LEARN
LINEAR ALGEBRA
IN ONE DAY

FOR UNDER GRADUATES of ALL UNIVERSITIES

DR. BINAYAK SAHU

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DEDICATION

In Bhagbad-GITA
Lord Krishna tells:

Your right is to work only
But never to its fruits;
Let not the fruits of action be thy motive,
Not let thy attachment be to inaction

Dedicated
In the Loving Memory
Of my beloved father
Late Suryanarayan Sahu

PREFACE

The book titled as Learn Linear Algebra in One Day is meant for students of Under Graduate Courses of B.Tech., B.E, B.Sc. & BCA students as well as for those appear for various competitive examinations. The treatment of problem solving is the ultimate aim of this book, has been done in a organized and interesting manner. All the care has been taken to explain the theory to the minimum level of understanding the basics. Each topic is followed with a precise example with solution. It is sincerely hoped that this book will satisfy the common needs of the students on the subject of differential equations. The students shall definitely like this method of learning & refer this book as a readymade tool to induct self for any examination.

I have of course used my own experiences realizing student's difficulty in referring subject books & consulted my conscience in each step to prepare this book.

I do not claim any originality but the presentation of the subject content is mine & unique. I do express here with my indebtedness to my teachers and students who inspired me to prepare this book.

I am very much thankful to the UAB Publishing House for their great efforts in editing & publishing this book. I am also thankful to my mother, wife and Aditya for their patience & support during preparation of this text. Special thanks to

Suggestions for improvement of this book will always be accepted with thanks.

Dr. Binayak Sahu

MATHEMATICS BASICS

$\frac{d}{dx}\left(\frac{x^n+1}{n+1}\right) = x^n$	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, n \neq 1$
$\frac{d}{dx}(x) = 1$	$\int dx = x + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\, dx = \sin x + C,$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\, dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = sec^2 x$	$\int sec^2 x\ dx = \tan x + C,$
$\frac{d}{dx}(\cot x) = -cosec^2 x$	$\int - cosec^2 x\ dx = \cot x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x\ dx = \sec x + C$
$\frac{d}{dx}(cosec\ x) =$ $- cosec\ x \cot x$	$\int cosec\ x \cot x\ dx =$ $-cosec\ x + C$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$
$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = -\cos^{-1} x + C$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = -\cot^{-1} x + C$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$
$\frac{d}{dx}(cosec^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = -cosec^{-1} x + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x\, dx = e^x + C$
$\frac{d}{dx}(\log\|x\|) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log \|x\| + C$
$\frac{d}{dx}(a^x) = a^x \log a$	$\int a^x\, dx = \frac{a^x}{\log a} + C$

CONTENTS

TIPS FOR STUDENTS

- **NEVER READ MATHEMATICS, BUT WRITE MATHEMATICS:** Do not believe in reading Mathematics. Always adopt pen and paper for doing Mathematics otherwise you will be simply deceiving yourself.
- **REGULARITY IN MATHEMATICS:** The most important thing in the study of mathematics is regularity in its studies. Do not postpone your daily work in mathematics.
- **REMEMBERING OF FORMULAE:** In any study of Mathematics, every student is expected to by rote certain basic and standard formulae in Mathematics, mainly from trigonometry, geometry, differential and integral calculus. Etc.
- **BEFORE GOING TO THE CLASS:** Before going to the class read the portion which is taught in the last class and the lot to be taught by your teacher in the next class and point out a few doubts, so that you will get the maximum benefits.
- **SOLVE UNIVERSITY PAPERS:** In order to generate confidence, the best method is that you have studied some part of the syllabus turn to the Previous year, University Question Papers & try to solve the questions and in case you are able to solve them, then ultimately the confidence level grows up in you, otherwise you can contact your teacher.

- **SELECTION OF SYLLABUS FOR ANY SUBJECT:** There is no Harm in making the selection of the portion of the syllabus which you are finalizing to prepare for the University examination. But whatever you decide to prepare, prepare it very thoroughly. DO ALL TYPES QUESTIONS FROM THIS SELECTION. But do not commit a blunder by making a selection out of selection.

☞ **REMEMBER:** Better KNOW EVERYTHING OF SOMETHING, RATHER THAN SOMETHING OF EVERYTHING AND NOTHING LIKE IF **YOU KNOW EVERY THING OF EVERYTHING**.

- **REVISION BEFORE EXAMINATION:** While preparing any book or any chapter, mark them with a pencil or copy those questions which you feel are difficult or require some special trick or method which you are likely to forget.

  Only these questions may be reviewed carefully and revised before examination.

FINAL EXAMINATION: Attempt, all the questions you know well, on a priority basis. Avoid overwriting and scratching in the answer script. Attempt all questions. Keep/create time, always to revise your answers.

- **CORRESPONDENCE WITH THE AUTHOR**: You are most welcome to meet me with regard to any difficulty with any formulae or answers or any method or any question which you may come across. big. That is, you can only use this method to solve differential equations with known constants. If you do have an equation without the known constants, then this method is useless and you will have to find another method.

CHAPTER 1

1. MATRIX ALGEBRA:

A rectangular array of functions or numbers (real or complex) enclosed in brackets is called a matrix. It can be of any order, **m x n**, where **m** stands for number of rows and **n** the number of columns, and we write **A** = $\mathbf{(a_{ij})_{m\,x\,n}}$. We also read it as A is a matrix of size **m** cross **n**.

EXAMPLE: Here is a matrix of size 2 x 3 ("2 by 3"), because it has 2 rows and 3 columns:

$\begin{bmatrix} 1 & 2 & 5 \\ 2 & -3 & 6 \end{bmatrix}$ & the matrix has 2 x 3 = 6 entries.

In general, an **m** x **n** matrix has **m** rows and **n** columns and has **mn** entries.

EXAMPLE: Consider, $\begin{bmatrix} \boldsymbol{a} & b \\ c & \boldsymbol{d} \end{bmatrix}$. Here is a matrix of size 2 x 2 (an order 2 square matrix). The boldfaced entries (**a** & **d**) lie on the **main diagonal** / **principal diagonal** of the matrix. The other diagonal is the **skew diagonal**.

An n x m matrix A, is of the form:

$$A_{m\,X\,n} = \left(a_{ij}\right)_{mXn} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

An element, $a_{32}$is located at the intersection of 3rd

and 2^{nd} column. These are also called, the entries of the matrix. Similarly, a_{ij} is at the intersection of i^{th} row and j^{th} column.

1.1 TYPES OF MATRICES:

RECTANGULAR MATRIX: A rectangular matrix is formed by a different number of rows and columns, and its dimension is noted as **m** x **n**. Basically, almost all matrices are rectangular matrix but if the rows and columns become the same then it won't be a rectangular matrix.

SQUARE MATRIX: A matrix A with same number of rows and columns is a square matrix. A matrix of n rows and n columns is a square matrix of order n.

VECTORS: A vector is a matrix with only one row or one column. Its entries are called the components of the vector/matrix.

COLUMN MATRIX: A matrix of order **m x 1** is called as a column vector/column matrix, that is a matrix with only one column is a column vector.

EXAMPLE: A vector in the form $\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$ can be looked as a 3×1 matrix, called a COLUMN MATRIX / COLUMN VECTOR.

ROW MATRIX: A matrix of order **1 x n** is called as a row vector, hat is a matrix with only row is a row vector. We denote them as bold letters like, a, b etc.

EXAMPLE: A vector in the form [2 4 6] can be looked as a 1 × 3 matrix, called a ROW MATRIX/ ROW VECTOR.

ZERO MATRIX: In a zero matrix, all the elements are zeros.

EXAMPLE:

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ *are zero matrices*. We call such matrices as null matrices as well.

UPPER TRIANGULAR MATRIX If all the elements below the diagonal are zero that means you have an upper triangular matrix. That is aij = 0 for all i≤j, that is: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$.

EXAMPLE: $\begin{bmatrix} 2 & -5 \\ 0 & 3 \end{bmatrix}$ *and* $\begin{bmatrix} -2 & 5 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ are upper triangular matrices.

LOWER TRIANGULAR MATRIX If all the elements above the diagonal are zero that means you have an upper triangular matrix. That is aij = 0 for al i≥j, that is: $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{31} & a_{33} \end{bmatrix}$.

EXAMPLE: $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ *and* $\begin{bmatrix} -2 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ are lower triangular matrices.

DIAGONAL MATRIX: In a diagonal matrix, all the elements above and below the diagonal are zeros. It is like a combination of an upper triangular matrix and a lower triangular matrix.

EXAMPLE: $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ *and* $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are diagonal matrices.

SCALAR MATRIX: A scalar matrix is a diagonal matrix in which the diagonal elements are equal.

EXAMPLE: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ *and* $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ are scalar matrices.

IDENTITY MATRIX: An identity matrix is a diagonal matrix in which the diagonal elements are equal to 1.

EXAMPLE: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity or unit matrices of order 2 and 3 respectively.

1.2 MATRIX OPERATIONS:

TRANSPOSE OF A MATRIX: If **A** is an **m** × **n** matrix, then the transpose of **A**, $\mathbf{A}^{\mathbf{T}}$, is the n × m matrix, obtain from A, with the rows and columns interchanged in A.

That is $(A_{ij})^T = A_{ji}$

NOTE: Transpose of a matrix A is A^T, obtained by interchanging rows into columns.

EXAMPLE: The transpose of $\begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix}$ *is* $\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$

$$\textit{and tranpoe of} \begin{bmatrix} -2 & 6 & 5 \\ 1 & 2 & 0 \\ 3 & 9 & 1 \end{bmatrix} \; is \; \begin{bmatrix} -2 & 1 & 3 \\ 6 & 2 & 9 \\ 5 & 0 & 1 \end{bmatrix}$$

PROPERTIES OF TRANSPOSE:

(i) $(A^T)^T = A$

Answer: We have $(A^T)^T = ((a_{ij})^T)^T = (a_{ji})^T = (a_{ij}) = A$.

EXAMPLE: The transpose of

$$A = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix} \; is \; A^T = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix} \& \left(A^T\right)^T = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix} = A$$

(ii) $(A+B)^T = A^T+B^T$

(iii) $(mA)^T = mA^T$.

(iv) TRANSPOSE OF A PRODUCT: The transpose of a product of two matrices is equal to the product of their transposes taken in reverse order, that is **$(AB)^T=B^TA^T$.**

It can be given any extension, like

$(ABC - - -M)^T= M^T - - - C^T B^T A^T$.

EQUALITY OF MATRICES: Two or more matrices are said to be ***EQUAL*** if they are of the same size & their corresponding entries are equal.

EXAMPLE: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$ implies, a=2, b=5, c=1 and =4.

NOTE: Matrices that are not equal are called different. Thus, matrices of different sizes are always different.

MATRIX ADDITION: Two or more matrices can be ad if they are of the same size, an then the corresponding entries added.

EXAMPLE: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} a+2 & b+5 \\ c+1 & d+4 \end{bmatrix}$

SCALAR MULTIPLICATION: (Multiplication by a Number).The product of any m x n matrix $\mathbf{A}_{\mathbf{m\,X\,n}} = \left(\mathbf{a}_{\mathbf{ij}}\right)_{\mathbf{mXn}}$ and any scalar **c** (number c) is written $\mathrm{c}\mathbf{A}_{\mathbf{m\,X\,n}} = \mathbf{c}\left(\mathbf{a}_{\mathbf{ij}}\right)_{\mathbf{mXn}} = \left(\mathbf{ca}_{\mathbf{ij}}\right)_{\mathbf{mXn}}$ and is the matrix obtained by multiplying each entry of **A** by **c**.

RULES FOR MATRIX ADDITION AND SCALAR MULTIPLICATION:

a) A + B = B + A (Commutative)

b) A + (B+C)= (A+B) +C (Associative)

c) A + 0 = A (0 is the additive identity)

d) A + (-A) = 0, is the zero matrix.
(- A is the additive inverse of A)

e) c(A + B) = cB + cA

f) (c+d)A = cA + dA

g) c(kA) =(ck)A

h) 1A = A

MATRIX MULTIPLICATION: The matrix multiplication of two matrices **A** & **B** is **AB**, said to be defined if **number of rows of B** equals to the **number of columns of A**. That is if A is of order m x **n** then B should be of order **n** x p, where p is any integer, so that the product, **AB** is defined & of order m x p.

It is also called as the row by column multiplication.

NOTE: A [m x n] B [n x p] = C [m x p]

EXAMPLE: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & h \\ f & g \end{bmatrix} = \begin{bmatrix} ae + bf & ah + bg \\ ce + df & ch + dg \end{bmatrix}$

We can see that the multiplication is conducted a row by column multiplication.

EXAMPLE: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f & g \\ h & i & j \end{bmatrix}$

$= \begin{bmatrix} ae + bh & af + bi & ag + bj \\ ce + dh & cf + di & cg + dj \end{bmatrix}$

The given matrices are of sizes **2** x 2 and 2 x **3**, and hence the resultant matrix is of the size **2** x **3**.

We can see that the multiplication is conducted a row by column multiplication.

PROPERTIES:

a. Matrix multiplication is not commutative, that is AB ≠ BA

b. A(BC) = (AB)C, (Associative)

c. AB = 0 does not mean necessarily that either A = 0 or B = 0.

d. (A + B) C = AC + BC (Distributive)

e. AC = BC does not necessarily mean that A = B.

1.3 REAL MATRICES:

SYMMETRIC MATRIX: If a square matrix A is equal to its transpose then A is called a symmetric matrix, that is if $\mathbf{A^T = A}$, that is if $a_{ji} = a_{ij}$.

A matrix A is said to be **symmetric if $\mathbf{A^T = A}$,** that is if $a_{ij} = a_{ji}$ for all i & j.

SKEW – SYMMETRIC MATRIX: If a square matrix A is equal to the negative of its transpose then A is called a skew – symmetric matrix, that is if $\mathbf{A^T = -A}$, that is if $a_{ji} = -a_{ij}$.

A matrix A is said to be **skew – symmetric if $\mathbf{A^T = -A}$,** that is if $a_{ij} = -a_{ji}$ for all i & j.

ORTHOGONAL MATRICES: A matrix A is said to be **orthogonal if $\mathbf{A^T = A^{-1}}$.**

Example: The matrix A = $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

RESULT: Any square matrix A can be expressed as the sum a symmetric matrix and a skew–symmetric matrix.

Proof: $(A + A^T), which\ is\ symmetric\ always, since$

$(A + A^T)^T = A^T + (A^T)^T = A^T + A$

$Similarly, (A - A^T), is\ kew - symmetric\ always,$

$since (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$

Now, $\boldsymbol{A} = \frac{\boldsymbol{1}}{\boldsymbol{2}}\{(\boldsymbol{A} + \boldsymbol{A^T}) + (\boldsymbol{A} - \boldsymbol{A^T})\}, in\ which\ (A + A^T)$

is symmetric and $(A - A^T)$ I skew-symmetric.

Example: Express the square matrix $A = \begin{bmatrix} 3 & -4 \\ 6 & 0 \end{bmatrix}$ **as sum of symmetric matrix and a skew symmetric matrix.**

Solution: $A = \dfrac{A + A'}{2} + \dfrac{A - A'}{2}$

$\Rightarrow \begin{bmatrix} 3 & -4 \\ 6 & 0 \end{bmatrix} =$

$$\frac{\begin{bmatrix} 3 & -4 \\ 6 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ -4 & 0 \end{bmatrix}}{2} + \frac{\begin{bmatrix} 3 & -4 \\ 6 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ -4 & 0 \end{bmatrix}}{2}$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

1.4 LINEAR COMBINATION

A **LINEAR COMBINATION of matrices** A, B, C - - , M, N of same order is an expression of the form:

aA + bB + cC + - - - +mM + nN, where a, b, c, - - - m, n are scalars/constants.

EXAMPLE: Show that if A, B, C - - , M, N are matrices of same order and are symmetric, then so is their linear combination.

Answer: A, B, C - - , M, N being symmetric, $A^T = A$, $B^T = B$, - - - , $N^T = N$. We show that the linear combination aA + bB + cC + - - - +mM + nN is also symmetric, where a, b, c, - - - m, n are scalars.

$(aA + bB + - - - + nN)^T = (aA^T + bB^T + - - - + nN^T)$
$=(aA + bB + - - - + nN)$
$\Rightarrow$ (aA + bB + - - - + nN) is symmetric

1.5 LINEAR SYSTEM OF EQUATIONS

An m ***by n linear system of equations*** is a system of m linear equations in n variables.

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$ **- - - (1)**

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_n$

In matrix form the above system can be written as A**x** = **b**, that is:

That is, $$\begin{bmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & -- & a_{2n} \\ -- & -- & -- & -- \\ a_{m1} & a_{m2} & -- & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_x \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_x \end{bmatrix}.$$

where $A = \begin{bmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & -- & a_{2n} \\ -- & -- & -- & -- \\ a_{m1} & a_{m2} & -- & a_{mn} \end{bmatrix}$ is the coefficient matrix and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_x \end{bmatrix}$ & $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_x \end{bmatrix}$ are column vectors.

If the system of equations has solution/s then the system is said to be consistent, else, it is inconsistent.

INNER PRODUCT OF VECTORS: The inner product of a row vector **a** and a column vector **b** with n components each is a 1 x 1 matrix defined by

$$a \cdot b = \begin{bmatrix} a_1 & a_2 & -- & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_x \end{bmatrix} = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

RESULTS:1. For symmetric matrices A & B, AB is symmetric if and only if AB = BA.

2. The linear transformation y = Ax with matrix A = $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and x = $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, y = $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is a counter clockwise rotation of the Cartesian x_1x_2 – coordinate system in the plane about the origin, where θ is the angle of rotation.

3. If A = $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then

$A^m = \begin{bmatrix} \cos m\theta & -\sin m\theta \\ \sin m\theta & \cos m\theta \end{bmatrix}$, where m is an Integer.

IDEMPOTENT, NILPOTENT & NILPOTENCY

A matrix is A is said to be **IDEMPOTENT** if $A^2 = A$.

A matrix A is said to be **NILPOTENT** if $A^m = 0$, where 0 is the null matrix. Here m is called the degree of nilpotency.

ELEMENTARY ROW OPERATIONS

By elementary row operations, we mean:

i) Interchange of any two rows in a matrix,

ii) a constant multiple of any row in a matrix and

iii) a sum (subtraction) of constant multiple of a row with (from) another row in a matrix do not alter the value of the matrix.

Definitions: A system $Ax = b$ of m linear equations in n unknowns is said to be **homogeneous** if $b = 0$. Otherwise the system is said to be nonhomogeneous.

1.6 GAUSS ELIMINATION METHOD:

SOLVING A SYSTEM OF LINEAR EQUATIONS: We use elementary row operations to go for the above method. Let a system of m-equations in n-unknowns is given by:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\dots \qquad \dots \qquad \dots \quad & \dots \\
a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n
\end{aligned}
$$

We write down the augmented matrix for of the above system of linear equations as:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & --- & a_{1n} & b_1 \\ a_{21} & a_{22} & --- & a_{2n} & b_2 \\ -- & -- & -- & -- & - \\ a_{n1} & a_{n2} & & a_{nn} & b_n \end{array}\right]$$

Consider the **1st entry of the 1st row as the pivot element**. Using the pivot element by means of elementary row operations reduce the above matrix unto the upper triangular form moving diagonally, and then by back substitution we can get the values of the unknowns. Three cases exist.

