14}$

$\Rightarrow \quad (h-3)^2 + (k-3)^2 = h^2 + 4 + 4h$

$\Rightarrow \quad k^2 - 10h - 6k + 14 = 0$

or $\quad y^2 - 10x - 6y + 14 = 0$

10. If $a > 2b > 0$ then the positive value of m for which $y = mx - b\sqrt{1+m^2}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$, is ***[IIT (Screening)-2002]***

(a) $\dfrac{2b}{\sqrt{a^2-4b^2}}$ (b) $\dfrac{\sqrt{a^2-4b^2}}{2b}$

(c) $\dfrac{2a}{a-2b}$ (d) $\dfrac{b}{a-2b}$

Solution

(a) Any tangent to $x^2 + y^2 = b^2$ is

$y = mx - b\sqrt{1+m^2}$. It touches $(x - a)^2 + y^2 = b^2$

if $\dfrac{ma - b\sqrt{1+m^2}}{\sqrt{m^2+1}} = b$ or $ma = 2b\sqrt{1+m^2}$

or $m^2a^2 = 4b^2 + 4b^2m^2, \therefore m = \pm\dfrac{2b}{\sqrt{a^2-4b^2}}$

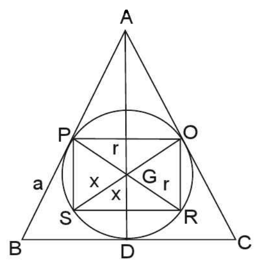
11. A circle is inscribed in an equilateral triangle of side a, the area of any square inscribed in the circle is ***[IIT-1994]***

(a) $\dfrac{a^2}{3}$ (b) $\dfrac{2a^2}{3}$

(c) $\dfrac{a^2}{6}$ (d) $\dfrac{a^2}{12}$

Solution

(c) If p be the altitude, then $p = a \sin 60° = \dfrac{a}{2}\sqrt{3}$



Since the triangle is equilateral, therefore centroid, orthocentre, circumcentre and incentre all coincide.

Hence, radius of the inscribed circle $= \dfrac{1}{3}p$

$= \dfrac{a}{2\sqrt{3}} = r$ or diameter $= 2r = \dfrac{a}{\sqrt{3}}$

Now if x be the side of the square inscribed, then angle in a semicircle being a right angle, hence

$$x^2 + x^2 = d^2 = 4r^2$$

$\Rightarrow \quad 2x^2 = \dfrac{a^2}{3} \quad \therefore$ Area $= x^2 = \dfrac{a^2}{6}$

12. The locus of centre of the circle of the which touches the circle $x^2 + (y - 1)^2 = 1$ externally and also touches x-axis is

(a) $\{(x, y): x^2 + (y-1)^2 = 4\} \cup \{(x, y): y < 0\}$
(b) $\{(x, y): x^2 = 4y\} \cup \{(0, y): y < 0\}$
(c) $\{(x, y): x^2 = y\} \cup \{(0, y): y < 0\}$
(d) $\{(x, y): x^2 = 4y\} \cup \{(x, y): y < 0\}$

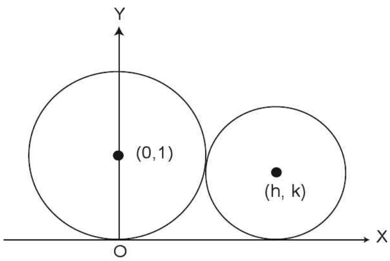
[IIT Screening-2005]

Solution

(b) According to given condition

$$\sqrt{(h-0)^2 + (k-1)^2} = 1 + |k|$$

$\Rightarrow \quad h^2 + (k-1)^2 = (1 + |k|)^2$



$\Rightarrow \quad h^2 = 2k + 2\,|\,k\,|$

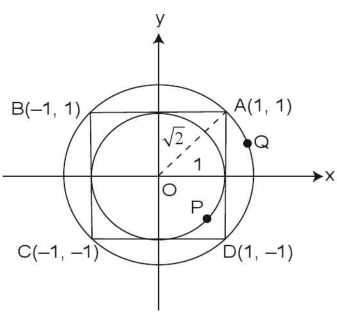
Hence locus is $x^2 = 2y + 2\,|y|,\ y > 0,\ x^2 = 4y,\ y < 0,\ x^2 = 0 \ \Rightarrow\ x = 0.$

13. Let $ABCD$ be a square of side length 2 units, C_2 is the circle through vertices A, B, C, D and C_1 is the circle touching all sides of the square. If P is a point on C_1 and Q is a point on C_2, then $\dfrac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$ is equal to ***[IIT JEE-2006]***

(a) 0.5 (b) 0.75
(c) 1 (d) 1.25

Solution

(b) Let centre of the square be origin and co-ordinate axes as shown in the diagram. Then obviously radi of circles c_1 and c_2 are 1 and $\sqrt{2}$ respectively.



So let $P \equiv (1\cos\theta, 1\sin\theta) = (\cos\theta, \sin\theta)$

$Q \equiv (\sqrt{2}\cos\phi, \sqrt{2}\sin\phi)$ Then,

$PA^2 + PB^2 + PC^2 + PD^2$

$= (\cos\theta - 1)^2 + (\sin\theta - 1)^2 + (\cos\theta + 1)^2$

$+ (\sin\theta - 1)^2 + (\cos\theta + 1)^2 + (\sin\theta + 1)^2$

$+ (\cos\theta - 1)^2 + (\sin\theta + 1)^2 = 12$

Similarly $QA^2 + QB^2 + QC^2 + QD^2 = 16$

Example = 12/16 = 0.75

14. Let C be any circle with centre $(0, \sqrt{2})$. Then, on the circle C, there can be ***[I.I.T.-1997]***

(a) at the most one rational point
(b) at the most two rational points
(c) at the most three rational points
(d) None of these

Solution

(b) Let the circle be $x^2 + (y - \sqrt{2})^2 = a^2$

$\Rightarrow \quad x^2 + y^2 - 2\sqrt{2}\,y = c$ where $c = a^2 - 2 = a$ rational no.

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be three distinct rational points on the circle

$\therefore \quad x_1^2 + y_1^2 - 2\sqrt{2}\,y_1 = c$ .... (1)

$x_2^2 + y_2^2 - 2\sqrt{2}\,y_2 = c$ .... (2)

$x_3^2 + y_3^2 - 2\sqrt{2}\,y_3 = c$ ....(3)

comparing the irrational parts of the equations, we get

$y_1 = y_2 = y_3$ .... (4)

Comparing the rational parts of the equations, we get

$x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2$

$\Rightarrow \quad x_1^2 = x_2^2 = x_3^2 \qquad [\because y_1 = y_2 = y_3]$

$\therefore$ the only possible values of x are $\pm x_1, \pm x_2, \pm x_3$

$\therefore$ there can be at the most two rational points on the circle C. Thus (b) is the correct answer.

15. Angle of intersection of the curves $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$ is equal to ***[MPPET-2006]***

(a) $\pi/2$ (b) $\pi/3$
(c) $\pi/4$ (d) None of these

Solution

(c) The equations of the given curves are

$r = \sin\theta + \cos\theta$

Multiplying r on both sides, we get

$r^2 = r\sin\theta + r\cos\theta$

Let $x = r\cos\theta,\ y = r\sin\theta$

$\Rightarrow \quad x^2 + y^2 = y + x$

$\Rightarrow \quad \left(x - \dfrac{1}{2}\right)^2 + \left(y - \dfrac{1}{2}\right)^2 = \left(\dfrac{1}{\sqrt{2}}\right)^2$ ....(i)

This equation represents a circle, whose centre is $\left(\frac{1}{2},\frac{1}{2}\right)$ and the radius = $\frac{1}{\sqrt{2}}$ unit.

and $r = 2\sin\theta$ on squaring, we get $r^2 = 2r\sin\theta$

$\Rightarrow \quad x^2 + y^2 = 2y$

$\Rightarrow \quad x^2 + (y-1)^2 = 1$ ....(ii)

This equation also represents a circle, whose centre is (0, 1) and radius = 1.

Distance between the centres of both circles

$$d = \sqrt{\left(0-\frac{1}{2}\right)^2 + \left(1-\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$$

Hence, the angle of intersection of two circles

$$= \cos^{-1}\left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right) = \cos^{-1}\left(\frac{\frac{1}{2}+1-\frac{1}{2}}{2\times\frac{1}{\sqrt{2}}\times 1}\right)$$

$$= \cos^{-1}\left(\frac{2}{\sqrt{2}}\right) = \frac{\pi}{4} \quad \left[\because \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

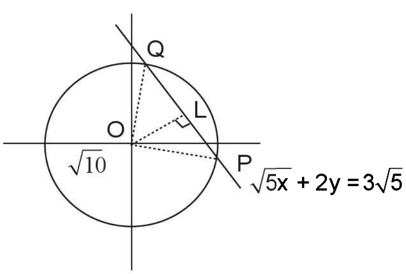
16. The area of the triangle formed by joining the origin to the points of intersection of the line $x\sqrt{5} + 2y = 3\sqrt{5}$ and circle $x^2 + y^2 = 10$ is

[Roorkee Qualification-1998]

(a) 3 (b) 4
(c) 5 (d) 6

Solution

(c) Length of perpendicular from origin to the line $x\sqrt{5} + 2y = 3\sqrt{5}$ is



$$OL = \frac{3\sqrt{5}}{\sqrt{(\sqrt{5})^2 + 2^2}} = \frac{3\sqrt{5}}{\sqrt{9}} = \sqrt{5}$$

Radius of the given circle = $\sqrt{10} = OQ = OP$

$$PQ = 2QL = 2\sqrt{OQ^2 - OL^2}$$

$$= 2\sqrt{10-5} = 2\sqrt{5}$$

Thus area of

$$\Delta\, OPQ = \frac{1}{2} \times PQ \times OL = \frac{1}{2} \times 2\sqrt{5} \times \sqrt{5} = 5.$$

Paragraph (Q.18 to 20)

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F respectively. The line PQ is given by the equation $\sqrt{3}\,x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2},\frac{3}{2}\right)$, Further, it is given that the origin and the centre of C are on the same side of the line PQ.

17. The equation of circle C is

[IIT-JEE-2008]

(a) $(x - 2\sqrt{3})^2 + (y-1)^2 = 1$

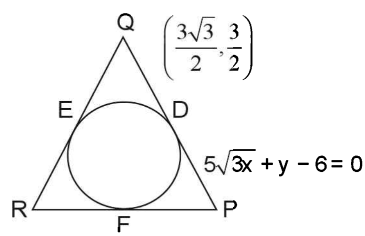
(b) $(x - 2\sqrt{3})^2 + \left(y + \frac{1}{2}\right)^2 = 1$

(c) $(x - \sqrt{3})^2 + (y+1)^2 = 1$

(d) $(x - \sqrt{3})^2 + (y-1)^2 = 1$

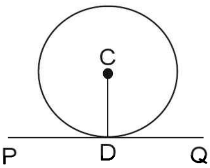
Solution

(d) We have to find the equation to the circle of radius 1 and touching the line $\sqrt{3} + y - 6 = 0$ at $D\left(\frac{3\sqrt{3}}{2},\frac{3}{2}\right)$



The equation to CD in parametric form is

$$\frac{x - \frac{3\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} = \frac{y - \frac{3}{2}}{\frac{1}{2}} = \pm 1$$



Which gives $C \equiv (\sqrt{3}, 1)$ or $(2\sqrt{3}, 2)$
But origin and C are on the same side of PQ.
Then, $C \equiv (\sqrt{3}, 1)$
The equation to circle is $(x - \sqrt{3})^2 + (y - 1)^2 = 1$.
Remark: Choices (b) and (c) can be at once eliminated because $D\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$ does'nt lie on the equation given by (b) and (c).

18. Points E and F are given by *[IIT-JEE-2008]*

(a) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), (\sqrt{3}, 0)$

(b) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (\sqrt{3}, 0)$

(c) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

(d) $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Solution

(a) C divides RD in the ratio 2 : 1

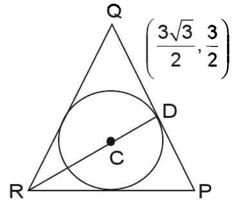
As $C \equiv (\sqrt{3}, 1)$ and $D \equiv \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

We have

$$R \equiv \left(\frac{3\sqrt{3} - 2.\frac{3\sqrt{3}}{2}}{3-2}, \frac{3.1 - 2.\frac{3}{2}}{3-2}\right) \equiv (0, 0)$$

The points P and Q are at a distance of $\sqrt{3}$ from D. The equation to PQ in parametric form is

$$\frac{x - \frac{3\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{y - \frac{3}{2}}{\frac{\sqrt{3}}{2}} = \pm\sqrt{3}$$



Which give two points $Q(\sqrt{3}, 3)$ and $P(2\sqrt{3}, 0)$

E is the mid-point of $QR \equiv \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$

F is the mid-point of $PR \equiv (\sqrt{3}, 0)$

19. Equations of the sides QR, RP are *[IIT JEE-2008]*

(a) $y = \frac{2}{\sqrt{3}}x + 1, y = -\frac{2}{\sqrt{3}}x - 1$

(b) $y = \frac{1}{\sqrt{3}}x, y = 0$

(c) $y = \frac{\sqrt{3}}{2}x + 1, y = -\frac{\sqrt{3}}{2}x - 1$

(d) $y = \sqrt{3}x, y = 0$

Solution

(d) The equation to RP is simply $y = 0$
The equation to QR is $y - 3$
$= \sqrt{3}(r - \sqrt{3})$ i.e., $y = \sqrt{3}x$

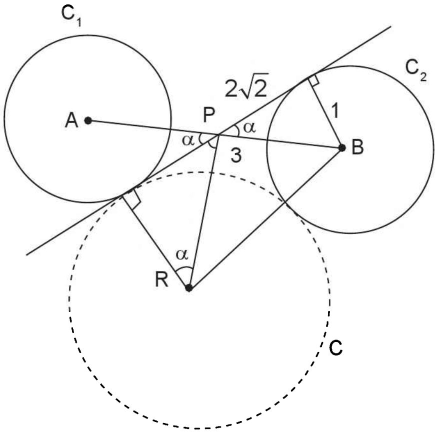
20. The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the mid-point of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is *[IIT-2009]*
(a) 2 (b) 4
(c) 6 (d) 8

Solution

(d) $\cos\alpha = \frac{2\sqrt{2}}{3}$

$\sin\alpha = \frac{1}{3}$

$\tan\alpha = \frac{2\sqrt{2}}{R}$



$\Rightarrow \quad R = \dfrac{2\sqrt{2}}{\tan\alpha} = 8$ units.

Alternate

$$(R+1)^2 = (R-1)^2 + (4\sqrt{2})^2$$

$\Rightarrow \quad R = 8$

21. If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and (1, 1) for ***[AIEEE-2009]***

(a) all values of p
(b) all except one value of p
(c) all except two values of p
(d) exactly one value of p

Solution

(a) Given circles

$$S = x^2 + y^2 + 3x + 7y + 2p - 5 = 0$$
$$S' = x^2 + y^2 + 2x + 2y - p^2 = 0$$

Equation of required circle is $S + \lambda S' = 0$

As it passes through (1,1) the value of

$$\lambda = \frac{-(7+2p)}{(6-p^2)}$$

If $7 + 2p = 0$, it becomes the second circle.
so, it is true for all value of p.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. A pair of tangents are drawn from the origin to the circle $x^2 + y^2 + 20(x + y) + 20 = 0$. The equation of the pair of tangents is ***[PET-90]***

(a) $x^2 + y^2 + 10xy = 0$
(b) $x^2 + y^2 + 5xy = 0$
(c) $2x^2 + 2y^2 + 5xy = 0$
(d) $2x^2 + 2y^2 - 5xy = 0$

2. If d is the distance between the centres of two circles r_1, r_2 are their radii and $d = r_1 + r_2$, then ***[PET-86]***

(a) The circles touch each other externally.
(b) The circles touch each other internally.
(c) The circles cut each other.
(d) The circles are disjoint.

3. The length of tangent from the point (5, 1) to the circle $x^2 + y^2 + 6x - 4y - 3 = 0$, is

(a) 81 (b) 29
(c) 7 (d) 21

4. An infinite number of tangents can be drawn from (1, 2) to the circle $x^2 + y^2 - 2x - 4y + \lambda = 0$, then λ is equal to ***[PET-89]***

(a) –20
(b) 0
(c) 5
(d) Cannot be determined

5. The equation of the circle which passes through the points (2, 3) and (4, 5) and the centre lies on the straight lines $y - 4x + 3 = 0$, is ***[RPET-85; PET-89]***

(a) $x^2 + y^2 + 4x - 10y + 25 = 0$
(b) $x^2 + y^2 - 4x - 10y + 25 = 0$
(c) $x^2 + y^2 - 4x - 10y + 16 = 0$
(d) $x^2 + y^2 - 14x + 8 = 0$

6. The two tangents to a circle from an external point are always ***[PET-86]***

(a) Equal
(b) Perpendicular to each other
(c) Parallel to each other
(d) None of these

7. The circle on the chord $x \cos \alpha + y \sin \alpha = p$ of the circle $x^2 + y^2 = a^2$ as diameter has the equation

[PET-93; Roorke-67]

(a) $x^2 + y^2 - a^2 - 2p(x \cos \alpha + y \sin \alpha - p) = 0$
(b) $x^2 + y^2 + a^2 + 2p(x \cos \alpha - y \sin \alpha + p) = 0$
(c) $x^2 + y^2 - a^2 + 2p(x \cos \alpha + y \sin \alpha + p) = 0$
(d) $x^2 + y^2 - a^2 - 2p(x \cos \alpha - y \sin \alpha - p) = 0$

8. Tangents drawn from origin to the circle $x^2 + y^2 - 2ax - 2by + b^2 = 0$ are perpendicular to each other, if

[PET-95]

(a) $a - b = 1$ (b) $a + b = 1$
(c) $a^2 = b^2$ (d) $a^2 = b^2 = 1$

9. The equation of a circle with origin as centre passing through the vertices of an equilateral triangle whose median is of length $3a$ is

[AIEEE-2002]

(a) $x^2 + y^2 = 9a^2$
(b) $x^2 + y^2 = 16a^2$
(c) $x^2 + y^2 = a^2$
(d) $x^2 + y^2 = 4a^2$

10. Any circle through the points of intersection of the lines $x + \sqrt{3}y = 1$ and $\sqrt{3}x - y = 2$ if intersects these lines at point P and Q, then the angle subtended by the are PQ at its centre is

[PET-98]

(a) 180°
(b) 90°
(c) 120°
(d) Depend on centre and radius

11. Circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g'x + 2f'y = 0$ touch externally, if

[PET-94]

(a) $f'g = g'f$
(b) $fg = f'g'$
(c) $f'g' + fg = 0$
(d) $f'g' + g'f = 0$

12. A circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passing through (4, –2) and concentric to the circle $x^2 + y^2 - 2x + 4y + 20 = 0$, then the value of c will be ***[RPET-84, 86]***

(a) –4 (b) 4
(c) 0 (d) 1

13. The square of the length of the tangent (power) from (3, –4) on the circle $x^2 + y^2 - 4x - 6y + 3 = 0$ ***[PET-2000]***

(a) 20 (b) 30
(c) 40 (d) 50

14. The centres of a set of circles, each of radius 3, lie on the circle $x^2 + y^2 = 25$. The locus of any point in the set is

[AIEEE-2002]

(a) $4 \le x^2 + y^2 \le 64$
(b) $x^2 + y^2 \le 25$
(c) $x^2 + y^2 \ge 25$
(d) $3 \le x^2 + y^2 \le 9$

15. The circle $x^2 + y^2 - 10x + 16 = 0$ and $x^2 + y^2 = r^2$ intersect each other at two distinct points, if

[DCE-1996; MPPET-2006]

(a) $r < 2$ (b) $r > 8$
(c) $2 < r < 8$ (d) $2 \le r \le 8$

16. The equation of a tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are

[IIT-88; DCE-98; UPSEAT-2005; Roorkee-89; RPET-96; MPPET-2006]

(a) $x = 0, y = 0$
(b) $x = 1, y = 0$
(c) $(h^2 - r^2)x - 2rhy = 0, y = 0$
(d) $(h^2 - r^2)x - 2rhy = 0, x = 0$

17. The number of common tangents to two circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 8x + 12 = 0$ is

[MPPET-2005]

(a) 1 (b) 2
(c) 5 (d) 3

18. The angle between a pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9 \sin^2\alpha + 13\cos^2\alpha = 0$ is 2α. The equation of the locus of the point P is ***[IIT-96; 1M]***

(a) $x^2 + y^2 + 4x - 6y + 4 = 0$
(b) $x^2 + y^2 + 4x - 6y - 9 = 0$
(c) $x^2 + y^2 + 4x - 6y - 4 = 0$
(d) $x^2 + y^2 + 4x - 6y + 9 = 0$

19. Equation of circle passing through intersection of two circles $x^2 + y^2 + 13x - 3y = 0$, $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ and also through (1, 1) is ***[MP PET-2007]***

(a) $4x^2 + 4y^2 - 30x - 10y - 25 = 0$
(b) $4x^2 + 4y^2 + 30x - 13y - 25 = 0$
(c) $4x^2 - 4y^2 - 17x - 10y + 25 = 0$
(d) None of these

20. If the circle $x^2 + y^2 = 4$ bisects the circumference of the circle $x^2 + y^2 - 2x + 6y + a = 0$, then a equals ***[RPET-1999]***
(a) 4 (b) –4
(c) 16 (d) –16

21. The condition that the circle $(x - 3)^2 + (y - 4)^2 = r^2$ lies entirely within the circle $x^2 + y^2 = R^2$, is ***[AMU-1999]***
(a) $R + r \le 7$ (b) $R^2 + r^2 < 49$
(c) $R^2 - r^2 < 25$ (d) $R - r > 5$

22. A variable circle passes through a fixed point $A(p, q)$ and touches x-axis. The locus of the other end of the diameter of this circle passing through A will be ***[AIEEE-2004]***
(a) $(x - p)^2 = 4qx$ (b) $(x - q)^2 = 4py$
(c) $(x - p)^2 = 4qy$ (d) $(y - q)^2 = 4px$

23. A circle passes through (0, 0) and (1, 0) and touches the circle $x^2 + y^2 = 9$, then the centre of circle is ***[IIT-1992; AIEEE-02]***
(a) $\left(\frac{3}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{3}{2}\right)$
(c) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (d) $\left(\frac{1}{2}, \pm\sqrt{2}\right)$

24. A diameter of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord of another circle C with centre (2, 1). The radius of this circle C is ***[IIT(Screening)-2004]***
(a) 1 (b) 3
(c) 2 (d) $\sqrt{3}$

25. Tangents drawn from the point $P(1, 8)$ to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points A and B.
The equation of the circumcircle of the triangle PAB is ***[IIT-2009]***
(a) $x^2 + y^2 + 4x - 6y + 19 = 0$
(b) $x^2 + y^2 - 4x - 10y + 19 = 0$
(c) $x^2 + y^2 - 2x + 6y - 29 = 0$
(d) $x^2 + y^2 - 6x - 4y + 19 = 0$

SOLUTIONS

1. (c) Equation of pair of tangent is given by $ss_1 = T^2$ or

$s = x^2 + y^2 + 20(x + y) + 20$, $s^1 = 20$, $T = 10(x + y) + 20 = 0$

$\therefore$ $ss_1 = T^2$

$\Rightarrow$ $20\{x^2 + y^2 + 20(x + y) + 20\} = 10^2 (x + y + 2)^2$

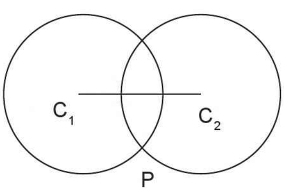
$\Rightarrow$ $4x^2 + 4y^2 + 10xy = 0$

$\Rightarrow$ $2x^2 + 2y^2 + 5xy = 0.$

OR

The equation of the pair of tangents from the origin (0, 0) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $c(x^2 + y^2) = (gx + fy)^2$.

2. (a)



$C_1C_2 = C_1P + C_2P, d = r_1 + r_2$

3. (c) Length of tangent is given by $L_T = \sqrt{S_1} = \sqrt{49} = 7$.

4. (c) Clearly the point (1, 2) is the centre of the given circle and infinite tangents can only be drawn from a point circle. Hence radius should be zero.

$\therefore$ $\sqrt{1^2 + 2^2 - \lambda} = 0 \Rightarrow \lambda = 5$

5. (b) First find the centre. Let centre be (h, k), then

$$\sqrt{(h-2)^2 + (k-3)^2} = \sqrt{(h-4)^2 + (k-5)^2} \qquad \text{........ (1)}$$

and $k - 4h + 3 = 0$ ....(2)

from (1), we get $-4h + 6k - 8h + 10k = 16 + 25 - 4 - 9$.

or $4h + 4k - 28 = 0$ or $h + k - 7 = 0$ ....(3)

from (3) and (2), we get (h, k) as (2, 5). Hence centre is (2, 5) and radius is 2.

OR

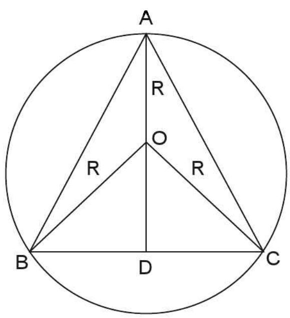
Obviousy circle $x^2 + y^2 - 4x - 10y + 25 = 0$ passing through (2, 3), (4, 5).

6. (a) From the definition: tangents drawn from an external point to a circle are of the same length.

7. (a) The equation of the circle on the chord $A = x \cos \alpha + y \sin \alpha - p = 0$ of the circles $= x^2 + y^2 - a^2 = 0$ as diameter has the equation $S + \lambda A = 0$ ....(i)
where λ is constant whose value is obtained by given condition. Comparing (i) with given option we find correct answer (a) only.

8. (c) We know that if tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are perpendicular if $g^2 + f^2 = 2c$ presently: $g = -a, f = -b, c = b^2 \Rightarrow a^2 = b^2$.

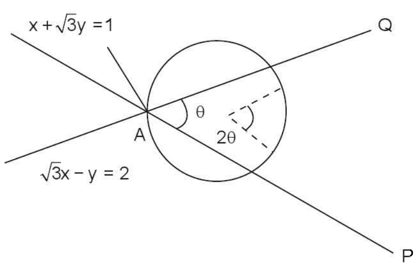
9. (d)



$$O\ (0, 0),\ R = \frac{2}{3}AD = \frac{2}{3} \times 3a = 2a$$

$\therefore$ Equation of desired circle is $x^2 + y^2 = (2a)^2$

10. (a) Let the point of intersect of two lines is A.



$\therefore$ The angle subtended by PQ on centre C.
= two times the angle subtended by PQ on point A.

for $x + \sqrt{3}y = 1, m_1 = -1/\sqrt{3}$ and for $\sqrt{3}x - y = 2, m_2 = \sqrt{3}$.

$\because \quad m_1 \times m_2 = \dfrac{-1}{\sqrt{3}} \times \sqrt{3} = -1$

$\therefore \quad \angle A = 90°$.

$\therefore$ The angle subtended by arc PQ at its centre $= 2 \times 90° = 180°$.

11. (a) Since both circles are passing through origin (constant terms are zero) therefore their common tangents will pass through origin and will also be identical which are as follows

$\therefore \quad gx + fy = 0$ (tangent to circle 1 at the origin) ....(i)

$g^1x + f^1y = 0$ (tangent to circle 2 at the origin) ....(ii)

On comparing (i) and (ii), we find $\dfrac{g}{g^1} = \dfrac{f}{f^1}$

i.e., $gf^1 = fg^1$

12. (a) We know that equation of concentric circles different only in constant term. Therefore, required equation of circle is $x^2 + y^2 - 2x + 4y + k = 0$....(i), where k is constant.
Now this circle also passes through the point $(4, -2)$

$\therefore \quad 16 + 4 - 8 - 8 + k = 0 \Rightarrow k = -4$

13. (c) Length of tangent

$$= \sqrt{3^2 + (-4)^2 - 4(3) - 6(-4) + 3} = \sqrt{40}$$

$\therefore$ Square of length of tangent = 40.

14. (a) Let (h, k) be any point in the set, then equation of circle is

$$(x - h)^2 + (y - k)^2 = 9$$

But (h, k) lies on $x^2 + y^2 = 25$, then $h^2 + k^2 = 25$

$\therefore \quad 2 \le$ Distance between the two circle ≤ 8

$$2 \le \sqrt{h^2 + k^2} \le 8 \Rightarrow 4 \le (h^2 + k^2) \le 64.$$

$\therefore$ Locus of (h, k) is $4 \le x^2 + y^2 \le 64$.

15. (c) The equation of the given circles are

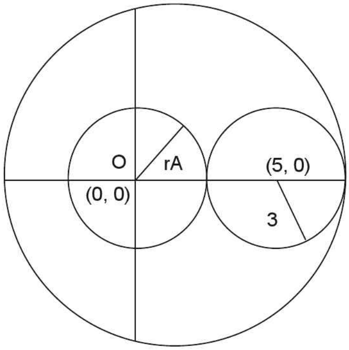
$$x^2 + y^2 - 10x + 16 = 0$$

or $\quad (x - 5)^2 + y^2 = 3^2$ ....(i)

whose centre is (5, 0) and radius = 3 unit and

$x^2 + y^2 = r^2$ .....(ii)

whose centre is (0, 0) and radius = r



Clearly, these two circles will intersect each other at two distinct points, if $r > OA$
$\Rightarrow r > 5-3 \Rightarrow r > 2$
and $r < OB \;\Rightarrow r < 2+3+3 \;\Rightarrow r < 8$
$\therefore 2 < r < 8$

16. (d) Since, the given equation of the circle is $x^2+y^2-2rx-2hy+h^2=0$ whose centre is (r, h) and radius is r.
Thus $x = 0$ is one of the tangent.
Let another tangent is $y = mx$ to the circle.

This line will be tangent, if $\dfrac{h-mr}{\sqrt{1+m^2}} = r$

$$\Rightarrow h^2+m^2r^2-2mhr = r^2+m^2r^2$$

$$\Rightarrow m = \left(\frac{h^2-r^2}{2hr}\right)$$

Therefore, equation of tangent is

$$y = \frac{(h^2-r^2)}{2hr}x \Rightarrow (h^2-r^2)x-2hry=0$$

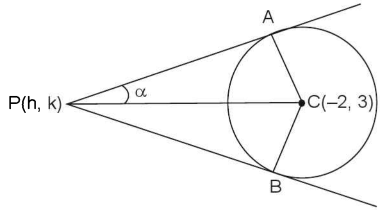
$\therefore$ Tangents are $x = 0$ and
$(h^2-r^2)x-2hry=0$

17. (d) $\because$ Centre and radius of circle $x^2+y^2=4$ are $C_1(0, 0)$ and 2 respectively and centre and radius of circle $x^2+y^2-8x+12=0$ are C_2 (4, 0) and 2 respectively.

$\because$ $C_1C_2 = r_1 + r_2$

$\therefore$ Circles touch each other externally, so the number of common tangents is 3.

18. (b) Centre of the circle
$x^2 + y^2 + 4x - 6y + 9\sin^2\alpha + 13\cos^2\alpha = 0$
is $C(-2, 3)$ and its radius is



$$\sqrt{(-2)^2+(+3)^2-9\sin^2\alpha-13\cos^2\alpha}$$
$$= \sqrt{4+9-9\sin^2\alpha-13\cos^2\alpha}$$
$$= \sqrt{13-13\cos^2\alpha-9\sin^2\alpha}$$
$$= \sqrt{13(1-\cos^2\alpha)-9\sin^2\alpha}$$
$$= \sqrt{13\sin^2\alpha-9\sin^2\alpha}$$
$$= \sqrt{4\sin^2\alpha} = 2\sin\alpha$$

Let (h, k) be any point P and

$$\angle APC = \alpha, \angle PAC = \pi/2$$

That is, triangle APC is a right triangle.

Thus, $\sin\alpha = \dfrac{AC}{PC} = \dfrac{2\sin\alpha}{\sqrt{(h+2)^2+(k-3)^2}}$

$\Rightarrow \quad \sqrt{h+2)^2+(k-3)^2} = 2$
$\Rightarrow \quad (h+2)^2 + (k-3)^2 = 4$
$\Rightarrow \quad h^2 + 4 + 4h + k^2 + 9 - 6k = 4$
$\Rightarrow \quad h^2 + k^2 + 4h - 6k + 9 = 0$

Thus, required equation of the locus is $x^2 + y^2 + 4x - 6y - 9 = 0$.

19. (b) $(x^2 + y^2 + 13x - 3y) + \lambda(2x^2 + 2y^2 + 4x - 7y - 25) = 0$ it passes through (1, 1) therefore $\lambda = \dfrac{1}{2}$
Hence the required equation will be
$4x^2 + 4y^2 + 30x - 13y - 25 = 0$.

20. (c) The common chord of given circles is
$2x - 6y - 4 - a = 0$ .... (i)
Since, $x^2 + y^2 = 4$ bisects the circumference of the circle $x^2 + y^2 - 2x + 6y + a = 0$, therefore, (i) passes through the centre of second circle i.e., $(1, -3)$.
$\therefore$ $2 + 18 - 4 - a = 0 \Rightarrow a = 16$.

21. (d) Clearly, $r_1 - r_2 > C_1C_2$
$r_1 = R, C_1(3, 4); r_2 = r; C_2(0, 0)$
$R - r > \sqrt{(3-0)^2+(4-0)^2} \Rightarrow R - r > 5$.

22. Let other end of the diameter through A be $B = (h, k)$.

Then centre of the circle $= \left(\frac{h+p}{2}, \frac{k+q}{2}\right)$

$\because$ Given circle touches x-axis

$\therefore$ Its radius $= \frac{k+q}{2}$

But radius $= \frac{1}{2}AB = \frac{1}{2}\sqrt{(h-p)^2 + (k-q)^2}$

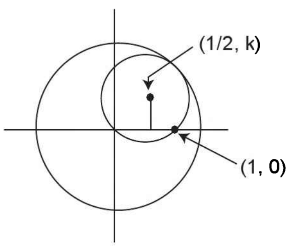
$\Rightarrow \left(\frac{k+q}{2}\right)^2 = \frac{1}{4}[(h-p)^2 + (k-q)^2]$

$\Rightarrow (h-p)^2 = 4qk$

Hence required locus of B is $(x-p)^2 = 4qy$

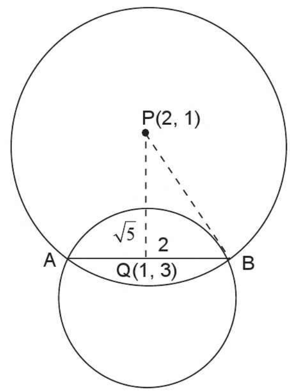
23. (d) Radius of the circle $r = 3/2$

$$\frac{1}{4} + k^2 = \frac{9}{4} \Rightarrow k = \pm\sqrt{2}$$



Hence centre is $\left(\frac{1}{2}, \pm\sqrt{2}\right)$

24. Let AB be given diameter of the given circle which is a chord of another circle C. Centre of the given circle $Q = (1, 3)$ and its radius $QB = \sqrt{1+9-6} = 2$



If P be the centre of the circle C, then

$$PQ = \sqrt{1+4} = \sqrt{5}$$

$\therefore$ radius of circle $C = PB$,

$$= \sqrt{PQ^2 + QB^2} = \sqrt{5+4} = 3$$

25 (b) : The centre of the circle is C (3, 2)

Since CA and CB are perpendicular to PA and PB, CP is the diameter of the circumcircle of triangle PAB. Its equation is

$$(x-3)(x-1) + (y-2)(y-8) = 0$$

or $\quad x^2 + y^2 - 4x - 10y + 19 = 0$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE) : FOR IMPROVING SPEED WITH ACCURACY

1. The number of common tangents of the circles $x^2 + y^2 - 2x - 1 = 0$ and $x^2 + y^2 - 2y - 7 = 0$ is ***[Orissa JEE-2007]***

(a) 1 (b) 2
(c) 3 (d) 4

2. The circles $x^2 + y^2 = 9$ and $x^2 + y^2 - 12y + 27 = 0$ touch each other. The equation of their common tangent is

(a) $4y = 9$ (b) $y = 3$
(c) $y = -3$ (d) $x = 3$

3. The two tangents to a circle from an external point are always ***[MPPET-86]***

(a) Equal
(b) Perpendicular to each other
(c) Parallel to each other
(d) None of these

4. The angle between the tangents from (α, β) to the circle $x^2 + y^2 = a^2$, is

(a) $\tan^{-1}\left(\frac{a}{\sqrt{\alpha^2+\beta^2-a^2}}\right)$

(b) $\tan^{-1}\left(\frac{\sqrt{\alpha^2+\beta^2-a^2}}{a}\right)$

(c) $2\tan^{-1}\left(\frac{a}{\sqrt{\alpha^2+\beta^2-a^2}}\right)$

(d) None of these

5. Two tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ will be perpendicular to each other, if

(a) $g^2 + f^2 = 2c$ (b) $g = f = c^2$
(c) $g + f = c$ (d) None of these

6. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y = 24$ is **[IIT-98; 2M; KCET-2007]**

(a) 0 (b) 1
(c) 3 (d) 4

7 The triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have co-ordinate (3, 4) and (–4, 3) respectively, then $\angle QPR$ is equal to **[IIT-2000]**

(a) $\pi/2$ (b) $\pi/3$
(c) $\pi/4$ (d) $\pi/6$

8. If circle $x^2 + y^2 + 6x - 2y + k = 0$ bisects the circumference of the circle $x^2 + y^2 + 2x - 6y - 15 = 0$, then k is equal to **[EAMCET-2003]**

(a) –23 (b) 23
(c) –21 (d) 21

9. The circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 4x + 6y + 4 = 0$ **[Kerala PET-2008]**

(a) touch externally
(b) do not intersect
(c) touch internally
(d) intersect at two points

10 If the circles $x^2 + y^2 = 9$ and $x^2 + y^2 + 2\alpha x + 2y + 1 = 0$ touch each other internally, then α = is equal to **[Karnataka CET-2008]**

(a) $\frac{4}{3}$ (b) $\frac{-4}{3}$
(c) $\pm\frac{4}{3}$ (d) 1

11. Two circles $x^2 + y^2 - 2x - 3 = 0$ and $x^2 + y^2 - 4x - 6y - 8 = 0$ are such that **[MPPET-2008]**

(a) they touch internally
(b) they touch externally
(c) they intersect in two points
(d) they are non-intersecting

12. The two circles $x^2 + y^2 - 4y = 0$ and $x^2 + y^2 - 8y = 0$ **[BIT Ranchi-1985]**

(a) Touch each other internally
(b) Touch each other externally
(c) Do not touch each other
(d) None of these

13. The equation of the circle which passes through the point of intersection of circles $x^2 + y^2 - 8x - 2y + 7 = 0$ and $x^2 + y^2 - 4x + 10y + 8 = 0$ and having its centre on y-axis, will be

(a) $x^2 + y^2 + 22x + 9 = 0$
(b) $x^2 + y^2 + 22x - 9 = 0$
(c) $x^2 + y^2 + 22y + 9 = 0$
(d) $x^2 + y^2 + 22y - 9 = 0$

14. From any point on the circle $x^2 + y^2 = a^2$ tangents are drawn to the circle $x^2 + y^2 = a^2 \sin^2 \alpha$, the angle between them is **[RPET-2002]**

(a) $\alpha/2$ (b) α
(c) 2α (d) None of these

15. If the circles $x^2 + y^2 = 4$, $x^2 + y^2 - 10x + \lambda = 0$ touch externally, then λ is equal to **[AMU-1999]**

(a) –16 (b) 9
(c) 16 (d) 25

16. The points of intersection of the circles $x^2 + y^2 = 25$ and $x^2 + y^2 - 8x + 7 = 0$ are **[MPPET-1988]**

(a) (4, 3) and (4, –3)
(b) (4, –3) and (–4, –3)
(c) (–4, 3) and (4, 3)
(d) (4, 3) and (3, 4)

17. If the squares of the lengths of the tangents drawn from a point P to circles $x^2 + y^2 = a^2$, $x^2 + y^2 = b^2$, $x^2 + y^2 = c^2$ are in A.P., then: **[MPPET-2010]**

(a) a, b, c are in A.P.
(b) $a, b, c,$ are in G.P.
(c) a^2, b^2, c^2 are in A.P.
(d) a^2, b^2, c^2 are in G.P.

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 17 minutes.
3. The worksheet consists of 17 questions. The maximum marks are 51.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The length of the tangent from the point (5, 4) to the circle $x^2 + y^2 + 2x - 6y = 6$ is

[DCE-1999]

(a) $\sqrt{21}$ (b) $\sqrt{38}$
(c) $2\sqrt{2}$ (d) $2\sqrt{13}$

2. The equation of the circle concentric to the circle $2x^2 + 2y^2 - 3x + 6y + 2 = 0$ and having area double the area of this circle, is

[DCE-2006]

(a) $8x^2 + 8y^2 - 24x + 48y - 13 = 0$
(b) $16x^2 + 16y^2 + 24x - 48y - 13 = 0$
(c) $16x^2 + 16y^2 - 24x + 48y - 13 = 0$
(d) $8x^2 + 8y^2 + 24x - 48y - 13 = 0$

3. The two circles $x^2 + y^2 - 2x + 6y + 6 = 0$ and $x^2 + y^2 - 5x + 6y + 15 = 0$

(a) Intersect (b) Are concentric
(c) Touch internally (d) Touch externally

4. If the circle $x^2 + y^2 + 4x + 22y + c = 0$ bisects the circumference of the circle $x^2 + y^2 - 2x + 8y - d = 0$, then $c + d$ is equal to

[Kerala PET-2007]

(a) 30 (b) 50
(c) 40 (d) 56

5. Two diameters of the circle $3x^2 + 3y^2 - 6x - 18y - 7 = 0$ are along the lines $3x + y = c_1$ and $x - 3y = c_2$. Then the value of c_1c_2 is

[Kerala PET-2007]

(a) –48 (b) 80
(c) –72 (d) 54

6. Length of the tangents from the point (1, 2) to the circles $x^2 + y^2 + x + y - 4 = 0$ and $3x^2 + 3y^2 - x - y - k = 0$ are in the ratio 4 : 3, then k is equal to

[Kerala PET-2007]

(a) 21/2 (b) 4/21
(c) 21 (d) 21/4

7. The equation of a circle concentric to the circle $x^2 + y^2 - 6y + 7 = 0$ is

[SCRA-2007]

(a) $x^2 + y^2 - 6y = 0$
(b) $x^2 + y^2 - 6y + 17 = 0$
(c) $x^2 + y^2 - 6x - 18 = 0$
(d) None of these

8. The equation of the circle passing through the intersection of $x^2 + y^2 - 4x + 6y + 8 = 0$, $x^2 + y^2 - 10x - 6y - 8 = 0$ and which passes through the origin is

[SCRA-2007]

(a) $x^2 + y^2 - 2x - 7y = 0$ (b) $x^2 + y^2 + 7x = 0$
(c) $x^2 + y^2 - 7x = 0$ (d) $x^2 + y^2 + 12y = 0$

9. The equation of the pair of tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is ***[SCRA-2007]***

(a) $(gx + fy)^2 = c(x^2 + y^2)$
(b) $(fx + gy)^2 = c(x^2 + y^2)$
(c) $(gx + fy)^2 = x^2 + y^2$
(d) $c(gx + fy)^2 = x^2 + y^2$

10. The equations of the tangents to the circle $x^2 + y^2 = 16$ parallel to the line $y = x$ are

[SCRA-2007]

(a) $y = x \pm 4\sqrt{2}$
(b) $y = x \pm 2\sqrt{3}$
(c) $y = x \pm \sqrt{8}$
(d) $y + x + 4\sqrt{2} = 0$

11. What is the equation of circle which touches the lines $x = 0$, $y = 0$ and $x = 2$?

[NDA-2007]

(a) $x^2 + y^2 + 2x + 2y + 1 = 0$
(b) $x^2 + y^2 - 4x - 4y + 1 = 0$
(c) $x^2 + y^2 - 2x - 2y + 1 = 0$
(d) None of the above

12. If the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 - 2gx + g^2 - b^2 = 0$ touch each other externally, then

(a) $g = ab$ (b) $g^2 = a^2 + b^2$
(c) $g^2 = ab$ (d) $g = a + b$

13. For an equilateral triangle the centre is the origin and the length of altitude is a. Then the equation of the circumcircle is ***[Kerala (Engg.)-2006]***

(a) $x^2 + y^2 = a^2$ (b) $3x^2 + 3y^2 = 2a^2$
(c) $x^2 + y^2 = 4a^2$ (d) $9x^2 + 9y^2 = 4a^2$

14. The equation of the circle having its centre on the line $x + 2y - 3 = 0$ and passing through the points of intersection of the circles $x^2 + y^2 - 2x - 4y + 1 = 0$ and $x^2 + y^2 - 4x - 2y + 4 = 0$ is ***[MNR-1992]***

(a) $x^2 + y^2 - 6x + 7 = 0$
(b) $x^2 + y^2 - 3y + 4 = 0$
(c) $x^2 + y^2 - 2x - 2y + 1 = 0$
(d) $x^2 + y^2 + 2x - 4y + 4 = 0$

15. If the circles $x^2 + y^2 - 9 = 0$ and $x^2 + y^2 + 2ax + 2y + 1 = 0$ touch each other, then a is equal to ***[Roorkee Qualifying-1998]***

(a) –4/3 (b) 0
(c) 1 (d) 4/3

16. The points of intersection of circles $x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2by$ are ***[AMU-2000]***

(a) (0, 0), (a, b)
(b) (0, 0), $\left(\frac{2ab^2}{a^2+b^2}, \frac{2ba^2}{a^2+b^2}\right)$
(c) (0, 0), $\left(\frac{a^2+b^2}{a^2}, \frac{a^2+b^2}{b^2}\right)$
(d) None of the above

17. If three circles are such that each inersects the remaining two, then their radical axes ***[MPPET-2010]***

(a) form a triangle (b) are coincident
(c) are concurrent (d) are parallel

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**16.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ	**17.** ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

2. (c) Given circle $2x^2 + 2y^2 - 3x + 6y + 2 = 0$

or $x^2 + y^2 - \frac{3}{2}x + 3y + 1 = 0$

$$\text{Area } = \pi\left(\sqrt{\left(\frac{-3}{4}\right)^2 + \left(\frac{3}{2}\right)^2 - 1}\right)^2 = \frac{29\pi}{16}$$

Required circle is $x^2 + y^2 - \frac{3}{2}x + 3y + \lambda = 0$

$$\text{Area } = 2\times\frac{29\pi}{16}$$

$$\pi\times\left(\sqrt{\left(\frac{-3}{4}\right)^2 + \left(\frac{3}{2}\right)^2 - \lambda}\right)^2 = \frac{38\pi}{16}$$

$$\lambda = \frac{-13}{16}$$

Put in equation to get $16x^2 + 16y^2 - 24x + 48y - 13 = 0$

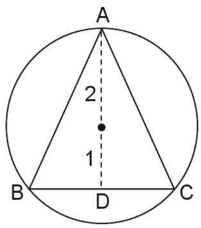
12. (d) $x^2 + y^2 = a^2$ and c_1 (0, 0); $r_1 = a$
$x^2 + y^2 - 2gx + g^2 - b^2 = 0$ touch each other

$C_2(g, 0)$; $r_2 = \sqrt{g^2 - (g^2 - b^2)} = b$

$C_1C_2 = r_1 + r_2 \Rightarrow g = a + b$

13. (d) In equilateral triangle centroid and circumcentre are coincident i.e., origin (Given)

$\therefore \quad OA = \dfrac{2AD}{3} \Rightarrow OA = \dfrac{2a}{3}$



$\therefore \quad$ Equation of circle $x^2 + y^2 = \left(\dfrac{2a}{3}\right)^2$

16. (b) $x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2by$ are 2 circle equation of common chord $ax - by = 0$ or

$y = \dfrac{ax}{b}$

Put in circle 1, $x^2 + \left(\dfrac{a}{b}x\right)^2 = 2ax$

$\Rightarrow \quad x = 0,$

$\Rightarrow \quad x = 0, x = \dfrac{2abh2}{a^2 + b^2}$

$\therefore$ Point are (0, 0) and $\left(\dfrac{2ab^2}{a^2+b^2}, \dfrac{2a^2b}{a^2+b^2}\right)$

(Put values of x in $y = \dfrac{b}{a}x$)

PART E

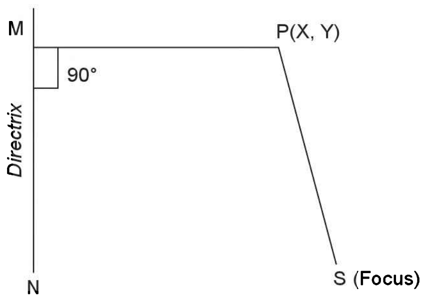
Conic Section

LECTURE 1

Parabola 1

BASIC CONCEPTS

1. **Conic** A conic is the locus of a point P which moves so that its distance from a fixed point S is in a constant ratio to its perpendicular distance from a fixed straight line MN.



$$\frac{PS}{PM} = e = \text{Eccentricity} = \text{ratio}$$

P-**Any** point on the conic

2. **Focus** The fixed point (S) is called the focus.
3. **Directrix** The fixed line (MN) is called the directrix.
4. **Eccentricity** The constant ratio is called the eccentricity and it is denoted by e.
5. (i) If $e = 0$ then conic is a circle.
   (ii) If $e = 1$ then conic is a parabola.
   (iii) If $e < 1$ then conic is ellipse.
   (iv) If $e > 1$ then conic is hyperbola
   (v) If $e = \sqrt{2}$ then conic is rectangular hyperbola.
6. **Axis of the Conic** The line passing through the focus and vertex. Every conic is symmetrical about its axis.

6.1 Parabola has only one axis. Axis is a line through focus and perpendicular to the directrix.

6.2 Ellipse has two axes namely major axis and minor axis.

6.3 Hyperbola has two axes namely Transverse and Conjugate axes.

7. **Vertex** The point of intersection of the conic and its axis is called the vertex of the conic.
8. **Latus Rectum** The latus rectum of a conic is the chord passing through the focus and perpendicular to the axis.
9. **Tangent at the Vertex** Line through vertex and perpendicular to the axis of conic.
10. **Double Ordinate** Any chord which is perpendicular to the axis of the conic is known as its double ordinate.
11. **Focal Distance of any Point $P(x, y)$ on the Conic** Distance of $P(x, y)$ from the focus S of a conic is called the focal distance of the point P.
12. **Focal Chord** Any chord through the focus of a conic is called its focal chord.
13. **General Equation of the Conic** General equation of the conic having focus at (a, b), eccentricity e and directrix the line $lx + my + n = 0$ is

$(l^2 + m^2)\{(x-a)^2 + (y-b)^2\} = e^2(lx + my + n)^2$ ....(1)

Equation (1) represents

(i) circle if $e = 0$
(ii) parabola if $e = 1$
(iii) ellipse if $e < 1$
(iv) Hyperbola if $e > 1$

13.1 If $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$... (i) and

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$ .... (ii)

then the equation (i) represents

(i) A pair of straight line if $\Delta = 0$
(ii) A circle if $\Delta \neq 0$, $a = b$ and $h = 0$.
(iii) A parabola if $\Delta \neq 0$ and $h^2 = ab$.
(iv) An ellipse if $\Delta \neq 0$ and $h^2 < ab$.
(v) A hyperbola if $\Delta \neq 0$ and $h^2 > ab$.
(vi) A rectangle hperbola if $\Delta \neq 0$, $h^2 > ab$ and $a + b = 0$.

14. Centre of the Conic The centre of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$$= \left(\frac{\begin{vmatrix} h & g \\ b & f \end{vmatrix}}{\begin{vmatrix} a & h \\ h & b \end{vmatrix}}, \frac{\begin{vmatrix} g & a \\ f & h \end{vmatrix}}{\begin{vmatrix} a & h \\ h & b \end{vmatrix}} \right)$$

where $\begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2 \neq 0$

The centre of the conic is the solution of following two equation $\frac{\partial f(x,y)}{\partial x} = \frac{\partial f(x,y)}{\partial y} = 0$ or

$ax + hy + g = 0$

$hx + by + f = 0$

NOTE

The conic is symmetrical about its centre and centre of parabola is not defined.

15. Position of Straight Line with Respect to a Conic Solving equations of conic and line we find quadratic in any one variable and interpret three different cases as follows. If both roots of quadratic are

(i) Real and distinct $B^2 > 4AC$, then straight line will be chord to the conic.
(ii) Equal ($B^2 = 4AC$), then straight line will be tangent to the conic.
(iii) Imaginary ($B^2 < 4AC$), then Straight line lies outside the conic i.e., neither touch nor intersect.

16. Double Ordinate A chord perpendicular to the axis of a conic is known as its double ordinate.

17. Parabola A parabola is the locus of a point which moves such that its distance from a fixed point is equal to its distance from a fixed straight line (directrix). Its eccentricity is always equal to unity ($e = 1$) and its centre is not defined.

18. General Equation of a Parabola Equation of the parabola having focus at $S(a, b)$ and directrix $lx + my + n = 0$ is

$(l^2 + m^2)\{(x-a)^2 + (y-b)^2\} = (lx + my + n)^2$

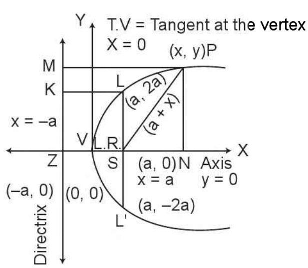
NOTE

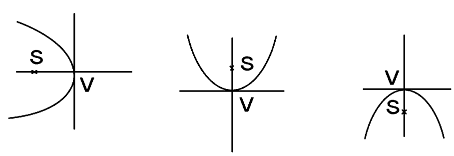
Second degree terms in the equation of parabola only form a perfect square. Therefore general equation of a parabola may be written as $(px - qy)^2 + 2gx + 2fy + c = 0$.

(i) Equation of axis of the parabola $px - qy = 0$
(ii) Equation of the tangent of the vertex $2gx + 2fy + c = 0$

Parabola: Importants results connected with standard forms of a Parabola

(i) Standard forms $y^2 = 4ax$ $y^2 = -4ax$ $x^2 = 4ay$ $x^2 = -4ay$





(ii) Coordinates of vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
(iii) Coordinates of focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
(iv) Equation of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
(v) Equation of the directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
(vi) Equation of the latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$
(vii) Length of the latus rectum	$4a$	$4a$	$4a$	$4a$
(viii) End points of latus rectum	$(a, 2a)$, $(a, -2a)$	$(-a, 2a)$, $(-a, -2a)$	$(2a, a)$, $(-2a, a)$	$(2a, -a)$, $(-2a, -a)$
(ix) Parametric equations	$x = at^2$, $y = 2at$	$x = -at^2$, $y = 2at$	$x = 2at$, $y = at^2$	$x = 2at$, $y = -at^2$
(x) Slope of the parabola	Right handed	Left handed	Upward	Downward
(xi) Equation of tangent at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
(xii) Area of triangle enclosed by the end points of latus rectum and the vertex of the parabola	$2a^2$	$2a^2$	$2a^2$	$2a^2$
(xiii) Parametric coordinates of a point t on the parabola	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
(xiv) Focal distance (PS) of a point $P(x, y)$ on the parabola	$a + x$	$a - x$	$a + y$	$a - y$
(xv) If the point (x_1, y_1) is	$y_1^2 - 4ax_1 < 0$	$y_1^2 + 4ax_1 < 0$	$x_1^2 - 4ay_1 < 0$	$x_1^2 + 4ay_1 < 0$ inside the parabola then
(xvi) If the point (x_1, y_1) is on the parabola then	$y_1^2 - 4ax_1 = 0$	$y_1^2 + 4ax_1 = 0$	$x_1^2 - 4ay_1 = 0$	$x_1^2 + 4ay_1 = 0$
(xvii) If the point (x_1, y_1) is out side the parabola then	$y_1^2 - 4ax_1 > 0$	$y_1^2 + 4ax_1 > 0$	$x_1^2 - 4ay_1 > 0$	$x_1^2 + 4ay_1 > 0$

(xviii) Equation of the tangent to the parabola at a point (x_1, y_1) is	$yy_1 = 2a(x + x_1)$	$yy_1 = -2a(x + x_1)$	$xx_1 = 2a(y + y_1)$	$xx_1 = -2a(y + y_1)$
(xix) If the line $y = mx + c$ is a tangent then the value of c is (condition of tangency) and the co-ordinates of the point of contact are	$c = \frac{a}{m}$ $\left(\frac{a}{m^2}, \frac{2\pi}{m}\right)$	$c = -\frac{a}{m}$ $\left(\frac{a}{m^2}, \frac{-2a}{m}\right)$	$c = -am^2$ $(2am, am^2)$	$c = am^2$ $(-2am, -am^2)$

NOTES

1. Two parabolas are said to be equal if their latus rectum are equal.
2. Distance between vertex and focus = a = one fourth of the L.R., for example $V(0, 0)$ $S(a, 0)$
3. Distance between tangent at the vertex and directrix = a, for example $x = 0$ and $x = -a$
4. Distance between tangent at the vertex and latus rectum is = a, for example $x = 0$ and $x = a$.
5. Distance between directrix and latus rectum = $2a$ = half of the latus rectum. for example $x = -a$ and $x = a$.
6. Directrix,tangent at the vertex and latus rectum are parallel straight lines. There fore distance between any two is obtained as distance between any two parallel lines for example ($x = -a$, $x = 0$, $x = a$).
7. Equation of parabola

7.1 Equation of Parabola reducible in Standard form:
Equation of parabola having vertex $v(h, k)$ and axis
   (i) Parallel to x-axis $(y - k)^2 = 4a(x - h)$ or $x = Ay^2 + By + C$
   (ii) Parallel to y-axis $(x - h)^2 = 4a(y - k)$ or $y = Ax^2 + Bx + C$

If origin be shifted at $v(h, k)$ then:
   (a) Transformed equation of $(y - k)^2 = 4a(x - h)$ is $Y^2 = 4aX$
   (b) Transformed equation of $(x - h)^2 = 4a(y - k)$ is $X^2 = 4aY$
   (c) For finding transformed equation of the parabola substitute $x + h$ for X and $y + k$ for Y in the equation.

7.2 (Perpendicular distance of point $P(x, y))^2$ = Length of L.R. × (perpendicular distance of point $P(x, y)$ from the tangent at the vertex)
   (i) If $V(x_1, y_1)$ and $S(x_2, y_1)$, then equation of parabola $(y - y_1)^2 = 4(x_2 - x_1)(x - x_1)$
   (ii) If $V(x_1, y_1)$ and $S(x_1, y_2)$, then equation of parabola $(x - x_1)^2 = 4(y_2 - y_1)(y - y_1)$

19. (i) **The equation of the chord joining the point** $t_1(at_1^2, 2at_1)$ and $t_2(at_2^2, 2at_2)$ is: $y(t_1 + t_2) = 2(x + at_1t_2)$

(ii) **Condition for the chord to be a focal chord** The chord joining the points $t_1(at_1^2, 2at_1)$ and $t_2(at_2^2, 2at_2)$ passes through focus $(a, 0)$ then $t_1t_2 = -1$ and coordinates of the ends of focal chord be $t(at^2, 2at)$ and $-\frac{1}{t}\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

(iii) **Length of the Focal chord** The length of a focal chord joining the points $t_1(at_1^2, 2at_1)$ and $t_2(at_2^2, 2at_2)$ is $a(t_2 - t_1)^2$

NOTE

The length of the focal chord through the point 't' on the parabola $y^2 = 4ax$ is $a(t + 1/t)^2$.

(iv) Semi Latus rectum is the harmonic mean between segments of a focal chord.

$$SL = 2a = \frac{2(SP)(SQ)}{SP + SQ}$$

(v) If any chord joining the points t_1 and t_2 on the parabola $y^2 = 4ax$ subtends 90° at the vertex then $t_1\ t_2 = -4$, (For normal chord then $t^2 = 2$).

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find the equation of the parabola whose focus is (3, –4) and equation of the directrix is $x + y - 2 = 0$. ***[MP-1995]***

Solution

Let $S(3, -4)$ be the focus and ZM be the directrix, whose equation is $x + y - 2 = 0$ .... (1)

Let $P(x, y)$ be any point on the parabola. Join SP and draw $PM \perp ZM$.

Then, $SP = PM$ .... (2)

$\because \quad SP = \sqrt{(x-3)^2 + (y+4)^2}$

PM = length of perpendicular drawn from $P(x, y)$ to equation (1)

$= \dfrac{x+y-2}{\sqrt{1+1}}$

$\therefore \quad PM = \dfrac{x+y-2}{\sqrt{2}}$

$\therefore$ By equation (2),

$$\sqrt{(x-3)^2 + (y+4)^2} = \frac{x+y-2}{\sqrt{2}}$$

Squaring both the sides,

$$(x-3)^2 + (y+4)^2 = \left(\frac{x+y-2}{\sqrt{2}}\right)^2$$

$\Rightarrow \quad x^2 + 9 - 6x + y^2 + 16 + 8y = \dfrac{(x+y-2)^2}{2}$

$\Rightarrow \quad 2(x^2 + y^2 - 6x + 8y + 25) = x^2 + y^2 + 4 + 2xy - 4x - 4y$

$\Rightarrow \quad x^2 - 2xy + y^2 - 8x + 20y + 46 = 0$

Which is the required equation of parabola.

2. Find the vertex, focus, equation of directrix, axis and tangent at the vertex of the parabola $x^2 + 4x + 4y + 16 = 0$. ***[MP-1997]***

Solution

Equation of parabola is $x^2 + 4x + 4y + 16 = 0$

$\Rightarrow \quad x^2 + 4x + 4 + 4y + 12 = 0$

$\Rightarrow \quad (x + 2)^2 + 4(y + 3) = 0$

$\Rightarrow \quad (x + 2)^2 = -4(y + 3)$ .... (1)

As the origin be transferred to (–2, –3) the reduced coordinates of (x, y) are (X, Y).

Then $x = X - 2$

$y = Y - 3$

Hence, equation (1) reduces to $X^2 = -4Y$

Compare above equation with $X^2 = -4aY$ .... (2)

Here, $4a = 4$

$\Rightarrow \quad a = 1$

(i) Focus of (2) is $(0, -a)$.

i.e., $X = 0, \quad Y = -1$

$\Rightarrow \quad x + 2 = 0$ and $y + 3 = -1$

$\Rightarrow \quad x = -2$ and $y = -4$

Therefore the focus of (1) is (–2, –4).

(ii) Vertex of (2) is (0, 0) .

i.e., $X = 0, \quad Y = 0$

$\Rightarrow \quad x + 2 = 0$ and $y + 3 = 0$

$\Rightarrow \quad x = -2$ and $y = -3$

Therefore, the vertex of (1) is (–2, –3)

(iii) Axis of (2) is $X = 0$

i.e. $\quad x + 2 = 0$

$\Rightarrow \quad x = -2$

Therefore, the axis of (1) is $x = -2$.

(iv) Directrix of (2) is $Y = 1$

Therefore, the directrix of (1) is

$y + 3 = 1 \Rightarrow y = -2$

$\Rightarrow \quad y + 2 = 0$

(v) Tangent at the vertex of (2) is $Y = 0$

Therefore, for (1), it is $y = -3$

3. Find the equation of parabola whose axis is parallel to x-axis and which passes through the points (3, 3), (6, 5) and (6, –3).

Solution

Let equation of the parabola be $x = Ay^2 + By + C$ .... (1)

Point (3, 3) lies on parabola

$\therefore \quad 3 = 9A + 3B + C$ .... (2)

Point (6, 5) lies on parabola

$\therefore \quad 6 = 25A + 5B + C$ ...... (3)

Point (6, –3) lies on parabola, $6 = 9A - 3B + C$ .... (4)

From equation (3) and (4),

$25A + 5B + C = 9A - 3B + C$

$\Rightarrow \quad 16A = -8B$

$\Rightarrow \quad B = -2A$

From equation (2) and (3), $-3 = -16A - 2B$

$\Rightarrow \quad 16A + 2B = 3$

$\Rightarrow \quad 16A + 2(-2A) = 3, (\because B = -2A)$

$\Rightarrow \quad 12A = 3$

$\Rightarrow \quad A = \frac{1}{4}$

$\therefore \quad B = -2 \times \frac{1}{4} = -\frac{1}{2}$

Putting the value of A and B in equation (2),

$$3 = 9 \times \frac{1}{4} - 3 \times \frac{1}{2} + C$$

$\Rightarrow \quad 12 = 9 - 6 + 4C$

$\Rightarrow \quad C = \frac{9}{4}$

Putting the value of A, B and C in equation (1),

$$x = \frac{1}{4}y^2 - \frac{1}{2}y + \frac{9}{4}$$

$\Rightarrow \quad 4x = y^2 - 2y + 9$

Which is the required equation of parabola.

4. A double ordinate of the parabola $y^2 = 4px$ is of length $8p$. Prove that the lines from the vertex to its ends are at right angles.

