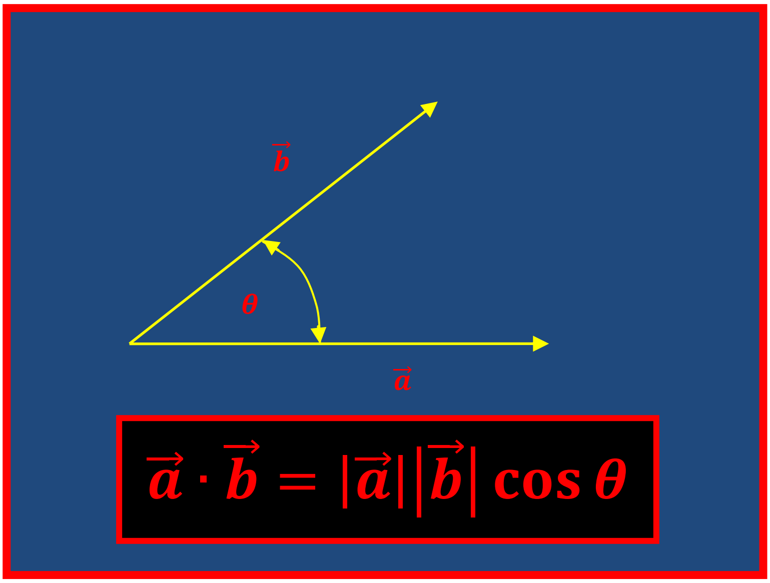
VECTOR ALGEBRA

for Engineers and Scientists

DEMETRIOS P. KANOUSSIS Ph.D

Vector Algebra for Engineers and Scientists

With Applications in Engineering, Physics and Geometry



1) An excellent supplementary textbook, applications oriented, for Engineering and Sciences students, ideal for independent study

2) Fundamental concepts and definitions, solving techniques and methods, with applications to a variety of problems in Engineering, Physics and Geometry

3) Covers in considerable depth and details vector Algebra and related topics and demonstrates the strength and generality of vector methods in solving miscellaneous problems

4) 72 fully worked illustrative examples and 145 graded problems

5) Odd numbered problems are provided with answers

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Dr. Kanoussis is a professional Electrical Engineer and Mathematician. He received his Ph.D degree in Engineering and his Master degree in Mathematics from Tennessee Technological University, U.S.A, and his Bachelor degree in Electrical Engineering from National Technical University of Athens (N.T.U.A), Greece.

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Additionally to his professional activities, teaching and research, Demetrios P. Kanoussis is **the author of several textbooks in Electrical Engineering and Applied Mathematics.**

A complete list of Dr. Kanoussis textbooks in Mathematics and Engineering can be found in the Author's page at Amazon Author Central (https://www.amazon.com/Demetrios-P.-Kanoussis/e/B071GZ215Z).

Vector Algebra for Engineers and Scientists

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PREFACE

The concept of the **vector** plays an extremely important role in Engineering, Physics and Geometry. **Vector quantities** have both magnitude **and** direction, as opposed to **scalar quantities** which have **only** magnitude. For example, the velocity, the acceleration, the force, the electric and magnetic fields, etc. are vector quantities, while mass, temperature, volume, etc. are scalar quantities.

Vectors are important in almost all branches of Engineering, Geometry and Physics and in particular in the study of Applied Mathematics. Using vectors, many important equations in Engineering and Physics are expressed in a compact and concise form, **independent** from the particular coordinate system being used.

In this book we lay out fundamental concepts and definitions, define the fundamental **vector operations** (equality of vectors, addition, subtraction, multiplication of a vector by a scalar, etc), define the various types of **vector products** (the dot or scalar product, the cross or outer product, the scalar triple product and the vector triple product), and show the strength of vector algebra in proving various important formulas in Geometry, Trigonometry, Engineering and Physics.

The book contains 11 chapters, as shown analytically in the Table of contents. The first two chapters are devoted to fundamental concepts, definitions, terminology and **vector operations**. Chapter 3 is devoted to the Cartesian systems and the coordinate expression of vectors. In chapter 4 we introduce the concept of linear independence of vectors and investigate a number of useful consequences. Chapters 5 up to 9 are devoted to the study of various types **of vector products**, i.e. the dot product, the cross product, the scalar triple product and the vector triple product, and investigate a considerable number of applications in Physics and Geometry. In chapter 10 we derive **the vector equations** of straight lines, planes, circles and spheres and prove various properties using the theory of vectors. Finally, in chapter 11 we derive and summarize some **fundamental formulas** of plane and solid analytic Geometry, (distance of a point from a straight line, distance of a point from a plane, the least distance between two skew lines, the area of a triangle, the volume of a parallelepiped formed by three concurrent vectors, the angle between two planes, etc).

The book contains 72 illustrative worked out examples and 145 graded problems for solution. The examples and the problems are designed to help students to develop a solid background in the algebra of vectors, to broaden their knowledge and sharpen their analytical skills and finally to prepare them to pursue successfully more advanced studies in Engineering and Mathematics.

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CHAPTER 1: Fundamental Concepts and Definitions

1-1) Scalars and Vectors

In Engineering and Physics we consider various types of physical quantities. One class of quantities is completely specified by just one single number (**the magnitude**) and the corresponding unit of measurement. Such quantities are called **scalar quantities or simply scalars**. For example, length, area, volume, mass, density, temperature, electric charge, electric potential, energy and work are all scalar quantities. If the area of a plane figure is $5\ m^2$, the number 5 (magnitude) together with the corresponding unit (m^2) **completely specifies the area of the figure**. If the mass of a body is $10\ Kg$, this completely specifies the mass of the body, etc.

Another class of physical quantities consists of quantities the complete determination of which requires both magnitude and direction. Such quantities are called **vector quantities or simply vectors**. For example, velocity, acceleration, force, torque of a force, momentum, electric and magnetic fields, etc, are vector quantities.

A vector is represented by **a directed line segment** as shown in Figure 1-1.

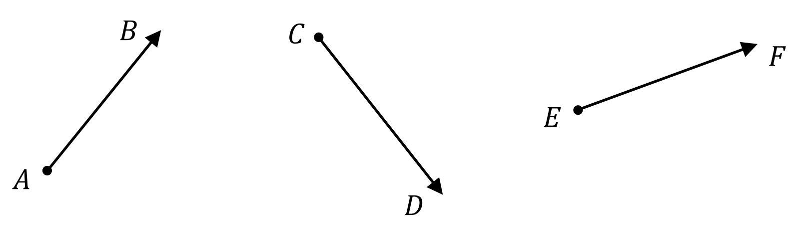

Fig. 1-1: The vector concept.

Symbolically the three vectors shown in Fig. 1-1 are denoted as $\overrightarrow{AB}$, $\overrightarrow{CD}$ and $\overrightarrow{EF}$ respectively. In the first vector $\overrightarrow{AB}$ the point A is the initial point or **the origin** of the vector and the point B is **the terminal point** of the vector. **The magnitude or the length** of the vector $\overrightarrow{AB}$ is denoted by $\left|\overrightarrow{AB}\right|$, and similarly $\left|\overrightarrow{CD}\right|$ and $\left|\overrightarrow{EF}\right|$ are the magnitudes (**lengths**) of the vectors $\overrightarrow{CD}$ and $\overrightarrow{EF}$ respectively. A second notation for vectors is often used, which consists of **single small letters beneath an arrow**, such as $\vec{a}$, $\vec{b}$, $\vec{c}$,...., and in this notation, $|\vec{a}|$, $\left|\vec{b}\right|$, $|\vec{c}|$,...., will represent the magnitudes of the corresponding vectors.

A vector of particular interest is **the null or zero vector**, which is defined as a vector whose magnitude (length) is zero. The null vector is denoted by $\vec{0}$. The null vector has no direction.

1-2) Equality of Vectors

Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be **equal**, if and only if, $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ with the same orientation and $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{CD}\right|$.

$$\overrightarrow{AB} = \overrightarrow{CD} \Leftrightarrow \begin{Bmatrix} \overrightarrow{AB} \parallel \overrightarrow{CD} \ \ and\ of\ the\ same\ orientation, \boldsymbol{and} \\ \left|\overrightarrow{AB}\right| = \left|\overrightarrow{CD}\right| \end{Bmatrix}$$

Note that two vectors which are not parallel cannot possibly be equal even if they have the same length (magnitude).

Two vectors $\overrightarrow{AB}$ and $\overrightarrow{KL}$ are said to be **opposite** if they are parallel, have the same magnitude but opposite orientations.

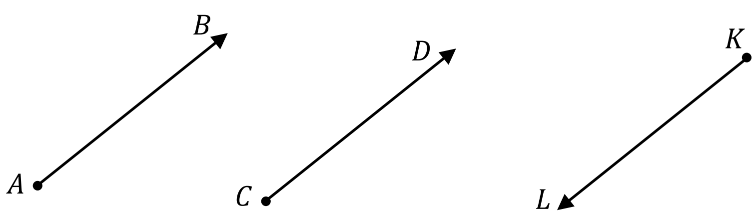

Fig. 1-2: Equal and opposite vectors.

The sum of two opposite vectors is the null vector, for example in Fig. 1-2, since $\overrightarrow{AB}$ and $\overrightarrow{KL}$ are opposite vectors,

$$\overrightarrow{AB} + \overrightarrow{KL} = \vec{0} \qquad (1-1)$$

Note: In equations involving vector quantities, it is customary to denote the null vector by 0, (instead of the full notation $\vec{0}$).

A vector which is restricted to pass through a given, fixed point is called **a bound or localized vector**. In this case the line of action of the vector is fixed. **A force acting on a body is a bound vector since its effect depends on the point of its application**.

On the other hand, a vector which is not restricted to pass through a fixed point in space is called **a free vector**. Free vectors have the same magnitude and direction and act at different points in the same line or parallel lines. **They are all equivalent to each other**. The moment of a couple of forces is a free vector, (see Example 7-10).

CHAPTER 2: Vector Operations

In order to be able to apply the theory of vectors in various problems arising in Engineering and Physics, we have to develop an **Algebra of Vectors**, i.e. how to add and subtract vectors, how to multiply a vector by a scalar (number), what is meant by " a linear combination of vectors", etc.

2-1) Vector addition and subtraction

Let us consider two vectors $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$ be two vectors as shown in Fig. 2-1.

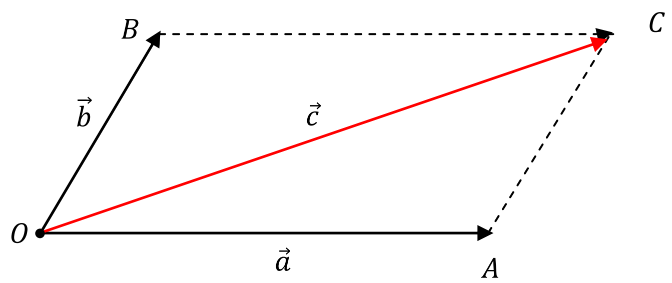

Fig. 2-1: Addition of two vectors, "The parallelogram law of addition".

The vector $\vec{c} = \overrightarrow{OC}$ which is **the diagonal** of the parallelogram $OACB$ is defined as **the sum** of the two vectors $\vec{a}$ and $\vec{b}$ and is written as

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} \quad or \quad \vec{c} = \vec{a} + \vec{b}$$

Alternatively, the sum of the two vectors can be obtained from the equivalent "**triangle law of addition**". In Fig. 2-1, the vectors $\overrightarrow{OA}$ and $\overrightarrow{BC}$ are **equal** (since they are parallel and have the same orientation and equal magnitudes). If we therefore consider the triangle OBC having sides $\vec{b} = \overrightarrow{OB}$ and $\vec{a} = \overrightarrow{BC}$, then **the third side** $\vec{c} = \overrightarrow{OC}$ of the triangle is the sum of the two vectors $\vec{a}$ and $\vec{b}$.

The "**triangle law of addition**" can be applied to obtain **the sum of more than two vectors**, as shown in Fig. 2-2.

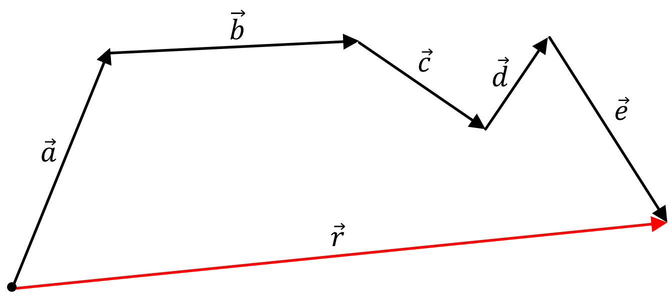

Fig. 2-2: The vector addition of five vectors.

In Fig. 2-2, the initial point of $\vec{b}$ coincides with the terminal point of $\vec{a}$, the initial point of $\vec{c}$ coincides with the terminal point of $\vec{b}$, the initial point of $\vec{d}$ coincides with the terminal point of $\vec{c}$, and the initial point of $\vec{e}$ coincides with the terminal point of $\vec{d}$. In this case **the vector $\vec{r}$ with initial point the initial point of the first vector $\vec{a}$ and terminal point the terminal point of the last vector $\vec{e}$ is defined as the sum of the five vectors** and is written as

$$\vec{r} = \vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e}$$

If the vectors are not arranged so that the terminal point of one is the initial point of the next one, then **we translate them so that the terminal point of one coincides with the initial point of the next**, and then perform the addition. Let us for definiteness consider four vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ as shown in Fig. 2-3. We consider a point O (**the origin**) and form the "**successive**" vectors $\overrightarrow{OA} = \vec{a},\ \overrightarrow{AB} = \vec{b},\ \overrightarrow{BC} = \vec{c}$ and $\overrightarrow{CD} = \vec{d}$. Then the vector $\overrightarrow{OD}$ is the sum of the four vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, i.e.

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \vec{a} + \vec{b} + \vec{c} + \vec{d}$$

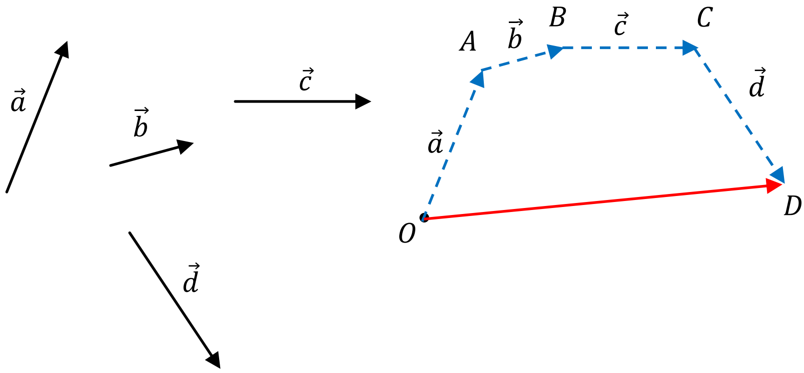

Fig. 2-3: The vector addition of four, arbitrary vectors.

Let $\vec{a}$ and $\vec{b}$ be any two vectors. The **subtraction of $\vec{\boldsymbol{b}}$ from $\vec{\boldsymbol{a}}$** is defined to be **the addition of the vector $(-\vec{\boldsymbol{b}})$ to $\vec{\boldsymbol{a}}$**, i.e. $\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}} = \vec{\boldsymbol{a}} + (-\vec{\boldsymbol{b}})$. The subtraction of two vectors is shown in Fig. 2-4.

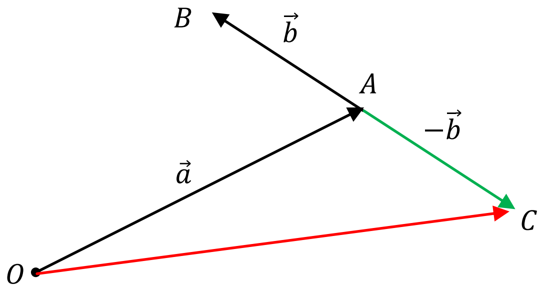

Fig. 2-4: The subtraction of two vectors.

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OA} - \overrightarrow{AB} = \vec{a} - \vec{b}$$

The **vector addition** has the following properties:

1) It is commutative, i.e.

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad (2-1)$$

2) It is associative, i.e.

$$\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c} \qquad (2-2)$$

3) There exists the null vector $\vec{0}$ such that for any vector $\vec{a}$,

$$\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a} \qquad (2-3)$$

4) For any vector $\vec{a}$, there is an "**additive inverse**" $(-\vec{a})$ such that

$$\vec{a}+(-\vec{a})=\vec{0} \qquad (2-4)$$

2-2) Multiplication of a vector by a scalar

Let k be any real number (**scalar**) and $\vec{a}$ be any vector ($\vec{a} \neq \vec{0}$). Then **the product $\boldsymbol{k\vec{a}}$ is a vector** such that,

a) The magnitude of $k\vec{a}$ is $|\boldsymbol{k\vec{a}}| = |\boldsymbol{k}||\boldsymbol{\vec{a}}|$,

b) If $\boldsymbol{k} > 0$ then the vector $k\vec{a}$ has **the same direction** with $\vec{a}$, while if $\boldsymbol{k} < 0$ then the vector $k\vec{a}$ has **direction opposite** to $\vec{a}$, (i.e. $k\vec{a}$ has the same direction with $(-\vec{a})$),

c) If $k=0$ then the vector $k\vec{a}=\vec{0}$.

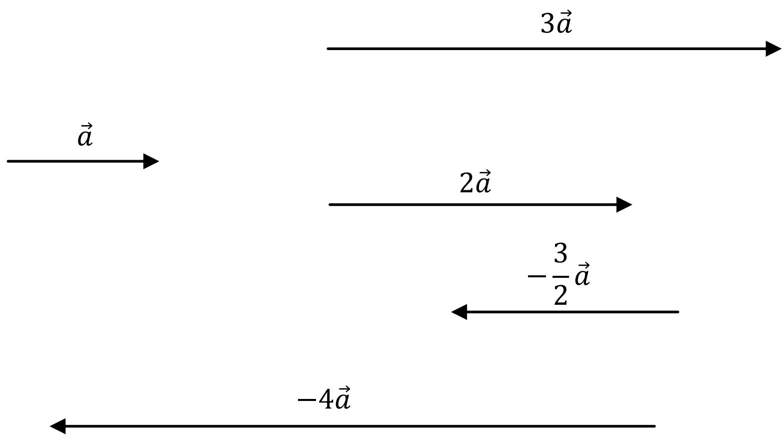

Fig. 2-5: Multiplication of a vector by a scalar.

The **multiplication of a vector by a scalar** has the following properties:

1) It is commutative, i.e.

$$k\vec{a} = \vec{a}\,k \qquad (2-5)$$

2) It is associative, i.e.

$$k(\lambda\vec{a}) = \lambda(k\vec{a}) = (k\lambda)\vec{a} \qquad (2-6)$$

3) It is distributive, i.e.

$$(k+\lambda)\vec{a} = k\vec{a} + \lambda\vec{a} \quad \boldsymbol{and} \quad k\left(\vec{a}+\vec{b}\right) = k\vec{a} + k\vec{b} \qquad (2-7)$$

2-3) The unit vector

If we divide a vector by its magnitude (**positive number**) then we obtain a new vector which has **the same direction and orientation with the original vector and magnitude equal to $\mathbf{1}$**. For this reason the thus obtained vector is called "**a unit vector**". To emphasize that a vector $\vec{u}$ is a unit vector we use the symbol $\hat{u}$. The magnitude of any unit vector $\hat{u}$ is (by definition) equal to 1, i.e. $|\hat{\boldsymbol{u}}| = \mathbf{1}$.

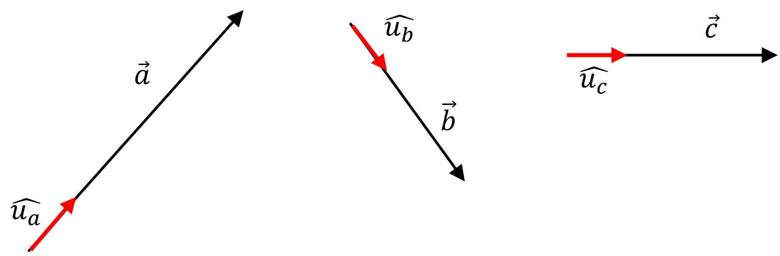

Fig. 2-6: Unit vectors.

In Fig. 2-7, $\widehat{u_a}$, $\widehat{u_b}$, $\widehat{u_c}$ are the unit vectors corresponding to the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.

$$\begin{Bmatrix} \widehat{u_a} = \dfrac{\vec{a}}{|\vec{a}|} & |\widehat{u_a}| = 1 \\ \widehat{u_b} = \dfrac{\vec{b}}{|\vec{b}|} & |\widehat{u_b}| = 1 \\ \widehat{u_c} = \dfrac{\vec{c}}{|\vec{c}|} & |\widehat{u_c}| = 1 \end{Bmatrix}$$

2-4) The position vector

The position vector of a point A with respect to **an arbitrarily chosen origin $\boldsymbol{O}$**, is the vector $\overrightarrow{OA}$. The vector $\overrightarrow{OA}$ **uniquely** specifies the position of the point A relative to the origin O.

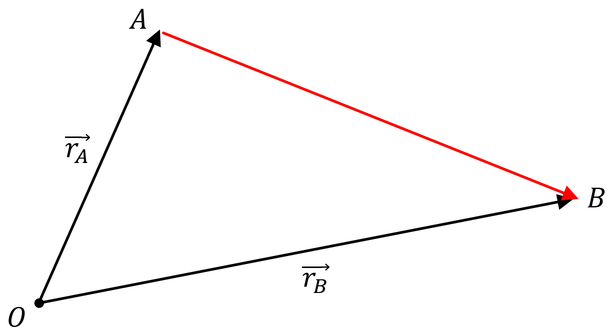

Fig. 2-7: The position vectors.

In Fig. 2-7, let $\overrightarrow{r_A}$ and $\overrightarrow{r_B}$ be the position vectors of the points A and B respectively, relative to the same origin O. Then,

$$\overrightarrow{AB} = \overrightarrow{r_B} - \overrightarrow{r_A} \qquad (2-8)$$

In words: **Every vector $\overrightarrow{\boldsymbol{AB}}$ is equal to the difference of the position vector of its terminal point $\boldsymbol{B}$ minus the position vector of its initial point $\boldsymbol{A}$.**

2-5) Collinear vectors

Two or more vectors are said to be **collinear** if they are **parallel to the same straight line**, (and as such are therefore **parallel to each other**).

If $\vec{a}$ is a given vector, then any vector $\vec{b}$ of the form $\boldsymbol{\vec{b} = k\vec{a}}$ will be collinear to $\vec{a}$, (k is a real number ranging from $-\infty$ up to $+\infty$). In other words, **the general expression of all vectors collinear to $\boldsymbol{\vec{a}}$ is $\boldsymbol{k\vec{a}}$, ($\boldsymbol{k}$ real)**.

The vectors whose directions are neither parallel nor coincident are said to be **non-collinear**.

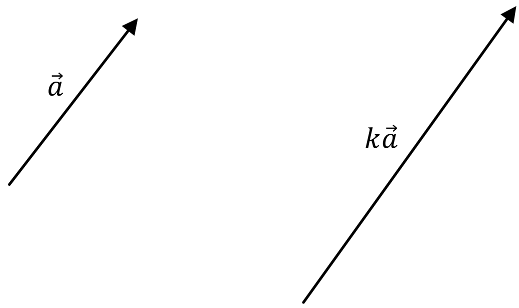

Fig. 2-8: Collinear vectors.

2-6) Coplanar vectors

Two or more vectors are said to be **coplanar** if they are **parallel to the same plane**.

If $\vec{a}$ and $\vec{b}$ are any two, **non collinear vectors**, then any vector $\vec{c}$ of the form $\boldsymbol{\vec{c} = k\vec{a} + \lambda\vec{b}}$**,** ($k$ and λ not zero simultaneously), **will be coplanar to $\boldsymbol{\vec{a}}$ and $\boldsymbol{\vec{b}}$**.

Three or more vectors which cannot be made to lie in the same plane are called **non coplanar** vectors.

Let us assume that two vectors $\vec{a}$ and $\vec{b}$ are parallel to the plane (Π) as shown in Fig. 2-9. Then we may translate the two vectors, so that they lie on the plane (Π) and have the same origin O. The vector $\vec{c} = k\vec{a} + \lambda\vec{b}$ lies also on the plane (Π), which means that the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

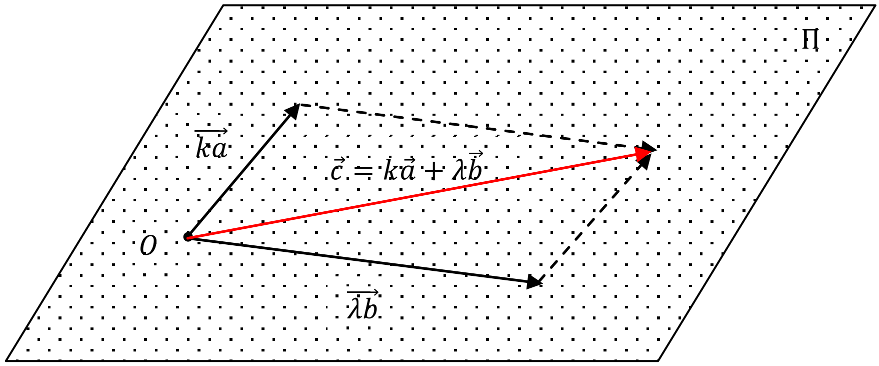

Fig. 2-9: Coplanar vectors.

Example 2-1

Simplify the following vector quantities:

a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$ **b)** $\overrightarrow{AB} - \overrightarrow{BC} + \overrightarrow{CA}$

Solution

It is known that every vector $\overrightarrow{AB}$ is equal to the difference of the position vector of its terminal point B minus the position vector of its initial point A (see equation (2-8)).

$$\boldsymbol{a})\ \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = (\overrightarrow{r_B} - \overrightarrow{r_A}) + (\overrightarrow{r_C} - \overrightarrow{r_B}) + (\overrightarrow{r_D} - \overrightarrow{r_C}) + (\overrightarrow{r_E} - \overrightarrow{r_D}) \Longrightarrow$$

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{r_E} - \overrightarrow{r_A} = \overrightarrow{AE}$$

$$\boldsymbol{b})\ \overrightarrow{AB} - \overrightarrow{BC} + \overrightarrow{CA} = (\overrightarrow{r_B} - \overrightarrow{r_A}) - (\overrightarrow{r_C} - \overrightarrow{r_B}) + (\overrightarrow{r_A} - \overrightarrow{r_C}) \Longrightarrow$$

$$\overrightarrow{AB} - \overrightarrow{BC} + \overrightarrow{CA} = 2(\overrightarrow{r_B} - \overrightarrow{r_C}) = 2\,\overrightarrow{CB}$$

Example 2-2

If ABCD is a quadrilateral and $\overrightarrow{OB} + \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{OA}$ show that ABCD is a parallelogram, (O is an arbitrary origin in space).

Solution

$$\overrightarrow{OB} + \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{OA} \Longrightarrow$$

$$(\overrightarrow{r_B} - \overrightarrow{r_O}) + (\overrightarrow{r_D} - \overrightarrow{r_O}) = (\overrightarrow{r_C} - \overrightarrow{r_O}) + (\overrightarrow{r_A} - \overrightarrow{r_O}) \Longrightarrow$$

$$\overrightarrow{r_B} + \overrightarrow{r_D} - 2\overrightarrow{r_0} = \overrightarrow{r_C} + \overrightarrow{r_A} - 2\overrightarrow{r_0} \Longrightarrow$$

$$\overrightarrow{r_B} + \overrightarrow{r_D} = \overrightarrow{r_C} + \overrightarrow{r_A} \Longrightarrow \overrightarrow{r_B} - \overrightarrow{r_C} = \overrightarrow{r_A} - \overrightarrow{r_D} \Longrightarrow \overrightarrow{CB} = \overrightarrow{DA}$$

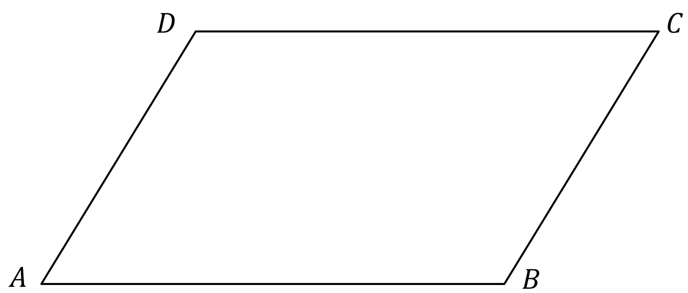

Fig. 2-10: Proof that ABCD is a parallelogram.

The vector equality $\overrightarrow{CB} = \overrightarrow{DA}$ implies that $CB \parallel DA$ **and** $CB = DA$, and this implies that ABCD is a parallelogram.

Example 2-3

Show that the sum of vectors from the center to the vertices of a square ABCD is zero.

Solution

Let O be the center of a square ABCD, as shown in Fig. 2-11. Then,

$$\overrightarrow{OA} = -\overrightarrow{OC} \quad \boldsymbol{and} \quad \overrightarrow{OB} = -\overrightarrow{OD} \Longrightarrow$$

$$\overrightarrow{OA} + \overrightarrow{OC} = 0 \quad \boldsymbol{and} \quad \overrightarrow{OB} + \overrightarrow{OD} = 0$$

and adding term wise we get, $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 0$.

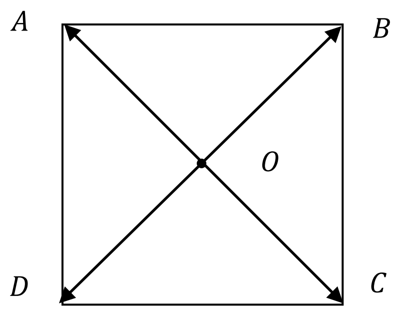

Fig. 2-11: A square ABCD with center O.

Example 2-4

Use the **triangular inequality** (known from Geometry) to prove the vector inequalities, $\left|\vec{a} + \vec{b}\right| \leq |\vec{a}| + \left|\vec{b}\right|$ and $\left||\vec{a}| - \left|\vec{b}\right|\right| \leq \left|\vec{a} - \vec{b}\right|$.

Solution

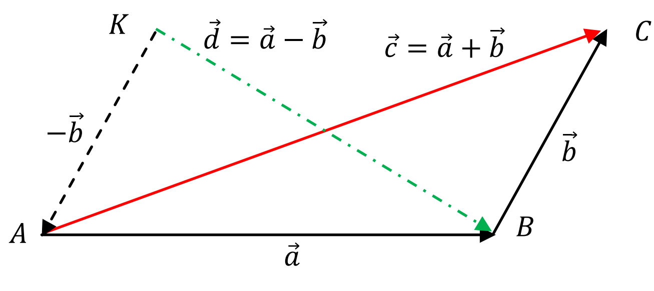

Fig. 2-12: Vector inequalities.

Let ABC a given triangle. From Geometry it is known that

$$AC < AB + BC \quad \boldsymbol{and} \quad KB > |AB - KA| \qquad (*)$$

From the first inequality in (*) we get,

$$|\overrightarrow{AC}| < |\overrightarrow{AB}| + |\overrightarrow{BC}| \Rightarrow |\vec{c}| < |\vec{a}| + |\vec{b}| \xrightarrow{(\vec{c}=\vec{a}+\vec{b})}$$

$$|\vec{a} + \vec{b}| < |\vec{a}| + |\vec{b}|$$

The equality holds only in case $\vec{a}$ and $\vec{b}$ are collinear.

From the second inequality in (*) we get,

$$|\overrightarrow{KB}| > \left||\overrightarrow{AB}| - |\overrightarrow{KA}|\right| \Rightarrow |d| > \left||\vec{a}| - |\vec{b}|\right| \xrightarrow{(\vec{d}=\vec{a}-\vec{b})}$$

$$|\vec{a} - \vec{b}| > \left||\vec{a}| - |\vec{b}|\right|$$

The equality holds only in case $\vec{a}$ and $\vec{b}$ are collinear.

Example 2-5

Show that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to one –half of the third side.

Solution

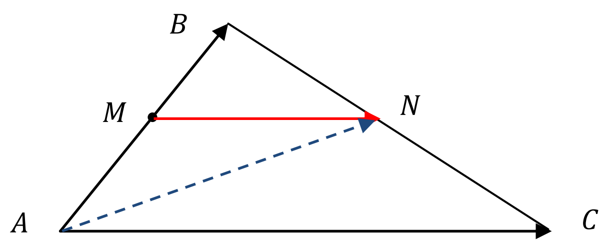

Fig. 2-13: A triangle ABC (M and N midpoints of AB and BC).

Let M and N be the mid-points of the sides AB and BC respectively. Then

$$\overrightarrow{AN} = \overrightarrow{AM} + \overrightarrow{MN} = \overrightarrow{AC} + \overrightarrow{CN} \Rightarrow$$

$$\overrightarrow{MN} = \overrightarrow{AC} + \overrightarrow{CN} - \overrightarrow{AM} = \overrightarrow{AC} + \overrightarrow{CN} + \overrightarrow{MA} \Longrightarrow$$

$$\overrightarrow{MN} = \overrightarrow{AC} + \frac{1}{2}\overrightarrow{CB} + \frac{1}{2}\overrightarrow{BA} = \overrightarrow{AC} + \frac{1}{2}\left(\overrightarrow{CB} + \overrightarrow{BA}\right) \Longrightarrow$$

$$\overrightarrow{MN} = \overrightarrow{AC} + \frac{1}{2}\overrightarrow{CA} = \overrightarrow{AC} - \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\overrightarrow{AC}$$

This means that $MN \parallel AC$ and that $MN = \left|\overrightarrow{MN}\right| = \frac{1}{2}AC = \frac{1}{2}\left|\overrightarrow{AC}\right|$.

Example 2-6

Show that the diagonals of any parallelogram bisect each other.

Solution

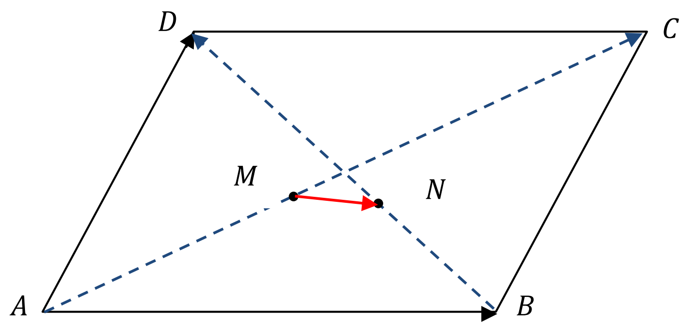

Fig. 2-14: A parallelogram ABCD.

Let ABCD be a given parallelogram. The diagonal $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{AB}$ (**the parallelogram law of addition**).The mid-point M of $\overrightarrow{AC}$ is determined from the equation

$$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{AB}\right) \qquad (*)$$

The diagonal $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$ and the mid-point N of $\overrightarrow{BD}$ is determined from the equation

$$\overrightarrow{BN} = \frac{1}{2}\overrightarrow{BD} = \frac{1}{2}\left(\overrightarrow{BA} + \overrightarrow{AD}\right) = \frac{1}{2}\left(-\overrightarrow{AB} + \overrightarrow{AD}\right) \quad (**)$$

From the vector equation

$$\overrightarrow{MN} + \overrightarrow{NB} + \overrightarrow{BA} + \overrightarrow{AM} = 0 \Longrightarrow \overrightarrow{MN} = -\left(\overrightarrow{NB} + \overrightarrow{BA} + \overrightarrow{AM}\right) \Longrightarrow$$

$$\overrightarrow{MN} = -\left(-\overrightarrow{BN} - \overrightarrow{AB} + \overrightarrow{AM}\right) = \overrightarrow{BN} + \overrightarrow{AB} - \overrightarrow{AM} \overset{(*)(**)}{\Longrightarrow}$$

$$\overrightarrow{MN} = \frac{1}{2}\left(-\overrightarrow{AB} + \overrightarrow{AD}\right) + \overrightarrow{AB} - \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{AB}\right) = 0 \Longrightarrow M \equiv N$$

This means that the **two mid-points coincide**, i.e. the diagonals bisect each other.

PROBLEMS

2-1) Show that the median of a trapezoid is parallel to the bases and equal to one-half of their sum.

2-2) A vector $\vec{a}$ belonging to two directions is the null vector.

2-3) In a parallelogram ABCD show that $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$ and $\overrightarrow{AB} - \overrightarrow{AD} = \overrightarrow{DB}$.

2-4) If AM is the median of a triangle ABC, show that $\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AM}$.

2-5) If AD, BE and CF are the medians of a triangle ABC, show that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = 0$.

2-6) If G is the centroid of a triangle ABC show that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = 0$.

2-7) If a line divides two sides of a triangle proportionally show that this line is parallel to the third side.

2-8) Let OABC be a tetrahedron and D be the midpoint of AB. Show that $2\,\overrightarrow{CD} = \overrightarrow{OA} + \overrightarrow{OB} - 2\,\overrightarrow{OC}$.

2-9) Let O be the center of a regular polygon with n-sides, $A_1A_2 \cdots A_n$. Show that:

1) $\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \cdots + \overrightarrow{OA_n} = 0.$

2) If M is an arbitrary point in the plane, then

$$\overrightarrow{MA_1} + \overrightarrow{MA_2} + \overrightarrow{MA_3} + \cdots + \overrightarrow{MA_N} = n\,\overrightarrow{MO}.$$

2-10) Show that the line segment joining the mid-points of the diagonals of a trapezoid is parallel to the bases of the trapezoid and equal to one-half of their difference.

CHAPTER 3: Right-handed and Left-handed systems, Cartesian Coordinates in Space

3-1) Right-handed and left-handed systems

Let us consider three **non coplanar** vectors $\vec{a}, \vec{b}$ and $\vec{c}$ emanating from the same origin O, as shown in Fig. 3-1.