CASE – I: UNIQUE SOLUTION

EXAMPLE: Solve the following system of equations by Gauss elimination method:

– x + y + 2z = 2
3x – y + z = 6
–x + 3y + 4z = 4.

Answer: The augmented matrix of the given system of equations is $\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ -1 & 3 & 4 & 4 \end{array}\right] \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \approx$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 2 & 2 & 2 \end{array}\right] R_3 \to R_3 - R_2$$

$$\approx \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 0 & -5 & -10 \end{array}\right]$$, this form is the upper triangular form / **row echelon form**.

Now we observe from the 3rd row, that

–5z = –10 ⇒ z = 2.

And then by back substitution from the **2nd row**, we have, 2y + 3z = 12, so that we get y = –1.

And from the **1st row**: –x + y + 2z = 3 we have that x = 1.

Hence the solution is x = 1, y = –1, z = 2, which is unique.

CASE – II: INFINITELY MANY SOLUTIONS

EXAMPLE: Solve 12x – 26 y + 34 z = 18, - 30 x + 65 y – 85 z = – 45.

Answer: The augmented matrix form will be

$$\left[\begin{array}{ccc|c} 12 & -26 & 34 & 18 \\ -30 & 65 & -85 & -45 \end{array}\right] \quad R_1 = \frac{R_s}{2}$$

$$\approx \left[\begin{array}{ccc|c} 6 & -13 & 17 & 9 \\ -30 & 65 & -85 & -45 \end{array}\right]$$ Consider 6 as the pivot element, $R_2 = R_2 - 5R_1$

$$\approx \left[\begin{array}{ccc|c} 6 & -13 & 17 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now from the 1st row, we get 6x – 13 y + 17 z = 9

$\Rightarrow$ x = $\frac{-13}{6}y - \frac{-17}{6}z + \frac{9}{6}$, where y and z remain arbitrary. So the system has infinitely many solutions.

CASE – III: NO SOLUTION

EXAMPLE: Solve 3x + 2y = 5 , 4y + 3z = 8, 2x – z= 2.

Answer: The augmented matrix form will be

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 4 & 0 & 3 & 8 \\ 2 & 0 & -1 & 2 \end{array}\right] \approx \text{Interchange } R_1 \text{ and } R_3.$$

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 4 & 0 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{array}\right] \approx \text{Divide } R_1 \text{ by 2.}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 1 \\ 4 & 0 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{array}\right] \approx \text{Perform: } R_2=R_2-4R_1,\ R_3=R_3-3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 2 & 3/2 & 2 \end{array}\right]$$

From the above augmented matrix, we see that, the 2n row represent 0 = 4, which is absurd. Hence, the system has no solution.

1.7 RANK OF A MATRIX

ROW ECHELON FORM: At the end of the Gauss elimination the form of the coefficient matrix, in the augmented matrix, and the system itself are said to be in the row echelon form.

The number of nonzero rows, r, in the row-reduced coefficient matrix R is called the **rank of R** and also the rank of A.

METHODS OF FINDING RANK:

Method – I Reduce the given matrix into upper triangular form by means of elementary row operations (use the process of pivoting), and in the form count the number of non-zero rows is the rank of the matrix.

Example: Find the rank of the matrix,

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ -1 & 0 & 3 & -2 \\ -1 & -1 & 8 & -3 \end{pmatrix}.$$

Solution: We try finding the rank by means of elementary row operations. Here

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ -1 & 0 & 3 & -2 \\ -1 & -1 & 8 & 3 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\approx \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -1 & 5 & -1 \\ 0 & -2 & 10 & 4 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$$\approx \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -1 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As maximum no. of non-zero rows is the no. of linearly independent row vector = 2.

So, the rank of matrix = 2.

Method – II The rank of a matrix will be said to be r if **(a)** there is at least one minor of size r x r which is non-zero and

(b) all the minors of order greater than r are zero.

Example: Find the rank of the matrix

$$A = \begin{bmatrix} 3 & -1 & 5 \\ 2 & -4 & 6 \\ 5 & -5 & 11 \end{bmatrix}.$$

Solution: First of all, we try the 3 x 3 minor,

$\begin{vmatrix} 3 & -1 & 5 \\ 2 & -4 & 6 \\ 5 & -5 & 11 \end{vmatrix} = 0$, so that the Rank $\neq$ 3. Next we try

the next lower order minor, say

$\begin{vmatrix} 3 & -1 \\ 2 & -4 \end{vmatrix} -12 + 2 = -10 \neq 0$, So the rank of the matrix = 2.

NOTE:

a. The rank of a matrix $A_{m \times n}$ will be $\leq$ min (m, n).

b. The rank of matrix is zero only when the given matrix is a null matrix.

c. The rank of a matrix is also equal to the maximum number of linearly independent columns in a matrix

d. The rank of a matrix A and its transpose A^T is same.

e. The rank of a non-singular matrix A of order n is n.

Vector Space : A non-empty is said to be a vector space if it is closed under scalar multiplication and vector addition (that is for any vectors $\vec{a}$ & $\vec{b}$ in V, and scalars α & β, $\alpha\vec{a}$ and $\alpha\vec{a} + \beta\vec{b}$ are also in V) and they satisfy the following laws :

For vector addition :–

(i) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(ii) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

(iii) $\vec{a} + \vec{0} = \vec{a}$

(iv) $\vec{a} + (-\vec{a}) = \vec{0}$

And for scalar multiplication :–

(i) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

(ii) $(k_1 + k_2)\vec{a} = k_1\vec{a} + k_2\vec{a}$.

(iii) $k_1(k_2\vec{a}) = (k_1k_2)\vec{a}$

(iv) $1\vec{a} = \vec{a}$.

BASIS AND DIMENSION:

The maximum number of linearly independent vectors in a vector space V form a basis, and the number of these independent vectors is called the dimension of the vector space V.

Example: Show that $V = \{(x, y, z) \in R^3 : x + y = 0\}$ is a vector space over R. Find its dimension.

Solution: $x + y = 0 \Rightarrow x = -y$

$V = \{(x, y, z)\} = \{(-y, y, z)\}$. It has two independent variables.

So dimension = 2. Basis is (-1,1,1), (1,-1,0).

Evaluation of determinants and Cramer's Rule:

Expansion of a determinant:

The expansion of a determinant is the product of the diagonal elements once the determinant is reduced to the upper triangular form by means of elementary row operations using the pivot elements in each step.

Cramer's Rule:

(i) If Ax = b describes a system of n-linear equations in n-unknowns, x_1, x_2, - - - , x_n & the system is non-homogenous and if the coefficient determinant

D = |A| $\neq$ 0, then the given system has the non-zero solution, namely, $x_1 = \frac{D_1}{D}$, $x_2 = \frac{D_2}{D}$, - - - , $x_n = \frac{D_n}{D}$;

where D_j 's are obtained from D by replacing the jth column with b.

Example: Solve the following system of equation by Cramer's rule:

x + 2y + 3z = 20, 7x + 3y + z = 13, x + 6y + 2z = 0

Solution: Here the coefficient determinant,

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 7 & 3 & 1 \\ 1 & 6 & 2 \end{vmatrix} = 91 \neq 0.$$

Now, $D_x = \begin{vmatrix} 20 & 2 & 3 \\ 13 & 3 & 1 \\ 0 & 6 & 2 \end{vmatrix} = 182,$

$$D_y = \begin{vmatrix} 1 & 20 & 3 \\ 7 & 13 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -273, D_z = \begin{vmatrix} 1 & 2 & 20 \\ 7 & 3 & 13 \\ 1 & 6 & 0 \end{vmatrix} = 728$$

So by Cramer's rule

$$x = \frac{D_x}{D} = \frac{182}{91} = 2, y = \frac{D_y}{D} = \frac{-273}{91} = -3,$$

$$z = \frac{D_z}{D} = \frac{728}{91} = 8.$$

NOTE: If the coefficient determinant D is zero and one of D_1 / D_2 / D_3 is not-zero, then the system of linear equations are said to be inconsistent.

Example: Test the consistency of the system:

x + 3 y = 2 and 2x + 6 y = 7.

Solution: Here

$$\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0., but\ D_1 = \begin{vmatrix} 2 & 3 \\ 7 & 6 \end{vmatrix} = 12 - 21 = -9 \neq 0.$$

Hence the system is inconsistent.

(ii) If Ax = 0, that is if the system is homogeneous, then the trivial solution is x = 0. But the system will have the non-zero solution only when the coefficient determinant D = |A| = 0.

1.8 INVERSE OF A MATRIX:

A. Gauss Jordan Method : We write [A | I], where I is the unit matrix of the same size as that of A. Then by means of elementary row operations we reduce the matrix A in the above augmented form to identity matrix, and when the same operations are executed simultaneously with I we get A^{-1}, that is [I | A^{-1}].

Example: Determine the inverse of the following matrix by using Gauss-Jordan elimination method or otherwise. $A=\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$.

Solution: First of all we see that

$A=\begin{vmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{vmatrix}=29\neq 0,$ so that inverse of the matrix exists.

$\left[\begin{array}{ccc|ccc} -1 & 2 & 2 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$ Consider (–1) as the pivot element, $R_1 = (-1)R_1$

$$\approx \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & -1 & 0 & 0 \\ 0 & 3 & 6 & 2 & 1 & 0 \\ 0 & 6 & 3 & 2 & 0 & 1 \end{array}\right], R_3 = \frac{1}{3} R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & -1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 6 & 3 & 2 & 0 & 1 \end{array}\right]$$ Consider 1 as the pivot

element, $R_3 = R_3 - 6R_2$

$$\approx \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & -1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -9 & -2 & -2 & 1 \end{array}\right] R_3 = \frac{1}{-9} R_3$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & -1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & \frac{2}{9} & \frac{-1}{9} \end{array}\right] R_1 = R_1 + 2R_3, R_2 = R_2 -$$

$2R_3$,

$$\approx \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & \frac{-5}{9} & \frac{4}{9} & \frac{-2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{-1}{9} & \frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & \frac{2}{9} & \frac{-1}{9} \end{array}\right] R_1 = R_1 + 2R_2$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{-1}{9} & \frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & \frac{2}{9} & \frac{-1}{9} \end{array}\right] \text{ So, } A^{-1} = \begin{bmatrix} \frac{-1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{-1}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{-1}{9} \end{bmatrix}$$

ADJOINT METHOD: We know it from 10 + 2, but to be recalled as now that $A^{-1} = \frac{Adj.A}{|A|}$, provided $|A| \neq 0$, where Adj. A is the transpose of the cofactor matrix of A.

Inverse of a diagonal matrix is the diagonal matrix whose diagonal elements are the reciprocals of the given diagonal matrix, provided all the diagonal elements of the given matrix are non-zero. For example the inverse of the diagonal matrix A = $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}$.

Related matters: (i) Show that $(AB)^{-1} = B^{-1}A^{-1}$

Proof: $(AB)\,(AB)^{-1} = I$

$\Rightarrow A^{-1}\,(AB)\,(AB)^{-1} = A^{-1}I = A^{-1}$

$\Rightarrow ((A^{-1}A)B)\,(AB)^{-1} = A^{-1}$

$\Rightarrow (IB)\,(AB)^{-1} = A^{-1}$

$\Rightarrow B(AB)^{-1} = A^{-1}$

$\Rightarrow B^{-1}B(AB)^{-1} = B^{-1}A^{-1}$

$\Rightarrow I(AB)^{-1} = B^{-1}A^{-1}$

$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

NOTE: Giving extension to the above proof, we can get:$(A\ B\ C\ -\ -\ -\ M\ )^{-1} = M^{-1}-\ -\ -\ C^{-1}\ B^{-1}\ A^{-1}$

*The inverse of the product of matrices is the product of their inverses taken in reverse order.

(ii) Show that $(A^2)^{-1} = (A^{-1})^2$

Proof: $(A^2)^{-1} = (AA)^{-1} = A^{-1}A^{-1} = (A^{-1})^2$

(iii) Show that $(A^{-1})^T = (A^T)^{-1}$

Proof: $I = I^T = (AA^{-1})^T = (A^{-1})^T(A)^T$

$\Rightarrow I(A^T)^{-1} = (A^{-1})^T(A)^T(A^T)^{-1} = (A^{-1})^T I$

$\Rightarrow (A^T)^{-1} = (A^{-1})^T$.

(iv) Show that $(A^{-1})^{-1} = A$

Proof: $I = (AA^{-1}) \Rightarrow I^{-1} = (AA^{-1})^{-1} = (A^{-1})^{-1} A^{-1}$ (since $(AB)^{-1} = B^{-1}A^{-1}$)

$\Rightarrow I = (A^{-1})^{-1} A^{-1} \Rightarrow IA = (A^{-1})^{-1} A^{-1}A = (A^{-1})^{-1}$

$\Rightarrow (A^{-1})^{-1} = A$.

CHAPTER 2

2. MATRIX EIGEN VALUE

The value or values of λ for which the system of equations given by $Ax = \lambda x$ has a non-zero solution are called Eigen values of A, where A is a n x n matrix, x is the column vector of n – unknowns. Here for each λ, the corresponding solution vector of the system $Ax = \lambda x$ is called the Eigen vector of A.

2.1EIGEN VALUES & EIGEN VECTORS:

Given $Ax = \lambda x$, which gives $(A - \lambda I) x = 0$, that describes a homogeneous system of linear equations. By Cramer's rule this system has a non-zero solution if the coefficient determinant $| A - \lambda I | = 0$.

NOTE: a. The coefficient determinant $|A - \lambda I |$ is called the characteristic determinant.

b. The equation $|A - \lambda I | = 0$ is called the characteristic equation.

c. The solution of the characteristic equation gives values of λ, called the Eigen values of A. The set of Eigen values of A is called the spectrum of A and the

Eigen value with largest magnitude is called the spectral radius.

d. The defect in each Eigen value is defined as $\Delta_\lambda = M_\lambda - m_\lambda$, where M_λ is called the algebraic multiplicity – the number of times λ is a root of the characteristic equation, and m_λ is the number of linearly independent Eigen vectors for the corresponding λ, called the geometric multiplicity of λ.

e. When $\Delta_\lambda = M_\lambda - m_\lambda > 0$, we say that λ has a positive defect. In case $\Delta_\lambda = M_\lambda - m_\lambda = 0$, then λ has zero defect.

f. The sum of the principal diagonal elements of a square matrix is called the trace of the matrix. The sum of the Eigen values of a matrix is equal to the trace of A.

g. If a matrix A has the Eigen values $\lambda_1, \lambda_2, - - - , \lambda_n$; none of which is zero, then the Eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, - - - , \frac{1}{\lambda_n}$.

EXAMPLE: Find the Eigen values and the corresponding eigenvectors of the following matrix:

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 5 & 0 & -1 \end{bmatrix}.$$

Solution: The **c**haracteristics equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 5 & 0 & -1-\lambda \end{vmatrix}=0$$

$$\Rightarrow (3-\lambda)(2-\lambda)(-1-\lambda)-5(2-\lambda)=0$$

$$\Rightarrow (2-\lambda)\left[(3-\lambda)(-1-\lambda)-5\right]=0$$

$$\Rightarrow (2-\lambda)\left[(\lambda^2-2\lambda-8)\right]=0$$

$$\Rightarrow (2-\lambda)(\lambda-4)(\lambda+2))=0$$

$$\Rightarrow \lambda=2,4,-2$$

For $\lambda=2$, we have

$$\begin{bmatrix} 3-2 & 0 & 1 \\ 0 & 2-2 & 0 \\ 5 & 0 & -1-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 5 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

$\Rightarrow x+z=0, 5x-3z=0$ $\Rightarrow$ x = z = 0.

Therefore, the Eigen vector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

And, for $\lambda = 4$, we have

$$\begin{bmatrix} 3-4 & 0 & 1 \\ 0 & 2-4 & 0 \\ 5 & 0 & -1-4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 5 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow -x + z = 0, -2y = 0$ $\Rightarrow$ y = 0.

Therefore, the Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = -2$, we have

$$\begin{bmatrix} 3+2 & 0 & 1 \\ 0 & 2+2 & 0 \\ 5 & 0 & -1+2 \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 5 & 0 & 1 \\ 0 & 4 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix} = 0$$

$\Rightarrow 5\mathrm{x} + \mathrm{z} = 0, 4\mathrm{y} = 0$

$\Rightarrow$ z = − 2 5x, y = 0.

Therefore, the Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$.

NOTE: a. The spectrum in the above problem is { 2, 4 , **–**2 }, and the spectral radius is 4.

b. The defect for each Eigen value is zero.

THE INNER PRODUCT of 2 column vectors **a** and **b** defined as $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T\mathbf{b}$.

Example: If a = $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_x \end{bmatrix}$ and b = $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_x \end{bmatrix}$, then

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T\mathbf{b} = \begin{bmatrix} a_1 & a_2 & -- & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_x \end{bmatrix}$$

$= a_1b_1 + a_2b_2 + - - - + a_nb_n$.

2.2ORTHOGONAL TRANSFORMATION:

A linear transformation is defined as an equation of the form y = Ax where A is a square matrix.

A linear transformation y = Ax, where A is orthogonal is called an orthogonal transformation.

PROPERTIES: An orthogonal transformation preserves the value of the inner product.

Proof: We show that if u and v are two orthogonal transformations then $\mathbf{u} \cdot \mathbf{v} = \mathbf{a} \cdot \mathbf{b}$.

Let $\mathbf{u} = A\mathbf{a}$ and $\mathbf{v} = A\mathbf{b}$ be two orthogonal transformations, where A is an orthogonal matrix, so that $A^T = A^{-1}$.

Now by the definition of inner product

$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = (A\mathbf{a})^T(A\mathbf{b})$

$= \mathbf{a}^T A^T A\mathbf{b} = \mathbf{a}^T A^{-1} A\mathbf{b} = \mathbf{a}^T I\mathbf{b} = \mathbf{a}^T\mathbf{b} = \mathbf{a} \cdot \mathbf{b}.$

A set of vectors { a_1, a_2, a_3, - - - , a_n } aqre said to form an orthonormal system if $a_i \cdot a_j = \begin{cases} 1, \text{for } i = j \\ 0, \text{for } i \neq j \end{cases}$

RESULTS:

1. Show that a matrix is said to be orthogonal if and only if column vectors form an orthonormal system.

2. Show that the determinant of an orthogonal matrix has the value ± 1.

TEST YOUR KNOWLEDGE:

a. Find t so that (1-t, 0, 0), (1, 1-t, 0) and (1, 1, 1-t) are linearly dependent

b. Solve the following system of equation by Cramer's rule:

$3x + 7y + 8z = -13$, $2x + 9z = -5$, $-4x + y - 26z = 2$.

c. Solve by Gauss Elimination : $2x - y + z = 10$, $x + 3y - 2z = 20$, $3x + y - 6z = 15$.

d. Find the rank of the matrix $\begin{bmatrix} 3 & -1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$.

e. Solve the following system of equation by Cramer's rule:

$3x + 7y + 8z = -13$, $2x + 9z = -5$, $-4x + y - 26z = 2$.

f. Find the inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, by both the Gauss Jordan Method & Adjoint method and check the result.

2.3 COMPLEX MATRICES: Hermitian, skew Hermitian and unitary matrices, Similarity of matrices.

Complex matrices: If the elements of a matrix are complex numbers (and or real numbers) then the matrix is called a complex matrix.