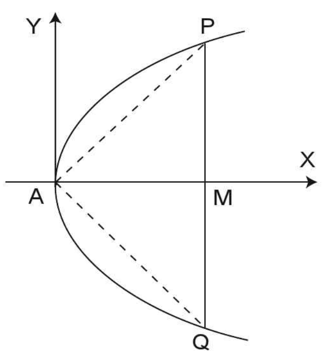
Solution

Let the double ordinate PQ meet the axis of the parabola $y^2 = 4px$ in M and $AM = h$, then $PQ = 8p$

$\therefore$ Coordinates of P are $(h, 4p)$

and coordinates of Q are $(h, -4p)$

Point $P(h, 4p)$ lies on parabola.



$\therefore \quad 16p^2 = 4ph \Rightarrow h = 4p$

$\therefore$ Coordinates of P and Q are $(4p, 4p)$ and $(4p, -4p)$ respectively.

$\because$ Coordinates of vertex A are $(0, 0)$,

$\therefore$ Slope of $AP = \frac{4p-0}{4p-0} = 1 = m_1$

and Slope of $AQ = \frac{4p-0}{-4p-0} = -1 = m_2$

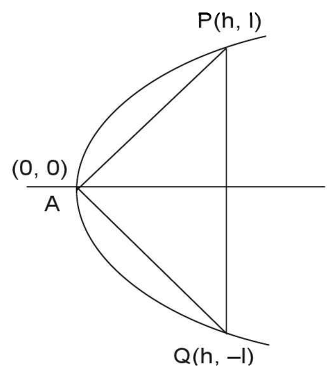
$\therefore \quad m_1 m_2 = -1$ Hence, AP is perpendicular to AQ. **Proved.**

5. An equilateral triangle is inscribed in the parabola $y^2 = 4ax$ whose vertex is at the vertex of parabola. Find the length of its side. ***[NCERT]***

Solution

Let APQ be an equilateral triangle. Coordinates of its vertices A, P, Q be taken as $A(0, 0)$, $P(h, l)$ and $Q(h, -l)$.

$AP^2 = (h - 0)^2 + (l - 0)^2 = h^2 + l^2$



$\Rightarrow \quad AP = \sqrt{h^2 + l^2}$,

Similarly, $AQ = \sqrt{h^2 + l^2}$

and $PQ = \sqrt{(h-h)^2 + (l+l)^2} = 2l$

$\because \quad AP = PQ \Rightarrow \sqrt{h^2 + l^2} = 2l$

$\Rightarrow \quad h^2 + l^2 = 4l^2$

$\Rightarrow \quad h^2 = 3l^2 \Rightarrow h = \sqrt{3}\, l$

P lies on parabola $y^2 = 4ax$

$\therefore \quad l^2 = 4ah \Rightarrow l^2 = 4a\sqrt{3}\, l \Rightarrow l = 4a\sqrt{3}$

but $h = \sqrt{3}\, l = 12a$.

Hence, length of $PQ = 2l = 8a\sqrt{3}$

6. Show that the area of the triangle inscribed in the parabola $y^2 = 4ax$ is $\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$ where y_1, y_2, y_3 are the ordinates of the angular points.

Solution

Let x_1, x_2, x_3 be the abscissa of the vertices of the triangle whose ordinates are y_1, y_2, y_3 respectively.

$\therefore$ Coordinates of the vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3).

Point (x_1, y_1) lies on parabola

$$\therefore \quad y_1^2 = 4ax_1 \Rightarrow x_1 = \frac{y_1^2}{4a}$$

Similarly, $x_2 = \frac{y_2^2}{4a}$ and $x_3 = \frac{y_3^2}{4a}$

$\therefore$ Vertices of the triangle are

$$\left(\frac{y_1^2}{4a}, y_1\right), \left(\frac{y_2^2}{4a}, y_2\right) \text{ and } \left(\frac{y_3^2}{4a}, y_3\right)$$

$$\therefore \quad \text{Area of the triangle} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} \frac{y_1^2}{4a} & y_1 & 1 \\ \frac{y_2^2}{4a} & y_2 & 1 \\ \frac{y_3^2}{4a} & y_3 & 1 \end{vmatrix}$$

Taking out $\frac{1}{4a}$ common from first column,

$$= \frac{1}{2}\cdot\frac{1}{4a}\begin{vmatrix} y_1^2 & y_1 & 1 \\ y_2^2 & y_2 & 1 \\ y_3^2 & y_3 & 1 \end{vmatrix}$$

Now, operating $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$,

$$= \frac{1}{8a}\begin{vmatrix} y_1^2 - y_2^2 & y_1 - y_2 & 0 \\ y_2^2 - y_3^2 & y_2 - y_3 & 0 \\ y_3^2 & y_3 & 1 \end{vmatrix}$$

Expanding along C_3,

$$= \frac{1}{8a}\begin{vmatrix} y_1^2 - y_2^2 & y_1 - y_2 \\ y_2^2 - y_3^2 & y_2 - y_3 \end{vmatrix}$$

$$= \frac{1}{8a}\begin{vmatrix} (y_1 + y_2)(y_1 - y_2) & y_1 - y_2 \\ (y_2 + y_3)(y_2 - y_3) & y_2 - y_3 \end{vmatrix}$$

$$= \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)\begin{vmatrix} y_1 + y_2 & 1 \\ y_2 + y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)[(y_1 + y_2) - (y_2 + y_3)]$$

$$= \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_1 - y_3)$$

$$= \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1),$$

(in magnitude). **Proved.**

7. Prove that the equation $y^2 + 2Ax + 2By + C = 0$ represents a parabola whose axis is parallel to the axis of X. Find the vertex and latus rectum of the parabola.

Solution

Given equation is $y^2 + 2Ax + 2By + C = 0$

$$\Rightarrow \quad y^2 + 2By = -2Ax - C$$

$$\Rightarrow \quad y^2 + B^2 + 2By = -2Ax + B^2 - C$$

$$\Rightarrow \quad (y + B)^2 = -2A\left(x - \frac{B^2 - C}{2A}\right)$$

Let $X = x - \frac{B^2 - C}{2A}$ and $Y = y + \text{B}$, then equation of the parabola be $Y^2 = -2AX$, which is of the form of $y^2 = 4ax$, where $4a = -2A$.

Obviously, given equation represents parabola. Axis of the parabola: $Y = 0$ **Proved**

$$\Rightarrow \quad y + B = 0$$

$$\Rightarrow \quad y = -B$$

Which is parallel to X-axis.

For vertex: $X = 0$, $Y = 0$

$$\Rightarrow \quad x - \frac{B^2 - C}{2A} = 0,\ y + B = 0$$

$$\Rightarrow \quad x = \frac{B^2 - C}{2A}, y = -B$$

$\therefore$ Vertex of the parabola is $\left(\frac{B^2 - C}{2A}, -B\right)$.

Latus rectum of the parabola $= 4a = -2A$

8. Find the equation of parabola whose focus is at the point (1, –1) and vertex is (2, 1).

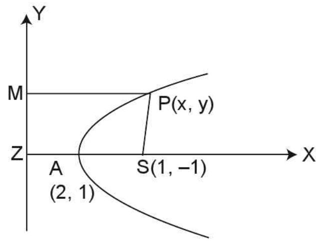
Solution

Vertex A is at (2, 1) and focus S is at (1, –1). Suppose coordinates of Z are (α, β).

$\therefore$ A is mid-point of SZ,

$$\therefore \quad \frac{\alpha+1}{2} = 2 \Rightarrow \alpha = 3$$

$$\text{and} \quad \frac{\beta-1}{2} = 1 \Rightarrow \beta = 3$$



$\therefore$ Coordinates of Z is (3, 3), Slope of $ZS = \frac{1+1}{2-1} = 2$

Slope of directrix $= -\frac{1}{2}$

Equation of directrix is $y - 3 = -\frac{1}{2}(x-3)$

$\Rightarrow$ $2y - 6 = -x + 3 \Rightarrow x + 2y - 9 = 0$

Suppose $P(x, y)$ is any point on the parabola, by definition of parabola $PS = PM$

$$\Rightarrow \sqrt{(x-1)^2+(y+1)^2} = \frac{x+2y-9}{\sqrt{1+4}}$$

$$\Rightarrow (x-1)^2 + (y+1)^2 = \frac{(x+2y-9)^2}{5}$$

$\Rightarrow$ $5[x^2 - 2x + 1 + y^2 + 2y + 1] = x^2 + 4y^2 + 81 + 4xy - 18x - 36y$

$\Rightarrow$ $4x^2 - 4xy + y^2 + 8x + 46y - 71 = 0$

Which is the required equation of parabola.

9. Find the vertex, axis, focus and latus rectum of parabola: $9y^2 - 16x - 12y + 57 = 0$

Solution

Given equation of the parabola is

$$9y^2 - 16x - 12y + 57 = 0$$

$$\Rightarrow 9y^2 - 12y = 16x - 57$$

$$\Rightarrow y^2 - \frac{12}{9}y = \frac{1}{9}(16x-27)$$

$$\Rightarrow y^2 - \frac{4}{3}y = \frac{16}{9}x - \frac{59}{9}$$

$$\Rightarrow y^2 - \frac{4}{3}y + \frac{4}{9} = \frac{16}{9}x - \frac{57}{9} + \frac{4}{9}$$

$$\Rightarrow \left(y-\frac{2}{3}\right)^2 = \frac{16}{9}x - \frac{53}{9}$$

$$\Rightarrow \left(y-\frac{2}{3}\right)^2 = \frac{16}{9}\left(x-\frac{53}{16}\right)$$

$$\Rightarrow x - \frac{53}{16} = X \text{ and } y - \frac{2}{3} = Y$$

Then new equation of the parabola is

$$Y^2 = \frac{16}{9}X^2$$

Which is of the form of $y^2 = 4ax$.

Here, $a = \frac{4}{9}$

For vertex: $X = 0$, $Y = 0$

$$\Rightarrow x - \frac{53}{16} = 0,\ y - \frac{2}{3} = 0$$

$$\Rightarrow x = \frac{53}{16},\ y = \frac{2}{3}$$

$\therefore$ Vertex is $\left(\frac{53}{16}, \frac{2}{3}\right)$

$\therefore$ For Axis $Y = 0$ $\Rightarrow y - \frac{2}{3} = 0$

$\therefore$ Axis is $y = \frac{2}{3}$

For focus: $X = a$, $Y = 0$

$$\Rightarrow x - \frac{53}{16} = \frac{4}{9},\ y - \frac{2}{3} = 0$$

$$\Rightarrow x = \frac{4}{9} + \frac{53}{16},\ y = \frac{2}{3}$$

$$\Rightarrow x = \frac{64+177}{144},\ y = \frac{2}{3}$$

$$\Rightarrow x = \frac{541}{144},\ y = \frac{2}{3}$$

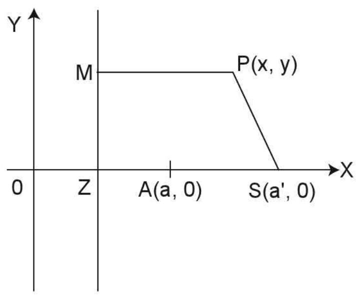
$\therefore$ Focus is $\left(\frac{541}{144}, \frac{2}{3}\right)$

Latus rectum $= 4a = 4 \times \frac{4}{9} = \frac{16}{9}$

10. Prove that the equation of the parabola whose vertex and focus are on the axis of X at distances a and a' from origin, respectively is $y^2 = 4(a' - a)(x - a)$.

Solution

Let A be the vertex and S the focus of the parabola, which lie on the axis of X. The directrix meets the axis of X at Z.



Let $Z \to (\alpha, \beta)$

Since, A is the mid-point of ZS,

$\therefore \quad \dfrac{\alpha + a'}{2} = a$ and $\dfrac{\beta + 0}{2} = 0$

Hence, $\alpha = 2a - a'$ and $\beta = 0$

Therefore the coordinates of Z are $(2a - a', 0)$.
Equation of the directrix is $x = 2a - a'$

$\Rightarrow \quad x - 2a + a' = 0$

Let $P(x, y)$ be any point on parabola. Draw $PM \perp ZM$.

$SP = PM$, (by definition of parabola)

$$\Rightarrow \quad \sqrt{(x - a')^2 + y^2} = x - 2a + a'$$

Squaring the both sides,

$\Rightarrow \quad (x - a')^2 + y^2 = (x - 2a + a')^2$

$\Rightarrow \quad y^2 = (x - 2a + a')^2 - (x - a')^2$

$= [x - 2a + a' + x - a'] [x - 2a + a' - x + a']$

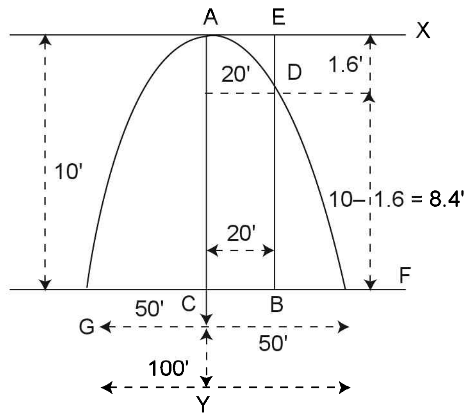
$= (2x - 2a)(2a' - 2a)$

$\Rightarrow \quad y^2 = 4(a' - a)(x - a)$ **Proved**.

11. The Girder of a bridge is in the form of parabola. Its height is 10 feet and the base is 100 feet wide. How high is it 20 feet from the centre ?

Solution

Let AX and AY be taken as the axis of X and Y respectively.



Equation of parabola, $x^2 = 4ay$ .... (1)

Given: $CA = 10'$, $FG = 100'$, $CF = CG = 50'$

$\therefore$ Coordinates of F (50, 10).

Since the parabola passes through (50, 10),

$\therefore \quad (50)^2 = 4a \cdot 10 \Rightarrow 4a = 250$

By equation (1), $x^2 = 250y$ .... (2)

When $x = 20$, then by equation (2), $(20)^2 = 250y$

or $\quad y = \dfrac{400}{250} = 1.6$

$\therefore$ Coordinates of D be (20, 1.6).

The height of girder at the distance $20'$ $(BD) = BE - DE = 10 - 1.6 = 8.4'$.

12. The axis of parabola is $3x - 4y + 5 = 0$, its vertex is (1, 2) and latus rectum is 8 in length, then prove that equation of the parabola is $9x^2 + 26y^2 - 24xy - 130x - 160y + 425 = 0$

Solution

Equation of the axis is $3x - 4y + 5 = 0$ .... (1)

Equation of the line perpendicular to line (1) is $4x + 3y + \lambda = 0$

If it passes through vertex $A(1, 2)$, then

$4 \times 1 + 3 \times 2 + \lambda = 0$

$\Rightarrow \quad 10 + \lambda = 0 \Rightarrow \lambda = -10$

$\therefore$ Equation of the line passing through vertex $A(1, 2)$ and perpendicular to the axis is $4x + 3y - 10 = 0$ ...... (2)

Let $P(x, y)$ be any point on the parabola.

∴ Length of the perpendicular drawn from $P(x, y)$ to the line (1) is,

$$PN = \frac{3x - 4y + 5}{\sqrt{9+16}} = \frac{3x + 4y + 5}{5}$$

Again, length of the perpendicular drawn from $P(x, y)$ to the line (2) is,

$$PM = \frac{4x + 3y - 10}{\sqrt{16+9}} = \frac{4x + 3y - 10}{5}$$

∴ Equation of the parabola, $Y^2 = 4aX$

⇒ $PN^2 = 8\,PM$,

$[\because 4a = 8]$

⇒ $\left(\frac{3x - 4y + 5}{5}\right)^2 = 8.\frac{4x + 3y - 10}{5}$

⇒ $(3x - 4y + 5)^2 = 40\,(4x + 3y - 10)$

⇒ $9x^2 + 16y^2 - 24xy + 30\,x - 40y + 25 = 160\,x + 120y - 400$

⇒ $9x^2 + 16y^2 - 24xy - 130x - 160y + 425 = 0$

Proved.

13. If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus. ***[NCERT]***

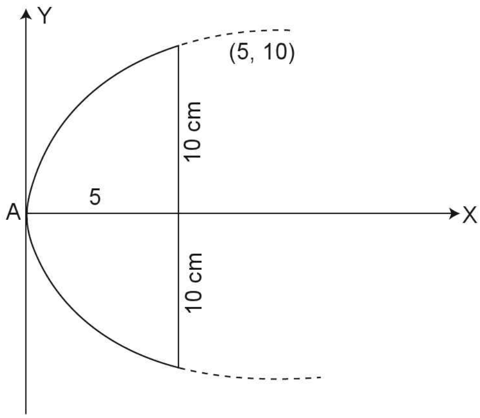
Solution

Let the equation of the parabola be taken as $y^2 = 4ax$, then the point (5, 10) must lie on it

⇒ $(10)^2 = 4a \times 5$

⇒ $a = \frac{100}{20} = 5.$

⇒ The focus of the reflector is at the mid-point of the given diameter.



14. An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola? ***[NCERT]***

Solution

In this case, we can take the equation of the parabolic arch as $x^2 = -4ay$ ....(i)

(Downward parabola)

According to given conditions, the point (2.5, –10),

i.e., $\left(\frac{5}{2}, -10\right)$ lies on the arch, therefore,

$$\left(\frac{5}{2}\right)^2 = -4a(-10)$$

⇒ $a = \frac{25}{4 \times 40} = \frac{5}{32}$

⇒ Equation (i) becomes

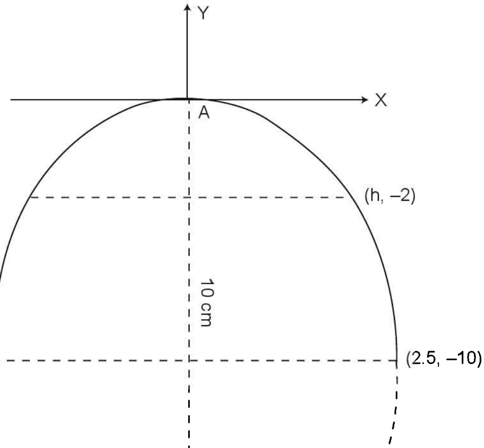
$$x^2 = -4\left(\frac{5}{32}\right)y \qquad \text{..........(ii)}$$

Let, $2h$ metre be the width of the arch 2 metre below the vertex, then the point $(h, -2)$ lies on (ii)

⇒ $h^2 = -\frac{5}{8}(-2) \Rightarrow h^2 = \frac{5}{4}$

⇒ $h = \frac{\sqrt{5}}{2} \Rightarrow 2h = \sqrt{5}$

Hence, the required width of the arch (2 m below the vertex) = $2h$ metre $= \sqrt{5}$ metre = 2.236 m.



15. The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle. ***[NCERT]***

Solution

Here, the cable is in the form of an upward parabola, whose equation (w.r.t axes as shown) can be written as $x^2 = 4ay$ ....(i)

Since the top of the longest wire above the lowest point A (vertex) of the cable is (30 – 6) m = 24 m, therefore, the points (–50, 24) and (50, 24) lie on the parabola (i).

$\Rightarrow \quad (50)^2 = 4a \times 24$

$\Rightarrow \quad a = \dfrac{2500}{96} = \dfrac{625}{24}.$

$\therefore$ The equation (i) becomes

$$x^2 = 4\left(\frac{625}{24}\right)y,$$

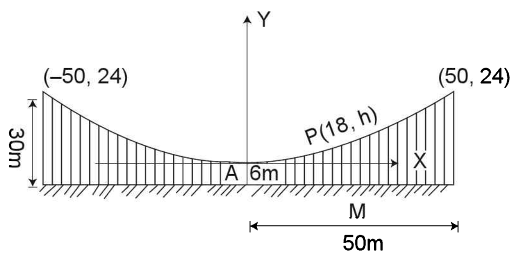
i.e., $x^2 = \dfrac{625}{6}y,$ ....(ii)

Let, M be the point 18m away from the middle and (6 + h) m be the vertical supporting at M, then the point $P(18, h)$ lies on the cable, i.e., on (ii).

$\therefore \quad (18)^2 = \left(\dfrac{625}{6}\right)h$

$\Rightarrow \quad h = \dfrac{(18)^2 \times 6}{625} = 3.11$ nearly.

$\therefore$ Height of the vertical supporting at M = (6 + h) m = (6 + 3.11) m = 9.11 m. nearly.



16. The focus of a parabolic mirror as shown in figure is at a distance of 6 cm from its vertex. If the mirror is 20 cm deep , find the distance LM.

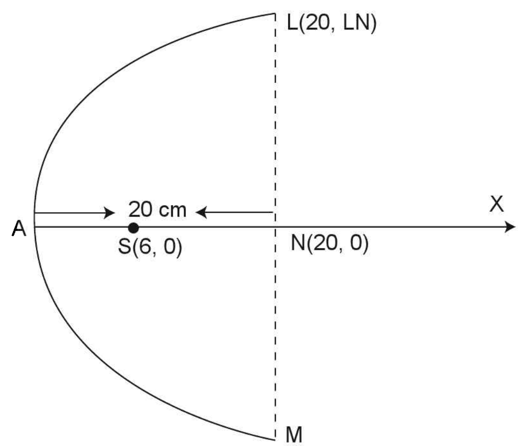
Solution

Let the axis of the mirror be along the positive direction of x-axis and the vertex A be the origin. Since the focus is at a distance of 6 cm from the vertex.Then the coordinates of the focus are (6, 0). Therefore, the equation of the parabolic section is $y^2 = 24x$ [Putting a = 6 in $y^2 = 4ax$] Since L (20, LN) lies on this parabola. Therefore,

$LN^2 = 24 \times 20$

$\Rightarrow \quad LN = 4\sqrt{30}$

$\therefore \quad LM = 2LN = 8\sqrt{30}$ cm.



17. The towers of a bridge, hung in the form of a parabola, have their tops 30 m above the roadway and are 200 metres apart. If the cable is 5 m above the roadway at the centre of the bridge, find the length of the vertical supporting cable 30 metres from the centre.

Solution

Let CAB be the bridge and $X'OX$ be the roadway. Let A be the centre of the bridge. Taking $X'OX$ as x-axis and y-axis along OA, we find that the coordinates of A are (0, 5). Clearly, the bridge is in the shape of a parabola having its vertex at A (0, 5). Let its equation be $x^2 = 4a(y - 5)$ ....(i)

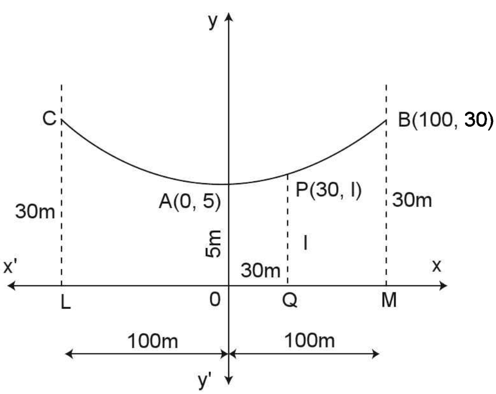
It passes through B (100, 30). Therefore, $(100)^2 = 4a(30 - 5) \Rightarrow a = 100$.

Putting the value of a in (i), we get $x^2 = 400(y - 5)$ ....(ii)

Let, l metres be the length of the vertical supporting cable 30 metres from the centre. Then, $P(30, l)$ lies on (ii). Therefore, $900 = 400(l - 5)$

$$\Rightarrow \quad l = \frac{9}{4} + 5 = \frac{29}{4} \text{ m.}$$

Hence, the length of the vertical supporting cable 30 metres from the centre of the bridge is $\frac{29}{4}$ m.

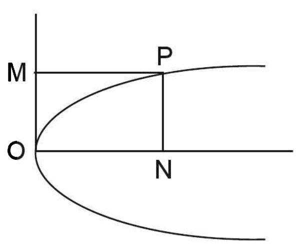


18. The axes of a parabola is $3x + 4y + 6 = 0$, its vertex is $(-2, 0)$ and latus rectum is 4 in length, then prove that equation of the parabola is $9x^2 + 24xy + 16y^2 - 44x + 108y - 124 = 0$.

Solution

The axis of the parabola is $3x + 4y + 6 = 0$ ...... (1)

Its gradient = – 3/4



$\therefore$ Gradient of its perpendicular line = 4/3

Equation of the line perpendicular to the line (1) and passing through vertex $A(-2, 0)$ is

$$y - 0 = \frac{4}{3}(x + 2)$$

$\Rightarrow$ $3y = 4x + 8$

$\Rightarrow$ $4x - 3y + 8 = 0$ ....(2)

Let, $P(x, y)$ be any point on the parabola.

$\therefore$ Length of the perpendicular drawn from $P(x, y)$ to line (1) is, $PN = \frac{2x + 4y + 6}{\sqrt{3^2 + 4^2}}$

$= \frac{3x + 4y + 6}{5}$ and length of the perpendicular drawn from $P(x, y)$ to the line (2) is, $PM = \frac{4x - 3y + 8}{\sqrt{4^2 + (-3)^2}} = \frac{4x - 3y + 8}{5}$

$\therefore$ Equation of the parabola is $Y^2 = 4aX$

$\Rightarrow$ $PN^2 = 4.\ PM$, [$\because$ here $4a = 4$]

$$\Rightarrow \left(\frac{3x + 4y + 6}{5}\right)^2 = 4\left(\frac{4x - 3y + 8}{5}\right)$$

$$\Rightarrow (3x + 4y + 6)^2 = 25 \times \frac{4}{5}(4x - 3y + 8)$$

$\Rightarrow$ $(3x + 4y + 6)^2 = 20(4x - 3y + 8)$

$\Rightarrow$ $9x^2 + 24xy + 16y^2 + 36x + 48y + 36 = 80x - 60y + 160$

$\Rightarrow$ $9x^2 + 24xy + 16y^2 - 44x + 108y - 124 = 0$

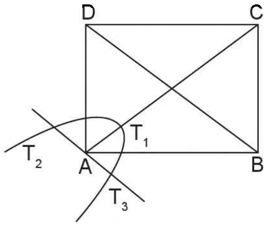
This is the required equation of the parabola. **Proved.**

19. Let $ABCD$ be a square of side length 2 units. A line M is drawn through A parallel to BD. A point S moves such that its distances from the line BD and the vertex A are equal. If focus of S cuts M at T_2 and T_3, and AC at T_1, then area of $\Delta\, T_1T_2T_3$ is ***[IIT-JEE-2006]***

(a) 2 (b) 1
(c) 1/2 (d) 4

Solution

(b) Obviously locus of S is a parabola with focus at A and directrix along BD. Obviously AC is the axis of this parabola and its vertex is at T_1 and line M is along its LR. So T_2, T_3 are end points of LR.



Since $AC = \sqrt{4+4} = 2\sqrt{2}$

$\therefore \quad AT_1 = \frac{1}{4}AC = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad T_2T_3 = 4\left(\frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{2}}$

$\therefore \quad$ area of $\Delta T_1T_2T_3 = \frac{1}{2}.\frac{1}{\sqrt{2}}.\frac{4}{\sqrt{2}} = 1$

20. Prove that the area of triangle inscribed in a parabola is twice the area of the triangle formed by the tangents at the vertices of the triangle. ***[IIT-1996]***

Solution

Let A, B, C be the points $(at_1^2, 2at_1)$ etc. Let P, Q, R be the points of intersection of tangents at B, C; C, A and A, B.

$\therefore \quad P = \{at_2t_3, a(t_2 + t_3)\}$ etc.

Area of $\Delta\, ABC = \frac{1}{2}\begin{vmatrix} at_1^2 & 2ay_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}$

Apply $R_1 - R_2$, $R_2 - R_3$

$$= \frac{1}{2}.a.2a\begin{vmatrix} t_1^2 - t_2^2 & t_1 - t_2 & 0 \\ t_2^2 - t_3^2 & t_2 - t_3 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}$$

$$= a^2.(t_1 - t_2)(t_2 - t_3)\begin{vmatrix} t_1 + t_2 & 1 \\ t_2 + t_3 & 1 \end{vmatrix}$$

$$= a^2(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)$$
$$= a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$$

Area of $\Delta\, PQR$

$$= \frac{1}{2}\begin{vmatrix} at_2t_3 & a(t_2 + t_3) & 1 \\ at_3t_1 & a(t_3 + t_1) & 1 \\ at_1t_2 & a(t_1 + t_2) & 1 \end{vmatrix}$$

Apply $R_1 - R_3$ and $R_2 - R_3$

$$= \frac{1}{2}a.a\begin{vmatrix} t_3(t_2 - t_1) & (t_2 - t_1) & 0 \\ t_1(t_3 - t_2) & (t_3 - t_2) & 0 \\ at_1t_2 & a(t_1 + t_2) & 1 \end{vmatrix}$$

$$= \frac{1}{2}a^2.(t_2 - t_1)(t_3 - t_2)\begin{vmatrix} t_3 & 1 \\ t_1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}a^2.(t_2 - t_1)(t_3 - t_2)(t_3 - t_1)$$

$$= \frac{1}{2}a^2.(t_1 - t_2)(t_2 - t_3)(t_3 - t_1) = \frac{1}{2}\Delta\, ABC$$

$\therefore \quad \Delta\, ABC = 2\Delta\, PQR$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

Dirction 1 to 4. Find the coordinates of the focus, axis, equation of the directrix and the length of the latus rectum of the following parabola:

1. $y^2 = 12x$ ***[NCERT]***

2. $x^2 = 6y$ ***[NCERT]***

3. $y^2 = -8x$ ***[NCERT]***

4. $x^2 = -16y$ ***[NCERT]***

Dirction 5 to 9. Find the equation of the parabola that satisfies the given conditions:

5. Focus (6, 0); directrix $x = -6$ ***[NCERT]***

6. Focus (0, –3); directrix $y = 3$ ***[NCERT]***

7. Vertex (0, 0); focus (–2, 0). ***[NCERT]***

8. Vertex (0, 0), passing through (2, 3) and axis is along x-axis. ***[NCERT]***

9. Vertex (0, 0), passing through (5, 2) and symmetric with respect to y-axis. ***[NCERT]***

10. Write the equation of the parabola with the line $x + y = 0$ as directrix and the point (1, 0) as focus.

11. The focus of a parabola is (1, 5) and its directrix is $x + y + 2 = 0$. Find the equation of the parabola, its vertex and length of latus rectum.

12. The parabola $y^2 = 4px$ goes through the point (3, –2). Obtain the length of latus rectum and the coordinates of the focus.

13. An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 1.5 m from the vertex of the parabola?

14. Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 24y$ to the ends of its latus-rectum.

15. Find the foci, vertices, directrices and axes of the following parabolas

(i) $y^2 - 8y - x + 19 = 0$

(ii) $y = x^2 - 2x + 3$

ANSWERS

1. $F(3, 0)$, axis-x-axis, directrix $x = -3$, length of the Latus rectum = 12

2. $F(0, 3/2)$, axis-y-axis, directrix $y = -3/2$, length of the Latus rectum = 6

3. $F(-2, 0)$, axis – x-axis, directrix $x = 2$, length of the Latus rectum = 8

4. $F(0, -4)$, axis y-axis, directrix $y = 4$, length of the Latus rectum = 16

5. $y^2 = 24x$

6. $x^2 = -12y$

7. $y^2 = -8x$

8. $2y^2 = 9x$

9. $2x^2 = 25y$

10. $(x - y)^2 - 4x + 2 = 0$

11. Equation of parabola $x^2 - 2xy + y^2 - 8x - 24y + 48 = 0$, vertex (–1, 3), latus rectum $= 8\sqrt{2}$

12. $\frac{4}{3}; \left(\frac{1}{3}, 0\right)$

13. 1.936 m

14. 72 sq. units

15. (i) $\left(\frac{3}{8}, \frac{1}{2}\right), \left(\frac{3}{8}, \frac{9}{16}\right)$, $y = \frac{5}{8}, x = \frac{3}{8}$

(ii) $\left(1, \frac{9}{4}\right)$, (1; 2), $y = \frac{7}{4}, x = 1$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The equation of the parabola with focus (a, b) and directrix $\frac{x}{a} + \frac{y}{b} = 1$ is given by ***[PET-97]***

(a) $(ax - by)^2 - 2a^3x - 2b^3y + a^4 + a^2b^2 + b^4 = 0$

(b) $(ax + by)^2 - 2a^3x - 2b^3y - a^4 + a^2b^2 - b^4 = 0$

(c) $(ax - by)^2 + a^4 + b^4 - 2a^3x = 0$

(d) $(ax - by)^2 - 2a^3x = 0$

Solution

(a) $(x - a)^2 + (y - b)^2 = \left(\frac{bx + ay - ab}{\sqrt{a^2 + b^2}}\right)^2$

2. The equation of the parabola whose focus is (5, 3) and directrix is $3x - 4y + 1 = 0$ is ***[PET-2002]***

(a) $(4x + 3y)^2 - 256x - 142y + 849 = 0$
(b) $(4x - 3y)^2 - 256x - 142y + 849 = 0$
(c) $(3x + 4y)^2 - 142x - 256y + 849 = 0$
(d) $(3x - 4y)^2 - 256x - 142y + 849 = 0$

Solution

(a) $PM^2 = PS^2$

$\Rightarrow \quad (x-5)^2 + (y-3)^2 = \left(\dfrac{3x-4y+1}{\sqrt{9+16}}\right)^2$

$\Rightarrow \quad 25(x^2 + 25 - 10x + y^2 + 9 - 6x) = 9x^2 + 16y^2 + 1 - 12xy + 6x - 8y - 12xy$

$\Rightarrow \quad 16x^2 + 9y^2 - 256x - 142y + 24xy + 849 = 0$

$\Rightarrow \quad (4x + 3y)^2 - 256x - 142y + 849 = 0$

NOTE

We know that the equation of parabola whose focus is (a, b) and the directrix $lx + my + n = 0$ is $(l^2 + m^2)\{(x-a)^2 + (y-b)^2\} = c^2\{lx + my + n)\}^2$.

3. For the parabola $y^2 + 6y - 2x + 5 = 0$
(I) The vertex is $(-2, -3)$
(II) The directrix is $y + 3 = 0$
Which of the following is correct?
[EAMCET-2007]

(a) both I and II are true
(b) I is true, II is false
(c) I is false, II is true
(d) both I and II are false

Solution

(b) $y^2 + 6y - 2x + 5 = 0$

$\Rightarrow \quad (y+3)^2 = 2(x+3)$

I: Vertex of parabola is $(-2, -3) = (h, k)$

II: Equation of the directrix is $x - h + a = 0$

$\Rightarrow \quad 2x + 5 = 0$

$\therefore \quad$ I is true and II is false.

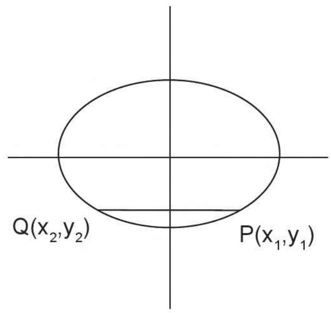
4. Let, $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are ***[IIT JEE-2008]***

(a) $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$
(b) $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$
(c) $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$
(d) $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$

Solution

(b, c) The ellipse is $x^2 + 4y^2 = 4$,

i.e., $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$



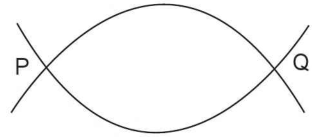
We have $b^2 = a^2(1 - e^2)$

$\Rightarrow \quad e = \dfrac{\sqrt{3}}{2}$

P and Q are obtained as $P \equiv \left(\sqrt{3}, -\dfrac{1}{2}\right)$,

$Q \equiv \left(-\sqrt{3}, -\dfrac{1}{2}\right)$

We have $PQ = 2\sqrt{3}$ with PQ as latus rectum two parabolas are possible, their vertices being $\left(0, \dfrac{\sqrt{3}-1}{2}\right)$ and $\left(0, -\dfrac{\sqrt{3}-1}{2}\right)$



The equation to the parabolas can be simply obtained as $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$ and $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$

5. Axis of a parabola is along the line $y = x$ and its focus and vertex are at distances $2\sqrt{2}$ and $\sqrt{2}$ respectively from the origin. If vertex and focus lie in first quadrant, then the equation of the parabola will be
[IIT JEE-2006]

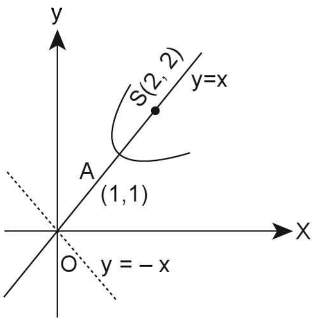
(a) $(x + y)^2 = 8(x + y - 2)$
(b) $(x + y)^2 = 4(x + y - 2)$

(c) $(x - y)^2 = 8(x + y - 2)$
(d) $(x - y)^2 = 4(x + y - 2)$

Solution

(c) Obviously vertex $A = (1, 1)$ and focus $S = (2, 2)$. Since $AS = 1$, so directrix passes through the origin. Since it is perpendicular to the axis $(y = x)$ so its equation is $y = -x$.

Hence, equation of the parabola will be



$$(x-2)^2 + (y-2)^2 = \left(\frac{x+y}{\sqrt{2}}\right)^2$$

$\Rightarrow$ $2(x^2 + y^2 - 4x - 4y + 8) = x^2 + y^2 + 2xy$

$\Rightarrow$ $x^2 + y^2 - 2xy = 8x + 8y - 16$

$\Rightarrow$ $(x - y)^2 = 8(x + y - 2)$

6. The locus of the vertices of the family of parabolas $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ is **[AIEEE-2006]**

(a) $xy = 3/4$
(b) $xy = 35/16$
(c) $xy = 64/105$
(d) $xy = 105/64$

Solution

(d) Given equation can be written as

$$\left(x + \frac{3}{4a}\right)^2 = \frac{3}{a^3}\left(y + \frac{35a}{16}\right)$$

If its vertex be (x, y), then

$$x = -\frac{3}{4a},\ y = -\frac{35a}{16}$$

Eliminating parameter a, we get the required locus of the vertices as $xy = 105/64$.

7. If the point $(at_1^2, 2at_1)$ is one extremity of a focal chord of the parabola $y^2 = 4ax$, then coordinates of the other extremity and that the length of the chord is:

(a) $a\left(t_1 + \frac{1}{t_1}\right)^2$
(b) $a\left(t_1 + \frac{t_1}{1}\right)^2$
(c) $a\left(t_1 - \frac{1}{t_1}\right)^2$
(d) $a\left(t_1 + \frac{t_1}{1}\right)^2$

Solution

(a) Let PSQ be a focal chord where the points P and Q be t_1 and t_2. Since, the chord is focal, we have $t_1t_2 = -1$ ....(1)

$$PSQ = PS + SQ = (a + x_p) + (a + x_Q)$$

$$= 2a + a(t_1^2 + t_2^2) = a\left[2 + t_1^2 + \frac{1}{t_1^2}\right] \text{ by (1)}$$

$$= a\left[t_1 + \frac{1}{t_1}\right]^2$$

8. Consider a circle with its centre lying on the focus of the parabola $y^2 = 2px$ such that it touches the directrix of the parabola. Then, a point of intersection of the circle and the parabola is **[IIT-1995]**

(a) $\left(\frac{p}{2}, p\right)$
(b) $\left(\frac{P}{2}, -P\right)$
(c) $\left(\frac{-p}{2}, p\right)$
(d) $\left(\frac{-p}{2}, -p\right)$

Solution

(a, b) Focus of parabola $y^2 = 2px$ is $(p/2, 0)$ ..... (i)

$\therefore$ Radius of circle whose centre is $(p/2, 0)$ and touching $x + (p/2) = 0$ is p.

Equation of circle is $\left(x - \frac{p}{2}\right)^2 + y^2 = p^2$ .. (ii)

From (i) and (ii), we get the point of intersection $\left(\frac{p}{2}, p\right), \left(\frac{p}{2}, -p\right)$.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The focus of the parabola $x^2 = -16y$ is *[RPET-87; PET-88, 92]*
(a) (4, 0) (b) (0, 4)
(c) (–4, 0) (d) (0, –4)

2. The length of the latus rectum of the parabola $9x^2 - 6x + 36y + 19 = 0$ is *[PET-94]*
(a) 36 (b) 9
(c) 6 (d) 4

3. The focus of the parabola $y = 2x^2 + x$ is *[PET-2000]*
(a) (0, 0) (b) (1/2, 1/4)
(c) (–1/4, 0) (d) (–1/4, 1/8)

4. The point of the parabola $y^2 = 18x$, for which the ordinate is three times the abscissa is *[PET-2003]*
(a) (6, 2) (b) (–2, –6)
(c) (3, 18) (d) (2, 6)

5. The equation of the parabola whose vertex is at (2, –1) and focus at (2, –3) is *[Kerala (Engg.)-2002]*
(a) $x^2 + 4x - 8y - 12 = 0$
(b) $x^2 - 4x + 8y + 12 = 0$
(c) $x^2 + 8y = 12$
(d) $x^2 - 4x + 12 = 0$

6. $x - 2 = t^2$, $y = 2t$ are the parametric equations of the parabola
(a) $y^2 = 4x$
(b) $y^2 = -4x$
(c) $x^2 = -4y$
(d) $y^2 = 4(x - 2)$

7. The area of the triangle formed by the line joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum is *[NCERT]*
(a) 12 sq. units (b) 16 sq. units
(c) 18 sq. units (d) 24 sq. units

8. If the vertex of a parabola be at origin and directrix be $x + 5 = 0$, then its latus rectum is *[RPET-99]*
(a) 5 (b) 10
(c) 20 (d) 40

9. A parabola passing through the point (–4, –2) has its vertex at the origin and y-axis as its axis. The latus rectum of the parabola is
(a) 6 (b) 8
(c) 10 (d) 12

10. The equation of the parabola with vertex at the origin and directrix $y = 2$ is *[MPPET-2005]*
(a) $y^2 = 8x$ (b) $y^2 = -8x$
(c) $y^2 = \sqrt{8}x$ (d) $x^2 = -8y$

11. If a focal chord of the parabola $y^2 = ax$ is $2x - y - 8 = 0$, then the equation of the directrix is *[MPPET-2005]*
(a) $x + 4 = 0$ (b) $x - 4 = 0$
(c) $y - 4 = 0$ (d) $y + 4 = 0$

12. The number of parabolas that can be drawn if two ends of the latus rectum are given, is *[DCE-2005]*
(a) 1 (b) 2
(c) 4 (d) 3

13. Vertex of the parabola $x^2 + 4x + 2y - 7 = 0$ is *[PET-90]*
(a) (–2, 11/2) (b) (–2, 2)
(c) (–2, 11) (d) (2, 11)

14. Focus of the parabola $(y - 2)^2 = 20(x + 3)$ is *[Karnataka CET-99]*
(a) (3, –2) (b) (2, –3)
(c) (2, 2) (d) (3, 3)

15. If the chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ of the parabola $y^2 = 4ax$ passes through the focus of the parabola, then *[PET-93]*
(a) $t_1t_2 = -1$ (b) $t_1 t_2 = 1$
(c) $t_1 + t_2 = -1$ (d) $t_1 - t_2 = 1$

16. If 'a' and 'c' are the segments of a focal chord of a parabola and b the semi-latus rectum, then: *[PET-95]*
(a) a, b, c are in A.P.
(b) a, b, c are in G.P.
(c) a, b, c are in H.P.
(d) None of these

17. The equation of latus rectum of a parabola is $x + y = 8$ and the equation of the tangent at the vertex is $x + y = 12$, then length of the latus rectum is *[PET-2002]*

(a) $4\sqrt{2}$ (b) $2\sqrt{2}$
(c) 8 (d) $8\sqrt{2}$

18. The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with the directrix *[IIT-2002; KCET-2008]*

(a) $x = -a$ (b) $x = -\frac{a}{2}$
(c) $x = 0$ (d) $x = \frac{a}{2}$

19. The line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$. Then one of the values of k is *[IIT-2000]*

(a) $\frac{1}{8}$ (b) 8
(c) 4 (d) $\frac{1}{4}$

20. The equation of the directrix of the parabola $x^2 + 8y - 2x = 7$ is *[MPPET-2006]*

(a) $y = 3$ (b) $y = -3$
(c) $y = 2$ (d) $y = 0$

21. The vertex of the parabola $x^2 + 8x + 2y + 8 = 0$ is *[DCE-1999]*

(a) (–4,4) (b) (4,– 1)
(c) (–4, –1) (d) (4,1)

22. Statement-1: The curve $y = \frac{-x^2}{2} + x + 1$ is symmetric with respect to the line $x = 1$. because
Statement-2: A parabola is symmetric about its axis.

(a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(c) Statement-1 is True, Statement-2 is False
(d) Statement-1 is False, Statement-2 is True *[IIT-2007]*

23. The point of contact between the line $y = x + 2$ and the parabola $y^2 = 8x$ has the coordinates *[MP PET-2007]*

(a) (2, 3) (b) (2, 2)
(c) (4, 2) (d) (2, 4)

24. The length of the chord of the parabola $y^2 = 4ax$ which passes through the vertex and makes an angle θ with the axis of the parabola, is

(a) $4a \cos\theta \operatorname{cosec}^2\theta$
(b) $4a \cos^2\theta \operatorname{cosec}\theta$
(b) $a \cos\theta \operatorname{cosec}^2\theta$
(d) $a \cos^2\theta \operatorname{cosec}\theta$

25. What is the equation to the parabola, whose vertex and focus are on the x-axis at distances a and b from the origin respectively? $(b > a > 0)$ *[NDA-2007]*

(a) $y^2 = 8(b - a)(x - a)$
(b) $y^2 = 4(b + a)(x - a)$
(c) $y^2 = 4(b - a)(x + a)$
(d) $y^2 = 4(b - a)(x - a)$

26. The equation of the parabola with its vertex at the origin, axis on the y-axis and passing through the point (6, –3) is *[PET-2001]*

(a) $y^2 = 12x + 6$
(b) $x^2 = 12y$
(c) $x^2 = -12y$
(d) $y^2 = -12x + 6$

27. Through the vertex of the parabola $y^2 = 4x$ chords OP and OQ are drawn at right angles to one another. The locus of middle point of PQ is *[Orissa JEE-2008]*

(a) $y^2 = x + 8$
(b) $y^2 = - 2x + 8$
(c) $y^2 = 2x - 8$
(d) $y^2 = x - 8$

28. The focal distance of a point on the parabola $y^2 = 12x$ is 4. What is the absicca of the point? *[NDA-2006]*

(a) 1 (b) –1
(c) $2\sqrt{3}$ (d) –2

SOLUTIONS

1. (d) Focus of parabola $x^2 = -16y$, $a = 4$
Vertex = (0, 0), Focus = (0, –a) = (0, –4)

2. (d) $9x^2 - 6x + 36y + 19 = 0$
$\Rightarrow \quad 9x^2 - 6x + 19 = -36y$
$\Rightarrow \quad (3x-1)^2 = -36y - 18 = -36\left(y+\frac{1}{2}\right)$
$\Rightarrow \quad 9\left(x-\frac{1}{3}\right)^2 = -36\left(y+\frac{1}{2}\right);$
$\left(x-\frac{1}{2}\right)^2 = -4\left(y+\frac{1}{2}\right)$
Latus rectum = 4
or
$4a = \text{L.R.} = \frac{\text{Coff.of } y}{\text{Coff.of } x^2} = \frac{36}{9} = 4$

3. (c) The given equation of parabola is
$y = 2x^2 + x$
$\Rightarrow \quad x^2 + \frac{x}{2} = \frac{y}{2}$
$\Rightarrow \quad \left(x+\frac{1}{4}\right)^2 = \frac{y}{2} + \frac{1}{16}$
It can be written as $X^2 = \frac{1}{2}Y$ ....(1)
Here $A = \frac{1}{8}$, focus of (1) is given by
$X = 0, Y = a$ i.e., $X = 0, Y = 1/8$
$\Rightarrow \quad x + \frac{1}{4} = 0, y + \frac{1}{8} = \frac{1}{8}$
$\Rightarrow \quad x = -\frac{1}{4}, y = 0$
i.e., focus of given parabola is $\left(-\frac{1}{4}, 0\right)$

4. (d) Let $y = 3x$, then $(3x)^2 = 18x$
$\Rightarrow \quad 9x^2 = 18x$
$\Rightarrow \quad x = 2$ and $y = 6$
or
Verification method. In option *B*, *C* and *D*, ordinate is three times its absciassa but only option *D* is satisfied its parabola.

5. (b) Equaton of the axis (a line joining vertex and focus) is $x = 2$ i.e., a line parallel to y-axis. Hence parabola is opening downwards its equation is $(x - h)^2 = -4a(y - k)$
$\Rightarrow \quad (x - 2)^2 = -4(2)\,(y - (-1))$
$(x - 2)^2 = -8(y + 1);$
i.e., $x^2 - 4x + 8y + 12 = 0$

6. (d) Here, $\frac{y}{2} = t$ and $x - 2 = t^2$
Eliminating t from both equation we find parabola.
$\Rightarrow \quad (x-2) = \left(\frac{y}{2}\right)^2 \Rightarrow y^2 = 4(x - 2)$

7. (c) Area $= \frac{1}{2}(12\times 3) = 18$ sq. unit

8. (c) Focus ≡ (5, 0). Therefore, latus rectum = $4a = 20$.

9. (b) Since y-axis is its axis therefore the equation of the parabola $x^2 = 4ay$ also parabola passes through given point
$\therefore \quad 16 = -8a$
$\Rightarrow \quad a = -2$
$\therefore \quad$ Latus rectum = $|4a| = 8$

10. (d) $\because$ Directrix of parabola is $y = 2$
$\therefore \quad a = -2$
$\therefore \quad$ Required equation of parabola is $x^2 = -4(2)y$
$\Rightarrow \quad x^2 = -8y$

11. (a) $\because$ Focal chord of parabola $y^2 = ax$ is $2x - y - 8 = 0$
$\because \quad$ This chord passes through focus i.e., $\left(\frac{a}{4}, 0\right)$
$\therefore \quad 2.\frac{a}{4} - 0 - 8 = 0 \Rightarrow a = 16$
$\therefore \quad$ Directrix is $x = -4$
$\Rightarrow \quad x + 4 = 0$

12. (b) If two ends of the latus ractum are given, then two parabolas can be drawn either on left side or right side of a latus rectum.

13. (a) $(x + 2)^2 = -2y + 7 + 4$

$\Rightarrow \quad (x+2)^2 = -2\left(y - \dfrac{11}{2}\right)$

Hence, vertex is $\left(-2, \dfrac{11}{2}\right)$

14. (c) For focus (x, y), $(x + 3, y - 2) \equiv (5, 0)$

($\because$ $4a = 20$) $(x, y) \equiv (+2, 2)$

15. (a) $\dfrac{(y - 2at_2)}{(2at_2 - 2at_1)} = \dfrac{(x - ay_2^2)}{(at_2^2 - at_1^2)}$

As focus i.e., $(a, 0)$ lies on it ,

$\Rightarrow \quad \dfrac{-2at}{2a(t_2 - t_1)} = \dfrac{a(1 - t_2^2)}{a(t_2 - t_1)(t_2 + t_1)}$

$\Rightarrow \quad -t_2 = \dfrac{(1 - t_2^2)}{(t_2 + t_1)}$

$\Rightarrow \quad -t_2^2 - t_1 t_2 = 1 - t_2^2$

$\Rightarrow \quad t_1 t_2 = -1$

or

Equation of the chord joining given two point is

$$\left(\frac{t_1 + t_2}{2}\right) y = x + at_1 t_2;$$

it passes through $(a, 0)$, then $0 = a + at_1 t_2$

$\Rightarrow \quad t_1 t_2 = -1$

16. (c) Semi-latus rectum is harmonic mean between segments of focal chord of a parabola.

$\therefore \quad b = \dfrac{2ac}{a + c}$

$\Rightarrow$ a, b, c are in H.P.

17. (d) We now that distance between latus rectum and the tangent at the vertex is a and length of L.R. is $4a$

Therefore, $a = \dfrac{12 - 8}{\sqrt{1+1}} = (\dfrac{c_2 - c_1}{\sqrt{a^2 + b^2}}$ distance between two parallel lines)

$\therefore$ L.R. $= 4a = 4 \times \dfrac{4}{\sqrt{2}} = 8\sqrt{2}$.

18. (c) $\alpha = \dfrac{at^2 + a}{2}$, $\beta = \dfrac{2at + 0}{2}$

$\Rightarrow \quad 2\alpha = at^2 + a,\ at = \beta$

$\therefore \quad 2\alpha = a.\dfrac{\beta^2}{a^2} + a$ or $2a\alpha = \beta^2 + a^2$

$\therefore$ The locus is $y^2 = \dfrac{4a}{2}\left(x - \dfrac{a}{2}\right)$

$= 4b(x - b), \left(b = \dfrac{a}{2}\right)$

$\therefore$ Directrix is $(x - b) + b = 0$ or $x = 0$.

19. (c) The parabola is $y^2 = 4.\dfrac{k}{4}\left(x - \dfrac{8}{k}\right)$.

Putting $y = Y, x - \dfrac{8}{k} = X$, the equation is $Y^1 = 4.\dfrac{k}{4}.X$.

$\therefore$ The directrix is $X + \dfrac{k}{4} = 0$,

i.e., $x - \dfrac{8}{k} + \dfrac{k}{4} = 0$.

But $x - 1 = 0$ is the directrix. So, $\dfrac{8}{k} - \dfrac{k}{4} = 1$

$\Rightarrow \quad k = -8, 4$.

20. (a) The given equation of the parabola is $x^2 + 8y - 2x = 7$

$\Rightarrow \quad x^2 - 2x + 1 = 8 - 8y$

$\Rightarrow (x - 1)^2 = 4 \times 2(1 - y)$

$\Rightarrow \quad (x - 1)^2 = -4 \times (2)(y - 1)$

$\Rightarrow \quad X^2 = -4ay$

where, $X = x - 1$, $Y = y - 1$

Hence, the equation of the directrix of the parabola will be

$Y = a \Rightarrow y - 1 = 2$ [$\because$ $Y = y - 1$] $\therefore$ $y = 3$

21. (a) Given parabola is $x^2 + 8x + 2y + 8 = 0$

$\Rightarrow x^2 + 8x = -2y - 8$

$\Rightarrow \quad (x - 4)^2 = -2(y - 4)$

$\Rightarrow \quad X^2 = -2y$

vertex is $(0,0), X = 0$, $x + 4 = 0$

$\Rightarrow \quad x = -4$

and $Y = 0, y - 4 = 0$

$\Rightarrow \quad y = 4$ Vertex is $(-4, 4)$

22. (a) The given curve is $y = -\frac{x^2}{2} + x + 1$ or $(x-1)^2 = -2(y-3/2)$ which is a parabola, so should be symmetric with respect to its axis $x - 1 = 0$.

23. (d) Point of contact be (h, k)

$k = h + 2, k^2 = 8h$

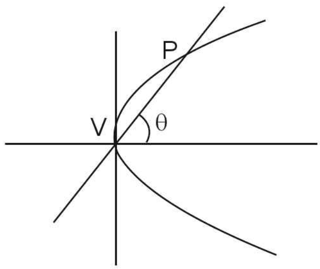
Eliminating k, $(h+2)^2 = 8h$

$h^2 - 4h + 4 = 0$

$h = 2, k = 4$

$\therefore$ (2, 4) is point of contact.

24. (a) Length of chord = VP



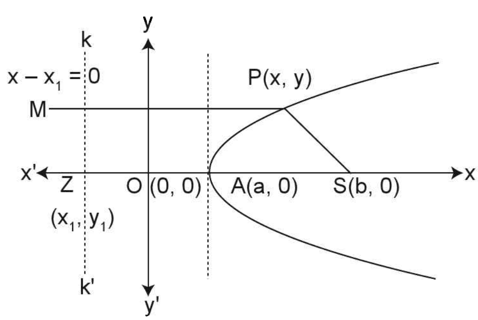
Equation of chord $y = (\tan\theta)x$,
put into $y^2 = 4ax$ to get point of intersection as $(4a\cot^2\theta, 4a\cot\theta)$ and (0, 0)
Hence, length of chord

$$= \sqrt{(4a\cot^2\theta - 0)^2 + (4a\cot\theta - 0)^2}$$

$$= 4a\cot\theta\sqrt{\cot^2\theta + 1} = 4a\cot\theta\,\mathrm{cosec}\,\theta$$

$$= 4a\cos\theta\,\mathrm{cosec}^2\theta$$

25. (d) Equation of a parabola, whose vertex and focus are on the x-axis at a distance a and b from origin respectively.



Let O, A and S be the origin, vertex and focus of parabola respectively then $OA = a$, $OS = b$

$\therefore$ Coordinates of S are $(b, 0)$

Let kk' be the directrix. Suppose SA produced meets the directrix at z. Let the coordinates of z be (x_1, y_1), then

$$\frac{x_1 + b}{2} = a \text{ and } \frac{y_1 + 0}{2} = 0$$

$\therefore$ $x_1 = 2a - b$ and $y_1 = 0$

So the equation of directrix kk' is $x = x_1$ if $x = 2a - b$

Let $P(x, y)$ be any point on the parabola then $SP = PM = \sqrt{(x-b)^2 + (y-0)^2} = x_1$ But $x_1 = 2a + b$

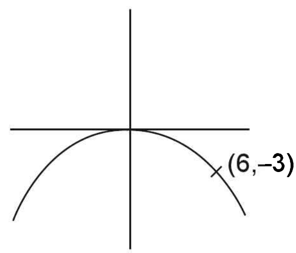
$\Rightarrow$ $(x-b)^2 + (y-0)^2 = (x - 2a + b)^2 = [(x-b) - 2(a-b)]^2$

$\Rightarrow$ $(x-b)^2 + y^2 = (x-b)^2 + 4(a-b)^2 - 4(x-b)(a-b)$

$\Rightarrow$ $y^2 = 4(a-b)[(a-b) - (x-b)]$

$\Rightarrow$ $y^2 = 4(a-b)(a-x) = 4(b-a)(x-a)$

26. Let equation be $x^2 = 4ay$

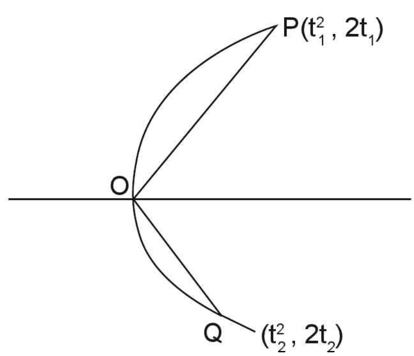


$6^2 = -4a.3 \Rightarrow a = -3$

$\therefore$ Equation is $x^2 = -12y$

27. (c) $m_{OP} \cdot m_{OQ} = -1$

$$\frac{2}{t_1} \cdot \frac{2}{t_2} = -1;\ t_1 t_2 = -4$$



Mid-point of $PQ(h,k) \equiv \left(\frac{t_1^2+t_2^2}{2}, t_1+t_2\right)$

$$K = t_1 + t_2, h = \frac{t_1^2+t_2^2}{2}$$

$$k^2 = t_1^2 + t_2^2 + 2t_1t_2 \text{ (squaring)}$$
$$k^2 = 2h + 2(-4) \text{ or } y^2 = 2x - 8$$

28. (a) Focal distance = 4 $(a + x_1) = 4$

$a = \frac{12}{4} = 3$; (x_1, y_1) is point on hyperbola.