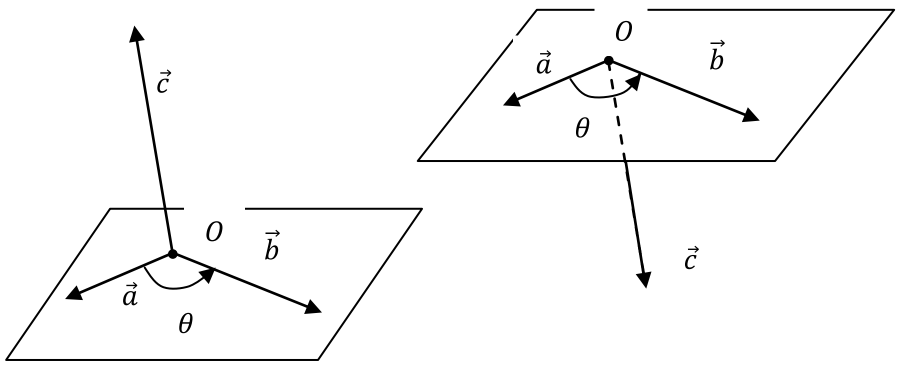

Fig. 3-1: Right-handed and left-handed systems.

Let (Π) be the plane defined by the two vectors $\vec{a}$ and $\vec{b}$ and θ be the **smallest angle** $(0 < \theta < \pi)$ at which when the vector $\vec{a}$ rotates about the origin O coincides with the vector $\vec{b}$. Now let us assume that we place **a right-hand screw** (which is used in everyday life) at the origin O and rotate it in the direction defined by the angle θ. If the right-hand screw advances towards that part of the space where the vector $\vec{c}$ lies, then we say that three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ (**in that order**) define **a right-handed coordinate system** $\{\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}\}$.

If in the aforesaid process, the right handed screw advances towards the part of the space where the vector $\vec{c}$ does not lie, then we say that the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ (**in that order**) define **a left-handed coordinate system** $\{\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}, \vec{\boldsymbol{c}}\}$.

The first figure in Fig. 3-1 shows a right-handed system while the second figure shows a left-handed system, (note that in both figures the right-hand screw advances from the **“bottom” towards the “top”**).

Assuming that the triad of vectors $\{\vec{a}, \vec{b}, \vec{c}\}$ (**in that order**) forms a right-hand coordinate system, then **any circular permutation** of these three vectors will still produce a right-handed coordinate system.

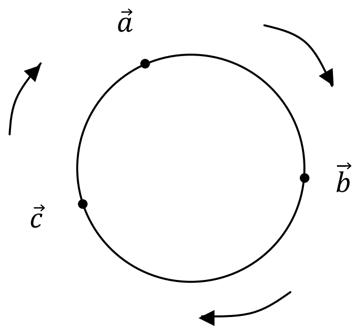

Fig. 3-2: Cyclic permutation.

In Fig. 3-2, if $\{\vec{a}, \vec{b}, \vec{c}\}$ form a right-handed coordinate system, then $\{\vec{b}, \vec{c}, \vec{a}\}$ and $\{c, \vec{a}, \vec{b}\}$ will also form right-handed coordinate systems. On the contrary, **if we interchange the position of two vectors retaining the third one in its original place**, then we change the orientation of the triad. For example, assuming that $\{\vec{a}, \vec{b}, \vec{c}\}$ is right-handed system, then $\{\vec{b}, \vec{a}, \vec{c}\}$ and $\{\vec{a}, \vec{c}, \vec{b}\}$ will be left-handed systems (let the reader check it).

Right-handed coordinate systems shall be assumed throughout this textbook.

3-2) Cartesian coordinates in space

Let us consider three axes in space $\{Ox, Oy, Oz\}$ which are **non coplanar and mutually perpendicular to each other**. Let also $\hat{x}, \hat{y}$ and $\hat{z}$ be the **unit vectors** along the axes Ox, Oy and Oz respectively. Assuming that

a) The three unit vectors $\{\hat{x}, \hat{y}, \hat{z}\}$ form a right-handed coordinate system and

b) They have equal lengths, i.e.

$$|\hat{x}| = |\hat{y}| = |\hat{z}| = 1 \qquad (3-1)$$

then we say that the three axes $\{Ox, Oy, Oz\}$ form **a Cartesian coordinate System** in space.

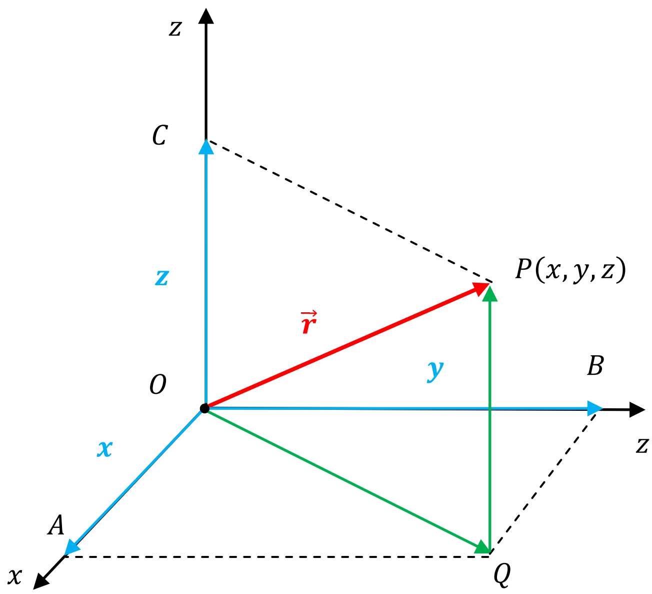

Fig. 3-3: A Cartesian Coordinates System in space.

Let $\vec{r} = \overrightarrow{OP}$ be **the position vector** of a point P in space, relative to the Cartesian system $\{Ox, Oy, Oz\}$ as shown in Fig. 3-3, and $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ be the vector projections of $\overrightarrow{OA}$ on Ox, Oy and Oz respectively.

We note that the vector radius $\overrightarrow{OP}$ is the vector sum of the vectors $\overrightarrow{OQ}$ and $\overrightarrow{QP}$, i.e.

$$\overrightarrow{OP} = \vec{r} = \overrightarrow{OQ} + \overrightarrow{QP} = \underbrace{\overrightarrow{OA} + \overrightarrow{OB}}_{=\overrightarrow{OQ}} + \underbrace{\overrightarrow{OC}}_{=\overrightarrow{QP}} \Longrightarrow$$

$$\overrightarrow{OP} = \vec{r} = \left|\overrightarrow{OA}\right|\hat{x} + \left|\overrightarrow{OB}\right|\hat{y} + \left|\overrightarrow{OC}\right|\hat{z}$$

or if we call $x = \left|\overrightarrow{OA}\right|$, $y = \left|\overrightarrow{OB}\right|$ and $z = \left|\overrightarrow{OC}\right|$,

$$\overrightarrow{OP} = \vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \qquad (3-2)$$

The real numbers x, y and z (each one ranging from $-\infty$ up to $+\infty$, depending on the position of the point P in space relative to the given system) are called the $x-$ coordinate, the $y-$ coordinate and the $z-$ coordinate respectively of the point P relative to the given Cartesian system. **We note that any point $\boldsymbol{P}$ in space, is uniquely determined from an ordered triad of numbers, its Cartesian coordinates $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$.** We may therefore use the notation $\boldsymbol{P(x, y, z)}$ to uniquely define the point P in space in terms of its **Cartesian coordinates**.

The **geometrical distance** of $P(x, y, z)$ from the origin $O(0,0,0)$ is the **magnitude** (absolute value) of the vector $\vec{r} = \overrightarrow{OP}$, i.e.

$$r = OP = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} \qquad (3-3)$$

Points lying on **the coordinate axes** have the following representation:

$$\begin{Bmatrix} (x, 0,0) & -\infty < x < \infty & x - axis \\ (0, y, 0) & -\infty < y < \infty & y - axis \\ (0,0, z) & -\infty < z < \infty & z - axis \end{Bmatrix} \qquad (3-4)$$

Points lying on **the coordinate planes** have the following representation:

$$\begin{Bmatrix} (x, y, 0) & -\infty < x < \infty, -\infty < y < \infty, & x - y\,plane \\ (0, y, z) & -\infty < y < \infty, -\infty < z < \infty, & y - z\,plane \\ (x, 0, z) & -\infty < x < \infty, -\infty < z < \infty, & x - z\,plane \end{Bmatrix} \qquad (3-5)$$

The following simple Theorems are useful when working with vectors.

Theorem 3-1: If $\vec{r_1} = x_1\hat{x} + y_1\,\hat{y} + z_1\,\hat{z}$ and $\vec{r_2} = x_2\,\hat{x} + x_2\,\hat{y} + z_2\,\hat{z}$ then

$$\vec{r_1} \pm \vec{r_2} = (x_1 \pm x_2)\,\hat{x} + (y_1 \pm y_2)\hat{y} + (z_1 \pm z_2)\,\hat{z} \qquad (3-6)$$

Theorem 3-2: If $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$ is a vector and k is a real number (scalar), then

$$k\,\vec{r} = (kx)\,\hat{x} + (ky)\,\hat{y} + (kz)\,\hat{z} \qquad (3-7)$$

3-3) Chaslse's Theorem

A line (ε) equipped with a unit vector $\hat{u}$ emanating from an arbitrary (but fixed) point O of the line (**the origin**) is called **an axis**.

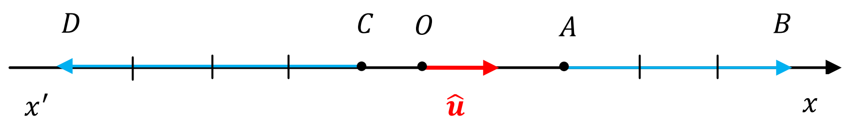

Fig. 3-4: The axis of real numbers.

The **relative value or relative magnitude or algebraic value** of a vector $\overrightarrow{AB}$ lying on the axis (or **being parallel** to the axis) is **the real number k (positive or negative)** which when multiplied by the unit vector $\hat{u}$ produces the vector $\overrightarrow{AB}$. We shall use the notation $\overline{\boldsymbol{AB}}$ to denote **the relative value** of the vector $\overrightarrow{AB}$. For example, in Fig. 3-4, $\overline{AB} = 3$ while $\overline{CD} = -4$, since $\overrightarrow{AB} = \overline{AB}\,\hat{u} = 3\,\hat{u}$ while $\overrightarrow{CD} = \overline{CD}\,\hat{u} = (-4)\,\hat{u}$. Note the difference between **the magnitude and the relative value of a vector**. The magnitude of a vector is always **a positive number**, (the geometrical length of the vector) whereas its relative value can be either **positive or negative**, depending on whether the vector is of the same or opposite orientation of the unit vector.

Based on the concept of the relative value of a vector, we may define **the ratio of two vectors lying on a given axis (or being parallel) to the same axis** as the ratio of their corresponding relative values, i.e.

$$\frac{\overrightarrow{AB}}{\overrightarrow{CD}} = \frac{\overline{AB}}{\overline{CD}} \qquad (3-8)$$

Theorem 3-3: The relative value of any vector $\overrightarrow{AB}$ lying on the $x'Ox$ axis, is the difference between the coordinate x_B of its terminal point B minus the coordinate x_A of its initial point A.

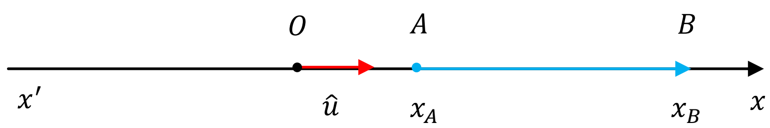

Fig. 3-5: The relative (or signed) value of a vector.

Indeed, in Fig. 3-5, $\overline{AB} = x_B - x_A$, while $\overline{BA}$ which is the relative value of the vector $\overrightarrow{BA}$ will be $\overline{BA} = x_A - x_B$.

Theorem 3-4: (Chaslse's Theorem): Let A, B and C be any three points on an axis $x'Ox$. No matter which is the position of these three points relative to each other, it is always true that

$$\overline{AB} + \overline{BC} = \overline{AC} \qquad (3-9)$$

Proof: By virtue of Theorem 3-3, we have,

$$\overline{AB} + \overline{BC} = (x_B - x_A) + (x_C - x_B) = x_C - x_A = \overline{AC} \qquad (3-10)$$

An **equivalent** expression of **Chaslse's Theorem** is the following:

$$\overline{AB} + \overline{BC} + \overline{CA} = 0 \qquad (3-11)$$

Indeed, equation (3-9) may be written equivalently as

$$\overline{AB} + \overline{BC} = \overline{AC} = -\overline{CA} \Leftrightarrow \overline{AB} + \overline{BC} + \overline{CA} = 0 \qquad (3-12)$$

Theorem 3-5: (A generalization of Chaslse's Theorem): For any n points $A_1, A_2, A_3, \cdots, A_{n-1}, A_n$ on an axis $x'Ox$, it is always true that

$$\overline{A_1A_2} + \overline{A_2A_3} + \overline{A_3A_4} + \cdots \overline{A_{n-1}A_n} = \overline{A_1A_n} \qquad (3-13)$$

or equivalently,

$$\overline{A_1A_2} + \overline{A_2A_3} + \overline{A_3A_4} + \cdots \overline{A_{n-1}A_n} + \overline{A_nA_1} = 0 \qquad (3-14)$$

The proof is similar to the proof of Theorem 3-4. Let the reader try to prove (3-13) and (3-14).

3-4) Equality of vectors expressed in Cartesian coordinates

Theorem 3-6: Let $\overrightarrow{r_1} = x_1\,\hat{x} + y_1\,\hat{y} + z_1\,\hat{z}$ and $\overrightarrow{r_2} = x_2\,\hat{x} + y_2\,\hat{y} + z_2\,\hat{z}$ be **the Cartesian expressions** of two vectors $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ in a given Cartesian coordinate system $\{Ox, Oy, Oz\}$. The following equivalency holds true:

$$\overrightarrow{r_1} = \overrightarrow{r_2} \Leftrightarrow \begin{Bmatrix} x_1 = x_2 \\ y_1 = y_2 \\ z_1 = z_2 \end{Bmatrix} \qquad (3-15)$$

Proof: The equality $\overrightarrow{r_1} = \overrightarrow{r_2}$ implies $\overrightarrow{r_1} - \overrightarrow{r_2} = 0$, i.e.

$$(x_1 - x_2)\,\hat{x} + (y_1 - y_2)\,\hat{y} + (z_1 - z_2)\,\hat{z} = 0 \qquad (*)$$

Since the left-hand side in (*) represents the null vector, **its length must be zero**, and equation (3-3) implies that

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = 0 \Leftrightarrow \begin{Bmatrix} x_1 - x_2 = 0 \\ y_1 - y_2 = 0 \\ z_1 - z_2 = 0 \end{Bmatrix}$$

and this completes the proof.

Equation (3-15) shows that in general, **any vector equality splits into three scalar equations**, **one for each coordinate axis**.

3-5) Division of a vector $\overrightarrow{AB}$ in a given ratio λ

The partial ratio (ABC) of three points A, B and C lying on an axis is defined to be the ratio $\frac{\overline{AC}}{\overline{CB}}$, i.e. the ratio of the relative values of the two vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$ whose sum is equal to the vector $\overrightarrow{AB}$, ($\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$).

The partial ratio is **positive** when C lies between A and B and **negative** when C lies outside the segment AB, since in this case the vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$ have opposite orientations. Also the partial ratio **vanishes** when C coincides with A and **tends to** ∞ when C approaches the point B.

Theorem 3-7: Let $\overrightarrow{r_A}$ and $\overrightarrow{r_B}$ be the position vectors of two points A and B respectively. Then the position vector $\overrightarrow{r_C}$ of a point C lying on the line AB and dividing the vector $\overrightarrow{AB}$ in partial ratio $(ABC) = \lambda$ is given by the formula

$$\overrightarrow{r_C} = \frac{\overrightarrow{r_A} + \lambda\,\overrightarrow{r_B}}{1+\lambda} \qquad (3-16)$$

Proof: Since $(ABC) = \frac{\overline{AC}}{\overline{CB}} = \lambda$, equation (3-8) implies that

$$\frac{\overline{AC}}{\overline{CB}} = \lambda \Longrightarrow \overrightarrow{AC} = \lambda\,\overrightarrow{CB} \overset{(2-8)}{\Longrightarrow} \overrightarrow{r_C} - \overrightarrow{r_A} = \lambda(\overrightarrow{r_B} - \overrightarrow{r_C}) \Longrightarrow \overrightarrow{r_C} = \frac{\overrightarrow{r_A} + \lambda\,\overrightarrow{r_B}}{1+\lambda}$$

and the proof is completed.

Vector equation (3-16) splits into the following **three scalar equations**,

$$\left\{x_C = \frac{x_A + \lambda x_B}{1+\lambda} \quad y_C = \frac{y_A + \lambda y_B}{1+\lambda} \quad z_C = \frac{z_A + \lambda z_B}{1+\lambda}\right\} \qquad (3-17)$$

Corollary 3-1: The position vector of **the midpoint M** of the vector $\overrightarrow{AB}$ is given by the formula

$$\overrightarrow{r_M} = \frac{\overrightarrow{r_A} + \overrightarrow{r_B}}{2} \qquad (3-18)$$

This formula is obtained directly from (3-16), since in this case the partial ratio $(ABC) = \lambda = 1$. This vector equation is split into three scalar equations

$$\left\{x_M = \frac{x_A + x_B}{2} \quad y_M = \frac{y_A + y_B}{2} \quad z_M = \frac{z_A + z_B}{2}\right\} \qquad (3-19)$$

3-6) Translation of Coordinate axes

Let a new coordinate system $\{O'x', O'y', O'z'\}$ be obtained by means of **a translation** of the axes Ox, Oy and Oz, the displacement vector being $\vec{R} = a\,\hat{x} + b\,\hat{y} + c\,\hat{z}$, (see Fig. 3-6).

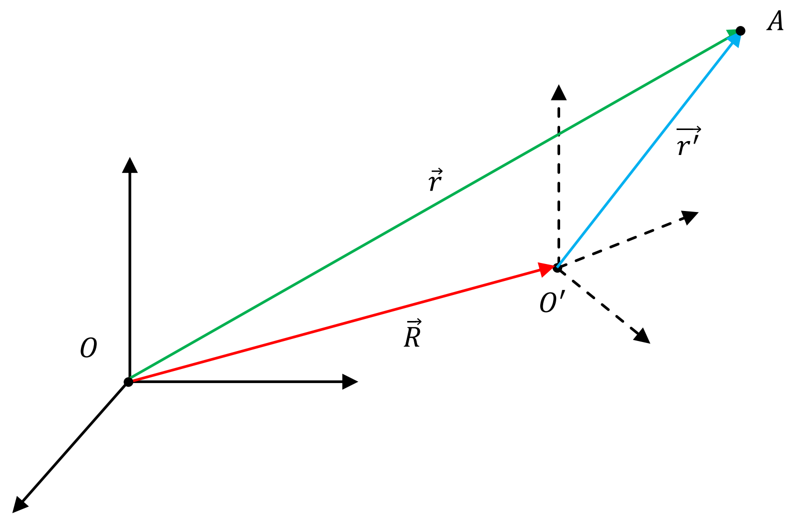

Fig. 3-6: Translation of Coordinate axes.

If the position vector of a point A in the $\{Ox, Oy, Oz\}$ system is $\vec{r}$ and the position vector of the same point A in the $\{O'x', O'y', O'z'\}$ system is $\overrightarrow{r'}$, then as it is seen in Fig. 3-6,

$$\vec{r} = \vec{R} + \overrightarrow{r'} \qquad (3-20)$$

Projecting this vector equality on the coordinate axes we obtain the formulas

$$\{x = a + x' \quad y = b + y' \quad z = c + z'\} \qquad (3-21)$$

which describes how the new coordinates are connected to the old ones.

Note 1: Since any vector $\vec{r}$ in Cartesian coordinates can be expressed as $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$, we may alternatively use the notation $\vec{\boldsymbol{r}} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ where (x, y, z) is an ordered triad of numbers. For example, $\vec{r} = (3,2,5)$ stands for the vector $\vec{r} = 3\,\hat{x} + 2\,\hat{y} + 5\,\hat{z}$, while $\vec{r} = (-2,7,-1)$ stands for $\vec{r} = -2\,\hat{x} + 7\,\hat{y} - \hat{z}$, etc.

Note 2: Let $\overrightarrow{r_A} = (x_A, y_A, z_A)$ and $\overrightarrow{r_B} = (x_B, y_B, z_B)$ be the position vectors of the two points A and B respectively. Then the vector

$$\vec{R} = \overrightarrow{r_B} - \overrightarrow{r_A} = (x_B - x_A, y_B - y_A, z_B - z_A) \Longrightarrow$$

$$\vec{R} = (x_B - x_A)\,\hat{x} + (y_B - y_A)\,\hat{y} + (z_B - z_A)\,\hat{z} \qquad (3-22)$$

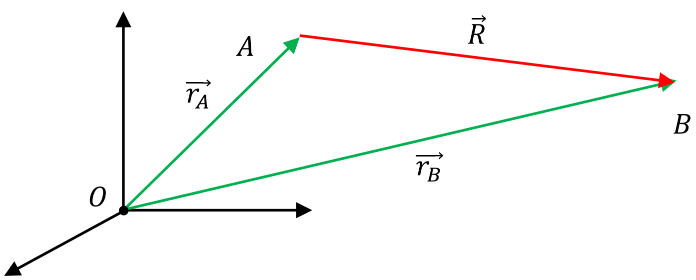

Fig. 3-7: The expression of a vector in Cartesian coordinates.

The **length of the vector $\vec{R}$**, (i.e. the **geometrical distance** between the points A and B) is (see formula (3-3)),

$$R = |\vec{R}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} \qquad (3-23)$$

Example 3-1

If $\vec{r} = 2\hat{x} + 3\hat{y} + \hat{z}$, find $|\vec{r}|$ and $|(-5)\,\vec{r}|$

Solution

$$|\vec{r}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

$$|(-5)\,\vec{r}| = |-5|\,|\vec{r}| = 5\,|\vec{r}| = 5\sqrt{14}$$

Example 3-2

If $\overrightarrow{OA} = 2\hat{x} + 3\hat{y} + 5\hat{z}$ and $\overrightarrow{OB} = \hat{x} - 2\hat{y} - 3\hat{z}$, find: $\overrightarrow{AB}$, $\left|\overrightarrow{AB}\right|$, $2\overrightarrow{OA} + 3\overrightarrow{OB}$ and $\left|2\overrightarrow{OA} + 3\overrightarrow{OB}\,\right|$.

Solution

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\hat{x} + 3\hat{y} + 5\hat{z}) - (\hat{x} - 2\hat{y} - 3\hat{z}) \Longrightarrow$$

$$\overrightarrow{AB} = (2-1)\hat{x} + \big(3-(-2)\big)\hat{y} + \big(5-(-3)\big)\hat{z} = \hat{x} + 5\hat{y} + 8\hat{z}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{1^2 + 5^2 + 8^2} = \sqrt{90}$$

$$2\overrightarrow{OA} + 3\overrightarrow{OB} = 2(2\hat{x} + 3\hat{y} + 5\hat{z}) + 3(\hat{x} - 2\hat{y} - 3\hat{z}) \Longrightarrow$$

$$2\overrightarrow{OA} + 3\overrightarrow{OB} = (4\hat{x} + 6\hat{y} + 10\hat{z}) + (3\hat{x} - 6\hat{y} - 9\hat{z}) \Longrightarrow$$

$$2\overrightarrow{OA} + 3\overrightarrow{OB} = 7\hat{x} + \hat{z}$$

$$\left|2\overrightarrow{OA} + 3\overrightarrow{OB}\right| = \sqrt{7^2 + 0^2 + 1^2} = \sqrt{50}$$

Example 3-3

Find the unit vector $\hat{r}$ associated with the vector $\vec{r} = -2\hat{x} + 3\hat{y} + \hat{z}$.

Solution

The unit vector $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ (see section 2-3), i.e.

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{-2\hat{x} + 3\hat{y} + \hat{z}}{|-2\hat{x} + 3\hat{y} + \hat{z}|} = \frac{-2\hat{x} + 3\hat{y} + \hat{z}}{\sqrt{(-2)^2 + 3^2 + 1^2}} = \frac{-2\hat{x} + 3\hat{y} + \hat{z}}{\sqrt{14}} \Longrightarrow$$

$$\hat{r} = -\frac{2}{\sqrt{14}}\hat{x} + \frac{3}{\sqrt{14}}\hat{y} + \frac{1}{\sqrt{14}}\hat{z}$$

Example 3-4

Given that $\vec{a} = a_1\hat{x} + 2\hat{y} - 7\hat{z}$ and $\vec{b} = 3\hat{x} + b_2\hat{y} + b_3\hat{z}$, determine a_1, b_2 and b_3 so that $\vec{a} = \vec{b}$.

Solution

Making use of equation (3-15) we have:

$$\vec{a} = \vec{b} \Leftrightarrow \left\{ \begin{matrix} a_1 = 3 \\ 2 = b_2 \\ -7 = b_3 \end{matrix} \right\} \Leftrightarrow \left\{ \begin{matrix} a_1 = 3 \\ b_2 = 2 \\ b_3 = -7 \end{matrix} \right\}$$

Example 3-5

If $\vec{r_A}$, $\vec{r_B}$ and $\vec{r_C}$ are the position vectors of the vertices A, B and C respectively of a triangle ABC, find the position vector of the centroid G of the triangle.

Solution

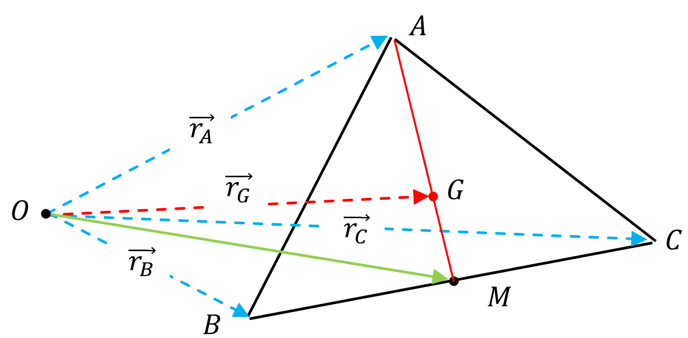

Fig. 3-8: The centroid G of a triangle ABC.

Let ABC be the triangle, AM the median corresponding to the vertex A, and G be the centroid. It is known from Geometry that $\overline{AG} = 2\overline{GM}$, (for the notation $\overline{AG}$ and $\overline{GM}$ see section 3-3). Let us consider the triangle OAM. The position vectors of A and M are $\vec{r_A} = \overrightarrow{OA}$ and $\overrightarrow{OM} = \frac{\vec{r_B}+\vec{r_C}}{2}$ (why ?), respectively. The point G (the centroid) divides the vector $\overrightarrow{AM}$ in partial ratio $(AMG) = \frac{\overline{AG}}{\overline{GM}} = 2 \div 1$ (for the definition of the partial ratio see section 3-5) and application of formula (3-18) yields,

$$\vec{r_G} = \frac{\vec{r_A} + 2\overrightarrow{OM}}{1+2} = \frac{\vec{r_A} + \vec{r_B} + \vec{r_C}}{3}$$

This shows that the position vector of the centroid G is the average of the three position vectors of the vertices.

Example 3-6

Let $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$ be two given vectors. Let also $a = |\vec{a}|$ and $b = |\vec{b}|$ be the magnitudes of $\vec{a}$ and $\vec{b}$ respectively. Show that the vector $a\vec{b} + b\vec{a}$ is collinear to the bisector of the angle formed by the two vectors.

Solution

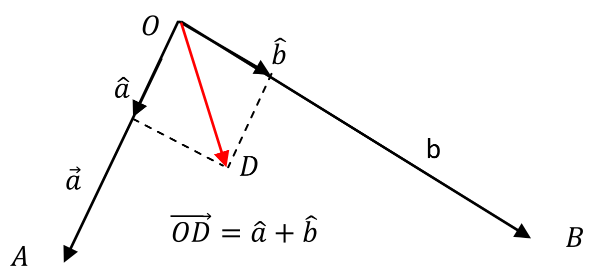

Fig. 3-9: The bisector of an angle.

Let $\hat{a}$ and $\hat{b}$ be the unit vectors of $\vec{a}$ and $\vec{b}$ respectively, as shown in Fig. 3-9. Then, $|\hat{a}| = |\hat{b}| = 1$, $\vec{a} = a\hat{a}$, $\vec{b} = b\hat{b}$, and

$$a\vec{b} + b\vec{a} = ab\hat{b} + ba\hat{a} = ab\left(\hat{a} + \hat{b}\right) = ab\,\overrightarrow{OD} \quad (*)$$

However $\overrightarrow{OD} = \hat{a} + \hat{b}$ is collinear to the bisector of the angle formed by the two vectors (**diagonal of a rhombus**), and therefore the product of $\overrightarrow{OD}$ by the positive number (ab) will be collinear to the bisector.

PROBLEMS

3-1) Let ABCD be a quadrilateral and P and Q be the mid-points of the diagonals AC and BD. Show that the sum of the vectors (**the resultant**) $\overrightarrow{AB}, \overrightarrow{CB}, \overrightarrow{CD}$ and $\overrightarrow{AD}$ is equal to 4 $\overrightarrow{PQ}$.

3-2) Let G_1, G_2, G_3 and G_4 are the centroids of the sides BCD, CDA, DAB and ABC respectively, of a tetrahedron ABCD. Show that the vector sum $\overrightarrow{AG_1} + \overrightarrow{BG_2} + \overrightarrow{CG_3} + \overrightarrow{DG_4} = 0$.

Hint: Let $\vec{r_A}$, $\vec{r_B}$, $\vec{r_C}$ and $\vec{r_D}$ will be the position vectors of the vertices A, B, C and D with respect to an arbitrary origin O. Then, according to Example 3-5, $\vec{r_{G_1}} = \frac{1}{3}(\vec{r_B} + \vec{r_C} + \vec{r_D})$, etc. The vector $\overrightarrow{AG_1} = \vec{r_{G_1}} - \vec{r_A}$, etc.

3-3) If $\overrightarrow{AB} = 2\hat{x} + \hat{y} + 3\hat{z}$, find its magnitude $AB = \left|\overrightarrow{AB}\right|$ and the unit vector $\hat{u}$ in the direction of $\overrightarrow{AB}$.

(**Answer:** $AB = \left|\overrightarrow{AB}\right| = \sqrt{14}$, and $\hat{u} = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \frac{1}{\sqrt{14}}(2\hat{x} + \hat{y} + 3\hat{z})$).

3-4) The position vectors of three points A, B and C in space are $\vec{r_A} = 2\hat{x} + \hat{y} + \hat{z}$, $\vec{r_B} = \hat{x} + \hat{z}$ and $\vec{r_C} = \hat{y} + 2\hat{z}$ respectively. Show that ABC is a right triangle.

Hint: Determine the lengths $AB = \left|\overrightarrow{AB}\right|, AC = \left|\overrightarrow{AC}\right|$ and $BC = \left|\overrightarrow{BC}\right|$ and verify that **the Pythagorean Theorem** is satisfied.

3-5) Consider the vectors $\vec{a} = (1,2,3)$ and $\vec{b} = (-2,1,-1)$ and find: **1)** The vector $\vec{c} = 2\vec{a} + 3\vec{b}$, **2)** The length $|\vec{c}|$ and **3)** The unit vector $\hat{c}$ in the direction of $\vec{c}$.

(**Answer: 1)** $\vec{c} = -4\hat{x} + 7\hat{y} + 3\hat{z}$, **2)** $|\vec{c}| = \sqrt{74}$, **3)** $\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{74}}(-4\hat{x} + 7\hat{y} + 3\hat{z})$).

3-6) If $\vec{a} = (2,3,1), \vec{b} = (0,1,2)$ and $\vec{c} = (1,1,0)$. Determine:

1) The vector $\vec{s} = 2\vec{a} + \vec{b} + \vec{c}$, **2)** The length $|\vec{s}|$ and **3)** The unit vector $\hat{s}$ in the direction of $\vec{s}$.

3-7) If three **coplanar** forces $\vec{F_1}, \vec{F_2}$ and $\vec{F_3}$ acting on the point P of a solid body are in equilibrium, show that

$$\frac{|\vec{F_1}|}{\sin\theta_1} = \frac{|\vec{F_2}|}{\sin\theta_2} = \frac{|\vec{F_3}|}{\sin\theta_3}$$

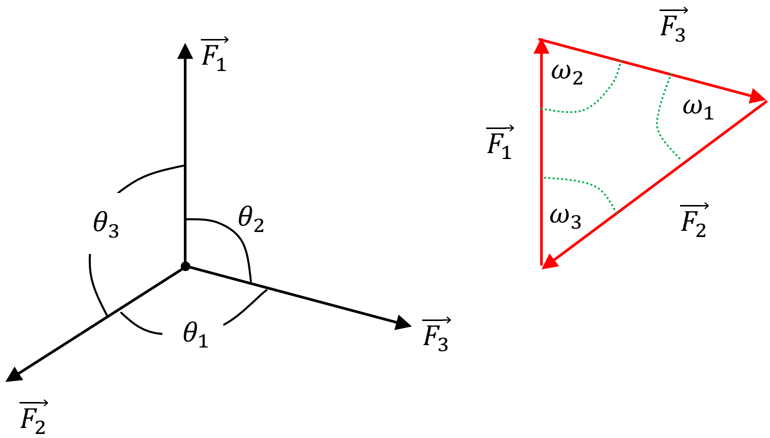

Fig. 3-10: Forces in equilibrium.

Hint: Since $\vec{F_1} + \vec{F_2} + \vec{F_3} = 0$ the "**polygon of the forces**" is closed, i.e. the vectors $\vec{F_1}, \vec{F_2}$ and $\vec{F_3}$ form a triangle (the red triangle in the Figure). Application of the "**law of sines**" yields,

$$\frac{|\vec{F_1}|}{\sin\omega_1} = \frac{|\vec{F_2}|}{\sin\omega_2} = \frac{|\vec{F_3}|}{\sin\omega_3}$$

and since $\theta_1 + \omega_1 = 180°, \theta_2 + \omega_2 = 180°$ and $\theta_3 + \omega_3 = 180°$, $\sin\theta_1 = \sin\omega_1, \sin\theta_2 = \sin\omega_2$ and $\sin\theta_3 = \sin\omega_3$, and this completes the proof.

3-8) In Fig. 3-10, the sum of the magnitudes of the forces is $200\ Nt$, while $\theta_1 = 130°$ and $\theta_2 = 160°$. Find the magnitude of each one of the forces.

Hint: The third angle $\theta_3 = 360° - (130° + 160°) = 70°$. Also from the equilibrium condition,

$$\frac{|\vec{F_1}|}{\sin 130°} = \frac{|\vec{F_2}|}{\sin 160°} = \frac{|\vec{F_3}|}{\sin 70°} = \frac{|\vec{F_1}| + |\vec{F_1}| + |\vec{F_1}|}{\sin 130° + \sin 160° + \sin 70°}$$
$$= \frac{200}{\sin 130° + \sin 160° + \sin 70°} = \frac{200}{2.047} \cong 97.70$$

and $|\vec{F_1}| = (97.70) \cdot (\sin 130°) \cong 7.Nt$, etc.

3-9) If $\vec{r_A}$, $\vec{r_B}$ and $\vec{r_C}$ are the position vectors of the vertices A, B and C respectively of a triangle ABC, show that the position vector of the **incenter** (the point where the angle bisectors intersect ≡ center of the inscribed circle) is

$$\vec{r} = \frac{a\vec{r_A} + b\vec{r_B} + c\vec{r_C}}{a+b+c}, \quad (a = BC, b = AC, c = AB)$$

3-10) Show that the segments joining the midpoints of opposite sides of a skew quadrilateral bisect each other.

3-11) Let us consider a tetrahedron ABCD whose position vectors of the vertices are $\vec{r_A}, \vec{r_B}, \vec{r_C}$ and $\vec{r_D}$. Find the position vector of the centroid G of the tetrahedron.

(**Answer:** $\vec{r_G} = \frac{\vec{r_A}+\vec{r_B}+\vec{r_C}+\vec{r_D}}{4}$).

Hint: The centroid G of the tetrahedron lies on the segment joining a vertex (say the vertex A) with the centroid G_A of the opposite side (triangle) and divides this segment in ratio $AG \div AG_A = 3 \div 4$. To derive the answer, you may use the method outlined in Example 3-5.

3-12) If the diagonals of a quadrilateral bisect each other show that the quadrilateral is a parallelogram.

3-13) If $A(3), B(-1), C(2)$ and $D(-4)$ are four points on an axis $x'x$, find: $\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{AD}$.

(**Answer:** $\overline{AB} = -4, \overline{BC} = 3, \overline{CD} = -6, \overline{AD} = -7$).

Hint: $\overline{AB} = (-1) - 3 = -4$, etc.

3-14) If $\overline{A_1A_2} = 2,\ \overline{A_2A_3} = (-4),\ \overline{A_3A_4} = 5,\ \overline{A_4A_5} = 3$, apply Chaslse's Theorem (see Section 3-3) to find: $\overline{A_1A_3}$, $\overline{A_2A_4}$ and $\overline{A_1A_5}$. Also, determine the point P which divides $\overline{A_1A_5}$ in partial ratio $(A_1A_5P) = 3 \div 1$, (see section 3-5), assuming that $x_{A_1} = 1$.

(**Answer:** $\overline{A_1A_3} = -2,\ \overline{A_2A_4} = 1,\ \overline{A_1A_5} = 6,\ x_P = {}^{11}/_{2}$).

3-15) If A, B, C and D are arbitrary points on an axis, show that

$$\overline{AB}\,\overline{CD} + \overline{AC}\,\overline{DB} + \overline{AD}\,\overline{BC} = 0$$

Hint: $\overline{AC} = \overline{AB} + \overline{BC}$ and $\overline{DB} = \overline{DC} + \overline{CB}$.

3-16) If for three points A, B and C on an axis it is true that $(\overline{AB})^3 + (\overline{BC})^3 + (\overline{CA})^3 = 0$, show that at least two points coincide.

3-17) If $O, A_1, A_2, \cdots, A_n$ are arbitrary points on an axis, find a point M such that $\overline{MA_1} + \overline{MA_2} + \cdots + \overline{MA_n} = 0$.

(**Answer:** $\overline{OM} = \frac{1}{n}(\overline{OA_1} + \overline{OA_2} + \cdots + \overline{OA_n})$).