NOTE:

a. If z = x + iy and $z = \bar{z}$ then x + iy = x – iy so that y = 0. Hence the complex number z = x is purely real.

b.If z = x + iy and $z = -\bar{z}$ then x + iy = – (x – iy) = –x + iy so that x = 0.

Hence the complex number z = iy is purely imaginary or zero according as y is non-zero or zero.

Types of Complex Matrices:

A. HERMITIAN MATRIX: A complex

square matrix **A** is said to be Hermitian if $\overline{A}^T = A$, that is if, $(\overline{a_{ij}})^T = (a_{ij})$

NOTE: The principal diagonal elements of a Hermitian matrix are pure real.

Explanation: We know that if $z = \overline{z}$ then z is real.

Here A being Hermitian $\overline{A}^T = A$, we have $(\overline{a_{ij}})^T = (a_{ij})$ so that $(\overline{a_{ji}}) = (a_{ij})$.

And along the principal diagonal, i = j, so that from the above line $(\overline{a_{ii}}) = (a_{ii}) \Rightarrow$ (a_{ii}) are real.

Example: $\begin{bmatrix} 3 & 1-i \\ 1+i & 0 \end{bmatrix}$, $\begin{bmatrix} -2 & -1+i & 5+i \\ -1-i & 6 & 2-i \\ 5-i & 2+i & 2 \end{bmatrix}$

are examples of hermitian matrices.

B. SKEW – HERMITIAN MATRIX: A complex square matrix **A** is said to be Skew-Hermitian if $\overline{A}^T = -A$, that is if, $(\overline{a_{ij}})^T = -(a_{ij})$

NOTE: The principal diagonal elements of a skew-Hermitian matrix are pure imaginary or zero.

EXPLANATION: We know that if $\overline{z} = -z$ then z is pure imaginary or zero.

Here A being Skew-Hermitian $\overline{A}^T = -A$, we have, $(\overline{a_{ij}})^T = -(a_{ij})$ so that $(\overline{a_{ji}}) = -(a_{ij})$ .Again along the principal diagonal, i = j, so that from the above line $(\overline{a_{ii}}) = -(a_{ii}) \Rightarrow$ (a $_{ii}$) are pure imaginary or zero.

Example: $\begin{bmatrix} i & 1-i \\ -1-i & 0 \end{bmatrix}$,

$\begin{bmatrix} -2i & -1+i & 2+i \\ 1+i & 0 & 2-i \\ -2+i & -2-i & i \end{bmatrix}$ are examples of skew-hermitian matrices.

C. UNITARY MATRIX: A square matrix A is said to be unitary if $\overline{A}^T A = I$ that is if $\overline{A}^T = A^{-1}$.

Example: $\frac{1}{2}\begin{bmatrix} 1+i & -1-i \\ 1+i & 1-i \end{bmatrix}$ is an example of a Unitary matrix, as $\overline{A}^T = A^{-1}$.

2.4 RESULTS related to Eigen Values of Complex Matrices

Show that the Eigen values of a hermitian matrix are real.

Solution: Given that matrix A is Hermitian so that $\overline{A}^T = A \Rightarrow A^T = \overline{A}$

Let λ be the Eigen value of A so that we have the equation: Ax = λx

$$\Rightarrow \bar{x}^T Ax = \bar{x}^T \lambda x = \lambda \bar{x}^T x$$

$\Rightarrow \lambda = \dfrac{\bar{x}^T Ax}{\bar{x}^T x}$. Where $\bar{x}^T x = |x_1^2| + |x_2^2| + - - - + |x_n^2|$, is a real number.

Now λ and the numerator on the right hand side being numbers, the numerator on the right hand side is also a number. The proof will be over if show that $\bar{x}^T Ax$ is real, for which we show that it is equal to its conjugate.

Now by the property of transpose, the transpose of a number is equal to itself.

$$(\bar{x}^T Ax) = (\bar{x}^T Ax)^T = x^T A^T \bar{x} = x^T \overline{A}\, \bar{x} = \overline{(\bar{x}^T Ax)}$$

$\Rightarrow \bar{x}^T Ax$ is real.

1. The Eigen values of a skew – hermitian matrix are pure imaginary or zero.

Solution: Given that matrix A is Hermitian so that $\overline{A}^T = -A \Rightarrow A^T = -\overline{A}$

Now, $\lambda = \dfrac{\overline{x}^T A x}{\overline{x}^T x}$, where $\overline{x}^T x = |x_1^2| + |x_2^2| + - - - +|x_n^2|$, is a real number.

Now λ and the numerator on the right hand side being numbers, the numerator on the right hand side is also a number. The proof will be over if show that $\overline{x}^T A x$ is pure imaginary or zero, for which we show that it is equal to its negative of its conjugate.

Now by the property of transpose, the transpose of a number is equal to itself.

$(\overline{x}^T A x) = (\overline{x}^T A x)^T = x^T A^T \overline{x}$

$= x^T (-\overline{A})\,\overline{x} = -(\overline{\overline{x}^T A x})$

$\Rightarrow \overline{x}^T A x$ is pure imaginary.

2. The Eigen values of a unitary matrix are such that their absolute value is 1.

Solution: Let A be unitary, so that $\overline{A}^T = A^{-1}$.

Let λ be the Eigen value of A so that we have the equation: Ax = λ x - - (1)

Now $\overline{(Ax)}^T = \overline{(\lambda x)}^T \Rightarrow \overline{x}^T \overline{A}^T = \overline{\lambda} \; \overline{x}^T$ - - - (2)

Equation (2) x (1) : $\overline{x}^T \overline{A}^T A x = \lambda x \; \overline{\lambda} \; \overline{x}^T = \lambda \overline{\lambda} \; x \; \overline{x}^T = |\lambda|^2 x \; \overline{x}^T$.

$\Rightarrow \overline{x}^T A^{-1} A x = |\lambda|^2 x \; \overline{x}^T$.

$\Rightarrow \overline{x}^T x = |\lambda|^2 x \; \overline{x}^T$.

$$\Rightarrow |\lambda|^2 = \frac{\overline{x}^T x}{\overline{x}^T x} = 1$$

$\Rightarrow |\lambda| = 1$.

2.5 QUADRATIC FORMS:

If **A** is a square matrix and **x** is a column vector then $x^T Ax = Q$, is called a quadratic form.

For **A** = (aij)$_{n \times n}$ and **x** = [x_1 x_2 - - - x_n], we have

$x^T Ax = a_{11}x_1^2 + +a_{12}x_2^2 + - - - + a_{1n}x_n^2$

$+ a_{21}x_1^2 + a_{22}x_2^2 + - - - + a_{2n}x_n^2$

- -

$+ a_{n1}x_1^2 + a_{n2}x_2^2 + - - - + a_{nn}x_n^2$.

Note:

1. If A is Hermitian / skew - Hermitian and x is complex, the above quadratic form is called Hermitian / skew - Hermitian form.

2. The value of a hermitian form is always real and that of a skew-hermitian form is pure imaginary or zero irrespective of the choice of x.

DEFINITIONS:

1. For complex vectors with n components , the inner product of two such vectors is defined by $\mathrm{a} \bullet \mathrm{b} = \bar{\mathrm{a}}^{T} b$.

2. The **length or norm** of such a complex vector is defined by

$\| a \| =$
$\sqrt{a \bullet a} = \sqrt{a^T a} = \sqrt{| a_1 |^2 + | a_2 |^2 + --- + | a_n |^2}$.

A few Important Results:

1. Unitary transformation y = Ax, where A is a unitary matrix preserves the value of the inner product defined by

Proof: We show that if u and v are two orthogonal transformations then **u** · **v** = **a** · **b**.

Let **u** = A**a** and **v** = A**b** be two orthogonal transformations, where A is an unitary matrix, so that $\overline{\mathrm{A}}^{T} = A^{-1}$

Now by the definition of inner product

u · **v** = $\overline{\mathrm{u}}^{T} v$ = $\overline{(\mathrm{Aa})}^{T}$ (A**b**) = $\overline{\mathrm{a}}^{T}\,\overline{\mathrm{A}}^{T}$ A**b** = $\overline{\mathrm{a}}^{T}$ A^{-1}A**b** = $\overline{\mathrm{a}}^{T}$ **b = a · b.**

2. The absolute value of the determinant of a Unitary matrix is always 1.

3. The inverse of a unitary matrix is unitary.

4. If a matrix is such that $A\overline{A}^{T} = \overline{A}^{T}A$, then A is said to be a **normal matrix**. Hermitian, skew- Hermitian and unitary matrices are normal.

2.6 SIMILARITY OF MATRICES AND DIAGONALISATION OF A MATRIX: A

transformation, which gives $\hat{A}$ from A through the relation $P^{-1}AP$ is called a similarity transformation, and $\hat{A}$ is called similar to A. Here $\hat{A}$ **= $P^{-1}AP$**.

Related Result:

$\hat{A}$ has the same Eigen values as that of A and $y = P^{-1}x$ is the Eigen vector of $\hat{A}$ corresponding to the same Eigen value.

DIAGONALISATION:

Let A be the given square matrix, which is to be diagonalised.

Step –1 : Find the Eigen values of A & find the corresponding linearly independent Eigen vectors.

Step –2 : Construct the matrix X, with its columns as the above Eigen vectors. Find its inverse.

Step – 3 : The required diagonal matrix is $D = X^{-1}AX$. Verify that the diagonal elements of D are the Eigen values of the given matrix A.

Example: Reduce the following matrix into diagonal form $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$.

Solution: The **c**haracteristics equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -19-\lambda & 7 \\ -42 & 16-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-19-\lambda)(16-\lambda)+294=0$$

$$\Rightarrow \lambda = 2,-5$$

For $\lambda = 2$, the Eigen vector is $\begin{bmatrix}1\\3\end{bmatrix}$ and for $\lambda = -5$ the Eigen vector is $\begin{bmatrix}1\\2\end{bmatrix}$

$$\therefore X = \begin{bmatrix}1 & 1\\3 & 2\end{bmatrix},\ |X| = 2-3=-1,$$

$$X^{-1} = \frac{\begin{bmatrix}2 & -1\\-3 & 1\end{bmatrix}}{-1} = \begin{bmatrix}-2 & 1\\3 & -1\end{bmatrix}$$

D = $X^{-1}AX = \begin{bmatrix}2 & 0\\0 & -5\end{bmatrix}$.

2.7 TRANSFORMATION OF FORMS TO PRINCIPAL AXES:

In this section we learn to reduce each quadratic form into some conic sections or some forms in the principal axes.

Let Q = x^TAx = Given number be the given quadratic form.

Step –1: Generate the corresponding symmetric coefficient matrix from the given quadratic form.

Step –2 : Find the Eigen values of this symmetric matrix. Let they be λ_1 and λ_2.

Step –3 : The required conic section is given by the relation

$\lambda_1 y_1^2 + \lambda_2 y_2^2$ = Given number.

NOTE: The Conic section may be an ellipse, an hyperbola or a pair of straight lines.

Example: Transform the quadratic form

$4x^2 + 12xy + 13y^2 = 16$ into the principal axes.

Solution: - $4x^2 + 12xy + 13y^2 = 16$

$$Q = 4x^2 + 12xy + 13y^2 = X^T AX$$

Where $A = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$

Characteristics equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 52 - 4\lambda - 13\lambda + \lambda^2 - 36 = 0$$

$$\Rightarrow \lambda^2 - 17\lambda + 16 = 0$$

$$\Rightarrow \lambda = 1,16$$

$$Q = \lambda_1 x^2 + \lambda_2 y^2 = 16$$

$$\Rightarrow x^2 + 16y^2 = 16$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{1} = 1$$,which represents an ellipse.

SOME DEFINITIONS:

1. The sum of the principal diagonal elements of a matrix is called trace the Trace of the matrix.

2. Trace of a matrix A is equal to the sum of the its Eigen values.

3. Similar matrices have equal traces.

CHAPTER 3

3. VECTOR DIFFERENTIAL CALCULUS:

3.1 INTRODUCTION TO SOME VECTOR TERMINOLOGY:

A. Scalar Quantity: A quantity having magnitude but no direction is called a scalar quantity, such as mass, temperature, volume, density etc.

B. Vector Quantity: A quantity having both magnitude and direction is called a vector quantity, such as velocity, displacement, force, acceleration etc. Vectors are represented by letters with an arrow over it, like $\vec{a}, \vec{b}, \vec{c}$ - - - etc., whereas scalars are written as ordinary letters without arrows.

C. Null Vector: A vector whose magnitude is zero is called a *null vector* or a zero vector, $\vec{o}$.

D. Unit Vector: A vector whose magnitude is unity is called a *unit vector.* A unit vector is denoted by the sign "^" over it. Thus $\hat{a}$ is a *unit vector* in the direction of the vector $\vec{a}$. Also a *unit vector* in the direction of a vector is obtained by dividing the vector by its magnitude.

Therefore, the *unit vector* along the vector, $\vec{a}$, is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}.$$

Unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$:

Another popular representation of vectors is $\vec{a}$ is

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

In this representation, are the unit vectors in the positive direction of the axes of a Cartesian coordinate system.

LIKE AND UNLIKE VECTORS: Vectors having same direction are called *like vectors* and vectors having opposite direction are called *unlike vectors*.

EQUAL VECTORS: Two or more vectors are said to be *equal* if they have the same magnitude and same direction.

COLLINEAR OR PARALLEL VECTORS: Vectors having the same line of action or having lines of action parallel to the same vector are called *collinear* or *parallel* vectors.

COPLANAR VECTORS: The vectors which lie in the same plane or are parallel to the same plane are called *coplanar vectors*.

3.2 VECTOR ALGEBRA:

A. Vector Addition: The sum or resultant of two vectors $\vec{a}$ and $\vec{b}$ is a vector obtained by placing the initial point of $\vec{b}$ on the terminal point of $\vec{a}$ and then joining the initial point of $\vec{a}$ to the terminal point of $\vec{b}$.

B. Scalar Multiplication: The product **c** of any vector $\vec{a}$ and any scalar **c** is the vector obtained by multiplying each component of $\vec{a}$ by **c**.

$\mathbf{c}\vec{a} = [\,\mathbf{c}a_1 , \mathbf{c}a_2 , \mathbf{c}a_3\,]$.

Geometrically, $\mathbf{c}\vec{a}$ with **c** > 0 has the same direction as that of $\vec{a}$ and with **c** < 0 the direction is opposite to that of $\vec{a}$.

B. Laws of Vector Algebra: If $\vec{a}, \vec{b}$ and $\vec{c}$ are vectors and m & n are scalars, then

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ Commutative Law for Addition

2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ Associative Law for Addition

3. $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$ Distributive Law

4. $(m + n)\vec{a} = m\vec{a} + n\vec{b}$ Distributive Law

Scalar Product / Inner Product of Two Vectors:

The scalar product or dot product of two vectors $\vec{a}$ and $\vec{b}$ with magnitude a and b respectively, is defined as $\vec{a} \cdot \vec{b} = a\,b\cos\theta = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the two vectors $\vec{a}$ *and* $\vec{b}$.

In component form $\vec{a} \,.\, \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

PROPERTIES OF SCALAR PRODUCT / INNER PRODUCT:

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ **(Scalar Product is Commutative)**

2. $(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (m\vec{b})$, where m is a scalar **(Scalar Product is Associative with respect to a scalar)**

3. $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$ **(Distributive Law)**

4. $(m + n)\vec{a} = m\vec{a} + n\vec{a}$ **(Distributive Law)**

5. If the vectors $\vec{a}$ *and* $\vec{b}$ are mutually perpendicular then $\vec{a} \cdot \vec{b} = 0.$,

since $\cos\frac{\pi}{2} = 0.$ Here $\vec{a}$ *and* $\vec{b}$ are said to be orthogonal to each other.

In particular, $\hat{i} \cdot \hat{j} = 0$, $\hat{j} \cdot \hat{k} = 0$ and $\hat{k} \cdot \hat{i} = 0$.

3.3 APPLICATIONS OF SCALAR PRODUCT:

1. Length and Angle in terms of dot / inner product:

If $\vec{a} = \vec{b}$ then $\vec{a} \cdot \vec{b}$ = |a| ^{2}.

Hence |a| = $\sqrt{\vec{a} \cdot \vec{a}}$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{\sqrt{\vec{a}\cdot\vec{a}}\sqrt{\vec{b}\cdot\vec{b}}}.$$

2. THE PROJECTION of vector $\vec{a}$ in the direction of a vector $\vec{b}$ is: p = |a| cosθ = $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

Example: Find the projection of $\vec{a}$ on $\vec{b}$ if $\vec{a} = 12\hat{i} - 3\hat{j} + 6\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} + 4\hat{k}$.

Solution: Scalar projection of $\vec{a}$ on $\vec{b}$ = $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)$

$$= \frac{(12\hat{i} - 3\hat{j} + 6\hat{k})\cdot(2\hat{i} + 4\hat{j} + 4\hat{k})}{\sqrt{4+16+16}}$$

$$= \frac{24-12+24}{\sqrt{36}} = 6.$$

3. ORTHOGONAL: THE inner product of two non-zero vectors is zero if and only if these vectors are perpendicular.

4. WORK DONE BY A FORCE: Let a constant force

$\vec{p}$ acts on a body and let it be given a displacement, $\vec{d}$ .Now the work done by the force $\vec{p}$ in the displacement is defined as: ***W*** = $|\vec{p}|\,|\vec{d}|\cos\alpha$., where α. is the angle between $\vec{p}$ and $\vec{d}$.

3.4 VECTOR PRODUCT (CROSS PRODUCT):

The vector or cross product of two vectors $\vec{a}$ *and* $\vec{b}$ is defined as $\vec{a}\ X\ \vec{b} = ab\sin\theta\,\hat{n}$, where θ is the angle between $\vec{a}$ *and* $\vec{b}$ and $\hat{n}$ is the unit vector normal to the plane containing the vectors $\vec{a}$ *and* $\vec{b}$. It clearly that $|\vec{a} \times \vec{b}| = ab\sin\theta$ and therefore, $\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$.

In component form , if where $\vec{a}$ = [a_1 a_2 a_3] and $\vec{b}$ = [b_1 b_2 b_3] then

$\vec{a}\ X\ \vec{b}$ =(a_2b_2–b_2a_2) $\hat{i}$ – (a_1b_3–b_1a_3) $\hat{j}$ +(a_1b_2 – b_1a_2) $\hat{k}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note: $\hat{i} \times \hat{j} = \hat{k}$, $\hat{k} \times \hat{i} = \hat{j}$, $\hat{j} \times \hat{k} = \hat{i}$.

Also $\hat{j} \times \hat{i} = -\hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}$, $\hat{i} \times \hat{k} = -\hat{j}$.

Properties of Vector Product:

It is distributive with respect to vector addition.

1. $\vec{a}\ X\ (\vec{b} + \vec{c}) = \vec{a}\ X\ \vec{b} + \vec{a}\ X\ \vec{c}$.

2. $(\vec{a} + \vec{b})X\ \vec{c} = (\vec{a}\ X\ \vec{c}) + (\vec{b}\ X\ \vec{c})$.