$x_1 = 1$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The latus rectum of the parabola $y^2 = 5x + 4y + 1$ is *[PET-96]*
(a) 5/4 (b) 10
(c) 5 (d) 5/2

2. Which of the following points lie on the parabola $x^2 = 4ay$ *[RPET-2002]*
(a) $x = at^2, y = 2at$ (b) $x = 2at, y = at$
(c) $x = 2at^2, y = at$ (d) $x = 2at, y = at^2$

3. If the parabola $y^2 = 4ax$ passes through (3,2) then the length of its latus rectum is *[DCE-1998]*
(a) 2/3 (b) 4/3
(c) 1/3 (d) 4

4. Equation of parabola whose focus is (–3, 0) and directrix is $x + 5 = 0$, will be *[NDA-2002; MP PET-2007]*
(a) $x^2 = 4(y + 4)$ (b) $x^2 = 4(y - 4)$
(c) $y^2 = 4(x + 4)$ (d) $y^2 = 4(x - 4)$

5. Vertex of the parabola $y^2 + 2y + x = 0$ lies in the quadrant *[PET-89]*
(a) First (b) Second
(c) Third (d) Fourth

6. The focus of the parabola $y^2 = 4y - 4x$ is
(a) (0, 2) (b) (1, 2)
(c) (2, 0) (d) (2, 1)

7. Focus and directrix of the parabola $x^2 = -8ay$ are *[RPET-2001]*
(a) $(0, -2a)$ and $y = 2a$
(b) $(0, 2a)$ and $y = -2a$
(c) $(2a, 0)$ and $x = -2a$
(d) $(-2a, 0)$ and $x = 2a$

8. The end points of latus rectum of the parabola $x^2 = 4ay$ are *[RPET-1997]*
(a) $(a, 2a), (2a, -a)$
(b) $(-a, 2a), (2a, a)$
(c) $(a, -2a), (2a, a)$
(d) $(-2a, a), (2a, a)$

9. The focus of the parabola $4y^2 - 6x - 4y = 5$ is *[RPET-97]*
(a) (–8/5, 2) (b) (–5/8, 1/2)
(c) (1/2, 5/8) (d) (5/8, –1/2)

10. $y^2 - 2x - 2y + 5 = 0$ represents:
(a) Acircle whose centre is (1, 1)
(b) A parabola whose focus is (1, 2)
(c) A parabola whose directrix is $x = 3/2$
(d) A parabola whose directrix is $x = -1/2$

11. If the axis of a parabola is horizontal and it passes through the points (0, 0), (0, –1) and (6, 1), then its equation is
(a) $y^2 + 3y - x - 4 = 0$
(b) $y^2 + 3y + x - 4 = 0$
(c) $y^2 - 3y - x - 4 = 0$
(d) None of these

12. The equation of a circle passes through the vertex and the extremities of the latus rectum of the parabola $y^2 = 8x$ is *[PET-1998]*
(a) $x^2 + y^2 + 10x = 0$ (b) $x^2 + y^2 + 10y = 0$
(c) $x^2 + y^2 - 10x = 0$ (d) $x^2 + y^2 - 5x = 0$

13. The length of the latus rectum of the parabola $x^2 - 4x - 8y + 12 = 0$ is *[PET-2000]*
(a) 4 (b) 6
(c) 8 (d) 10

14. The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is *[IIT-2001]*
(a) $x = -1$ (b) $x = 1$
(c) $x = \frac{-3}{2}$ (d) $x = \frac{3}{2}$

15. The focus of the parabola $y^2 - x - 2y + 2 = 0$ is ***[MPPET-2005]***

(a) (1/4, 0) (b) (1,2)
(c) (5/4, 1) (d) (3/4, 5/2)

16. The curve described parametrically by $x = t^2 + t + 1, y = t^2 - t + 1$ represents ***[IIT-1999]***

(a) a pair of straight lines
(b) an ellipse
(c) a parabola
(d) a hyperbola

17. Let a focal chord of parabola $y^2 = 16x$ cuts it at points (f, g) and (h, k). The $f \cdot h$ is ***[Orissa JEE-2007]***

(a) –8 (b) –16
(c) 16 (d) None of these

18. The directrix of the parabola $y^2 + 4x + 3 = 0$ is ***[VITEEE-2008]***

(a) $x - \frac{4}{3} = 0$ (b) $x + \frac{1}{4} = 0$
(c) $x - \frac{3}{4} = 0$ (d) $x - \frac{1}{4} = 0$

19. A parabola has the origin as its focus and the line $x = 2$ as the directrix. Then the vertex of the parabola is at ***[AIEEE-2008]***

(a) (2,0) (b) (0, 2)
(c) (1, 0) (d) (0, 1)

20. The length of the latus rectum of the parabola whose focus is (3, 3) and directrix is $3x - 4y - 2 = 0$ is ***[UPSEAT-2001]***

(a) 2 (b) 1
(c) 4 (d) None of these

21. If a double ordinate of the parabola $y^2 = 4ax$ be of length $8a$, then the angle between the lines joining the vertex of the parabola with the ends of this double ordinate is

(a) 30° (b) 60°
(c) 90° (d) 120°

22. The vertex (V) and focus (S) of the parabola $x^2 + 4x + 2y - 7 = 0$ ***[MPPET-90]***

(a) $V(-2\ 11/2), S(-2, 5)$
(b) $V(2, -11/2), S(2, -5)$
(c) $V(2, 11/2), S(2, 5)$
(d) $V(-2, -11/2), S(-2, -5)$

23. The ends of latus rectum of parabola $x^2 + 8y = 0$ are ***[PET-95]***

(a) (–4, –2) and (4, 2)
(b) (4, –2) and (–4, 2)
(c) (–4, –2) and (4, –2)
(d) (4, 2) and (–4, 2)

24. The parabola $y^2 = x$ is symmetric about ***[Kerala (Engg.)-2002]***

(a) x-axis
(b) y-axis
(c) both x-axis and y-axis
(d) The line $y = x$

25. Let P be the point (1, 0) and Q a point of the locus $y^2 = 8x$. The locus of the mid-point of PQ is ***[AIEEE-2005]***

(a) $x^2 + 4y + 2 = 0$ (b) $x^2 - 4y + 2 = 0$
(c) $y^2 - 4x + 2 = 0$ (d) $y^2 + 4x + 2 = 0$

26. The parametric equations of the curve $y^2 = 8x$ are

(a) $x = t^2, y = 2t$ (b) $x = 2t^2, y = 4t$
(c) $x = 2t, y = 4t^2$ (d) None of these

27. The locus of the middle points of the chords of the parabola $y^2 = 4ax$ which passes through the origin is ***[RPET-97-UPSEAT-99]***

(a) $y^2 = ax$ (b) $y^2 = 2ax$
(c) $y^2 = 4ax$ (d) $x^2 = 4ay$

28. The equation of the parabola whose focus is (3, –4) and directrix $6x - 7y + 5 = 0$ is ***[MPPET-2008]***

(a) $(7x + 6y)^2 - 570x + 750y + 2100 = 0$
(b) $(7x + 6y)^2 + 570x - 750y + 2100 = 0$
(c) $(7x - 6y)^2 - 570x + 750y + 2100 = 0$
(d) $(7x - 6y)^2 + 570x - 750y + 2100 = 0$

29. The equation $x^2 + 4xy + y^2 + 2x + 4y + 2 = 0$ represents

(a) An ellipse
(b) A pair of straight lines
(c) A hyperbola
(d) None of these

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 17 minutes.
3. The worksheet consists of 17 questions. The maximum marks are 51.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The directrix of the parabola $x^2 - 4x - 8y + 12 = 0$ is ***[Karnataka CET-03]***
(a) $x = 1$ (b) $y = 0$
(c) $x = -1$ (d) $y = -1$

2. If the parabola $y^2 = 4ax$ passes through $(-3, 2)$, then length of its latus rectum is
(a) 2/3 (b) 1/3
(c) 4/3 (d) 4

3. The coordinates of the extremities of the latus rectum of the parabola $5y^2 = 4x$ are
(a) (1/ 5, 2/5) , (–1/5, 2/5)
(b) (1/5, 2/5), (1/5, –2/5)
(c) (1/5, 4/5),(1/5, –4/5)
(d) None of these

4. The focal distance of a point on the parabola $y^2 = 16x$ whose ordinate is twice the abscissa, is
(a) 6 (b) 8
(c) 10 (d) 12

5. The length of latus rectum of the parabola $4y^2 + 2x - 20y + 17 = 0$ is ***[PET-1999]***
(a) 3 (b) 6
(c) 1/2 (d) 9

6. Equation of the parabola with its vertex at (1, 1) and focus (3, 1) is ***[KCET-2001, 2002]***
(a) $(x - 1)^2 = 8(y - 1)$
(b) $(y - 1)^2 = 8(x - 3)$
(c) $(y - 1)^2 = 8(x - 1)$
(d) $(x - 3)^2 = 8(y - 1)$

7. If the vertex of the parabola $y = x^2 - 8x + c$ lies on x-axis, then the value of c is
(a) –16 (b) –4
(c) 4 (d) 16

8. The equations $x = \frac{t}{4}, y = \frac{t^2}{4}$ represents
(a) A circle (b) A parabola
(c) An ellipse (d) A hyperbola

9. The equation of the line joining the vertex of the parabola $y^2 = 6x$ to the points on it whose abscissa is 24, is
(a) $y \pm 2x = 0$ (b) $2y \pm x = 0$
(c) $x \pm 2y = 0$ (d) $2x \pm y = 0$

10. The latus rectum of a parabola whose directrix is $x + y - 2 = 0$ and focus is $(3, -4)$, is
(a) $-3\sqrt{2}$ (b) $3\sqrt{2}$
(c) $-3/\sqrt{2}$ (d) $3/\sqrt{2}$

11. The focus of the parabola $x^2 + 2y + 6x = 0$ is ***[MPPET-2004]***
(a) (–3,4) (b) (3,4)
(c) (3,–4) (d) (–3,–4)

12. The curve described parametrically by $x = t^2 + 2t - 1, y = 3t + 5$ represents ***[VIT-2007]***
(a) an ellipse
(b) a hyperbola
(c) a parabola
(d) a circle

13. The equation of the axis of the parabola $x^2 - 4y + 8 = 0$, is ***[NDA-2002]***
(a) $y = 0$ (b) $y = 2$
(c) $x = 0$ (d) $x = 2$

14. For the parabola $y^2 - 8y - x + 19 = 0$; the focus and directrix are ***[NDA-2002]***
(a) $\left(\frac{13}{4}, 4\right)$ and $x = \frac{11}{4}$
(b) $\left(\frac{19}{7}, 8\right)$ and $y = 7$
(c) $\left(\frac{7}{3}, 3\right)$ and $y = 9$
(d) (6,3) and $x = 7$

15. Equation of the directrix of the conic $x^2 + 4y + 4 = 0$ is ***[Kerala PET-2008]***
(a) $y = 1$ (b) $y = -1$
(c) $y = 0$ (d) $x = 0$

16. If (2, 0)is the vertex and the y-axis is the directrix of a parabola, then where is its focus? ***[NDA-2006]***

(a) (0, 0) (b) (–2, 0)
(c) (4, 0) (d) (–4, 0)

17. The focus of the parabola $x^2 = -4y$ is ***[MPPET-2009]***

(a) (1, 0) (b) (–1, 0)
(c) (0, 1) (d) (0, –1)

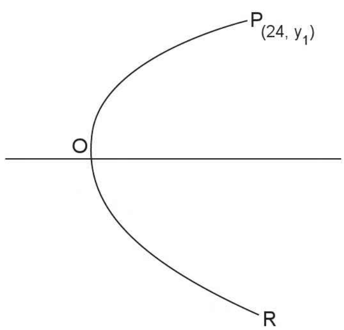
ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**16.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ	**17.** ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

9. (b) Let, $(24, y_1)$ be a point on parabola $y^2 = 6x$

$\therefore$ Point is $y^2 = 6.24 \Rightarrow y = \pm 12$



Here, P and Q are point (24, 12), (24, –12)

$\therefore$ Equation OP and OQ are $y = \dfrac{\pm 12 - 0}{24 - 0}x$

$2y \pm x = 0$

12. $x = t^2 + 2t - 1$ ....(1)

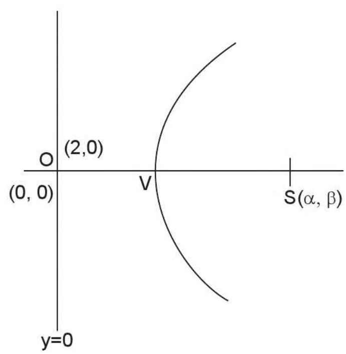
$y = 3t + 5$ or $t = \dfrac{y-5}{3}$

Put in (1) $x = \left(\dfrac{y-5}{3}\right)^2 + 2\left(\dfrac{y-5}{3}\right) = -1$

$9x = y^2 - 10y + 25 + 6y - 30 - 9$

$y^2 - 4y - 14 = 9x$ which is parabola.

16. Origin is point of intersection of axis and directrix.



Let, $S(\alpha, \beta)$ be focus mid-point of $OS = V$

$$\left(\frac{\alpha+0}{2}, \frac{\beta+0}{2}\right) \equiv (2, 0) \quad (\alpha, \beta) \equiv (4, 0)$$

17. (d) Given equation of parabola is $x^2 = -4y$

Here, $a = -1$

$\therefore$ Focus is (0, –1)

LECTURE

2

Parabola 2

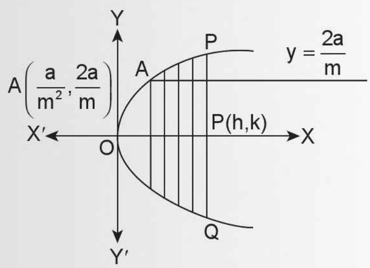
BASIC CONCEPTS

1. Position of a Straight Line with Respect to a Parabola)

1.1 If straight line $y = mx + c$ is a chord to the parabola $y^2 = 4ax$ then $c > mc$ and length of the chord cut on the line $= \frac{4}{m^2}\sqrt{(a^2 - mca)(1 + m^2)}$

NOTES

1. The length of the chord of the parabola $y^2 = 4ax$ through vertex and making angle θ with axis of the parabola is
$= 4\ a\cos\theta\ \text{cosec}^2\theta = 4a\cot\theta\ \text{cosec}\ \theta.$
2. The length of the focal chord of the parabola $y^2 = 4ax$ making angle θ with axis of the parabola is $= 4a\ \text{cosec}^2\ \theta$.
3. Circle described on focal chord as diameter touches directrix.
4. The equation of the chord of the parabola $y^2 = 4ax$ if mid-point of the chord be $P(x, y)$ is $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$
5. Diameter of a Parabola
Diameter of a conic is the locus of middle points of a series of its parallel chords. The equation of the diameter, bisecting chords of slope m, of the parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$.
Vertex of Diamater: $A\left(\frac{a}{m^2}, \frac{2a}{m}\right)$



6. Angle subtended by a double ordinate of length $8a$ at the vertex is 90°.

1.2 If straight line $y = mx + c$ is a tangent to the parabola $y^2 = 4ax$ then $c = a/m$.

NOTES

1. Gradient form of the TANGENT: For all values of m straight line $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$ and the coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.
2. If $lx + my + n = 0$ be a tangent to the parabola $y^2 = 4ax$ then $nl = am^2$ and the coordinates of the point of contact are $\left(\frac{n}{1}, \frac{-2am}{l}\right)$

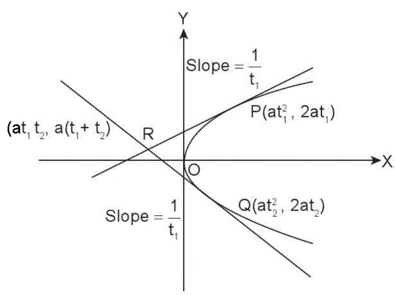
3. If straight lines $lx + my + n = 0$ and $xx_1 = 2a(y + y_1)$ be indentical then $\frac{x_1}{l} = \frac{-2a}{m} = \frac{-2ay_1}{n}$ and $x_1 = -\frac{-2al}{m}, y_1 = \frac{n}{m}$
4. If straight line $lx + my + n = 0$ be a tangent to the parabola $x^2 = 4ay$ then $nm = al^2$ and coordinates of the point of contact are: $\left(\frac{-2al}{m}, \frac{n}{m}\right)$
5. The equation of tangent to $(y - k)^2 = 4a(x - h)$ may be given as: $y - k = m(x - h) + \frac{a}{m}$.
6. The equation of tangent to $(x - h)^2 = 4a(y - k)$ may be given as: $x - h = m(y - k) + \frac{a}{m}$.
7. Equation of the tangent to the parabola at vertex is obtained equation the lowest degree term to zero.

1.3 The straight line $y = mx + c$ lies outside the parabola $y^2 = 4ax$ i.e., both are separate (i.e., they both neither touch nor intersect) if $a < mc$.

2. Parametric Form The equation of the tangent to the parabola $y^2 = 4ax$ at the point $t(at^2, 2at)$ is $ty = x + at^2$

2.1 The point of intersection of tangents drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $R = (at_1t_2, a(t_1 + t_2))$

Point of intersection of tangents to the parabola $y^2 = 4ax$ taken in pair at point t_1, t_2 and t_3 are $[at_1t_2, a(t_1 + t_2)]$ $[at_1t_3, a(t_1 + t_3)]$, $[at_2t_3, a(t_2 + t_3)]$



2.2 Angle between tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

$$\theta = \tan^{-1}\left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$$

2.3 The G.M. of the x-coordinates of P and Q (i.e., $\sqrt{at_1^2 \times at_2^2} = at_1t_2$) is the x-coordinate of the point of intersection of tangents at P and Q on the parabola.

2.4 The A.M. of the y-coordinates of P and Q $\left(\text{i.e., } \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ is the y-coordinate of the point of intersection of tangents at P and Q on the parabola.

2.5 The orthocentre of the triangle formed by three tangents to the parabola lies on the directrix.

3. Equations of Normal in Different Forms

3.1 Cartesian form The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

3.2 Parametric form The equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is $y + tx = 2at + at^3$.

3.3 Slope Form The equation of the normal to the parabola $y^2 = 4ax$ in terms of slope 'm' is $y = mx - 2am - am^3$.

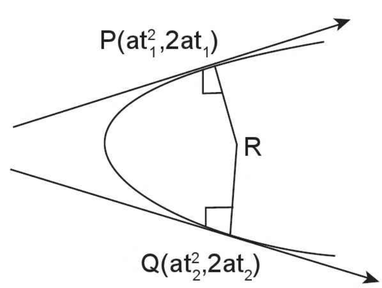
NOTE

The coordinates of the point of contact are $(am^2, -2am)$.

3.4 Condition for normality The line $y = mx + c$ is a normal to the parabola $y^2 = 4ax$ if $c = -2am - am^3$.

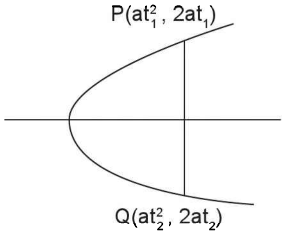
3.5 Point of intersection of normals

The point of intersection of normals drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is



$R = [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$

3.6 If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $Q(at_2^2, 2at_2)$, then $t_2 = -t_1 - \frac{2}{t_1}$.



NOTE

That PQ is normal to the parabola at P and not at Q.

3.7 If the normals at the points $t_1(at_1^2, 2at_1)$ and $t_2(at_2^2, 2at_2)$ meet on some point of the parabola $y^2 = 4ax$, then $t_1t_2 = 2$.

3.8 Equation of normal to the parabola $x^2 = 4ay$ at (x_1, y_1) is $y - y_1 = -\frac{2a}{x_1}(x - x_1)$

3.9 Equation of normal to the parabola $x^2 = 4ay$ in terms of gradient is $y = mx + 2a + \frac{a}{m^2}$ and coordinates of the feet of normals are: $\left(\frac{-2a}{m}, \frac{a}{m^2}\right)$

3.10 Parametric equation of normal to the parabola $x^2 = 4ay$ at $t(2at_1, at^2)$ is $x + yt = 2at + at^3$.

3.11 Area of the triangle formed by joining any three points $t_1(at_1^2, 2at_1)$, $t_2(2at_2^2, 2at_2)$ and $t_3(2at_3^2, 2at_3)$ on the parabola is $a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$.

3.12 Area of the triangle formed by the tangents to the parabola $y^2 = 4ax$ at the points t_1, t_2 and t_3 is $\frac{a^2}{2}(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$ (Orthocentre of such Δ lies on the directrix)

3.13 Area of the triangle formed by the normals to the parabola $y^2 = 4ax$ at the points t_1, t_2 and t_3 is $\frac{a^2}{2}(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)(t_1 + t_2 + t_3)^2$

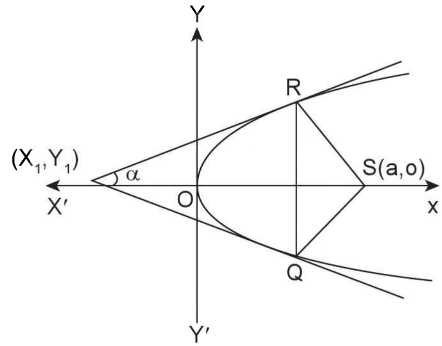
NOTE

Area of a triangle inscribed in a parabola is twice the area of triangle formed by tangents at the vertices of triangle.

4. Number of Tangents Drawn from a Point to a Parabola

Two tangents can be drawn from a point to a parabola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.

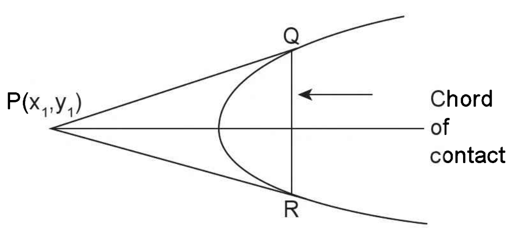
5. Equation of the pair of tangents darwn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $SS_1 = T^2$, where $S \equiv y^2 - 4ax$, $S_1 = y_1^2 - 4ax_1$ and $T \equiv yy_1 - 2a(x + x_1)$



NOTES

1. Angle (α)between the pair of tangents drawn from $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is given by $\alpha = \tan^{-1}\frac{\sqrt{y_1^2 - 4ax_1}}{a + x_1}$
2. Director circle of the parabola: $x + a = 0$ (directrix)

6. Chord of Contact The equation of chord of contact of a pair of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $T = 0$ where $T \equiv yy_1 - 2a(x + x_1)$.



6.1 Length of the chord of contact of a pair of tangents drawn from point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $= \frac{\sqrt{(y_1^2 + 4a^2)(y_1^2 - 4ax_1)}}{a}$

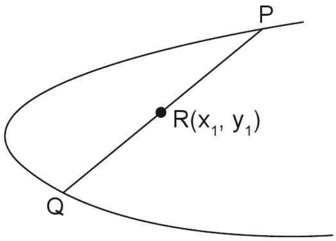
6.2 Area of the triangle formed by the tangents from the point $P(x_1, y_1)$ to the parabola $y = 4ax$ and their chord of contact is

$$\frac{1}{2a}(y_1^2 - 4ax_1)^{3/2}$$

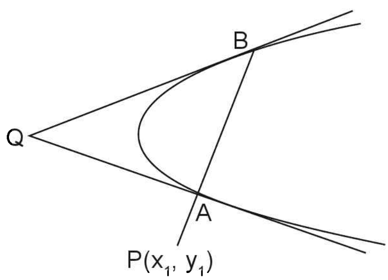
7. Chord with a Given Mid-Point

The equation of the chord of the parabola $y^2 = 4ax$ with $P(x_1, y_1)$ as its middle point is given by $T = S_1$.

where, $T \equiv yy_1 - 2a(x + x_1)$ and $S_1 \equiv y_1^2 - 4ax$.



8. Pole and Polar Let P be a given point. Let a line through P intersect the parabola at two points A and B. Let the tangents at A and B intersect at Q. The locus of point Q is a straight line called the polar and point P with respect to the parabola and the point P is called the pole of the polar.



9. Equation of Polar of a Point The polar of a point $P(x_1, y_1)$ with respect to the parabola $y^2 = 4ax$ is $T = 0$ where, $T \equiv yy_1 - 2a(x + x_1)$.

9.1 Remarks (i) Polar of the focus is the directrix.

(ii) Any tangent is the polar of its point of contact.

(iii) Pole is a given line $lx + my + n = 0$ with respect to the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, \frac{-2am}{l}\right)$.

(iv) Pole of the chord joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{y_1y_2}{4a}, \frac{y_1 + y_2}{2}\right)$

(v) The point of intersection of the polars of two points Q and R is the pole of QR.

(vi) If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$, then the polar of Q will pass through P and such points are said to be **conjugate points**.

(vii) If the pole of a line $ax + by + c = 0$ lies on the another line $a_1x + b_1y + c_1 = 0$ then the pole of the second line will lie on the first and such lines are said to the **conjugate lines**.

(viii) The locus of pole of focal chord of the parabola $y^2 = 4ax$ is the directrix.

10. Some Important Points Connected with Two Parabolas $y^2 = 4ax$ and $x^2 = 4by$.

10.1 Points of intersection of two parabolas $V(\mathbf{0}, 0)\ P(4a^{1/3} b^{2/3}, 4a^{2/3}, b^{1/3})$.

10.2 Angle between two parabolas at (i) $V(0, 0)$ is 90°

(ii) $P(4a^{1/3} b^{2/3}, 4a^{2/3} b^{1/3})$ is

$$\theta = \tan^{-1}\left(\frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}\right)$$

(iii) for $a = b$, $\theta = \tan^{-1}\frac{3}{4}$, $x^2 = a(y - 3a), y^2 = a(x - 3a)$

10.3 Equation of the common tangent of two parabolas is $a^{1/3}x + b^{1/3}y + a^{2/3}b^{2/3} = 0$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find the equation of tangent to the parabola $y^2 = 4x + 5$ which is parallel to the line $y = 2x + 7$.

Solution

Any line parallel to given line is $y = 2x + \lambda$. If it is a tangent to the parabola then it will meet it in two coincident points.

Eliminating x.

$\therefore$ $y^2 - 5 = 4x = 2.2x = 2(y - \lambda)$

or $y^2 - 2y + 2\lambda - 5 = 0$. Its roots are equal

$\therefore$ $B^2 - 4AC = 0$ gives $4 - 4(2\lambda - 5) = 0$

or $24 = 8\lambda$ $\therefore$ $\lambda = 3$

$\therefore$ $y = 2x + 3$ is the required tangent.

2. Find the equation of the common tangent to the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

Solution

Any tangent to the parabola $y^2 = 4ax$ is

$$y = mx + \frac{a}{m} \qquad \text{..................(1)}$$

If it is a tangent to the parabola $x^2 = 4by$ then it will cut it in two coincident points.

Eliminating y, we get $x^2 = 4b\left(mx + \frac{a}{m}\right)$ or

$$x^2 - 4bmx - 4b\frac{a}{m} = 0 \qquad \text{..............(2)}$$

The roots of (2) will be equal if $B^2 - 4AC = 0$

or $16b^2m^2 + \frac{16ab}{m} = 0$

or $m^3 = -\frac{a}{b}$ $\therefore$ $m = -\frac{a^{1/3}}{b^{1/3}}$

On putting the value of m in (1), the equation of the common tangent is $y = -\frac{a^{1/3}}{b^{1/3}}x - a\frac{b^{1/3}}{a^{1/3}}$

or $yb^{1/3} + xa^{1/3} + a^{2/3}b^{2/3} = 0$

3. Prove that the two parabolas $y^2 = 4ax$ and $x^2 = 4by$ intersect at an angle $\tan^{-1}\frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}$

Solution

Angle between two curves is equal to the angle between the tangents at their common point.

Solving the two, we get $y = \frac{x^2}{4b}$ from $x^2 = 4by$

Putting the value of y in $y^2 = 4ax$, we get $\left(\frac{x^2}{4b}\right)^2 = 4ax$ or $x(x^3 - 64ab^2) = 0$

$x = 0$ or $4a^{1/3}b^{2/3}$ and hence $y = 0$ or $4a^{2/3}b^{1/3}$

Therefore, the points of intersection are $(0, 0)$, $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$.

But tangents at the vertex $(0, 0)$ to the two parabolas are axes of x and y respectively and angle between them is $\pi/2$. If the other point be taken as (x_1, y_1) then tangents to the parabolas are $yy_1 = 2a(x + x_1)$ ....(1)

$$\therefore \quad m_1 = \frac{2a}{y_1} = \frac{2a}{4a^{2/3}b^{1/3}} = \frac{1a^{1/3}}{2b^{1/3}}$$

$$\therefore \quad xx_1 = 2b(y + y_1) \qquad \text{...........(2)}$$

$$\therefore \quad m_2 = \frac{x_1}{2b} = \frac{4a^{1/3}b^{2/3}}{2b} = 2\frac{a^{1/3}}{b^{1/3}}$$

If θ be the angle between the tangents, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{\frac{a^{1/3}}{b^{1/3}}\left(-\frac{1}{2} + 2\right)}{1 + \frac{a^{2/3}}{b^{2/3}}}$$

$$= \frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}$$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. Three normals are drawn to the parabola $y^2 = x$ through point $(a, 0)$. Then

[IIT-1991; Orissa JEE-2008]

(a) $a = 1/2$ (b) $a = 1/4$
(c) $a > 1/2$ (d) None of these

Solution

(c) Equation of normal in slope form on $y^2 = 4Ax$ is $y = mx - 2Am - Am^3$ .... (i)

$$= mx - 2\left(\frac{1}{4}\right)m - \left(\frac{1}{4}m^3\right)$$

$$\left(\because y^2 = x \therefore A = \frac{1}{4}\right)$$

$\Rightarrow$ $4mx - 4y - m^3 - 2m = 0$
$\because$ $(a, 0)$ lies on the normal
So $4m \times a - 4 \times 0 - m^3 - 2m = 0$
$\Rightarrow$ $m(m^2 + 2 - 4a) = 0$
$\Rightarrow$ $m = 0$ or $m^2 + 2 - 4a = 0$
If $m = 0$, then from (i) $y = 0$ i.e., x-axis is one normal
If $m^2 + 2 - 4a = 0 \Rightarrow m^2 = 4a - 2 \because m^2 > 0$

$$\Rightarrow \quad 4a - 2 > 0 \Rightarrow a > \frac{1}{2}$$

2. The tangent to parabola $y^2 = 16x$ is perpendicular to a line $3y - 9x - 1 = 0$ is

[Orissa JEE-2008]

(a) $3y + x + 36 = 0$ (b) $3y - x - 36 = 0$
(c) $x + y - 36 = 0$ (d) None of these

Solution

(a) slope of given line = 3
slope of the tangent = – 1/3
($\because$ tangent is perpendicular to the line)
equation of tangent in slope form (m) on $y^2 = 4ax$ is

$$y = mx + \frac{a}{m} \qquad [\text{here } y^2 = 16x]$$

$$y = -\frac{1}{3}x + \frac{4}{-\frac{1}{3}} = -\frac{1}{3}x - 12$$

$\Rightarrow$ $3y = -x - 36 \Rightarrow x + 3y + 36 = 0$

3. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the point of intersection of parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then ***[AIEEE-2004]***

(a) $d^2 + (2b - 3c)^2 = 0$
(b) $d^2 + (3b + 2c)^2 = 0$
(c) $d^2 + (2b + 3c)^2 = 0$
(d) $d^2 + (3b - 2c)^2 = 0$

Solution

(c) Points of intersection of given parabolas are $(0, 0)$ and $(4a, 4a)$. Equation of the line passing through these points is $y = x$. Comparing it with the given line, we have $d = 0$ and $2b = -3c$.
$\Rightarrow$ $d^2 + (2b + 3c)^2 = 0$

4. The angle of intersection between the curves $x^2 = 4(y + 1)$ and $x^2 = -4(y + 1)$ is

[UPSEAT-2002]

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$

(c) 0 (d) $\frac{\pi}{2}$

Solution

(c) Point of intersection = $(0, -1)$; $\frac{dy}{dx} = \frac{2x}{4}$

and $\frac{-2x}{4}$

$\therefore$ $m_1 = 0, m_2 = 0 \Rightarrow \theta = 0°$.

5. Show that the locus of a point such that two of the three normals drawn from it to the parabola $y^2 = 4ax$ coincide is $27ay^2 = 4(x - 2a)^3$

Solution

Any normal to the parabola is $y = mx - 2am - am^3$
If it passes through the point (h, k), then
$k = mh - 2am - am^3$
or $am^3 + 0m^2 + m(2a - h) + k = 0$ ...(1)
Above is a cubic in m showing that there will be three normals passing through the point (h, k). If their slopes be m_1, m_2 and m_3 then $m_1 + m_2 + m_3 = 0$ ....(2)

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - h}{a} \quad .....(3)$$

$$m_1m_2m_3 = -k/a \quad ....(4)$$

If two of the normals coincide, then $m_2 = m_3$ ......(5)

In order to find the locus of the point of intersection (h, k) we have to eliminate the three quantities m_1, m_2 and m_3 between the four equations (2), (3), (4) and (5) $m_1 + 2m_2 = 0$ from (2) and (5)

$\therefore \quad m_1 = -2m_2$ ....(6)

$$m_1(m_2 + m_3) + m_2m_3 = \frac{2a-h}{a} \text{ by (3)}$$

or $\quad m_1(2m_2) + m_2^2 = \frac{2a-h}{a}$ by (5) as $m_3 = m_2$

or $\quad (-2m_2)(2m_2) + m_2^2 = \frac{2a-h}{a}$ by (6)

$\therefore \quad m_2^2 = \frac{h-2a}{3a}$ ....(7)

Again $\therefore \; m_1m_2^2 = -\frac{k}{a}$, by (4) and (5)

$$(-2m_2)m_2 = -\frac{k}{a}, \text{ by (6)}$$

$\therefore \quad m_2^3 = \frac{k}{2a}$ ....(8)

Now $(m_2^2)^3 = (m_2^3)^2$

$\therefore \quad \left(\frac{h-2a}{3a}\right)^3 = \frac{k^2}{4a^2}$ or $27ak^2 = 4(h - 2a)^3$

Hence, the required locus of the point (h, k) is $27ay^2 = 4(x - 2a)^3$

6. If the three normals drawn at three different points on the parabola $y^2 = 4x$ pass through the point (h, k), then prove that $h > 2$.

Solution

For real values of $m_1, m_2, m_3, \sum m_1^2 = +ive$

or $\quad (\sum m_1)^2 - 2\sum m_1m_2 > 0,\ y^2 = 4x \therefore a = 1$

or $\quad 0 - \frac{2(2a-h)}{a} > 0$ Put $a = 1$

or $\quad (h - 2) > 0 \therefore h > 2$

7. Normals are drawn at points P, Q, R lying on the parabola $y^2 = 4x$ which intersect at (3, 0). Then ***[IIT JEE-2006]***

(a) area of $\Delta PQR = 2$
(b) circumradius of $\Delta PQR = 5/2$
(c) circum centre of $\Delta PQR = (5/2, 0)$
(d) All of the above

Solution

(d) Any normal to $y^2 = 4x$ is $y = mx - 2m - m^3$
If it passes through (3, 0), then $0 = 3m - 2m - m^3$

$\Rightarrow \quad m^3 - m = 0 \Rightarrow \quad m = 0, 1, -1$

$\because \quad P, Q, R \equiv (m^2, -2m), m = 0, 1, -1$

$\therefore \quad P = (0, 0), Q = (1, -2), R = (1, 2)$

Now area of $\Delta PQR = \frac{1}{2}[(1\times 2) - (-2\times 1)] = 2$

If (h, k) be circum centre, then

$h^2 + k^2 = (h - 1)^2 + (k + 2)^2 = (h - 1)^2 + (k - 2)^2$

$\Rightarrow \quad 0 = -2h + 4k + 5 = -2h - 4k + 5$

$\Rightarrow \quad 2h - 4k = 5, 2h + 4k = 5 \Rightarrow h = 5/2, k = 0$

$\therefore \quad$ circum centre = (5/ 2, 0) and circum radius $= \sqrt{\frac{25}{4} + 0} = \frac{5}{2}$

Hence all given option are correct.

8. The equation of the common tangent to the parabola $y = x^2$ and $y = -(x - 2)^2$ is/are ***[IIT-JEE-2006]***

(a) $y = 0$ (b) $y = 4(x + 1)$
(c) $y = -4(x - 1)$ (d) $y = 4(x - 1)$

Solution

(a,d) Equation of a tangent to $y = x^2$ is

$y = mx - \frac{1}{4}m^2$ .... (1)

Also equation of a tangent with slope m to $y = -(x - 2)^2$ is

$y = m(x-2) + \frac{1}{4}m^2$ .... (2)

If (1) and (2) are identical, then $-\frac{1}{4}m^2 = -2m + \frac{1}{4}m^2$

$\Rightarrow \quad m = 0, 4$

Hence common tangents are $y = 0$ and $y = 4x - 4$.

9. Find the locus of the point of intersection of those normals to the parabola $x^2 = 8y$ which are at right angles to each other. ***[IIT-1997]***

Solution

Any normal to $y^2 = 4ax$ is $y = mx - 2am - am^3$

$\therefore \quad$ Any normal to $x^2 = 4ay$ is obtained from above by interchanging x and y.

$\therefore$ Normal to $x^2 = 8y$, $(a = 2)$ is $x = my - 4m - 2m^3$

If it passes through (h, k) then $h = mk - 4m - 2m^3$

or $2m^3 + 0m^2 + m(4 - k) + h = 0$ ....(1)

Above shows that three normals will pass through (h, k). If two of them are perpendicular then $m_2m_3 = -1$.

But $m_1m_2m_3 = -h/2$

$\therefore$ $-m_1 = -h/2$ or $m_1 = h/2$

Since, m_1 is a root of (1), it will satisfy it.

$\therefore$ $2\left(\frac{h^3}{8}\right) + \frac{h}{2}(4-k) + h = 0$

or $h^2 + 2(4 - k) + 4 = 0$

$\therefore$ Locus of (h, k) is $x^2 - 2y + 12 = 0$

10. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of P is **[AIEEE-2010]**

(a) $2x + 1 = 0$ (b) $x = -1$
(c) $2x - 1 = 0$ (d) $x = 1$

Solution

Locus of P will be diectrix of parabola which is $x = -1$

Hence correct option is (b)

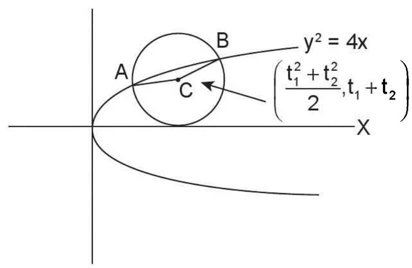
11. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be **[IIT (1)-2010]**

(a) $-\frac{1}{r}$ (b) $\frac{1}{r}$
(c) $\frac{2}{r}$ (d) $-\frac{2}{r}$

Solution

(C, D) $A(t_1^2, 2t_1)$, $B(t_2^2, 2t_2)$

Centre of circle $\left(\frac{t_1^2 + t_2^2}{2}, t_1 + t_2\right)$



$\Rightarrow$ $|t_1 + t_2| = r$, slope of $AB = \frac{2}{t_1 + t_2} = \pm\frac{2}{r}$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The equation of the tangent to the parabola $y = x^2 - x$ at the point where $x = 1$, is **[PET-92]**

(a) $y = -x - 1$ (b) $y = -x + 1$
(c) $y = x + 1$ (d) $y = x - 1$

2. The line $lx + my + n = 0$ will touch the parabola $y^2 = 4ax$, if **[MNR-77]**

(a) $mn = al^2$ (b) $lm = an^2$
(c) $ln = am^2$ (d) $mn = al$

3. The maximum number of normal that can be drawn from a point to a parabola is **[PET-90]**

(a) 0 (b) 1
(c) 2 (d) 3

4. The point on the parabola $y^2 = 8x$ at which the normal is inclined at 60° to the x-axis has the coordinates **[PET-93]**

(a) $(6, -4\sqrt{3})$ (b) $(6, 4\sqrt{3})$
(c) $(-6, -4\sqrt{3})$ (d) $(-6, 4\sqrt{3})$

5. The straight line $y = 2x + \lambda$ does not meet the parabola $y^2 = 2x$, if **[PET-93, MNR-77]**

(a) $\lambda < \frac{1}{4}$ (b) $\lambda > \frac{1}{4}$
(c) $\lambda = 4$ (d) $\lambda = 1$

6. The line $y = 2x + c$ is tangent to the parabola $y^2 = 4x$, then c is equal to **[PET-96]**

(a) $-1/2$ (b) $1/2$
(c) $1/3$ (d) 4

7. The equation of the tangent to the parabola $y^2 = 9x$ which goes through the point (4, 10) is *[PET-2000]*

(a) $x + 4y + 1 = 0$ (b) $9x + 4y + 4 = 0$
(c) $x - 4y + 36 = 0$ (d) $9x - 4y + 4 = 0$

8. The normal to the parabola $y^2 = 8x$ at the point (2, 4) meets the parabola again at the point *[Orissa JEE-2003]*

(a) (–18, –12) (b) (–18, 12)
(c) (18, 12) (d) (18, –12)

9. The normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again in the point $(bt_2^2, 2bt_2)$, then *[AIEEE-2003]*

(a) $t_2 = -t_1 - \frac{2}{t_1}$ (b) $t_2 = -t_1 + \frac{2}{t_1}$
(c) $t_2 = t_1 - \frac{2}{t_1}$ (d) $t_2 = t_1 + \frac{2}{t_1}$

10. The points of the parabola $y^2 = 12x$ whose focal distance is 4, are

(a) $(2, \sqrt{3}), (2, \sqrt{3})$
(b) $(1, 2\sqrt{3}), (1, -2\sqrt{3})$
(c) (1, 2)
(d) None of these

11. The centroid of the triangle formed by joining the feet of the normals drawn from any point to the parabola $y^2 = 4ax$, lies on *[PET-99]*

(a) Axis
(b) Directrix
(c) Latus rectum
(d) Tangent at vertex

12. Two common tangents to the circle $x^2 + y^2 = 2a^2$ and parabola $y^2 = 8ax$ are *(AIEEE-2002)*

(a) $x = \pm(y + 2a)$ (b) $y = \pm(x + 2a)$
(c) $x = \pm(y + a)$ (d) $y = \pm(x + a)$

13. Two equal parabolas have the same vertex and their axes are at right angles. What is the angle between the tangents drawn to them at their point of intersection (other than vertex)? *[NDA-2004]*

(a) $\frac{\pi}{4}$ (b) $\tan^{-1} 2$
(c) $\frac{\pi}{3}$ (d) $\tan^{-1}\left(\frac{3}{4}\right)$

14. The length of the normal chord to the parabola $y^2 = 4x$, which subtends right angle at the vertex is *[RPET-1999]*

(a) $6\sqrt{3}$ (b) $3\sqrt{3}$
(c) 2 (d) 1

15. From the point (–1, –60) two tangents are drawn to the parabola $y^2 = 4x$. Then the angle between the two tangents is *[J & K-2005]*

(a) 30° (b) 45°
(c) 60° (d) 90°

16. Equation of the chord of the parabola $y^2 = 6x$ which is bisected at (–1, 1) is *[EAMCET-96]*

(a) $y - 3x = 4$ (b) $y - 3x + 4 = 0$
(c) $3x - y = 0$ (d) $3x - y = 1$

17. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$
The point on this line from which the other tangent to the parabola is perpendicular to the given tangent is *[AIEEE-2007]*

(a) (–1,1) (b) (0, 2)
(c) (2, 4) (d) (–2, 0)

18. Tangent to the parabola $y = x^2 + 6$ at (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at the point. *[IIT Screening-2005]*

(a) (–6, –9) (b) (–13, –9)
(c) (–6, –7) (d) (13, 7)

19. If $x + y = k$ is a normal of the parabola $y^2 = 12x$, then k is equal to *[IIT (Screening)-2000]*

(a) 3 (b) 9
(c) –9 (d) –3

20. The polar of focus of a parabola is *[RPET-99]*

(a) x = axis
(b) y-axis
(c) Directrix
(d) Latus rectum

21. Equation of any normal to the parabola $y^2 = 4a(x - a)$ is

(a) $y = mx - 2am - am^3$
(b) $y = m(x + a) - 2am - am^3$
(c) $y = m(x - a) + \frac{a}{m}$
(d) $y = m(x - a) - 2am - am^3$

22. Position of point (4, 3) with respect to the parabola $4y^2 - 6x - 4y = 5$ is
(a) Out side (b) Inside
(c) On the parabola (d) None of these

23. The value of C, for which the line $y = 2x + C$ is tangent to the parabola $y^2 = 4a(x + a)$, is **[MPPET-2008]**
(a) a (b) $\frac{3a}{2}$
(c) $2a$ (d) $\frac{5a}{2}$

24. The tangents drawn at the extremeties of a focal chord of the parabola $y^2 = 16x$ **[Karnataka CET-2008]**
(a) intersect at an angle of 60°
(b) intersect at an angle of 45°
(c) intersect on $x = 0$
(d) intersect on the line $x + 4 = 0$

25. The focal chord to $y^2 = 16x$ is tangent to $(x - 6)^2 + y^2 = 2$, then the possible values of the slope of this chord, are **[IIT-2003]**
(a) $\{-1, 1\}$ (b) $\{-2, 2\}$
(c) $\{-2, 1/2\}$ (d) $\{2, -1/2\}$

26. Angle between tangents drawn from the point (1, 4) to the parabola $y^2 = 4x$ is **[IIT scr.-2004]**
(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) $\pi/2$

27. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose **[IIT-2009]**
(a) vertex is $\left(\frac{2a}{3}, 0\right)$
(b) directrix is $x = 0$
(c) latus rectum is $\frac{2a}{3}$
(d) focus is $(a, 0)$

28. The point of intersection of common tangents to the conic $2(x^2 + y^2) = a^2$ and $y^2 = 4ax$ is the focus of the parabola **[MPPET-2010]**
(a) $x^2 = 4ay$
(b) $x^2 = -4ay$
(c) $y^2 = -4ax$
(d) $y^2 = -4a(x + a)$

SOLUTIONS

1. (d) Equation of tangent at point (x_1, y_1)

$$\frac{y + y_1}{2} = xx_1 - \left(\frac{x + x_1}{2}\right)$$

at $x_1 = 1; y_1 = 1^2 - 1 = 0$

$\therefore$ Equation is $\frac{y+0}{2} = x \times 1 - \left(\frac{x+1}{2}\right)$

or $y = x - 1$

2. (c) $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$

$\therefore$ $lx + my + n = 0$ or $y = \frac{-l}{m}x - \frac{n}{m}$ touch

$y^2 = 4ax$ then $-\frac{n}{m} = \frac{a}{\frac{-l}{m}} \Rightarrow ln = am^2$

3. (d) $\because$ Equation of normal is $y = mx - 2am - am^3$

$\because$ It is cubic in m,
$\therefore$ maximum number of normals = 3

4. (a) Point of contact $(am^2, -2am)$

$$a = 2; m = \tan 60° = \sqrt{3}$$

$\therefore$ point of contact $(6, -4\sqrt{3})$

5. (b) If $y = mx + c$ does not meet $y^2 = 4ax$

$\therefore$ $a < mc$ or $\frac{1}{2} < 2\lambda \Rightarrow \lambda > \frac{1}{4}$

6. (b) $c = \frac{a}{m} \Rightarrow c = \frac{1}{2}$

7. (c, d) Equation of tangent to the parabola $y = mx + \frac{9}{4m}$

$\because$ It pases through point (4, 10)

$$10 = 4m + \frac{9}{4m}$$

or $16m^2 - 40m + 9 = 0$

$16m^2 - 36m - 4m + 9 = 0$

or $4m(4m - 9) - (4m - 9) = 0$

$$(4m - 1)(4m - 9) = 0 \Rightarrow m = \frac{1}{4}, \frac{9}{4}$$

$\therefore$ Equation of tangent $y = \frac{x}{4} + 9$

or $y = \frac{9}{4}x + 1$

$x - 4y + 36 = 0$ or $9x - 4y + 4 = 0$

8. (d) If normal at t_1 meets the curve again at t_2 then $t_2 = -t_1 - \frac{2}{t_1}$

Parametric point on parabola $y^2 = 8x$ is

$$(2t_1^2, 4t_1) \equiv (2, 4) \text{ or } t_1 = 1$$

$$t_2 = -1 - \frac{2}{1} = -3$$

$\therefore$ other point $(2t_2^2, 4t_2)$

$(2 \times (-3)^2, 4(-3)) \equiv (18, -12)$

9. (a) Slope of normal at point $t_1 = -t_1$

$$\frac{2bt_2 - 2bt_1}{bt_2^2 - bt_1^2} = -t_1$$

$$\Rightarrow \frac{2}{t_1 + t_2} = -t_1$$

$$\Rightarrow t_2 = -t_1 - \frac{2}{t_1}$$

10. **(b)** Let point be $(3t^2, 6t)$; focal distance = 4

$$3 + 3t^2 = 4 \Rightarrow t = \pm\frac{1}{\sqrt{3}}$$

$\therefore$ Point $(1, \pm 2\sqrt{3})$

11. **(a)** Let m_1, m_2, m_3 are slope of normal meeting at (x, y)

$\therefore$ $y = mx - 2am - am^3$

$am^3 + m(2a - x) + y = 0$

$m_1 + m_2 + m_3 = 0;$

centroid is

$$\left(\frac{a(m_1^2 + m_2^2 + m_3^2)}{3}, \frac{-2a}{3}(m_1 + m_2 + m_3)\right)$$

$\because$ y coordinate = 0,

$\therefore$ centroid lie on x axis.

12. (b) Equation of tangent to circle $x^2 + y^2 = 2a^2$ is

$$y = mx \pm a\sqrt{2(1 + m^2)},$$

$\because$ It is also tangent to $y^2 = 4ax$

$\therefore$ $a\sqrt{2(1 + m^2)} = \frac{2a}{m} \Rightarrow m^4 + m^2 - 2 = 0$

$(m^2 + 2)(m^2 - 1) = 0 \Rightarrow m = \pm 1$

$\therefore$ Equation is $y = \pm (x + 2a)$

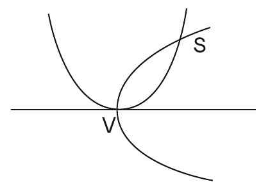
13. (d) Let equation of parabola be $y^2 = 4ax$ and $x^2 = 4ay$

For point of intersection on $\left(\frac{x^2}{4a}\right)^2 = 4ax$

$\Rightarrow$ $x = 0, x = 4a$

$\therefore$ Point are (0, 0) $(4a, 4a)$ at $(4a, 4a)$ slope of tangent to parabola $y^2 = 4ax$ is

$$m_1 = \frac{2a}{y_1} = \frac{2a}{4a} = \frac{1}{2};\ m_2 = \frac{x_1}{2a} = \frac{4a}{2a} = 2$$



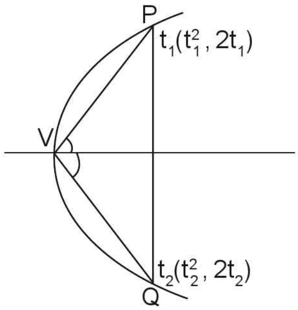
$\therefore$ Angle between them $= \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

$$= \tan^{-1}\left(\frac{2 - \frac{1}{2}}{1 + 2.\frac{1}{2}}\right)$$

$$= \tan^{-1}\frac{3}{4}$$

14. **(a)** Let the normal chord at t_1 intersect the chord again at t_2 $m_{VP}.m_{VQ} = -1;\ \frac{2t_1 - 0}{t_1^2 - 0} \times \frac{2t_2 - 0}{t_2^2 - 0} = -1$

$$t_1 t_2 = -4 \Rightarrow t_1\left(-t_1 - \frac{2}{t_1}\right) = -4$$



$\Rightarrow \quad t_1^2 = 2; t_1 = V_2$

$t_2 = \dfrac{-4}{\sqrt{2}} - 2\sqrt{2}$; Point are $= (2, 2\sqrt{2})$ and $(8, -4\sqrt{2})$

Length $= \sqrt{(8-2)^2 + (-6\sqrt{2})^2} = \sqrt{108} = 6\sqrt{3}$

15. (d) Equation of directrix to the parabola $y^2 = 4x$ is $x = -1$

$\because$ Point $(-1, -60)$ lies on directrix,
Angle between 2 tangents = 90°

16. (a) Equation of chord $T = S_1$

$yy_1 - 3(x + x_1) = y_1^2 - 6x_1$

$y \times 1 - 3(x - 1) = 1 - 6(-1)$

$\Rightarrow \quad y - 3x + 3 = 7$

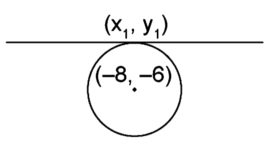
$y = 3x + 4$

17. (d) Let point be (x_1, y_1), $\therefore y_1 = x_1 + 2$

$\because$ tangent are perpendicular point will lie on directrics

i.e., $x_1 = -2$, $\therefore y_1 = -2 + 2 = 0$ Point is $(-2, 0)$

18. (c) Equation of tangent to the parabola $y = x^2 + 6$ at $(1, 7)$ $\dfrac{y+7}{2} = x \times 1 + 6 \Rightarrow y + 7 = 2x + 12$



$\Rightarrow \quad y = 2x + 5$ or $2x - y + 5 = 0$

$\therefore$ For point (x_1, y_1)

$$\frac{x_1 + 8}{2} = \frac{y_1 + 6}{-1} = -\left(\frac{2(-8) - (-6) + 5}{2^2 + 1^2}\right)$$

$(x_1, y_1) \equiv (-6, -7)$

19. (b) $y = -x + k$; is normal to the parabola $y^2 = 12x$

$k = -2am - am^3 = -2 \times 3 \times (-1) - 3(-1)^3$

$k = 6 + 3 = 9$

20. (c) Polar of any point (x_1, y_1) to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$; Here, $(x_1, y_1) \equiv (a, 0)$

$0 = 2a(x + a) \Rightarrow x = -a$ is directrix.

21. (d) Equation of normal $y = m(x - a) - 2am - am^3$

22. (b) Put (4, 5) is equation $4.3^2 - 6.4 - 4.3 - 5 = -5 < 0$

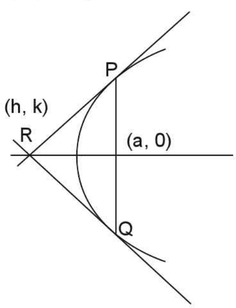
$\therefore$ Point is inside the parabola.

23. **(d)** Here $y = m(x + a) + \dfrac{a}{m}$ is equation of tangent

$y = mx + am + \dfrac{a}{m}$, Here $c = am + \dfrac{a}{m}$

$c = a.2 + \dfrac{a}{2} \Rightarrow c = \dfrac{5a}{2}$

24. (d) Let (h, k) be point of intersection



$\therefore$ Equation of PQ is $ky = 2a(x + h)$, it passes through $(a, 0)$

$\therefore$ $h = -a$, which lies on directrix, $x = -4$.

25. (a) Focus of $y^2 = 16x$ is (4, 0)

$\therefore$ Equation of straight line is $y - 0 = m(x - 4)$

$mx - y - 4m = 0$; it is tangent to $(x - 6)^2 + y^2 = 2$

$\left|\dfrac{6m - 0 - 4m}{\sqrt{m^2 + 1}}\right| = \sqrt{2}$ (Distance from centre = radius)

$\sqrt{2}m = \sqrt{m^2 + 1}$

$\Rightarrow$ Squaring we get $m^2 + 1 = 2m^2$

$m = \pm 1$

26. (c) $\tan\theta = \dfrac{\sqrt{y_1^2 - 4x_1}}{x_1 + a} = \dfrac{\sqrt{4^2 - 4.1}}{1+1} = \sqrt{3}$

$\theta = \dfrac{\pi}{3}$ or 60°

27. (a, d) $G = (h, k)$

$\Rightarrow \quad h = \dfrac{2a + at^2}{3}, k = \dfrac{2at}{3}$

$\Rightarrow \quad \left(\dfrac{3h - 2a}{a}\right) = \dfrac{9k^2}{4a^2}$

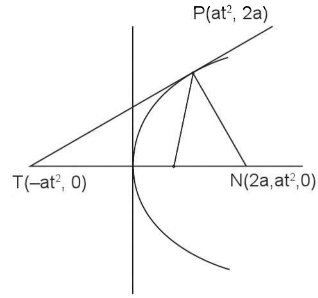
$\Rightarrow$ required parabola is

$\dfrac{9y^2}{4a^2} = \dfrac{(3x - 2a)}{a}$

$= \dfrac{3}{a}\left(x - \dfrac{2a}{3}\right)$

$\Rightarrow \quad y^2 = \dfrac{4a}{3}\left(x - \dfrac{2a}{3}\right)$

Vertex $= \left(\dfrac{2a}{3}, 0\right)$; Focus = $(a, 0)$



28. (c) Equation of tangent to $x^2 + y^2 = \dfrac{a^2}{2}$ is

$y = mx \pm \dfrac{a}{\sqrt{2}}\sqrt{1 + m^2}$

which is common to tangent of $y^2 = 4ax$ i.e.,

$y = mx + \dfrac{a}{m}, \quad \therefore \quad \dfrac{a}{\sqrt{2}}\sqrt{1+m^2} = \dfrac{a}{m}$

Solving $m = \pm 1$

$\therefore$ Equation of tangent is $y = x + a$ and $y = -(x + a)$

Point of intersection of both lines $(-a, 0)$.

We know that focus of $y^2 = -4ax$ is $(-a, 0)$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The line $y = mx + c$ touches the parabola $x^2 = 4ay$ if ***[MNR-73, PET-94]***

(a) $c = -am$

(b) $c = -a/m$

(c) $c = -am^2$

(d) $c = a/m^2$

2. If the tangent to the parabola $y^2 = ax$ makes an angle of 45° with x-axis, then the point of contact is ***[RPET-85, 90, 2003]***

(a) $\left(\dfrac{a}{2}, \dfrac{a}{2}\right)$ (b) $\left(\dfrac{a}{4}, \dfrac{a}{4}\right)$

(c) $\left(\dfrac{a}{2}, \dfrac{a}{4}\right)$ (d) $\left(\dfrac{a}{4}, \dfrac{a}{2}\right)$

3. The line $y = mx + 1$ is a tangent to the parabola $y^2 = 4x$ if ***[DCE-2000]***

(a) $m = 1$ (b) $m = 2$

(c) $m = 4$ (d) $m = 3$

4. The point on parabola $2y = x^2$, which is nearest to the point (0, 3) is ***[J & K-2005]***

(a) $(\pm 4, 8)$ (b) $(\pm 1, 1/2)$

(c) $(\pm 2, 2)$ (d) None of these

5. If the angle subtended by a normal chord of the parabola $y^2 = 4ax$ at its vertex is $\pi/2$, then slope of the normal is ***[PET (Raj.)-1999]***

(a) 1 (b) –1

(c) $\sqrt{2}$ (d) 2

6. The equation of the normal to the parabola $y^2 = 4ax$ at the point $(a/m^2, 2a/m)$ is ***[PET (Raj.)-1987]***

(a) $y = mx - 2am - am^3$

(b) $m^3y = m^2x - 2am^2 - a$

(c) $m^3y = 2am^2 - xm^2 + a$

(d) None of these

7. If $3x - 4y + 5 = 0$ is a tangent to the parabola $y^2 = 4ax$, then a is equal to
[Haryana (CEE)-1994]
(a) 5/4 (b) –5/4
(c) 15/16 (d) –4/3

8. Two parabolas $y^2 = 4x$ and $x^2 = 4y$ intersect at a point P other than the origin, such that
[Haryana (CEE)-96]
(a) their tangents at P make complementary angle with x-axis.
(b) they touch each other at P.
(c) they intersect at right angles at P.
(d) None of these

9. If $x + y - a = 0$ is a tangent to the parabola $y^2 - y + x = 0$, then its point of contact is
[Kerala (CEE)-2003]
(a) $(a, 0)$ (b) $(1, 0)$
(c) $(0, 1)$ (d) $(0, -1)$

10. The equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is ***[IIT Screening-2001]***
(a) $\sqrt{3}y = 3x + 1$ (b) $\sqrt{3}y = -(x + 3)$
(c) $\sqrt{3}y = x + 3$ (d) $\sqrt{3}y = -(3x + 1)$

11. The point of intersection of tangents drawn at the end points of LR of the parabola $y^2 = 4x$ is
[IIT-1994]
(a) $(-2, 0)$ (b) $(-3, 0)$
(c) $(-1, 0)$ (d) None of these

12. Tangents are drawn from the point $(-1, 2)$ to the parabola $y^2 = 4x$. The area of the triangle formed by these tangents and their chord of contact is ***[Roorkee-94]***
(a) 8 (b) $8\sqrt{2}$
(c) $8\sqrt{3}$ (d) None of these

13. The equation of the tangent to the parabola $y^2 = 4ax$ at point $(a/t^2, 2a/t)$ is ***[RPET-96]***
(a) $ty = xt^2 + a$ (b) $ty = x + at^2$
(c) $y = tx = at^2$ (d) $y = tx + (a / t^2)$

14. If the tangent drawn at any point P on the parabola cuts directrix at K and if S is the focus of the parabola then $\angle PSK$ is
(a) $\pi/3$ (b) $\pi/4$
(c) $\pi/2$ (d) π

15. The point of contact of the tangent to the parabola $y^2 = 4ax$ which makes an angle of 60° with x-axis, is
(a) $(a/3, 2a/\sqrt{3})$ (b) $(2a/\sqrt{3}, a/3)$
(c) $(a/\sqrt{3}, 2a/3)$ (d) None of these

16. The point on the parabola $y^2 = 8x$ at which the normal is parallel to the line $x - 2y + 5 = 0$ is
(a) $(-1/2, 2)$ (b) $(1/2, -2)$
(c) $(2, -1/2)$ (d) $(-2, 1/2)$

17. If the normal to the parabola $y^2 = 4ax$ at the point $t(at^2, 2at)$ cuts the parabola again at T $(aT^2, 2aT)$, then
(a) $-2 \leq T \leq 2$
(b) $T \in (-\infty, -8) \cup (8, \infty)$
(c) $T^2 < 8$
(d) $T^2 \geq 8$

18. Two perpendicular tangents to $y^2 = 4ax$ always intersect on the line, if
[Karnataka CET-2000]
(a) $x = a$ (b) $x + a = 0$
(c) $x + 2a = 0$ (d) $x + 4a = 0$

19. The equation of the locus of a point which moves so as to be at equal distances from the point $(a, 0)$ and the y-axis is
(a) $y^2 - 2ax + a^2 = 0$ (b) $y^2 + 2ax + a^2 = 0$
(c) $x^2 - 2ay + a^2 = 0$ (d) $x^2 + 2ay + a^2 = 0$

20. If the line $y = mx + c$ is a tangent to the parabola $y^2 = 4a(x + a)$ then $ma + \frac{a}{m}$ is equal to
(a) c (b) $2c$
(c) $-c$ (d) $3c$

21. If point (x_1, y_1) is inside the parabola $y^2 - 14x + 6y - 12 = 0$, then $y_1^2 - 14x_1 + 6y_1 - 12$ is
(a) < 0 (b) > 0
(c) $= 0$ (d) None of these

22. Straight line $y = 2x + 9$ does not intersect parabola $y^2 = ax$ if
(a) $a < 1/4$ (b) $a < 3/4$
(c) $a = 0$ (d) None of these

23. Pole of straight line $3x + 5y + 7 = 0$ with respect to the parabola $y^2 = 12x$ is
(a) $(5/3, -8)$ (b) $(5/3, -10)$
(c) $(-7/3, 10)$ (d) $(7/3, -10)$

24. Locus of the middle points of the chords of parabola $y^2 = 8x$ if slope of the chords be 4
(a) $y = 2$ (b) $y + x = 1$
(c) $y = 1$ (d) $y = 0$

25. If one extremily of a focal chord of a parabola $y^2 = x$ is $(4, -2)$, then the slope of tangent at the other end is

(a) $-1/4$
(b) -4
(c) $1/4$
(d) 4

26. If the normal to $y^2 = 12x$ at $(3, 6)$ meets the parabola again in $(27, -18)$ and the circle on the normal chord as diameter is

[Kurukshetra CEE-1998]

(a) $x^2 + y^2 + 30x + 12y - 27 = 0$
(b) $x^2 + y^2 + 30x + 12y + 27 = 0$
(c) $x^2 + y^2 - 30x - 12y - 27 = 0$
(d) $x^2 + y^2 - 30x + 12y - 27 = 0$

27. The line $lx + my + n = 0$ is a normal to the parabola $y^2 + 4ax = 0$, if

[MPPET-2010]

(a) $al(l^2 + 2m^2) + m^2n = 0$
(b) $al(l^2 + 2m^2) = m^2n$
(c) $al(2l^2 + m^2) + m^2n = 0$
(d) $al(2l^2 + m^2) = 2m^2n$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 17 minutes.
3. The worksheet consists of 17 questions. The maximum marks are 51.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If line $x = my + k$ touches the parabola $x^2 = 4ay$, then k is equal to ***[PET-95]***
(a) a/m (b) am
(c) am^2 (d) $-am^2$

2. On the parabola $y = x^2$, the point least distance from the straight line $y = 2x - 4$ is ***[AMU-2001]***
(a) (1, 1) (b) (1, 0)
(c) (1, –1) (d) (0, 0)

3. For the parabola $y^2 + 4ax = 0$ $(a > 0)$, which of the following is false? ***[Haryana (CEE)-2000]***
(a) Tangent at the vertex is $x = 0$
(b) Directrix of the parabola is $x = a$
(c) Vertex of the parabola is (0, 0)
(d) Focus of the parabola is $(a, 0)$

4. An equilateral triangle is inscribed in the parabola $y^2 = 4x$ one of whose vertex is at the vertex of the parabola. The length of each side of the triangle is ***[VIT-2006]***
(a) $\frac{\sqrt{3}}{2}$ (b) $4\frac{\sqrt{3}}{2}$
(c) $8\frac{\sqrt{3}}{2}$ (d) $8\sqrt{3}$

5. $P(2, 2)$ is a point on the parabola $y^2 = 2x$ and A is its vertex. If Q is another point on the parabola such that PQ is perpendicular to AP, then the length PQ is equal to ***[NDA-2006]***
(a) $\sqrt{2}$ (b) $2\sqrt{2}$
(c) $4\sqrt{2}$ (d) $6\sqrt{2}$

6. The point of intersection of tangents of the parabola at the points t_1 and t_2 is ***[RPET-02]***
(a) $(at_1t_2, a(t_1 + t_2))$
(b) $(2at_1t_2, a(t_1 + t_2))$
(c) $(2at_1t_2, 2a(t_1 + t_2))$
(d) None of these

7. The line $x - y + 2 = 0$ touches the parabola $y^2 = 8x$ at the point ***[Roorkee-98]***
(a) (2, –4) (b) $(1, 2\sqrt{2})$
(c) $(4, -4\sqrt{2})$ (d) (2, 4)

8. The equation of a tangent to the parabola $y^2 = 4ax$ making an angle θ with x-axis is
(a) $y = x \cot\theta + a \tan\theta$
(b) $x = y \tan\theta + a \cot\theta$
(c) $y = x \tan\theta + a \cot\theta$
(d) None of these

9. The equation of the common tangent of the parabolas $x^2 = 108y$ and $y^2 = 32x$ is
(a) $2x + 3y = 36$
(b) $2x + 3y + 36 = 0$
(c) $3x + 2y = 36$
(d) $3x + 2y + 36 = 0$

10. The tangent to the parabola $y^2 = 4ax$ at the point $(a, 2a)$ makes with x-axis an angle equal to: ***[SCRA-96]***
(a) $\pi/3$ (b) $\pi/4$
(c) $\pi/2$ (d) $\pi/6$

11. If $x = my + c$ is a normal to the parabola $x^2 = 4ay$, then the value of c is
(a) $-2am - am^3$ (b) $2am + am^3$
(c) $-\frac{2a}{m} - \frac{a}{m^3}$ (d) $\frac{2a}{m} + \frac{a}{m^3}$

12. If y_1, y_2 are the ordinates of two points P and Q on the parabola and y_3 is the ordinate of the point of intersection of tangents at P and Q, then
(a) y_1, y_2, y_3 are in A.P.
(b) y_1, y_3, y_3 are in A.P.
(c) y_1, y_2, y_3 are in G.P.
(d) y_1, y_3, y_2 are in G.P.