CHAPTER 4: Linear Dependence of Vectors

4-1) Linear vector equations

Any equation of the form

$$\lambda_1\overrightarrow{r_1} + \lambda_2\overrightarrow{r_2} + \lambda_3\overrightarrow{r_3} + \cdots + \lambda_n\overrightarrow{r_n} = 0 \qquad (4-1)$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ are **scalars (real numbers)** is called **a linear vector equation**. The vectors $\overrightarrow{r_1}$, $\overrightarrow{r_2}$, $\overrightarrow{r_3}, \cdots$, $\overrightarrow{r_n}$ are said to be **linearly independent** when equation (4-1) holds true **if and only if** $\boldsymbol{\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_n = 0}$, i.e.

$$\begin{Bmatrix} \overrightarrow{r_1}, \overrightarrow{r_2}, \overrightarrow{r_3}, \cdots, \overrightarrow{r_n} \\ Linearly\ Independent \end{Bmatrix} \Leftrightarrow \{\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_n = 0\} \qquad (4-2)$$

Otherwise the vectors are said to be **linearly dependent**.

Assuming that $\{\overrightarrow{r_1}, \overrightarrow{r_2}, \overrightarrow{r_3}, \cdots, \overrightarrow{r_n}\}$ are **linearly dependent vectors**, then **at least one of them** can be expressed as a linear combination of the remaining $(n-1)$ vectors. Indeed, assuming for example that $\lambda_2 \neq 0$, then equation (4-1) implies that

$$\overrightarrow{r_2} = -\frac{\lambda_1}{\lambda_2}\overrightarrow{r_1} - \frac{\lambda_3}{\lambda_2}\overrightarrow{r_3} - \cdots - \frac{\lambda_n}{\lambda_2}\overrightarrow{r_n} \qquad (4-3)$$

which shows that $\overrightarrow{r_2}$ is expressed as a linear combination of the remaining vectors $\overrightarrow{r_1}, \overrightarrow{r_3}, \cdots, \overrightarrow{r_n}$. **The converse statement is also true**. If a vector $\vec{y}$ is a linear combination of the $(n-1)$ vectors $\{\overrightarrow{r_1}, \overrightarrow{r_2}, \cdots, \overrightarrow{r_{n-1}}\}$ then the set of vectors $\{\vec{y}, \overrightarrow{r_1}, \overrightarrow{r_2}, \cdots, \overrightarrow{r_{n-1}}\}$ is a set **of linearly dependent** vectors (Why?).

4-2) Linear dependence of collinear vectors

Theorem 4-1: Two collinear vectors $\vec{x}$ and $\vec{y}$ always satisfy a linear vector equation of the form $a\vec{x} + b\vec{y} = 0, (ab \neq 0)$.

Proof: a) Assuming that $\vec{y}$ is collinear to $\vec{x}$, then $\vec{y} = k\vec{x}$, where k is a real number, (see section 2-5), i.e.

$$\vec{y} = k\vec{x} \Longrightarrow \vec{y} - k\vec{x} = 0 \Longrightarrow \vec{y} + (-k)\vec{x} = 0 \quad (*)$$

or, if we multiply both sides of (*) by a number $L \neq 0$,

$$L\vec{y} + (-kL)\vec{x} = 0, \quad or \quad a\vec{x} + b\vec{y} = 0, where\ \ a = (-kL)\ \ and\ b = L$$

b) The converse is also true, i.e. assuming that $a\vec{x} + b\vec{y} = 0$, where a and b not zero simultaneously, then $\vec{x}$ and $\vec{y}$ are collinear. Indeed from the given equation we get $\vec{y} = -(a/b)\vec{x}$, which shows that $\vec{x}$ and $\vec{y}$ are collinear.

Theorem 4-2: If two non collinear (not parallel) vectors $\vec{x}$ and $\vec{y}$ satisfy the linear equation $a\vec{x} + b\vec{y} = 0$, then necessarily, we must have, $a = 0$ and $b = 0$.

Proof: Since assuming otherwise, for example that $b \neq 0$, then the given equation would imply that $\vec{y} = -(a/b)\vec{x}$, i.e. $\vec{x}$ and $\vec{y}$ would be collinear, which however contradicts our hypothesis. We are therefore forced to assume that $b = 0$, and then from the given equation $a = 0$, and this completes the proof.

Note: Theorem 4-2 is important in proving various Theorems of plane Euclidean Geometry with the aid of vectors.

Theorem 4-3: Three points A, B and C are collinear (lie on the same straight line), if and only if, for any point O in space, the three vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ satisfy a linear equation $a\,\overrightarrow{OA} + b\,\overrightarrow{OB} + c\,\overrightarrow{OC} = 0$, where the sum of the scalar coefficients $a + b + c = 0$.

Proof: a) Let us consider the three vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ shown in Fig. 4-1. Assuming that A, B and C lie on the same straight line (ε), the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are collinear, i.e.

$$\overrightarrow{AB} = \lambda\,\overrightarrow{BC}, \qquad for\ a\ suitable\ value\ of\ \lambda \quad (*)$$

Since $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ and $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$, equation (*) implies,

$$\overrightarrow{OB} - \overrightarrow{OA} = \lambda\left(\overrightarrow{OC} - \overrightarrow{OB}\right) \Rightarrow -\overrightarrow{OA} + (\lambda + 1)\overrightarrow{OB} - \lambda\overrightarrow{OC} = 0 \quad or$$

$$a\,\overrightarrow{OA} + b\,\overrightarrow{OB} + c\,\overrightarrow{OC} = 0, where\ a = (-1), b = (\lambda + 1), c = -\lambda \quad (**)$$

In equation (**) the sum of the coefficients is

$$a + b + c = (-1) + (\lambda + 1) + (-\lambda) = 0.$$

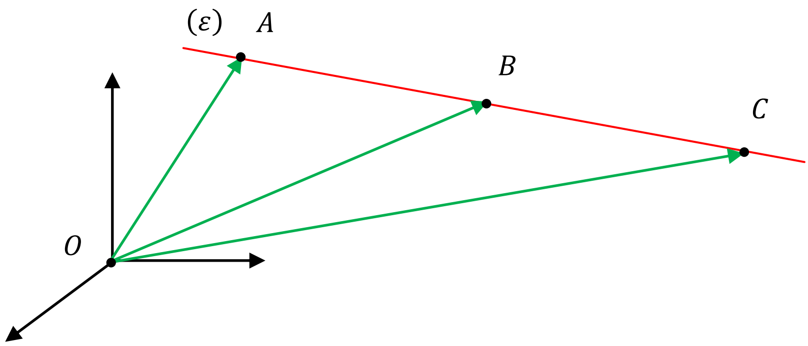

Fig. 4-1: Three collinear points A, B, C.

b) The converse statement is also true. Let us assume that $a\,\overrightarrow{OA} + b\,\overrightarrow{OB} + c\,\overrightarrow{OC} = 0$, where $a + b + c = 0$. Then the vector equation is written as,

$$(-b - c)\overrightarrow{OA} + b\,\overrightarrow{OB} + c\,\overrightarrow{OC} = 0 \Rightarrow b\left(\overrightarrow{OB} - \overrightarrow{OA}\right) = c\left(\overrightarrow{OA} - \overrightarrow{OC}\right) \Rightarrow$$

$$b\overrightarrow{AB} = c\overrightarrow{AC} \Rightarrow \overrightarrow{AB} \parallel \overrightarrow{AC} \Rightarrow A, B, C \quad lie\ on\ (\varepsilon)$$

4-3) Linear dependence of coplanar vectors

Theorem 4-4: Three coplanar vectors $\vec{x}, \vec{y}$ and $\vec{z}$ always satisfy a linear vector equation of the form $a\vec{x} + b\vec{y} + c\vec{z} = 0$. Coplanar vectors mean either parallel to the same plane or lying on the same plane, but having different directions.

Proof: a) Let $\vec{x}, \vec{y}$ and $\vec{z}$ be any three coplanar vectors, as shown in Fig. 4-2, and let us resolve $\vec{z}$ in the directions defined by $\vec{x}$ and $\vec{y}$.

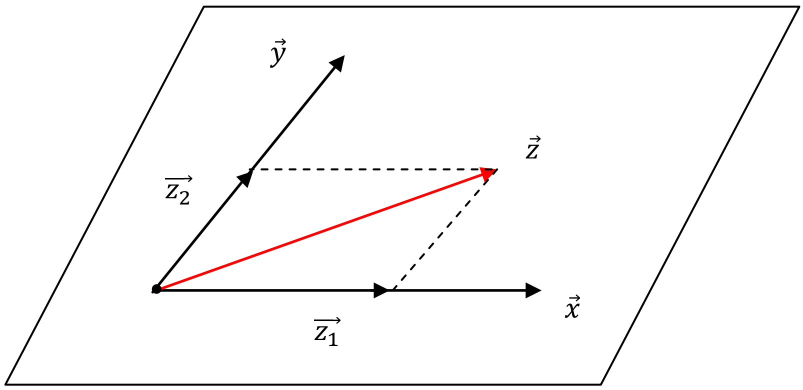

Fig. 4-2: Coplanar vectors $\vec{x}, \vec{y}, \vec{z}$.

Since $\overrightarrow{z_1} \parallel \vec{x}$ (i.e. these two vectors are **collinear**), $\overrightarrow{z_1} = k\vec{x}$ and for similar reasons, $\overrightarrow{z_2} = \lambda\vec{y}$. The vector $\vec{z}$ is the sum of $\overrightarrow{z_1}$ and $\overrightarrow{z_2}$, i.e.

$$\vec{z} = \overrightarrow{z_1} + \overrightarrow{z_2} \Longrightarrow \vec{z} = k\vec{x} + \lambda\vec{y} \Longrightarrow k\vec{x} + \lambda\vec{y} - \vec{z} = 0 \quad (*)$$

or if we multiply through by a number $L \neq 0$,

$$(kL)\vec{x} + (\lambda L)\vec{y} + (-L)\vec{z} = 0, \quad or \quad a\vec{x} + b\vec{y} + c\vec{z} = 0 \quad (**)$$

where $a = kL$, $b = \lambda L$ and $c = (-L)$.

b) The converse is also true, i.e. assuming $a\vec{x} + b\vec{y} + c\vec{z} = 0$, where a, b and c not all zero simultaneously, the vectors $\vec{x}, \vec{y}$ and $\vec{z}$ are coplanar. Indeed, assuming for example $c \neq 0$, the given equation implies $\vec{z} = (-a/c)\vec{x} + (-b/c)\vec{y}$, which in turn implies that $\vec{z}$ is coplanar with $\vec{x}$ and $\vec{y}$ (see section 2-6), and this completes the proof.

Theorem 4-5: If three non coplanar vectors $\vec{x}, \vec{y}$ and $\vec{z}$ satisfy the linear equation $a\vec{x} + b\vec{y} + c\vec{z} = 0$, then necessarily, we must have, $a = 0$ and $b = 0$ and $c = 0$.

Proof: The proof is similar to the proof of Theorem 4-2. Let the reader try to prove it.

Note: Theorem 4-5 is important in proving various Theorems of solid Euclidean Geometry with the aid of vectors.

Theorem 4-6: Four points A, B, C and D are coplanar (lie on the same plane), if and only if, for any point O in space, the four vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ and $\overrightarrow{OD}$ satisfy a linear equation $a\,\overrightarrow{OA} + b\,\overrightarrow{OB} + c\,\overrightarrow{OC} + d\,\overrightarrow{OD} = 0$, where the sum of the scalar coefficients $a + b + c + d = 0$.

Proof: a) Let the four points A, B, C and D be coplanar, as shown in Fig. 4-3.

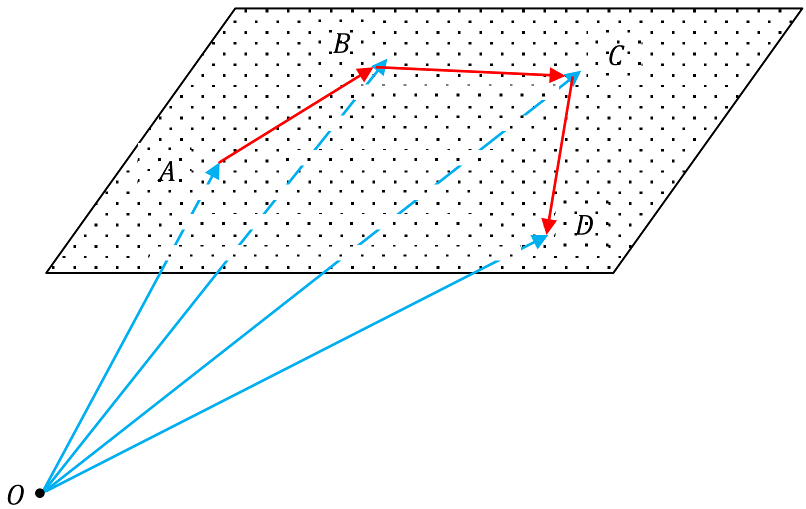

Fig. 4-3: Four coplanar points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$.

Since the vectors $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{CD}$ are coplanar, we can find two numbers k and λ such that, (see section 2-6),

$$\overrightarrow{BC} = k\,\overrightarrow{AB} + \lambda\,\overrightarrow{CD} \qquad (*)$$

However, since $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$ equation (*) implies,

$$\overrightarrow{OC} - \overrightarrow{OB} = k\left(\overrightarrow{OB} - \overrightarrow{OA}\right) + \lambda\left(\overrightarrow{OD} - \overrightarrow{OC}\right) \Longrightarrow$$

$$\underbrace{k}_{a}\,\overrightarrow{OA} + \underbrace{(-k-1)}_{b}\,\overrightarrow{OB} + \underbrace{(\lambda+1)}_{c}\,\overrightarrow{OC} + \underbrace{(-\lambda)}_{d}\,\overrightarrow{OD} = 0 \qquad (**)$$

We note that the sum of the coefficients is

$$a + b + c + d = k + (-k-1) + (\lambda+1) + (-\lambda) = 0$$

b) The converse statement is also true. Let us assume that $a\,\overrightarrow{OA} + b\,\overrightarrow{OB} + c\,\overrightarrow{OC} + d\,\overrightarrow{OD} = 0$ where the $a + b + c + d = 0$. Then the given vector equation is written equivalently as,

$$(-b-c-d)\,\overrightarrow{OA} + b\,\overrightarrow{OB} + c\,\overrightarrow{OC} + d\,\overrightarrow{OD} = 0 \Longrightarrow$$

$$b\left(\overrightarrow{OB} - \overrightarrow{OA}\right) + c\left(\overrightarrow{OC} - \overrightarrow{OA}\right) + d\left(\overrightarrow{OD} - \overrightarrow{OA}\right) = 0 \Longrightarrow$$

$$b\,\overrightarrow{AB} + c\,\overrightarrow{AC} + d\,\overrightarrow{AD} = 0$$

and according to Theorem 4-4 the three vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar, and since they all have the same origin A, the four points A, B, C and D lie on the same plane.

Example 4-1

Investigate whether there are two vectors $\vec{a}$ and $\vec{b}$ collinear to $\vec{A} = \hat{x} + 2\hat{y} + \hat{z}$ and $\vec{B} = 3\hat{x} - \hat{y} + 4\hat{z}$ respectively, such that $\vec{a} + \vec{b} = 8\hat{x} + 2\hat{y} + 10\hat{z}$.

Solution

Any vector collinear to $\vec{A}$ can be expressed as $k\vec{A}$ $(k \neq 0)$ and similarly any vector collinear to $\vec{B}$ can be expressed as $\lambda\vec{B}$ $(\lambda \neq 0)$, (see section 2-5). According to the problem, we must have,

$$\vec{a} + \vec{b} = 8\hat{x} + 2\hat{y} + 10\hat{z} \xrightarrow{(\vec{a}=k\vec{A})(\vec{b}=\lambda\vec{B})} k\vec{A} + \lambda\vec{B} = 8\hat{x} + 2\hat{y} + 10\hat{z} \Longrightarrow$$

$$k(\hat{x} + 2\hat{y} + \hat{z}) + \lambda(3\hat{x} - \hat{y} + 4\hat{z}) = 8\hat{x} + 2\hat{y} + 10\hat{z} \Longrightarrow$$

$$(k + 3\lambda)\hat{x} + (2k - \lambda)\hat{y} + (k + 4\lambda)\hat{z} = 8\hat{x} + 2\hat{y} + 10\hat{z} \Longrightarrow$$

$$\begin{Bmatrix} k + 3\lambda = 8 \\ 2k - \lambda = 2 \\ k + 4\lambda = 10 \end{Bmatrix} \qquad (*)$$

System (*) is a system **of three equations in two unknowns** (k and λ). Systems of this form (**more equations than unknowns**) in general do not have a solution. However, in our case, the system does have a solution. Solving the first two equations, we obtain, $k = 2$ and $\lambda = 2$. These values of k and λ do satisfy the third equation and this shows that $k = 2,\ \lambda = 2$ is a solution of system (*). The sought for vectors $\vec{a}$ and $\vec{b}$ are,

$$\begin{Bmatrix} \vec{a} = k\vec{A} = 2\vec{A} = 2\hat{x} + 4\hat{y} + 2\hat{z} \\ \vec{b} = \lambda\vec{B} = 2\vec{B} = 6\hat{x} - 2\hat{y} + 8\hat{z} \end{Bmatrix}$$

(Verify that $\vec{a} + \vec{b} = 8\hat{x} + 2\hat{y} + 10\hat{z}$).

Example 4-2

Show that the vectors $\vec{x} = \vec{b} + \vec{c} - 2\vec{a},\ \vec{y} = \vec{a} + \vec{c} - 2\vec{b}$ and $\vec{z} = \vec{a} + \vec{b} - 2\vec{c}$ are coplanar.

Solution

Since $1 \cdot \vec{x} + 1 \cdot \vec{y} + 1 \cdot \vec{z} = \left(\vec{b} + \vec{c} - 2\vec{a}\right) + \left(\vec{a} + \vec{c} - 2\vec{b}\right) + \left(\vec{a} + \vec{b} - 2\vec{c}\right) = 0$, the three vectors $\vec{x}, \vec{y}$ and $\vec{z}$ are coplanar, (see Theorem 4-4).

Example 4-3

If AD is the bisector of the angle formed by the two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, show that $\frac{AD}{DB} = \frac{OA}{OB} = \frac{a}{b}$ (This is a known **property of the bisector**, which is proved in Geometry, using geometrical considerations. In this Example we shall prove this property using vectors).

Solution

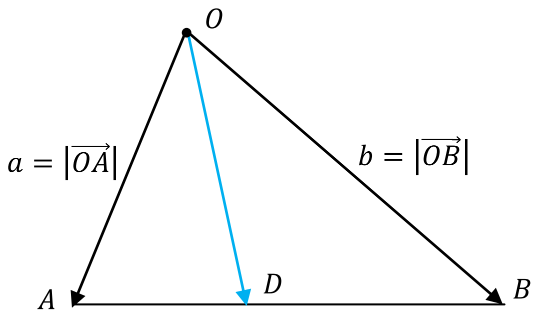

Fig. 4-4: The Theorem of bisectors.

Let $(ABD) = \frac{AD}{DB} \triangleq \lambda$, be **the partial ratio** at which the point D divides the vector $\overrightarrow{AB}$. Then (see formula (3-16)),

$$\overrightarrow{OD} = \frac{\overrightarrow{OA} + \lambda\overrightarrow{OB}}{1+\lambda} = \frac{\vec{a} + \lambda\vec{b}}{1+\lambda} \qquad (*)$$

On the other hand, $\overrightarrow{OD} \parallel \left(a\vec{b} + b\vec{a}\right)$, (see Example 3-6), and therefore, since $\overrightarrow{OD}$ and $\left(a\vec{b} + b\vec{a}\right)$ are collinear,

$$\overrightarrow{OD} = k\left(a\vec{b} + b\vec{a}\right) \qquad (**)$$

From (*) and (**) we get,

$$\frac{\vec{a} + \lambda\vec{b}}{1+\lambda} = k\left(a\vec{b} + b\vec{a}\right) \Longrightarrow$$

$$\{k(1+\lambda)b - 1\}\vec{a} + \{k(1+\lambda)a - \lambda\}\vec{b} = 0 \qquad (***)$$

and since $\vec{a}$ and $\vec{b}$ **are not collinear**, then necessarily, we must have (Theorem 4-2),

$$\{k(1+\lambda)b - 1 = 0 \quad \boldsymbol{and} \quad k(1+\lambda)a - \lambda = 0\} \qquad (****)$$

From the first equation in (****) we obtain, $(1+\lambda) = {}^{1}/_{b}$, and substituting in the second equation yields,

$$\lambda = \frac{AD}{DB} = \frac{a}{b}$$

and this completes the proof.

Example 4-4

Show that the three medians of a triangle are concurrent at a point G, called the centroid of the triangle. The distance of G from each vertex is two-thirds of the length of the corresponding median.

Solution

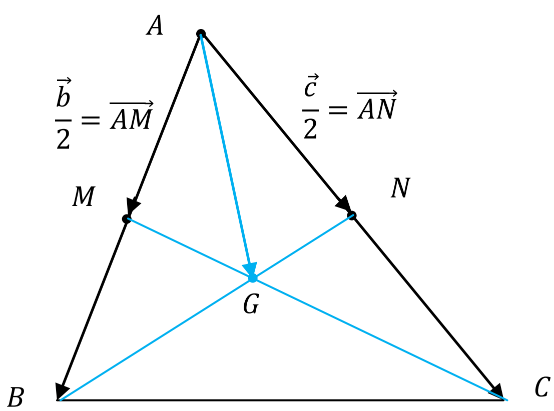

Fig. 4-5: The centroid of a triangle.

Let us for simplicity call $\vec{b} = \overrightarrow{AB}$ and $\vec{c} = \overrightarrow{AC}$. The two medians BN and CM intersect at the point G. The point G divides $\overrightarrow{BN}$ in partial ratio $(BNG) = k$ and $\overrightarrow{CM}$ in partial ratio $(CMG) = \lambda$, i.e.

$$\left\{(BNG) = \frac{\overline{BG}}{\overline{GN}} = k \quad \boldsymbol{and} \quad (CMG) = \frac{\overline{CG}}{\overline{GM}} = \lambda\right\} \qquad (*)$$

The vector $\overrightarrow{AG}$ divides $\overrightarrow{BN}$ in partial ratio k, and by virtue of formula (3-16),

$$\overrightarrow{AG} = \frac{\vec{b} + k\frac{\vec{c}}{2}}{1+k} \qquad (**)$$

and similarly, since $\overrightarrow{AG}$ divides $\overrightarrow{CM}$ in partial ratio λ,

$$\overrightarrow{AG} = \frac{\vec{c} + \lambda\frac{\vec{b}}{2}}{1+\lambda} \qquad (***)$$

From (**) and (***) we obtain the following equation,

$$\frac{\vec{b} + k\frac{\vec{c}}{2}}{1+k} = \frac{\vec{c} + \lambda\frac{\vec{b}}{2}}{1+\lambda} \Longrightarrow$$

$$(2 + \lambda - k\lambda)\vec{b} + (k\lambda - k - 2)\vec{c} = 0$$

and since $\vec{b}$ and $\vec{c}$ are **not collinear**, we must necessarily have,

$$\begin{Bmatrix} 2 + \lambda - k\lambda = 0 \\ k\lambda - k - 2 = 0 \end{Bmatrix} \qquad (****)$$

Adding term wise these two equations we get $\boldsymbol{k = \lambda}$, and substituting into the first one yields,

$$k^2 - k - 2 = 0 \Longrightarrow k = -1 \quad \boldsymbol{or} \quad k = 2$$

The root $k = -1$ **is rejected**, since $(1 + k)$ appears in the denominator in (**), and finally the only solution of system (****) is $\boldsymbol{k = \lambda = 2}$. This means that (see equation (*)),

$$\frac{\overline{BG}}{\overline{GN}} = 2, \qquad \frac{\overline{BG}}{\overline{BN}} = \frac{2}{3} \quad \boldsymbol{and} \quad \frac{\overline{CG}}{\overline{GM}} = 2, \qquad \frac{\overline{CG}}{\overline{CM}} = \frac{2}{3}$$

Example 4-5

The position vectors (relative to an arbitrary origin O) of three points A, B and C are $\overrightarrow{r_A} = 3\vec{x} + 2\vec{y} + \vec{z}$, $\overrightarrow{r_B} = 2\vec{x} + 2\vec{y} + 3\vec{z}$ and $\overrightarrow{r_C} = \vec{x} + 2\vec{y} + 5\vec{z}$ respectively. Show that the points A, B and C are collinear.

Solution

It suffices to show that there are constants a, b and c (**not all zero simultaneously**) having sum zero, ($a + b + c = 0$), such that $a\overrightarrow{r_A} + b\overrightarrow{r_B} + c\overrightarrow{r_C} = 0$, (Theorem 4-3).

$$a\overrightarrow{r_A} + b\overrightarrow{r_B} + c\overrightarrow{r_C} = a(3\vec{x} + 2\vec{y} + \vec{z}) + b(2\vec{x} + 2\vec{y} + 3\vec{z}\,) + c(\vec{x} + 2\vec{y} + 5\vec{z}) \Longrightarrow$$

$$a\overrightarrow{r_A} + b\overrightarrow{r_B} + c\overrightarrow{r_C} = \vec{x}(3a + 2b + c) + \vec{y}(2a + 2b + 2c) + \vec{z}(a + 3b + 5c)$$

The vector sum $a\overrightarrow{r_A} + b\overrightarrow{r_B} + c\overrightarrow{r_C} = 0$ if we choose

$$\begin{Bmatrix} 3a + 2b + c = 0 \\ 2a + 2b + 2c = 0 \\ a + 3b + 5c = 0 \end{Bmatrix} \qquad (*)$$

The linear system (*) is homogeneous and therefore $a = 0, b = 0$ and $c = 0$ is **a trivial solution**. However, **the system admits non-trivial solutions**. Let the reader verify that $a = c,\ b = -2c,\ c = arbitrary$ is also **a valid solution** (for any value of c). For example, we may choose $a = 1, b = -2, c = 1$. Note that $a + b + c = 0$, and Theorem 4-3 implies that A, B and C are collinear.

Example 4-6

Show that the vectors $\vec{a} = 7\vec{x} - 8\vec{y} + 9\vec{z}$, $\vec{b} = 3\vec{x} + 20\vec{y} + 5\vec{z}$ and $\vec{c} = 5\vec{x} + 6\vec{y} + 7\vec{z}$ are coplanar.

Solution

It suffices to show that one of the vectors, say the vector $\vec{a}$, can be expressed as a linear combination of the other two vectors $\vec{b}$ and $\vec{c}$, (see section 4-3, Theorem 4-4). This means that we seek two constants k and λ, **not zero simultaneously**, such that

$$\vec{a} = k\vec{b} + \lambda\vec{c} \Longrightarrow$$

$$7\vec{x} - 8\vec{y} + 9\vec{z} = k(3\vec{x} + 20\vec{y} + 5\vec{z}) + \lambda(5\vec{x} + 6\vec{y} + 7\vec{z}) \Longrightarrow$$

$$7\vec{x} - 8\vec{y} + 9\vec{z} = (3k + 5\lambda)\vec{x} + (20k + 6\lambda)\vec{y} + (5k + 7\lambda)\vec{z} \Longrightarrow$$

$$\left\{\begin{matrix} 3k + 5\lambda = 7 \\ 20k + 6\lambda = -8 \\ 5k + 7\lambda = 9 \end{matrix}\right\} \qquad (*)$$

The reader may verify easily that $k = -1,\ \lambda = 2$ is a solution of the system, and this shows that $\vec{a},\ \vec{b}$ and $\vec{c}$ are coplanar.

PROBLEMS

4-1) Show that the three points A, B and C with position vectors $\vec{r_A} = \vec{x}, \vec{r_B} = \vec{y}$ and $\vec{r_C} = 3\vec{x} - 2\vec{y}$ are collinear.

Hint: Apply Theorem 4-3.

4-2) Show that the line segments joining the mid-points of the consecutive sides of a quadrilateral form a parallelogram.

4-3) The position vectors of the vertices A, B and C of a triangle ABC are $\vec{r_A} = -\hat{x} + 6\hat{y} + 6\hat{z},\ \vec{r_B} = 7\hat{y} + 10\hat{z}$ and $\vec{r_C} = -4\hat{x} + 9\hat{y} + 6\hat{z}$. Show that ABC is an isosceles and right angled triangle.

Hint: Find the lengths AB, BC, AC. Note that $AB = \left|\overrightarrow{AB}\right| = |\vec{r_B} - \vec{r_A}|$, etc.

4-4) Show that the four points $3\vec{x} + 2\vec{y} - 5\vec{z},\ -3\vec{x} + 8\vec{y} - 5\vec{z}$, $-3\vec{x} + 2\vec{y} + \vec{z}$ and $-\vec{x} + 4\vec{y} - 3\vec{z}$ are coplanar.

Hint: Apply Theorem 4-6.

4-5) Let ABCD be the parallelogram shown in Fig. 4-6, and M be the mid-point of BC. If DM intersects the diagonal AC at F, show that AC=3FC.

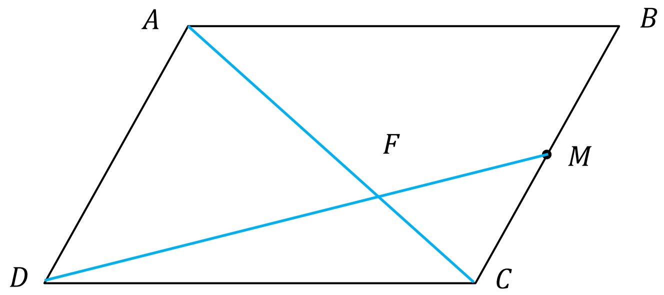

Fig. 4-6: Parallelogram ABCD (M is the midpoint of BC).

Hint: Let $\overline{AF} = \lambda\, \overline{AC}$ and $\overline{DF} = k\, \overline{DM}$. Starting with $\overrightarrow{DF} + \overrightarrow{FA} = \overrightarrow{DA}$, show that $\left(\frac{k}{1-k} - 2\right)\overrightarrow{FM} + \left(\frac{\lambda}{1-\lambda} - 2\right)\overrightarrow{CF} = 0$, and since $\overrightarrow{FM}$ and $\overrightarrow{CF}$ are not collinear, $\left(\frac{k}{1-k} - 2\right) = 0$ and $\left(\frac{\lambda}{1-\lambda} - 2\right) = 0$, etc.

4-6) Let $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}, \cdots, \vec{r_n}$ be the position vectors of the n points $A_1, A_2, A_3, \cdots, A_n$ respectively, with respect to an arbitrary origin O. If the position vectors satisfy the vector equation $a_1\vec{r_1} + a_2\vec{r_2} + a_3\vec{r_3} + \cdots + a_n\vec{r_n} = 0$, where $a_1, a_2, a_3, \cdots, a_n$ are real numbers having sum zero ($a_1 + a_2 + a_3 + \cdots + a_n = 0$), show that the vector equation is retained when the reference system is changed.

Hint: If O' is the new origin, then $\vec{r_k} = \overrightarrow{OO'} + \overrightarrow{r'}_k\,, k = 1,2,3, \cdots, n$ (see equation 3-20).

4-7) Show that the sum of the vectors associated with the medians of a triangle is zero.

4-8) Show that the line segments joining the mid-points of the opposite edges of a tetrahedron bisect each other.

4-9) If the diagonal AC of a parallelogram ABCD bisects the angle formed by AD and AB, show that the parallelogram is a rhombus.

4-10) Let O be the center of a regular hexagon ABCDEF. Show that $\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}+\overrightarrow{AE}+\overrightarrow{AF}=6\overrightarrow{AO}$.

4-11) Show that the bisector of an exterior angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.

Hint: See Example 4-3.

4-12) Resolve the force $\overrightarrow{OF}=3\hat{x}+2\hat{y}-7\hat{z}$ into three components, along the direction of the vectors $\vec{a}=2\hat{x}-3\hat{y}+\hat{z}$, $\vec{b}=3\hat{x}+\hat{y}+4\hat{z}$ and $\vec{c}=-\hat{x}+5\hat{y}-2\hat{z}$.

Hint: It suffices to find three numbers k,λ and m such that $\overrightarrow{OF}=k\vec{a}+\lambda\vec{b}+m\vec{c}$.

4-13) If $\vec{a}=\hat{x}+2\hat{y}+\hat{z}$, $\vec{b}=2\hat{x}-\hat{y}+3\hat{z}$ and $\vec{c}=\hat{x}+3\hat{y}+5\hat{z}$, find: **1)** the vector $\vec{x}$ in the direction of $\vec{a}$ but having magnitude that of $\vec{b}$, **2)** The vector $\vec{y}$ of magnitude $\left|\vec{a}-\vec{b}+\vec{c}\right|$ in the direction of $\vec{c}$.

(**Answer: 1)** $\vec{x}=\sqrt{\frac{7}{3}}\,\vec{a}$, **2)** $\vec{y}=\frac{3}{\sqrt{7}}\,\vec{c}$).

Hint: $\vec{x}=k\vec{a}, |\vec{x}|=|k\,\vec{a}|=\left|\vec{b}\right|$, i.e. $|k|=\frac{|\vec{b}|}{|\vec{a}|}$, etc.

CHAPTER 5: The Inner or Dot Product of two Vectors

There are two ways in which two vectors $\vec{a}$ and $\vec{b}$ can be multiplied. The first one, denoted by $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}$ or $(\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}})$ is **a real number (scalar)** and is called the **dot-product or the inner product of $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$**, while the second one denoted by $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$ or $[\vec{\boldsymbol{a}}, \vec{\boldsymbol{b}}]$ is **a vector and is called the cross-product or the outer product of $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$.**

Let $\vec{a}$ and $\vec{b}$ be two vectors and $\theta = \measuredangle(\vec{a}, \vec{b})$ be **the smallest angle** between the two vectors, $(0 \leq \theta \leq \pi)$.

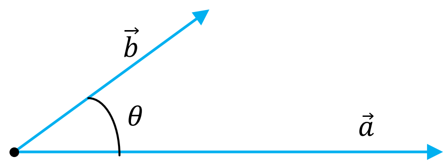

Fig. 5-1: The dot (inner) product of two vectors.

The inner product or dot product of the two vectors $\vec{a}$ and $\vec{b}$ is defined to be **the scalar quantity (real number)**,

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = a\, b \cos\theta \qquad (5-1)$$

where $\boldsymbol{a} = |\vec{\boldsymbol{a}}|$ and $\boldsymbol{b} = |\vec{\boldsymbol{b}}|$ are the magnitudes of the vectors $\vec{a}$ and $\vec{b}$ respectively.

1) If $\vec{a}$ and $\vec{b}$ are parallel and have the same orientation ($\boldsymbol{\theta} = \boldsymbol{0}$), then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| = ab$. If $\vec{a}$ and $\vec{b}$ are parallel and of opposite orientation ($\boldsymbol{\theta} = \boldsymbol{\pi}$), then $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}| = -ab$.

2) If $\vec{a}$ and $\vec{b}$ are perpendicular to each other ($\theta = \frac{\pi}{2}$), then $\vec{a} \cdot \vec{b} = 0$, and **conversely**, if $\vec{a} \cdot \vec{b} = 0$ then $\theta = \frac{\pi}{2}$, meaning that $\boldsymbol{\vec{a}} \perp \boldsymbol{\vec{b}}$.

3) If $\vec{a} = \vec{b}$ then $\theta = 0°$, $\cos 0° = 1$ and formula (5-1) implies,

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2 = a^2 \qquad (5-2)$$

4) The dot product of two vectors is **commutative** and **distributive**, i.e.

$$\begin{Bmatrix} \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \\ \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \end{Bmatrix} \qquad (5-3)$$

The first one follows directly from the definition of the dot product (formula 5-1). The second one is proved with the aid of Fig. 5-2.

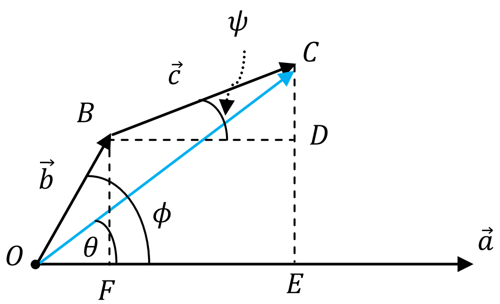

Fig. 5-2: The distributive law of the dot multiplication.

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \overrightarrow{OC} = |\vec{a}||\overrightarrow{OC}| \cos\theta = |\vec{a}|(OE) = |\vec{a}|(OF + FE) \Longrightarrow$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = |\vec{a}|(OF) + |\vec{a}|(FE) = |\vec{a}||\vec{b}| \cos\phi + |\vec{a}|(BD) \Longrightarrow$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = |\vec{a}||\vec{b}| \cos\phi + |\vec{a}||\vec{c}| \cos\psi = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

and this completes the proof.

5) If $\vec{a}$ and $\vec{b}$ are any two vectors and k and λ be any two real numbers, then it is easily shown that

$$(k\vec{a}) \cdot \left(\lambda\vec{b}\right) = (k\lambda)\, \vec{a} \cdot \vec{b} \qquad (5-4)$$

6) Let $\hat{x}, \hat{y}$ and $\hat{z}$ be the unit vectors along the Ox, Oy and Oz axes respectively, in an orthonormal Cartesian system $Oxyz$. The magnitude of each one of these vectors is one, (**unit vectors**), and also these vectors are **pair-wise perpendicular**, (see section 3-2), i.e.

$$\begin{Bmatrix} \hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0 \\ \hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1 \end{Bmatrix} \qquad (5-5)$$

7) The Cartesian expression of the dot product of two vectors.

Let $\vec{a} = a_1\hat{x} + a_2\hat{y} + a_3\hat{z}$ and $\vec{b} = b_1\hat{x} + b_2\hat{y} + b_3\hat{z}$ be the Cartesian expression of two vectors $\vec{a}$ and $\vec{b}$. Since the dot product **is distributive**, we have,

$$\vec{a} \cdot \vec{b} = (a_1\hat{x} + a_2\hat{y} + a_3\hat{z}) \cdot (b_1\hat{x} + b_2\hat{y} + b_3\hat{z}) \qquad (*)$$

and taking into account formulas (5-5) equation (*) finally simplifies to the following, (let the reader check it),

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad (5-6)$$

Formula (5-6) is called **the Cartesian expression of the dot product** of the two vectors $\vec{a}$ and $\vec{b}$. Formula (5-6) implies that two vectors $\vec{a} = a_1\hat{x} + a_2\hat{y} + a_3\hat{z}$ and $\vec{b} = b_1\hat{x} + b_2\hat{y} + b_3\hat{z}$ are **perpendicular**, if and only if

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = 0 \qquad (5-7)$$

8) The length of a vector in a Cartesian system of axes.