3. It is anti-commutative, that is $\vec{a}\ X\ \vec{b} = -\vec{b}\ X\ \vec{a}$

4. It is not associative, $(\vec{a}\ X\ \vec{b})X\ \vec{c} \neq \vec{a}\ X(\vec{b}\ X\ \vec{c})$.

Scalar Triple Product : The scalar triple product of 3 – vectors $\vec{a}$ =[a_1 a_2 a_3], $\vec{b}$ = [b_1 b_2 b_3] and $\vec{c}$ = [c_1 c_2 c_3] is defined by $(\vec{a}\ \vec{b}\ \vec{c}) = \vec{a} \cdot (\vec{b}\ X\ \vec{c})$ and

$$(\vec{a}\ \vec{b}\ \vec{c}) = \vec{a} \cdot (\vec{b}\ X\ \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

PROPERTIES:

1. $(\vec{a}\ \vec{b}\ \vec{c}) = \vec{a} \cdot (\vec{b}\ X\ \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c}\ X\ \vec{a})$,

the value of the scalar triple product remains unchanged if the cyclic order of the vectors remains unchanged.

2. $(k\vec{a}\ \vec{b}\ \vec{c}) = k(\vec{a}\ \vec{b}\ \vec{c})$, where k is any scalar.

3. It may be observed that

$(\hat{i}\ \hat{j}\ \hat{k}) = (\hat{j}\ \hat{k}\ \hat{i}) = (\hat{k}\ \hat{i}\ \hat{j}) = 1.$

3.5 APPLICATIONS OF VECTOR PRODUCT:

1. The volume of the tetrahedron whose three co terminus edges are given by $\vec{a}, \vec{b}$ and $\vec{c}$, being $\frac{1}{6}$ th of the volume of the parallelepiped, is $\frac{1}{6}(\vec{a}\ \vec{b}\ \vec{c})$.

2. Linear Independence of three vectors: Three vectors are said to form a linearly independent set if their scalar triple product is non-zero, that is if $(\vec{a}\ \vec{b}\ \vec{c}) \neq 0$.

3.6 VECTOR VALUED FUNCTION:

If t is a scalar variable and if to each value of t in some interval there corresponds a value of a vector $\vec{V}$, we say that $\vec{V}$ is a vector function and we write

$\vec{V} = \vec{V}(t) = [v_1(t)\ \ v_2(t)\ \ v_3(t)]$

Derivative of a vector function is

$$\frac{d\vec{V}}{dt} = [v_1'(t)\ v_2'(t)\ \ v_3'(t)].$$

Partial Derivatives of a Vector Function:

If $\vec{V} = [v_1\ \ v_2\ \ v_3]$ are differentiable function of n – variables $t_1, t_2, - - - t_n$, then the partial derivative of

$\vec{V}$ with respect to t_1 is defined as :

$$\frac{\partial \vec{V}}{\partial t_1} = \frac{\partial v_1}{\partial t_1}\hat{i} + \frac{\partial v_2}{\partial t_1}\hat{j} + \frac{\partial v_3}{\partial t_1}\hat{k}.$$

3.7 GRADIENT OF A SCALAR FIELD:

Let the scalar function f(x, y, z) be continuous and differentiable, then the gradient of f(x, y, z) , written as grad f, is a vector, defined as: grad f = $\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$.

Let us introduce the differential operator

∇ = $\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$, read as **del** or **nabla**,

so that ∇f = $\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$.

APPLICATIONS:

Directional Derivative: The directional derivative of f at any given point P in the direction of a given vector $\vec{a}$ is the rate of change of 'f' at P in the direction of the vector $\vec{a}$.

It is defined as $D_a f$ = $\frac{1}{|\vec{a}|}\vec{a}\cdot grad\ f$.

Example: Find the directional derivative of function $\phi(x, y, z) = x^2 + y^2 - 2z^2$ at the point (2, -2, 2) in the direction of $2\hat{i}-4\hat{j}-4\hat{k}$.

Solution: $grad(\phi)=2x\hat{i}+2y\hat{j}-4z\hat{k}$

$grad(\phi)$ at (2, -2, 2) = $=4\hat{i}-4\hat{j}-8\hat{k}$

$$D_a\phi=\frac{\vec{a}}{|\vec{a}|}\cdot grad(\phi) = \frac{(2\hat{i}-4\hat{j}-4\hat{k})\cdot(4\hat{i}-4\hat{j}-8\hat{k})}{\sqrt{4+16+16}}$$

$$=\frac{8+16+32}{6}=\frac{28}{3}.$$

2. Characterization of Maximum increase by Gradient: If the gradient of f at any point P is not the zero vector, then it has the direction of maximum increase of f at P.

3. Gradient as Surface Normal Vector: If a surface S :f(x,y,z) = constant is such that f is differentiable and if at a point P on S grad f is not zero then it is a normal vector of S at P. So that the unit normal vector at P is $\frac{1}{|\,grad\,\mathrm{f}\,|}grad\,\mathrm{f}$, where grad f is calculated at P.

4. Vector fields given by vector $\vec{V}$ (P) is obtained as the gradient of a scalar valued function, say, $\vec{V}$ (P)= grad f(P). Here f(P) is called a potential function.

5. The **Laplace equation** $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$

$= \nabla^2 f = \nabla\cdot\nabla f.$

Here the **Laplacian** of f is denoted by $\nabla^2 f$ or Δf.

The differential operator

$\nabla^2 = \Delta = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$, is read as " nabla squared" or "delta".

3.8 DIVERGENCE OF A VECTOR FUNCTION:

Let $\vec{V}$ = [v_1 v_2 v_3] be a vector function of variables x, y and z and is differentiable. Then div $\vec{V}$ = $\nabla\cdot\vec{V}$ = $\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$ .is called the divergence of $\vec{V}$.

Note : The **Laplacian** of f, denoted by $\nabla^2 f$ can also be stated as div (grad f) = $\nabla\cdot\nabla f = \nabla^2 f$.

APPLICATIONS:

1. Condition of Incompressibility: Let $\vec{V}$ be the velocity vector of a fluid motion. The flow is **incompressible** if **div** $\vec{V}$ **= 0**.

Example: Test whether the vector v = (x, y, -z) is incompressible.

Solution:

$$div\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(-z) = 1 + 1 - 1 = 1 \neq 0.$$

So the given set of vectors are not incompressible.

3.9 CURL OF A VECTOR FUNCTION:

The curl of a vector valued function $\vec{V}$ = [v_1 v_2 v_3] is defined as : Curl $\vec{V}$ = ∇ x $\vec{V}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$.

$$= \left(\frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)\hat{i} + \left(\frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)\hat{j} + \left(\frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)\hat{k}.$$

APPLICATIONS:

1. The curl of a vector field is connected with rotational properties of the vector field and justifies the name rotation used as curl. So **rot** $\vec{V}$ is also used as notation for **curl** $\vec{V}$.

2. Since curl (grad f) = 0, we say that gradient fields describing a motion are **irrotational**. .

3. A fluid motion given by vector is irrotational if Curl $\vec{V}$ = 0.

Some useful results of Grad, Div, Curl:

1. $\nabla (f^{n}) = n f^{n-1} \nabla f$

2. div (f$\vec{V}$) = f div$\vec{V}$ + $\vec{V} \cdot \nabla$ f.

3. div (f∇g) = f∇^2g + ∇f·∇g

4. div (f∇g) – div (g∇f) = f ∇^2 g – g∇^2f.

5. curl (f$\vec{V}$) = (grad f) x $\vec{V}$ + f curl $\vec{V}$.

6. div ($\vec{U}$ x$\vec{V}$) = $\vec{V}$ · curl $\vec{U}$ + $\vec{U}$ · curl $\vec{V}$

7. div (curl $\vec{V}$) = 0 , curl (grad f) = 0.

VECTOR DIFFERENTIAL CALCULUS:

EXAMPLES & SOLUTION IN GRAD, DIV & CURL

1. Find the first partial derivatives of the vector function $\vec{V} = \mathrm{xy}\hat{\mathrm{i}} + \mathrm{yz}\hat{\mathrm{j}} + \mathrm{zx}\hat{\mathrm{k}}$.

Solution:

$$\frac{\partial \vec{V}}{\partial x} = \frac{\partial}{\partial \mathrm{x}}(\mathrm{xy}\hat{\mathrm{i}}) + \frac{\partial}{\partial \mathrm{x}}(\mathrm{yz}\hat{\mathrm{j}}) + \frac{\partial}{\partial \mathrm{x}}(\mathrm{zx}\hat{\mathrm{k}}) = \mathrm{y}\hat{\mathrm{i}} + \mathrm{z}\hat{\mathrm{k}}$$

$$\frac{\partial \vec{V}}{\partial y} = \frac{\partial}{\partial \mathrm{y}}(\mathrm{xy}\hat{\mathrm{i}}) + \frac{\partial}{\partial \mathrm{y}}(\mathrm{yz}\hat{\mathrm{j}}) + \frac{\partial}{\partial \mathrm{y}}(\mathrm{zx}\hat{\mathrm{k}}) = \mathrm{x}\hat{\mathrm{i}} + \mathrm{z}\hat{\mathrm{j}}$$

$$\frac{\partial \vec{\mathrm{V}}}{\partial \mathrm{z}} = \frac{\partial}{\partial \mathrm{z}}(\mathrm{xy}\hat{\mathrm{i}}) + \frac{\partial}{\partial \mathrm{z}}(\mathrm{yz}\hat{\mathrm{j}}) + \frac{\partial}{\partial \mathrm{z}}(\mathrm{zx}\hat{\mathrm{k}}) = \mathrm{y}\hat{\mathrm{j}} + \mathrm{x}\hat{\mathrm{k}}$$

2. Let, $\vec{V} = \mathrm{yz}\hat{\mathrm{i}} + 3\mathrm{zx}\hat{\mathrm{j}} + \mathrm{z}\hat{\mathrm{k}}$. Find curl $\vec{V}$

Solution: $$culr\vec{V} = \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ \frac{\partial}{\partial \mathrm{x}} & \frac{\partial}{\partial \mathrm{y}} & \frac{\partial}{\partial \mathrm{z}} \\ \mathrm{yz} & 3\mathrm{zx} & \mathrm{z} \end{vmatrix}$$

$= \hat{i}(0-3x) - \hat{j}(0-y) + \hat{k}(3z-z)$

$= -3x\hat{i} + y\hat{j} + 2z\hat{k}$.

3. Let $\vec{F} = x^3\hat{i} + x^2y\hat{j} + x^2z\hat{k}$. Find $div\vec{F}$

Solution: $Div\vec{F} =$

$$\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial x}\hat{j} + \frac{\partial}{\partial x}\hat{k}\right).\left(x^3\hat{i} + x^2y\hat{j} + x^2z\hat{k}\right)$$

$$= \left(\frac{\partial}{\partial x}x^3 + \frac{\partial}{\partial x}x^2y + \frac{\partial}{\partial x}x^2z\right)$$

4. Prove that $div(\vec{u}\times\vec{v}) = \vec{v}.curl\vec{u} - \vec{u}.curl\vec{v}$.

Solution: $div\left(\vec{u}\times\vec{v}\right) = \nabla\cdot(\vec{u}\times\vec{v})$

$$= \sum \hat{i}\cdot\frac{\partial}{\partial x}(\vec{u}\times\vec{v})$$

$$= \sum \hat{i}\cdot\left(\frac{\partial\vec{u}}{\partial x}\times\vec{v} + \vec{u}\times\frac{\partial\vec{v}}{\partial x}\right)$$

$$= \sum \hat{i}\cdot\frac{\partial\vec{u}}{\partial x}\times\vec{v} + \sum \hat{i}\cdot\vec{u}\times\frac{\partial\vec{v}}{\partial x}$$

$$= \sum \hat{i}\times\frac{\partial\vec{u}}{\partial x}\cdot\vec{v} - \sum \hat{i}\times\frac{\partial\vec{v}}{\partial x}\cdot\vec{u}$$

$= curl\left(\vec{u}\right)\cdot\vec{v} - culr\left(\vec{v}\right).\vec{u}$.

5. Find the directional derivative of

$f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ at a point P(3,0,4) in the

direction of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.

Solution: Given, $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, so that

$$grad(f) = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{i} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(\frac{-1}{2}\right)\frac{-2x}{\left(x^2 + y^2 + z^2\right)^{3/2}}\hat{i} + \left(\frac{-1}{2}\right)\frac{-2y}{\left(x^2 + y^2 + z^2\right)^{3/2}}\hat{j}$$

$$+ \left(\frac{-1}{2}\right)\frac{-2z}{\left(x^2 + y^2 + z^2\right)^{3/2}}\hat{k}$$

$$= \frac{-x}{\left(x^2 + y^2 + z^2\right)^{3/2}}\hat{i} + \frac{-y}{\left(x^2 + y^2 + z^2\right)^{3/2}}\hat{j}$$

$$+ \frac{-z}{\left(x^2 + y^2 + z^2\right)^{3/2}}\hat{k}$$

$= \dfrac{-3}{125}\hat{i} + 0\hat{j} + \dfrac{-4}{125}\hat{k}$ at P(3, 0, 4).

And the directional derivative in the direction of given vector is: $D_a f = \dfrac{\vec{a}}{|\vec{a}|} \cdot grad(f)$

$$= \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}) \cdot \frac{-3}{125}\hat{i} + 0\hat{j} + \frac{-4}{125}\hat{k}$$

$$= \frac{1}{\sqrt{3}}\left(\frac{-3}{125} + \frac{-4}{125}\right)$$

$$= \frac{-7}{125\sqrt{3}}.$$

6. Find a unit vector perpendicular to both $\vec{a} = 3\hat{i} - 2\hat{j}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Solution: $\vec{a} \times \vec{b} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 0 \\ 1 & -2 & 1 \end{bmatrix} = -2\hat{i} - 3\hat{j} - 4\hat{k}$

$\Rightarrow \left|\vec{a} \times \vec{b}\right| = \left|-2\hat{i} - 3\hat{j} - 4\hat{k}\right| = \sqrt{4+9+16} = \sqrt{29}$

$\therefore \hat{n} = \dfrac{\vec{a} \times \vec{b}}{\left|\vec{a} \times \vec{b}\right|} = \dfrac{-2\hat{i} - 3\hat{j} - 4\hat{k}}{\sqrt{29}}$.

7. Find the unit vector normal to the surface $x^2y + yz = 6$ at the point (2,3,-2).

Solution: $x^2y + yz = 6$

$\Rightarrow x^2y + yz - 6 = 0$

$f = x^2y + yz - 6$

$grad(f) = 2xy\hat{i} + (x^2 + z)\hat{j} + y\hat{k}$

$grad(f)$ at (2, 3, -2) $= 6\hat{i} + 2\hat{j} + 3\hat{k}$

Unit normal vector $= \dfrac{grad(f)}{|grad(f)|} = \dfrac{6\hat{i} + 2\hat{j} + 3\hat{k}}{\left|6\hat{i} + 2\hat{j} + 3\hat{k}\right|}$

$= \dfrac{6\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{36+4+9}} = \dfrac{6\hat{i} + 2\hat{j} + 3\hat{k}}{7}$

8. Determine the vector $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$ is solenoidal and/or irrotational.

Solution: $culr\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$

So vectors are not irrotational.

$$div\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

So it is not solenoidal.

9. Find the projection of $\vec{a}$ on $\vec{b}$ if $\vec{a} = 12\hat{i} - 3\hat{j} + 6\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} + 4\hat{k}$.

Solution: Scalar projection of $\vec{a}$ on $\vec{b}$ = $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)$

$$= \frac{\left(12\hat{i} - 3\hat{j} + 6\hat{k}\right)\cdot\left(2\hat{i} + 4\hat{j} + 4\hat{k}\right)}{\sqrt{4+16+16}}$$

$$= \frac{24-12+24}{\sqrt{36}} = 6.$$

Vector projection of $\vec{a}$ on $\vec{b}$ = $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\left(\dfrac{\vec{b}}{|\vec{b}|}\right)$

$$= 6\frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4+16+16}}$$

$$= \frac{6\left(2\hat{i} + 4\hat{j} + 4\hat{k}\right)}{6} = \left(2\hat{i} + 4\hat{j} + 4\hat{k}\right).$$

10. Find $curl(xy\hat{i} + y^2\hat{j} + xz\hat{k})$ at (-2, 4, 1).

Solution: $\vec{F} = xy\hat{i} + y^2\hat{j} + xz\hat{k}$

$$culr\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 & xz \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(z-0) + \hat{k}(0-y) = -z\hat{j} - y\hat{k}$$

$$culr\vec{F}\Big]_{(-2,4,1)} = -1\hat{j} - 4\hat{k} = -\hat{j} - 4\hat{k}$$

11. Determine the scalar point function ϕ if it exists for $\vec{u} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$.

Solution: $grad(\phi) = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

$$\Rightarrow \left(\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} \right) = \left(x^2\hat{i} + y^2\hat{j} + z^2\hat{k} \right)$$

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} + \text{constant.}$$

12. Find a unit vector perpendicular to $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = 4\hat{i} - 2\hat{j}$.

Solution: $\vec{a} \times \vec{b} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & -1 \\ 4 & -2 & 0 \end{bmatrix} = -2\hat{i} - 4\hat{j} - 8\hat{k}$

$$\Rightarrow \left| \vec{a} \times \vec{b} \right| = \left| -2\hat{i} - 4\hat{j} - 8\hat{k} \right| = \sqrt{4+16+64} = 2\sqrt{21}$$

$$\therefore \hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{-2\hat{i} - 4\hat{j} - 8\hat{k}}{2\sqrt{21}} = \frac{-\hat{i} - 2\hat{j} - 4\hat{k}}{\sqrt{21}}$$

13. Find the directional derivative of function $\phi(x, y, z) = x^2 + y^2 - 2z^2$ at the point (2, -2, 2) in the direction of $2\hat{i} - 4\hat{j} - 4\hat{k}$.

Solution: $grad(\phi) = 2x\hat{i} + 2y\hat{j} - 4z\hat{k}$

$grad(\phi)$ at (2, -2, 2) = $= 4\hat{i} - 4\hat{j} - 8\hat{k}$

$$D_a\phi = \frac{\vec{a}}{|\vec{a}|} \cdot grad(\phi) =$$

$$\frac{(2\hat{i} - 4\hat{j} - 4\hat{k}) \cdot (4\hat{i} - 4\hat{j} - 8\hat{k})}{\sqrt{4 + 16 + 16}} = \frac{8 + 16 + 32}{6} = \frac{28}{3}$$

14. If $\phi(x, y, z) = x^2 + y^2 - z^2$, find the value of $curl(grad\phi)$ at any point (x, y, z).

Solution: $grad(\phi) = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$

$$curl(grad(\phi)) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & -2z \end{vmatrix} = \vec{0}$$

15. Find the volume of the tetrahedron whose vertices are (1,3,6), (3,7,12), (8,8,9), (2,2,8).

Solution: O(1, 3, 6), A(3, 7, 12), B(8, 8, 9), C(2, 2, 8)

$\overrightarrow{OA} = \vec{a} = (2,4,6)$, $\overrightarrow{OB} = \vec{b} = (7,5,3)$, $\overrightarrow{OC} = (1,-1,2)$

Volume = $\frac{1}{6}\left[\vec{a},\vec{b},\vec{c}\right] = \left|\frac{1}{6}\begin{vmatrix}2 & 4 & 6\\ 7 & 5 & 3\\ 1 & -1 & 2\end{vmatrix}\right| = \left|\frac{-90}{6}\right| = 15$

cubic units.