13. If the line $2x + y + k = 0$ is normal to the parabola $y^2 = -8x$, then the value of k will be
(a) –16 (b) –8
(c) –24 (d) 24

14. The angle of $\cap$ between the curves $y^2 = 4x$ and $x^2 = 32y$ at point (16, 8) and (0, 0) are

(a) $\tan^{-1}\frac{3}{5}, \frac{\pi}{2}$ (b) $\tan^{-1}\frac{4}{5}, \frac{\pi}{2}$

(c) $\pi, \frac{\pi}{2}$ (d) $\frac{\pi}{2}, \frac{\pi}{2}$

15. The condition for which the straight line $y = mx + c$ touches the parabola $y^2 = 4ax$ is ***[PET-97]***

(a) a/c (b) $\frac{a}{c} = m$

(c) $m = a^2 c$ (d) $m = ac^2$

16. If the tangent and normal at any point P of a parabola meet the axes in T and G respectively, then ***[RPET-01]***

(a) $ST \neq SG = SP$

(b) $ST = SG \neq SP$

(c) $ST = SG = SP$

(d) $ST = SG \,.\, SP$

17. If for the parabola $y^2 = 4x, \frac{dy}{dx} = 1$ then the abscissa of the point is ***[Orissa JEE-2007]***

(a) -1 (b) 1

(c) 0 (d) None of these

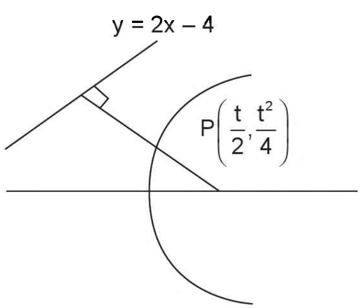
ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**16.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ	**17.** ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

2. (a) Least distance is along normal to the parabola and straight line $m \times 2 = -1; m = \frac{-1}{2}$

(m = slope of normal)



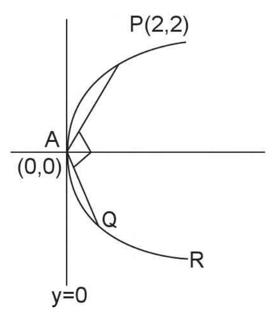
$\therefore$ Slope of tangent at $P = 2 = t$

$P(1, 1)$

5. Let $Q\left(\frac{1}{2}t^2, t\right)$ be point

$\therefore m_{AQ} \,.\, m_{AP} = -1$

$\frac{t}{\frac{t^2}{2}} \cdot \frac{2}{1} = -1 \Rightarrow t = -4$



$\therefore \quad Q \equiv \left(\frac{16}{2}, -4\right) \equiv (8, -4);$

$PQ = \sqrt{(8-2)^2 + (-4-2)^2} = 6\sqrt{2}$

9. (b) $x^2 = 108y$ ....(i) , $4a = 108 \Rightarrow a = 27$

$y^2 = 32y$ ....(ii), $4b = 32 \Rightarrow b = 8$

Equation of tangent to parabola (1)

$y = mx - am^2$ or $y = mx - 27m^2$

Equation of tangent to parabola (2)

$y = mx + \frac{b}{m}$; $y = mx + \frac{8}{m}$

for common tangent $-27m^2 = \frac{8}{m} \Rightarrow m = \frac{-2}{3}$

Put in equation (1) $y = \frac{-2}{3}x - 27 \times \frac{4}{9}$

$\Rightarrow \quad 2x + 3y + 36 = 0$

LECTURE

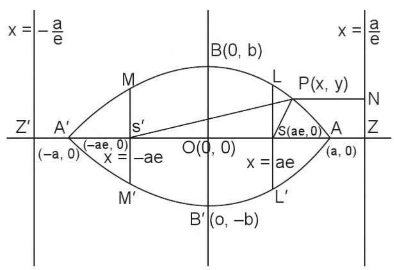
3

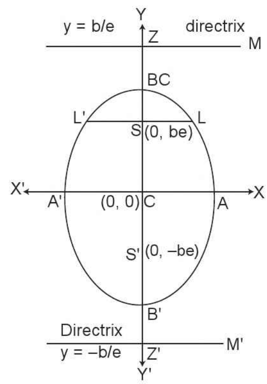
Ellipse 1

BASIC CONCEPTS

1. Standard forms

	Horizontal form of an ellipse	Vertical form of an ellipse
	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1; a>b$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1; b>a$





(i) Symmetry	Centre or both axes	Centre or both axes
(ii) Centre	(0, 0)	(0, 0)
(iii) Eccentricity	$b^2 = a^2(1-e^2)$ or $e=\sqrt{1-\frac{b^2}{a^2}}$	$a^2 = b^2(1-e^2)$ or $e=\sqrt{1-\frac{a^2}{b^2}}$
	or $e=\sqrt{1-\frac{(\text{Minor axis})^2}{(\text{Major axis})^2}}$	or $e=\sqrt{1-\frac{(\text{Minor axis})^2}{(\text{Major axis})^2}}$
(iv) Vertices of major axis	$A'(-a, 0)$ and $A(a, 0)$	$B(0, b)$ and $B'(0, -b)$
(v) Length of major axis	$2a$ = distance between vertices $= A'A$	$2b$ = distance between vertices $= BB'$
(vi) Equation of the major axis	$y = 0$ (horizontal)	$x = 0$ (vertical)

(vii)	Vertices of minor axis	$B(0, b)$, $B'(0, -b)$	$A(a, 0)$ and $A'(-a, 0)$
(viii)	Length of minor axis	$2b$ = distance between vertices $= BB'$	$2a$ = distance between vertices $= A'A$
(ix)	Equation of minor axis	$x = 0$ (vertical)	$y = 0$ (horizontal)
(x)	Number of foci, directrices, latus rectum and focal radii	02 (Ellipse is symmetrical about both axes)	02
(xi)	Focii	$s'(-ae, 0)$, $s(ae, 0)$ or $\left(\pm\sqrt{a^2-b^2}, 0\right)$	$s(0, be)$, $s'(0, -be)$ or $\left(0, \pm\sqrt{b^2-a^2}\right)$
(xii)	Distance between focii	$s's = 2ae$	$s's = 2be$
(xiii)	Equation of latus rectum	$x = \pm\, ae$	$y = \pm\, be$
(xiv)	Ends of latusrectum	$\left\{\left(ae, \frac{b^2}{a}\right), \left(-ae, \frac{b^2}{a}\right), \left(-ae, -\frac{b^2}{a}\right), \left(ae, -\frac{b^2}{a}\right)\right\}$	$\left\{\left(\frac{a^2}{b}, be\right), \left(-\frac{a^2}{b}, be\right), \left(-\frac{a^2}{b}, -be\right), \left(\frac{a^2}{b}, -be\right)\right\}$
(xv)	Length of latus-rectum	$\frac{2b^2}{a}$	$2\frac{a^2}{b}$
(xvi)	Focal radii of point $P(x_1, y_1)$	$a + ex_1$, $a - ex_1$	$b + ey_1$, $b - ey_1$
(xvii)	Sum of focal radii of point $P(x_1, y_1)$	$2a$ = Length of major axis	$2b$ = Length of major axis
(xviii)	Equation of directrices	$x = -\frac{a}{e}$, $x = \frac{a}{e}$	$y = \frac{b}{e}$, $y = -\frac{b}{e}$
(xix)	Distance between directrices	$2a/e$	$2b/e$
(xx)	Area	πab	πab
(xxi)	Parametric equations	$x = a\cos\theta$, $y = b\sin\theta$	$x = a\cos\theta$, $y = b\sin\theta$
(xxii)	Coordinates of point θ. Here θ is eccentric angle point $(a\cos\theta, b\sin\theta)$	$\theta\ (a\cos\theta, b\sin\theta)$	$\theta(a\cos\theta, b\sin\theta)$

(xxiii) Auxilliary circle: A circle having centre same as that of centre of ellipse described on the major axis as its diameter. e.g., Auxilliary circle of $x^2/a^2 + y^2/b^2 = 1$ is $x^2 + y^2 = a^2$.

(xxiv) Position of a point with respect to an ellipse: The point $P(x_1, y_1)$ lies outside, on or, inside the ellipse $x^2/a^2 + y^2/b^2 = 1$ according as $x_1^2/a^2 + y_1^2/b^2 - 1 > 0$, = or, < 0.

1.1 Two ellipse are said to be concentric and uniform if ratio of their axis be same i.e., their eccentricities are same.

NOTE

Major Axis: It is a straight line joining two foci.

1.2 Minor axis It is a straight line perpendicular to the major axis passing through the centre.

1.3 Centre: The point of intersection of major axis and minor axis of an ellipse is called its centre.

1.4 Distance between centre and focii = ae

1.5 Distance between centre and vertices = a

1.6 Distance between two vertices = $2a$ if major axis is vertical

1.7 Distance between centre and directrix = a/e

1.8 Distance between latus rectums = $2ae$

1.9 Distance between directrix and latus rectum

$$= \frac{a}{e} - ae$$

2. The Ellipse

An ellipse is the locus of a point which moves in a plane in such a way that its distance from a fixed point (focus) of the plane bears a constant ratio, which is less than 1, to its distance from a fixed line (directrix) lying in that plane. i.e., $\frac{PS}{PN} = e < 1$ from the figure of basic concepts

3. Ellipse Reducible to Standard Form Equation of the ellipse if the centre of the ellipse lies at (h, k) and the equation of the majoraxis is

(i) Parallel to x-axis

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1; a > b$$

$$b^2 = a^2 (1 - e^2)$$

(ii) Parallel to y-axis

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \; b > a$$

$$a^2 = b^2 (1 - e^2)$$

NOTE

If origin be translated at (h, k) then transformed equation of $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ is $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$

wherea $x = X + h$ and $y = Y + k$

4. General Equation of an Ellipse

Equation of the ellipse having focus at (a, b) and directrix $lx + my + n = 0$ is

$(l^2 + m^2) \{ (x - a)^2 + (y + b)^2 \} = e^2 \{lx + my + n\}^2$;

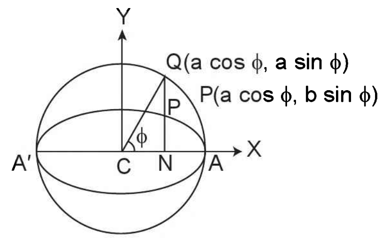
where, $0 < e < 1$

5. Auxiliary Circle of the Ellipse

The circle drawn on major axis AA' as diameter is known as the Auxilarly circle.

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then the equation of its auxiliary circle is



$x^2 + y^2 = a^2$.

Let Q be a point on auxiliary circle so that QN, perpendicular to major axis meets the ellipse at P. The points P and Q are called as corresponding points on the ellipse and auxiliary circle respectively.

The angle ϕ is known as eccentric angle of the point P on the ellipse.

It may be noted that the CQ and not CP is inclined at ϕ with x-axis.

NOTE

Relation between perpendicular distances of corresponding points on the ellipse and auxiliary circle from the major axis $= \frac{b \sin\phi}{a \sin\phi} = \frac{b}{a}$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Prove that the equation $4x^2 + y^2 - 8x + 2y + 1 = 0$ represents an ellipse. Find its eccentricity, foci, equations of directrices and the length of latus rectum.

Solution

Given equation is $4x^2 + y^2 - 8x + 2y + 1 = 0$

$\Rightarrow \quad 4x^2 - 8x + y^2 + 2y + 1 = 0$

$\Rightarrow \quad 4(x^2 - 2x + 1) + y^2 + 2y + 1 = 4$

$\Rightarrow \quad 4(x - 1)^2 + (y + 1)^2 = 4$

$$\Rightarrow \quad \frac{(x-1)^2}{1} + \frac{(y+1)^2}{4} = 1 \quad \text{......... (1)}$$

As the origin moved through (1, –1) the x and y becomes to X and Y.

$x = X + 1$

$y = Y - 1$

$$\text{Equation (1) becomes } \frac{X^2}{1} + \frac{Y^2}{4} = 1 \quad \text{....... (2)}$$

Here, $a^2 = 1$, $b^2 = 4$ hence, equation (2) represents an ellipse of the form $b^2 > a^2$. **Proved.**

Now, $a^2 = b^2(1 - e^2)$

$$\Rightarrow \quad 1 = 4(1 - e^2) \Rightarrow \frac{1}{4} = 1 - e^2 \Rightarrow e^2 = \frac{3}{4}$$

$$\therefore \quad e = \frac{\sqrt{3}}{2}$$

$$\text{The length of latus rectum } = \frac{2a^2}{b} = \frac{2.1}{2} = 1$$

The coordinates of foci $(0, \pm be)$

$$= \left(0, \pm 2.\frac{\sqrt{3}}{2}\right) = (0, \pm\sqrt{3})$$

$\because \quad X = 0$

$\therefore \quad x - 1 = 0 \Rightarrow x = 1$

and $Y = \pm\sqrt{3} \ \therefore \ y + 1 = \pm\sqrt{3} \Rightarrow y = \pm\sqrt{3} - 1$

Hence, the foci are $(1, \pm\sqrt{3} - 1)$.

$$\text{Equations to the directrices } y + 1 = \pm\frac{4}{\sqrt{3}}$$

$$\Rightarrow \quad y = \pm\frac{4}{\sqrt{3}} - 1$$

$$\Rightarrow \quad \sqrt{3}.y + \sqrt{3} \pm 4 = 0$$

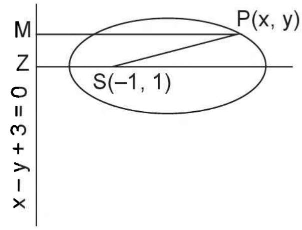
2. Find the equation of the ellipse, whose eccentricity is $\frac{1}{2}$, focus is the point (–1, 1) and directrix is the line $x - y + 3 = 0$.

Solution

Let $P(x, y)$ be any point on the ellipse. Let S be (–1, 1) and equation of ZM be $x - y + 3 = 0$ .... (1)

Join S and P. Drawn $PM \perp ZM$

By definition, $SP = ePM$



$\Rightarrow \quad SP^2 = e^2 PM^2$

$$\Rightarrow \quad (x + 1)^2 + (y - 1)^2 = \frac{(x - y + 3)^2}{8}$$

$$\Rightarrow \quad x^2 + y^2 + 2x - 2y + 2 = \frac{x^2 + y^2 + 9 - 2xy + 6x - 6y}{8}$$

$\Rightarrow \quad 8[x^2 + y^2 + 2x - 2y + 2]$

$= x^2 + y^2 + 9 - 2xy + 6x - 6y$ .... (2)

$\Rightarrow \quad 7x^2 + 7y^2 + 10x - 10y + 2xy + 7 = 0$

This is the required equation.

3. Find the equation of the locus of all points, the sum of whose distances from (3, 0) and (9, 0) is 12.

Solution

Let the variable point be $P(h, k)$ and given points are $A(3, 0)$ and $B(9, 0)$.

According to question, $PA + PB = 12$

$$\Rightarrow \quad \sqrt{(h-3)^2 + (k-0)^2} + \sqrt{(h-9)^2 + (k-0)^2} = 12$$

$$\Rightarrow \quad \sqrt{(h-3)^2 + k^2} = 12 - \sqrt{(h-9)^2 + k^2}$$

$\Rightarrow \quad (h - 3)^2 + k^2$

$$= 144 - 24\sqrt{(h-9)^2 + k^2} + (h-9)^2 + k^2$$

$\Rightarrow \quad (h-3)^2-(h-9)^2$
$= 144 - 24\sqrt{(h-9)^2+k^2}$

$\Rightarrow \quad (h^2-6h+9)-(h^2-18h+81)$
$= 144 - 24\sqrt{(h-9)^2+k^2}$

$\Rightarrow \quad 12h - 72 = 144 - 24\sqrt{(h-9)^2+k^2}$

$\Rightarrow \quad h - 6 = 12 - 2\sqrt{(h-9)^2+k^2}$

$\Rightarrow \quad h - 18 = -2\sqrt{(h-9)^2+k^2}$

$\Rightarrow \quad 2\sqrt{(h-9)^2+k^2} = 18 - h$

$\Rightarrow \quad 4[(h-9)^2+k^2] = (18-h)^2$

$\Rightarrow \quad 4(h^2+k^2-18h+81) = 324 - 36h + h^2$

$\Rightarrow \quad 4h^2+4k^2-72h+324 = 324-36h+h^2$

$\Rightarrow \quad 4h^2+4k^2-72h = -36h+h^2$

$\Rightarrow \quad 3h^2+4k^2-36h = 0$

$\therefore$ Locus of the point $P(h, k)$ is $3x^2 + 4y^2 - 36x = 0$

4. The coordinates of the foci of an ellipse are (2, 0) and (– 2, 0) and length of latus rectum is 6, then prove that
$2e^2 + 3e - 2 = 0.$

Solution

Length of the latus rectum $= \dfrac{2b^2}{a} \Rightarrow 6 = \dfrac{2b^2}{a}$

$\Rightarrow b^2 = 3a$

Distance between the foci = $2ae$

$\Rightarrow \quad \sqrt{(2+2)^2+(0-0)^2} = 2ae \Rightarrow 4 = 2ae$

$\Rightarrow \quad ae = 2 \Rightarrow a = 2/e$

Again, $b^2 = a^2(1-e^2) \Rightarrow 3a = a^2(1-e^2)$

$\Rightarrow \quad 3 = a(1-e^2)$

$\Rightarrow \quad 3 = \dfrac{2}{e}(1-e^2)$

$\Rightarrow \quad 3e = 2 - 2e^2$

$\Rightarrow \quad 2e^2 + 3e - 2 = 0.$

5. If the lines joining the foci and the ends of minor axis are at right angle, what is the eccentricity? If length of major axis is $2\sqrt{2}$, then find the equation of an ellipse.

Solution

Let equation of an ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Given that: $\angle SBS' = 90°$

$\therefore$ Slope of SB × slope of $S'B = -1$

$\Rightarrow \quad \dfrac{b-0}{0+ae} \times \dfrac{b-0}{0-ae} = -1$

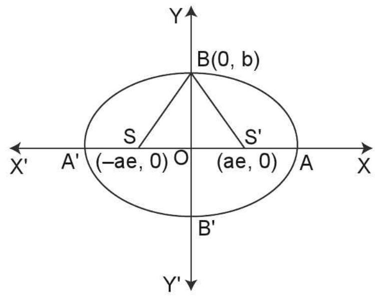
$\Rightarrow \quad \dfrac{b^2}{-a^2e^2} = -1$

$\Rightarrow \quad b^2 = a^2e^2 \qquad$ .... (1)

But, $b^2 = a^2(1-e^2) \therefore a^2e^2 = a^2(1-e^2)$

or $\quad e^2 = 1 - e^2$ or $2e^2 = 1$ or $e = \dfrac{1}{\sqrt{2}}$

Given that: $2a = 2\sqrt{2}$



$\Rightarrow \quad a = \sqrt{2} \Rightarrow a^2 = 2$

$\therefore$ From equation (1), $b^2 = a^2e^2 = 2 \times \dfrac{1}{2} = 1$

Equation of an ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$\Rightarrow \quad \dfrac{x^2}{2}+\dfrac{y^2}{1}=1$

$\Rightarrow \quad x^2 + 2y^2 = 2$

6. Prove that the point (1, 3) lies on the major axis of the ellipse $4x^2 + 9y^2 - 16x - 54y + 61 = 0$.

Solution

Equation of the ellipse is $4x^2 + 9y^2 - 16x - 54y + 61 = 0$

$\Rightarrow \quad (4x^2 - 16x) + (9y^2 - 54y) = -61$

$\Rightarrow \quad 4(x^2 - 4x) + 9(y^2 - 6y) = -61$

$\Rightarrow \quad 4(x^2 - 4x + 4) + 9(y^2 - 6y + 9) = 16 + 81 - 61$

$\Rightarrow \quad 4(x-2)^2 + 9(y-3)^2 = 36$

$\Rightarrow \quad \dfrac{(x-2)^2}{9}+\dfrac{(y-3)^2}{4}=1 \Rightarrow \dfrac{X^2}{9}+\dfrac{Y^2}{4}=1$

Where $x - 2 = X$ and $y - 3 = Y$

Equation of the major axis is $Y = 0$

$\Rightarrow \quad y - 3 = 0 \Rightarrow y = 3$

Obviously, the point (1, 3) lies on the line $y = 3$ i.e., the point lies on the major axis of the ellipse.

7. Find the centre of the ellipse

$$\frac{(x+y-2)^2}{9}+\frac{(x-1)^2}{16}=1.$$

Solution

Given ellipse is $\frac{(x+y-2)^2}{9}+\frac{(x-1)^2}{16}=1$

Let, $X = x + y - 2$ and $Y = x - 1$, then equation of the ellipse be $\frac{X^2}{9}+\frac{Y^2}{16}=1$

For centre: $X = 0$ and $Y = 0$

$\Rightarrow$ $x + y - 2 = 0$ and $x - 1 = 0$

$\Rightarrow$ $x + y - 2 = 0$ and $x = 1$

$\Rightarrow$ $1 + y - 2 = 0$ and $x = 1$

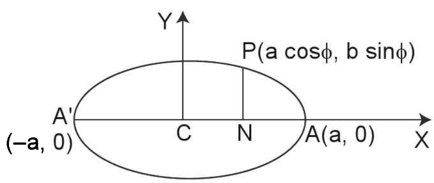
$\Rightarrow$ $y = 1$ and $x = 1$

$\therefore$ Centre of the ellipse is (1, 1)

8. If P be any point on the ellipse, C is its centre. PN the perpendicular from it is the major axis, A and A' vertices, then prove that: $PN^2 : AN \times A'N :: b^2 : a^2$.

Solution

Let $P(a \cos \phi, b \sin \phi)$ be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



$\therefore$ $PN = b \sin \phi$

$\therefore$ and $AN = CA - CN = a - a \cos \phi$

and $A'N = CA' + CN = a + a \cos \phi$

$$\therefore \quad \frac{PN^2}{AN.A'N} = \frac{(b\sin\phi)^2}{(a-a\cos\phi)(a+a\cos\phi)}$$

$$= \frac{b^2\sin^2\phi}{a^2(1-\cos^2\phi)} = \frac{b^2\sin^2\phi}{a^2\sin^2\phi} = \frac{b^2}{a^2}$$

So, $PN^2 : AN . A'N = b^2 : a^2$. **Proved.**

9. An ellipse is described by using an endless string which is passed over two pins. If the axes are 6 cm and 4 cm. Find the necessary length of the string and the distance between the pins.

Solution

Major axis of the ellipse = $2a = 6 \Rightarrow a = 3$

Minor axis of the ellipse = $2b = 4 \Rightarrow b = 2$

Obviously, two pins S and S' are foci of the ellipse.

Let P be any point on the ellipse, then PS and PS' are focal length of the point P.

$\therefore$ Length of the string = $PS + PS'$

= length of the major axis = 6 cm

Let eccentricity of the ellipse be e.

Then, $b^2 = a^2(1 - e^2) \Rightarrow 4 = 9(1 - e^2)$

$\Rightarrow \frac{4}{9}=1-e^2 \Rightarrow e^2=1-\frac{4}{9}$

$\Rightarrow e^2=\frac{5}{9} \Rightarrow e=\frac{\sqrt{5}}{3}$

Therefore, distance between the pins = SS' = $2ae = 2\times3\times\frac{\sqrt{5}}{3} = 2\sqrt{5}$ cm.

10. A man running round a race course notes that the sum of the distances of two flag posts from him is always 10 metre and the distance between the flag posts is 8 metre. Find the area of the path he enclosed in m².

Solution

Obviously, running path is an ellipse whose foci are respectively two flag posts. Let major axis is $2a$, minor axis is $2b$, eccentricity is e and P is any point on the ellipse . Then according to question, $2a = ps + ps' = 10$

$\Rightarrow$ $a = 5$ metre and $2ae = SS' = 8 \Rightarrow ae = 4$

$\therefore$ $e=\frac{ae}{a}=\frac{4}{5}$ $\therefore b^2 = a^2(1 - e^2)$

$$=25\left(1-\frac{16}{25}\right)=25\left(\frac{9}{25}\right)=9$$

$\Rightarrow$ $b = 3$ metre.

$\therefore$ Required area = area of the ellipse = πab

$= \pi \times 5 \times 3 = 15\pi$ m².

11. An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end. ***[NCERT]***

Solution

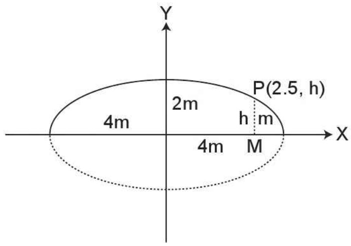
The arch is in the form of a horizontal ellipse.

Its equation can be taken as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ....(i)

According to given conditions $a = 4$, $b = 2$

$\therefore$ (i) becomes $\dfrac{x^2}{4^2}+\dfrac{y^2}{2^2}=1$

or $x^2 + 4y^2 = 16$ ......(ii)



Let, M be the point on the base (major axis) 1.5 m from one end , i.e., at a distance of (4 – 1.5) m = 2.5 m from the centre and let P be the point on the arch vertically above M, then the point (2.5, h) lies on (ii).

$\therefore$ $(2.5)^2 + 4h^2 = 16 \Rightarrow \left(\dfrac{5}{2}\right)^2+4h^2-16=16$

$\Rightarrow \quad 4h^2=16-\dfrac{25}{4}=\dfrac{39}{4} \Rightarrow h^2=\dfrac{39}{4\times 4}$

$\Rightarrow \quad h=\dfrac{\sqrt{39}}{4}=\dfrac{6.245}{4}=1.56$

$\therefore$ Required height of the arch at a point 1.5 m from one end = 1.56 m.

12. A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the x-axis. ***[NCERT]***

Solution

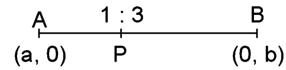
Let l be the length of the rod and let at any position meet x-axis at $A(a, 0)$ and y-axis at $B(0, b)$ so that $l^2 = a^2 + b^2 \Rightarrow (12)^2 = a^2 + b^2$. ....(i)

($\because$ $l = 12$ is given)

Let P be the point on AB which is 3 cm from A and hence. 9 cm from B.

This means that P divides $[AB]$ in the ratio 3: 9, i.e., 1: 3.

If $P = (x, y)$, then by section formula, we have



$$(x,y)=\left(\frac{1\times 0+3\times a}{1+3},\frac{1\times b+3\times 0}{1+3}\right)$$

$\Rightarrow \quad x=\dfrac{3a}{4}$ and $y=\dfrac{b}{4}$

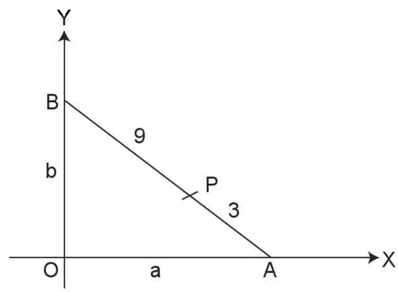
$\Rightarrow \quad a=\dfrac{4x}{3}$ and $b = 4y$.

Substituting these values of a and b in (i), we get

$$144=\left(\frac{4x}{3}\right)^2+(4y)^2$$

$\Rightarrow \quad 1=\dfrac{x^2}{9\times 9}+\dfrac{y^2}{9}$ (Dividing by 144)

or $\dfrac{x^2}{81}+\dfrac{y^2}{9}=1$, which is the required equation to the locus of P.



UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. Find the locus of the point whose distance from (0, 4) is $\dfrac{2}{3}$ times of their distance from the line $y = 9$. Name the curve.

2. Find the equation of the ellipse whose focus is $(1, -1)$, the corresponding directrix is $x - y + 3 = 0$ and $e=\dfrac{1}{2}$.

3. A rod PQ of length 16 units moves with its ends P and Q always touching x-axis and y-axis respectively. Determine the equation of the locus of a point K on $[PQ]$ which is always at a distance of 4 units from P.

4. An arch is in the form of a semi ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.6 m from the end.

Directions 5-12

Find the equation for the ellipse that satisfies the given conditions:

5. Vertices $(\pm 5, 0)$, foci $(\pm 4, 0)$ ***[NCERT]***

6. Vertices $(0, \pm 13)$, foci $(0, \pm 5)$. ***[NCERT]***

7. Ends of major axis $(\pm 3, 0)$, ends of minor axis $(0, \pm 2)$ ***[NCERT]***

8. Length of major axis 26, foci $(\pm 5, 0)$ ***[NCERT]***

9. Length of minor axis 16, foci $(0, \pm 6)$. ***[NCERT]***

10. Centre at (0, 0), major axis on the y-axis and passes through the points (3, 2) and (1, 6). ***[NCERT]***

11. Centre at (0, 0), major axis on the x-axis and passes through the points (4, 3) and (6, 2). ***[NCERT]***

12. Find the coordinates of the foci, the vertices, the length of major axis, minor axis, the eccentricity and the length of latus rectum of the following ellipse:
    (i) $\frac{x^2}{25}+\frac{y^2}{100}=1$
    (ii) $36x^2 + 4y^2 = 144$
    (iii) $16x^2 + y^2 = 16$

ANSWERS

1. $9x^2 + 5y^2 = 180$, ellipse
2. $7(x^2 + y^2) + 2xy - 22x + 22y + 7 = 0$
3. $\frac{x^2}{144}+\frac{y^2}{16}=1$
4. 1.6 m
5. $\frac{x^2}{25}+\frac{y^2}{9}=1$
6. $\frac{x^2}{144}+\frac{y^2}{169}=1$
7. $\frac{x^2}{9}+\frac{y^2}{4}=1$
8. $\frac{x^2}{169}+\frac{y^2}{144}=1$
9. $\frac{x^2}{64}+\frac{y^2}{100}=1$
10. $\frac{x^2}{10}+\frac{y^2}{40}=1$
11. $x^2 + 4y^2 = 52$ or $\frac{x^2}{52}+\frac{y^2}{13}=1$
12. (i) $F(0, \pm\sqrt{75})$; $V = (0, \pm 10)$; Major axis = 20; Minor axis = 10, $e = \sqrt{3}/2$; Latus rectum = 5.
    (ii) $F(0, \pm 4\sqrt{2})$; $V = (0, \pm 6)$; Major axis = 12; Minor axis = 4, $e = 2\sqrt{2}/3$; Latus rectum = 4/3
    (iii) $F(0, \pm\sqrt{15})$; $V = (0, \pm 4)$; Major axis = 8; Minor axis = 2, $e = \sqrt{15}/4$; Latus rectum = 1/2.

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The equation $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$ represents: ***[BIT Ranchi-1986]***
   (a) A circle
   (b) An ellipse
   (c) A hyperbola
   (d) A rectangular hyperbola

Solution

(b) Check $\Delta \neq 0$ and $h^2 < ab$.

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

2. The position of the point (4, –3) with respect to the ellipse $2x^2 + 5y^2 = 20$ is
(a) Outside the ellipse (b) On the ellipse
(c) On the major axis (d) None of these

Solution

(a) Since $S_1 > 0$. Hence the point is outside the ellipse.
Note here that
$S_1 = 2x_1^2 + 5y_1^2 - 20, x_1 = 4, y = -3$
$= 2 \times (4)^2 + 5(-3)^2 - 20$
$= 32 + 45 - 20$
$= 77 - 20 = 57 > 0.$

3. In a model, it is shown that an arc of a bridge is semi elliptical with major axis horizontal. If the length of the base is 9 m and the highest part of the bridge is 3 m from horizontal, the best approximation of the height of the arc, 2 m from the centre of base is ***[VIT-2006]***
(a) 11/4 m (b) 8/3 m
(c) 7/2 m (d) 2 m

Solution

(b) Equation of the elliptical arc is

$$\frac{4x^2}{81} + \frac{y^2}{9} = 1$$

$\Rightarrow \quad 4x^2 + 9y^2 = 81$

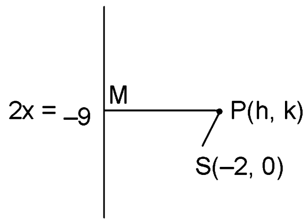
If required height be h, then $(2, h)$ lies on it.
So $\quad 16 + 9h^2 = 81$

$$\Rightarrow \quad h = \frac{\sqrt{65}}{3} = \frac{8}{3} \text{ m (approx.)}$$

4. A point is such that ratio of its distance from a fixed point and line $x = 9/2$ is always 2 : 3. Then locus of the point will be ***[DCE-2005]***
(a) Hyperbola (b) Ellipse
(c) Parabola (d) Circle

Solution

(b) **Trick**: Here $x = 9/2$ is directrix and in question, $PS = \frac{2}{3}PM$ (Given)



$\therefore \quad e = 2/3 < 1$ which can be eccentricity of only a ellipse.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If the latus rectum of an ellipse be equal to half of its minor axis, then its eccentricity is ***[PET-91]***
(a) 3/2 (b) $\sqrt{3}/2$
(c) 2/3 (d) $\sqrt{2}/3$

2. The centre of the ellipse $4x^2 + 9y^2 - 16x - 54y + 61 = 0$ is ***[PET-92]***
(a) (1, 3) (b) (2, 3)
(c) (3, 2) (d) (3, 1)

3. The equation of the ellipse whose latus rectum is 8 and whose eccentricity is $1/\sqrt{2}$, referred to the principal axes of coordinates, is ***[PET-93]***

(a) $\frac{x^2}{18} + \frac{y^2}{32} = 1$

(b) $\frac{x^2}{8} + \frac{y^2}{9} = 1$

(c) $\frac{x^2}{64} + \frac{y^2}{32} = 1$

(d) $\frac{x^2}{16} + \frac{y^2}{24} = 1$

4. The equation $\frac{x^2}{2-r} + \frac{y^2}{r-5} + 1 = 0$ represents an ellipse, if ***[PET-95; KCET-2008]***
(a) $r > 2$
(b) $2 < r < 5$
(c) $r > 5$
(d) None of these

5. The length of the latus rectum of the ellipse $9x^2 + 4y^2 = 1$, is ***[PET-99]***
(a) 3/2 (b) 8/3
(c) 4/9 (d) 8/9

6. The eccentricity of the ellipse $25x^2 + 16y^2 = 400$ is ***[PET-2001]***

(a) 3/5 (b) 1/3
(c) 2/5 (d) 1/5

7. If distance between the directrices be thrices the distance between the foci, then eccentricity of ellipse is

(a) 1/2 (b) 2/3
(c) $1/\sqrt{3}$ (d) 4/5

8. If the foci and vertices of an ellipse be (± 1, 0) and (± 2, 0), then the minor axis of the ellipse is

(a) $2\sqrt{5}$ (b) 2
(c) 4 (d) $2\sqrt{3}$

9. An ellipse has *OB* as semi minor axis, *F* and *F'* its foci and the angle *FBF* is a right angle. Then the eccentricity of the ellipse is ***[AIEEE-2005]***

(a) $\frac{1}{4}$ (b) $\frac{1}{\sqrt{3}}$
(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2}$

10. The eccentricity of an ellipse, with its centre at the origin, is $\frac{1}{2}$. If one of the directrices is $x = 4$, then the equation of the ellipse is ***[AIEEE-2004; MP PET-2009]***

(a) $4x^2 + 3y^2 = 1$
(b) $3x^2 + 4y^2 = 12$
(c) $4x^2 + 3y^2 = 12$
(d) $3x^2 + 4y^2 = 1$

11. In an ellipse, the distances between its foci is 6 and minor axis is 8. Then its eccentricity is ***[AIEEE-2006]***

(a) $\frac{1}{2}$ (b) $\frac{4}{5}$
(c) $\frac{1}{\sqrt{5}}$ (d) $\frac{3}{5}$

12. The equation of the ellipse whose foci are (±2, 0) and eccentricity is 1/2, is ***[AIEEE-2002]***

(a) $\frac{x^2}{12}+\frac{y^2}{16}=1$ (b) $\frac{x^2}{16}+\frac{y^2}{12}=1$
(c) $\frac{x^2}{16}+\frac{y^2}{8}=1$ (d) None of these

13. *P* be a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci F_1 and F_2. If *A* is the area of the triangle PF_1F_2, then the maximum value of *A* is ..... ***[DCE-1996]***

(a) ea/b (b) $ae.b$
(c) ab/e (d) e/ab

14. Let *E* be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and *C* be the circle $x^2 + y^2 = 4$. Let *P* and *Q* be the point (1,2) and (2,1) respectively. Then ***[DCE-1996]***

(a) *Q* lies inside *C* but outside *E*
(b) *P* lies outside both *C* and *E*
(c) *P* lies inside both *C* and *E*
(d) *P* lies inside *C* but outside *E*

15. The curve represented by the equation $4x^2 + 16y^2 - 24x - 32y - 12 = 0$ is ***[MPPET-2005]***

(a) a parabola
(b) a pair of straight line
(c) an ellipse with eccentricity 1/2
(d) an ellipse with eccentricity $\sqrt{3}/2$.

16. Equation of the ellipse whose foci are (2,2) and (4,2) and the major axis is of length 10 is ***[MPPET-2006]***

(a) $\frac{(x+3)^2}{24}+\frac{(y+2)^2}{25}=1$
(b) $\frac{(x-3)^2}{24}+\frac{(y-2)^2}{25}=1$
(c) $\frac{(x+3)^2}{25}+\frac{(y+2)^2}{24}=1$
(d) $\frac{(x-3)^2}{25}+\frac{(y-2)^2}{24}=1$

17. The foci of an ellipse are (0, ±4) and the equations for the directrices are $y = \pm 9$. The equations for the ellipse is ***[MPPET-2006]***

(a) $5x^2 + 9y^2 = 4$
(b) $2x^2 - 6y^2 = 28$
(c) $6x^2 + 3y^2 = 45$
(d) $9x^2 + 5y^2 = 180$

18. If $P(x,y)$, $F_1 = (3,0)$, $F_2 = (-3,0)$ and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ is equal to ***[IIT-1998]***

(a) 8 (b) 6
(c) 10 (d) 12

19. The equation of a directrix of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$, is **[VIT-2007]**

(a) $3y = 5$ (b) $y = 5$
(c) $3y = 25$ (d) $y = 3$

20. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and having its centre (0, 3) is **[MPPET-2008, 2010]**

(a) 4 (b) 3
(c) $\sqrt{12}$ (d) 7/2

21. The parametric representation of a point on the ellipse whose foci are (3, 0) and (– 1, 0) and eccentricity 2/3 is **[Kerala PET-2007]**

(a) $(1+3\cos\theta, \sqrt{3}\sin\theta)$
(b) $(1 + 3\cos\theta, 5\sin\theta)$
(c) $(1+3\cos\theta, 1+\sqrt{5}\sin\theta)$
(d) $(1+3\cos\theta, \sqrt{5}\sin\theta)$

22. The foci of the ellipse $25(x + 1)^2 + 9(y + 2)^2 = 225$ are at **[MNR-91; UPSEAT-2000; PET-98]**

(a) (–1, 2) and (–1, –6) (b) (–1, 2) and (6, 1)
(c) (1, –2) and (1, –6) (d) (–1, –2) and (1, 6)

23. The eccentric angles of the extremities of latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given by

(a) $\tan^{-1}\left(\pm\frac{ae}{b}\right)$ (b) $\tan^{-1}\left(\pm\frac{be}{a}\right)$
(c) $\tan^{-1}\left(\pm\frac{b}{ae}\right)$ (d) $\tan^{-1}\left(\pm\frac{a}{be}\right)$

24. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is **[AIEEE-2005; MP PET-2010]**

(a) a/b (b) $\sqrt{ab}$
(c) ab (d) $2ab$

25. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the mid-point of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the points. **[IIT-2009]**

(a) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ (b) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$
(c) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (d) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

26. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is **[IIT-2009]**

(a) $\frac{31}{10}$ (b) $\frac{29}{10}$
(c) $\frac{21}{10}$ (d) $\frac{27}{10}$

27. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is **[AIEEE-2009]**

(a) $x^2 + 16y^2 = 16$ (b) $x^2 + 12y^2 = 16$
(c) $4x^2 + 48y^2 = 48$ (d) $4x^2 + 64y^2 = 48$

SOLUTIONS

1. (b) $\frac{2b^2}{a}=b \Rightarrow \frac{b}{a}=\frac{1}{2}\Rightarrow\frac{b^2}{a^2}=\frac{1}{4}$

Hence, $e=\sqrt{1-\frac{b^2}{a^2}}=\frac{\sqrt{3}}{2}$.

2. (b) $4(x – 2)^2+ 9(y – 3)^2= 36$. Hence the center is (2, 3)

3. (c) $\frac{2b^2}{a}=8,\ e=\frac{1}{\sqrt{2}}$

$\Rightarrow a^2 = 64, b^2= 32$

Hence, required equation of ellipse is

$\frac{x^2}{64}+\frac{y^2}{32}=1$

4. (b) $\dfrac{x^2}{2-r}+\dfrac{y^2}{r-5}+1=0$

$\Rightarrow \quad \dfrac{x^2}{r-2}+\dfrac{y^2}{5-r}=1$

Hence $r > 2$ and $r < 5$

$\Rightarrow \quad 2 < r < 5.$

5. (c) Given ellipse is $\dfrac{x^2}{\left(\frac{1}{3}\right)^2}+\dfrac{y^2}{\left(\frac{1}{2}\right)^2}=1$

Here, $b > a$

$\therefore \quad$ Latus rectum $=\dfrac{2a^2}{b}=\dfrac{2\times\frac{1}{9}}{\frac{1}{2}}=\dfrac{4}{9}.$

6. (a) $\dfrac{x^2}{16}+\dfrac{y^2}{25}=1 \Rightarrow a^2 = b^2(1-e^2)$

$\Rightarrow \quad e^2 = 1-\dfrac{16}{25} \Rightarrow e=\dfrac{3}{5}$

7. (c) Accoding to the condition, $\dfrac{2a}{e}=bae$

$\Rightarrow \quad e=\dfrac{1}{\sqrt{3}}.$

8. (d) $ae = 1, a = 2, e = 1/2$

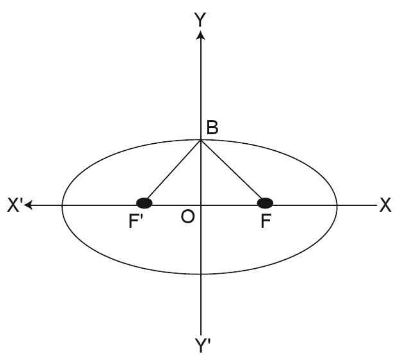
$\Rightarrow \quad b=\sqrt{4\left(1-\dfrac{1}{4}\right)}=\sqrt{3}$

Hence minor axes is $2\sqrt{3}$.

9. (c) $\angle F9\,BF = 90°, F9\,B \perp FB$

i.e., slope of $(F'B)$ × slope of $(FB) = -1$

$\Rightarrow \quad \dfrac{b}{ae}\times\dfrac{b}{-ae}=-1, b^2=a^2e^2$ ....(1)



We know that, $e=\sqrt{1-\dfrac{b^2}{a^2}}$

$=\sqrt{1-\dfrac{a^2e^2}{a^2}}=\sqrt{1-e^2}$

$e^2 = 1 - e^2, 2e^2 = 1, e^2=\dfrac{1}{2}, e=\dfrac{1}{\sqrt{2}}.$

10. (b) Since directrix is parallel to y-axis, hence axis of the ellipse are parallel to x-axis.

Let the equation of the ellipse be

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1, (a>b)$

$e^2=1-\dfrac{b^2}{a^2} \Rightarrow \dfrac{b^2}{a^2}=1-e^2=1-\dfrac{1}{4} \Rightarrow \dfrac{b^2}{a^2}=\dfrac{3}{4}.$

Also, one of the directrices is $x = 4$

$\Rightarrow \quad \dfrac{a}{e}=4$

$\Rightarrow \quad a=4e=4.\dfrac{1}{2}=2;\ b^2=\dfrac{3}{4}a^2=\dfrac{3}{4}.4=3$

$\therefore \quad$ Required ellipse is $\dfrac{x^2}{4}+\dfrac{y^2}{3}=1$ or $3x^2 + 4y^2 = 1$.

11. (d) Given that $2ae = 6$ and $2b = 8 \Rightarrow ae = 3$ and $b = 4$

$\Rightarrow \quad \dfrac{ae}{b}=\dfrac{3}{4} \Rightarrow e^2=\dfrac{9}{16}\left(\dfrac{b^2}{a^2}\right)$

We know that $\dfrac{b^2}{a^2}=1-e^2$

$\Rightarrow \quad e^2=\dfrac{9}{16}(1-e^2)$

$\Rightarrow \quad \left(\dfrac{16+9}{9}\right)e^2=1 \Rightarrow e=\dfrac{3}{5}$

12. (b) The foci of an ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is given by $(\pm ae, 0)$

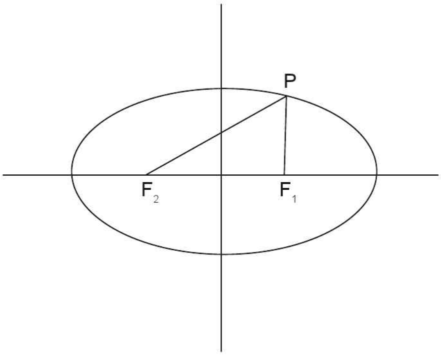
Since, $e=\dfrac{1}{2}, ae=2 \Rightarrow a=4$

$\therefore \quad b^2 = a^2(1-e^2)=16\left(1-\dfrac{1}{4}\right)=12$

Thus, the equation of an ellipse is $\dfrac{x^2}{16}+\dfrac{y^2}{12}=1.$

13. (b) Given ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let, point $P(a\cos\theta, b\sin\theta), F_1=(ae,0), (-ae,0)$



$$A=\frac{1}{2}[a\cos\theta(0)-ae(0-b\cos\theta)+ae(b\sin\theta-0)]$$

$$A=\frac{1}{2}[0+abe\sin\theta+abe\sin\theta]$$

$\Rightarrow$ $A=abe\sin\theta$

Maximum A, $\sin\theta$ should be maximum

Maximum $\sin\theta=1$ $A=ae\,.\,b\times 1=ae\,.\,b$

14. (b) Let S be the equation of curve and (x_1, y_1) is any point, then if $S_1=0$ x_1, y_1 lies on the curve, $S_1>0$ (x_1, y_1) outside the curve

$S_1<0$ (x_1, y_1) inside the curve. Given that ellipse $\frac{x^2}{9}+\frac{y^2}{4}-1$

$\Rightarrow$ Standard equation of ellipse,

$$S_E=\frac{x^2}{9}+\frac{y^2}{4}-1$$

Circle $x^2+y^2=4 \Rightarrow S_C=x^2+y^2-4$ point $P(1, 2)$ and $Q(2, 1)$

$$S_E \text{ at } P_{(1,2)}=\frac{1}{9}+\frac{4}{4}-1=\frac{1}{9}>0$$

$\Rightarrow$ P outside E ....(i)

$$S_E \text{ at } Q_{(2,1)}=\frac{4}{9}+\frac{1}{4}-1=\frac{-11}{36}<0$$

$\Rightarrow$ Q outside E ....(ii)

S_C at $Q_{(2,1)}=2^2+1^2-4=1>0$

$\Rightarrow$ Q outside C .....(iii)

S_C at $P_{(1,2)}=1+2^2-4=1>0$

$\Rightarrow$ P outside C ....(iv)

By the above conclusion, only (b) is correct.

15. (d) The given equation is $4x^2-24x+36+16y^2-32y-12-36+16-16=0$

$\Rightarrow$ $(2x-6)^2+(4y-4)^2=64$

$$\Rightarrow \quad \frac{(x-3)^2}{16}+\frac{(y-1)^2}{4}=1$$

This represents an ellipse and $a^2=16, b^2=4$

$$\therefore \quad e=\sqrt{1-\frac{4}{16}}=\frac{\sqrt{3}}{2}.$$

16. (d) Since, given that foci of an ellipse are (2,2) and (4,2) and major axis is of length 10.

$\therefore$ Focal distance = 2

$\Rightarrow$ $2ae=2$ ....(i)

and $2a=10 \Rightarrow a=5$ ....(ii)

From (i) and (ii), $2\times 5\times e=2 \Rightarrow e=\frac{1}{5}$

We know that, $b^2=a^2(1-e^2)$

$$\therefore \quad b^2=25\left(1-\frac{1}{25}\right)=\frac{24}{25}\times 25=24$$

and centre of an ellipse = (3,2)

$\therefore$ Equation of an ellipse is

$$\frac{(x-3)^2}{25}+\frac{(y-2)^2}{24}=1$$

17. (d) Foci of an ellipse are (0,4) and (0,–4) and equations of directrices are $y=\pm 9$. So, focal distance = 8

$\Rightarrow$ $2be=8 \Rightarrow be=4$ ....(i)

and $\frac{b}{e}=9$ ....(ii)

From equation (i) and (ii), we get,

$b^2=36 \Rightarrow b=6$ and $e=\frac{2}{3}$ [using (i)]

$$\text{Now, } a^2=b^2(1-e^2)=36\left(1-\frac{4}{9}\right)$$

$$=\frac{36\times 5}{9}=20$$

So, equation of ellipse is, $\frac{x^2}{20}+\frac{y^2}{36}=1$

$\Rightarrow$ $9x^2+5y^2=180.$

18. (c) $16x^2+25y^2=400$ (given)

$$\Rightarrow \quad \frac{x^2}{25}+\frac{y^2}{16}=1$$

Here, $a^2=25, b^2=16$

But, $b^2=a^2(1-e^2)$

$\Rightarrow \quad 16 = 25\,(1 - e^2)$

$\Rightarrow \quad \frac{16}{25} = 1 - e^2$

$\Rightarrow \quad e^2 = 1 - \frac{16}{25} = \frac{9}{25}$

$\Rightarrow \quad e = 3/5$

Now foci of the ellipse are $(\pm\, ae\,,\, 0) \equiv (\pm\, 3,\, 0)$

we have $3 = a.\frac{3}{5} \Rightarrow a = 5$

Now , $PF_1 + PF_2$ = focal distance = $2a = 2 \times 5 = 10$

Therefore, (c) is the answer.

19. (c) The equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{25} = 1$,

where $a = 4$, $b = 5$ and $e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$

$\therefore$ Equation of directrix is

$$y = \pm\left(\frac{5}{3/5}\right) = \pm\frac{25}{3}$$

$\Rightarrow \quad 3y = 25$ is a equation of the directrix.

20. (a) Foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{5} = 1$ is

$(\pm\sqrt{a^2 + b^2}, 0)$ i.e., $(\pm\sqrt{7}, 0)$

$\therefore$ equation of circle passing through F_1 $(\sqrt{7}, 0)$ and centre $C(0, 3)$ is $(x - 0)^2 + (y - 3)^2 = (r)^2$

$$r = CF_1 = \sqrt{(\sqrt{7})^2 + 3^2} = 4$$

21. (d) Centre of ellipse (1, 0)

distance of the foci = $2ae = 4$

$ae = 2$; $a = 3$

$b = \sqrt{5}$; $\therefore$ parametric form $x - 1 = 3\cos\theta$; $y = \sqrt{5}\sin\theta$

$x = 1 + 3\cos\theta$; $y = \sqrt{5}\sin\theta$

22. (a) $25(x + 1)^2 + 9(y + 2)^2 = 225$

$$\frac{(x+1)^2}{9} + \frac{(y+2)^2}{25} = 1$$

focii $(x + 1, y + 1) \equiv (0, \pm\sqrt{25 - 9}) \equiv (0, \pm 4)$

$(x, y) \equiv (-1, 2)$ and $(-1, -6)$

23. (c) Coordinates of extremities of latus rectum

$$= \left(\pm ae,\, \pm\frac{b^2}{a}\right) \equiv (a\cos\theta, b\sin\theta)$$

$\therefore \quad \frac{y \text{ coordinate}}{x \text{ coordinate}} = \frac{b}{a}\tan\theta = \pm\frac{b^2}{a^2 e}$

$$\theta = \tan^{-1}\left(\frac{b}{ae}\right)$$

24. (d) For greatest area, vertices are $(a\cos\theta, b\sin\theta)$ $(-a\cos\theta, b\sin\theta)$ $(a\cos\theta, -b\sin\theta)$ $(-a\cos\theta, -b\sin\theta)$

Area = $(2a\cos\theta)(2b\sin\theta) = 2ab\sin 2\theta$

Maximum area = $2ab$

25. (c) Normal is $4x\sec\phi - 2y\,\text{cosec}\,\phi = 12$

$Q = (3\cos\phi, 0)$

$M = (\alpha, \beta)$

$$\alpha = \frac{3\cos\phi + 4\cos\phi}{2} = \frac{7}{2}\cos\phi$$

$\Rightarrow \quad \cos\phi = \frac{2}{7}\alpha$

$\beta = \sin\phi$

$\cos^2\phi + \sin^2\phi = 1$

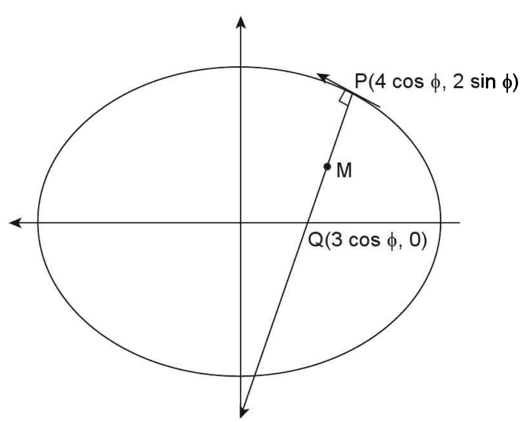
$\Rightarrow \quad \frac{4}{49}\alpha^2 + \beta^2 = 1$

$\Rightarrow \quad \frac{4}{49}x^2 + y^2 = 1$

$\Rightarrow$ latus rectum $x = \pm 2\sqrt{3}$

$$\frac{48}{49} + y^2 = 1 \Rightarrow y = \pm\frac{1}{7}$$

$(\pm 2\sqrt{3},\ \pm 1/7)$



26. (d) Equation of line AM is $x + 3y - 3 = 0$

Perpendicular distance of line from origin

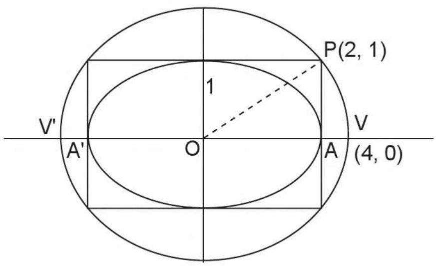
$= \frac{3}{\sqrt{10}}$

Length of $AM = 2\sqrt{9 - \frac{9}{10}} = 2 \times \frac{9}{\sqrt{10}}$

$\Rightarrow$ Area $= \frac{1}{2} \times 2 \times \frac{9}{\sqrt{10}} \times \frac{3}{\sqrt{10}} = \frac{27}{10}$ sq. units.

27. (b) Given that $x^2 + 4y^2 = 4$

or $\frac{x^2}{4} + \frac{y^2}{1} = 1 \Rightarrow a = 2, b = 1, \Rightarrow p = (2, 1)$



Required ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{x^2}{4^2} + \frac{y^2}{b^2} = 1$

(2, 1) lies on it

$\Rightarrow \frac{4}{16} + \frac{1}{b^2} = 1$

$\Rightarrow \frac{1}{b^2} = 1 - \frac{1}{4} = \frac{3}{4}$

$\Rightarrow b^2 = \frac{4}{3}$

$\therefore \frac{x^2}{16} + \frac{y^2}{\left(\frac{4}{3}\right)} = 1$

$\Rightarrow \frac{x^2}{16} + \frac{3y^2}{4} = 1$

$\Rightarrow x^2 + 12y^2 = 16$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The position of the point (1, 3) with respect to the ellipse $4x^2 + 9y^2 - 16x - 54y + 61 = 0$ is ***[PET-91]***

(a) outside the ellipse
(b) on the ellipse
(c) on the major axis
(d) on the minor axis

2. The equation $2x^2 + 3y^2 = 30$ represents ***[PET-98]***

(a) A circle (b) An ellipse
(c) A hyperbola (d) A parabola

3. Equation of the ellipse with eccentricity 1/2 and foci at (± 1, 0) is ***[PET-2002]***

(a) $\frac{x^2}{3} + \frac{y^2}{4} = 1$ (b) $\frac{x^2}{4} + \frac{y^2}{3} = 1$

(c) $\frac{x^2}{4} + \frac{y^2}{3} = \frac{4}{3}$ (d) None of these

4. The equations of the directrices of the ellipse $16x^2 + 25y^2 = 400$ are

(a) $2x = \pm 25$ (b) $5x = \pm 9$
(c) $3x = \pm 10$ (d) None of these

5. Eccentricity of the ellipse whose latus rectum is equal to the distance between two focus points, is

(a) $\frac{\sqrt{5}+1}{2}$ (b) $\frac{\sqrt{5}-1}{2}$

(c) $\frac{\sqrt{5}}{2}$ (d) $\frac{\sqrt{3}}{2}$

6. The eccentricity of the conic $4x^2 + 16y^2 - 24x - 32y = 1$ is ***[MPPET-2004]***

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$

(c) $\frac{\sqrt{3}}{4}$ (d) $\sqrt{3}$

7. A focus of an ellipse is at the origin. The directrix is the line $x = 4$ and the eccentricity is 1/2. Then the length of the semi-major axis is ***[AIEEE-2008]***

(a) 5/3 (b) 8/3
(c) 2/3 (d) 4/3

8. If P is any point on the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$, and S and S' are the foci, then $PS + PS'$ is equal to ***[Karnataka CET-2007]***

(a) 4 (b) 8
(c) 10 (d) 12

9. The length of the latus rectum of an ellipse is 1/3 of the major axis. Its eccentricity is ***[EAMCET-91]***

(a) 2/3 (b) $\sqrt{2/3}$
(c) $5 \times 4 \times 3/7^3$ (d) $(3/4)^4$

10. The equation of the ellipse whose centre is at origin and which passes through the points (– 3, 1) and (2, – 2) is

(a) $5x^2 + 3y^2 = 32$
(b) $3x^2 + 5y^2 = 32$
(c) $5x^2 - 3y^2 = 32$
(d) $3x^2 + 5y^2 + 32 = 0$

11. If the eccentricity of the two ellipse $\frac{x^2}{169}+\frac{y^2}{25}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are equal, then the value of a/b is ***[UPSEAT-2001]***

(a) 5/13 (b) 6/13
(c) 13/5 (d) 13/6

12. The eccentricity of the conic $\frac{(x+2)^2}{7}+(y-1)^2=14$ is ***[Kerala PET-2007]***

(a) $\sqrt{\frac{7}{8}}$ (b) $\sqrt{\frac{6}{17}}$
(c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{\frac{6}{7}}$

13. Latus rectum of ellipse $4x^2 + 9y^2 - 8x - 36y + 4 = 0$ is ***[PET-89]***

(a) 8/3 (b) 4/3
(c) $\sqrt{5}/3$ (d) 16/3

14. The eccentricity of the ellipse $4x^2 + 9y^2 = 36$, is ***[PET-2000]***

(a) $1/2\sqrt{3}$ (b) $1/\sqrt{3}$
(c) $\sqrt{5}/3$ (d) $\sqrt{5}/6$

15. If the eccentricity of an ellipse be 5/8 and the distance between its foci of 10, then its latus rectum is

(a) 39/4 (b) 12
(c) 15 (d) 27/2

16. The equation of the ellipse whose centre is at origin, eccentricity is $1/\sqrt{2}$ and latus rectum is 8, is

(a) $2x^2 + y^2 = 64$
(b) $2x^2 + y^2 = 8$
(c) $x^2 + 2y^2 = 64$
(d) $x^2 + 2y^2 = 8$

17. For the ellipse $3x^2 + 4y^2 = 12$, the length of latus rectum is ***[MNR-73]***

(a) 3/2 (b) 3
(c) 8/3 (d) $\sqrt{3/2}$

18. The equation $\frac{x^2}{1-r}-\frac{y^2}{1+r}=1, r>1$ represents:

(a) An ellipse
(b) A hyperbola
(c) A circle
(d) An imaginary ellipse

19. Length of major axis of ellipse $9x^2 + 7y^2 = 63$ is ***[DCE-2003]***

(a) 3 (b) 9
(c) 6 (d) $2\sqrt{7}$

20. The equation of the ellipse whose foci are at (± 2, 0) and eccentricity is $\frac{1}{2}$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where ***[Orissa JEE-2008]***

(a) $a^2 = 16, b^2 = 12$
(b) $a^2 = 12, b^2 = 16$
(c) $a^2 = 16, b^2 = 4$
(d) $a^2 = 4, b^2 = 16$

21. If the lines joining the foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a > b$, and an extremity of its minor axis are inclined at an angle 60°, then the eccentricity of the ellipse is ***[Kerala PET-2008]***

(a) $\frac{2}{3}$ (b) $\frac{1}{2}$
(c) $\frac{\sqrt{3}}{4}$ (d) $\frac{3}{5}$

22. Let A and B are two fixed points in a plane then locus of another point C on the same plane such that $CA + CB$ = constant, ($> AB$) is ***[VITEEE-2008]***

(a) circle (b) ellipse
(c) parabola (d) hyperbola

23. The locus of a point which moves such that the sum of its distances from two fixed points is a constant is ***[Karnataka CET-2008]***

(a) an ellipse (b) a hyperbola
(c) a circle (d) a parabola

24. Suppose S and S' are foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$. If P is a variable point on the ellipse and if Δ is the area of triangle PSS', then maximum value of Δ is ***[Kerala (CEE)-2005]***

(a) 12 (b) 20
(c) 24 (d) 16

25. If the area of the auxillary circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a > b)$ is twice the area of the ellipse, then the eccentricity of the ellipse is ***[Karnataka CET-2007]***

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$
(c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{2}$

26. The curve represented by $x = 3(\cos t + \sin t)$, $y = 4(\cos t - \sin t)$ is ***[EAMCET-88; DEC-2000]***

(a) Ellipse (b) Parabola
(c) Hyperbola (d) Circle

27. Which one of the following is correct? The eccentricity of the conic

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, (\lambda \geq 0)$$

[NDA-2008]

(a) increases with increase in λ
(b) decreases with increase in λ
(c) does not change with λ
(d) None of the above is correct

28. What is the eccentricity of an ellipse if its latus rectum is equal to one-half of its minor axis? ***[NDA-2009]***

(a) 1/4 (b) 1/2
(c) $\sqrt{3}/4$ (d) $\sqrt{3}/2$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 16 minutes.
3. The worksheet consists of 16 questions. The maximum marks are 48.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The eccentricity of the ellipse $4x^2 + 9y^2 + 8x + 36y + 4 = 0$ is ***[PET-96]***

(a) 5/6 (b) 3/5

(c) $\sqrt{2}/3$ (d) $\sqrt{5}/3$

2. If distance between the directrices be thrices the distance between the foci, then eccentricity of ellipse is

(a) 1/2 (b) 2/3

(c) $1/\sqrt{3}$ (d) 4/5

3. The equation of the ellipse whose foci are (± 5, 0) and one of its directrix is $5x = 36$, is

(a) $\frac{x^2}{36}+\frac{y^2}{11}=1$ (b) $\frac{x^2}{6}+\frac{y^2}{\sqrt{11}}=1$

(c) $\frac{x^2}{6}+\frac{y^2}{11}=1$ (d) None of these

4. If the eccentricity of an ellipse be $1/\sqrt{2}$, then its latus rectum is equal to its:

(a) Minor axis

(b) Semi-minor axis

(c) Major axis

(d) Semi-major axis

5. If the eccentricity of an ellipse becomes zero, then it takes the form of

(a) A circle (b) A parabola

(c) A straight line (d) None of these

6. The eccentricity of an ellipse is 2/3, latus rectum is 5 and centre is (0, 0). The equation of the ellipse is

(a) $\frac{x^2}{81}+\frac{y^2}{45}=1$ (b) $\frac{4x^2}{81}+\frac{4y^2}{45}=1$

(c) $\frac{x^2}{9}+\frac{y^2}{5}=1$ (d) $\frac{x^2}{81}+\frac{y^2}{45}=5$

7. The distance between the directrices of the ellipse $\frac{x^2}{36}+\frac{y^2}{20}=1$ is

(a) 8 (b) 12

(c) 18 (d) 24

8. If the distance between the foci of an ellipse be equal to its minor axis, then its eccentricity is

(a) 1/2 (b) $1/\sqrt{2}$

(c) 1/3 (d) $1/\sqrt{3}$

9. P is any point on the ellipse $81x^2 + 144y^2 = 1944$ whose foci are S and S'. The $SP + S'P$ equals ***[DCE-1999]***

(a) 3 (b) $4\sqrt{6}$

(c) 36 (d) 324

10. The distance of a point on the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ from the centre is 2. The eccentric angle of the point is ***[NDA-2003]***

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{4}$ (d) π

11. The eccentricity of the ellipse $9x^2 + 5y^2 - 30y = 0$ is ***[MPPET-2008; DCE-1998]***

(a) 1/3 (b) 2/3

(c) 3/4 (d) 4/5

12. If the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ are $(0, \sqrt{7})$ and $(0, -\sqrt{7})$, then the foci of the ellipse $\frac{x^2}{9+t^2}+\frac{y^2}{16+t^2}=1, t \in R,$ are ***[Kerala PET-2008]***

(a) $(0, \sqrt{7}), (0, -\sqrt{7})$ (b) $(0, 7), (0, -7)$

(c) $(0, 2\sqrt{7}), (0, -2\sqrt{7})$ (d) $(\sqrt{7}, 0), (-\sqrt{7}, 0)$

13. The sum of focal distances of any point on the ellipse with major and minor axes as $2a$ and $2b$ respectively, is equal to ***[PET-2003]***

(a) $2a$ (b) $2\frac{a}{b}$

(c) $2\frac{b}{a}$ (d) b^2/a

14. The latus rectum of an ellipse is 10 and the minor axis is equal to the distance between the foci. The equation of the ellipse is

(a) $x^2 + 2y^2 = 100$ (b) $x^2 + \sqrt{2}y^2 = 10$

(c) $x^2 - 2y^2 = 100$ (d) None of these

15. The lengths of major and minor axis of an ellipse are 10 and 8 respectively and its major axis along the y-axis. The equation of the ellipse referred to its centre as origin is *[Pb. (CET)-2003]*

(a) $\frac{x^2}{25}+\frac{y^2}{16}=1$ (b) $\frac{x^2}{16}+\frac{y^2}{25}=1$

(c) $\frac{x^2}{100}+\frac{y^2}{64}=1$ (d) $\frac{x^2}{64}+\frac{y^2}{100}=1$

16. What is the sum of focal radii of any point on an ellipse equal to? *[NDA-2009]*

(a) Length of latus rectum
(b) Length of major axis
(c) Length of minor axis
(d) Length of semi-latus rectum

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
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15. ⓐ ⓑ ⓒ ⓓ
16. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. (d) $4x^2 + 9y^2 + 8x + 36y + 4 = 0$

$$4(x^2 + 2x + 1) + 9(y^2 + 4y + 4) = 36$$

$$\frac{(x+1)^2}{9}+\frac{(y+2)^2}{4}=1$$

eccentricity $=\sqrt{1-\frac{4}{9}}=\frac{\sqrt{5}}{3}$

2. (c) Given $\frac{2a}{e}=3.2ae \quad e=\sqrt{\frac{1}{3}}$

5. (a) If $e = 0$; then equation will be $(x - \alpha)^2 + (y + \beta)^2 = 0$

Where (α, β) is focus. Here equation is form of a circle.

9. (b) $81x^2 + 144y^2 = 1944$

or $\frac{x^2}{24}+\frac{y^2}{\frac{27}{2}}=1$

$SP + S'P$ = major axis = $2a$

$= 2\sqrt{24} = 4\sqrt{6}$

16. (b) We know that, the sum of focal radii of any point or, an ellipse is equal length of major axis.