If $\vec{a} = \vec{b}$ formula (5-7) implies

$$\vec{a}\cdot\vec{a} = |\vec{a}|^2 = a^2 = {a_1}^2 + {a_2}^2 + {a_3}^2 \Longrightarrow$$

$$a = |\vec{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2} \qquad (5-8)$$

9) Angle between two vectors expressed by their Cartesian coordinates, ($\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$).

Since in general, $\vec{a}\cdot\vec{b} = ab\cos\theta$, (where $a = |\vec{a}|$, $b = |\vec{b}|$),

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab} \xRightarrow{(5-6)(5-8)}$$

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}\,\sqrt{{b_1}^2 + {b_2}^2 + {b_3}^2}} \qquad (5-9)$$

10) The direction cosines of a position vector $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$.

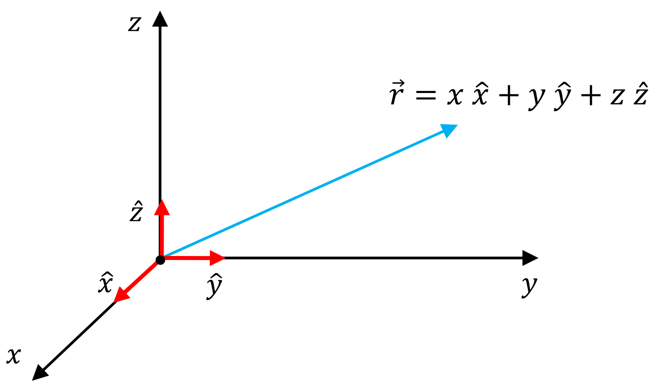

Fig. 5-3: Direction angles and direction cosines.

Let $\hat{x}, \hat{y}$ and $\hat{z}$ be the unit vectors along the axes Ox, Oy and Oz respectively, as shown in Fig. 5-3.The angles between the vector $\vec{r}$ and the unit vectors are called **direction angles**, while the cosines of the direction angles are called **direction cosines**. Let for definiteness θ_x be the angle between $\vec{r}$ and $\hat{x}$, θ_y the angle between $\vec{r}$ and $\hat{y}$ and θ_z the angle between $\vec{r}$ and $\hat{z}$. From the fundamental definition of the dot product,

$$\vec{r} \cdot \hat{x} = |\vec{r}||\hat{x}| \cos\theta_x \xRightarrow{(5-6),(5-8),(|\hat{x}|=1)}$$

$$\cos\theta_x = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

and similarly for the other two direction cosines. In summary, the **direction cosines** of the vector $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$, are given by the formulas

$$\left\{\begin{aligned} \cos\theta_x &= \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{|\vec{r}|} \\ \cos\theta_y &= \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{|\vec{r}|} \\ \cos\theta_z &= \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{|\vec{r}|} \end{aligned}\right\} \qquad (5-10)$$

It is evident, (from (5-10) that the sum of the squares of the direction cosines is equal to 1, i.e.

$$(\cos\theta_x)^2 + \left(\cos\theta_y\right)^2 + (\cos\theta_z)^2 = 1 \qquad (5-11)$$

11) Work done by a force.

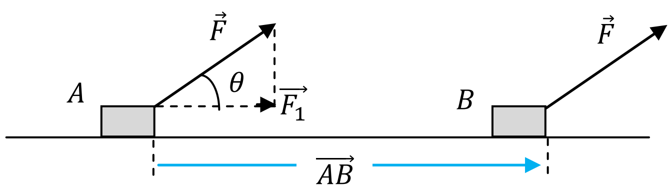

Fig. 5-4: Work done by a force.

In Fig. 5-4 the object moves from the point A to the point B, under the action of **a constant** force $\vec{F}$. The work W done by the force $\vec{F}$, as known from Physics, is

$$W = |\vec{F_1}|AB = \left(|\vec{F}|\cos\theta\right)AB = \vec{F}\cdot\overrightarrow{AB} \qquad (5-12)$$

Formula (5-12) applies when **the force $\vec{F}$ (as a vector) remains constant** while moving its point of application from A to B. If $\vec{F}$ varies, then **an appropriate integration** should be carried out, in order to obtain the work W.

Example 5-1

If $\vec{a} = 2\hat{x} + 3\hat{y} + \hat{z}$ and $\vec{b} = -\hat{x} + 5\hat{y} - \hat{z}$ find:

1) The dot product $\vec{a}\cdot\vec{b}$, **2)** The magnitudes $a = |\vec{a}|$ and $b = |\vec{b}|$, and **3)** The angle θ between $\vec{a}$ and $\vec{b}$.

Solution

1) The dot product is expressed by formula (5-6), i.e.

$$\vec{a}\cdot\vec{b} = 2\cdot(-1) + 3\cdot 5 + 1\cdot(-1) = -2 + 15 - 1 = 12$$

2) The magnitudes of the vectors are obtained from (5-8),

$$\left\{\begin{matrix} a = |\vec{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14} \\ b = |\vec{b}| = \sqrt{(-1)^2 + 5^2 + (-1)^2} = \sqrt{27} \end{matrix}\right\}$$

3) The angle between $\vec{a}$ and $\vec{b}$ is given from (5-9), i.e.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab} = \frac{12}{\sqrt{14}\cdot\sqrt{27}} \cong 0.617 \Longrightarrow \theta \cong \cos^{-1}(0.617) = 51.9°$$

Example 5-2

For what value of k the vectors $\vec{a} = 2\hat{x} + k\hat{y} + 3\hat{z}$ and $\vec{b} = k\hat{x} - 4\hat{y} + 7\hat{z}$ are perpendicular?

Solution

The two vectors will be perpendicular, **if and only if their dot product is zero**, i.e.

$$\vec{a}\cdot\vec{b}=0 \Longrightarrow 2k-4k+3\cdot 7=0 \Longrightarrow -2k+21=0 \Longrightarrow k=\frac{21}{2}$$

Example 5-3

Derive the "**Law of Cosines**".

Solution

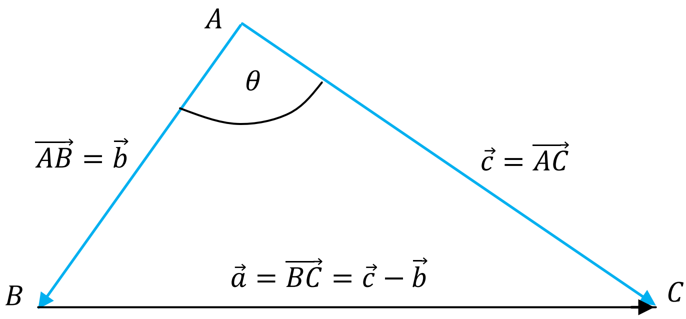

Fig. 5-5: The "Law of cosines".

Let $a=|\vec{a}|, b=|\vec{b}|$ and $c=|\vec{c}|$. Then,

$$\vec{a}=\vec{c}-\vec{b} \Longrightarrow \vec{a}\cdot\vec{a}=(\vec{c}-\vec{b})\cdot(\vec{c}-\vec{b}) \Longrightarrow$$

$$\vec{a}\cdot\vec{a}=\vec{c}\cdot\vec{c}-\vec{b}\cdot\vec{c}-\vec{c}\cdot\vec{b}+\vec{b}\cdot\vec{b} \Longrightarrow$$

$$a^2=b^2+c^2-2\vec{b}\cdot\vec{c}=b^2+c^2-2bc\cos\theta \qquad (*)$$

Note: Formula (*) is known as "**the law of cosines**" and is a fundamental formula in Trigonometry. If $\theta=90°$, then $\cos 90°=0$ and formula (*) reduces to **the Pythagorean Theorem**.

Example 5-4

Show that an angle inscribed in a semicircle is a right angle.

Solution

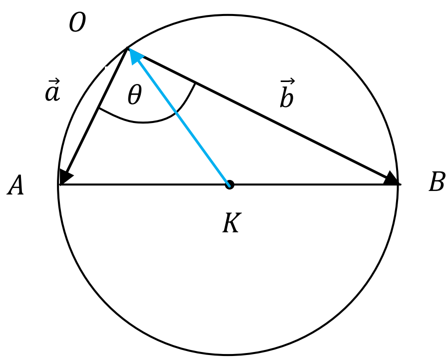

Fig. 5-6: Angle inscribed in a semicircle.

Consider an angle θ inscribed in a semicircle, as shown in Fig. 5-6.

$$\vec{a} = \overrightarrow{KA} - \overrightarrow{KO} \quad and \quad \vec{b} = \overrightarrow{OK} + \overrightarrow{KB} \qquad (*)$$

and taking the dot product of $\vec{a}$ and $\vec{b}$ we get,

$$\vec{a} \cdot \vec{b} = \left(\overrightarrow{KA} - \overrightarrow{KO}\right) \cdot \left(\overrightarrow{OK} + \overrightarrow{KB}\right) \Longrightarrow$$

$$\vec{a} \cdot \vec{b} = \overrightarrow{KA} \cdot \overrightarrow{OK} - \overrightarrow{KO} \cdot \overrightarrow{OK} + \overrightarrow{KA} \cdot \overrightarrow{KB} - \overrightarrow{KO} \cdot \overrightarrow{KB} \xRightarrow{(\overrightarrow{OK}=-\overrightarrow{KO})(\overrightarrow{KB}=-\overrightarrow{KA})}$$

$$\vec{a} \cdot \vec{b} = \overrightarrow{KA} \cdot \overrightarrow{OK} + \overrightarrow{KO} \cdot \overrightarrow{KO} - \overrightarrow{KA} \cdot \overrightarrow{KA} - \overrightarrow{KO} \cdot \overrightarrow{KB} \Longrightarrow$$

$$\vec{a} \cdot \vec{b} = \overrightarrow{KA} \cdot \overrightarrow{OK} + R^2 - R^2 - \overrightarrow{KO} \cdot \overrightarrow{KB} \Longrightarrow$$

$$\vec{a} \cdot \vec{b} = \overrightarrow{KA} \cdot \overrightarrow{OK} + \overrightarrow{KB} \cdot \overrightarrow{OK} = \underbrace{\left(\overrightarrow{KA} + \overrightarrow{KB}\right)}_{0} \cdot \overrightarrow{OK} = 0$$

and this shows that $\vec{a}$ and $\vec{b}$ are perpendicular, i.e. $\boldsymbol{\theta = 90°}$.

Example 5-5

Show that the altitudes of a triangle are concurrent to a point O, called “**the orthocenter**”.

Solution

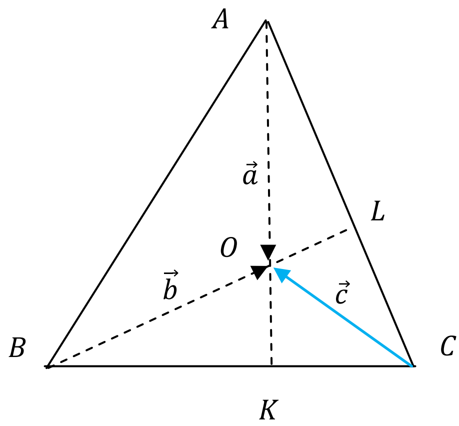

Fig. 5-7: The Theorem of altitudes.

Let O be the point of intersection of the two altitudes, AK and BL. For convenience we set, $\vec{a} = \overrightarrow{AO}$, $\vec{b} = \overrightarrow{BO}$ and $\vec{c} = \overrightarrow{CO}$. Since $\vec{a} \perp \overrightarrow{BC}$ and $\vec{c} \perp \overrightarrow{AC}$, we have

$$\vec{a} \cdot \overrightarrow{BC} = 0 \quad \boldsymbol{and} \quad \vec{b} \cdot \overrightarrow{AC} = 0 \xRightarrow{(\overrightarrow{BC}=\vec{b}-\vec{c})(\overrightarrow{AC}=\vec{a}-\vec{c})}$$

$$\vec{a} \cdot (\vec{b} - \vec{c}) = 0 \quad \boldsymbol{and} \quad \vec{b} \cdot (\vec{a} - \vec{c}) = 0 \Longrightarrow$$

$$\left\{\begin{matrix} \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = 0 \quad \boldsymbol{and} \\ \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{c} = 0 \end{matrix}\right\} \Longrightarrow \left\{\begin{matrix} \vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} \quad \boldsymbol{and} \\ \vec{b} \cdot \vec{a} = \vec{b} \cdot \vec{c} \end{matrix}\right\} \xRightarrow{(\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a})}$$

$$\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c} \Longrightarrow (\vec{a} - \vec{b}) \cdot \vec{c} = 0 \Longrightarrow \overrightarrow{AB} \cdot \vec{c} = 0$$

which shows that $\overrightarrow{CO} \perp \overrightarrow{AB}$, and this completes the proof.

Example 5-6

Find the direction cosines of the vector $\vec{r} = 2\hat{x} + \hat{y} - 2\hat{z}$.

Solution

The direction cosines are given from (5-10), i.e.

$$\left\{\begin{aligned} \cos\theta_x &= \frac{2}{\sqrt{2^2+1^2+(-2)^2}} = \frac{2}{3} \\ \cos\theta_y &= \frac{1}{\sqrt{2^2+1^2+(-2)^2}} = \frac{1}{3} \\ \cos\theta_z &= \frac{-2}{\sqrt{2^2+1^2+(-2)^2}} = \frac{-2}{3} \end{aligned}\right\} \Longrightarrow$$

$$\left\{\begin{aligned} \theta_x &= \cos^{-1}\left(\frac{2}{3}\right) \cong 48.18^\circ \\ \theta_y &= \cos^{-1}\left(\frac{1}{3}\right) \cong 70.52^\circ \\ \theta_z &= \cos^{-1}\left(\frac{-2}{3}\right) \cong 131.81^\circ \end{aligned}\right\}$$

Example 5-7

Three forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ and $\overrightarrow{F_3}$ with magnitudes 2 Nt, 3 Nt and 1 Nt, act on a particle in the directions of $2\hat{x} - 3\hat{y} + \hat{z}$, $-\hat{x} + 4\hat{y} + 5\hat{z}$ and $\hat{x} + \hat{y} - \hat{z}$ respectively. If the particle is displaced from the point $P(1,1,1)$ to the point $Q(-2,7,6)$ find the work done by the forces, (distances are expressed in m).

Solution

Let us call F_1, F_2, F_3 the magnitudes of the forces and $\widehat{u_1}$, $\widehat{u_2}$, $\widehat{u_3}$ the unit vectors along the direction of the forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ and $\overrightarrow{F_3}$ respectively.

The **resultant** (**vector sum**) of the three forces is $\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$, and the work done by $\vec{F}$ is (see formula (5-12)),

$$W = \vec{F} \cdot \overrightarrow{PQ} = \left(\vec{F_1} + \vec{F_2} + \vec{F_3}\right) \cdot \overrightarrow{PQ} = \vec{F_1} \cdot \overrightarrow{PQ} + \vec{F_2} \cdot \overrightarrow{PQ} + \vec{F_3} \cdot \overrightarrow{PQ} \quad (*)$$

The vector $\overrightarrow{PQ}$ is (see formula (3-22)),

$$\overrightarrow{PQ} = (-2-1)\hat{x} + (7-1)\hat{y} + (6-1)\hat{z} = -3\hat{x} + 6\hat{y} + 5\hat{z} \quad (**)$$

The force $\vec{F_1}$ is,

$$\vec{F_1} = F_1 \widehat{u_1} = 2 \cdot \frac{2\hat{x} - 3\hat{y} + \hat{z}}{\sqrt{2^2 + (-3)^2 + 1^2}} = \frac{2}{\sqrt{14}}(2\hat{x} - 3\hat{y} + \hat{z})$$

and

$$\vec{F_1} \cdot \overrightarrow{PQ} = \frac{2}{\sqrt{14}}(2\hat{x} - 3\hat{y} + \hat{z}) \cdot (-3\hat{x} + 6\hat{y} + 5\hat{z}) \Longrightarrow$$

$$\vec{F_1} \cdot \overrightarrow{PQ} = \frac{2}{\sqrt{14}}(-6 - 18 + 5) = -\frac{38}{\sqrt{14}} J (= Joules) \quad (***)$$

Similarly we find,

$$\vec{F_2} \cdot \overrightarrow{PQ} = \frac{156}{\sqrt{42}} J \quad and \quad \vec{F_3} \cdot \overrightarrow{PQ} = -\frac{2}{\sqrt{3}} J \quad (****)$$

The total work done by $\vec{F}$ is

$$W = -\frac{38}{\sqrt{14}} + \frac{156}{\sqrt{42}} - \frac{2}{\sqrt{3}} \cong 12.76\, J$$

Note: The unit of work (energy) in the SI system of units is the **Joule** $(1\, J = (1Nt)(1\, m))$.

PROBLEMS

5-1) Find the angle between the two vectors, $\vec{a} = 2\hat{x} + \hat{y} - 3\hat{z}$ and $\vec{b} = 3\hat{x} - 2\hat{y} - \hat{z}$.

(**Answer:** $\theta = 60°$).

5-2) Show that if $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ then either $\vec{a} = 0$, or $\vec{b} = \vec{c}$ or $\vec{a} \perp \left(\vec{b} - \vec{c}\right)$.

5-3) Find the angle between the vectors $\vec{a} = \hat{x} + 2\hat{y} - \hat{z}$ and $\vec{b} = 3\hat{x} - \hat{y}$.

(**Answer:** $\theta \cong 82.6°$).

5-4) Show that the vectors $\vec{a} = (1,3,-2)$ and $\vec{b} = (1,-1,-1)$ are perpendicular. What are the magnitudes $a = |\vec{a}|$ and $b = \left|\vec{b}\right|$?

5-5) Show the **Cauchy's-Schwarz inequality**, $\left|\vec{a} \cdot \vec{b}\right| \le |\vec{a}|\left|\vec{b}\right|$. Then assuming that $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, show that

$$(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \le ({a_1}^2 + {a_2}^2 + {a_3}^2)\left({b_1}^2 + {b_2}^2 + {b_3}^2\right)$$

Hint: Since $\vec{a} \cdot \vec{b} = ab\cos\theta$ and $|\cos\theta| \le 1$, Cauchy's-Schwarz inequality follows immediately.

5-6) Show that the vector $\vec{x} = \vec{b} - \frac{(\vec{a} \cdot \vec{b})}{a^2}\vec{a}$ is perpendicular to $\vec{a}$.

Hint: It suffices to show that $\vec{a} \cdot \vec{x} = 0$.

5-7) Consider a vector $\vec{x}$ and a unit vector $\hat{u}$ not collinear to $\vec{x}$. Resolve $\vec{x}$ into two vectors, perpendicular to each other, one of which is collinear to $\hat{u}$.

(**Answer:** $\vec{x} = (\vec{x} \cdot \hat{u})\hat{u} + \{\vec{x} - (\vec{x} \cdot \hat{u})\hat{u}\}$).

5-8) Find the angle between the vectors $\vec{a} = 4\hat{x} + 3\hat{y}$ and $\vec{b} = -3\hat{x} + 5\hat{z}$.

5-9) If $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$ are three concurrent edges of a parallelepiped and $\overrightarrow{OP}$ is its diagonal, show that

$$OP^2 = a^2 + b^2 + c^2 + 2ab\cos\left(\widehat{\vec{a}, \vec{b}}\right) + 2bc\cos\left(\widehat{\vec{b}, \vec{c}}\right) + 2ca\cos\left(\widehat{\vec{c}, \vec{a}}\right)$$

Hint: $OP^2 = \overrightarrow{OP} \cdot \overrightarrow{OP} = \left(\vec{a} + \vec{b} + \vec{c}\right) \cdot \left(\vec{a} + \vec{b} + \vec{c}\right)$.

5-10) If $\vec{a} = (1,3,2)$, $\vec{b} = (2,1,5)$, $\vec{c} = (2,1,7)$ and $\vec{d} = (2,0,9)$ find the vector

$$\vec{x} = \frac{(\vec{c} \cdot \vec{d})\vec{a} - (\vec{a} \cdot \vec{b})\vec{c}}{\left(\vec{a} + \vec{b}\right) \cdot \left(\vec{c} + \vec{d}\right)}$$

5-11) Show that the diagonals of a rhombus intersect at right angles.

5-12) Let $\vec{a} = 2\hat{x} + 3\hat{y} + \hat{z}$ and $\vec{b} = \hat{x} + \lambda\hat{y} - 2\hat{z}$. Find λ so that $\left(2\vec{a} + \vec{b}\right) \perp \left(\vec{a} - \vec{b}\right)$.

Hint: $\left(2\vec{a} + \vec{b}\right) \cdot \left(\vec{a} - \vec{b}\right) = 0$.

5-13) Show that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its four sides.

5-14) Show that the perpendicular bisectors of the sides of a triangle intersect at a common point, called "**the circum-center**" of the triangle. The circum-center is **the center of the circum-circle of the triangle**.

5-15) In any quadrilateral, the sum of the squares of its two diagonals is equal to twice the sum of the squares of the line segments joining the mid-points of the opposite sides.

5-16) If a line is perpendicular to two intersecting lines at their point of intersection P, then this line will be perpendicular to any other line of the plane determined by these two intersecting lines, passing through P, i.e. **it will be perpendicular to the plane determined by the two lines**.

5-17) A force $\vec{F} = 2\hat{x} + 3\hat{y} + \hat{z}\ (Nt)$ displaces its point of application from $P(1,1,1)$ to $Q(2,3,5)$. Find the work done by the force.

(**Answer:** $W = 12\,J$).

5-18) Find the angle between the line joining $A(1,2,3)$ to $B(2,-1,-2)$ and the line joining $A(1,2,3)$ to $D(-2,6,10)$.

Hint: Find the dot product $\overrightarrow{AB} \cdot \overrightarrow{AD}$.

5-19) Show that the vectors $\vec{a} = (2,-1,1)$, $\vec{b} = (3,-4,-4)$ and $\vec{c} = (1,-3,-5)$ form a right-angled triangle and verify the Pythagorean Theorem.

Hint: $\vec{a} \cdot \vec{c} = 0$.

5-20) Consider the parallelepiped formed by the three concurrent vectors $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$. If $a = 10\ cm,\ b = 7\ cm,\ c = 9\ cm,\ \measuredangle(\vec{a}, \vec{b}) = 75°$, $\measuredangle(\vec{b}, \vec{c}) = 50°, \measuredangle(\vec{c}, \vec{a}) = 80°$, find the length of its diagonal OP.

Hint: See Problem 5-9.

CHAPTER 6: The Projection of a Vector on an Axis

6-1) The signed projection of a vector on an axis

Let us consider an axis $(\ell'\ell)$ defined by its **unit vector** $\hat{\ell}$ and a vector $\overrightarrow{OA}$, in space. The vector and the axis are not, necessarily, coplanar, i.e. the lines $(\ell'\ell)$ and (OA) may be skew lines, (two nonintersecting, nonparallel lines in space are called **skew lines**).

Angle between a vector and an axis in space is defined to be the angle θ, $(0^\circ \leq \theta \leq 180^\circ)$ which is formed by two rays emanating from an arbitrary point P and such that the first one is parallel and in the positive direction of the axis (as defined by the unit vector of the axis), while the second one is parallel and of the same orientation with the vector.

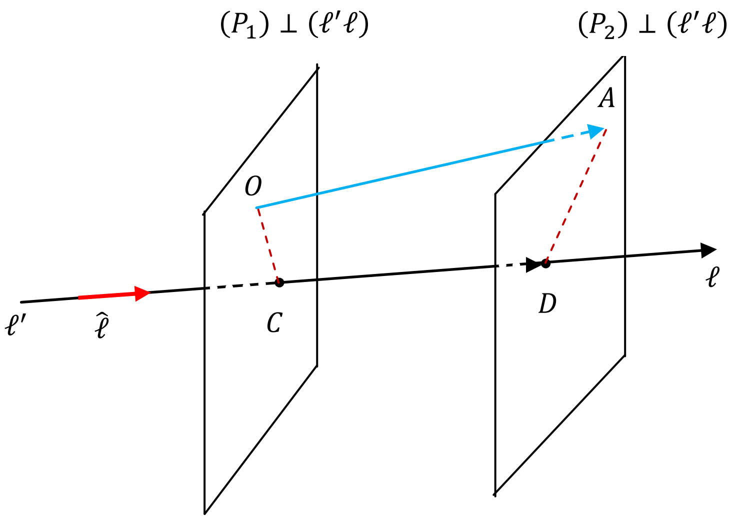

Fig. 6-1: The vector projection of a vector on an arbitrary axis.

Let us consider the axis $(\ell'\ell)$ and the vector $\overrightarrow{OA}$ shown in Fig. 6-1, (**not necessarily coplanar**). The plane (P_1) passes through O and is **perpendicular** to $(\ell'\ell)$, while the plane (P_2) passes through A and is

perpendicular to $(\ell'\ell)$. The point C is the vertical projection of the origin O of the vector $\overrightarrow{OA}$ on the axis $(\ell'\ell)$ and likewise the point D is the vertical projection of the terminal point A of the vector $\overrightarrow{OA}$ on the axis $(\ell'\ell)$. The vector $\overrightarrow{CD}$ is the "**vector projection**" of the vector $\overrightarrow{OA}$ on the axis $(\ell'\ell)$. The relative magnitude (signed value) of the vector $\overrightarrow{CD}$, denoted by $\overline{CD}$ (see section 3-3), is called **the signed or algebraic projection** of the vector $\overrightarrow{OA}$ on the axis $(\ell'\ell)$, i.e.

$$Proj.of\ \overrightarrow{OA}\ on\ (\ell'\ell) = \overline{CD} \qquad (6-1)$$

Theorem 6-1: The signed projection of a vector $\overrightarrow{OA}$ on an axis $(\ell'\ell)$ is equal to the dot product of the vectors $\overrightarrow{OA}$ and $\hat{\ell}$ (the unit vector of the axis), i.e.

$$Proj.of\ \overrightarrow{OA}\ on\ (\ell'\ell) = \overline{CD} = \overrightarrow{OA}\cdot\hat{\ell} \qquad (6-2)$$

Proof: Any vector $\overrightarrow{OA}$ in space, is resolved into two components, one **parallel** to $(\ell'\ell)$, denoted by $\overrightarrow{OA_{\parallel}}$, and another one **perpendicular** to $(\ell'\ell)$, denoted by $\overrightarrow{OA_{\perp}}$, i.e.

$$\overrightarrow{OA} = \overrightarrow{OA_{\parallel}} + \overrightarrow{OA_{\perp}} \qquad (6-3)$$

Taking the dot product of both sides of (6-3) with $\hat{\ell}$, we obtain,

$$\overrightarrow{OA}\cdot\hat{\ell} = \left(\overrightarrow{OA_{\parallel}} + \overrightarrow{OA_{\perp}}\right)\cdot\hat{\ell} = \overrightarrow{OA_{\parallel}}\cdot\hat{\ell} + \overrightarrow{OA_{\perp}}\cdot\hat{\ell} \qquad (*)$$

In equation (*) the term $\overrightarrow{OA_{\perp}}\cdot\hat{\ell} = 0$, since the two vectors $\overrightarrow{OA_{\perp}}$ and $\hat{\ell}$ are perpendicular. Also, since $\overrightarrow{OA_{\parallel}} = \overrightarrow{CD}$ (see Fig. 6-1), the term $\overrightarrow{OA_{\parallel}}\cdot\hat{\ell} = \overrightarrow{CD}\cdot\hat{\ell} = \overline{CD}$, (see section 3-3), and this completes the proof.

Theorem 6-2: Equal vectors have equal signed projections on the same axis.

Proof: If $\overrightarrow{OA} = \overrightarrow{KL}$ then $\overrightarrow{OA}\cdot\hat{\ell} = \overrightarrow{KL}\cdot\hat{\ell}$ and this completes the proof, (see formula (6-2)).

Note: Theorem 6-2 implies that **we may project both sides of a vector equality on an arbitrary axis and get a corresponding algebraic equality**.

Theorem 6-3: The projection of a vector $\overrightarrow{OA}$ on an axis $(\ell'\ell)$ is

$$Proj.of\ \overrightarrow{OA}\ on\ (\ell'\ell) = \left|\overrightarrow{OA}\right| \cos\theta \qquad (6-4)$$

where θ is the angle between $\overrightarrow{OA}$ and the axis $(\ell'\ell)$.

Proof: From equation (6-2) we have,

$$Proj.of\ \overrightarrow{OA}\ on\ (\ell'\ell) = \overrightarrow{OA}\cdot\hat{\ell} = \left|\overrightarrow{OA}\right|\left|\hat{\ell}\right| \cos\theta = \left|\overrightarrow{OA}\right| \cos\theta$$

since $\left|\hat{\ell}\right| = 1$.

Theorem 6-4: The projection of the sum of vectors on an arbitrary axis is equal to the sum of the projections of the individual vectors on the same axis.

Let for example $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be four vectors in space and $(\ell'\ell)$ be an arbitrary axis. Then,

$$\begin{aligned} &Proj.of\ \left(\vec{a}+\vec{b}+\vec{c}+\vec{d}\right)\ on\ (\ell'\ell) \\ &\qquad = Proj.of\ \vec{a}\ on\ (\ell'\ell) + Proj.of\ \vec{b}\ on\ (\ell'\ell) \\ &\qquad + Proj.of\ \vec{c}\ on\ (\ell'\ell) + Proj.of\ \vec{d}\ on\ (\ell'\ell) \qquad (6-5) \end{aligned}$$

Proof: According to equation (6-2),

$$Proj.of\ \left(\vec{a}+\vec{b}+\vec{c}+\vec{d}\right)\ on\ (\ell'\ell) = \left(\vec{a}+\vec{b}+\vec{c}+\vec{d}\right)\cdot\hat{\ell} =$$

$$\begin{aligned} &\vec{a}\cdot\hat{\ell}+\vec{b}\cdot\hat{\ell}+\vec{c}\cdot\hat{\ell}+\vec{d}\cdot\hat{\ell} \\ &\qquad = Proj.of\ \vec{a}\ on\ (\ell'\ell) + Proj.of\ \vec{b}\ on\ (\ell'\ell) \\ &\qquad + Proj.of\ \vec{c}\ on\ (\ell'\ell) + Proj.of\ \vec{d}\ on\ (\ell'\ell) \end{aligned}$$

(by virtue of (6-2)) and this completes the proof.

Corollary 6-1: If the sum of n vectors (coplanar or not) is equal to zero, then the projection of this sum on an arbitrary axis shall also be equal to zero.

This follows immediately from Theorem 6-4.

6-2) Some interesting applications of the Theory of projections

a) Application in Mechanics (Resultant of coplanar forces)

Let us for definiteness consider three coplanar and concurrent forces $\overrightarrow{F_1}, \overrightarrow{F_2}$ and $\overrightarrow{F_3}$ as shown in Fig. 6-2.

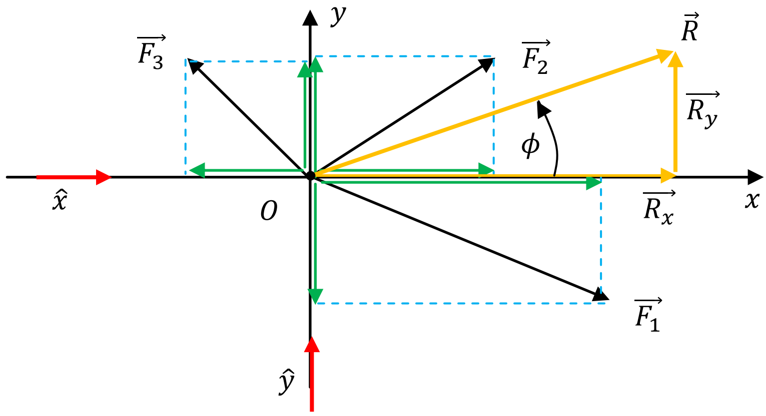

Fig. 6-2: Resultant of coplanar forces.

The main problem in Mechanics, is **to determine the resultant (vector sum)** $\vec{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$. If R_x and R_y are the signed projections of $\vec{R}$ on the x and the y axis respectively, then

$$\vec{R} = R_x\hat{x} + R_y\hat{y} \qquad (6-6)$$

According to Theorem 6-4,

$$\begin{Bmatrix} R_x = \overline{F_{1x}} + \overline{F_{2x}} + \overline{F_{3x}} \\ R_y = \overline{F_{1y}} + \overline{F_{2y}} + \overline{F_{3y}} \end{Bmatrix} \qquad (6-7)$$

where $\overline{F_{1x}},\ \overline{F_{2x}},\ \overline{F_{3x}}$ are **the signed projections** of the forces on the x axis and similarly, $\overline{F_{1y}},\ \overline{F_{2y}},\ \overline{F_{3y}}$ are **the signed projections** of the forces on the y axis.

Quite often, the forces are described **by their magnitudes and the angles they form with the two axes**. For example if $\overrightarrow{F_1}$ is described by its magnitude (**strength**) $F_1 = \left|\overrightarrow{F_1}\right|$ and its angle θ_1 with respect to the positive x axis, then $\overline{F_{1x}} = F_1 \cos\theta_1$, $\overline{F_{1y}} = F_1 \sin\theta_1$ and similarly for the other forces.

Once R_x and R_y are determined from (6-7) then **the magnitude $\boldsymbol{R}$ of the resultant and the angle it makes with the positive $\boldsymbol{x}$ axis**, are determined from the equations,

$$R = \left|\vec{R}\right| = \sqrt{{R_x}^2 + {R_y}^2}, \qquad \tan\phi = \frac{R_y}{R_x} \qquad (6-8)$$

The method obviously can be extended to any number of forces, and also applies in cases where the concurrent forces are not coplanar (forces in space). In this case there will be a third component R_z, i.e. a projection of the resultant $\vec{R}$ on the z axis.

Equilibrium conditions:

A system of planar, concurrent forces is in equilibrium if $R_x = 0$ and $R_y = 0$ (since then $\vec{R} = 0$), i.e.

$$Equilibrium\ Condition\ \ \vec{R} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} R_x = 0 \\ R_y = 0 \end{Bmatrix} \qquad (6-9)$$

b) Angle between two vectors in space, determined by their direction cosines.

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be two vectors in space, having direction cosines $(\cos\theta_x, \cos\theta_y, \cos\theta_z)$ and $(\cos\phi_x, \cos\phi_y, \cos\phi_z)$ respectively, and let also ***ω* be the angle between the two vectors**. Then the angle ω is determined from the equation,

$$\cos\omega = \cos\theta_x \cos\phi_x + \cos\theta_y \cos\phi_y + \cos\theta_z \cos\phi_z \qquad (6-10)$$

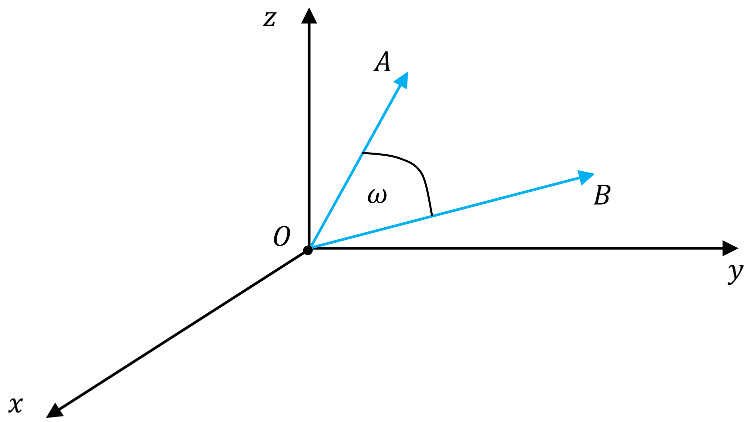

Fig. 6-3: Angle between two vectors in space.

Taking the dot product of the two vectors we get,

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = (OA)(OB)\cos\omega \Longrightarrow$$

$$\cos\omega = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{(OA)(OB)} = \frac{OA_x OB_x + OA_y OB_y + OA_z OB_z}{(OA)(OB)} \Longrightarrow$$

$$\cos\omega = \frac{OA_x}{OA}\frac{OB_x}{OB} + \frac{OA_y}{OA}\frac{OB_y}{OB} + \frac{OA_z}{OA}\frac{OB_z}{OB} \xRightarrow{(5-10)}$$

$$\cos\omega = \cos\theta_x \cos\phi_x + \cos\theta_y \cos\phi_y + \cos\theta_z \cos\phi_z$$

and this completes the proof.

Example 6-1

Find the angle ω between the two vectors $\overrightarrow{OA} = 2\hat{x} - \hat{y} + 2\hat{z}$ and $\overrightarrow{OB} = 4\hat{x} - 3\hat{z}$.

Solution

The direction cosines of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are, (see formula (5-10)),

$$\left\{\begin{aligned} \cos\theta_x &= \frac{2}{\sqrt{2^2+(-1)^2+2^2}} = \frac{2}{3} \\ \cos\theta_y &= \frac{-1}{\sqrt{2^2+(-1)^2+2^2}} = -\frac{1}{3} \\ \cos\theta_z &= \frac{2}{\sqrt{2^2+(-1)^2+2^2}} = \frac{2}{3} \end{aligned}\right\} \quad and$$

$$\left\{\begin{aligned} \cos\phi_x &= \frac{4}{\sqrt{4^2+0^2+(-3)^2}} = \frac{4}{5} \\ \cos\phi_y &= \frac{0}{\sqrt{4^2+0^2+(-3)^2}} = 0 \\ \cos\phi_z &= \frac{-3}{\sqrt{4^2+0^2+(-3)^2}} = -\frac{3}{5} \end{aligned}\right\}$$

Application of formula (6-20) yields,

$$\cos\omega = \left(\frac{2}{3}\right)\cdot\left(\frac{4}{5}\right) + \left(-\frac{1}{3}\right)\cdot 0 + \left(\frac{2}{3}\right)\cdot\left(-\frac{3}{5}\right) = \frac{2}{15} \Longrightarrow$$

$$\omega = \cos^{-1}\left(\frac{2}{15}\right) \cong 82.33°$$

Alternative method:

Taking the dot product of the two vectors, we get,

$$\overrightarrow{OA}\cdot\overrightarrow{OB} = \left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\cos\omega \overset{(5-6)}{\Longrightarrow}$$

$$2\cdot 4 + (-1)\cdot 0 + 2\cdot(-3) = \sqrt{2^2+(-1)^2+2^2}\cdot\sqrt{4^2+0^2+(-3)^2}\,\cos\omega \Longrightarrow$$

$$2 = 3\cdot 5\cos\omega \Longrightarrow \cos\omega = \frac{2}{15} \Longrightarrow \omega = \cos^{-1}\left(\frac{2}{15}\right) \cong 82.33°$$

Example 6-2

Find the vector projection and the signed projection (algebraic value) of the vector $\vec{a} = 2\hat{x} + 3\hat{y} + \hat{z}$ on the vector $\vec{\ell} = \hat{x} + \hat{y} + 2\hat{z}$.