16. Verify whether the following set of vectors is linearly independent. (4,2,9), (3,2,1), (-4,6,9).

Solution: $\begin{vmatrix}4 & 2 & 9\\ 3 & 2 & 1\\ -4 & 6 & 9\end{vmatrix} = 220$. So vectors are linearly independent.

17. Find $div\vec{F}$ and $curl\vec{F}$ at the point P(1,2,3) if $\vec{F} = x^2yz\hat{i} + xy^2z\hat{j} + xyz^2\hat{k}$.

Solution: $\vec{F} = x^2yz\hat{i} + xy^2z\hat{j} + xyz^2\hat{k}$

$$div\vec{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2)$$

= 2 xyz+2 xyz+ 2 xyz = 6 xyz

$$(div\vec{F})_{(1,2,3)} = 36$$

$$culr\vec{F} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2yz & xy^2z & xyz^2\end{vmatrix}$$

$$= x(z^2 - y^2)\hat{i} + y(x^2 - z^2)\hat{j} + z(y^2 - x^2)\hat{k}$$

$$\left(curl\vec{F}\right)_{(1,2,3)} = 5\hat{i} - 16\hat{j} + 9\hat{k}$$

18. Find the divergence of $\vec{u} = \frac{\vec{r}}{r^3}$ at (-2,4,1).

Solution: $\vec{u} = \frac{\vec{r}}{r^3} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}$

$$Div\vec{F} = \left(\frac{\partial}{\partial x}\left(\frac{x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial x}\left(\frac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial x}\left(\frac{z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)\right)$$

$$= \frac{\left(x^2+y^2+z^2\right)^{\frac{3}{2}} - \frac{3}{2}2x^2\left(x^2+y^2+z^2\right)^{\frac{1}{2}}}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} + \frac{\left(x^2+y^2+z^2\right)^{\frac{3}{2}} - \frac{3}{2}2y^2\left(x^2+y^2+z^2\right)^{\frac{1}{2}}}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} + \frac{\left(x^2+y^2+z^2\right)^{\frac{3}{2}} - \frac{3}{2}2z^2\left(x^2+y^2+z^2\right)^{\frac{1}{2}}}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}.$$

$$= \frac{\left(x^2+y^2+z^2\right)^{\frac{3}{2}} - 3\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}.$$

$$= \frac{-2\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} = -2$$

19. Find the curl of the vector field $\vec{u} = xy\hat{i} + y^2\hat{j} + xz\hat{k}$ at the point (-2,4,1).

Solution: $culr\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & y^2 & zx \end{vmatrix}$

$= \hat{i}(0-0) - \hat{j}(z-0) + \hat{k}(0-x)$

$= -z\hat{j} - x\hat{k}$

Hence, $curl\vec{V}\Big]_{(-2,4,1)} = -\hat{j} + 2\hat{k}$

20. Find a vector of length 10, which is perpendicular to $\vec{a} = 2\hat{i} + 3\hat{j}$ and $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$.

Solution: $\vec{a} \times \vec{b} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix} = 6\hat{i} - 4\hat{j} - 5\hat{k}$

$\Rightarrow \left|\vec{a} \times \vec{b}\right| = \left|6\hat{i} - 4\hat{j} - 5\hat{k}\right| = \sqrt{36+16+25} = \sqrt{77}$

$\therefore$ Vector of length 10 which is perpendicular to $\vec{a}, \vec{b}$ is

$$10\frac{\vec{a} \times \vec{b}}{\left|\vec{a} \times \vec{b}\right|} = \frac{6\hat{i} - 4\hat{j} - 5\hat{k}}{\sqrt{77}}$$

21. Find the directional derivative of function $f(x, y, z) = x^2 + 3y^2 + 4z^2$ at the point (1, 0, 1) in the direction of $-\hat{i} - \hat{j} + \hat{k}$.

Solution: $grad(f) = 2x\hat{i} + 6y\hat{j} + 8z\hat{k}$

$grad(f)$ at (1, 0, 1) $= 2\hat{i} + 8\hat{k}$

$$D_a f = \frac{\vec{a}}{|\vec{a}|} \cdot grad(f)$$

$$= \frac{(2\hat{i} + 8\hat{k}) \cdot (-\hat{i} - \hat{j} + \hat{k})}{\sqrt{1+1+1}} = \frac{-2+8}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

22. Find a scalar point function Φ such that $grad(\phi) = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution: $grad(\phi) = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow \left(\frac{\partial \varphi}{\partial x}\hat{i} + \frac{\partial \varphi}{\partial y}\hat{j} + \frac{\partial \varphi}{\partial z}\hat{k} \right) = \left(x\hat{i} + y\hat{j} + z\hat{k} \right)$$

$$\phi = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + \text{constant.}$$

23. Verify the identity $\nabla.\nabla \times \vec{u} = 0$ when $\vec{u} = e^x \sin y\hat{i} + e^x \cos y\hat{j} + e^x\hat{k}$.

Solution: $$\nabla \times \vec{U} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & e^x \cos y & e^x \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(e^x - 0) + \hat{k}(e^x \cos y - e^x \cos y)$$

$$= -\hat{j}e^x$$

$$\nabla \cdot (\nabla \times \vec{U}) =$$

$$\Rightarrow \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left(-e^x \hat{j} \right) = 0 + 0 + 0 = 0$$

1. Find the vector projection of $\vec{a} = (-3,4,7)$ in the direction of $\vec{b} = (2,5,2)$.

Solution: Vector projection of $\vec{a}$ on $\vec{b}$ = $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\left(\frac{\vec{b}}{|\vec{b}|}\right)$

= $\frac{-6+20+14}{\sqrt{4+25+4}} = \frac{2\hat{i}+5\hat{j}+2\hat{k}}{\sqrt{4+25+4}}$

= $\frac{28}{33}\left(2\hat{i}+5\hat{j}+2\hat{k}\right)$

2. Test whether the vector v = (x, y, -z) is irrotational and incompressible.

Solution: $culr\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & -z \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$

So the vectors are irrotational.

$div\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(-z) = 1+1-1 = 1 \neq 0$.

So the given set of vectors are not incompressible.

3. Find the unit normal vector at (2, 2, 3) to the surface $x^2 + y^2 + 2z^2 = 26$.

Solution: $f = x^2 + y^2 + 2z^2 - 26$

$grad(f) = 2x\hat{i} + 2y\hat{j} + 4z\hat{k}$

$grad(f)$ at (2, 2, 3) $= 4\hat{i} + 4\hat{j} + 12\hat{k}$

Unit normal vector $= \frac{grad(f)}{|grad(f)|} = \frac{4\hat{i}+4\hat{j}+12\hat{k}}{|4\hat{i}+4\hat{j}+12\hat{k}|}$

$$= \frac{4\hat{i}+4\hat{j}+12\hat{k}}{\sqrt{16+16+144}} = \frac{\hat{i}+\hat{j}+3\hat{k}}{\sqrt{11}}.$$

4. Find the angle between the two planes x + y + z = 8, 2x + y – z = 3.

Solution: x + y + z = 8, 2x + y – z = 3.

$\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}+\hat{j}-\hat{k}$

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}\right) = \cos^{-1}\left(\frac{2+1-1}{\sqrt{3}\cdot\sqrt{6}}\right) = \cos^{-1}\left(\frac{2}{3\sqrt{2}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{3}\right).$$

5. If $f(x, y, z) = x^2 + y^2 - z$, calculate Curl (grad f).

Solution: $grad(\phi) = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$$curl(grad(\phi)) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & -1 \end{vmatrix} = \vec{0}$$

6. If $f = x^2 + y^2 + z^2$ find the value of Curl (grad f).

Solution: $f = x^2 + y^2 + z^2$

$$\text{grad } f = 2x\hat{i} + 2\hat{j} + 2z\hat{k}$$

$$curl(grad\ f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & 2z \end{vmatrix}$$

$$= (0-0)\hat{i} + (0-0)\hat{j} + (0-0)\hat{k} = \vec{0}$$

7. Find the unit vector normal to the surface $z = x^2 + y^2$ at the point (1, 1, 2).

Solution: Here $z = x^2 + y^2$

$$\Rightarrow x^2 + y^2 - z = 0$$

$$\therefore f = x^2 + y^2 - z$$

$$grad(f) = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$grad(f)$ at (1, 1, 2) $= 2\hat{i} + 2\hat{j} - \hat{k}$

Unit normal vector $= \dfrac{grad(f)}{|grad(f)|} = \dfrac{2\hat{i} + 2\hat{j} - \hat{k}}{\left|2\hat{i} + 2\hat{j} - \hat{k}\right|}$

$$= \frac{2\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{4+4+1}} = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{3}$$

8. Show that $div(f\nabla g) = f\nabla^2 g + \nabla f . \nabla g$.

Solution: $div(f\nabla g) = \nabla \cdot (f\nabla g)$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(f\frac{\partial g}{\partial x}\hat{i} + f\frac{\partial g}{\partial y}\hat{j} + f\frac{\partial g}{\partial z}\hat{k}\right)$$

$$= \left(\frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)\right)$$

$$= f\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}\right)$$

$$= f\nabla^2 g + \nabla f \cdot \nabla g$$

9. Show that $Curl(Curl\vec{v}) = grad(div\vec{v}) - \nabla^2\vec{v}$

Solution: $curl(curl\vec{V}) = \nabla \times (\nabla \times \vec{V})$

$$= (\nabla \cdot \vec{V})\nabla - (\nabla \cdot \nabla)\vec{V}$$

$$= \nabla(\nabla \cdot \vec{V}) - \nabla^2\vec{V}$$

$$= grad(div\vec{V}) - \nabla^2\vec{V}$$

(a) $\left|\vec{a}+\vec{b}\right|^2 + \left|\vec{a}-\vec{b}\right|^2 = (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) + (\vec{a}-\vec{b}).(\vec{a}-\vec{b})$

$\because \left[\left|\vec{a}\right|^2 = \vec{a}.\vec{a}\right]$

$$= \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} + \vec{a}.\vec{a} - \vec{a}.\vec{b} - \vec{b}.\vec{a} + \vec{b}.\vec{b}$$

$$= 2\vec{a}.\vec{a} + 2\vec{b}.\vec{b}$$

$$= 2\left(\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2\right).$$

(b) O (1, 3, 6), A(3, 7, 12), B(8, 8, 9), C(2, 2, 8)

$\overrightarrow{OA} = \vec{a} = (2,4,6)$, $\overrightarrow{OB} = \vec{b} = (7,5,3)$, $\overrightarrow{OC} = (1,-1,2)$

Volume= $\frac{1}{6}\left[\vec{a},\vec{b},\vec{c}\right]$= $\left|\frac{1}{6}\begin{vmatrix} 2 & 4 & 6 \\ 7 & 5 & 3 \\ 1 & -1 & 2 \end{vmatrix}\right|$

$= \left|\dfrac{-90}{6}\right| = 15$ cubic units.

(c) Find the components of $\vec{a} = [4,0,-4]$ in the direction of $\vec{b} = [1,1,1]$.

b. If $\vec{a} = [4,0,-4]$, $\vec{b} = [1,1,1]$, $\vec{c} = [1,2,1]$ find $[\vec{a}\ \ \vec{b}\ \ \vec{c}]$.

c. Prove that curl(grad f) = 0.

a. If $\vec{a}$ and $\vec{b}$ are any two vectors prove that $\left|\vec{a}+\vec{b}\right|^2 + \left|\vec{a}-\vec{b}\right|^2 = 2\left(\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2\right)$.

b. Find the volume of the tetrahedron of its vertices are (1, 3, 6), (3, 7, 12), (8, 8, 9) and (2, 2, 8).

1.

(a) $$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$grad(f) = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{i} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(\frac{-1}{2}\right)\frac{-2x}{\left(x^2+y^2+z^2\right)^{3/2}}\hat{i} + \left(\frac{-1}{2}\right)\frac{-2y}{\left(x^2+y^2+z^2\right)^{3/2}}\hat{j}$$

$$+ \left(\frac{-1}{2}\right)\frac{-2z}{\left(x^2+y^2+z^2\right)^{3/2}}\hat{k}$$

$$= \frac{-x}{\left(x^2+y^2+z^2\right)^{3/2}}\hat{i} + \frac{-y}{\left(x^2+y^2+z^2\right)^{3/2}}\hat{j} + \frac{-z}{\left(x^2+y^2+z^2\right)^{3/2}}\hat{k}$$

$= \frac{-3}{125}\hat{i} + 0\hat{j} + \frac{-4}{125}\hat{k}$ at P(3, 0, 4)

$$D_a f = \frac{\vec{a}}{|\vec{a}|} \cdot grad(f)$$

$$= \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}) \cdot \frac{-3}{125}\hat{i} + 0\hat{j} + \frac{-4}{125}\hat{k}$$

$$= \frac{1}{\sqrt{3}}\left(\frac{-3}{125} + \frac{-4}{125}\right) = \frac{-7}{125\sqrt{3}}$$

a. Find the unit vector perpendicular to $\hat{i} + \hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} - \hat{k}$.

b. Find the directional derivatives of $\phi(x, y, z) = x^2 + y^2 - z^2$ at the point $(1,1,\sqrt{2})$ in the direction of $\hat{i} + \hat{j} + \hat{k}$.

c. Determine whether the vector is $x\hat{i} + y\hat{j} + z\hat{k}$ solenoidal; irrotational.

a.

(i) Find a scalar point function ϕ such that $grad(\phi) = 2x\hat{i} - y\hat{j} + z\hat{k}$.

(ii) Verify whether the vectors (-4, 6, 9), (3, 2, 1) and (4, 2, 9) are linearly independent.

(a) Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$

$$\Rightarrow \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & -1 \end{vmatrix}$$

$$= \hat{i}(-1+1) - \hat{j}(-1-2) + \hat{k}(-1-2) = 3\hat{j} - 3\hat{k}$$

So, unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is

$$\frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|} = \frac{3\hat{j}-3\hat{k}}{\left|3\hat{j}-3\hat{k}\right|} = \frac{3\hat{j}-3\hat{k}}{\sqrt{9+9}} = \frac{3\hat{j}-3\hat{k}}{3\sqrt{2}} = \frac{\hat{j}-\hat{k}}{\sqrt{2}} \; .$$

(b)

$$grad(\phi) = \nabla\phi = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{i} + \frac{\partial f}{\partial z}\hat{k} = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$$

$$grad(\phi)\big]_{(1,1,\sqrt{2})} = 2\hat{i} + 2\hat{j} - 2\sqrt{2}\hat{k}$$

$$D_a\phi = \frac{\vec{a}}{|\vec{a}|}\cdot grad(\phi)$$

$$= \frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})\cdot\left(2\hat{i}+2\hat{j}-2\sqrt{2}\hat{k}\right)$$

$$= \frac{2+2-2\sqrt{2}}{\sqrt{3}} = \frac{4-2\sqrt{2}}{\sqrt{3}}$$

(c) $div(\vec{v}) = 1+1+1 = 3 \neq 0$

So, It is not solenoidal.

$$curl(\vec{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}.$$

$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0) = \vec{0}$

So, it is irrotational.

(i) We have $grad(\phi) = 2x\hat{i} - y\hat{j} + z\hat{k}$.

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = -y, \frac{\partial f}{\partial z} = z$$

$$\Rightarrow \frac{\partial f}{\partial x} = 2x \Rightarrow f = x^2 + g(y,z)$$

$$\Rightarrow \frac{\partial f}{\partial y} = -y \Rightarrow \frac{\partial g(y,z)}{\partial y} = -y$$

$$\Rightarrow g(y,z) = -\frac{y^2}{2} + h(z)$$

$$\Rightarrow f = x^2 - \frac{y^2}{2} + h(z)$$

$$\Rightarrow \frac{\partial f}{\partial z} = z \Rightarrow h'(z) = z \Rightarrow h(z) = \frac{z^2}{2} + c$$

$$\Rightarrow f = x^2 - \frac{y^2}{2} + \frac{z^2}{2} + c$$

(ii) $\begin{vmatrix} -4 & 6 & 9 \\ 3 & 2 & 1 \\ 4 & 2 & 9 \end{vmatrix}$

$= -4(18-2) - 6(27-4) + 9(6-8)$

$= -64 - 138 - 18 = -210 \neq 0$

So, the vectors (-4, 6, 9), (3, 2, 1) and (4, 2, 9) are linearly independent.

CHAPTER 4

SOLVED EXAMPLES TO GROW CONFIDENCE

1. Find t so that (1-t, 0, 0), (1, 1-t, 0) and (1, 1, 1-t) are linearly dependent.

Solution: As they are linearly dependent, the determinant of the given vectors is zero.

$$\Rightarrow \begin{vmatrix} 1-t & 1 & 1 \\ 0 & 1-y & 1 \\ 0 & 0 & 1-t \end{vmatrix} = 0 \Rightarrow (1-t)^3 = 0 \Rightarrow t = 1.$$

2. Solve the following system of equations by Gauss elimination method:

- x + y + 2 z = 2, 3x – y + z = 6, -x + 3 y + 4z = 4.

Solution: Augmented matrix is $\left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 4 \\ -1 & 3 & 4 & 6 \end{array}\right]$

$$\begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - R_1 \end{array} = \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 2 & 2 & 2 \end{array}\right] R_3 \to R_3 - R_2$$

$$= \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

So $-x+y+2z=3$, $2y+7z=12$, $-5z=-10$

Solution is z = 2, y = -1, x = 1.

3. Find the rank of the matrix $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$

Solution: rank is the maximum no. of linearly independent rows in a matrix.

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - 3R_4 \\ R_3 \to R_3 + 3R_4 \end{matrix} \quad = \begin{bmatrix} 0 & -5 & -8 \\ 0 & 5 & 8 \\ 0 & 10 & 16 \\ 1 & 2 & 4 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 + 2R_1$

$$= \begin{bmatrix} 0 & -5 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

As two rows are linearly independent, so Rank of the matrix = 2.

4. Solve the following system of equation by Cramer's rule:
2x+7y+8z = -13, 2x+9z = -5, -4x+y-26z = 2.

Solution: $D = \begin{vmatrix} 3 & 7 & 8 \\ 2 & 0 & 9 \\ -4 & 1 & -26 \end{vmatrix} = 101 \neq 0$,

$$D_x = \begin{vmatrix} -13 & 7 & 8 \\ -5 & 0 & 9 \\ 2 & 1 & -26 \end{vmatrix} = -707$$

$$D_y = \begin{vmatrix} 3 & -13 & 8 \\ 2 & -5 & 9 \\ -4 & 2 & -26 \end{vmatrix} = 0,$$

$$D_z = \begin{vmatrix} 3 & 7 & -13 \\ 2 & 0 & -5 \\ -4 & 1 & 2 \end{vmatrix} = 101$$

So by Cramer's rule

$$x = \frac{D_x}{D} = \frac{-707}{101} = -7, \quad y = \frac{D_y}{D} = \frac{0}{101} = 0,$$

$$\mathrm{z} = \frac{\mathrm{D_z}}{\mathrm{D}} = \frac{101}{101} = 1$$

5. If $A=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$**, find the value of** $3A^2-9A+6I$ **where I is the 2 X 2 identity matrix.**

Solution:

$$3A^2-9A+6I=3\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}-9\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}+6\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$=3\begin{bmatrix}1 & 4\\0 & 1\end{bmatrix}-\begin{bmatrix}9 & 18\\0 & 9\end{bmatrix}+\begin{bmatrix}6 & 0\\0 & 6\end{bmatrix}$$

$$=\begin{bmatrix}3-9+6 & 12+-18+0\\0-0+0 & 3-9+6\end{bmatrix}=\begin{bmatrix}0 & -6\\0 & 0\end{bmatrix}$$

6. Find the rank of the matrix $A=\begin{bmatrix}3 & -1 & 5\\2 & -4 & 6\\5 & -5 & 11\end{bmatrix}$**.**

Solution: $\begin{vmatrix}3 & -1 & 5\\2 & -4 & 6\\5 & -5 & 11\end{vmatrix}=0$, So Rank $\neq 3$.