LECTURE

4

Ellipse 2

(Position of line with respect to an ellipse, diameter, tangents and normals, chord of content)

BASIC CONCEPTS

1. Position of a Straight Line with Respect to an Ellipse

1.1 The line $y = mx + c$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at two real points (chord) if $c^2 < a^2m^2 + b^2$, then length of the chord

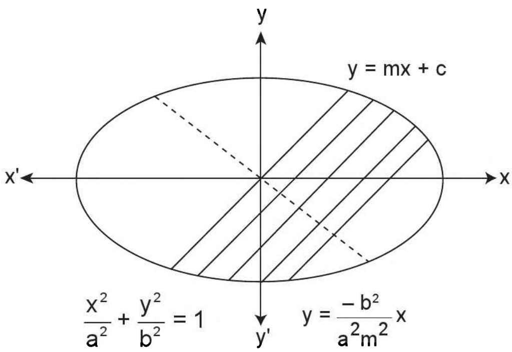
$$= \frac{2ab}{a^2m^2+b^2}\sqrt{(a^2m^2+b^2-c^2)(1+m^2)}$$

1.2 Equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of the chord joining points θ and ϕ on this ellipse is

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right)$$

1.3 Equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with (x_1, y_1) as its mid-point is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$

1.4 Diameter The locus of the mid-points of a system of parallel chords of an ellipse is called a diameter (or diameter corresponding to that system of chords) of the ellipse. Every diameter of an ellipse passes through its centre. The points of intersection of the ellipse and a diameter are known as the end points of that diameter.



1.5 The diameter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to the system of chords $y = mx + c$ (c is parameter) $y=-\frac{b^2}{a^2m}x$

1.6 Conjugate diameter Two diameters of an ellipse are said to be its conjugate diameters if each bisects all chords parallel to the other.

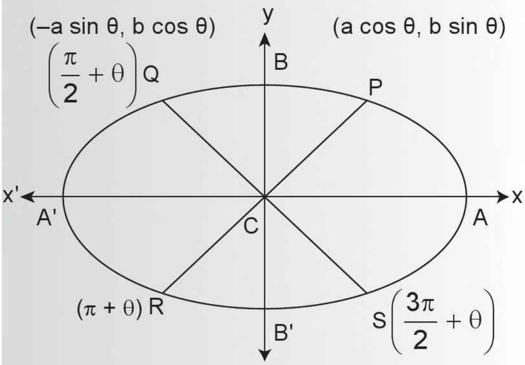
1.7 Equation of conjugate diameter of $y = mx$ with respect to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=-\frac{b^2}{a^2m}x$.

1.8 $y = m_1x$ and $y = m_2x$ are conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $m_1m_2=-\frac{b^2}{a^2}$

Also, $e=\sqrt{1+m_1m_2}$

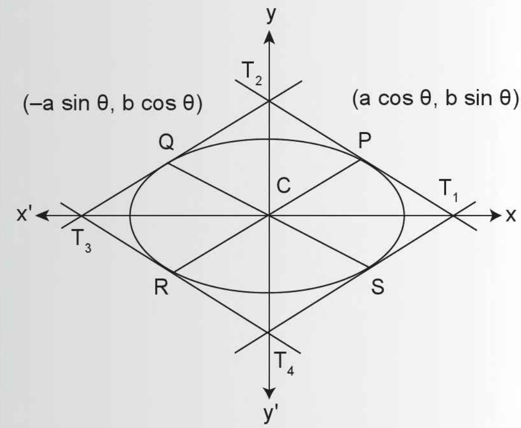
NOTES

1. Major and minor axes of ellipse are mutually conjugate diameters but product of their slopes does not satisfy the condition $m_1m_2 = \frac{-b^2}{a^2}$
2. Eccentric angles of extremities of any two semi conjugate diameters differ by right angle (90°).
3. The difference of eccentric angles of end points of two conjugate diameters of an ellipse is equal to $\pi/2$.



4. The sum of the squares of two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of its semi axes. Hence, if CP and CQ are two conjugate semi-diameters, then: $CP^2 + CQ^2 = a^2 + b^2$
5. Sum of the squares of the two semi conjugate diameters of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $a^2 + b^2$
6. Length of the Equi-Conjugate Diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $=\sqrt{2(a^2+b^2)}$

   If the lengths of two conjugate diameters of an ellipse be equal to each other then they are called equi conjugate diameters.
7. The area of the parallelogram formed by the tangents at the extremities of two conjugate diameters of an ellipse is constant ($4ab$) and is equal to product of the major and minor axis. (Area of parallelogram $PQRS$)



2. If $y = mx + c$ be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then $c=\pm\sqrt{a^2m^2+b^2}$ and coordinates of the point of contact are $\left(\frac{-a^2m}{c},\frac{b^2}{c}\right)$

2.1 Gradient form equation of the tangent

For all values of m straight line $y = mx \pm \sqrt{a^2m^2+b^2}$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and coordinates of points of contact are $\left(\frac{-a^2m}{\sqrt{a^2m^2+b^2}},\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$ and $\left(\frac{a^2m}{\sqrt{a^2m^2+b^2}},\frac{-b^2}{\sqrt{a^2m^2+b^2}}\right)$

2.2 The line $y = mx + c$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at two imaginary points (Line and ellipse are separate) if $c^2 > a^2m^2 + b^2$

2.3 Cartesian tangent

The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

or $\frac{x}{\frac{a^2}{x_1}}+\frac{y}{\frac{b^2}{y_1}}=1$

2.4 Parametric form equation of thetangent

The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point θ ($a\cos\theta$, $b\sin\theta$) is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

NOTES

1. The Product of the perpendiculars drawn from foci of the ellipse on the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $= b^2$.
2. If straight line $lx + my + n = 0$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then, $n^2 = a^2l^2 + b^2m^2$ and coordinates of the point of contact are $\left(-l\frac{a^2}{n},-m\frac{b^2}{n}\right)$

3. **Normal** The Equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(i) at point (x_1, y_1) : $\frac{x}{\frac{x_1}{a^2}}-\frac{y}{\frac{y_1}{b^2}}=a^2-b^2$ (cartesian form)

(ii) at point 'ϕ': $ax\sec\phi - by\,\mathrm{cosec}\,\phi = a^2 - b^2$. (paramatric form)

(iii) with slope m : $y=mx-\frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$ (gradient form)

(iv) From a given point, four normals can be drawn to an ellipse. if 'ϕ_1', 'ϕ_2', 'ϕ_3', and 'ϕ_4' be normal points of these normals, then $\phi_1+\phi_2+\phi_3+\phi_4=(2n+1)\pi\ n\in N$

(v) If the line $lx + my + n = 0$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then

$$\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$$

(vi) **Conormal points** In general form normals can be drawn to an ellipse from any point and the four points on the ellipse the normals at which pass thrugh the fixed point(x, y) are called **Conormal Points**.

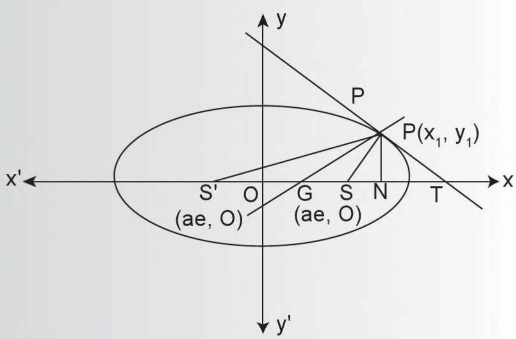
NOTE

The sum of eccntric angles of these conormal point is equal to an odd multiple of two right angles i.e., the sum of the eccentric angles of the feet of the normals drawn from any point to an ellipse is $(2n + 1)\pi$

(vii) **Concurrent normals** If α, β, γ be the eccentric angle of three points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at which the normals are concurrent then $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$

NOTE

The length of subtangent and subnormal for the point (x_1, y_1) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ subtangent $NT=\frac{a^2}{x_1}-x_1$ and Subnormal $GN=\frac{b^2}{a^2}x_1$



4. **Number of Tangents drawn from a point**

4.1 Two tangents can be drawn from a point to an ellipse. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the ellipse.

Example

Find the locus of the foot of the perpendicular drawn from centre upon any tangent to the ellipse

Solution

An tangent to given ellipse is $y=mx+\sqrt{a^2m^2+b^2}$ ....(i)

The equation of the line through the centre (0,0) of the ellipse perpendicular to (1) is $y = 1 1/m\, x$ or $m = -x/y$ ......(II)

Eliminate m between (1) and (II) to set the regard locus

$$(x^2 + y^2)^2 = a^2x^2 + b^2y^2$$

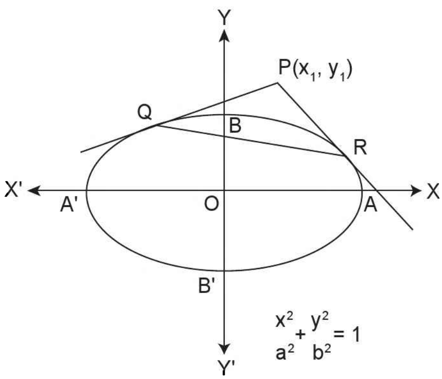
4.5 Equation of the pair of tangents

The equation of the pair of tangents drawn from an external point (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$SS_1 = T^2 \text{ where } S=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1,$$

$$S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$

$$T=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$



NOTE

The angle θ between the pair of tangents drawn from an external point (x_1, y_1) to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \tan\theta=\frac{2ab\sqrt{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1}}{x_1^2+y_1^2-a^2-b^2}$$

(i) Condition of perpendicularity for the pair of tangent is $x_1^{\,2} + y_1^{\,2} = a^2 + b^2$

5. Director Circle

It is the locus of the points from which two perpendicular tangents are drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The equation of Director circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x^2 + y^2 = a^2 + b^2$

6. Pole and Polar

6.1 The equation of the chord of contact/polar of the point $P(x_1, y_1)$ with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $T = 0$ or $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

6.2 The pole of the line $lx + my + n = 0$ with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\left(-l\frac{a^2}{n}, -m\frac{b^2}{n}\right)$

7. Conjugate Points

Two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are said to be conjugate points with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=1$

Example

The value of k if (1, 2), $(k, -1)$ are conjugate points with respect to the ellipse $2x^2 + 3y^2 = 6$ is

[EAMCET-2007]

(a) 2 (b) 4
(c) 6 (d) 8

Solution

(c) Polar of $P(1, 2)$ with respect to $\frac{x^2}{3}+\frac{y^2}{2}=1$ is $S_1 = 0$

i.e., $x + 3y - 1 = 0$

(1, 2), and $(k, -1)$ are conjugate one passes through the polar of the other $k - 3 - 1 = 0$ or $k = 4$

8. Conjugate Lines Two lines $L_1 = l_1x + m_1y + n_1 = 0$ and $L_2 = l_2x + m_2y + n_2 = 0$ are said to be conjugate lines with respect to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $a^2 l_1l_2 + b^2 m_1m_2 = n_1n_2$

9. Important Results Connected with the Ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1; b > a$

(i) Centre $C(0, 0)$

(ii) $a^2 = b^2(1 - e^2)$ or $e=\sqrt{1-\frac{a^2}{b^2}}$

(iii) Vertices: (± b, 0)

(iv) Length of Major axis = $2b$ = distance between vertices

(v) Equation of major axis: $y = 0$

(vi) vertices of minor axis: $(0, \pm a)$

(vii) Length of minor axis = $2a$ = distance between vertices

(viii) Equation of minor axis: $x = 0$

(ix) Foci: $(\pm be, o)$

(x) Distance between foci = $2be$

(xi) Equation of Latus rectum: $x = \pm be$

(xii) Equation of directrices: $x = \pm b/e$

(xiii) Distance between directrices = $2b/e$

10. Some Properties of an Ellipse

(i) At any point of an ellipse, the normal and tangent bisect the internal and external angles between the focal distances of that point.

(ii) The locus of the feet of perpendiculars drawn form foci on any tangent to an ellipse is its auxiliary circle.

(iii) The product of perpendiculars drawn from foci of any tangent to an ellipse is constant and equal to the square of its semi minor axis.

(iv) The poles of the directrices of an ellipse are its corresponding foci.

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$ intersects it again at the point $Q(2\theta)$, then $\cos\theta$ is equal to

(a) $\frac{2}{3}$ (b) $-\frac{2}{3}$

(c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

Solution

(b) The normal at $P(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$, where $a^2 = 14$, $b^2 = 5$

It meets the curve again at $Q(2\theta)$ i.e., $(a\cos 2\theta, b\sin 2\theta)$.

$\therefore \quad \frac{a}{\cos\theta}a\cos 2\theta - \frac{b}{\sin\theta}(b\sin 2\theta) = a^2 - b^2$

$\Rightarrow \quad \frac{14}{\cos\theta}\cos 2\theta - \frac{5}{\sin\theta}(\sin 2\theta) = 14 - 5$

$\Rightarrow \quad 18\cos^2\theta - 9\cos\theta - 14 = 0$

$\Rightarrow \quad (6\cos\theta - 7)(3\cos\theta + 2) = 0$

$\Rightarrow \quad \cos\theta = -\frac{2}{3}$

2. If the normal at any point P on the ellipse cuts the major and minor axes in G and g respectively and C be the centre of the ellipse, then ***[Kurukshetra CEE-1998]***

(a) $a^2(CG)^2 + b^2(Cg)^2 = (a^2 - b^2)^2$

(b) $a^2(CG)^2 - b^2(Cg)^2 = (a^2 - b^2)^2$

(c) $a^2(CG)^2 - b^2(Cg)^2 = (a^2 + b^2)^2$

(d) None of these

Solution

(a) Let at a point (x_1, y_1) normal will be

$$\frac{(x-x_1)a^2}{x_1} = \frac{(y-y_1)b^2}{y_1}$$

At $G, y = 0 \Rightarrow x = CG = \frac{x_1(a^2-b^2)}{a^2}$

At $g, x = 0 \Rightarrow y = Cg = \frac{y_1(b^2-a^2)}{b^2}$

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$

$\Rightarrow \quad a^2(CG)^2 + b^2(Cg)^2 = (a^2 - b^2)^2.$

3. The locus of the poles of normal chords of an ellipse is given by ***[UPSEAT-2001]***

(a) $\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2$

(b) $\frac{a^3}{x^2}+\frac{b^3}{y^2}=(a^2-b^2)^2$

(c) $\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2+b^2)^2$

(d) $\frac{a^3}{x^2}+\frac{b^3}{y^2}=(a^2+b^2)^2$

Solution

(a) Let the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ .... (i)

Let (h, k) be the poles.

Now polar of (h, k) w.r.t. the ellipse is given by

$\frac{xh}{a^2}+\frac{yk}{b^2}=1$ .... (ii)

It is a normal to the ellipse then it must be identical with $ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$ .... (iii)

Hence comparing (ii) and (iii), we get

$$\frac{(h/a^2)}{a\sec\theta}=\frac{(k/b^2)}{-b\operatorname{cosec}\theta}=\frac{1}{(a^2-b^2)}$$

$\Rightarrow\quad \cos\theta=\frac{a^3}{h(a^2-b^2)}$

and $\quad \sin\theta=\frac{b^3}{k(a^2-b^2)}$

Squaring and adding we get,

$$1=\frac{1}{(a^2-b^2)^2}\left(\frac{a^6}{h^2}+\frac{b^6}{k^2}\right)$$

$\therefore$ Required locus of (h, k) is

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

4. The locus of mid-points of parts in between axes and tangents of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will be *[UPSEAT-1999]*

(a) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$ (b) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=2$

(c) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=3$ (d) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$

Solution

(d) Let mid-point of part PQ which is in between the axis is $R(x_1, y_1)$ then co-ordinates of P and Q will be $(2x_1, 0)$ and $(0, 2y_1)$ respectively.

$\therefore$ equation of line PQ is $\frac{x}{2x_1}+\frac{y}{2y_1}=1$

or $\quad y=-\left(\frac{y_1}{x_1}\right)x+2y_1$

If this line touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then it will satisfy the condition, $c^2 = a^2m^2 + b^2$

i.e., $\quad (2y_1)^2=a^2\left(\frac{-y_1}{x_1}\right)^2+b^2$

or $\quad 4y_1^2=\left\{\frac{a^2y_1^2}{x_1^2}\right\}+b^2$

or $\quad 4=\left(\frac{a^2}{x_1^2}\right)+\left(\frac{b^2}{y_1^2}\right)$ or $\left(\frac{a^2}{x_1^2}\right)+\left(\frac{b^2}{y_1^2}\right)=4$

$\therefore$ Required locus of (x_1, y_1) is

$$\left(\frac{a^2}{x^2}+\frac{b^2}{b^2}\right)=4$$

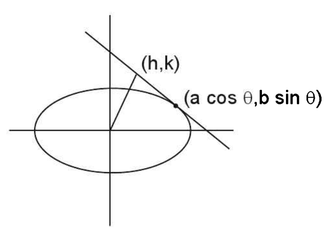
5. Locus of the foot of the perpendicular drawn from the centre upon any tangent to the ellipse $\frac{x^2}{a^2}+\frac{b^2}{y^2}=1$, is

(a) $(x^2 + y^2)^2 = b^2x^2 + a^2y^2$
(b) $(x^2 + y^2)^2 = b^2x^2 - a^2y^2$
(c) $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$
(d) $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$

Solution

(d) $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1,$

Slope $\equiv \frac{b\cot\theta}{a}\times\frac{k}{h}=-1$



$$\frac{h\cos\theta}{a}+\frac{k\sin\theta}{b}=1$$

Since, $\operatorname{cosec}\theta=\sqrt{1+\frac{a^2h^2}{k^2b^2}}$

$\Rightarrow \quad \frac{h^2}{kb} + \frac{k}{b} = \text{cosec}\ \theta$

$\Rightarrow \quad (h^2 + k^2) = bk \text{ cosec } \theta = \frac{bk\left(\sqrt{k^2b^2 + a^2h^2}\right)}{bk}$

$\Rightarrow \quad (x^2 + y^2)^2 = a^2 x^2 + b^2 y^2.$

6. If the normal at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the coordinate axes in G and g respectively, then $PG : Pg$ is equal to

(a) $a : b$ (b) $a^2 : b^2$
(c) $b^2 : a^2$ (d) $b : a$

Solution

(c) Let $P(a \cos \theta, b \sin \theta)$ be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then the equation of the normal at P is $ax \sec \theta - by \text{ cosec } \theta = a^2 - b^2$. It meets the coordinate axes at

$$G\left(\frac{a^2 - b^2}{a}\cos\theta, 0\right)$$

and $g\left(0, -\frac{a^2 - b^2}{b}\sin\theta\right)$.

$$\Rightarrow PG^2 = \left(a\cos\theta - \frac{a^2 - b^2}{a}\cos\theta\right)^2 + b^2\sin^2\theta$$

$$= \frac{b^2}{a^2}(b^2 \cos^2\theta + a^2 \sin^2\theta)$$

and $Pg^2 = \frac{a^2}{b^2}(b^2 \cos^2\theta + a^2 \sin^2\theta)$,

$\therefore \quad PG : Pg = b^2 : a^2.$

7. A chord PQ of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ subtends right angle at its centre. The locus of the point of intersection of tangents drawn at P and Q is ***[EAMCET-1991]***

(a) a circle (b) a parabola
(c) an ellipse (d) a hyperbola

Solution

(c) Let the point of intersection be $R(x_1, y_1)$. Then PQ is the chord of contact of the ellipse with respect to R and its equation will be

$$\frac{xx_1}{9} + \frac{yy_1}{4} = 1 \qquad \text{........ (1)}$$

Now combined equation of lines joining P, Q to centre $O(0, 0)$ is [It is obtained by making $x^2/9 + y^2/4 = 1$ homogeneous with help of (1)]

$$\frac{x^2}{9} + \frac{y^2}{4} = \left(\frac{xx_1}{9} + \frac{yy_1}{4}\right)^2$$

As given $OP \perp OQ$, so coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \left(\frac{x_1^2}{81} - \frac{1}{9}\right) + \left(\frac{y_1^2}{16} - \frac{1}{4}\right)$$

Hence locus of (x_1, y_1) is $\frac{x^2}{81} + \frac{y^2}{16} = \frac{13}{36}$, which is an ellipse.

8. If the polar of a point with respect to the parabola $y^2 = 4x$ touches the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, then the locus of this point is ***[EAMCET-95]***

(a) $\alpha^2x^2 + \beta^2y^2 = 1$
(b) $\alpha^2x^2 - \beta^2y^2 = 1$
(c) $\frac{x^2}{\alpha^2} - \frac{\beta^2y^2}{4\alpha^2} = 1$
(d) $\frac{x^2}{\alpha^2} - \frac{\beta^2y^2}{4} = 1$

Solution

(c) Let given point be (x_1, y_1). Then its polar with respect to $y^2 = 4x$ is $yy_1 = 2(x + x_1)$

$$\Rightarrow y = \frac{2}{y_1}x + \frac{2x_1}{y_1}$$

If it touches given ellipse, then

$$c^2 = a^2m^2 + b^2$$

$$\Rightarrow \left(\frac{2x_1}{y_1}\right)^2 = \alpha^2\left(\frac{2}{y_1}\right)^2 + \beta^2$$

Hence required locus is $4x^2 = 4\alpha^2 + y^2 \beta^2$

$$\Rightarrow \frac{x^2}{\alpha^2} - \frac{\beta^2y^2}{4\alpha^2} = 1$$

9. The locus of the middle point of the intercept of the tangents drawn from an external point to the ellipse $x^2 + 2y^2 = 2$ between the coordinates axes, is **[IIT Screening-2004]**

(a) $\frac{1}{x^2}+\frac{1}{2y^2}=1$ (b) $\frac{1}{4x^2}+\frac{1}{2y^2}=1$

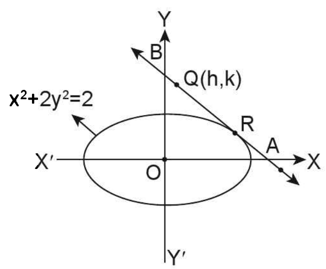
(c) $\frac{1}{2x^2}+\frac{1}{4y^2}=1$ (d) $\frac{1}{2x^2}+\frac{1}{y^2}=1$

Solution

(c) Let the point of contact be $R \equiv (\sqrt{2}\cos\theta, \sin\theta)$

Equation of tangent AB is

$$\frac{x}{\sqrt{2}}\cos\theta + y\sin\theta = I$$



$\Rightarrow$ $A \equiv (\sqrt{2}\sec\theta, 0)$; $B \equiv (0, \text{cosec }\theta)$

Let the middle point Q of AB be (h, k)

$$\Rightarrow \quad h = \frac{\sec\theta}{\sqrt{2}},\ k = \frac{\text{cosec }\theta}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{h\sqrt{2}},\ \sin\theta = \frac{1}{2k}$$

$$\Rightarrow \quad \frac{1}{2h^2}+\frac{1}{4k^2}=1,$$

$\therefore$ Required locus is $\frac{1}{2x^2}+\frac{1}{4y^2}=1$

Trick: The locus of mid-points of the portion of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intercepted between axes is $a^2y^2 + b^2x^2 = 4x^2y^2$.

i.e., $\frac{a^2}{4x^2}+\frac{b^2}{4y^2}=1$ or $\frac{1}{2x^2}+\frac{1}{4y^2}=1$

10. The locus of the point of intersection of tangents drawn at two points of an ellipse the difference of whose eccentric angles is constant, will be **[IIT (Hyderabad)-2001]**

(a) a circle
(b) a parabola
(c) an ellipse
(d) a hyperbola

Solution

(c) Let (x, y) be the point of intersection of tangents drawn at points ϕ_1, and ϕ_2. Then

$$x = a\frac{\cos\left(\frac{\phi_1+\phi_2}{2}\right)}{\cos\left(\frac{\phi_1-\phi_2}{2}\right)},\quad y = b\frac{\sin\left(\frac{\phi_1+\phi_2}{2}\right)}{\sin\left(\frac{\phi_1-\phi_2}{2}\right)}$$

As given $\phi_1 - \phi_2$ = constant = 2λ (say)

$$\frac{x\cos\lambda}{a} = \cos\left(\frac{\phi_1+\phi_2}{2}\right),$$

$$\frac{y\sin\lambda}{b} = \sin\left(\frac{\phi_1+\phi_2}{2}\right)$$

$$\Rightarrow \quad \frac{x^2\cos^2\lambda}{a^2}+\frac{y^2\sin^2\lambda}{b^2}=1$$

which is an ellipse.

Directions: Questions 12 to 14

Tangents are drawn from the point P (3, 4) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at points A and B.

11. The coordinates of A and B are **[IIT-2010]**

(a) (3, 0) and (0, 2)

(b) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}\frac{8}{5}\right)$

(c) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and (0, 2)

(d) (3, 0) and $\left(-\frac{9}{5}\frac{8}{5}\right)$

Solution

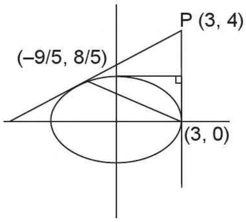
(d) Equation of chord of contact

$$\frac{x}{3}+y=1 \Rightarrow x = 3(1-y)$$

Solving with ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$

$$(1-y)^2+\frac{y^2}{4}=1$$

$\Rightarrow$ $4(y^2+1-2y)+y^2=4 \Rightarrow 5y^2-8y=0$



$$y=0 \; \& \; \frac{8}{5}$$

$\Rightarrow$ $x=3 \; \& \; 3\left(1-\frac{8}{5}\right) \Rightarrow x=3,-\frac{9}{5}$

$\Rightarrow$ Points are (3, 0) and $\left(-\frac{9}{5},\frac{8}{5}\right)$

12. The orthocentre of the triangle *PAB* is ***[IIT-2010]***

(a) $\left(5,\frac{8}{7}\right)$ (b) $\left(\frac{7}{5},\frac{25}{8}\right)$

(c) $\left(\frac{11}{5},\frac{8}{5}\right)$ (d) $\left(\frac{8}{25},\frac{7}{5}\right)$

Solution

(d) y coordinate of the orthocentre must be 8/5.

13. The equation of the locus of the point whose distances from the Point P and line AB are equal, is ***[IIT-2010]***

(a) $9x^2+y^2-6xy-54x-62y+241=0$
(b) $x^2+9y^2+6xy-54x+62y-241=0$
(c) $9x^2+9y^2-6xy-54x-62y-241=0$
(d) $x^2+y^2-2xy+27x+31y-120=0$

Solution

(a) $\sqrt{(x-3)^2+(y-4)^2}=\frac{|x+3y-3|}{\sqrt{1+9}}$

$\Rightarrow$ $10\{(x^2+9-6x)+[y^2+16-8y]\}=(x+3y-3)^2=x^2+9y^2+9-6xy-6x-18y$

$\Rightarrow$ $9x^2+y^2-6xy-54x-62y+241=0$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The equations of the tangents of the ellipse $9x^2+16y^2=144$ which passes through the point (2 ,3) is ***[PET-96]***

(a) $y=3, x+y=5$ (b) $y=-3, x-y=5$
(c) $y=4, x+y=3$ (d) $y=-4, x-y=3$

2. The locus of the point of intersection of perpendicular tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is ***[PET-95]***

(a) $x^2+y^2=a^2-b^2$ (b) $x^2-y^2=a^2-b^2$
(c) $x^2+y^2=a^2+b^2$ (d) $x^2-y^2=a^2+b^2$

3. The equation of normal at the point (0, 3) of the ellipse $9x^2+5y^2=45$ is ***[PET-98]***

(a) $y-3=0$ (b) $y+3=0$
(c) x-axis (d) y-axis

4. The equation of the tangents drawn at the ends of the major axis of the ellipse $9x^2+5y^2-30y=0$ are ***[PET-99]***

(a) $y=\pm 3$
(b) $x=\pm\sqrt{5}$
(c) $y=0, y=6$
(d) None of these

5. If the line $x\cos\alpha+y\sin\alpha=p$ be normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then ***[PET-2001]***

(a) $p^2(a^2\cos^2\alpha+b^2\sin^2\alpha)=a^2-b^2$
(b) $p^2(a^2\cos^2\alpha+b^2\sin^2\alpha)=(a^2-b^2)^2$
(c) $p^2(a^2\sec^2\alpha+b^2\operatorname{cosec}^2\alpha)=a^2-b^2$
(d) $p^2(a^2\sec^2\alpha+b^2\operatorname{cosec}^2\alpha)=(a^2-b^2)^2$

6. If the line $y=2x+c$ be a tangent to the ellipse $\frac{x^2}{8}+\frac{y^2}{4}=1$, then $c=$ ***[MP PET-2009; MNR-99; DCE-2000]***

(a) ± 4 (b) ± 6
(c) ± 1 (d) ± 8

7. The equation of the tangent at the point (1/4, 1/4) of the ellipse $\frac{x^2}{4}+\frac{y^2}{12}=1$ is

(a) $3x + y = 48$
(b) $3x + y = 3$
(c) $3x + y = 16$
(d) None of these

8. What will be the equation of that chord of ellipse $\frac{x^2}{36}+\frac{y^2}{9}=1$ which passes from the point (2, 1) and bisected on the point?

[UPSEAT-99]

(a) $x + y = 2$ (b) $x + y = 3$
(c) $x + 2y = 1$ (d) $x + 2y = 4$

9. The equation of the tangent to the ellipse $x^2 + 16y^2 = 16$ making an angle of 60° with x-axis is

(a) $\sqrt{3}x-y+7=0$
(b) $\sqrt{3}x-y-7=0$
(c) $\sqrt{3}x-y\pm 7=0$
(d) None of these

10. The number of maximum normals which can be drawn from a point to ellipse is

[MPPET-2004]

(a) 4 (b) 2
(c) 1 (d) 3

11. The value of λ, for which the line $2x-\frac{8}{3}\lambda y = -3$ is a normal to the conic $x^2+\frac{y^2}{4}=1$, is

[MPPET-2004]

(a) $-\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$
(c) –3 (d) $\frac{\sqrt{3}}{2}$

12. The product of perpendiculars on a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ drawn from its foci is

[CET (Karnataka)-92; EAMCET-2000]

(a) a^2 (b) b^2
(c) $a^2 + b^2$ (d) $\sqrt{a^2+b^2}$

13. Minimum area of the triangle by any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with the coordinate axes is ***[IIT Screening-05]***

(a) $\frac{a^2+b^2}{2}$ (b) $\frac{(a+b)^2}{2}$
(c) ab (d) $\frac{(a-b)^2}{2}$

14. The area of the quadrilateral formed by the tangents at the end points of latus rectum to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is

[IIT Screening-2003]

(a) 27/4 sq. unit (b) 9 sq. unit
(c) 27/2 sq. unit (d) 27 sq. unit

15. Tangent is drawn to ellipse $\frac{x^2}{27}+y^2=1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ where $\theta \in (0, \pi/2)$. Then the value of θ such that sum of intercepts on axes made by this tangent is minimum, is

[IIT Screening-2003]

(a) π/3 (b) π/6
(c) π/8 (d) π/4

16. In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation of diameter conjugate to the diameter $y=\frac{b}{a}x$, is

(a) $y=-\frac{b}{a}x$ (b) $y=-\frac{a}{b}x$
(c) $x=-\frac{b}{a}y$ (d) None of these

17. The pole of the straight line $x + 4y = 4$ with respect to ellipse $x^2 + 4y^2 = 4$ is

(a) (1, 4) (b) (1, 1)
(c) (4, 1) (d) (4, 4)

18. If the normal at any end of a tatus rectum of an ellipse passes through one end of the miner axis, then $e^4 + e^2$ is equal to

(a) 1 (b) 1/2
(c) $\sqrt{3}/2$ (d) 2

19. The line $3x+5y=15\sqrt{2}$ is a tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$, at a point whose eccentric angle is ***[MPPET-2008]***

(a) π/6 (b) π/4
(c) π/3 (d) 2π/3

20. The greatest distance from the point $(\sqrt{7}, 0)$ to the curve $9x^2 + 16y^2 = 144$ is **[Kerala PET-2008]**

(a) $\sqrt{7}$ (b) $2+\sqrt{7}$
(c) $3+\sqrt{7}$ (d) $4+\sqrt{7}$

21. The equation of that diameter of the ellipse $4x^2 + 9y^2 = 36$ which is conjugate to its diameter parallel to the line $x + 3y - 7 = 0$ is **[MPPET-2010]**

(a) $4x - 3y = 0$ (b) $4x + 3y = 0$
(c) $3x - 4y = 0$ (d) $3x + 4y = 0$

SOLUTIONS

1. (a) $9x^2 + 16y^2 = 144$ or $\frac{x^2}{16}+\frac{y^2}{9}=1$

Equation of tangent is $y = mx \pm \sqrt{16m^2+9}$

$\because$ It passes through the point (2, 3), $(3, -2m)^2 = 16m^2 + 9$

$4m^2 - 12m + 9 = 16m^2 + 9 \Rightarrow m = 0, -1$

$\therefore$ Equations are $y = \pm 3; y = -x \pm 5$

2. (c) Locus of point of intersection of perpendicular tangents is knows as director circle $x^2 + y^2 = a^2 + b^2$

3. (d) Equation of normal at point (x_1, y_1) to ellipse $\frac{x^2}{5}+\frac{y^2}{9}=1$

$$\frac{5x}{x_1}-\frac{9y}{y_1}=1 \Rightarrow 5xy_1 - 9yx_1 = x_1y_1$$

(0, 3); $15x = 0 \Rightarrow x = 0$ i.e., y-axis.

4. (c) $9x^2 + 5(y - 3)^2 = 45$ or $\frac{x^2}{5}+\frac{(y-3)^2}{9}=1$

Extremities of major axis $(x, y - 3) \equiv (0, \pm 3)$

$\therefore$ $(x, y) \equiv (0, 6)\ (0, 0)$

Equation of tangent is $9xx_1 + 5yy_1 - 30\left(\frac{y+y_1}{2}\right)=0$

$9xx_1 + 5yy_1 - 15y - 15y_1 = 0$

at (0, 0) $y = 0$; at (0, 6), $30y - 15y - 90 = 0$

$y = 6$

5. (d) Equation of normal at point $(a \cos \theta, b \sin \theta)$ is $ax \sec \theta + by \operatorname{cosec} \theta = a^2 - b^2$; $x \cos \alpha + y \sin \alpha = p$

Comparing both equation $\frac{a\sec\theta}{\cos\alpha}=\frac{-b\operatorname{cosec}\theta}{\sin\alpha}=\frac{a^2-b^2}{p}$

$$\cos\theta=\frac{ap}{(a^2-b^2)\cos\alpha};$$

$$\sin\theta=\frac{-bp}{(a^2-b^2)\sin\alpha}=1$$

$\cos^2 \theta + \sin^2 \theta = 1$

$$\Rightarrow \frac{a^2p^2}{(a^2-b^2)^2\cos^2\alpha}+\frac{b^2p^2}{\sin^2\alpha(a^2-b^2)^2}=1$$

$p^2[a^2 \sec^2 \alpha + b^2 \operatorname{cosec}^2 \alpha] = (a^2 - b^2)^2$

6. (b) $c=\pm\sqrt{a^2m^2+b^2}=\pm\sqrt{8\times 2^2+4}=\pm 6$

7. (a) Equation of tangent at (x_1, y_1) is $\frac{xx_1}{4}+\frac{yy_1}{12}=1$

$$(x_1, y_1) \equiv \left(\frac{1}{4},\frac{1}{4}\right)$$

$\Rightarrow \frac{x}{16}+\frac{y}{48}=1$ or $3x + y = 48$

8. (d) Mid-point is (2, 1), equation is $T = S_1$

$$\frac{xx_1}{36}+\frac{yy_1}{9}-1=\frac{x_1^2}{36}+\frac{y_1^2}{9}-1$$

$(x_1, y_1) \equiv (2, 1)$

$\Rightarrow \frac{2x}{36}+\frac{y}{9}=\frac{4}{36}+\frac{1}{9} \Rightarrow x + 2y = 4$

9. (c) $x^2 + 16y^2 = 16$ or $\frac{x^2}{16}+\frac{y^2}{1}=1$

Equation of tangent $y = mx \pm \sqrt{16m^2+1}$

$m = \tan 60°$; $y=\sqrt{3}x\pm 7$

or $\sqrt{3}x - y \pm 7 = 0$

10. (a) $\because$ Equation of normal is

$$y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$$

degree of equation in m is 4.

$\therefore$ No. of normal = 4

11. (*a, d*) $2x - \frac{8}{3}\lambda = -3 \Rightarrow y = \frac{3}{4\lambda}x + \frac{9}{8x}$ is normal.

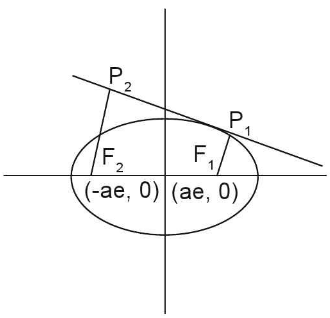
$$\frac{9}{8\lambda} = \frac{-(1-4)\frac{3}{4}\lambda}{\sqrt{1 + 4\frac{9}{16}\lambda^2}}$$

$$\Rightarrow \frac{9}{8\lambda} = \frac{9}{\sqrt{16\lambda^2 + 36}} \Rightarrow 16\lambda^2 + 36 = 64\lambda^2$$

$$\lambda = \pm\frac{\sqrt{3}}{2}$$

12. (b) $P_1F_1 - P_2F_2 = ?$

Let equation of tangent be $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$



$$P_1F_1 = \left|\frac{\left(\frac{ae}{a}\cos\theta\right) + 0 - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right|$$

$$= ab\left|\frac{e\cos\theta - 1}{b^2\cos^2\theta + a^2\sin^2\theta}\right|$$

$$\Rightarrow P_2F_2 = ab\left|\frac{e\cos\theta + 1}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right|$$

$$\Rightarrow (P_1F_1)(P_2F_2)$$

$$= a^2b^2\left|\frac{e^2\cos^2\theta - 1}{\sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)^2}}\right|$$

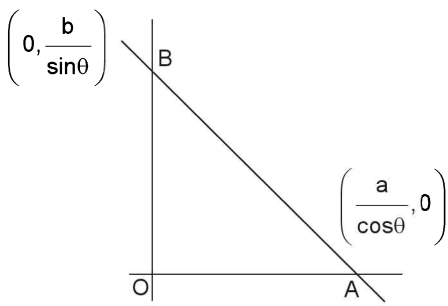
$$= a^2b^2\left|\frac{\left(1 - \frac{b^2}{a^2}\right)\cos^2\theta - 1}{a^2\sin^2\theta + b^2\cos^2\theta}\right|$$

$$= b^2\left|\frac{a^2\cos^2\theta - b^2\cos^2\theta - a^2}{a^2\sin^2\theta + b^2\cos^2\theta}\right|$$

$$= b^2\left(\frac{a^2(1 - \cos^2\theta) + b^2\cos^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}\right)$$

$$= b^2\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}\right) = b^2$$

13. (c) Let equation of tangent be $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$



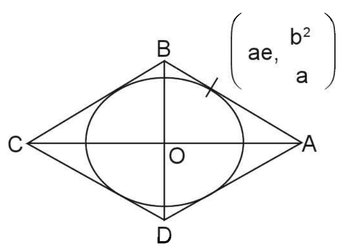
Area of $\Delta\, AOB = \frac{1}{2} \times \frac{a}{\cos\theta} \times \frac{b}{\sin\theta} = \frac{ab}{\sin 2\theta}$

Minimum value of area = ab

14. (d) Area of quadrilateral = 4($\Delta\, AOB$)

Equation of tangent $\frac{xx_1}{a^2} + \frac{yy_2}{b^2} = 1$

$$\frac{xe}{a} + \frac{y}{a} = 1$$



A and B are $\left(\frac{a}{e}, 0\right)$ and $(0, a)$ respectively.

Area of $\Delta\, AOB = \left|\frac{1}{2} \times \frac{a}{e} \times a\right| = \frac{a^2}{2e}$

$$= \frac{9}{2 \times \sqrt{1 - \frac{5}{9}}} = \frac{9}{2 \times \frac{2}{3}} = \frac{27}{4}$$

Area of quadrilateral $= 4.\frac{27}{4} = 27$

15. (b) Equation of tangent $\frac{x\cos\theta}{3\sqrt{3}} + \frac{y\sin\theta}{1} = 1$

$\therefore$ Intercepts are $\left(\frac{3\sqrt{3}}{\cos\theta}, 0\right)\left(0, \frac{1}{\sin\theta}\right)$

$\therefore$ Sum of intercept $= \frac{3\sqrt{3}}{\cos\theta} + \frac{1}{\sin\theta}$

For minimum $\frac{d}{d\theta}(3\sqrt{3}\sec\theta + \operatorname{cosec}\theta) = 0$

$3\sqrt{3}\sec\theta\tan\theta - \operatorname{cosec}\theta\cos\theta = 0$

$\tan^3\theta = \frac{1}{3\sqrt{3}} \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

16. (a) For conjugate diameter $m_1 m_2 = -\frac{b^2}{a^2}$

$m_1\left(\frac{b}{a}\right) = \frac{-b^2}{a^2} \Rightarrow m_1 = \frac{-b}{a}x$

$\therefore$ Equation of diameter $y = \frac{-b}{a}x$

17. (b) Let pole be (x_1, y_1);

$\therefore$ equation of polar is $xx_1 + 4yy_1 = 4$

Comparing with $x + 4y = 4$, we get

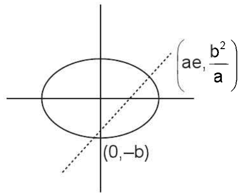
$\frac{x_1}{1} = \frac{4y_1}{4} = \frac{4}{4}$

$(x_1, y_1) \equiv (1, 1)$

18. (a) Equation of normal $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$

$(x_1, y_1) \equiv \left(ae, \frac{b^2}{a}\right)$

$\frac{x}{e} - y = ae^2$, passes through $(0, b)$



$-b = ae^2$ squaring and putting $b^2 = a^2(1 - e^2)$

$a^2(1 - e^2) = a^2e^4 \Rightarrow e^4 + e^2 = 1$

19. (b) Equation tangent of point $(5\cos\theta, 3\sin\theta)$

$\frac{x\cos\theta}{5} + \frac{y\sin\theta}{3} = 1$

Comparing with $3x + 5y = 15\sqrt{2}$

$$\frac{\frac{\cos\theta}{5}}{3} = \frac{\frac{\sin\theta}{3}}{5} = \frac{1}{15\sqrt{2}}$$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}};\ \sin\theta = \frac{1}{\sqrt{2}}\ \therefore \theta = 45°$

20. (d) $9x^2 + 16y^2 = 144$ or $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Equation of normal is $\frac{16x}{x_1} - \frac{9y}{y_1} = 16 - 9$

or $16xy_1 - 9yx_1 = 7x_1y_1$

$(\sqrt{7}, 0)$ lies on if, $16\sqrt{7}y_1 = 7x_1y_1$

$y_1 = 0;\ x_1 = \frac{16}{\sqrt{7}}$

$y_1 = 0; x_1 = \pm 4;\ x_1 = \frac{16}{\sqrt{7}};\ y_1 = \pm 3 - \sqrt{\frac{6}{7}}$

distance between $(\pm 4, 0)$ and $(\sqrt{7}, 0)$

$= 4 - \sqrt{7}$ or $4 + \sqrt{7}$

$\therefore$ maximum distance $= 4 + \sqrt{7}$

21. (a) Let equation of diameter $y = mx$

$\because$ It is conjugate diameter, $m_1 m_2 = \frac{-b^2}{a^2}$

Equation of ellipse is $4x^2 + 9y^2 = 36$

or $\frac{x^2}{9} + \frac{y^2}{4} = 1 \quad m_1 \times \left(-\frac{1}{3}\right) = \frac{-4}{9}$

$\Rightarrow m = \frac{4}{3}\ \therefore$ Equation $y = \frac{4}{3}x$

or $4x - 3y = 0$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The distance from the major axis of any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its corresponding point on the auxiliary circle are in the ratio ***[NDA-2003]***

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$
(c) $\frac{a^2}{b^2}$ (d) $\frac{b^2}{a^2}$

2. The equation of the normal at the point (2, 3) on the ellipse $9x^2 + 16y^2 = 180$, is ***[PET-2000]***

(a) $3y = 8x - 10$ (b) $3y - 8x + 7 = 0$
(c) $8y + 3x + 7 = 0$ (d) $3x + 2y + 7 = 0$

3. If any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cuts off intercepts of length h and k on the axes, then $\frac{a^2}{h^2}+\frac{b^2}{k^2}$ is equal to

(a) 0 (b) 1
(c) –1 (d) None of these

4. The angle between the pair of tangents drawn from the point (1, 2) to the ellipse $3x^2 + 2y^2 = 5$ is ***[UPSEAT-2001; DCE-2003]***

(a) $\tan^{-1}(12/5)$ (b) $\tan^{-1}(6/\sqrt{5})$
(c) $\tan^{-1}(12/\sqrt{5})$ (d) $\tan^{-1}(6/5)$

5. What will be the equation of that chord of ellipse $\frac{x^2}{36}+\frac{y^2}{9}=1$ which passes from the point (2, 1) and bisected on the point ***[UPSEAT-1999]***

(a) $x + y = 2$ (b) $x + y = 3$
(c) $x + 2y = 1$ (d) $x + 2y = 4$

6. If line $\frac{x}{a}+\frac{y}{b}=\sqrt{2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then eccentric angle of the point of contact is ***[EAMCET-95]***

(a) 60° (b) 45°
(c) 90° (d) 0°

7. The number of real tangents through (3, 5) that can be drawn to the ellipses $3x^2 + 5y^2 = 32$ and $25x^2 + 9y^2 = 450$ is ***[VIT-2006]***

(a) 0 (b) 2
(c) 3 (d) 4

8. Given a point (a cos θ, b sin θ) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the equation $ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2$ represents ***[MP PET-2007]***

(a) Tangent (b) Normal
(c) Directrix (d) None of these

9. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are ***[IIT-1999]***

(a) $\left(\frac{2}{5},\frac{1}{5}\right)$ (b) $\left(-\frac{2}{5},\frac{1}{5}\right)$
(c) $\left(-\frac{2}{5},-\frac{1}{5}\right)$ (d) $\left(\frac{2}{5},-\frac{1}{5}\right)$

10. The locus of the point of intersection of the perpendicular tangents to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is ***[Karnataka CET-03]***

(a) $x^2 + y^2 = 9$ (b) $x^2 + y^2 = 4$
(c) $x^2 + y^2 = 13$ (d) $x^2 + y^2 = 5$

11. If the line $x + 2y + k = 0$ is a tangent to the ellipse $3x^2 + 4y^2 = 12$, then k is equal to

(a) ± 4 (b) ± 8
(c) ± 12 (d) ± 16

12. The equation of tangent to the ellipse $x^2 + 3y^2 = 3$ which is perpendicular to the line $4y = x - 5$ is

(a) $4x + y + 7 = 0$ (b) $4x + y - 7 = 0$
(c) $4x + y + 3 = 0$ (d) $4x + y + 43 = 0$

13. The eccentricity of an ellipse whose pair of conjugate diameters are $y = x$ and $3y = -2x$

(a) $2/\sqrt{3}$ (b) $1/\sqrt{3}$
(c) 3/4 (d) $1/\sqrt{2}$

14. The number of real tangents that can be drawn to the ellipse $3x^2 + 5y^2 = 32$ passing through (3, 5) is

(a) 0 (b) 1
(c) 2 (d) 4

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 9 minutes.
3. The worksheet consists of 9 questions. The maximum marks are 27.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If the line $y = mx + c$ touches the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, then c is equal to ***[RPET-94, 95, 99]***

(a) $\pm\sqrt{b^2m^2 + a^2}$ (b) $\pm\sqrt{a^2m^2 + b^2}$
(c) $\pm\sqrt{b^2m^2 - a^2}$ (d) $\sqrt{a^2m^2 - b^2}$

2. The line $lx + my + n = 0$ will be tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if

(a) $a^2l^2 + b^2m^2 = n^2$ (b) $al^2 + bm^2 = n^2$
(c) $a^2l + b^2m = n$ (d) None of these

3. The equation $x^2 - 2xy + y^2 + 3x + 2 = 0$ represents

(a) A parabola (b) An ellipse
(c) A hyperbola (d) A circle

4. The centre of the ellipse $\frac{(x+y-2)^2}{9} + \frac{(x-y)^2}{16} = 1$, is ***[EAMCET-94]***

(a) (0, 0) (b) (1, 1)
(c) (1, 0) (d) (0, 1)

5. The number of values of c such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + y^2 = 1$ is ***[IIT-1998]***

(a) 0 (b) 1
(c) 2 (d) Infinite

6. The length of the axes of the ellipse represented by $4x^2 + 9y^2 + 8x + 36y + 4 = 0$ are

(a) 6, 4 (b) 12, 8
(c) 14, 64 (d) 18, 8

7. The equation of ellipse whose foci are (0, 1) and (0, –1) and minar axis of length 1 is

(a) $\frac{x^2}{4} + \frac{y^2}{5} = 1$
(b) $\frac{x^2}{1/4} + \frac{y^2}{5/4} = 1$
(c) $4x^2 + 5y^2 = 1$
(d) None of these

8. If $2x + 3y = n$ is a tangent to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 2$, then n is equal to

(a) 12 (b) 8
(c) 6 (d) 5

9. The equation of the normal at the point (2, 3) on the ellipse $9x^2 + 16y^2 = 180$, is ***[MPPET-2000]***

(a) $3y = 8x - 10$
(b) $3y - 8x + 7 = 0$
(c) $8y + 3x + 7 = 0$
(d) $3x + 2y + 7 = 0$

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

3. (a) $\Delta = \begin{vmatrix} 1 & -1 & 3/2 \\ -1 & 1 & 0 \\ 3/2 & 0 & 2 \end{vmatrix} \neq 0$

Also $h^2 - ab = (-1)^2 - 1 = 0$

$\therefore$ Equation is parabola.

4. (b) Centre is point of intersection of $x + y - 2 = 0$ and $x - y = 0$ i.e., $(1, 1)$

6. (a) $4(x + 1)^2 + 9(y + 2)^2 = 36$ is equation of ellipse

or $\dfrac{(x+1)^9}{9} + \dfrac{(y+2)^2}{4} = 1$

$\therefore$ Length of major axis = 2.3 = 6

Length of minor axis = 2.2 = 4

7. (b) Equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; a < b$$

$$2be = 2 \Rightarrow be = 1$$

Also minor axis = $2a = 1$

$\Rightarrow \quad a = \dfrac{1}{2}$

$$a^2 = b^2(1 - e^2)$$

$$\left(\frac{1}{2}\right)^2 = b^2 - (1)^2 \Rightarrow b^2 = \frac{5}{4}$$

$\therefore$ Equation is $\dfrac{x^2}{1/4} + \dfrac{y^2}{5/4} = 1$

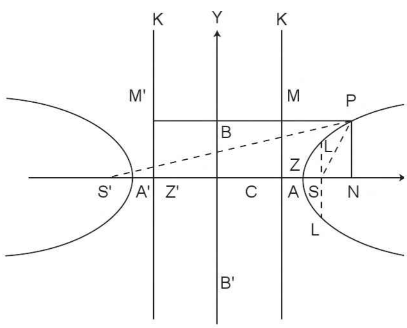
LECTURE 5

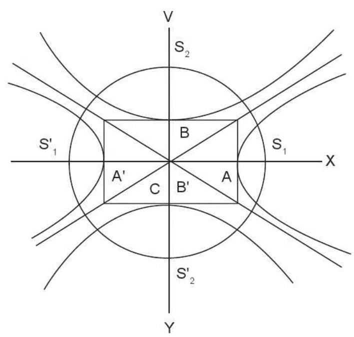
Hyperbola

BASIC CONCEPTS

1. Important Results Connected with Standard Forms of a Hyperbola

1. Standard Forms	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$





2. Shape	Hyperbola	Conjugate hyperbola
3. Centre	$C(0, 0)$	$C(0, 0)$
4. Symmetry	Centre or about both the axis	Centre or about both the axis.
5. Eccentricity	$b^2 = a^2(e^2-1)$ or $e=\sqrt{1+\frac{b^2}{a^2}}$	$a^2 = b^2(e'^2-1)$ or $e'=\sqrt{1+\frac{a^2}{b^2}}$
	or $e=\sqrt{1+\frac{(\text{conjugate axis})^2}{(\text{transverse axis})^2}}$	or $e'=\sqrt{1+\frac{(\text{conjugate axis})^2}{(\text{transverse axis})^2}}$
6. Vertices of Hyperbola (or vertices of transverse axis)	$A'(-a, 0); A(a, 0)$	$B'(0, -b), B(0, b)$

7. Length of transverse axis	$2a$ = Distance between vertices	$2b$ = Distance between vertices
8. Vertices of conjugate axis	$B(0, b), B'(0, -b)$	$A'(-a, 0)$ and $A(a, 0)$
9. Length of conjugate axis	$2b$ = distance between vertices = BB'	$2a$ = distance between vertices = $A'A$
10. Equation of transverse axis	$y = 0$ (Horizontal)	$x = 0$ (Vertical)
11. Equation of conjugate axis	$x = 0$	$y = 0$
12. The number of foci, directrices, Latus rectum and focal radii	02 (Because the hyperbola is symmetric about both the axis)	02
13. Foci	$s'(-ae, 0), s(ae, 0)$	$s'(0, -be), s(0, be)$
14. Distance between Foci	$2ae = s's$	$2be = s's$
15. Equation of latus rectum	$x = \pm ae$	$y = \pm be$
16. Vertices of latus rectum	$\left\{\left(ae, \frac{b^2}{a}\right), \left(-ae, \frac{b^2}{a}\right), \left(-ae, -\frac{b^2}{a}\right), \left(ae, -\frac{b^2}{a}\right)\right\}$	$\left\{\left(\frac{a^2}{b}, be\right), \left(-\frac{a^2}{b}, be\right), \left(-\frac{a^2}{b}, -be\right), \left(\frac{a^2}{b}, -be\right)\right\}$
17. Length of Latus rectum	$\frac{2b^2}{a}$	$2\frac{a^2}{b}$
18. Focal radii (distances) of point $P(x_1, y_1)$ [PS' and PS]	$ex_1 \pm a$	$ey_1 \pm b$
19. Difference of focal radii of point $P(x_1, y_1)$	$2a$ = Length of transverse axis = constant	$2b$ = Length of transverse axis = constant
20. Equation of directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$
21. Distance between directrices	$2a/e$	$2b/e$
22. Parametric equation	$x = a \sec\theta, y = b\tan\theta$	$x = a\tan\theta, y = b\sec\theta$
23. Coordinates of point θ on hyperbola where θ is an eccentric angle.	$\theta(a\sec\theta, b\tan\theta)$	$\theta(a\tan\theta, b\sec\theta)$
24. Point $P(x_1, y_1)$ lies the hyperbola		
(a) Inside, then	$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 > 0$	$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} + 1 > 0$

(b) in side, then	$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1=0$	$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}+1=0$
(c) out side, then	$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1<0$	$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}+1<0$
25. $y = mx + c$	$c=\sqrt{a^2m^2-b^2}$	$c=\sqrt{b^2-a^2m^2}$

NOTES

1. $\frac{1}{e^2}+\frac{1}{e'^2}=1$
2. The point A and A^1 are called vertices of the hyperbola and the line joining them is called the transverse axis whose length $AA^1 = 2a$
3. **Conjugate Axis** The axis of y i.e., $x = 0$ does not cut the hyperbola in real points as $y^2 = -b^2$ (on putting $x = 0$). But if B and B^1 be two points on y-axis such that $CB = CB^1 = b$ then the line BB^1 is called its conjugate axis.
4. Most of the formulas proved for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are defined for the circle $x^2 + y^2 = a^2$ if we replace b^2 by a^2.
5. Most of the results proved for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are true for the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if we replace b^2 by $-b^2$ follwing are the list of corresponding results applicable in the case of hyperbola.

2. Hyperbola Reducible in Standard Form

Equation of hyperbola when its centre C(h, k) and the transverse axis

2.1 Parallel to the x-axis

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1,\ b^2 = a^2 (e^2 - 1)$$

2.2 Parallel to y-axis $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=-1;$

$a^2 = b^2 (e'^2 - 1)$

NOTE

If the origin is shifted at the point (h, k) then the equation $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ is transformed into the form $\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$, where $x = X + h$, $y = Y + k$.

3. General Equation of an Hyperbola

Equation of the hyperbola having focus at (a, b) and directrix $lx + my + n = 0$ is

$(l^2 + m^2) \{(x - a)^2 + (y - b)^2\} = e^2 \{lx + my + n\}^2$; where $e^2 > 1$.

4. Auxiliary Circle of the Hyperbola

The circle described on transverse axis of the hyperbola as diameter is called auxiliary circle. Thus for the hyperbola

(i) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, its equation of auxiliary circle is $x^2 + y^2 = a^2$

(ii) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$, its equation of auxiliary circle is $x^2 + y^2 = b^2$

5. Rectangular Hyperbola

If the transverse and conjugate axes of hyperbola are equal then it is called a rectangular hyperbola.

Thus the equation of the rectangular hyperbola is

$x^2 - y^2 = a^2$ or $xy = c^2$,

where $c^2=\frac{a^2}{2}$ and its eccentricity $=\sqrt{2}$.

6. Eccentric Angle

The eccentric angle of a particular point on the hyperbola is not real. Thus the eccentric angle is not as important for points lying on the hyperbola

7. Position of a Straight Line with Respect to a Hyperbola

7.1. If the straight line $y = mx + c$ be a chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $c^2 > a^2m^2 - b^2$ and the length of chord is

$$\left|\frac{2ab}{a^2m^2-b^2}\sqrt{\{c^2-(a^2m^2-b^2)\}(1+m^2)}\right|$$

7.2 The equation of the chord joining any two points θ and ϕ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{x}{a}\cos\frac{\theta-\phi}{2}-\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta+\phi}{2}\right).$$

7.3 The equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with (x_1, y_1) as its middle point is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$$

8. Diameter

The locus of the middle points of a system of parallel chords of the hyperbola is a straight line through the centre of the hyperbola and is called a diameter of the hyperbola.

8.1 The equation of diameter with respect to the system of parallel chords of line $y = mx + c$ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $y=+\frac{b^2}{a^2m}x$

m = gradient of parallel chords.

8.2 Conjugate diameter

The diameters of the hyperbola are said to be conjugate diameter to each other if each bisects the chords parallel to the other.

8.3 The equation of conjugate diameter of diameter $y = mx$ of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$y=\frac{b^2}{a^2m}x$$

8.4 If $y = m_1 x$ and $y = m_2 x$ be a pair of cojugate diameters of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Then condition is $m_1m_2=\frac{b^2}{a^2}$ and eccentricity $=e=\sqrt{1+m_1m_2}$, if both conjugate diameters are co-incident, then $m=\pm\frac{b}{a}$.

8.5 Of a pair of conjugate diamters of a hyperbola one meets it in real points, and the other meets the conjugate hyeprbola in real points.

8.6 If a pair of conjugate diameter diameters cuts two conjugate hyperbola in P and Q respectively, then $(CP)^2 - (CQ)^2 = a^2 - b^2$, C = centre of the hyperbola.

8.7 The tangents drawn at the points where a pair of conjugate diameters meet a hyperbola and its conjugate, form a parallelogram whose vertices lie on the asymptotes and whose area is constant and equal to $4ab$.

9. If the straight line $y = mx + c$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $c=\pm\sqrt{a^2m^2-b^2}$ and the coordinate of the point of contact are %

$$\left(-\frac{a^2m}{c},\frac{b^2}{c}\right).$$

9.1 Equation of tangent in slope form

The straight line $y=mx\pm\sqrt{a^2m^2-b^2}$ is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for all values of m and the coordinates of the point of contact are

$$\left(\frac{-a^2m}{\sqrt{a^2m^2-b^2}};\frac{-b^2}{\sqrt{a^2m^2-b^2}}\right)$$

and $$\left(\frac{a^2m}{\sqrt{a^2m^2-b^2}};\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$$

10. If the straight line $y = mx + c$ and the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ both intersect in imaginary point that is both are separate or the straight line lies outside the hyperbola, then condition is $c^2 < a^2 m^2 - b^2$

11. Cartesian Tangent

The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point (x_1, y_1) on it is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$

12. Parametric Equation of the Tangent

The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point θ (a sec θ, b tan θ) is

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1.$$

NOTES

1. If the straight line $lx + my + n = 0$ is tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ then condition is

$n^2 = a^2l^2 - b^2m^2$ and the coordinate of the point of contact is

$$\left(-l\frac{a^2}{n}, m\frac{b^2}{n}\right)$$

2. The equation of conjugate diameter of diameter $y = mx$ of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is $y = \frac{a^2 m}{b^2}x$.

13. Normal

The equation of cartesian normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at a point (x_1, y_1) on it is

$$\frac{x}{\frac{x_1}{a^2}} + \frac{y}{\frac{y_1}{b^2}} = a^2 + b^2 \text{ or } \frac{x - x_1}{\frac{x_1}{a^2}} + \frac{y - y_1}{\frac{y_1}{b^2}} = 0$$

14. Parametric Equation of Normal

$ax \cos\theta + by \cot\theta = a^2 + b^2$ or $ax + by \operatorname{cosec}\theta = (a^2 + b^2)\sec\theta$

14.1 Four normals can be drawn to a hyperbola from a point in its plane.

14.2 If the straight line $lx + my + n = 0$ is a normal of the hyperbola, then condition is

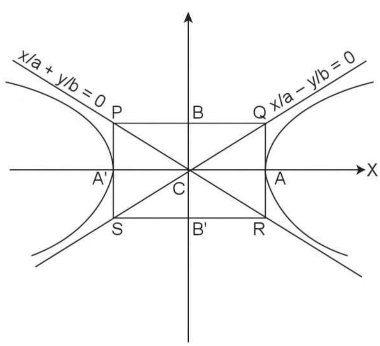
$$\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$$

15 **Asymptote** An asymptote of hyperbola is a straight line, which meets a hyperbola in two coincident points at distance of infinite but it is not altogether at infinity.

Equation of asymptotes of the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is

$\frac{x}{a} \pm \frac{y}{b} = 0$ or $y = \pm\frac{b}{a}x$ and the angle between the asymptotes is $= 2\tan^{-1}(b/a)$.



NOTES

1. The combined equation of the asymptotes differs from the equation of the hyperbola by simply a constant term.
2. The angle between asymptotes of hyperbola is 90°, then it is called the rectangular hyperbola.
3. The product of the perpendiculars from any point on the hyperbola to its asymptotes is $= \frac{a^2b^2}{a^2 + b^2}$

16. The equation of the pair of tangents drawn from a outside point $P(x_1, y_1)$ to the hyperbola is given by $SS_1 = T^2$ where

$$S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1,$$

$$S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1,$$

$$T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$$

The equation of the chord of contact of its tangents is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

NOTE

Two tangents (Real) can be drawn from external point (x_1, y_1) to the hyperbola whose slopes are given by following equation $m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 + b^2) = 0$ and whose joint equation is defined by $SS_1 = T^2$

17. Important results about Rectangular hyperbola $xy = c^2$

17.1 Conjugate hyperbola $xy = -c^2$

17.2 Parametric equation $x = ct$, $y = (c/t)$

17.3 Equation of tangent at point (x_1, y_1) is $xy_1 + yx_1 = 2c^2$

or $\frac{x}{x_1} + \frac{y}{y_1} = 2$

17.4 Equation of normal at point (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$

17.5 The normal drawn on point t of the rectangular hyperbola $xy = c^2$ again meet the hyperbola at point t^1 if $t^3t^1 = -1$.

17.6 The foci of hyperbola $xy = c^2$ is $(c\sqrt{2}, c\sqrt{2})$; $(-c\sqrt{2}, -c\sqrt{2})$.

18. $\sqrt{(x-ae)^2+y^2}+\sqrt{(x+ae)^2+y^2}=2a,\ e<1$ (ellipse)

19. $\sqrt{(x+ae)^2+y^2}-\sqrt{(x-ae)^2+y^2}=2a,\ e>1$ (hyperbola)

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find the foci, centre, eccentricity and the axes of the hyperbola $\dfrac{(x-1)^2}{9}-\dfrac{(y-2)^2}{16}=1.$

Solution

Equation of hyperbola $\dfrac{(x-1)^2}{9}-\dfrac{(y-2)^2}{16}=1$ ....(1)

Let $x - 1 = X$ and $y - 2 = Y$

Now equation (1) reduces to, $\dfrac{X^2}{9}-\dfrac{Y^2}{16}=1$

Here $a^2 = 9$, $b^2 = 16$

$\therefore$ $a = 3$, $b = 4$ By $b^2 = a^2(e^2 - 1)$

$\Rightarrow$ $16 = 9(e^2-1) \Rightarrow \dfrac{16}{9}=e^2-1$

$\Rightarrow$ $\dfrac{16}{9}+1=e^2 \Rightarrow e^2=\dfrac{25}{9}$ $\quad\therefore\ e=\dfrac{5}{3}$

$\left.\begin{array}{l}\text{Length of transverse axis} = 2a = 6\\ \text{Length of conjugate axis} = 2b = 8\end{array}\right\}$

Centre is (0, 0)

Here $\left.\begin{array}{ll}X=x-1 & \therefore x-1=0\\ Y=y-2 & \therefore y-2=0\end{array}\right\}$

$\therefore$ Centre is (1, 2)

Coordinates of foci $(\pm ae, 0)$

Hence, $X = ae$

$\Rightarrow$ $x-1=3\times\dfrac{5}{3}=5\ \therefore x=6$

and $Y=0 \Rightarrow y-2=0 \Rightarrow y=2$

$\therefore$ The coordinates of focus is (6, 2).