Solution

The unit vector $\hat{\ell}$ associated with $\vec{\ell}$ is

$$\hat{\ell} = \frac{\vec{\ell}}{|\vec{\ell}|} = \frac{\hat{x} + \hat{y} + 2\hat{z}}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{1}{\sqrt{6}}(\hat{x} + \hat{y} + 2\hat{z}) \qquad (*)$$

The **signed projection** of $\vec{a}$ on $\hat{\ell}$ is, (see formula (6-2)),

$$Proj.of\ \vec{a}\ on\ (\ell'\ell) = \bar{a} = \vec{a} \cdot \hat{\ell} \Longrightarrow$$

$$\bar{a} = \frac{1}{\sqrt{6}}(2\hat{x} + 3\hat{y} + \hat{z}) \cdot (\hat{x} + \hat{y} + 2\hat{z}) = \frac{2 + 3 + 2}{\sqrt{6}} = \frac{7}{\sqrt{6}}$$

The **vector projection** (say $\overrightarrow{a_\ell}$) of $\vec{a}$ on $\hat{\ell}$ is,

$$\overrightarrow{a_\ell} = \bar{a}\,\hat{\ell} = \frac{7}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}}(\hat{x} + \hat{y} + 2\hat{z}) = \frac{7\hat{x} + 7\hat{y} + 14\hat{z}}{6}$$

Example 6-3

Consider the vectors $\vec{a} = 2\hat{x} - \hat{y} + \hat{z}$, $\vec{b} = -\hat{x} + 2\hat{y} - 3\hat{z}$ and $\vec{c} = 5\hat{x} - 3\hat{y} + 4\hat{z}$. Verify that the projection of $\left(\vec{a} + \vec{b} + \vec{c}\right)$ on the axis $\ell'\ell$ defined by the vector $\overrightarrow{AB} = 3\hat{x} - 2\hat{y} + \hat{z}$ is equal to the sum of the signed values of the projections of the individual vectors on the same axis (Theorem 6-4).

Solution

The vector $\vec{a} + \vec{b} + \vec{c} = 6\hat{x} - 2\hat{y} + 2\hat{z}$, while the unit vector $\hat{\ell}$ along the $\ell'\ell$ axis is

$$\hat{\ell} = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \frac{3\hat{x} - 2\hat{y} + \hat{z}}{\sqrt{3^2 + (-2)^2 + 1^2}} = \frac{1}{\sqrt{14}}(3\hat{x} - 2\hat{y} + \hat{z}) \qquad (*)$$

1) The projection of $\left(\vec{a} + \vec{b} + \vec{c}\right)$ on the axis $\ell'\ell$ is (see Theorem 6-2),

$$Proj.of\ \left(\vec{a} + \vec{b} + \vec{c}\right)\ on\ \ell'\ell = \left(\vec{a} + \vec{b} + \vec{c}\right) \cdot \hat{\ell} =$$

$$(6\hat{x} - 2\hat{y} + 2\hat{z}) \cdot \frac{(3\hat{x} - 2\hat{y} + \hat{z})}{\sqrt{14}} = \frac{18 + 4 + 2}{\sqrt{14}} = \frac{24}{\sqrt{14}} \qquad (**)$$

2) If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are the signed projections of $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively on the $\ell'\ell$ axis, then,

$$\bar{a} = \vec{a} \cdot \hat{\ell} = (2\hat{x} - \hat{y} + \hat{z}) \cdot \frac{(3\hat{x} - 2\hat{y} + \hat{z})}{\sqrt{14}} = \frac{6 + 2 + 1}{\sqrt{14}} = \frac{9}{\sqrt{14}}$$

$$\bar{b} = \vec{b} \cdot \hat{\ell} = (-\hat{x} + 2\hat{y} - 3\hat{z}) \cdot \frac{(3\hat{x} - 2\hat{y} + \hat{z})}{\sqrt{14}} = \frac{-3 - 4 - 3}{\sqrt{14}} = -\frac{10}{\sqrt{14}}$$

$$\bar{c} = \vec{c} \cdot \hat{\ell} = \left(5\hat{x} - \widehat{3y} + 4\hat{z}\right) \cdot \frac{(3\hat{x} - 2\hat{y} + \hat{z})}{\sqrt{14}} = \frac{15 + 6 + 4}{\sqrt{14}} = \frac{25}{\sqrt{14}}$$

We notice that

$$Proj.of\ \vec{a}\ on\ \ell'\ell + Proj.of\ \vec{b}\ on\ \ell'\ell + Proj.of\ \vec{c}\ on\ \ell'\ell =$$

$$\bar{a} + \bar{b} + \bar{c} = \frac{9 - 10 + 25}{\sqrt{14}} = \frac{24}{\sqrt{14}} \overset{(**)}{\Longrightarrow}$$

$$Proj.of\ \vec{a}\ \ on\ \ell'\ell + Proj.of\ \vec{b}\ \ on\ \ell'\ell + Proj.of\ \vec{c}\ \ on\ \ell'\ell$$
$$= Proj.of\ \left(\vec{a} + \vec{b} + \vec{c}\right) on\ \ell'\ell$$

and this verifies Theorem 6-4.

Example 6-4

Find the resultant $\vec{R}$ of the forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, $\overrightarrow{F_3}$ and $\overrightarrow{F_4}$ shown in Fig. 6-4. Given: $F_1 = 5\ Nt,\ F_2 = 2\ Nt,\ F_3 = 3\ Nt,\ F_4 = 3\ Nt$ and $a = 20°,\ \beta = 55°,\ \gamma = 40°,\ \delta = 50°$.

Solution

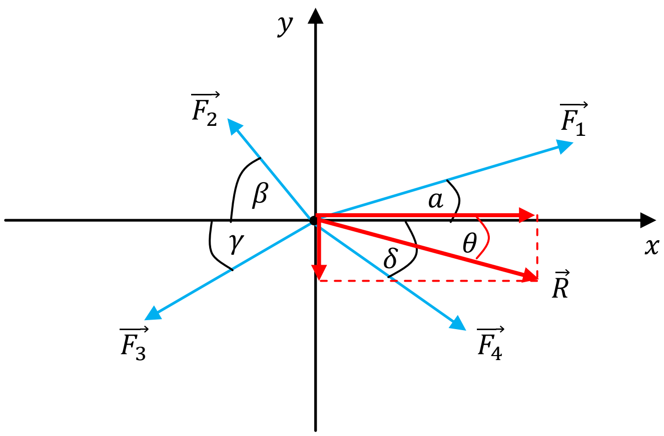

Fig. 6-4: Resultant of four coplanar forces.

If R_x and R_y are the components of the resultant $\vec{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} + \overrightarrow{F_4}$, then (see formula (6-7)),

$$\begin{Bmatrix} R_x = \overline{F_{1x}} + \overline{F_{2x}} + \overline{F_{3x}} + \overline{F_{4x}} \\ R_y = \overline{F_{1y}} + \overline{F_{2y}} + \overline{F_{3y}} + \overline{F_{4x}} \end{Bmatrix} \qquad (*)$$

From the first equation in (*) we get,

$$R_x = F_1 \cos\alpha - F_2 \cos\beta - F_3 \cos\gamma + F_4 \cos\delta \Longrightarrow$$

$$R_x = 5 \cos 20° - 2 \cos 55° - 3 \cos 40° + 3 \cos 50° \Longrightarrow$$

$$R_x \cong 3.18\ Nt \qquad (**)$$

Similarly, the y- component of the resultant is,

$$R_y = F_1 \sin\alpha + F_2 \sin\beta - F_3 \sin\gamma - F_4 \sin\delta \Longrightarrow$$

$$R_y = 5\sin 20° + 2\sin 55° - 3\sin 40° - 3\sin 50° \Longrightarrow$$

$$R_y \cong -0.88\ Nt \qquad (***)$$

The resultant, as **a vector** is

$$\vec{R} = R_x\hat{x} + R_y\hat{y} = 3.18\hat{x} - 0.88\hat{y}\ \ Nt$$

The magnitude $R = |\vec{R}| = \sqrt{3.18^2 + (-0.88)^2} \cong 3.30\ Nt$, while the angle θ is determined from

$$\tan\theta = \frac{R_y}{R_x} = \frac{-0.88}{3.18} \cong -0.276 \Longrightarrow \theta = \tan^{-1}(-0.276) \cong -15.46°$$

(negative angle θ means that R_y points towards the negative direction of the y axis, (see Fig. 6-4)).

Example 6-5

Consider a regular polygon with n sides $A_1A_2A_3 \cdots A_n$.Projecting the **closed line** determined by the consecutive vectors $\overrightarrow{A_1A_2},\ \overrightarrow{A_2A_3}, \overrightarrow{A_3A_4}, \cdots, \overrightarrow{A_{n-1}A_n},\ \overrightarrow{A_nA_1}$ on an arbitrary axis, lying in the plane of the polygon, show that

$$\cos\theta + \cos\left(\theta + \frac{2\pi}{n}\right) + \cos\left(\theta + \frac{4\pi}{n}\right) + \cdots + \cos\left(\theta + \frac{(n-1)2\pi}{n}\right) = 0$$

Solution

Since the vector sum

$$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \overrightarrow{A_3A_4} + \cdots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_1} = \overrightarrow{A_1A_1} = 0$$

the sum of the projections of the individual vectors on any axis will be zero as well, (see Corollary 6-1), i.e.

$$\overline{A_1A_2} + \overline{A_2A_3} + \overline{A_3A_4} + \cdots + \overline{A_{n-1}A_n} + \overline{A_nA_1} = 0 \qquad (*)$$

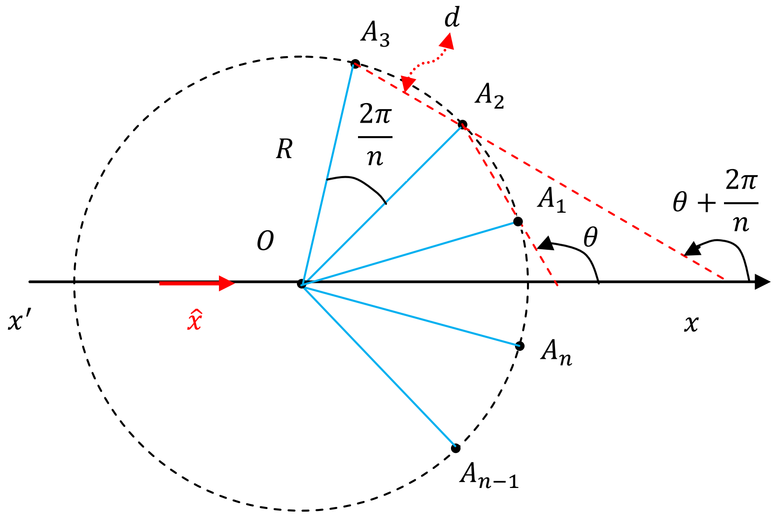

Fig. 6-5: Projection of a closed regular polygon line on an axis.

If we call d the side of the normal polygon, ($d = A_1A_2 = A_2A_3 = \cdots = A_{n-1}A_n = A_nA_1$, θ is the angle between $\overrightarrow{A_1A_2}$ and the positive x axis, i.e. $\theta = \measuredangle(\overrightarrow{A_1A_2}, \hat{x})$, as shown in Fig. 6-5 and $\overline{A_1A_2}, \overline{A_2A_3}, \overline{A_3A_4}, \cdots$ the **signed projections** of the vectors $\overrightarrow{A_1A_2}, \overrightarrow{A_2A_3}, \overrightarrow{A_3A_4}, \cdots$ on the x axis, then formula (6-4) implies,

$$\overline{A_1A_2} = d\cos\theta\,, \overline{A_2A_3} = d\cos\left(\theta + \frac{2\pi}{n}\right), \overline{A_3A_4} = d\cos\left(\theta + \frac{4\pi}{2n}\right), \cdots$$

and substituting into equation (*) the sought for formula follows easily.

PROBLEMS

6-1) The direction cosines of two vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are $(0.4, 0.6, 0.8)$ and $(0.18, 0.37, 0.70)$ respectively. Find the angle θ between the two vectors.

(**Answer:** $\theta \cong 31.35°$).

6-2) Consider n vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, \cdots, \overrightarrow{a_n}$. If the sum of the signed projections on each one of the axes Ox, Oy, Oz of a Cartesian system is zero, what is the resultant of the vectors?

6-3) Starting with the identity derived in Example 6-5, evaluate the sum

$$\mathrm{S} = \cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \cdots + \cos\left(\frac{(n-1)2\pi}{n}\right)$$

(**Answer:** $S = -1$).

Hint: Apply the identity in Example 6-5, for $\theta = 0$.

6-4) Show that

$$\cos 5° + \cos 77° + \cos 149° + \cos 221° + \cos 293° = 0$$

Hint: Apply the identity proved in Example 6-5, for $\theta = 5°$ and $n = 5$.

6-5) Following the procedure outlined in Example 6-5, show that for any $n \geq 2$,

$$\sum_{k=1}^{n} \sin\left(\frac{2k\pi}{n}\right) = 0 \qquad \sum_{k=1}^{n} \cos\left(\frac{2k\pi}{n}\right) = 0$$

6-6) In a triangle ABC consider a point M on BC, such that the partial ratio $(BCM) = \overline{BM} \div \overline{MC} = \mu \div \nu$. If $\overline{AM}, \overline{AB}, \overline{AC}$ are the projections of the vectors $\overrightarrow{AM}, \overrightarrow{AB}, \overrightarrow{AC}$ respectively on the axis determined by the vector $\overrightarrow{BC}$, show that $(\mu + \nu)\overline{AM} = \nu\overline{AB} + \mu\overline{AC}$.

Hint: Use formula (3-16) to express $\overrightarrow{AM}$ in terms of $\overrightarrow{AB}, \overrightarrow{AC}, \mu$ and ν.

6-7) In a Cartesian system $Oxyz$, a vector $\overrightarrow{OA}$ has a length $OA = \left|\overrightarrow{OA}\right| = 10$. The angle between the vector and the axes Ox and Oy are $\theta_x = 65°$ and $\theta_y = 58°$ respectively. Determine the angle θ_z the vector forms with the Oz axis (assuming $0 < \theta_z < 90°$), and the coordinates of the vector.

(**Answer:** $\theta_z \cong 42.67°, \overline{OA_x} = 4.22, \overline{OA_y} = 5.30, \overline{OA_z} = 7.35$).

Hint: Use formula (5-11).

6-8) Find the resultant $\vec{R}$ of the forces $\overrightarrow{F_1}, \overrightarrow{F_2}$ and $\overrightarrow{F_3}$ shown in Fig. 6-6. Assume that $F_1 = 10\ Nt,\ F_2 = 5\ Nt,\ F_3 = 15\ Nt,\ a = 50°,\ \beta = 40°,$ $\gamma = 70°$.

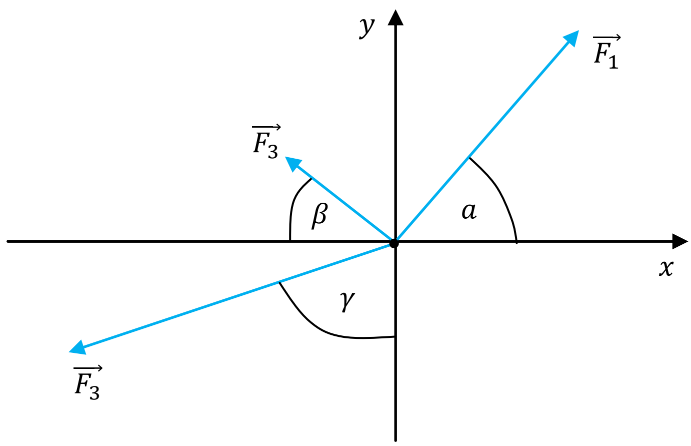

Fig. 6-6: Resultant of three coplanar forces.

CHAPTER 7: The Outer or Cross Product of two Vectors

A second type of product of two vectors is the so called **outer or cross product** and is another **vector**, whose exact definition and properties are given in the sequel. The cross product of two vectors has some very interesting applications in Engineering, Physics, Geometry, etc.

7-1) Definition and properties of the cross product

Let $\vec{a}$ and $\vec{b}$ be two vectors and $\theta = \measuredangle(\vec{a}, \vec{b})$ be the smallest angle between the two vectors, $(0 \leq \theta \leq \pi)$.

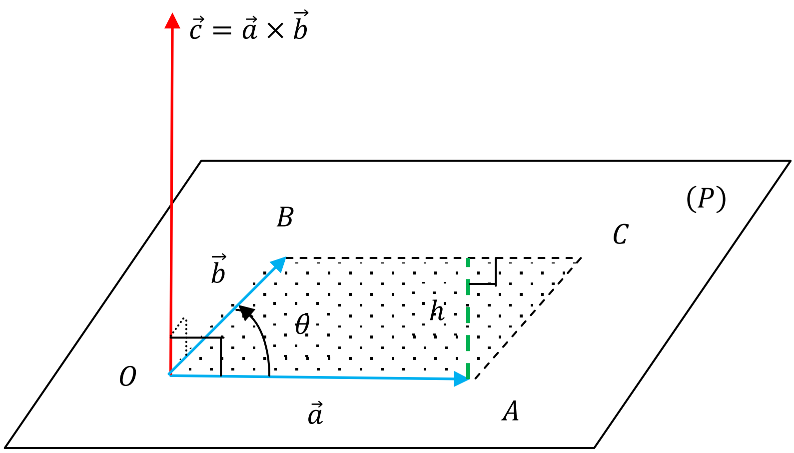

Fig. 7-1: The cross (outer) product of two vectors.

The outer or cross product of two vectors $\vec{a}$ and $\vec{b}$, designated $\vec{a} \times \vec{b}$ and read $\vec{a}$ cross $\vec{b}$, is **a third vector** $\boldsymbol{\vec{c} = \vec{a} \times \vec{b}}$ such that:

a) Is perpendicular to the plane (P), determined by the two vectors $\vec{a}$ and $\vec{b}$, i.e. $\boldsymbol{\vec{c}}$ **is perpendicular to both** $\boldsymbol{\vec{a}}$ **and** $\boldsymbol{\vec{b}}$,

b) Its direction is such that the system $\{\vec{a}, \vec{b}, \vec{c}\}$ (in this order) **is right-handed**, (see section 3-1), i.e. the vector $\vec{c}$ points in the direction a right-handed screw advances when its head is rotated from $\vec{a}$ to $\vec{b}$ through the angle θ,

c) Its magnitude is

$$|\vec{c}| = |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta = ab \sin\theta \qquad (7-1)$$

where $a = |\vec{a}|$ and $b = |\vec{b}|$ are the magnitudes of the vectors $\vec{a}$ and $\vec{b}$ respectively.

1) The cross product of two **collinear (parallel)** vectors ($\theta = 0$ or $\theta = \pi$) is **zero**, since in both cases $\sin\theta = 0$. In particular, the cross product of any vector by itself is zero, i.e. $\vec{\boldsymbol{a}} \times \vec{\boldsymbol{a}} = \mathbf{0}$. And conversely, if the cross product of two non zero vectors $\vec{a}$ and $\vec{b}$ is zero, then the two vectors **are parallel**, (since in this case $\sin\theta = 0$).

2) The magnitude $|\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}|$ is equal to the geometric area of the parallelogram $\boldsymbol{OACB}$ formed by the two vectors $\vec{\boldsymbol{a}} = \overrightarrow{\boldsymbol{OA}}$ and $\vec{\boldsymbol{b}} = \overrightarrow{\boldsymbol{OB}}$, (see Fig. 7-1). Indeed, if h is the perpendicular distance between the two parallel lines OA and BC, then $h = (OB)\sin\theta$, and the area of the parallelogram is

$$(OACB) = (OA)h = (OA)(OB)\sin\theta = |\overrightarrow{OA} \times \overrightarrow{OB}| = |\vec{a} \times \vec{b}|$$

3) The cross product is **anti commutative**, i.e.

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \qquad (7-2)$$

Indeed, the vectors $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ have the same magnitude but opposite directions. This implies that when working with cross products of vectors, **the order of the vectors is important**.

4) If k is any real number (not zero, $k \in \mathbb{R} - \{0\}$), then

$$(k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b}) \qquad (7-3)$$

5) The cross product **is distributive** with respect to the addition, i.e.

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \qquad (7-4)$$

Let us for example consider the cross product $(\vec{a} + 3\vec{b}) \times (4\vec{c} - \vec{d})$. We may expand this cross product with the aid of (7-3) and (7-4), i.e.

$$(\vec{a} + 3\vec{b}) \times (4\vec{c} - \vec{d}) = 4\vec{a} \times \vec{c} + 12\vec{b} \times \vec{c} - \vec{a} \times \vec{d} - 3\vec{b} \times \vec{d}$$

6) Let $\hat{x}, \hat{y}$ and $\hat{z}$ be the unit vectors along the Ox, Oy and Oz axes respectively, of an orthonormal right-handed Cartesian system $Oxyz$. The magnitude of each one of these vectors is one, (unit vectors), and also these vectors are pair wise perpendicular, (see section 3-2).

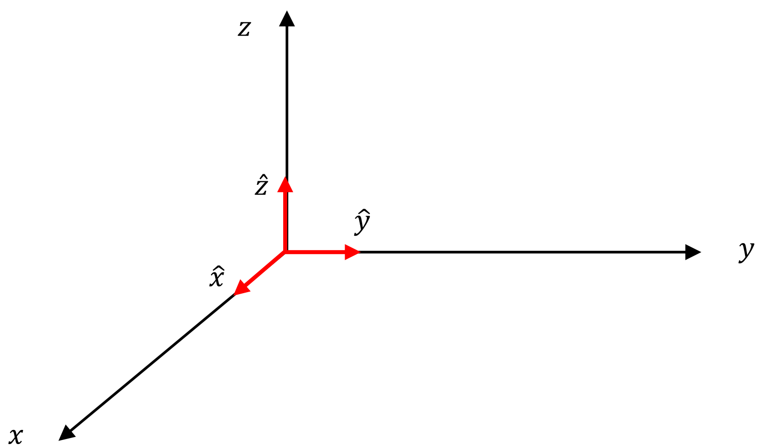

Fig. 7-2: Cross products of the unit vectors.

The following vector identities hold true, (let the reader check it):

$$\begin{Bmatrix} \hat{x} \times \hat{x} = 0 & \hat{y} \times \hat{y} = 0 & \hat{z} \times \hat{z} = 0 \\ \hat{x} \times \hat{y} = \hat{z} & \hat{y} \times \hat{z} = \hat{x} & \hat{z} \times \hat{x} = \hat{y} \end{Bmatrix} \qquad (7-5)$$

Since the cross product is anti commutative, $\hat{y} \times \hat{x} = -\hat{x} \times \hat{y} = -\hat{z}$, etc.

7) The Cartesian expression of the cross product.

Let us consider the two vectors $\vec{a} = a_1\hat{x} + a_2\hat{y} + a_3\hat{z}$ and $\vec{b} = b_1\hat{x} + b_2\hat{y} + b_3\hat{z}$. The cross product of these two vectors is

$$\vec{a} \times \vec{b} = (a_1\hat{x} + a_2\hat{y} + a_3\hat{z}) \times (b_1\hat{x} + b_2\hat{y} + b_3\hat{z}) \qquad (*)$$

which by virtue of (7-3) and (7-4) yields,

$$\vec{a} \times \vec{b} = a_1\hat{x} \times (b_1\hat{x} + b_2\hat{y} + b_3\hat{z}) + a_2\hat{y} \times (b_1\hat{x} + b_2\hat{y} + b_3\hat{z}) + a_3\hat{z} \times (b_1\hat{x} + b_2\hat{y} + b_3\hat{z}) \xRightarrow{(7-3)\,(7-4)(7-5)}$$

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\hat{x} - (a_1b_3 - a_3b_1)\hat{y} + (a_1b_2 - a_2b_1)\hat{z} \quad (**)$$

(let the reader verify the calculations). Formula (**) can be expressed, equivalently, in a convenient form, using **determinants notation**, i.e.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad (7-6)$$

(The reader is supposed to be familiar with the elementary properties of determinants).

7-2) Turning moments (Torques)

The original motivation for the cross product of two vectors came from Physics. While forces (in general) are responsible for the translation of objects, **torques are responsible for the rotation of rigid bodies**. Let us for definiteness consider a force $\vec{F}$ applied at a point P of a rigid body, whose position vector with respect to an origin O is $\vec{r}$, (see Fig. 7-3) Then **the torque (or turning moment) $\vec{T}$** of the force $\vec{F}$ with respect to the origin O, is defined to be,

$$\vec{T} = \vec{r} \times \vec{F} = \overrightarrow{OP} \times \vec{F} \qquad (7-7)$$

The physical unit of the torque is $(Nt) \cdot (m)$.

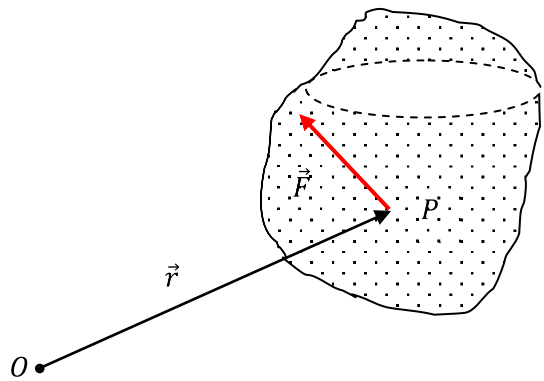

Fig. 7-3: The torque of a force.

If a number of forces $\overrightarrow{F_1}, \overrightarrow{F_1}, \cdots, \overrightarrow{F_n}$ are applied at the points $P_1, P_2, \cdots, P_n$ determined by the vectors $\overrightarrow{r_1}, \overrightarrow{r_2}, \cdots, \overrightarrow{r_n}$ with respect to the origin O, then the total torque of the forces (denoted by $\sum \vec{T}$) with respect to the same origin O is defined to be,

$$\sum \vec{T} = \overrightarrow{T_1} + \overrightarrow{T_2} + \cdots + \overrightarrow{T_n} \Rightarrow$$

$$\sum \vec{T} = \overrightarrow{r_1} \times \overrightarrow{F_1} + \overrightarrow{r_2} \times \overrightarrow{F_2} + \cdots + \overrightarrow{r_n} \times \overrightarrow{F_n} \qquad (7-8)$$

Theorem 7-1: (The Varignon Theorem of Torques).

Let us consider n forces $\overrightarrow{F_1}, \overrightarrow{F_2}, \cdots, \overrightarrow{F_n}$ applied at the same point P of a rigid body. The vector sum $\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2} + \cdots + \overrightarrow{F_n}$ is called **the resultant** of the n forces. The **Varignon's Theorem** may be stated as follows:

The torque of the resultant of n concurring forces $\overrightarrow{F_1}, \overrightarrow{F_2}, \cdots, \overrightarrow{F_n}$ about any point O in space (the origin), is equal to the algebraic sum of the torques of the components $\overrightarrow{F_1}, \overrightarrow{F_2}, \cdots, \overrightarrow{F_n}$.

Varignon's Theorem follows immediately from the distributive law of the cross product with respect to the addition. Indeed, if $\vec{r}$ is the position vector of the point P, at which all the forces concur, then the torque of the resultant $\vec{F}$ with respect to the origin O is, (by definition),

$$\vec{T} = \vec{r} \times \vec{F} = \vec{r} \times \left(\overrightarrow{F_1} + \overrightarrow{F_2} + \cdots + \overrightarrow{F_n}\right) \xRightarrow{(7-4)}$$

$$\vec{T} = \vec{r} \times \overrightarrow{F_1} + \vec{r} \times \overrightarrow{F_2} + \cdots + \vec{r} \times \overrightarrow{F_n} \qquad (7-9)$$

and this completes the proof.

7-3) Equilibrium of a rigid body

Let us consider n forces $\overrightarrow{F_1}, \overrightarrow{F_2}, \cdots, \overrightarrow{F_n}$ applied at the points $\overrightarrow{r_1}, \overrightarrow{r_2}, \cdots, \overrightarrow{r_n}$ of a rigid body, respectively. From Mechanics we know that **the body is in equilibrium when both the vector sum of the forces (resultant) vanishes and the sum of all the turning moments (torques) of the forces about the origin *O* vanishes**. In symbols,

$$\begin{Bmatrix} \overrightarrow{F_1} + \overrightarrow{F_2} + \cdots + \overrightarrow{F_n} = 0 \\ \overrightarrow{r_1} \times \overrightarrow{F_1} + \overrightarrow{r_2} \times \overrightarrow{F_2} + \cdots + \overrightarrow{r_n} \times \overrightarrow{F_n} = 0 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \sum \vec{F} = 0 \\ \sum \vec{T} = 0 \end{Bmatrix} \qquad (7-10)$$

Equations (7-10) are **the necessary and sufficient conditions** for the equilibrium of a rigid body.

7-4) Rotation of a rigid body about a fixed axis

Let us consider a rigid body rotating about an axis $\ell'\ell$ with angular frequency ω (rad/sec) as shown in Fig. 7-4. The linear velocity $\vec{v}$ of a point M of the body, determined by its position vector $\vec{r}$ with respect to an arbitrary point O lying on the axis, is given by the formula

$$\vec{v} = \vec{\omega} \times \vec{r} \qquad (7-11)$$

Note that $\vec{v}$ does **not** depend on the choice of the arbitrary origin O. In formula (7-11) $\boldsymbol{\vec{\omega}}$ **is a vector** with magnitude ω (rad/sec) and direction that of the axis.

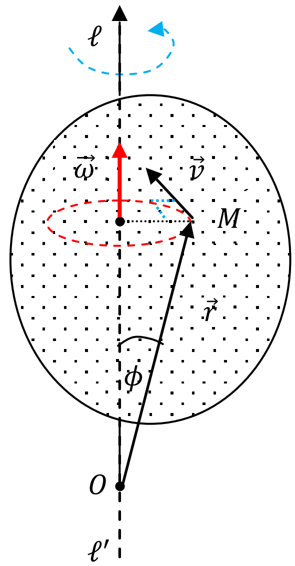

Fig. 7-4: Rotation of a body about an axis.

The magnitude of the linear velocity $v = |\vec{v}| = (r \sin \phi)\omega$, while the vector $\vec{v}$ at any point M of the rigid body, is perpendicular to the plane of $\vec{\omega}$ and $\vec{r}$, that is $\vec{v} = \vec{\omega} \times \vec{r}$. **Note that $\vec{v}$ is independent of the origin O**, since if O' is another point on the axis and $\overrightarrow{r'}$ is the position vector of M with respect to the new origin, then

$$\overrightarrow{r'} = \overrightarrow{O'M} = \overrightarrow{O'O} + \overrightarrow{OM} = \overrightarrow{O'O} + \vec{r} \Longrightarrow$$

$$\vec{\omega} \times \overrightarrow{r'} = \vec{\omega} \times \left(\overrightarrow{O'O} + \vec{r}\right) = \vec{\omega} \times \overrightarrow{O'O} + \vec{\omega} \times \vec{r} = \vec{\omega} \times \vec{r} = \vec{v}$$

since $\vec{\omega} \times \overrightarrow{O'O} = 0$, (the vectors $\vec{\omega}$ and $\overrightarrow{O'O}$ are collinear).

Example 7-1

Find the cross product of the vectors $\vec{a} = 2\hat{x} + \hat{y} + \hat{z}$ and $\vec{b} = \hat{x} + 3\hat{y} + 2\hat{z}$.

Solution

The cross product is given by formula (7-6), i.e.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \hat{x} - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \hat{y} + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \hat{z} \Longrightarrow$$

$$\vec{a} \times \vec{b} = (2-3)\hat{x} - (4-1)\hat{y} + (6-1)\hat{z} = -\hat{x} - 3\hat{y} + 5\hat{z}$$

Example 7-2

In Example 7-1, find the unit vector $\hat{n}$ perpendicular to both $\vec{a}$ and $\vec{b}$.

Solution

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{-\hat{x} - 3\hat{y} + 5\hat{z}}{\sqrt{(-1)^2 + (-3)^2 + 5^2}} = \frac{1}{\sqrt{35}}(-\hat{x} - 3\hat{y} + 5\hat{z})$$

Note: The thus obtain unit vector $\hat{n}$ forms a right-handed system with $\vec{a}$ and $\vec{b}$, i.e. **the system $\{\vec{a}, \vec{b}, \hat{n}\}$ is right-handed** (see Fig. 7-1).

Example 7-3

Find the area of the triangle formed by the two vectors, $\overrightarrow{AB} = 3\hat{x} + 2\hat{y}$ and $\overrightarrow{AC} = \hat{x} + 2\hat{z}$.

Solution

The sought for area A is half of the area of the parallelogram formed by the two vectors (see Fig. 7-1), i.e.

$$A = \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| \qquad (*)$$

The cross product is

$$\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 4\hat{x} - 6\hat{y} - 2\hat{z} \Longrightarrow$$

$$A = \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| = \frac{1}{2}\sqrt{4^2 + (-6)^2 + (-2)^2} = \frac{\sqrt{56}}{2} \ square \ units$$

Note: The square unit, is considered to be the area of a square formed by the unit vectors $\hat{x}$ and $\hat{y}$, ($|\hat{x}| = |\hat{y}| = 1$).

Example 7-4

Show that $\left|\vec{a} \times \vec{b}\right|^2 + \left|\vec{a} \cdot \vec{b}\right|^2 = a^2 b^2$, where $a = |\vec{a}|$ and $b = \left|\vec{b}\right|$.

Solution

$$\left|\vec{a} \times \vec{b}\right|^2 = (ab \sin\theta)^2 = a^2 b^2 (\sin\theta)^2$$

$$\left|\vec{a} \cdot \vec{b}\right|^2 = (ab \cos\theta)^2 = a^2 b^2 (\cos\theta)^2$$

and adding term wise we get,

$$\left|\vec{a} \times \vec{b}\right|^2 + \left|\vec{a} \cdot \vec{b}\right|^2 = a^2 b^2 \{(\sin\theta)^2 + (\cos\theta)^2\} = a^2 b^2$$

Example 7-5

Derive the “**Law of Sines**”.

Solution

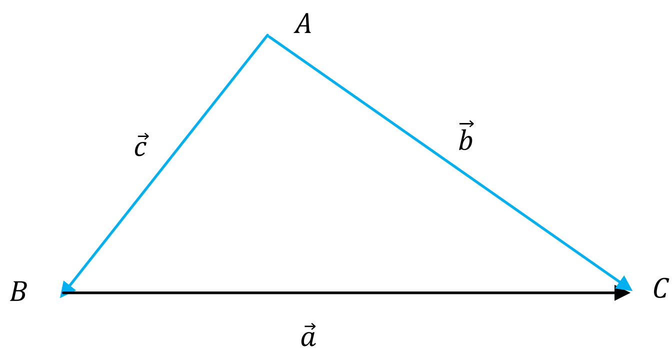

Fig. 7-5: The “Law of Sines”.

Let us consider a triangle formed by the vectors $\vec{b}$ and $\vec{c}$ as shown in Fig. 7-5. We notice that $\vec{a} = \vec{b} - \vec{c}$ and cross multiplying both sides by $\vec{a}$ we get,

$$\vec{a} \times \vec{a} = \vec{a} \times (\vec{b} - \vec{c}) \xRightarrow{(\vec{a}\times\vec{a}=0)}$$

$$\vec{a} \times (\vec{b} - \vec{c}) = 0 \Longrightarrow \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = 0 \Longrightarrow \vec{a} \times \vec{b} = \vec{a} \times \vec{c} \Longrightarrow$$

$$|\vec{a} \times \vec{b}| = |\vec{a} \times \vec{c}| \Longrightarrow ab \sin C = ac \sin B \xRightarrow{(a\neq 0)}$$

$$b \sin C = c \sin B \Longrightarrow \frac{b}{\sin B} = \frac{c}{\sin C} \qquad (*)$$

Similarly we can show that

$$\frac{b}{\sin B} = \frac{a}{\sin A} \qquad (**)$$

From (*) and (**) we get,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad (***)$$

Note: Formula (***) is known as the "**Law of Sines**" and is a fundamental formula in Trigonometry.

Example 7-6 (The Heron's formula for the area of a triangle)

If a, b and c are the sides of a triangle, show that its area E is given by the formula

$$E = \sqrt{\tau(\tau - a)(\tau - b)(t - c)} \quad \textit{where} \quad 2\tau = a + b + c$$

This is known as **the Heron's formula**, named after **the Heron of Alexandria**. It is an important formula, since it gives **the area of a triangle in terms of its sides only**.

Solution

Let us consider the triangle ABC shown in Fig. 7-5. The area E of the triangle will be,

$$E = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| \Longrightarrow 2E = \left|\vec{a} \times \vec{b}\right| \Longrightarrow 4E^2 = \left|\vec{a} \times \vec{b}\right|^2 \quad (*)$$

and taking into account the identity shown in Example 7-4, equation (*) yields,

$$\begin{aligned} 4E^2 &= a^2b^2 - \left|\vec{a} \cdot \vec{b}\right|^2 = a^2b^2 - a^2b^2(\cos A)^2 \\ &= a^2b^2(1 - (\cos A)^2) \qquad (**) \end{aligned}$$

From the "**Law of Cosines**" (see Example 5-3),

$$c^2 = a^2 + b^2 - 2ab\cos A \Longrightarrow \cos A = \frac{a^2 + b^2 - c^2}{2ab} \quad (***)$$

Substituting the expression of $\cos A$ into (**) and simplifying yields the desired result (the Heron's formula). For detailed calculations, see Problem 7-7.

Example 7-7

The three sides of a triangle are $a = 6\ m, b = 8\ m$ and $c = 12\ m$. What is the area E of the triangle?