$\begin{vmatrix}3 & -1\\2 & -4\end{vmatrix}-12+2=-10\neq 0$, So Rank = 2.

7. Verify the linear dependence or independence of the following set of vectors: (1,0,1), (1,1,-1), (-

1,1,-3).Solution:

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -3 \end{vmatrix} = 1(-3+1) - 0 + 1(1+1) = 0$$

So vectors are linearly dependent.

8. Evaluate the determinant $\begin{vmatrix} 1 & 0 & 3 & 7 \\ 4 & 2 & 0 & 1 \\ 7 & 7 & 3 & 0 \\ 5 & 0 & 6 & 8 \end{vmatrix}$**.**

Solution: $\begin{vmatrix} 1 & 0 & 3 & 7 \\ 4 & 2 & 0 & 1 \\ 7 & 7 & 3 & 0 \\ 5 & 0 & 6 & 8 \end{vmatrix}$

$$= 1 \times \begin{vmatrix} 2 & 0 & 1 \\ 7 & 3 & 0 \\ 0 & 6 & 8 \end{vmatrix} - 0 + 3 \times \begin{vmatrix} 4 & 2 & 1 \\ 7 & 7 & 0 \\ 5 & 0 & 8 \end{vmatrix} - 7 \times \begin{vmatrix} 4 & 2 & 0 \\ 7 & 7 & 3 \\ 5 & 0 & 6 \end{vmatrix}$$

= 1[2(24-0)+0+1(42-0)]+3[4(56-0)-2(56-0)+1(0-35)]-7[4(42-0)-2(42-15)+0]

= -617 $\neq 0$. So Rank = 4.

9. Solve the following system of equation by Cramer's rule: x+2y+3z = 20, 7x+3y+z = 13, x+6y+2z = 0

Solution: $D = \begin{vmatrix} 1 & 2 & 3 \\ 7 & 3 & 1 \\ 1 & 6 & 2 \end{vmatrix} = 91 \neq 0$,

$$D_x = \begin{vmatrix} 20 & 2 & 3 \\ 13 & 3 & 1 \\ 0 & 6 & 2 \end{vmatrix} = 182$$

$$D_y = \begin{vmatrix} 1 & 20 & 3 \\ 7 & 13 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -273, \quad D_z = \begin{vmatrix} 1 & 2 & 20 \\ 7 & 3 & 13 \\ 1 & 6 & 0 \end{vmatrix} = 728$$

So by Cramer's rule

$$x = \frac{D_x}{D} = \frac{182}{91} = 2, \; y = \frac{D_y}{D} = \frac{-273}{91} = -3,$$

$$\mathrm{z} = \frac{\mathrm{D_z}}{\mathrm{D}} = \frac{728}{91} = 8$$

10. For which value of x is the matrix $\begin{pmatrix} x & 2 \\ 1 & x-1 \end{pmatrix}$ **singular? Justify your answer.**

Solution: For matrix to be singular

$$\begin{vmatrix} x & 2 \\ 1 & x-1 \end{vmatrix} = 0 \Rightarrow x^2 - x - 2 = 0 \Rightarrow x = -1, 2$$

11. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$**, find the value of** A^3**.**

Solution: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then

$$A^3 = \begin{bmatrix} \cos 3\theta & -\sin 3\theta \\ \sin 3\theta & \cos 3\theta \end{bmatrix}$$

12. Solve the following system of equations by Gauss elimination method: x+y-z = 9, 8y+6z = -6, -2x+4y-6z = 40.

Solution: Augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ -2 & 4 & -6 & 40 \end{array}\right]$

$$R_3 \to R_3 + 2R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 6 & -8 & 58 \end{array}\right] R_3 \to R_3 - \frac{6}{8}R_2 \qquad =$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 0 & \frac{-100}{8} & \frac{500}{8} \end{array}\right]$$

So $x - y - z = 9, 8y + 6z = -6$

$$\frac{-100z}{8} = \frac{500}{8}$$

Solution is z = -5, y = 3, x = 1

13. Convert the following matrix into triangular form and hence determine its determinant.

$$A = \begin{bmatrix} 3 & 1 & -1 & 0 \\ 2 & 2 & 2 & 1 \\ -1 & 3 & 0 & 4 \\ 8 & 6 & -2 & 2 \end{bmatrix}$$

Solution: $\begin{bmatrix} 3 & 1 & -1 & 0 \\ 2 & 2 & 2 & 1 \\ -1 & 3 & 0 & 4 \\ 8 & 6 & -2 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - \frac{2}{3}R_1 \\ R_3 \to R_3 + \frac{1}{3}R_1 \\ R_4 \to R_4 - \frac{8}{3}R_1 \end{matrix}$

$$= \begin{bmatrix} 3 & 1 & -1 & 0 \\ 0 & \frac{4}{3} & \frac{8}{3} & 1 \\ 0 & \frac{10}{3} & \frac{-1}{3} & 4 \\ 0 & \frac{10}{3} & \frac{2}{3} & 2 \end{bmatrix} \begin{matrix} R_3 \to R_3 - \frac{10}{4}R_2 \\ R_4 \to R_4 - \frac{10}{4}R_2 \end{matrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 & 0 \\ 0 & \frac{4}{3} & \frac{8}{3} & 1 \\ 0 & 0 & \frac{-21}{3} & \frac{6}{4} \\ 0 & 0 & \frac{-18}{3} & \frac{-2}{4} \end{bmatrix} R_4 \to R_4 - \frac{18}{21}R_3$$

$$= \begin{bmatrix} 3 & 1 & -1 & 0 \\ 0 & \frac{4}{3} & \frac{8}{3} & 1 \\ 0 & 0 & \frac{-21}{3} & \frac{6}{4} \\ 0 & 0 & 0 & \frac{-50}{28} \end{bmatrix} = 3 \times \frac{4}{3} \times \frac{-21}{3} \times \frac{-50}{21} = 50$$

14. Use Cramer's rule to solve the following system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ x_1 - x_2 + x_3 &= 2 \\ x_1 - 2x_2 &= 0 \end{aligned}$$

Solution: $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -2 & 0 \end{vmatrix} = 2 \neq 0$,

$$D_x = \begin{vmatrix} 4 & 1 & 1 \\ 2 & -1 & 1 \\ 0 & -2 & 0 \end{vmatrix} = 4$$

$$D_y = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 2 \qquad D_z = \begin{vmatrix} 1 & 1 & 4 \\ 1 & -1 & 2 \\ 1 & -2 & 0 \end{vmatrix} = 2$$

So by Cramer's rule

$$x = \frac{D_x}{D} = \frac{4}{2} = 2, \; y = \frac{D_y}{D} = \frac{2}{2} = 1, \; z = \frac{D_z}{D} = \frac{2}{2} = 1$$

15. Reduce the following matrix into a matrix in its upper triangular form $\begin{pmatrix} 5 & 4 & 0 \\ 10 & 9 & 6 \\ 0 & 2 & 1 \end{pmatrix}$**.**

Solution: $\begin{bmatrix} 5 & 4 & 0 \\ 10 & 0 & 8 \\ 0 & 2 & 1 \end{bmatrix} R_2 \to R_2 - 2R_1$

$$= \begin{bmatrix} 5 & 4 & 0 \\ 0 & -8 & 8 \\ 0 & 2 & 1 \end{bmatrix} R_3 \to R_3 + \frac{R_2}{4}$$

$$= \begin{bmatrix} 5 & 4 & 0 \\ 0 & -8 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

16. Solve by Cramer's rule: x+2y+5z = 23, 3x+y+4z = 26, 6x+y+7z = 47.

Solution: $D = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 5 \\ 6 & 1 & 7 \end{vmatrix} = -6 \neq 0$,

$$D_x = \begin{vmatrix} 23 & 2 & 5 \\ 26 & 1 & 5 \\ 47 & 1 & 7 \end{vmatrix} = -24$$

$$D_y = \begin{vmatrix} 1 & 23 & 5 \\ 3 & 26 & 5 \\ 6 & 47 & 7 \end{vmatrix} = -12,$$

$$D_z = \begin{vmatrix} 1 & 2 & 23 \\ 3 & 1 & 26 \\ 6 & 1 & 47 \end{vmatrix} = -18$$

So by Cramer's rule $x = \frac{D_x}{D} = \frac{-24}{-6} = 4$,

$$y = \frac{D_y}{D} = \frac{-12}{-6} = 2, \ z = \frac{D_z}{D} = \frac{-18}{-6} = 3$$

17. Solve by Gauss elimination method: x+y-z = 9, 8y+6z = -6, -2x+4y-6z =40. [5]

Solution: Augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ -2 & 4 & -6 & 40 \end{array}\right]$

$$R_3 \to R_3 + 2R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 6 & -8 & 40 \end{array}\right] R_3 \to R_3 - \frac{6}{8} R_2$$

$$=\left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 0 & \frac{-100}{8} & \frac{500}{8} \end{array}\right]$$

So, $x+y-z=9$, $8y+6z=-6$, $\frac{-100}{8}z=\frac{500}{8}$

Solution is z = -5y = 3, x = 1

18. Find the value of $A^2-6A+8I$ where A = $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

Solution:

$$A^2-6A+8I=\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}-6\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}+8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix}-\begin{bmatrix} 18 & 12 \\ 12 & 18 \end{bmatrix}+\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}=\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

19. Check whether the following vectors are linearly independent: (-1, 5, 0), (16, 8, -3), (-64, 56, 9).

Solution: $\begin{vmatrix} -1 & 5 & 0 \\ 16 & 8 & -3 \\ -64 & 56 & 9 \end{vmatrix}=0$. So vectors are linearly dependent.

20. $V = \{(v_1, v_2, v_3) \in R^3; v_1 + v_2 = 0\}$ **is a vector space. Find a basis for V; hence find its dimension.**

Solution: $v_1 + v_2 = 0 \Rightarrow v_1 = -v_2$

$V = \{(v_1, v_2, v_3)\} = \{(-v_2, v_2, v_3)\}$. It is of two variables.

So dimension = 2.   Basis are (-1,1,1), (1,-1,0).

21. Find the rank of the matrix $\begin{pmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{pmatrix}$

Solution: $\begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix} R_1 \leftrightarrow R_3$

$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 9 & 3 & 1 & 0 \\ 0 & -6 & 1 & 9 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 9R_1 \end{matrix}$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -3 & -2 & -9 \\ 0 & -6 & 1 & 9 \\ 0 & -6 & 1 & 9 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -3 & -2 & -9 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 5 & 9 \end{bmatrix} \quad R_4 \to R_4 + \frac{5}{4} R_3$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -3 & -2 & -9 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & \frac{81}{4} \end{bmatrix}$$

As there are 4 no. of Linearly independent vectors, so Rank = 4.

22. Solve the following system of equations by Gauss elimination method:
x+ y - z = 9, 8y+6z = -6, -2x+4y-6z = 40.

Solution: Augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ -2 & 4 & -6 & 40 \end{array}\right]$

$R_3 \to R_3 + 2R_1$

$$=\left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 6 & -8 & 58 \end{array}\right] R_3 \to R_3 - \frac{6}{8}R_2$$

$$=\left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 0 & \frac{-100}{8} & \frac{500}{8} \end{array}\right]$$

So $x - y - z = 9$, $8y + 6z = -6$, $\frac{-100z}{8} = \frac{500}{8}$

Solution is z = -5, y = 3, x = 1

23. If $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, find the value of A^4.

Solution: $A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

24. Find the degree of nilpotency of the matrix $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$.

Solution: $A^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So degree of nil-potency is 2.

25. Do the vectors (1, 1) and (-1, 1) form a basis of the vector space R^2? Justify your answer.

Solution: (1,1) and (-1,1) form a basis if they are linearly independent.

$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1+1 = 2 \neq 0$, So they are linearly independent. Hence they form a basis.

26. Solve the following system of equations by Gauss elimination method:
2x - y + z = 10, x + 3y - 2z = 20, 3x + y - 6z = 15.

Solution: Augmented matrix is $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 20 \\ 2 & -1 & 1 & 10 \\ 3 & 1 & -6 & 15 \end{array}\right]$

$R_2 \rightarrow R_2 - 2R_1$
$R_3 \rightarrow R_3 - 3R_1$

$$= \left[\begin{array}{ccc|c} 1 & 3 & -2 & 20 \\ 0 & -7 & 5 & -30 \\ 0 & -8 & 0 & -45 \end{array}\right]$$

So $x + 3y - 2z = 20$, $-7y + z = -30$, $-8y = -45$

Solution is $y = \frac{45}{8}, z = \frac{75}{8}, x = \frac{55}{8}$

27. Find the rank of the matrix:

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ -1 & 0 & 3 & -2 \\ -1 & -1 & 8 & -3 \end{pmatrix}$$

Solution: $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ -1 & 0 & 3 & -2 \\ -1 & -1 & 8 & 3 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$

$$= \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -1 & 5 & -1 \\ 0 & -2 & 10 & 4 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$$= \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -1 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As maximum no. of linearly independent row vector = 2. So Rank of matrix = 2.

28. Test whether the following set of equations is consistent:

x – 2y + z = 4, 3x + y – 2z = -7, x – 4y + 2z = 6.

Solution: Augmented matrix is $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 3 & 1 & -2 & -7 \\ 1 & -4 & -2 & 6 \end{array}\right]$

$R_2 \rightarrow R_2 - 3R_1$
$R_3 \rightarrow R_3 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -7 & -5 & -19 \\ 0 & -2 & -3 & 2 \end{array}\right] R_3 \rightarrow R_3 - \frac{2}{7}R_2$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & -7 & -5 & -19 \\ 0 & 0 & \frac{-11}{7} & \frac{52}{7} \end{array}\right]$$

As left side and right side both are non zero hence the set of equation is consistent.

29. Show that $V = \{(x, y, z) \in R^3 : x + y = 0\}$ **is a vector space over R. Find its dimension.**

Solution: (i) Closure on addition: -

Let $\vec{a} = (x_1, y_1, z_1), \vec{b} = (x_2, y_2, z_2) \in V$

Then $x_1 + y_1 = 0, x_2 + y_2 = 0$

Now $\vec{a} + \vec{b} = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in V$ as

$x_1 + x_2 + y_1 + y_2 = 0$

It satisfies closure law.

(ii) Commutative law on addition: -

$\vec{a}+\vec{b}=(x_1+x_2, y_1+y_2, z_1+z_2)$

$\vec{b}+\vec{a}=(x_2+x_1, y_2+y_1, z_2+z_1)$

$=(x_1+x_2, y_1+y_2, z_1+z_2)=\vec{a}+\vec{b}$

It satisfies commutative law.

(iii) Associative law on addition: -

Let

$\vec{a}=(x_1, y_1, z_1), \vec{b}=(x_2, y_2, z_2), \vec{c}=(x_3, y_3, z_3)\in V$

$\vec{a}+(\vec{b}+\vec{c})=(x_1, y_1, z_1)+\left((x_2, y_2, z_2)+(x_3, y_3, z_3)\right)$

$=(x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$

$(\vec{a}+\vec{b})+\vec{c}=\left((x_1, y_1, z_1)+(x_2, y_2, z_2)\right)++(x_3, y_3, z_3)$

$=(x_1+x_2+x_3, y_1+y_2+y_3, z_1+z_2+z_3)$

It satisfies associative law.

(iv) Identity: - $\vec{0}=(0,0,0)$, $\vec{a}=(x_1, y_1, z_1)\in V$

Then $\vec{0}+\vec{a}=(0+x_1, 0+y_1, 0+z_1)=(x_1, y_1, z_1)=\vec{a}$

$\vec{a}+\vec{0}=(x_1+0, y_1+0, z_1+0)=(x_1, y_1, z_1)=\vec{a}$

$\therefore \vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$

So $\vec{0}$ is the identity element.

(v) Inverse: -

$\vec{a}=(x_1, y_1, z_1)\in V$, $x_1+y_1=0$

$-\vec{a}=(-x_1, -y_1, -z_1)$

$\vec{a}+\left(-\vec{a}\right)=(x_1-x_1, y_1-y_1, z_1-z_1)=(0,0,0)=\vec{0}$

So $-\vec{a}$ is the inverse of $\vec{a}$.

(vi)Let α be any scalar,

$\vec{a} = (x_1, y_1, z_1), \vec{b} = (x_2, y_2, z_2) \in V$

Such that $x_1 + y_1 = 0, x_2 + y_2 = 0$

$\alpha\left(\vec{a} + \vec{b}\right) = \alpha(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= (\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2, \alpha z_1 + \alpha z_2)$

$= (\alpha x_1, \alpha y_1, \alpha z_1) + (\alpha x_2, \alpha y_2, \alpha z_2)$

$= \alpha(x_1, y_1, z_1) + \alpha(x_2, y_2, z_2) = \alpha\vec{a} + \alpha\vec{b}$

(vii) Let α and β are any scalar, $\vec{a} = (x_1, y_1, z_1) \in V$

$(\alpha + \beta)\vec{a} = (\alpha + \beta)(x_1, y_1, z_1)$

$= (\alpha x_1 + \beta x_1, \alpha y_1 + \beta y_1, \alpha z_1 + \beta z_1)$

$= \alpha(x_1, y_1, z_1) + \beta(x_1, y_1, z_1) = \alpha\vec{a} + \beta\vec{a}$

(viii) Let α and β are any scalar, $\vec{a} = (x_1, y_1, z_1) \in V$

$\alpha\left(\beta\vec{a}\right) = \beta(\alpha\vec{a}) = \alpha\beta\vec{a}$

(ix) $1 \in R$ So $1 \cdot \vec{a} = \vec{a} \cdot 1 = \vec{a}$

Since it satisfies all the properties of vector space, so it is a vector space.

30. Find the value of k, so that the vectors (1, -1, 2), (2, 1, -3) and (7, -1, k) are linearly independent.

Solution: $\begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & -3 \\ 7 & -1 & k \end{vmatrix} = 0$

$\Rightarrow 1(k - 3) + 1(2k + 21) + (-2 - 9) = 0 \Rightarrow 3k = 0$

$\Rightarrow k = 0$

31. Give the example of two matrices A and B such that $A \neq 0, B \neq 0$ but AB = 0.

Solution:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq 0, \quad AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

32. Are the vectors (1, 0, 1), (0, 1, 1) and (1, 1, 1) linearly independent?

Solution:

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1(1-1) - 0 + 1(0-1) = 0 - 0 - 1 = -1 \neq 0.$$

So vectors (1, 0, 1), (0, 1, 1) and (1, 1, 1) are linearly independent.