Also, $X=-ae \Rightarrow x-1=-3\times\dfrac{5}{3}=-5$

$\Rightarrow$ $x = -4$ and $Y = 0$ or $y = 2$

The second focus is (–4, 2)

2. Find the centre, eccentricity, foci and directrix of the hyperbola $9x^2 - 16y^2 + 18x + 32y - 151 = 0$.

Solution

Given equation of the hyperbola is

$9x^2 - 16y^2 + 18x + 32y - 151 = 0$

$\Rightarrow$ $9x^2 + 18x - 16y^2 + 32y = 151$

$\Rightarrow$ $9(x^2 + 2x) - 16(y^2 - 2y) = 151$

$\Rightarrow$ $9(x + 1)^2 - 16(y - 1)^2 = 151 - 16 + 9$

$\Rightarrow$ $9(x + 1)^2 - 16(y - 1)^2 = 144$

$\Rightarrow$ $\dfrac{9(x+1)^2}{144}-\dfrac{16(y-1)^2}{144}=1$

$\Rightarrow$ $\dfrac{(x+1)^2}{16}-\dfrac{(y-1)^2}{9}=1$

Putting $x + 1 = X$ and $y - 1 = Y$, then equation of the hyperbola be $\dfrac{X^2}{16}-\dfrac{Y^2}{9}=1$

Here, $a^2 = 6 \Rightarrow a = 4$ and $b^2 = 9 \Rightarrow b = 3$

For centre, $X = 0, Y = 0$

$\Rightarrow$ $x + 1 = 0, y - 1 = 0$

$\Rightarrow$ $x = -1, y = 1$

$\therefore$ (–1, 1) is the centre.

For eccentricity, $e : b^2 = a^2(e^2 - 1)$

$\Rightarrow$ $9 = 16(e^2 - 1)$

$\Rightarrow$ $\dfrac{9}{16}=e^2-1$

$\Rightarrow$ $e^2=1+\dfrac{9}{16}=\dfrac{25}{16} \Rightarrow e=\dfrac{5}{4}$

For foci $X = \pm ae$, $Y = 0$

$\Rightarrow$ $x+1=\pm 4\times\dfrac{5}{4}, y-1=0$

$\Rightarrow$ $x + 1 = \pm 5, y = 1$

$\Rightarrow$ $x = 4, -6; y = 1$

$\therefore$ Foci are (4, 1) and (–6, 1)

For directrix $X = \pm\frac{a}{e}$

$\Rightarrow \quad x+1 = \pm\frac{4}{5/4} \Rightarrow x = \pm\frac{16}{5} - 1$

$\Rightarrow \quad x = \frac{11}{5}$ and $x = -\frac{21}{5}$

$\Rightarrow \quad 5x - 11 = 0$ and $5x + 21 = 0$

3. The difference of the distance of a variable point from the points (3, 0) and (–3, 0) is 4. Find the locus of that point.

Solution

Let, $P(h, k)$ be variable point and given points are $S(3, 0)$ and $S'(-3, 0)$, then according to question : $PS' - PS = 4$

$\Rightarrow \quad \sqrt{(h+3)^2 + k^2} - \sqrt{(h-3)^2 + k^2} = 4 \quad .. (1)$

But for every value of h and k,

$[(h+3)^2 + k^2] - [(h-3)^2 + k^2] = 12h \quad ..(2)$

$\therefore$ Dividing equation (2) by equation (1), we get

$\sqrt{(h+3)^2 + k^2} + \sqrt{(h-3)^2 + k^2} = 3h \quad .. (3)$

Adding equation (1) and equation (3), we have

$2\sqrt{(h+3)^2 + k^2} = 4 + 3h$

$\Rightarrow \quad 4[(h+3)^2 + k^2] = (4+3h)^2$

$\Rightarrow \quad 4(h^2 + 6h + 9 + k^2) = 16 + 24h + 9h^2$

$\Rightarrow \quad 4h^2 + 24h + 36 + 4k^2 = 16 + 24h + 9h^2$

$\Rightarrow \quad 4h^2 + 4k^2 + 20 = 9h^2$

$\Rightarrow \quad 5h^2 - 4k^2 = 20$

$\Rightarrow \quad \frac{h^2}{4} - \frac{k^2}{5} = 1$

Therefore, required locus is $\frac{x^2}{4} - \frac{y^2}{5} = 1$

4. The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. Find the equation of the hyperbola of its eccentricity is 3.

Solution

Equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{16} = 1$

Here, $a^2 = 25$ and $b^2 = 16$. Obviously, $a^2 > b^2$

$\therefore \quad b^2 = a^2(1 - e^2)$

$\Rightarrow \quad 16 = 25(1 - e^2) \Rightarrow \frac{16}{25} = 1 - e^2$

$\Rightarrow \quad e^2 = 1 - \frac{16}{25} = \frac{9}{25} \Rightarrow e = \frac{3}{5}$

$\therefore$ Foci of the ellipse be $(\pm ae, 0)$

or $\left(\pm 5 \times \frac{3}{5}, 0\right)$ or $(\pm 3, 0)$

Let equation of the hyperbola be $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$ .... (1)

Then, its foci are $(\pm\alpha e, 0)$ or $(\pm 3\alpha, 0)$ ($\because e = 3$)

According to question $(\pm 3\alpha, 0) = (\pm 3, 0)$

$\Rightarrow \quad \alpha = 1$ and $\beta^2 = \alpha^2(e^2 - 1) = 1(3^2 - 1) = 8$

Putting the values of α^2 and β^2 in equation (1)

$\frac{x^2}{1} - \frac{y^2}{8} = 1$

Which is the required hyperbola.

5. The two foci of a hyperbola $x^2 - y^2 = a^2$ are S and S'. C is the centre. For any point P on the hyperbola, prove that $SP.\ S'P = CP^2$.

Solution

Given equation of the hyperbola is $x^2 - y^2 = a^2$

$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ Here, $a^2 = a^2$ and $b^2 = a^2$

$\therefore \quad b^2 = a^2(e^2 - 1) \Rightarrow a^2 = a^2(e^2 - 1)$

$\Rightarrow \quad 1 = e^2 - 1 \Rightarrow e^2 = 2 \Rightarrow e = \sqrt{2}$

$\therefore$ Foci are $S(a\sqrt{2}, 0)$ and $S'(-a\sqrt{2}, 0)$

Let $P(x, y)$ be the any point on the hyperbola, then

$x_1^2 - y_1^2 = a^2 \quad .... (1)$

Now, $SP = \sqrt{(x_1 - a\sqrt{2})^2 + (y_1 - 0)^2}$

$= \sqrt{x_1^2 - 2a\sqrt{2}x_1 + 2a^2 + y_1^2}$

$= \sqrt{x_1^2 - 2a\sqrt{2}x_1 + 2a^2 + (x_1^2 - a^2)}$ [from equation (1)]

$= \sqrt{2x_1^2 - 2a\sqrt{2}x_1 + a^2} = \sqrt{(\sqrt{2}x_1 - a)^2}$

$\Rightarrow \quad SP = \sqrt{2}x_1 - a$

Similarly, $S'P = \sqrt{2}x_1 + a$

Therefore, $SP.S'P = (\sqrt{2}x_1 - a)(\sqrt{2}x_1 + a)$

$= 2x_1^2 - a^2 = x_1^2 + (x_1^2 - a^2)$

$= x_1^2 + y_1^2,$ [from equation (1)]

$= (x_1 - 0)^2 + (y_1 - 0)^2$

$= CP^2$, [$\because C(0, 0)$ is centre] **Proved**.

6. Find the equation of the hyperbola of given transverse axis, whose vertex bisects the distance between the centre and the focus.

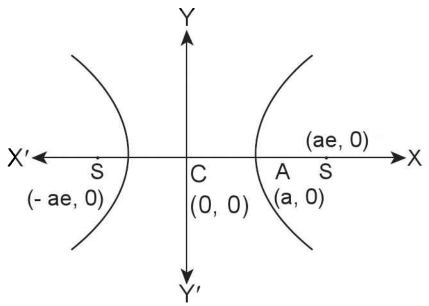
Solution

Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad ......(1)$$

Length of the transverse axis = $2a$

Let C be the centre, S the focus and A be one of the vertices of the hyperbola.



$\therefore$ Distance between the centre and the focus = $CS = ae$

$\therefore$ By the hypothesis, $CA=\dfrac{CS}{2}$

$\Rightarrow \quad a=\dfrac{ae}{2}$

$\Rightarrow \quad e=2$

Now, $b^2 = a^2(e^2 - 1) = a^2(4 - 1) = 3a^2$

Substituting the value of b^2 is equation (1), we get

$$\frac{x^2}{a^2}-\frac{y^2}{3a^2}=1 \Rightarrow 3x^2-y^2=3a^2.$$

7. Referred to the principal axes as the axes of coordinates, find the equation of the hyperbola, whose foci are $(0, \pm\sqrt{10})$ and which passes through the point (2, 3).

Solution

Since, the foci are on y-axis, so let the equation of the hyperbola be $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ ......(1)

It passes through the point (2, 3) $\dfrac{9}{a^2}-\dfrac{4}{b^2}=1$

$$\Rightarrow \frac{9}{a^2}-\frac{4}{a^2(e^2-1)}=1 \quad \{\because b^2=a^2(e^2-1)\} \qquad ....(2)$$

The coordinates of foci are

$(0, \pm ae) = (0, \pm\sqrt{10})$

$\therefore \quad ae=\sqrt{10}$

Substituting $ae=\sqrt{10}$ in Eq. (2), we get

$$\frac{9}{a^2}-\frac{4}{10-a^2}=1$$

$\Rightarrow \quad 90 - 9a^2 - 4a^2 = a^2(10 - a^2)$

$\Rightarrow \quad a^4 - 23a^2 + 90 = 0$

$\Rightarrow \quad (a^2 - 18)(a^2 - 5) = 0$

$\Rightarrow \quad a^2 = 18$ or $a^2 = 5$

Now, $b^2 = a^2(e^2 - 1) = a^2e^2 - a^2$

$\Rightarrow \quad b^2 = 10 - a^2$

When $a^2 = 18$, $b^2 = 10 - 18 = -8$, (which is not possible)

$\therefore \quad a^2 = 5$ thus, $b^2 = 10 - 5 = 5$.

Hence, the equation of the required hyperbola is $\dfrac{y^2}{5}-\dfrac{x^2}{5}=1$.

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$ ***[NCERT]***

2. $16x^2 - 9y^2 = 576$ ***[NCERT]***

3. $5y^2 - 9x^2 = 36$ ***[NCERT]***

Directions: Questions 4 to 8 find the equations of the hyperbola satisfying the given conditions.

4. Vertices (± 2, 0), foci (± 3, 0). ***[NCERT]***

5. Vertices (0, ± 5), foci (0, ± 8) ***[NCERT]***

6. Foci (± 5, 0), the transverse axis is of length 8. ***[NCERT]***

7. Foci (0, ± 13), the conjugate axis is of length 24. ***[NCERT]***

8. Vertices $(\pm 7, 0), e = \frac{4}{3}$ *[NCERT]*
9. Write down the equation of a conic with eccentricity e, focus. $(a, 0)$ and directrix Y-axis.
10. Find the equation of the hyperbola with directrix $x + \sqrt{2}y = 1$, focus at (0, 0) and eccentricity 2.
11. Find the equation of the hyperbola with vertices at (0, ± 6) and $e = \frac{5}{3}$. Also find its foci.
12. Show that the equation $16x^2 - 3y^2 - 32x - 12y - 44 = 0$ represents a hyperbola. Find the lengths of axes and eccentricity.

ANSWERS

1. Foci (± 5, 0), Vertices (± 4, 0); $e = \frac{5}{4}$; Latus rectum $= \frac{9}{2}$
2. Foci (± 10, 0), Vertices (± 6, 0); $e = \frac{5}{3}$; Latus rectum $= \frac{64}{3}$
3. Foci $\left(0, \pm\frac{2\sqrt{14}}{\sqrt{5}}\right)$, Vertices $\left(0, \pm\frac{6}{\sqrt{5}}\right)$; $e = \frac{\sqrt{14}}{3}$; Latus rectum $= \frac{4\sqrt{5}}{3}$
4. $\frac{x^2}{4} - \frac{y^2}{5} = 1$
5. $\frac{y^2}{25} - \frac{x^2}{39} = 1$
6. $\frac{x^2}{16} - \frac{y^2}{9} = 1$
7. $\frac{y^2}{25} - \frac{x^2}{144} = 1$
8. $\frac{x^2}{49} - \frac{9y^2}{343} = 1$
9. $x^2(1 - e^2) + y^2 - 2ax + a^2 = 0$
10. $x^2 + 5y^2 + 8\sqrt{2}xy - 8x - 8\sqrt{2}y + 4 = 0$
11. $16y^2 - 9x^2 = 576$; (0, ± 10)
12. $2\sqrt{3}, 8; \sqrt{\frac{19}{3}}$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. What is the slope of the tangent line drawn to the hyperbola $xy = a (a \neq 0)$ at the point $(a, 1)$ *[AMU-2000]*

(a) $1/a$ (b) $-1/a$
(c) a (d) $-a$

Solution

(b) Given equation of hyperbola $xy = a$
Slope of tangent at point (x_1, y_1) is

$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)},$$

$$\therefore \quad \frac{xdy}{dx} + y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x}$$

At point $(a, 1)$; $m = \left(\frac{dy}{dx}\right)_{(a,1)} = -\frac{1}{a}$

2. What will be equation of that chord of hyperbola $25x^2 - 16y^2 = 400$, whose mid point is (5, 3) ***[UPSEAT-1999]***
(a) $115x - 117y = 17$
(b) $125x - 48y = 481$
(c) $127x + 33y = 341$
(d) $15x + 121y = 105$

Solution

(b) According to question, $S \equiv 25x^2 - 16y^2 - 400 = 0$
Equation of required chord is $S_1 = T$ .... (i)
Here, $S_1 = 25\,(5)^2 - 16\,(3)^2 - 400 = 625 - 144 - 400 = 81$ and $T = 25xx_1 - 16yy_1 - 400$,
where $x_1 = 5, y_1 = 3$
$= 25(x)\,(5) - 16\,(y)\,(3) - 400 = 125x - 48y - 400$
So from (i), required chord is $125x - 48y - 400 = 81$ or $125x - 48y = 481$

3. The product of the lengths of perpendiculars drawn from any point on the hyperbola $x^2 - 2y^2 - 2= 0$ to its asymptotes is ***[EXMCET-2003]***
(a) 1/2 (b) 2/3
(c) 3/2 (d) 2

Solution

(b) Given equation is $\frac{x^2}{2} - \frac{y^2}{1} = 1$ ...... (i)
Product of length of perpendiculars drawn from any point on the hyperbola (i) to the asymptotes is $\frac{a^2b^2}{a^2+b^2} = \frac{2\times 1}{2+1} = \frac{2}{3}$.

4. The coordinates of the foci of the rectangular hyperbola $xy = c^2$ are
(a) $(\pm c, \pm c)$ (b) $(\pm c\sqrt{2}, \pm c\sqrt{2})$
(c) $\left(\pm\frac{c}{\sqrt{2}}, \pm\frac{c}{\sqrt{2}}\right)$ (d) None of these

Solution

(b) $xy = c^2$ as $c^2 = \frac{a^2}{2}$. Here, coordinates of focus are $(ae\cos 45°, ae\sin 45°) \equiv (c\sqrt{2}, c\sqrt{2})$,
$\{\because e = \sqrt{2}, a = c\sqrt{2}\}$
Similarly, other focus is $(-c\sqrt{2}, -c\sqrt{2})$

NOTE
Students should remember this result as *a* fact.

5. If θ is the acute angle of intersection at a real point of intersection of the circle $x^2 + y^2 = 5$ and the parabola $y^2 = 4x$ then tan θ is equal to: ***[Karnataka CET-2005]***
(a) 1 (b) $\sqrt{3}$
(c) 3 (d) $1/\sqrt{3}$

Solution

(c) Solving equations $x^2 + y^2 = 5$ and $y^2 = 4x$ we get $x^2 + 4x - 5 = 0$ i.e., $x = 1, -5$
For $x = 1$; $y^2 = 4 \Rightarrow y = \pm 2$
For $x = -5$; $y^2 = -20$ (imaginary values)
∴ Points are (1, 2) (1, –2); m_1 for $x^2 + y^2 = 5$ at (1, 2)
$\frac{dy}{dx} = -\frac{x}{y}\Big|_{(1,2)} = -\frac{1}{2}$ Similarly, m_2 for $y^2 = 4x$ at (1, 2) is 1.

$$\therefore \quad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{-\frac{1}{2} - 1}{1 - \frac{1}{2}}\right| = 3$$

6. If a circle cuts a rectangular hyperbola $xy = c^2$ in *A*, *B*, *C*, *D* and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively. Then ***[Kurukshetra CEE-1998]***
(a) $t_1t_2 = t_3t_4$ (b) $t_1t_2t_3t_4 = 1$
(c) $t_1 = t_2$ (d) $t_3 = t_4$

Solution

(b) Let equation of circle is $x^2 + y^2 = a^2$
Parametric form of $xy = c^2$ are $x = ct,\ y = \frac{c}{t}$

$$\Rightarrow \quad c^2t^2 + \frac{c^2}{t^2} = a^2$$

$$\Rightarrow \quad c^2t^4 - a^2t^2 + c^2 = 0$$

Product of roots will be, $t_1t_2t_3t_4 = \frac{c^2}{c^2} = 1$.

7. The locus of the intersection point of $x\cos\alpha - y\sin\alpha = a$ and $x\sin\alpha - y\cos\alpha = b$ is ***[UPSEAT-2003]***
(a) Ellipse (b) Hyperbola
(c) Parabola (d) None of these

Solution

(d) $x \cos \alpha - y \sin \alpha = a$, $x \sin \alpha - y \cos \alpha = b$

Intersection points are

$$h = \frac{a\cos\alpha - b\sin\alpha}{\cos 2\alpha},$$

$$k = \frac{a\sin\alpha - b\cos\alpha}{\cos 2\alpha}$$

Then the locus of point (h, k) is $x^2 + y^4 - 2x^2 y^2 = (a^2 + b^2)(x^2 + y^2) + 4abxy$, which is not a locus of any given curves.

8. The locus of the middle points of the chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$ is ***[EAMCET-1989]***

(a) $3x - 4y = 4$ (b) $3y - 4x + 4 = 0$
(c) $4x - 4y = 3$ (d) $3x - 4y = 2$

Solution

(a) Let $P(x_1, y_1)$ be the middle point of the chord of the hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$

$\therefore$ Equation of the chord is $T = S_1$

i.e., $3xx_1 - 2yy_1 + 2(x + x_1) - 3(y + y_1)$
$= 3x_1^2 - 2y_1^2 + 4x_1 - 6y_1$

or

$(3x_1 + 2)x - (2y_1 + 3)y + 2x_1 - 3y_1$

$\Rightarrow$ $3x_1^2 - 2y_1^2 + 4x_1 - 6y_1$

If this chord is parallel to line $y = 2x$, then

$$m_1 = m_2 \Rightarrow -\frac{3x_1 + 2}{-(2y_1 + 3)} = 2$$

$\Rightarrow$ $3x_1 - 4y_1 = 4$

Hence, the locus of the middle point (x_1, y_1) is $3x - 4y = 4$.

9. Eccentricity of the rectangular hyperbola $\int_0^1 e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$ is ***[UPSEAT-2002]***

(a) 2 (b) $\sqrt{2}$
(c) 1 (d) $1/\sqrt{2}$

Solution

(b) Eccentricity of rectangular hyperbola is $\sqrt{2}$.

10. Slopes of the common tangent to the hyperbolas $\frac{x^2}{9} - \frac{y^2}{16} = 1$ and $\frac{y^2}{9} - \frac{x^2}{16} = 1$ are ***[Roorkee-97]***

(a) $2, -2$ (b) $1, -1$
(c) $1, 2$ (d) $-1, -2$

Solution

(b) Any tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is $y = mx + \sqrt{9m^2 - 16}$ .... (1)

Any tangent to the hyperbola $\frac{x^2}{(-16)} - \frac{y^2}{(-9)} = 1$ parallel to (1) is $y = mx + \sqrt{-16m^2 + 9}$ ... (2)

If (1) and (2) are identical then $9m^2 - 16 = -16m^2 + 9$

$\Rightarrow$ $m^2 = 1$

$\Rightarrow$ $m = \pm 1$

11. The number of normals which can be drawn from an external point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is ***[EAMCET-95]***

(a) 2 (b) 4
(c) 6 (d) 8

Solution

(b) Equation of any normal to the hyperbola is

$$y = mx - \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2m^2}}$$

$\Rightarrow$ $(a^2 - b^2m^2)(y - mx)^2 = m^2(a^2 + b^2)^2$

If it passes through the point (x_1, y_1), then $(a^2 - b^2m^2)(y_1 - mx_1)^2 = m^2(a^2 + b^2)^2$

It is 4 degree equation in m, so it gives 4 values of m.

Corresponding to there 4 values, four normals can be drawn from the point (x_1, y_1).

12. The curve for which the slope of the tangent at any point equals the ratio of the abscissa to the ordinate of the point is ***[CET (Kernataka)-1997]***

(a) a circle (b) a parabola
(c) an ellipse (d) a hyperbola

Solution

(d) Given $\frac{dy}{dx} = \frac{x}{y}$

$\Rightarrow$ $x\,dx = y\,dy$

$\Rightarrow$ $x^2 - y^2 = c$ which is a hyperbola.

13. If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangents is *[IIT-99]*

(a) $9x^2 - 8y^2 + 18x - 9 = 0$
(b) $9x^2 - 8y^2 - 18x + 9 = 0$
(c) $9x^2 - 8y^2 - 18x - 9 = 0$
(d) $9x^2 - 8y^2 + 18x + 9 = 0$

Solution

(b) The equation of chord of contact at point (h, k) is $xh - yk = 9$, Comparing with $x = 9$, we have $h = 1, k = 0$

Hence, equation of pair of tangent at point (1, 0) is $SS_1 = T^2$

$\Rightarrow \quad (x^2 - y^2 - 9)(1^2 - 0^2 - 9) = (x - 9)^2$

$\Rightarrow \quad -8x^2 + 8y^2 + 72 = x^2 - 18x + 81$

$\Rightarrow \quad 9x^2 - 8y^2 - 18x + 9 = 0$

14. Let $P(a \sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$, where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of the normals at P and Q, then k is equal to *[IIT-1999; MP PET-2002]*

(a) $\frac{a^2+b^2}{a}$ (b) $-\left(\frac{a^2+b^2}{a}\right)$

(c) $\frac{a^2+b^2}{b}$ (d) $-\left(\frac{a^2+b^2}{b}\right)$

Solution

(d) Given $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$

The equation of tangent at point P is

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

m of tangent $= \frac{b}{\tan\theta} \times \frac{\sec\theta}{a} = \frac{b}{a\sin\theta}$

Hence, the equation of perpendicular at P is

$$y - b\tan\theta = -\frac{a\sin\theta}{b}(x - a\sec\theta)$$

or $\quad by - b^2\tan\theta = -a\sin\theta\, x + a^2\tan\theta$

or $\quad a\sin\theta\, x + by = (a^2 + b^2)\tan\theta \quad$ .... (i)

Similarly, the equation of perpendicular at Q is $a\sin\phi\, x + by = (a^2 + b^2)\tan\phi \quad$ .... (ii)

On multiplying (i) by $\sin\phi$ and (ii) by $\sin\theta$

$a\sin\theta\sin\phi\, x + b\sin\phi\, y = (a^2 + b^2)\tan\theta\sin\phi$

$a\sin\phi\sin\theta\, x + b\sin\theta\, y = (a^2 + b^2)\tan\phi\sin\theta$

On subtraction by,

$(\sin\phi - \sin\theta) = (a^2 + b^2)(\tan\theta\sin\phi - \tan\phi\sin\theta)$

$$\therefore \quad y = k = \frac{a^2+b^2}{b}, \frac{\tan\theta\sin\phi - \tan\phi\sin\theta}{\sin\phi - \sin\theta}$$

$$\because \quad \theta + \phi = \frac{\pi}{2} \Rightarrow \phi = \frac{\pi}{2} - \theta$$

$\Rightarrow \quad \sin\phi = \cos\theta$ and $\tan\phi = \cot\theta$

$$\therefore \quad y = k = \frac{a^2+b^2}{b}, \frac{\tan\theta\cos\theta - \cot\theta\sin\theta}{\cos\theta - \sin\theta}$$

$$= \frac{a^2+b^2}{b}\left(\frac{\sin\theta - \cos\theta}{\cos\theta - \sin\theta}\right) = \frac{(a^2+b^2)}{b}$$

15. An ellipse has eccentricity $\frac{1}{2}$ and one focus at the point $P\left(\frac{1}{2}, 1\right)$. Its one directrix is the common tangent nearer to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. The equation of the ellipse in the standard form, is *[IIT-96]*

(a) $\frac{(x-1/3)^2}{1/9} + \frac{(y-1)^2}{1/2} = 1$

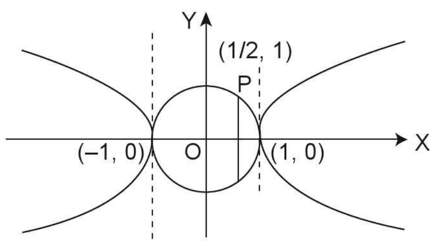
(b) $\frac{(x-1/3)^2}{1/9} + \frac{(y+1)^2}{1/12} = 1$

(c) $\frac{(x-1/3)^2}{1/9} - \frac{(y-1)^2}{1/12} = 1$

(d) $\frac{(x-1/3)^2}{1/9} - \frac{(y+1)^2}{1/2} = 1$

Solution

(a) There are two common tangents to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$. There are $x = 1$ and $x = -1$



Out of these, $x = 1$ is nearer to the point $P(1/2, 1)$. Thus a directrix of the required ellipse is $x = 1$.

If $Q(x, y)$ is any point on the ellipse, then its distance from the focus is $QP = \sqrt{\left(x-\frac{1}{2}\right)^2 + (y-1)^2}$ and its distance from the directrix $x = 1$ is $|x - 1|$. By definition of ellipse, $QP = e\,|x - 1|$

$$\Rightarrow \sqrt{\left(x-\frac{1}{2}\right)^2 + (y-1)^2} = \frac{1}{2}|x-1|$$

$$\Rightarrow 3x^2 - 2x + 4y^2 - 8y + 4 = 0$$

or $$\frac{\left(x-\frac{1}{3}\right)^2}{1/9} + \frac{(y-1)^2}{1/12} = 1$$

16. If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, $S(x_4, y_4)$, then ***[IIT-1998]***

(a) $x_1 + x_2 + x_3 + x_4 = 0$
(b) $y_1 + y_2 + y_3 + y_4 = 0$
(c) $x_1 x_2 x_3 x_4 = c^4$
(d) $y_1y_2y_3y_4 = c^4$

Solution

(a, b, c, d) On solving, we get

$$x^2 + \frac{c^4}{x^2} = a^2 \Rightarrow x^4 - a^2x^2 + c^4 = 0$$

$\therefore \quad x_1 + x_2 + x_3 + x_4 = 0; x_1x_2x_3x_4 = c^4$ ... (i)

Since both the curves are symmetrical in x and y

$\therefore \quad y_1 + y_2 + y_3 + y_4 = 0; y_1y_2y_3y_4 = c^4$ .. (ii)

NOTE

Result (ii) can also be obtained by eliminating x.

17. The locus of the mid points of the chords of the circle $x^2 + y^2 = 16$ which are tangent to the hyperbola $9x^2 - 16y^2 = 144$ is ***[Roorkee-97]***

(a) $(x^2 + y^2)^2 = 9x^2 - 16y^2$
(b) $(x^2 + y^2)^2 = 16x^2 - 9y^2$
(c) $(x^2 - y^2)^2 = 9x^2 - 16y^2$
(d) $(x^2 - y^2)^2 = 16x^2 - 9y^2$

Solution

(b) Let (x_1, y_1) be the mid-point of a chord of given circle.

Then its equation is given by

$$T = S_1 \Rightarrow xx_1 + yy_1 = x_1^2 + y_1^2$$

$$\Rightarrow \quad y = -\frac{x_1}{y_1}x + \frac{x_1^2 + y_1^2}{y_1}$$

If it is a tangent to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, then from condition of tangency $c^2 = a^2m^2 - b^2$, we have $\left(\frac{x_1^2 + y_1^2}{y_1}\right)^2 = 16\left(-\frac{x_1}{y_1}\right)^2 - 9$

Hence, required locus is $(x^2 + y^2)^2 = 16x^2 - 9y^2$.

18. A hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and its transverse and conjugate axes coincide with the major and minor axes of the ellipse. If the product of their eccentricities is 1, then ***[IIT-JEE-2006]***

(a) the equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$
(b) the equation of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$
(c) focus of the hyperbola is $(5\sqrt{3}, 0)$
(d) focus of the hyperbola is $(5, 0)$

Solution

(b, d) Let equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...... (1)}$$

Since it passes through the focus $(\pm\sqrt{25-16}, 0) = (\pm 3, 0)$ of the ellipse, so $\frac{9}{a^2} = 1 \Rightarrow a^2 = 9$ .... (2)

Also product of their eccentricities = 1

$$\Rightarrow \sqrt{1-\frac{16}{25}} \times \sqrt{1+\frac{b^2}{a^2}} = 1$$

$$\Rightarrow 1 + \frac{b^2}{9} = \frac{25}{9}$$

$$\Rightarrow b^2 = 16$$

$\therefore$ equation of the hyperbola will be

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

Its focus $= (\pm\sqrt{a^2 + b^2}, 0) = (\pm 5, 0)$

Hence (b) and (d) are correct answers.

19. The point where the line $2x+\sqrt{6}y=2$ touches the curve $x^2 - 2y^2 = 4$ is *[IIT (Screening)-2004]*

(a) $(\sqrt{6},1)$ (b) $(4,-\sqrt{6})$

(c) $(1/2,1/\sqrt{6})$ (d) $(1,0)$

Solution

(b) Given hyperbola is $\frac{x^2}{4}-\frac{y^2}{2}=1$. Let given line be a tangent to it at point $(2\sec\theta, \sqrt{2}\tan\theta)$

But equation of tangent at this point is

$\frac{x}{2}\sec\theta-\frac{y}{\sqrt{2}}\tan\theta=1$.

Comparing it with given line equation

$\frac{\sec\theta}{2}=1, \frac{\tan\theta}{\sqrt{2}}=-\frac{\sqrt{6}}{2}$

$\therefore$ Required point is $(4,-\sqrt{6})$.

20. Consider *a* branch of the hyperbola $x^2-2y^2-2\sqrt{2}x-4\sqrt{2}y-6=0$ with vertex at the point *A*. Let *B* be one of the end points of its latus rectum. If *C* is the focus of the hyperbola nearest to the point *A*, then the area of the triangle *ABC* is *[IIT JEE-2008]*

(a) $1-\sqrt{\frac{2}{3}}$ (b) $\sqrt{\frac{3}{2}}-1$

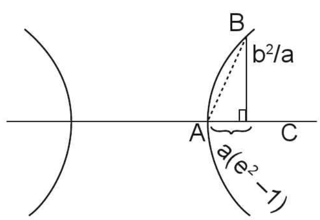
(c) $1+\sqrt{\frac{2}{3}}$ (d) $\sqrt{\frac{3}{2}}+1$

Solution

(b) The hyperbola

$x^2-2y^2-2\sqrt{2}x-4\sqrt{2}y-6=0$ can be reduced to $\frac{(x-\sqrt{2})^2}{4}-\frac{(y+\sqrt{2})^2}{2}=1$

We have $a=2, b=\sqrt{2}$ $\quad e=\sqrt{\frac{b^2}{a^2}+1}=\sqrt{\frac{3}{2}}$



The area of the triangle $ABC=\frac{1}{2}=a(e-1)$,

$\frac{b^2}{a}=\frac{b^2(e-1)}{2}$

$=\frac{2\left(\sqrt{\frac{3}{2}}-1\right)}{2}=\sqrt{\frac{3}{2}}-1$

21. If the tangent at the point $(2\sec\theta, 3\tan\theta)$ of the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$ is parallel to the line $3x - y + 4 = 0$. Then value of θ is: *[Orissa JEE-2008]*

(a) 30° (b) 60°

(c) 90° (d) 45°

Solution

(a) Equation of tangent at (x', y') on hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{xx'}{a^2}-\frac{yy'}{b^2}=1$

$\therefore \quad \frac{2(\sec\theta)x}{4}-\frac{(3\tan\theta)y}{9}=1$

$\Rightarrow \quad y=\frac{3\sec\theta}{2\tan\theta}x-\frac{3}{\tan\theta}$

slope of tangent $=\frac{3\sec\theta}{2\tan\theta}$

$\because$ tangent is parallel to the line $3x-y+4=0$

$\therefore \quad \frac{3\sec\theta}{2\tan\theta}=3$

$\Rightarrow \quad \sin\theta=\frac{1}{2}\Rightarrow\theta=30°$

22. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinates axes, then *[IIT-2009]*

(a) equation of ellipse is $x^2 + 2y^2 = 2$

(b) the foci of ellipse are $(\pm1, 0)$

(c) equation of ellipse is $x^2 + 2y^2 = 4$

(d) the foci of ellipse are $(\pm\sqrt{2},2)$

Solution

(a, b) Ellipse and hyperbola will be confocal

$\Rightarrow \quad (\pm ae, 0) = (\pm 1, 0)$

$\Rightarrow \quad \left(\pm a \times \frac{1}{\sqrt{2}}, 0\right) = (\pm 1, 0)$

$\Rightarrow \quad a = \sqrt{2}$ and $e = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad b^2 = a^2(1 - e^2)$

$\Rightarrow \quad b^2 = 1$

$\therefore$ Equation of ellipse $\frac{x^2}{2} + \frac{y^2}{1} = 1$

23. Match the conics in Column–I with the statements/expressions in Column–II. ***[IIT-2009]***

Column-I	Column-II
(A) Circle	(p) The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
(B) Parabola	(q) Points z in the complex plane satisfying $\|z + 2\| - \|z - 2\| = \pm 3$
(C) Ellipse	(r) Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2t}{1+t^2}$
(D) Hyperbola	(s) The eccentricity of the conic lies in the interval $1 \le x < \infty$
	(t) Points z in the complex plane satisfying $\text{Re}(z + 1)^2 = \|z\|^2 + 1$

Solution

(A) $\rightarrow$ (p), (B) $\rightarrow$ (s, t), (C) $\rightarrow$ (r), (D) $\rightarrow$ (q, s)

(p) $\frac{1}{k^2} = 4\left(1 + \frac{h^2}{k^2}\right) \Rightarrow 1 = 4(k^2 + h^2)$

$\therefore \quad h^2 + k^2 = \left(\frac{1}{2}\right)^2$ which is a circle.

(q) If $|z - z_1| - |z - z_2| = k$ where $k < |z_1 - z_2|$ the locus is a hyperbola.

(r) Let, $t = \tan\alpha$

$\Rightarrow \quad x = \sqrt{3}\cos 2\alpha$ and $y = \sin 2\alpha$

or $\quad \cos 2\alpha = \frac{x}{\sqrt{3}}$ and $\sin 2\alpha = y$

$\therefore \quad \frac{x^2}{3} + y^2 = \sin^2 2\alpha + \cos^2\alpha = 1$, which is an ellipse.

(s) If eccentricity is $[1, \infty)$, then the conic can be a parabola (if $e = 1$) and a hyperbola if $e \in (1, \infty)$

(t) Let, $z = x + iy$, $x, y \in R$

$\Rightarrow \quad (x + 1)^2 - y^2 = x^2 + y^2 + 1$

$\Rightarrow \quad y = x$; which is a parabola.

Instruction for Question 24 and 25

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

24. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is ***[IIT (1) 2010]***

(a) $2x - \sqrt{5}y - 20 = 0$ (b) $2x - \sqrt{5}y + 4 = 0$

(c) $3x - 4y + 8 = 0$ (d) $4x - 3y + 4 = 0$

Solution

(b) Let equation of tangent to ellipse

$$\frac{x\sec\theta}{3} - \frac{\tan\theta}{2}y = 1$$

$2x\sec\theta - 3\tan\theta\, y = 6$

It is also tangent to circle $x^2 + y^2 - 8x = 0$

$\Rightarrow \quad \frac{|8\sec\theta - 6|}{\sqrt{4\sec^2\theta + 9\tan^2\theta}} = 4$

$(8\sec\theta - 6)^2 = 16(13\sec^2\theta - 9)$

$\Rightarrow \quad 12\sec^2\theta + 8\sec\theta - 15 = 0$

$\Rightarrow \quad \sec\theta = \frac{5}{6}$ and $-\frac{3}{2}$ but $\sec \neq \frac{5}{6}$

$\Rightarrow \quad \sec\theta = -\frac{3}{2} \Rightarrow \tan\theta = \frac{\sqrt{5}}{2}$,

$\therefore$ slope is positive

Equation of tangent is $= 2x - \sqrt{5}y + 4 = 0$

25. Equation of the circle with AB as its diameter is ***[IIT (1) 2010]***

(a) $x^2 + y^2 - 12x + 24 = 0$

(b) $x^2 + y^2 + 12x + 24 = 0$

(c) $x^2 + y^2 + 24x - 12 = 0$

(d) $x^2 + y^2 - 24x - 12 = 0$

Solution

(a) $x^2 + y^2 - 8x = 0$

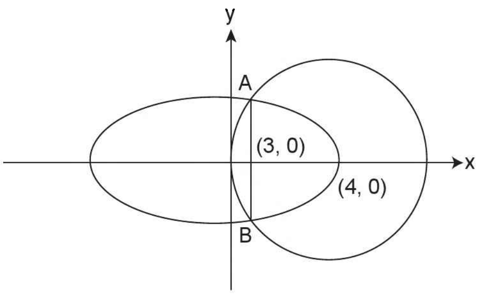
$\frac{x^2}{9} - \frac{y^2}{4} = 1 \Rightarrow 4x^2 - 9y^2 = 36$

$\Rightarrow \quad 4x^2 - 9(8x - x^2) = 36$

$13x^2 - 72x - 36 = 0$

$13x^2 - 78x + 6x - 36 = 0$

$(13x + 6)(x + 6) = 0$



$\Rightarrow \quad x = -\dfrac{6}{13}$ and $x = 6$

But, $x > 0 \Rightarrow x = 6$

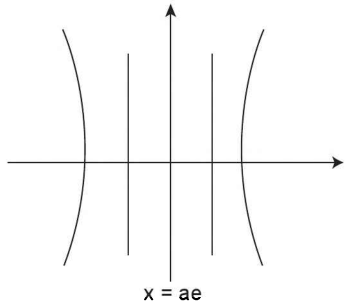
$\Rightarrow \quad A(6, \sqrt{2})$ and $B(6, -\sqrt{2})$

$\Rightarrow$ Equation of circle with AB as a diameter

$x^2 + y^2 - 12x + 24 = 0$

26. The line $2x + y = 1$ is tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is **[IIT (1)-2010]**

Solution



Put (1) in

$y = -2x + 1$ is tangent, $e^2 = 1 + \dfrac{b^2}{a^2}$

It passes through $\left(\dfrac{-a}{e}, 0\right)$

$0 = -\dfrac{2a}{e} + 1$

$\Rightarrow \quad \dfrac{a}{e} = \dfrac{1}{2}$

$e = 2a$

$c^2 = a^2m^2 - b^2$

$\Rightarrow 1 = 4a^2 - b^2$

$\Rightarrow 1 + b^2 - 4a^2 = 0$

$b^2 = 4a^2 - 1(-1)$

$= 1 + \dfrac{(4a^2 - 1)}{a^2}$

$e^2 = 1 + 4 - \dfrac{1}{a^2}$

$e^2 = 5 - \dfrac{4}{e^2}$

$\Rightarrow e^4 - 5e^2 + 4 = 0$

$\Rightarrow (e^2 - 1)(e^2 - 4) = 0$

$e^2 - 1 \neq 0$ $e = 2$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The equation of the hyperbola whose directrix is $x + 2y + 1 = 0$, focus $(2, 1)$ and eccentricity 2 will be **[MP PET-88, 89]**

(a) $x^2 - 16xy - 11y^2 - 12x + 6y + 21 = 0$
(b) $3x^2 + 16xy + 15y^2 - 4x - 14y - 1 = 0$
(c) $x^2 + 16xy + 11y^2 - 12x - 6y + 21 = 0$
(d) None of these

2. The locus of the point of intersection of the lines $\dfrac{x}{a} + \dfrac{y}{b} = m$ and $\dfrac{x}{a} - \dfrac{y}{b} = \dfrac{1}{m}$, where m is a parameter, is always

(a) A circle
(b) A parabola
(c) An ellipse
(d) A hyperbola

3. The difference of the focal distances of any point on the hyperbola $9x^2 - 16y^2 = 144$, is **[PET-95]**

(a) 8 (b) 7
(c) 6 (d) 4

4. The latus rectum of the hyperbola $9x^2 - 16y^2 - 18x - 32y - 151 = 0$ is **[MP PET-96]**

(a) 9/4 (b) 9
(c) 3/2 (d) 9/2

5. If transverse and conjugate axes of a hyperbola are equal, then its eccentricity is ***[MP PET-2003]***

(a) $\sqrt{3}$ (b) $\sqrt{2}$

(c) $1/\sqrt{2}$ (d) 2

6. The foci of the hyperbola $9x^2 - 16y^2 = 144$ are ***[PET-2001]***

(a) $(\pm 4, 0)$ (b) $(0, \pm 4)$

(c) $(\pm 5, 0)$ (d) $(0, \pm 5)$

7. The directrix of the hyperbola is $\frac{x^2}{9} - \frac{y^2}{4} = 1$ ***[UPSEAT-2003]***

(a) $x = 9/\sqrt{13}$ (b) $y = 9/\sqrt{13}$

(c) $x = 6/\sqrt{13}$ (d) $y = 6/\sqrt{13}$

8. If e and e' are the eccentricities of the ellipse $5x^2 + 9y^2 = 45$ and the hyperbola $5x^2 - 4y^2 = 45$ respectively, then ee' is equal to

(a) 9 (b) 4

(c) 5 (d) 1

9. The equation of the hyperbola referred to the axis an axes of coordinate and whose distance between the foci is 16 and eccentricity is $\sqrt{2}$, is

(a) $x^2 - y^2 = 16$ (b) $x^2 - y^2 = 32$

(c) $x^2 - 2y^2 = 16$ (d) $y^2 - x^2 = 16$

10. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of b^2

(a) 1 (b) 5

(c) 7 (d) 9

11. Centre of hyperbola $9x^2 - 16y^2 + 18x + 32y - 151 = 0$ is

(a) $(1, -1)$ (b) $(-1, 1)$

(c) $(-1, -1)$ (d) $(1, 1)$

12. If P is a point on the hyperbola $16x^2 - 9y^2 = 144$ whose foci are S_1 and S_2, then $PS_1 - PS_2$ is equal to

(a) 4 (b) 6

(c) 8 (d) 12

13. The eccentricity of the hyperbola conjugate to $x^2 - 3y^2 = 2x + 8$ is ***[MPPET-2006]***

(a) $\frac{2}{\sqrt{3}}$ (b) $\sqrt{3}$

(c) 2 (d) None of these

14. The distance between the directries of a rectangular hyperbola is 10 units, then distance between its foci is ***[MPPET-2002]***

(a) $10\sqrt{2}$ (b) 5

(c) $5\sqrt{2}$ (d) 20

15. For the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant when α varies? ***[AIEEE-2007; IIT Sc.-2003]***

(a) abscissae of vertices (b) abscissae of foci

(c) eccentricity (d) directrix

16. A hyperbola, having the transverse axis of the length $2 \sin \theta$, is confocal (foci of both conic coincide) with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is ***[IIT-JEE-2007]***

(a) $x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 1$

(b) $x^2 \sec^2\theta - y^2 \operatorname{cosec}^2\theta = 1$

(c) $x^2 \sin^2\theta - y^2 \cos^2\theta = 1$

(d) $x^2 \cos^2\theta - y^2 \sin^2\theta = 1$

17. The line $lx + my + n = 0$ will be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if

(a) $a^2l^2 + b^2m^2 = n^2$ (b) $a^2l^2 - b^2m^2 = n^2$

(c) $am^2 - b^2n^2 = a^2l^2$ (d) None of these

18. The equation of the normal to the hyperbola $\frac{x^2}{16} - \frac{y^2}{b^2} = 1$, at the point $(8, 3\sqrt{3})$ is ***[PET-96]***

(a) $\sqrt{3}x + 2y = 25$ (b) $x + y = 25$

(c) $y + 2x = 25$ (d) $2x + \sqrt{3}y = 25$

19. If the tangent of the point $(2 \sec \phi, 3 \tan \phi)$ of the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ is parallel to $3x - y + 4 = 0$, then the value of ϕ is

(a) 45° (b) 60°

(c) 30° (d) 75°

20. The locus of a point which moves such that the difference of its distances from two fixed points is always a constant is ***[KCET-2003]***

(a) A straight line (b) A circle

(c) An ellipse (d) A hyperbola

21. The point of contact of the line $y = x - 1$ with $3x^2 - 4y^2 = 12$ is

(a) (4, 3) (b) (3, 4)

(c) (4, –3) (d) None of these

22. The value of m for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{49} = 1$, is **[KCET-93]**

(a) $\sqrt{\frac{17}{20}}$ (b) $\sqrt{\frac{20}{17}}$

(c) $\sqrt{\frac{3}{20}}$ (d) $\sqrt{\frac{20}{3}}$

23. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is

(a) $3y = 9x + 2$ (b) $y = 2x + 1$

(c) $2y = x + 8$ (d) $y = x + 2$

24. The locus of a point $P(\alpha, \beta)$ moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is **(AIEEE-2005)**

(a) A parabola (b) A hyperbola

(c) An ellipse (d) A circle

25. The value of m, for which the line $y = mx + \frac{25\sqrt{3}}{3}$ is a normal to the conic $\frac{x^2}{16} - \frac{y^2}{9} = 1$, is **[MPPET-2004]**

(a) $-\frac{2}{\sqrt{3}}$ (b) $\sqrt{3}$

(c) $-\frac{\sqrt{3}}{2}$ (d) None of these

26. The combined equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$ is **[DCE-2007]**

(a) $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

(b) $2x^2 + 5xy + 2y^2 + 4x + 5y - 2 = 0$

(c) $2x^2 + 5xy + 2y^2 = 0$

(d) None of these

27. The slopes of the common tangent to the hyperbolas $\frac{x^2}{9} - \frac{y^2}{16} = 1$ and $\frac{y^2}{9} - \frac{x^2}{16} = 1$ are **[UPSEE-07]**

(a) 1 (b) –1

(c) 2 (d) None of these

28. The equation of the director circle of the hyperbola $9x^2 - 16y^2 = 144$ is **[MPPET-2008]**

(a) $x^2 + y^2 = 7$ (b) $x^2 + y^2 = 9$

(c) $x^2 + y^2 = 16$ (d) $x^2 + y^2 = 25$

29. If e be the eccentricity and α be the angle between asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\sec\frac{\alpha}{2}$ is equal to **[MPPET 2010]**

(a) 0 (b) e

(c) e^2 (d) $\frac{e}{2}$

30. If the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{l^2} - \frac{y^2}{m^2} = 1$ cut each other orthogonally, then **[MPPET 2010]**

(a) $a^2 + b^2 = l^2 + m^2$ (b) $a^2 - b^2 = l^2 - m^2$

(c) $a^2 - b^2 = l^2 + m^2$ (d) $a^2 + b^2 = l^2 - m^2$

SOLUTIONS

1. (a) $(x-2)^2 + (4-1)^2 = 4\left[\frac{(x+24-1)^2}{5}\right]$

$\Rightarrow$ $5[x^2 + y^2 - 4x - 2y + 5]$

$= 4[x^2 + 4y^2 + 1 + 4xy - 2x - 4y]$

$\Rightarrow$ $x^2 - 11y^2 - 16xy - 12x + 6y + 21 = 0$

2. (d) $\frac{x}{a} + \frac{y}{b} = m$ and $\frac{x}{a} - \frac{y}{b} = \frac{1}{m}$ multiplying both, we get, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which is a hyperbola.

3. (a) The hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

we have difference of focal distances = $2a = 8$

4. (d) $9x^2 - 18x + 9 - 16y^2 - 32y - 16 = 144$

$\Rightarrow$ $\frac{(x-1)^2}{16} - \frac{(y+1)^2}{9} = 1$

$\Rightarrow$ latus rectum $= \frac{2b^2}{a} = \frac{2\times 9}{4} = \frac{9}{2}$

5. (b) Hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Here, transverse and conjugate axis of a hyperbola is equal .

$\therefore$ $a = b$ $\therefore$ $x^2 - y^2 = a^2$, which is a rectangular hyperbola,

Eccentricity $e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{2}$

6. (c) The equation of hyperbola is $\frac{x^2}{16}-\frac{y^2}{9}=1$

now, $b^2 = a^2 (e^2 - 1) \Rightarrow e=\frac{5}{4}$

Hence, foci are $(\pm ae, 0)$

$\Rightarrow$ $\left(\pm 4,\frac{5}{4},0\right)$ i.e., $(\pm 5, 0)$

7. (a) Directrix of hyperbola $x = a/e$ where

$$e=\sqrt{\frac{b^2+a^2}{a^2}}=\frac{\sqrt{b^2+a^2}}{a}$$

directrix $x=\frac{a^2}{\sqrt{a^2+b^2}}=\frac{9}{\sqrt{9+4}}$

$\Rightarrow$ $x=\frac{9}{\sqrt{13}}$

8. (d) Clearly, $e=\frac{2}{3}$ and $e'=\frac{3}{2}$ $\therefore$ $ee' = 1$

9. (b) According to question, transverse axis = conjugate axis.

Given that, $e=\sqrt{2}, 2ae=16$ $\therefore$ $a=4\sqrt{2}$.

Therefore, equation of hyperbola is $x^2 - y^2 = 32$.

10. (c) Hyperbola is $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$

$$a=\sqrt{\frac{144}{25}}, b=\sqrt{\frac{81}{25}},\ e_1=\sqrt{1+\frac{81}{144}}$$

$$=\sqrt{\frac{255}{144}}=\frac{15}{2}=\frac{5}{4}.$$

therefore, foci $= (ae_{1,} 0) = \left(\frac{12}{5}\times\frac{5}{4},0\right)=(3,0)$

$= (3, 0)$ therefore, focus of ellipse i.e., $(ae, 0) = (3, 0)$

$\Rightarrow$ $e=\frac{3}{4}$. Hence, $b^2=16\left(1-\frac{9}{16}\right)=7$

11. (b) Centre is given by $\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right)$

$$=\left(\frac{+16.9}{-9.16},\frac{-9(16)}{-9(16)}\right)=(-1,1)$$

12. (b) $\frac{x^2}{32}-\frac{y^2}{42}=1$. Therefore, $ps_1 \sim ps_2 = 2(3) = 6$

13. (c) Equation of the given hyperbola is $x^2 - 3y^2 = 2x + 8$

$\Rightarrow$ $x^2 - 2x + 1 - 3y^2 = 9 \Rightarrow \frac{(x-1)^2}{9}-\frac{y^2}{3}=1$ ....(i)

Then, the equation of its conjugate hyperbola will be $\frac{y^2}{3}-\frac{(x-1)^2}{9}=1$ ....(ii).

Here, $a^2 = 9$, $b^2 = 3$.

For eccentricity of hyperbola, we know

$a^2 = b^2 (e^2 - 1)$

$\Rightarrow$ $9 = 3(e^2 - 1)$

$\Rightarrow$ $e^2-1=\frac{9}{3}=3$

$\Rightarrow$ $e^2 = 3 + 1 = 4$ $\therefore$ $e = 2$

14. (d) Eccentricity of rectangular hyperbola, $e=\sqrt{2}$

Distance between direction (given)

$$2\left(\frac{a}{e}\right)=10 \Rightarrow \frac{2a}{e}=10$$

$\Rightarrow$ $2a = 10e \Rightarrow 2a=10\sqrt{2}$

Distance between foci

$=2ae=10\sqrt{2}\times\sqrt{2}=20$

15. (b) $\because$ $b^2 = a^2 (e^2 - 1) \Rightarrow \sin^2\alpha = \cos^2\alpha\,(e^2 - 1)$

$\Rightarrow$ $\tan^2\alpha + 1 = e^2$

$\Rightarrow$ $e^2 = \sec^2\alpha$, Vertices $= (\pm \cos\alpha, 0)$

Coordinate of focii are $(\pm ae, 0) \equiv (\pm 1, 0)$

$\Rightarrow$ if α varies then the abscissa of foci remain constant.

16. (a) $3x^2 + 4y^2 = 12$

$\Rightarrow$ $\frac{x^2}{4}+\frac{y^2}{3}=1$

$\therefore$ $a=2, b=\sqrt{3}$

$$e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{3}{4}}=\frac{1}{2} \Rightarrow ae = 1$$

$\therefore$ (sin θ) $e = 1$ ($a = \sin\theta$; given)

$e = \operatorname{cosec}\theta$,

$$b = \sin\theta\sqrt{\operatorname{cosec}^2\theta - 1} = \cos\theta$$

Hence, equation of hyperbola is

$$\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1.$$

17. (b) $lx + my + n = 0$ or $y = \frac{-l}{m}x - \frac{n}{m}$ is tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore$ $c^2 = a^2m^2 - b^2$

$\Rightarrow$ $\left(\frac{-n}{m}\right)^2 = a^2\left(-\frac{l}{m}\right)^2 - b^2$

$n^2 = a^2l^2 - b^2m^2$

18. (d) Equation of normal at point (x_1, y_1) is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

$\Rightarrow$ $\frac{16x}{8} + \frac{9y}{3\sqrt{3}} = 16 + 9$

$\Rightarrow$ $2x + \sqrt{3}y = 25$

19. (c) Equation of tangent to parabola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ at point (2 sec ϕ, 3 tan ϕ) is $\frac{x\sec\phi}{2} - \frac{y\tan\phi}{3} = 1$

Slope $= \frac{\frac{\sec\phi}{2}}{\frac{\tan\phi}{3}} = \frac{3}{1} \Rightarrow \sec\phi = \frac{1}{2} \Rightarrow \phi = 30°$

20. (d) If $PA - PB$ = const, then locus of P can be hyperbola or portion of a line through A and B but excluding line segment AB.

By elimination $\therefore$ (d) option is correct.

21. (a) Point of contact $\left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$

$m = 1$; $c = -1$; $\frac{x^2}{4} - \frac{y^2}{3} = 1$; $a^2 = 4$, $b^2 = 3$

$$\left(\frac{-4\times 1}{-1}; \frac{-3}{-1}\right) \equiv (4, 3)$$

22. (a) $y = mx + 6$ is tangent to hyperbola $\frac{x^2}{100} - \frac{y^2}{49} = 1$

$6 = \sqrt{100\times m^2 - 49} \Rightarrow 100m^2 = 36 + 49$

$m^2 = \frac{85}{100} \Rightarrow m = \sqrt{\frac{17}{20}}$

23. (d) Equation of tangent to the parabola $y^2 = 8x$ is $y = mx + \frac{2}{m}$

This is also tangent to hyperbola $xy = -1$

$\therefore$ $x\left(mx + \frac{2}{m}\right) = -1$ will have equal roots.

$mx^2 + \frac{2}{m}x + 1 = 0; D = 0$

$\left(\frac{2}{m}\right)^2 - 4, m = 0, m = 1$

$\therefore$ Equation $y = x + 2$

24. (b) If $y = \alpha x + \beta$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$C^2 = a^2m^2 - b^2 \Rightarrow \beta^2 = \alpha^2 a^2 - b^2$

$\therefore$ Locus is $a^2x^2 - y^2 = b^2$ which is hyperbola.

25. (a) Here, for normal $c = -\frac{(a^2+b^2)m}{\sqrt{a^2 - b^2m^2}}$

$$\frac{25\sqrt{3}}{3} = \frac{-(16+9)\times m}{\sqrt{16 - 9m^2}}$$

$(\sqrt{16 - 9m^2}) = (-\sqrt{3}m)^2$

$\Rightarrow$ $16 - 9m^2 = 3m^2$

$m = \frac{-2}{\sqrt{3}}$ is

26. (a) Equation of asymptotes will be $2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0$

$\because$ It is pair of straight line

$\therefore$ $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

$\begin{vmatrix} 2 & 5/2 & 2 \\ 5/2 & 2 & 5\lambda \\ 2 & 5/2 & \lambda \end{vmatrix} = 0; \lambda = 2$ $\therefore$ (a) is

27. (a, b) Equation of tangent to the hyperbola.

$$\frac{x^2}{9}-\frac{y^2}{16}=1 \Rightarrow y=mx\pm\sqrt{9m^2-16}$$

$$\frac{x^2}{-16}-\frac{y^2}{-9}=1;\ y=mx\pm\sqrt{-16m^2+9}$$

for common tangent $9m^2 - 16 = -16m^2 + 9 \Rightarrow m = \pm 1$

28. (a) Equation of directrix circle is

$x^2 + y^2 = a^2 - b^2$, or $x^2 + y^2 = 16 - 9 = 7$

($\because$ Equation of hyperbola is $9x^2 - 16y^2 = 144$ or $\frac{x^2}{16}-\frac{y^2}{9}=1$)

29. (b) Angle between asymptotes

$$= 2\tan^{-1}\frac{b}{a} = \alpha$$

$$\frac{b}{a} = \tan\frac{\alpha}{2};\ e=\sqrt{1+\frac{b^2}{a^2}}$$

$$=\sqrt{1+\tan^2\frac{\alpha}{2}} = \sec\frac{\alpha}{2}$$

30. (c) A hyperbola and ellipse are orthogonal if they are con focal. (i.e., have same focus).

$\therefore\quad (\pm\sqrt{a^2-b^2}, 0) \equiv (\sqrt{l^2+m^2}, 0)$

$\therefore\quad a^2 - b^2 = l^2 + m^2$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. A point on the curve $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$ is ***[PET-1988]***

(a) $(A\cos\theta, B\sin\theta)$ (b) $(A\sec\theta, B\tan\theta)$
(c) $(A\cos^2\theta, B\sin^2\theta)$ (d) None of these

2. The eccentricity of the hyperbola $x^2 - y^2 = 25$ is ***[PET-87]***

(a) $\sqrt{2}$ (b) $1/\sqrt{2}$
(c) 2 (d) $1+\sqrt{2}$

3. The equation of the hyperbola whose foci are (6, 4) and (–4, 4) and eccentricity 2 is given by ***[MP PET-93]***

(a) $12x^2 - 4y^2 - 24x + 32y - 127 = 0$
(b) $12x^2 + 4y^2 + 24x - 32y - 127 = 0$
(c) $12x^2 - 4y^2 - 24x - 32y + 127 = 0$
(d) $12x^2 - 4y^2 + 24x + 32y + 127 = 0$

4. The latus rectum of the hyperbola $16x^2 - 9y^2 = 144$, is ***[PET-2000]***

(a) 16/3 (b) 32/3
(c) 8/3 (d) 4/3

5. The eccentricity of the hyperbola $9x^2 - 16y^2 - 18x - 64y - 199 = 0$ is ***[MPPET-2006]***

(a) $\frac{16}{9}$ (b) $\frac{5}{4}$
(c) $\frac{25}{16}$ (d) zero

6. If $\frac{x^2}{36}-\frac{y^2}{k^2}=1$ is a hyperbola, then which of the following statements can be true? ***[Karnataka CET-2007]***

(a) (–3, 1) lies on the hyperbola
(b) (3, 1) lies on the hyperbola
(c) (10, 4) lies on the hyperbola
(d) (5, 2) lies on the hyperbola

7. If the eccentricity and length of the latus rectum of a hyperbola are $\frac{\sqrt{13}}{3}$ and $\frac{10}{3}$ units respectively, then what is the length of the transverse axis? ***[NDA-2007]***

(a) $\frac{7}{2}$ units (b) 12 units
(c) $\frac{15}{2}$ units (d) $\frac{15}{4}$ units

8. The distance between the directries of the hyperbola $x = 8\sec\theta$, $y = 8\tan\theta$ is ***[Karnataka CET-03]***

(a) $16\sqrt{2}$ (b) $\sqrt{2}$
(c) $8\sqrt{2}$ (d) $4\sqrt{2}$

9. $x^2 - 4y^2 - 2x + 16y - 40 = 0$ represents ***[DCE-99]***

(a) A pair of straight lines
(b) An ellipse
(c) A hyperbola
(d) A parabola

10. The equation $2x^2 + 3y^2 - 8x - 18y + 35 = k$ represents

(a) No locus $k > 0$

(b) An ellipse if $k < 0$

(c) A point if $k = 0$

(d) A hyperbola if $k > 0$

11. If e and e' be the eccentricities of a hyperbola and its conjugate, then $\frac{1}{e^2}+\frac{1}{e'^2}$ is equal to

[DCE-04, 2000, 1998; MPPET-2008]

(a) 0 (b) 1

(c) 2 (d) None of these

12. The equation of the hyperbola whose conjugate axis is 5 and the distance between the foci is 13, is

(a) $25x^2 - 144y^2 = 900$

(b) $144x^2 - 25y^2 = 900$

(c) $144x^2 + 25y^2 = 900$

(d) $25x^2 + 144y^2 = 900$

13. The equation of the hyperbola whose foci are the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ and the eccentricity is 2, is

(a) $\frac{x^2}{4}+\frac{y^2}{12}=1$ (b) $\frac{x^2}{4}-\frac{y^2}{12}=1$

(c) $\frac{x^2}{12}+\frac{y^2}{4}=1$ (d) $\frac{x^2}{12}-\frac{y^2}{4}=1$

14. If the latus rectum of an hyperbola be 8 and eccentricity be $3/\sqrt{5}$, then the equation of the hyperbola is:

(a) $4x^2 - 5y^2 = 100$

(b) $5x^2 - 4y^2 = 100$

(c) $4x^2 + 5y^2 = 100$

(d) $5x^2 + 4y^2 = 100$

15. The one which does not represent a hyperbola is ***[PET-92]***

(a) $xy = 1$

(b) $x^2 - y^2 = 5$

(c) $(x - 1)(y - 3) = 3$

(d) $x^2 - y^2 = 0$

16. The radius of the director circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is ***[PET-99]***

(a) $a - b$ (b) $\sqrt{a-b}$

(c) $\sqrt{a^2-b^2}$ (d) $\sqrt{a^2+b^2}$

17. The line $3x - 4y = 5$ is a tangent to the hyperbola $x^2 - 4y^2 = 5$. The point of contact is

(a) (3, 1) (b) (2, 1/4)

(c) (1, 3) (d) None of these

18. The value of m, for which the line $y = mx + 2$ becomes a tangent to the conic $4x^2 - 9y^2 = 36$, are ***[MPPET-2004]***

(a) $\pm\frac{2}{3}$ (b) $\pm\frac{2\sqrt{2}}{3}$

(c) $\pm\frac{8}{9}$ (d) $\pm\frac{4\sqrt{2}}{3}$

19. The ratio of the eccentricities of the two hyperbolas $2x^2 - 3y^2 = 1$ and $3x^2 - 2y^2 = 1$ is ***[SCRA-2007]***

(a) 5 : 3 (b) 4 : 3

(c) 3 : 2 (d) $\sqrt{2}:\sqrt{3}$

20. The equation to the normal to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at (–4, 0) is

[Karnataka CET-2008]

(a) $x = 1$ (b) $y = 0$

(c) $2x - 3y = 1$ (d) $x = 0$

21. The eccentricity of the conic $x^2 - 4y^2 = 1$, is ***[PET-1999]***

(a) $2/\sqrt{3}$ (b) $\sqrt{3}/2$

(c) $2/\sqrt{5}$ (d) $\sqrt{5}/2$

22. The equation of the normal at the point (a sec θ, b tan θ) of the curve $b^2x^2 - a^2y^2 = a^2b^2$ is

(a) $\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2+b^2$

(b) $\frac{ax}{\tan\theta}+\frac{by}{\sec\theta}=a^2+b^2$

(c) $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

(d) $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2-b^2$

23. The equation of the normal at the point (6,4) on the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 3$, is

(a) $3x + 8y = 50$ (b) $3x - 8y = 50$
(c) $8x + 3y = 50$ (d) $8x - 3y = 50$

24. The eccentricity of the hyperbola can never be equal to

(a) $\sqrt{9/5}$ (b) $2/\sqrt{9}$
(c) $3/\sqrt{8}$ (d) $\sqrt{2}$

25. Equation of the common tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is

[MPPET-2010]

(a) $y = x \pm \sqrt{a^2 - b^2}$

(b) $y = x \pm \sqrt{b^2 - a^2}$

(c) $y = \pm x \pm \sqrt{a^2 - b^2}$

(d) $y = \pm x \pm \sqrt{b^2 - a^2}$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The eccentricity of curve $x^2 - y^2 = 1$ is ***[PET-95]***

(a) 1/2 (b) $1/\sqrt{2}$
(c) 2 (d) $\sqrt{2}$

2. The reciprocal of the eccentricity of rectangular hyperbola, is

(a) 2 (b) 1/2
(c) $\sqrt{2}$ (d) $1/\sqrt{2}$

3. The eccentricity of the conjugate hyperbola of the hyperbola $x^2 - 3y^2 = 1$, is ***[MP PET-99]***

(a) 2 (b) $2/\sqrt{3}$
(c) 4 (d) 4/3

4. The focii of the hyperbola $2x^2 - 3y^2 = 5$, is ***[MP PET-2000]***

(a) $(\pm 5/\sqrt{6}, 0)$ (b) $(\pm 5/6, 0)$
(c) $(\pm\sqrt{5}/6, 0)$ (d) None of these

5. The eccentricity of the hyperbola $\frac{\sqrt{1999}}{3}(x^2 - y^2) = 1$ is ***[KCET-1999]***

(a) $\sqrt{3}$ (b) $\sqrt{2}$
(c) 2 (d) $2\sqrt{2}$

6. The latus rectum of the hyperbola $9x^2 - 16y^2 + 72x - 32y - 16 = 0$ is ***[Pb. CET-2004]***

(a) 9/2 (b) –9/2
(c) 32/3 (d) –32/3

7. The equation of the hyperbola in the standard form with transverse axis along x-axis, the length of $LR = 9$ and eccentricity $= \frac{5}{4}$, is ***[Kerala (CEE)-2005]***

(a) $\frac{x^2}{36} - \frac{y^2}{64} = 1$ (b) $\frac{x^2}{36} - \frac{y^2}{27} = 1$
(c) $\frac{x^2}{64} - \frac{y^2}{36} = 1$ (d) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

8. If $x = h + a\sec\theta$ and $y = k + b\,\text{cosec}\,\theta$, then ***[Kerala PET-2008]***

(a) $\frac{a^2}{(x+h)^2} - \frac{b^2}{(y+k)^2} = 1$
(b) $\frac{a^2}{(x-h)^2} + \frac{b^2}{(y-k)^2} = 1$
(c) $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
(d) $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

9. If e_1 and e_2 are the eccentricities of a hyperbola $3x^2 - 3y^2 = 25$ and its conjugate, then ***[Karnataka CET-2008]***

(a) $e_1 + e_2 = 4$ (b) $e_1 + e_2 = \sqrt{2}$
(c) $e_1^2 + e_2^2 = 2$ (d) $e_1^2 + e_2^2 = 4$

10. The length of transverse axis of the hyperbola $3x^2 - 4y^2 = 32$ is ***[KCET-2001]***

(a) $8\sqrt{2}/\sqrt{3}$ (b) $16\sqrt{2}/\sqrt{3}$
(c) 3/32 (d) 64/3

11. The standard equation of the hyperbola having the distance between foci as 32 and eccentricity $2\sqrt{2}$, is ***[NDA-2001]***

(a) $7x^2 - y^2 = 56$
(b) $x^2 - 7y^2 = 56$
(c) $7x^2 - y^2 = 224$
(d) $x^2 - 7y^2 = 224$

12. The equation of the normal to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at (–4, 0) is ***[UPSEAT-02]***

(a) $y = 0$ (b) $y = x$
(c) $x = 0$ (d) $x = -y$

13. The equation of the tangent to the hyperbola $4y^2 = x^2 - 1$ at the point (1, 0) is ***[Karnataka CET-94]***

(a) $x = 1$ (b) $y = 1$
(c) $y = 4$ (d) $x = 4$

14. If the line $lx + my = 1$ is a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then $\frac{a^2}{l^2} - \frac{b^2}{m^2}$ is equal to

[EAMCET-2007; Orissa JEE-2007]

(a) $a^2 - b^2$
(b) $a^2 + b^2$
(c) $(a^2 + b^2)^2$
(d) $(a^2 - b^2)^2$

15. The equation of the tangent at the point (a sec θ, b tan θ) of the conic $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is

(a) $x \sec^2\theta - y \tan^2\theta = 1$
(b) $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$
(c) $\frac{x + a\sec\theta}{a^2} - \frac{y + b\tan\theta}{b^2} = 1$
(d) None of these

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**6.** ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

3. (a) Conjugate hyperbola of $x^2 - 3y^2 = 1$ is $-x^2 + 3y^2 = 1$ or $\frac{-x^2}{1} + \frac{y^2}{1/3} = 1$

$$e = \sqrt{1 + \frac{1}{1/3}} = 2$$

7. (c) Equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{2b^2}{a} = 9 \Rightarrow \frac{2a^2(e^2-1)}{a} = 9$$

$$2a\left(\left(\frac{5}{4}\right)^2 - 1\right) = 9$$

$\Rightarrow$ $a = 8$, $b^2 = a^2(e^2 - 1)$

$$b^2 = 8^2\left(\left(\frac{5}{4}\right)^2 - 1\right) = 6^2,$$

∴ Equation is $\frac{x^2}{64} - \frac{y^2}{36} = 1$

12. (a) Equation of normal is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$$

$a^2xy_1 + b^2yx_1 = (a^2 + b^2)x_1y_1$

$\Rightarrow$ $0 + 9y(-4) = 0$

$y = 0$

Test Your Skills

ASSERTION/REASONING

ASSERTION AND REASONING TYPE QUESTIONS

Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct.