Solution

The sought for area is obtained readily from **Heron's formula**.

$$2\tau = a + b + c = 6 + 8 + 12 = 26\ m \Longrightarrow \tau = 13\ m$$

$$E = \sqrt{\tau(\tau - a)(\tau - b)(t - c)} = \sqrt{13(13 - 6)(13 - 8)(13 - 12)} \Longrightarrow$$

$$E = \sqrt{13 \cdot 7 \cdot 5 \cdot 1} = \sqrt{455} \cong 21.33\ \ m^2$$

Example 7-8

If $\vec{x}\cdot\vec{y}=\vec{x}\cdot\vec{z}$ and $\vec{x}\times\vec{y}=\vec{x}\times\vec{z}$, show that $\vec{y}=\vec{z}$, (provided that $\vec{x}\neq 0$).

Solution

$$\vec{x}\cdot\vec{y}=\vec{x}\cdot\vec{z}\Longrightarrow\vec{x}\cdot\vec{y}-\vec{x}\cdot\vec{z}=0\Longrightarrow\vec{x}\cdot(\vec{y}-\vec{z})=0\overset{(\vec{x}\neq 0)}{\Longrightarrow}$$

$$\begin{Bmatrix}\vec{y}-\vec{z}=0, \quad \boldsymbol{or}\\ \vec{x}\perp(\vec{y}-\vec{z})\end{Bmatrix} \qquad (*)$$

From the second equation we get,

$$\vec{x}\times\vec{y}=\vec{x}\times\vec{z}\Rightarrow\vec{x}\times\vec{y}-\vec{x}\times\vec{z}=0\Longrightarrow\vec{x}\times(\vec{y}-\vec{z})=0\overset{(\vec{x}\neq 0)}{\Longrightarrow}$$

$$\begin{Bmatrix}\vec{y}-\vec{z}=0, \quad \boldsymbol{or}\\ \vec{x}\parallel(\vec{y}-\vec{z})\end{Bmatrix} \qquad (**)$$

Since the vectors $\vec{x}$ and $(\vec{y}-\vec{z})$ cannot be perpendicular and parallel at the same time, the only alternative, from (*) and (**) is $(\vec{y}-\vec{z})=0$, i.e. $\vec{y}=\vec{z}$ and this completes the proof.

Example 7-9

A force $\vec{F}=2\hat{x}-3\hat{y}+5\hat{z}$ is applied at the point P whose position vector is $\overrightarrow{r_P}=3\hat{x}+4\hat{y}+\hat{z}$. Find the torque $\vec{T}$ of the force, **a)** about the origin and **b)** about the point $K(-1,-4,7)$.

Solution

a) The torque of $\vec{F}$ about the origin is (see formula (7-7)),

$$\vec{T}=\overrightarrow{r_P}\times\vec{F}=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\3&4&1\\2&-3&5\end{vmatrix}=\begin{vmatrix}4&1\\-3&5\end{vmatrix}\hat{x}-\begin{vmatrix}3&1\\2&5\end{vmatrix}\hat{y}+\begin{vmatrix}3&4\\2&-3\end{vmatrix}\hat{z}\Longrightarrow$$

$$\vec{T}=23\hat{x}-13\hat{y}-17\hat{z}$$

b) The torque of $\vec{F}$ about the point $K(-1,-4,7)$ is

$$\vec{T} = \overrightarrow{KP} \times \vec{F} = \left(\overrightarrow{OP} - \overrightarrow{OK}\right) \times \vec{F} = (4\hat{x} + 8\hat{y} - 6\hat{z}) \times (2\hat{x} - 3\hat{y} + 5\hat{z}\,)$$

and expressing the cross product into its Cartesian form, we find,

$$\vec{T} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 4 & 8 & -6 \\ 2 & -3 & 5 \end{vmatrix} = \begin{vmatrix} 8 & -6 \\ -3 & 5 \end{vmatrix} \hat{x} - \begin{vmatrix} 4 & -6 \\ 2 & 5 \end{vmatrix} \hat{y} + \begin{vmatrix} 4 & 8 \\ 2 & -3 \end{vmatrix} \hat{z} \Longrightarrow$$

$$\vec{T} = 22\hat{x} - 32\hat{y} - 28\hat{z}$$

Example 7-10

A couple consists of a pair of opposite forces $\vec{F}$ and $-\vec{F}$ applied at two different points P and Q. Show that the torque of the couple is independent from the origin O.

Solution

Let (P) be the plane defined by the two forces $\vec{F}$ and $-\vec{F}$, (note that two parallel lines define a plane).

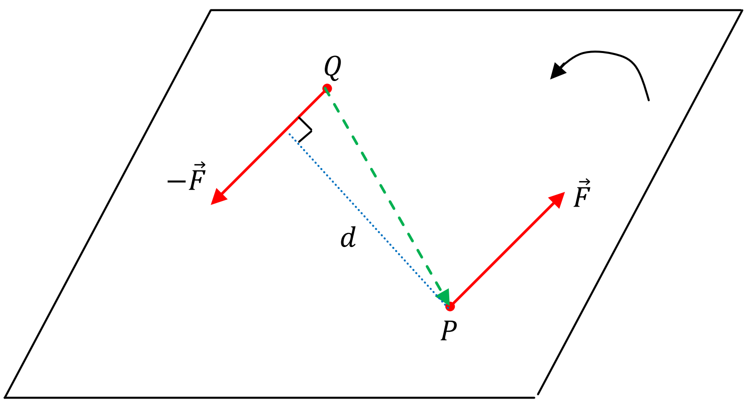

Fig. 7-6: The torque of a couple.

If $\overrightarrow{r_P}$ and $\overrightarrow{r_Q}$ are the position vectors of P and Q respectively with respect to **an arbitrary origin $\boldsymbol{O}$**, then the torque of the couple with respect to the arbitrary origin is, (see formula (7-8)),

$$\vec{T} = \overrightarrow{r_P} \times \vec{F} + \overrightarrow{r_Q} \times \overrightarrow{(-F)} = \left(\overrightarrow{r_P} - \overrightarrow{r_Q}\right) \times \vec{F} = \overrightarrow{QP} \times \vec{F} \qquad (*)$$

Formula (*) shows that the torque of the couple does not depend on the arbitrarily chosen origin O, on the contrary depends solely on the force $\vec{F}$ and the distance between the two forces, i.e. it is a characteristic of the couple, (as we say, **the torque of a couple is a free vector**). The couple has a turning effect, i.e. tends to rotate (and not to translate) the body on which it applies. It is easily shown that the magnitude T of the couple torque, (as expressed in (*)), is given by the formula,

$$T = F \cdot d \qquad (**)$$

where d is the **perpendicular** distance between the two parallel forces, (let the reader show it).

Example 7-11

Assume that n forces $\overrightarrow{F_1},\ \overrightarrow{F_2}, \cdots,\ \overrightarrow{F_n}$, each one of the same magnitude F act downward (in the $-\hat{z}$ direction), at the points $P_1, P_2, \cdots, P_n$ of the horizontal xOy plane. Let us further assume that another force $\vec{X}$ acts at another point P of the plane, so that the system of forces is in equilibrium. Find $\vec{X}$ and the coordinates of P.

Solution

Let $\overrightarrow{r_1}, \overrightarrow{r_2}, \cdots, \overrightarrow{r_n}$ be the position vectors of the points $P_1, P_2, \cdots, P_n$ respectively and $\vec{r}$ be the point of application of P. The first formula in (7-10), (**equilibrium of forces**) implies that,

$$\vec{X} = -nF\hat{z} \qquad (*)$$

The second equilibrium condition in (7-10), (**equilibrium of moments**), implies that,

$$\vec{r} \times \vec{X} = -\left(\overrightarrow{r_1} \times \overrightarrow{F_1} + \overrightarrow{r_2} \times \overrightarrow{F_2} + \cdots + \overrightarrow{r_n} \times \overrightarrow{F_n}\right) \overset{(*)}{\Rightarrow}$$

$$\vec{r} \times (-nF\hat{z}) = -\left\{\overrightarrow{r_1} \times (-F\hat{z}) + \left(\overrightarrow{r_2} \times (-F\hat{z})\right) + \cdots + \left(\overrightarrow{r_n} \times (-F\hat{z})\right)\right\} \Longrightarrow$$

$$-nF(\vec{r} \times \hat{z}) = F(\overrightarrow{r_1} + \overrightarrow{r_2} + \cdots + \overrightarrow{r_n}) \times \hat{z} \Longrightarrow$$

$$(-n\vec{r}) \times \hat{z} = (\overrightarrow{r_1} + \overrightarrow{r_2} + \cdots + \overrightarrow{r_n}) \times \hat{z} \Longrightarrow$$

$$\vec{r} = -\frac{\overrightarrow{r_1} + \overrightarrow{r_2} + \cdots + \overrightarrow{r_n}}{n} \qquad (**)$$

Example 7-12

A rigid body rotates about the $z'z$ axis with angular frequency $\omega = 10\ rad/sec$. Find the linear velocity of the body at the point $M(1,2,3)$.

Solution

$$\vec{v} = \vec{\omega} \times \vec{r} = 10\hat{z} \times (\hat{x} + 2\hat{y} + 3\hat{z}) \Longrightarrow$$

$$\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 10 \\ 1 & 2 & 3 \end{vmatrix} = -20\hat{x} + 10\hat{y} \quad (m/\sec)$$

PROBLEMS

7-1) Show that the necessary and sufficient condition that two vectors $\vec{a}$ and $\vec{b}$ are parallel, is the vanishing of the cross product $\vec{a} \times \vec{b}$.

7-2) Find the area of the triangle ABC with vertices $A(1,2,-3)$, $B(-1,0,4)$ and $C(2,7,5)$.

Hint: Area is equal to $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$.

7-3) If $\vec{a} + \vec{b} + \vec{c} = 0$ show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

7-4) Find the cross product of $\vec{a} = \hat{x} - \hat{y} + 2\hat{z}$ and $\vec{b} = -2\hat{x} + \hat{y} - \hat{z}$.

7-5) In Problem 7-4, find the unit vector $\hat{n}$ perpendicular to the plane determined by $\vec{a}$ and $\vec{b}$.

(**Answer:** $\hat{n} = \frac{1}{\sqrt{11}}(-\hat{x} - 3\hat{y} - \hat{z})$).

Hint: $\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

7-6) Find the area of a triangle having sides $a = 12\ m, b = 8\ m, c = 15\ m$.

Hint: Apply Heron's formula, (Example 7-6).

7-7) In Example 7-6 perform detailed calculations to derive the Heron's formula.

7-8) Show that the area of a plane quadrilateral ABCD is given by the formula $\frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{BD}\right|$, i.e. is equal to one-half the magnitude of the cross product of its two diagonals.

7-9) Assuming that $\hat{x}, \hat{y}, \hat{z}$ are the unit vectors of a right-handed, orthogonal Cartesian system $\{Oxyz\}$, show that:

$$(\hat{x} \times \hat{y}) \times \hat{z} = 0$$
$$(\hat{x} \times \hat{y}) \cdot \hat{z} = 1$$
$$\hat{x} \cdot (\hat{x} \times \hat{y}) = 0$$

7-10) If $\vec{a} = (2,1,-3)$, $\vec{b} = (1,1,2)$ and $\vec{c} = (-1,3,5)$ find:

a) $\vec{a} \cdot (\vec{b} \times \vec{c})$, **b)** $(\vec{a} + 2\vec{b}) \times \vec{c}$ **c)** $\vec{a} \times \vec{b} + \vec{b} \times \vec{c}$

7-11) If $\overrightarrow{r_A}, \overrightarrow{r_B}, \overrightarrow{r_C}$ are the position vectors of the vertices A, B and C of a triangle ABC, show that its area E is given by the formula

$$E = \frac{1}{2}|\overrightarrow{r_A} \times \overrightarrow{r_B} + \overrightarrow{r_B} \times \overrightarrow{r_C} + \overrightarrow{r_C} \times \overrightarrow{r_A}|$$

7-12) Find the projection of the vector $\vec{a} = (2,1,3)$ in a direction perpendicular to the plane formed by the two vectors $\vec{b} = (1,1,3)$ and $\vec{c} = (-1,2,4)$.

Hint: The unit vector perpendicular to the plane determined by the vectors $\vec{b}$ and $\vec{c}$ is $\hat{n} = \frac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|}$ and the projection of $\vec{a}$ on $\hat{n}$ is $\vec{a}\cdot\hat{n}$.

7-13) Given a tetrahedron with vertices A, B, C and D, a vector is constructed perpendicularly to each face, pointing outwards and having length equal to the area of the face. Show that the sum of these four outwards vectors is equal to zero.

Hint: If we choose the vertex A to be the origin, then the position vectors of B, C and D will be $\overrightarrow{r_B}, \overrightarrow{r_C}$ and $\overrightarrow{r_D}$ respectively. The vector normal to the face ABC, pointing outwards and having length equal to the area of the face is $\widehat{n_1} = \frac{1}{2}(\overrightarrow{r_B} \times \overrightarrow{r_C})$, etc.

7-14) Find the torque of the force $\vec{F} = 2\hat{x} - 3\hat{y} + 5\hat{z}$ applied at the point P determined by its position vector $\overrightarrow{r_P} = \hat{x} + 2\hat{y} + 7\hat{z}$ with respect: **a)** To the origin and **b)** To the point $Q(-1,-2,4)$.

Hint: a) $\vec{T} = \overrightarrow{r_P} \times \vec{F}$, **b)** $\vec{T} = \overrightarrow{QP} \times \vec{F} = \left(\overrightarrow{r_P} - \overrightarrow{r_Q}\right) \times \vec{F}$.

7-15) A rigid body is spinning with angular velocity $\omega = 10\ rad/sec$ about an axis parallel to the vector $2\hat{x} - \hat{y} + 5\hat{z}$ and passing through the point $P(4,8,-3)$. Find the linear velocity of the point A of the body whose position vector is $\vec{r} = (1,2,3)$, (lengths are measured in m).

(**Answer:** $\vec{v} = \frac{10}{\sqrt{30}}(-13\hat{x} - \hat{y} + 5\hat{z})\ \ m/sec$).

7-16) Three concurrent forces $\overrightarrow{F_1} = 2\hat{x} + 3\hat{y} + \hat{z}$, $\overrightarrow{F_2} = -\hat{x} + \hat{y} + 2\hat{z}$ and $\overrightarrow{F_3} = \hat{x} - \hat{y} + \hat{z}$ are applied at the point $P(2,-1,5)$ of a rigid body. **a)** Find the resultant $\vec{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$, **b)** Find the moments $\overrightarrow{T_1}, \overrightarrow{T_2}, \overrightarrow{T_3}$ of the forces with respect to the origin, **c)** Find the torque of the resultant $\vec{R}$ with respect to the origin and verify thus the Varignon's Theorem.

7-17) The system of forces in Fig. 7-7 is in equilibrium. Find $\overrightarrow{F_3}$ and d.

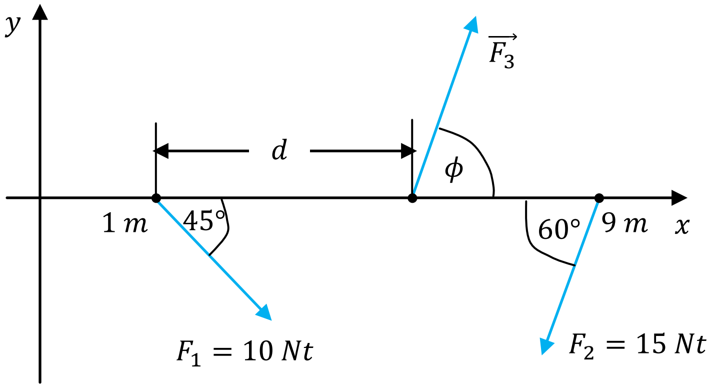

Fig. 7-7: Equilibrium of forces.

(**Answer:** $F_3 \cong 20\ Nt,\ \ \phi \cong 88.77°,\ \ d \cong 5.20\ m$).

Hint: The net resultant $\sum \vec{F} = 0$, which implies $\sum F_x = 0$ and $\sum F_y$, and also the total torque about any point must be zero. Choose as a point the point of application of $\overrightarrow{F_1}$, whose distance from the origin is $1\ m$. Note that only the y −components of the forces produce torques with respect to this particular point (why?).

7-18) The torque of a couple is $20\ Nt \cdot m$. If $F = 5\ Nt$ find the perpendicular distance between the two forces.

7-19) If the four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar, show that

$$\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = 0$$

7-20) Show the vector identity

$$\left(\vec{a} \times \vec{b}\right) \cdot \left(\vec{c} \times \vec{d}\right) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} \\ \vec{a} \cdot \vec{d} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

CHAPTER 8: The Scalar Triple Product (or Dot Cross Product)

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any three given vectors. The **scalar vector product** of these three vectors is defined to be $(\vec{a} \times \vec{b}) \cdot \vec{c}$ and is denoted $(\vec{a}, \vec{b}, \vec{c})$. So, by definition,

$$(\vec{a}, \vec{b}, \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} \qquad (8-1)$$

Formula (8-1) implies that $(\vec{a}, \vec{b}, \vec{c})$ is **a scalar** quantity (as being the dot product of the two vectors $(\vec{a} \times \vec{b})$ and $\vec{c}$).

1) If $\{\vec{a}, \vec{b}, \vec{c}\}$ (**in this order**) forms a right-handed system, then the scalar cross product $(\vec{a}, \vec{b}, \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ is **a positive number**, while if $\{\vec{a}, \vec{b}, \vec{c}\}$ forms a left-handed system, then $(\vec{a}, \vec{b}, \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ is **a negative number**.

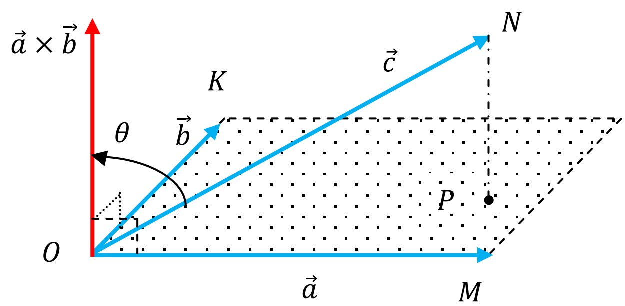

Fig. 8-1: Scalar cross product of $\vec{a} = \overrightarrow{OM}$, $\vec{b} = \overrightarrow{OK}$, $\vec{c} = \overrightarrow{ON}$.

In Fig. 8-1, assuming that $\{\vec{a}, \vec{b}, \vec{c}\}$ forms a right-handed system, then the vectors $(\vec{a} \times \vec{b})$ and $\vec{c}$ will lie on the same side of the space, relative to the plane determined by the vectors $\vec{a}$ and $\vec{b}$. This means that the angle θ between the vectors $(\vec{a} \times \vec{b})$ and $\vec{c}$ is an acute angle, i.e. $0 < \theta < 90°$ and therefore $\cos\theta > 0$. The scalar cross product

$$(\vec{a}, \vec{b}, \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = |\vec{a} \times \vec{b}||\vec{c}| \cos\theta > 0$$

and this proves the assertion.

If the system $\{\vec{a}, \vec{b}, \vec{c}\}$ (**in this order**) is a left-handed system, then $90° < \theta < 180°$, the $\cos\theta < 0$ and $(\vec{a}, \vec{b}, \vec{c}) < 0$.

2) The **geometrical significance** of the scalar vector product is that **the magnitude** of $(\vec{a}, \vec{b}, \vec{c})$ gives the volume of the parallelepiped formed by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

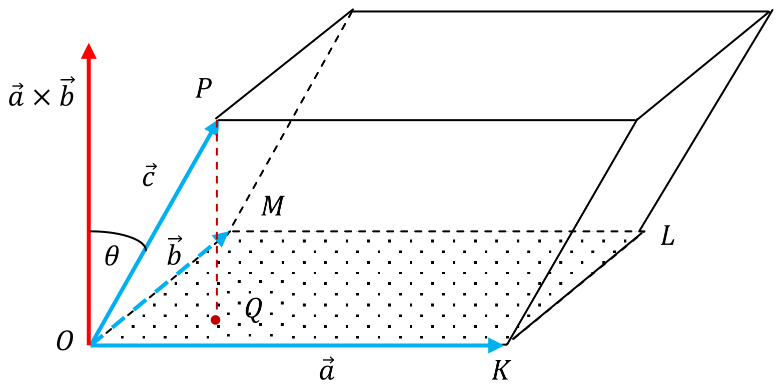

Fig. 8-2: Geometrical meaning of the scalar vector product.

In Fig. 8-2, let PQ be the height of the parallelepiped formed by the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$. The area of the parallelogram $OKLM$ is

$$Area\ of\ Parallelogram\ OKLM = |\vec{a} \times \vec{b}| \qquad (*)$$

as proved in chapter 7, while the height of the parallelepiped is

$$PQ = |\overrightarrow{OP}| \cos\theta = |\vec{c}| \cos\theta \qquad (**)$$

The volume V of the parallelepiped formed by the three given vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is

$$V = (Area\ OKLM) \cdot (Height\ PQ) = \left|\vec{a} \times \vec{b}\right| |\vec{c}| \cos\theta \quad (***)$$

The magnitude of the scalar vector product $\left(\vec{a}, \vec{b}, \vec{c}\right)$ is

$$\left|\left(\vec{a}, \vec{b}, \vec{c}\right)\right| = \left|\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}\right| = \left|\vec{a} \times \vec{b}\right| |\vec{c}| \cos\theta \overset{(***)}{\Longrightarrow}$$

$$Volume\ V = \left|\left(\vec{a}, \vec{b}, \vec{c}\right)\right| \qquad (8-2)$$

and this completes the proof.

3) Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are **coplanar** if and only if the scalar vector product $\left(\vec{a}, \vec{b}, \vec{c}\right)$ vanishes, ($\left(\boldsymbol{\vec{a}}, \boldsymbol{\vec{b}}, \boldsymbol{\vec{c}}\right) = \mathbf{0}$).

Since in this case the volume formed by the three vectors must be zero.

4) The Cartesian expression of the scalar vector product

Let $\vec{a} = a_1\hat{x} + a_2\hat{y} + a_3\hat{z}$, $\vec{b} = b_1\hat{x} + b_2\hat{y} + b_3\hat{z}$ and $\vec{c} = c_1\hat{x} + c_2\hat{y} + c_3\hat{z}$ be three given vectors. Then

$$\left(\vec{a}, \vec{b}, \vec{c}\right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad (8-3)$$

The proof is easy, (see Problem 8-11).

5) The scalar vector product is distributive, i.e.

$$\left(\vec{a} + \vec{d}, \vec{b}, \vec{c}\right) = \left(\vec{a}, \vec{b}, \vec{c}\right) + \left(\vec{d}, \vec{b}, \vec{c}\right) \qquad (8-4)$$

Proof: From the definition of the dot cross product we have,

$$\left(\vec{a} + \vec{d}, \vec{b}, \vec{c}\right) = \left\{\left(\vec{a} + \vec{d}\right) \times \vec{b}\right\} \cdot \vec{c} = \left\{\vec{a} \times \vec{b} + \vec{d} \times \vec{b}\right\} \cdot \vec{c} =$$

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}+\left(\vec{d}\times\vec{b}\right)\cdot\vec{c}=\left(\vec{a},\vec{b},\vec{c}\right)+\left(\vec{d},\vec{b},\vec{c}\right)$$

and this completes the proof.

6) The scalar vector product remains unchanged if we perform a cyclic permutation on the vectors, (see Fig. 3-2), i.e.

$$\left(\vec{a},\vec{b},\vec{c}\right)=\left(\vec{b},\vec{c},\vec{a}\right)=\left(\vec{c},\vec{a},\vec{b}\right) \qquad (8-5)$$

Formula (8-5) is proved easily, if we consider the Cartesian expression (8-3) of $\left(\vec{a},\vec{b},\vec{c}\right)$ and take into account well known properties of determinants, (recall that **the value of a determinant does not change if two rows or two columns are interchanged**).

Example 8-1

Find the scalar vector product $\left(\vec{a},\vec{b},\vec{c}\right)$ if,

1) $\vec{a}=2\hat{y}-\hat{z},\ \vec{b}=\hat{x}+\hat{z},\ \vec{c}=\hat{x}+\hat{y}+\hat{z}$, and

2) $\vec{a}=-\hat{x}+\hat{y}+\hat{z},\ \vec{b}=2\hat{x}-\hat{z},\ \vec{c}=\hat{x}-\hat{y}+2\hat{z}$.

Solution

Application of formula (8-3) yields:

1) In the first case,

$$\left(\vec{a},\vec{b},\vec{c}\right)=\begin{vmatrix}0 & 2 & -1\\ 1 & 0 & 1\\ 1 & 1 & 1\end{vmatrix}=0\cdot\begin{vmatrix}0 & 1\\ 1 & 1\end{vmatrix}-2\cdot\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}+(-1)\cdot\begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix}=-1$$

2) In the second case,

$$\left(\vec{a},\vec{b},\vec{c}\right)=\begin{vmatrix}-1 & 1 & 1\\ 2 & 0 & -1\\ 1 & -1 & 2\end{vmatrix}$$
$$=(-1)\cdot\begin{vmatrix}0 & -1\\ -1 & 2\end{vmatrix}-1\cdot\begin{vmatrix}2 & -1\\ 1 & 2\end{vmatrix}+1\cdot\begin{vmatrix}2 & 0\\ 1 & -1\end{vmatrix}=-6$$

Example 8-2

Find the volume of the parallelepiped formed by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ of the foregoing Example.

Solution

The sought for volume is obtained from formula (8-2).

1) In the first case,

$$V = \left|\left(\vec{a}, \vec{b}, \vec{c}\right)\right| = |-1| = 1 \;\; (cubic\ unit)$$

2) In the second case,

$$V = \left|\vec{a}, \vec{b}, \vec{c}\right| = |-6| = 6 \;\; (cubic\ units)$$

Note: The cubic unit is by definition the volume of the cube formed by the three unit vectors $\hat{x}, \hat{y}, \hat{z}$.

Example 8-3

For which value(s) of the parameter λ the three vectors $\vec{a}(1, \lambda, -2), \vec{b}(-1,2,-\lambda)$ and $\vec{c}(3,-1,1)$ shall be coplanar?

Solution

The three given vectors shall be coplanar if and only if their scalar vector product is zero, i.e.

$$\left(\vec{a}, \vec{b}, \vec{c}\right) = 0 \Leftrightarrow \begin{vmatrix} 1 & \lambda & -2 \\ -1 & 2 & -\lambda \\ 3 & -1 & 1 \end{vmatrix} = 0 \Leftrightarrow$$

$$1 \cdot \begin{vmatrix} 2 & -\lambda \\ -1 & 1 \end{vmatrix} - \lambda \cdot \begin{vmatrix} -1 & -\lambda \\ 3 & 1 \end{vmatrix} + (-2) \cdot \begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} = 0 \Leftrightarrow$$

$$3\lambda^2 = 12 \Leftrightarrow \lambda^2 = 4 \Leftrightarrow \lambda = \pm 2.$$

Example 8-4

In the foregoing Example, determine the values of λ for which the system $\{\vec{a}, \vec{b}, \vec{c}\}$ is a right-handed system.

Solution

The system of vectors $\{\vec{a}, \vec{b}, \vec{c}\}$ is right-handed if $(\vec{a}, \vec{b}, \vec{c}) > 0$, i.e.

$$-3\lambda^2 + 12 > 0 \Leftrightarrow \lambda^2 < 4 \Leftrightarrow -2 < \lambda < 2$$

Example 8-5

Resolve a given vector $\vec{x}$ in the directions of three given vectors $\vec{a}, \vec{b}, \vec{c}$, (non-coplanar).

Solution

We want to find three real numbers x_1, x_2, x_3 such that

$$\vec{x} = x_1\vec{a} + x_2\vec{b} + x_3\vec{c} \qquad (*)$$

Dot multiplying both sides of (*) by $(\vec{b} \times \vec{c})$ and noting that $\vec{b} \cdot (\vec{b} \times \vec{c}) = 0$ and $\vec{c} \cdot (\vec{b} \times \vec{c}) = 0$, (since $(\vec{b} \times \vec{c})$ is normal to both $\vec{b}$ and $\vec{c}$) yields,

$$(\vec{b} \times \vec{c}) \cdot \vec{x} = x_1(\vec{b} \times \vec{c}) \cdot \vec{a} \Longrightarrow (\vec{b}, \vec{c}, \vec{x}) = x_1(\vec{b}, \vec{c}, \vec{a}) \overset{(8-5)}{\Longrightarrow}$$

$$(\vec{x}, \vec{b}, \vec{c}) = x_1(\vec{a}, \vec{b}, \vec{c}) \Longrightarrow x_1 = \frac{(\vec{x}, \vec{b}, \vec{c})}{(\vec{a}, \vec{b}, \vec{c})}$$

and similarly, by cyclic permutation,

$$x_2 = \frac{(\vec{a}, \vec{x}, \vec{c})}{(\vec{a}, \vec{b}, \vec{c})} \qquad x_3 = \frac{(a, \vec{b}, \vec{x})}{(\vec{a}, \vec{b}, \vec{c})}$$

and finally,

$$\vec{x} = \frac{(\vec{x},\vec{b},\vec{c})}{(\vec{a},\vec{b},\vec{c})}\vec{a} + \frac{(\vec{a},\vec{x},\vec{c})}{(\vec{a},\vec{b},\vec{c})}\vec{b} + \frac{(a,\vec{b},\vec{x})}{(\vec{a},\vec{b},\vec{c})}\vec{c} \qquad (**)$$

Note that $(\vec{a},\vec{b},\vec{c}) \neq 0$, since $\vec{a},\vec{b},\vec{c}$ have been assumed to be non-coplanar.

PROBLEMS

8-1) Find the scalar vector product of $\vec{a} = (1,2,3), \vec{b} = (2,0,-1)$ and $\vec{c} = (3,-1,0)$. Is the system $\{\vec{a},\vec{b},\vec{c}\}$ right-handed or left handed?

(**Answer:** $(\vec{a},\vec{b},\vec{c}) = -13$, Left-handed).

8-2) Find the scalar vector product of the following vectors and determine whether these systems form a right-handed or a left-handed system.

1) $\vec{a} = (2,-1,5),\ \vec{b} = (3,4,7),\ \vec{c} = (0,2,-3)$

2) $\vec{a} = (3,-5,2),\ \vec{b} = (4,-2,3),\ \vec{c} = (2,6,8)$

3) $\vec{a} = (-5,1,3),\ \vec{b} = (-3,4,-1),\ \vec{c} = (7,3,5)$

8-3) Find the volume of a tetrahedron ABCD in terms of the coordinates of its vertices. Assume:
$A(x_a,y_a,z_a), B(x_b,y_b,z_b), C(x_c,y_c,z_c), D(x_d,y_d,z_d)$

(**Answer:** $V = \frac{1}{6}\begin{vmatrix} x_b - x_a & y_b - y_a & z_b - z_a \\ x_c - x_a & y_c - y_a & z_c - z_a \\ x_d - x_a & y_d - y_a & z_d - z_a \end{vmatrix}$).

Hint: The sought for volume of the tetrahedron is $\left({}^{1}/_{6}\right)$ of the volume of the parallelepiped formed by the vectors $\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}$, (see formula (8-2). In case the determinant turns out to be a negative number, we have to take the absolute value of this, since the volume is always a positive number.

8-4) Apply the result obtained in the foregoing Problem to find the volume of a tetrahedron with vertices
$A = (1,2,3), B(-1,-2,8), C(7,9,-5)$ and $D(3,5,13)$.

8-5) Determine whether the four points $A(1,2,3), B(4,3,6), C(0,1,1)$ and $D(1,0,2)$ are coplanar or not.

(**Answer:** Non-coplanar).

Hint: Consider the vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ and check whether they are coplanar or not, (see property 3).

8-6) Find the volume of the parallelepiped formed by the vectors $\overrightarrow{OA} = (1,-3,1), \overrightarrow{OB} = (-2,5,7), \overrightarrow{OC} = (4,1,2)$.

Hint: See formula (8-2).

8-7) For which values of the parameter λ the vectors $\overrightarrow{OA} = (\lambda, 1,0)$, $\overrightarrow{OB} = (0,-1,3)$ and $\overrightarrow{OC} = (2,5,\lambda + 1)$ are coplanar?

(**Answer:** $\lambda = -8 \pm \sqrt{70}$).

Hint: See Example 8-3.

8-8) For which values of λ the vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ of the foregoing Problem form a right-handed system?

Hint: See Example 8-4.

8-9) Show the vector identity,

$$\left(\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}\right) = 2\left(\vec{a}, \vec{b}, \vec{c}\right)$$

8-10) Assuming that $\vec{a} \cdot \vec{y} = A,\ \vec{b} \cdot \vec{y} = B,\ \vec{c} \cdot \vec{y} = C$, show that

$$\vec{y} = A\frac{\vec{b} \times \vec{c}}{\left(\vec{a}, \vec{b}, \vec{c}\right)} + B\frac{\vec{c} \times \vec{a}}{\left(\vec{a}, \vec{b}, \vec{c}\right)} + C\frac{\vec{a} \times \vec{b}}{\left(\vec{a}, \vec{b}, \vec{c}\right)}$$

Hint: Resolve $\vec{y}$ in the directions of $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$, $\vec{c} \times \vec{a}$, (see Example 8-5).

8-11) Prove formula (8-3).

Hint: Find the Cartesian expression of the cross product first, and then dot multiply by the third vector $\vec{c}$.

CHAPTER 9: The Vector Triple Product

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any three given vectors. We may consider the following vector products,

$$\vec{a} \times (\vec{b} \times \vec{c}) \quad or \quad (\vec{a} \times \vec{b}) \times \vec{c} \tag{9-1}$$

which are called **vector triple products**. Note that **a vector triple product is a vector**. For example,

$$\hat{x} \times (\hat{y} \times \hat{z}) = \hat{x} \times \hat{x} = 0$$

while

$$\hat{y} \times (\hat{x} \times \hat{y}) = \hat{y} \times \hat{z} = \hat{x}$$

(see equation (7-5)).

1) Assuming that $\vec{a} = a_1\hat{x} + a_2\hat{y} + a_3\hat{z},\ \vec{b} = b_1\hat{x} + b_2\hat{y} + b_3\hat{z}$ and $\vec{c} = c_1\hat{x} + c_2\hat{y} + c_3\hat{z}$ then we may easily show that, (see Problem 9-14),

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix} \tag{9-2}$$

2) The vector triple product is **not** associative, i.e.

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

3) The following two **vector identities** are useful when working with vector triple products.

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \tag{9-3}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a} \tag{9-4}$$

Proof: Since the vector $\vec{d} = \vec{a} \times (\vec{b} \times \vec{c})$ is perpendicular to both $\vec{a}$ and $(\vec{b} \times \vec{c})$ this means that $\vec{d}$ lies in the plane determined by the vectors $\vec{b}$ and $\vec{c}$, and therefore (see section 4-3),

$$\vec{d} = \vec{a} \times (\vec{b} \times \vec{c}) = k\vec{b} + \lambda\vec{c} \qquad (*)$$

where k and λ are constants to **be determined**. Dot multiplying both sides of (*) by $\vec{a}$, and taking into consideration that $\vec{a} \perp \vec{d}$, (i.e. that $\vec{a} \cdot \vec{d} = 0$), we get,

$$\vec{a} \cdot \vec{d} = 0 = k(\vec{a} \cdot \vec{b}) + \lambda(\vec{a} \cdot \vec{c}) \Longrightarrow$$

$$\{k = -m(\vec{a} \cdot \vec{c}) \quad and \quad \lambda = m(\vec{a} \cdot \vec{b})\} \qquad (**)$$

where m is a not zero constant, and equation (*) implies,

$$\vec{d} = \vec{a} \times (\vec{b} \times \vec{c}) = -m(\vec{a} \cdot \vec{c})\vec{b} + m(\vec{a} \cdot \vec{b})\vec{c} \qquad (***)$$

Formula (***) should be valid for all vectors $\vec{a}, \vec{b}$ and $\vec{c}$, and if for simplicity take $\vec{a} = \hat{y}, \vec{b} = \hat{y}$ and $\vec{c} = \hat{z}$, then equation (***) implies,

$$\hat{y} \times (\hat{y} \times \hat{z}) = -m(\hat{y} \cdot \hat{z})\hat{y} + m(\hat{y} \cdot \hat{y})\hat{z} \xRightarrow{(\hat{y}\cdot\hat{z}=0),(\hat{y}\cdot\hat{y}=1)}$$

$$\hat{y} \times \hat{x} = 0 + m\hat{z} \Longrightarrow -\hat{z} = m\hat{z} \Longrightarrow m = -1$$

and formula (***) becomes,

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

and this completes the proof.

Regarding formula (9-4) we note that

$$(\vec{a} \times \vec{b}) \times \vec{c} = -\vec{c} \times (\vec{a} \times \vec{b}) \xRightarrow{(9-3)}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = -\{(\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}\} = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a}$$

Example 9-1

If $\vec{a} = \hat{x} + \hat{y} + \hat{z}$, $\vec{b} = 2\hat{x} - \hat{y}$, $\vec{c} = \hat{x} + 2\hat{z}$, find $\vec{a} \times (\vec{b} \times \vec{c})$.

Solution

First we find the cross product $\vec{b} \times \vec{c}$,

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & -1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -2\hat{x} - 4\hat{y} + \hat{z}$$

and then,

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 1 \\ -2 & -4 & 1 \end{vmatrix} = 5\hat{x} - 3\hat{y} - 2\hat{z}$$

Example 9-2

Work previous Example, with the aid of formula (9-3).

Solution

$$\vec{a} \cdot \vec{b} = (\hat{x} + \hat{y} + \hat{z}) \cdot (2\hat{x} - \hat{y}) = 2 - 1 + 0 = 1 \quad (*)$$

$$\vec{a} \cdot \vec{c} = (\hat{x} + \hat{y} + \hat{z}) \cdot (\hat{x} + 2\hat{z}) = 1 + 0 + 2 = 3 \quad (**)$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \xRightarrow{(*)(**)}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = 3(2\hat{x} - \hat{y}) - (\hat{x} + 2\hat{z}) = 5\hat{x} - 3\hat{y} - 2\hat{z}$$

which is identical to the result found in Example 9-1.