33. For what value of c the equations x + y = 7 and 2x – cy = 14 have infinitely many solutions?

Solution: For many solution

$$\begin{vmatrix} 1 & 1 \\ 2 & -c \end{vmatrix} = 0 \Rightarrow -c - 2 = 0 \Rightarrow c = -2.$$

34. If $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}$ **and** $B^T = \begin{bmatrix} 2 & 5 & 1 & 8 \\ 3 & 0 & 1 & 2 \\ 2 & 1 & 3 & 6 \end{bmatrix}$

verify that $(AB)^T = B^T A^T$**.**

Solution: $A=\begin{bmatrix}1&0&1&2\\2&3&0&1\\1&1&2&2\end{bmatrix}$ and

$$B^T=\begin{bmatrix}2&5&1&8\\3&0&1&2\\2&1&3&6\end{bmatrix}$$

$A^T=\begin{bmatrix}1&2&1\\0&3&1\\1&0&2\\2&1&2\end{bmatrix}$ and $B=\begin{bmatrix}2&3&2\\5&0&1\\1&1&3\\8&2&6\end{bmatrix}$

$$AB=\begin{bmatrix}1&2&1\\0&3&1\\1&0&2\\2&1&2\end{bmatrix}\begin{bmatrix}2&5&1&8\\3&0&1&2\\2&1&3&6\end{bmatrix}=\begin{bmatrix}19&8&17\\27&8&13\\25&9&21\end{bmatrix}$$

$$(AB)^T=\begin{bmatrix}19&27&25\\8&8&9\\17&13&21\end{bmatrix}\text{- - - (1)}$$

$$B^TA^T=\begin{bmatrix}2&5&1&8\\3&0&1&2\\2&1&3&6\end{bmatrix}\begin{bmatrix}1&2&1\\0&3&1\\1&0&2\\2&1&2\end{bmatrix}=\begin{bmatrix}19&27&25\\8&8&9\\17&13&21\end{bmatrix}\text{(2)}$$

From (1) and (2) it verifies that $(AB)^T = B^T A^T$.

35. Find the rank of the matrix

$$A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix}$$

Solution: $A = \begin{vmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{vmatrix} = -459 \neq 0$, so rank of the matrix is 4. **OR**

$$A = \begin{bmatrix} 9 & 3 & 1 & 0 \\ 3 & 0 & 1 & -6 \\ 1 & 1 & 1 & 1 \\ 0 & -6 & 1 & 9 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 1 & -6 \\ 9 & 3 & 1 & 0 \\ 0 & -6 & 1 & 9 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\approx \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -2 & -9 \\ 0 & -6 & -8 & -9 \\ 0 & -6 & 1 & 9 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 9R_1 \end{matrix}$$

$$\approx \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -2 & -9 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 5 & 27 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$\approx \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -2 & -9 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & \frac{153}{4} \end{bmatrix} \quad R_4 \to R_4 + \frac{5}{4} R_3$$

As here 4 no. of linearly independent row vectors, So its rank = 4.

36. Solve the following systems of equations by Cramer's rule:
x+2y+3z = 20, 7x+3y+z = 13, x+6y+2z = 0.

Solution: $D = \begin{vmatrix} 1 & 2 & 3 \\ 7 & 3 & 1 \\ 1 & 6 & 2 \end{vmatrix} = 91 \neq 0,$

$$D_x = \begin{vmatrix} 20 & 2 & 3 \\ 13 & 3 & 1 \\ 0 & 6 & 2 \end{vmatrix} = 182$$

$$D_y = \begin{vmatrix} 1 & 20 & 3 \\ 7 & 13 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -273,$$

$$D_z = \begin{vmatrix} 1 & 2 & 20 \\ 7 & 3 & 13 \\ 1 & 6 & 0 \end{vmatrix} = 728$$

So by Cramer's rule

$$x = \frac{D_x}{D} = \frac{182}{91} = 2,\ y = \frac{D_y}{D} = \frac{-273}{91} = -3,$$

$$z = \frac{D_z}{D} = \frac{728}{91} = 8$$

37. If $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$**, find the value of** $A^2 - 2A$**.**

Solution: $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, then

$$A^2 - 2A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

38. Determine rank of the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix}$**.**

Solution: $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$

$$\approx \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} R_3 \to R_3 - 2R_2 \approx \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

As here maximum no. of linearly independent row vectors = 2.

So, rank of the matrix = 2.

39. If $A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$**, find the value of** A^4**.**

Solution:

$$A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

40. Solve the following system of equations by Gauss elimination method:
3x + y + 2z = 7, x – y + z = 6, 2x + 3y + 3z = 5.

Solution: Augmented matrix is $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 3 & 1 & 2 & 7 \\ 2 & 3 & 3 & 5 \end{array}\right]$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 4 & -1 & -11 \\ 0 & 5 & 1 & -7 \end{array}\right] R_3 \to R_3 - \frac{5}{4} R_2$

$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 4 & -1 & -11 \\ 0 & 0 & \frac{9}{4} & \frac{27}{4} \end{array}\right]$

So $x - y + z = 6$, $4y - z = -11$, $\frac{9}{4} z = \frac{27}{4}$

Solution is z = 3, y = -2, x = 1

41. Solve the following system of equations using Cramer's rule: 3x – y + z = 11, x + y + 3z = 13, 2x – 3y + 4z = 23.

Solution: $D = \begin{vmatrix} 3 & -1 & 1 \\ 1 & 1 & 3 \\ 2 & -3 & 4 \end{vmatrix} = 32 \neq 0$,

$$D_x = \begin{vmatrix} 11 & -1 & 1 \\ 13 & 1 & 3 \\ 23 & -3 & 4 \end{vmatrix} = 64$$

$$D_y = \begin{vmatrix} 3 & 11 & 1 \\ 1 & 13 & 3 \\ 2 & 23 & 4 \end{vmatrix} = -32,$$

$$D_z = \begin{vmatrix} 3 & -1 & 11 \\ 1 & 1 & 13 \\ 2 & -3 & 23 \end{vmatrix} = 128$$

So by Cramer's rule

$$x = \frac{D_x}{D} = \frac{64}{32} = 2, \quad y = \frac{D_y}{D} = \frac{-32}{32} = -1,$$

$$\mathrm{z} = \frac{\mathrm{D_z}}{\mathrm{D}} = \frac{128}{32} = 4$$

42. Find the rank of the matrix $\begin{bmatrix} 4 & 3 & 1 \\ 8 & 0 & 5 \\ 4 & -3 & 4 \\ 4 & 1 & 2 \end{bmatrix}$

Solution: $\begin{bmatrix} 4 & 3 & 1 \\ 8 & 0 & 5 \\ 4 & -3 & 4 \\ 4 & 1 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$

$\approx \begin{bmatrix} 4 & 3 & 1 \\ 0 & -6 & 3 \\ 0 & -6 & 3 \\ 0 & -2 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_4 \\ R_3 \to R_3 - 3R_4 \end{matrix}$

$\approx \begin{bmatrix} 4 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ Here it has 2 no. of linearly independent row vectors.

So, Rank = 2.

43. Verify whether the vectors (-4, 6, 9), (3, 2, 1) and (4, 2, 9) are linearly independent.

Solution:

$$\begin{vmatrix} -4 & 6 & 9 \\ 3 & 2 & 1 \\ 4 & 2 & 9 \end{vmatrix} = -4(18-2) - 6(27-4) + 9(6-8)$$

$= -64 - 138 - 18 = -210 \neq 0$

So, the vectors (-4, 6, 9), (3, 2, 1) and (4, 2, 9) are linearly independent.

44. How many solutions does the following linear systems have and why :

$x + 2y = 2, \qquad 2x + 4y = 1.$

Solution:

$$D = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0, D_x = \begin{vmatrix} 2 & 2 \\ 7 & 4 \end{vmatrix} = 8 - 14 = -6 \neq 0,$$

$$D_y = \begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} 7 - 4 = 3 \neq 0$$

As $D = 0, D_x \neq 0, D_y \neq 0$, So by Cramer's rule it has no solution.

45. If A is any 5X5 matrix, and a polynomial associated with the matrix A is defined by $|\lambda I - A| = \sum_{i=0}^{5} \alpha_i \lambda^i$, then what is the determinant of the matrix A.

Solution: $|A| = \alpha_0$.

46. Find the parameter α such that the vectors $v_1 = (1,1,\alpha), \ v_2 = (1,0,\alpha), \ v_3 = (0,\alpha,0)$ are linearly dependent.

Solution: As the three vectors are linearly dependent. So their determinant is zero.

$$\begin{vmatrix} 1 & 1 & \alpha \\ 1 & 0 & \alpha \\ 0 & \alpha & 0 \end{vmatrix} = 0 \Rightarrow 1(0-\alpha^2) - 1(0-0) + \alpha(\alpha - 0) = 0$$

$$\Rightarrow -\alpha^2 + \alpha^2 = 0$$

Which satisfies all values of $\alpha \in R$.

47. Find the dimensions of the vector space $V = \{v = (v_1, v_2, v_3) : v_1 + v_2 - v_3 = 0\}$.

Solution: $V = (v_1, v_2, v_3) = (v_1, v_2, v_1 + v_2)$. So dimension of V = 2.

48. If the set of vectors $v_1 = (1,1,1), \quad v_2 = (1,0,1), \quad v_3 = (0,1,0)$ are dependent, find their dependency.

Solution: The vectors $v_1 (1,1,1), v_2 = (1,0,1), and \quad v_3 = (0,1,0)$ are dependent. Their dependency is $v_1 = v_2 + v_3$.

49. Solve the following systems of equations by Gauss elimination method. x+z = 1, 2x+z = 0, x+y+z = 1.

Solution: Its Augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{array}\right]$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & -2 & -1 & -1 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

So, $x + y + z = 1, \qquad -y = -1, \qquad -z = 1$

$\Rightarrow z = -1, y = 1, x = 1$

50. If $A = (a_{ij})$ is any n x n matrix with $a_{ij} = 0$ for $j \le i$, then show that $A^n = 0$. [5]

Solution: $A = [a_{ij}]_{n\times n}$ where $a_{ij} = 0$ for $j \le i$

$A^2 = A \times A = [c_{ij}]_{n\times n}$, where $c_{ij} = \sum_{k=1}^{n} a_{ik} a_{kj} = 0$

$\Rightarrow A^2 = 0 \qquad \Rightarrow A^n = A^{n-2}.A^2 = 0$

51. Find the Eigen value of the matrix

$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

Solution: Characteristics equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow 10 + 7\lambda + \lambda^2 - 6 = 0$$

$$\Rightarrow \lambda = -1, -6$$

52. Prove that the Eigen values of a Hermitian matrix are real.

Solution: Given matrix A is Hermitian matrix i.e

$$\overline{A}^T = A$$

Let λ be the eigen value of A.

Then Ax = λ x

$$\Rightarrow \overline{Ax} = \overline{\lambda x} \Rightarrow \overline{A}\,\overline{x} = \overline{\lambda}\,\overline{x} \Rightarrow \left(\overline{A}\,\overline{x}\right)^T = \left(\overline{\lambda}\,\overline{x}\right)^T$$

$$\Rightarrow \overline{x}^T \overline{A}^T = \overline{\lambda}\,\overline{x}^T \Rightarrow \overline{x}^T A = \overline{\lambda}\,\overline{x}^T$$

By multiplying x both side

$$\Rightarrow \overline{x}^T Ax = \overline{\lambda}\,\overline{x}^T x \Rightarrow \overline{\lambda} = \frac{\overline{x}^T Ax}{\overline{x}^T x}$$

As Numerator and denominator both are real

$\Rightarrow \overline{\lambda} = real \Rightarrow \lambda = real$ (Proved)

53. Let $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots, \lambda_k$ be the distinct eigenvalues of a n x n matrix. Prove that corresponding eigenvectors $x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_k$ form a linearly independent set.

Solution: Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_k$ are distinct Eigen values and $x_1, x_2, x_3, ..., x_k$ are corresponding Eigen vectors.

If possible they are not linearly independent.

Let r < k the largest integer for which $x_1, x_2, x_3, ..., x_r$ is linearly independent and $x_1, x_2, x_3, ..., x_{r+1}$ are linearly dependent. Hence $\exists$ scalars $c_1, c_2, c_3, ..., c_{r+1}$ not all zero, such that

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + ...+ c_{r+1} x_{r+1} = 0$$

$$\Rightarrow A(c_1 x_1 + c_2 x_2 + c_3 x_3 + ...+ c_{r+1} x_{r+1}) = A.0$$

$$\Rightarrow (c_1 A x_1 + c_2 A x_2 + c_3 A x_3 + ...+ c_{r+1} A x_{r+1}) = 0$$

$$\Rightarrow c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + c_3 \lambda_3 x_3 + ...+ c_{r+1} \lambda_{r+1} x_{r+1} = 0$$

Now (2) - λ_{r+1}(1)

$$\Rightarrow c_1(\lambda_1 - \lambda_{r+1}) x_1 + c_2(\lambda_2 - \lambda_{r+1}) x_2 + ...$$

$$+ c_r(\lambda_r - \lambda_{r+1}) x_r = 0$$

$$\Rightarrow c_1(\lambda_1 - \lambda_{r+1}) = 0, c_2(\lambda_2 - \lambda_{r+1}) = 0, ...,$$

$$c_r(\lambda_r - \lambda_{r+1}) = 0$$

$\Rightarrow c_1 = c_2 = c_3 = ... = c_r = 0$ as

$\lambda_1, \lambda_2, \lambda_3, ..., \lambda_k$ are distinct.

$\Rightarrow c_{r+1} x_{r+1} = 0 \Rightarrow c_{r+1} = 0$

$\Rightarrow c_1 = c_2 = c_3 = ... = c_r = c_{r+1} = 0$ this is a contradiction

$\Rightarrow x_1, x_2, x_3, ..., x_k$ are linearly independent.

54. Express the matrix $A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 4 & -2 \\ -4 & 2 & 2 \end{bmatrix}$ as a sum of a symmetric and a skew-symmetric matrix.

Solution: $A = \frac{A + A'}{2} + \frac{A - A'}{2}$

$$= \frac{\begin{bmatrix} 2 & 4 & 6 \\ 0 & 4 & -2 \\ -4 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & -4 \\ 4 & 4 & 2 \\ 6 & -2 & 2 \end{bmatrix}}{2}$$

$$+ \frac{\begin{bmatrix} 2 & 4 & 6 \\ 0 & 4 & -2 \\ -4 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & -4 \\ 4 & 4 & 2 \\ 6 & -2 & 2 \end{bmatrix}}{2}$$

$$= \frac{\begin{bmatrix} 4 & 4 & 2 \\ 4 & 8 & 0 \\ 2 & 0 & 4 \end{bmatrix}}{2} + \frac{\begin{bmatrix} 0 & 4 & 10 \\ -4 & 0 & -4 \\ -10 & 4 & 0 \end{bmatrix}}{2}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 5 \\ -2 & 0 & -2 \\ -5 & 2 & 0 \end{bmatrix}$$

55.

$A = \begin{bmatrix} 1 & i\sqrt{3} \\ -i\sqrt{3} & 1 \end{bmatrix}$**. Verify whether A is Hermitian, Skew-Hermitian.**

Solution: $A = \begin{bmatrix} 1 & i\sqrt{3} \\ -i\sqrt{3} & 1 \end{bmatrix}$, $\overline{A} = \begin{bmatrix} 1 & -i\sqrt{3} \\ i\sqrt{3} & 1 \end{bmatrix}$

$$\overline{A}^T = \begin{bmatrix} 1 & i\sqrt{3} \\ -i\sqrt{3} & 1 \end{bmatrix} = A \neq -A$$

So A is Hermitian matrix but not skew Hermitian matrix.

56. A nonsingular 3 X 3 matrix has Eigen values 2, 3 and 4. What are the Eigen values of A^{-1}? Justify your answer.

Solution: For non singular matrix A if

$\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots.., \lambda_k$ are the Eigen values then Eigen values of A^{-1} are $\frac{1}{\lambda_1},\frac{1}{\lambda_2},\frac{1}{\lambda_3},..........\frac{1}{\lambda_n}$.

If a non singular matrix A has Eigen values 2,3,4 the Eigen value of A^{-1} are $\frac{1}{2},\frac{1}{3},\frac{1}{4}$.

57. Transform the quadratic form $4x^2+12xy+13y^2=16$ into the principal axes.

Solution: $4x^2+12xy+13y^2=16$

$$Q=4x^2+12xy+13y^2=X^TAX$$

Where $A=\begin{bmatrix}4 & 6\\ 6 & 13\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix}$

Characteristics equation is $|A-\lambda I|=0$

$$\Rightarrow\begin{vmatrix}4-\lambda & 6\\ 6 & 13-\lambda\end{vmatrix}=0\Rightarrow 52-4\lambda-13\lambda+\lambda^2-36=0$$

$$\Rightarrow \lambda^2-17\lambda+16=0\Rightarrow\lambda=1,16$$

$$Q=\lambda_1x^2+\lambda_2y^2=16$$

$\Rightarrow x^2+16y^2=16\ \Rightarrow\frac{x^2}{16}+\frac{=y^2}{1}=1$ which represents an ellipse.

58. Is the matrix $A=\frac{1}{3}\begin{bmatrix}-2 & 1 & 2\\ 2 & 2 & 1\\ 1 & -2 & 2\end{bmatrix}$ **orthogonal?**

Justify your answer.

Solution:

$$A^T\times A=\frac{1}{3}\begin{bmatrix}-2 & 1 & 2\\ 2 & 2 & 1\\ 1 & -2 & 2\end{bmatrix}\times\frac{1}{3}\begin{bmatrix}-2 & 2 & 1\\ 1 & 2 & -2\\ 2 & 1 & 2\end{bmatrix}$$

$$=\frac{1}{9}\begin{bmatrix}4+1+4 & -4+2+2 & -2-2+4\\ -4+2+2 & 4+4+1 & 2-4+2\\ -2-2+4 & 2-4+2 & 1+4+4\end{bmatrix}$$

$$=\frac{1}{9}\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$$

So A is orthogonal.

59. If $A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$ **find the Eigen value of** A^{-1}**.**

Solution: Characteristics equation is $|A-\lambda I|=0$

$$\Rightarrow\begin{vmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{vmatrix}=0$$

$\Rightarrow (2-\lambda)^2 - 1 = 0$

$\Rightarrow \lambda = 1,3$

So Eigen values of A are 1 and 3

$\Rightarrow$ Eigen values of A^{-1} are 1 and $\frac{1}{3}$.

60. Is the matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2}i \\ \frac{\sqrt{3}}{2}i & \frac{1}{2} \end{bmatrix}$ **unitary?**

Justify your answer.

Solution: If $\overline{A}^T A = I$ then A is unitary matrix.

$$\overline{A}^T A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}i}{2} \\ \frac{\sqrt{3}i}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}i}{2} \\ \frac{-\sqrt{3}i}{2} & \frac{1}{2} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{4} + \frac{3}{4} & \frac{-\sqrt{3}i}{4} + \frac{\sqrt{3}i}{4} \\ \frac{\sqrt{3}i}{4} - \frac{\sqrt{3}i}{4} & \frac{1}{4} + \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

So given matrix is Unitary matrix.