(a) **Assertion** is True, **Reason** is True and **Reason** is a correct explanation for **Assertion.**

(b) **Assertion** is True, **Reason** is True and **Reason** is not a correct explanation for **Assertion.**

(c) **Assertion** is True and **Reason** is False.

(d) **Assertion** is False and **Reason** is True.

1. **Assertion:** The points (2, 1) and (–3, 5) lie on opposite side of the line $3x - 2y + 1 = 0$.
Reason: The algebraic perpendicular distance from the given point to the line have opposite sign.

2. **Assertion:** The line $2x + y + 6 = 0$ is perpendicular to the line $x - 2y + 5 = 0$ and second line passes through (1, 3).
Reason: Product of the slopes of the lines is equal to –1.

3. **Assertion:** If $3a - 2b + 5c = 0$, family of line $ax + by + c = 0$ are always concurrent at a point.
Reason: If $L_1 = 0$ and $L_2 = 0$ be two lines, then family of lines is $L_1 + \lambda L_2 = 0, \lambda \neq 0$.

4. **Assertion:** The area of triangle formed by the points $A(20, 22)$, $B(21, 24)$, $C(22, 23)$ is same as are formed by $P(0, 0)$, $Q(1, 2)$ and R (2, 1)
Reason: The area of triangle is constant with respect to transition of coordinate axes.

5. **Assertion:** If (0, 3), (1, 1) and (–1, 2) be the mid-points of the sides of a triangle, then centroid of the original triangle is (0, 2).
Reason: The centroids of the triangle and joins the mid-points of the sides of triangle are same.

6. **Assertion:** If the points (1, 2) and (3, 4) be on the same side of the line $3x - 5y + \lambda = 0$ then $\lambda < 7$ or $\lambda > 11$.
Reason: If the points (x_1, y_1) and (x_2, y_2) be on thesame side of the line $f(x, y) \equiv ax + by + c = 0$, then $\dfrac{f(x_1, y_1)}{f(x_2, y_2)} < 0$.

7. **Assertion:** The equation $2x^2 + 3xy - 2y^2 + 5x - 5y + 3 = 0$ represents a pair of perpendicular straight lines.
Reason: A pair of lines given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are perpendicular, if $a + b = 0$.

8. **Assertion:** The joint equation of lines $y = x$ and $y = -x$ is $y^2 = -x^2$ i.e., $x^2 + y^2 = 0$.
Reason: The joint equation of lines $ax + by = 0$ and $cx + dy = 0$ is $(ax + by)(cx + dy) = 0$ where a, b, c, d are constants.

9. **Assertion:** The combined equation of L_1, L_2 is $2x^2 + 6xy + y^2 = 0$ and that of L'_1, L'_2 is $3x^2 +$

$8xy + y^2 = 0$. If the angle between L_1, L'_2 is θ, then angle between L_2, L'_1 is also θ.
Reason: If the pairs of line $L_1L_2 = 0, L'_1 \times L'_2 = 0$ are equally inclined, then angle between $L_1 \times L'_2$ = angle between L_2, L'_1.

10. Assertion: The point of intersection of the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$.

Reason: The point of intersection is $\left(\sqrt{\left(\frac{f^2 - bc}{h^2 - ab}\right)}, \sqrt{\left(\frac{g^2 - ac}{h^2 - ab}\right)}\right)$.

11. Assertion: The distance between the lines represented by $x^2 + 2\sqrt{2}xy + 2y^2 + 4\sqrt{2}x + 4y + 1 = 0$ is 2.
Reason: Distance between the lines $ax + by + c = 0$ and $ax + by + c_1 = 0$ is $\frac{|c - c_1|}{\sqrt{a^2 + b^2}}$

12. Assertion: The line $(x - 3)\cos\theta + (y - 3)\sin\theta = 1$ touches a circle $(x - 3)^2 + (y - 3)^2 = 1$ for all values of θ.
Reason: $x\cos\theta + y\sin\theta = a$ is a tangent of circle $x^2 + y^2 = a^2$ for all values of θ.

13. Assertion: Two tangents are drawn from a point on the circle $x^2 + y^2 = 2a^2$ to the circle $x^2 + y^2 = a^2$, then tangents always perpendicular.
Reason: $x^2 + y^2 = 2a^2$ is the director circle of $x^2 + y^2 = a^2$.

14. Assertion: The power of the point (2, 4) with respect to the circle $x^2 + y^2 - 6x + 4y - 8 = 0$ is 16.
Reason: Power of circle = (length of tangent)^2.

15. Assertion: If the perpendicular tangents to the circle C_1 meet at P. Then the locus of P has a circle C_2. Then $\frac{\text{radius of } C_1}{\text{radius of } C_2} = \frac{1}{\sqrt{2}}$.

Reason: Circles C_1 and C_2 are concentric.

16. Assertion: The line $2x + y = 5$ is a diameter of the circle $x^2 + y^2 - 6x + 2y = 0$.
Reason: Normal of a circle always pass through centre of circle.

17. Assertion: The number of common tangents to the circle $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y = 24$ is 4.
Reason: Circles with centre C_1, C_2 and radii r_1, r_2 and, if $|C_1C_2| > r_1 + r_2$, then circles have 4 common tangents.

18. Assertion: Equation of polar to the circle $x^2 + y^2 = 4$ w.r.t the point (3, 5) is $3x + 5y = 4$.
Reason: The point (3, 5) lies outside the circle.

19. Assertion: Tangents cannot be drawn from the point $(1, \lambda)$ to the circle $x^2 + y^2 + 5x - 5y = 0$, if $2 < \lambda < 3$.

Reason: $\left(1 + \frac{5}{2}\right)^2 + \left(\lambda - \frac{5}{2}\right)^2 < \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2$.

20. Assertion: Number of circles passing through (–2, 1), (–1, 0), (–4, 3) is 1.
Reason: Through three non collinear points in a plane only one circle can be drawn.

21. Assertion: Circle $x^2 + y^2 - 6x - 4y + 9 = 0$ bisects the circumference of the circle $x^2 + y^2 - 8x - 6y + 23 = 0$.
Reason: Centre of first circle lie on the second circle.

22. Assertion: The smallest possible radius of circle which pass through (1, 0) and (0, 1) is $\frac{1}{\sqrt{2}}$.

Reason: Circle passes through origin.

23. Assertion: Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Then the locus of the centre of C is an ellipse.
Reason: If A and B are foci and P be any point on the ellipse, then $AP + BP$ = Constant.

24. Assertion: If two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other, then $f'g = fg'$.
Reason: Two circles touch each other, if line joining their centres is perpendicular to all possible common tangents.

25. Assertion (a): Length of latus rectum of parabola $(6x + 8y + 7)^2 = 4(8x + 6y + 3)$ is 4
Reason (R): Length of latus rectum of the parabola $y^2 = 4ax$ is $4a$.

26. Assertion (a): If normal at the ends of double ordinate $x = 4$ of parabola $y^2 = 4x$ meet the curve again at P and P' respectively, then PP' = 12 unit.

Reason (R): If normal at t_1 of $y^2 = 4ax$ meet the parabola again at t_2, then $t_2 = t_1 - \frac{2}{t_1}$.

27. **Assertion (a):** The equation of the director circle to the ellipse $9x^2 + 16y^2 = 144$ is $x^2 + y^2 = 25$.
Reason (R): Director circle is the locus of point of intersection of perpendicular tangents to an ellipse.

28. **Assertion (a):** The equation $13x^2 - 18xy + 37y^2 + 2x + 14y - 2 = 0$ represents an ellipse.
Reason (R): The square of the coefficient of xy is less than the product of coefficient of x^2 and y^2 is only condition.

29. **Assertion (a):** The sum of focal distances of a point on the ellipse $9x^2 + 4y^2 - 18x - 24y + 9 = 0$ is 4
Reason (R): The equation $9x^2 + 4y^2 - 18x - 24y + 9 = 0$ can be expressed as $9(x - 1)^2 + 4(y - 3)^2 = 36$.

30. **Assertion (a):** From the point $(\lambda, 3)$ tangents are drawn to $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and are perpendicular to each other then $\lambda = \pm 2$.
Reason (R): The locus of the point of intersection of perpendicular tangents to $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is $x^2 + 3y = 13$.

31. **Assertion (a):** Any chord of the ellipse $x^2 + y^2 + xy = 1$ through (0, 0) is bisected at (0, 0).
Reason (R): The centre of an ellipse is a point through which every chord is bisected.

32. **Assertion (a):** In hyperbola the distance between foci is always greater than the difference of focal distances of any point on it
Reason (R): If e be the eccentricity of the hyperbola, then $e > 1$.

33. **Assertion (a):** $\frac{5}{3}$ and $\frac{5}{4}$ are the eccentricities of two conjugate hyperbolas.
Reason (R): If e and e_1 are the eccentricities of two conjugate hyperbolas than $ee_1 > 1$.

34. **Assertion (a):** The point (5, –4) inside the hyperbola $y^2 - 9x^2 + 1 = 0$.
Reason (R): The point (x_1, y_1) inside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 < 0$

ASSERTION/REASONING: SOLUTIONS

1. (a) The algebraic perpendicular distance from (2, 1) to the line $3x - 2y + 1 = 0$ is $\frac{3(2) - 2(1) + 1}{\sqrt{(3)^2 + (-2)^2}} = \frac{5}{\sqrt{13}} = L_1$ (say) and the algebraic perpendicular distance from (–3, 5) to the line $3x - 2y + 1 = 0$ is $\frac{3(-3) - 2(5) + 1}{\sqrt{(3)^2 + (-2)^2}} = -\frac{18}{\sqrt{3}} = L_2$ (say)

Here, $\frac{L_1}{L_2} < 0$

$\therefore$ Given, points lie on opposite side of the line $3x - 2y + 1 = 0$.

2. (a) Slope of $2x + y + 6 = 0$ is $-2 = m_1$ (say) and slope of $x - 2y + 5 = 0$ is $\frac{1}{2} = m_2$ (say)

$\because \quad m_1 m_2 = -1$

Also, (1, 3) lies on $x - 2y + 5 = 0$.

3. (b) $\because$ $ax + by + c = 0$ .... (i)

and $3a - 2b + 5c = 0$

$\Rightarrow \quad \frac{3}{5}a - \frac{2}{5}b + c = 0$ .... (ii)

From Equation (i) and (ii),

$a\left(x - \frac{3}{5}\right) + b\left(y + \frac{2}{5}\right) = 0$ is a family of lines always concurrent at $x - \frac{3}{5} = 0$ and $y + \frac{2}{5} = 0$

i.e., $\left(\frac{3}{5}, -\frac{2}{5}\right)$.

4. (a) The area of triangle unchanged by shifting origin to any point. If origin shifted to

(20, 22) then A, B, C becomes $P(20 - 20, 22 - 22)$, $Q(21 - 20, 24 - 22)$ and $R(22 - 20, 23 - 22)$
i.e., $P(0, 0)$, $Q(1, 2)$ and $R(2, 1)$ both are true.

5. (a) Let $A \equiv (\alpha, \beta)$, then $B \equiv (-2 - \alpha, 4 - \beta)$ and $C \equiv (2 - \alpha, 2 - \beta)$

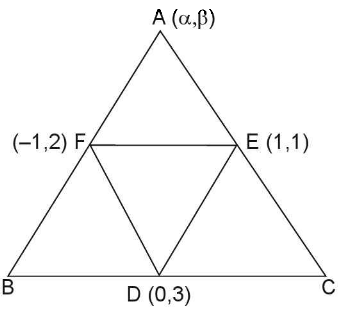
$\because$ D is mid point of B and C.

$\therefore$ $0 = -2 - \alpha + 2 - \alpha$ and $6 = 4 - \beta + 2 - \beta$

$\therefore$ $\alpha = 0$ and $\beta = 0$

$\therefore$ $A \equiv (0, 0)$, $B \equiv (-2, 4)$ and $C \equiv (2, 2)$

Centroid of ΔABC is (0, 2) and centroid of ΔDEF is (0, 2) which are same.



6. (c) Let $f(x, y) = 3x - 5y + \lambda$

$\because$ Points (1, 2) and (3, 4) be on the same side of the line $3x - 5y + \lambda = 0$ then,

$$\frac{f(1,2)}{f(3,4)} > 0$$

$$\Rightarrow \frac{3 - 10 + \lambda}{9 - 20 + \lambda} > 0$$

$$\Rightarrow \frac{\lambda - 7}{\lambda - 11} > 0$$

$\therefore$ $\lambda < 7$ or $\lambda > 11$.

7. (d) Here, $\Delta \neq 0$, $h^2 > ab$

$\therefore$ $2x^2 + 3xy - 2y^2 + 5x - 5y + 3 = 0$ represents a rectangular hyperbola.

8. (d) The joint equation of $y = x$ and $y = -x$ is $(x - y)(x + y) = 0$
i.e., $x^2 - y^2 = 0$

9. (a) Pair of bisectors of $2x^2 + 6xy + y^2 = 0$ and $3x^2 + 8xy + y^2 = 0$ are coincides.

$$\left(i.e., \frac{x^2 - y^2}{2 - 1} = \frac{xy}{3} \text{ and } \frac{x^2 - y^2}{3 - 1} = \frac{xy}{4}\right)$$

$\therefore$ Angle between L_1, L'_2 = angle between L_2, L'_1.

10. (a) $x = \sqrt{\left(\frac{f^2 - bc}{h^2 - ab}\right)}$

$$= \frac{\sqrt{(f^2 - bc)(h^2 - ab)}}{(h^2 - ab)}$$

$$= \frac{\sqrt{(r^2h^2 - abf^2 - bch^2 + ab^2c)}}{(h^2 - ab)}$$

$$= \frac{\sqrt{(f^2h^2 - abf^2 - bch^2 + b(af^2 + bg^2 + ch^2 - 2fgh)}}{(h^2 - ab)}$$

$(\because abc + 2fgh - af^2 - bg^2 - ch^2 = 0)$

$$= \frac{\sqrt{(f^2h^2 + b^2g^2 - 2bfgh)}}{(h^2 - ab)}$$

$$= \left(\frac{bg - hf}{h^2 - ab}\right)$$

Similarly, $y = \left(\frac{af - gh}{h^2 - ab}\right)$

Hence, $(x, y) = \left(\frac{bg - hf}{h^2 - ab}, \frac{af - gh}{h^2 - ab}\right)$.

11. (d) $\because \Delta = 2 \times 2 \times 1 + 2 \times 2 \times 2\sqrt{2} \times \sqrt{2} - 1 \times 4 - 2 \times 8 - 1 \times 2 = 4 + 16 - 4 - 16 - 2 = -2 \neq 0$ and $h^2 = ab$
The, given conic is a parabola not a pair of parallel lines.

12. (a) $\because$ Now, replacing x by $x - 3$, y by $y - 3$ and a by 1, then $x \cos\theta + y \sin\theta = a$ reduce in $(x - 3)\cos\theta + (y - 3)\sin\theta = 1$, and $x^2 + y^2 = a^2$ reduce in $(x - 3)^2 + (y - 3)^2 = 1$.

13. (a) $\because$ Locus of point of intersection of perpendicular tangents is director circle.

14. (a) Let $S = x^2 + y^2 - 6x + 4y - 8$

$\therefore$ $S_1 = 2^2 + 4^2 - 6 \cdot 2 + 4.4 - 8$
$= 4 + 16 - 12 + 16 - 8$
$= 16$

$(\sqrt{S_1})^2 = 16$

Power of circle $= 16$.

15. (b) C_2 is the director circle of C_1
If radius of C_1 is r, then radius of C_2 is $r\sqrt{2}$ and circles are concentric.

16. (b) $\because$ Diameter of circle pass through centre
$\because$ Centre = (3, 1) lie on $2x + y = 5$.

17. (d) $C_1 \equiv (0, 0), r_1 = 2$

$C_2 \equiv (3, 4),\ r_2 = \sqrt{(9+16+24)} = 7$

$\therefore\ |C_1C_2| < |r_2 - r_1|$

i.e., One tangent.

18. (b) Here pole (3, 5)

$\therefore$ Equation of polar is

$x \cdot 3 + y \cdot 5 = 4$

$\Rightarrow\ 3x + 5y = 4$

and equation of polar is same point outside or inside.

19. (a) Let $P = (1, \lambda)$ $C = \left(-\frac{5}{2}, \frac{5}{2}\right)$ and radius

$$= \sqrt{\left(-\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2}$$

$$= \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2}$$

Point $P(1, \lambda)$ must be inside the circle

$\therefore\ CP < r$

$\Rightarrow\ (CP)^2 < r^2$

$$\Rightarrow\ \left(1+\frac{5}{2}\right)^2 + \left(\lambda - \frac{5}{2}\right)^2 < \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2$$

$$\frac{49}{4} + \lambda^2 + \frac{25}{4} - 5\lambda < \frac{25}{4} + \frac{25}{4}$$

$\Rightarrow\ \lambda^2 - 5\lambda + 6 < 0$

$\Rightarrow\ (\lambda - 2)(\lambda - 3) < 0$

$\therefore\ 2 < \lambda < 3.$

20. (d) Let $P \equiv (-2, 1)$, $Q \equiv (-1, 0)$ and $R \equiv (-4, 3)$

$$\because\ \text{Slope of } PQ = \frac{0-1}{-1+2} = -1$$

$$\text{and Slope of } QR = \frac{3-0}{-4+1} = -1$$

$\because$ Slope of PQ = Slope of QR

$\Rightarrow$ P, Q, R are collinear.

i.e., no circle is drawn.

21. (b) Equation of common chord of these circles is $(x^2 + y^2 - 6x - 4y + 9) - (x^2 + y^2 - 8x - 6y + 23) = 0$

$\Rightarrow\ 2x + 2y - 14 = 0$

or $\ x + y - 7 = 0$

Since centre of the second circle ie, (4, 3) lie on it.

Hence $x + y - 7 = 0$ is a diameter of second circle and hence first circle bisects the circumference of the second circle.

22. (a) Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Since its passes through (1, 0) and (0, 1), then

$1 + 0 + 2g + 0 + c = 0$

$$\Rightarrow\ g = -\frac{(1+c)}{2}$$

and $\ 0 + 1 + 0 + 2f + c = 0$

$$\Rightarrow\ f = -\frac{(1+c)}{2}$$

$$\therefore\ \text{Radius } = \sqrt{(g^2 + f^2 - c)}$$

$$= \sqrt{\left(\frac{(1-c)^2}{4} + \frac{(1+c)^2}{4} - c\right)}$$

$$= \sqrt{\left(\frac{1+c^2}{2}\right)}$$

For minimum radius, c must be equal to zero.

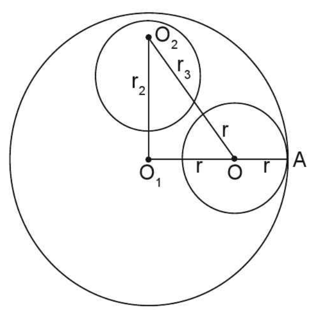
$$\therefore\ \text{Radius } = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}},$$

$$\text{then } g = -\frac{1}{2}, f = -\frac{1}{2}, c = 0$$

$\therefore$ Circle is $x^2 + y^2 - x - y = 0$

which pass through origin.

23. (a) Let the given circles C_1 and C_2 have centres O_1 and O_2 with radii r_1 and r_2 respectively. Let centre of circle C is at O radius is r.



$\because\ OO_2 = r + r_2$

$OO_1 = r_1 - r$

$\Rightarrow\ OO_1 + OO_2 = r_1 + r_2$

which is greater than O_1O_2 as $O_1O_2 < r_1 + r_2$.

$\therefore$ Locus of O is an ellipse with foci O_1 and O_2

Alternative Method: Let $O_1 \equiv (0, 0)$, $O_2 \equiv (a, b)$ and $O \equiv (h, k)$

$\therefore$ $C_1 : x^2 + y^2 = r_1^2$

$C_2 : (x - a)^2 + (y - b)^2 = r_2^2$

$C : (x - h)^2 + (y - k)^2 = r^2$

$\Rightarrow$ $OO_2 = r + r_2$

$\Rightarrow$ $\sqrt{(h-a)^2 + (k-b)^2} = r_1 + r_2$ ....(i)

and

$OO_1 = r_1 - r$

$\sqrt{(h^2 + k^2)} = r_i - r$ ....(ii)

Adding equation (i) and (ii), we get

$\sqrt{(h-a)^2 + (k-b)^2}$

$+\sqrt{(h^2 + k^2)} = r_1 + r_2$

$\therefore$ Locus of O is $\sqrt{(x-a)^2 + (y-b)^2}$ $+\sqrt{(x^2 + y^2)} = r_1 + r_2$ which represents an ellipse with foci are at (a, b) and $(0, 0)$.

24. (c) Let $C_1 \equiv (-g, -f)$, $r_1 = \sqrt{(g^2 + f^2)}$

and $C_2 \equiv (-g', -f')$, $r_2 = \sqrt{(g'^2 + f'^2)}$

$\therefore$ $|C_1C_2| = r_2 \pm r_2$

$\Rightarrow$ $\sqrt{(g-g')^2 + (f-f')^2}$

$= \sqrt{(g^2 - f^2)} \pm \sqrt{(g'^2 + f'^2)}$

$\Rightarrow$ $(g - g')^2 + (f - f')^2 = g^2 + f^2 + g'^2 + f'^2$

$\pm 2\sqrt{(g^2 + f^2)(g'^2 + f'^2)}$

or $(-gg' - ff')^2 = (g^2 + f^2)(g'^2 + f'^2)$

$\Rightarrow$ $g^2f'^2 + g'^2f^2 - 2gg'ff' = 0$

or $(gf' - g'f)^2 = 0$

$\therefore$ $gf' = g'f$

and line joining centres may not be parallel to common tangents.

25. (d) $\because$ $6x + 8y + 7 = 0$ and $8x + 6y + 3 = 0$ are not perpendicular to each other.

$\therefore$ Latus rectum $\neq 4$

And length of latus rectum of the parabola $y^2 = 4ax$ is $4a$

26. (c) End points of double ordinate $x = 4$ of parabola $y^2 = 4x$ are $(4, \pm 4)$

$\Rightarrow$ $t_1 = \pm 2$

$\Rightarrow$ $t_2 = -t_1 - \dfrac{2}{t_1} = \pm 3$

$\Rightarrow$ $P(9, 6)$ and $P'(9, -6)$

$\therefore$ $PP' = 12$ unit

27. (a) Any tangent in terms of slope of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$y = mx \pm \sqrt{(a^2m^2 + b^2)}$

$(y - mx)^2 = a^2m^2 + b^2$

$\Rightarrow$ $(x^2 - a^2)m^2 - 2xym + y^2 - b^2 = 0$

$\therefore$ $m_1m_2 = -1$

$\Rightarrow$ $\dfrac{y^2 - b^2}{x^2 - a^2} = -1$

$\Rightarrow$ $x^2 + y^2 = a^2 + b^2$ is director circle of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Now, $9x^2 + 16y^2 = 144$

$\Rightarrow$ $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

Then, director circle is $x^2 + y^2 = 16 + 9 = 25$

28. (c) For ellipse $\Delta \neq 0$, $h^2 < ab$

Here, $a = 13$, $h = -9$, $b = 37$, $g = 1$, $f = 7$, $c = -2$

$\therefore$ $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= (13)(37)(-2) + 2 \times 7 \times 1 \times -9 - 13 \times 49 - 37 \times 1 + 2 \times 81 = -1600 \neq 0$ and $h^2 = 81 < ab \Rightarrow h^2 < ab$

29. (d) $9x^2 + 4y^2 - 18x - 24y + 9 = 0$

$\Rightarrow$ $9(x - 1)^2 + 4(y - 3)^2 = 36$

$\Rightarrow$ $\dfrac{(x-1)^2}{2^2} + \dfrac{(y-3)^2}{3^2} = 1$

Here, $b > a$

$\therefore$ Sum of focal distances of a point is $2b = 6$

30. (c) $\because$ Locus of point of intersection of perpendicular tangents to $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is director circle $x^2 + y^2 = a^2 + b^2$

$\therefore$ $(\lambda, 3)$ lie on $x^2 + y^2 = 4 + 9$

$\Rightarrow$ $\lambda^2 + 9 = 4 + 9$

$\therefore$ $\lambda = \pm 2$

31. (a) Let $y = mx$ be any chord through $(0, 0)$, then solving $y = mx$ and $x^2 + y^2 + xy = 1$

$\Rightarrow \quad x^2 + m^2 x^2 + x(mx) = 1$

$\Rightarrow \quad x^2(1 + m + m^2) - 1 = 0$

$\therefore \quad x_1 + x_2 = 0$

$\Rightarrow \quad \dfrac{x_1 + x_2}{2} = 0$

Also, $\dfrac{y_1 + y_2}{2} = \dfrac{mx_1 + mx_2}{2} = m\left(\dfrac{x_1 + x_2}{2}\right) = 0$

$\Rightarrow$ Mid-point of chord is (0, 0) for all m.

32. (a) $\because$ Distance between foci $SS' = 2ae$ and Difference of focal distances of any point on hyperbola $S'P - SP = 2a$
Given $SS' > S'P - SP$
$\Rightarrow \quad 2ae > 2a \Rightarrow e > 1$

33. (b) Two conjugate hyperbolas are $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

and $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$

Hence, $\quad e^2 = 1 + \dfrac{b^2}{a^2} = \dfrac{a^2 + b^2}{a^2}$

and $\quad e_1^2 = 1 + \dfrac{a^2}{b^2} = \dfrac{a^2 + b^2}{b^2}$

$\therefore \quad \dfrac{1}{e^2} + \dfrac{1}{e_1^2} = \dfrac{a^2}{a^2 + b^2} + \dfrac{b^2}{a^2 + b^2} = 1$

$\because \quad \dfrac{1}{(5/3)^2} + \dfrac{1}{(5/4)^2} = \dfrac{9}{25} + \dfrac{16}{25} = 1$

and $\quad ee_1 = \dfrac{5}{3} \times \dfrac{5}{4} = \dfrac{25}{12} > 1$

34. (c) $\because$ $y^2 - 9x^2 + 1 = 0 \Rightarrow 9x^2 - y^2 - 1 = 0$
Point is (5, –4) then $9(5)^2 - (-4)^2 - 1 = 225 - 17 = 208 > 0$

$\therefore$ Point (5, –4) inside the hyperbola.

If (x_1, y_1) inside the hyperbola

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1,$

then $\dfrac{x_1^2}{a^2} - \dfrac{y_1^2}{b^2} - 1 > 0$

MENTAL PREPARATION TEST

1. A (1, 2) and B(5, 3) are two fixed points. P is a movable point such that $PA : PB = 2 : 3$. Find its locus of P.
2. The distance of a movable point from (3, 4) is three times of its distance from (–3, –4). Find the locus.
3. The coordinate of A, B and C are (2, 0), (–2, 0) and (3, –3) respectively. Find the locus of P, where $PA^2 + PB^2 = 2PC^2$.
4. Find the locus of the point whose distance from the point (1, –2) is double the distance from the point (–3, 5).
5. Find out the distance between the points $(p, q + r)$ and $(q, p + r)$.
6. If Y ordinate of a point is 4 units more than its abscissa and this point is equidistant from point (–2, 3) and (6, –3) then find the coordinates of the point.
7. Prove that points (2, 5), (6, 8), (9, 12) and (5, 9) are coordinates of a rhombus.
8. A point moves so that the sum of its distances from two fixed points $(ae, 0)$ and $(-ae, 0)$ is $2a$. Show that equation of locus is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where $b^2 = a^2(1 - e^2)$
9. Using distance formula show that the points (–1, 2), (5, 0) and (2, 1) are collinear.
10. Slope of a line joining the points (7, 3) and $(k, 2)$ is – 4. Find the value of k.
11. Find the area of the triangle formed by the points $(p + 1, 1)$, $(2p + 1, 3)$ and $(2p + 2, 2p)$ and show that these points are collinear if $p = 2$ or $-\dfrac{1}{2}$
12. The co-ordinates of A, B, C and D are (6, 3), (–3, 5), (4, –2) and $(x, 3x)$ respectively and $\dfrac{\text{area of } \Delta\, DBC}{\text{area of } \Delta\, ABC} = \dfrac{1}{2}$, find x

13. Find the co-ordinates of the orthocentre of the triangle whose angular points are (1, 0), (2, –4) and (–5, –2).

14. Find the slope of a line whose inclination is

(i) $\frac{\pi}{6}$ (ii) $\frac{4\pi}{3}$ (iii) $\frac{11\pi}{6}$

15. State whether the two lines in each of the following problems are parallel, perpendicular or neither

(i) through (5, 6) and (2, 3); through (9, –2) and (6, –5)

(ii) through (2, –5) and (–2, 5); through (6, 3) and (1, 1)

(iii) through (8, 2) and (–5, 3); through (16, 6) and (3, 15).

16. Find the equation of a line parallel to y-axis, at a distance of $\frac{5}{2}$ units to the left of y-axis.

17. Find the lengths of the perpendicular segments drawn from the vertices $A(2, 1)$, $B(5, 2)$ and $C(4, 4)$ of the ΔABC to the opposite sides.

18. A straight line passes through the point (1, 1) and the portion of the line intercepted between the x-axis and y-axis is divided at this point in the ratio 3 : 4. Find the equation of the line.

19. Find the equations of the straight lines cutting off an intercept –1 from the y-axis and equally inclined to the axes.

20. Find the equation of the straight line joining the points : (2, 3), (2, –4)

21. Find the equation of the straight line cutting off intercepts 2 and 3 from the axes.

22. Prove that the middle point of the line segment joining (2, 5) and (4, 3) lies on the straight line $2x - 3y + 6 = 0$.

23. Find the coordinates of the points which are at a distance $4\sqrt{2}$ from the point (3, 4) and which lie on the line passing through the point (3, 4) and inclined at angle of 45° with the positive direction of x-axis.

24. Find the equation of a line which passes through the point (2, 9) and making an angle of $\frac{\pi}{4}$ with x-axis. Also find the points on the line which are at a distance of (i) 2 units, (ii) 5 units from (2, 9).

25. Find the direction in which a straight line must be drawn through the point (1, 2) so that the point of intersection with the line $x + y = 4$ may be at a distance of $\sqrt{\frac{2}{3}}$ units from (1, 2).

26. Find the equation of the line which passes through the point (6, 7) and cuts off intercepts on the axes which are equal in magnitude but opposite in sign.

27. The perpendicular distance of a line from the origin is 5 units and its slope is –1, find its equation.

28. Find the equation to the straight line passing through the point (2, 3) and perpendicular to the straight line $4x - 3y = 10$.

29. Find the equation of the right bisector of the line segment joining the points (1, 3) and (3, –5).

30. What is the value of y so that the line through $(3, y)$ and (4, 9) is parallel to the line through (–1, 6) and (0, 8)?

31. Find the equation of the line that has y-intercept 4 and is perpendicular to the line $y = 3x - 2$.

32. Find the equation of the line passing through the intersection of the lines $2x + 3y - 2 = 0$ and $x - 2y + 1 = 0$ and having x-intercept equal to 3.

33. Find the distance of the point (0, –1) from the line joining the points (1, 3) and (–2, 6).

34. Find the points on the line $y = x$ which are at a distance of 5 units from the line $4x + 3y - 1 = 0$

35. Find the bisector of the acute angle between the lines $3x + 4y = 11$ and $12x - 5y = 2$.

36. A triangle is formed by the lines $x + y - 6 = 0$, $3y - x + 2 = 0$ and $3y = 5x + 2$, find the coordinates of its orthocentre.

37. Find the bisectors of the angles between the lines $3x + 4y = 11$ and $12x - 5y = 2$ and distinguish between them

38. Find the bisector of the acute angle between the lines $2x - y - 4 = 0$ and $x - 2y + 10 = 0$.

39. Find the equation of a straight line passing through the point of intersection of the lines $3x + y - 9 = 0$ and $4x + 3y - 7 = 0$ and perpendicular to the line $5x - 4y + 1 = 0$.

MENTAL PREPARATION TEST ANSWERS

1. $5x^2 + 5y^2 + 22x - 12y - 91 = 0$
2. $2x^2 + 2y^2 + 15x + 20y + 50 = 0$
3. $3x - 3y = 7$
4. $3x^2 + 3y^2 + 26x - 44y + 131 = 0$
5. $(p - q)\sqrt{2}$ unit
6. (20, 24)
10. $\frac{29}{4}$
11. $\frac{1}{2}\,|2p^2 - 3p - 2|$ square units
12. $\frac{11}{8}$ or $-\frac{3}{8}$
13. $\left(\frac{11}{13}, -\frac{7}{13}\right)$
14. (i) $\frac{1}{\sqrt{3}}$ (ii) $\sqrt{3}$ (iii) $-\frac{1}{\sqrt{3}}$
15. (i) Parallel
(ii) Perpendicular
(iii) Neither
16. $2x + 5 = 0$
17. $\frac{7}{\sqrt{5}}, \frac{7}{\sqrt{13}}, \frac{7}{\sqrt{10}}$
18. $4x + 3y = 7$
19. $x - y - 1 = 0$ and $x + y + 1 = 0$
20. $x - 2 = 0$
21. $3x + 2y - 6 = 0$
23. (7, 8); (–1, 0)
24. $x - y + 7 = 0$; $(2 \pm \sqrt{2}, 9 \pm \sqrt{2})$; $\left(2 \pm \frac{5}{\sqrt{2}}, 9 \pm \frac{5}{\sqrt{2}}\right)$
26. $x - y + 1 = 0$
27. $x + y = \pm 5\sqrt{2}$
28. $3x + 4y - 18 = 0$
29. $x - 4y - 6 = 0$
30. 7
31. $x + 3y - 12 = 0$
32. $x + 5y - 3 = 0$
33. $\frac{5}{\sqrt{2}}$ units
34. $\left(\frac{26}{7}, \frac{26}{7}\right)$ and $\left(-\frac{24}{7}, -\frac{24}{7}\right)$
35. $11x + 3y - 17 = 0$
36. $\left(\frac{5}{2}, \frac{5}{2}\right)$
37. Bisector of the angle containing the origin is $3x - 11y + 19 = 0$ and that of the angle not containing the origin is $11x + 3y - 17 = 0$
38. $x - y + 2 = 0$
39. $4x + 5y - 1 = 0$

TOPICWISE WARMUP TEST 1

1. Let ABC be a triangle, two of whose vertices are (15, 0) and (0, 10). If the orthocentre is (6, 9) then the third vertex is ***[Orissa JEE-2007]***
(a) (10, 0) (b) (0, 0)
(c) (10, 10) (d) None of these

2. If $P(1, 2)$, $Q(4, 6)$, $R(5, 7)$ and $S(a, b)$ are the vertices of a parallelogram PQRS then ***(IIT-1998)***
(a) $a = 2, b = 4$ (b) $a = 3, b = 4$
(c) $a = 2, b = 3$ (d) $a = 3, b = 5$

3. A triangle with vertices (4, 0), (–1, 1), (3, 5) is ***(AIEEE-02)***
(a) Isosceles and right angled
(b) Isosceles but not right angled
(c) Right angled but not isosceles
(d) Neither right angled nor Isosceles

4. Orthocentre of triangle with verticws (0,0), (3,4) and (4,0) is ***(IIT-2003)***
(a) $\left(3, \frac{5}{4}\right)$
(b) (3, 12)
(c) $\left(3, \frac{3}{4}\right)$
(d) (3, 9)

5. The orthocentre of the triangle with vertices $\left(2, \frac{\sqrt{3}-1}{2}\right)$, $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(2, -\frac{1}{2}\right)$ is ***(IIT-1993)***

(a) $\left(\frac{3}{2}, \frac{\sqrt{3}-3}{6}\right)$ (b) $\left(2, -\frac{1}{2}\right)$

(c) $\left(\frac{5}{4}, \frac{\sqrt{3}-2}{4}\right)$ (d) $\left(\frac{1}{2}, -\frac{1}{2}\right)$

6. Locus of centroid of the triangle whose vertices are $(a \cos t, a \sin t)$, $(b \sin t, -b \cos t)$ and $(1, 0)$, where t is a parameter; is ***(AIEEE-2003)***

(a) $(3x-1)^2 + (3y)^2 = a^2 - b^2$
(b) $(3x-1)^2 + (3y)^2 = a^2 + b^2$
(c) $(3x+1)^2 + (3y)^2 = a^2 + b^2$
(d) $(3x+1)^2 + (3y)^2 = a^2 - b^2$

7. If the points (1,1), (–1,–1), $(-\sqrt{3}, \sqrt{3})$ are the vertices of a triangle, then this triangle is ***[MPPET-2004]***

(a) Equilateral (b) Right–angled
(c) Isosceles (d) None of these

8. The area of a triangle whose vertices are (1,–1), (–1,1) and (–1,–1) is given by ***[MPPET-2003]***

(a) 2 (b) 1/2
(c) 1 (d) 3

9. The number of integer values of m, for which the x-coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer, is ***[IIT-2001]***

(a) 2 (b) 0
(c) 4 (d) 1

10. The equation of the straight line passing through the point (4,3) and making intercepts on the co-ordinate axes whose sum is –1 is ***[AIEEE-2004]***

(a) $\frac{x}{2} - \frac{y}{3} = 1$ and $\frac{x}{-2} + \frac{y}{1} = 1$

(b) $\frac{x}{2} - \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$

(c) $\frac{x}{2} - \frac{y}{3} = 1$ and $\frac{x}{2} + \frac{y}{1} = 1$

(d) $\frac{x}{2} + \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$

11. The line parallel to the x-axis and passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$, where $(a, b) \neq (0,0)$ is ***[AIEEE-2005]***

(a) Above the x-axis at a distance of 3/2 from it
(b) Above the x-axis at a distance of 2/3 from it
(c) Below the x-axis at a distance of 3/2 from it
(d) Below the x-axis at a distance of 2/3 from it

12. If x_1, x_2, x_3 and y_1, y_2, y_3 are both in G.P. with the same common ratio, then the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) ***[AIEEE-2003]***

(a) Lie on a straight line
(b) Lie on an ellipse
(c) Lie on a circle
(d) Are vertices of a triangle

13. What is the equation of the line which passes through (4,–5) and is parallel to the line $3x + 4y + 5 = 0$? ***[NDA-2005]***

(a) $3x - 4y - 32 = 0$ (b) $3x + 4y + 8 = 0$
(c) $4x - 3y - 31 = 0$ (d) $3x + 4y - 8 = 0$

14. If the mid point of the section of a straight line intercepted between the axes is (1,1), then what is the equation of this line? ***[NDA-05]***

(a) $2x + y = 3$ (b) $2x - y = 1$
(c) $x - y = 0$ (d) $x + y = 2$

15. Which one of the following is the nearest point on the line $3x - 4y = 25$ from the origin? ***[NDA-2004]***

(a) (–1,–7) (b) (3,–4)
(c) (–5,–8) (d) (3,4)

16. The perpendicular form of the straight line $\sqrt{3}x + 2y = 7$ is ***[NDA-2002]***

(a) $y = -\frac{\sqrt{3}}{2}x + \frac{7}{2}$

(b) $\frac{x}{\frac{7}{\sqrt{3}}} + \frac{y}{\frac{7}{2}} = 1$

(c) $\frac{\sqrt{3}}{\sqrt{7}}x + \frac{2}{\sqrt{7}}y = \sqrt{7}$

(d) $\frac{\sqrt{3}}{\sqrt{7}}x + \frac{2}{\sqrt{7}}y = \sqrt{7}$

17. If β is the acute angle between the lines $px + qy = p + q$; and $p(x - y) + q(x + y) = 2q$, then the value of sin β is ***[NDA-2002]***

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{3}{4}$

(c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

18. The perpendicular segment from the origin to a line is of length 4 units and is inclined to the positive direction of x-axis at an angle of 300. The equation of the line is

[NDA-2001]

(a) $\sqrt{3}x+y=8$ (b) $\frac{\sqrt{3}}{2}x+y=4$

(c) $x+\sqrt{3}y=8$ (d) $x+\frac{\sqrt{3}}{2}y=4$

19. The equation of the line passing through (–1,–2) and having a slope of 4/7 is

[NDA-2001]

(a) $7y + 10 = 4x$ (b) $y=\frac{4}{7}y+\frac{10}{7}$

(c) $x=\frac{4}{7}y+\frac{10}{7}$ (d) $4x + 7y = 10$

20. The angle between the lines $x \cos \alpha + y \sin \alpha = a$ and $x \sin \beta - y \cos \beta = a$ is

[NDA-2001]

(a) $\beta - \alpha$ (b) $\pi + \beta - \alpha$

(c) $\frac{\pi}{2}+\alpha+\beta$ (d) $\frac{\pi}{2}-\beta+\alpha$

21. The perpendicular distance between two parallel lines $3x + 4y - 6 = 0$ and $6x + 8y + 7 = 0$ is

[NDA-2001]

(a) 1/5 units (b) 13/5 units
(c) 19/10 units (d) 1/2 units

22. The locus of the mid-point of the distance between the axes of the variable line $x \cos \alpha + y \sin \alpha = p$, where p is constant, is

(AIEEE-2002)

(a) $x^2 + y^2 = 4p^2$ (b) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{4}{p^2}$

(c) $x^2+y^2=\frac{4}{p^2}$ (d) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{2}{p^2}$

23. The intercepts on the straight line $y = mx$ by the line $y = 2$ and $y = 6$ is less than 5, than m belongs to

[DCE-2004]

(a) $\left]-\frac{4}{3},\frac{4}{3}\right[$ (b) $\left]\frac{4}{3},\frac{3}{8}\right[$

(c) $\left]-\infty,-\frac{4}{3}\right[,\left]\frac{4}{3},\infty\right[$ (d) $\left]\frac{4}{3},\infty\right[$

24. A line passes through (2, 2) and is perpendicular to the line $3x + y = 3$ its y intercept is

[DCE-2001]

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) 1 (d) $\frac{4}{3}$

25. Equation of a straight line passing through the point of intersection of $x - y + 1 = 0$ and $3x + y - 5 = 0$ are perpendicular to one of them is

[DCE-2002]

(a) $x + y + 3 = 0$ (b) $x + y -3 = 0$
(c) $x - 3y - 5 = 0$ (d) $x - 3y + 5 = 0$

26. A point P moves so that its distance from the point $(a, 0)$ is always equal to its distance from the line $x + a = 0$. The locus of the point is

[MPPET-1982]

(a) $y^2 = 4ax$ (b) $x^2 = 4ay$
(c) $y^2 + 4ax = 0$ (d) $x^2 + 4ay = 0$

27. Let $A(2,-3)$ and $B(-2,1)$ be vertices of a triangle ABC. If the centroid of this triangle moves on the line $2x + 3y =1$, then the locus of the vertex C is the line

[AIEEE-2004]

(a) $3x - 2y = 3$ (b) $2x - 3y = 7$
(c) $3x + 2y = 5$ (d) $2x + 3y = 9$

28. If the lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent $(a + b + c \neq 0)$ then

[DCE-01]

(a) $a^3 + b^3 + c^3 - 3abc = 0$
(b) $a = b$
(c) $a = b = c$
(d) $a^2 + b^2 + c^2 - bc - ca - ab = 0$

29. The straight line $5x + 4y = 0$, $x + 2y - 10 = 0$ and $2x + y + 5 = 0$ are

[DCE-1997]

(a) concurrent
(b) the sides of an equilateral triangle
(c) the sides of a right angled triangle
(d) None of these

30. In a triangle ABC, if A is (1, 2) and equation of the medians through B and C are $x + y = 5$ and $x = 4$ respectively, then B is ***[DCE-1996]***

(a) (1, 4)
(b) (7, –2)
(c) (4, 1)
(d) (–2, 7)

TOPICWISE WARMUP TEST 2

1. The line $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 square units. The equation of the circle is ***[IIT-1989; AIEEE-2003, Kerala Engg.-2005]***

(a) $x^2 + y^2 + 2x - 2y = 62$
(b) $x^2 + y^2 - 2x + 2y = 47$
(c) $x^2 + y^2 + 2x - 2y = 47$
(d) $x^2 + y^2 - 2x + 2y = 62$

2. The centre of circle inscribed in square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$, is ***[IIT-2003]***

(a) (4, 7) (b) (7, 4)
(c) (9, 4) (d) (4, 9)

3. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along diameters of a circle of circumference 10π, then the equation of the circle is ***[AIEEE-04]***

(a) $x^2 + y^2 + 2x - 2y - 23 = 0$
(b) $x^2 + y^2 - 2x - 2y - 23 = 0$
(c) $x^2 + y^2 + 2x + 2y - 23 = 0$
(d) $x^2 + y^2 - 2x + 2y - 23 = 0$

4. The equation of the circle with origin as centre passing the vertices of an equilateral triangle whose median is of length $3a$ is ***(AIEEE; 2002)***

(a) $x^2 + y^2 = 9a^2$ (b) $x^2 + y^2 = 16a^2$
(c) $x^2 + y^2 = a^2$ (d) None of these

5. The centres of a set of circles, each of radius 3, lie on the circle $x^2 + y^2 = 25$. The locus of any point in the set is ***(AIEEE-2002)***

(a) $4 \le x^2 + y^2 \le 64$ (b) $x^2 + y^2 \le 25$
(c) $x^2 + y^2 \ge 25$ (d) $3 \le x^2 + y^2 \le 9$

6. If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the y-axis, then the length of PQ is ***(IIT-2002)***

(a) 4 (b) $2\sqrt{5}$
(c) 5 (d) $3\sqrt{5}$

7. If the circles $x^2 + y^2 + 2x + 2ky + 6 = 0$ and $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, then k is ***(IIT-2000)***

(a) 2 or $-\frac{3}{2}$ (b) –2 or $\frac{3}{2}$
(c) 2 or $\frac{3}{2}$ (d) –2 or $-\frac{3}{2}$

8. The equation of the circle concentric to the circle $2x^2 + 2y^2 - 3x + 6y + 2 = 0$ and having area double the area of this circle, is ***[DCE-2006]***

(a) $8x^2 + 8y^2 - 24x + 48y - 13 = 0$
(b) $16x^2 + 16y^2 + 24x - 48y - 13 = 0$
(c) $16x^2 + 16y^2 - 24x + 48y - 13 = 0$
(d) $8x^2 + 8y^2 + 24x - 48y - 13 = 0$

9. One of the diameter of the circle $x^2 + y^2 - 12x + 4y + 6 = 0$ is given by ***[DCE-2006]***

(a) $x + y = 0$ (b) $x + 3y = 0$
(c) $x = y$ (d) $3x + 2y = 0$

10. If $2x - 4y = 9$ and $6x - 12y + 7 = 0$ are common tangents to the circle, then radius of the circle is ***[DCE-2005]***

(a) $\frac{\sqrt{3}}{5}$ (b) $\frac{17}{6\sqrt{5}}$
(c) $\frac{\sqrt{2}}{3}$ (d) $\frac{17}{3\sqrt{5}}$

11. How many common tangents can be drawn to the following circles $x^2 + y^2 = 6x$ and $x^2 + y^2 + 6x + 2y + 1 = 0$? ***[DCE-2004]***

(a) 4 (b) 3
(c) 2 (d) 1

12. The slope of the tangents at the point (h, h) on the circle $x^2 + y^2 = a^2$ is ***[DCE-2003]***

(a) 0 (b) 1
(c) –1 (d) will depend on h

13. The locus of the centres of the circles which cut the circles $x^2 + y^2 + 4x – 6y + 9 = 0$ and $x^2 + y^2 – 5x + 4y + 2 = 0$ orthogonally is ***[DCE-2002]***

(a) $3x + 4y – 5 = 0$
(b) $9x – 10y + 7 = 0$
(c) $9x + 10y – 7 = 0$
(d) $9x – 10y + 11 = 0$

14. The equation of the image of the circle $(x – 3)^2 + (y – 2)^2 = 1$ by the mirror $x + y' = 19$ is ***[DCE-2002]***

(a) $(x – 14)^2 + (y – 13)^3 = 1$
(b) $(x – 15)^2 + (y – 14)^2 = 1$
(c) $(x – 16)^2 + (y – 15)^2 = 1$
(d) $(x – 17)^2 + (y – 16)^2 = 1$

15. The equation of circle whose centre lies on $3x – y – 4 = 0$ and $x + 3y + 2 = 0$ and has an area 154 square units is ***[DCE-2001]***

(a) $x^2 + y^2 – 2x + 2y – 47 = 0$
(b) $x^2 + y^2 – 2x + 2y + 47 = 0$
(c) $x^2 + y^2 + 2x – 2y – 47 = 0$
(d) None of these

16. The locus of the mid-point of the chords of a circle $x^2 + y^2 = 4$, which subtended a right angle at the centre is ***[IIT-1984; MPPET-90; RPET-1997; DCE-2000, 2001]***

(a) $x + y = 2$ (b) $x^2 + y^2 = 1$
(c) $x^2 + y^2 = 2$ (d) $x – y =0$

17. The angle between the tangents drawn from the origin to the circle $(x – 7)^2 + (y + 1)^2 = 25$ is ***[DCE-2000]***

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{8}$

18. The two circles $x^2 + y^2–2x + 6y + 6 = 0$ and $x^2 + y^2– 5x + 6y + 15 = 0$ touch each other the equation of their common tangent is ***[DCE-99]***

(a) $x = 3$
(b) $y = 6$
(c) $7x – 12y – 21 = 0$
(d) $7x + 12y + 21 = 0$

19. The length of the tangent from the point (5,4) to the circle $x^2 + y^2 + 2x – 6y = 6$ is ***[DCE-1999]***

(a) $\sqrt{21}$ (b) $\sqrt{38}$
(c) $2\sqrt{2}$ (d) $2\sqrt{13}$

20. Which one of the following statements is correct ? The circles $x^2 + y^2+ x – 2y = 0$ and $x^2 + y^2 – 2x + y = 0$ ***[NDA-2005]***

(a) Touch each other
(b) Intersect each other
(c) Do not intersect each other
(d) Are concentric

21. If r_1, r_2 and r_3 are the radii of the circles $x^2 + y^2 – 4x + 6y = 5$, $x^2 + y^2 + 6x – 4y = 3$ and $x^2 + y^2 – 2x + 4y = 8$, respectively, then ***[NDA-2002]***

(a) $r_1> r_2 > r_3$
(b) $r_2> r_3 > r_1$
(c) $r_3> r_1 > r_2$
(d) $r_1> r_3 > r_2$

22. The coordinates of the centre and the radius of the circle $x^2 + y^2 + 4x – 6y – 36 = 0$ are, respectively, given by ***[NDA-2001]***

(a) (–4,6) and 6 (b) (4,–6) and 7
(c) (2,–3) and 6 (d) (–2,3) and 7

23. An equilateral triangle is inscribed in the circle $x^2 + y^2 = a^2$ with one of the vertices at $(a, 0)$. What is the equation of the side opposite to this vertex? ***[NDA-2006]***

(a) $2x – a = 0$ (b) $x + a = 0$
(c) $2x + a = 0$ (d) $3x – 2a = 0$

24. What is the radius of the circle passing through the points (0, 0), $(a, 0)$ and $(0, b)$? ***[NDA-2006]***

(a) $\sqrt{a^2 - b^2}$ (b) $\sqrt{a^2 + b^2}$
(c) $\frac{1}{2}\sqrt{a^2 + b^2}$ (d) $2\sqrt{a^2 + b^2}$

25. The radius of any circle touching the lines $3x – 4y + 5 = 0$ and $6x – 8y – 9 = 0$ is ***[MPPET-2005]***

(a) 1.9 (b) 0.95
(c) 2.9 (d) 1.45

26. If two circles of the same radius r and centres at (2, 3)and (5,6) respectively, then the value of r is *[MPPET-2005]*
(a) 3 (b) 2
(c) 1 (d) 5

27. The limiting points of the system of circles represented by the equation $2(x^2 + y^2) + \lambda x + 9/2 = 0$ are *[MPPET-2004]*
(a) $\left(\pm\frac{3}{2},0\right)$
(b) (0,0) and $\left(\frac{9}{2},0\right)$
(c) $\pm\left(\frac{9}{2},0\right)$
(d) (±3, 0)

28. The value of c, for which the line $y = 2x + c$ is a tangent to the circle $x^2 + y^2 = 16$, is *[MPPET-2004]*
(a) $-16\sqrt{5}$ (b) $4\sqrt{5}$
(c) $16\sqrt{5}$ (d) 20

29. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y = 24$ is *[IIT-1998]*
(a) 0 (b) 1
(c) 3 (d) 4

30. The triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinate (3, 4) and (–4, 3) respectively, then $\angle QPR$ is equal to *[IIT-2000]*
(a) $\pi/2$ (b) $\pi/3$
(c) $\pi/4$ (d) $\pi/6$

TOPICWISE WARMUP TEST 3

1. The focus of the parabola $x^2 + 2y + 6x = 0$ is
(a) (–3, 4) (b) (3, 4)
(c) (3, –4) (d) (–3, –4)

2. The equation of the parabola with vertex at the origin and directrix $y = 2$ is
(a) $y^2 = 8x$ (b) $y^2 = -8x$
(c) $y^2 = \sqrt{8}x$ (d) $x^2 = -8y$

3. The equation of the parabola whose focus is (3, 0) and directrix is $x + 2 = 0$ will be
(a) $y^2 = 5(2x - 1)$ (b) $x^2 = 5(2y - 1)$
(c) $y^2 = 5(2x + 1)$ (d) $x^2 = 5(2y + 1)$

4. If (2, 0) and (5, 0) are respectively the vertex and focus of a parabola, then its equation will be
(a) $y^2 = 12x + 24$ (b) $y^2 = 12x - 24$
(c) $y^2 = -12x - 24$ (d) $y^2 = -12x + 24$

5. The equation of the axis of the parabola $x^2 - 4x - 3y + 10 = 0$ is
(a) $x + 2 = 0$ (b) $y + 2 = 0$
(c) $x - 2 = 0$ (d) $y - 2 = 0$

6. The focus of the parabola $(y - 2)^2 = 20(x + 3)$ is
(a) (–3, 2) (b) (3, –2)
(c) (2, –3) (d) (2, 2)

7. The equation of the parabola with focus (0, 0) and directrix $x + y = 4$ will be
(a) $x^2 + y^2 - 2xy + 8x + 8y - 16 = 0$
(b) $x^2 + y^2 - 2xy + 8x + 8y = 0$
(c) $x^2 + y^2 + 8x + 8y - 16 = 0$
(d) $x^2 - y^2 + 8x + 8y - 16 = 0$

8. The latus rectum of a parabola is a line
(a) through the focus
(b) parallel to the directrix
(c) perpendicular to the axis
(d) all of these

9. The focal distance of a point P on the parabola $y^2 = 12x$, if the ordinate of P is 6, is
(a) 4 (b) 5
(c) 6 (d) None of these

10. The eccentricity 'e' of a parabola is
(a) < 1 (b) > 1
(c) = 1 (d) 0

11. The parabola $y^2 = 4ax$ passes thro' the point (2, –6), then the length of its latus rectum is
(a) 18 (b) 9
(c) 6 (d) 16

12. In a parabola semi-latus rectum is the harmonic mean of the

(a) segments of a focal chord
(b) segments of the directrix
(c) segments of a chord
(d) None of these

13. The equation of the ellipse with respect to coordinate axes whose minor axis is equal to the distance between its foci and whose LR = 10, will be
(a) $2x^2 + y^2 = 100$
(b) $x^2 + 2y^2 = 100$
(c) $2x^2 + 3y^2 = 80$
(d) None of these

14. The eccentricity of the conic represented by the equation $x^2 + 2y^2 - 2x + 3y + 2 = 0$ is
(a) 0 (b) 1/2
(c) $1/\sqrt{2}$ (d) $\sqrt{2}$

15. The eccentricity of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is
(a) 2 (b) $2/\sqrt{3}$
(c) 4 (d) 4/3

16. If eccentricities of a hyperbola and its conjugate hyperbola are e and e' respectively, then $\frac{1}{e^2}+\frac{1}{e'^2}$ is equal to
(a) 1 (b) 2
(c) 0 (d) 3

17. The centre of the hyperbola $9x^2 - 16y^2 - 18x - 32y = 151$ is
(a) (1, 1) (b) (–1, 1)
(c) (1, –1) (d) (–1, –1)

18. The equation of the conic with focus at (1, –1) directrix along $x - y + 1 = 0$ and eccentricity $\sqrt{2}$ is
(a) $xy = 1$
(b) $x^2 - y^2 = 1$
(c) $2xy - 4x + 4y + 1 = 0$
(d) $2xy + 4x - 4y - 1 = 0$

19. If distance between foci of an ellipse is 16 and its eccentricity is 1/2, then length of its major axis will be
(a) 64 (b) 8
(c) 32 (d) 16

20. Eccentricity of the conic $16x^2 + 7y^2 = 112$ is
(a) 3/4 (b) 4/3
(c) 7/16 (d) $3/\sqrt{7}$

21. If the foci are at the points $S(2, 0)$ and $S'(-2, 0)$ and latus rectum is 6, then the equation of the ellipse in the standard form is
(a) $\frac{x^2}{16}+\frac{y^2}{12}=1$ (b) $\frac{x^2}{4}+\frac{y^2}{3}=1$
(c) $\frac{x^2}{4}+\frac{y^2}{12}=1$ (d) None of these

22. The latus rectum of the conic $3x^2 + 4y^2 - 6x + 8y - 5 = 0$ is
(a) 3 (b) $\sqrt{3}/2$
(c) $2/\sqrt{3}$ (d) None of these

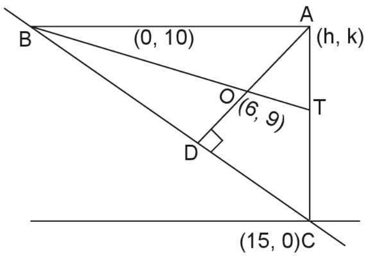
23. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents an ellipse if
(a) $\Delta = 0, h^2 < ab$ (b) $\Delta \neq 0, h^2 < ab$
(c) $\Delta \neq 0, h^2 > ab$ (d) $\Delta \neq 0, h^2 = ab$

24. The equation of the ellipse with foci at (± 3, 0) and vertices at (± 5, 0) is
(a) $\frac{x^2}{9}+\frac{y^2}{25}=1$ (b) $\frac{x^2}{25}+\frac{y^2}{9}=1$
(c) $\frac{x^2}{16}+\frac{y^2}{25}=1$ (d) $\frac{x^2}{25}+\frac{y^2}{16}=1$

25. The foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide, then the value of b^2 is
(a) 1 (b) 5
(c) 7 (d) 9

TOPICWISE WARMUP TEST 1: SOLUTIONS

1. (d): Let third vertex be $A(h, k)$ then $AD \perp BC$ through (6, 9)
$\Rightarrow$ $OA \perp BC$ (O is orthocentre)
$\Rightarrow$ $m_1 \times m_2 = -1$
(m_1 = slope of BC, m_2 = slope of OA)
$\Rightarrow$ $\left(-\frac{10}{15}\right)\left(\frac{k-9}{h-6}\right)=-1$



$\Rightarrow \quad 2k - 18 = 3h - 18$

$\Rightarrow \quad 2k = 3h$ ....(I)

Similarly, $BT \perp AC \Rightarrow BO \perp AC$

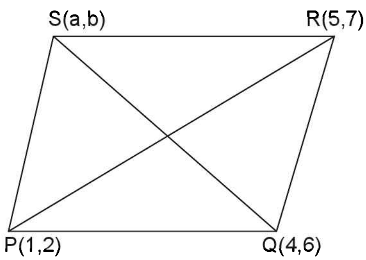
$\Rightarrow \quad m_1 \times m_2 = -1$ (m_1 = slope of BO, m_2 = slope of AC)

$$\Rightarrow \quad \left(-\frac{1}{6}\right)\left(\frac{k}{h-15}\right) = -1$$

$\Rightarrow \quad k = 6h - 90$ ....(II)

By (I) and (II) $\therefore h = 20, k = 30.$

2. (c) Diagonals cut each other at middle points.