Example 9-3

If $\hat{x}, \hat{y}, \hat{z}$ are the unit vectors of the $Oxyz$ orthogonal Cartesian system and $\vec{a}$ is an arbitrary vector, show that

$$\hat{x} \times (\vec{a} \times \hat{x}) + \hat{y} \times (\vec{a} \times \hat{y}) + \hat{z} \times (\vec{a} \times \hat{z}) = 2\vec{a}$$

Solution

Application of formula (9-3) yields,

$$\hat{x} \times (\vec{a} \times \hat{x}) = (\hat{x} \cdot \hat{x})\vec{a} - (\hat{x} \cdot \vec{a})\hat{x} = \vec{a} - (\hat{x} \cdot \vec{a})\hat{x}$$

$$\hat{y} \times (\vec{a} \times \hat{y}) = (\hat{y} \cdot \hat{y})\vec{a} - (\hat{y} \cdot \vec{a})\hat{y} = \vec{a} - (\hat{y} \cdot \vec{a})\hat{y}$$

$$\hat{z} \times (\vec{a} \times \hat{z}) = (\hat{z} \cdot \hat{z})\vec{a} - (\hat{z} \cdot \vec{a})\hat{z} = \vec{a} - (\hat{z} \cdot \vec{a})\hat{z}$$

and adding term wise we get,

$$\begin{aligned}\hat{x} \times (\vec{a} \times \hat{x}) + \hat{y} \times (\vec{a} \times \hat{y}) + \hat{z} \times (\vec{a} \times \hat{z}) \\ = 3\vec{a} - \underbrace{\{(\hat{x} \cdot \vec{a})\hat{x} + (\hat{y} \cdot \vec{a})\hat{y} + (\hat{z} \cdot \vec{a})\hat{z}\}}_{\vec{a}} \Longrightarrow\end{aligned}$$

$$\hat{x} \times (\vec{a} \times \hat{x}) + \hat{y} \times (\vec{a} \times \hat{y}) + \hat{z} \times (\vec{a} \times \hat{z}) = 3\vec{a} - \vec{a} = 2\vec{a}$$

and this completes the proof.

Example 9-4

Show that the vector

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) + (\vec{a} \times \vec{c}) \times (\vec{d} \times \vec{b}) + (\vec{a} \times \vec{d}) \times (\vec{b} \times \vec{c})$$

is parallel to the vector $\vec{a}$.

Solution

If we call $\vec{x} = \vec{a} \times \vec{b}$, then

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{x} \times (\vec{c} \times \vec{d}) \overset{(9-3)}{\Longrightarrow}$$

$$\begin{aligned}(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) &= (\vec{x} \cdot \vec{d})\vec{c} - (\vec{x} \cdot \vec{c})\vec{d} \\ &= \{(\vec{a} \times \vec{b}) \cdot \vec{d}\}\vec{c} - \{(\vec{a} \times \vec{b}) \cdot \vec{c}\}\vec{d} \Longrightarrow\end{aligned}$$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a}, \vec{b}, \vec{d})\vec{c} - (\vec{a}, \vec{b}, \vec{c})\vec{d} \qquad (*)$$

Similarly we have,

$$(\vec{a} \times \vec{c}) \times (\vec{d} \times \vec{b}) = -(\vec{a}, \vec{b}, \vec{d})\vec{c} - (\vec{b}, \vec{c}, \vec{d})\vec{a} \qquad (**)$$

$$(\vec{a} \times \vec{d}) \times (\vec{b} \times \vec{c}) = (\vec{a}, \vec{b}, \vec{c})\vec{d} - (\vec{b}, \vec{c}, \vec{d})\vec{a} \qquad (***)$$

and adding term wise we get,

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$$
$$=-2(\vec{b},\vec{c},\vec{d})\vec{a}$$

which is **a scalar multiple of $\vec{a}$**, and this implies that the given vector quantity is a vector parallel to the vector $\vec{a}$.

Example 9-5

If $\vec{a},\vec{b},\vec{c}$ are given vectors, with $\vec{a}$ not perpendicular to $\vec{b}$, "solve" for the unknown vector $\vec{y}$ the system of equations

$$\begin{Bmatrix} \vec{a}\cdot\vec{y}=\lambda \ \ (\lambda\in\mathbb{R}-\{0\}) \\ \vec{b}\times\vec{y}=\vec{c} \end{Bmatrix}$$

Solution

Cross multiplying the second equation by $\vec{a}$ we get,

$$\vec{a}\times(\vec{b}\times\vec{y})=\vec{a}\times\vec{c}\xRightarrow{(9-3)}$$

$$(\vec{a}\cdot\vec{y})\vec{b}-(\vec{a}\cdot\vec{b})\vec{y}=\vec{a}\times\vec{c}\xRightarrow{(\vec{a}\cdot\vec{y}=\lambda)}\lambda\vec{b}-(\vec{a}\cdot\vec{b})\vec{y}=\vec{a}\times\vec{c}\xRightarrow{(\vec{a}\cdot\vec{b}\neq 0)}$$

$$\vec{y}=\frac{\lambda\vec{b}-\vec{a}\times\vec{c}}{(\vec{a}\cdot\vec{b})}$$

PROBLEMS

9-1) Find the vector triple products

$$\hat{x}\times(\hat{y}\times\hat{z}),\qquad \hat{y}\times(\hat{z}\times\hat{y}),\quad \hat{z}\times(\hat{x}\times\hat{y})$$

(**Answer:** $0,\hat{z},0$).

9-2) If $\vec{a}=(1,2,3),\vec{b}=(2,-3,1),\vec{c}=(3,-2,5)$ find by direct computations the vector triple products $\vec{a}\times(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b})\times\vec{c}$.

9-3) Work Problem 9-2 with the aid of formulas (9-3) and (9-4).

Answer: $31\hat{x} - 44\hat{y} + 19\hat{z},\ 11\hat{x} - 76\hat{y} - 37\hat{z}$).

9-4) Prove the vector identity

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$

Hint: Use formula (9-3).

9-5) Prove the vector identity

$$(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c}) = (\vec{b} \cdot \vec{c})|\vec{a}|^2 - (\vec{a} \cdot \vec{b})(\vec{a} \cdot \vec{c})$$

Hint: Show first that $(\vec{a} \times \vec{b}) \cdot \vec{y} = \vec{a} \cdot (\vec{b} \times \vec{y})$, (see formula (8-5)). Then, if we call $\vec{y} = \vec{a} \times \vec{c}$, the term $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{y} = \vec{a} \cdot (\vec{b} \times \vec{y}) = \vec{a} \cdot \left(\vec{b} \times (\vec{a} \times \vec{c})\right)$=$\vec{a} \cdot \{(\vec{b} \cdot \vec{c})\vec{a} - (\vec{b} \cdot \vec{a})\vec{c}\}$, etc.

9-6) If $\vec{a} = (2, -1, 3), \vec{b} = (-1, 1, 1), \vec{c} = (4, 1, -1)$, find the triple products $\vec{a} \times (\vec{b} \times \vec{c})$ and $(\vec{a} \times \vec{b}) \times \vec{c}$.

9-7) Show that, in general, $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$.

9-8) Prove the vector identity

$$\vec{a} \times \{\vec{b} \times (\vec{c} \times \vec{d})\} = (\vec{b} \cdot \vec{d})(\vec{a} \times \vec{c}) - (\vec{b} \cdot \vec{c})(\vec{a} \times \vec{d})$$

9-9) Show that if the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar, then $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 0$.

9-10) Evaluate the quantities $\{\vec{a} \times (\vec{b} \times \vec{c})\} \times \vec{d}$ and $\{\vec{a} \times (\vec{b} \times \vec{c})\} \cdot \vec{d}$ if $\vec{a} = 2\hat{x} + 3\hat{y} - \hat{z}, \vec{b} = -\hat{x} - \hat{y} + 2\hat{z}, \vec{c} = \hat{x} - \hat{y} + 3\hat{z}, \vec{d} = \hat{x} + 2\hat{y} + \hat{z}$.

9-11) If the vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{y}$ satisfy the conditions

$$\{\vec{a} \parallel \vec{c} \quad \vec{y} \perp \vec{c} \quad \vec{a} = \vec{b} + \vec{c} \times \vec{y}\}$$

express $\vec{y}$ in terms of the other vectors.

(**Answer:** $\vec{y} = \frac{\vec{c}\times\vec{b}}{|\vec{c}|^2}$).

Hint: Cross multiply both sides of $\vec{a} = \vec{b} + \vec{c} \times \vec{y}$ by $\vec{c}$ and apply formula (9-3).

9-12) Show that from the vector relations $\vec{a} \cdot \vec{y} = A,\ \vec{b} \cdot \vec{y} = B$ it follows,

$$\vec{y} = \frac{A}{\left|\vec{a} \times \vec{b}\right|^2}\left(\vec{b} \times \left(\vec{a} \times \vec{b}\right)\right) + \frac{B}{\left|\vec{a} \times \vec{b}\right|^2}\left(\vec{a} \times \left(\vec{b} \times \vec{a}\right)\right) + \lambda\left(\vec{a} \times \vec{b}\right)$$

where λ is an arbitrary real number.

Hint: Resolve $\vec{y}$ in the directions of $\vec{a},\ \vec{b}, \vec{a} \times \vec{b}$, i.e. find the constants p, q, λ such that $\vec{y} = p\vec{a} + q\vec{b} + \lambda\left(\vec{a} \times \vec{b}\right)$, from which,

$$\begin{Bmatrix} \vec{a} \cdot \vec{y} = \vec{a} \cdot \left(p\vec{a} + q\vec{b} + \lambda\left(\vec{a} \times \vec{b}\right)\right) = A \\ \vec{b} \cdot \vec{y} = \vec{b} \cdot \left(p\vec{a} + q\vec{b} + \lambda\left(\vec{a} \times \vec{b}\right)\right) = B \end{Bmatrix} \xrightarrow{\left(\vec{a}\cdot\left(\vec{a}\times\vec{b}\right)=0\right)\left(\vec{b}\cdot\left(\vec{a}\times\vec{b}\right)=0\right)}$$

$$\begin{Bmatrix} p|\vec{a}|^2 + q\left(\vec{a} \cdot \vec{b}\right) = A \\ p\left(\vec{a} \cdot \vec{b}\right) + q\left|\vec{b}\right|^2 = B \end{Bmatrix}$$

Solving this equation for p and q and substituting into $\vec{y} = p\vec{a} + q\vec{b} + \lambda\left(\vec{a} \times \vec{b}\right)$ and making use of formula (9-3) leads to the desired result.

9-13) If $\vec{a}, \vec{b}, \vec{x}, \vec{y}$ are given vectors with $\vec{b} \perp \vec{y}$ and $\vec{z}$ and $\vec{w}$ are arbitrary vectors, simplify the expression

$$\vec{y} \cdot \left\{\vec{a} \times \left(\vec{b} \times \vec{x}\right) - \vec{y} \times (\vec{z} \times \vec{w})\right\}$$

(**Answer:** $-\left(\vec{a} \cdot \vec{b}\right)(\vec{x} \cdot \vec{y})$).

9-14) Prove formula (9-2).

CHAPTER 10: Vector Equation of Lines, Planes and Spheres in Parametric Form

In this chapter we shall study some fundamental properties of various geometrical figures (lines, planes and spheres) and solve a variety of problems with the aid of the theory of vectors, developed thus far.

10-1) The vector equation of a line in parametric form

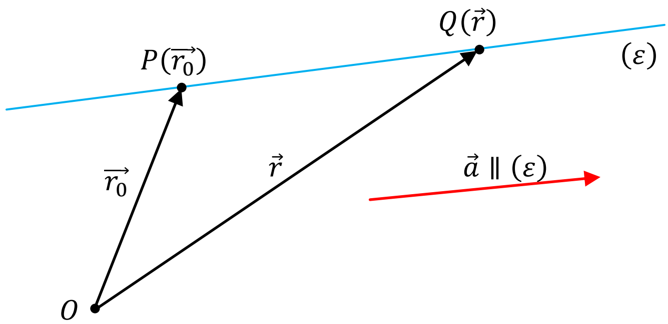

Fig. 10-1: The vector equation of a line.

a) Let us consider a straight line (ε) passing through a given point P determined by its position vector $\vec{r_0}$ and being parallel to a given vector $\vec{a}$. If $Q(\vec{r})$ is another, **arbitrary point of the line** $(\boldsymbol{\varepsilon})$, then the vector $\overrightarrow{PQ}$ will be parallel to $\vec{a}$, which in turn means that (see section 2-5),

$$\overrightarrow{PQ} = \lambda\vec{a}, \qquad -\infty < \lambda < \infty$$

or, since $\overrightarrow{PQ} = \vec{r} - \vec{r_0}$, $\vec{r} - \vec{r_0} = \lambda\vec{a}$, and therefore,

$$\vec{r} = \vec{r_0} + \lambda\vec{a} \qquad -\infty < \lambda < \infty \qquad (10-1)$$

For each value of the parameter λ, equation (10-1) represents the position vector of a point Q belonging to the line (ε), and as λ varies from $-\infty$ up to $+\infty$, equation (10-1) represents the position vectors of all the points of the line (ε), and is therefore called **the vector equation**,

in parametric form, of a line (ε) passing through the point $P(\overrightarrow{r_0})$ and being parallel to a given vector $\vec{a}$.

b) In case the line (ε) passes through two points $A(\overrightarrow{r_1})$ and $B(\overrightarrow{r_2})$, (and is therefore uniquely determined), we may consider $\vec{a} = \overrightarrow{AB} = \overrightarrow{r_2} - \overrightarrow{r_1}$ and in this case equation (10-1) is written equivalently,

$$\vec{r} = \overrightarrow{r_1} + \lambda(\overrightarrow{r_2} - \overrightarrow{r_1}) \qquad -\infty < \lambda < \infty \qquad (10-2)$$

c) In a Cartesian system $\{Ox, Oy, Oz\}$ the equation of a straight line passing through the point $P(\overrightarrow{r_0}) = P(x_0, y_0, z_0)$ and being parallel to a vector $\vec{a} = (a_1, a_2, a_3)$ is

$$\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3} = \lambda \Leftrightarrow \begin{Bmatrix} x = x_0 + \lambda a_1 \\ y = y_0 + \lambda a_2 \\ z = z_0 + \lambda a_3 \end{Bmatrix} \qquad (10-3)$$

where the parameter λ varies from $-\infty$ up to $+\infty$. Equation (10-3) follows directly from (10-1), (let the reader check it).

d) Similarly the equation of a straight line determined by its two points $A(\overrightarrow{r_1}) = A(x_1, y_1, z_1)$ and $B(\overrightarrow{r_2}) = B(x_2, y_2, z_2)$ is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2} = \lambda \Leftrightarrow \begin{Bmatrix} x = x_1 + \lambda(x_1 - x_2) \\ y = y_1 + \lambda(y_1 - y_2) \\ z = z_1 + \lambda(z_1 - z_2) \end{Bmatrix} \ (10-4)$$

Formula (10-4) follows easily from (10-2), (let the reader check it).

10-2) The vector equation of a plane in parametric form

a) Let us consider a plane (Π) passing through a point $P(\overrightarrow{r_0})$ and being parallel to two given vectors $\vec{a}$ and $\vec{b}$. Let also $Q(\vec{r})$ be an arbitrary point of the plane determined by its position vector $\vec{r}$. The vector $\overrightarrow{PQ} = \vec{r} - \overrightarrow{r_0}$ shall be **coplanar to $\vec{a}$ and $\vec{b}$**, which means that (see section 2-6),

$$\overrightarrow{PQ} = \lambda\vec{a} + t\vec{b} \Longrightarrow \vec{r} - \overrightarrow{r_0} = \lambda\vec{a} + t\vec{b} \Longrightarrow$$

$$\vec{r} = \overrightarrow{r_0} + \lambda\vec{a} + t\vec{b} \qquad -\infty < \lambda < \infty, \quad -\infty < t < \infty \qquad (10-5)$$

Note that the equation of the plane (Π) involves **two independent parameters** λ and t, each one varying from $-\infty$ up to $+\infty$. To each pair (λ, t) there corresponds one $\vec{r}$, i.e. one point of the plane (Π). Formula (10-5) represents therefore the vector equation of a plane passing through $\overrightarrow{r_0}$ and being parallel to the two vectors $\vec{a}$ and $\vec{b}$.

b) Based on (10-5) we may obtain the vector equation of a plane determined by three points $A(\overrightarrow{r_1}), B(\overrightarrow{r_2})$ and $C(\overrightarrow{r_3})$.

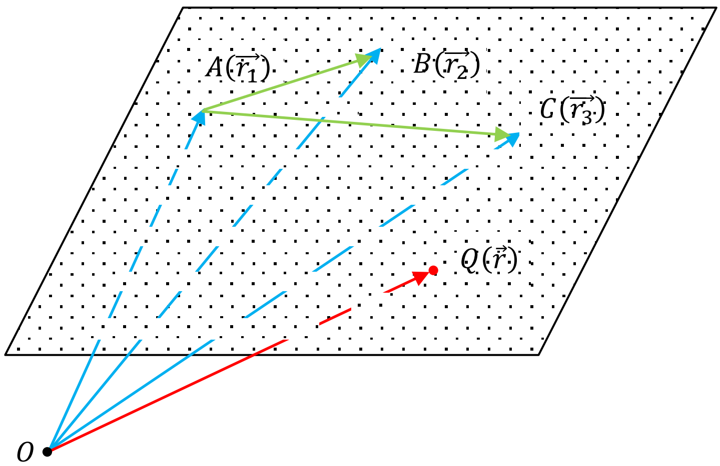

Fig. 10-2: The vector equation of a plane.

If $Q(\vec{r})$ is an arbitrary point of the plane, then we may consider that the plane passes through $A(\overrightarrow{r_1})$ and is parallel to the vectors $\vec{a} = \overrightarrow{AB} = \overrightarrow{r_2} - \overrightarrow{r_1}$ and $\vec{b} = \overrightarrow{AC} = \overrightarrow{r_3} - \overrightarrow{r_1}$, and according to (10-5) the equation of the plane will be,

$$\vec{r} = \overrightarrow{r_1} + \lambda\vec{a} + t\vec{b} \xRightarrow{(10-5)}$$

$$\vec{r} = \overrightarrow{r_1} + \lambda(\overrightarrow{r_2} - \overrightarrow{r_1}) + t(\overrightarrow{r_3} - \overrightarrow{r_1}), \quad \begin{Bmatrix} -\infty < \lambda < \infty \\ -\infty < t < \infty \end{Bmatrix} \qquad (10-6)$$

Equation of a surface in space: Equation of a surface in space is an equation in three variables x, y and z, satisfied by the coordinates of **all** the points of the surface. **In general, an equation of the form $\boldsymbol{\phi(x, y, z) = 0}$ represents a surface in space**. The coordinates of any point $M(x_0, y_0, z_0)$ of the surface satisfies the equation of the surface, i.e. $\phi(x_0, y_0, z_0) = 0$, **and conversely**, if $\phi(x_1, y_1, z_1) = 0$ then the point $N(x_1, y_1, z_1) = 0$ belongs to the surface.

c) Starting with equation (10-6) and expressing the vectors with their Cartesian representations, we may show that the general equation of a plane (surface in space) is (see Problem 10-1),

$$Ax + By + Cz + D = 0, \quad where \quad |A| + |B| + |C| \neq 0 \qquad (10-7)$$

In (10-7) D is a constant and similarly A, B and C are constants, not all zero simultaneously. In general, **every linear equation in $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ represents a plane**.

10-3) Equation of a sphere

A sphere is the set of points in space which are equidistant from a **fixed** point $M(\overrightarrow{r_0}) = M(x_0, y_0, z_0)$, **the center**. All the points of the sphere have the same distance R from the center, (R is called the **radius** of the sphere).

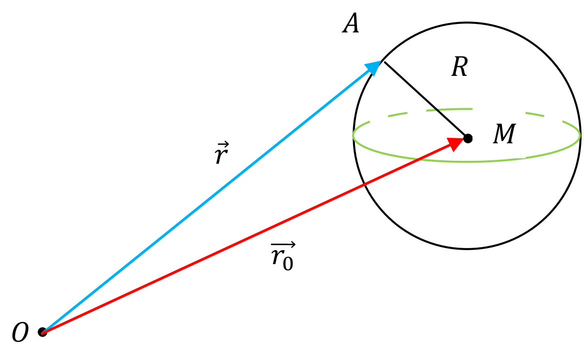

Fig. 10-3: The vector equation of a sphere.

Since the distance of all points of the sphere from the center O is constant and equal to R, we have,

$$|\vec{r} - \overrightarrow{r_0}| = AM = R \ (constant) \qquad (10-8)$$

Equation (10-8) is the vector equation of a sphere centered at $M(\overrightarrow{r_0})$ and having radius R. Setting $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ and $\overrightarrow{r_0} = x_0\hat{x} + y_0\hat{y} + z_0\hat{z}$, equation (10-8) yields,

$$|(x - x_0)\hat{x} + (y - y_0)\hat{y} + (z - z_0)\hat{z}| = R \xRightarrow{(5-8)}$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2 \qquad (10-9)$$

Equation (10-9) is **the equation of a sphere in Cartesian coordinates**, **centered at $\boldsymbol{M(x_0, y_0, z_0)}$ and having radius $\boldsymbol{R}$**.

Example 10-1

Find the parametric equation of a line through $(3,2,5)$ parallel to $2\hat{x} - 4\hat{y} + \hat{z}$.

Solution

Application of formula (10-1) with $\overrightarrow{r_0} = 3\hat{x} + 2\hat{y} + 5\hat{z}$ and $\vec{a} = 2\hat{x} - 4\hat{y} + \hat{z}$, yields,

$$\vec{r} = \overrightarrow{r_0} + \lambda\vec{a} \Rightarrow \vec{r} = 3\hat{x} + 2\hat{y} + 5\hat{z} + \lambda(2\hat{x} - 4\hat{y} + \hat{z}) \Rightarrow$$

$$\vec{r} = (3 + 2\lambda)\hat{x} + (2 - 4\lambda)\hat{y} + (5 + \lambda)\hat{z}\,, \qquad -\infty < \lambda < \infty \qquad (*)$$

Equation (*) is the vector equation of the line in parametric form. Since $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, equation (*) is written equivalently as

$$x\hat{x} + y\hat{y} + z\hat{z} = (3 + 2\lambda)\hat{x} + (2 - 4\lambda)\hat{y} + (5 + \lambda)\hat{z} \Leftrightarrow$$

$$\begin{Bmatrix} x = 3 + 2\lambda \\ y = 2 - 4\lambda \\ z = 5 + \lambda \end{Bmatrix} \Leftrightarrow \left\{\frac{x-3}{2} = \frac{y-2}{-4} = \frac{z-5}{1} = \lambda\right\} \qquad (**)$$

Example 10-2

Find the parametric equation of a plane through $(1,2,3)$ and parallel to the vectors $\vec{a} = 2\hat{x} - \hat{y} + \hat{z}$ and $\vec{b} = \hat{x} + 5\hat{y} - 7\hat{z}$.

Solution

Application of formula (10-5) yields,

$$\vec{r} = \overrightarrow{r_0} + \lambda\vec{a} + t\vec{b} \Longrightarrow$$

$$\vec{r} = \hat{x} + 2\hat{y} + 3\hat{z} + \lambda(2\hat{x} - \hat{y} + \hat{z}) + t(\hat{x} + 5\hat{y} - 7\hat{z}) \Longrightarrow$$

$$\vec{r} = (1 + 2\lambda + t)\hat{x} + (2 - \lambda + 5t)\hat{y} + (3 + \lambda - 7t)\hat{z} \qquad (*)$$

where each one of the parameters λ and t varies from $-\infty$ up to ∞. Since $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, equation (*) may be written equivalently,

$$x\hat{x} + y\hat{y} + z\hat{z} = (1 + 2\lambda + t)\hat{x} + (2 - \lambda + 5t)\hat{y} + (3 + \lambda - 7t)\hat{z} \Longrightarrow$$

$$\begin{Bmatrix} x = 1 + 2\lambda + t \\ y = 2 - \lambda + 5t \\ z = 3 + \lambda - 7t \end{Bmatrix} \qquad (**)$$

Elimination of λ and t yields the following expression for the equation of the sought for plane,

$$2x + 15y + 11z - 65 = 0 \qquad (***)$$

which is the standard **Cartesian equation of a plane**, (see eq. (10-7)). To eliminate λ and t in (***) it suffices to solve the system of the first two equations for λ and t (in terms of x and y), and substitute into the third equation. For detailed calculations see Problem 10-2.

Example 10-3

Find the point where the line $\{x = 3 + 5\lambda, y = -2 + \lambda, z = 3 + \lambda\}$ pierces the xy $-$plane, the yz $-$plane and the $3x - 2y + z = 7$ plane.

Solution

a) The z −coordinate of the point where the given line pierces the xy −plane is $z = 0$, and this implies that $z = 3 + \lambda = 0$, i.e. $\boldsymbol{\lambda = -3}$, and therefore,

$$\begin{Bmatrix} x = 3 + 5\lambda = 3 + 5 \cdot (-3) = -12 \\ y = -2 + \lambda = -2 - 3 = -5 \end{Bmatrix}$$

The line pierces the xy −plane at the point $(x, y) = (-12, -5)$.

b) Similarly the x −coordinate of the point where the line pierces the yz −plane is $x = 0$, and this implies that $x = 3 + 5\lambda = 0$, i.e. $\boldsymbol{\lambda = -{}^{3}/_{5}}$ and hence,

$$\begin{Bmatrix} y = -2 + \lambda = -2 - \dfrac{3}{5} = -\dfrac{13}{5} \\ z = 3 + \lambda = 3 - \dfrac{3}{5} = \dfrac{12}{5} \end{Bmatrix}$$

The line pierces the yz −plane at the point $(y, z) = \left(-\frac{13}{5}, \frac{12}{5}\right)$.

c) The point (x_0, y_0, z_0) where the given line pierces the plane belongs to the plane and its coordinates must therefore **satisfy** the equation of the plane i.e.

$$3x_0 - 2y_0 + z_0 = 7 \Longrightarrow$$

$$3(3 + 5\lambda) - 2(-2 + \lambda) + (3 + \lambda) = 7 \Longrightarrow 14\lambda = -9 \Longrightarrow \lambda = -\frac{9}{14}$$

The point where the line pierces the plane is

$$\begin{Bmatrix} x_0 = 3 + 5\lambda = 3 + 5 \cdot \left(-\dfrac{9}{14}\right) = -\dfrac{3}{14} \\ y_0 = -2 + \lambda = -2 + \left(-\dfrac{9}{14}\right) = -\dfrac{37}{14} \\ z_0 = 3 + \lambda = 3 + \left(-\dfrac{9}{14}\right) = \dfrac{33}{14} \end{Bmatrix}$$

Example 10-4

Find the equation of a line through $M(1,2,3)$ and parallel to the line $\{2x-y+3z=5,\quad x+3y+z=2\}$.

Solution

The intersection of two planes is a straight line, in general, (unless the two planes are parallel). Solving the system for x and y, we may express x and y in terms of z, i.e.

$$\begin{Bmatrix} 2x-y=5-3z \\ x+3y=2-z \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} x=\dfrac{-10z+17}{7} \\ y=\dfrac{z-1}{7} \end{Bmatrix} \qquad (*)$$

The parametric equation of **the line of intersection** of the two planes is therefore,

$$\begin{Bmatrix} x=\dfrac{-10t+17}{7} \\ y=\dfrac{t-1}{7} \\ z=t \end{Bmatrix} \quad where \quad -\infty<t<\infty \qquad (**)$$

For each value of t, x, y and z , as expressed in (**) satisfy the equation of the first **and** the second plane, i.e. these points belong on the first **and** on the second plane, in other words equation (**) is the equation of the line of intersection of these two planes. For example for $t=0$ we get one point of the line, which is the point $A\left(\frac{17}{7},-\frac{1}{7},0\right)$ while for $t=1$ we get another point $B(1,0,1)$. The vector $\overrightarrow{AB}$ is

$$\begin{aligned}\overrightarrow{AB} &= \left(1-\frac{17}{7}\right)\hat{x}+\left(0-\left(-\frac{1}{7}\right)\right)\hat{y}+(1-0)\hat{z} \\ &= -\frac{10}{7}\hat{x}+\frac{1}{7}\hat{y}+\hat{z} \qquad (***)\end{aligned}$$

The problem thus is reduced to the following:

Find the equation of a line through $M(1,2,3)$ and parallel to the vector $\overrightarrow{AB} = -\frac{10}{7}\hat{x} + \frac{1}{7}\hat{y} + \hat{z}$, which according to formula (10-1) is

$$\vec{r} = (\hat{x} + 2\hat{y} + 3\hat{z}) + \lambda\overrightarrow{AB} \Longrightarrow$$

$$\vec{r} = \left(1 - \frac{10}{7}\lambda\right)\hat{x} + \left(2 + \frac{1}{7}\lambda\right)\hat{y} + (3 + \lambda)\hat{z}, \qquad -\infty < \lambda < \infty$$

Example 10-5

Find the distance of $M(2,3,1)$ from the line $\frac{x-3}{1} = \frac{y+2}{4} = \frac{z-4}{2}$.

Solution

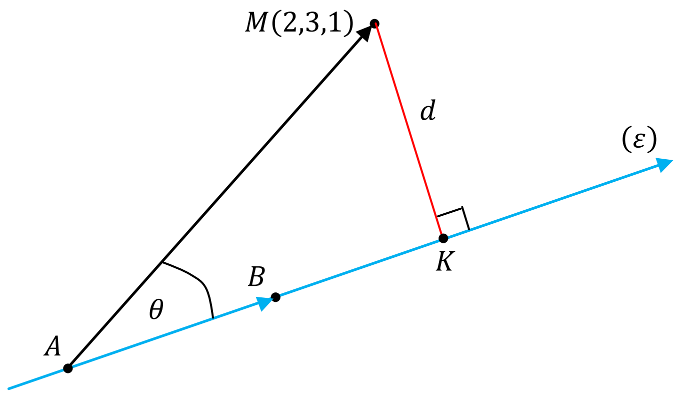

Fig. 10-4: Shortest distance between a point and a line in space.

Let (ε) be the straight line $\frac{x-3}{1} = \frac{y+2}{4} = \frac{z-4}{2} = \lambda$, i.e.

$$\begin{Bmatrix} x = 3 + \lambda \\ y = -2 + 4\lambda \\ z = 4 + 2\lambda \end{Bmatrix} \qquad -\infty < \lambda < \infty \qquad (*)$$

For $\lambda = 0$ we obtain the point $A(3,-2,4)$ of the line, while for another value of λ, say $\lambda = 1$, we obtain another point $B(4,2,6)$ of the line (ε). The distance $d = MK$, (in Fig. 10-4), between $M(2,3,1)$ from the line (ε) is

$$d = AM \cdot \sin\theta \Longrightarrow d^2 = (AM)^2 \cdot (\sin\theta)^2$$
$$= (AM)^2 \cdot (1 - (\cos\theta)^2) \qquad (**)$$

where $AM = \left|\overrightarrow{AM}\right|$ is the magnitude of the vector $\overrightarrow{AM}$. We note that

$$\overrightarrow{AB} = (4\hat{x} + 2\hat{y} + 6\hat{z}) - (3\hat{x} - 2\hat{y} + 4\hat{z}) = \hat{x} + 4\hat{y} + 2\hat{z}$$

$$\overrightarrow{AM} = (2\hat{x} + 3\hat{y} + \hat{z}) - (3\hat{x} - 2\hat{y} + 4\hat{z}) = -\hat{x} + 5\hat{y} - 3\hat{z}$$

and hence,

$$\overrightarrow{AB} \cdot \overrightarrow{AM} = (AB)(AM)\cos\theta \Longrightarrow$$

$$(\hat{x} + 4\hat{y} + 2\hat{z}) \cdot (-\hat{x} + 5\hat{y} - 3\hat{z})$$
$$= \sqrt{1^2 + 4^2 + 2^2} \cdot \sqrt{(-1)^2 + 5^2 + (-3)^2}\,\cos\theta \Longrightarrow$$

$$-1 + 20 - 6 = \sqrt{21} \cdot \sqrt{35}\cos\theta \Longrightarrow \cos\theta = \frac{13}{\sqrt{735}} \overset{(**)}{\Longrightarrow}$$

$$d^2 = 35 \cdot \left(1 - \frac{13^2}{735}\right) = 35 \cdot \frac{566}{735} = \frac{566}{21} \Longrightarrow d = \sqrt{\frac{566}{21}}$$

Alternative solution:

The distance ℓ between $M(2,3,1)$ and an arbitrary point $P(x, y, z)$ of (ε) is,

$$\ell^2 = (x-2)^2 + (y-3)^2 + (z-1)^2 \overset{(*)}{\Rightarrow}$$

$$\ell^2 = (\lambda + 1)^2 + (4\lambda - 5)^2 + (2\lambda + 3)^2 \Longrightarrow$$

$$\ell^2 = 21\lambda^2 - 26\lambda + 35 \qquad (***)$$

Equation (***) expresses the distance between M and $P(x, y, z) \in (\varepsilon)$ as a function of λ, I.e. $\ell = \ell(\lambda)$. The shortest distance is the minimum value of ℓ, i.e. $d = \ell_{min}$. To find the minimum value of $\ell(\lambda)$, we set the derivative equal to zero, i.e.

$$\frac{d}{d\lambda}\big(\ell(\lambda)\big) = 0 \overset{(***)}{\Longrightarrow} \frac{d}{d\lambda}(21\lambda^2 - 26\lambda + 35) = 0 \Rightarrow$$

$$42\lambda - 26 = 0 \Rightarrow \lambda = \frac{13}{21}$$

and the minimum value of ℓ is,

$$d^2 = {\ell_{min}}^2 = 21 \cdot \left(\frac{13}{21}\right)^2 - 26 \cdot \frac{13}{21} + 35 = \frac{566}{21} \Rightarrow d = \sqrt{\frac{566}{21}}$$

Note: The coordinates of the point K (see Fig. 10-4) are found from equation (*), if we set $\lambda = {}^{13}/_{21}$.

Example 10-6

Examine whether the line $\vec{r} = (2\hat{x} - \hat{y}) + \lambda(3\hat{x} - \hat{y} - \hat{z})$ intersects the line $\vec{r} = \hat{x} + t(\hat{x} + 2\hat{y} - \hat{z})$.

Solution

Assuming that these two lines intersect, it must be a point P belonging to the first **and** the second line. This in turn means that there are two numbers λ and t, such that

$$(2\hat{x} - \hat{y}) + \lambda(3\hat{x} - \hat{y} - \hat{z}) = \hat{x} + t(\hat{x} + 2\hat{y} - \hat{z}) \Rightarrow$$

$$(2 + 3\lambda)\hat{x} + (-1 - \lambda)\hat{y} + (-\lambda)\hat{z} = (1 + t)\hat{x} + (2t)\hat{y} + (-t)\hat{z} \Leftrightarrow$$

$$\left\{\begin{matrix} 2 + 3\lambda = 1 + t \\ -(1 + \lambda) = 2t \\ \lambda = t \end{matrix}\right\} \qquad (*)$$

However, the given system **has no solution** (let the reader check it), and this means that there are no common points between the two lines. The two lines do not intersect, they are **skew lines**.

Example 10-7

Find the equation of a sphere with center at $M(1,-2,3)$ and radius equal to $R=5$.

Solution

The vector equation of the sphere is (see equation (10-8)),

$$|\vec{r}-\overrightarrow{r_0}|=R \Leftrightarrow |\vec{r}-(\hat{x}-2\hat{y}+3\hat{z})|=5, \qquad or$$

$$|(x-1)\hat{x}+(y+2)\hat{y}+(z-3)\hat{z}|=5 \quad (*)$$

In Cartesian coordinates the equation of the sphere is

$$(x-1)^2+(y+2)^2+(z-3)^2=25$$

Example 10-8

Find the center and the radius of the sphere whose equation is $x^2+y^2+z^2-4x+2y-2z=3$.

Solution

$$x^2+y^2+z^2-4x+2y-2z=3 \Longrightarrow$$

$$\{(x^2-4x+4)-4\}+\{(y^2+2y+1)-1\}+\{(z^2-2z+1)-1\}=3$$

or equivalently,

$$(x-2)^2+(y+1)^2+(z-1)^2=3+4+1+1=9=3^2$$

This is the equation of a sphere centered at $(2,-1,1)$ and having radius $R=3$.

Example 10-9

Show that the equation of the plane tangent to the spherical surface of the sphere $(x-a)^2+(y-b)^2+(z-c)^2=R^2$ at the point $P(x_1,y_1,z_1)$ is

$$(x-a)(x_1-a)+(y-b)(y_1-b)+(z-c)(z_1-c)=R^2$$

Solution

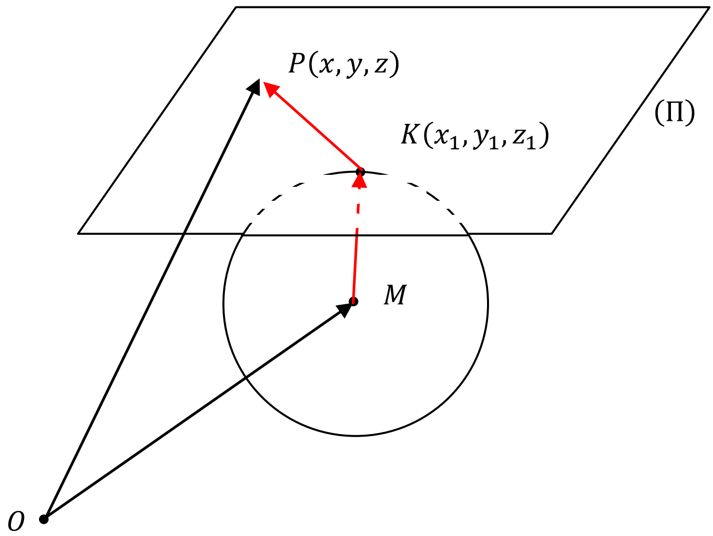

Fig. 10-5: Plane tangent to a given sphere.