61. Reduce the following matrix into diagonal form $A=\begin{bmatrix}-19 & 7\\ -42 & 16\end{bmatrix}$.

Solution: Characteristics equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}-19-\lambda & 7\\ -42 & 16-\lambda\end{vmatrix}=0$$

$$\Rightarrow (-19-\lambda)(16-\lambda)+294=0 \Rightarrow \lambda=2,-5$$

For $\lambda=2$, Eigen vector is $\begin{bmatrix}1\\3\end{bmatrix}$ and

For $\lambda=-5$, Eigen vector is $\begin{bmatrix}1\\2\end{bmatrix}$

$$\therefore X=\begin{bmatrix}1 & 1\\ 3 & 2\end{bmatrix},\ |X|=2-3=-1,$$

$$X^{-1}=\frac{\begin{bmatrix}2 & -1\\ -3 & 1\end{bmatrix}}{-1}=\begin{bmatrix}-2 & 1\\ 3 & -1\end{bmatrix}$$

$$D=X^{-1}AX=\begin{bmatrix}-2 & 1\\ 3 & -1\end{bmatrix}\begin{bmatrix}-19 & 7\\ -42 & 16\end{bmatrix}\begin{bmatrix}1 & 1\\ 3 & 2\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & -5\end{bmatrix}$$

62. Find the Eigen values of the matrix $\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.

Solution: Characteristics equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix}=0 \Rightarrow (1-\lambda)^2-4=0 \Rightarrow \lambda=-1,3$$

63. Is the matrix $\begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$ orthogonal?

Solution: $A=\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A^T=\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

$$\Rightarrow AA^T=\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{3}{4}+\frac{1}{4} & \frac{-\sqrt{3}}{4}+\frac{\sqrt{3}}{4} \\ \frac{-\sqrt{3}}{4}+\frac{\sqrt{3}}{4} & \frac{1}{4}+\frac{3}{4} \end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I$$

So given matrix is orthogonal.

64. Find the value of λ so that

$$[1,\lambda,-1]\begin{bmatrix}1 & 3 & 2\\0 & 2 & 1\\0 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\1\\\lambda\end{bmatrix}=0$$

Solution: $\begin{bmatrix}1 & \lambda & -1\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\0 & 2 & 1\\0 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\1\\\lambda\end{bmatrix}=0$

$$\Rightarrow\begin{bmatrix}1 & \lambda & -1\end{bmatrix}\begin{bmatrix}1+3+2\lambda\\2+\lambda\\3+2\lambda\end{bmatrix}=0$$

$$\Rightarrow 1+3+2\lambda+2\lambda+\lambda^2-3-2\lambda=0$$

$$\Rightarrow \lambda^2+2\lambda+1=0 \qquad \Rightarrow \lambda=-1$$

65. Find the Eigen values and the corresponding eigenvectors of the following matrix:

$$\begin{bmatrix}3 & 0 & 1\\0 & 2 & 0\\5 & 0 & -1\end{bmatrix}$$

Solution: Characteristics equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 5 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(2-\lambda)(-1-\lambda) - 5(2-\lambda) = 0$$

$$\Rightarrow (2-\lambda)\left[(3-\lambda)(-1-\lambda) - 5\right] = 0$$

$$\Rightarrow (2-\lambda)\left[(\lambda^2 - 2\lambda - 8)\right] = 0$$

$$\Rightarrow (2-\lambda)(\lambda-4)(\lambda+2)) = 0 \Rightarrow \lambda = 2,4,-2$$

For $\lambda = 2$ $\therefore$ Eigen vector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

For $\lambda = 4$ $\therefore$ Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = -2$ $\therefore$ Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$.

66. Is the following matrix Skew-Hermitian? Find $\bar{x}Ax$ where $A = \begin{bmatrix} 1 & i & 4 \\ -i & 3 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ and x = [1, i, -i].

Solution: $A = \begin{bmatrix} 1 & i & 4 \\ -i & 3 & 0 \\ 4 & 0 & 2 \end{bmatrix}$, $\overline{A} = \begin{bmatrix} 1 & -i & 4 \\ i & 3 & 0 \\ 4 & 0 & 2 \end{bmatrix}$

$\Rightarrow \overline{A}^T = \begin{bmatrix} 1 & i & 4 \\ -i & 3 & 0 \\ 4 & 0 & 2 \end{bmatrix} = A$. So it is not Skew Hermitian.

Now $\overline{X}^T AX = [1 \quad -i \quad i] \begin{bmatrix} 1 & i & 4 \\ -i & 3 & 0 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix}$

$= [1 - 1 + 4i \quad i - 3i + 0 \quad 4 + 0 + 2i] \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix}$

$= [4i \quad -2i \quad 4 + 2i] \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix} = 4i + 2 - 4i + 2 = 4$

67. Find the basis of eigenvectors and diagonalise the following matrix. $\begin{bmatrix} 8 & -1 \\ 5 & 2 \end{bmatrix}$**.**

Solution: Characteristics equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (16-10\lambda+\lambda^2)+5=0$$

$$\Rightarrow \lambda^2 - 10\lambda + 21 = 0 \Rightarrow \lambda = 3,7$$

For $\lambda = 3$ Eigen vector is $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ and

For $\lambda = 7$ Eigen vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\therefore X = \begin{bmatrix} 1 & 1 \\ 5 & 1 \end{bmatrix}, \; |X| = 1-5 = -4, \; X^{-1} = \frac{\begin{bmatrix} 1 & -1 \\ -5 & 1 \end{bmatrix}}{-4}$$

$$D = X^{-1}AX = \frac{\begin{bmatrix} 1 & -1 \\ -5 & 1 \end{bmatrix}}{-4} \begin{bmatrix} 8 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 5 & 1 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & -1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 8-5 & 8-1 \\ 5+10 & 5+2 \end{bmatrix}}{-4}$$

$$= \frac{\begin{bmatrix} 1 & -1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 15 & 7 \end{bmatrix}}{-4}$$

$$= \frac{\begin{bmatrix} -12 & 0 \\ 0 & -28 \end{bmatrix}}{-4} = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$$

68. Find spectral radius of the matrix A =
$\begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$**.**

Solution: Characteristics equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 10+7\lambda+\lambda^2-6=0$$
$$\Rightarrow \lambda=-1,-6$$

Spectral radius = $\max(|-1|,|-6|)=6$

69. Find the Eigen values of $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ **and the corresponding eigenvectors.**

Solution: Characteristics equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = -3,-3,5$$

For λ = 5, Eigen vector is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, for λ =3, Eigen vector is $\begin{bmatrix}1\\1\\1\end{bmatrix}$

70. Find the type of conic section represented by the following quadratic form
$7x^2+6xy+7y^2=200.$

Solution: Symmetric matrix is $A=\begin{bmatrix}7 & 3\\3 & 7\end{bmatrix}$

Characteristics equation is $|A-\lambda I|=0$

$$\Rightarrow\begin{vmatrix}7-\lambda & 3\\3 & 7-\lambda\end{vmatrix}=0\Rightarrow(7-\lambda)^2-9=0\Rightarrow\lambda=4,10$$

$$\therefore\lambda_1x^2+\lambda_2y^2=200$$

$$\Rightarrow 4x^2+10y^2=200$$

$$\Rightarrow\frac{x^2}{50}+\frac{y^2}{20}=1$$

71. Express the matrix $\begin{pmatrix}1 & 2\\4 & 2\end{pmatrix}$ as a sum of a symmetric and skew symmetric matrix.

Solution: $A=\dfrac{A+A'}{2}+\dfrac{A-A'}{2}$

$$\Rightarrow A = \frac{\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix}}{2} + \frac{\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 2 \end{bmatrix}}{2}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

72. Find the characteristic polynomial of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ hence find the value of $A^2 + 4A + 3I$.

Solution: Characteristics equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$$\therefore A^2 - 2A - 3I = 0$$

So, $A^2 + 4A + 3I = A^2 - 2A - 3I + 6A + 6I$

$$= 0 + 6\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 12 \\ 12 & 12 \end{bmatrix}$$

73. A nonsingular matrix has eigenvalues 2 and 3; find the eigenvalues of A^{-1}.

Solution: $|A| \neq 0$, A has Eigen values 2 and 3 then

A^{-1} has Eigen values $\frac{1}{2}$ and $\frac{1}{3}$.

74. If A is Hermitian matrix, prove that diagonal elements are real.

Solution: As A is Hermitian matrix,

$$\Rightarrow \overline{A}^T = A \Rightarrow \left[\overline{a}_{ij}\right]^T = \left[a_{ij}\right] \Rightarrow \left[\overline{a}_{ji}\right] = \left[a_{ij}\right] \Rightarrow \overline{a}_{ji} = a_{ij}$$

$\Rightarrow \overline{a}_{ii} = a_{ii}$ (for diagonal elements $i = j.$) $\Rightarrow a_{ii}$ is purely real number.

All the diagonal elements are real.

75. Find the Eigen values and Eigen vectors of the matrix $A = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

Its characteristics equation is

$$\begin{vmatrix} 3-\lambda & 5 & 3 \\ 0 & 4-\lambda & 6 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(4-\lambda)(1-\lambda) = 0 \Rightarrow \lambda = 1,3,4$$

For $\lambda = 3$ corresponding Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = 1$ corresponding Eigen vector is $\begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}$

For $\lambda = 4$ corresponding Eigen vector is $\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

76. Find a matrix A such that
$x^T Ax = (x_1 - x_2 + 4x_3)^2 - 4(x_2 - x_4)^2$.

Solution: $(x_1 - x_2 + 4x_3)^2 - 4(x_2 - x_4)^2$

$= x_1{}^2 + x_2{}^2 + 16x_3{}^2 - 2x_1x_2 - 8x_2x_3$

$+ 8x_1x_3 - 4x_2{}^2 - 4x_4{}^2 + 8x_2x_4$

$= x_1{}^2 - 3x_2{}^2 + 16x_3{}^2 - 4x_4{}^2 - 2x_1x_2$

$- 8x_2x_3 + 8x_1x_3 + 8x_2x_4$

Then $A = \begin{bmatrix} 1 & -1 & 4 & 0 \\ -1 & -3 & -4 & 4 \\ 4 & -4 & 16 & 0 \\ 0 & 4 & 0 & -4 \end{bmatrix}$

77. Find the Eigen values of $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

Solution: Characteristics equation is

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)^2 - 1 = 0.$$

$\Rightarrow \lambda = 4,2$. So, Eigen values are 4 and 2.

78. Transform the quadratic form $4x^2 + 12xy + 13y^2 = 169$ into its principal axes.

Solution: $4x^2 + 12xy + 13y^2 = 169$

Here symmetric matrix $A = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$

Its characteristics equation is $\begin{bmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{bmatrix} = 0$

$\Rightarrow 52 - 17\lambda + \lambda^2 - 36 = 0$

$\Rightarrow \lambda^2 - 17\lambda + 16 = 0 \Rightarrow \lambda = 16,1$

So, $\lambda_1 {y_1}^2 + \lambda_2 {y_2}^2 = Q \Rightarrow 16{y_1}^2 + {y_2}^2 = 169$

Which represents an ellipse.

79. Find the Eigen values and corresponding Eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution: $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Its Characteristics equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = -3, -3, 5$$

For λ = 5, Eigen vector is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$,

For λ = -3, Eigen vector is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

80. Let $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots, \lambda_k$ be the distinct Eigen values of a n x n matrix. Prove that corresponding eigenvectors $x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_k$ form a linearly independent set.

Solution: Let $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots, \lambda_k$ are distinct Eigen values and $x_1, x_2, x_3, \ldots\ldots\ldots, x_k$ are corresponding Eigen vectors.

If possible they are not linearly independent.

Let r < k the largest integer for which $x_1, x_2, x_3, \ldots\ldots\ldots, x_r$ is linearly independent and $x_1, x_2, x_3, \ldots\ldots\ldots, x_{r+1}$ are linearly dependent. Hence $\exists$ scalars $c_1, c_2, c_3, \ldots\ldots\ldots, c_{r+1}$ not all zero, such that

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + \ldots\ldots\ldots + c_{r+1} x_{r+1} = 0$$

$$\Rightarrow A(c_1 x_1 + c_2 x_2 + c_3 x_3 + \ldots\ldots\ldots + c_{r+1} x_{r+1}) = A.0$$

$$\Rightarrow (c_1 A x_1 + c_2 A x_2 + c_3 A x_3 + \ldots\ldots\ldots + c_{r+1} A x_{r+1}) = 0$$

$$\Rightarrow c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + c_3 \lambda_3 x_3 + \ldots\ldots\ldots + c_{r+1} \lambda_{r+1} x_{r+1} = 0$$

Now (2) - λ_{r+1}(1)

$$\Rightarrow c_1(\lambda_1 - \lambda_{r+1}) x_1 + c_2(\lambda_2 - \lambda_{r+1}) x_2 + \ldots\ldots\ldots$$

$$+ c_r(\lambda_r - \lambda_{r+1}) x_r = 0$$

$$\Rightarrow c_1(\lambda_1 - \lambda_{r+1}) = 0, c_2(\lambda_2 - \lambda_{r+1}) = 0, \ldots\ldots\ldots$$

$$, c_r(\lambda_r - \lambda_{r+1}) = 0$$

$\Rightarrow c_1 = c_2 = c_3 = \ldots\ldots\ldots = c_r = 0$ $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots, \lambda_k$ are distinct.

$$\Rightarrow c_{r+1} x_{r+1} = 0$$

$\Rightarrow c_{r+1} = 0$

$\Rightarrow c_1 = c_2 = c_3 = ...= c_r = c_{r+1} = 0$ this is a contradiction

$\Rightarrow x_1, x_2, x_3, ..., x_k$ are linearly independent.

81. If $\lambda = 0$ is an Eigen values of the matrix A, then what one can say about $|A|$.

Solution: $\lambda = 0, |A| = 0$, as determinant of a matrix is same as product of all the Eigen values.

82. Find the eigenvalues of the matrix $\begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$.

Solution: $A = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$

Its characteristics equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 2 & 4-\lambda \end{vmatrix} = 0.$$

$\Rightarrow (1-\lambda)(4-\lambda) - 0 = 0 \Rightarrow \lambda = 1,4$

83. If A is any square matrix, then show that $\dfrac{A + A^T}{2}$ is symmetric and $\dfrac{A - A^T}{2}$ is skew symmetric.

Solution: Let $B = \frac{A + A^T}{2}$

$$\Rightarrow B^T = \left(\frac{A + A^T}{2}\right)^T = \frac{1}{2}\left(A + A^T\right)^T$$

$$= \frac{1}{2}\left((A)^T + \left(A^T\right)^T\right) = \frac{1}{2}\left(A^T + A\right) = \frac{1}{2}\left(A + A^T\right)$$

$\Rightarrow B^T = B$ So, B is Symmetric matrix.

$\Rightarrow \frac{A + A^T}{2}$ is a symmetric matrix.

Let $C = \frac{A - A^T}{2}$

$$\Rightarrow C^T = \left(\frac{A - A^T}{2}\right)^T = \frac{1}{2}\left(A - A^T\right)^T$$

$$= \frac{1}{2}\left((A)^T - \left(A^T\right)^T\right) = \frac{1}{2}\left(A^T - A\right) = -\frac{1}{2}\left(A - A^T\right)$$

$\Rightarrow C^T = -C$ So, C is a Skew Symmetric matrix.

$\Rightarrow \frac{A - A^T}{2}$ is a skew symmetric matrix.

84. Find the Eigen values and set of all linearly independent Eigen vectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Its characteristics equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^3 = 0 \Rightarrow \lambda = 1$$

So, it's Eigen vector is:

$$[A - \lambda I]X = 0 \quad \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix} = 0$$

$$\Rightarrow \mathrm{x} = \mathrm{n}_1, \mathrm{y} = \mathrm{n}_2, \mathrm{z} = \mathrm{n}_3$$

So, Eigen vector is $\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ where $n_1, n_2, n_3 \in R$

85. Find the matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix containing Eigen values of the matrix A along the diagonal an

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution: A= $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Its characteristics equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)((1-\lambda)^2 - 1) - 1(1 - \lambda - 1) + 1(1 - 1 + \lambda) = 0$$

$$\Rightarrow (1-\lambda)^3 - (1-\lambda) + \lambda + \lambda = 0$$

$$\Rightarrow 1 - 3\lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda + \lambda + \lambda = 0$$

$$\Rightarrow 3\lambda^2 - \lambda^3 = 0 \Rightarrow \lambda^2(3-\lambda) = 0 \Rightarrow \lambda = 0,3$$

For $\lambda = 0$, Eigen vectors are $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 3$ Eigen vectors is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

So, $P = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

TRIGONOMETRIC FORMULAE

I. PYTHAGOREAN IDENTITIES:

For any angle θ

$$sin^2\theta + cos^2\theta = 1$$

$$sec^2\theta = 1 + tan^2\theta$$

$$cosec^2\theta = 1 + cot^2\theta$$

II. ANGLE ADDITION FORMULAS

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$sin\ (\alpha - \beta) = sin\ \alpha\ cos\ \beta - cos\ \alpha\ sin\ \beta$$

$$cos\ (\alpha + \beta) = cos\ \alpha\ cos\ \beta + sin\ \alpha\ sin\ \beta$$

$$cos\ (\alpha - \beta) = cos\ \alpha\ cos\ \beta - sin\ \alpha\ sin\ \beta$$

$$tan\ (\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

$$tan\ (\alpha - \beta) = \frac{tan\ \alpha - tan\ \beta}{1 + tan\ \alpha\ tan\ \beta}$$

III. DOUBLE ANGLE FORMULAS

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = cos^2\theta - sin^2\theta = 2cos^2\theta - 1 = 1 - 2sin^2\theta$$

$$tan\ 2\theta = \frac{2\tan\theta}{1-tan^2\theta}$$

IV.HALF ANGLE FORMULAS

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad cos\frac{\theta}{2} = \pm\sqrt{\frac{1+cos\,\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-cos\,\theta}{1+cos\,\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+cos\,\theta}$$

V.POWER REDUCING FORMULAS

$$sin^2\theta = \frac{1-\cos 2\theta}{2},\ cos^2\theta = \frac{1+cos\,2\theta}{2},\ tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

VI.PRODUCT TO SUM FORMULAS

$$\cos\alpha\cos\beta = \frac{1}{2}[cos\,(\alpha+\beta) + cos\,(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = \frac{1}{2}[cos\,(\alpha-\beta) - cos\,(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[sin\,(\alpha+\beta) + sin\,(\alpha-\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[sin\,(\alpha+\beta) - sin\,(\alpha-\beta)]$$

VII.SUM TO PRODUCT FORMULAS

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

ABOUT THE AUTHOR

Dr. Binayak Sahu,
M.Sc., M.Phil, Ph.D, DIT, LL.B.

The author is a highly experienced professor of Mathematics with more than 28 years of teaching, research & administrative experience. He has the specialization in mapping students mind & their learning attitude, taken all into analysis in preparing this book. He has taught Mathematics at various levels starting from higher secondary to Post Graduates across B.Tech. BCA, and other UG & PG students.

He is well qualified with M.Sc., M.Phil., Ph.D. (Mathematics), DIT, DCA – CEDTI (GoI) &LL.B as his degrees / Certificates. He is the life member of Indian Society of Theoretical &Applied Mechanics, IIT Kharagpur; SAE India, Chennai; Indian Science Congress Association, New Delhi; ISRD; INAEG; Indian Mathematical Society, Aurangabad. He has various other publications in journals & editorials. He has already authored five books..

He has also authored four books: Engineering Mathematics – I, II, Learn 51 Tricks of Multiplication in One Day, Learn Differential Equation in One Day, Learn Laplace Transformation in One Day, 99 Friends to win over Integration.