Hence $\dfrac{a+4}{2} = \dfrac{1+5}{2} \Rightarrow a = 2,$

$\dfrac{b+6}{2} = \dfrac{2+7}{2} \Rightarrow b = 3$

3. (a) Let (4, 0), B (–1, 1), C (3, 5) then $AB = \sqrt{26}$, $AC = \sqrt{26}$ and $BC = \sqrt{52}$ i.e., $AB = AC$.

Hence triangle is isosceles, also, $(BC)^2 = AB^2 + AC^2$.

Hence triangle is right angled also .

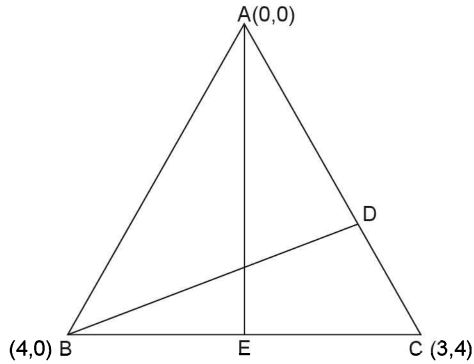
4. (c) $BD \perp AC$. Slope of $BD = -\dfrac{3}{4}$

Equation of BD, $3x + 4y - 12 = 0$;

$AE \perp BC$ .... (i)

Slope of $AE = \dfrac{1}{4}$, Equation of AE,

$x - 4y = 0$ .... (ii)



From equation (i) and (ii), $x = 3$, $y = \dfrac{3}{4}$ orthocentre of the triangle is $\left(3, \dfrac{3}{4}\right)$.

5. (b) Here the given triangle is a right angled triangle at the vertex $\left(2, -\dfrac{1}{2}\right)$. Hence the orthocentre is at $\left(2, -\dfrac{1}{2}\right)$.

6. (b) $3h = a \cos t + b \sin t + 1$, $3k = a \sin t - b \cos t$ $a^2 + b^2 = (3h-1)^2 + (3k)^2 = (3x-1)^2 + (3y)^2 = a^2 + b^2$.

7. (b) Given, vertices of triangle are $A(1,1)$, $B(-1,-1)$ and $C(-\sqrt{3}, \sqrt{3})$

Now,

$$AB = \sqrt{(-1-1)^2 + (-1-1)^2} = \sqrt{4+4} = \sqrt{8}$$

$$BC = \sqrt{(-\sqrt{3}+1)^2 + (\sqrt{3}+1)^2} = \sqrt{8}$$

$$CA = \sqrt{(1+\sqrt{3})^2 + (1-\sqrt{3})^2} = \sqrt{8}$$

i.e., $AB = BC = CA$. So, it is an equilateral triangle

8. l (a) $\Delta = \dfrac{1}{2}$ [1 (1 + 1) – 1 (–1 + 1) –1 (–1 – 1)] = 2.

9. (a) $\left.\begin{array}{r} 3x + 4y = 9 \\ y = mx + 1 \end{array}\right\}$ solving for x, $x = \dfrac{5}{3+4m}$

Now, for x to be an integer, $3 + 4m = \pm 5$ or ± 1

The integral values of m satisfying these conditions are –2 and –1. Therefore (a) is the answer.

10. (a) Here $a + b = -1$. Required line is

$$\frac{x}{a} - \frac{y}{1+a} = 1 \qquad \text{........... (i)}$$

Since line (i) passes through (4,3)

$\therefore \quad \frac{4}{a} - \frac{3}{1+a} = 1$

$\Rightarrow \quad 4 + 4a - 3a = a + a^2$

$\Rightarrow \quad a^2 = 4 \Rightarrow a = \pm 2$

$\therefore$ Required line are $\frac{x}{2} - \frac{y}{3} = 1$

and $\frac{x}{-2} + \frac{y}{1} = 1$.

11. (c) The lines passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$ is $ax + 2by + .b + \lambda\,(bx - 2ay - 3a) = 0$

$\Rightarrow (a + b\lambda)x + (2b - 2a\lambda)y + 3b - 3\lambda a = 0$ ....(i)

Line (i) is parallel to x-axis,

$\therefore \quad a + b\lambda = 0$

$\Rightarrow \lambda = \frac{-a}{b} = 0$ Put the value of λ in (i)

$$ax + 2by + 3b - \frac{a}{b}(bx - 2ay - 3a) = 0$$

$$\Rightarrow \quad y\left(2b + \frac{2a^2}{b}\right) + 3b + \frac{3a^2}{b} = 0$$

$$\Rightarrow \quad y\left(\frac{2b^2 + 2a^2}{b}\right) = -\left(\frac{3b^2 + 3a^2}{b}\right)$$

$$y = \frac{-3(a^2 + b^2)}{2(b^2 + a^2)} = \frac{-3}{2}, \ y = -\frac{3}{2}.$$

So, it is 3/2 unit below x-axis.

12. (a) Taking coordinates as $\left(\frac{3}{r}, \frac{y}{r}\right)$; ; (x,y) and (xr, yr)

Above co-ordinates satisfy the relation $y = mx$, so lie on a straight line.

13. (b) $3x + 4y + 5 = 0 \therefore \ y = -\frac{3x}{4} - \frac{5}{4}$

$\because \quad m$ for the given line $= -3/4$

$\therefore \quad m$ for the required line $= -3/4$

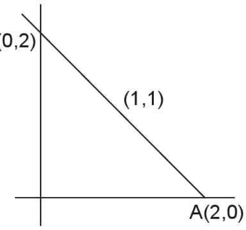
$\therefore$ Equation for the required parallel line is

$\Rightarrow \quad y + 5 = \frac{-3}{4}(x - 4)$

$\Rightarrow \quad 4y + 20 = -3x + 12$

$\therefore \quad 3x + 4y + 8 = 0$

14. (d) P is the mid-point of AB and $P = (1,1)$



$\therefore \quad A = (2,0)$ and $B = (0,2)$

$\therefore$ Equation of the line AB is $\frac{x}{2} + \frac{y}{2} = 1$

$\Rightarrow \quad x + y = 2$

15. (b) The required point is the foot of the perpendicular from the origin on the line $3x - 4y = 25$. The equation of a line passing through the origin and perpendicular to $3x - 4y = 25$ is $4x + 3y = 0$

Solving the equations, $3x - 4y = 25$ ...... (i),

$4x + 3y = 0$ .... (ii)

$\therefore \quad x = 3$ and $y = -4$

$\therefore$ Required point is $(3, -4)$ which is nearest point.

16. (c) Given $\sqrt{3}x + 2y = 7$

Dividing by, $\sqrt{3+4} = \sqrt{7}$, we get

$\frac{\sqrt{3}}{\sqrt{7}}x + \frac{2}{\sqrt{7}}y = \sqrt{7}$ is normal form or perpendicular form of the straight line.

17. (d) True for all values of p and q

$\therefore$ take $p = 1, q = 0$.

Now given line s are as follows :

$x = 1$ ....(i)

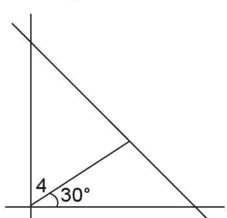
$x - y = 0$ ....(ii)

$$\cos\beta = \frac{(1)(0) + (0)(-1)}{\sqrt{1^2 + 0^2}\sqrt{1+1}} = \frac{-1}{\sqrt{2}}$$

$\cos\beta = \cos 135° \Rightarrow \theta = 135°$

$\therefore \quad \sin\beta = \sin 135° = \frac{1}{\sqrt{2}}$

18. (a) $\alpha = 30°$ and $p = 4$



∴ Required equation is $x \cos 30° + y \sin 30° = 4$

$\Rightarrow \quad \frac{x\sqrt{3}}{2} + \frac{y+1}{2} = 4$

$\Rightarrow \quad x\sqrt{3} + y = 8$

19. (a) ∵ $y - y_1 = m(x - x_1)$.

Here, $x_1 = -1, y_1 = -2$ and $m = \frac{4}{7}$

∴ Required equation of line is

$$y + 2 = \frac{4}{7}(x+1)$$

$\Rightarrow \quad 7y + 14 = 4x + 4$

$\Rightarrow \quad 7y + 10 = 4x$

20. (d) $x \cos \alpha + y \sin \alpha = a$

$\Rightarrow \quad y = \tan \beta . x - a \sec \beta$

∴ $m_2 = \tan \beta$ y $= x \cot \alpha + a \operatorname{cosec} \alpha$

∴ $m_1 = -\cot \alpha$ and $x \sin \beta - y \cos \beta = a \Rightarrow$ y

If θ be the angle between the given lines, then

$$\tan\theta = \frac{-\cot\alpha - \tan\beta}{1 - \cot\alpha . \tan\beta}$$

$$= \frac{\frac{1+\tan\alpha\tan\beta}{\tan\alpha}}{\frac{\tan\alpha - \tan\beta}{\tan\alpha}} = -\frac{1+\tan\alpha\tan\beta}{\tan\alpha - \tan\beta}$$

$$= -\cot(\alpha - \beta) = \tan\left[\frac{\pi}{2} + (\alpha - \beta)\right]$$

∴ $\theta = \frac{\pi}{2} - \beta + \alpha$

21. (c) The distance between two parallel lines

$$= \frac{6 - \left(-\frac{7}{2}\right)}{5} = \frac{6 + \frac{7}{2}}{5} = \frac{19}{10} \text{ units.}$$

22. (b) The straight line $x \cos \alpha + y \sin \alpha = p$ meets the x-axis at the point $A\left(\frac{p}{\cos\alpha}, 0\right)$ and the y-axis at the point $B\left(0, \frac{p}{\sin\alpha}\right)$. Let (h, k) be the coordinates of the middle point of the line segment AB.

Then, $h = \frac{p}{2\cos\alpha}$ and $k = \frac{p}{2\sin\alpha}$

$\Rightarrow \quad \cos\alpha = \frac{p}{2h}$ and $\sin\alpha = \frac{p}{2k}$

$\Rightarrow \quad \sin^2\alpha + \cos^2\alpha = \frac{p^2}{4h^2} + \frac{p^2}{4k^2} = 1$

Hence locus of the point (h, k) is

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}.$$

23. (c) We have $y = mx$, $y = 2$ and $y = 6$

Let the point of intersection of line $y = mx$ by the lines $y = 2$ and $y = 6$ are $p(2/m, 2)$ and $Q(6/m, 6)$

∴ Intercepts on $y = mx$ is

$$PQ = \sqrt{\left(\frac{2}{m} - \frac{6}{m}\right)^2 + (2-6)^2} \; PQ < 5$$

(given)

∴ $\sqrt{\left(\frac{2}{m} - \frac{6}{m}\right)^2 + 16} < 5$

$\Rightarrow \quad \left(\frac{2}{m} - \frac{6}{m}\right)^2 + 16 < 25$

$\Rightarrow \quad -3 < \frac{2}{m} - \frac{6}{m} < 3$

$\Rightarrow \quad -\frac{4}{3} > m > \frac{4}{3}$

∴ $m \in \left]-\infty, -\frac{4}{3}\right[\cup \left]\frac{4}{3}, \infty\right[$

24. (d) The line perpendicular to given line $3x + y = 3$ is given by, $x - 3y = \lambda$, which passes through the points (2, 2) i.e., $2 - 6 = \lambda$

$\Rightarrow \lambda = -1$

∴ $x - 3y = -4$ or $x - 3y + 4 = 0$ ...(i)

compare (i) with $ax + by + c = 0$, we get if

intercept $= -\frac{c}{b} = \frac{-4}{-3} = \frac{4}{3}$

25. (d) Equation of any line through the point of intersection of the given lines in $(3x + y - 5) + \lambda(x - y + 1) = 0$ since this lines is perpendicular to one of the given lines $\frac{3+\lambda}{\lambda - 1} = -1$ or $\frac{1}{3}$

$\Rightarrow \quad \lambda = -1$ or -5

Therefore, the required straight line is $x + y - 3 = 0$ or $x - 3y + 5 = 0$

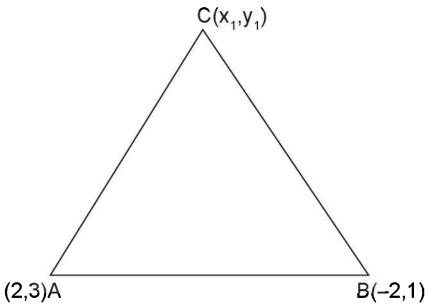
26. (a) $(x - a)^2 + y^2 = (x + a)^2$

$\Rightarrow \quad y^2 = 4ax.$

27. (d) Let the third vertex be (x_1, y_1) then

Centroid (G) $= \left(\dfrac{x_1 + 2 - 2}{2}, \dfrac{y_1 - 3 + 1}{3}\right)$

i.e., $G\left(\dfrac{x_1}{3}, \dfrac{y_1 - 2}{3}\right)$



given centroid of triangle moves on the line $2x + 3y = 1$

$\therefore \quad 2\left(\dfrac{x_1}{3}\right) + 3\left(\dfrac{y_1 - 2}{3}\right) = 1$ i.e., $2x_1 + 3y_1 = 9$

Locus of (x_1, y_1) is $2x + 3y = 9$

28. (c) Since $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$\Rightarrow \quad 3abc - a^3 - b^3 - c^3 = 0$

$\Rightarrow \quad -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$ as $a + b + c = 0$

$\therefore \quad a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\dfrac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$

As a, b, c are real number

$\therefore \quad b - c = 0, c - a = 0, a - b = 0, a = b = c.$

29. (a) Given straight lines are, $5x + 4y = 0$ ...(i)

$x + 2y - 10 = 0$ ....(ii)

and $2x + y + 5 = 0$ ....(iii)

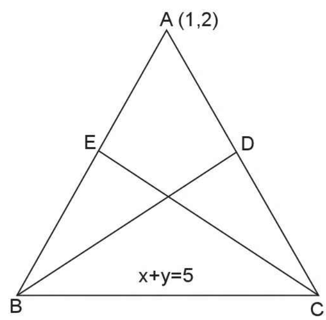
$$D = \begin{vmatrix} 5 & 4 & 0 \\ 1 & 2 & -10 \\ 2 & 1 & 5 \end{vmatrix}$$

$\Rightarrow \quad 5(10 + 10) - 4(5 + 20) + 0$

$\Rightarrow \quad 5 \times 20 - 4 \times 25 = 100 - 100 = 0$

Since the value of determinant of coefficient and constant is zero, therefore the given lines are concurrent.

30. (b) Let coordinate of B is (x_1, y_1).
Let point E is $(4, y_2)$ $A = (1, 2)$ (given)



E is the mid-point of $AB = \dfrac{1 + x_1}{2} = 4$

$\Rightarrow \quad x_1 = 7$ Point (x_1, y_1) lies on the line $x + y = 5$

$\Rightarrow \quad 7 + y_1 = 5$

$\Rightarrow \quad y_1 = -2.$

Therefore, point is $(7, -2)$

TOPICWISE WARMUP TEST 2: SOLUTIONS

1. (b) Centre of circle = Point of intersection of diameters = $(1, -1)$

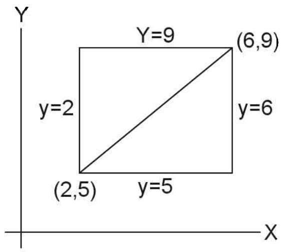
Now area $= 154 \Rightarrow \pi r^2 = 154 \Rightarrow r = 7$

Hence the equation of required circle is

$(x - 1)^2 + (y + 1)^2 = 7^2$

$\Rightarrow \quad x^2 + y^2 - 2x + 2y = 47.$

2. (a) Centre is (4, 7) .



3. (d) According to question two diameters of the circle are $2x + 3y + 1 = 0$
and $3x - y - 4 = 0$
Solving, we get $x = 1, y = -1$
$\therefore$ Centre of the circle is $(1, -1)$
Given $2\pi r = 10\pi \Rightarrow r = 5$
Required circle is $(x - 1)^2 + (y + 1)^2 = 5^2$
or $x^2 + y^2 - 2x + 2y - 23 = 0$.

4. (d) Centre (0,0), radius $= 3a \times \frac{2}{3} = 2a$.
Hence circle $x^2 + y^2 = 4a^2$ as centroid divides median in the ratio of 2 : 1 .

5. (a) Let (h, k) be any point in the set, then equation of circle is $(x - h)^2 + (y - k)^2 = 9$
But (h, k) lies on $x^2 + y^2 = 25$, then $h^2 + k^2 = 25$
$\therefore$ $2 \le$ Distance between the two circle ≤ 8
$2 \le \sqrt{h^2 + k^2} \le 8 \Rightarrow 4 \le (h^2 + k^2) \le 64$.
$\therefore$ Locus of (h, k) is $4 \le x^2 + y^2 \le 64$.

6. (c) Let $P = (x_1, y_1)$. The tangent at P is $xx_1 + yy_1 + 3(x + x_1) + 3(y + y_1) - 2 = 0$ ....(i)
Coordinates of Q satisfy (i), $5x - 2y + 6 = 0$, $x = 0$.
So, $3x_1 + 6y_1 + 7 = 0$ and $Q = (0, 3)$
$\therefore$ $PQ^2 = x_1^2 + (y_1 - 3)^2 = x_1^2 + y_1^2 - 6y_1 + 9$
$= 11 - 6x_1 - 12y_1$, ($\because x_1^2 + y_1^2 + 6x_1 + 6y_1 - 2 = 0$)
$= 11 - 2(3x_1 + 6y_1) = 11 - 2(-7) = 25$.
So, $PQ = 5$.

7. (a) $2gg' + 2ff' = c + c'$ i.e., $2.1.0 + 2.k.k. = 6 + k$
or $2k^2 - k - 6 = 0$ or $(2k + 3)(k - 2) = 0$
$\therefore$ $k = 2, -3/2$.

8. (c) The equation of given circle is $x^2 + y^2 - \frac{3}{2}x + 3y + 1 = 0$
whose centre is $\left(\frac{3}{4}, -\frac{3}{2}\right)$ and radius,

$$r = \sqrt{\frac{9}{16} + \frac{9}{4} - 1}$$
$$= \sqrt{\frac{9 + 36 - 16}{16}} = \sqrt{\frac{29}{16}}$$

$\therefore$ Area of circle $= \pi r^2 = \pi\left(\frac{29}{16}\right) = \frac{29\pi}{16}$

$\Rightarrow$ Area of required circle $= 2 \times \frac{29\pi}{16} = \frac{29\pi}{8}$

Let R be the radius of required circle.

$\therefore$ $R^2 = \frac{29}{8}$. Now equation of circle is

$$\left(x - \frac{3}{4}\right)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{29}{8}$$

$\Rightarrow$ $x^2 - \frac{3x}{2} + \frac{9}{16} + y^2 + 3y + \frac{9}{4} - \frac{29}{8} = 0$

$\Rightarrow$ $16x^2 + 16y^2 - 24x + 48y - 13 = 0$.

9. (b) The cordinates of the centre of the circle $x^2 + y^2 - 12x + 4y + 6 = 0$ are $(6, -2)$.
Clearly the line $x + 3y = 0$ passes through this point.
Hence, $x + 3y = 0$ is a diameter of the given circle.

10. (b) Given that $2x - 4y = 9$
and $6x - 12y + 7 = 0$
or $2x - 4y + \frac{7}{3} = 0$
Therefore these are parallel and at the end points of diameter of the circle.

$\therefore$ $d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}} = \frac{\left|-9 - \frac{7}{3}\right|}{\sqrt{2^2 + (-4)^2}}$

$$= \frac{\left|-\frac{34}{3}\right|}{\sqrt{20}} = \frac{34}{3.2\sqrt{5}} = \frac{17}{3\sqrt{5}}$$

Since the distance between these two tangents is equal to the diameter of the circle.

$\therefore$ diameter of circle $= \frac{17}{3\sqrt{5}}$

Hence radius of circle $= \frac{17}{6\sqrt{5}}$.

11. (a) We have, $x^2 + y^2 = 6x$ ....(i)

and $x^2 + y^2 + 6x + 2y + 1 = 0$ ....(ii)

$\therefore$ Centres and radii of both circles are $C_1(3,0)$, 3 and $C_2(-3,-1)$, 3 respectively

Now, for common tangents, we know $C_1C_2 = r_1 + r_2$

$\therefore\ C_1C_1 = \sqrt{(3+3)^2 + (0+1)^2} = \sqrt{36+1} = \sqrt{37}$

and $r_1 + r_2 = 3 + 3 = 6$

$\Rightarrow\ C_1C_2 > r_1 + r_2$

$\therefore$ Number of common tangents are 4.

12. (c) Given that $x^2 + y^2 = a^2$

$\Rightarrow\ y^2 = a^2 - x^2$

Differentiate w.r.t x, $2y\dfrac{dy}{dx} = -2x$

$\Rightarrow\ \dfrac{dy}{dx} = -\dfrac{x}{y}$

$\therefore$ Slope at point (h,h) is $\left(\dfrac{dy}{dx}\right)$

at $(h,h) = -\dfrac{h}{h} = -1$

13. (b) Let circle $x^2 + y^2 + 2gx + 2fy + c = 0$ according to question $2g \times 2 + 2f \times -3 = c + 9$

$4g - 6f = c + 9$ ....(i)

and $2g \times \dfrac{-5}{2} + 2f \times 2 = c + 2,\ -5g + 4f = c + 2$

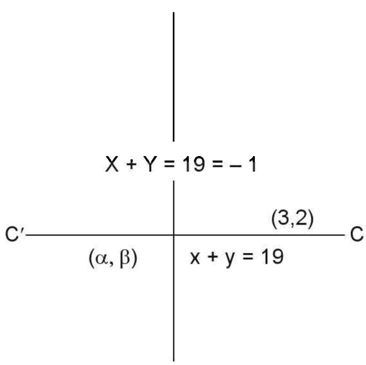
...............(ii)

Substracting (ii) from (i), $9g - 10f = 7$

$\Rightarrow\ -9(-g) + 10(-f) = 7$

$\therefore$ Locus of centre is $-9x + 10y = 7$ or $9x - 10y + 7 = 0$

14. (d) The image of the circle has same radius but centre different. If centre is (α, β) then slope of CC_1 × slope of $(x + y = 19) = -1$



$\dfrac{\beta-2}{\alpha-3} \times -1 = -1,\ \beta - 2 = \alpha - 3,\ \alpha - \beta = 1$ ...(i)

and mid-point of CC_1 lie on $n + y = 19$ then

$\dfrac{\alpha+3}{2} + \dfrac{\beta+2}{2} = 19,\ \alpha + \beta = 33$ ....(ii)

From (i) and (ii) we get $\alpha = 17$ and $\beta = 16$.

$\therefore$ Required circle is $(x - 17)^2 + (y - 16)^2 = 1$

15. (a) Point of intersection of $(3x - y - 4) = 0$

$x + 3y = -2,\ 9x - 3y = 12$ ....(i)

$x + 3y = -2$ ....(ii)

substracting the equation (i) and (ii), we get $10x = 10$

$\Rightarrow\ x = 1;\ 1 + 3y = -2,\ 3y = -3,\ y = -1$

Centre is $(1, -1)$, Area of the circle is πr^2

$154 = \pi r^2$

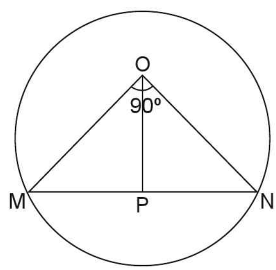
$\Rightarrow\ r^2 = \dfrac{154}{\pi} = \dfrac{154}{22} \times 7 = \dfrac{154}{22} \times 7 \Rightarrow r = 7$

Equation of the circle is $(x - 1)^2 + (y + 1)^2 = 7^2$

$x^2 + y^2 - 2x + 2y + 2 = 49,$

$x^2 + y^2 - 2x + 2y - 47 = 0$

16. (c) Circle is $x^2 + y^2 = 4$



Let MN is the one chord and P is the mid-point and the co-ordinate of P is (h,k) $OM = ON =$ radius $= 2$

$\angle MON = 910° \Rightarrow \angle MOP = \angle PON = 45°$

In ΔMOP, $OP = OM \cos 45 = 2 \times \dfrac{1}{\sqrt{2}} = \sqrt{2}$

$\Rightarrow\ OP = \sqrt{2}$ ....(i)

Centre of circle O is (0,0) and P is (h,k)

$OP = \sqrt{(h-0)^2 + (k-0)^2},$

$OP = \sqrt{h^2 + k^2}$ ......(ii)

By equation (i) and (ii), $\sqrt{h^2 + k^2} = \sqrt{2},$

$h^2 + k^2 = 2$

$\Rightarrow$ Locus of (h, k) is $x^2 + y^2 = 2$

17. (c) Given circle is $(x-7)^2+(y+1)^2=25$ then centre is (7,–1), radius $r=5$ Let $y=mx$

$\Rightarrow \quad mx-y=0$ is the tangent of circle

Therefore, length of perpendicular from centre is equal to the radius of circle, $\dfrac{7m+1}{\sqrt{1+m^2}}=\pm 5$

$\Rightarrow \quad (7m+1)^2=25(1+m^2)$

$\Rightarrow \quad 49m^2+1+14m=25m^2+25$

$\Rightarrow \quad 12m^2+7m-12=0$

$\Rightarrow \quad (3m+4)(4m-3)=0$

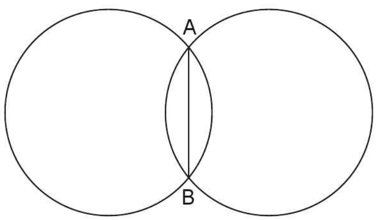
$m_1=-\frac{4}{3},\ m_2=\frac{3}{4},$ Here $m_1\times m_2=-1$

Therefore, both the tangent will be perpendicular, and angle between them is $\frac{\pi}{2}$

18. (a) Equation of common chord is $S_1-S_2=0$

$=(x^2+y^2-2x+6y+6)-(x^2+y^2-5x+6y+15)=0$

$\Rightarrow \quad 3x-9=0, \Rightarrow x=3$



This equation is also the common tangent of the given circles because when circles are touch each other, then common chord (AB) will also be the common tangent.

19. (a) Given form is (5,4), Circle is $x^2+y^2+2x-6y=6$

Length of tangent

$=\sqrt{S_1}=\sqrt{5^2+4^2+2\times 5-6\times 4-6}$

$=\sqrt{25+16+10-24-6}=\sqrt{21}$

20. (b) $x^2+y^2+x-2y=0$ ....(i)

and $x^2+y^2-2x+y=0$ ....(ii)

On solving these two equations, we get $x-y=0$ and $x=y=0$ and 1/2

Hence these two circles intersect each other at the points (0,0) and $\left(\frac{1}{2},\frac{1}{2}\right)$

21. (a) Equation of circles are

$x^2+y^2-4x+6y=5$ ....(i),

$x^2+y^2+6x-4y=3$ ....(ii)

and $x^2+y^2-2x+4y=8$ ....(iii)

$\because \quad r=\sqrt{g^2+f^2-c}$

$\therefore \quad r_1=\sqrt{4+9+5}=\sqrt{18}$

$r_2=\sqrt{9+4+3}=\sqrt{16},$

$r_3=\sqrt{1+4+8}=\sqrt{13}$

Hence, $r_1>r_2>r_3$

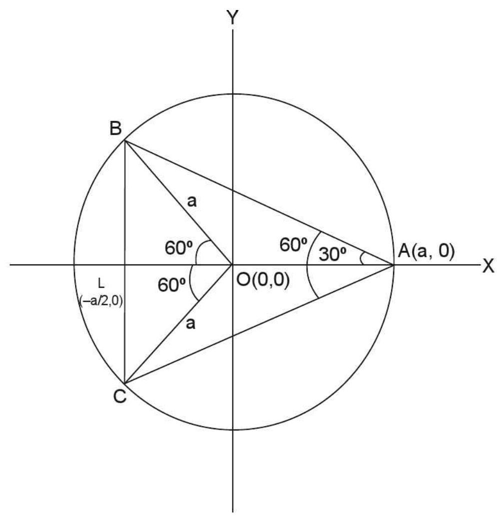
22. (d) $x^2+y^2+4x-6y-36=0$

$\therefore \quad g=2, f=-3$ and $c=-36$

$\therefore$ Centre = (–2,3) and Radius

$=\sqrt{4+9+36}=7.$

23. (c) $\angle BAX=180°-30°=150°$



Central angle = 2 × circumference angle

In ΔOLB, cos 60°

$=\cos 60°=\frac{OL}{a}\Rightarrow \frac{1}{2}=\frac{OL}{a}\Rightarrow OL=\frac{a}{2}$

BC is a straight line passing through point L $(-a/2, 0)$ and parallel to y-axis $\therefore$ its equation is $x=-a/2 \Rightarrow 2x+a=0$

24. (c) Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$

$\therefore$ The point will satisfy the given equation of the circle.

$\therefore \quad c=0$ ....(i), $a^2+2ga+c=0$.....(ii) and $b^2+2fb+c=0$ ...(iii) then from equation (i), (ii) and (iii), $g-\frac{a}{2},\ f=-\frac{b}{2}$

$\therefore \quad$ centre $=\left(\frac{a}{2},\frac{b}{2}\right)$

$\therefore$ The radius of the circle = The distance of any point or the circle from the angle

$$= \sqrt{\left(0-\frac{a}{2}\right)^2 + \left(0-\frac{b}{2}\right)^2}$$

$$= \sqrt{\frac{a^2}{4}+\frac{b^2}{4}} = \frac{1}{2}\sqrt{a^2+b^2}$$

25. (b) Distance between the lines $3x - 4y + 5 = 0$ and $6x - 8y - 9 = 0$ is $\left|\frac{5+\frac{9}{2}}{\sqrt{3^2+4^2}}\right| = \left(\frac{19}{10}\right) = 1.9$

1.9 is the diameter of circle

$\therefore$ Radius of circle $= \frac{1.9}{2} = 0.95$

26. (a) The equation of the circle are $(x-2)^2 + (y-3)^2 = r^2$

$\Rightarrow x^2 + y^2 - 4x - 6y - r^2 + 13 = 0$ ......(i)

and $(x-5)^2 + (y-6)^2 = r^2$

$\Rightarrow x^2 + y^2 - 10x - 12y - r^2 + 61 = 0$ ...(ii)

The given circles cut orthogonally, if

$2.(-2)(-5) + 2(-3)(-6) = r^2 + 13 - r^2 + 61$

$\Rightarrow 20 + 36 - 13 - 61 = -2r^2 \Rightarrow r = 3$

27. (a) We have, $2(x^2 + y^2) + \lambda x + 9/2 = 0$

On comparing with standard equation of circle,

We get centre $\left(-\frac{\lambda}{4}, 0\right)$ and radius

$$r = \sqrt{\frac{\lambda^2}{16} - \frac{9}{4}}$$

We know that for limiting point $\frac{\lambda^2}{16} - \frac{9}{4} = 0$

$\Rightarrow \lambda^2 = 36 \Rightarrow \lambda = \pm 6$

$\therefore$ limiting point are $\left(-\frac{6}{4}, 0\right)$ and $\left(\frac{6}{4}, 0\right)$

or $\left(\pm\frac{3}{2}, 0\right)$

28. (b) Given that $y = 2x + c$ ....(i)

and $x^2 + y^2 = 16$ ....(ii)

We know that $y = mx + c$ is tangent to the circle

$x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1+m^2}$,

$m = 2, a = 4$

$\therefore c = \pm 4\sqrt{1+2^2} = \pm 4\sqrt{5}$

29. (b) $x^2 + y^2 = 4$ (given)

Centre $\equiv C_1 \equiv (0, 0)$ and $R_1 = 2$

Again $x^2 + y^2 - 6x - 8y - 24 = 0$, then $C_2 \equiv (3, 4)$

and $R_2 = 7$

Again $C_1C_2 = 5 = R_2 - R_1$

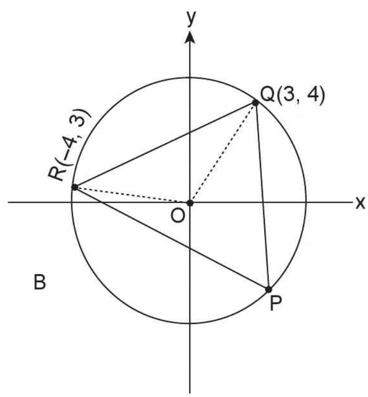
Threrfore, the given circles touch internally such that they can have just one common tangents at the point of contact.

Therefore, (b) is the answer.

30. (c) O is the point at centre and P is the point at circumference.

Therefore, angle QOR is double the angle QPR.

So it is sufficient to find the angle QOR.



Now slope of $OQ = 4/3$ slope of $OR = -3/4$

again $m_1 m_2 = -1$ therefore, $\angle QOR = 90°$

which implies that $\angle QPR = 45°$

Therefore, (c) is the answer.

TOPICWISE WARMUP TEST 3: SOLUTIONS

1. (a) Given that $x^2 + 2y + 6x = 0$

or $x^2 + 6x = -2y$ ....(1)

On adding 9 both sides of equation (1), we get

$x^2 + 6x + 9 = -2y + 9$ or $(x+3)^2 = -2y + 9$ which is of the form of parabola $X^2 = 4aY$ where $4a = -2 \Rightarrow a = -1/2$

∴ focus of the parabola $X^2 = 4aY$ is $(0, a)$

$$= \left(0, -\frac{1}{2}\right)$$

but, $X = x + 3$ and $Y = y - \frac{9}{2}$

∴ $x = -3$ and $y = 4$

[on putting $X = 0$ and $Y = -1/2$]

∴ focus of the parabola $x^2 + 2y + 6x = 0$ is $(-3, 4)$

2. (d) ∵ Directrix of parabola is $y = 2$

∴ $a = -2$

∴ Required equation of parabola is $x^2 = -4 \,.\, 2 \,.\, y \Rightarrow x^2 = -8y$

3. (a) Let $(x - y)$ be a point on the parabola, then by definition $(x - 3)^2 + (y - 0)^2 = (x + 2)^2$

$\Rightarrow$ $y^2 - 6x + 9 = 4x + 4 \Rightarrow y^2 = 5(2x - 1)$

4. (b) $a = 3$ and axis is along x-axis, so on taking vertex as origin equation of the parabola is $y^2 = 12x$.

Now shifting the origin to its original position, the required equation will be $y^2 = 12(x - 2)$

or $y^2 = 12x - 24$.

5. (c) $x^2 - 4x = 3y - 10$

$\Rightarrow$ $(x - 2)^2 = 3(y - 2)$

∴ Axis is $x - 2 = 0$

6. (d) Here vertex $=(-3, 2)$ and $a = 5$. So focus is $(5 - 3, 0 + 2) = (2, 2)$

7. (a) By definition equation will be

$$x^2 + y^2 = \left(\frac{x + y - 4}{\sqrt{2}}\right)^2$$

$\Rightarrow$ $x^2 + y^2 - 2xy + 8x + 8y - 16 = 0$

8. (d) By definition.

9. (c) Parabola is $y^2 = 12x$. Since ordinate of P is 6,

∴ $36 = 12x$

$\Rightarrow$ $x = 3$

∴ P is (3, 6). Focal distance of P = x co-ordinate + $a = 3 + 3 = 6$

[∵ $4a = 12$, ∴ $a = 3$]

10. (c) $e = 1$ for parabola.

11. (a) Since $y^2 = 4\,ax$ passes thro' (2, –6)

∴ $36 = 4a\,(2)$

$\Rightarrow$ $a = \frac{36}{8} = \frac{9}{2}$

∴ $4a = 4\left(\frac{9}{2}\right) = 18$

∴ latus rectum = 18.

12. (a) In a parabola semilatus rectum is the harmonic mean of the segments of the focal chord.

13. (b) As given $2b = 2ae \Rightarrow b = ae$

$\Rightarrow$ $b^2 = a^2e^2 = a^2 - b^2 \Rightarrow 2b^2 = a^2$ ....(1)

Also $\frac{2b^2}{a} = 10$

$\Rightarrow$ $b^2 = 5a$ ....(2)

(1), (2) $\Rightarrow a^2 = 100, b^2 = 50$

Hence equation of the ellipse will be

$$\frac{x^2}{100} + \frac{y^2}{50} = 1 \Rightarrow x^2 + 2y^2 = 100$$

14. (c) $x^2 + 2y^2 - 2x + 3y + 2 = 0$

$\Rightarrow$ $(x - 1)^2 + 2\,(y + 3/4)^2 = 1/8$

$\Rightarrow$ $\frac{(x-1)^2}{1/8} + \frac{(y+3/4)^2}{1/16} = 1$

It an ellipse with $a^2 = 1/8$, $b^2 = 1/16$.

Hence its eccentricity

$$e = \sqrt{1 - b^2/a^2} = \sqrt{1 - 8/16} = 1/\sqrt{2}$$

15. (a) Equation of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is $-x^2 + 3y^2 = 1$

$\Rightarrow$ $-\frac{x^2}{1} + \frac{y^2}{1/3} = 1$

Here $a^2 = 1$, $b^2 = 1/3$

∴ eccentricity $e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + 3} = 2$

16. (a) Let equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The equation of its conjugate hyperbola will be $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since their eccentricities are e and e',

so $e^2 = 1 + \frac{b^2}{a^2} = \frac{a^2 + b^2}{a^2}$ ....(1)

$e'^2 = 1 + \frac{a^2}{b^2} = \frac{a^2 + b^2}{b^2}$ ....(2)

(1), (2) $\Rightarrow \frac{1}{e^2} + \frac{1}{e'^2} = 1$

17. (c) $9x^2 - 16y^2 - 18x - 32y = 151$

$\Rightarrow \quad 9(x-1)^2 - 16(y+1)^2 = 144$

$$\Rightarrow \quad \frac{(x-1)^2}{16} - \frac{(y+1)^2}{9} = 1$$

$\therefore \quad$ centre $= (1, -1)$

18. (c) By definition, equation will be $(x-1)^2 + (y+1)^2 = 2\left(\frac{x-y+1}{\sqrt{2}}\right)^2$

$\Rightarrow \quad 2xy - 4x + 4y + 1 = 0$

19. (c) $2ae = 16, e = 1/2$

$\Rightarrow \quad a = 16$

$\Rightarrow \quad$ length of major axis $= 32$

20. (a) Conic is $\frac{x^2}{7} + \frac{y^2}{16} = 1$

$(a^2 = 7, b^2 = 16, a^2 < b^2)$

$$\therefore \quad e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{7}{16}} = \frac{3}{4}$$

21. (a) Clearly, $ae = 2$ and $\frac{2b^2}{a} = 6$

$$\Rightarrow \quad \frac{2a^2(1-e^2)}{a} = 6$$

$\Rightarrow \quad 2a^2(1-e^2) = 6a$

$\Rightarrow \quad 2a^2 - 2a^2e^2 = 6a$

$\Rightarrow \quad 2a^2 - 2.4 = 6a$

$\Rightarrow \quad a^2 - 3a - 4 = 0$

$\Rightarrow \quad (a-4)(a+1) = 0$

$\Rightarrow \quad a = 4, -1$

When $a = 4, 4e = 2 \quad \Rightarrow e = 1/2$

$a = -1, -e = 2 \quad \Rightarrow e = -2$

But $e > 0 \quad \therefore e = 1/2$

$$\therefore \quad a = \frac{2}{e} = \frac{2}{\frac{1}{2}} = 4.$$

Now $b^2 = a^2(1-e^2) = 16\left(1 - \frac{1}{4}\right) = \frac{16 \times 3}{4} = 12$

Hence ellipse is $\frac{x^2}{16} + \frac{y^2}{12} = 1.$

22. (a) Given conic is $3(x-1)^2 + 4(y+1)^2 = 3 + 4 + 5 = 12$

$$\Rightarrow \quad \frac{(x-1)^2}{4} + \frac{(y+1)^2}{3} = 1$$

$$\Rightarrow \quad \frac{X^2}{4} + \frac{Y^2}{3} = 1$$

$[X = x-1, Y = y+1]$

which is an ellipse.

Here $a^2 = 4, b^2 = 3$.

Latus Rectum $= 2.\frac{b^2}{a} = 2.\frac{3}{2} = 3$

23. (b) Given equation represents an ellipse if $\Delta \neq 0, h^2 < ab$.

24. (d) Foci are $(\pm ae, 0)$, Vertices are $(\pm a, 0)$

$\therefore \quad a = 5, ae = 3$

Since $b^2 = a^2(1-e^2) = a^2 - a^2e^2 = 25 - 9 = 16$

$\therefore \quad$ ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{x^2}{25} + \frac{y^2}{16} = 1.$$

25. (c) For the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1, a^2 = 16$

$$\therefore \quad e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{b^2}{16}} = \frac{\sqrt{16-b^2}}{4}$$

$\therefore \quad$ foci of the ellipse are $(\pm ae, 0)$ i.e., $(\pm\sqrt{16-b^2}, 0)$

For the hyperbola $a^2 = \left(\frac{12}{5}\right)^2, b^2 = \left(\frac{9}{5}\right)^2$

$\therefore \quad$ eccentricity e is given by

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{81}{144}}$$

$$= \sqrt{\frac{225}{144}} = \frac{15}{12} = \frac{5}{4}$$

$\therefore \quad$ foci of the hyperbola are $(\pm ae, 0)$ i.e., $(\pm 3, 0)$

By the given condition $\sqrt{16 - b^2} = 3$

$\Rightarrow \quad b^2 = 16 - 9 = 7 \quad \Rightarrow \quad b^2 = 7.$

QUESTION BANK : SOLVE THESE TO MASTER

1. The equations of the internal bisector of $\angle BAC$ of Δ ABC with vertices $A(5, 2)$, $B(2, 3)$ and C (6, 5) is

(a) $2x + y + 12 = 0$ (b) $x + 2y - 12 = 0$
(c) $2x + y - 12 = 0$ (d) None of these

2. If the orthocentre and centroid of a triangle are (–3, 5) and (3, 3) respectively then the circumcentre is

(a) (0, 4) (b) (6, –2)
(c) (0, 8) (d) (6, 2)

3. If a,c, b are three terms of a G.P., then the line $ax + by + c = 0$

(a) has a fixed direction
(b) always passes through a fixed point
(c) forms a triangle with the axes whose area is constant
(d) always cuts intercepts on the axes such that their sum is zero.

4. A line through the point $A(2, 0)$, which makes an angle of 30° with the positive direction of x-axis is rotated about A in clockwise direction through an angle 15°. The equation of the straight line in the new position is

(a) $(2-\sqrt{3})x - y - 4 + 2\sqrt{3} = 0$

(b) $(2-\sqrt{3})x + y - 4 + 2\sqrt{3} = 0$

(c) $(2-\sqrt{3})x - y + 4 + 2\sqrt{3} = 0$

(d) None of these

5. A rectangle has two opposite vertices at the points (1, 2) and (5, 5). If the other vertices lie on the line $x = 3$, then the coordinates of the other vertices are

(a) (3, –1), (3, –6) (b) (3, 1), (3, 5)
(c) (3, 2), (3, 6) (d) (3, 1), (3, 6)

6. If $P(1, 0)$, $Q(-1, 0)$, $R(2, 0)$ are three given points, then the locus of point S satisfying the relation $SQ^2 + SR^2 = 2SP^2$ is

(a) a straight line || to x-axis
(b) a straight line || to y-axis
(c) circle through the origin
(d) circle with centre at the origin

7. If the lines $ax + 12y + 1 = 0$, $bx + 13y + 1 = 0$ and $cx + 14y + 1 = 0$ are concurrent, then a, b, c are in

(a) H.P. (b) G.P.
(c) A.P. (d) None of these

8. The base of a triangle lies along the line $x = a$ and is of length a. The area of the triangle is a^2, if the vertex lies on the line

(a) $x = 0$ (b) $x = -a$
(c) $x = 3a$ (d) $x = -3a$

9. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is

(a) square
(b) circle
(c) straight line
(d) two intersecting lines

10. Consider a triangle PQR with $P \equiv (0,0)$, $Q \equiv (a, 0)$, $R \equiv (0, b)$. Then the centroid, orthocentre and circumcentre

(a) lie on a straight line
(b) form a scalene triangle with area $\frac{1}{2}|ab|$
(c) form a right angled triangle with area $\frac{1}{2}|ab|$
(d) None of these

11. Two sides of a square lie on the lines $x + y = 1$ and $x + y + 2 = 0$. Then its area is

(a) 4.5 sq. units (b) 4 sq. units
(c) 9 sq. units (d) 5 sq. units

12. If $A(\cos\alpha, \sin\alpha)$, $B(\sin\alpha, -\cos\alpha)$, $C(1, 2)$ are the vertices of a ΔABC, then as α varies, the locus of its centroid is

(a) $x^2 + y^2 - 2x - 4y + 1 = 0$
(b) $3(x^2 + y^2) - 2x - 4y + 1 = 0$
(c) $x^2 + y^2 - 2x - 4y + 3 = 0$
(d) None of these

13. If the point $P(a^2, a)$ lies in the region corresponding to the acute angle between the lines $2y = x$ and $4y = x$, then $a \in (p, q)$ where $p + q =$

(a) 8 (b) 10
(c) 6 (d) 0

14. If the intercept made on the line $y = mx$ by lines $y = 2$ and $y = 6$ is less than 5 then the range of values of m is

(a) $\left(-\infty,-\frac{4}{3}\right)\cup\left(\frac{4}{3},\infty\right)$
(b) $\left(-\frac{4}{3},\frac{4}{3}\right)$
(c) $\left(-\frac{3}{4},\frac{3}{4}\right)$
(d) None of these

15. The lines $x + (a - 1)y + 1 = 0$ and $2x + a^2 y - 1 = 0$ are perpendicular if
(a) $|a| = 2$ (b) $0 < a < 1$
(c) $-1 < a < 0$ (d) $a = -1$

16. The equation of a straight line passing through the point (–5, 4) and which cuts off an intercept of $\sqrt{2}$ units between the lines $x + y + 1 = 0$ and $x + y - 1 = 0$ is
(a) $x - 2y + 13 = 0$
(b) $2x - y + 14 = 0$
(c) $x - y + 9 = 0$
(d) $x - y + 10 = 0$

17. The number of integral (integral point means both the coordinates should be integer) exactly in the interior of the triangle with vertices (0, 0) (0, 21) and (21, 0) is
(a) 133 (b) 190
(c) 233 (d) 105

18. If a vertex of a triangle is (1, 1) and the mid-points of two sides through this vertex are (–1, 2) and (3, 2), then the centroid of the triangle is
(a) $\left(\frac{-1}{3},\frac{7}{3}\right)$ (b) $\left(-1,\frac{7}{3}\right)$
(c) $\left(\frac{1}{3},\frac{7}{3}\right)$ (d) $\left(1,\frac{7}{3}\right)$

19. Let $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ be the vertices of a right-angled triangle with AC as its hypotenuse. If the area of the triangle is 1, then the set of the values which 'k' can take is given by
(a) {0, 2} (b) {–1, 3}
(c) {–3, –2} (d) {1, 3}

20. The equation $2x^2 + kxy + 2y^2 = 0$ represents a pair of real and distinct lines if
(a) $k \in (-4, 4)$
(b) $k \in R$
(c) $k \in (-\infty, -4) \cup (4, \infty)$
(d) None of these

21. If the lines joining the origin to the intersection of the line $y = mx + 2$ and the curve $x^2 + y^2 = 1$ are at right angles, then
(a) $m^2 = 1$ (b) $m^2 = 3$
(c) $m^2 = 7$ (d) $2m^2 = 1$

22. If pair of lines represented by $ax^2 + 2hxy + by^2 = 0$, $b \neq 0$, are such that the sum of the slopes of the lines is three times the product of their slopes then
(a) $3b + 2h = 0$ (b) $3a + 2h = 0$
(c) $3h + 2a = 0$ (d) None of these

23. If (2, 0) is the vertex and y-axis is the directrix of a parabola; then its focus will be
(a) (2, 0) (b) (4, 0)
(c) (–2, 0) (d) (–4, 0)

24. If (0, 4) and (0, 2) are vertex and focus of a parabola, then its equation is
(a) $x^2 + 8y = 32$ (b) $y^2 + 8y = 32$
(c) $x^2 - 8y = 32$ (d) $y^2 - 8x = 32$

25. The equation of the parabola whose vertex and focus are (1, 1) and (3, 1) respectively, will be
(a) $(y - 1)^2 = 8(x - 3)$
(b) $(x - 1)^2 = 8(y - 1)$
(c) $(x - 3)^2 = 8(y - 1)$
(d) $(y - 1)^2 = 8(x - 1)$

26. The equation of the parabola with vertex at (0, 0), axis along x-axis and passing through $\left(\frac{5}{3},\frac{10}{3}\right)$ is
(a) $y^2 = 20x$ (b) $y^2 = \frac{20x}{3}$
(c) $y^2 = \frac{10x}{3}$ (d) None of these

27. For the parabola $y^2 = 4x$, the point P whose focal distance is 17, is
(a) (16, 8) or (16, –8) (b) (8, 8) or (8, –8)
(c) (4, 8) or (4, –8) (d) None of these

28. The latus rectum of a parabola whose focal chord is PSQ such that $SP = 3$ and $SQ = 2$ is given by
(a) $\frac{24}{5}$ (b) $\frac{12}{5}$
(c) $\frac{6}{5}$ (d) None of these

29. The curve described parametrically by $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents
(a) a pair of st. lines (b) an ellipse
(c) a parabola (d) a hyperbola

30. Equation of the ellipse whose eccentricity is 1/2 and foci are (± 1, 0) will be
(a) $\frac{x^2}{3}+\frac{y^2}{4}=1$ (b) $\frac{x^2}{4}+\frac{y^2}{3}=1$
(c) $\frac{x^2}{4}+\frac{y^2}{3}=\frac{4}{3}$ (d) None of these

31. The eccentricity of the hyperbola $4x^2 - 9y^2 - 8x = 32$ is
(a) $\sqrt{5}/3$ (b) $\sqrt{13}/3$
(c) $\sqrt{13}/2$ (d) 3/2

32. If for an ellipse, its centre is (0, 0); focus is (0, 3) and semimajor axis is 5, then its equation will be
(a) $\frac{x^2}{9}+\frac{y^2}{25}=1$ (b) $\frac{x^2}{9}+\frac{y^2}{16}=1$
(c) $\frac{x^2}{16}+\frac{y^2}{25}=1$ (d) None of these

33. If major axis of an ellipse is thrice of its minor axis, then its eccentricity is
(a) 1/3 (b) $1/\sqrt{3}$
(c) $1/\sqrt{2}$ (d) $2\sqrt{2}/3$

34. Eccentricity of the ellipse $4x^2 + 9y^2 + 8x + 36y + 4 = 0$ is
(a) 3/5 (b) 5/6
(c) $\sqrt{2}/3$ (d) $\sqrt{5}/3$

35. The eccentricity of an ellipse whose major axis is twice its minor axis is given by
(a) $\sqrt{2}/3$ (b) $\sqrt{3}$
(c) $\sqrt{2}$ (d) $\sqrt{3}/2$

36. The latus rectum of the hyperbola $16x^2 - 9y^2 = 144$ is
(a) 8/3 (b) 16/3
(c) 32/3 (d) 4/3

37. If the distance between foci is 2 and the distance between the directrices is 5, then equation of the ellipse in the standard form is
(a) $6x^2 + 10y^2 = 5$
(b) $6x^2 + 10y^2 = 15$
(c) $x^2 + 3y^2 = 10$
(d) None of these

38. If S' and S are the foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $P(x, y)$ be a point on it, then the value of $SP + S'P$ is
(a) $2b$ (b) $2a$
(c) $a - b$ (d) $a + b$

39. In an ellipse the distance between its foci is 6 and its minor axis is 8. Then its eccentricity is
(a) 4/5 (b) $1/\sqrt{52}$
(c) 3/5 (d) 1/2

40. The equation $x = \frac{e' + e^{-1}}{2}$; $y = \frac{e' - e^{-1}}{2}$; $t \in R$ represents
(a) an ellipse (b) a parabola
(c) a hyperbola (d) None of these

ANSWERS

PART A CARTESIAN COORDINATES

Lecture 1
Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (d)	**5.** (b)	**9.** (c)	**13.** (d)
2. (d)	**6.** (b)	**10.** (a)	**14.** (b)
3. (a)	**7.** (c)	**11.** (a)	**15.** (d)
4. (a)	**8.** (b)	**12.** (a)	

Worksheet: To Check the Preparation Level

1. (b)	**5.** (c)	**9.** (b)	**13.** (b)
2. (b)	**6.** (b)	**10.** (c)	**14.** (d)
3. (d)	**7.** (a)	**11.** (b)	
4. (b)	**8.** (a)	**12.** (b)	

Lecture 2
Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (c)	**8.** (b)	**15.** (a)	**22.** (d)
2. (d)	**9.** (b)	**16.** (c)	**23.** (a)
3. (c)	**10.** (a)	**17.** (b)	**24.** (c)
4. (b)	**11.** (c)	**18.** (b)	**25.** (b)
5. (b)	**12.** (c)	**19.** (b)	**26.** (a)
6. (a)	**13.** (a)	**20.** (a)	**27.** (c)
7. (a)	**14.** (a)	**21.** (b)	

Worksheet: To Check the Preparation Level

1. (a)	**7.** (c)	**13.** (a)	**19.** (b)
2. (d)	**8.** (c)	**14.** (b)	**20.** (a)
3. (c)	**9.** (a)	**15.** (b)	**21.** (c)
4. (a)	**10.** (d)	**16.** (d)	
5. (c)	**11.** (a)	**17.** (a)	
6. (b)	**12.** (c)	**18.** (d)	

Lecture 3
Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (c)	**7.** (b)	**13.** (d)	**19.** (b)
2. (c)	**8.** (a)	**14.** (d)	**20.** (c)
3. (b)	**9.** (b)	**15.** (c)	**21.** (d)
4. (a, c, d)	**10.** (c)	**16.** (a)	
5. (a)	**11.** (b)	**17.** (d)	
6. (d)	**12.** (b)	**18.** (b)	

Worksheet: To Check the Preparation Level

1. (d)	**5.** (c)	**9.** (d)	**13.** (c)
2. (c)	**6.** (b)	**10.** (c)	**14.** (d)
3. (c)	**7.** (b)	**11.** (b)	**15.** (c)
4. (b)	**8.** (b)	**12.** (a)	

PART B STRAIGHT LINE

Lecture 1
Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (b)	**5.** (d)	**9.** (b)
2. (a)	**6.** (d)	**10.** (a)
3. (a)	**7.** (a)	
4. (b)	**8.** (a)	

Worksheet: To Check the Preparation Level

1. (a)	**5.** (c)	**9.** (b)
2. (b)	**6.** (a)	**10.** (a)
3. (a)	**7.** (a)	
4. (c)	**8.** (c)	

Lecture 2
Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (c)	**6.** (c)	**12.** (b)	**18.** (c)
2. (d)	**7.** (a)	**13.** (a)	**19.** (b)
3. (b)	**8.** (d)	**14.** (c)	**20.** (a)
4. (b)	**9.** (a)	**15.** (b)	**21.** (b)
5. (c)	**10.** (a)	**16.** (b)	
6. (a)	**11.** (a)	**17.** (b)	

Worksheet: To Check the Preparation Level

1. (b)	**6.** (b)	**11.** (d)	**16.** (a)
2. (b)	**7.** (b)	**12.** (b)	**17.** (b)
3. (b)	**8.** (c)	**13.** (c)	
4. (c)	**9.** (a)	**14.** (a)	
5. (a)	**10.** (b)	**15.** (a)	

Lecture 3
Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (b)	**5.** (c)	**9.** (c)	**13.** (b)
2. (c)	**6.** (b)	**10.** (c)	**14.** (c)
3. (c)	**7.** (b)	**11.** (b)	**15.** (d)
4. (c)	**8.** (c)	**12.** (d)	

Worksheet: To Check the Preparation Level

1. (a)	**5.** (b)	**9.** (b)	**13.** (b)
2. (b)	**6.** (c)	**10.** (a)	**14.** (b)
3. (d)	**7.** (b)	**11.** (c)	
4. (a)	**8.** (b)	**12.** (b)	

Lecture 4
Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (a)	**5.** (c)	**9.** (a)
2. (a)	**6.** (a)	**10.** (a)
3. (c)	**7.** (a)	
4. (d)	**8.** (d)	

Worksheet: To Check the Preparation Level

1. (b)	**5.** (d)	**9.** (d)
2. (b)	**6.** (d)	**10.** (a)
3. (a)	**7.** (b)	
4. (d)	**8.** (a)	

Lecture 5
Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (b)	**4.** (d)	**7.** (a)	**10.** (c)
2. (a)	**5.** (d)	**8.** (b)	
3. (c)	**6.** (d)	**9.** (a)	

Worksheet: To Check the Preparation Level

1. (b)	**5.** (d)	**9.** (c)	**13.** (c)
2. (c)	**6.** (c)	**10.** (b)	
3. (a)	**7.** (a)	**11.** (d)	
4. (b)	**8.** (c)	**12.** (a)	

PART C PAIR OF STRAIGHT LINES

Lecture 1
Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (d)	**9.** (b)	**17.** (d)	**25.** (c)
2. (b)	**10.** (a)	**18.** (b)	**26.** (b)
3. (c)	**11.** (c)	**19.** (a)	**27.** (a)
4. (d)	**12.** (c)	**20.** (a)	**28.** (c)
5. (b)	**13.** (b)	**21.** (a)	**29.** (d)
6. (a)	**14.** (c)	**22.** (c)	**30.** (a)
7. (a)	**15.** (b)	**23.** (b)	
8. (a)	**16.** (c)	**24.** (d)	

Worksheet: To Check the Preparation Level

1. (c)	**5.** (b)	**9.** (c)	**13.** (c)
2. (c)	**6.** (b)	**10.** (a)	**14.** (c)
3. (d)	**7.** (a)	**11.** (d)	**15.** (c)
4. (a)	**8.** (d)	**12.** (d)	**16.** (b)

Lecture 2

Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1. (a)	**6.** (c)	**11.** (a)	**16.** (c)
2. (c)	**7.** (d)	**12.** (d)	**17.** (c)
3. (b)	**8.** (d)	**13.** (b)	**18.** (c)
4. (c)	**9.** (d)	**14.** (a)	**19.** (d)
5. (a)	**10.** (b)	**15.** (c)	

Worksheet: To Check the Preparation Level

1. (a)	**6.** (a)	**11.** (d)
2. (c)	**7.** (a)	**12.** (d)
3. (a)	**8.** (d)	**13.** (d)
4. (b)	**9.** (c)	**14.** (c)
5. (b)	**10.** (d)	**15.** (d)

PART D CIRCLE

Lecture 1

Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1. (a)	**5.** (c)	**9.** (a)	**13.** (d)
2. (a)	**6.** (a)	**10.** (a)	**14.** (a)
3. (d)	**7.** (d)	**11.** (c)	
4. (a)	**8.** (c)	**12.** (d)	

Worksheet: To Check the Preparation Level

1. (d)	**5.** (d)	**9.** (a)	**13.** (d)
2. (a)	**6.** (a)	**10.** (b)	**14.** (a)
3. (a)	**7.** (a)	**11.** (a)	**15.** (a)
4. (b)	**8.** (c)	**12.** (a)	

Lecture 2

Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1. (c)	**7.** (a)	**13.** (b)	**19.** (c)
2. (c)	**8.** (c)	**14.** (a)	**20.** (c)
3. (b)	**9.** (d)	**15.** (b)	**21.** (c)
4. (b)	**10.** (b)	**16.** (d)	
5. (a)	**11.** (c)	**17.** (a)	
6. (c)	**12.** (b)	**18.** (d)	

Worksheet: To Check the Preparation Level

1. (a)	**7.** (a)	**13.** (c)	**19.** (d)
2. (a)	**8.** (d)	**14.** (a)	**20.** (b)
3. (d)	**9.** (c)	**15.** (d)	**21.** (b)
4. (b)	**10.** (d)	**16.** (b)	
5. (d)	**11.** (c)	**17.** (b)	
6. (a)	**12.** (d)	**18.** (a)	

Lecture 3

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (a)	**6.** (b)	**11.** (c)	**16.** (a)
2. (b)	**7.** (c)	**12.** (a)	**17.** (c)
3. (a)	**8.** (a)	**13.** (c)	
4. (c)	**9.** (d)	**14.** (c)	
5. (a)	**10.** (c)	**15.** (a)	

Worksheet: To Check the Preparation Level

1. (a)	**6.** (d)	**11.** (d)	**16.** (b)
2. (c)	**7.** (a)	**12.** (d)	**17.** (c)
3. (c)	**8.** (c)	**13.** (d)	
4. (b)	**9.** (a)	**14.** (a)	
5. (a)	**10.** (a)	**15.** (a d)	

PART E CONIC SECTION

Lecture 1

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (c)	**9.** (b)	**17.** (c)	**25.** (c)
2. (d)	**10.** (c)	**18.** (d)	**26.** (b)
3. (b)	**11.** (d)	**19.** (c)	**27.** (b)
4. (c)	**12.** (c)	**20.** (a)	**28.** (c)
5. (d)	**13.** (c)	**21.** (c)	**29.** (c)
6. (a)	**14.** (d)	**22.** (a)	
7. (a)	**15.** (c)	**23.** (c)	
8. (d)	**16.** (c)	**24.** (a)	

Worksheet: To Check the Preparation Level

1. (d)	**6.** (c)	**11.** (a)	**16.** (c)
2. (c)	**7.** (d)	**12.** (c)	**17.** (d)
3. (b)	**8.** (b)	**13.** (b)	
4. (b)	**9.** (b)	**14.** (a)	
5. (c)	**10.** (b)	**15.** (c)	

Lecture 2

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (c)	**8.** (a)	**15.** (a)	**22.** (d)
2. (d)	**9.** (c)	**16.** (b)	**23.** (d)
3. (a)	**10.** (c)	**17.** (d)	**24.** (c)
4. (c)	**11.** (c)	**18.** (b)	**25.** (d)
5. (c)	**12.** (b)	**19.** (a)	**26.** (d)
6. (c)	**13.** (a)	**20.** (a)	**27.** (b)
7. (c)	**14.** (c)	**21.** (a)	

Worksheet: To Check the Preparation Level

1. (a)	**6.** (a)	**11.** (a)	**16.** (c)
2. (a)	**7.** (d)	**12.** (b)	**17.** (b)
3. (d)	**8.** (c)	**13.** (d)	
4. (d)	**9.** (b)	**14.** (a)	
5. (d)	**10.** (b)	**15.** (b)	

Lecture 3

Unsolved Objective Problems (Identical Problems For Practice): For Improving Speed With Accuracy

1. (c)	**8.** (d)	**15.** (a)	**22.** (b)
2. (b)	**9.** (b)	**16.** (c)	**23.** (a)
3. (b)	**10.** (b)	**17.** (b)	**24.** (a)
4. (d)	**11.** (c)	**18.** (d)	**25.** (b)
5. (b)	**12.** (d)	**19.** (d)	**26.** (a)
6. (a)	**13.** (a)	**20.** (a)	**27.** (b)
7. (b)	**14.** (c)	**21.** (b)	**28.** (d)

Worksheet: To Check the Preparation Level

1. (d)	**5.** (a)	**9.** (b)	**13.** (a)
2. (c)	**6.** (b)	**10.** (b)	**14.** (a)
3. (a)	**7.** (c)	**11.** (b)	**15.** (b)
4. (d)	**8.** (b)	**12.** (a)	**16.** (b)

Lecture 4

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (b)	**5.** (d)	**9.** (b, *d*)	**13.** (b)
2. (b)	**6.** (b)	**10.** (c)	**14.** (c)
3. (b)	**7.** (c)	**11.** (a)	
4. (c)	**8.** (b)	**12.** (a, *b*)	

Worksheet: To Check the Preparation Level

1. (a)	**4.** (b)	**7.** (b)
2. (a)	**5.** (c)	**8.** (a)
3. (a)	**6.** (a)	**9.** (b)

Lecture 5

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (b)	**8.** (c)	**15.** (d)	**22.** (c)
2. (a)	**9.** (c)	**16.** (c)	**23.** (a)
3. (a)	**10.** (c)	**17.** (a)	**24.** (b)
4. (b)	**11.** (b)	**18.** (b)	**25.** (c)
5. (b)	**12.** (a)	**19.** (d)	
6. (c)	**13.** (b)	**20.** (b)	
7. (c)	**14.** (a)	**21.** (d)	

Worksheet: To Check the Preparation Level

1. (d)	**5.** (b)	**9.** (d)	**13.** (a)
2. (d)	**6.** (a)	**10.** (a)	**14.** (c)
3. (a)	**7.** (c)	**11.** (c)	**15.** (b)
4. (a)	**8.** (b)	**12.** (a)	

TEST YOUR SKILLS

Question Bank

1. (c)	**11.** (a)	**21.** (c)	**31.** (b)
2. (a)	**12.** (b)	**22.** (b)	**32.** (c)
3. (c)	**13.** (c)	**23.** (b)	**33.** (d)
4. (a)	**14.** (a)	**24.** (a)	**34.** (d)
5. (d)	**15.** (a)	**25.** (d)	**35.** (d)
6. (c)	**16.** (c)	**26.** (b)	**36.** (c)
7. (c)	**17.** (b)	**27.** (a)	**37.** (b)
8. (c)	**18.** (d)	**28.** (a)	**38.** (b)
9. (a)	**19.** (b)	**29.** (c)	**39.** (c)
10. (a)	**20.** (c)	**30.** (b)	**40.** (c)