Let $M(a,b,c)$ be the center of the sphere with radius R, and (Π) be a plane tangent to the spherical surface at the point $K\left(x_1,y_1,z_1\right)$. If $P(x,y,z)$ be an arbitrary point of the plane, then the vector $\overrightarrow{MK}$, (the radius from M to K), will be **perpendicular** to $\overrightarrow{KP}$, and hence

$$\overrightarrow{MK}\cdot\overrightarrow{KP}=0\Longrightarrow$$

$$\{(x_1-a)\hat{x}+(y_1-b)\hat{y}+(z_1-c)\hat{z}\}\\ \cdot\{(x-x_1)\hat{x}+(y-y_1)\hat{y}+(z-z_1)\hat{z}\}=0\Longrightarrow$$

$$(x-x_1)(x_1-a)+(y-y_1)(y_1-b)+(z-z_1)(z_1-c)=0\Longrightarrow$$

$$\{(x-a)-(x_1-a)\}(x_1-a)+\{(y-b)-(y_1-b)\}(y_1-b)\\ +\{(z-c)-(z_1-c)\}(z_1-c)=0\Longrightarrow$$

$$(x-a)(x_1-a)+(y-b)(y_1-b)+(z-c)(z_1-c)\\ =(x_1-a)^2+(y_1-b)^2+(z_1-c)^2=R^2$$

since the point $K(x_1, y_1, z_1)$ belongs to the spherical surface of the sphere with center $M(a, b, c)$ and radius R, and this completes the proof.

Example 10-10

1) What is the relative position of the two spheres:

$$\begin{Bmatrix} x^2 + y^2 + z^2 - 10x - 14y - 2z + 66 = 0 \\ x^2 + y^2 + z^2 - 22x - 20y - 1z + 234 = 0 \end{Bmatrix}$$

2) Find the equation of the straight line determined by the centers M and K of the spheres.

Solution

1) The equation of the first sphere is

$$x^2 + y^2 + z^2 - 10x - 14y - 2z + 66 = 0 \Longrightarrow$$

$$\{x^2 - 10x + 5^2\} - 5^2 + \{y^2 - 14y + 7^2\} - 7^2 + \{z^2 - 2z + 1^1\} - 1^2 + 66 = 0 \Longrightarrow$$

$$(x-5)^2 + (y-7)^2 + (z-1)^2 = 5^2 + 7^2 + 1^2 - 66 \Longrightarrow$$

$$(x-5)^2 + (y-7)^2 + (z-1)^2 = 9 = 3^2 \qquad (*)$$

Equation (*) is the equation of a sphere with center at $M(5{,}7{,}1)$ and radius $R_1 = 3$.

Similarly, from the second equation we have,

$$x^2 + y^2 + z^2 - 22x - 20y - 14z + 234 = 0 \Longrightarrow$$

$$\{x^2 - 22x + 11^2\} - 11^2 + \{y^2 - 20y + 10^2\} - 10^2 + \{z^2 - 14z + 7^2\} - 7^2 + 234 = 0 \Longrightarrow$$

$$(x-11)^2 + (y-10)^2 + (z-7)^2 = 11^2 + 10^2 + 7^2 - 234 \Longrightarrow$$

$$(x-11)^2 + (y-10)^2 + (z-7)^2 = 36 = 6^2 \qquad (**)$$

Equation (**) is the equation of a sphere with center $K(11,10,7)$ and radius $R_2 = 6$.

We note that the distance MK, between the two centers is

$$MK = \sqrt{(11-5)^2 + (10-7)^2 + (7-1)^2} = \sqrt{6^2 + 3^2 + 6^2} = 9 \Longrightarrow$$

$$MK = 9 = R_1 + R_2$$

and this shows that the two spheres **are tangent to each other externally**.

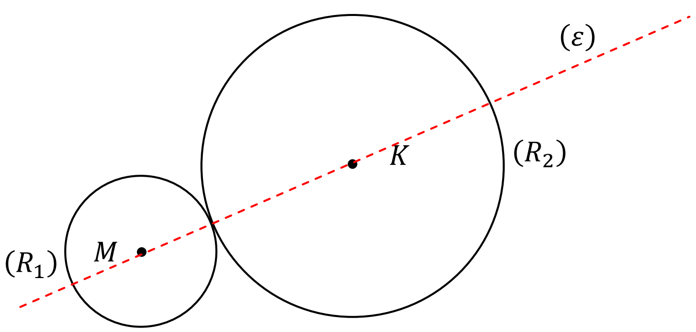

Fig. 10-6: Relative position of two spheres.

2) The line (ε) is determined by the two centers $M(5,7,1)$ and $K(11,10,7)$, and its equation is (see (10-4)),

$$\begin{Bmatrix} x = 11 + 6\lambda \\ y = 10 + 3\lambda \\ z = 7 + 6\lambda \end{Bmatrix}, \qquad \lambda \in \mathbb{R}$$

PROBLEMS

10-1) Starting with equation (10-6) show that the general equation of a plane is given by equation (10-7).

10-2) In Example 10-2, eliminate λ and t between the equations of the system (**), to derive equation (***).

10-3) Find the vector equation of the line determined by the two points $M(\overrightarrow{r_1} = 3\hat{x} + 2\hat{y} - \hat{z})$ and $N(\overrightarrow{r_2} = 5\hat{x} - 2\hat{y} + 2\hat{z})$.

(**Answer:** $\vec{r} = (3 - 2\lambda)\hat{x} + (2 + 4\lambda)\hat{y} - (1 + 3\lambda)\hat{z}$).

Hint: Use equation (10-2).

10-4) Find the parametric equation of the plane determined by the three points $M_1(1,-1,1)$, $M_2 = (3,-7,4)$ and $M_3(8,-4,1)$.

Hint: Use equation (10-6).

10-5) A line (ε) passes through $M(3,0,0)$ and is parallel to the vector $\vec{a} = \hat{x} + 2\hat{y} + 3\hat{z}$. A second line (η) passes through $N(0,9,0)$ and is parallel to the vector $\vec{b} = 5\hat{x} - 5\hat{y} + 8\hat{z}$. Determine whether (ε) and (η) are coplanar or skew lines.

(**Answer:** Skew lines).

Hint: See Example 10-6.

10-6) Find the parametric equation of a line through $M(1,1,1)$ parallel to $\vec{a} = 9\hat{x} - 3\hat{y} + 5\hat{z}$.

Hint: Use equation (10-1).

10-7) Find the parametric equation of the plane determined by the three points $M_1(1,0,0), M_2(0,1,0)$ and $M_3(0,0,1)$.

(**Answer:** $\vec{r} = (1 - \lambda - t)\hat{x} + \lambda\hat{y} + t\hat{z}$).

10-8) If a plane intersect the coordinate axes Ox, Oy, Oz at A, B and C respectively, and if we set $\overline{OA} = a$, $\overline{OB} = b$ and $\overline{OC} = c$, show that the equation of the plane can be put in the form,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

10-9) Find the equation of a plane which is parallel to the $x'x$ axis and passes through $M(0,1,3)$ and $N(2,4,5)$.

(**Answer:** $2y-3z+7=0$).

Hint: If a plane is parallel to the $x'x$ axis, then the variable x is missing from the equation of the plane, (why?).

10-10) Find the Cartesian equation of the plane determined by the three points $M(1,-1,2), N(2,1,2)$ and $P(1,1,4)$.

10-11) Find the equation of a plane passing through $M(2,-1,3)$ and cutting equal line segments on the coordinate axes.

(**Answer:** $x+y+z=4$).

Hint: See Problem 10-8.

10-12) Find the point of intersection of the planes

$$\begin{Bmatrix} 2x-y+3z=9 \\ x+2y+2z=3 \\ 3x+y-4z=-6 \end{Bmatrix}$$

Hint: It suffices to solve the system of equations for x, y and z. The answer is $(x,y,z)=(1,-1,2)$.

10-13) Find the equation of the line determined by the two points $M(-1,2,3)$ and $N(2,6,-2)$.

(**Answer:** $\frac{x+1}{3}=\frac{y-2}{4}=\frac{z-3}{-5}$).

10-14) Find the center and the radius of the sphere $x^2+y^2+z^2-2x-4y-6z=2$.

(**Answer:** Center $K(1,2,3)$ and radius $R=4$).

Hint: See Example 10-8.

10-15) Find the center and the radius of the circle determined by the cut of the spherical surface $x^2 + y^2 + z^2 - 10y = 0$ with the plane $x + 2y + 2z = 19$.

(**Answer:** Center $K(1,7,2)$ and radius $R = 4$).

Hint: The center of the sphere is $K(0,5,0)$ and its radius is $R = 5$. The projection of K on the given plane will be the center of the circle.

10-16) Show that the planes $4x + 3y - 5z = 8$ and $4x + 3y - 5z = -12$ are parallel, and determine their distance d.

(**Answer:** $d = 2\sqrt{2}$).

10-17) Find the distance h between $M(2,-1,3)$ and the line $\frac{x+1}{3} = \frac{y+2}{4} = \frac{z-1}{5}$.

(**Answer:** $h = 0.3\sqrt{38}$).

Hint: See Example 10-5.

CHAPTER 11: Miscellaneous Applications

In this chapter we develop some rather interesting applications and derive a number of useful formulas in plane and solid geometry, with the aid of the theory of vectors and their related properties.

1) Distance between a point and a line

The shortest distance h of a point $M(\xi, \eta)$ from the line (ε) having equation $ax + by + c = 0$, is given by the formula,

$$h = \frac{|a\xi + b\eta + c|}{\sqrt{a^2 + b^2}} \qquad (11-1)$$

The absolute value in the numerator signifies the fact that the (geometrical) distance must **be positive**.

Proof: Let h be the shortest distance of $M(\xi, \eta)$ from the line (ε).

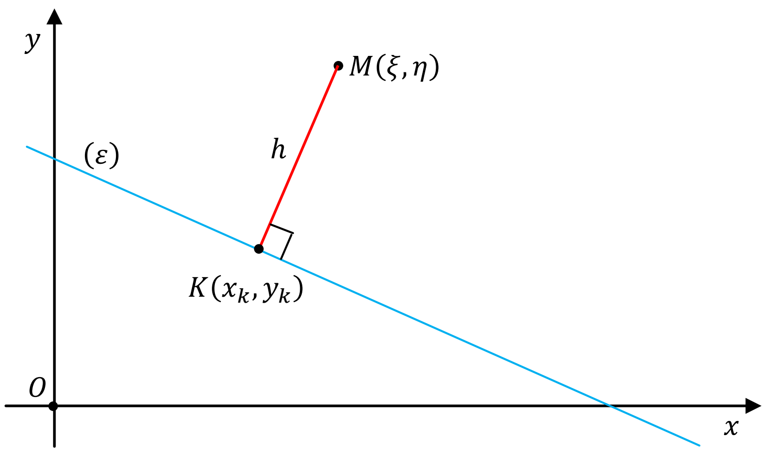

Fig. 11-1: Distance between a point and a line.

If $MK \perp (\varepsilon)$ then the shortest distance h between M and (ε) is $h = MK$. The slope of the line (ε) is $(-{}^{a}/_{b})$ and since $MK \perp (\varepsilon)$ the slope of MK will be $\left({}^{b}/_{a}\right)$, (**the product of slopes of two perpendicular lines is** $(-\mathbf{1})$). The line determined by the two points M and K is

$$y-\eta=\frac{b}{a}(x-\xi) \Leftrightarrow bx-ay=b\xi-a\eta \qquad (*)$$

Since K is the point of intersection of (ε) and MK, the coordinates (x_k, y_k) of the point K must satisfy both equations

$$\left\{\begin{matrix} ax_k+by_k=-c \\ bx_k-ay_k=b\xi-a\eta \end{matrix}\right\} \qquad (**)$$

Solving the system (**) for x_k and y_k we get,

$$\left\{\begin{matrix} x_k=\dfrac{b^2\xi-ab\eta-ac}{a^2+b^2} \\ y_k=\dfrac{a^2\eta-ab\xi-bc}{a^2+b^2} \end{matrix}\right\} \qquad (***)$$

The distance $h=MK=\sqrt{(\xi-x_k)^2+(\eta-y_k)^2}$, which by virtue of equations (***), yields formula (11-1). For detailed calculations, see Problem 11-1. For an alternative proof, see Problem 11-2.

2) Unit vector perpendicular to a plane

The unit vector $\hat{n}$ perpendicular to the plane $ax+by+cz+d=0$, is

$$\hat{n}=\frac{a\hat{x}+b\hat{y}+c\hat{z}}{\sqrt{a^2+b^2+c^2}} \qquad (11-2)$$

where as usual, $\hat{x}, \hat{y}, \hat{z}$ are the unit vectors along the x, y and z axes respectively.

Proof: Let $O(x_0, y_0, z_0)$ be a given point on the plane. Let us now consider the points $P\left(0,0,-\frac{d}{c}\right)$ and $Q\left(0,-\frac{d}{b},0\right)$. Both points P and Q belong on the given plane, since the coordinates of each point satisfy the equation of the plane, (let the reader check it). The vector $\overrightarrow{OP}\times\overrightarrow{OQ}$ is perpendicular to the plane determined by these two vectors, i.e. is perpendicular to the plane $ax + by + cz + d = 0$ and the sought for unit vector is given by the formula,

$$\hat{n} = \frac{\overrightarrow{OP} \times \overrightarrow{OQ}}{\left|\overrightarrow{OP} \times \overrightarrow{OQ}\right|} \qquad (*)$$

The vector $\overrightarrow{OP} = -x_0\hat{x} - y_0\hat{y} - \left(\frac{d}{c} + z_0\right)\hat{z}$, while $\overrightarrow{OQ} = -x_0\hat{x} - \left(\frac{d}{b} + y_0\right)\hat{y} - z_0\hat{z}$ and the cross product is

$$\overrightarrow{OP} \times \overrightarrow{OQ} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -x_0 & -y_0 & -\left(\dfrac{d}{c} + z_0\right) \\ -x_0 & -\left(\dfrac{d}{b} + y_0\right) & -z_0 \end{vmatrix} \Longrightarrow$$

$$\overrightarrow{OP} \times \overrightarrow{OQ} = -\frac{d(by_0 + cz_0 + d)}{bc}\hat{x} + \frac{dx_0}{c}\hat{y} + \frac{dx_0}{b}\hat{z}$$

or, since $ax_0 + by_0 + cz_0 + d = 0$ (the point $O(x_0, y_0, z_0)$ lies on the plane and hence its coordinates satisfy the equation of the plane),

$$\overrightarrow{OP} \times \overrightarrow{OQ} = \frac{dx_0}{bc}(a\hat{x} + b\hat{y} + c\hat{z}) \qquad (**)$$

By virtue of formulas (*) and (**) the unit vector is

$$\hat{n} = \frac{a\hat{x} + b\hat{y} + c\hat{z}}{\sqrt{a^2 + b^2 + c^2}}$$

and this completes the proof.

3) Distance between a point and a plane

The shortest distance h of a point $M(\xi, \eta, \zeta)$, in space, from the plane $ax + by + cz + d = 0$, is given by the formula,

$$h = \frac{|a\xi + b\eta + c\zeta + d|}{\sqrt{a^2 + b^2 + c^2}} \qquad (11 - 3)$$

Proof: Let $M(\xi,\eta,\zeta)$ be a point in space and $ax+by+cz+d=0$ be the equation of a plane (Π).

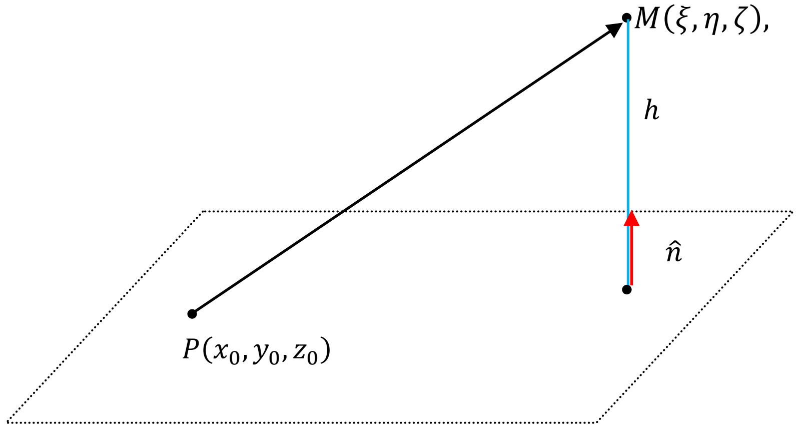

Fig. 11-2: Distance between a point and a plane.

If $P(x_0,y_0,z_0)$ is a point on the plane $ax+by+cz+d=0$, then the sought for distance h is the magnitude (**positive number**) of the projection of the vector $\overrightarrow{PM}$ on the unit vector $\hat{n}$, which is perpendicular to the plane, i.e.

$$h=\left|\overrightarrow{PM}\cdot\hat{n}\right|\xRightarrow{(11-2)}$$

$$h=\left|\left((\xi-x_0)\hat{x}+(\eta-y_0)\hat{y}+(\zeta-z_0)\hat{z}\right)\cdot\frac{a\hat{x}+b\hat{y}+c\hat{z}}{\sqrt{a^2+b^2+c^2}}\right|\Longrightarrow$$

$$h=\left|\frac{1}{\sqrt{a^2+b^2+c^2}}\{a(\xi-x_0)+b(\eta-y_0)+c(\zeta-z_0)\}\right|\Longrightarrow$$

$$h=\left|\frac{1}{\sqrt{a^2+b^2+c^2}}\{a\xi+b\eta+c\zeta-(ax_0+by_0+cz_0)\}\right|\quad(*)$$

and since $ax_0+by_0+cz_0+d=0$, (recall that $P(x_0,y_0,z_0)\in\Pi$), equation (*) finally yields,

$$h = \frac{|a\xi + b\eta + c\zeta + d|}{\sqrt{a^2 + b^2 + c^2}}$$

and this completes the proof.

4) The signed (or algebraic area) of a triangle ABC

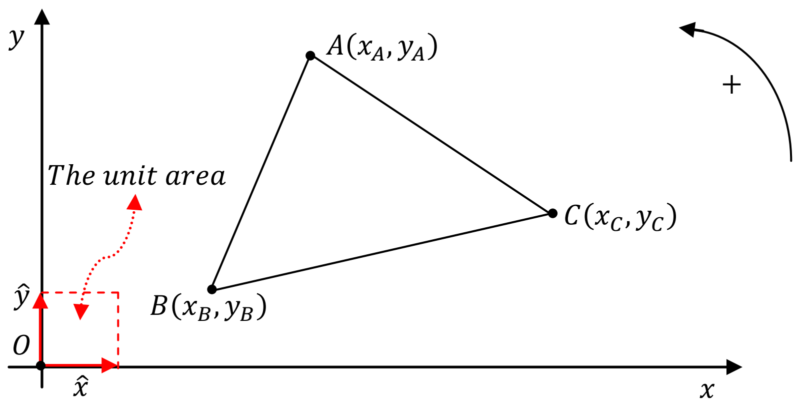

Fig. 11-3: Area of a triangle.

The **positive** direction on the plane $\{Ox, Oy\}$ is the direction in which when the unit vector $\hat{x}$ is rotated about the origin O by 90°, coincides with the unit vector $\hat{y}$. The positive direction on the plane $\{Ox, Oy\}$ is the one shown in Fig. 11-3.

The signed or algebraic area of the triangle ABC is the geometrical area of ABC (which by definition is a positive number) bearing the "+" sign when moving along the $A - B - C - A$ path we move in the **positive** direction, or the "-" sign when moving along the $A - B - C - A$ path we move in the **negative** direction.

The algebraic area E of the triangle ABC is given by the formula,

$$E = \frac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} \quad square\ units \qquad (11-4)$$

Note: The geometrical area of the triangle (which is always positive) is the absolute value of E, i.e. is equal to $|E|$.

The unit area is the area of the square formed by the two unit vectors $\hat{x}$ and $\hat{y}$, (see Fig. 11-4).

Proof: See Problem 11-3.

Corollary: The necessary and sufficient condition that three points $A(x_A, y_A), B(x_B, y_B), C(x_C, y_C)$ lie on the same straight line is,

$$\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix} = 0 \qquad (11-5)$$

(since in this case the area of the triangle will be zero).

5) The volume of a parallelepiped formed by three vectors $\vec{a} = \overrightarrow{OA}, \vec{b} = \overrightarrow{OB}$ and $\vec{c} = \overrightarrow{OC}$, in terms of the lengths $a = |\vec{a}|, b = |\vec{b}|, c = |\vec{c}|$ and the angles $\theta_1 = \measuredangle(\vec{b}, \vec{c}), \theta_2 = \measuredangle(\vec{a}, \vec{c}), \theta_3 = \measuredangle(\vec{a}, \vec{b})$.

The sought for volume V is given by the formula,

$$V = 2abc\sqrt{\sin\tau \sin(\tau - \theta_1) \sin(\tau - \theta_2) \sin(\tau - \theta_3)} \qquad (11-6)$$

where

$$2\tau = \theta_1 + \theta_2 + \theta_3 \qquad (11-7)$$

Formula (11-6) gives the volume of the parallelepiped **in cubic units**. A cubic unit is by definition the volume of the unit cube, i.e. the cube formed by the three unit vectors $\hat{x}, \hat{y}$ and $\hat{z}$.

Corollary: The volume of **the tetrahedron $\boldsymbol{OABC}$** is

$$V_{OABC} = \frac{1}{3}abc\sqrt{\sin\tau \sin(\tau - \theta_1) \sin(\tau - \theta_2) \sin(\tau - \theta_3)} \qquad (11-8)$$

Proof: See Problem 11-4.

6) The shortest distance between two skew lines

Two nonparallel, nonintersecting lines in space are called **skew lines**. Let $\vec{r} = \vec{r_1} + \lambda\vec{a}$ and $\vec{r} = \vec{r_2} + t\vec{b}$ be the vector parametric equations of two skew lines (ε_1) and (ε_2) respectively. Then the shortest distance h between these two lines is given by the formula,

$$h = \frac{\left|\left(\vec{a}, \vec{b}, \vec{r_2} - \vec{r_1}\right)\right|}{\left|\vec{a} \times \vec{b}\right|} \qquad (11-9)$$

Proof: Let two skew lines (ε_1) and (ε_2) as shown in Fig. 11-4.

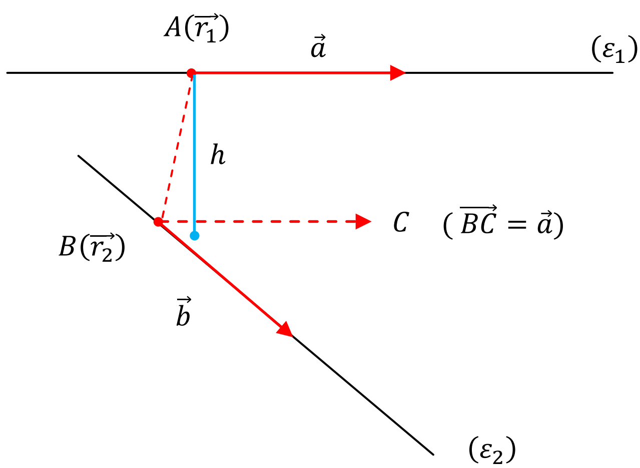

Fig. 11-4: Shortest distance between two skew lines.

If we consider the vector $\overrightarrow{BC} = \vec{a}$, then the shortest distance h between (ε_1) and (ε_2) shall be the distance of A from the plane determined by $\overrightarrow{BC}$ and $\vec{b}$. The volume of the parallelepiped formed by the vectors $\vec{b}, \overrightarrow{BC} = \vec{a}$ and $\overrightarrow{AB} = \vec{r_2} - \vec{r_1}$ is, (see equation (8-2)),

$$V = \left|\left(\vec{a}, \vec{b}, \vec{r_2} - \vec{r_1}\right)\right| \qquad (*)$$

The area of the base (of the parallelepiped) formed by the vectors $\vec{b}$ and $\overrightarrow{BC} = \vec{a}$, is equal to the magnitude of the cross product $\vec{a} \times \vec{b}$, i.e.

$$Area\ of\ base = \left|\vec{a} \times \vec{b}\right| \qquad (**)$$

However, since the volume is equal to the area of the base multiplied by the altitude corresponding to this base, we have,

$$V = (Area\ of\ base) \cdot h \xRightarrow{(*)(**)} h = \frac{\left|\left(\vec{a}, \vec{b}, \vec{r_2} - \vec{r_1}\right)\right|}{\left|\vec{a} \times \vec{b}\right|}$$

and this completes the proof.

7) Angle between two planes

The angle θ between two planes (Π_1) and (Π_2) with equations $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ respectively, is given by the formula,

$$\cos\theta = \pm \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1{}^2 + b_1{}^2 + c_1{}^2}\ \sqrt{a_2{}^2 + b_2{}^2 + c_2{}^2}} \qquad (11 - 10)$$

Note: The two planes are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Proof: The angle between two intersecting planes is considered to be the angle between the two unit vectors $\widehat{n_1}$ and $\widehat{n_2}$, perpendicular to the planes (Π_1) and (Π_2) respectively. Application of formula (11-2) yields,

$$\left\{\widehat{n_1} = \frac{a_1\hat{x} + b_1\hat{y} + c_1\hat{z}}{\sqrt{a_1{}^2 + b_1{}^2 + c_1{}^2}} \quad \widehat{n_2} = \frac{a_2\hat{x} + b_2\hat{y} + c_2\hat{z}}{\sqrt{a_2{}^2 + b_2{}^2 + c_2{}^2}}\right\} \qquad (*)$$

and since $\widehat{n_1} \cdot \widehat{n_2} = |\widehat{n_1}||\widehat{n_2}|\cos\theta = 1 \cdot 1 \cdot \cos\theta = \cos\theta$, formula (11-10) follows immediately.

8) Straight line tangent to a circle

The equation of the tangent (ε) to the circle $(x - a)^2 + (y - b)^2 = r^2$, at the point $M(x_1, y_1)$, (belonging to the circle) is,

$$(x_1 - a)(x - a) + (y_1 - b)(y - b) = r^2 \qquad (11-11)$$

Proof: See Problem 11-5.

9) Plane tangent to a sphere

The equation of a plane tangent to the sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$, at the point $P(x_1, y_1, z_1)$, (belonging to the spherical surface) is,

$$(x-a)(x_1-a) + (y-b)(y_1-b) + (z-c)(z_1-c) = R^2 \quad (11-12)$$

Proof: See Example 10-9.

Example 11-1

On the xOy plane, find the distance of $M(3,4)$ from the straight line $2x + y - 5 = 0$.

Solution

Application of formula (11-1) yields,

$$h = \frac{|2 \cdot 3 + 1 \cdot 4 + (-5)|}{\sqrt{2^2 + 1^2}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

Example 11-2

Find the distance of $M(1,2,3)$ from the plane $2x - 4y + z + 7 = 0$.

Solution

Application of formula (11-3) yields,

$$h = \frac{|2 \cdot 1 + (-4) \cdot 2 + 1 \cdot 3 + 7|}{\sqrt{2^2 + (-4)^2 + 1^2}} = \frac{4}{\sqrt{21}}$$

Example 11-3

Show that the planes $2x + 2y - 4z - 7 = 0$ and $3x - y + z - 10 = 0$ are perpendicular.

Solution

Since $2 \cdot 3 + 2 \cdot (-1) + (-4) \cdot 1 = 0$, the two planes are perpendicular, (see formula (11-10)).

Example 11-4

Find the measure of the acute angle between the planes $2x + 3y + 4z - 5 = 0$ and $3x - 6y + 5z + 11 = 0$.

Solution

Application of equation (11-10) yields,

$$\cos\theta = \pm \frac{2 \cdot 3 + 3 \cdot (-6) + 4 \cdot 5}{\sqrt{2^2 + 3^2 + 4^2} \cdot \sqrt{3^3 + (-6)^2 + 5^2}} = \pm \frac{8}{\sqrt{29} \cdot \sqrt{70}}$$

For the acute angle, we choose the plus sign, i.e.

$$\cos\theta = \frac{8}{\sqrt{29} \cdot \sqrt{70}} \Longrightarrow \theta = \cos^{-1}\left(\frac{8}{\sqrt{29} \cdot \sqrt{70}}\right) \cong 79.8°$$

Example 11-5

Find the geometrical area of a triangle ABC, on the xOy plane, if the vertices are $A(1,5), B(3,6)$ and $C(-2,4)$.

Solution

Application of equation (11-4) yields,

$$E = \frac{1}{2}\begin{vmatrix} 1 & 5 & 1 \\ 3 & 6 & 1 \\ -2 & 4 & 1 \end{vmatrix} \Longrightarrow$$

$$E = \frac{1}{2}\left\{1 \cdot \begin{vmatrix} 6 & 1 \\ 4 & 1 \end{vmatrix} - 5 \cdot \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 3 & 6 \\ -2 & 4 \end{vmatrix}\right\} \Rightarrow$$

$$E = \frac{1}{2}\{1 \cdot (6 - 4) - 5 \cdot (3 + 2) + 1 \cdot (12 + 12)\} = 0.5 \;\; square\ units$$

Example 11-6

Find the shortest distance h between the line $\vec{r} = \vec{r_1} + \lambda\vec{a}$, where $\vec{r_1} = 2\hat{x} + 3\hat{y} + \hat{z}$, $\vec{a} = 4\hat{x} - 3\hat{y} + 2\hat{z}$, and the $x'Ox$ axis.

Solution

The vector equation of the Ox axis is $\vec{r} = 0 + t\hat{x}$, where t is a parameter. Application of formula (11-9) with $\vec{r_1} = 2\hat{x} + 3\hat{y} + \hat{z}$, $\vec{r_2} = 0$ and $\vec{a} = 4\hat{x} - 3\hat{y} + 2\hat{z}$, $\vec{b} = \hat{x}$ yields,

$$h = \frac{\left|\left(\vec{a}, \vec{b}, \vec{r_2} - \vec{r_1}\right)\right|}{\left|\vec{a} \times \vec{b}\right|} = \frac{|(4\hat{x} - 3\hat{y} + 2\hat{z}, \hat{x}, -2\hat{x} - 3\hat{y} - \hat{z})|}{|(4\hat{x} - 3\hat{y} + 2\hat{z}) \times (\hat{x})|} \qquad (*)$$

The numerator in (*) is the scalar vector product of the three shown vectors, and according to equation (8-3) is,

$$(4\hat{x} - 3\hat{y} + 2\hat{z}, \hat{x}, -2\hat{x} - 3\hat{y} - \hat{z}) = \begin{vmatrix} 4 & -3 & 2 \\ 1 & 0 & 0 \\ -2 & -3 & -1 \end{vmatrix} = (-1)\begin{vmatrix} -3 & 2 \\ -3 & -1 \end{vmatrix}$$
$$= -9$$

Note that due to the zeros in the second row, the determinant has been expanded along the elements of this row.

The denominator in (*) yields $2\hat{y} + 3\hat{z}$ and the magnitude of this vector is $\sqrt{2^2 + 3^2} = \sqrt{13}$, and finally the sought for distance h is

$$h = \frac{|-9|}{\sqrt{13}} = \frac{9}{\sqrt{13}}$$

Example 11-7

Find the volume of a parallelepiped formed by three vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ assuming that $a = |\overrightarrow{OA}| = 10, b = |\overrightarrow{OB}| = 7, c = |\overrightarrow{OC}| = 12$, while $\theta_1 = \measuredangle(\overrightarrow{OB}, \overrightarrow{OC}) = 60°, \theta_2 = \measuredangle(\overrightarrow{OA}, \overrightarrow{OC}) = 40°, \theta_3 = \measuredangle(\overrightarrow{OA}, \overrightarrow{OB}) = 80°$.

Solution

If $\theta_1 + \theta_2 + \theta_3 = 2\tau = 180°$, i.e. $\tau = 90°$, then the sought for volume is (see formula (11-6)),

$$V = 2abc\sqrt{\sin\tau\sin(\tau - \theta_1)\sin(\tau - \theta_2)\sin(\tau - \theta_3)} \Longrightarrow$$

$$V = 2 \cdot 10 \cdot 7 \cdot 12 \cdot \sqrt{\sin 90° \cdot \sin 30° \cdot \sin 50° \cdot \sin 10°} \Longrightarrow$$

$$V \cong 433.26 \qquad \textit{cubic units}$$

Example 11-8

Find a unit vector perpendicular to the plane $3x + 2y - 7z + 5 = 0$.

Solution

Application of formula (11-2) yields,

$$\hat{n} = \frac{3\hat{x} + 2\hat{y} - 7\hat{z}}{\sqrt{3^2 + 2^2 + (-7)^2}} = \frac{1}{\sqrt{62}}(3\hat{x} + 2\hat{y} - 7\hat{z})$$

Example 11-9

What equation should the constants a, b, c and r satisfy, if the straight line $ax + by + c = 0$ is to be tangent to the circle having equation $x^2 + y^2 = r^2$?

Solution

The coordinates of the **common points** of the given two lines are solutions of the system of equations,

$$\begin{Bmatrix} ax + by + c = 0 \\ x^2 + y^2 = r^2 \end{Bmatrix} \qquad (*)$$

Solving the first equation for y and substituting into the second, we get, (let the reader check it),

$$(a^2 + b^2)x^2 + 2acx + c^2 - b^2r^2 = 0 \qquad (**)$$

Equation (**) is quadratic in x, and in general has **two solutions**. If the straight line is to be **tangent** to the circle, then the **two solutions should coincide**, i.e. $x_1 = x_2$, and this occurs when the discriminant D of the quadratic equation is zero, i.e. when

$$D = (2ac)^2 - 4(a^2 + b^2)(c^2 - b^2r^2) = 0$$

which eventually simplifies to $(a^2 + b^2)r^2 = c^2$, and this is the sought for equation.

Example 11-10

Find the equation of the plane tangent to the sphere $(x-1)^2 + (y-2)^2 + z^2 = 3^2$ at the point $M(2,4,2)$ of the spherical surface.

Solution

We first note that the point $M(2,4,2)$ belongs to the spherical surface, since its coordinates satisfy the equation of the sphere. Application of formula (11-12) yields the sought for equation,

$$(2-1)(x-1) + (4-2)(y-2) + (2-0)(z-0) = 3^2 \Longrightarrow$$

$$x + 2y + 2z - 14 = 0$$

PROBLEMS

11-1) In section 1, (the distance between a point and a straight line), start with the equation $h = MK = \sqrt{(\xi - x_k)^2 + (\eta - y_k)^2}$, use the expressions for x_k and y_k as given in (***), and derive equation (11-1).

11-2) Alternative proof of formula (11-1).

Assuming that $P(x_p, y_p)$ is a point on the line (ε), find the distance MP^2 as function of x_p, i.e. $MP^2 = MP^2(x_p)$, and then determine x_p which minimizes the distance MP.

Hint: See Alternative proof in Example 10-5.

11-3) Prove formula (11-4).

Hint: The area of the triangle is $E = \frac{1}{2}\left|\overrightarrow{BC} \times \overrightarrow{BA}\right|$, where $\overrightarrow{BC} = (x_C - x_B)\hat{x} + (y_C - y_B)\hat{y}$ and $\overrightarrow{BA} = (x_A - x_B)\hat{x} + (y_A - y_B)\hat{y}$.

11-4) Prove formula (11-6).

Hint: The volume V of the parallelepiped is $V = \left|\left(\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}\right)\right|$.

11-5) Prove formula (11-11).

Hint: See Example 10-9.

11-6) Find the measure of the acute angle between the planes $3x + y - 2z = 5$ and $x + 3y + 4z = 8$.

Hint: Apply formula (11-10).

11-7) Find the area of a triangle ABC, assuming that $A = A(1, -1), B = B(3,3)$ and $C = C(2,4)$.

(**Answer:** $E = 3\ \ square\ units$).

11-8) Show that the planes $2x - y + 7z - 5 = 0$ and $6x - 3y + 21z - 10 = 0$ are parallel, and find their (perpendicular) distance.

11-9) Find the tangent to the circle $(x-1)^2 + (y-2)^2 = 8$ at the point $M(3,4)$.

(**Answer:** $x + y - 7 = 0$).

Hint: Apply formula (11-11).

11-10) Find the area of the triangle formed by the three lines $x + y - 5 = 0$, $2x + 3y - 7 = 0$ and $3x - y - 4 = 0$.

Hint: Find the vertices A, B and C of the triangle. The vertex A is the intersection of the first two lines, i.e. the solution of the system $x + y - 5 = 0$ and $2x + 3y - 7 = 0$, etc.

11-11) Let us consider the circle $(x-5)^2 + (y-4)^2 = 6^2$, which lies entirely on the xOy $(z = 0)$ plane. Find the equation of a sphere whose intersection with the $z = 0$ plane is the given circle and being tangent to the plane $3x + 2y + 6z - 1 = 0$.

Answer: The problem has two solutions, **a)** Sphere centered at $K(5,4,8)$ and radius $R = 10$, **b)** Sphere centered at $L\left(5,4, {}^{160}/_{3}\right)$ and radius $r = {}^{178}/_{13}$.

11-12) Find the volume of a tetrahedron formed by three vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ assuming that $a = \left|\overrightarrow{OA}\right| = 6, b = \left|\overrightarrow{OB}\right| = 7, c = \left|\overrightarrow{OC}\right| = 10$, while $\theta_1 = \measuredangle\left(\overrightarrow{OB}, \overrightarrow{OC}\right) = 45°, \theta_2 = \measuredangle\left(\overrightarrow{OA}, \overrightarrow{OC}\right) = 70°, \theta_3 = \measuredangle\left(\overrightarrow{OA}, \overrightarrow{OB}\right) = 55°$.

Hint: Apply formula (11-8).

11-13) Consider a tetrahedron $OABC$, where $\left|\overrightarrow{OA}\right| = \left|\overrightarrow{OB}\right| = \left|\overrightarrow{OC}\right| = L$, while $\measuredangle\left(\overrightarrow{OB}, \overrightarrow{OC}\right) = \measuredangle\left(\overrightarrow{OA}, \overrightarrow{OC}\right) = \measuredangle\left(\overrightarrow{OA}, \overrightarrow{OB}\right) = \theta$. Find the volume of the tetrahedron in terms of L and θ.

(**Answer:** $V = \frac{1}{3}L^3\left(\sin\frac{\theta}{2}\right)^2\sqrt{1+2\cos\theta}$).

11-14) Find the shortest distance between the two skew lines in Example 10-6.

Hint: Apply formula (11-9).

11-15) Consider the straight line $ax + by + c = 0$ and the circle $x^2 + y^2 = r^2$. What inequality should a, b, c and r satisfy, if **a)** the line intersects the circle at two points? **b)** If the line does not intersects the circle?

(**Answer:** **a)** $(a^2 + b^2)r^2 > c^2$, **b)** $(a^2 + b^2)r^2 < c^2$).

Hint: See Example 11-9.

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